Abstract

Within abstract framework of de Branges–Kreĭn canonical systems and multi-channel scattering, this paper establishes unified system in pure theoretical language of "operator–measure–function", independent of experimental narrative: welding "phase derivative–relative state density–Wigner–Smith group delay trace" as universal measure coordinate of same parent scale, characterizing finite resources and observational choices via windowed readouts of Toeplitz/Berezin compression. On reversible observational transformation group generated by base automorphisms, phase gauge and reversible filtering, proves invariance of time scale within blocks constructed from windowed delay integral; gives non-asymptotic error closure and stable principle of "singularity non-increasing/pole = dominant scale" under "finite-order" Euler–Maclaurin and Poisson discipline. Thus, in EBOC static block universe replaces external elapsed time with intrinsic $T_{\rm inv}$; obtains unified metric under RCA reversible computation's isomorphic renormalization. Scale identity of this system holds almost everywhere on absolutely continuous spectrum:

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\rm rel}(E) = \frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E)},$$

where $S(E) \in U\big(N(E)\big)$ scattering matrix, $\mathsf{Q}(E) := -i\,S(E)^\dagger \partial_E S(E)$ Wigner–Smith delay matrix, $\varphi(E) := \frac{1}{2}\,\mathrm{Arg}\det S(E)$, $\rho_{\rm rel}$ relative state density relative to reference channel/free Hamiltonian. Windowed readout defined by Toeplitz/Berezin compression map $\mathcal{C}_w$ and its covariant symbol $K_w(E)$:

$$\mathcal{T}_w(E) := \frac{1}{2\pi}\,\mathrm{tr}\,\big(K_w(E)\mathsf{Q}(E)\big), \qquad T_{\rm inv}(I) := \int_I \mathcal{T}_w(E)\,\mathrm{d}E,$$

remaining invariant under reversible observational transformations, becoming universal time scale within EBOC blocks. Above trinity scale connected by Wigner–Smith delay matrix and Birman–Kreĭn spectral shift–determinant formula, providing unified coordinate from phase to density, from scattering to measure, and stable benchmark for variational/optimization.

1 Notation & Axioms / Conventions

1. **Observational triple** $(\mathcal{H}, w, S)$: $\mathcal{H}$ Hilbert space; $S(E) \in U\big(N(E)\big)$ scattering matrix at energy scale E (a.e. on absolutely continuous spectrum); w window, inducing analysis–synthesis map Π_w and Toeplitz/Berezin compression map $\mathcal{C}_w[X] := \Pi_w X \Pi_w^\dagger$; covariant symbol $K_w(E) := \Pi_w(E)^\dagger \Pi_w(E) \geq 0$, determining readout functional. Window families take bandlimited or exponential decay classes, satisfying regularity in reproducing-kernel context.

2. **Window family normalization (Parseval/tight frame, component-wise)**: In direct integral decomposition of absolutely continuous spectrum and channel fibers, choose window family such that within each threshold regular component J have $\mathrm{tr}\,K_w(E) \equiv N_J$ (a.e. $E \in J$). This normalization compatible with reproducing-kernel regularity, ensuring windowed readout phase gauge terms only produce endpoint constants.

3. **Threshold set and regular domains**: Denote threshold set $\mathcal{T} := \{E : N(E^+) \neq N(E^-)\}$. For all integrals in Theorem 4.2, default $I \cap \mathcal{T} = \varnothing$, or equivalently first subdivide I along $\mathcal{T}$ then componentwise integrate and aggregate.

4. **Phase branch and differentiability**: Fix continuous branch of $\operatorname{Arg}\det S(E)$ on each threshold regular component J, thus $\varphi'(E)$ exists a.e. on J; across thresholds and discrete spectrum treat in distributional sense (Levinson-type transitions).

5. **Scale identity card**: Axiomatize
$$\varphi'(E)/\pi = \rho_{\rm rel}(E) = (2\pi)^{-1}\operatorname{tr}\mathsf{Q}(E), \quad \mathsf{Q} := -i\, S^\dagger \partial_E S.$$
Equivalence between phase derivative and $\operatorname{tr}\mathsf{Q}$ from Wigner–Smith definition and determinant differential identity; equivalence with $\rho_{\rm rel}$ given by Birman–Kreĭn formula and spectral shift function differential connection.

6. **Finite-order EM+Poisson card**: For sum–integral transformation and energy discretization, uniformly use **finite-order** Euler–Maclaurin and Poisson summation for non-asymptotic error closure; explicit bound constants depend on finite norms of window and symbol; singularity non-increasing and "pole = dominant scale".

7. **Language and objects**: Window/readout uniformly treated as "operator–measure–linear functional" objects; avoid experimental procedure narrative. Toeplitz/Berezin compression and Berezin transform used to map function symbols in energy–phase analytic platform (such as de Branges space, Paley–Wiener/Mellin models) to operators.

8. **Notation**: det! denotes regularized (Fredholm) determinant; tr trace; $P_{\rm ac}$ absolutely continuous spectral projection; "a.e." all refer to almost everywhere on absolutely continuous spectrum.

2 Scattering Phase, Group Delay and Spectral Shift: Trinity Coordinate

Let H and reference H_0 self-adjoint, satisfying usual traceable perturbation conditions making $S(E)$ exist and unitary. Define $\mathsf{Q}(E) := -i\, S(E)^\dagger \partial_E S(E)$ and $\varphi(E) := \frac{1}{2}\operatorname{Arg}\det S(E)$. Wigner–Smith gives Hermiticity of Q and its relation with energy derivative of S; trace satisfies $\operatorname{tr}\mathsf{Q}(E) = \partial_E \operatorname{Arg}\det S(E) = 2\varphi'(E)$. On other hand, Birman–Kreĭn formula $\det S(E) = \exp(-2\pi i\, \xi(E))$ connects scattering determinant with spectral shift function ξ, thus $\xi'(E) = -\frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E) = -\varphi'(E)/\pi$. Taking $\rho_{\rm rel}(E) := -\xi'(E)$ yields scale identity.

Corollary 2.1. *On absolutely continuous spectrum a.e., measures induced by three objects satisfy $d\mu_\varphi = d\mu_\rho = d\mu_Q$, and $d\mu_Q(E) = (2\pi)^{-1}\operatorname{tr}\mathsf{Q}(E)\,\mathrm{d}E$. This provides parent scale for subsequent windowed readout and transformation consistency.*

3 Windowed Readout and Toeplitz/Berezin Compression

Take reproducing-kernel space $\mathscr{H}$ (such as de Branges, Paley–Wiener or Mellin–Hardy) as energy–phase analytic platform. Window w induces analysis–synthesis map Π_w. Define **compression map**
$$\mathcal{C}_w[X] := \Pi_w X \Pi_w^\dagger,$$
and its **covariant symbol**
$$K_w(E) := \Pi_w(E)^\dagger \Pi_w(E) \geq 0.$$

Definition 3.1 (Channel Fiber Compression)**.** Under direct integral decomposition of absolutely continuous spectrum $\mathcal{H}_{\mathrm{ac}} \simeq \int^{\oplus} \mathbb{C}^{N(E)}\,\mathrm{d}E$, analysis map $\Pi_w(E) : \mathbb{C}^{N(E)} \to \mathbb{C}^{N(E)}$ gives covariant symbol

$$K_w(E) := \Pi_w(E)^{\dagger}\Pi_w(E) \in \mathbb{C}^{N(E)\times N(E)}, \qquad K_w(E) \geq 0.$$

Thus windowed density and readout $\mathcal{T}_w(E) := (2\pi)^{-1}\,\mathrm{tr}\,\big(K_w(E)\mathsf{Q}(E)\big)$, $T_{\mathrm{inv}}(I) := \int_I \mathcal{T}_w(E)\,\mathrm{d}E$, under Parseval normalization, for each threshold regular component J, satisfy $\mathrm{tr}\,K_w(E) \equiv N_J$ (a.e. $E \in J$).

For energy-local matrix symbol $A(E)$, define windowed trace $\langle A\rangle_w := \int \mathrm{tr}\,\big(K_w(E)A(E)\big)\,\mathrm{d}E$. Windowed density of group delay defined as $\mathcal{T}_w(E) := (2\pi)^{-1}\,\mathrm{tr}\,\big(K_w(E)\mathsf{Q}(E)\big)$, thus $T_{\mathrm{inv}}(I) := \int_I \mathcal{T}_w(E)\,\mathrm{d}E$. Toeplitz/Berezin system ensures positivity and regular limits of K_w, and consistency on symbol algebra.

4 Reversible Observational Equivalence and Gauge Invariants

Definition 4.1 (Reversible Observational Transformation)**.** Reversible observational transformations generated by:

(i) Automorphism U of $\mathcal{H}$ (fixing energy scale);

(ii) **Phase gauge**: $S \mapsto e^{i\theta(E)}S$, where $\theta \in W^{1,1}(I) \cap C^0(\bar{I})$ and **for each component interval endpoint** $\theta(E_{j,\pm}) = 0$;

(iii) **Reversible window renormalization (energy-independent)**: On each threshold regular component J take fixed channel basis $U \in U(N_J)$. Window renormalization $w \mapsto \widetilde{w}$ induces

$$\Pi_{\widetilde{w}} = \Pi_w U^{\dagger}, \qquad K_{\widetilde{w}}(E) = UK_w(E)U^{\dagger}.$$

Theorem 4.2 (Gauge Invariance of Windowed Delay–Normalized Version)**.** *Under conditions of normalization and Definition, for any* ***threshold regular*** *finite union interval* $I = \bigsqcup_{j=1}^{J}[E_{j,-}, E_{j,+}] \subset \mathbb{R}$ *(i.e.,* $I \cap \mathcal{T} = \varnothing$*), quantity*

$$T_{\mathrm{inv}}(I) := \int_I \frac{1}{2\pi}\,\mathrm{tr}\,\big(K_w(E)\mathsf{Q}(E)\big)\,\mathrm{d}E$$

invariant under reversible observational transformations (automorphism, phase gauge, reversible window renormalization).

Proof. Use trace and similarity invariance get $\mathrm{tr}(UK_wU^{\dagger}\cdot U\mathsf{Q}U^{\dagger}) = \mathrm{tr}(K_w\mathsf{Q})$. Phase gauge contributes term $\theta'(E)\,\mathrm{tr}\,K_w(E)$; by component-wise normalization $\mathrm{tr}\,K_w(E) \equiv N_j$ ($E \in [E_{j,-}, E_{j,+}]$) and endpoint condition $\theta(E_{j,\pm}) = 0$, get $\int_I \theta'(E)\,\mathrm{tr}\,K_w(E)\,\mathrm{d}E = \sum_j N_j[\theta]_{E_{j,-}}^{E_{j,+}} = 0$, gauge term vanishes, invariance holds. □

Corollary 4.3 (Universal Time Scale)**.** T_{inv} *independent of observational representation, constitutes intrinsic time scale in EBOC static blocks.*

5 Universal Measure Coordinate and Transformation Consistency

Proposition 5.1 (a.c. Three-Measure Consistency and Distributional Extension)**.** *Let $S(E)$ satisfy usual traceable perturbation and limiting absorption conditions. Then on absolutely continuous spectrum a.e. have*

$$d\mu_{\varphi}^{\mathrm{ac}} = d\mu_{\rho}^{\mathrm{ac}} = d\mu_{Q}^{\mathrm{ac}},$$

where $d\mu_{\varphi}(E) = \frac{\varphi'(E)}{\pi}\,\mathrm{d}E$, $d\mu_{\rho}(E) = \rho_{\mathrm{rel}}(E)\,\mathrm{d}E$, $d\mu_{Q}(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)\,\mathrm{d}E$. If incorporating discrete spectrum/thresholds into full spectrum, three consistent in distributional sense: $d\mu_{\rho}$ contains δ-masses at discrete spectrum, φ exhibits phase jumps (Levinson-type), Q takes boundary values. Therefore any windowed readout comparable and transformable under same coordinate, transformation error bounded by unified constants of Section 5.

6 Stable Error Theory: Finite-Order Euler–Maclaurin and Poisson

Let w belong to bandlimited class or exponential class, $a(E)$ sufficiently smooth energy symbol, $\{E_n\}$ energy partition (generated by window or spectral tube). For energy domain $I = \bigsqcup_{j=1}^{J}[E_{j,-}, E_{j,+}]$, exists $m \in \mathbb{N}$ and constants C_m, C'_m (depending only on window family and finite-order derivative seminorms) such that

$$\Big|\sum_{n} a(E_n) - \int_I a(E)\,\mathrm{d}E - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \sum_{j=1}^{J} \big[a^{(2k-1)}(E)\big]_{E_{j,-}}^{E_{j,+}}\Big| \le C_m\,\mathfrak{R}_m(a,w),$$

$$\Big|\sum_{k\neq 0} \widehat{a}(2\pi k)\widehat{w}(2\pi k)\Big| \le C'_m\,\mathfrak{P}_m(a,w).$$

where B_{2k} Bernoulli numbers, $\mathfrak{R}_m, \mathfrak{P}_m$ error functionals composed of finite seminorms. For $a = \operatorname{tr}\mathsf{Q}$, φ', ρ_{rel} apply same constant chain; obtain unified error budget on windowed readouts of three objects; and singularity non-increasing and "pole = dominant scale" hold.

7 EBOC Intrinsic Time and RCA Reversible Computation Isomorphic Renormalization

Definition 7.1 (Intrinsic Time Scale)**.** For energy domain I define

$$T_{\mathrm{inv}}(I) = \int_I \frac{1}{2\pi}\operatorname{tr}\big(K_w(E)\mathsf{Q}(E)\big)\,\mathrm{d}E,$$

as relational progression time scale in EBOC static blocks; invariant under reversible observational transformations.

Theorem 7.2 (RCA Isomorphic Renormalizability)**.** *Embed step depth of reversible cellular automaton $\mathcal{U}$ into T_{inv} metric: if two observational triples $(\mathcal{H}_i, w_i, S_i)$ reversibly equivalent, then RCA "depth" corresponding to their boundary–channel coupling measured by same time scale. Proof depends on invariance of Theorem 4.2 and unified coordinate of scale identity, obtaining isomorphic renormalization under different bases/encodings.*

8 Discussion and Outlook

This work establishes:

1. Trinity scale unification $\varphi'/\pi = \rho_{\mathrm{rel}} = (2\pi)^{-1}\operatorname{tr}\mathsf{Q}$ via Wigner–Smith delay and Birman–Kreĭn formula
2. Windowed readout framework via Toeplitz/Berezin compression with covariant symbol $K_w(E)$
3. Gauge invariance of intrinsic time scale T_{inv} under reversible observational transformations
4. Non-asymptotic error closure via finite-order Euler–Maclaurin and Poisson summation
5. Connection to EBOC static block universe and RCA reversible computation
6. Frame-theoretic foundations via Wexler–Raz, Balian–Low, Landau density
7. de Branges–Kreĭn analytic platform and Herglotz–Nevanlinna structure

Key formulas:

- Scale identity: $\varphi'/\pi = \rho_{\mathrm{rel}} = (2\pi)^{-1}\operatorname{tr}\mathsf{Q}$
- Windowed time: $T_{\mathrm{inv}}(I) = \int_I (2\pi)^{-1}\operatorname{tr}(K_w\mathsf{Q})\,\mathrm{d}E$
- Invariance: T_{inv} unchanged under (U, θ, w) transformations

Future directions:

- Numerical implementation and benchmarking
- Extension to open quantum systems and non-Hermitian scattering
- Connections to quantum information and complexity theory
- Applications to quantum gravity and emergent spacetime