Positive operators on L^p is order continuous

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Abstract

In this note, we follow the approach in [Zaa12] to show that positive operators on L^p spaces for $1 \leq p < \infty$ is order continuous (we exclude the $p = \infty$ case, see subsection 1.4 for detail). The logic flow is demonstrated in the following Figure 1. The structure of this note is as follows:

Contents

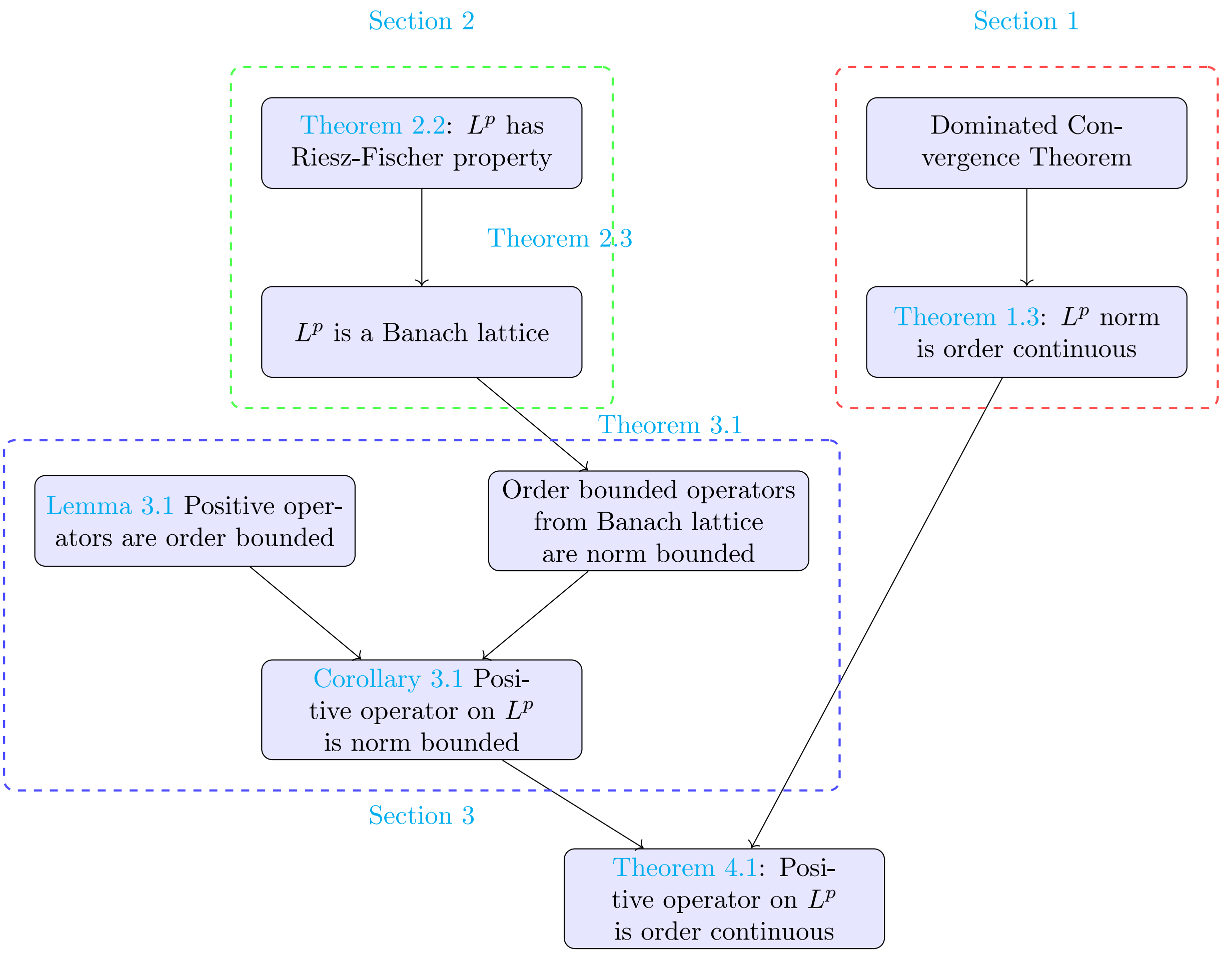

Figure 1: Logical structure of the proof

1 Prove L^p norm is order continuous

In this section, we prove that L^p norm is order continuous, we first present related definitions and theorems that we take for granted.

1.1 Related definitions

Definition 1.1 (L^p norm)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 \le p < \infty$. For a measurable function $f : X \to \mathbb{C}$ (or $\mathbb{R}$), the L^p norm of f is defined as

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p}$$

provided that $\int_X |f(x)|^p \, d\mu(x) < \infty$. The space $L^p(X, \mu)$ consists of all equivalence class of measurable functions $[f]$, $f : X \to \mathbb{C}$ (or $\mathbb{R}$) for which $\|f\|_p < \infty$, with functions that differ only on a set of measure zero identified.

Definition 1.2 (Monotone sequence)**.** Let E be a Riesz space. The sequence (f_n) in E is said to be increasing if

$$f_1 \le f_2 \le \cdots$$

and denoted as $f_n \uparrow$; is said to be decreasing if

$$f_1 \ge f_2 \ge \cdots$$

and denoted as $f_n \downarrow$.

Definition 1.3 (Monotone convergence)**.** If $f_n \uparrow$ and $f = \sup_n f_n$ exists in E, we write $f_n \uparrow f$.

If $f_n \downarrow$ and $f = \inf_n f_n$ exists in E, we write $f_n \downarrow f$.

If $f_n \uparrow f$ or $f_n \downarrow f$, we say f_n converges monotonically to f.

Remark. Let $[f_n] \downarrow [f]$ in L^p. This means we have (with pointwise a.e. partial order see Definition 1.5)

$$[f_1] \ge_{a.e} [f_2] \ge_{a.e} \cdots$$

and $[f] =_{a.e.} \inf_n [f_n]$. In particular, this mean $[f_n] \ge_{a.e} [f]$ for all $n \in \mathbb{N}$ and if there exists $[g]$ such that $[f_n] \ge_{a.e} [g]$ for all $n \in \mathbb{N}$ implies $[f] \ge_{a.e} [g]$.

Definition 1.4 (Convergence in order)**.** Let (f_n) be a sequence in a normed Riesz space E. We say f_n converges to f in order, denoted as $f_n \to f(ord)$ if there exists a sequence $p_n \downarrow 0$ in E such that

$$|f_n - f| \le p_n, \quad \forall n \in \mathbb{N}$$

Remark. It is evident that $f_n \uparrow f$ or $f_n \downarrow f$ implies $f_n \to f(ord)$.

Definition 1.5 (pointwise almost everywhere convergence in L^p)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, the space $L^p(X, \mu)$ consists of all equivalence classes of measurable functions $[f]$, $f : X \to \mathbb{C}$ (or $\mathbb{R}$) for which $\|f\|_p < \infty$, with functions that differ only on a set of measure zero identified.

Let $\{[f_n]\}_{n=1}^{\infty}$ be a sequence in $L^p(X, \mu)$. We say $[f_n]$ converge pointwise almost everywhere to

$[f] \in L^p(X, \mu)$, and denoted as $[f_n] \to [f](\mu\text{-}a.e.)$ if for any $f_n \in [f_n]$, $f \in [f]$, we have

$$\mu(\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}) = \mu(X)$$

and $[f] \in L^p(X, \mu)$.

Definition 1.6 (Almost everywher partial order on L^p)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, the space $L^p(X, \mu)$ consists of all equivalence classes of measurable functions $[f]$, $f : X \to \mathbb{C}$ (or $\mathbb{R}$) for which $\|f\|_p < \infty$, with functions that differ only on a set of measure zero identified.

Fix $[f], [g] \in L^p(X, \mu)$, we say that $[f] \leq_{a.e} [g]$ if for any $f \in [f], g \in [g]$, we have,

$$\mu(\{x \in X : f(x) > g(x)\}) = 0$$

Remark. The Axiom of Choice grants us the ability to choose arbitary representatives from an equivalence class in L^p.

Definition 1.7 (Order continuous norm)**.** Let E be a normed Riesz space. A norm $\|\cdot\|$ on E is called *order continuous* if for every sequence $(f_n)_{n=1}^{\infty}$ in E such that $f_n \downarrow 0$ (i.e., f_n is decreasing and $\inf_n f_n = 0$), we have

$$\lim_{n\to\infty} \|f_n\| = 0$$

1.2 Related theorems

Lemma 1.1 (Countable intersection of full measure set has full measure)**.** Let $(X, \mathcal{M}, \mu)$ be a measurable space. Let $\{X_n\}_{n\in\mathbb{N}}$ be a countable collection of full measure subset of X, i.e.,

$$\mu(X_n) = \mu(X), \quad \forall n \in \mathbb{N}$$

Then

$$\mu\left(\bigcap_{n\in\mathbb{N}} X_n\right) = \mu(X)$$

Proof. For all $n \in \mathbb{N}$, we have

$$\mu(X \setminus X_n) = 0$$

By De Morgan's law, we have

$$X \setminus \bigcap_{n\in\mathbb{N}} X_n = \bigcup_{n\in\mathbb{N}} X \setminus X_n$$

Hence, by countable subadditivity, we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} X \setminus X_n\right) \leq \sum_{n\in\mathbb{N}} \mu(X \setminus X_n) = 0$$

■

Proposition 1.1 shows that the almost everywhere partial order in L^p is indeed a partial order.

Proposition 1.1 (Almost everywhere partial order in L^p)**.** This proposition verifies that almost everywhere partial order in L^p is a partial order, see Definition 1.6

Proof. First, we show reflexivity. Let $[f] \in L^p(X, \mu)$, where $(X, \mathcal{M}, \mu)$ is a measurable space. Fix $f_1, f_2 \in [f]$, we have

$$0 \leq \mu(\{x \in X : f_1(x) > f_2(x)\}) \leq \mu(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$$

Hence, we have $[f] \leq_{a.e} [f]$.

Second, we prove anti-symmetry. Let $[f], [g] \in L^p(X, \mu)$, such that $[f] \leq_{a.e} [g]$ and $[g] \leq_{a.e} [f]$. This implies for any $f \in [f], g \in [g]$,

$$\mu(\{x \in X : f(x) > g(x)\}) = 0$$

and for any $f \in [f], g \in [g]$, we have

$$\mu(\{x \in X : f(x) < g(x)\}) = 0$$

Hence, this concludes that

$$0 \leq \mu(\{x \in X : f(x) \neq g(x)\}) \leq \mu(\{x \in X : f(x) > g(x)\}) + \mu(\{x \in X : f(x) < g(x)\}) = 0$$

Hence, by definition, $[f] =_{a.e.} [g]$.

Last, we prove transitivity. Let $[f], [g], [h] \in L^p(X, \mu)$ such that $[f] \leq_{a.e} [g]$ and $[g] \leq_{a.e} [h]$. Fix any $f \in [f], g \in [g]$, and $h \in [h]$, we have

$$\mu(\{x \in X : f(x) > g(x)\}) = \mu(\{x \in X : g(x) > h(x)\}) = 0$$

Hence,

$$\mu(\{x \in X : f(x) > h(x)\}) = 0$$

as

$$\{x \in X : f(x) > h(x)\} \subset \{x \in X : f(x) > g(x)\} \cup \{x \in X : g(x) > h(x)\}$$

hence, this shows transitivity. ■

Lemma 1.2 (Infimum in L^p). Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $[f] =_{a.e.} \inf_n [f_n]$ for all $n \in \mathbb{N}$ and $[f], \{[f_n]\}_{n \in \mathbb{N}} \in L^p(X, \mu)$. Then,

$$\inf_n [f_n] =_{a.e.} [\inf_n f_n]$$

In other words, $[f] = a.e. \inf_n [f_n]$ implies

$$\mu(\{x \in X : f(x) = \inf_n f_n(x)\}) = \mu(X)$$

for any $f \in [f]$ and $f_n \in [f_n]$ for all $n \in \mathbb{N}$.

Proof. Since $[f] =_{a.e.} \inf_n [f_n]$, this implies

- $[f] \leq_{a.e.} [f_n]$ for all $n \in \mathbb{N}$
- If $[g] \in L^p(X, \mu)$ such that $[g] \leq_{a.e.} [f_n]$ for all $n \in \mathbb{N}$, then $[g] \leq_{a.e.} [f]$.

Fix a representative $f \in [f]$, $g \in [g]$, and $f_n \in [f_n]$ for all $n \in \mathbb{N}$. Define

$$X_f = \{x \in X : f(x) \leq f_n(x), \quad \forall n \in \mathbb{N}\}$$

and

$$X_g = \{x \in X : g(x) \leq f_n(x), \quad \forall n \in \mathbb{N}\}$$

and

$$X_{fg} = \{x \in X : g(x) \leq f(x)\}$$

By $[f] =_{a.e.} \inf_n [f_n]$, we have

$$\mu(X_f) = \mu(X_g) = \mu(X_{fg}) = \mu(X)$$

Hence, by Lemma 1.1, we have

$$\mu\{X_f \cap X_g \cap X_{fg}\} = \mu(\{x \in X : f(x) = \inf_n f_n(x)\}) = \mu(X)$$

Therefore, this concludes that $f = \inf_n f_n$ almost everywhere, i.e., $[f] =_{a.e.} [\inf_n f_n]$. Combining $[f] =_{a.e.} \inf_n [f_n]$, we have

$$\inf_n [f_n] =_{a.e.} [\inf_n f_n]$$

■

Theorem 1.1 (Dominated Convergence Theorem in L^p)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Suppose that:

1. $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions from X to $\mathbb{C}$ (or $\mathbb{R}$);
2. $f_n \to f$ μ-almost everywhere on X as $n \to \infty$;
3. There exists a function $g \in L^p(X, \mu)$ such that $|f_n(x)| \leq g(x)$ for μ-almost all $x \in X$ and all $n \in \mathbb{N}$.

Then:

(a) $f \in L^p(X, \mu)$;

(b) $\lim_{n\to\infty} \|f_n - f\|_p = 0$, i.e., $f_n \to f$ in $L^p(X, \mu)$;

(c) $\lim_{n\to\infty} \int_X |f_n|^p \, d\mu = \int_X |f|^p \, d\mu$.

Lemma 1.3 (Equivalence of Order and Pointwise Convergence for Monotone Sequences)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. If $\{[f_n]\}_{n=1}^{\infty}$ is a monotone sequence in $L^p(X, \mu)$ with respect to the almost everywhere partial order, then $[f_n] \to [f](ord)$ if and only if $[f_n] \to [f]$ μ-*a.e.*.

Proof. We will prove the result for a decreasing sequence $\{[f_n]\}_{n=1}^{\infty}$ such that $[f_n] \downarrow [f]$. The case where $[f_n] \uparrow [f]$ follows by similar arguments.

Part 1: ($[f_n] \downarrow [f] \Rightarrow [f_n] \to [f]$ μ-*a.e.*)

Assume $[f_n] \downarrow [f]$ in $L^p(X, \mu)$. By definition, this means that:

- $[f_1] \geq_{a.e.} [f_2] \geq_{a.e.} \cdots$ (the sequence is decreasing in the a.e. partial order)
- $[f] =_{a.e.} \inf_n [f_n]$ (the infimum exists and equals $[f]$)

We need to prove that $[f_n] \to [f]$ μ-*a.e.*, which means showing that for any choice of representatives $f_n \in [f_n]$ and $f \in [f]$, we have

$$\mu(\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}) = \mu(X)$$

Let us fix arbitrary representatives $f_n \in [f_n]$ for all $n \in \mathbb{N}$ and $f \in [f]$. We will analyze the set where the desired convergence property fails:

$$\begin{aligned} E &= \{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\} \\ &= X \setminus \{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\} \end{aligned}$$

We can decompose this set as:

$$\begin{aligned} E = &(X \setminus \{x \in X : f_n(x) \text{ is decreasing}\}) \\ &\cup (X \setminus \{x \in X : f(x) = \inf_n f_n(x)\}) \end{aligned}$$

We will show that both parts of this union have measure zero.

For the first part, let $D_n = \{x \in X : f_n(x) < f_{n+1}(x)\}$ for each $n \in \mathbb{N}$. Since $[f_n] \geq_{a.e.} [f_{n+1}]$, we have $\mu(D_n) = 0$ for all $n \in \mathbb{N}$. The set where the sequence is not decreasing can be written as:

$$X \setminus \{x \in X : f_n(x) \text{ is decreasing}\} = \bigcup_{n=1}^{\infty} D_n$$

By the countable sub-additivity of the measure:

$$\mu(X \setminus \{x \in X : f_n(x) \text{ is decreasing}\}) \leq \sum_{n=1}^{\infty} \mu(D_n) = 0$$

For the second part, we need to show that $\mu(X \setminus \{x \in X : f(x) = \inf_n f_n(x)\}) = 0$. This is directly followed by Lemma 1.2.

Thus, $\mu(E) = 0$, which means $\mu(\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}) = \mu(X)$, proving that $[f_n] \to [f]$ μ-*a.e.*.

Part 2: (Given $[f_n] \downarrow$, $[f_n] \to [f]$ μ-$a.e.$ $\Rightarrow [f_n] \downarrow [f]$)

Assume that $[f_n] \to [f]$ μ-*a.e.* and $[f_n]$ is decreasing. We need to show that $[f] =_{a.e.} \inf_n [f_n]$.

First, we verify that $[f]$ is a lower bound for the sequence $\{[f_n]\}_{n=1}^{\infty}$. We know:

1. Since $[f_n]$ is decreasing, $[f_n] \geq_{a.e.} [f_m]$ for all $m \geq n$.
2. $[f_n] \to [f]$ μ-*a.e.* means for any representatives $f_n \in [f_n]$ and $f \in [f]$, $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in X$.

Let us fix representatives $f_n \in [f_n]$ and $f \in [f]$. Define:

$$A = \{x \in X : f_n(x) \text{ is decreasing}\}$$
$$B = \{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}$$

From our assumptions, $\mu(A) = \mu(B) = \mu(X)$, so $\mu(A \cap B) = \mu(X)$ as well by Lemma 1.1.

For any $x \in A \cap B$ and any $n \in \mathbb{N}$, we have:

$$f_n(x) \geq \lim_{m\to\infty} f_m(x) = f(x)$$

since $\{f_m(x)\}_{m=n}^{\infty}$ is decreasing. This implies $[f_n] \geq_{a.e.} [f]$ for all $n \in \mathbb{N}$.

Now we need to show that $[f]$ is the greatest lower bound. Let $[g]$ be any lower bound for $\{[f_n]\}_{n=1}^{\infty}$, i.e., $[g] \leq_{a.e.} [f_n]$ for all $n \in \mathbb{N}$. Fix representatives $g \in [g]$, $f_n \in [f_n]$, and $f \in [f]$.

For almost all $x \in X$, we have $g(x) \leq f_n(x)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, and using the fact that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in X$, we get $g(x) \leq f(x)$ for almost all $x \in X$. This means $[g] \leq_{a.e.} [f]$.

Therefore, $[f] =_{a.e.} \inf_n [f_n]$, proving that $[f_n] \downarrow [f]$.

The proof for an increasing sequence $[f_n] \uparrow [f]$ follows similar reasoning, with appropriate reversal of inequalities. ■

Theorem 1.2 (Order convergence in L^p implies pointwise almost everywhere convergence)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. If $\{[f_n]\}_{n=1}^{\infty}, [f]$ in $L^p(X, \mu)$ and $[f_n] \to [f](ord)$, then

$[f_n] \to [f]$ μ-*a.e.*.

Proof. By Definition 1.4, there exists $\{[p_n]\}_{n\in\mathbb{N}}$ in L^p such that $[p_n] \downarrow [0]$ and

$$|[f_n] - [f]| \leq_{a.e.} [p_n], \quad \forall n \in \mathbb{N}$$

Fix arbitrary representatives $f \in [f], f_n \in [f_n]$, and $p_n \in [p_n]$ for all $n \in \mathbb{N}$.

Define X_n for all $n \in \mathbb{N}$ as follows

$$X_n := \{x \in X : |f_n(x) - f(x)| \leq p_n(x)\}$$

It is clear from the definition, we have

$$\mu(X_n) = \mu(X), \quad \forall n \in \mathbb{N}$$

Moreover, define

$$\tilde{X} = \{x \in X : p_n(x) \downarrow 0\}$$

By Lemma 1.3, we get $\mu(\tilde{X}) = \mu(X)$. Hence, by Lemma 1.1, we have

$$\begin{aligned}\mu\left(\tilde{X} \cap \bigcap_{n\in\mathbb{N}} X_n\right) &= \mu(\{x \in X : |f_n(x) - f(x)| \leq p_n(x), \forall n \in \mathbb{N} \text{ and } p_n(x) \downarrow 0\}) \\ &= \mu(\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}) \\ &= \mu(X)\end{aligned}$$

Hence, we have $[f_n] \to [f]$ μ-a.e. by Definition 1.5. ■

Example (Pointwise a.e. convergence in L^p does not implies order convergence)**.** Let $X = [0,1]$ with Lebesgue measure and for $n \in \mathbb{N}$, let

$$\Delta_n = [2^{-n}, 2^{-(n-1)}]$$

For $1 \leq p < \infty$, let

$$f_n(x) = \begin{cases} 2^{n/p} & \text{if } x \in \Delta_n \\ 0 & \text{if } x \notin \Delta_n \end{cases}, \quad \forall n \in \mathbb{N}$$

Then we have

$$\int |f_n|^p \, dx = 1$$

For $p = \infty$, let

$$f_n(x) = \begin{cases} n & \text{if } x \in \Delta_n \\ 0 & \text{if } x \notin \Delta_n \end{cases}$$

Hence, we have $\|f_n\|_\infty = n$. As $n \to \infty$, we have $f_n \to 0$ pointwise. We can show that this convergence is not in order.

Proof. For $1 \leq p < \infty$ and $0 \leq f_n \leq q_n$ for some sequence $q_n \downarrow 0$ in L^p. Then $q_n \geq f_n$ for all $n \in \mathbb{N}$. So $q_n \geq q_m \geq f_m$ for all $m \geq n$, which implies

$$\int (q_n)^p \, dx \geq \sum_{n=m}^{\infty} \int (f_m)^p \, dx = \infty$$

This contradicts to $q_n \downarrow 0$.

For $p = \infty$, we get similarly

$$\|q_n\|_\infty \geq \|f_m\|_\infty = m, \quad \forall m \geq n$$

Hence, $\|q_n\|_\infty = \infty$ which is also a contradiction. ⊛

1.3 Proof of L^p norm is order continuous

Theorem 1.3 (L^p norm is order continuous)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p < \infty$[a]. Then the L^p-norm is order continuous. That is, if $(f_n)_{n=1}^\infty$ is a sequence in $L^p(X, \mu)$ such that $f_n \downarrow 0$ μ-almost everywhere (i.e., f_n is decreasing and $\inf_n f_n = 0$ μ-almost everywhere), then $\|f_n\|_p \downarrow 0$ as $n \to \infty$.

[a] L^∞ norm is not order continuous, see subsection 1.4

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence in $L^p(X, \mu)$ such that $f_n \downarrow 0$ μ-almost everywhere. We need to show that $\|f_n\|_p \to 0$ as $n \to \infty$.

First, we observe that since $f_n \downarrow 0$ μ-almost everywhere, we have $f_n \ge 0$ μ-almost everywhere for all $n \in \mathbb{N}$. Therefore, $|f_n| = f_n$ μ-almost everywhere.

We'll now demonstrate that the sequence $\{\|f_n\|_p\}$ is monotone decreasing. For any $n \in \mathbb{N}$, since $f_n \ge f_{n+1} \ge 0$ μ-almost everywhere, we have $f_n^p \ge f_{n+1}^p$ μ-almost everywhere (because $t \mapsto t^p$ is an increasing function on $[0, \infty)$ for $p > 0$). Therefore

$$\begin{aligned} \|f_n\|_p^p &= \int_X |f_n(x)|^p \, d\mu(x) \\ &= \int_X f_n(x)^p \, d\mu(x) \\ &\ge \int_X f_{n+1}(x)^p \, d\mu(x) \\ &= \int_X |f_{n+1}(x)|^p \, d\mu(x) \\ &= \|f_{n+1}\|_p^p \end{aligned}$$

Since $p > 0$, we can take the p-th root of both sides to obtain $\|f_n\|_p \ge \|f_{n+1}\|_p$. Thus, the sequence $\{\|f_n\|_p\}$ is monotone decreasing.

Now, let's define $g = f_1$. Since $f_1 \in L^p(X, \mu)$, we have $g \in L^p(X, \mu)$.

Furthermore, since $f_n \downarrow$, we have $f_n \le f_1 = g$ for all $n \in \mathbb{N}$. Thus, $|f_n| \le g$ μ-almost everywhere for all $n \in \mathbb{N}$.

We also have $f_n \to 0$ μ-almost everywhere as $n \to \infty$ (since $f_n \downarrow 0$).

Therefore, we can apply the Theorem 1.1(Dominated convergence theorem) for L^p spaces:

Since

1. $(f_n)_{n=1}^\infty$ is a sequence of measurable functions;
2. $f_n \to 0$ μ-almost everywhere as $n \to \infty$;
3. $|f_n| \le g$ μ-almost everywhere for all $n \in \mathbb{N}$, where $g \in L^p(X, \mu)$;

Theorem 1.1 implies that

$$\lim_{n\to\infty} \|f_n\|_p = \|0\|_p = 0$$

Hence, the L^p-norm is order continuous for $1 \le p < \infty$. ■

1.4 L^∞ norm is not order continuous

First, we show the definition of L^∞ norm.

Definition 1.8 (L^∞ norm). For a measurable function f on a measure space $(X, \mathcal{M}, \mu)$, the L^∞ norm is defined as

$$\|f\|_\infty = \operatorname{ess\,sup}_{x\in X} |f(x)| = \inf\{M \geq 0 : |f(x)| \leq M \text{ for } \mu\text{-almost every } x \in X\}$$

where ess sup denotes the essential supremum.

Now, we give a counterexample to show that L^∞ norm is not order continuous.

Example (L^∞ norm is not order continuous). Consider the sequence of functions $\{f_n\}_{n=1}^\infty$ on $[0, 1]$ defined by

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$$

This sequence is decreasing pointwise to the zero function $f(x) = 0$, i.e., $f_n \downarrow 0$ as $n \to \infty$. However,

$$\|f_n\|_\infty = 1$$

for all $n \in \mathbb{N}$, while $\|f\|_\infty = 0$. Thus

$$\lim_{n\to\infty} \|f_n\|_\infty = 1 \neq 0 = \|f\|_\infty$$

This demonstrates that the L^∞ norm does not preserve order limits, and hence is not order continuous.

2 Prove that L^p has Riesz Fischer property and is Banach lattice

In this section, we first present some related definitions and theorem we take for granted. Then we prove in section 2.2 L^p has the Riesz Fischer property. In section 2.3 we prove that a normed Riesz space is Banach lattice if and only if it has the Riesz Fischer property. This concludes that L^p is a Banach lattice.

2.1 Related definitions

Definition 2.1 (Normed Riesz Space). A *normed Riesz space* (or *normed vector lattice*) is a triple $(X, \leq, \|\cdot\|)$ where:

1. $(X, \leq)$ is a Riesz space (i.e., a vector space equipped with a partial ordering $\leq$ that makes it a lattice, and such that the vector space operations are compatible with the ordering);
2. $\|\cdot\|$ is a norm on X;
3. The norm is *lattice norm* (or *Riesz norm*), meaning it satisfies:

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$$

for all $x, y \in X$, where $|x| = x \vee (-x)$ denotes the absolute value of x.

Definition 2.2 (Banach Lattice). A Banach lattice is a complete normed Riesz space, i.e., a normed Riesz space that is complete with respect to its norm.

Definition 2.3 (Riesz-Fischer Property). A normed Riesz space $(E, \|\cdot\|)$ is said to have the *Riesz-Fischer property* if for every sequence $(f_n)_{n=1}^{\infty}$ in the positive cone E^+ satisfying

$$\sum_{n=1}^{\infty} \|f_n\| < \infty,$$

the series $\sum_{n=1}^{\infty} f_n$ converges in E.

2.2 Related theorems

Theorem 2.1 (Monotone Convergence Theorem in L^p). Let $(X, \mathcal{M}, \mu)$ be a measure space, let $1 \leq p < \infty$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from X to $[0, \infty]$. If $\{f_n\}$ is monotonically increasing (i.e., $0 \leq f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots$ a.e.) and $f_n \to f$ pointwise a.e. as $n \to \infty$, then:

$$\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$$

or equivalently,

$$\lim_{n\to\infty} \int_X |f_n|^p \, d\mu = \int_X |f|^p \, d\mu$$

2.3 Prove that L^p has Riesz-Fischer property

Theorem 2.2 (Riesz-Fischer Property for L^p Spaces). Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. The space $L^p(X, \mathcal{M}, \mu)$ has the Riesz-Fischer property. That is, for any sequence $(f_n)_{n=1}^{\infty}$ of non-negative functions in L^p such that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$, the series $\sum_{n=1}^{\infty} f_n$ converges in L^p to a function $f \in L^p$.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative functions in L^p such that $\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$.

Define the partial sums $s_n = \sum_{k=1}^{n} f_k$ for $n = 1, 2, \ldots$. We first show that the sequence $(s_n)_{n=1}^{\infty}$ is bounded in L^p norm.

For each n, by the triangle inequality for the L^p norm, we have:

$$\|s_n\|_p = \left\| \sum_{k=1}^{n} f_k \right\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p = M < \infty$$

Thus, the sequence $(s_n)_{n=1}^{\infty}$ is bounded in L^p norm.

Furthermore, since each f_n is non-negative, the sequence $(s_n)_{n=1}^{\infty}$ is monotonically increasing pointwise, i.e., $s_1 \leq s_2 \leq \ldots \leq s_n \leq \ldots$ for all $x \in X$.

Let $s(x) = \lim_{n\to\infty} s_n(x)$ for each $x \in X$. This pointwise limit exists (possibly $s(x) = +\infty$ at some points) because the sequence is monotonically increasing.

By the Theorem 2.1 (Monotone Convergence Theorem) for L^p spaces, since $\{s_n\}$ is a monotonically increasing sequence of non-negative functions with $s_n \to s$ pointwise, we have:

$$\|s\|_p = \lim_{n\to\infty} \|s_n\|_p \leq M < \infty \tag{1}$$

Hence, $s \in L^p$ and $s = \sum_{n=1}^{\infty} f_n$ in the L^p sense. This proves that L^p has the Riesz-Fischer property.

Alternatively, we can view this as follows: Let $\alpha = \sum_{n=1}^{\infty} \|f_n\|_p$ and $s_n = \sum_{k=1}^{n} f_k$ for $n = 1, 2, \ldots$. Then $\|s_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq \alpha$ for all n, so the pointwise limit $s = \sum_{n=1}^{\infty} f_n$ satisfies $\|s\|_p = \lim_{n\to\infty} \|s_n\|_p \leq \alpha$. This shows that $s = \sum_{n=1}^{\infty} f_n$ is a member of L^p. ∎

2.4 Related lemmas

First we need to use the following three lemmas for on normed Riesz space: infinite triangle inequality, property of convergence in normed Riesz space, and relationship between norm and order convergence.

Lemma 2.1 (Infinite Triangle Inequality)**.** Let $(E, \|\cdot\|)$ be a normed Riesz space with the Riesz-Fischer property. Then the following inequality holds:

$$\left\| \sum_{n=1}^{\infty} u_n \right\| \leq \sum_{n=1}^{\infty} \|u_n\|$$

Proof. We proceed by contradiction. Suppose there exists a sequence $(u_n)_{n=1}^{\infty} \subset E^+$ with $\sum_{n=1}^{\infty} \|u_n\| < \infty$ such that the inequality fails. Then there exists $\varepsilon > 0$ satisfying:

$$\left\| \sum_{n=1}^{\infty} u_n \right\| > \sum_{n=1}^{\infty} \|u_n\| + \varepsilon$$

For each $m \in \mathbb{N}$, define the tail sum $t_m := \sum_{n=m+1}^{\infty} u_n$. By the triangle inequality and our assumption:

$$\begin{aligned}
\|t_m\| &= \left\|\sum_{n=1}^{\infty} u_n - \sum_{n=1}^{m} u_n\right\| \\
&\geq \left\|\sum_{n=1}^{\infty} u_n\right\| - \left\|\sum_{n=1}^{m} u_n\right\| \\
&> \sum_{n=1}^{\infty} \|u_n\| + \varepsilon - \sum_{n=1}^{m} \|u_n\| \\
&= \sum_{n=m+1}^{\infty} \|u_n\| + \varepsilon
\end{aligned}$$

Thus, for all $m \in \mathbb{N}$, we have $\|t_m\| > \sum_{n=m+1}^{\infty} \|u_n\| + \varepsilon$.

Since $\sum_{n=1}^{\infty} \|u_n\| < \infty$, we can choose a strictly increasing sequence of indices $(n_k)_{k=1}^{\infty}$ such that:

$$\sum_{n=n_k+1}^{\infty} \|u_n\| < \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}$$

Define $s_k := \sum_{n=n_k+1}^{\infty} u_n$ for each $k \in \mathbb{N}$. This forms a decreasing sequence in E^+.

Now, construct a new sequence $(w_j)_{j=1}^{\infty} \subset E^+$ by arranging all elements u_n that appear in any s_k, with the property that each u_n appears at least k times if $n > n_k$. By construction:

$$\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{k=1}^{\infty} \frac{2}{k^2} < \infty$$

By the Riesz-Fischer property of E, the sum $s := \sum_{j=1}^{\infty} w_j$ exists in E. Furthermore, for each $k \in \mathbb{N}$:

$$s \geq k \cdot s_k$$

since each element appearing in s_k appears at least k times in the sequence (w_j).

Therefore, by monotonicity of the norm and our earlier inequality:

$$\|s\| \geq k \cdot \|s_k\| > k \cdot \varepsilon \quad \text{for all } k \in \mathbb{N}$$

This implies $\|s\| = \infty$, contradicting the fact that $s \in E$. Thus, the infinite triangle inequality must hold for any sequence in E^+ with summable norms. ■

Lemma 2.2 (Properties of Convergence in Normed Riesz Spaces). Let E be a normed Riesz space. If $(f_n)_{n=1}^{\infty} \subset E$ converges to f (in order, in norm) and $f_n \geq g$ for all $n \in \mathbb{N}$, then $f \geq g$. In particular, if $(f_n)_{n=1}^{\infty} \subset E^+$ converges to f and $f_n \geq 0$ for all $n \in \mathbb{N}$, then $f \geq 0$. Thus, the positive cone E^+ is closed.

Proof. It may be assumed that $g = 0$. Since $|f^- - f_n^-| \leq |f - f_n|$, the sequence $(f_n^- : n = 1, 2, \ldots)$ converges to f^-. But $f_n^- = 0$ for all n, so $f^- = 0$. In other words, $f \geq 0$. ■

Lemma 2.3 (Relation between Norm and Order Convergence). Let E be a normed Riesz space. If $(f_n)_{n=1}^{\infty} \subset E$ is an increasing sequence (i.e., $f_n \uparrow$) and $f_n \to f$ in norm, then $f_n \uparrow f$ (similar for decreasing sequence)

Proof. Let $f_n \uparrow$ and $f_n \to f$ (norm). Since $f_n \geq f_m$ for all $n \geq m$, we have $f \geq f_m$ by Lemma 2.2. This holds for all m, so f is an upper bound of the sequence.

Let g be another upper bound. Then $f_n \leq g$ for all n, so $f \leq g$ (once more by Lemma 2.2)). It follows that f is the least upper bound of the sequence, i.e., $f_n \uparrow f$. ■

2.5 Prove a normed Riesz space is a Banach lattice if and onlf it has the Riesz Fischer property

Now we use this lemma to prove that a normed Riesz space is a Banach lattice if and only if it has the Riesz-Fischer property.

Theorem 2.3 (Characterization of Banach Lattices)**.** Let $(E, \|\cdot\|)$ be a normed Riesz space. Then E is a Banach lattice if and only if E has the Riesz-Fischer property.

Proof. We prove both directions of the equivalence.

($\Rightarrow$) Assume E is a Banach lattice:

Let $(u_n)_{n=1}^{\infty} \subset E^+$ be a sequence of positive elements such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Define the sequence of partial sums $(s_n)_{n=1}^{\infty}$ by:

$$s_n = \sum_{k=1}^{n} u_k \quad \text{for } n \in \mathbb{N}$$

For any integers $m > n$, using the triangle inequality:

$$\begin{aligned} \|s_m - s_n\| &= \left\| \sum_{k=n+1}^{m} u_k \right\| \\ &\leq \sum_{k=n+1}^{m} \|u_k\| \end{aligned}$$

Since $\sum_{n=1}^{\infty} \|u_n\| < \infty$, we have $\sum_{k=n+1}^{m} \|u_k\| \to 0$ as $n, m \to \infty$. Thus, $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence in E. Since E is a Banach lattice, there exists $s \in E$ such that $\|s_n - s\| \to 0$ as $n \to \infty$.

Moreover, since $(s_n)_{n=1}^{\infty}$ is an increasing sequence of positive elements ($s_n \leq s_{n+1}$ for all n), by Lemma 2.3 , the order limit $\sup_n s_n$ exists and equals s. Therefore:

$$\sum_{n=1}^{\infty} u_n = \sup_{n \in \mathbb{N}} s_n = s \in E$$

This establishes that E has the Riesz-Fischer property.

($\Leftarrow$) Assume E has the Riesz-Fischer property:

To prove E is a Banach lattice, we need to show that E is norm complete. By the standard characterization of Banach spaces, it suffices to show that any absolutely convergent series in E is convergent in norm.

Let $(f_n)_{n=1}^{\infty} \subset E$ be a sequence such that $\sum_{n=1}^{\infty} \|f_n\| < \infty$. We need to prove that the series $\sum_{n=1}^{\infty} f_n$ converges in norm in E.

First, observe that $\sum_{n=1}^{\infty} \|f_n^+\| \leq \sum_{n=1}^{\infty} \|f_n\| < \infty$ and $\sum_{n=1}^{\infty} \|f_n^-\| \leq \sum_{n=1}^{\infty} \|f_n\| < \infty$, where f_n^+ and f_n^- are the positive and negative parts of f_n, respectively.

By our hypothesis that E has the Riesz-Fischer property, the order limits $s^+ = \sum_{n=1}^{\infty} f_n^+$ and $s^- = \sum_{n=1}^{\infty} f_n^-$ exist in E. Define $s = s^+ - s^-$, which represents the sum $\sum_{n=1}^{\infty} f_n$.

For each $m \in \mathbb{N}$, let $s_m = \sum_{k=1}^{m} f_k$. We need to show that $\|s - s_m\| \to 0$ as $m \to \infty$.

Note that $s - s_m = \sum_{k=m+1}^{\infty} f_k$. By the infinite triangle inequality established in Lemma 2.1

$$\|s - s_m\| = \left\| \sum_{k=m+1}^{\infty} f_k \right\| \tag{2}$$

$$\leq \sum_{k=m+1}^{\infty} \|f_k\| \tag{3}$$

Since $\sum_{n=1}^{\infty} \|f_n\| < \infty$, we have $\sum_{k=m+1}^{\infty} \|f_k\| \to 0$ as $m \to \infty$. Therefore, $\|s - s_m\| \to 0$ as $m \to \infty$, which proves that the sequence of partial sums $(s_m)_{m=1}^{\infty}$ converges in norm to s.

This confirms that E is a Banach lattice, completing the proof. ■

2.6 L^p is a Banach lattice

Corollary 2.1 (L^p Spaces are Banach Lattices)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is a Banach lattice.

Proof. By Theorem 2.2, L^p spaces have the Riesz-Fischer property, and by Theorem 2.3, this concludes that L^p spaces are Banach lattice. ■

3 Prove positive operator on L^p are norm bounded

In this section, we prove that positive operator on L^p is norm bounded. We first present some related definitions, and then we prove that every positive operator on Riesz space is order bounded. Then, we prove the main theorem that every positive order bounded operator between a Banach lattice and a normed Riesz space is norm bounded. Since we know that L^p spaces are Banach lattice, this concludes that positive operators on L^p are norm bounded.

3.1 Related Definitions

Definition 3.1 (Positive Operator). The linear operator $T : V \to W$ is called a *positive operator* if T maps the positive cone of V into the positive cone of W, i.e., if $f \geq 0$ in V implies $Tf \geq 0$ in W.

Definition 3.2 (Order Bounded Operator). Let E and F be Riesz spaces. A operator $T : E \to F$ is called *order bounded* if it maps any order interval in E into an order interval on F

Definition 3.3 (Norm Bounded Operator). Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces over the same field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). An operator $T : V \to W$ is said to be *norm bounded* if there exists a constant $M \in [0, \infty)$ such that

$$\|Tx\|_W \leq M\|x\|_V \quad \text{for all } x \in V$$

3.2 Positive operator on Riesz space is order bounded

In this section, we show that every positive operator between Riesz spaces is order bounded.

Lemma 3.1 (Positive Operators are Order Bounded). Let V and W be Riesz spaces, and let $T : V \to W$ be a positive operator. Then T is order bounded.

Proof. Let $[f_1, f_2]$ be an arbitrary order interval in V. We need to show that T maps this interval into an order interval in W.

Fix $f \in [f_1, f_2]$, we have $f_1 \leq f \leq f_2$. Since T is positive, we have

$$Tf_1 \leq Tf \leq Tf_2$$

Since f is arbitrary, this implies T maps the order interval $[f_1, f_2]$ into $[Tf_1, Tf_2]$

This shows that T is order bounded. ∎

3.3 Positive order bounded operator on Banach lattice to normed Riesz space is norm bounded

In this section, we present the main theorem as follows.

Theorem 3.1 (Positive order bounded operator on Banach lattice is normed bounded). If E is a Banach lattice and F is a normed Riesz space, and $T : E \to F$ is a positive order bounded operator, then T is norm bounded.

Proof. We prove this theorem by contradiction. Assume that $T : E \to F$ is positive, order bounded, but not norm bounded. Then there exists a sequence $(f_n)_{n=1}^{\infty}$ in E such that $\|Tf_n\| > 2n^3\|f_n\|$ for all n.

Note that $f_n \neq 0$ for all n, so (replacing f_n by $f_n/\|f_n\|$) we may assume that $\|f_n\| = 1$ and $\|Tf_n\| > 2n^3$ for all n. Since $\|Tf_n\|$ is less than or equal to $\|Tf_n^+\| + \|Tf_n^-\|$, it follows that at least one of $\|Tf_n^+\|$ and $\|Tf_n^-\|$ exceeds n^3. Hence, we may assume that

$$f_n > 0, \|f_n\| \leq 1 \text{ and } \|Tf_n\| > n^3 \text{ for } n = 1, 2, \dots$$

Writing $u_n = n^{-2} f_n$, we get

$$u_n \in E^+, \|u_n\| \leq n^{-2} \text{ and } \|Tu_n\| > n \text{ for } n = 1, 2, \dots$$

It follows that $\sum_{n=1}^{\infty} \|u_n\|$ converges, so (since E is a Banach lattice and, therefore, has the Riesz-Fischer property by Theorem 2.3) the element $s = \sum_{n=1}^{\infty} u_n$ exists in E.

This shows that all u_n are contained in the order interval $[0, s]$. Since T is order bounded, all Tu_n are then contained in some order interval in F. But then the sequence of all Tu_n is norm bounded, which contradicts $\|Tu_n\| > n$ for all n.

Hence, the operator T is norm bounded. ■

3.4 Positive operator between L^p spaces are norm bounded

Corollary 3.1 (Positive Operators on L^p Spaces are Norm Bounded)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p < \infty$, and let $T : L^p \to L^p$ be a positive operator. Then T is norm bounded.

Proof. We know from Corollary 2.1 that $L^p(X)$ is a Banach lattice for $1 \leq p < \infty$.

Since T is a positive operator, it is order bounded by Lemma 3.1. Now, by Theorem 3.1, every positive order bounded operator from a Banach lattice to a normed Riesz space is norm bounded.

Therefore, the positive operator $T : L^p \to L^p$ is norm bounded. ■

4 Prove positive operator on L^p is order continuous

In this section, we combine everything we have showned before and prove that every positive operator on L^p is order continuous. First, we present the related definition.

4.1 Related Definitions

Definition 4.1 (Order Continuous Operator). Let E and F be Riesz spaces. A linear operator $T : E \to F$ is called *order continuous* if for every sequence $(f_n)_{n=1}^{\infty}$ in E such that $f_n \downarrow 0$ (i.e., f_n is decreasing and $\inf_n f_n = 0$), we have

$$Tf_n \downarrow 0 \quad \text{in} \quad F$$

4.2 Proof of the main theorem

Theorem 4.1 (Positive Operators on L^p are Order Continuous). Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p < \infty$, and let $T : L^p(X) \to L^p(X)$ be a positive operator. Then T is order continuous.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a monotone decreasing sequence in $L^p(X)$ such that $f_n \downarrow 0$. We need to show that $Tf_n \downarrow 0$.

From Theorem 1.3, L^p norm is order continuous, i.e., $f_n \downarrow 0$ implies $\|f_n\| \downarrow 0$.

Since T is a positive operator, by Corollary 3.1, we know that T is norm bounded, i.e.,

$$\|Tf_n\| \leq \|T\| \|f_n\|$$

where $\|T\|$ is the operator norm.

This implies that $\|Tf_n\| \to 0$ in norm as $n \to \infty$.

Since T is a positive operator, and $f_n \downarrow$, we have $Tf_n \downarrow$. By Lemma 2.3, we know $Tf_n \downarrow 0$.

Hence, T is order continuous. ■

References

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