Du's Theorem and Contraction

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This note aims to understand the relationship between Du's Theorem and Contraction.

1 Preliminary

In this section, we first present related definitions, and some lemmas for further applications.

1.1 Cone-related Definitions

Definition 1.1 (Cone). Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following two conditions:

- $x \in P, \lambda \geq 0$ implies $\lambda x \in P$
- $x \in P, -x \in P$ implies $x = \theta$, where θ denoes the zero element of E

Remark. Every cone $P \subset E$ defines a partial ordering in E given by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$.

Definition 1.2 (Solid Cone). A cone P is called solid if it contains interior point, i.e., $\mathrm{int}(P) \neq \emptyset$ or denote $\mathring{P} \neq \emptyset$.

Remark. If cone P is solid and $y - x \in \mathring{P} \neq \emptyset$, we write $x \ll y$.

Definition 1.3 (Normal Cone). Let P be a cone on a real Banach space, we say P is normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Remark. We choose definition for normal cone as it most intuitively shows normality is for the order structure to be compatible with the norm structure.

1.2 Operator-related Definitions

Definition 1.4 (Increasing operator). Let E be a real Banach space, θ is the zero element of E, $P \subset E$ is a cone. $A : D \to E$ is called an increasing operator, if $\forall x_1, x_2 \in D \subset E$, $x_1 \leq x_2 \Rightarrow Ax_1 \leq Ax_2$.

Definition 1.5 (Convex operator). An operator $T : D(T) \subset E \to E$ is said to be convex if for $x, y \in D(T)$ with $x \leq y$ and every $t \in [0, 1]$, we have

$$T(tx + (1-t)y) \leq tT(x) + (1-t)T(y)$$

Remark. T is concave if $-T$ is convex.

1.3 Useful lemmas

Lemma 1.1 (Zorn's Lemma)**.** Let M be a nonempty poset. If every chain in E has an upper bound, then M has at least one maximal element.

Lemma 1.2 (Relative Compactness in metric space)**.** A relatively compact subset K of a metric space (X, d) is separable, i.e., it contains a countable dense subset.

Lemma 1.3 (Subset of Separable set in Metric space)**.** Every subset of a separable subset of a metric space is separable.

Lemma 1.4 (Monotone sequence)**.** Let E be a real Banach space with a partial order induced by a normal solid cone P. If (x_n) is a monotone sequence with a convergent subsequence in X. Then the entire sequence converges to the same limit.

Lemma 1.5 (Properties of Concave and Convex operators)**.** Let $T : D(T) \subset E \to E$ be a convex operator on a real Banach space E. Then for $\theta \leq y \leq x$ and $t \in [0, 1]$, we have

$$Tx - T(x - ty) \geq t[Tx - T(x - y)] \tag{1}$$

Remark. For a concave operator S, we have

$$Sx - S(x - ty) \geq t[Sx - S(x - y)] \tag{2}$$

for $\theta \leq y \leq x, t \in [0, 1]$.

2 Related Theorems

In this section, we present some useful and related theorems for Du's theorem. The proof is very detailed with motivations included.

Theorem 2.1 (Sun 1998). Suppose $u_0, v_0 \in E$, $u_0 < v_0$, $A : [u_0, v_0] \to E$ is an increasing operator satisfing

$$u_0 \leq Au_0, \quad Av_0 \leq v_0 \tag{3}$$

If $A([u_0, v_0])$ is relatively compact (sequentially compact), then A have a minimal fixed point x_* and maximal fixed point x^* satisfying

$$u_0 \leq u_1 \leq u_n \leq x_* \leq x^* \leq v_n \leq v_1 \leq v_0 \tag{4}$$

where $u_n := A^n u_0$ and $v_n := A^n v_0$.

Proof. **Step 0: Understand the Problem**.

The key condition of A is increasing and satisfy Equation 3 gives A leaves its domain invariant, i.e.,

$$A([u_0, v_0]) \subset [u_0, v_0] \tag{5}$$

Since we are looking for fixed points, i.e., $Ax = x$, instead of find them directly, we look into two types of points:

- Points where the operator maps them up, i.e., $x \leq Ax$
- Points where the operator maps them down, i.e, $Ax \leq x$.

Then, fixed points must be in both sets. This motivates the following definitions:

Define $F = \{x \in [u_0, v_0] : x \leq Ax\}$, and we know $u_0 \leq Au_0$, we know $u_0 \in F$, hence $F \neq \emptyset$.

Define $G = \{x \in [u_0, v_0] : Ax \leq x\}$, and we know $Av_0 \leq v_0$, hence $v_0 \in G$, $G \neq \emptyset$.

Moreover, fixed points should behave like the "boundary" of F and G. In particular, for the set F, the fixed point should behave like the maximal element of F. This motivates the use of Zorn's lemma to prove the existence of maximal elements in F.

Step 1.1: Start with F and use Zorn's lemma

We want to use Zorn's Lemma (Lemma 1.1) to find a maximal element, hence, we need to show that every chain in F has an upper bound. Hence we let H be an arbitary chain in F and we want to find some $z^* \in F$ such that

1. $x \leq z^*$ for all $x \in H$ (it is an upper bound of H)
2. $z^* \leq Az^*$ (so $z^* \in F$)

But the problem is: how to contruct such z^*, as we cannot just take the supremum of H, it may not exists.

One property of $x \in H \subset F$ is that

$$x \leq Ax \quad \text{for all } x \in H$$

This means that for each element H, applying A moves it up (or keep it fixed). So if we could find an upper bound for $A(H)$, it would also be an upper bound for H.

Why is it more desirable to find an upper bound for $A(H)$? Because, we know that $A([u_0, v_0])$ is relatively compact. This will help us to find an upper bound.

Step 1.2: Use $A([u_0, v_0])$ is relatively compact

Since H is a totally ordered set and A is increasing, we know that $A(H)$ is also a totally ordered set. To find the supremum of $A(H)$, we can do it in an iterative way, i.e., construct a sequence

$$(z_n) = (\max\{y_1, y_2, \cdots, y_n\})_{n\in\mathbb{N}}$$

But there is one problem, $A(H)$ is potentially uncountable. **The key point here is to use relative compactness of $A([u_0, v_0])$.** Using $A([u_0, v_0])$ is relatively compact, from Lemma 1.2, we know $A([u_0, v_0])$ is separable. Moreover, by Lemma 1.3, we know $A(H) \subset A([u_0, v_0])$ is separable. Hence, there exists $V = (y_i)_{i\in\mathbb{N}}$ a countable dense subset of $A(H)$.

The we define

$$z_n = \max\{y_1, \cdots, y_n\} \tag{6}$$

and we have two important observations,

- $z_1 \le z_2 \le \cdots \le z_n \le z_{n+1} \le \cdots$
- $V \subset A([u_0, v_0])$ is a sequence in a relatively compact subset, hence contains a convergent subsequence.

Hence, we know there exists a convergent subsequence (z_{n_i}), i.e., $z_{n_i} \to z^*$. Moreover, since (z_n) is a monotonically increasing sequence, z^* is also the supremum of (z_n) by Lemma 1.4.

Since $[u_0, v_0]$ is closed, $z^* \in [u_0, v_0]$ and

$$y_n \le z_n \le z^*, \quad \forall n \in \mathbb{N}$$

Since V is dense in $A(H)$, we have

$$x \le Ax \le z^*, \quad \forall x \in H \tag{7}$$

This gives us the first piece of the puzzle, and we still need to show that $z^* \in F$, i.e., $z^* \le Az^*$. Using Equation 7 and A is increasing, we have

$$Ax \le Az^*$$

and by density of V, we know

$$y_i \le Az^*, \quad \forall i \in \mathbb{N} \tag{8}$$

This implies

$$z_n = \max\{y_1, \cdots, y_n\} \le Az^*, \quad \forall n \in \mathbb{N} \tag{9}$$

taking the limit, we know

$$z^* \le Az^*$$

Hence, $z^* \in F$. This completes the proof that for any totally ordered subset $H \subset F$, there exists an upper bound.

Step 1.3 Get the existence of Fixed points

Hence, by Zorn's lemma, F has at least one maximal element $v^* \in F$, i.e.,

$$v^* \le Av^*$$

and by the definition of maximal element, this implies

$$v^* = Av^*$$

Similarly, we can obtain G has a minimal element $u^* \in G$ and $Au^* = u^*$.

Step 2.1

In **Step 1**, we have proved that the fixed points exist, but we want something stronger, we want to find the minimal and maximal fixed points.

The main idea is to look for order intervals that "trap" all the fixed points and have the same boundary behaviors as our original order interval $[u_0, v_0]$. Intuitively, we are finding the tightest order interval that contains all the fixed points.

This motivates the definition of the set of all such order intervals, i.e., we define

$$S = \{[u, v] : u, v \in [u_0, v_0], \quad \underbrace{u \leq v, u \leq A_u, Av \leq v}_{\text{same boundary behavior like } [u_0, v_0]} \quad, \mathrm{Fix}(A) \subset [u, v]\} \tag{10}$$

where $\mathrm{Fix}(A)$ is the set of all fixed points of A.

Step 2.2 Minimal element of S as the tightest interval

The definition of S indicates that the tightest order interval that contains all the fixed points of A should be the minimal element of S with respect to the following order

$$I_1 \leq_S I_2 \Leftrightarrow I_1 \supset I_2$$

Then, the maximal element under this order $\leq_S$ is the tightest order interval that contains all the fixed points.

Step 2.3 Zorn's lemma on S

Since we also cares about the maximal element, its existence can also be shown through Zorn's lemma.

Fix $S_1 = \{I_r : r \in \Gamma\}$ be a totally ordered subset of S with respect to $\leq_S$, where Γ is arbitrary index set and $I_r := [u_r, v_r]$.

Step 2.4: Use the result from Step 1:

One key observation is that the set

$$H_1 = \{u_r : r \in \Gamma\} \tag{11}$$

is a totally ordered subset of $[u_0, v_0]$ such that

$$u_r \leq Au_r$$

by definition of S. Hence, using the result of **Step 1**, we know there exists $z^* \in D$ and $(z_n) \subset A(H_1)$ such that $z_n \to z^*$ as $n \to \infty$ and

$$u_r \leq Au_r \leq z^*, \quad \forall r \in \Gamma \tag{12}$$

$$z^* \leq Az^*$$

Then by definition of S, we have for $x \in \mathrm{Fix}(A)$, $u_r \leq x$, for all $r \in \Gamma$, then

$$u_r \leq Au_r \leq Ax = x$$

Hence,

$$z_n \leq x, \quad \forall n \in \mathbb{N}$$

by the same reasoning as Equation 8 and Equation 9. As $z_n \to z^*$, we get

$$z^* \leq x, \quad x \in \text{Fix }(A) \tag{13}$$

Similarly, considering $G_1 = \{v_r : r \in \Gamma\}$, we can obtain there exists $w^* \in [u_0, v_0]$ that satisfies

$$w^* \leq v_r, \quad \forall r \in \Gamma \tag{14}$$

$$w^* \geq x, \quad x \in \text{Fix }(A) \tag{15}$$

Hence, combining Equation 13 and Equation 15, we have $z^* \leq x \leq w^*$ for all $x \in \text{Fix}(A)$.

Step 2.5 Conclude that $I^* = [z^*, w^*]$ is an upper bound

Let $I^* = [z^*, w^*]$, by above reasoning, $I^* \subset S$. Moreover, by Equation 12 and Equation 14, we have I^* is an upper bound of S_1 under $\leq_S$, i.e., the lower bound under set inclusion.

Hence, by Zorn's lemma, S has a maximal element under $\leq_S$ or a minimal element under $\subset$, i.e., $I_* = [x_*, x^*]$. In other words, for all $x \in \text{Fix}(A)$,

$$x_* \leq x \leq x^*$$

Step 2.6 Conclude I_*'s boundary are the minimal and maximal fixed points

By the definition of S and $I_* \in S$, we have

$$x_* \leq Ax_* \leq A(Ax_*) \leq A(Ax^*) \leq Ax^* \leq x^*$$

Hence the order interval $\bar{I} = [Ax_*, Ax^*] \in S$ and $\bar{I} \subset I_*$.

Since I_* is the minimal element under set inclusion, this implies $\bar{I} = I_*$, i.e.,

$$x_* = Ax_*, \quad x^* = Ax^*$$

Hence, by definition of S, x_* is the minimal fixed point and x^* is the maximal fixed point. ■

3 Du's Lemma, Theorem, Corollary

We present Du's convergence lemma, main theorem and corollary with proof. The proof is very detailed with motivation for each step.

3.1 Du's Convergence Lemma

In Theorem 2.1, we know the existence of minimal and maximal fixed point exists under such conditions. But how to construct them? Du's Convergence Lemma provides one way to construct the minimal fixed point.

Lemma 3.1 (Du's Convergence Lemma)**.** Suppose P is a normal cone, $v > \theta$, $A : [\theta, v] \to E$ is a concave and increasing operator. If there exists $0 < \epsilon < 1$ such that $A\theta \geq \epsilon v$, $Av \leq v$, then A has minimal fixed point u^* in $[\theta, v]$, $u^* > \theta$. Let $u_0 = \theta, u_n = Au_{n-1}$ $(n = 1, 2, 3, \cdots)$, then we have

$$\|u_n - u^*\| \leq N\|A\theta\|\epsilon^{-2}(1-\epsilon)^n, \quad (n = 1, 2, 3, \cdots) \tag{16}$$

Here N is the normal constant of P.

Proof. Step 1: Geometric convergence

The key part of the proof relies on proving the iterative sequence has a geometric convergence.

First, we want to prove by induction that

$$u_{n+1} - u_n \leq (1-\epsilon)^{n-1}(u_{n+1} - u_1) \qquad (n = 1, 2, 3, \cdots) \tag{17}$$

For $n = 1$, we have $u_2 - u_1 \leq u_2 - u_1$ trivially.

Now we suppose that for $n = k$,

$$u_{k+1} - u_k \leq (1-\epsilon)^{k-1}(u_{k+1} - u_1)$$

is true.

Now let $x = u_{k+1}$ and $y = \frac{u_{k+1}-u_1}{1-\epsilon}$. Then, we have by $u_1 = A\theta \geq \epsilon v \geq \epsilon u_{k+1}$

$$\theta \leq y = \frac{u_{k+1} - u_1}{1-\epsilon} \leq \frac{u_{k+1} - \epsilon u_{k_1}}{1-\epsilon} = x \leq z \tag{18}$$

Hence, using the Induction Hypothesis, we have

$$u_k \geq x - (1-\epsilon)^{k-1}(u_{k+1} - u_1) = x - (1-\epsilon)^k y \tag{19}$$

Using Lemma 1.5 and Equation 2, we have

$$\begin{aligned}
u_{k+2} - u_{k+1} &= Ax - Au_k \\
&\leq Ax - A(x - (1-\epsilon)^k y) && \text{(Equation 19)} \\
&\leq (1-\epsilon)^k [Ax - A(x - y)] && \text{(Equation 2)} \\
&= (1-\epsilon)^k [Ax - A\theta] && (x - y = A\theta)
\end{aligned}$$

This completes the induction.

Step 2: Prove the iterative sequence is Cauchy

By $u_1 = A\theta \geq \epsilon v > \theta$, and $Av \leq v$, we have

$$\theta = u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v \tag{20}$$

and

$$u_1 = A\theta \geq \epsilon v \geq \epsilon A v \geq \epsilon u_{n+1} \tag{21}$$

Using Equation 21 and Equation 17, we can get

$$u_{n+1} - u_n \leq (1-\epsilon)^{n-1}\left(\frac{u_1}{\epsilon} - u_1\right) = \frac{(1-\epsilon)^n}{\epsilon} u_1 \tag{22}$$

Hence, for all $m > n$, we have

$$\begin{aligned}
\theta &\leq u_m - u_n && \text{(Equation 20)}\\
&\leq (u_m - u_{m-1}) + \cdots + (u_{n+1} - u_n) && \text{(Telescope)}\\
&\leq \frac{u_1}{\epsilon}((1-\epsilon)^{n+1} + \cdots + (1-\epsilon)^m) && \text{(Equation 22)}\\
&\leq \frac{(1-\epsilon)^n}{\epsilon^2} u_1 && \text{(Geom. Sum)}
\end{aligned}$$

Since P is normal, this gives $\{u_n\}$ is a Cauchy sequence.

As E is a real Banach space, we get there exists $u^* \in E$ such that $u_n \to u^*$ as $n \to \infty$.

Step 3: Prove that the limit is the minimal fixed point

Since $u_n \leq v$ for all $n \in \mathbb{N}$, and since A is increasing, we have

$$u_n \leq u^* \leq v$$

then, $Au^* \geq Au_n$ for all $n \in \mathbb{N}$, let $n \to \infty$, we get

$$Au^* \geq u^* \tag{23}$$

Using the induction result, we have when $m \to \infty$,

$$\theta \leq u^* - u_n \leq \frac{(1-\epsilon)^n}{\epsilon^2} u_1 \tag{24}$$

We can fix $n \in \mathbb{N}$ such that $(1-\epsilon)^n \epsilon^{-2} < 1$. This give,

$$\begin{aligned}
\theta &\leq Au^* - u^*\\
&\leq Au^* - u_{n+1}\\
&= Au^* - Au_n\\
&\leq Au^* - A\left(u^* - \frac{(1-\epsilon)^n}{\epsilon^2} u_1\right) && \text{(Equation 24)}\\
&\leq \frac{(1-\epsilon)^n}{\epsilon^2}(Au^* - A(u^* - u_1)) && \text{(Lemma 1.5)}\\
&\leq \frac{(1-\epsilon)^n}{\epsilon^2}(Au^* - A\theta) && (\theta \leq u^* - u_1)\\
&\to \theta \quad (n \to \infty)
\end{aligned}$$

By P is normal, we have $Au^* - u^* = \theta$, i.e., $u^* = Au^*$.

Moreover, Equation 24 gives Equation 16.

Last, we show that u^* is the minimal fixed point of A in $[\theta, v]$. Suppose $\theta \leq \bar{x} \leq v$ such that $\bar{x} = A\bar{x}$. Since A is increasing, we have $u_n \leq \bar{x}$, as $n \to \infty$, we have $u^* \leq \bar{x}$. This shows that u^* is the minimal fixed point. ∎

Theorem 3.1 (Du's Theorem)**.** Suppose that the cone P is normal, $u_0, v_0 \in E$ and $u_0 < v_0$. Moreover, $A : [u_0, v_0] \to E$ is an increasing operator. Let $h_0 = v_0 - u_0$. If one of the following assumptions holds:

- A is a concave operator, $Au_0 \geq u_0 + \epsilon h_0$, $Av_0 \leq v_0$ where $\epsilon \in (0, 1)$ is a constant
- A is a convex operator, $Au_0 \geq u_0, Av_0 \leq v_0 - \epsilon h_0$ where $\epsilon \in (0, 1)$ is a constant

Then, A has a unique fixed point x^* in $[u_0, v_0]$. Moreover, for any $x_0 \in [u_0, v_0]$, the iterative sequence $\{x_n\}$ given by $x_n = Ax_{n-1} \quad (n = 1, 2, \cdots)$ satisfying that

$$\|x_n - x^*\| \to 0 \text{ as } n \to \infty$$

$$\|x_n - x^*\| \leq M(1 - \epsilon)^n \quad \forall n = 1, 2, \cdots$$

Proof. Step 1: Transform to a simpler domain:

To use Lemma 3.1, we can transform A via

$$Bx = A(x + u_0) - u_0 \tag{25}$$

This transform the domain to $[\theta, h_0]$ where $h_0 = v_0 - u_0$. This transformation gives $B : [\theta, h_0] \to E$ increasing and concave. The boundary conditions become

- $B\theta = A(u_0) - u_0 \geq u_0 + \epsilon h_0 - u_0 = \epsilon h_0$
- $Bh_0 = A(v_0) - u_0 \leq v_0 - u_0 = h_0$

Step 2: Finding the minimal fixed point

From **Step 1**, we applies Lemma 3.1, we can conclude that B has a minimal fixed point u^* in $[\theta, h_0]$ with error bound

$$\|u_n - u^*\| \leq N\|B\theta\|\epsilon^{-2}(1 - \epsilon)^n$$

where $u_0 = \theta, u_n = Bu_{n-1}$. N is the normal coefficient of P. Let $M_0 := N\|B\theta\|\epsilon^{-2}$, we get

$$\|u_n - u^*\| \leq M_0(1 - \epsilon)^n$$

Step 3: The Sandwich argument

To have unique fixed point, we need two sequences, one from above, one from below:

- One from above: $h_n = Bh_{n-1}$ starting from h_0 and decreasing from above
- One from below: u_n as descibed in **Step 2**

These two sequences sandwich any sequence y_n where $y_0 \in [\theta, h_0]$ and $y_n = By_{n-1}$. Since B is increasing, we have

$$u_0 \leq y_0 \leq h_0 \Rightarrow u_n \leq y_n \leq h_n$$

Now, we are sandwiching an arbitrary sequence between two sequences and we know (u_n) converges to u^* from below. To obtain a unique fixed point, all we need to show is that (h_n) also converges to u^* from above.

Step 4: Analyzing the Upper sequence For this sequence (h_n), we know

$$h_0 \geq h_1 \geq h_2 \geq \cdots \geq u^*$$

Hence, we know $h_n - u^* \geq \theta$ but we don't know this difference. So this motivates us to create a measure about how different between h_n and u^*. We let

$$t_n = \sup\{t > 0 : u^* > th_n\}$$

which is the fraction of h_n is contained in u^*. Moreover, since $u^* \geq B\theta \geq \epsilon h_0 \geq \epsilon h_n$, this gives $t_n \geq \epsilon > 0$.

Using concavity of B, we have

$$\begin{aligned} u^* = Bu^* \geq B(t_n h_n) &\geq (1-t_n)B\theta + t_n B h_n \\ &\geq (1-t_n)\epsilon h_0 + t_n h_{n+1} \\ &\geq [(1-t_n)\epsilon + t_n] h_{n+1} \end{aligned}$$

Since $t_{n+1} = \sup\{t > 0 : u^* \geq t_{n+1} h_{n+1}\}$, we get $t_{n+1} \geq (1-t_n)\epsilon + t_n$, hence we have

$$1 - t_{n+1} \leq (1-t_n)(1-\epsilon), \quad n = 1, 2, \cdots$$

Hence

$$1 - t_n \leq (1-t_1)(1-\epsilon)^{n-1} \leq (1-\epsilon)^n$$

Hence

$$\theta \leq h_n - u^* \leq h_n - t_n h_n \leq (1-t_n)h_0 \leq (1-\epsilon)^n h_0$$

This gives $\|h_n - u^*\| \leq N\|h_0\|(1-\epsilon)^n$.

Step 5: Combine everything together

For any $y_0 \in [\theta, h_0]$, let $y_n = By_{n-1}$, then $u_n \leq y_n \leq h_n$, therefore,

$$\begin{aligned} \|y_n - u^*\| &\leq \|y_n - u_n\| + \|u_n - u^*\| \\ &\leq N\|h_n - u_n\| + \|u_n - u^*\| \\ &\leq N\|h_n - u^*\| + (N+1)\|u_n - u^*\| \\ &\leq M(1-\epsilon)^n \end{aligned}$$

where $M = N^2\|h_0\| + (N+1)M_0$ is a constant. This gives global stability, hence we have unique fixed point.

The convex case is similar. ■

4 Lemmas from Du's Theorem

In this section, we present some lemmas from Du's Theorem.

The objective of this section is to relate Du's Theorem with contraction. This first lemma starts with contraction on $\mathbb{R}$. The motivation is simple, $\mathbb{R}$ is a totally ordered set, this makes positivity very powerful.

To make everything clear, here we are interested in a weaker form of contraction, i.e., contracting to the unique fixed point. Here we defined it as weak contraction as follows:

Definition 4.1 (Weak contraction). Let $X = (X, d)$ be a metric space. Let $T : X \to X$ be a self-map with unique fixed point $x^* \in X$, such that there exists $\alpha \in (0, 1)$, for all $x \in X$, we have

$$d(Tx, x^*) \leq \alpha d(x, x^*) \tag{26}$$

Definition 4.2 (Strongly contractive mapping). Let $T : X \to X$ be a self map on a metric space X. We call T a strongly contractive mapping if for all x, y, we have

$$d(Tx, Ty) < d(x, y)$$

Definition 4.3 (Weak strongly contractive mapping). Let $T : X \to X$ be a self map on a metric space X with a unique fixed point $x^* \in X$. We call T a weak strongly contractive mapping if for all $x \in X$, we have

$$d(Tx, x^*) < d(x, x^*)$$

Lemma 4.1 (Du's Theorem and Weak Contraction on $\mathbb{R}$). Let $\mathbb{R} = (\mathbb{R}, \leq)$, i.e., paired with the usual total order $\leq$. Let $u_0, v_0 \in \mathbb{R}$ and $u_0 < v_0$.

Let $T : [u_0, v_0] \to \mathbb{R}$ be an increasing operator. Let $h_0 = v_0 - u_0$. If one of the following assumptions holds:

- T is concave operator, $Tu_0 \geq u_0 + \epsilon h_0$, $Tv_0 \leq v_0$ where $\epsilon \in (0, 1)$ is a constant
- T is convex operator, $Tu_0 \geq u_0$, $Tv_0 \leq v_0 - \epsilon h_0$, where $\epsilon \in (0, 1)$ is a constant

Then, T has a unique fixed point $x^* \in [u_0, v_0]$. Moreover, T is a weak strongly contractive mapping on $[u_0, v_0]$.

Proof. By Theorem 3.1, we know that T is a self-map on $[u_0, v_0]$ with unique fixed point $x^* \in [u_0, v_0]$.

We need to show that for all $x \in [u_0, v_0]$, we have

$$\|Tx - x^*\| < \|x - x^*\| \tag{27}$$

We now prove the first case, when T is concave.

We can partition the order interval into two parts $[u_0, x^*]$ and $[x^*, v_0]$.

For $x \in [u_0, x^*]$, we have by Theorem 3.1, $\|T^n x - x^*\| \to 0$ as $n \to \infty$.

Since T is increasing, this implies $x_n := T^n x$ must be increasing sequence. Similarly, we have for $x \in [x^*, v_0]$, (x_n) must be a decreasing sequence. Hence, this implies on $[u_0, v_0]$

$$\|Tx - x^*\| < \|x - x^*\|, \quad x \neq x^* \tag{28}$$

where the strict inequality is from the fact that x is not the unique fixed point. ■