Finite Throughput–Group Delay Dilation Principle: Unifying Bandwidth, Time Delay, and Redshift via WSIG–EBOC–RCA's "Window–Scale–Gauge"

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Abstract

We establish a rigorous theory unifying "finite throughput–queueing delay–apparent dilation (redshift)" within windowed scattering and information geometry (WSIG) scale system. In pure theoretical language of operator–measure–function, using "trinity" scale identity $\varphi'(E)/\pi = \rho_{\text{rel}}(E) = -\frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)$ as mother scale, define observer window's Toeplitz/Berezin compression $K_{w,h}$ and its readout functional, characterize "finite throughput" abstract form via passivity and causality-preserving scattering axioms. Prove three main theorems: (T1) **Throughput–Delay Dilation Theorem**: Under passive load monotonicity, any windowed group delay readout monotonically non-decreasing with load; when baseline readout $\Phi_w[\lambda_0] > 0$, induces Mellin dilation of energy and time scales scale-equivalent to redshift factor $1 + z_w = \Delta_w \geq 1$; (T2) **Group Delay–Bandwidth Product Upper Bound**: Windowed energy spectrum total readout upper-bounded by product of "group delay flux" and "effective bandwidth constant", with non-asymptotic error bound from Nyquist–Poisson–Euler–Maclaurin (NPE) three-term closure; (T3) **Scale Gauge Equivalence**: Cosmological expansion and internal resolution enhancement scale-equivalent to gauge pairing $a(t) = R(t)^{-1}$, all readouts remaining invariant on same mother scale. Further provide "front slope–group delay" correspondence at reversible cellular automaton (RCA) dynamics layer, embedding queueing–throughput–delay discrete dynamics into EBOC static block encoding. Throughout use only finite-order Euler–Maclaurin and Poisson discipline, maintaining "singularity non-increasing/poles = principal scales" error control; any numerical and experimental interfacing relegated to appendices.

Keywords: Finite throughput; Group delay dilation; WSIG; EBOC; RCA; NPE error closure; Redshift equivalence; Scale gauge

Notation & Axioms / Conventions

1. **Scale Identity (Trinity):** For absolutely continuous spectrum a.e.

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\rm rel}(E) = -\frac{1}{2\pi} \operatorname{tr} \mathsf{Q}(E)}, \qquad \mathsf{Q}(E) = -i\, S(E)^\dagger S'(E),\ S(E) \in U(N), \tag{1}$$

where φ is total scattering phase, $\rho_{\rm rel}$ relative state density (Birman–Kreĭn perspective), Q Wigner–Smith group delay matrix. By Birman–Kreĭn formula $\det S(E) = \exp(-2\pi i\, \xi(E))$ and $-i\, \partial_E \log \det S(E) = \operatorname{tr} \mathsf{Q}(E)$ obtain $\operatorname{tr} \mathsf{Q}(E) = -2\pi\, \xi'(E)$, thus above holds ($\xi$ spectral shift function).

2. **Observer Window and Compression:** Take non-negative weight window $w \in L^1(\mathbb{R}) \cap L^\infty$ and smoother h (approximating identity, Carleson regular). Define Toeplitz/Berezin compression $K_{w,h}$ acting on energy spectrum symbol $F(E)$ windowed readout

$$\langle F \rangle_w := \int_{\mathbb{R}} w(E)\, \operatorname{tr} F(E)\, dE, \qquad \langle F \rangle_{w,h} := \operatorname{tr}(K_{w,h} F), \tag{2}$$

differing under NPE discipline only by finite-order Poisson+Euler–Maclaurin error. Characterization of Toeplitz/Berezin and Carleson embedding see recent developments in weighted Bergman/Hardy systems.

3. **Passivity and Monotonicity:** Load represented by unitarity-preserving scattering family $S_\lambda(E) \in U(N)$ ($\lambda \in [0,1]$). Passivity axiom written as

$$\partial_\lambda \mathsf{Q}_\lambda(E) \succeq 0 \ \text{ a.e.} \tag{3}$$

Throughout derivations use only this Loewner monotonicity, no longer claim sign equivalence with $\partial_\lambda \rho_{\rm rel}$. Supportable by monotonicity of spectral shift function $\xi(E; \lambda)$ for monotonically coupled self-adjoint spectral pairs ($H_\lambda = H_0 + \lambda V$, $V \geq 0$) with λ.

4. **NPE Three-Term Closure (Finite-Order):** Discrete–continuous error decomposition

$$\mathcal{E}_{\rm total} = \mathcal{E}_{\rm alias} + \mathcal{E}_{\rm BL} + \mathcal{E}_{\rm tail}, \tag{4}$$

where $\mathcal{E}_{\rm alias}$ from Poisson spectral aliasing, $\mathcal{E}_{\rm BL}$ finite-order Euler–Maclaurin boundary layer term, $\mathcal{E}_{\rm tail}$ controlled by window decay and target function step. Standard references for Poisson and Euler–Maclaurin (including Bernoulli constants) see cited sources.

5. **Scale Gauge:** Dual gauge of external scale factor $a(t)$ and internal resolution constant $R(t)$

$$a(t) = R(t)^{-1}, \qquad \kappa(t) := \dot{a}/a = -\dot{R}/R, \tag{5}$$

implements Mellin dilation on energy mother scale, redshift satisfies $1 + z = a(t_0)/a(t_e) = R(t_e)/R(t_0)$.

1 Windowed Readout, Group Delay Flux, and Effective Bandwidth

Define windowed group delay flux

$$\Phi_w := \frac{1}{2\pi} \int_{\mathbb{R}} w(E) \operatorname{tr} \mathsf{Q}(E)\, dE = -\langle \rho_{\mathrm{rel}} \rangle_w. \tag{6}$$

Define window's effective bandwidth constant

$$B_w := |w|_{L^1} \cdot |\widehat{w}|_{\mathrm{pack}}, \tag{7}$$

where $\widehat{w}$ corresponding Fourier–Mellin transform, $|\cdot|_{\mathrm{pack}}$ controlled by Nyquist/Landau density (for bandlimited or tight frame window families can be given by upper/lower bound constants). If $0 \preceq F(E) \preceq \alpha(E)\mathbf{1}$, then

$$\langle F \rangle_w \le N\, |\alpha|_{L^\infty(\operatorname{supp} w)} \cdot |w|_{L^1}. \tag{8}$$

General case Q not necessarily positive, thus cannot take $F = (2\pi)^{-1}\mathsf{Q}$; can use

$$\boxed{\left| \left\langle \tfrac{1}{2\pi}\mathsf{Q} \right\rangle_w \right| \le \frac{N}{2\pi} |\mathsf{Q}|_{L^\infty(\operatorname{supp} w)} |w|_{L^1}} \tag{9}$$

as crude upper bound. Necessary conditions for sampling/frame density and bandwidth packing see Landau and subsequent generalizations.

2 Throughput Abstraction: Operator–Measure–Function Paradigm

Let observation triple $(\mathcal{H}, w, S)$. Write scale measure $d\mu_{\mathsf{Q}} := (2\pi)^{-1} \operatorname{tr} \mathsf{Q}(E)\, dE$. Define energy readout

$$\mathscr{R}_w := \int w(E)\, \rho_{\mathrm{rel}}(E)\, dE = -\frac{1}{2\pi} \int w(E) \operatorname{tr} \mathsf{Q}(E)\, dE = -\int w(E)\, d\mu_{\mathsf{Q}}(E) = -\Phi_w. \tag{10}$$

"Finite throughput" abstracted as Carleson-type constraint on μ_{Q}: exists constant C_{th} and energy domain partition $\{\Omega_k\}$ such that

$$|\mu_{\mathsf{Q}}|(\Omega_k) \le C_{\mathrm{th}} \Lambda(\Omega_k), \tag{11}$$

where Λ reference measure induced by window family (given by Nyquist density or frame density). Corresponding Toeplitz/Berezin type embedding in weighted function spaces characterizable by Carleson condition.

3 Main Theorem I: Throughput–Delay Dilation Theorem

Theorem 3.1 (Load Monotonicity and Dilation Factor)**.** *Let $S_\lambda(E)$ satisfy passivity axiom $\partial_\lambda \mathsf{Q}_\lambda(E) \succeq 0$ a.e., and window $w \ge 0$ compactly supported or satisfying frame regularity and NPE conditions. Then for any $0 \le \lambda_0 < \lambda_1 \le 1$,*

$$\Phi_w[\lambda_1] - \Phi_w[\lambda_0] = \frac{1}{2\pi} \int w(E) \operatorname{tr}\big(\mathsf{Q}_{\lambda_1}(E) - \mathsf{Q}_{\lambda_0}(E)\big)\, dE\ \ge 0. \tag{12}$$

*Define scale's "delay dilation factor" (**when** $\Phi_w[\lambda_0] > 0$)*

$$\Delta_w(\lambda_1, \lambda_0) := \frac{\Phi_w[\lambda_1]}{\Phi_w[\lambda_0]} \geq 1. \tag{13}$$

General case always-valid statement is

$$\Phi_w[\lambda_1] - \Phi_w[\lambda_0] \geq 0, \tag{14}$$

equivalent to $\mathscr{R}_w$'s non-increasing (because $\mathscr{R}_w = -\Phi_w$). By Trinity, Δ_w simultaneously characterizes phase density derivative and relative state density elongation, inducing energy scale–time scale Mellin dilation, equivalent to redshift factor $1 + z_w = \Delta_w(\lambda_1, \lambda_0) \geq 1$. Root lies in spectral shift function $\xi(E; \lambda)$ monotonic with non-negative coupling, and $\operatorname{tr} \mathsf{Q}(E) = -2\pi\, \partial_E \xi(E)$.

Proof. Loewner order monotonicity at each E and $w \geq 0$ integration immediately yields non-decreasing. By Birman–Kreĭn and Trinity linear relations, monotonicity equivalently transmitted between φ' and $\operatorname{tr} \mathsf{Q}$; on ρ_{rel} differs only by fixed negative sign ($\rho_{\mathrm{rel}} = -\frac{1}{2\pi} \operatorname{tr} \mathsf{Q}$), thus above non-decreasing unified expressed as Φ_w. $\square$

Corollary 3.2 (Window–Band Domain Locality)**.** *If w supported on bandlimited region Ω, then z_w is local redshift for that domain; for mutually non-overlapping $\{\Omega_j\}$ with decomposition windows $\{w_j\}$ forming tight frame,*

$$1 + z_{\mathrm{global}} = \sum_j \omega_j\,(1 + z_{w_j}), \qquad \omega_j = \frac{\Phi_{w_j}[\lambda_0]}{\sum_k \Phi_{w_k}[\lambda_0]}, \tag{15}$$

giving global dilation as weighted combination of local dilations; if baseline state $\mu_{\mathsf{Q}}[\lambda_0]$ is non-negative measure (equivalent to $\Phi_{w_j}[\lambda_0] \geq 0$ for all j), this combination is convex combination.

4 Main Theorem II: Group Delay–Bandwidth Product Upper Bound

Theorem 4.1 (Finite Window Energy Spectrum Conservation Upper Bound)**.** *Under §2's Carleson-type throughput constraint, window family tight frame, and NPE validity, for any $S(E)$ and window w*

$$\boxed{\left|\Phi_w\right| \leq C_{\mathrm{frame}}\, C_{\mathrm{th}}\, B_w\left(1 + \left|\varepsilon_{\mathrm{NPE}}\right|\right)}, \tag{16}$$

where C_{frame} given by window family upper/lower bound constants, B_w effective bandwidth constant, $\varepsilon_{\mathrm{NPE}}$ finite-order error term satisfying

$$\left|\varepsilon_{\mathrm{NPE}}\right| \leq C_1\, \mathcal{E}_{\mathrm{alias}} + C_2\, \mathcal{E}_{\mathrm{BL}}^{(m)} + C_3\, \mathcal{E}_{\mathrm{tail}}^{(\beta)}. \tag{17}$$

Proof Sketch. Along Berezin compression commutator inequalities and Carleson embedding control μ_{Q}, window family frame upper/lower bounds connect local energy spectrum with global readout, finally use finite-order Poisson dealiasing and Euler–Maclaurin boundary layer absorb discrete–continuous difference; singularity non-increasing ensures principal scale poles not amplified. $\square$

Corollary 4.2 (Optimal Window's Variational Tendency). *Under fixed C_{th} and resource constraints $|w|_{L^1}, |\widehat{w}|_{\mathrm{pack}}$, $w^\star$ maximizing Φ_w tends toward tight frame and near-minimum uncertainty window; if bandlimited domain invariant, $w^\star$ nearly constant amplitude within domain, minimum boundary layer cost at boundaries. Its frame/density conditions compatible with Wexler–Raz biorthogonality and Balian–Low obstructions.*

5 Main Theorem III: Scale Gauge Equivalence and Redshift = Resolution Enhancement

Theorem 5.1 (Gauge Equivalence). *Let $a(t), R(t)$ satisfy $a(t) = R(t)^{-1}$ and $\kappa(t) = \dot{a}/a = -\dot{R}/R$. On Trinity mother scale, for any window w and energy readout*

$$\mathscr{R}_w(t) = \int w(E)\, \rho_{\mathrm{rel}}(E;t)\, dE = -\frac{1}{2\pi} \int w(E)\, \mathrm{tr}\, \mathsf{Q}(E;t)\, dE, \qquad (18)$$

exists Mellin dilation invariance

$$\mathscr{R}_w(t_0) = \mathscr{R}_{(1+z)^{-1}\,w\circ\mathcal{D}_{1+z}}(t_e), \qquad \mathcal{D}_{1+z} : E \mapsto \frac{E}{1+z}, \quad 1+z = \frac{a(t_0)}{a(t_e)} = \frac{R(t_e)}{R(t_0)}. \quad (19)$$

Thus "expansion" and "resolution enhancement" merely gauge restatements under same mother scale, not changing any windowed readout's ontological meaning.

Corollary 5.2 (Redshift–Delay Pairing). *If load λ related to gauge a such that $\partial_\lambda \mathsf{Q} \succeq 0$ and $d\lambda$ induces $da/a = d\lambda \cdot \eta(\lambda)$ gain, then T1's Δ_w and T3's $1+z$ unify: $1 + z_w = \Delta_w$.*

6 Discrete Dynamics Interface: RCA Front Slope and Group Delay

Under EBOC static block encoding, let one-dimensional reversible cellular automaton $\mathcal{A}$'s spacetime diagram as two-dimensional SFT. Let its front slope c_{RCA} be average velocity of coherent defect propagation. WSIG scale embedding yields energy axis–lattice scale isomorphism, making

$$c_{\mathrm{RCA}} = \left(\frac{dx}{dt}\right)_{\mathrm{eff}} \propto \left(\tfrac{1}{N}\,\mathrm{tr}\,\mathsf{Q}\right)^{-1}, \qquad (20)$$

i.e., group delay trace reciprocal determines RCA's effective advancement rate; passive load increases $\mathrm{tr}\,\mathsf{Q}$, thus lowering c_{RCA}, manifesting as discrete dynamics "deceleration" and continuous scale "redshift" isomorphism. Foundational properties of reversibility, information velocity, and light cone bounds see classical CA theory surveys.

7 Error Theory and "Singularity Non-Increasing/Poles = Principal Scales"

Proposition 7.1 (Finite-Order NPE Closure). *If window w has m-order smoothness and β-order decay, under Landau density condition,*

$$\left|\mathscr{R}^{\mathrm{disc}}*w - \mathscr{R}^{\mathrm{cont}}*w\right| \le C(m,\beta)\left(\sum_{\ell\neq 0}|\widehat{w}(\ell f_s)| + \sum_{j=1}^{m}\frac{|B_{2j}|}{(2j)!}|\partial^{(2j-1)}w|_{L^1} + \int_{|E|>E_{\max}} |w(E)\,\rho_{\mathrm{rel}}(E)|\, dE\right),$$

$$(21)$$

where B_{2j} Bernoulli constants; if ρ_{rel} contains only finite poles and window operation does not excite new singularities, error terms do not elevate singularity, principal scale determined by poles.

8 Multi-Port Structure and Statistical Characterization

Let $\{\tau_j(E)\}_{j=1}^N$ be eigenvalues of $\mathsf{Q}(E)$ (proper delay times), then

$$\Phi_w = \frac{1}{2\pi} \sum_{j=1}^N \int w(E)\, \tau_j(E)\, dE, \qquad \partial_\lambda \tau_j(E) \geq 0 \ \text{ a.e.} \tag{22}$$

Statistical level distributions and moment generating functions in chaotic cavities and non-ideal coupling given by random matrix theory, supporting "total group delay = eigentime sum" decomposition and its extreme fluctuation laws.

9 Counterpoint with Communication Capacity Intuition (Pure Theoretical Restatement)

Using μ_{Q}'s Carleson constraint to represent "finite throughput", corresponding to finite total group delay per unit resource block. T1 gives "load $\uparrow \Rightarrow$ delay readout $\uparrow$"; T2 upper-bounds total readout as "group delay flux $\times$ effective bandwidth constant". If using scale time constant $\mathcal{T}_w := \Phi_w / |w|_{L^1}$ as apparent period elongation, when $\Phi_w[\lambda_0] > 0$ have

$$\frac{\mathcal{T}_w[\lambda_1]}{\mathcal{T}_w[\lambda_0]} \;=\; \Delta_w(\lambda_1, \lambda_0) \;=\; 1 + z_w. \tag{23}$$

General case, only $\mathcal{T}_w[\lambda_1] - \mathcal{T}_w[\lambda_0] \geq 0$ (equivalent to Φ_w non-decreasing) holds, giving scale equivalence "redshift = effective delay stretching", no probabilistic assumptions or experimental vocabulary needed.

10 Gauge Group and Invariance

Define Mellin–Heisenberg gauge group $\mathcal{G}$ action on window and readout

$$g \cdot w(E) = \chi\, w(\chi E), \qquad g \cdot \rho_{\text{rel}}(E) = \rho_{\text{rel}}(E/\chi), \tag{24}$$

then windowed readout under Trinity mother scale satisfies

$$\langle \rho_{\text{rel}} \rangle_{g \cdot w} = \langle g \cdot \rho_{\text{rel}} \rangle_w. \tag{25}$$

Scale gauge $a = R^{-1}$ is one-dimensional subgroup of $\mathcal{G}$, equivalent to T3.

Appendix A: Theorem T2 Proof Outline

Start with Berezin compression $K_{w,h}$ and self-adjoint symbol $F(E) = (2\pi)^{-1}\mathsf{Q}(E)$. Carleson embedding yields

$$\left| \operatorname{tr}(K_{w,h}F) \right| \leq C_{\mathrm{th}}\, |K_{w,h}|_{\mathrm{Car}} \lesssim C_{\mathrm{th}}\, C_{\mathrm{frame}}\, B_w. \tag{26}$$

where $|K_{w,h}|_{\mathrm{Car}}$ controlled by window family frame density and $\widehat{w}$ packing degree; then use finite-order Poisson dealiasing, Euler–Maclaurin absorb boundary layer and tail term,

$$\Phi_w = \langle F \rangle_w = \operatorname{tr}(K_{w,h}F) + O(\varepsilon_{\mathrm{NPE}}), \qquad \left| \Phi_w \right| \leq C_{\mathrm{th}}\, C_{\mathrm{frame}}\, B_w \left(1 + \left| \varepsilon_{\mathrm{NPE}} \right| \right). \tag{27}$$

When w bandlimited and window family nearly tight, $|K_{w,h}|_{\mathrm{Car}}$ upper/lower bounds merge, bound can be tight.

Appendix B: RCA's Constructive Scale

In one-dimensional reversible CA's local rule and SFT encoding framework, introduce energy axis scale map $\Theta : \mathbb{Z}^2 \to \mathbb{R}$, making each characteristic direction's discrete velocity satisfy with scale group delay

$$c_{\mathrm{RCA}}(\theta) = \frac{\Delta x}{\Delta t} \propto \left(\frac{1}{N} \operatorname{tr} \mathsf{Q}\big(\Theta(\theta) \big) \right)^{-1}. \tag{28}$$

(If and only if choosing unified space/time unit gauge, proportionality constant can be normalized.) Passivity makes $\operatorname{tr} \mathsf{Q}$ monotonically increasing, thus c_{RCA} monotonically decreasing; this decrease isomorphic with T1's $\Delta_w \geq 1$, giving "non-acceleration–non-increase" type reversible dynamics readout. Foundational theorems on reversibility, information light cone, and velocity upper bounds see cited sources.

Appendix C: NPE Error Budget (Finite-Order Recipe)

1. **Poisson Dealiasing:** Choose sampling rate f_s satisfying Landau density condition, control
$$\mathcal{E}_{\mathrm{alias}} \leq \sum_{\ell \neq 0} \left| \widehat{w}(\cdot + \ell f_s) \right|. \tag{29}$$

2. **Euler–Maclaurin Boundary Layer (m order):** Take m such that $\partial^{(2m)}w$ and $\partial^{(2m)}\rho_{\mathrm{rel}}$ integrable, error
$$\mathcal{E}_{\mathrm{BL}}^{(m)} \lesssim \sum_{j=1}^{m} \frac{|B_{2j}|}{(2j)!} \, |\partial^{(2j-1)}w|_{L^1} \, |\partial^{(2j-1)}\rho_{\mathrm{rel}}|_{L^\infty(\operatorname{supp} w)}. \tag{30}$$

3. **Tail Term:** Energy domain truncation $E_{\max}$, $\mathcal{E}_{\mathrm{tail}}^{(\beta)} \leq |w \cdot \mathbf{1}_{|E|>E_{\max}}|_{L^1} \, |\rho_{\mathrm{rel}}|_{L^\infty}$, controlled by w's β-order decay to required precision.

Terminology Cards (Fixed)

- **Card A (Scale Identity):** $\dfrac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = -\dfrac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)$; readouts uniformly executed on mother scale.

- **Card B (Finite-Order EM and "Poles = Principal Scales"):** Use only finite-order Euler–Maclaurin and Poisson; error theory follows "singularity non-increasing/poles = principal scales".

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