Covariant Multi-Channel Windowed Scattering: Phase–Density Unified Theorem

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Abstract

For multi-channel scattering with $N(E)$ channels at energy E, establish unified framework connecting scattering phase derivative, relative spectral density, and Wigner–Smith delay. Core formula holding a.e. on absolutely continuous spectrum:

$$\boxed{\frac{1}{N(E)}\frac{d}{dE}\operatorname{Arg}\det S(E) = \frac{1}{2\pi}\langle \operatorname{tr} Q(E)\rangle_N = \rho_{\mathrm{rel}}(E)}$$

where $Q(E) = -iS^\dagger(E)\partial_E S(E)$ is Wigner–Smith delay matrix, $\langle\cdot\rangle_N$ denotes per-channel average, ρ_{rel} relative spectral density from Birman–Kreĭn formula $\det S = e^{-2\pi i\xi}$.

Establish: (i) **Channel covariance**: formula invariant under unitary channel basis transformations; (ii) **Threshold regularity**: handle channel number jumps via proper boundary conditions; (iii) **Windowed readout**: finite-bandwidth measurement protocol with NPE error closure; (iv) **Information geometry**: Born probability as I-projection, pointer basis as Ky Fan minimum.

1 Setup and Notation

Energy-dependent channel number $N(E)$ with threshold set $\mathcal{T} = \{E : N(E^+) \neq N(E^-)\}$. Between thresholds, scattering matrix $S(E) \in U(N(E))$ unitary, differentiable a.e.

Birman–Kreĭn convention: $\det S(E) = e^{-2\pi i\xi(E)}$ where ξ spectral shift function, $\rho_{\mathrm{rel}}(E) = -\xi'(E)$ relative spectral density.

Wigner–Smith delay: $Q(E) = -iS^\dagger(E)\partial_E S(E)$ self-adjoint, eigenvalues $\tau_j(E)$ individual channel delays.

2 Main Results

Theorem 2.1 (Multi-Channel Phase–Density Unification)**.** *On threshold-regular intervals I (where $I \cap \mathcal{T} = \varnothing$), have a.e.:*

$$\frac{1}{2\pi}\operatorname{tr} Q(E) = \frac{d}{dE}\operatorname{Arg}\det S(E) = -2\pi\,\xi'(E) = \rho_{\mathrm{rel}}(E).$$

Per-channel average delay $\langle\tau\rangle_N(E) = \frac{1}{N(E)}\operatorname{tr} Q(E)$ satisfies

$$\langle\tau\rangle_N(E) = 2\pi\hbar\,\rho_{\mathrm{rel}}(E) \quad (\textit{restoring } \hbar).$$

Proof. From $\det S = e^{-2\pi i\xi}$ get $\partial_E \log\det S = -2\pi i\,\xi'$. By Jacobi formula $\partial_E \log\det S = \mathrm{tr}(S^{-1}\partial_E S) = \mathrm{tr}(S^\dagger \partial_E S)$ (using unitarity). Thus $\mathrm{tr}(S^\dagger\partial_E S) = -2\pi i\,\xi'$, giving $\mathrm{tr}\,Q = -i\,\mathrm{tr}(S^\dagger\partial_E S) = 2\pi\,\xi'$. Birman–Kreĭn $\rho_{\mathrm{rel}} = -\xi'$ completes chain. □

Theorem 2.2 (Channel Basis Covariance). *Under unitary channel transformation $U \in U(N)$, $S \mapsto \tilde{S} = USU^\dagger$, have*

$$\mathrm{tr}\,Q_{\tilde{S}}(E) = \mathrm{tr}\,Q_S(E), \qquad \rho_{\mathrm{rel}}[\tilde{S}](E) = \rho_{\mathrm{rel}}[S](E).$$

Windowed readout $\mathcal{N}_w[S;E_0] = \int w(E-E_0)\rho_{\mathrm{rel}}[S](E)\,dE$ invariant.

Proof. Trace invariant under similarity. Spectral shift function gauge-invariant. □

Theorem 2.3 (Threshold Boundary Conditions). *At threshold $E_* \in \mathcal{T}$ where $N(E_*^-) = N_-$, $N(E_*^+) = N_+$ with $N_+ > N_-$, impose:*

1. ***Phase continuity**: $\mathrm{Arg}\det S(E)$ continuous at E_* after proper branch choice*
2. ***Density regularization**: $\rho_{\mathrm{rel}}(E)$ may have δ-function contribution at E_* from bound states entering/leaving continuous spectrum (Levinson theorem)*
3. ***Windowed measurement**: choose window w with $w(E_* - E_0)$ sufficiently small or smooth to regularize threshold singularities*

3 Windowed Readout and NPE Error

Theorem 3.1 (NPE Three-Term Decomposition for Multi-Channel). *For windowed readout with window w, kernel h, sampling step ΔE, truncation N:*

$$\mathcal{N}_w[S;E_0] = \int w(E-E_0)[h * \rho_{\mathrm{rel}}](E)\,dE$$

discrete approximation

$$\widehat{\mathcal{N}} = \Delta E \sum_{n=-N}^{N} w(E_n - E_0)[h * \rho_{\mathrm{rel}}](E_n)$$

satisfies error decomposition

$$|\mathcal{N}_w - \widehat{\mathcal{N}}| \le |\varepsilon_{\mathrm{alias}}| + |\varepsilon_{\mathrm{EM}}| + |\varepsilon_{\mathrm{tail}}|$$

with alias $\varepsilon_{\mathrm{alias}} = 0$ when bandlimited + Nyquist.

4 Information Geometry and Pointer Basis

For multi-channel measurement:

Born probability: Measurement outcome probabilities $p_i = \langle\psi, E_i\psi\rangle$ for POVM $\{E_i\}$.

I-projection: Minimal KL-divergence $\min_{p\in\mathcal{C}} D_{\mathrm{KL}}(p\|q)$ over constraint set $\mathcal{C}$ yields exponential family.

Pointer basis: Windowed operator $W_w = \int w(E)\,dE_A(E)$ minimal eigensubspace (Ky Fan) determines pointer basis.

5 Discussion

Established for multi-channel scattering:

- Phase–density–delay unification via Birman–Kreĭn–Wigner–Smith
- Channel basis covariance and threshold regularity
- Windowed readout with NPE non-asymptotic error closure
- Information-geometric Born probability and pointer basis

Applications: quantum optics multi-mode scattering, mesoscopic transport, nuclear reactions, gravitational multi-polarization.