

# Foundations of Sequence-to-Sequence Modeling for Time Series

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## Abstract

The availability of large amounts of time series data, paired with the performance of deep-learning algorithms on a broad class of problems, has recently led to significant interest in the use of sequence-to-sequence models for time series forecasting. We provide the first theoretical analysis of this time series forecasting framework. We include a comparison of sequence-to-sequence modeling to classical time series models, and as such our theory can serve as a quantitative guide for practitioners choosing between different modeling methodologies.

## 1 Introduction

Time series analysis is a critical component of real-world applications such as climate modeling, web traffic prediction, neuroscience, as well as economics and finance. We focus on the fundamental question of time series forecasting. Specifically, we study the task of forecasting the next  $\ell$  steps of an  $m$ -dimensional time series  $\mathbf{Y}$ , where  $m$  is assumed to be very large. For example, in climate modeling,  $m$  may correspond to the number of locations at which we collect historical observations, and more generally to the number of sources which provide us with time series.

Often, the simplest way to tackle this problem is to approach it as  $m$  separate tasks, where for each of the  $m$  dimensions we build a model to forecast the univariate time series corresponding to that dimension. Auto-regressive and state-space models [8, 4, 6, 5, 12], as well as non-parametric approaches such as RNNs [3], are often used in this setting. To account for correlations between different time series, these models have also been generalized to the multivariate case [23, 24, 35, 13, 14, 1, 2, 35, 2, 38, 31, 41, 25]. In both univariate and multivariate settings, an observation at time  $t$  is treated as a single sample point, and the model tries to capture relations between observations at times  $t$  and  $t + 1$ . Therefore, we refer to these models as *local*.

In contrast, an alternative methodology based on treating  $m$  univariate time series as  $m$  samples drawn from some unknown distribution has also gained popularity in recent years. In this setting, each of the  $m$  dimensions of  $\mathbf{Y}$  is treated as a separate example and a single model is learned from these  $m$  observations. Given  $m$  time series of length  $T$ , this model learns to map past vectors of length  $T - \ell$  to corresponding future vectors of length  $\ell$ . LSTMs and RNNs [15] are a popular choice of model class for this setup [9, 11, 42, 22, 21, 43].<sup>1</sup> Consequently, we refer to this framework as *sequence-to-sequence modeling*.

While there has been progress in understanding the generalization ability of local models [40, 27, 28, 29, 17, 18, 19, 20, 44], to the best of our knowledge the generalization properties of sequence-to-sequence modeling have not yet been studied, raising the following natural questions:

- What is the generalization ability of sequence-to-sequence models and how is it affected by the statistical properties of the underlying stochastic processes (e.g. non-stationarity, correlations)?
- When is sequence-to-sequence modeling preferable to local modeling, and vice versa?

We provide the first generalization guarantees for time series forecasting with sequence-to-sequence models. Our results are expressed in terms of simple, intuitive measures of non-stationarity and correlation strength between different time series and hence explicitly depend on the key components of the learning problem.

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<sup>1</sup>Sequence-to-sequence models are also among the winning solutions in the recent time series forecasting competition: <https://www.kaggle.com/c/web-traffic-time-series-forecasting>.

Table 1: Summary of local, sequence-to-sequence, and hybrid models.

LEARNING MODEL	TRAINING SET	HYPOTHESIS	EXAMPLE
UNIVAR. LOCAL	$\mathbf{Z}_i = \{(Y_{t-1-p}^{t-1}(i), Y_t(i)) : p \leq t \leq T\}$	$h_i : \mathcal{Y}^p \rightarrow \mathcal{Y}$	ARIMA
MULTIVAR. LOCAL	$\mathbf{Z} = \{(Y_{t-1-p}^{t-1}(\cdot), Y_t(\cdot)) : p \leq t \leq T\}$	$h : \mathcal{Y}^{m \times p} \rightarrow \mathcal{Y}^m$	VARMA
SEQ-TO-SEQ	$\mathbf{Z} = \{(Y_1^{T-1}(i), Y_T(i)) : 1 \leq i \leq m\}$	$h : \mathcal{Y}^T \rightarrow \mathcal{Y}$	Neural nets
HYBRID	$\mathbf{Z} = \{(Y_{t-p}^{t-1}(i), Y_t(i)) : 1 \leq i \leq m, p \leq t \leq T\}$	$h : \mathcal{Y}^p \rightarrow \mathcal{Y}$	Neural nets

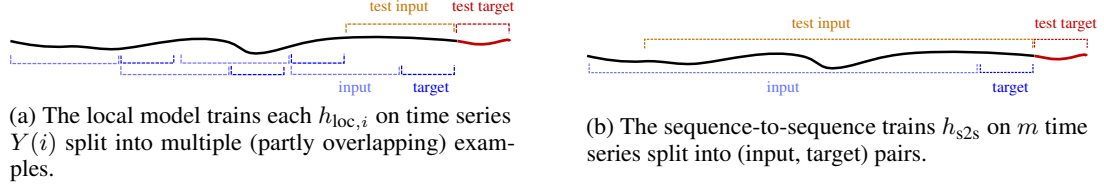


Figure 1: Local and sequence-to-sequence splits of a one dimensional time series into training and test examples.

We compare our generalization bounds to guarantees for local models and identify regimes under which one methodology is superior to the other. Therefore, our theory may also serve as a quantitative guide for a practitioner choosing the right modeling approach.

The rest of the paper is organized as follows: in Section 2, we formally define sequence-to-sequence and local modeling. In Section 3, we define the key tools that we require for our analysis. Generalization bounds for sequence-to-sequence models are given in Section 4. We compare sequence-to-sequence and local models in Section 5. Section 6 concludes this paper with a study of a setup that is a hybrid of the local and sequence-to-sequence models.

## 2 Sequence-to-sequence modeling

We begin by providing a formal definition of sequence-to-sequence modeling. The learner receives a multi-dimensional time series  $\mathbf{Y} \in \mathcal{Y}^{m \times T}$  which we view as  $m$  time series of same length  $T$ . We denote by  $Y_t(i)$  the value of the  $i$ -th time series at time  $t$  and write  $Y_a^b(i)$  to denote the sequence  $(Y_a(i), Y_{a+1}(i), \dots, Y_b(i))$ . Similarly, we let  $Y_t(\cdot) = (Y_t(1), \dots, Y_t(m))$  and  $Y_a^b(\cdot) = (Y_a(\cdot), \dots, Y_b(\cdot))$ . In particular,  $\mathbf{Y} \equiv Y_1^T(\cdot)$ . In addition, the sequence  $Y_1^{T-1}(\cdot)$  is of a particular importance in our analysis and we denote it by  $\mathbf{Y}'$ .

The goal of the learner is to predict  $Y_{T+1}(\cdot)$ .<sup>2</sup> We further assume that our input  $\mathbf{Y}$  is partitioned into a training set of  $m$  examples  $\mathbf{Z} = \{Z_1, \dots, Z_m\}$ , where each  $Z_i = (Y_1^{T-1}(i), Y_T(i)) \in \mathcal{Y}^T$ . The learner's objective is to select a hypothesis  $h : \mathcal{Y}^T \rightarrow \mathcal{Y}$  from a given hypothesis set  $\mathcal{H}$  that achieves a small generalization error:

$$\mathcal{L}(h \mid \mathbf{Y}) = \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{\mathcal{D}} [L(h(Y_1^T(i)), Y_{T+1}(i)) \mid \mathbf{Y}],$$

where  $L : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, M]$  is a bounded<sup>3</sup> loss function and  $\mathcal{D}$  is the distribution of  $Y_{T+1}$  conditioned on the past  $\mathbf{Y}$ .

In other words, the learner seeks a hypothesis  $h$  that maps sequences of past  $Y_1(i), \dots, Y_T(i)$  values to sequences of future values  $Y_{T+1}(i), \dots, Y_{T+\ell}(i)$ , justifying our choice of “sequence-to-sequence” terminology.<sup>4</sup> Incidentally, the machine translation problem studied by Sutskever et al. [36] under the same name represents a special case of our problem when sequences (sentences) are independent and data is

<sup>2</sup>We are often interested in long term forecasting, i.e. predicting  $Y_{T+1}^{T+\ell}(\cdot)$  for  $\ell \geq 1$ . For simplicity, we only consider the case of  $\ell = 1$ . However, all our results extend immediately to  $\ell \geq 1$ .

<sup>3</sup>Most of the results in this paper can be straightforwardly extended to unbounded case assuming  $Y$  is sub-Gaussian.

<sup>4</sup>In practice, each  $Z_i$  may start at a different, arbitrary time  $t_i$ , and may furthermore include some additional features  $X_i$ , i.e.  $Z_i = (Y_{t_i}^{T-1}(i), X_i, Y_T(i))$ . Our results can be extended to this case as well using an appropriate choice of the hypothesis set.

stationary. In fact, LSTM-based approaches used in aforementioned translation problem are also common for time series forecasting [9, 11, 42, 22, 21, 43]. However, feed-forward NNs have also been successfully applied in this framework [34] and in practice, our definition allows for any set of functions  $\mathcal{H}$  that map input sequences to output sequences. For instance, we can train a feed-forward NN to map  $Y_1^{T-1}(i)$  to  $Y_T(i)$  and at inference time use  $Y_2^T(i)$  as input to obtain a forecast for  $Y_T(i)$ .<sup>5</sup>

We contrast sequence-to-sequence modeling to *local* modeling, which consists of splitting each time series  $Y(i)$  into a training set  $\mathbf{Z}_i = \{Z_{i,1}, \dots, Z_{i,T}\}$ , where  $Z_{i,t} = (Y_{t-p}^{t-1}(i), Y_t(i))$  for some  $p \in \mathbb{N}$ , then learning a separate hypothesis  $h_i$  for each  $\mathbf{Z}_i$ . Each  $h_i$  models relations between observations that are close in time, which is why we refer to this framework as local modeling. As in sequence-to-sequence modeling, the goal of a local learner is to achieve a small generalization error for  $h_{\text{loc}} = (h_1, \dots, h_m)$ , given by:

$$\mathcal{L}(h_{\text{loc}} | \mathbf{Y}) = \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{\mathcal{D}} [L(h_i(Y_{T-p}^T(i)), Y_{T+1}(i)) | \mathbf{Y}].$$

In order to model correlations between different time series, it is also common to split  $\mathbf{Y}$  into one single set of multivariate examples  $\mathbf{Z} = \{Z_1, \dots, Z_T\}$ , where  $Z_i = (Y_{t-p}^{t-1}(\cdot), Y_t(\cdot))$ , and to learn a single hypothesis  $h$  that maps  $\mathcal{Y}^{m \times p} \rightarrow \mathcal{Y}^m$ . As mentioned earlier, we consider this approach a variant of local modeling, since  $h$  in this case again models relations between observations that are close in time.

Finally, *hybrid* or *local sequence-to-sequence* models, which interpolate between local and sequence-to-sequence approaches, have also been considered in the literature [43]. In this setting, each local example is split across the temporal dimension into smaller examples of length  $p$ , which are then used to train a single sequence-to-sequence model  $h$ . We discuss bounds for this specific case in Section 6.

Our work focuses on the statistical properties of the sequence-to-sequence model. We provide the first generalization bounds for sequence-to-sequence and hybrid models, and compare these to similar bounds for local models, allowing us to identify regimes in which one methodology is more likely to succeed.

Aside from learning guarantees, there are other important considerations that may lead a practitioner to choose one approach over others. For instance, the local approach is trivially parallelizable; on the other hand, when additional features  $X_i$  are available, sequence-to-sequence modeling provides an elegant solution to the *cold start problem* in which at test time we are required to make predictions on time series for which no historical data is available.

### 3 Correlations and non-stationarity

In the standard supervised learning scenario, it is common to assume that training and test data are drawn *i.i.d.* from some unknown distribution. However, this assumption does not hold for time series, where observations at different times as well as across different series may be correlated. Furthermore, the data-generating distribution may also evolve over time.

These phenomena present a significant challenge to providing guarantees in time series forecasting. To quantify non-stationarity and correlations, we introduce the notions of mixing coefficients and discrepancy, which are defined below.

The final ingredient we need to analyze sequence-to-sequence learning is the Rademacher complexity  $\mathfrak{R}_m(\mathcal{F})$  of a family of functions  $\mathcal{F}$  on a sample of size  $m$ , which has been previously used to characterize learning in the *i.i.d.* setting [16, 30]. In App. A, we include a brief discussion of its properties.

#### 3.1 Expected mixing coefficients

To measure the strength of dependency between time series, we extend the notion of  $\beta$ -mixing coefficients [7] to *expected  $\beta$ -mixing coefficients*, which are a more appropriate measure of correlation in sequence-to-sequence modeling.

**Definition 1** (Expected  $\beta_{\text{s2s}}$  coefficients). Let  $i, j \in [m] \triangleq \{1, \dots, m\}$ . We define

$$\beta_{\text{s2s}}(i, j) = \mathbb{E}_{\mathbf{Y}'} [\|P(Y_T(i)|\mathbf{Y}')P(Y_T(j)|\mathbf{Y}') - P(Y_T(i), Y_T(j)|\mathbf{Y}')\|_{TV}],$$

<sup>5</sup>As another example, a runner-up in the Kaggle forecasting competition (<https://www.kaggle.com/c/web-traffic-time-series-forecasting>) used a combination of boosted decision trees and feed-forward networks, and as such employs the sequence-to-sequence approach.

where  $TV$  denotes the total variations norm. For a subset  $I \subseteq [m]$ , we define

$$\beta_{s2s}(I) = \sup_{i,j \in C} \beta_{s2s}(i, j).$$

The coefficient  $\beta_{s2s}(i, j)$  captures how close  $Y_{T+1}(i)$  and  $Y_{T+1}(j)$  are to being independent, given  $\mathbf{Y}'$  (and averaged over all realizations of  $\mathbf{Y}'$ ). We further study these coefficients in Section 4, where we derive explicit upper bounds on expected  $\beta_{s2s}$ -mixing coefficients for various standard classes of stochastic processes, including spatio-temporal and hierarchical time series.

We also define the following related notion of  $\bar{\beta}$ -coefficients.

**Definition 2** (Unconditional  $\bar{\beta}$  coefficients). Let  $i, j \in [m] \triangleq \{1, \dots, m\}$ . We define

$$\begin{aligned} \bar{\beta}(i, j) &= \|\Pr(Y_1^T(i), Y_1^T(j)) - \Pr(Y_1^T(i)) \Pr(Y_1^T(j))\|_{TV} \\ \bar{\beta}'(i, j) &= \|\Pr(Y_1^{T-1}(i), Y_1^{T-1}(j)) - \Pr(Y_1^{T-1}(i)) \Pr(Y_1^{T-1}(j))\|_{TV} \end{aligned}$$

and as before, for a subset  $I$  of  $[m]$ , write  $\bar{\beta}(I) = \sup_{i,j \in I} \bar{\beta}(i, j)$  (and similarly for  $\bar{\beta}'$ ).

Note that  $\beta_{s2s}$  coefficients measure the strength of dependence between time series conditioned on the history observed so far, while  $\bar{\beta}$  coefficients measure the (unconditional) strength of dependence between time series. The following result relates these two notions.

**Lemma 1.** For  $\bar{\beta}$  (and  $\bar{\beta}'$  similarly), we have the following upper bound:

$$\bar{\beta}(i, j) \leq \beta_{s2s}(i, j) + \mathbb{E}_{\mathbf{Y}'} \left[ \text{Cov} \left( \Pr(Y_T(i) \mid \mathbf{Y}'), \Pr(Y_T(j) \mid \mathbf{Y}') \right) \right]$$

The proof of this result (as well as all other proofs in this paper) is deferred to the supplementary material.

Finally, we require the notion of *tangent collections*, within which time series are independent.

**Definition 3** (Tangent collection). Given a collection of time series  $C = \{Y(1), \dots, Y(c)\}$ , we define the tangent collection  $\tilde{C}$  as  $\{\tilde{Y}(1), \dots, \tilde{Y}(c)\}$  such that  $\tilde{Y}(i)$  is drawn according to the marginal  $\Pr(Y(i))$  and such that  $\tilde{Y}(i)$  and  $\tilde{Y}(i')$  are independent for  $i \neq i'$ .

The notion of tangent collections, combined with mixing coefficients, allows us to reduce the analysis of correlated time series in  $C$  to the analysis of independent time series in  $\tilde{C}$  (see Prop. 6 in the appendix).

### 3.2 Discrepancy

Various notions of discrepancy have been previously used to measure the non-stationarity of the underlying stochastic processes with respect to the hypothesis set  $\mathcal{H}$  and loss function  $L$  in the analysis of local models [18, 44]. In this work, we introduce a notion of discrepancy specifically tailored to sequence-to-sequence modeling scenario, taking into account both the hypothesis set and the loss function.

**Definition 4** (Discrepancy). Let  $\mathcal{D}$  be the distribution of  $Y_{T+1}$  conditioned on  $\mathbf{Y}$  and let  $\mathcal{D}'$  be the distribution of  $Y_T$  conditioned on  $\mathbf{Y}'$ . We define the discrepancy  $\Delta$  as  $\Delta = \sup_{h \in \mathcal{H}} |\mathcal{L}(h \mid \mathbf{Y}) - \mathcal{L}(h \mid \mathbf{Y}')|$  where  $\mathcal{L}(h \mid \mathbf{Y}') = \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{\mathcal{D}'} [L(h(Y_1^{T-1}(i)), Y_T(i)) \mid \mathbf{Y}']$ .

The discrepancy forms a pseudo-metric on the space of probability distributions and can be completed to a Wasserstein metric (by extending  $\mathcal{H}$  to all Lipschitz functions). This also immediately implies that the discrepancy can be further upper bounded by the  $l_1$ -distance and by relative entropy between conditional distributions of  $Y_T$  and  $Y_{T+1}$  (via Pinsker's inequality). However, unlike these other divergences, the discrepancy takes into account both the hypothesis set and the loss function, making it a finer measure of non-stationarity.

However, the most important property of the discrepancy is that it can be upper bounded by the related notion of *symmetric discrepancy*, which can be estimated from data.

**Definition 5** (Symmetric discrepancy). We define the symmetric discrepancy  $\Delta_s$  as

$$\Delta_s = \frac{1}{m} \sup_{h, h' \in \mathcal{H}} \left| \sum_{i=1}^m L(h(Y_1^T(i)), h'(Y_1^T(i))) - L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))) \right|.$$

**Proposition 1.** Let  $\mathcal{H}$  be a hypothesis space and let  $L$  be a bounded loss function which respects the triangle inequality. Let  $h \in \mathcal{H}$  be any hypothesis. Then,  $\Delta \leq \Delta_s + \mathcal{L}(h \mid \mathbf{Y}) + \mathcal{L}(h \mid \mathbf{Y}')$ .

We do not require test labels to evaluate  $\Delta_s$ . Since  $\Delta_s$  only depends on the observed data,  $\Delta_s$  can be computed directly from samples, making it a useful tool to assess the non-stationarity of the learning problem.

Another useful property of  $\Delta_s$  is that, for certain classes of stochastic processes, we can provide a direct analysis of this quantity.

**Proposition 2.** *Let  $I_1, \dots, I_k$  be a partition of  $\{1, \dots, m\}$ ,  $C_1, \dots, C_k$  be the corresponding partition of  $\mathbf{Y}$  and  $C'_1, \dots, C'_k$  be the corresponding partition of  $\mathbf{Y}'$ . Write  $c = \min_j |C_j|$ , and define the expected discrepancy*

$$\Delta_e = \sup_{h, h' \in \mathcal{H}} \left[ \mathbb{E}_Y [L(h(Y_1^T), h'(Y_1^T))] - \mathbb{E}_Y [L(h(Y_1^{T-1}), h'(Y_1^{T-1}))] \right].$$

*Then, writing  $\mathfrak{R}$  the Rademacher complexity (see Appendix A) we have with probability  $1 - \delta$ ,*

$$\Delta_s \leq \Delta_e + \max \left( \max_j \mathfrak{R}_{|C_j|}(\tilde{C}'_j), \max_j \mathfrak{R}_{|C_j|}(\tilde{C}_j) \right) + \sqrt{\frac{1}{2c} \log \frac{2k}{\delta - \sum_j (|I_j| - 1)[\bar{\beta}(I_j) + \bar{\beta}'(I_j)]}}.$$

The expected discrepancy  $\Delta_e$  can be computed analytically for many classes of stochastic processes. For example, for stationary processes, we can show that it is negligible. Similarly, for covariance-stationary<sup>6</sup> processes with linear hypothesis sets and the squared loss function, the discrepancy is once again negligible. These examples justify our use of the discrepancy as a natural measure of non-stationarity. In particular, the covariance-stationary example highlights that the discrepancy takes into account not only the distribution of the stochastic processes but also  $\mathcal{H}$  and  $L$ .

**Proposition 3.** *If  $Y(i)$  is stationary for all  $1 \leq i \leq m$ , and  $\mathcal{H}$  is a hypothesis space such that  $h \in \mathcal{H} : \mathcal{Y}^{T-1} \rightarrow \mathcal{Y}$  (i.e. the hypotheses only consider the last  $T - 1$  values of  $Y$ ), then  $\Delta_e = 0$ .*

**Proposition 4.** *If  $\mathbf{Y}$  is covariance stationary for all  $1 \leq i \leq m$ ,  $L$  is the squared loss, and  $\mathcal{H}$  is a linear hypothesis space  $\{x \rightarrow w \cdot x \mid \|w\| \in \mathbb{R}^p \leq \Lambda\}$ ,  $\Delta_e = 0$ .*

Another insightful example is the case when  $\mathcal{H} = \{h\}$ : then,  $\Delta = 0$  even if  $\mathbf{Y}$  is non-stationary, which illustrates that learning is trivial for trivial hypothesis sets, even in non-stationary settings.

The final example that we consider in this section is the case of non-stationary periodic time series. Remarkably, we show that the discrepancy is still negligible in this case provided that we observe all periods with equal probability.

**Proposition 5.** *If the  $Y(i)$  are periodic with period  $p$  and the observed starting time of each  $Y(i)$  is distributed uniformly at random in  $[p]$ , then  $\Delta_e = 0$ .*

## 4 Generalization bounds

We now present our generalization bounds for time series prediction with sequence-to-sequence models. We write  $\mathcal{F} = \{L \circ h : h \in \mathcal{H}\}$ , where  $f = L \circ h$  is the loss of hypothesis  $h$  given by  $f(h, Z_i) = L(h(Y_1^{T-1}(i)), Y_T)$ . To obtain bounds on the generalization error  $\mathcal{L}(h \mid \mathbf{Y})$ , we study the gap between  $\mathcal{L}(h \mid \mathbf{Y})$  and the empirical error  $\hat{\mathcal{L}}(h)$  of a hypothesis  $h$ , where

$$\hat{\mathcal{L}}(h) = \frac{1}{m} \sum_{i=1}^m f(h, Z_i).$$

That is, we aim to give a high probability bound on the supremum of the empirical process  $\Phi(\mathbf{Y}) = \sup_h [\mathcal{L}(h \mid \mathbf{Y}) - \hat{\mathcal{L}}(h)]$ . We take the following high-level approach: we first partition the training set  $\mathbf{Z}$  into  $k$  collections  $C_1, \dots, C_k$  such that within each collection, correlations between different time series are as weak as possible. We then analyze each collection  $C_j$  by comparing the generalization error of sequence-to-sequence learning on  $C_j$  to the sequence-to-sequence generalization error on the *tangent* collection  $\tilde{C}_j$ .

<sup>6</sup>Recall that a process  $X_1, X_2, \dots$  is called stationary if for any  $l, k, m$ , the distributions of  $(X_k, \dots, X_{k+l})$  and  $(X_{k+m}, \dots, X_{k+m+l})$  are the same. Covariance stationarity is a weaker condition that requires that  $\mathbb{E}[X_k]$  be independent of  $k$  and that  $\mathbb{E}[X_k X_m] = f(k - m)$  for some  $f$ .

**Theorem 4.1.** Let  $C_1, \dots, C_k$  form a partition of the training input  $\mathbf{Z}$  and let  $I_j$  denote the set of indices of time series that belong to  $C_j$ . Assume that the loss function  $L$  is bounded by 1. Then, we have for any  $\delta > \sum_j (|I_j| - 1)\beta(I_j)$ , with probability  $1 - \delta$ ,

$$\Phi(\mathbf{Y}) \leq \max_j \left[ \widehat{\mathfrak{R}}_{\tilde{C}_j}(\mathcal{F}) \right] + \Delta + \frac{1}{\sqrt{2 \min_j |I_j|}} \sqrt{\log \left( \frac{k}{\delta - \sum_j (|I_j| - 1)\beta_{s2s}(I_j)} \right)}.$$

Theorem 4.1 illustrates the trade-offs that are involved in sequence-to-sequence learning for time series forecasting. As  $\sum_j (|I_j| - 1)\beta(I_j)$  is a function of  $m$ , we expect it to decrease as  $m$  grows (*i.e.* more time series we have), allowing for smaller  $\delta$  as  $m$  increases.

Assuming that the  $C_j$  are of the same size, if  $\mathcal{H}$  is a collection of neural networks of bounded depth and width then  $\mathfrak{R}_{\tilde{C}_j}(\mathcal{F}) = \mathcal{O}(\sqrt{kT/m})$  (see Appendix A). Therefore,

$$\mathcal{L}(h | \mathbf{Y}) \leq \widehat{\mathcal{L}}(h) + \Delta + \mathcal{O}\left(\sqrt{\frac{kT}{m}}\right)$$

with high probability uniformly over  $h \in \mathcal{H}$ , provided that  $\frac{m}{k} \sum_{j=1}^k \beta_{s2s}(I_j) = o(1)$ . This shows that extremely high-dimensional ( $m \gg 1$ ) time series are beneficial for sequence-to-sequence models, whereas series with a long histories  $T \gg m$  will generally not benefit from sequence-to-sequence learning. Note also that correlations in data reduce the effective sample size from  $m$  to  $m/k$ .

Furthermore, Theorem 4.1 indicates that balancing the complexity of the model (e.g. depth and width of a neural net) with the fit it provides to the data is critical for controlling both the discrepancy and Rademacher complexity terms. We further illustrate this bound with several examples below.

#### 4.1 Independent time series

We begin by considering the case where all dimensions of  $\mathbf{Y}$  are independent. Although this may seem a restrictive assumption, it arises in a variety of applications: in neuroscience, different dimensions may represent brain scans of different patients; in reinforcement learning, they may correspond to different trajectories of a robotic arm.

**Theorem 4.2.** Let  $\mathcal{H}$  be a given hypothesis space with associated function family  $\mathcal{F}$  corresponding to a loss function  $L$  bounded by 1. Suppose that all dimensions of  $\mathbf{Y}$  are independent and let  $I_1 = [m]$ ; then  $\beta(I_1) = 0$  and so for any  $\delta > 0$ , with probability at least  $1 - \delta$  and for any  $h \in \mathcal{H}$ :

$$\mathcal{L}(h | \mathbf{Y}) \leq \widehat{\mathcal{L}}(h) + 2\mathfrak{R}_m(\mathcal{F}) + \Delta + \sqrt{\frac{\log(1/\delta)}{m}}.$$

Theorem 4.2 shows that when time series are independent, learning is not affected by correlations in the samples and can only be obstructed by the non-stationarity of the problem, captured via  $\Delta$ .

Note that when examples are drawn *i.i.d.*, we have  $\Delta = 0$  in Theorem 4.2: we recover the standard standard generalization results for regression problems.

#### 4.2 Correlated time series

We now consider several concrete examples of high-dimensional time series in which different dimensions may be correlated. This setting is common in a variety of applications including stock market indicators, traffic conditions, climate observations at different locations, and energy demand.

Suppose that each  $Y(i)$  is generated by the auto-regressive (AR) processes with correlated noise

$$y_{t+1}(i) = \Theta_i(y_0^t(i)) + \varepsilon_{t+1}(i) \quad (4.1)$$

where the  $w_i \in \mathbb{R}^p$  are unknown parameters and the noise vectors  $\varepsilon_t \in \mathbb{R}^m$  are drawn from a Gaussian distribution  $\mathcal{N}(0, \Sigma)$  where, crucially,  $\Sigma$  is not diagonal. The following lemma is key to our analysis.

**Lemma 2.** Two AR processes  $Y(i), Y(j)$  generated by (4.1) such that  $\sigma = \text{Cov}(Y(i), Y(j)) \leq \sigma_0 < 1$  verify  $\beta_{s2s}(i, j) = \max\left(\frac{3}{2(1-\sigma_0^2)}, \frac{1}{1-2\sigma_0}\right) \sigma = \mathcal{O}(\sigma)$ .



**Hierarchical time series.** As our first example, we consider the case of hierarchical time series that arises in many real-world applications [39, 37]. Consider the problem of energy demand forecasting: frequently, one observes a sequence of energy demands at a variety of levels: single household, local neighborhood, city, region and country. This imposes a natural hierarchical structure on these time series.

Formally, we consider the following hierarchical scenario: a binary tree of total depth  $D$ , where time series are generated at each of the leaves. At each leaf,  $Y(i)$  is given by the AR process (4.1) where we impose  $\Sigma_{ij} = (\frac{1}{m})^{d(i,j)}$  given  $d(i, j)$  the length of the shortest path from either leaf to the closest common ancestor between  $i$  and  $j$ . Hence, as  $d(i, j)$  increases,  $Y(i)$  and  $Y(j)$  grow more independent.

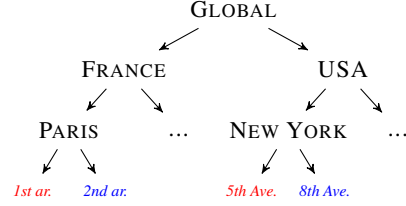


Figure 2: Hierarchical dependence. Collections for  $d = 1$  are given by  $C_1$  which contains the red leaves and  $C_2$  which contains the blue right-side leaves.

For the bound of Theorem 4.1 to be non-trivial, we require a partition  $C_1, \dots, C_k$  of  $\mathbf{Z}$  such that within a given  $C_j$  the time series are close to being independent. One such construction is the following: fix a depth  $d \leq D$  and construct  $C_1, \dots, C_{2^d}$  such that each  $C_i$  contains exactly one time series from each sub-tree of depth  $D - d$ ; hence,  $|C_i| = 2^{D-d}$ . Lemma 2 shows that for each  $C_i$ , we have  $\beta(C_i) = \mathcal{O}(m^{d-D})$ . For example, setting  $d = \frac{D}{2} = \frac{\log m}{2}$ , it follows that for any  $\delta > 0$ , with probability  $1 - \delta$ ,

$$\mathcal{L}(h|\mathbf{Y}) \leq \widehat{\mathcal{L}}(h) + \max_j \left[ \mathfrak{R}_{\tilde{C}_j}(\mathcal{F}) \right] + \Delta + \frac{1}{\sqrt{2} \sqrt[4]{m}} \sqrt{\log \left( \frac{\sqrt{m}}{\delta - \frac{m}{\mathcal{O}(\sqrt{m \log m})}} \right)}.$$

Furthermore, suppose the model is a linear AR process given by  $y_{t+1}(i) = w_i \cdot (y_{t-p}^t(i)) + \varepsilon_{t+1}(i)$ . Then, the underlying stochastic process is weakly stationary and by Prop. 3 our bound reduces to:  $\mathcal{L}(h|\mathbf{Y}) \leq \widehat{\mathcal{L}}(h) + \max_j \left[ \mathfrak{R}_{\tilde{C}_j}(\mathcal{F}) \right] + \mathcal{O}\left(\frac{\sqrt{\log m}}{m^{1/4}}\right)$ . By Proposition 5, similar results holds when  $\Theta_i$  is periodic.

**Spatio-temporal processes.** Another common task is spatio-temporal forecasting, in which historical observation are available at different locations. These observations may represent temperature at different locations, as in the case of climate modeling [26, 10], or car traffic at different locations [22].

It is natural to expect correlations between time series to decay as the geographical distance between them increases. As a simplified example, consider that the sphere  $\mathbb{S}^3$  is subdivided according to a geodesic grid and a time series is drawn from the center of each patch according to (4.1), also with  $\Sigma_{ij} = m^{-d(i,j)}$  but this time with  $d(i, j)$  equal to the (geodesic) distance between the center of two cell centers. We choose subsets  $C_i$  with the goal of minimizing the strength of dependencies between time series within each subsets. Assuming we divide the sphere into  $\sqrt{m}$  collections size approximately  $c = \sqrt{m}$  such that the minimal distance between two points in a set is  $d_0$ , we obtain

$$\mathcal{L}(h | \mathbf{Y}) \leq \widehat{\mathcal{L}}(h) + \max_j \left[ \mathfrak{R}_{\tilde{C}_j}(\mathcal{F}) \right] + \Delta + \frac{1}{\sqrt{2} \sqrt[4]{m}} \sqrt{\log \left( \frac{\sqrt{m}}{\delta - \mathcal{O}(m^{1-d_0})} \right)}.$$

As in the case of hierarchical time series, Proposition 3 or Proposition 5 can be used to remove the dependence on  $\Delta$  for certain families of stochastic processes.

## 5 Comparison to local models

This section provides comparison of learning guarantees for sequence-to-sequence models with those of local models. In particular, we will compare our bounds on the generalization gap  $\Phi(\mathbf{Y})$  for sequence-to-sequence models and local models, where the gap is given by

$$\Phi_{\text{loc}}(\mathbf{Y}) = \sup_{(h_1, \dots, h_m) \in H^m} \left[ \mathcal{L}(h_{\text{loc}} | \mathbf{Y}) - \widehat{\mathcal{L}}(h_{\text{loc}}) \right] \quad (5.1)$$

where  $\widehat{\mathcal{L}}(h_{\text{loc}})$  is the average empirical error of  $h_i$  on the sample  $\mathbf{Z}_i$ , defined as  $\widehat{\mathcal{L}}(h_{\text{loc}}) = \frac{1}{mT} \sum_{i=1}^m \sum_{t=1}^T f(h_i, Z_{t,i})$  where  $f(h_i, Z_{t,i}) = L(h_i(Y_{t-p}^{t-1}(i)), Y_t(i))$ .

To give a high probability bound for this setting, we take advantage of existing results for the single local model  $h_i$  [18]. These results are given in terms of a slightly different notion of discrepancy  $\Delta$ , defined by

$$\Delta(\mathbf{Z}_i) = \sup_{h \in \mathcal{H}} \left[ \mathbb{E} \left[ L(h(Y_{t-p+1}^T), Y_{T+1}) \mid Y_1^T \right] - \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[ L(h(Y_{t-p}^{t-1}), Y_t) \mid Y_1^{t-1} \right] \right].$$

Another required ingredient to state these results is the expected sequential covering number  $\mathbb{E}_{v \sim T(\mathbb{P})}[\mathcal{N}_1(\alpha, \mathcal{F}, v)]$  [18]. For many hypothesis sets, the log of the sequential covering number admits upper bounds similar to those presented earlier for the Rademacher complexity. We provide some examples below and refer the interested reader to [33] for a details.

**Theorem 5.1.** *For  $\delta > 0$  and  $\alpha > 0$ , with probability at least  $1 - \delta$ , for any  $(h_1, \dots, h_m)$ , and any  $\alpha > 0$ ,*

$$\Phi_{\text{loc}}(\mathbf{Y}) \leq \frac{1}{m} \sum_{i=1}^m \Delta(\mathbf{Z}_i) + 2\alpha + \sqrt{\frac{2}{T} \log \frac{m \max_i \mathbb{E}_{v \sim T(\mathbf{Z}_i)}[\mathcal{N}_1(\alpha, \mathcal{F}, v)]}{\delta}}.$$

Choosing  $\alpha = 1/\sqrt{T}$ , we can show that, for standard local models such as the linear hypothesis space  $\{x \rightarrow w^\top x, w \in \mathbb{R}^p, \|w\|_2 \leq \Lambda\}$ , we have

$$\sqrt{\frac{1}{T} \log \frac{2m \mathbb{E}_{v \sim T(\mathbf{Z})}[\mathcal{N}_1(\alpha, \mathcal{F}, v)]}{\delta}} = \mathcal{O}\left(\sqrt{\frac{\log m}{T}}\right).$$

In this case, it follows that  $\Phi_{\text{loc}}(\mathbf{Y}) \leq \frac{1}{m} \sum_{i=1}^m \Delta(\mathbf{Z}_i) + \mathcal{O}\left(\sqrt{\frac{\log m}{T}}\right)$ , where the last term in this bound should be compared with corresponding (non-discrepancy) terms in the bound of Theorem 4.1, which, as discussed above, scales as  $\mathcal{O}(\sqrt{T/m})$  for a variety of different hypothesis sets.

Hence, when we have access to relatively few time series compared to their length ( $m \ll T$ ), learning to predict each time series as its own independent problem will with high probability lead to a better generalization bound. On the other hand, in extremely high-dimensional settings when we have significantly more time series than time steps ( $m \gg T$ ), sequence-to-sequence learning will (with high probability) provide superior performance. We also expect the performance of sequence-to-sequence models to deteriorate as the correlation between time series increases.

A direct comparison of bounds in Theorem 4.1 and Theorem 5.1 is complicated by the fact that discrepancies that appear in these results are different. In fact, it is possible to design examples where  $\frac{1}{m} \sum_{i=1}^m \Delta(\mathbf{Z}_i)$  is constant and  $\Delta$  is negligible, and vice-versa.

Consider a tent function  $g_b$  such that  $g_b(s) = 2bs/T$  for  $s \in [0, T/2]$  and  $g_b(s) = -2bs/T + 2b$  for  $s \in [T/2, T]$ . Let  $f_b$  be its periodic extension to the real line, and define  $\mathcal{S} = \{f_b : b \in [0, 1]\}$ . Suppose that we sample uniformly  $b \in [0, 1]$  and  $s \in \{0, T/2\}$   $m$  times, and observe time series  $f_{b_i}(s_i), \dots, f_{b_i}(s_i + T)$ . Then, as we have shown in Proposition 5,  $\Delta$  is negligible for sequence-to-sequence models. However, unless the model class is trivial, it can be shown that  $\Delta(\mathbf{Z}_i)$  is bounded away from zero for all  $i$ .

Conversely, suppose we sample uniformly  $b \in [0, 1]$   $m$  times and observe time series  $f_{b_i}(0), \dots, f_{b_i}(T/2 + 1)$ . Consider a set of local models that learn an offset from the previous point  $\{h : x \mapsto x + c, c \in [0, 1]\}$ . It can be shown that in this case  $\Delta(\mathbf{Z}_i) = 0$ , whereas  $\Delta$  is bounded away from zero for any non-trivial class of sequence-to-sequence models.

From a practical perspective, we can simply use  $\Delta_s$  and empirical estimates of  $\Delta(\mathbf{Z}_i)$  to decide whether to choose sequence-to-sequence or local models.

We conclude this section with an observation that similar results to Theorem 4.1 can be proved for multivariate local models with the only difference that the sample complexity of the problem scales as  $\mathcal{O}(\sqrt{m/T})$ , and hence these models are even more prone to the curse of dimensionality.

## 6 Hybrid models

In this section, we discuss models that interpolate between local and sequence-to-sequence models. This hybrid approach trains a single model  $h$  on the union of *local* training sets  $\mathbf{Z}_1, \dots, \mathbf{Z}_m$  used to train



$m$  models in the local approach. The bounds that we state here require the following extension of the discrepancy to  $\Delta_t$ , defined as

$$\Delta_t = \frac{1}{m} \sup_{h \in \mathcal{H}} \left| \sum_{i=1}^m \mathbb{E}_{\mathcal{D}}[L(h(Y_{t-p-1}^{t-1}(i)), Y_t(i)) | Y_1^{t-1}] - \mathbb{E}_{\mathcal{D}'}[L(h(Y_{T-p}^T(i)), Y_{T+1}(i)) | \mathbf{Y}] \right|$$

Many of the properties that were discussed for the discrepancy  $\Delta$  carry over to  $\Delta_t$  as well. The empirical error in this case is the same as for the local models:

$$\hat{\mathcal{L}}(h) = \frac{1}{mT} \sum_{i=1}^m \sum_{t=1}^T f(h, Z_{t,i}).$$

Observe that one straightforward way to obtain a bound for hybrid models is to apply Theorem 5.1 with  $(h, \dots, h) \in \mathcal{H}^m$ . Alternatively, we can apply Theorem 4.1 at every time point  $t = 1, \dots, T$ .

Combining these results via union bound leads to the following learning guarantee for hybrid models.

**Theorem 6.1.** *Let  $C_1, \dots, C_k$  form a partition of the training input  $\mathbf{Z}$  and let  $I_j$  denote the set of indices of time series that belong to  $C_j$ . Assume that the loss function  $L$  is bounded by 1. Then, for any  $\delta > 0$ , with probability  $1 - \delta$ , for any  $h \in \mathcal{H}$  and any  $\alpha > 0$*

$$\mathcal{L}(h | \mathbf{Y}) \leq \hat{\mathcal{L}}(h) + \min(B_1, B_2),$$

where

$$B_1 = \frac{1}{T} \sum_{t=1}^T \Delta_t + \max_j \hat{\mathfrak{R}}_{\tilde{C}_j}(\mathcal{F}) + \frac{1}{\sqrt{2 \min_j |I_j|}} \sqrt{\log \left( \frac{2Tk}{\delta - 2 \sum_j (|I_j| - 1) \beta_{2s}(I_j)} \right)}$$

$$B_2 = \frac{1}{m} \sum_{i=1}^m \Delta(\mathbf{Z}_i) + 2\alpha + \sqrt{\frac{2}{T} \log \frac{2m \max_i \mathbb{E}_{v \sim T(\mathbf{Z}_i)}[\mathcal{N}_1(\alpha, \mathcal{F}, v)]}{\delta}}.$$

Using the same arguments for the complexity terms as in the case of sequence-to-sequence and local models, this result shows that hybrid models are successful with high probability when  $m \gg T$  or correlations between time series are strong, as well as when  $T \gg m$ .

Potential costs for this model are hidden in the new discrepancy term  $\frac{1}{T} \sum_{t=1}^T \Delta_t$ . This term leads to different bounds depending on the particular non-stationarity in the given problem. As before this trade-off can be accessed empirically using the data-dependent version of discrepancy.

Note that the above bound does not imply that hybrid models are superior to local models: using  $m$  hypotheses  $h_1, \dots, h_m$  can help us achieve a better trade-off between  $\hat{\mathcal{L}}(h)$  and  $B_2$ , and vice versa.

## 7 Conclusion

We formally introduce sequence-to-sequence learning for time series, a framework in which a model learns to map past sequences of length  $T$  to their next values. We provide the first generalization bounds for sequence-to-sequence modeling. Our results are stated in terms of new notions of discrepancy and expected mixing coefficients. We study these new notions for several different families of stochastic processes including stationary, weakly stationary, periodic, hierarchical and spatio-temporal time series.

Furthermore, we show that our discrepancy can be computed from data, making it a useful tool for practitioners to empirically assess the non-stationarity of their problem. In particular, the discrepancy can be used to determine whether the sequence-to-sequence methodology is likely to succeed based on the inherent non-stationarity of the problem.

Furthermore, compared to the local framework for time series forecasting, in which independent models for each one-dimensional time series are learned, our analysis shows that the sample complexity of sequence-to-sequence models scales as  $\mathcal{O}(\sqrt{T/m})$ , providing superior guarantees when the number  $m$  of time series is significantly greater than the length  $T$  of each series, provided that different series are weakly correlated.

Conversely, we show that the sample complexity of local models scales as  $\mathcal{O}(\sqrt{\log(m)/T})$ , and should be preferred when  $m \ll T$  or when time series are strongly correlated. We also study hybrid models for

which learning guarantees are favorable both when  $m \gg T$  and  $T \gg m$ , but which have a more complex trade-off in terms of discrepancy.

As a final note, the analysis we have carried through is easily extended to show similar results for the sequence-to-sequence scenario when the test data includes *new* series not observed during training, as is often the case in a variety of applications.

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## A Rademacher complexity

**Definition 6** (Rademacher complexity). Given a family of functions  $\mathcal{F}$  and a training set  $\mathbf{Z} = \{Z_1, \dots, Z_m\}$ , the Rademacher complexity of  $\mathcal{F}$  conditioned on  $\mathbf{Y}'$  is given by

$$\widehat{\mathfrak{R}}_{\mathbf{Z}}(\mathcal{F}) = \mathbb{E}_{\mathbf{Z}, \sigma} \left[ \max_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^m \sigma_i f(Z_i) \mid \mathbf{Y}' \right]$$

where  $\sigma_1, \dots, \sigma_m$  are i.i.d. random variables uniform on  $\{-1, +1\}$ . The Rademacher complexity of  $\mathcal{F}$  for sample size  $m$  is given by

$$\mathfrak{R}_m(\mathcal{F}) = \mathbb{E}_{\mathbf{Y}'} \left[ \widehat{\mathfrak{R}}_{\mathbf{Z}}(\mathcal{F}) \right].$$

The Rademacher complexity has been studied for a variety of function classes. For instance, for the linear hypothesis space  $\mathcal{H} = \{x \rightarrow w^\top x, \|w\|_2 \leq \Lambda\}$ ,  $\widehat{\mathfrak{R}}_{\mathbf{Z}}$  can be upper bounded by  $\widehat{\mathfrak{R}}_{\mathbf{Z}}(\mathcal{H}) \leq \frac{\Lambda}{\sqrt{m}} \max_i \|Z_i\|_2$ . As another example, the hypothesis class of ReLU feed-forward neural networks with  $d$  layers and weight matrices  $W_k$  such that  $\prod_{k=1}^d \|W_k\|_F \leq \gamma$  verifies  $\widehat{\mathfrak{R}}_{\mathbf{Z}}(\mathcal{H}) \leq \frac{2^{d-1/2}\gamma}{\sqrt{m}} \max_i \|Z_i\|_2$  [32].

## B Discrepancy analysis

**Proposition 1.** Let  $\mathcal{H}$  be a hypothesis space and let  $L$  be a bounded loss function which respects the triangle inequality. Let  $h' \in \mathcal{H}$ . Then,

$$\Delta \leq \Delta_s + \mathcal{L}(h \mid \mathbf{Y}) + \mathcal{L}(h \mid \mathbf{Y}')$$

*Proof.* Let  $h, h' \in \mathcal{H}$ . For ease of notation, we write

$$\Delta_s(h, h', \mathbf{Y}') = \frac{1}{m} \sum_i L(h(Y_1^T(i)), h'(Y_1^T(i))) - \frac{1}{m} \sum_i L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))).$$

Applying the triangle inequality to  $L$ ,

$$\begin{aligned} \mathcal{L}(h \mid \mathbf{Y}) &= \frac{1}{m} \sum_i \mathbb{E}[L(h(Y_1^T(i)), Y_{T+1}(i)) \mid \mathbf{Y}] \\ &\leq \frac{1}{m} \sum_i L(h(Y_1^T(i)), h'(Y_1^T(i))) + \frac{1}{m} \sum_i \mathbb{E}[L(h'(Y_1^T(i)), Y_{T+1}(i)) \mid \mathbf{Y}] \\ &= \frac{1}{m} \sum_i L(h(Y_1^T(i)), h'(Y_1^T(i))) + \mathcal{L}(h' \mid \mathbf{Y}). \end{aligned}$$

Then, by definition of  $\Delta_s(h, h', \mathbf{Y}')$ , we have

$$\begin{aligned} \mathcal{L}(h \mid \mathbf{Y}) &\leq \frac{1}{m} \sum_i L(h(Y_1^T(i)), h'(Y_1^T(i))) - \frac{1}{m} \sum_i L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))) \\ &\quad + \frac{1}{m} \sum_i L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))) + \mathcal{L}(h' \mid \mathbf{Y}) \\ &\leq \Delta_s(h, h', \mathbf{Y}') + \mathcal{L}(h' \mid \mathbf{Y}) + \frac{1}{m} \sum_i L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))). \end{aligned}$$

By an application of the triangle inequality to  $L$ ,

$$\begin{aligned} \mathcal{L}(h, \mathcal{D}) &\leq \Delta_s(h, h', \mathbf{Y}') + \mathcal{L}(h' \mid \mathbf{Y}) + \frac{1}{m} \sum_i \mathbb{E}[L(h(Y_1^{T-1}(i)), Y_T(i)) \mid \mathbf{Y}'] \\ &\quad + \frac{1}{m} \sum_i \mathbb{E}[L(h'(Y_1^{T-1}(i)), Y_T(i)) \mid \mathbf{Y}'] \\ &= \Delta_s(h, h', \mathbf{Y}') + \mathcal{L}(h' \mid \mathbf{Y}) + \mathcal{L}(h \mid \mathbf{Y}') + \mathcal{L}(h' \mid \mathbf{Y}'). \end{aligned}$$

Finally, we obtain

$$\mathcal{L}(h \mid \mathbf{Y}) - \mathcal{L}(h \mid \mathbf{Y}') \leq \Delta_s(h, h', \mathbf{Y}') + \mathcal{L}(h' \mid \mathbf{Y}) + \mathcal{L}(h' \mid \mathbf{Y}')$$

and the result announced in the theorem follows by taking the supremum over  $\mathcal{H}$  on both sides.  $\square$

**Proposition 2.** Let  $I_1, \dots, I_k$  be a partition of  $\{1, \dots, m\}$ , and  $C_1, \dots, C_k$  be the corresponding partition of  $\mathbf{Y}$ . Write  $c = \min_j |C_j|$ . Then we have with probability  $1 - \delta$ ,

$$\Delta_s \leq \Delta_e + \max \left( \max_j \mathfrak{R}_{|C_j|}(\tilde{C}'_j), \max_j \mathfrak{R}_{|C_j|}(\tilde{I}_j) \right) + \sqrt{\frac{1}{2c} \log \frac{2k}{\delta - \sum_j (|I_j| - 1)[\bar{\beta}(I_j) + \bar{\beta}'(I_j)]}}.$$

*Proof.* By definition of  $\Delta_s$ ,

$$\begin{aligned} \Delta_s &= \sup_{h, h' \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \left[ L(h(Y_1^T(i)), h'(Y_1^T(i))) - L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))) \right] \\ &\leq \sup_{h, h' \in \mathcal{H}} \left[ \frac{1}{m} \sum_{i=1}^m L(h(Y_1^T(i)), h'(Y_1^T(i))) - \mathbb{E}_Y[L(h(Y_1^T), h'(Y_1^T))] \right] \\ &\quad + \sup_{h, h' \in \mathcal{H}} \left[ \mathbb{E}_Y[L(h(Y_1^T), h'(Y_1^T))] - \mathbb{E}_Y[L(h(Y_1^{T-1}), h'(Y_1^{T-1}))] \right] \\ &\quad + \sup_{h, h' \in \mathcal{H}} \left[ \mathbb{E}_Y[L(h(Y_1^{T-1}), h'(Y_1^{T-1}))] - \frac{1}{m} \sum_{i=1}^m L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))) \right] \end{aligned}$$

by sub-additivity of the supremum. Now, define

$$\begin{aligned} \phi(\mathbf{Y}) &\triangleq \sup_{h, h' \in \mathcal{H}} \left[ \frac{1}{m} \sum_{i=1}^m L(h(Y_1^T(i)), h'(Y_1^T(i))) - \mathbb{E}_Y[L(h(Y_1^T), h'(Y_1^T))] \right] \\ \psi(\mathbf{Y}') &\triangleq \sup_{h, h' \in \mathcal{H}} \left[ \mathbb{E}_Y[L(h(Y_1^{T-1}), h'(Y_1^{T-1}))] - \frac{1}{m} \sum_{i=1}^m L(h(Y_1^{T-1}(i)), h'(Y_1^{T-1}(i))) \right]. \end{aligned}$$

By definition of  $\Delta_e$ , we have from the previous inequality

$$\Delta_s \leq \Delta_e + \phi(\mathbf{Y}_1^T) + \psi(\mathbf{Y}_1^{T-1}).$$

We now proceed to give a high-probability bound for  $\phi$ ; the same reasoning will yield a bound for  $\psi$ . By sub-additivity of the max,

$$\begin{aligned} \phi(\mathbf{Y}) &\leq \sum_j \frac{|C_j|}{m} \sup_{h \in \mathcal{H}} \left[ \mathbb{E}_Y[f(h, Y_1^T)] - \frac{1}{|C_j|} \sum_{Y \in C_j} f(h, Y_1^T) \right] \\ &\leq \sum_j \frac{|C_j|}{m} \phi(C_j) \end{aligned}$$

and so by union bound, for  $\epsilon > 0$

$$\Pr(\phi(\mathbf{Y}) > \epsilon) \leq \sum_j \Pr(\phi(C_j) > \epsilon).$$

Let  $\epsilon > \max_j \mathbb{E}[\phi(\tilde{C}_j)]$  and set  $\epsilon_j = \epsilon - \mathbb{E}[\phi(\tilde{C}_j)]$ .

Define for time series  $Y(i), Y(j)$  the mixing coefficient

$$\bar{\beta}(i, j) = \|\Pr(Y_1^T(i), Y_1^T(j)) - \Pr(Y_1^T(i)) \Pr(Y_1^T(j))\|_{TV}$$

where we also extend the usual notation to  $\bar{\beta}(C_j)$ .

$$\begin{aligned} \Pr(\phi(C_j) > \epsilon) &= \Pr\left(\phi(C_j) - \mathbb{E}[\phi(\tilde{C}_j)] > \epsilon_j\right) \\ &\stackrel{(a)}{\leq} \Pr\left(\phi(\tilde{C}_j) - \mathbb{E}[\phi(\tilde{C}_j)] > \epsilon_j\right) + (|I_j| - 1)\bar{\beta}(I_j) \\ &\stackrel{(b)}{\leq} e^{-2c\epsilon_j^2} + (|I_j| - 1)\bar{\beta}(I_j), \end{aligned}$$



where (a) follows by applying Prop. 6 to the indicator function of the event  $\Pr(\phi(C_j) - \mathbb{E}[\phi(\tilde{C}_j)] \geq \epsilon)$ , and (b) is a direct application of McDiarmid's inequality to  $\phi(\tilde{C}_j) - \mathbb{E}[\phi(\tilde{C}_j)]$ .

Hence, by summing over  $j$  we obtain

$$\Pr(\phi(\mathbf{Y}) > \epsilon) \leq k e^{-2 \min_j |C_j| (\epsilon - \max_j \mathbb{E}[\phi(\tilde{C}_j)])^2} + \sum_j (|I_j| - 1) \bar{\beta}(I_j)$$

and similarly

$$\Pr(\psi(\mathbf{Y}') > \epsilon) \leq k e^{-2 \min_j |C'_j| (\epsilon - \max_j \mathbb{E}[\psi(\tilde{C}'_j)])^2} + \sum_j (|I'_j| - 1) \bar{\beta}'(I'_j),$$

which finally yields

$$\begin{aligned} \Pr(\Delta_s - \Delta_e > \epsilon) &\leq \Pr(\phi(\mathbf{Y}) > \epsilon) + \Pr(\psi(\mathbf{Y}') > \epsilon) \\ &\leq 2k \exp(-2c(\epsilon - \max_j \mathbb{E}[\phi(\tilde{C}_j)], \max_j \mathbb{E}[\psi(\tilde{C}'_j)]))^2 + \sum_j (|I_j| - 1) [\bar{\beta}(I_j) + \bar{\beta}'(I'_j)], \end{aligned}$$

where we recall that we write  $c = \min_j |C_j|$ . We invert the previous equation by setting

$$\epsilon = \max(\max_j \mathbb{E}[\phi(\tilde{C}_j)], \max_j \mathbb{E}[\psi(\tilde{C}'_j)]) + \sqrt{\frac{1}{2c} \log \frac{2k}{\delta - \sum_j (|I_j| - 1) [\bar{\beta}(I_j) + \bar{\beta}'(I'_j)]}},$$

yielding with probability  $1 - \delta$ ,

$$\Delta_s \leq \Delta_e + \max(\max_j \mathbb{E}[\phi(\tilde{C}_j)], \max_j \mathbb{E}[\psi(\tilde{C}'_j)]) + \sqrt{\frac{1}{2c} \log \frac{2k}{\delta - \sum_j (|I_j| - 1) [\bar{\beta}(I_j) + \bar{\beta}'(I'_j)]}}.$$

We now bound  $\mathbb{E}[\phi(\tilde{C}_j)]$  by  $\mathfrak{R}_{|C_j|}(\tilde{C}_j)$ . A similar argument yields the bound for  $\psi$ . By definition, we have

$$\begin{aligned} \mathbb{E}[\phi(\tilde{C}_j)] &= \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \frac{1}{|C_j|} \sum_{Z \in \tilde{C}_j} f(h, Y_1^T(i)) - \mathbb{E}_Y[f(h, Y_1^T)] \right] \\ &= \frac{1}{|C_j|} \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \sum_{Z \in \tilde{C}_j} \underbrace{f(h, Y_1^T(i)) - \mathbb{E}_Y[f(h, Y_1^T)]}_{g(h, Y_1^T(i))} \right] \\ &= \frac{1}{|C_j|} \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \sum_{Z \in \tilde{C}_j} g(h, Y_1^T(i)) \right] \end{aligned}$$

Standard symmetrization arguments as those used for the proof of the famous result by [16], which hold also when data is drawn independently but not identically at random, yield

$$\mathbb{E}[\phi(\tilde{C}_j)] \leq \mathfrak{R}_{|C_j|}(\tilde{C}_j).$$

The same argument yields for  $\psi$

$$\mathbb{E}[\psi(\tilde{C}'_j)] \leq \mathfrak{R}_{|C'_j|}(\tilde{C}'_j).$$

To conclude our proof, it only remains to prove the bound

$$\bar{\beta}(i, j) \leq \beta_{s2s}(i, j) + \mathbb{E}_{\mathbf{Y}'} \left[ \text{Cov} \left( \Pr(Y_T(i) \mid \mathbf{Y}'), \Pr(Y_T(j) \mid \mathbf{Y}') \right) \right]$$

Let  $Y(i), Y(j)$  be two time series, and write  $X_i = \mathbb{E}[\Pr(Y_1^T(i) \mid \mathbf{Y}')]$ . Then the following bound holds

$$\begin{aligned} \bar{\beta}(i, j) &= \|\Pr(Y_1^T(i), Y_1^T(j)) - \Pr(Y_1^T(i)) \Pr(Y_1^T(j))\|_{TV} \\ &= \|\mathbb{E}[\Pr(Y_1^T(i), Y_1^T(j) \mid \mathbf{Y}')] - \mathbb{E}[X_i] \mathbb{E}[X_j]\|_{TV} \\ &= \|\mathbb{E}[\Pr(Y_1^T(i), Y_1^T(j) \mid \mathbf{Y}_1^{T-1})] - \mathbb{E}[X_i, X_j] - \mathbb{E}[\text{Cov}(X_i, X_j)]\|_{TV} \\ &\leq \beta_{s2s}(i, j) + \mathbb{E}_{\mathbf{Y}'}[\text{Cov}(X_i, X_j)], \end{aligned}$$

which is the desired inequality.  $\square$

We now show two useful lemmas for various specific cases of time series and hypothesis spaces.

**Proposition 3.** *If  $Y(i)$  is stationary for all  $1 \leq i \leq m$ , and  $\mathcal{H}$  is a hypothesis space such that  $h \in \mathcal{H} : \mathcal{Y}^{T-1} \rightarrow \mathcal{Y}$  (i.e. the hypotheses only consider the last  $T-1$  values of  $Y$ ), then  $\Delta_e = 0$ .*

*Proof.* Let  $h, h' \in \mathcal{H}$ . For stationary  $Y(i)$ , we have  $\Pr(Y_1^T(i)) = \Pr(Y_2^T(i))$ , and so

$$\mathbb{E}[L(h(Y_2^T), h'(Y_2^T))] - \mathbb{E}[L(h(Y_1^{T-1}), h'(Y_1^{T-1}))] = 0$$

and so taking the supremum over  $h, h'$  yields the desired result.  $\square$

**Proposition 4.** *If  $Y(i)$  is covariance stationary for all  $1 \leq i \leq m$ ,  $L$  is the squared loss, and  $\mathcal{H}$  is a linear hypothesis space  $\{x \rightarrow w \cdot x \mid \|w\| \in \mathbb{R}^p \leq \Lambda\}$ , then  $\Delta_e = 0$ .*

*Proof.* Recall that a time series  $Y$  is covariance stationary if  $\mathbb{E}_Y[Y_t]$  does not depend on  $t$  and  $\mathbb{E}_Y[Y_t Y_s] = f(t-s)$  for some function  $f$ .

Let now  $(h, h') \in \mathcal{H} \equiv (w, w') \in \mathbb{R}^p$ . We write  $\Sigma = \Sigma_2^T(Y) = \Sigma_1^T(Y)$  the covariance matrix of  $Y$  where the equality follows from covariance stationarity. Without loss of generality, we consider  $p = T-1$ . Then,

$$\begin{aligned} & \mathbb{E}[L(h(Y_2^T), h'(Y_2^T))] - \mathbb{E}[L(h(Y_1^{T-1}), h'(Y_1^{T-1}))] \\ &= \mathbb{E}[(w - w')^\top \Sigma_2^T(Y)(w - w')] - \mathbb{E}[(w - w')^\top \Sigma_1^T(Y)(w - w')] \\ &= 0. \end{aligned}$$

Taking the supremum over  $h, h'$  yields the desired result.  $\square$

**Proposition 5.** *If the  $Y(i)$  are periodic of period  $p$  and the observed starting time of each  $Y(i)$  is distributed uniformly at random in  $[p]$ , then  $\Delta_e = 0$ .*

*Proof.* This proof is similar to the stationary case: indeed, we can write  $\Pr(Y_1^{T-1}(i)) = \frac{1}{p} \Pr(Y(i))$  due to the uniform distribution on starting times. Then, by the same reasoning, we have also

$$\Pr(Y_2^T(i)) = \frac{1}{p} \Pr(Y(i)) = \Pr(Y_1^{T-1}(i)),$$

from which the result follows.  $\square$

## C Generalization bounds

**Proposition 6.** Yu [40, Corollary 2.7]. *Let  $f$  be a real-valued Borel measurable function such that  $0 \leq f \leq 1$ . Then, we have the following guarantee:*

$$\left| \mathbb{E}[f(\tilde{C})] - \mathbb{E}[f(C)] \right| \leq (|C|-1)\beta,$$

where  $\beta$  is the total variation distance between joint distributions of  $C$  and  $\tilde{C}$ .

**Theorem 4.1.** *Let  $\mathcal{H}$  be a hypothesis space, and  $h \in \mathcal{H}$ . Let  $C_1, \dots, C_k$  form a partition of the training input  $\mathbf{Y}_1^T$ , and consider that the loss function  $L$  is bounded by 1. Then, we have for  $\delta > 0$ , with probability  $1 - \delta$ ,*

$$\Phi_{s2s}(h) \leq \Delta + \max_j \left[ \mathfrak{R}_{|C_j|}(\tilde{C}_j \mid \mathbf{Y}) \right] + \frac{1}{\sqrt{2 \min_j |I_j|}} \sqrt{\log \left( \frac{k}{\delta - \sum_j (|I_j|-1)\beta_{s2s}(I_j)} \right)}.$$

For ease of notation, we write

$$\begin{aligned} \phi(\mathbf{Y}) &= \sup_{h \in \mathcal{H}} \mathcal{L}(h \mid \mathbf{Y}') - \hat{\mathcal{L}}(h, \mathbf{Y}) \\ &= \sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^m \mathbb{E}[f(h, Y_1^T(i)) \mid \mathbf{Y}'] - \frac{1}{m} \sum_{i=1}^m f(h, Y_1^T(i)). \end{aligned}$$

We begin by proving the following lemma.

**Lemma 3.** Let  $\bar{\mathbf{Y}}$  be equal to  $\mathbf{Y}$  on all time series except for the last, where we have  $\bar{Y}(m) = Y(m)$  at all times except for time  $t = T$ . Then

$$|\phi(\mathbf{Y}) - \phi(\bar{\mathbf{Y}})| \leq \frac{1}{m}$$

*Proof.* Fix  $h^* \in \mathcal{H}$ . Then,

$$\begin{aligned} & \mathcal{L}(h^* | \mathbf{Y}') - \hat{\mathcal{L}}(h^*, \mathbf{Y}) - \sup_{h \in \mathcal{H}} [\mathcal{L}(h | \bar{\mathbf{Y}}') - \hat{\mathcal{L}}(h, \bar{\mathbf{Y}})] \\ & \leq \mathcal{L}(h^* | \mathbf{Y}') - \hat{\mathcal{L}}(h^*, \mathbf{Y}) - [\mathcal{L}(h^*, | \bar{\mathbf{Y}}') - \hat{\mathcal{L}}(h^*, \bar{\mathbf{Y}})] \\ & \stackrel{(a)}{\leq} \hat{\mathcal{L}}(h^*, \bar{\mathbf{Y}}) - \hat{\mathcal{L}}(h^*, \mathbf{Y}) \\ & \leq \frac{1}{m} [f(h^*, \bar{Y}_1^T(m)) - f(h^*, Y_1^T(m))] \leq \frac{1}{m}. \end{aligned}$$

where (a) follows from the fact that  $\mathbf{Y}' = \bar{\mathbf{Y}}'$  and the last inequality follows from the fact that  $f$  is bounded by 1.

By taking the supremum over  $h^*$ , the previous calculations show that  $\phi(\mathbf{Y}) - \phi(\bar{\mathbf{Y}}) \leq 1/m$ ; by symmetry, we obtain  $\phi(\bar{\mathbf{Y}}) - \phi(\mathbf{Y}) \leq 1/m$  which proves the lemma.  $\square$

We now prove the main theorem.

*Proof.* Observe that the following bounds holds

$$\begin{aligned} \Phi_{\text{s2s}}(\mathbf{Y}) &= \mathcal{L}(h | \mathbf{Y}) - \hat{\mathcal{L}}(h, \mathbf{Y}) \\ &\leq \sup_{h \in \mathcal{H}} [\mathcal{L}(h | \mathbf{Y}) - \mathcal{L}(h | \mathbf{Y}')] + \sup_{h \in \mathcal{H}} [\mathcal{L}(h | \mathbf{Y}') - \hat{\mathcal{L}}(h, \mathbf{Y})]. \end{aligned}$$

and so

$$\Phi_{\text{s2s}}(\mathbf{Y}) - \Delta \leq \underbrace{\sup_{h \in \mathcal{H}} \mathcal{L}(h, | \mathbf{Y}') - \hat{\mathcal{L}}(h, \mathbf{Y})}_{\phi(\mathbf{Y})}.$$

Define  $M = \max_j \mathbb{E}[\phi(\tilde{C}_j) | \bar{\mathbf{Y}}']$ . Then,

$$\Pr(\Phi_{\text{s2s}}(\mathbf{Y}) - \Delta - M > \epsilon | \mathbf{Y}') \leq \Pr(\phi(\mathbf{Y}) - M > \epsilon | \mathbf{Y}'). \quad (\text{C.1})$$

By sub-additivity of the supremum, we have

$$\phi(\mathbf{Y}) - M \leq \sum_j \frac{|C_j|}{m} \sup_{h \in \mathcal{H}} [\mathcal{L}(h | \mathbf{Y}) - \hat{\mathcal{L}}(h, C_j) - M]$$

and so by union bound,

$$\Pr(\phi(\mathbf{Y}) - M \geq \epsilon | \mathbf{Y}') \leq \sum_j \Pr(\phi(C_j) - M \geq \epsilon | \mathbf{Y}').$$

By definition of  $M$ ,

$$\begin{aligned} \Pr(\phi(C_j) - M \geq \epsilon | \mathbf{Y}') &\leq \Pr(\phi(C_j) - \mathbb{E}[\phi(\tilde{C}_j) | \bar{\mathbf{Y}}'] \geq \epsilon | \mathbf{Y}') \\ &\stackrel{(a)}{\leq} \Pr(\phi(\tilde{C}_j) - \mathbb{E}[\phi(\tilde{C}_j) | \bar{\mathbf{Y}}'] \geq \epsilon | \mathbf{Y}') + (|I_j| - 1) \beta_{\text{s2s}}(I_j | \mathbf{Y}') \\ &\stackrel{(b)}{\leq} e^{-2|C_j|\epsilon^2} + (|I_j| - 1) \beta_{\text{s2s}}(I_j | \mathbf{Y}'). \end{aligned}$$

where (a) follows by applying Prop. 6 to the indicator function of the event  $\Pr(\phi(C_j) - \mathbb{E}[\phi(\tilde{C}_j) | \bar{\mathbf{Y}}'] \geq \epsilon)$ , and (b) is a direct application of McDiarmid's inequality, following Lemma 3. The notation  $\beta_{\text{s2s}}(I_j | \mathbf{Y}')$

indicates the total variation distance between the joint distributions of  $C_j$  and  $\tilde{C}_j$  conditioned on  $\mathbf{Y}'$ . In particular, we have  $\mathbb{E}_{\mathbf{Y}'} \beta_{\text{s2s}}(C_j \mid \mathbf{Y}') = \beta_{\text{s2s}}(C_j)$ .

Finally, taking the expectation of the previous term over all possible  $\mathbf{Y}'$  values and summing over  $j$ , we obtain

$$\Pr(\mathcal{L}(h \mid \mathbf{Y}) - \hat{\mathcal{L}}(h, \mathbf{Y}) - \mathbb{E}_{\tilde{C}_j}[\phi(\tilde{C}_j) \mid \tilde{\mathbf{Y}}] \geq \epsilon) \leq \sum_j e^{-2|C_j|\epsilon^2} + \sum_j (|I_j|-1)\beta_{\text{s2s}}(I_j).$$

Combining this bound with (C.1), we obtain

$$\begin{aligned} \Pr\left(\Phi_{\text{s2s}}(\mathbf{Y}) - \Delta - M > \epsilon\right) &\leq \sum_j e^{-2|C_j|\epsilon^2} + \sum_j (|I_j|-1)\beta_{\text{s2s}}(I_j) \\ &\leq k e^{-2 \min_j |C_j| \epsilon^2} + \sum_j (|I_j|-1)\beta_{\text{s2s}}(I_j) \end{aligned}$$

We invert the previous equation by choosing  $\delta > \sum_j (|I_j|-1)\beta_{\text{s2s}}(I_j)$  and setting

$$\epsilon = \sqrt{\frac{\log \frac{k}{\delta - \sum_j (|I_j|-1)\beta_{\text{s2s}}(I_j)}}{2 \min_j |I_j|}},$$

which yields that with probability  $1 - \delta$ , we have

$$\Phi_{\text{s2s}}(Z) \leq M + \Delta + \sqrt{\frac{\log \left( \frac{k}{\delta - \sum_j (|I_j|-1)\beta_{\text{s2s}}(I_j)} \right)}{2 \min_j |I_j|}}.$$

To conclude our proof, it remains to show that

$$M \leq \mathfrak{R}_{|C_j|}(\tilde{C}_j \mid \tilde{\mathbf{Y}}').$$

$$\begin{aligned} \mathbb{E}[\phi(\tilde{C}_j) \mid \tilde{\mathbf{Y}}'] &= \mathbb{E}\left[\sup_{h \in \mathcal{H}} \mathcal{L}(h \mid \tilde{\mathbf{Y}}') - \frac{1}{|C_j|} \sum_{i=1}^m f(h, \tilde{Y}_1^T(i)) \mid \tilde{\mathbf{Y}}'\right] \\ &= \frac{1}{|C_j|} \mathbb{E}\left[\sup_{h \in \mathcal{H}} \sum_{\tilde{Y}_1^T \in \tilde{C}_j} \mathbb{E}[f(h, \tilde{Y}_1^T) \mid \tilde{\mathbf{Y}}'] - f(h, \tilde{Y}_1^T(i)) \mid \tilde{\mathbf{Y}}'\right] \\ &\leq \frac{1}{|C_j|} \mathbb{E}\left[\sup_{h \in \mathcal{H}} \sum_{\tilde{Y}_1^T \in \tilde{C}_j} g(h, \tilde{Y}_1^T(i)) \mid \tilde{\mathbf{Y}}'\right] \end{aligned}$$

where we've defined

$$g(h, \tilde{Y}_1^T(i)) \triangleq \mathbb{E}[f(h, \tilde{Y}_1^T(i)) \mid \tilde{\mathbf{Y}}'] - f(h, \tilde{Y}_1^T(i)).$$

Similar arguments to those used at the end of Appendix B yield the desired result, which concludes the proof of Theorem 4.1.  $\square$

## D Generalization bounds for local models

**Theorem 5.1.** *Let  $h = (h_1, \dots, h_m)$  where each  $h_i$  is a hypothesis learned via a local method to predict the univariate time series  $Z_i$ . For  $\delta > 0$  and any  $\alpha > 0$ , we have w.p. with  $1 - \delta$*

$$\Phi_{\text{loc}}(\mathbf{Z}) \leq \frac{1}{m} \sum_i \Delta(Y(i)) + 2\alpha + \sqrt{\frac{2}{T} \log \frac{m \max_i (\mathbb{E}_{v \sim T(Y(i))} [\mathcal{N}_1(\alpha, \mathcal{F}, v)])}{\delta}}$$

*Proof.* Write

$$\Phi(Y_1^T(i)) = \sup_{h \in \mathcal{H}} \mathbb{E}[f(h, Y_1^{T+1}) \mid Y_1^T] - \frac{1}{T} \sum_{t=1}^T f(h, Y_t^{t+T}(i)).$$

By [18, Theorem 1], we have that for  $\epsilon > 0$ , and  $1 \leq i \leq m$ ,

$$\Pr(\Phi(Y_1^T(i)) - \Delta(Y(i)) > \epsilon) \leq \mathbb{E}_{v \sim T(p)}[\mathcal{N}_1(\alpha, \mathcal{F}, v)] \exp\left(-\frac{T(\epsilon - 2\alpha)^2}{2}\right).$$

By union bound,

$$\Pr\left(\frac{1}{m} \sum_i \Phi(Y_1^T(i)) - \Delta(Y(i)) > \epsilon\right) \leq m \max_i (\mathbb{E}_{v \sim T(Y(i))}[\mathcal{N}_1(\alpha, \mathcal{F}, v)]) \exp\left(-\frac{T(\epsilon - 2\alpha)^2}{2}\right)$$

We invert the previous equation by letting

$$\epsilon = 2\alpha + \sqrt{\frac{2}{T} \log \frac{m \max_i (\mathbb{E}_{v \sim T(Y(i))}[\mathcal{N}_1(\alpha, \mathcal{F}, v)])}{\delta}}.$$

which yields the desired result.  $\square$

## E Analysis of expected mixing coefficients

**Lemma 2.** *Two AR processes  $Y(i), Y(j)$  generated by (4.1) such that  $\sigma = \text{Cov}(Y(i), Y(j)) \leq \sigma_0 < 1$  verify  $\beta_{s2s}(i, j) = \max\left(\frac{3}{2(1-\sigma_0^2)}, \frac{1}{1-2\sigma_0}\right) \sigma$ .*

*Proof.* For simplicity, we write  $U = Y(i)$  and  $V = Y(j)$ .

Write

$$\begin{aligned} \beta &= \|P(U_T | \mathbf{Y}') P(V_T | \mathbf{Y}') - P(U_T, V_T | \mathbf{Y}')\|_{TV} \\ &= \sup_{u,v} |P(U_T = u) P(V_T = v) - P(U_T = u, V_T = v)| \\ &= \sup_{u,v} \left| P(U_T = u | U_0^{T-1}) P(V_T = v | V_0^{T-1}) - P(U_T = u, V_T = v | U_0^{T-1}, V_0^{T-1}) \right| \\ &= \sup_{u,v} \left| \left[ P(u, v | U_0^{T-1}, V_0^{T-1}) + f(\sigma, \delta, \epsilon) \right] - P(u, v | U_0^{T-1}, V_0^{T-1}) \right| \end{aligned}$$

where we've written  $\delta = u - \Theta_i(U_0^{T-1})$  (and  $\epsilon$  similarly for  $v$ ), and we've defined

$$\begin{aligned} f(\sigma, \delta, \epsilon) &= P(u | U_0^{T-1}) P(v | V_0^{T-1}) - P(u, v | U_0^{T-1}, V_0^{T-1}) \\ &= e^{-\frac{1}{2}(\delta^2 + \epsilon^2)} - \frac{1}{1 - \sigma^2} e^{-\frac{1}{2} \frac{1}{1 - \sigma^2} (\delta^2 + \epsilon^2 - 2\sigma\epsilon\delta)}. \end{aligned}$$

Assuming we can bound  $f(\sigma, \delta, \epsilon)$  by a function  $g(\sigma)$  independent of  $\delta, \epsilon$ , we can then derive a bound on  $\beta$ .

Let  $x = \sqrt{\delta^2 + \epsilon^2}$  be a measure of how far the AR process noises lie from their mean  $\mu = 0$ . Using the inequality

$$|\delta\epsilon| \leq \delta^2 + \epsilon^2,$$

we proceed to bound  $|f(\sigma, \delta, \epsilon)|$  by bounding  $f$  and  $-f$ .

$$\begin{aligned} f(\sigma, \delta, \epsilon) &\leq e^{-\frac{1}{2}(\delta^2 + \epsilon^2)} - e^{-\frac{1}{2} \frac{1}{1 - \sigma^2} (\delta^2 + \epsilon^2 + 2\sigma|\delta\epsilon|)} \\ &\leq e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2} \frac{1}{1 - \sigma^2} (1 + 2\sigma)x^2} \\ &\leq e^{-\frac{1}{2}x^2} \left(1 - e^{-\frac{1}{2} \frac{2\sigma + \sigma^2}{1 - \sigma^2} x^2}\right) \end{aligned}$$

Using the inequality  $1 - x \leq e^{-x}$ , it then follows that

$$\begin{aligned} f(\sigma, \delta, \epsilon) &\leq e^{-\frac{1}{2}x^2} \left(1 - \left(1 - \frac{1}{2} \frac{2\sigma + \sigma^2}{1 - \sigma^2} x^2\right)\right) \\ &\leq \frac{1}{2} \frac{3}{1 - \sigma^2} \sigma x^2 e^{-\frac{1}{2}x^2} \\ &\stackrel{(a)}{\leq} \frac{3}{e(1 - \sigma^2)} \sigma \end{aligned} \tag{E.1}$$

where inequality (a) follows from the fact that  $y \rightarrow ye^{-y}$  is bounded by  $1/e$ .

Similarly, we now bound  $-f$ :

$$\begin{aligned} -f(\sigma, \delta, \epsilon) &\leq \frac{1}{1-\sigma^2} e^{-\frac{1}{2} \frac{1}{1-\sigma^2} (\delta^2 + \epsilon^2 - 2\sigma|\epsilon\delta|)} - e^{-\frac{1}{2}(\delta^2 + \epsilon^2)} \\ &\leq \frac{1}{1-\sigma^2} e^{-\frac{1}{2} \frac{1-2\sigma}{1-\sigma^2} x^2} - e^{-\frac{1}{2}x^2} \\ &\leq \frac{1}{1-\sigma^2} e^{-\frac{1}{2}(1-2\sigma)x^2} - e^{-\frac{1}{2}x^2}. \end{aligned}$$

One shows easily that this last function reaches its maximum for  $x_0^2 = \frac{1}{\sigma} \log(\frac{1-\sigma^2}{1-2\sigma})$ , at which point it verifies

$$-f(\sigma, x_0) = \frac{2\sigma}{1-2\sigma} e^{-\frac{1}{2\sigma} \log(\frac{1-\sigma^2}{1-2\sigma})} \leq \frac{2\sigma}{1-2\sigma} \quad (\text{E.2})$$

Putting (E.1) and (E.2) together, we obtain

$$\begin{aligned} |f(\sigma, \delta, \epsilon)| &\leq \sigma \max\left(\frac{3}{e(1-\sigma^2)}, \frac{1}{1-2\sigma}\right) \\ &\leq \max\left(\frac{3}{2(1-\sigma_0^2)}, \frac{1}{1-2\sigma_0}\right) \sigma \end{aligned}$$

Taking the expectation over all possible realizations of  $\mathbf{Y}'$  yields the desired result.  $\square$

*Proof.* Recall that  $\mathbf{Y}$  contains  $m' = mT$  examples, which we denote  $Y_{t-p}^t(i)$  for  $1 \leq i \leq m$  and  $1 \leq t \leq T$  (when  $t-p < 0$ , we truncate the time series appropriately). We define

$$\begin{aligned} \mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}) &= \frac{1}{m} \sum_{i=1}^m \mathbb{E}[L(h(Y_{T-p+1}^T(i)), Y_{T+1}(i)) \mid \mathbf{Y}] \\ \mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}') &= \frac{1}{m} \sum_{i=1}^m \frac{1}{T} \sum_{t=1}^T \mathbb{E}[L(h(Y_{t-p}^{t-1}(i)), Y_t(i)) \mid \mathbf{Y}'] \\ \widehat{\mathcal{L}}_{\text{hyb}}(h) &= \frac{1}{m} \sum_{i=1}^m \frac{1}{T} \sum_{t=1}^T L(h(Y_{t-p}^{t-1}(i)), Y_t(i)) \end{aligned}$$

where we note that here  $\mathbf{Y}'$  indicates each of the  $mT$  training samples excluding their last time point.

Observe that the following chain of inequalities holds:

$$\begin{aligned} \Phi_{\text{hyb}}(\mathbf{Y}) &= \sup_{h \in \mathcal{H}} \mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}) - \widehat{\mathcal{L}}_{\text{hyb}}(h) \\ &\leq \sup_{h \in \mathcal{H}} [\mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}) - \mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}')] + \sup_{h \in \mathcal{H}} [\mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}') - \widehat{\mathcal{L}}_{\text{hyb}}(h, \mathbf{Y})] \\ &\leq \frac{1}{T} \sum_{t=1}^T \sup_{h \in \mathcal{H}} \left[ \mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}) - \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{\mathcal{D}'}[L(h(Y_{t-p}^{t-1}(i)), Y_t(i)) \mid \mathbf{Y}'] \right] \\ &\quad + \sup_{h \in \mathcal{H}} [\mathcal{L}_{\text{hyb}}(h \mid \mathbf{Y}') - \widehat{\mathcal{L}}_{\text{hyb}}(h, \mathbf{Y})]. \end{aligned}$$

and so

$$\Phi_{\text{hyb}}(\mathbf{Y}) - \frac{1}{T} \sum_t \Delta_t \leq \underbrace{\sup_{h \in \mathcal{H}} \mathcal{L}_{\text{hyb}}(h, \mid \mathbf{Y}') - \widehat{\mathcal{L}}_{\text{hyb}}(h, \mathbf{Y})}_{\phi(\mathbf{Y})}.$$

Then, following the exact same reasoning as above for  $\Phi_{\text{s2s}}$  shows that for  $\delta > 0$ , we have with probability  $1 - \delta/2$

$$\Phi_{\text{hyb}}(\mathbf{Y}) \leq \underbrace{\max_j \widehat{\mathfrak{R}}_{\tilde{C}_j}(\mathcal{F}) + \frac{1}{T} \sum_t \Delta_t}_{B_1} + \sqrt{\frac{\log\left(\frac{2k}{\delta - \sum_j (|I_j| - 1)\beta(I_j)}\right)}{2 \min_j |I_j|}}$$



However, upper bounding  $\Phi_{\text{hyb}}$  can also be approached using the same techniques as Kuznetsov and Mohri [18], which we now describe. Let  $\alpha > 0$ . For a given  $h$ , computing  $\mathcal{L}_{\text{hyb}}(h, \mathbf{Y})$  is similar in expectation to running  $h$  on each of the  $m$  time series, yielding for each time series  $Y_{T-p+1}^T(i)$  the bound

$$\begin{aligned} & \mathbb{E}[L(h(Y_{T-p+1}^T(i)), Y_{T+1}(i)) \mid \mathbf{Y}] \\ & \leq \frac{1}{T} \sum_{t=1}^T L(h(Y_{t-p}^{t-1}(i)), Y_t(i)) + \Delta(\mathbf{Y}_i) + 2\alpha + \sqrt{\frac{2}{T} \log \frac{\max_i \mathbb{E}_{v \sim T(\mathbf{Y}_i)}[\mathcal{N}_i(\alpha, \mathcal{F}, v)]}{\delta}} \end{aligned}$$

and so by union bound, as above, we obtain with probability  $1 - \delta/2$

$$\begin{aligned} \Phi_{\text{hyb}}(\mathbf{Y}) & \leq \frac{1}{m} \sum \Delta(\mathbf{Y}_i) + 2\alpha + \sqrt{\frac{2}{T} \log \frac{2m \max_i \mathbb{E}_{v \sim T(\mathbf{Y}_i)}[\mathcal{N}_i(\alpha, \mathcal{F}, v)]}{\delta}} \\ & \leq B_2 \end{aligned}$$

We conclude by a final union bound on the event  $\{\Phi_{\text{hyb}}(\mathbf{Y}) \geq B_1 \cup \Phi_{\text{hyb}}(\mathbf{Y}) \geq B_2\}$ , we obtain with probability  $1 - \delta$ ,

$$\Phi_{\text{hyb}}(\mathbf{Y}) \leq \min(B_1, B_2)$$

□