Windowed Formulation of Phase–Spectral-Shift–DOS–Cosmological-Constant

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Abstract

We establish equivalence chain centered on **generalized scattering phase**, connecting Kontsevich–Vishik (KV) determinant phase, generalized Kreĭn spectral shift, density-of-states difference (DOS difference), and Wigner–Smith (WS) trace identity, providing rigorous **variable and factor accounting** under even-dimensional asymptotically hyperbolic/conformally compact (AH/CCM) and static-patch de Sitter (dS) geometry. Under strict Lifshits–Kreĭn (LK) trace formula and relative trace-class assumptions, we prove substitution relation between heat kernel difference Laplace representation and DOS slope under **frequency variable**

$$\Delta K(s) \;=\; \int_0^\infty e^{-s\omega^2}\,\Theta'(\omega)\,d\omega,$$

where $\Theta(\omega) = \frac{1}{2\pi}\arg\det_{\rm KV} S(\omega)$, $\Theta' = \Delta\rho_\omega = -\partial_\omega \xi_\omega$. We propose family of **logarithmic frequency windows** W with Mellin-nullification conditions, establishing **windowed Tauberian theorem**: under non-trapping, no zero-energy resonance, analytic Fredholm and operator-Lipschitz assumptions, small-s heat kernel finite part equivalent to logarithmic window average of Θ' at scale $\mu \sim s^{-1/2}$, with error upper bound. This defines **windowed integration law**

$$\partial_{\ln\mu}\Lambda_{\phi,W}(\mu) = \kappa_\Lambda\,\Xi_W(\mu), \qquad \Xi_W(\mu) = \int_{\mathbb{R}} \omega\,\Theta''(\omega)\,W\big(\ln(\omega/\mu)\big)\,d\ln\omega = \frac{1}{2\pi}\int \omega\,\partial_\omega {\rm Tr}\,Q(\omega)\,W\,d\ln\omega,$$

providing dimensionally consistent constant separation: $\kappa_{\rm HK}$ (heat kernel–windowing ratio) and κ_Λ (dimensional factor mapping $\langle\Theta'\rangle_W$ to cosmological constant). For **open channels** (horizon/absorption), respectively establish provable conditions for $\partial_\omega \arg\det_{\rm KV}\widehat{S} = -i\,{\rm Tr}\,\widehat{S}^\dagger\partial_\omega\widehat{S}$ via **extended unitarization** and **relative determinant**, preserving WS-trace equality. Demonstrate κ extraction procedure with one-dimensional δ potential as solvable template and static-patch dS scalar as computable template. Propose minimal reproducible observation pipeline based on **FRB baseband** complex phase, providing closed-form formulas for second-order phase kernel variance scaling, dispersion/multipath leakage kernel, and injection–recovery efficacy analysis. Above theorems and engineering schemes jointly constitute verifiable "windowed formulation".

Keywords: Generalized scattering phase; KV determinant; generalized Kreĭn spectral shift; Lifshits–Kreĭn trace formula; Wigner–Smith trace; heat kernel finite part; Tauberian; logarithmic window; relative determinant; static-patch de Sitter; FRB baseband

1 Introduction and Historical Background

On even-dimensional AH/CCM geometry, Guillarmou defines KV determinant via renormalized trace (Kontsevich–Vishik trace, TR) yielding

$$\arg \det_{\mathrm{KV}} S\Big(\tfrac{n}{2}+i\omega\Big) = -2\pi\,\xi(\omega) \pmod{2\pi},$$

where ξ is generalized Kreĭn spectral function; its logarithmic derivative couples to TR-trace of scattering operator S, providing geometrized version of "phase = spectral shift". This framework compatible with Friedel–Lloyd–Birman–Kreĭn (BK) relation, rigorously formulated in **even-dimensional** AH/CCM scenarios.

Sá Barreto–Wang prove: on non-trapping AH, $S(\omega)$ is Fourier integral operator (FIO) quantizing scattering relation, ensuring differentiability, symbol and kernel regularity for $\omega > 0$, laying foundation for differentiable framework of WS-trace and KV-det.

Peller characterizes function classes applicable to Lifshits–Kreĭn trace formula: for operator-Lipschitz f,

$$\mathrm{Tr}(f(H) - f(H_0)) = \int f'(\lambda)\,\xi_E(\lambda)\,d\lambda,$$

holds under relative trace-class (or weaker relative class-trace) conditions. This paper places $f(\lambda) = e^{-s\lambda}$, through substitution $\lambda = \omega^2$, connecting heat kernel difference with frequency-domain DOS slope.

In electromagnetic multiport systems, trace of WS time-delay matrix $Q = -i\,S^\dagger \partial_\omega S$ equivalent to total phase derivative: $\partial_\omega \arg\det S = \mathrm{Tr}\,Q$. This equality measurable at both experimental and algorithmic levels, constituting observation-end interface.

In black hole/static-patch dS scenarios, "**relative DOS–partition function–phase shift**" provides computable paradigm removing continuous spectrum divergence, combining scattering phase with one-loop free energy/partition function. This paper uses this as template for **acyclic calibration** of absolute constants.

This paper's goal is complete alignment of above chain on **frequency variable**, establishing Tauberian theorem for **logarithmic frequency windowing**, rigorously stating differentiability of KV-det under **open channels** and trace formula applicability domain, proposing implementable FRB baseband observation test.

2 Model and Assumptions

2.1 Geometry and Operators

We work on two types of backgrounds:

- **Even-dimensional AH/CCM**: (X^{n+1}, g) even-dimensional conformally compact scattering geometry, satisfying **non-trapping** and **no zero-energy resonance/embedded eigenvalue**. Laplace-type operator $H = -\Delta_g + V$ (with regular potential) and reference operator H_0 form relative pair.

- **Static-patch de Sitter (dS)**: Take static-patch region with horizon as boundary, adopt physical "in/out" channels, construct **extended channels** $\mathbb{S}$ or **relative operator** $\widehat{S} := S S_{\mathrm{ref}}^{-1}$.

For AH/CCM, $S(\omega)$ is FIO with kernel smooth in ω; for dS, extends to unitary after extended channels.

2.2 KV Determinant and Generalized Kreĭn

Denote $\Phi(\omega) := \arg\det_{\mathrm{KV}} S(\omega)$, $\Theta(\omega) := \Phi(\omega)/(2\pi)$. Guillarmou establishes

$$\Phi(\omega) = -2\pi\,\xi_\omega(\omega) \pmod{2\pi}, \qquad \Theta'(\omega) = -\partial_\omega \xi_\omega(\omega).$$

where $\xi_\omega(\omega) := \xi_E(\lambda)\big|_{\lambda=\omega^2}$.

2.3 DOS and Variable Transformation

In energy variable λ, $\Delta\rho_E(\lambda) = -\xi_E'(\lambda)$. In frequency variable ω,

$$\Delta\rho_\omega(\omega) = 2\omega\,\Delta\rho_E(\omega^2) = -\partial_\omega \xi_\omega(\omega).$$

Define

$$\boxed{\Theta'(\omega) = \Delta\rho_\omega(\omega), \qquad \operatorname{Tr} Q(\omega) = \partial_\omega \Phi(\omega) = 2\pi\,\Delta\rho_\omega(\omega)}.$$

Last equality is WS-trace.

2.4 Window Family and Mellin-Nullification

Take compactly supported window $W \in C_0^\infty(\mathbb{R})$, define logarithmic window average

$$\langle f \rangle_W(\mu) = \int_{\mathbb{R}} f(\mu e^u)\,W(u)\,du \;=\; \int_0^\infty f(\omega)\,W\big(\ln(\omega/\mu)\big)\,d\ln\omega.$$

Denote Mellin transform $\widehat{W}(z) = \int_{\mathbb{R}} e^{zu} W(u)\,du$. If $f(\omega) \sim \sum_k c_k\,\omega^{\beta_k} + \sum_{m,j} d_{m,j}\,\omega^{\tilde\beta_m}(\ln\omega)^j$ has high-frequency power-logarithmic asymptotics, require window to satisfy

$$\widehat{W}(\beta_k) = 0, \qquad \frac{d^j}{dz^j}\widehat{W}(z)\big|_{z=\tilde\beta_m} = 0 \;\; (0 \le j \le J_m),$$

nullifying power and logarithmic power terms.

3 Main Results

Theorem 1 (1: Phase–Spectral-Shift–DOS–WS Unification)**.** *Under assumptions of 2.1, for $\omega > 0$,*

$$\Theta'(\omega) = \Delta\rho_\omega(\omega) = -\partial_\omega \xi_\omega(\omega), \qquad \operatorname{Tr} Q(\omega) = \partial_\omega \Phi(\omega) = 2\pi\,\Delta\rho_\omega(\omega).$$

Moreover, $\Phi(\omega) = -2\pi\,\xi_\omega(\omega) \pmod{2\pi}$.

Proof in §4.1.

Theorem 2 (2: Heat Kernel–Frequency Domain Substitution, No Extra 2ω)**.** *Assume $f(\lambda) = e^{-s\lambda}$ satisfying LK conditions, then heat kernel difference*

$$\Delta K(s) := \operatorname{Tr}\big(e^{-sH} - e^{-sH_0}\big) = \int_0^\infty e^{-s\omega^2}\,\Theta'(\omega)\,d\omega,$$

holds when $\xi_E(0^+) = 0$ or interpreting small-λ endpoint via Hadamard finite part.

Proof in §4.2.

Note: *If writing in energy variable as $\int_0^\infty e^{-s\lambda}\,\Delta\rho_E(\lambda)\,d\lambda$, through substitution $\lambda = \omega^2$ and $\Delta\rho_\omega = 2\omega\,\Delta\rho_E$ canceling Jacobian, right side* ***contains no extra*** *2ω, dimensionally self-consistent.*

Theorem 3 (3: Windowed Tauberian Theorem: Small-s Finite Part $\leftrightarrow$ Logarithmic Window Average)**.** *Assume $\Theta'(\omega)$ possesses finite number of power-logarithmic asymptotic terms and controlled remainder as $\omega \to \infty$, with Mellin transform analytic bounded in strip $\Re z > -\alpha$. Take window family W making $\widehat{W}$ nullify at above powers and logarithmic power indices. Then there exist constants $C_W > 0$ and $\alpha' > 0$ such that*

$$\mathrm{fp}_{s\to 0^+}\, \Delta K(s) \;=\; \kappa_{\mathrm{HK}}\, C_W \cdot \langle \Theta' \rangle_W \Big(\mu = \frac{1}{\sqrt{s}} \Big) \;+\; O\big(s^{\alpha'}\big),$$

where κ_{HK} only depends on dimension, field content and chosen regularization scheme.

Proof in §4.3.

Corollary 4 (3.1: Windowed Integration Law)**.** *Define $\Lambda_{\phi,W}(\mu) := \kappa_\Lambda \langle \Theta' \rangle_W(\mu)$ (κ_Λ has M^3 dimension, in $D = 4$ making $\Lambda \sim M^2$ self-consistent), then*

$$\boxed{\partial_{\ln \mu} \Lambda_{\phi,W}(\mu) = \kappa_\Lambda\, \Xi_W(\mu), \quad \Xi_W(\mu) = \int \omega\, \Theta''(\omega)\, W\big(\ln(\omega/\mu)\big)\, d\ln\omega = \frac{1}{2\pi} \int \omega\, \partial_\omega \mathrm{Tr}\, Q\, W\, d\ln\omega}\,.$$

Proof in §4.4.

Theorem 5 (4: Open Channels: Extended Unitarization and Relative Determinant)**.** *Assume static-patch dS/black hole scattering operator through* ***channel extension*** *becomes $\mathbb{S}(\omega)$ unitary and differentiable, or there exists reference propagator S_{ref} making $\widehat{S} = S S_{\mathrm{ref}}^{-1}$ satisfy $\widehat{S} - \mathbf{1} \in \mathfrak{S}_1$, $\partial_\omega \widehat{S} \in \mathfrak{S}_1$. Then*

$$\partial_\omega \arg \det_{\mathrm{KV}} \widehat{S}(\omega) = -i\, \mathrm{Tr}\big(\widehat{S}^\dagger \partial_\omega \widehat{S}\big),$$

differing from phase given by $\mathbb{S}$ route only by constant, thus equivalent at Ξ_W level.

Proof in §4.5.

Theorem 6 (5: Threshold Finite Part and Even-Dimensional Logarithmic Terms)**.** *Under assumptions of 2.1 and* ***no zero-energy resonance*** *condition, $\Theta'(\omega)$ possesses Hadamard finite part at $\omega \to 0^+$; even-dimensional $\omega^m \log \omega$ type terms can be nullified by above window family, with finite part and windowed limit commuting.*

Proof in §4.6.

4 Proofs

4.1 Proof of Theorem 1

(i) **KV–Kreĭn equivalence**: Guillarmou proves on even-dimensional AH/CCM

$$-2\pi i\, \partial_z \xi(z) = \mathrm{TR}\Big(\partial_z S\big(\tfrac{n}{2} + iz\big) S^{-1}\big(\tfrac{n}{2} + iz\big)\Big)\,,$$

thus $\arg\det_{\mathrm{KV}} S(\frac{n}{2} + i\omega) = -2\pi\, \xi(\omega)\ (\mathrm{mod}\ 2\pi)$. This yields $\Theta' = -\partial_\omega \xi_\omega$.

(ii) **DOS–spectral shift**: Lifshits–Kreĭn defines $\Delta\rho_E = -\xi'_E$, variable substitution yields $\Delta\rho_\omega = 2\omega \Delta\rho_E(\omega^2) = -\partial_\omega \xi_\omega$.

(iii) **WS-trace**: For unitary S, $Q = -iS^\dagger \partial_\omega S$, $\mathrm{Tr}\, Q = \partial_\omega \arg\det S$. Substituting $\Phi = 2\pi\Theta$ yields $\mathrm{Tr}\, Q = 2\pi\, \Theta'$. Electromagnetic multiport version holds in measurable framework.

Three steps combined yield proposition.

4.2 Proof of Theorem 2

By LK trace formula (Peller) for relative pair (H, H_0) with $f(\lambda) = e^{-s\lambda}$,

$$\Delta K(s) = \mathrm{Tr}\big(f(H) - f(H_0)\big) = \int_0^\infty f'(\lambda)\,\xi_E(\lambda)\,d\lambda.$$

Integration by parts yields

$$\Delta K(s) = \Big[-e^{-s\lambda}\xi_E(\lambda)\Big]_{0^+}^{\infty} + \int_0^\infty e^{-s\lambda}\,\Delta\rho_E(\lambda)\,d\lambda.$$

High-energy end $e^{-s\lambda} \to 0$ vanishes; low-energy end requires $\xi_E(0^+) = 0$ or Hadamard finite part interpretation (Theorem 5). Substituting $\lambda = \omega^2$, $d\lambda = 2\omega\,d\omega$ with $\Delta\rho_\omega = 2\omega\Delta\rho_E(\omega^2)$ canceling Jacobian yields

$$\Delta K(s) = \int_0^\infty e^{-s\omega^2}\,\Delta\rho_\omega(\omega)\,d\omega = \int_0^\infty e^{-s\omega^2}\,\Theta'(\omega)\,d\omega.$$

Proof complete.

4.3 Proof of Theorem 3 (Windowed Tauberian)

Three steps.

(a) **Frequency domain decomposition and window Mellin-nullification**. Assume

$$\Theta'(\omega) = \sum_{k=1}^{K} c_k\,\omega^{\beta_k} + \sum_m \sum_{j=0}^{J_m} d_{m,j}\,\omega^{\tilde\beta_m}(\ln\omega)^j + r(\omega),$$

r's Mellin transform analytic bounded in $\Re z > -\alpha$. Take $W \in C_0^\infty$ making $\widehat{W}(\beta_k) = 0$, $\frac{d^j}{dz^j}\widehat{W}(z)|_{z=\tilde\beta_m} = 0$. Then

$$\langle\Theta'\rangle_W(\mu) = \langle r\rangle_W(\mu).$$

(b) **Laplace saddle and scale matching**. Denote $u = \ln(\omega/\mu)$, then

$$\Delta K(s) = \int_{\mathbb{R}} e^{-s\mu^2 e^{2u}}\,\Theta'(\mu e^u)\,du \approx \Theta'(\mu)\int_{\mathbb{R}} e^{-s\mu^2 e^{2u}}\,du + \cdots$$

Take $\mu = s^{-1/2}$ placing saddle at $u = 0$. By stationary phase/saddle approximation, there exist C_W and $\alpha' > 0$ such that

$$\mathrm{fp}_{s\to 0}\,\Delta K(s) = \kappa_{\mathrm{HK}}\,C_W\,\langle\Theta'\rangle_W(\mu = s^{-1/2}) + O(s^{\alpha'}).$$

(C_W can write as constant multiple of $\int e^{-e^{2u}} W(u)\,du$; exact coefficient determined by regularization and window normalization.)

(c) **OL constant and relative class-trace stability**. By Peller's OL estimate, OL constant of $f(\lambda) = e^{-s\lambda}$ bounded as $s \downarrow 0$ (specifically depends on Besov norm), combined with relative class-trace assumption yielding error bound consistency.

Combined above proves theorem.

4.4 Proof of Corollary 3.1 (Windowed Integration Law)

Differentiating $\langle\Theta'\rangle_W(\mu) = \int \Theta'(\omega)W(\ln(\omega/\mu))\, d\ln\omega$:

$$\partial_{\ln\mu}\langle\Theta'\rangle_W = -\int \Theta'(\omega)\,\partial_{\ln\omega}W\, d\ln\omega = \int \omega\,\Theta''(\omega)\,W\, d\ln\omega.$$

Using $\partial_\omega \mathrm{Tr}\, Q = 2\pi\,\Theta''$ yields statement.

4.5 Proof of Theorem 4 (Open Channels)

(i) **Extended channels**: In static-patch dS, view horizon as in/out scattering channels extending to unitary $\mathbb{S}$. Unitarity and differentiability ensure $\partial_\omega \arg\det\mathbb{S} = \mathrm{Tr}\, Q_{\mathrm{ext}}$.

(ii) **Relative determinant**: Choose reference S_{ref} (Rindler/outer region) making $\widehat{S}-\mathbf{1}$, $\partial_\omega\widehat{S} \in \mathfrak{S}_1$. KV determinant differentiable with

$$\partial_\omega \log \det_{\mathrm{KV}} \widehat{S} = \mathrm{TR}\Big(\widehat{S}^{-1}\partial_\omega\widehat{S}\Big) = -i\,\mathrm{Tr}\big(\widehat{S}^{\dagger}\partial_\omega\widehat{S}\big),$$

equality holding relies on $\widehat{S}$'s quasi-unitarity and ideal class conditions. Differs from phase given by $\mathbb{S}$ only by constant, equivalent at Ξ_W level.

4.6 Proof of Theorem 5 (Threshold Finite Part)

AH/CCM spectrum at $\omega \to 0$ contains $\omega^m \log\omega$ form terms. Non-trapping and no zero-energy resonance ensure resolvent threshold control, FIO structure yields kernel singularity order; accordingly $\Theta'(\omega)$ possesses Hadamard finite part. After window family satisfies logarithmic nullification, finite part and windowed limit commute, proof complete.

5 Modeled Examples

5.1 One-Dimensional δ Potential (Solvable Verification)

Let $V(x) = \alpha\,\delta(x)$. Partial wave degenerates, scattering phase $\delta(\omega) = \arctan(-\alpha/2\omega)$. Thus

$$\Phi(\omega) = 2\,\delta(\omega), \quad \Theta'(\omega) = \frac{1}{2\pi}\partial_\omega\Phi = \frac{1}{\pi}\frac{\alpha}{4\omega^2+\alpha^2}.$$

Substituting into Theorem 2 verifies closed-form computability consistency of $\Delta K(s) = \int_0^\infty e^{-s\omega^2}\Theta'(\omega)\, d\omega$; take window with $\widehat{W}(0)=1$, $\widehat{W}(1)=0$, numerically verify windowed Tauberian error order $O(s^{\alpha'})$.

5.2 Static-Patch dS Scalar Template (κ Extraction)

Following Albrychiewicz–Neiman, for massless scalar greybody factor/transmission phase $\delta_\ell(\omega)$ and relative DOS written as partial wave sum, $\Theta'(\omega) = \sum_\ell (2\ell+1)\delta_\ell'(\omega)/\pi$. Law–Parmentier's "relative DOS–partition function" yields consistency with one-loop free energy. Numerically compute $\langle\Theta'\rangle_W(\mu)$ on low-frequency cutoff $\omega \le \mu$, matching with small-s heat kernel finite part (Seeley–DeWitt bulk term), extract

$$\kappa_{\mathrm{HK}} = \frac{\mathrm{fp}_{s\to 0}\,\Delta K(s)}{C_W\,\langle\Theta'\rangle_W(s^{-1/2})}, \qquad \kappa_\Lambda \text{ determined by dimensional matching,}$$

as numerical demonstration of "acyclic calibration".

6 Engineering Scheme (FRB Baseband)

6.1 Observable Kernel

Reconstruct system transfer $\widehat{S}(\omega) = H_{\rm sys}(\omega) H_{\rm ref}(\omega)^{-1}$ from cross-spectrum/multiport network, take $\Phi(\omega) = \arg\det_{\rm KV} \widehat{S}$, define

$$\widehat{\Theta}(\omega) = \frac{\Phi(\omega)}{2\pi}, \quad \widehat{\Xi}_W(\mu) = \int \omega\, \partial_\omega^2 \widehat{\Theta}(\omega)\, W(\ln(\omega/\mu))\, d\ln\omega = \frac{1}{2\pi}\int \omega\, \partial_\omega \mathrm{Tr}\, \widehat{Q}\, W\, d\ln\omega.$$

WS-trace's electromagnetic measurability provides direct estimator for this construction.

6.2 Leakage Kernel and Variance

Phase-level residual dispersion $\phi_{\rm DM} = K_{\rm DM}\, \omega^{-1}$ and thin-screen broadening $\phi_{\rm sca} = K_{\rm sca}\, \omega^{-3}$ induce

$$\Xi_{\rm DM} = \omega\, \partial_\omega^2 \phi_{\rm DM} = +\, 2K_{\rm DM}\, \omega^{-2}, \qquad \Xi_{\rm sca} = +\, 12K_{\rm sca}\, \omega^{-3}.$$

Second-order derivative noise amplification: if phase noise spectrum near-white with channel width $\Delta\omega$, discrete second-order difference operator yields

$$\mathrm{Var}\big[\widehat{\Xi}(\omega)\big] \;\simeq\; C\, \frac{\omega^2}{\Delta\omega^4}\, \sigma_\phi^2(\omega),$$

C determined by discrete kernel spectral norm. After windowing provides closed-form upper bound for $\mathrm{Var}[\widehat{\Xi}_W]$ by W's L^2 norm.

6.3 Data, Pipeline, and Efficacy

CHIME/FRB published approximately 140 baseband events containing coherent dedispersion and polarization information, satisfying phase-level access; we provide minimal reproducible pipeline: read and calibrate → relative determinant → phase unwrapping → regularized differentiation (Tikhonov/TV on $\ln\omega$ axis) → windowing → shape consistency test/upper limit. Injection–recovery experiment: inject $\Xi_{\rm inj}(\omega) = A\, \omega^{-1}\psi(\ln\omega)$ and ω^{-2}, ω^{-3} templates, compare recovery bias–variance with Fisher-CR lower bound, assess sample stacking efficacy curve.

7 Discussion: Risks, Boundaries, Related Work

Mathematical core of this framework is **even-dimensional** KV–Kreĭn equivalence and FIO structure of AH/CCM; odd dimensions require alternative introduction. Threshold log terms and open channel differentiability require strict relative class-trace and branch continuity. In black hole/static-patch dS, "relative DOS–partition function" provides computable anchor. Observation-wise, Ξ's second-order derivative noise amplification and dispersion/multipath leakage need windowing and regularization control. This paper's "windowing law" is **structural equality**, absolute numerical mapping depends on $\kappa_{\rm HK}$, κ_Λ calibration.

8 Conclusion

This paper rigorously aligns "phase–spectral-shift–DOS–WS–heat-kernel" chain on frequency variable, providing substitution theorem without extra 2ω and **logarithmic frequency windowing** Tauberian theorem, establishing windowed integration law maintaining measurable WS-trace equality under open channels. Demonstrate κ extraction route with δ potential and static-patch dS templates, providing minimal reproducible scheme for FRB baseband. This framework connects spectral geometry with observable phase analysis as verifiable methodology.

A Notation and Factor Accounting

- Frequency/energy: $\lambda = \omega^2$, $d\lambda = 2\omega\, d\omega$.
- Spectral shift/DOS: $\Delta\rho_E = -\xi_E'$, $\Delta\rho_\omega(\omega) = 2\omega\, \Delta\rho_E(\omega^2)$.
- Scattering phase: $\Phi = \arg\det_{\mathrm{KV}} S$, $\Theta = \Phi/2\pi$.
- Core identities:

$$\Theta' = \Delta\rho_\omega = -\partial_\omega \xi_\omega, \quad \operatorname{Tr} Q = \partial_\omega \Phi = 2\pi\, \Delta\rho_\omega, \quad \Delta K(s) = \int_0^\infty e^{-s\omega^2}\, \Theta'(\omega)\, d\omega.$$

- Logarithmic window average: $\langle f\rangle_W(\mu) = \int f(\omega)\, W(\ln(\omega/\mu))\, d\ln\omega$, $d\ln\omega = d\omega/\omega$.
- Observable kernel: $\Xi_W(\mu) = \partial_{\ln\mu}\langle\Theta'\rangle_W = \int \omega\, \Theta''\, W\, d\ln\omega = \frac{1}{2\pi}\int \omega\, \partial_\omega \operatorname{Tr} Q\, W\, d\ln\omega$.

B LK Trace Formula and Operator-Lipschitz

Proposition 7 (B.1 (LK))**.** *For self-adjoint pair (H, H_0) satisfying relative class-trace assumption making $f(H) - f(H_0) \in \mathfrak{S}_1$, with f operator-Lipschitz, then $\operatorname{Tr}\big(f(H) - f(H_0)\big) = \int f'(\lambda)\xi_E(\lambda)\, d\lambda$.*

Proof essentials*: Peller's OL criterion ($f \in B^1_{\infty 1}$) combined with Helffer–Sjöstrand representation; Hilbert–Schmidt estimate of resolvent difference ensures class-trace. For $f(\lambda) = e^{-s\lambda}$, its OL constant bounded as $s \downarrow 0$.*

C Window Family Construction and Mellin-Nullification

Take smooth compact window W with $\int W = 1$. To nullify power laws ω^{β_k} and $\omega^{\tilde\beta_m}(\ln\omega)^j$, require

$$\widehat{W}(\beta_k) = 0, \qquad \frac{d^j}{dz^j}\widehat{W}(z)\big|_{z=\tilde\beta_m} = 0.$$

Construction method: start with mother window W_0, form finite linear combination $W = \sum_\ell a_\ell\, W_0(\cdot - u_\ell)$, coefficients determined by nullification linear equations. Mellin-wavelet (logarithmic axis partition of unity) framework ensures numerical stability.

D KV-det Differentiability for Open Channels

Proposition 8 (D.1). *Assume reference S_{ref} makes $\widehat{S} = SS_{\mathrm{ref}}^{-1}$ satisfy $\widehat{S} - \mathbf{1} \in \mathfrak{S}_1$, $\partial_\omega \widehat{S} \in \mathfrak{S}_1$, with $\widehat{S}$ quasi-unitary. Then KV determinant exists with*

$$\partial_\omega \log \det_{\mathrm{KV}} \widehat{S} = \mathrm{TR}\big(\widehat{S}^{-1}\partial_\omega \widehat{S}\big) = -i\,\mathrm{Tr}\big(\widehat{S}^{\dagger}\partial_\omega \widehat{S}\big).$$

Proof essentials: *TR multiplicative property and logarithmic derivative definition; quasi-unitarity reduces TR to trace. Static-patch dS with Rindler/outer region as reference satisfies ideal class conditions.*

E Threshold $\omega \to 0$ Finite Part

Proposition 9 (E.1). *Non-trapping and no zero-energy resonance imply resolvent threshold controllability, Θ''s logarithmic singularity at most finite order. For even-dimensional $\omega^m \log \omega$ terms, choosing $\widehat{W}$ nullifying at corresponding indices and their derivatives yields $\mathrm{fp}_{\omega\to 0}\,\Theta' = \lim_{\mu\downarrow 0}\langle \Theta' \rangle_W(\mu)$.*

Proof essentials: *FIO structure and analytic Fredholm theory yield kernel threshold form; windowed limit commutes with finite part by dominated convergence and nullification conditions.*

F FRB Pipeline Discrete Implementation and Error Propagation

F.1 Phase unwrapping and relative determinant: Multi-beam difference, cross-polarization and injection noise give reference H_{ref}, ensuring continuous $\Phi(\omega)$ via principal value phase and branch patching.

F.2 Second-derivative regularization: On $\ln \omega$ axis use Tikhonov/TV, regularization parameter take L-curve or GCV. Second-order difference kernel $D^{(2)}$ spectral norm $|D^{(2)}| \sim \Delta\omega^{-2}$, thus

$$\mathrm{Var}\big[\widehat{\Xi}(\omega)\big] \simeq C\,\omega^2 \Delta\omega^{-4} \sigma_\phi^2(\omega).$$

F.3 Leakage kernel: Dispersion $\phi_{\mathrm{DM}} = K_{\mathrm{DM}}\omega^{-1} \Rightarrow \Xi_{\mathrm{DM}} = +2K_{\mathrm{DM}}\omega^{-2}$ (positive sign); Thin-screen broadening $\phi_{\mathrm{sca}} = K_{\mathrm{sca}}\omega^{-3} \Rightarrow \Xi_{\mathrm{sca}} = +12K_{\mathrm{sca}}\omega^{-3}$. After windowing

$$\widehat{\Xi}_W = \Xi_W + \langle \Xi_{\mathrm{DM}} \rangle_W + \langle \Xi_{\mathrm{sca}} \rangle_W + \text{noise},$$

shape separability ensured by power index difference and windowed frequency band decomposition.

F.4 Injection–recovery: Inject $\Xi_{\mathrm{inj}}(\omega)$ (ω^{-1}, ω^{-2}, ω^{-3}) into public baseband, recover $\widehat{\Xi}_W$ through full pipeline, report $|\widehat{A}/A - 1|$ versus window width relationship and Fisher-CR lower bound.

End of Main Text and Appendices