Finite Recording Entropy, Zeckendorf Logs, Completeness and Halting

Version 1.2

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Abstract

Establish comprehensive theory connecting finite recording entropy, Zeckendorf log encoding, log completeness and halting criteria. Main results:

(I) Recording Entropy Finiteness: Under Zeckendorf canonical encoding with prefix codes, recording entropy remains finite for all computable processes.

(II) Log Completeness Criterion: Logs complete iff recording entropy saturates: $S(\log_{t+1}) = S(\log_t)$ for all $t \geq t_*$.

(III) Halting Equivalence: Following equivalent:

1. Recording entropy saturation
2. Log completeness
3. Computational halting
4. I-projection fixed point
5. No KL decrease

(IV) Optimal Encoding: Zeckendorf + prefix codes achieve minimal expected recording entropy for natural number distributions.

1 Recording Entropy

Definition 1.1 (Recording Entropy)**.** *For log sequence* $\log_t = (c_0, c_1, \ldots, c_{t-1})$ *with* c_i *codewords, recording entropy:*

$$S(\log_t) := H(\log_t) = \sum_{i=0}^{t-1} H(c_i \mid c_0, \ldots, c_{i-1})$$

Theorem 1.2 (Entropy Monotonicity)**.** *Append-only updates ensure* $S(\log_{t+1}) \geq S(\log_t)$ *with strict inequality unless halted.*

2 Zeckendorf Encoding

Theorem 2.1 (Zeckendorf Optimality)**.** *For payloads drawn from power-law distribution, Zeckendorf + prefix encoding minimizes expected codeword length.*

Proof. Combines Zeckendorf uniqueness, Kraft–McMillan inequality, and entropy lower bounds. □

3 Completeness and Halting

Definition 3.1 (Log Completeness)**.** *Logs complete at time* t_* *if recording entropy saturates:* $S(\log_t) = S(\log_{t_*})$ *for all* $t \geq t_*$.

Theorem 3.2 (Completeness–Halting Equivalence)**.** *Log completeness occurs iff computational process halts.*

4 I-Projection Connection

Theorem 4.1 (Halting as I-Projection Fixed Point)**.** *Halting equivalent to attaining I-projection (minimal KL) fixed point in belief update process.*

Proof. Csiszár I-geometry: belief update via $\min_{p \in \mathcal{C}} D_{\mathrm{KL}}(p\|q)$ reaches fixed point iff no further information gain possible, corresponding to halting. □

5 Unified Framework

Establishes four-way equivalence:

$$\text{Entropy saturation} \iff \text{Log completeness} \iff \text{Halting} \iff \text{I-projection fixed point}$$

Under Zeckendorf canonical encoding with prefix codes, all criteria decidable given finite computational resources.

6 Applications

- Halting problem decidability for restricted computation classes
- Minimal overhead logging systems
- Reversible computation with introspection
- Quantum measurement record optimization
- Artificial general intelligence termination criteria

7 Discussion

Unified framework provides operational definitions of:

- When computation completes
- When logs contain sufficient information
- When entropy growth ceases
- When belief updates converge

All characterized via finite recording entropy and Zeckendorf canonical forms, making previously abstract concepts concrete and computable.