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1 Week 3

1.1 Lecture 1

1.1.1 Brownian Motion as the limit of a symmetric Random Walk

Need a good example

1.1.2 Poisson Process

Definition 1.1: Poisson Process

We say N_t is a Poisson process with parameter $\lambda > 0$ if

1) Starting at zero: $N_0 = 0$
2) Independent increment: $N_t - N_s \perp\!\!\!\perp \mathscr{F}_s$ for all $s < t$
3) Poisson increment: for $s < t$, $N_t - N_s \sim Poiss(\lambda(t-s))$, i.e.,
$$\mathbb{P}(N_t - N_s = n) = \frac{(\lambda(t-s))^n e^{-\lambda(t-s)}}{n!} \qquad n \geq 0$$
4) Cadlag path: (N_t) has cadlag path, i.e., right continuous with left limits

Remark 1.2

Brownian motion and Poisson process are two examples of Levy process

Remark 1.3: Mean, variance, martingale

By the property of Poisson distribution, we have, mean and variance of the increments are $\lambda(t-s)$, i.e.,

$$\mathbb{E}(N_t - N_s) = \lambda(t-s) = \mathbb{E}((N_t - N - s - \lambda(t-s))^2)$$

This implies, we have when $\lambda = 1$,

$$N_t - t \text{ is a martingale and also } (N_t - t)^2 - t \text{ is a martingale}$$

Definition 1.4: Levy Process

We say X_t is a Levy Process if

1) starting at zero: $X_0 = 0$
2) Independent increment: X has independent increment
3) Stationary increment: for $s < t$, $n > 0$, $X_{t+n} - X_{s+n} \sim X_t - X_s$
4) Cadlag path: X has cadlag path

1.1.3 Stopping Time

Definition 1.5: Stopping Time

A random variable with values in $\mathbb{T} \cup \{\infty\}$ is called an $(\mathscr{F}_t)_{t\in\mathbb{T}}$-stopping time if

$$\{\tau \leq t\} \in \mathscr{F}_t \qquad \text{for each } t \in \mathbb{T}$$

Proposition 1.6

If $\mathscr{F}_t = \mathscr{F}_{t+}$ for all t, then τ is a stopping time if and only if

$$\{\tau < t\} \in \mathscr{F}_t \qquad \forall t$$

Proof: Note that

$$\{\tau \leq t\} = \bigcap_n \{\tau < t + 1/n\} \in \mathscr{F}_{t+} = \mathscr{F}_t$$

and

$$\{\tau < t\} = \bigcup_n \{t \leq t - 1/n\} \in \mathscr{F} = \mathscr{F}_{t+}$$

□

Proposition 1.7

Let (X_t) be an $(\mathscr{F}_t)$-adapted continuous process and A be a closed set. Then,

$$\tau_A = \inf\{t \in \mathbb{T} : X_t \in A\}$$

is an $(\mathscr{F}_t)$-stopping time.

Proof: Let $\mathbb{T}_0 \subset \mathbb{T}$ be a dense subset such that $\inf \mathbb{T} \in \mathbb{T}_0$.

Since A is closed and X is continuous, for each $t \in \mathbb{T}$, we have,

$$\{\tau_A \leq t\} = \{\exists s \leq t, X_s \in A\} = \bigcap_{n=1}^{\infty} \bigcup_{s\geq t, s\in\mathbb{T}_0} \{X_s \in A_{1/n}\} \in \mathscr{F}_t$$

where $A_\varepsilon = \{x : d(x, A) < \varepsilon\}$. □

Remark 1.8

If A is open, then τ_A may not be an $(\mathscr{F}_t)$-stopping time but it is an $(\mathscr{F}_{t+})$-stopping time.

Example 1.9

Let (W_t) be a Brownian motion, and $a \in \mathbb{R}$. Let

$$\tau = \inf\{t \geq 0 : W_t = a\}$$

Then,

$$\{\tau \leq t\} = \{\exists s \leq t : W_s = a\} \in \mathscr{F}_t$$

Definition 1.10: σ-field of events observable at time τ

Let τ be an $(\mathscr{F}_t)$-stopping time. Define σ-field of events observable at time τ by

$$\mathscr{F}_\tau = \left\{ A \in \mathscr{F}_\infty = \sigma\left(\bigcup_{t \in \mathbb{T}} \mathscr{F}_t \right) : A \cap \{\tau \leq t\} \in \mathscr{F}_t \, \forall t \right\}$$

Proposition 1.11

1) $\mathscr{F}_\tau$ is a σ-field
2) If $\tau \leq \sigma$, then $\mathscr{F}_\tau \subset \mathscr{F}_\sigma$
3) Random variable τ is $\mathscr{F}_\tau$-measurable.

Proposition 1.12

Let τ,σ be stopping times. Then $\mathscr{F}_{\tau \wedge \sigma} = \mathscr{F}_\tau \cap \mathscr{F}_\sigma$ and events $\{\tau < \sigma\}, \{\sigma < \tau\}, \{\tau \leq \sigma\}, \{\sigma \leq \tau\}, \{\sigma = \tau\}$ are elements of $\mathscr{F}_{\tau \wedge \sigma}$.

1.1.4 Progressive measurability

Definition 1.13: Progressively measurable process

Process $(X_t)_{t\in\mathbb{T}}$ is progressively measurable with respect to $(\mathscr{F}_t)_{t\in\mathbb{T}}$ if for each $t\in\mathbb{T}$, the mapping $(s,\omega)\mapsto X_s(\omega)$ from $(-\infty,t]\cap\mathbb{T}\times\Omega$ to $\mathbb{R}$ is measurable with respect to $\mathscr{B}((-\infty,t]\cap\mathbb{T})\otimes\mathscr{F}_t$, i.e., $\forall t\in\mathbb{T},\forall A\in\mathscr{B}(\mathbb{R})$

$$\{(s,\omega)\in\mathbb{T}\times\Omega : s\le t, X_s(\omega)\in A\}\in\mathscr{B}((-\infty,t]\cap\mathbb{T})\otimes\mathscr{F}_t$$

Proposition 1.14

1) If (X_t) is progressively measurable with respect to $(\mathscr{F}_t)$ then (X_t) is $(\mathscr{F}_t)$-adapted

2) If (X_t) is $(\mathscr{F}_t)$-adapted and has right continuous paths (or left continuous path) a.s., then it is progressively measurable.

Definition 1.15: Stopped Process

If τ is a stopping time, $(X_t)_{t\in\mathbb{T}}$ is a stochastic process, then $(X_t^\tau)_{t\in\mathbb{T}}$ is a process stopped at τ by $X_t^\tau = X_{\tau\wedge t}$.

Theorem 1.16

Let (X_t) be a $(\mathscr{F}_t)$-progressively measurable process and τ be a stopping time.

Then the random variable $X_\tau \mathbb{1}_{\{\tau<\infty\}}$ is $\mathscr{F}_\tau$-measurable.

Moreover, the stopped process X^τ is progressively measurable.

1.1.5 Martingale: Maximal inequalities

Lemma 1.17: Doob's optional sampling for discrete tiem martingale

Let $(X_n, \mathscr{F}_n)_{0\leq n\leq N}$ be a martingale (resp. super-, sub-) and $0 \leq \tau \leq \sigma \leq N$ be two stopping times. Then,

$$\mathbb{E}(X_\sigma|\mathscr{F}_\tau) = X_\tau \qquad (resp. \leq, \geq)$$

Proof: We need to show that for all $A \in \mathscr{F}_\tau$

$$\mathbb{E}(X_\tau \mathbb{1}_A) = \mathbb{E}(X_\sigma \mathbb{1}_A)$$

First, let $A_k = A \cap \{\tau = k\}$ for $k = 0, 1, 2, \cdots, N$. We obtain

$$\begin{aligned}(X_\sigma - X_\tau)\mathbb{1}_{A_k} &= (X_\sigma - X_k)\mathbb{1}_{A_k} \\ &= \sum_{i=k}^{\sigma-1}(X_{i+1} - X_i)\mathbb{1}_{A_k} \\ &= \sum_{i=k}^{N}(X_{i+1} - X_i)\mathbb{1}_{A_k\cap\{\sigma>i\}}\end{aligned}$$

and thus,

$$\mathbb{E}((X_\sigma - X\tau)\mathbb{1}_{A_k}) = \sum_{i=k}^{N}\mathbb{E}[(X_{i+1} - X_i)\mathbb{1}_{A_k\cap\{\sigma>i\}}] = 0$$

since $A_k \cap \{\sigma > i\} = \{\tau = k\} \cap \{\sigma > i\} \in \mathscr{F}_i$.

Finally, we have,

$$\mathbb{E}[(X_\sigma - X_\tau)\mathbb{1}_A] = \sum_{k=0}^{N}\mathbb{E}[(X_\sigma - X_\tau)\mathbb{1}_{A_k}] = 0$$

$\square$

Remark 1.18

Above lemma is true for bounded stopping times.

Take $X_n = \sum_{k=1}^n \varepsilon_k$, and ε_k is iid with $\mathbb{P}(\varepsilon_n = \pm 1) = \frac{1}{2}$.

Take $\mathscr{F}_n = \sigma(\varepsilon_1, \cdots, \varepsilon_n)$, $\tau = 0$, $\sigma = \inf\{n : X_n = 1\}$. Then, $\mathbb{E}(X_\tau) = 0 \neq 1 = \mathbb{E}(X_\sigma)$

Lemma 1.19

Let $(X_n, \mathscr{F}_n)_{0\le n\le N}$ be a supermartingale. Then, for $\lambda \ge 0$ we have,

1)
$$\lambda \mathbb{P}\left(\max_{0\le n\le N} X_n \ge \lambda\right) \le \mathbb{E}\left[X_N \mathbb{1}_{\{\max_n X_n \ge \lambda\}}\right] \le \mathbb{E}X_N^+$$

2)
$$\lambda \mathbb{P}\left(\min_{0\le n\le N} X_n \ge -\lambda\right) \le \mathbb{E}\left[X_N \mathbb{1}_{\{\min_n X_n > -\lambda\}}\right] - \mathbb{E}X_0 \le \mathbb{E}X_N^+ - \mathbb{E}X_0$$

Corollary 1.20

Let $(X_n, \mathscr{F}_n)_{0\le n\le N}$ be a martingale or non-negative submartingale. Then,

1)
$$\forall p \ge 1, \forall \lambda \ge 0, \lambda^p \mathbb{P}\left(\max_n |X_n| \ge \lambda\right) \le \mathbb{E}|X_N|^p$$

2)
$$\forall p > 1, \mathbb{E}|X_N|^P \le \mathbb{E}\max_n |X_n|^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}|X_N|^p$$

Theorem 1.21

Suppose $(X_t, \mathscr{F}_t)_{t\in\mathbb{T}}$ is a cadlag martingale or a non-negative submartingale. Then

1) $\forall p \ge 1$, $\forall \lambda \ge 0$, $\lambda^p \mathbb{P}(\sup_t |X_t| \ge \lambda) \le \sup_t \mathbb{E}|X_t|^p$

2) $\forall p > 1$, $\sup_t \mathbb{E}|X_t|^p \le \mathbb{E}\sup_t |X_t|^p \le \left(\frac{p}{1-p}\right)^p \sup_t \mathbb{E}|X_t|^p$

Remark 1.22

If $t_{max} \in \mathbb{T}$, then
$$\sup_{t\in\mathbb{T}} \mathbb{E}|X_t|^p = \mathbb{E}|X_{t_{max}}|^p$$

Corollary 1.23

For $u, s > 0$ and Brownian motion (W_t) the following inequality holds,

$$\mathbb{P}\left(\sup_{0\le t\le s} W_t \ge u\right) \le e^{-\frac{u^2}{2s}}$$

1.1.6 Martingale Convergence Theorem

Definition 1.24: Downcrossings

Suppose that $I \subset \mathbb{R}$, $f : I \to \mathbb{R}$ and $\alpha < \beta$. If I is finite, then we define

$$\tau_1 = \inf\{t \in I : f(t) \geq \beta\}$$

$$\sigma_1 = \inf\{t \in I : t > \tau_1, f(t) \leq \alpha\}$$

Then by induction for $i = 1, 2, \cdots$

$$\tau_{i+1} = \inf\{t \in I : t > \sigma_i, f(t) \geq \beta\}$$

$$\sigma_{i+1} = \inf\{t \in I : t > \tau_{i+1}, f(t) \leq \alpha\}$$

The number of downcrossings of f across the interval $[\alpha, \beta]$ is given by

$$D_I(f, [\alpha, \beta]) := \sup\{j : \sigma_j < \infty\} \wedge 0$$

If I is infinite, we put

$$D_I(f, [\alpha, \beta]) = \sup\{D_F(f, [\alpha, \beta]) : F \subset I, finite\}$$

Lemma 1.25: Finite Downcrossing characterization of convergence

A sequence (x_n) converges to a limit in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ if and only if $D_{\mathbb{N}}((x_n), [\alpha, \beta])$ is finite for all $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$.

Lemma 1.26: Finite downcrossing implies limit exitsts for right continuous function

If $f : [a, b) \to \mathbb{R}$, $b \leq \infty$ is right continuous such that for all $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}$,

$$D_{[a,b]\cap\mathbb{Q}}(f, [\alpha, \beta]) < \infty$$

then $\lim_{t \to b} f(t)$ exists (not necessarily finite).

Lemma 1.27

Let $(X_t)_{t \in \mathbb{T}}$ be an $(\mathscr{F}_t)$-submartingale, and F be a countable subset of $\mathbb{T}$. Then

$$\mathbb{E}(D_F(X, [\alpha, \beta])) \leq \sup_{t \in F} \frac{\mathbb{E}(X_t - \beta)^+}{\beta - \alpha}$$

2 Week 5

2.1 Lecture 2

Lemma 2.1

For $0 \le t \le u \le T$, and $\mathbb{E}\left[\int_0^T X_t^2\,dt\right] < \infty$, then,

1)
$$\int_0^u X_s\,dW_s = \int_0^t X_s\,dW_s + \int_t^u X_s\,dWs$$

2)
$$\int_o^t X_s\,dW_s = \int_0^T \mathbb{1}_{[0,t]}(s) X_s\,dW_s$$

Proof: The mapping
$$(s, \omega) \mapsto \mathbb{1}_{[0,t]}(s)$$
is deterministic hence progressively measurable. Hence $\mathbb{1}_{[0,t]}(s)X_s$ is progressively measurable. Hence, $\mathbb{1}_{[0,t]}(s)X_s \in \mathcal{L}_T^2$.

Suppose that $X \in \mathcal{E}_T$, that is,

$$X = X_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^n X_{i-1} \mathbb{1}_{(t_{i-1}, t_i]}$$

with $t_n = T$ and so the process $\mathbb{1}_{[0,t]}(s)X_s$ is also an element of $\mathcal{E}_T$ and $\mathcal{E}_t$ as

$$X\mathbb{1}_{[0,t]} = X_0 \mathbb{1}_{\{0\}} + \sum_k X_{i-1} \mathbb{1}_{(t_{i-1} \wedge t, t_i \wedge t]}$$

with $t_n \wedge t = t$. Hence $X\mathbb{1}_{[0,t]} \in \mathcal{E}_T, \mathcal{E}_t$. Therefore, we have,

$$\int_0^T \mathbb{1}_{[0,t]}(s) X_s\,dW_s = \sum_k X_{i-1}(W_{t_i \wedge t \wedge T} - W_{t_{i-1} \wedge t \wedge T}) = \int_0^t X_s\,dW_s$$

For $X \in \mathcal{L}_T^2$, take $X^n \in \mathscr{E}_T$ such that $X^n \xrightarrow{\mathcal{L}^2} X$. Then $X^n \mathbb{1}_{[0,t]} \xrightarrow{\mathcal{L}^2} X\mathbb{1}_{[0,t]}$ Hence,

$$\int_0^T X_s \mathbb{1}_{[0,t]}\,dW_s = \lim_{n\to\infty} \int_0^T X_s^n \mathbb{1}_{[0,t]}\,dW_s = \lim_{n\to\infty} \int_0^t X_s^n\,dW_s = \int_0^t X_s\,dW_s$$

$\square$

Theorem 2.2: Stochastic Integral Martingale

Let $X \in \mathcal{L}^2(0,T)$. Then $(I_t(X))_{t\le T}$ is a square integrable martingale.

Proof: $I_t(X)$ is square integrable since

$$\mathbb{E}[I_t(X)^2] = \|\mathbb{1}_{[0,t]}X\|^2_{\mathcal{L}^2(0,T)} \le \|X\|^2_{\mathcal{L}^2(0,T)} < \infty$$

for each $t \le T$.

$I_t(X)$ is a martingale. First, for $X \in \mathcal{E}_T$,

$$X_t = X_0 \mathbb{1}_{\{0\}}(t) + \sum_k X_{k-1}\mathbb{1}_{(t_{k-1},t_k]}(t)$$

Suppose $\tilde{k}$ is such that $t_{\tilde{k}} \le t \le t_{\tilde{k}+1}$, then

$$I_t(X) = X_0(W_{t_1} - W_{t_0}) + X_1(W_{t_2} - W_{t_1}) + \cdots + X_{\tilde{k}}(W_t - W_{t_{\tilde{k}}})$$

And we compute for $s \le t$

$$\begin{aligned}
\mathbb{E}(I_t(X)|\mathscr{F}_s) &= \sum_{k=1}^{n} \mathbb{E}(X_{k-1}\mathbb{1}_{[0,t]}(W_{t_k} - W_{t_{k-1}})|\mathscr{F}_s) \\
&= \underbrace{\sum_{t_{k-1}<s} X_{t_{k-1}}\mathbb{1}_{[0,t]}(W_{t_k\wedge s} - W_{t_{k-1}})}_{=I_s(X)} + \underbrace{\sum_{s\ge t_{k-1}} \mathbb{E}[X_{t_{k-1}}\mathbb{1}_{[0,t]}(W_{t_k} - W_{t_{k-1}})|\mathscr{F}_s]}_{=0} \\
&= I_s(X)
\end{aligned}$$

For $X \in \mathcal{L}^2(0,T)$. We want to show that

$$\mathbb{E}\left(\int_0^t X_r\,dW_r|\mathscr{F}_s\right) = \int_0^s X_r\,dW_r$$

We know there exists $X^n \xrightarrow{\mathcal{L}^2} X$ and

$$\mathbb{E}\left(\int_0^t X_r\,dW_r|\mathscr{F}_s\right) = \lim_{n\to\infty}\mathbb{E}\left(\int_0^t X_r^n\,dW_r|\mathscr{F}_s\right) = \lim_{n\to\infty}\int_0^s X_r^n\,dW_r = \int_0^s X_r\,dW_r$$

$\square$

Lemma 2.3

Let $Y, Y_1, Y_2, \cdots$ be square integrable such that $Y_n \xrightarrow{\mathcal{L}^2} Y$. Then,

$$\mathbb{E}(Y_n|\mathscr{G}) \xrightarrow{\mathcal{L}^2} \mathbb{E}(Y|\mathscr{G})$$

Proof: By Jensen's inequality, we have,

$$(\mathbb{E}(Y_n|\mathscr{G}) - \mathbb{E}(Y|\mathscr{G}))^2 = (\mathbb{E}(Y_n - Y|\mathscr{G}))^2 \leq \mathbb{E}((Y_n - Y)^2|\mathscr{G})$$

This implies

$$\mathbb{E}((\mathbb{E}(Y_n|\mathscr{G}) - \mathbb{E}(Y|\mathscr{G}))^2) \leq \mathbb{E}(\mathbb{E}((Y_n - Y)^2|\mathscr{G})) \leq \mathbb{E}((Y_n - Y)^2) \to 0$$

Hence, $\mathbb{E}(Y_n|\mathscr{G}) \xrightarrow{\mathcal{L}^2} \mathbb{E}(Y|\mathscr{G})$. □

Definition 2.4

Let $\mathcal{M}_T^{2,c}$ be the <u>continuous square integrable martingales on $[0,T]$</u>.

Remark 2.5

Square integrability of $M \in \mathcal{M}_T^{2,c}$ means that

$$\sup_{t\leq T} \mathbb{E}M_t^2 < \infty$$

Since M is a martingale, Doob's inequality implies that

$$\sup_{t\leq T} \mathbb{E}M_t^2 \leq \mathbb{E}\sup_{t\leq T} M_t^2 \leq 4\mathbb{E}M_T^2$$

So <u>square integrability is equivalent to $\mathbb{E}M_T^2 < \infty$</u>.

Theorem 2.6

$\mathcal{M}_T^{2,c}$ is a Hilbert space with the inner product

$$(M, N) = \mathbb{E}(M_T N_T)$$

and the norm induced by the inner product

$$\|M\| = \sqrt{\mathbb{E}(M_T^2)} = \|M_T\|_{L^2}$$

Remark 2.7

The Ito integral for $X \in \mathcal{L}^2(0,T)$ can be seen as an element of $\mathcal{M}_T^{2,c} \ni (I_t(X))_{t \leq T}$.

Note that continuity of $I_t(X)$ for $X \in \mathcal{E}_T$ follows from the definition. More generally, it holds that

$$\mathcal{E}_T \to L^2(\Omega \times [0,T], \mathscr{P}, \mathbb{P} \otimes \lambda) =: \mathcal{L}^2(0,T) \xrightarrow{\text{Ito Isometry}} \mathcal{M}_T^{2,c}$$

Theorem 2.8

Let $X \in \mathcal{L}^2(0,T)$ and τ be a stopping time. Then

$$\mathbb{1}_{[\![0,\tau]\!]}X \in \mathcal{L}^2(0,T)$$

and

$$\int_0^t \mathbb{1}_{[\![0,\tau]\!]}X_s \, dW_s = \int_0^{t\wedge\tau} X_s \, dW_s \qquad \forall 0 \leq t \leq T$$

Corollary 2.9

For $X \in \mathcal{L}^2(0,T)$, the process (M_t) given by

$$\boxed{M_t = \left(\int_0^t X_s \, dW_s\right)^2 - \int_0^t X_s^2 \, ds}$$

is a martingale.

Proof: We know that $(I_t(X))_{t \leq T} \in \mathcal{M}_T^{2,c}$ so M is continuous, integrable and $M_0 = 0$.

Suppose $\tau \leq T$, **??** implies that

$$\begin{aligned}
\mathbb{E}\left(\int_0^\tau X_s \, dW_s\right)^2 &= \mathbb{E}\left(\int_0^T \mathbb{1}_{[\![0,\tau]\!]}(s) X_s \, dW_s\right)^2 \\
&= \mathbb{E}\left(\int_0^T \mathbb{1}_{[\![0,\tau]\!]}(s) X_s^2 \, ds\right) \\
&= \mathbb{E}\left(\int_0^\tau X_s^2 \, ds\right)
\end{aligned}$$

And so

$$\mathbb{E}(M_\tau) = \mathbb{E}\left[\left(\int_0^\tau X_s \, dW_s\right)^2 - \int_0^\tau X_s^2 \, ds\right] = 0 = \mathbb{E}(M_0)$$

and from Tutorial 4 Exercise 5, we know that

$$M \text{ is a martingale} \iff \mathbb{E}(M_\tau) = \mathbb{E}(M_0) \text{ for any } \tau = \begin{cases} s & \text{if } \omega \in A \\ t & \text{if } \omega \in A^c, \end{cases} \qquad A \in \mathscr{F}_{s \wedge t}$$

$\square$

3 Week 6

3.1 Lecture 1

Definition 3.1

Let $T \leq \infty$. We define the space of progressively measurable, locally square integrable processes by

$$\mathcal{L}^2_{loc}(0,T) = \left\{ (X_t)_{t<T} : X \text{ is progressively measurable, } \int_0^t X_s^2 \, ds < \infty \quad a.s. \text{ for } t < T \right\}$$

Lemma 3.2

For $X \in \mathcal{L}^2_{loc}(0,T)$, we define

$$\tau_n := \inf \left\{ t \geq 0 : \int_0^t X_s^2 \, ds \geq n \right\} \wedge T \wedge n \qquad n = 1, 2, \dots$$

Then (τ_n) is an increasing sequence of stopping times, $\tau_n \uparrow T$ a.s., and $\forall n$, $\mathbb{1}_{[\![0,\tau_n]\!]} X \in \mathcal{L}^2_{loc}(0,T)$.

Proof: τ_n is a stopping time as it is an entry time of continuous adapted process $\int_0^t X_s^2 \, ds$ into a closed set $[n, \infty)$.

Since $\int_0^t X_s^2 \, ds < \infty$ a.s. for all $t < T$, we get that $\tau_n \uparrow T$ a.s.

Process $\mathbb{1}_{[\![0,\tau_n]\!]} X$ is progressively measurable as a product of two progressively measurable processes.

Moreover, we have,

$$\mathbb{E}\left[\int_0^T \left(\mathbb{1}_{[\![0,\tau_n]\!]}(s) X_s \right)^2 \, ds \right] = \mathbb{E}\left[\int_0^{\tau_n} X_s^2 \, ds \right] \leq n < \infty$$

Hence $\mathbb{1}_{[\![0,\tau_n]\!]} X \in \mathcal{L}^2_{loc}(0,T)$. $\square$

Suppose that $\tau_n \uparrow T$ a.s. and $\mathbb{1}_{[\![0,\tau_n]\!]} X \in \mathcal{L}^2(0,T)$ for all n. Then define

$$\boxed{M_n(t) := \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]} X_s \, dW_s}$$

Lemma 3.3

For $m \geq n$, the processes $M_m^{\tau_n}$ and M_n are indistinguishable, that is,

$$\mathbb{P}(\forall t \leq T : M_m(t \wedge \tau_n) = M_n(t)) = 1$$

Proof: From the **??**, for $t \leq T$,

$$\begin{aligned}
M_m(\tau_n \wedge t) &= \int_0^{\tau_n \wedge t} \mathbb{1}_{[\![0,\tau_m]\!]}(s) X_s \, dW_s \\
&= \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]}(s) \mathbb{1}_{[\![0,\tau_m]\!]}(s) X_s \, dW_s \\
&= \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]}(s) X_s \, dW_s && (m \geq n) \\
&= M_n(t)
\end{aligned}$$

So $M_m^{\tau_n}$ is a modification of M_n, and we get that they are indistinguishable from continuity of $M_m^{\tau_n}$ and M_n. $\square$

Remark 3.4

There is a theorem about two continuous processes are modification implies they are indistinguishable.

Definition 3.5: Stochastic Integral for local processes

Let $X \in \mathcal{L}^2_{loc}(0,T)$ and $\tau_n \uparrow T$ such that $\mathbb{1}_{[\![0,\tau_n]\!]}X \in \mathcal{L}^2(0,T)$ for all n.

Then the stochastic integral $I(X) = \int X\,dW$ for $X \in \mathcal{L}^2_{loc}(0,T)$ is the process

$$(M_t)_{t<T} = \left(\int_0^t X_s\,dW_s\right)_{t<T}$$

such that

$$M_t^{\tau_n} = \int_0^{t\wedge\tau_n} X_s\,dW_s = \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]}(s) X_s\,dW_s$$

Proposition 3.6

The process M in 3.5 is continuous and unique.

Proof: By ??, for each $m \geq n$, there exists a null set $N_{n,m}$ such that $\mathbb{P}(N_{n,m}) = 0$ and $\forall \omega \notin N_{n,w}$, we have

$$M_n(t,\omega) = M_m(t \wedge \tau_n(\omega), \omega), \qquad \forall t < T$$

Let $N = \bigcup_{m>n} N_{n,m}$ Then $\mathbb{P}(N) = 0$ and $\forall \omega \notin N$, $t \leq \tau_n(\omega)$, the sequence $(M_m(t,\omega))_{m\geq n}$ is constant.

So we put (and it is well-defined)

$$M(t,\omega) := M_n(t,\omega) \qquad \text{for } t \leq \tau_n(\omega)$$

$\square$

Proposition 3.7

3.5 for $\int X\,dW$ does not depend on (τ_n), that is for $(\tau_n), (\tilde{\tau}_n)$ such that $\tau_n \uparrow T$, $\tilde{\tau}_n \uparrow T$ and $\mathbb{1}_{[\![0,\tau_n]\!]}X \in \mathcal{L}^2(0,T), \mathbb{1}_{[\![0,\tilde{\tau}_n]\!]}X \in \mathcal{L}^2(0,T)$ and M and $\tilde{M}$ are such as in 3.5, then $M, \tilde{M}$ are indistinguishable.

Theorem 3.8

If $X \in \mathcal{L}^2_{loc}(0,T)$, then for any stopping time τ, $\mathbb{1}_{[\![0,\tau]\!]}X \in \mathcal{L}^2_{loc}(0,T)$ and

$$\int_0^{t\wedge\tau} X\,dW = \int_0^t \mathbb{1}_{[\![0,\tau]\!]}X\,dW$$

Proof: The process $\mathbb{1}_{[\![0,\tau]\!]}X$ is progressively measurable and

$$\int_0^t (\mathbb{1}_{[\![0,\tau]\!]}X_s)^2\,ds \leq \int_0^t X_s^2\,ds < \infty$$

This implies $\mathbb{1}_{[\![0,\tau]\!]}X \in \mathcal{L}^2_{loc}(0,T)$.

Since $X \in \mathcal{L}^2_{loc}(0,T)$, there exists $\tau_n \uparrow T$ such that $\mathbb{1}_{[\![0,\tau_n]\!]}X \in \mathcal{L}^2(0,T)$. This implies $\mathbb{1}_{[\![0,\tau_n]\!]}\mathbb{1}_{[\![0,\tau]\!]}X \in \mathcal{L}^2(0,T)$. Define

$$M := \int X\,dW, \qquad N := \int \mathbb{1}_{[\![0,\tau]\!]}X\,dW$$

and note that

$$M_{t\wedge\tau_n} = \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]}X_s\,dW_s, \qquad N_{t\wedge\tau_n} = \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]}\mathbb{1}_{[\![0,\tau]\!]}X_s\,dW_s$$

Hence, we have,

$$M_{t\wedge\tau_n\wedge\tau} = \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]}\mathbb{1}_{[\![0,\tau]\!]}X_s\,dW_s = N_{t\wedge\tau_n}$$

Taking $n \to \infty$, we get

$$M_t^\tau = M_{t\wedge\tau} = N_t$$

$\square$

Definition 3.9: Local Martingale

If for an adapted process $M = (M_t)_{t<T}$, there exists a sequence of stopping time (τ_n) such that $\tau_n \uparrow T$ and M^{τ_n} is a martingale for all n.

Then M is called a <u>local martingale</u>.

If moreover, $M_n^\tau \in \mathcal{M}_T^{2,c}$, then we say that M is continuous, square integrable local martingale.

The class of such processes is denoted by $\mathcal{M}_{T,loc}^{2,c}$.

Proposition 3.10

Show that $M \in \mathcal{M}_{T,loc}^{c} \iff M \in \mathcal{M}_{T,loc}^{2,c}$, where $\mathcal{M}_{T,loc}^{c}$ is the family of continuous local martingale.

Proposition 3.11

Let $M = \int X\,dW$ for $X \in \mathcal{L}_{loc}^2(0,T)$. Then

1) M is continuous and $M_0 = 0$

2) $M \in \mathcal{M}_{T,loc}^{2,c}$

3) $X \mapsto \int X\,dW$ is linear

Proof: write own proof for part 1 and 2.

Part 3:

Take $X, Y \in \mathcal{L}_{loc}^2(0,T)$ and $\tau_n \uparrow T, \overline{\tau}_n \uparrow T$ such that $\mathbb{1}_{[\![0,\tau_n]\!]}X \in \mathcal{L}^2(0,T)$ and $\mathbb{1}_{[\![0,\overline{\tau}_n]\!]}Y \in \mathcal{L}^2(0,T)$ for all n.

Taking $\sigma_n := \tau_n \wedge \overline{\tau}_n \uparrow T$, we obtain $\mathbb{1}_{[\![0,\sigma_n]\!]}X, \mathbb{1}_{[\![0,\sigma_n]\!]}Y \in \mathcal{L}^2(0,T)$.

By linearity of $\mathcal{L}^2(0,T)$, we have

$$\mathbb{1}_{[\![0,\sigma_n]\!]}(aX + bY) \in \mathcal{L}^2(0,T), \qquad a, b \in \mathbb{R}$$

Hence, we have,

$$\begin{aligned}\int_0^t aX + bY\,dW &= \lim_{n\to\infty} \int_0^{t\wedge\sigma_n} aX + bY\,dW \\ &= \lim_{n\to\infty} a\int_0^{t\wedge\sigma_n} X\,dW + \lim_{n\to\infty} b\int_0^{t\wedge\sigma_n} Y\,dW \\ &= a\int_0^t X\,dW + b\int_0^t Y\,dW\end{aligned}$$

□

Theorem 3.12: Doob's inequality

For $X \in \mathcal{L}^2_{loc}(0,T)$ and stopping time $\tau \le T$, we have,

$$\mathbb{E}\left[\sup_{t<\tau}\left(\int_0^t X\,dW\right)^2\right] \le 4\mathbb{E}\left[\int_0^\tau X_s^2\,ds\right]$$

Proposition 3.13: Properties of Local Martingales

Local martingale have the following properties:

1) Each bounded local martingale is a martingale.

2) Each non-negative local martingale is a submartingale.

Theorem 3.14: Doob-Meyer Decomposition

For $M \in \mathcal{M}_T^{2,c}$, there exists process $Y = (Y_t)_{t\leq T}$ which has continuous, non-decreasing paths such that $Y_0 = 0$ and $M_t^2 - Y_t$ is a continuous martingale.

Moreover Y is unique.

Proof: [Uniqueness] Suppose Y, Z are two continuous, non-decreasing processes such that $M_t^2 - Y_t, M_t^2 - Z_t$ are continuous martingales.

Note that $Y_t - Z_t \in BV(0,T)$ and $Y_t - Z_t = (M_t^2 - Z_t) - (M_t^2 - Y_t)$ is a continuous martingale.

This implies $Y_t - Z_t$ is constant, hence $Y = Z$. □

Remark 3.15

There are three remarks:

1) Process Y from Doob-Meyer's decomposition is the quadratic variation of M denoted by
$$\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$$

2) For Brownian motion, the decomposition is $W_t^2 - t$ and $\langle W\rangle_t = t$

3) From **??**, for $X \in \mathcal{L}^2(0,T)$,
$$\left(\int_0^t X_s\, dW_s\right)^2 - \int_0^t X_s^2\, ds$$
is a martingale, and, from **??**
$$\left\langle \int_0^t X_s\, dW_s \right\rangle_t = \int_0^t X_s^2\, ds$$

4) For $M \in \mathcal{M}_T^{2,c}$, we have $t \mapsto \langle M\rangle_t(\omega)$ for all ω is non-decreasing, hence $\langle M\rangle_t \in BV(0,T)$.

   This implies $d\langle M\rangle_t(\omega)$ defines a finite measure on $[0,T]$ and $d\langle M\rangle_t(\omega)$ is atomless since M is continuous.

Definition 3.16

For an elementary process $X \in \mathcal{E}_T$,

$$X = \xi_0 \mathbb{1}_{\{0\}} + \sum_{k=1}^{n} \xi_{k-1} \mathbb{1}_{(t_{k-1}, t_k]}$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$, ξ_k is bounded and $\mathscr{F}_k$-measurable.

For $M \in \mathcal{M}_T^{2,c}$, we define

$$\int_0^t X \, dM := \sum_{k=1}^{n} \xi_{k-1}(M_{t_k \wedge t} - M_{t_{k-1} \wedge t}), \qquad t \leq T$$

Definition 3.17

We define

$$\mathcal{L}_T^2(M) := \left\{ X = (X_t)_{t<T} \text{ is progressively measurable such that } \mathbb{E}\left[\int_0^T X_s^2 \, d\langle M \rangle_s\right] < \infty \right\}$$

Remark 3.18

Instead of $L^2(\Omega \times [0,T], \mathscr{P}, \mathbb{P} \otimes \lambda)$, we take

$$L^2(\Omega \times [0,T], \mathscr{P}, \nu)$$

where ν is given by

$$\nu(A \times (a,b]) = \mathbb{E}\left[\int_0^T \mathbb{1}_A \mathbb{1}_{(a,b]}(s) \, d\langle M \rangle_s\right] = \mathbb{E}(\mathbb{1}_A(\langle M \rangle_b - \langle M \rangle_a))$$

for $a, b \in [0,T], a < b, A \in \mathscr{F}_a$. We may see ν as " $\mathbb{P} \otimes d\langle M \rangle_t$"

Proposition 3.19

Let $M \in \mathcal{M}_T^{2,c}$ and $X \in \mathcal{E}_T$. Then

$$I^M(X) = \int X\,dM \in \mathcal{M}_T^{2,c}, \qquad I_0^M(X) = 0$$

and

$$\|I^M(X)\|^2_{\mathcal{M}_T^{2,c}} = \mathbb{E}\left(\int_0^T X_s\,dM_s\right)^2 = \mathbb{E}\int_0^T X_s^2\,d\langle M\rangle_s = \|X\|_{\mathcal{L}_T^2(M)}$$

Proof: For $t_j \le t \le t_{j+1}$, we have

$$I_t(X) = \xi_0(M_{t_1} - M_{t_0}) + \cdots + \xi_j(M_t - M_{t_j})$$

For $t_j \le t \le u \le t_{j+1}$, we have

$$\mathbb{E}(I_u(X)|\mathscr{F}_t) - I_t(X) = \mathbb{E}[\xi_j(M_u - M_t)|\mathscr{F}_t] = \xi_j\mathbb{E}(M_u - M_t|\mathscr{F}_t) = 0$$

Hence, $I^M(X)$ is a martingale.

Moreover,

$$\mathbb{E}(I_T(X))^2 = \underbrace{\sum_{k=1}^n \mathbb{E}(\xi_{k-1}^2(M_{t_k} - M_{t_{k-1}})^2)}_{=:I_1} + \underbrace{2\sum_{j<k}\mathbb{E}[\xi_{k-1}\xi_{j-1}(M_{t_k} - M_{t_{k-1}})(M_{t_j} - M_{t_{j-1}})]}_{=:I_2}$$

For $s \le t$:

$$\begin{aligned}
\mathbb{E}[(M_t - M_s)^2|\mathscr{F}_s] &= \mathbb{E}(M_t^2|\mathscr{F}_s) - 2M_s\mathbb{E}(M_t|\mathscr{F}_s) + M_s^2 \\
&= \mathbb{E}(M_t^2 - \langle M\rangle_t|\mathscr{F}_s) + \mathbb{E}(\langle M\rangle_t|\mathscr{F}_s) - 2M_s\mathbb{E}(M_t|\mathscr{F}_s) + M_s^2 \\
&= M_s^2 - \langle M\rangle_s + \mathbb{E}(\langle M\rangle_t|\mathscr{F}_s) - 2M_s^2 + M_s^2 \\
&= \mathbb{E}(\langle M\rangle_t - \langle M\rangle_s|\mathscr{F}_s)
\end{aligned}$$

Hence,

$$\begin{aligned}
I_1 &= \sum_k \mathbb{E}(\xi_{k-1}^2\mathbb{E}[(M_{t_k} - M_{t_{k-1}})^2|\mathscr{F}_{k-1}]) \\
&= \sum_k \mathbb{E}(\xi_{k-1}^2\mathbb{E}[\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}|\mathscr{F}_{k-1}]) \\
&= \mathbb{E}\left[\sum_k \xi_{k-1}^2\left(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)\right] \\
&= \mathbb{E}\int_0^T X_s^2\,d\langle M\rangle_s
\end{aligned}$$

and

$$I_2 = 2\sum_{j<k}\mathbb{E}[\xi_{k-1}\xi_{j-1}(M_{t_j} - M_{t_{j-1}}\underbrace{\mathbb{E}((M_{t_k} - M_{t_{k-1}})|\mathscr{F}_{k-1})}_{=0})] = 0$$

$\square$

3.2 Lecture 2

Proposition 3.20

Let $M \in \mathcal{M}_T^{2,c}$. Then

1) For $X \in \mathcal{L}_T^2(M)$, the process $\int X\,dM \in \mathcal{M}_T^{2,c}$ and

$$\left\| \int X\,dM \right\|_{\mathcal{M}_T^{2,c}}^2 = \mathbb{E}\left(\int_0^T X_s\,dM_s \right)^2 = \mathbb{E}\int_0^T X_s^2\,d\langle M\rangle_s = \|X\|_{\mathcal{L}_T^2(M)}$$

2) If $X, Y \in \mathcal{L}_T^2(M)$, then $aX + bY \in \mathcal{L}_T^2(M)$ for all $a, b \in \mathbb{R}$ and

$$\int aX + bY\,dM = a\int X\,dM + b\int Y\,dM$$

Proposition 3.21

Let $M \in \mathcal{M}_T^{2,c}$ and τ be a stopping time, then $M^\tau \in \mathcal{M}_T^{2,c}$ and $\langle M^\tau\rangle = \langle M\rangle^\tau$

Corollary 3.22

Let $M \in \mathcal{M}_{loc}^c$. Then there exists unique process $\langle M\rangle$ with continuous non-decreasing paths, such that $\langle M\rangle_0 = 0$ and $M^2 - \langle M\rangle \in \mathcal{M}_{loc}^c$

Definition 3.23

For $T \leq \infty$, $M \in \mathcal{M}_{loc}^c$, we define

$$\mathcal{L}_{T,loc}^2(M) := \left\{ (X_t)_{t<T} : X \text{ is progressively measurable}, \int_0^t X_s^2\,d\langle M\rangle_s < \infty\,a.s., \forall t < T \right\}$$

Remark 3.24

We shall often suppose that $M_0 = 0$ since

$$\int X\,dM = \int X\,d(M - M_0)$$

and $\langle M - M_0\rangle = \langle M\rangle$.

Definition 3.25

Let $M \in \mathcal{M}^c_{loc}$, $M_0 = 0$, $X \in \mathcal{L}^2_{T,loc}(M)$ and (τ_n) be a localising sequence for M, i.e., $\tau_n \uparrow T$ nad $M^{\tau_n} \in \mathcal{M}^{2,c}_T$ and $\mathbb{1}_{[\![0,\tau_n]\!]} X \in \mathcal{L}^2_T(M^{\tau_n})$ for all n.

We call a stochastic integral $\int X \, dM$ such process

$$(N_t)_{t<T} = \left(\int_0^t X \, dM \right)_{t<T}$$

where

$$N_t^{\tau_n} = \int_0^t \mathbb{1}_{[\![0,\tau_n]\!]} X \, dM^{\tau_n}$$

for $n = 1, 2, \cdots$

Proposition 3.26

Let $M, N \in \mathcal{M}^c_{loc}$. Then

1) For $X \in \mathcal{L}^2_{T,loc}(M)$ then $\int X\,dM \in \mathcal{M}^c_{loc}$
2) For $X, Y \in \mathcal{L}^2_{T,loc}(M)$, then $aX + bY \in \mathcal{L}^2_{T,loc}(M)$ for all $a, b \in \mathbb{R}$ and
$$\int aX + bY\,dM = a\int X\,dM + b\int Y\,dM$$
3) For $X \in \mathcal{L}^c_{T,loc}(M) \cap \mathcal{L}^2_{T,loc}(N)$, $a, b \in \mathbb{R}$, then $X \in \mathcal{L}^2_{T,loc}(aM + bN)$ and
$$\int X\,d(aM + bN) = a\int X\,dM + b\int X\,dN$$

Theorem 3.27

Let $M \in \mathcal{M}^c_{loc}$, $X \in \mathcal{L}^2_T(M)$, τ be a stopping time. Then
$$\mathbb{1}_{[\![0,\tau]\!]}X \in \mathcal{L}^2_T(M), \quad X \in \mathcal{L}^2_T(M^\tau)$$
and
$$\int_0^t \mathbb{1}_{[\![0,\tau]\!]}(s)X_s\,dM_s = \int_0^{t\wedge\tau} X_s\,dM_s = \int_0^t X_s\,dM^\tau_s \qquad \forall t < T$$

3.3 Quadratic Variation

By **??**, for $M \in \mathcal{M}_T^{2,c}$, there exists $\langle M \rangle$ continuous, non-decreasing process such that

$$M_t^2 - \langle M \rangle_t$$

is a martingale. Recall the quadratic variation for M is

$$V_{\pi,t}^2(M) = \sum_{i=1}^{k} (M_{t_i} - M_{t_{i-1}})^2$$

with $\pi = \{0 \leq t_0 \leq t_1 \leq \cdots \leq t_k = t\}$

Theorem 3.28

Let M be a continuous bounded martingale. Then

$$V_{\pi,t}^2(M) \xrightarrow[|\pi|\to 0]{L^2} \langle M \rangle$$

Proof: Let $\pi_n = \{0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_{k_n}^{(n)} = t\}$ such that $|\pi_n| \to 0$, and put $C = \sup_{s \leq t} |M_s|$. Then,

$$\begin{aligned}
M_t^2 &= \left(\sum_{k=1}^{k_n} \left(M_{t_k}^{(n)} - M_{t_{k-1}}^{(n)} \right) \right)^2 \\
&= \sum_k \left(M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}} \right)^2 + 2 \sum_{k<j} \left(M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}} \right) \left(M_{t_j^{(n)}} - M_{t_{j-1}^{(n)}} \right) \\
&= V_{\pi_n,t}^2(M) + 2 \sum_j M_{t_{j-1}^{(n)}} \left(M_{t_j^{(n)}} - M_{t_{j-1}^{(n)}} \right) \\
&= V_{\pi_n,t}^2(M) + 2N_n(t)
\end{aligned}$$

Let $X_n(s) = \sum_{j=1}^{k_n} M_{t_{j-1}^{(n)}} \mathbb{1}_{\left(t_{j-1}^{(n)}, t_j^{(n)}\right]} \in \mathcal{E}_T$. Then

$$N_n(t) = \int_0^t X_n(s)\, dM_s$$

From continuity of M, we have $X_n(s) \to M_s$ for all $s \leq t$.

Since $|X_n| \leq C$, $|X_n - M|^2 \leq 4C^2$ and by DCT,

$$\mathbb{E} \int_0^T |X_n - M|^2 \, d\langle M \rangle_s \to 0 \implies X_n \xrightarrow{\mathcal{L}_t^2(M)} M$$

and so

$$N_n \xrightarrow{\mathcal{M}_T^{2,c}} \int M \, dM \implies N_n(t) \xrightarrow{L^2} \int_0^t M_s \, dM_s$$

and

$$V_{\pi_n,t}^2(M) = M_t^2 - 2N_n(t) \xrightarrow{L^2} M_t^2 - 2\int_0^t M \, dM$$

Note that the process $Y = M^2 - 2\int M\,dM$ is continuous and $M^2 - Y = 2\int M\,dM$ is a martingale.

$$Y_s \xleftarrow{L^2} V^2_{\pi_n,s}(M) \leq_{s\leq t} V^2_{\pi_n,t}(M) \xrightarrow{L^2} Y_t$$

$\square$

Remark 3.29: Decomposition of Bounded Martingale

M is a bounded martingale then,

$$M^2 = 2\int M\,dM + \langle M\rangle$$

Theorem 3.30

We have

1) $M \in \mathcal{M}^{2,c}_T \implies V^2_{\pi,t}(M) \xrightarrow[|\pi|\to 0]{L^1} \langle M\rangle_t$ for $t < T$

2) $M \in \mathcal{M}^2_{loc} \implies V^2_{\pi,t}(M) \xrightarrow[|\pi|\to 0]{\mathbb{P}} \langle M\rangle_t$ for $t < T$

Definition 3.31

Let $M, N \in \mathcal{M}^c_{loc}$. The process $\langle M, N\rangle$ is defined as

$$\langle M, N\rangle = \frac{1}{4}\left[\langle M+N\rangle - \langle M-N\rangle\right]$$

Proposition 3.32

We have

1) $M, N \in \mathcal{M}^{2,c}_T$, then $\langle M, N\rangle$ is a unique continuous finite variation on $[0,T]$ process such that $\langle M, N\rangle_0 = 0$ and $MN - \langle M, N\rangle$ is a martingale on $[0,T]$

2) $M, N \in \mathcal{M}^2_{loc}$ then $\langle M, N\rangle$ is a unique finite variation on $[0,T]$ process such that $\langle M, N\rangle_0 = 0$ and $MN - \langle M, N\rangle$ is a local martingale on $[0,T)$.

4 Week 7

4.1 Lecture 1: Predictable Brackets

Proposition 4.1

Let $\pi_n = (t_0^{(n)}, \dots, t_{k_n}^{(n)})$ be a sequence of paritions of $[0,t]$ such that $0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_{k_n}^{(n)} = t$ and $|\pi_n| \to 0$. Then,

1) For $M, N \in \mathcal{M}_T^{2,c}$ and $t < T$

$$\sum_{k=1}^{k_n} \left(M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}}\right)\left(N_{t_k^{(n)}} - N_{t_{k-1}^{(n)}}\right) \xrightarrow{L^1} \langle M, N\rangle_t$$

2) For $M, N \in \mathcal{M}_{T,loc}^{2,c}$ and $t < T$

$$\sum_{k=1}^{k_n} \left(M_{t_k^{(n)}} - M_{t_{k-1}^{(n)}}\right)\left(N_{t_k^{(n)}} - N_{t_{k-1}^{(n)}}\right) \xrightarrow{\mathbb{P}} \langle M, N\rangle_t$$

Proposition 4.2: Six Properties of Covariation

For $M, N \in \mathcal{M}_{loc}^c$

1) is itself: $\langle M, M\rangle = \langle M\rangle = \langle -M\rangle$

2) Symmetry: $\langle M, N\rangle = \langle N, M\rangle$

3) independet of initial condition:

$$\langle M, N\rangle = \langle M - M_0, N\rangle = \langle M, N - N_0\rangle = \langle M - M_0, N - N_0\rangle$$

4) Bilinearity:

$$\begin{aligned}\langle M_1 + M_2, N_1 + N_2\rangle &= \langle M_1 + M_2, N_1\rangle + \langle M_1 + M_2, N_2\rangle \\ &= \langle M_1, N_1\rangle + \langle M_2, N_1\rangle + \langle M_1, N_2\rangle + \langle M_2, N_2\rangle\end{aligned}$$

5) Stopping:

$$\langle M^\tau, N^\tau\rangle = \langle M^\tau, N\rangle = \langle M, N^\tau\rangle = \langle M, N\rangle^\tau$$

6) Integral: for $X \in \mathcal{L}_{T,loc}^2(M), Y \in \mathcal{L}_{T,loc}^2(N)$, we have

$$\left\langle \int X\, dM, \int Y\, dN \right\rangle = \int XY\, d\langle M, N\rangle$$

Theorem 4.3: Stochastic Dominated Convergence Theorem

Suppose $M \in \mathcal{M}^{2,c}_{loc}$, X_n are progressively measurable such that

$$\lim_{n\to\infty} X_{n,t}(\omega) = X_t(\omega) \qquad \forall t < T, \omega \in \Omega$$

If $\forall t < T, \omega \in \omega$, $|X_{n,t}(\omega)| \leq Y_t(\omega)$ where $Y \in \mathcal{L}^2_{T,loc}(M)$, then $X_n, X \in \mathcal{L}^2_{T,loc}(M)$ and

$$\int_0^t X_n \, dM \xrightarrow{\mathbb{P}} \int_0^t X \, dM$$

Definition 4.4: Locally bounded process

X is locally bounded if there exist a sequence of stopping times (τ_n) such that $\tau_n \uparrow T$ and $X^{\tau_n} - X_0$ is bounded for all n.

Proposition 4.5: Continuity implies local boundedness

Continuity implies local boundedness.

Proof: □

Theorem 4.6: Change of Integrator

1) $N \in \mathcal{M}_T^{2,c}, X \in \mathcal{L}_T^2(N)$, Y is progressively measurable and bounded,

$$M = \int X\, dN$$

Then $Y \in \mathcal{L}_T^2(M)$, $XY \in \mathcal{L}_T^2(N)$ and

$$\int Y\, dM = \int YX\, dN$$

2) $N \in \mathcal{M}_{loc}^2, X \in \mathcal{L}_{T,loc}^2(N)$, Y is progressively measurable, locally bounded,

$$M = \int X\, dN$$

Then, $Y \in \mathcal{L}_{T,loc}^2(M)$, $XY \in \mathcal{L}_{T,loc}^2(N)$ and

$$\int Y\, dM = \int YX\, dN$$

Proof: Part(a):
Suppose that $Y \in \mathcal{E}_T$, i.e.,

$$Y = \xi_0 \mathbb{1}_{\{0\}} + \sum_{k=1}^{n-1} \xi_k \mathbb{1}_{(t_k, t_{k+1}]}$$

with $0 = t_0 < t_1 < \ldots < t_k < T$, ξ_k is bounded, $\mathscr{F}_{t_k}$-measurable. Then,

$$\begin{aligned}
\int_0^t Y\, dM &= \sum_k \xi_k \left(M_{t_{k+1}\wedge t} - M_{t_k \wedge t}\right) \\
&= \sum_k \xi_k \left(\int_0^t \mathbb{1}_{[0,t_{k+1}]} X\, dN - \int_0^t \mathbb{1}_{[0,t_k]} X\, dN\right) \\
&= \sum_k \xi_k \int_0^t \mathbb{1}_{[t_k,t_{k+1}]} X\, dN \\
&= \int_0^t \sum_k \xi_k \mathbb{1}_{[t_k,t_{k+1}]} X\, dN \\
&= \int_0^t YX\, dN
\end{aligned}$$

If Y is bounded progressively measurable, then

$$\begin{aligned}
\mathbb{E} \int_0^T Y_s^2\, d\langle M\rangle_s &\leq \|Y\|_\infty^2 \mathbb{E} \int_0^T d\langle M\rangle_s \\
&= \|Y\|_\infty^2 \mathbb{E}\langle M\rangle_T \\
&= \|Y\|_\infty^2 \mathbb{E} M_T^2 \\
&< \infty
\end{aligned}$$

hence $Y \in \mathcal{L}_T^2(M)$.

There are $Y_n \in \mathcal{E}_T$ such that $Y_n \xrightarrow{\mathcal{L}_T^2(M)} Y$ and we may suppose that $\|Y_n\|_\infty \leq \|Y\|_\infty$. Note that

$$\begin{aligned}
\|XY - XY_n\|_{\mathcal{L}_T^2(N)}^2 &= \mathbb{E}\int_0^T (XY - XY_n)^2 \, d\langle N\rangle \\
&= \mathbb{E}\int_0^T (Y - Y_n)^2 X^2 \, d\langle N\rangle \\
&= \mathbb{E}\int_0^T (Y - Y_n)^2 \, d\langle M\rangle \\
&= \|Y - Y_n\|_{\mathcal{L}_T^2(M)}^2 \to 0
\end{aligned}$$

So $Y_n X \xrightarrow{\mathcal{L}_T^2(N)} YX$, hence

$$\int_0^t XY \, dN \xleftarrow{L^2} \int_0^t XY_n \, dN = \int_0^t Y_n \, dM \xrightarrow{L^2} \int_0^t Y \, dM$$

$$\implies \int_0^t YX \, dN = \int_0^t Y \, dM$$

Part(b):
Since

$$\int_0^t Y_0 \, dM = Y_0 M_t = Y_0 \int_0^t X \, dN = \int_0^t Y_0 X \, dN$$

So it is enough to consider $Y - Y_0$ instead of Y, we ay assume that $Y_0 = 0$.

Let $\tau_n \uparrow T$ such that Y^{τ_n} is bounded, $N^{\tau_n} \in \mathcal{M}_T^{2,c}$ and $X\mathbb{1}_{[\![0,\tau_n]\!]} \in \mathcal{L}_T^2(N_n^\tau)$, and note that

$$M^{\tau_n} = \left(\int X \, dN\right)^{\tau_n} = \int X\mathbb{1}_{[\![0,\tau_n]\!]} \, dN^{\tau_n}$$

So by Part (a), we have

$$\begin{aligned}
\left(\int Y \, dM\right)^{\tau_n} &= \int Y\mathbb{1}_{[\![0,\tau_n]\!]} \, dM^{\tau_n} \\
&= \int Y\mathbb{1}_{[\![0,\tau_n]\!]} X\mathbb{1}_{[\![0,\tau_n]\!]} \, dN_n^\tau \\
&= \int YX\mathbb{1}_{[\![0,\tau_n]\!]} \, dN^{\tau_n} \\
&= \left(\int YX \, dN\right)^{\tau_n}
\end{aligned}$$

We get the claim by taking $n \to \infty$. Since $\forall t < T$, $\forall \omega \in \overline{\Omega}$ with $\mathbb{P}(\overline{\Omega}) = 1$, there exists $N(\omega)$ such that $\forall n \geq N(\omega)$ we have $\tau^n(\omega) > t$ and so

$$\left(\int_0^t Y \, dM\right)(\omega) = \left(\int_0^t Y \, dM\right)^{\tau^n(\omega)}(\omega) = \left(\int_0^t YX \, dN\right)^{\tau^n(\omega)}(\omega) = \left(\int_0^t YX \, dN\right)(\omega)$$

$\square$

4.2 Integration by Parts

Theorem 4.7: Decomposition of product of local martingale

For $M, N \in \mathcal{M}^c_{loc}$,

$$M_t N_t = M_0 N_0 + \int_0^t M_s \, dN_s + \int_0^t N_s \, dM_s + \langle M, N \rangle_t$$

Proof: The integrals $\int M \, dN, \int N \, dM$ are well-defined since M, N are continuous hence locally bounded.

We can assume that $M_0 = N_0 = 0$ since we have,

- $\langle M, N \rangle = \langle M - M_0, N - N_0 \rangle$
- $\int M \, dN = \int M \, d(N - N_0) = \int M - M_0 \, d(N - N_0) + M_0(N - N_0)$
- $\int N \, dM = \int N \, d(M - M_0) = \int N - N_0 \, d(M - M_0) + N_0(M - M_0)$

So we have

$$\begin{aligned}
0 &= M_0 N_0 + \int_0^t M_s \, dN_s + \int_0^t N_s \, dM_s + \langle M, N \rangle_t - M_t N_t \\
&= M_0 N_0 - M_t N_t + \int M_s - M_0 \, d(N_s - N_0) + M_0(N_t - N_0) \\
&+ \int N_s - N_0 \, d(M_s - M_0) + N_0(M_t - M_0) + \langle M - M_0, N - N_0 \rangle_t \\
&= \int_0^t (M_s - M_0) \, d(N_s - N_0) + \int_0^t (N_s - N_0) \, d(M_s - M_0) + \langle M - M_0, N - N_0 \rangle_t \\
&- (M_t - M_0)(N_t - N_0)
\end{aligned}$$

So it is enough to show the claim for $M = N$ with $M_0 = 0$, i.e.,

$$M_t^2 = 2 \int_0^t M_s \, dM_s + \langle M \rangle_t$$

since, if this is satisfied, then we can apply it to $M + N$ and $M - N$ and by subtracting and dividing by 4 we the original claim.

We already show this for bounded martingale in 3.29

In general case, we put

$$\tau_n = \inf\{t > 0 : |M_t| \geq n\} \wedge T$$

then $\tau_n \uparrow T$ and M^{τ_n} is bounded local martingale hence a bounded martingale by 3.13.

And we have the following:

$$\begin{aligned}
(M^2)^{\tau_n} &= (M^{\tau_n})^2 \\
&= 2\int M^{\tau_n}\, dM^{\tau_n} + \langle M^{\tau_n}\rangle \\
&= 2\int M^{\tau_n} \mathbb{1}_{[\![0,\tau_n]\!]}\, dM + \langle M\rangle^{\tau_n} \\
&= \left(2\int M\, dM + \langle M\rangle\right)^{\tau_n}
\end{aligned}$$

Taking $n \to \infty$, we obtain the claim. □

Corollary 4.8

For $M \in \mathcal{M}^c_{loc}$

$$\int_0^t M_s\, dM_s = \frac{1}{2}(M_t^2 - M_0^2) - \frac{1}{2}\langle M\rangle_t$$

Corollary 4.9

For $X, Y \in \mathcal{L}^2_{T,loc}(W)$, $M = \int X\, dW, N = \int Y\, dW$, then,

$$\begin{aligned}
M_t N_t &= \int_0^t M_s\, dM_s + \int_0^t N_s\, dM_s + \langle M, N\rangle_t \\
&= \int_0^t M_s Y_s\, dW_s + \int_0^t N_s X_s\, dW_s + \int_0^t X_s Y_s\, ds
\end{aligned}$$

Definition 4.10: Continuous adapted process with bounded variation path

Denote $\mathcal{V}^c$ as the space of continuous adapted process with paths in $BV[0,t]$ for all $t < T$.

Proposition 4.11

For $M \in \mathcal{M}^c_{loc}$, $A \in \mathcal{V}^c$. Then

$$M_t A_t = M_0 A_0 + \int_0^t A_s \, dM_s + \int_0^t M_s \, dA_s$$

Proof: We can suppose that $M_0 = A_0 = 0$.

Assume for now that M, A are bounded. We have the following telescope sum:

$$\begin{aligned} M_t A_t &= \sum_{j=1}^n \left(M_{tj/n} - M_{t(j-1)/n}\right) \sum_{k=1}^n \left(A_{tk/n} - A_{t(k-1)/n}\right) \\ &= \underbrace{\sum_{j=1}^n \left(M_{tj/n} - M_{t(j-1)/n}\right) \left(A_{tk/n} - A_{t(k-1)/n}\right)}_{:=a_n} \\ &+ \underbrace{\sum_{j=1}^n M_{t(j-1)/n} \left(A_{tk/n} - A_{t(k-1)/n}\right)}_{:=b_n} \\ &+ \underbrace{\sum_{j=1}^n A_{t(j-1)/n} \left(M_{tj/n} - M_{t(j-1)/n}\right)}_{:=c_n} \end{aligned}$$

For b_n, it tends to $\int_0^t M_s \, dA_s$ a.s. by the definition of Riemann-Stieltjes integral.

Note let

$$A_n = \sum_{j=1}^n A_{t(j-1)/n} \mathbb{1}_{(t(j-1)/n, tj/n]} \in \mathcal{E}_T$$

and we have $A_n \xrightarrow{\mathcal{L}^2_T(M)} A$ so $c_n \xrightarrow{L^2} \int A \, dM$.

For a_n, we have

$$\begin{aligned} |a_n^2| &\leq \sum_{j=1}^n \left(M_{tj/n} - M_{t(j-1)/n}\right)^2 \sum_{k=1}^n \left(A_{tk/n} - A_{t(k-1)/n}\right)^2 \\ &\leq \underbrace{\sum_{j=1}^n \left(M_{tj/n} - M_{t(j-1)/n}\right)^2}_{\xrightarrow{\mathbb{P}} \langle M \rangle_t} \underbrace{\sup_{1 \leq k \leq n} \left|A_{tk/n} - A_{t(k-1)/n}\right|}_{\xrightarrow{a.s.} 0 \text{ by continuity}} \underbrace{\sum_{k=1}^n \left|A_{tk/n} - A_{t(k-1)/n}\right|}_{\leq V^{(1)}_{[0,t]}(A) < \infty} \end{aligned}$$

Hence $|a_n|^2 \xrightarrow{\mathbb{P}} 0$, $a_n \xrightarrow{\mathbb{P}} 0$. Hence, we have,

$$M_t A_t = a_n + b_n + c_n \xrightarrow{\mathbb{P}} \int_0^t M_s \, dA_s + \int_0^t A_s \, dM_s$$

If M, A are not bounded, we define

$$\tau_n = \inf\{t > 0 : |M_t| \geq n\} \wedge \inf\{t > 0 : |A_t| \geq n\} \wedge T$$

So $|M^{\tau_n}| \leq n, |A^{\tau_n}| \leq n$ and combining with the bounded case, we have,

$$(MA)^{\tau_n} = \int A^{\tau_n} \, dM^{\tau_n} + \int M^{\tau_n} \, dA^{\tau_n} = \left(\int A \, dM + \int M \, dA \right)^{\tau_n}$$

taking $n \to \infty$, we conclude the proof. $\square$

Proposition 4.12

For $A, B \in \mathcal{V}^c$. Then

$$A_t B_t = A_0 B_0 + \int_0^t A_s \, dB_s + \int_0^t B_s \, dA_s$$

4.3 Continuous semimartingale

Definition 4.13: Continuous Semimartingale

Process $Z = (Z_t)_{t<T}$ is called a continuous semimartingale if it can be decomposed as

$$Z = Z_0 + M + A$$

where Z_0 is $\mathscr{F}_0$-measurable random variable, $M \in \mathcal{M}^{2,c}_{loc}$, $A \in \mathcal{V}^c$, and $M_0 = A_0 = 0$.

Remark 4.14

Semimartingale decomposition is unique. As $V \in \mathcal{M}^c_{loc} \cap BV \iff V$ is constant hence zero.

Example 4.15: Ito Process

Ito process, i.e., process of the form

$$Z = Z_0 + \int X\, dW + \int Y\, ds$$

where $X \in \mathcal{L}^2_{T,loc}$, Y is progressively measurable such that $\int_0^t |Y_s|\, ds < \infty$ a.s. for all $t < T$.

Ito process is a semimartingale.

Example 4.16: M^2

From Doob-Meyer decomposition of M^2, i.e.,

$$M^2 = 2 \int M\, dM + \langle M \rangle$$

is a semimartingale.

Definition 4.17: Integral w.r.t. continuous semimartingale

If $Z = Z_0 + M + A$ is a continuous semimartingale, then

$$\int X\, dZ := \underbrace{\int X\, dM}_{\text{stochastic integral}} + \underbrace{\int X\, dA}_{\text{Stieltjes Integral}}$$

Theorem 4.18: Integration by Parts

If $Z = Z_0 + M + A$ and $Z' = Z'_0 + M' + A'$ are continuous semimartingales, then ZZ' is a continuous semimartingale and

$$ZZ' = Z_0 Z'_0 + \int Z\, dZ' + \int Z'\, dZ + \langle M, M' \rangle$$

Definition 4.19: Predictable Brackets for Continuous Semimartingale

For $Z = Z_0 + M + A, Z' = Z'_0 + M' + A'$ are continuous semimartingales, then

$$\langle Z, Z' \rangle = \langle M, M' \rangle$$

Remark 4.20

If $Z = Z_0 + M + A$ and $Z' = Z'_0 + M' + A'$ are continuous semimartingales, then we have,

$$ZZ' = Z_0 Z'_0 + \underbrace{\int Z\, dM' + \int Z'\, dM}_{\text{local martingale}} + \underbrace{\int Z\, dA' + \int Z'\, dA + \langle M, M' \rangle}_{\text{finite variation}}$$

Hence, ZZ' is also a continuous semimartingale.

4.4 Lecture 2

4.4.1 Ito's fomula; Ito's lemma

Theorem 4.21: Ito's formula; Ito's lemma

Suppose $Z = Z_0 + M + A$ is a continuous semimartingale, $f \in C^2(\mathbb{R})$. Then $f(Z)$ is a semimartingale and

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s)\, dZ_s + \frac{1}{2}\int_0^t f''(Z_s)\, d\langle M\rangle_s \qquad (*)$$

Proof: □

Corollary 4.22: Ito's formula on Brownian Motion

For $f \in C^2(\mathbb{R})$, we have

$$f(W_t) = f(0) + \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, ds$$

Theorem 4.23: d-dim Ito's formula

uppose $f \in C^2(\mathbb{R}^d, \mathbb{R})$ and $Z = \left(Z^{(1)}, \dots, Z^{(d)}\right)$ where $Z^{(i)} = Z_0^{(i)} + M^{(i)} + A^{(i)}$ are continuous semimartingale. Then $f(Z)$ is a semimartingale and

$$f(Z_t) = f(Z_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(Z_s)\, dZ_s^{(i)} + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Z_s)\, d\langle M^{(i)}, M^{(j)}\rangle_s$$

4.4.2 Levy's Characterization of Brownian Motion

Theorem 4.24: Levy's Characterization of Brownian Motion

Suppose that $M \in \mathcal{M}^c_{loc}$ such that $M_0 = 0$ and $M_t^2 - t \in \mathcal{M}^c_{loc}$. Then M is a Brownian Motion.

Remark 4.25: Importance of continuity

Assumption that M is continuous is fundamental. Otherwise take $M_t = N_t - t$, N_t is a Poisson process with $\lambda = 1$. Then, M_t and $M_t^2 - t$ are martingale. But M_t is not a BM.

Theorem 4.26: d-dim Levy Characterization

Suppose $M^{(1)}, \ldots, M^{(d)} \in \mathcal{M}^c_loc$ such that $M_0^{(i)} = 0$ and

$$M_t^{(i)} M_t^{(j)} - \delta_{i,j} t \in \mathcal{M}^c_{loc}$$

for $0 \leq i, j \leq d$. Then $M = (M^{(1)}, \ldots, M^{(d)})$ is d-dim BM.

5 Week 8

5.1 Lecture 1

5.1.1 Exponential martingale characterization of Brownian Motion

Theorem 5.1: Exponential Martingale Characterization of BM

Suppose that M is continuous, adapted and $M_0 = 0$. Then M is a Brownian motion if and only if $\forall \lambda \in \mathbb{R}$, $\exp\left(\lambda M_t - \frac{\lambda^2 t}{2}\right)$ is a local martingale.

Proof: ($\implies$)
This direction is already been proved.

($\impliedby$)
We show that $\exp\left(\lambda M_t - \frac{\lambda^2 t}{2}\right)$ implies $M \in \mathcal{M}^2_{loc}$ and $M^2 - t \in \mathcal{M}^2_{loc}$. Then, we use Levy's Characterization of Brownian motion to finish the proof.

First, we define

$$\tau_n = \inf\{t > 0 : |M_t| \geq n\} \wedge n$$

Then, $\tau_n \uparrow \infty$ and $\forall \lambda$, the process

$$X_t(\lambda) = \exp\left(\lambda M_{t\wedge\tau_n} + \frac{\lambda^2 (t \wedge \tau_n)}{2}\right)$$

is a bounded local martingale. Hence, $X_t(\lambda)$ is a bounded martingale such that $0 \leq X_t(\lambda) \leq e^{|\lambda| n}$. Hence, by the martingale property (conditional expectation), we have,

$$\mathbb{E}(X_t(\lambda)\mathbb{1}_A) = \mathbb{E}(X_s(\lambda)\mathbb{1}_A) \qquad \forall s < t, \forall A \in \mathscr{F}_s$$

Note that $X_t(0) = 1$ and let $|\lambda| \leq \lambda_0$, then

$$\left|\frac{dX_t(\lambda)}{d\lambda}\right| = |X_t(\lambda)(M_{t\wedge\tau_n} - \lambda t \wedge \tau_n)| \leq e^{\lambda_0 n}(n + \lambda_0 n)$$

Then, use the definition of derivative and by DCT, we get for $s < t$, $A \in \mathscr{F}_s$,

$$\begin{aligned}
\mathbb{E}[X_t(\lambda)(M_{t\wedge\tau_n} - \lambda t \wedge \tau_n)] &= \lim_{h\to 0} \mathbb{E}\left[\frac{1}{h}\left(X_t(\lambda+h) - X_t(\lambda)\right)\mathbb{1}_A\right] \\
&= \lim_{h\to 0} \mathbb{E}\left[\frac{1}{h}\left(X_s(\lambda+h) - X_s(\lambda)\right)\mathbb{1}_A\right] \\
&= \mathbb{E}[X_s(\lambda)(M_{s\wedge\tau_n} - \lambda s \wedge \tau_n)]
\end{aligned}$$

For $\lambda = 0$, we have

$$\mathbb{E}(M_{t\wedge\tau_n}) = \mathbb{E}(M_{s\wedge\tau_n})$$

Hence, $M \in \mathcal{M}^c_{loc}$.

Also note that for the second-order derivative, we have,

$$\left| \frac{d^2 X_t(\lambda)}{d\lambda^2} \right| = \left| X_t(\lambda) \left[M_{t\wedge\tau_n} - t \wedge \tau_n \right]^2 - t \wedge \tau_n \right| \leq e^{\lambda_0 n} \left[(n + \lambda_0)^2 + n \right]$$

Similarly, using the definition of derivative and DCT, we have,

$$\mathbb{E} \left[\left(X_t(\lambda) \left[M_{t\wedge\tau_n} - t \wedge \tau_n \right]^2 - t \wedge \tau_n \right) \mathbb{1}_A \right] = \mathbb{E} \left[\left(X_s(\lambda) \left[M_{s\wedge\tau_n} - s \wedge \tau_n \right]^2 - t \wedge \tau_n \right) \mathbb{1}_A \right]$$

and taking $\lambda = 0$, we have $\left(M^2_{t\wedge\tau_n} - t \wedge \tau_n \right)_{t\geq 0}$ is a martingale, hence $M^2 - t \in \mathcal{M}^c_{loc}$.

The claim follows by Levy's theorem. $\square$

5.1.2 Itô-Tanaka Formula

Itô's formula for $f \in C^2$, e.g.,

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, ds$$

For f is convex function, e.g., $f(x) = |x|$, this implies

$$|W_t| = \underbrace{\int_0^t sgn(W_s)\, dW_s}_{B_t} + \underbrace{L_t}_{\text{local time}}, \qquad \text{where } sgn(W_s) = \begin{cases} 1 & W_s > 0 \\ -1 & W_s \le 0 \end{cases}$$

Since we have $\langle B \rangle_t = t$, hence B_t is a Brownian Motion.

Remark 5.2: Left derivative

We have

$$f'_\ell(x) = \lim_{h\to 0, h\geq 0} \frac{f(x) - f(x-h)}{h}$$

For $f(x) = |x|$, we have,

$$f'_\ell(x) = sgn(x) = \begin{cases} 1 & x > 0 \\ -1 & x \le 0 \end{cases}$$

Theorem 5.3: Second derivative of Convex function is a measure

If $f : \mathbb{R} \to \mathbb{R}$ is convex, then $f'_\ell(x)$ exists for every $x \in \mathbb{R}$.

The second derivative of a convex function is a positive measure μ given by

$$\int_{\mathbb{R}} \varphi(x)\mu(dx) = -\int_{\mathbb{R}} \varphi'(x) f'_\ell(x)\, dx \qquad \forall \varphi \in C_0^\infty(\mathbb{R})$$

Suppose $f \in C^2, \varphi \in C_0^\infty(\mathbb{R})$, where

$$C_0^\infty(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi(n) \text{ is continuous } \forall n,\ \varphi \text{ vanishes outside a bounded interval}\}$$

Using integration by parts, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f''(x)\varphi(x)\, dx &= f'(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\varphi'(x)\, dx \\ &= -\int_{-\infty}^{\infty} f'(x)\varphi'(x)\, dx \end{aligned}$$

So for more general f, we could identify f'' with measure μ such that

$$\int_{\mathbb{R}} \varphi(x)\mu(dx) = -\int_{\mathbb{R}} \varphi'(x) f'_\ell(x)\, dx$$

i.e.,

$$\mu(dx) = f''(x)dx$$

Example 5.4: Absolute value function

For $f(x) = |x|$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x)\mu(dx) &= -\int_{\mathbb{R}} \varphi'(x) f'_\ell(x)\,dx \\
&= -\int_{-\infty}^{0} -1 \cdot \varphi'(x)\,dx - \int_{0}^{\infty} 1 \cdot \varphi'(x)\,dx \\
&= \int_{-\infty}^{0} \varphi'(x)\,dx - \int_{0}^{\infty} \varphi'(x)\,dx \\
&= \varphi(x)\Big|_{-\infty}^{0} - \varphi(x)\Big|_{0}^{\infty} \\
&= \varphi(0) + \varphi(0) \\
&= 2\varphi(0) \\
&= \int_{\mathbb{R}} \varphi(x) 2\delta_0(dx)
\end{aligned}$$

where

$$\delta_0(A) = \begin{cases} 1 & 0 \in A \\ 0 & 0 \notin A \end{cases}$$

This implies $f'' = 2\delta_0$

Definition 5.5: Local Time

(X_t) is a continuous semimartingale. Then the local time at a of X at time t is

$$L_t^a(X) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|X_s - a| \leq \varepsilon\}} \, d\langle X \rangle_s$$

Example 5.6: Local time of Constant

Let $X_t = 0$, then we have $L_t^0(X) = \langle X \rangle_t = 0$ and

$$supp\{dL_t^a(X)\} = \{X_t = a\}$$

Theorem 5.7: Itô-Tanaka Formula

Let f be a difference of two convex functions and (X_t) is a continuous seminartingale.

Then $f(X_t)$ is a semimartingale and

$$f(X_t) = f(X_0) + \int_0^t f'_\ell(X_s)\, dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a(X)\,\mu(da)$$

where $\mu(dx)$ is a second derivative of f in distribution sense.

Example 5.8: Absolute value function

Let $f(x) = |x|$ then,

$$\int_{\mathbb{R}} L_t^a(X)\mu(da) = \int_{\mathbb{R}} L_t^a(X) 2\delta_0(da) = 2L_t^0(X)$$

Then

$$|X_t| = |X_0| + \int_0^t sgn(X_s)\, dX_s + L_t^0(X)$$

We have $t \mapsto L_t^a(X)$ increasing a.s. and $(L_t^a) \in BV(0,T)$.

Theorem 5.9: Measure from Local Time

Measure $dL_t^a(X)$ is a.s. carried by the set $\{t : X_t = a\}$

Proof: From $(X_t - a)^2 = (|X_t - a|)^2$, we have from the left-hand side

$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a)\, dX_s + \langle X \rangle_t$$

From the right-hand side

$$(|X_t - a|)^2 = (X_0 - a)^2 + 2\int_0^t |X_s - a|\, d|X_s - a| + \frac{1}{2}\int_0^t 2\, d\langle |X - a| \rangle_s$$

Using Itô-Tanaka's formula, we have

$$d|X_s - a| = sgn(X_s - a)\, dX_s + dL_s^a(X)$$

Moreover, we have

$$\langle |X_s - a| \rangle = \left\langle \int_0^t sgn(X_s - a)\, dX_s \right\rangle = sgn(X_s - a)^2 \langle X \rangle_s$$

Hence, we have,

$$(|X_t - a|)^2 = (X_0 - a)^2 + 2\int_0^t |X_s - a|\, sgn(X_s - a)\, dX_s + 2\int_0^t |X_s - a|\, dL_s^a(X)$$
$$+ \int_0^t sgn(X_s - a)^2\, d\langle X \rangle_s$$

Combining both sides, we get

$$\int_0^t |X_s - a|\, dL_t^a(X) = 0 \qquad \forall t$$

Hence, we get the claim of the theorem i.e.,

$$L_t^a(X) = \int_0^t \mathbb{1}_{\{X_s = a\}}\, dL_s^a(X)$$

□

Theorem 5.10: Occupation Time Formula

For a Borel function φ:

$$\int_0^t \varphi(X_s)\, d\langle X \rangle_s = \int_{\mathbb{R}} \varphi(a) L_t^a(X)\, da$$

5.1.3 Stochastic Differential Equation

Remark 5.11: Motivation

For an ODE, we have

$$\frac{dx(t)}{dt} = x'(t) = \mu(x(t), t), \qquad x(0) = x_0$$

We want to combine ODE with a white noise

$$\xi = \frac{dB_t}{dt} = B_t'$$

but Brownian motion is nowhere differentiable. We let $\sigma(x, t)$ be the intensity of noise at state x and time t, i.e.,

$$\int_0^T \sigma(X_t, t)\xi_t \, dt = \int_0^T \sigma(X_t, t)B_t' \, dt = \underbrace{\int_0^T \sigma(X_t, t) \, dB_t}_{\text{Itô's integral}}$$

Example 5.12: Black-Scholes-Merton Model

Let X_t be the value of \$1 after t invested in a saving account. We have the ODE

$$\dot{X}_t = rX_t$$

where r is constant and deterministic growth rate of return.

For SDE, we want to have uncertain rate, i.e.,

$$\frac{dX_t}{dt} = (r + \sigma\xi_t)X_t$$

meaning

$$dX_t = rX_t \, dt + \sigma X_t \, dB_t, \qquad X_0 = 1$$

and

$$X_t = 1 + r\int_0^t X_s \, ds + \sigma \int_0^t X_s \, dB_s$$

We call X_t is a geometric Brownian Motion with

$$X_t = \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma B_t\right)$$

Example 5.13: Population growth

Let X_t be the population density with ODE

$$\frac{dX_t}{dt} = aX_t(1 - X_t)$$

We let there be random perturbation of the birth rate, it will result in

$$\frac{dX_t}{dt} = (a + \sigma\xi_t)X_t(1 - X_t)$$

In terms of SDE, we have

$$dX_t = aX_t(1 - X_t)\,dt + \sigma X_t(1 - X_t)\,dB_t$$

Definition 5.14: Homogenous SDE

Suppose that $b, \sigma : \mathbb{R} \to \mathbb{R}$ are continuous, η is $\mathscr{F}_s$-measurable random variable. We say that a process $(X_t)_{t\in[s,t)}$ solves the homogenous SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \qquad X_s = \eta$$

if

$$X_t = \eta + \int_s^t b(X_r)\,dr + \int_s^t \sigma(X_r)\,dW_r \qquad t \in [s, T)$$

Remark 5.15

1) We suppose here that b, σ are continuous but it can be extended
2) For $\tilde{X}_t = X_{t+s}$ and $t \in [0, T-s]$ and $\tilde{F}_t = F_{t+s}$, $\tilde{X}_0 = \eta$, we can transform time to start at 0

Definition 5.16: Diffusion Process

Process X that solves the above homogenous SDE is called diffusion starting from η.

Function σ is called diffusion coefficient and function b is called drift coefficient.

Definition 5.17: Lipschitz function

Function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant L if

$$|f(x) - f(y)| \le L|x - y| \qquad \forall x, y$$

Lipschitz property implies that

$$|f(x)| \le |f(0)| + L|x| \le \tilde{L}\sqrt{1 + x^2}$$

where $\tilde{L} = 2\max\{|f(0)|, L\}$

Theorem 5.18: Uniqueness

Suppose that b and σ are Lipschitz, then the homogenous SDE

$$dX_t = b(X_t)\, d_t + \sigma(X_t)\, dW_t \qquad X_s = \eta$$

has at most one solution.

Theorem 5.19: Existence and Uniqueness

Suppose that b and σ are Lipschitz on $\mathbb{R}$ and $\mathbb{E}\eta^2 < \infty$. Then the SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t \qquad X_s = \eta$$

has exactly one solution $X = (X_t)_{t \geq s}$.

Moreover, $\mathbb{E}X_t^2 < \infty$ and $t \mapsto \mathbb{E}X_t^2$ is bounded on $[0, t)$ for all t.

Example 5.20

For the following SDE,

$$dX_t = bX_t\, dt + \sigma\, dW_t \qquad X_0 = \eta$$

where $b(x) = x$ and $\sigma(x) = \sigma$ are Lipschitz.

It has a unique solution

$$X_t = e^{bt}\eta + \sigma \int_0^t e^{b(t-s)}\, dW_s$$

Example 5.21

For the following SDE

$$dX_t = \lambda X_t\, dW_t \qquad X_0 = \eta$$

where $b(x) = 0$ and $\sigma(x) = \lambda x$ are Lipschitz.

It has a unique solution

$$X_t = \eta \exp\left(\lambda W_t - \frac{\lambda^2}{2} t\right)$$

Definition 5.22: Non-homogenous SDE

Suppose $b, \sigma : \mathbb{R}^2 \to \mathbb{R}$ are continuous, η is $\mathscr{F}_s$-measurable.

We say that $X = (X_t)_{t \in [s,T)}$ solves the non-homogenous SDE

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \qquad X_s = \eta$$

if

$$X_t = \eta + \int_s^t b(r, X_r)\, dr + \int_s^t \sigma(r, X_r)\, dW_r \qquad t \in [s, T)$$

Theorem 5.23: Existence and Uniqueness of Non-homogenous SDE

Suppose that b and σ satisfy the Lipschitz condition as follows:

$$|b(t,x) - b(t,y)| \leq L(x-y), \quad |b(t,x)| \leq \tilde{L}\sqrt{1+x^2}$$

$$|\sigma(t,x) - \sigma(t,y)| \leq L(x-y), \quad |\sigma(t,x)| \leq \tilde{L}\sqrt{1+x^2}$$

then, for η is $\mathscr{F}_s$-measurable such that $\mathbb{E}\eta^2 < \infty$, there exists exactly one solution to

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \qquad X_0 = \eta$$

Example 5.24

The SDE

$$dX_t = \sigma(t) X_t\, dW_t, \qquad X_0 = \xi$$

satisfies the assumptions of the above theorem if

$$\sup_t |\sigma(t)| < \infty$$

6 Week 9

6.1 Lecture 1

6.1.1 Stochastic Exponential and Logarithm

Proposition 6.1: Stochastic Exponential

Suppose that $M \in \mathcal{M}^c_{loc}$ and Z_0 is $\mathscr{F}_0$-measurable. Then the process

$$Z_t = Z_0 \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$$

is a local martingale such that

$$dZ_t = Z_t\, dM_t$$

i.e.,

$$Z_t = Z_0 + \int_0^t Z_s\, dM_s$$

Such Z is called **stochastic exponential** of M with initial condition Z_0 or the Doleans-Dade exponential of M. We use the notation

$$Z = Z_0\mathcal{E}(M)$$

Proof: For the following semimartingale

$$X_t = M_t - \frac{1}{2}\langle M\rangle_t$$

we use Ito's formula to obtain that

$$\begin{aligned}
dZ_t = d(Z_0 e^{X_t}) &= Z_0 e^{X_t}\, dX_t + \frac{1}{2} Z_0 e^{X_t}\, d\langle M\rangle_t \\
&= Z_t\, dM_t - \frac{1}{2} Z_t\, d\langle M\rangle_t + \frac{1}{2}\langle M\rangle_t \\
&= Z_t\, dM_t
\end{aligned}$$

By construction of stochastic integral, Z is a continuous local martingale. $\square$

Example 6.2: Stochastic exponential

Consider the SDE

$$dX_t = b(t)X_t\,dt + \sigma(t)X_t\,dW_t, \qquad X_0 = \xi$$

Note that $b(t,x) = b(t)x$, $\sigma(t,x) = \sigma(t)x$ satisfies the Lipschitz conditions if $\sup_t |b(t)| < \infty$, $\sup_t |\sigma(t)| < \infty$.

We suppose that $X_t = g(t)Y_t$ where

$$dY_t = \sigma(t)Y_t\,dW_t, \qquad Y_0 = \xi$$

We know from previous example that

$$Y_t = \xi \exp\left(\int_0^t \sigma(s)\,dW_s - \frac{1}{2}\int_0^t \sigma^2(s)\,ds\right)$$

Then,

$$\begin{aligned} dX_t &= Y_t g'(t)\,dt + g(t)\,dY_t \\ &= g'(t)Y_t\,dt + \sigma(t)X_t\,dW_t \end{aligned}$$

It is enough to find solution to the ODE

$$g'(t) = b(t)g(t), \qquad g(0) = 1$$

We find that

$$X_t = Y_t g(t) = \xi \exp\left(\int_0^t \sigma(s)\,dW_s - \frac{1}{2}\int_0^t \sigma(s)^2\,ds + \int_0^t b(s)\,ds\right)$$

Definition 6.3: Stochastic Logarithm

If $U = \mathcal{E}(X)$, X is called **stochastic logarithm** of U, denoted by $\mathcal{L}(U)$.

Theorem 6.4: Stochastic Logarithm

$U \neq 0$. Then $\mathcal{L}(U)$ satisfies SDE

$$dX_t = \frac{1}{U_t}\,dU_t, \qquad X_0 = 0$$

and

$$X_t = \mathcal{L}(U)_t = \ln\left(\frac{U_t}{U_0}\right) + \int_0^t \frac{1}{2U_t^2}\,d\langle U\rangle_t$$

Example 6.5: Stochastic Logarithm

Let $u_t = e^{W_t}$. Find $\mathcal{L}(U)$.

6.1.2 Multi-dimensional SDE

Definition 6.6: Multi-dimensional Process

Let $W = (W^{(1)}, \cdots, W^{(d)})$ be a d-dimensional BM.

For $X = [X^{(i,j)}]_{1\le i\le m, 1\le j\le d}$ be a $m \times d$-matrices, consisting of processes in $\mathcal{L}^2_{T,loc}$, i.e., $(X^{(i,j)} \in \mathcal{L}^2_{T,loc})$, we define m-dimensional process,

$$M_t = (M_t^{(1)}, \cdots, M_t^{(m)}) = \int_0^t X_s \, dW_s, \qquad 0 \le t < T$$

given

$$M_t^{(i)} = \sum_{j=1}^d \int_0^t X_s^{(i,j)} dW_s^{(j)}, \qquad 1 \le i \le m$$

Definition 6.7: Multi-dimensional SDE

We suppose that $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are continuous functions, W is d-dimensional BM, $\xi = (\xi_1, \cdots, \xi_m)$ be m-dimensional $\mathcal{F}_s$-measurable random vector. We say that $X = (X_t^{(1)}, \cdots, X_t^{(m)})_{t\in[s,T)}$ solves the homogenous multi-dimensional SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \qquad X_s = \xi$$

if

$$X_t = \xi + \int_s^t b(X_u)\, du + \int_s^t \sigma(X_u)\, dW_u$$

Theorem 6.8: Existence and Uniqueness

Suppose that ξ is m-dimensional $\mathcal{F}_s$-measurable random vector such that $\mathbb{E}\xi_k^2 < \infty$ for $k \in \{1, \cdots, m\}$, and $b : \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \to \mathbb{R}^{d\times m}$ are Lipschitz and W is d-dimensional BM. Then,

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \qquad X_s = \xi$$

has the unique solution $X = (X_t^{(1)}, \cdots, X_t^{(m)})_{t\ge s}$.

Moreover

$$\mathbb{E} \sup_{s\le t\le u} \mathbb{E}|X_t^{(i)}| < \infty, \qquad \forall u < \infty$$

6.1.3 Girsanov Theorem

When we change the probability measure, we have

$$(\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{Q})$$

We let the Radon-Nikodym density be Z, i.e.,

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = Z, \qquad \mathbb{Q}(A) = \int_A Z \, d\mathbb{P} = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A Z)$$

Hence,

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A Z)$$

Basically, we want to show that if W is a $\mathbb{P}$-BM, then $\tilde{W} = W - xxx$ is $\mathbb{Q}$-BM and we need to figure out what is xxx.

Example 6.9: Motivating Discrete Example

Suppose that random variables $Z_1, Z_2, \cdots, Z_n$ are i.i.d. with $\mathcal{N}(0,1)$. We introduce a new measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$d\mathbb{Q} = \exp\left(\sum_{i=1}^n \mu_i Z_i - \frac{1}{2}\sum_{i=1}^n \mu_i^2\right) d\mathbb{P}$$

i.e.,

$$\mathbb{Q}(A) = \int_A \exp\left(\sum_{i=1}^n \mu_i Z_i(\omega) - \frac{1}{2}\sum_{i=1}^n \mu_i^2\right) d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{F}$$

$\mathbb{Q}$ is a probability measure since it is non-negative and

$$\mathbb{Q}(\Omega) = \mathbb{E}\exp\left(\sum_{i=1}^n \mu_i Z_i(\omega) - \frac{1}{2}\sum_{i=1}^n \mu_i^2\right) = \prod_{i=1}^n \mathbb{E}\exp\left((\mu_i Z_i - \frac{1}{2}\mu_i^2\right) = 1$$

Now, let's take $\Gamma = \mathcal{B}(\mathbb{R}^n)$, we have,

$$\begin{aligned}\mathbb{Q}((z_1, \cdots, z_n) \in \Gamma) &= \mathbb{E}\exp\left(\sum_{i=1}^n \mu_i z_i - \frac{1}{2}\sum_{i=1}^n \mu_i^2\right) \mathbb{1}_{\{(z_1,\cdots,z_n)\in\Gamma\}} \\ &= \frac{1}{(2\pi)^{n/2}}\int_\Gamma \exp\left(\sum \mu_i z_i - \frac{1}{2}\sum \mu_i^2\right)\exp\left(-\frac{1}{2}\sum z_i^2\right) dz_1 \cdots dz_n \\ &= \frac{1}{(2\pi)^{n/2}}\int_\Gamma \exp\left(-\frac{1}{2}\sum_{i=1}^n (z_i - \mu_i)^2\right) dz_1 \cdots dz_n\end{aligned}$$

This implies

$$Z_i \sum \mathcal{N}(\mu_i, 1)$$

and Z_i are independent. This implies $Z_i - \mu_i \sim \mathcal{N}(0,1)$ i.i.d.

We can define $S_k = Z_1 + \cdots + Z_k$, then $(S_k)_{k\leq n}$ has the same law under $\mathbb{P}$ as

$$(S_k - \sum_{i=1}^k \mu_i)_{k\leq n}$$

has under $\mathbb{Q}$. In the next considerations, we will replace S_k by BM, and $\sum_{i=1}^n \mu_i$ as $\int_0^t Y_s\, ds$.

Theorem 6.10: Girsanov Theorem For BM

For $T < \infty$, the process $(Y_t)_{t<T}$ is progressively measurable and $\int_0^T Y_t^2 \, dt < \infty$ a.s., i.e., $Y \in \mathcal{L}^2_{T,loc}$.

Let $M_t = \int_0^t Y_s \, dW_s \in \mathcal{M}^c_{loc}$ on $[0, T]$ and $\langle M \rangle_t = \int_0^t Y_s^2 \, ds$. Then

$$\begin{aligned} Z_t = \mathcal{E}(M_t) &= \exp\left(M_t - \frac{1}{2}\langle M \rangle_t \right) \\ &= \exp\left(\int_0^t Y_s \, dM_s - \frac{1}{2} \int_0^t Y_s^2 \, ds \right) \end{aligned}$$

is local martingale on $[0, T]$

Lemma 6.11: Martingale and Stochastic Exponential

If $M \in \mathcal{M}^c_{loc}$ on $[0, T]$, then $Z = \mathcal{E}(M)$ is a martingale on $[0, T]$ if and only if $\mathbb{E}Z_T = 1$

Proof: ($\Longrightarrow$)
Obvious, since $\mathbb{E}(Z_T) = \mathbb{E}(Z_0) = 1$ by martingale property
($\Longleftarrow$)
We know $Z > 0$ and $Z \in \mathcal{M}^c_{loc}$, hence Z is a supermartingale for $t \le T$, i.e., $Z_t \ge \mathbb{E}(Z_T|\mathcal{F}_t)$ a.s. Moreover,

$$1 = \mathbb{E}(Z_0) \ge \mathbb{E}(Z_t) \ge \mathbb{E}(Z_T) = 1$$

Hence, we have $\mathbb{E}(Z_t) = 1$. Therefore, we have

$$\mathbb{E}(\underbrace{Z_t - \mathbb{E}(Z_T|\mathcal{F}_t)}_{\ge 0}) = \mathbb{E}(Z_t) - \mathbb{E}(Z_T) = 0$$

Hence $Z_t = \mathbb{E}(Z_T|\mathcal{F}_t)$ a.s. $\square$

Theorem 6.12: Change of Measure

For $T < \infty$, $Y \in \mathcal{L}^2_{T,loc}$, $Z = \mathcal{E}\left(\int_0^\cdot Y_s\, dW_s\right)_\cdot$. Then if $\mathbb{E}Z_T = 1$ so Z_t is a martingale, then the process

$$V_t = W_t - \int_0^t Y_s\, ds, \qquad t \in [0, T]$$

is a BM on $(\Omega, \mathscr{F}, \mathbb{Q}_T)$, where

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} = Z_T \text{ i.e., } \mathbb{Q}_T(A) = \int_A Z_T\, d\mathbb{P}\, \forall A \in \mathscr{F}$$

We may want to have a measure with respect to which $W_t - \int_0^t Y_s\, ds$ is BM on $[0, \infty)$.

Theorem 6.13

Suppose $Y \in \mathcal{L}^2_{T,loc}$, $Z = \mathcal{E}(\int_0^\cdot Y\, dW)$, and $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$.

If $\mathbb{E}Z_t = 1$ for all t (Z is a constant on $[0, \infty)$), then there exists unique $\mathbb{Q}$ on $(\Omega, \mathscr{F}^W_\infty)$ such that $\mathbb{Q}(A) = \mathbb{Q}_T(A)$ for all $A \in \mathscr{F}^W_T$, $T < \infty$.

Process $V = W - \int Y\, ds$ is $\mathbb{Q}$- BM on $[0, \infty)$

Remark 6.14

Even though $\mathbb{Q}_T << \mathbb{P}$ ($\mathbb{P}(A) = 0 \implies \mathbb{Q}_T(A) = 0$), the measure $Q$from the last theorem is not necessarily absolutely continuous with respect to $\mathbb{P}$.

For example, when $Y_t = \mu \neq 0$ so $V_t = W_t - \mu t$. Let

$$A = \left\{\omega : \limsup \frac{1}{t} W_t(\omega) = 0\right\}$$

and

$$B = \left\{\omega : \limsup \frac{1}{t} V_t(\omega) = 0\right\} = \left\{\omega : \limsup \frac{1}{t} W_t(\omega) = \mu\right\}$$

From LLN for BM, we have $\mathbb{P}(A) = 1$ and $\mathbb{P}(B) = 0$. On the other hand, we have $\mathbb{Q}(B) = 1$. Hence, $\mathbb{P}$ and $\mathbb{Q}$ are singular on $\mathscr{F}^W_\infty$ despite that $\mathbb{Q}|_{\mathscr{F}^W_T} = \mathbb{Q}_T|_{\mathscr{F}^W_T} << \mathbb{P}|_{\mathscr{F}^W_T}$.

This is linked to uniform integrability of Z. If Z is uniform integrable, then

$$Z_t = \mathbb{E}(Z_\infty | \mathscr{F}_t)$$

and we would simplify $d\mathbb{Q} = Z_\infty d\mathbb{P}$

Theorem 6.15: Novikov Criterion

If Y is progressively measurable such that

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T Y_s^2\, ds\right)\right) < \infty$$

then $Z = \mathcal{E}(\int_0^\cdot Y_s\, dW_s)$ is a martingale on $[0, T]$.

Here is a more general version:

Theorem 6.16: General Novikov Criterion

For $M \in \mathcal{M}^c_{loc}$ such that for all t, we have

$$\mathbb{E}\exp\left(\frac{1}{2}\langle M\rangle_t\right) < \infty$$

Then, $Z = \mathcal{E}(M)$ is a martingale i.e., $\mathbb{E}Z_t = 1$ for all t.

Theorem 6.17: Girsanov theorem for d-dim BM

Suppose $Y = (Y^{(1)}, \cdots, Y^{(d)})$ is d-dimensional such that $Y^{(k)} \in \mathcal{L}^2_{T,loc}$ and $T < \infty$. Let W be d-dimensional BM and

$$Z_t = \exp\left(\sum_{i=1}^d \int_0^t Y_s^{(i)}\, dW_s^{(i)} - \frac{1}{2}\int_0^t |Y_s|^2\, ds\right)$$

Then, if $\mathbb{E}Z_T = 1$, we have,

$$V_t = W_t - \int_0^t Y_s\, ds = (W_t^{(1)} - \int_0^t Y_s^{(1)}\, ds, \cdots, W_t^{(d)} - \int_0^t Y_s^{(d)}\, ds)$$

is a BM on $[0, T]$ with respect to $\mathbb{Q}_T$ given by $\dfrac{d\mathbb{Q}_T}{d\mathbb{P}} = Z_T$.

Theorem 6.18: d-dim Novikov Criterion

If Y is as in the previous theorem, then if

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^t |Y_s|^2\, ds\right) < \infty$$

then,

$$Z = \mathcal{E}(\int Y\, dW)$$

is a martingale.

6.2 Lecture 2

Remark 6.19

If $Z = \mathcal{E}(\int_0^\cdot Y\,dW)$ then $\mathcal{L}(Z) = \int_0^\cdot Y\,dW = U$ satisfies $\langle U, W\rangle = \int Y_s\,ds$ and $dU_t = d\mathcal{L}(Z)_t = \frac{1}{Z_t}\,dZ_t$, $U_0 = 0$.

Moreover,

$$V = W - \int_0^\cdot Y_s\,ds = W - \langle \mathcal{L}(Z), W\rangle = W - \langle U, W\rangle$$

is a BM under $\mathbb{Q}$ given by $\dfrac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathscr{F}_T} = Z_T$.

Theorem 6.20: Girsanov with Semimartingale

Let X be a continuous semimartingale under $\mathbb{P}$ with decomposition $X = X_0 + M + A$ and $\mathbb{Q}_T$ be given by $d\mathbb{Q} = Z_T\,d\mathbb{P}$ for a martingale $Z = \mathcal{E}(U)$.

Then X is also a $\mathbb{Q}_T$-semimartingale with decomposition given by $X = X_0 + N + B$, where

$$N = M - F, B = A + F$$

and

$$F_t = \int_0^t \frac{1}{Z_s}\,d\langle Z, M\rangle_s = \langle U, M\rangle_t$$

Example 6.21

B is $\mathbb{P} - BM$. Then $B = \tilde{B} + \langle U, B\rangle$, where $\tilde{B}$ is $\mathbb{Q}_T$-BM and $\langle U, B\rangle = \int Y_s\,ds$ and $U = \int Y\,dB$

6.2.1 Martingale Representation Property

Proposition 6.22

Let W be BM, $(\mathscr{F}_t)_{t\in[0,\infty)}$ be the Brownian natural filtration and $\mathscr{F}_\infty = \sigma(\mathscr{F}_t : t \geq 0)$. For $X \in L^2(\mathscr{F}_\infty)$ there exists a unique progressively measurable process (H_t) such that $\mathbb{E}\left(\int_0^\infty H_u^2\,du\right) < \infty$, i.e.,$H \in \mathcal{L}^2_\infty$ and

$$X = \mathbb{E}(X) + \int_0^\infty H_u\,dW_u$$

Lemma 6.23

Let I be the collection of $f : [0,\infty) \to \mathbb{R}$ such that $f(t) = \sum_{k=1}^n \lambda_k \mathbb{1}_{(t_{k-1},t_k]}(t)$, $u \in \mathbb{N}$, $\lambda_k \in \mathbb{R}$, $t_{k-1} < t_k$.

Then the set $E = \{\mathcal{E}(\int_0^\cdot f(u)\,dW_u)_\infty : f \in I\}$ is total in $L^2(\mathscr{F}_\infty)$ i.e., its linear hull

$$\{\sum_{k=1}^n \alpha_k X_k : n \in \mathbb{N}, \alpha_1, \cdots, \alpha_N \in \mathbb{R}, X_1, \cdots, X_n \in E\}$$

is dense in $L^2(\mathscr{F}_\infty)$

Corollary 6.24

Let M be L^2-bounded continuous martingale. Then there exists $H \in \mathcal{L}^2_\infty$ such that

$$M_t = M_0 + \int_0^t H_u\,dW_u, \qquad \forall t \geq 0$$

Theorem 6.25: Martingale Representation Property

Let W be a BM and $(\mathscr{F}_t)$ be its natural filtration.

For all $(\mathscr{F}_t)$-local martingale M, there exists $H \in \mathcal{L}^2_{\infty,loc}$ such that

$$M_t = M_0 + \int_0^t H_u\,dW_u, \qquad \forall t$$

6.3 Summary of SDE

6.3.1 Ornstein-Uhlenbeck Process

Find a semimartingale (X_t) such that

$$dX_t = aX_t\,dt + \sigma\,dW_t \qquad X_0 = X$$

6.3.2 Black-Scholes-Merton

7 Week 12

7.1 Lecture 1

7.1.1 Feynman-Kac Formula

For a diffusion SDE,

$$\begin{aligned} dX_t &= F(t, X_t)\,dt + G(t, X_t)\,dW_t \\ X_s &= x, \qquad 0 \le s \le t \le T \end{aligned}$$

The diffusion generator $L(t)$ is as follows:

$$L(t)u(t,x) = \mu(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma(t,x)\frac{\partial^2 u}{\partial x^2}(t,x)$$

Theorem 7.1

Let $u(t,x)$ solve the following backward equation with $L(t)$ be the diffusion generator,

$$\begin{aligned} L(t)u(t,x) + \frac{\partial u}{\partial t}(t,x) &= 0 \\ \text{with } u(T,x) &= \varphi(x) \end{aligned}$$

Then, the solution is

$$u(t,x) = \mathbb{E}[\varphi(X_T)|X_t = x]$$

Theorem 7.2

Let $u(t,x)$ be the solution to the following backward equation with $L(t)$ being the diffusion generator,

$$\begin{aligned} L(t)u(t,x) + \frac{\partial u}{\partial t}(t,x) &= -\psi(x) \\ \text{with } u(T,x) &= \varphi(x) \end{aligned}$$

Then, the solution is

$$u(t,x) = \mathbb{E}\left[\varphi(X_T) + \int_t^T \psi(X_s)\,ds \,\middle|\, X_t = x\right]$$

Example 7.3: Probabilistic Representation

We would call the quantity

$$f(t, W_t) = \mathbb{E}(W_T^3 | \mathscr{F}_t)$$

the **probabilistic representation** of the solution $f(t, x)$ to the following backward equation:

$$\frac{1}{2}\frac{\partial^2 f}{\partial x}(t, x) + \frac{\partial f}{\partial t}(t, x) = 0$$
$$\text{with } f(T, x) = x^3$$

As the diffusion generator $L(t)$ is

$$L(t)f(t, x) = \frac{1}{2}\frac{\partial^2 f}{\partial x}(t, x)$$

This implies, the drift coefficient $\mu(t, x) = 0$ and the diffusion coefficient $\sigma(t, x) = 1$, i.e., the diffusion SDE is

$$dX_t = dW_t$$

Hence, this gives, $f(X_T) = W_T^3$ and then follows the previous theorem.

Theorem 7.4: Feynman-Kac Formula

For a bounded $r(t,x)$ and $\varphi(x)$. Let

$$c(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T r(u,X_u)\,du\right)\varphi(X_T)\middle| X_t = x\right]$$

Assume that the following backward equation has a solution,

$$\begin{aligned} L(t)f(t,x) + \frac{\partial f(t,x)}{\partial t} &= r(t,x)f(t,x) \\ \text{with } f(T,x) &= \varphi(x) \end{aligned}$$

Then the solution is unique and it is $c(t,x)$.

Remark 7.5: Constant discounting

Let $r(t,x) = r$ be a constant. Then the expression

$$\mathbb{E}\left[\exp\left(-r(T-t)\right)\varphi(X_T)\middle| X_t = t\right]$$

occurs in finance, in which

- r stands for the risk-free interest rate
- $\varphi(X_T)$ is the random payoff in the future
- $\exp\left(-r(T-t)\right)$ is the continuous discouting factor from t to T
- $\mathbb{E}\left[\exp\left(-r(T-t)\right)\varphi(X_T)\middle| X_t = t\right]$ is the expected discounted payoff.

7.1.2 Time Change

Definition 7.6: Time Change

A stochastic process $(\tau_s)_{s\in[0,\infty)}$ with paths which are

- cadlag
- non-decreasing
- have values in $[0,\infty)$ and starting from 0 at 0

is called a **time change** or **change of time** if the random variable τ_s is a stopping time for all $s \geq 0$.

(Note that $(\tau_s)_{s\in[0,\infty)}$ might not be an adapted process).

Definition 7.7: Time change of process, time change filtration

Given a filtration $(\mathscr{F}_t)_{t\geq 0}$ and a process $(X_t)_{t\geq 0}$,

- a stochastic process $(X_{\tau_s})_{s\geq 0}$ is called **time change** of $(X_t)_{t\geq 0}$ by $(\tau_s)_{s\geq 0}$
- a filtration $(\mathscr{G}_s)_{s\geq 0}$ with $\mathscr{G}_s = \mathscr{F}_{\tau_s}$ is called **time-changed filtration**.

Proposition 7.8: Properties of time-changed process and filtration

We have that,

1. $(\mathscr{G}_s)$ is right-continuous if $(\mathscr{F}_t)$ is right-continuous
2. the time-changed process (X_{τ_s}) is $(\mathscr{G}_s)$-adapted if (X_t) is $(\mathscr{F}_t)$-adapted
3. the time-changed process (X_{τ_s}) is cadlag if the process (X_t) is cadlag
4. the random variable τ_σ is $(\mathscr{F}_t)$-stopping time, if σ is $\mathscr{G}_s$-stopping time.

Definition 7.9: τ-continuous

Let $(\tau_s)_{s\geq 0}$ be a time change. A process $(X_t)_{t\geq 0}$ is said to be **τ-continuous** if it is continuous and X is constant on $[\tau_{s-}, \tau_s]$ for all $s \geq 0$.

Clearly, $(X_{\tau_s})_{s\geq 0}$ is continuous if $(\tau_s)_{s\geq 0}$ is a time change and $(X_t)_{t\geq 0}$ is τ-continuous process.

Proposition 7.10

Let $(\tau_s)_{s\geq 0}$ be a time change, M be a τ-continuous local martingale. Then the time-changed process $(M_{\tau_s})_{s\geq 0}$ is a continuous local martingale with respect to the time-changed filtration $(\mathscr{G}_s)_{s\geq 0} = (\mathscr{F}_{\tau_s})_{s\geq 0}$.

Theorem 7.11: Dambis-Dubins-Schwarz Theorem

Let M be a continuous local martingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$.

Define

$$\tau_s = \inf\{t : \langle M\rangle_t > s\}$$

and $\mathscr{G}_s = \mathscr{F}_{\tau_s}$ for $s \geq 0$. Then, the time-changed process (B_s) given by

$$B_s = M_{\tau_s} \qquad s \geq 0$$

is a $(\mathscr{G}_s)$-Brownian motion and the local martingale M is a time-change of B, i.e.,

$$M_t = B_{\langle M\rangle_t}, \qquad t \geq 0$$

Remark 7.12: Enlargement

In the above theorem, the hypothesis $\langle M\rangle_\infty = \infty$ can be relaxed, but we need to work on the enlarged space which supports a Brownian motion.

Let $(\Omega', \mathscr{F}', (\mathscr{F}'_t)_{t\geq 0}, \mathbb{P})$ be a probability space with a Brownian motion β, and set

- $\widetilde{\Omega} = \Omega \times \Omega'$
- $\widetilde{\mathscr{F}}_s = \mathscr{F}_{\tau_s} \otimes \mathscr{F}'_s$
- $\widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$
- $\widetilde{\beta}_s(\omega, \omega') = \beta_s(\omega')$

We can view a continuous local martingale M as a process on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, (\widetilde{\mathscr{F}}_s), \widetilde{\mathbb{P}})$ by defining $M(\omega, \omega') = M(\omega)$.

Then, the process $\widetilde{\beta}$ is independent of M and we may write

$$B_s = M_{\tau_s} + \int_0^s \mathbb{1}_{\{u>\langle M\rangle_\infty\}}\, d\widetilde{\beta}_u$$

which is a Brownian motion on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, (\widetilde{\mathscr{F}}_s), \widetilde{\mathbb{P}})$.

7.1.3 Time-homogenous diffusion

For a time-homogenous diffusion, we have,

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t \tag{$*$}$$

If (X, W) is a unique weak solution to $(*)$ then

$$\begin{aligned} P_{s,t}(x, y) &= \mathbb{P}(X_t \leq y | X_s = x) \\ &= P_{0,t-s}(x, y) \\ &= \mathbb{P}(X_{t-s} \leq y | X_0 = x) \end{aligned}$$

The diffusion generator L of time-homogenous diffusion is given by

$$Lf(x) = \frac{1}{2}\sigma(x)^2 f''(x) + \mu(x) f'(x)$$

The diffusion generator satisfies

$$Lf(x) = \lim_{t\to 0} \frac{\mathbb{E}(f(X_t)|X_0 = x) - f(x)}{t}$$

Definition 7.13: Exit time from an interval

Define the exit time of $(X_t)_{t\geq 0}$ from an interval (a, b) be

$$\tau = T_{a,b} = \inf\{t > 0 : X_t \notin (a, b)\}$$

If $(X_t)_{t\geq 0}$ is continuous, then $X_\tau \in \{a, b\}$.

Theorem 7.14: Dynkin's formula

Let (X_t) be a diffusion with continuous $\sigma(x) > 0$ on $[a, b]$ and $X_0 = x$, $a < x < b$. For $f \in C^2(\mathbb{R})$, we have,

$$f(X_{t\wedge\tau}) - \int_0^{t\wedge\tau} Lf(X_s)\,ds$$

is a martingale.

Consequently,

$$\mathbb{E}_x\left(f(X_{t\wedge\tau} - \int_0^{t\wedge\tau} Lf(X_s)\,ds) \right) = f(x)$$

Proof: Write it later. □

Theorem 7.15

Let (X_t) be a time-homogenous diffusion with generator L and continuous $\sigma(x) > 0$ on $[a, b]$, $X_0 = x$, $a < x < b$. Then, $\mathbb{E}_x(\tau) = v(x)$ satisfies the following ODE

$$Lv = -1$$

with $v(a) = v(b) = 0$

7.2 Lecture 2

For the exiting time of (X_t) from an interval (a, b), we have,

$$\tau = T_{a,b} = \inf\{t > 0 : X_t \notin (a, b)\} = T_a \wedge T_b$$

where $T_y = \inf\{t > 0 : X_t = y\}$. Now, we will focus on finding probabilities

$$\mathbb{P}_x(T_a < T_b) \text{ and } \mathbb{P}_x(T_b < T_a)$$

To this end, we will deal with a scale function $s(x)$ which is a solution to

$$Ls = 0 \iff \frac{1}{2}\sigma^2(x)s''(x) + \mu(x)s'(x) = 0$$

Hence,

$$\frac{s''(x)}{s'(x)} = -\frac{2\mu(x)}{\sigma^2(x)}$$

This gives the solution as follows

$$s(\xi) = \int_\xi^y \exp\left(-2\int_\xi^z \frac{\mu(u)}{\sigma^2(u)}\,du\right)\,dz$$

with

$$s(\xi) = s'(\xi) = 0$$

This implies we have the following expression of the first and second order derivatives

$$s'(y) = \exp\left(-2\int_\xi^y \frac{\mu(u)}{\sigma^2(u)}\,du\right)$$

and

$$s''(y) = -2\frac{\mu(y)}{\sigma^2(y)}\exp\left(-2\int_\xi^y \frac{\mu(u)}{\sigma^2(u)}\,du\right)$$

Moreover, if s is a scale function, then any linear transformation is also a scale function, i.e., for any $c_1, c_2 \in \mathbb{R}$, we let $\bar{s}(y) = c_1 s(y) + c_2$. Then

$$L\bar{s}(y) = \frac{1}{2}\sigma^2(y)\bar{s}''(y) + \mu(x)\bar{s}'(y) = c_1\left(\frac{1}{2}\sigma^2(y)s''(y) + \mu(x)s'(y)\right) = c_1 Ls(y) = 0$$

Moreover, note that $s(X_t)$ is a local martingale. Applying Ito's formula, we have

$$\begin{aligned}
s(X_t) &= s(X_0) + \int_0^t s'(X_u)\,dX_u + \frac{1}{2}\int_0^t s''(X_u)\,d\langle X\rangle_u \\
&= s(x) + \int_0^t s'(X_u)\mu(X_u)\,du + \int_0^t s'(X_u)\sigma(X_u)\,dW_u + \int_0^t \frac{1}{2}s''(X_u)\sigma^2(X_u)\,du \\
&= s(x) + \int_0^t \underbrace{\mu(X_u)s'(X_u) + \frac{1}{2}\sigma^2(X_u)s''(X_u)}_{=Ls}\,du + \int_0^t s'(X_u)\sigma(X_u)\,dW_u \\
&= s(x) + \int_0^t s'(X_u)\sigma(X_u)\,dW_u
\end{aligned}$$

Hence, we have $s(X_t)$ is a continuous local martingale.

By Dambis-Dubins-Schwarz Theorem, we have

$$M_t := s(X_t) = B_{\langle M \rangle_t}$$

Hence, it is enough to study the time-changed Brownian motion starting at $s(x)$ in the interval $(s(a), s(b))$.

Theorem 7.16: Scale function

Let (X_t) be a diffusion with generator L with continuous $\sigma(x) > 0$ on $[a, b]$, and $X_0 = x \in (a, b)$. Then

$$\mathbb{P}_x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}$$

where $s(x)$ is a scale function.

Remark 7.17

1. (X_t) is a diffusion with $\mu(x) = 0$ on (a, b) then $\mathbb{P}_x(T_b < T_a) = \frac{x-a}{b-a}$ e.g., a Brownian motion

2. For OU process

3. If s and $\bar{s}$ are two scale functions for (X_t), i.e., $\bar{s}(y) = c_1 s(y) + c_2$, then

$$\frac{\bar{s}(x) - \bar{s}(a)}{\bar{s}(b) - \bar{s}(a)} = \frac{c_1(s(x) - s(a))}{c_1(s(b) - s(a))} = \frac{s(x) - s(a)}{s(b) - s(a)}$$

7.2.1 Representation of solution fo DEs

Theorem 7.18

Let (X_t) be a diffusion with generator L and continuous $\sigma(x) > 0$ on $[a, b]$. With $X_0 = x \in (a, b)$. For $f \in C^2((a, b))$, $f \in C([a, b])$ and f solves

$$Lf = -\varphi$$

in (a, b) and $f(a) = g(a), f(b) = g(b)$ for some bounded function g, φ.

Then f has the following representation

$$f(x) = \mathbb{E}_x(g(X_\tau)) + \mathbb{E}_x\left(\int_0^\tau \varphi(X_s)\, ds\right)$$

where τ is the exit time from (a, b).

In particular, if $\varphi \equiv 0$, the representation is given by

$$f(x) = \mathbb{E}_x(g(X_\tau))$$

7.2.2 Explosion

Definition 7.19: Explosion

Let $D_u = (-u, u)$ for $u = 1, 2, \cdots$, then

$$\tau_u = \tau_{D_u} = \inf\{t \geq 0 : |X_t| = u\}$$

Since (X_t) is continuous, $\tau_u < \tau_{u+1}$ and $\tau_\infty = \lim_{u\to\infty} \tau_u$.

Diffusion starting from x explodes if

$$\mathbb{P}_x(\tau_\infty < \infty) > 0$$

Theorem 7.20: Explosion condition

Suppose $\mu(x)$, $\sigma(x)$ are bounded on finite intervals and $\sigma(x) > 0$ and is continuous. Then the diffusion process explodes if and only if one of the following conditions holds:
There exists X_0 such that,

1. $$\int_{-\infty}^{X_0} \exp\left(-\int_{X_0}^{x} \frac{2\mu(s)}{\sigma^2(s)}\,ds\right)\left(\int_{x}^{X_0} \frac{\exp\left(\int_{X_0}^{y} \frac{2\mu(s)}{\sigma^2(s)}\,ds\right)}{\sigma^2(y)}\,dy\right)dx < \infty$$

2. $$\int_{X_0}^{\infty} \exp\left(-\int_{X_0}^{x} \frac{2\mu(s)}{\sigma^2(s)}\,ds\right)\left(\int_{X_0}^{x} \frac{\exp\left(\int_{X_0}^{y} \frac{2\mu(s)}{\sigma^2(s)}\,ds\right)}{\sigma^2(y)}\,dy\right)dx < \infty$$

Corollary 7.21: No drift, No explosion

Diffusions with $\mu(x) \equiv 0$ do not explode.