

Self-Referential Scattering and the Birth of Fermions: Riccati Square Roots, Spinor Double Cover, and a \mathbb{Z}_2 Exchange Phase

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Abstract

Standard quantum field theory explains the relation between spin and statistics through the spin–statistics theorem, derived from Lorentz covariance and microcausality on a continuous spacetime background. In topological approaches, the antisymmetry of fermionic wavefunctions can also be understood via the nontrivial topology of configuration spaces and associated line bundles, as in the Finkelstein–Rubinstein construction for solitons in nonlinear field theories. However, these frameworks typically assume relativistic quantum fields as the ontological starting point.

Within a discrete, causal quantum cellular automaton (QCA) ontology, the universe is described as a lattice of local Hilbert spaces updated by a global unitary step. In previous work, massive excitations were interpreted as localized, self-sustained interference structures whose internal dynamics consume part of a global information-update budget, giving mass an interpretation as “topological impedance” in an underlying scattering network. Building on this picture, the present work proposes a dynamical and geometric origin of fermionic statistics in terms of **self-referential scattering**.

We consider localized excitations realized as feedback loops in an effective one-dimensional scattering problem obtained from coarse-graining the QCA. The boundary response of such a loop is encoded in an impedance function or reflection coefficient solving a nonlinear Riccati equation, a structure well known in wave propagation and scattering theory. We show that the fixed-point condition for a stable, self-referential loop forces the physical state of the excitation to live on a **square-root branch** of the underlying scattering data. This branch structure induces a canonical double cover of the configuration space of N identical excitations. We prove that the generator corresponding to exchanging two excitations lifts to a nontrivial loop on this double cover with holonomy (-1) , so that the N -body wavefunction transforms in the sign representation of the permutation group and satisfies fermionic exchange statistics.

In this construction, spinor behavior and the $\text{Spin}(3)$ double cover of $\text{SO}(3)$ are not postulated but emerge from the necessity of taking square roots of self-referential scattering data. The internal “Riccati square-root” variable plays the role of a spinor amplitude whose squared modulus reproduces observable scattering characteristics. In this sense, “spin $1/2$ ” is reinterpreted as the topological fingerprint of information self-reference in a QCA-based universe. We outline an explicit realization in Dirac-type QCA models and propose engineered scattering networks in photonic and superconducting platforms to test the predicted \mathbb{Z}_2 exchange phase.

Keywords: Quantum cellular automata; self-referential scattering; Riccati equation; impedance; spin–statistics theorem; spinor double cover; Finkelstein–Rubinstein constraint; fermionic exchange phase

1 Introduction & Historical Context

1.1 Spin–Statistics Theorem in Continuous Field Theory

The spin–statistics theorem states that in three-plus-one-dimensional relativistic quantum field theory, half-integer spin particles must obey Fermi–Dirac statistics, while integer spin particles must obey Bose–Einstein statistics. Modern textbook derivations rely on the following ingredients: Lorentz covariance, vacuum stability, positive-definite probabilities of local observables, and (anti)commutation relations of operators at spacelike separations (microcausality). In this framework, the distinction between fermions and bosons is one of the input conditions of field operator commutation relations, and consistency with spin is subsequently guaranteed by structural theorems.

This proof structure is highly rigorous, yet brings a widely recognized puzzle: why is there such a profound connection between spin and many-body exchange statistics, rather than being mutually independent structures? Feynman once lamented that an “intuitive explanation” of this simple statement remains elusive.

1.2 Topological Spin–Statistics Relations and Solitons

An alternative line of thought, advocated by Finkelstein and Rubinstein, examines the homotopy properties of topological soliton configuration spaces in nonlinear field theories. They point out that when the fundamental group of the soliton configuration space is nontrivial, one can relate 2π rotations to particle exchanges via nontrivial line bundles, thus obtaining a spin–statistics correspondence. In this perspective, the wavefunction is no longer a single-valued function on configuration space, but rather a section of a line bundle; the homotopy class of soliton winding paths determines the exchange phase through the holonomy of the bundle. Further work has applied this idea to Yang–Mills solitons and Hopf topological invariants, demonstrating geometric relationships between linking numbers and statistics.

Topological methods provide geometric intuition for the spin–statistics theorem, but still take intrinsic continuous fields as fundamental objects, and typically start from known soliton models.

1.3 Quantum Cellular Automata and Discrete Ontology

In recent years, quantum cellular automata (QCA) have emerged as an attempt to reconstruct quantum field theory, and even the entire physical universe, within a framework of discrete, local unitary evolution. In the QCA picture, spacetime consists of a discrete lattice Λ with a local Hilbert space \mathcal{H}_x at each site, and dynamics are given by a global unitary operator U acting on finite neighborhoods. In the continuum limit, appropriate choices of local “coin” operators can reproduce Dirac, Weyl, or Maxwell equations, thus recovering standard quantum field theory on a discrete ontology.

In a series of previous works, mass was interpreted as a geometric “impedance” of information propagation rates: massless excitations correspond to feedforward propagation along the light cone, while massive excitations correspond to local structures with feedback and looping, whose internal evolution speed v_{int} and external group velocity v_{ext} satisfy an information rate conservation constraint. The corresponding microscopic picture is: a particle is a kind of self-referential feedback loop in the QCA network, whose stable existence depends on impedance matching between input and output.

1.4 Goals and Claims of This Work

This paper attempts to take a further step in the above QCA and “mass = topological impedance” picture, proposing the following claims:

1. Any localized excitation that realizes a stable rest mass in QCA can, after appropriate coarse-graining, be viewed as a **self-referential feedback loop** in an effective one-dimensional scattering problem.
2. The spatial evolution of the boundary response (impedance or reflection coefficient) of this feedback loop is governed by a nonlinear Riccati equation; its steady-state solution is a fixed point of a certain Möbius transformation and has a square-root discriminant structure.
3. For a system of N identical self-referential excitations, the natural quantization of the total configuration space is no longer a single-valued wavefunction, but rather a section of a **double cover line bundle** induced by the above Riccati structure. Particle exchange paths have \mathbb{Z}_2 holonomy on this double cover, leading to an exchange phase of (-1) .
4. Therefore, **massive self-referential excitations automatically realize Fermi–Dirac statistics** in this construction; bosons correspond to pure feedforward modes without self-reference or composites of several self-referential loops.

The starting point of this research is not to attempt to “re-prove” the spin–statistics theorem, but to construct a microscopic mechanism in discrete ontology that establishes an explicit connection between self-referential scattering, Riccati square roots, and spinor double covers, thus providing a dynamical–geometric explanation for the existence of fermions.

2 Model & Assumptions

2.1 Underlying QCA Structure

Let $\Lambda \subset \mathbb{Z}^d$ be a regular lattice; this paper primarily considers cases $d = 1, 3$. Each site $x \in \Lambda$ is associated with a finite-dimensional Hilbert space $\mathcal{H}_x \cong \mathbb{C}^q$, and the global Hilbert space is

$$\mathcal{H} = \bigotimes_{x \in \Lambda} \mathcal{H}_x.$$

Time evolution is given by a local unitary operator with finite neighborhood. That is, there exists a finite range R such that the single-step evolution U can be written as a finite-depth quantum circuit of local gates, respecting causality: $U^\dagger \mathcal{A}_O U \subset \mathcal{A}_{O^+}$, where \mathcal{A}_O is the local operator algebra of region O , and O^+ is a finite-thickness neighborhood of O .

To connect to the continuum limit, we consider a class of Dirac-type QCA, whose single-step evolution in momentum representation can be written as

$$U(k) = \exp(-iH_{\text{eff}}(k)\Delta t),$$

where $H_{\text{eff}}(k)$ in the long-wavelength limit $ka \ll 1$ approaches the Dirac Hamiltonian

$$H_{\text{eff}}(k) \approx \alpha k + \beta m,$$

with α, β matrices satisfying the Clifford algebra, a the lattice spacing, and m the effective mass parameter.

2.2 Self-Referential Scattering Unit and Effective One-Dimensional Model

Consider introducing a local structure in a finite region $D \subset \Lambda$ such that there exists a feedback channel within the region: part of the incident amplitude, after undergoing local scattering, is re-injected into the same region. For modes with wavelength much larger than the size of D , an effective one-dimensional description can be adopted, compressing the entire region D into an

equivalent transmission line or scattering center, with incident and outgoing signals propagating along a one-dimensional coordinate x .

In the frequency domain representation, suppose that for a fixed frequency ω , the mode satisfies an effective wave equation on the one-dimensional coordinate. Its propagation in the half-space $x > 0$ can be characterized by a position-dependent impedance $Z(x; \omega)$. According to electromagnetic wave and acoustic wave propagation theory, under appropriate one-dimensional approximations, the spatial evolution of $Z(x; \omega)$ satisfies a nonlinear Riccati equation:

$$\frac{dZ}{dx} = A(x; \omega) + B(x; \omega)Z + C(x; \omega)Z^2,$$

where A, B, C are determined by medium parameters. For layered media or discrete lattice models, Z jumps between layers according to Möbius transformations:

$$Z_{n+1} = \frac{a_n Z_n + b_n}{c_n Z_n + d_n}, \quad \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \text{SL}(2, \mathbb{C}).$$

The core requirement of the self-referential feedback structure is: at an effective boundary point $x = 0$, the input impedance $Z_{\text{in}}(\omega)$ externally exhibited by the local structure must equal the load impedance $Z_{\text{loop}}(\omega)$ “seen” by the internal loop. This self-consistency condition can be written in discrete representation as

$$Z_{\text{in}} = \Phi(Z_{\text{in}}),$$

where Φ is a composite mapping consisting of Möbius transformations and feedback phases from multiple layers.

Definition 2.1 (Self-Referential Scattering Unit). *At a given frequency ω , a local structure is called a self-referential scattering unit if its external equivalent input impedance $Z_{\text{in}}(\omega)$ is a fixed point of some complex Möbius transformation Φ :*

$$Z_{\text{in}}(\omega) = \Phi(Z_{\text{in}}(\omega)), \quad \Phi(z) = \frac{Az + B}{Cz + D}, \quad AD - BC = 1.$$

The stability of this fixed point is determined by the modulus of $\Phi'(Z_{\text{in}})$.

2.3 Riccati Fixed Point and Square-Root Discriminant

In general, the Möbius transformation fixed-point equation

$$Z = \frac{AZ + B}{CZ + D}$$

can be reduced to a quadratic equation

$$CZ^2 + (D - A)Z - B = 0,$$

whose solution is

$$Z_{\pm} = \frac{A - D \pm \sqrt{(A - D)^2 + 4BC}}{2C},$$

provided $C \neq 0$. Thus, the equivalent impedance of any self-referential scattering unit naturally carries a square-root branch structure, whose discriminant

$$\Delta = (A - D)^2 + 4BC$$

determines the physical properties of the two branch solutions through its phase and modulus. For lossless systems, (A, B, C, D) belong to an appropriate representation of $\text{SU}(1, 1)$ or $\text{SL}(2, \mathbb{R})$,

Δ lies on a certain curve in the complex plane, and the choice of square-root function $\sqrt{\Delta}$ corresponds to two types of boundary conditions.

We view this square-root structure as the embryonic form of a “spinor”: the observable impedance Z corresponds to the square of some “amplitude” variable ζ , i.e.,

$$Z = F(\zeta^2),$$

and the multivaluedness of ζ under closed paths in parameter space will determine the exchange statistics.

2.4 Configuration Space of Identical Self-Referential Excitations

Consider N well-separated self-referential scattering units in three-dimensional space, with center positions $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^3$. Ignoring internal structural details, the geometric configuration space is

$$Q_N = \frac{(\mathbb{R}^3)^N \setminus \Delta}{S_N},$$

where Δ is the diagonal subset where particle positions coincide, and S_N is the permutation group. For $d \geq 3$ dimensions, the fundamental group of Q_N is S_N , whose elements can be generated by exchanges of adjacent particles.

In standard quantum mechanics, the many-body wavefunction Ψ is viewed as a complex-valued single-valued function on Q_N . In topological spin–statistics schemes, Ψ is viewed as a section of a line bundle or vector bundle, with different bundle structures corresponding to different exchange statistics.

In our scheme, we will use the Riccati square-root structure internal to each self-referential scattering unit to construct a natural double cover space \tilde{Q}_N over Q_N , and show that exchange paths have \mathbb{Z}_2 holonomy on \tilde{Q}_N .

3 Main Results (Theorems and Alignments)

This section presents the core results of this paper, focused on three levels:

1. The square-root structure of self-referential scattering units and the Riccati equation;
2. The spinor double cover induced by the square-root structure;
3. The \mathbb{Z}_2 phase from particle exchange and its correspondence with Fermi statistics.

3.1 Self-Referential Scattering and Riccati Square Root

Theorem 3.1 (Square-Root Discriminant of Self-Referential Scattering Unit). *Suppose the equivalent transfer matrix of a self-referential scattering unit is*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}(2, \mathbb{C}),$$

and its external input impedance Z_{in} is a fixed point of the Möbius transformation, i.e., $Z_{\text{in}} = \Phi(Z_{\text{in}})$, with $\Phi(z) = (Az + B)/(Cz + D)$. Assume $C \neq 0$ and the system is lossless, so that the spectrum of M lies on the unit circle. Then:

1. Z_{in} satisfies the quadratic equation

$$CZ^2 + (D - A)Z - B = 0.$$

2. There exists a discriminant $\Delta = (A - D)^2 + 4BC$ such that

$$Z_{\text{in}} = Z_{\pm} = \frac{A - D \pm \sqrt{\Delta}}{2C},$$

where $\sqrt{\Delta}$ is the two-valued square-root function on the complex plane.

3. If M belongs to $\text{SU}(1, 1)$ or an equivalent representation, then exactly one of the two branch solutions Z_{\pm} corresponds to a stable fixed point ($|\Phi'(Z)| < 1$), and the other to an unstable fixed point ($|\Phi'(Z)| > 1$).

Therefore, the external response of any stable self-referential scattering unit can be equivalently characterized by a choice of one of a pair of square-root variables $\pm\sqrt{\Delta}$.

Proof Sketch. Writing the fixed-point condition as a quadratic equation immediately yields the discriminant and two-valued solution; the stability condition is determined by the modulus of the derivative of the Möbius transformation. The lossless condition constrains the spectrum of M , thereby constraining the phase and modulus of Δ , so that only one branch is stable. See Appendix A for details. \square

3.2 Emergence of Spinor Double Cover

Definition 3.2 (Spinor Internal Variable). *For each self-referential scattering unit, we introduce an internal variable ζ such that*

$$\sqrt{\Delta} = \Lambda\zeta^2,$$

where $\Lambda \in \mathbb{C}^{\times}$ is a nonzero constant related to the specific implementation. Define a normalized “spinor” variable

$$\chi = \frac{\zeta}{\|\zeta\|},$$

whose overall phase redundancy is regarded as a gauge freedom.

Thus, a stable self-referential scattering unit can be described by either the equivalent impedance Z_{in} or the spinor variable χ , which satisfy

$$Z_{\text{in}} = F(\chi^2),$$

where F is an explicit rational function. Note that χ and $-\chi$ correspond to the same Z_{in} .

Theorem 3.3 (Internal Spinor Double Cover). *Under the above assumptions, the internal state space of a single self-referential scattering unit can be viewed as a quotient space*

$$\mathcal{S}_{\text{int}} \cong \mathbb{C}^2 \setminus \{0\} / \{\chi \sim -\chi\},$$

and there exists a natural mapping

$$\pi_{\text{int}} : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathcal{S}_{\text{int}}, \quad \pi_{\text{int}}(\chi) = [\chi],$$

whose kernel is $\{\pm 1\}$. Under appropriate gauge choices and coarse-graining, the $\text{SU}(2)$ rotation representation on \mathbb{C}^2 projects via π_{int} to the $\text{SO}(3)$ rotation representation on \mathcal{S}_{int} , and thus the internal degrees of freedom are naturally organized into a double cover structure of $\text{Spin}(3)$.

This structure is completely isomorphic to standard spinor theory: χ plays the role of a spin-1/2 spinor, while Z_{in} and observable scattering phases correspond to quadratic invariants.

3.3 Particle Exchange and \mathbb{Z}_2 Exchange Phase

Theorem 3.4 (Fermionic Exchange Statistics of Self-Referential Excitations). *Consider N identical self-referential scattering units in three-dimensional space, with configuration space Q_N as defined in Section 2.4. Let \tilde{Q}_N be the double cover induced by the internal spinor variables:*

$$\tilde{Q}_N = \left\{ (\mathbf{x}_1, \dots, \mathbf{x}_N; \chi_1, \dots, \chi_N) \right\} / \sim,$$

where the equivalence relation identifies $\chi_j \sim -\chi_j$ for each particle, while requiring invariance of the external Z_{in} . Then:

1. \tilde{Q}_N is a double cover space of Q_N , whose covering transformation group is generated by simultaneously changing the sign of all χ_j , forming a \mathbb{Z}_2 .
2. For any pair of particles i, j , their exchange operation corresponds to a closed path γ_{ij} on Q_N . On \tilde{Q}_N , γ_{ij} lifts to two paths $\tilde{\gamma}_{ij}^\pm$, whose endpoints differ by a global sign change:

$$\tilde{\gamma}_{ij}^+(1) = -\tilde{\gamma}_{ij}^-(1).$$

3. If we regard the many-body state as a section of a line bundle on \tilde{Q}_N and require that this section changes sign under the covering transformation, then parallel transport along γ_{ij} gives the wavefunction a phase of (-1) :

$$\Psi(\gamma_{ij}(1)) = -\Psi(\gamma_{ij}(0)).$$

Therefore, in this quantization scheme, the geometric realization of particle exchange necessarily corresponds to fermionic antisymmetric statistics.

Proof Sketch. This is a concrete realization of the Finkelstein–Rubinstein scheme on self-referential spinor internal degrees of freedom. The key is: viewing particle winding paths as closed curves on \tilde{Q}_N , their lift in the covering space has nontrivial \mathbb{Z}_2 holonomy. Choosing a line bundle where the covering transformation corresponds to wavefunction sign change yields Fermi statistics. See Appendix B for details. \square

4 Proofs

This section provides proof outlines for the above theorems, with technical details expanded in the appendices.

4.1 Proof of Theorem 3.1

From the fixed-point condition

$$Z = \frac{AZ + B}{CZ + D}$$

we obtain

$$CZ^2 + (D - A)Z - B = 0.$$

This is a quadratic equation in Z ; as long as $C \neq 0$ we can write the explicit solution

$$Z_\pm = \frac{A - D \pm \sqrt{\Delta}}{2C}, \quad \Delta = (A - D)^2 + 4BC.$$

Assuming the system is lossless, i.e., $M \in \text{SU}(1,1)$ or a similar group, means that the eigenvalues of M lie on the unit circle, and M is closely related to the intrinsic phase of the corresponding scattering matrix S . The Möbius transformation

$$\Phi(z) = \frac{Az + B}{Cz + D}$$

has derivative

$$\Phi'(z) = \frac{1}{(Cz + D)^2}.$$

Substituting Z_{\pm} , we can evaluate $|\Phi'(Z_{\pm})|$. Under the lossless condition, the moduli of $(CZ_{\pm} + D)$ are reciprocals, so exactly one of the two branch solutions satisfies $|\Phi'(Z)| < 1$, corresponding to a stable fixed point and a stable self-referential scattering structure; the other branch is unstable, corresponding to a nonphysical solution or excited state. A detailed analysis comparing with the variable phase method and Levinson theorem is given in Appendix A.

4.2 Proof of Theorem 3.3

The discriminant Δ is an invariant of the trace and determinant of M ; in the lossless case, Δ typically lies on a complex plane curve passing through the origin. For each frequency ω and momentum k , we can write

$$\Delta(\omega, k) = \Lambda^2(\omega, k)\zeta^4(\omega, k),$$

where $\Lambda \neq 0$ is a gauge choice. Thus

$$\sqrt{\Delta} = \Lambda\zeta^2,$$

and Z_{in} can be rewritten as a rational function $F(\zeta^2)$. Viewing ζ as coordinates of a two-dimensional complex vector, we introduce normalization

$$\chi = \frac{\zeta}{\|\zeta\|} \in \mathbb{C}^2 \setminus \{0\},$$

and identify $\chi \sim -\chi$ to obtain the quotient structure of the internal state space. Standard group theory results show that the natural representation of $\text{SU}(2)$ on \mathbb{C}^2 projects via the quotient map to the representation of $\text{SO}(3)$ on S^2 , and $\text{SU}(2)$ is the double cover of $\text{SO}(3)$. Thus, the internal state has a double cover structure completely equivalent to a spin-1/2 spinor.

4.3 Proof of Theorem 3.4

The topological properties of Q_N have mature conclusions: in three-dimensional space, the fundamental group of Q_N is the permutation group S_N , whose generators can be viewed as adjacent particle exchange paths. The Finkelstein–Rubinstein proof shows: if there exists a double cover $\tilde{Q}_N \rightarrow Q_N$ with covering transformation group \mathbb{Z}_2 , and 2π spatial rotations and particle exchange paths are related to nontrivial closed loops in \tilde{Q}_N , then one can construct a line bundle such that the many-body state, as a section of this line bundle, changes sign under the covering transformation, thus realizing Fermi statistics.

In our construction, each particle carries an internal spinor variable χ_j , with $\chi_j \sim -\chi_j$ corresponding to the same physical impedance. Combining the internal variables of all particles naturally yields \tilde{Q}_N . Evolution along the exchange path γ_{ij} not only winds in position space, but also winds around the square-root branch cut in internal parameter space, causing the overall phase of (χ_i, χ_j) to undergo a 2π winding. This corresponds to nontrivial \mathbb{Z}_2 holonomy on \tilde{Q}_N . Choosing a line bundle where the covering transformation corresponds to wavefunction sign change yields

$$\Psi(\dots, \mathbf{x}_i, \mathbf{x}_j, \dots) = -\Psi(\dots, \mathbf{x}_j, \mathbf{x}_i, \dots).$$

Therefore, self-referential scattering excitations naturally realize fermionic exchange statistics.

5 Model Apply

This section shows how to implement the above self-referential scattering structure in concrete QCA models, and relate it to Dirac mass and topological impedance.

5.1 Self-Referential Defects in One-Dimensional Dirac-QCA

Consider a one-dimensional Dirac-type QCA, whose single-step evolution can be written as a quantum walk:

$$U = S_+ \otimes |\uparrow\rangle\langle\uparrow| + S_- \otimes |\downarrow\rangle\langle\downarrow| \circ (\mathbb{I} \otimes C),$$

where S_{\pm} shift the state left or right by one step, and C is a 2×2 coin matrix, e.g.,

$$C(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

In momentum representation, the eigenvalues of $U(k)$ are $e^{\mp i\omega(k)}$, satisfying the dispersion relation

$$\cos \omega(k) = \cos \theta \cos(ka),$$

which in the long-wavelength limit yields an effective Dirac equation, with mass m related to θ .

Against this background, modifying the coin matrix or introducing a local loop at a single site or finite sub-chain can realize an effective scattering center. For modes with wavelength much larger than the defect region size, their scattering is characterized by a 2×2 single-channel scattering matrix

$$S(k) = \begin{pmatrix} r(k) & t'(k) \\ t(k) & r'(k) \end{pmatrix}.$$

By explicitly modeling the internal loop of the defect region as additional boundary and feedback channels, its equivalent impedance can be written as a function $Z(x; k)$ satisfying a Riccati equation, whose value at the defect region periphery gives $r(k)$. Related techniques are closely related to the variable phase method.

5.2 Mass, Bound States, and Self-Referential Feedback

For defects with self-referential feedback, there exist certain frequencies ω and momenta k such that the pole condition

$$1 - r(\omega)e^{i\theta_{\text{loop}}} = 0$$

holds, where θ_{loop} is the additional loop phase. These poles correspond to bound or quasi-bound states, which in the continuum limit of QCA manifest as localized “particles” whose frequency deviates from the massless mode dispersion relation, defining an effective mass m .

On the other hand, the reflection coefficient can be written as

$$r(\omega) = e^{i\delta(\omega)},$$

where $\delta(\omega)$ is the scattering phase. Through the Levinson theorem and variable phase method, $\delta(\omega)$ can be related to the number of bound states and Riccati phase functions. The pole condition can be rewritten as a self-consistent equation for impedance, whose solution has a two-valued square-root structure, thus introducing a natural internal spinor variable for the bound state.

5.3 Comparison of Internal Spinor with Dirac Spinor

In the continuum limit of Dirac-QCA, fermionic states are typically described by four-component spinors or, in the one-dimensional case, two-component spinors. This construction shows that even starting from scalar scattering data, as long as there exists a self-referential feedback structure, the square-root discriminant contained in the equivalent impedance is sufficient to introduce a two-component internal degree of freedom χ , and under appropriate gauge choices recover the $SU(2)$ spinor transformation property.

Intuitively: the local coin of QCA determines the propagation direction and internal degrees of freedom; self-referential scattering feedback introduces topological memory for this internal degree of freedom, causing “half-angle” phases to appear under spatial rotations and particle exchanges. What is ultimately observed macroscopically is an excitation satisfying the Dirac equation and obeying Fermi-Dirac statistics, whose underlying microscopic picture is a square-root fixed point of a Riccati equation.

6 Engineering Proposals

Although this paper primarily focuses on theoretical structure, the tools of self-referential scattering and Riccati impedance equations are highly mature in engineering, and have been widely used to model electromagnetic and acoustic wave propagation in inhomogeneous media. This fact provides a natural stage for experimental tests.

6.1 Self-Referential Loops in One-Dimensional Waveguides and Superconducting Microwave Circuits

In one-dimensional electromagnetic waveguides or superconducting transmission lines, self-referential scattering units can be constructed through coupled resonators, branches, and feedback loops. Measurable quantities include:

- The reflection coefficient $r(\omega)$ and equivalent impedance $Z_{\text{in}}(\omega)$ in the frequency domain;
- Continuous transition from no bound state to a single bound state by adjusting feedback phase and coupling strength;
- Impedance branch structure predicted by Riccati equations, e.g., observing impedance variation with depth through slow modulation of medium parameters.

If two identical self-referential units can be constructed in parallel on the same waveguide, and a controllable geometric path designed to “exchange” their positions (e.g., through a reconfigurable switch network or timing control simulating exchange), then the overall phase (-1) effect could hopefully be observed in interference experiments.

6.2 Photonic QCA and Linear Optical Networks

Linear optical platforms based on waveguide arrays and beam-splitter-phase-shifter networks can realize two-dimensional QCA. By introducing ring optical path feedback at certain nodes, a discrete version of photonic “self-referential scattering” can be realized. By measuring correlation functions, interference fringes, and effective scattering phases, the Riccati square-root structure can be indirectly verified in the optical frequency range, thus laying the foundation for more complex many-body exchange experiments.

6.3 Integration with Superconducting Qubit Platforms

In superconducting quantum circuits, single-step QCA can be realized through controlled phase gates and SWAP gates, and self-referential feedback can be simulated through tunable couplers and delay lines. By encoding internal spinor variables χ in multi-qubit Hilbert space and using Ramsey interferometry to precisely measure global phase changes before and after exchange operations, prototypes of self-referential fermions could hopefully be realized on quantum information platforms.

7 Discussion (risks, boundaries, past work)

7.1 Relationship with Standard Spin–Statistics Theorem

The self-referential scattering mechanism proposed in this paper does not attempt to replace the standard spin–statistics theorem, but rather provides a complementary path from the perspective of discrete ontology and scattering networks. The core assumptions of the standard theorem are local, Lorentz-covariant quantum fields on continuous spacetime and their microcausality; this paper starts from QCA, views local unitarity as a more fundamental structure, and constructs self-referential feedback and Riccati equations upon it.

The correspondence between the two is as follows:

- In the standard theorem, particle exchange and 2π rotation are related through Lorentz group representation theory and field operator commutation relations; in this paper, particle exchange and 2π rotation are related through the square-root discriminant internal to self-referential scattering and the double cover configuration space.
- The standard theorem takes field operators as fundamental objects; this paper takes QCA local unitaries and scattering networks as fundamental objects.

Under appropriate continuum limits and coarse-graining, the two should be compatible: the spinor structure produced by self-referential scattering coincides with standard Dirac spinors, and Fermi statistics are guaranteed by both.

7.2 Connection with Topological Spin–Statistics Schemes

Finkelstein–Rubinstein and subsequent work have already shown that in soliton models with nontrivial topology, the geometric origin of spin and statistics can be obtained through double covers and line bundle holonomy of configuration spaces. This paper can be viewed as a concrete realization of this idea in QCA and scattering networks:

- Topological charges in soliton models correspond to feedback winding numbers in self-referential scattering networks;
- Nontrivial fundamental groups of soliton configuration spaces correspond to configuration spaces Q_N of multiple self-referential units;
- Line bundle holonomy corresponds to the square-root multivaluedness of internal spinor variables χ .

The difference is: soliton models typically rely on nonlinear equations of continuous fields, whereas the self-referential structure in this paper can be realized in discrete QCA networks through linear single-step unitaries and topological feedback loops, making it more suitable for interfacing with information theory and engineering implementations.

7.3 Applicable Boundaries and Potential Risks

The assumptions and derivations of this work have several boundary conditions and potential risks:

1. **Assumption of effective one-dimensional description:** compressing local structures in multi-dimensional QCA into one-dimensional transmission lines and impedance models requires mode wavelength much larger than the self-referential region scale. This approximation may fail in strong coupling or highly anisotropic situations.
2. **Applicability of Riccati equation:** impedance Riccati equations in classical wave theory rely on linear wave equations and certain boundary conditions; in QCA, although single-step unitary evolution can approximate linear equations in the continuum limit, high-frequency modes or deep discrete regions may introduce corrections.
3. **Many-body effects and interactions:** this paper primarily analyzes many-body configuration spaces and exchange paths in the weak-coupling and dilute-particle limit. Strong interactions or multi-particle clusters may change the topological structure of configuration space, thus affecting statistical properties.
4. **Dimensional limitations:** this paper focuses on three-plus-one dimensions. In two-plus-one dimensions, the fundamental group of configuration space is the braid group, allowing anyons and fractional statistics; the square-root structure of self-referential scattering networks may produce richer statistical behavior in two dimensions, requiring separate analysis.

8 Conclusion

Within the framework of quantum cellular automata and scattering networks, this paper proposes a mechanism linking Fermi statistics with self-referential scattering. The core conclusions can be summarized as:

1. **Mass as stable fixed point of self-referential scattering:** Massive particles can be viewed as self-referential scattering units in QCA networks, whose equivalent input impedance satisfies a Riccati equation arising from Möbius transformations, giving stable fixed points in square-root discriminant form.
2. **Self-reference produces square roots, square roots produce spinors:** The discriminant square root of self-referential scattering introduces an internal two-valued degree of freedom, which can be organized into a double cover structure of $SU(2)$ spinor space, with χ and $-\chi$ corresponding to the same observable impedance.
3. **Spinor double cover produces Fermi statistics:** Incorporating the internal spinors of N self-referential units into configuration space yields a natural double cover \tilde{Q}_N of Q_N . Particle exchange paths have \mathbb{Z}_2 holonomy on this double cover. Choosing a quantization scheme where the covering transformation corresponds to wavefunction sign change automatically yields Fermi–Dirac statistics.
4. **Engineering realizability:** Because Riccati impedance equations have been widely applied in electromagnetic waves, acoustic waves, and geophysics, constructing self-referential scattering networks and experimentally detecting square-root multivaluedness and exchange phases is feasible in engineering.

Future work includes: exploring anyonic statistics induced by self-referential scattering on two-plus-one-dimensional platforms; completing the derivation from underlying single-step unitaries to continuum field theory effective Lagrangians in concrete Dirac-QCA models; realizing minimal self-referential fermion units in superconducting quantum circuits and photonic networks and measuring their exchange phases.

Acknowledgements, Code Availability

This work has benefited from extensive discussions on the spin-statistics theorem, topological solitons, and QCA continuum limits. Riccati equation solving and QCA evolution simulation required for numerical verification can be implemented through standard scientific computing libraries, and corresponding example codes can be implemented and shared on general open-source platforms.

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A Riccati Impedance Equation and Möbius Fixed Points

This appendix provides a traditional derivation of the impedance Riccati equation and Möbius fixed-point structure, and clarifies the physical meaning of the parameters used in this paper.

A.1 One-Dimensional Waves and Impedance Definition

Consider a scalar wave equation in one-dimensional medium:

$$\frac{\partial^2 u}{\partial x^2} + k^2 n^2(x) u = 0,$$

where $u(x)$ is the field amplitude, $k = \omega/c$, and $n(x)$ is the position-dependent refractive index. In the frequency domain, define the local impedance

$$Z(x) = \frac{1}{i\omega\rho} \frac{\partial_x u(x)}{u(x)},$$

where ρ is a constant (such as medium density). Substituting into the wave equation yields

$$\frac{dZ}{dx} + Z^2 + k^2 n^2(x) = 0,$$

a standard form of the Riccati equation. Through appropriate scaling and variable transformations, it can be written as

$$\frac{dZ}{dx} = A(x) + B(x)Z + C(x)Z^2,$$

where

$$A = -k^2 n^2(x), \quad B = 0, \quad C = 1.$$

This derivation appears in various literature, such as the variable phase method and impedance propagation theory in geophysics.

A.2 Layered Media and Möbius Transformations

For layered media, each layer has constant $n(x)$ and thickness d_j . Boundary conditions between layers are satisfied, and the solution within each layer is a superposition of plane waves:

$$u_j(x) = A_j e^{ik_j x} + B_j e^{-ik_j x}.$$

Reflection and transmission within layers can be represented by 2×2 transfer matrices $M_j \in \text{SL}(2, \mathbb{C})$:

$$\begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} = M_j \begin{pmatrix} A_j \\ B_j \end{pmatrix}.$$

The impedance $Z_j = B_j/A_j$ transitions between layers according to

$$Z_{j+1} = \frac{a_j Z_j + b_j}{c_j Z_j + d_j}, \quad M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

This is a typical Möbius transformation. After multiple layers, the total transfer matrix M is the product of individual layer matrices, whose fixed-point solutions satisfy a quadratic equation, with the discriminant naturally exhibiting square-root structure.

A.3 Self-Referential Feedback and Fixed-Point Condition

If a feedback loop is introduced at a position $x = 0$, causing part of the outgoing wave to return to the input with phase θ_{loop} , the equivalent boundary condition is

$$u(0^-) = u(0^+) + e^{i\theta_{\text{loop}}} u(0^+).$$

The corresponding effect on impedance can be equivalently represented as an additional Möbius transformation. The self-referential condition requires that the impedance Z_{in} “seen” externally remains unchanged after one round of internal propagation and feedback, i.e.,

$$Z_{\text{in}} = \Phi(Z_{\text{in}}),$$

where Φ is a Möbius transformation composed of propagation through various layers and feedback phase. Solving this fixed-point equation yields the quadratic equation and square-root discriminant structure described in Theorem 3.1.

B Finkelstein–Rubinstein Construction and Self-Referential Loops

This appendix reviews the main ideas of the Finkelstein–Rubinstein (FR) topological spin–statistics scheme and explains how to embed self-referential scattering excitations into this framework.

B.1 Soliton Configuration Space and Double Cover

In nonlinear field theories, such as the Skyrme model, the topological soliton configuration space

$$\mathcal{C}_Q = \{\text{field configuration } \phi(\mathbf{x}) \mid Q[\phi] = Q\}$$

typically has nontrivial fundamental group $\pi_1(\mathcal{C}_Q)$. FR points out that one can artificially construct a double cover space $\tilde{\mathcal{C}}_Q$ and stipulate that closed paths corresponding to 2π spatial rotations lift to nontrivial closed curves on $\tilde{\mathcal{C}}_Q$ with holonomy (-1) . On this basis, regarding quantum states as sections of line bundles on $\tilde{\mathcal{C}}_Q$ and requiring wavefunctions to change sign under covering transformations yields spin-1/2 and Fermi statistics.

B.2 Multi-Soliton Configuration Space and Exchange Paths

For the configuration space Q_N of N solitons, FR analysis shows that its fundamental group is generated by particle exchanges and overall rotations, satisfying certain relations. By relating 2π rotations to pair exchanges and assigning \mathbb{Z}_2 values to these generators, different double covers can be constructed on Q_N , corresponding to bosonic, fermionic, or more general statistics.

B.3 FR Data of Self-Referential Scattering Excitations

In this paper’s construction, the Riccati discriminant square root internal to each self-referential scattering unit provides a natural two-valued degree of freedom $\chi \sim -\chi$. Combining the positions and internal spinors of all particles yields

$$\tilde{Q}_N = \left\{ (\mathbf{x}_1, \dots, \mathbf{x}_N; \chi_1, \dots, \chi_N) \right\} / \sim,$$

whose projection onto Q_N forgets all χ_j . The lift of exchange paths on \tilde{Q}_N has the following property:

- Performing spatial exchange alone without changing internal spinors corresponds to a closed path;
- However, due to the existence of the Riccati square-root branch cut, adiabatic evolution along this path causes the overall phase of (χ_i, χ_j) to wind around the origin once, thus corresponding to a nontrivial element on \tilde{Q}_N .

Therefore, the self-referential scattering structure automatically provides the double cover and holonomy data required by the FR scheme. Choosing a quantization rule where the covering transformation corresponds to wavefunction sign change yields Fermi statistics; choosing the covering transformation as trivial action yields Bose statistics. This paper emphasizes: under the natural dynamics of massive self-referential excitations, stability and unitarity require choosing the former, thus locking “massive” and “Fermi statistics” together in this construction.

C Remarks on 2+1 Dimensions and Anyonic Generalizations

Although this paper primarily focuses on three-plus-one dimensions, in two-plus-one dimensions, self-referential scattering and Riccati square-root structure may produce richer statistical behavior.

In two-plus-one dimensions, the fundamental group of the N -particle configuration space is the braid group B_N , whose representations can yield anyonic statistics, allowing continuous interpolation of phase or matrix representations during particle exchange between bosons and fermions. In self-referential scattering networks, the combination of the multivaluedness of internal spinor variables χ and braid group representations promises to give a concrete realization of a class of “self-referential anyons”: their statistical phase is no longer limited to $\pm\pi$, but is related to the winding number of the Riccati discriminant in parameter space.

Systematic analysis of this case requires extending this paper’s QCA-scattering construction to two-dimensional lattices and performing unified modeling of the topology of multi-particle braid paths and the geometric structure of internal self-referential feedback, left for future work.