Transition Dynamics in the PBMA Aiyagari Model

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1 Model Specification

1.1 Households

Households are infinitely lived with two distinct discount factors:

- Short-run discount factor $\beta\delta \in (0, 1)$
- Long-run discount factor $\delta \in (0, 1)$

The household optimization problem is:

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}\left[u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t)\right] \tag{1}$$

Subject to the budget constraint:

$$a_{t+1} + c_t \leq wz_t + (1 + r)a_t \tag{2}$$

$$c_t \geq 0 \tag{3}$$

$$a_t \geq -B \tag{4}$$

Where c_t is consumption, a_t is asset holdings, z_t is an exogenous component of labor income capturing stochastic employment risk, w is the wage, r is the interest rate, and B is the borrowing constraint.

We assume log utility $u(c) = \ln(c)$ and that labor productivity follows a Markov process with transition matrix Π and state space $\{z_1, z_2, \ldots, z_n\}$.

1.2 Firms

Production occurs according to a Cobb-Douglas technology:

$$Y = AK^{\alpha}N^{1-\alpha} \tag{5}$$

Firms maximize profits by choosing capital and labor:

$$\max_{K,N}\{AK^{\alpha}N^{1-\alpha} - (r + \eta)K - wN\} \tag{6}$$

The first-order conditions yield:

$$r = A\alpha \left(\frac{N}{K}\right)^{1-\alpha} - \eta \tag{7}$$

$$w = A(1-\alpha)\left(\frac{K}{N}\right)^{\alpha} \tag{8}$$

2 Equilibrium Concepts

There are three types of equilibrium concepts:

- Agents keep using short-run policy $\implies$ Short-Run Policy Equilibrium (SRPE)
- Agents keep using continuation policy $\implies$ Continuation Policy Equilibrium (CPE)
- Agents keep mixing these two policies $\implies$ PBMA

Remark: The PBMA equilibrium is constructed numerically. And it will be discussed separately in Section 4.

2.1 Computation Algorithm

Here is the outline of our algorithm:

1. Compute the optimal continuation policy σ_c and continuation value v_c by solving the following Bellman equation with discount factor δ

$$v_c(a,z) = \max_{a' \in \Gamma(a,z)} \left\{ u(wz + (1+r)a - a') + \delta \sum_{z'} v_c(a', z')\Pi(z, z') \right\}$$

   using Howard Policy Iteration

2. Compute the short-run policy σ_s and lifetime value v_s by solving a two-period DP problem as follows with discount factor $\beta\delta$

$$v_s(a,z) = \max_{a' \in \Gamma(a,z)} \left\{ u(wz + (1+r)a - a') + \beta\delta \sum_{z'} v_c(a', z')\Pi(z, z') \right\}$$

3. Compute the equilibriums using standard algorithm to find the equilibrium aggregate capital for solely using short-run policy and solely using continuation policy for further analysis.

2.2 Comparing the Aggregate Capital

For our baseline calibration ($\beta = 0.5$, $\delta = 0.96$, $\alpha = 0.33$, $\delta = 0.05$), we compute:

$$K_{SRPE} = 3.89 \tag{9}$$

$$K_{CPE} = 8.09 \tag{10}$$

And we varies the value of $\beta = 0.2, 0.4, 0.6, 0.8, 1.0$ and compute the aggregate equilibrium capital. See Figure 1

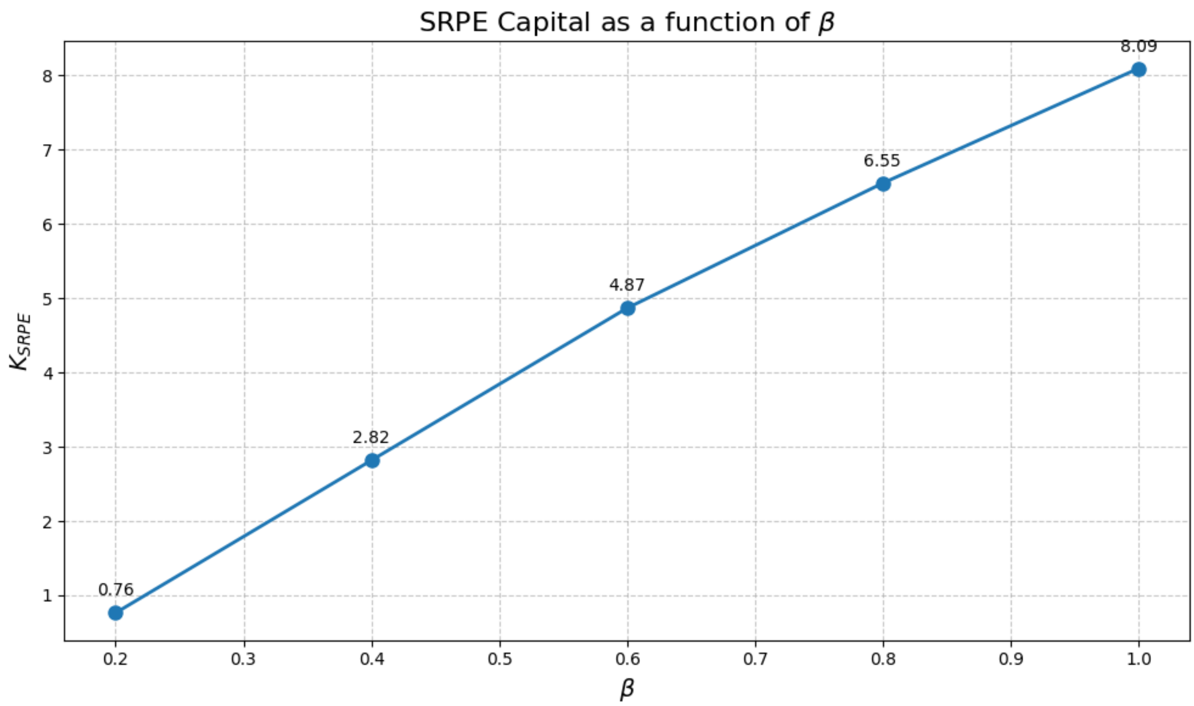

Figure 1: SRPE capital for different β

3 Transition Dynamics Methodology

We adapt Mukoyama's methodology to compute transitions between these equilibria.

First, we discuss the transition path from SRPE to CPE, i.e., agents is originally at SRPE and then there is a sudden change in the discount factor β to 1.

Second, we discuss the transition path from CPE to SRPE, i.e., agents are originally at CPE and then there is a sudden change in the discount factor β from 1 to some lower value.

The algorithm proceeds as follows:

1. Compute initial steady state capital K^s and terminal steady state capital K^c
2. Guess a time series for capital $\{K_t\}_{t=1}^T$ where T is sufficiently large
3. Derive prices $\{r_t, w_t\}_{t=1}^T$ from firm's optimization conditions
4. Use backward induction to compute value functions and policy functions for each period
5. Simulate the economy forward starting from the initial distribution
6. Compare the implied capital path to the initial guess and update
7. Iterate until convergence

3.1 Backward Induction

For the transition from SRPE to CPE, we:

1. Start with terminal value function $v_T = v_c$
2. For $t = T-1, T-2, \dots, 1$, we solve a two-period DP problem

$$v_t(a, z) = \max_{a'}\{u(w_t z + (1 + r_t)a - a') + \delta \sum_{z'} v_{t+1}(a', z')\Pi(z, z')\} \tag{11}$$

3. Store the resulting policy functions $\sigma_t(a, z)$

For the transition from CPE to SRPE, the backward induction use v_s as terminal value function and uses the short-run discount factor $\beta\delta$ between periods t and $t-1$.

3.2 Forward Simulation

Starting with the initial stationary distribution ψ_0:

1. For $t = 1, 2, \dots, T-1$:

$$\psi_t(a', z') = \sum_{a,z} \mathbb{I}[a' = \sigma_{t-1}(a, z)]\psi_{t-1}(a, z)\Pi(z, z') \tag{12}$$

2. Compute aggregate capital for each period:

$$K_t = \sum_{a,z} a \cdot \psi_t(a, z) \tag{13}$$

3.3 Transition dynamics from SRPE to CPE

The above algorithm converges and we can find a smooth transition path from SRPE to CPE. The transition path is plotted below with different beta, see Figure 2

3.4 Transition dynamics from CPE to SRPE

Does not converge. By backward induction, the aggregate capital drops to zero very quickly.

This might be possible, as keep using short-run policy is unstable. See Figure 3 the illustrative plot. I get this plot by setting a larger tolerance.

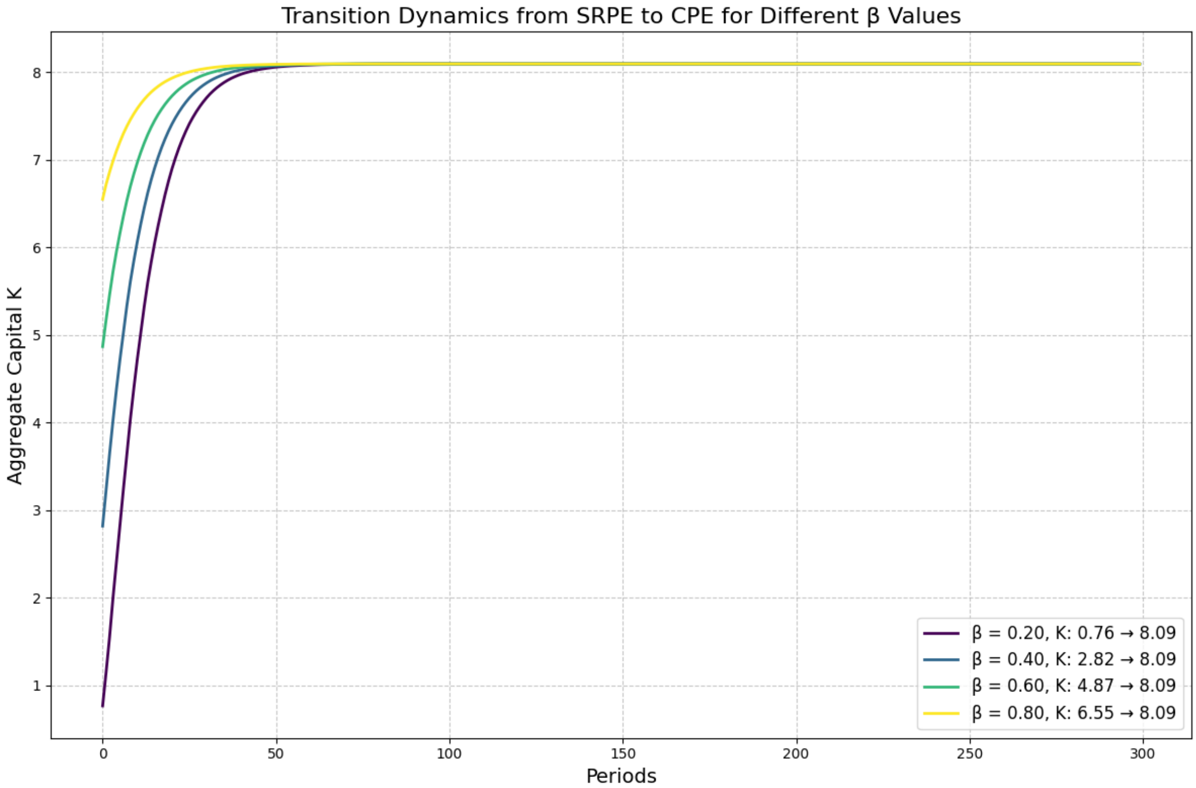

Figure 2: Transition Dynamics from SRPE to CPE with different β

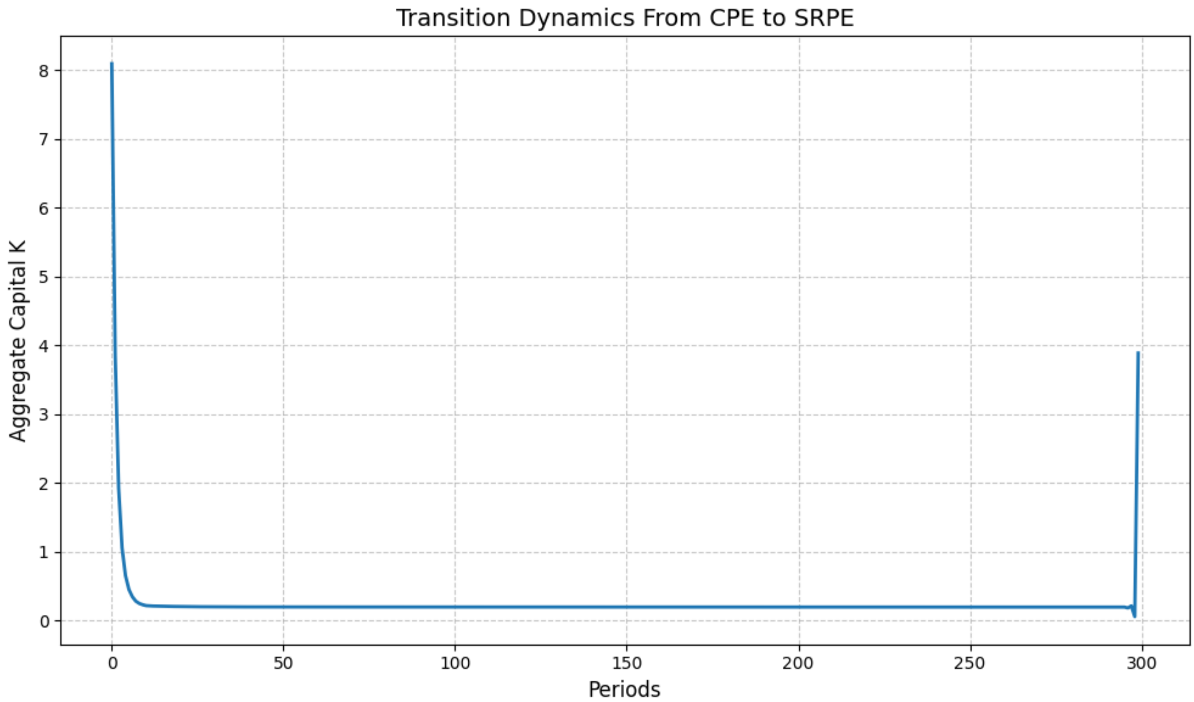

Figure 3: Fail to converge from CPE to SRPE

4 PBMA

This section discuss PBMA in this model. We let ω be the probability of using the short-run policy.

4.1 Stationary Asset Distribution of PBMA

Since we can get the Markov matrix for both short-run and continuation policies, we can also obtain the Markov matrix for the PBMA policies as a weighted average. This induces a stationary asset distribution for PBMA policies and a corresponding aggregate capital. Figure 4 is a plot for the aggregate capital under different ω.

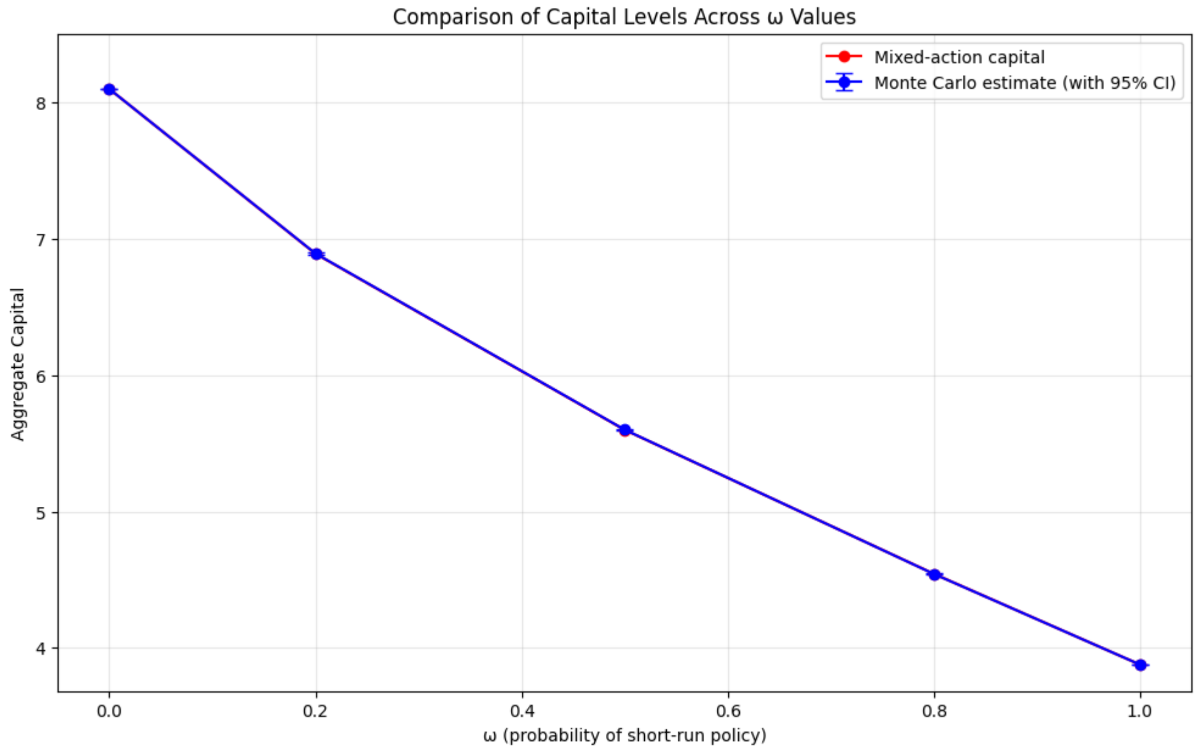

Figure 4: PBMA aggregate capital across ω

4.2 Use Monte Carlo to verify

Then I use Monte Carlo method to verify. The procedures are as follows:

1. calculate theoretical mixed-action capital levels for comparison
2. start with the CPE distribution as the initial state.
3. For each simulation, generate a random sequence of policy choices based on probability ω (1=short-run, 0=continuation)
4. For each time period, calculate current aggregate capital from the distribution and update the distribution using either the short-run or continuation policy based on the random sequence
5. After a burn-in period, average the capital path to get the mean capital level
6. aggregate results across multiple simulations to calculate the average capital

7. compare these Monte Carlo results with the theoretical mixed-action capital levels across different values of ω to validate the equilibrium predictions.

4.3 Result of Monte Carlo

As shown in the previous figure, at every level, the simulation result is very similar to the one from stationary asset distribution of PBMA. Figure 5 and 6 are detailed result for $\omega = 0.2$. (Under other ω, the results are similar)

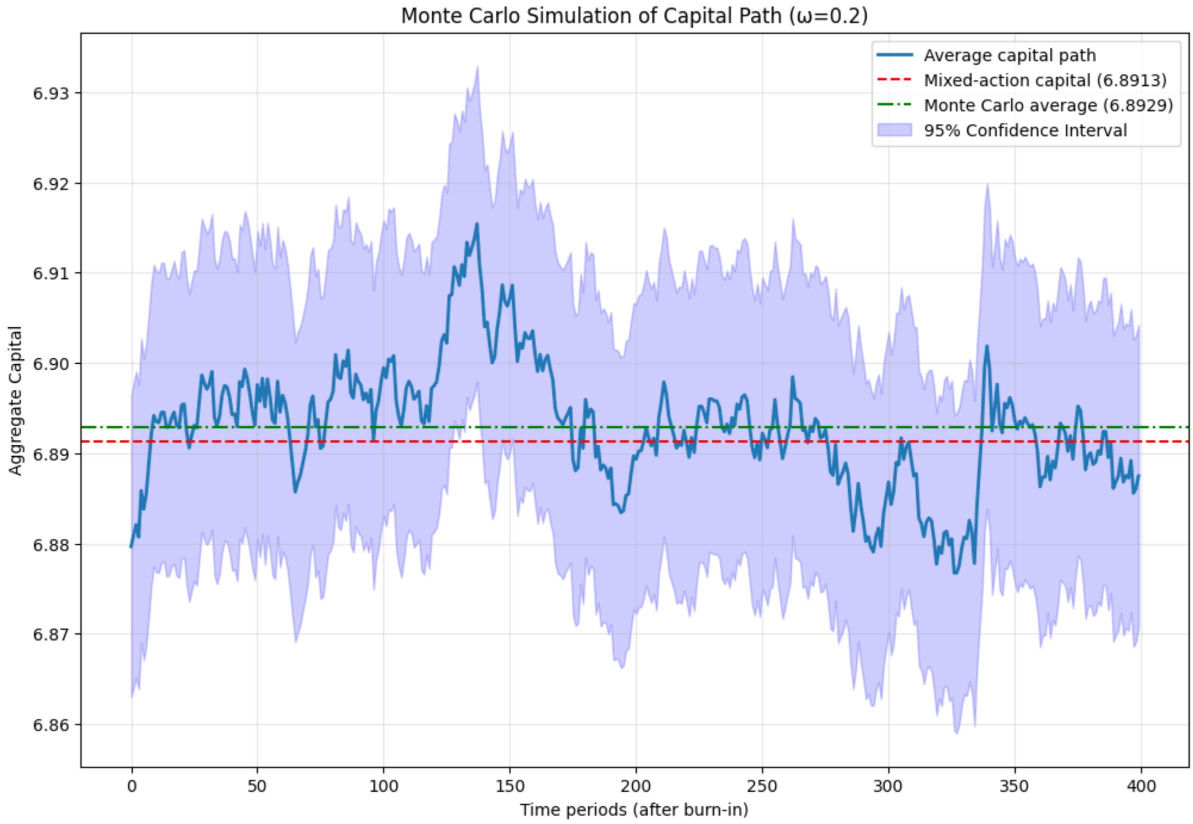

Figure 5: Simulation Result for $\omega = 0.2$

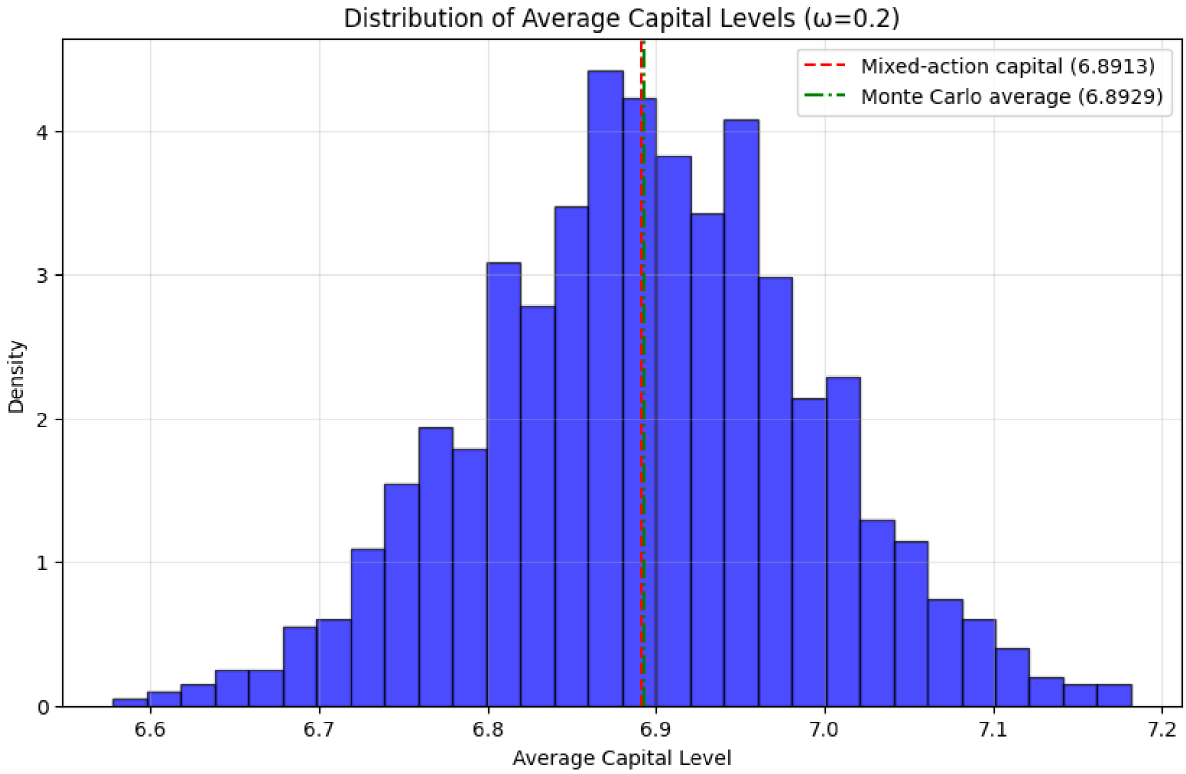

Figure 6: Simulation Distribution for $\omega = 0.2$

References

[1] Aiyagari, S. Rao (1994). "Uninsured Idiosyncratic Risk and Aggregate Saving." *Quarterly Journal of Economics*, 109(3): 659–684.

[2] Mukoyama, Toshihiko. "Transition Dynamics in the Aiyagari Model, with an application to Wealth Tax." Working Paper.