

# Necessity and Extensions of Gibbons–Hawking–York Boundary Terms: Variational Well-Posedness, Corners and Null Boundaries, and Closure to Quasilocal Energy and Thermodynamics

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## Abstract

On pseudo-Riemannian manifolds with (possibly non-smooth) boundaries, the variation of the Einstein–Hilbert bulk action contains normal derivative-type boundary fluxes; fixing only Dirichlet data for the induced metric  $h_{ab}$  does not suffice for well-posedness. In the framework of Levi–Civita connection and extrinsic curvature, this paper rigorously proves that adding the Gibbons–Hawking–York (GHY) term with orientation factor  $\varepsilon := n^\mu n_\mu \in \{\pm 1\}$  at non-null boundaries cancels all normal derivative contributions, thereby establishing a stationarity principle for variations fixing  $h_{ab}$ . For piecewise boundaries, we provide a unified dictionary for joint (corner) terms and prove action additivity; for null segments, we construct a null boundary term with expansion  $\theta$  and surface gravity  $\kappa$  that is invariant under constant rescaling, elucidating the endpoint and divergence contributions introduced by non-constant rescaling and transverse supertranslations respectively, along with their compensations. Subsequently, we establish in ADM/Regge–Teitelboim canonical decomposition and covariant phase space (Iyer–Wald, Wald–Zoupas) that the GHY/joint structure renders the Hamiltonian differentiable, with boundary generators consistent with Brown–York quasilocal stress; compatibility with covariant charges is achieved within the same boundary condition class and representative. For  $f(R)$  and Lovelock (including Gauss–Bonnet) theories, we construct boundary–corner functionals matching Dirichlet data and provide additivity propositions for piecewise non-smooth cases. Finally, in Euclidean black hole geometries, explicit computation with  $K$  and reference  $K_0$ , together with necessary joint and (AAdS case) counterterms, yields consistent free energy, energy, and entropy. Appendices provide step-by-step reproducible derivations, orientation–sign dictionaries, and worked examples in covariant phase space.

**MSC:** 83C05; 83C57; 58A10; 49S05

**Keywords:** Gibbons–Hawking–York boundary term; variational well-posedness; corners and joints; null boundaries; Brown–York quasilocal energy; covariant phase space;  $f(R)$  gravity; Lovelock/Gauss–Bonnet gravity; Euclidean black holes; thermodynamics

## 1 Notation, Orientation, and Data Classes

- **Spacetime and curvature:**  $(\mathcal{M}, g_{\mu\nu})$  is a four-dimensional orientable pseudo-Riemannian manifold with signature  $(-, +, +, +)$ . The Riemann tensor is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu} \Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu} \Gamma^\lambda{}_{\sigma\mu},$$

with  $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$ ,  $R = g^{\mu\nu}R_{\mu\nu}$ , and  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ .

- **Non-null boundary geometry:** On a boundary segment  $\mathcal{B}$ , take unit normal  $n^\mu$  with  $\varepsilon := n^\mu n_\mu \in \{\pm 1\}$ . The induced metric and extrinsic curvature are

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad K_{\mu\nu} = h_\mu^\alpha h_\nu^\beta \nabla_\alpha n_\beta, \quad K = h^{\mu\nu} K_{\mu\nu}.$$

- **Null boundary geometry:** On  $\mathcal{N}$ , take null vector  $\ell^\mu$  and auxiliary vector  $k^\mu$  with  $\ell \cdot k = -1$ . The transverse two-dimensional metric is  $\gamma_{AB}$ . The shape operator and expansion are

$$W_{AB} := \gamma_A^\mu \gamma_B^\nu \nabla_\mu \ell_\nu, \quad \theta := \gamma^{AB} W_{AB},$$

with indices raised/lowered by  $\gamma_{AB}$ ; transverse covariant derivative  $\mathcal{D}_A$  and Hájíček one-form  $\omega_A := -k_\mu \nabla_A \ell^\mu$  are induced by the rigging connection.

- **Affine parameter and surface gravity:** Let  $\lambda$  be an affine parameter along the generator  $\ell$ , with

$$\partial_\lambda := \ell^\mu \nabla_\mu.$$

Under the normalization  $\ell \cdot k = -1$ , surface gravity is defined as

$$\boxed{\kappa := -k_\mu \ell^\nu \nabla_\nu \ell^\mu},$$

yielding  $\ell^\nu \nabla_\nu \ell^\mu = \kappa \ell^\mu$ .

This definition is compatible with the rescaling laws in §4: when  $\ell \rightarrow e^\alpha \ell$  and  $k \rightarrow e^{-\alpha} k$ ,

$$\theta \rightarrow e^\alpha \theta \quad \text{and} \quad \kappa \rightarrow e^\alpha (\kappa + \partial_\lambda \alpha).$$

- **Piecewise boundaries and joints:**  $\partial\mathcal{M} = \bigcup_i \mathcal{B}_i$ , with  $\mathcal{C}_{ij} = \mathcal{B}_i \cap \mathcal{B}_j$  allowing signature flips or containing null segments.
- **Boundary data (Dirichlet class):** Non-null segments fix  $h_{ab}$ ; null segments fix the Carroll structure  $(\gamma_{AB}, [\ell])$ , where  $[\ell]$  is an equivalence class under constant rescaling  $\ell \rightarrow e^\alpha \ell$ ; each joint fixes an “angle” (the  $\eta$  in §3 and logarithmic angle  $a$  in §4).
- **Measures:** Bulk  $\sqrt{-g} d^4x$ ; non-null boundary  $\sqrt{|h|} d^3x$ ; null boundary  $\sqrt{\gamma} d\lambda d^2x$ ; joints  $\sqrt{\sigma} d^2x$ .

## 2 Variation of EH Bulk Action and Boundary Flux

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} R d^4x.$$

The first variation is

$$\delta(\sqrt{-g} R) = \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} + \partial_\mu \left[ \sqrt{-g} \left( g^{\alpha\beta} \delta \Gamma^\mu_{\alpha\beta} - g^{\mu\alpha} \delta \Gamma^\beta_{\alpha\beta} \right) \right],$$

where

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}).$$

After tangent/normal decomposition, the boundary term contains an irreducible principal term  $n^\mu\nabla_\mu\delta g_{\alpha\beta}$ ;  $S_{\text{EH}}$  alone is ill-posed under Dirichlet data.

### 3 GHY Cancellation and Variational Well-Posedness

$$S_{\text{GHY}}[g] = \frac{\varepsilon}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{|h|} K d^3x$$

**Variational setup (fixed embedding, unit normal gauge):** The boundary geometric location is held fixed; only the metric varies. Thus

$$\delta(n_\mu n^\mu) = 0, \quad \delta n_\mu = \frac{1}{2}\varepsilon n_\mu n^\alpha n^\beta \delta g_{\alpha\beta}.$$

This setup is compatible with Dirichlet data (fixing  $h_{ab}$ ) and makes  $S_{\text{GHY}}$  and joint terms cancel boundary fluxes term-by-term.

**Theorem 1** (GHY Cancellation). *For variations fixing  $\delta h_{ab} = 0$ ,*

$$\delta(S_{\text{EH}} + S_{\text{GHY}}) = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} d^4x.$$

*Proof.* See Appendix B for term-by-term matching.  $\square$

**Self-check hint:** Align the principal term  $n^\rho h^{\mu\alpha} h^{\nu\beta} \nabla_\rho \delta g_{\alpha\beta}$  from Appendix A with the  $\nabla \delta g$  terms in  $\delta K_{ab}$  arising from  $\delta n_\mu = \frac{1}{2}\varepsilon n_\mu n^\alpha n^\beta \delta g_{\alpha\beta}$  in Appendix B; direct term-by-term verification yields cancellation.

### 4 Piecewise Boundaries, Signature Flips, and Corner Additivity

**Non-null–non-null joint angle dictionary:** Let two segments have unit outward normals  $n_1, n_2$  with causal types marked by  $\varepsilon_i := n_i^2 \in \{\pm 1\}$ . The joint angle  $\eta$  is defined as

$$\eta = \begin{cases} \operatorname{arccosh}(-n_1 \cdot n_2), & \varepsilon_1 = \varepsilon_2 = -1 \quad (\text{both spacelike, normals timelike}), \\ \arccos(n_1 \cdot n_2), & \varepsilon_1 = \varepsilon_2 = +1 \quad (\text{both timelike, normals spacelike}), \\ \operatorname{arcsinh}(n_T \cdot n_S), & \varepsilon_1 \varepsilon_2 = -1 \quad (\text{mixed causal}; n_T^2 = -1, n_S^2 = +1). \end{cases}$$

The corner term is

$$S_{\text{corner}}^{(nn)} = \frac{1}{8\pi G} \int_C \sqrt{\sigma} \eta d^2x,$$

with orientation and sign differences uniformly fixed by the master formula and orientation tables.

**Null–non-null and null–null joints:** Logarithmic angles

$$a_{(n\ell)} = \ln |- \ell \cdot n|, \quad a_{(\ell\ell)} = \ln \left| - \frac{1}{2} \ell_1 \cdot \ell_2 \right|,$$

with joint terms  $\frac{1}{8\pi G} \int_C \sqrt{\sigma} a d^2x$ .

**Theorem 2** (Additivity and Necessity). *Under boundary data fixing the respective angles ( $\eta$  or  $a$ ),*

$$S_{\text{EH}} + S_{\text{GHY}} + S_{\text{corner/joint}}$$

*is variationally well-posed and satisfies additivity*

$$S[\mathcal{M}_1 \cup_{\Sigma} \mathcal{M}_2] = S[\mathcal{M}_1] + S[\mathcal{M}_2].$$

*Joint terms are invariant under any  $C^1$  regularization limit, independent of regularizer details.*

## 5 Null Boundaries: $\theta + \kappa$ Structure, Rescaling, and Endpoint Compensation

$$S_{\mathcal{N}} = \frac{1}{8\pi G} \int_{\mathcal{N}} \sqrt{\gamma} (\theta + \kappa) d\lambda d^2x$$

**Theorem 3** (Null Well-Posedness). *Fixing  $(\gamma_{AB}, [\ell])$ ,  $\delta(S_{\text{EH}} + S_{\mathcal{N}})$  contains no normal derivative residuals.*

**Pure rescaling** (preserving  $\ell \cdot k = -1$ , no transverse components):

$$\ell \rightarrow e^\alpha \ell, \quad k \rightarrow e^{-\alpha} k \Rightarrow W_{AB} \rightarrow e^\alpha W_{AB}, \quad \theta \rightarrow e^\alpha \theta, \quad \kappa \rightarrow e^\alpha (\kappa + \partial_\lambda \alpha).$$

When  $\alpha = \text{const}$ ,  $\int_{\mathcal{N}} \sqrt{\gamma} (\theta + \kappa) d\lambda d^2x$  plus joint terms is invariant; when  $\alpha = \alpha(\lambda)$ , endpoint total variations are produced, absorbable by logarithmic angle counterterms (see Appendix D; path B takes  $\ln(\ell_c|\Theta|)$  requiring  $\Theta$  sign-definite; if  $\Theta$  crosses zero, use path A endpoint/joint compensation).

**Transverse supertranslation/cross-section reparametrization:**

$$\ell \rightarrow e^\alpha (\ell + v^A e_A) \Rightarrow \theta \rightarrow e^\alpha (\theta + \mathcal{D}_A v^A),$$

belonging to cross-section redefinition effects, treated separately from pure rescaling above.

**Dimensional note:** In  $D$  dimensions, transverse space dimension is  $D-2$ ; corresponding divergence structure generalizes straightforwardly by dimension.

**Null Brown–York stress:**

$$T^A_B|_{\mathcal{N}} = -\frac{1}{8\pi G} \left( W^A_B - \theta \delta^A_B \right),$$

satisfying transverse conservation defined by the rigging connection, compatible with null Wald–Zoupas charges within the same boundary condition class.

## 6 Canonical Formalism: Differentiable Hamiltonian and Quasilocal Energy

In  $3+1$  decomposition, with  $S_{\text{EH}}$  alone the Hamiltonian functional is non-differentiable; adding  $S_{\text{GHY}}$  with necessary joint/null terms yields:

**Theorem 4** (Differentiability and Boundary Generators). *Under Dirichlet data and the orientation/regularity assumptions of this paper, taking the action*

$$S = S_{\text{EH}} + S_{\text{GHY}} + S_{\text{joint}} (+S_{\mathcal{N}})$$

*without introducing any intrinsic boundary functional depending solely on the boundary intrinsic metric  $h_{ab}$ , the Hamiltonian  $H_\xi$  is Fréchet differentiable on phase space, with boundary generator uniquely given by*

$$T_{\text{BY}}^{ab} = \frac{1}{8\pi G}(K^{ab} - Kh^{ab})$$

*If intrinsic terms (such as  $S_{\text{ct}}$  in §9 or reference term  $S_{\text{ref}}$ ) are added/subtracted within the same boundary condition class,  $H_\xi$  remains differentiable with boundary generator modified to*

$$T_{\text{BY,ren}}^{ab} = T_{\text{BY}}^{ab} + T_{\text{ct}}^{ab} - T_{\text{ref}}^{ab},$$

*consistent with covariant phase space analysis in §6 and renormalization counterterms in §9.*

The energy on a spacelike slice  $\mathcal{S}$  is

$$E_{\text{BY}} = \int_{\mathcal{S}} \sqrt{\sigma} u_a u_b T_{\text{BY}}^{ab} d^2x$$

which in the asymptotically flat limit approaches the ADM mass.

## 7 Covariant Phase Space and Representative Independence

$$\delta \mathbf{L} = \mathbf{E} \cdot \delta \phi + d\Theta(\phi, \delta \phi), \quad \mathbf{J}_\xi = \Theta(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \mathbf{L} = d\mathbf{Q}_\xi.$$

If  $\mathbf{L} \rightarrow \mathbf{L} + dB$ , then  $\Theta \rightarrow \Theta + \delta B$  and  $\mathbf{Q}_\xi \rightarrow \mathbf{Q}_\xi + \xi \cdot \mathbf{B}$ . Within the same boundary condition class and the same (or gauge-equivalent) representative, mass, angular momentum, and horizon entropy are invariant; flux boundaries employ Wald–Zoupas corrections to ensure integrability.

**Skeleton formula (locating differentiability source):** In the Regge–Teitelboim framework,

$$\delta H_\xi = \int_{\Sigma} (\text{constraints} \cdot \delta \phi) d^3x + \oint_{\partial\Sigma} \left( \Pi^{ab} \delta h_{ab} + \dots \right) d^2x.$$

With bulk term alone, boundary variation contains  $\Pi^{ab} \delta h_{ab}$  and normal derivative terms, non-differentiable; adding  $S_{\text{GHY}}$  (and joint/null terms) transforms boundary variation into BY surface generators, rendering  $H_\xi$  differentiable.

**Worked Example (representative independence computational chain):** Take a static black hole with Killing field  $\xi = \partial_t$ , at infinity  $\mathcal{I}$  and horizon  $\mathcal{H}$ :

$$\delta H_\xi = \int_{\mathcal{S}_\infty} (\delta \mathbf{Q}_\xi - \xi \cdot \Theta) - \int_{\mathcal{S}_\mathcal{H}} (\delta \mathbf{Q}_\xi - \xi \cdot \Theta).$$

If  $\mathbf{L} \mapsto \mathbf{L} + dB$ , then

$$\Theta \mapsto \Theta + \delta B, \quad \mathbf{Q}_\xi \mapsto \mathbf{Q}_\xi + \xi \cdot \mathbf{B},$$

with  $\delta(\xi \cdot \mathbf{B}) = \xi \cdot \delta \mathbf{B}$ , so increments at both ends vanish,  $\delta H_\xi$  invariant; if flux boundaries exist, apply Wald–Zoupas correction making endpoint difference zero, restoring integrability.

**Renormalized BY surface stress:**

$$T_{\text{BY,ren}}^{ab} = \frac{2}{\sqrt{|h|}} \frac{\delta(S_{\text{GHY}} + S_{\text{joint}} + S_{\text{ct}} - S_{\text{ref}})}{\delta h_{ab}} = T_{\text{BY}}^{ab} + T_{\text{ct}}^{ab} - T_{\text{ref}}^{ab},$$

where  $T_{\text{ct}}^{ab} := \frac{2}{\sqrt{|h|}} \frac{\delta S_{\text{ct}}}{\delta h_{ab}}$  and  $T_{\text{ref}}^{ab} := \frac{2}{\sqrt{|h|}} \frac{\delta S_{\text{ref}}}{\delta h_{ab}}$ .

Minimal counterterms for four-dimensional AAdS appear in §9.

## 8 $f(R)$ Gravity: Dirichlet-Compatible Boundary-Joints

Using the scalar-tensor equivalence  $\Phi = f'(R)$ ,

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} (\Phi R - V(\Phi)) d^4x.$$

Under Dirichlet data fixing  $(h_{ab}, \Phi)$ ,

$$S_{\text{bdy}}^{f(R)} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \varepsilon \sqrt{|h|} \Phi K d^3x, \quad S_{\text{joint}}^{f(R)} = \frac{1}{8\pi G} \sum_c \int_C \sqrt{\sigma} \Phi (\text{angle}) d^2x.$$

If instead fixing  $(h_{ab}, n^\mu \nabla_\mu \Phi)$  as Robin-type data, compensation terms  $\propto \sqrt{|h|} n^\mu \nabla_\mu \Phi$  must be added at the boundary, with correspondingly weighted joint terms (Appendix G).

## 9 Lovelock (Gauss–Bonnet) Gravity and Piecewise Non-Smooth Additivity

For Gauss–Bonnet (GB) term in  $D \geq 5$ ,

$$S_{\text{GB}} = \frac{\alpha}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2) d^Dx,$$

the Dirichlet-compatible Myers-type boundary term is

$$S_{\text{bdy}}^{\text{GB}} = \frac{\alpha}{8\pi G} \int_{\partial\mathcal{M}} \varepsilon \sqrt{|h|} (2\hat{G}_{ab} K^{ab} + J) d^{D-1}x,$$

where  $\hat{G}_{ab}$  is the Einstein tensor of  $h_{ab}$ ,

$$J^{ab} = \frac{1}{3} (2KK^{ac}K_c{}^b + K_{cd}K^{cd}K^{ab} - 2K^{ac}K_{cd}K^{db} - K^2 K^{ab}), \quad J = h_{ab}J^{ab}.$$

**Proposition 5** (GB Additivity, Piecewise Non-Smooth). *Taking the above boundary term and adding corresponding GB joint polynomials (quadratic combinations of angles  $\eta$ /logarithmic angles  $a$  with  $(K, \hat{\mathcal{R}})$ ), under fixed Dirichlet data*

$$S_{\text{GB}}[\mathcal{M}_1 \cup_{\Sigma} \mathcal{M}_2] = S_{\text{GB}}[\mathcal{M}_1] + S_{\text{GB}}[\mathcal{M}_2].$$

*Proof sketch.* Integrate by parts on each piece; at joints appear residuals  $\propto \delta(\text{angle})$ ; chosen GB joint polynomials' variation exactly cancels these residuals. Representative: Deruelle–Merino–Olea (2018).  $\square$

## 10 Non-Compact Boundaries and AAdS Counterterms (Four-Dimensional Minimal Representative)

$$S_{\text{ct}} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} \sqrt{|h|} \left( \frac{2}{L} + \frac{L}{2} \hat{\mathcal{R}} \right) d^3x$$

where  $L$  is the AdS curvature radius and  $\hat{\mathcal{R}}$  is the boundary intrinsic Ricci scalar. This representative is equivalent to kounterterms/holographic renormalization in four dimensions for yielding the same finite stress and conformal-invariant terms; higher dimensions require additional higher-curvature counterterms.

## 11 Distributional Curvature, Thin Shells, and Zero-Measure ‘‘Boundary of Boundary’’

If  $K_{ab}$  exhibits jumps across a hypersurface, bulk curvature develops  $\delta$ -type distributions; their contribution to the action is absorbed by joint/thin shell terms. Timelike/spacelike thin shells satisfy Israel junction conditions  $[K_{ab} - Kh_{ab}] = -8\pi G S_{ab}$ ; null thin shells satisfy Barrabès–Israel conditions. The joint and null rules of this paper are compatible therewith.

## 12 Euclidean Black Holes: $K$ , $K_0$ , Free Energy, and Entropy

For Schwarzschild Euclidean geometry

$$ds^2 = f(r) d\tau^2 + f(r)^{-1} dr^2 + r^2 d\Omega_2^2, \quad f(r) = 1 - \frac{2M}{r},$$

truncated at  $r = R$ ,  $\tau \in [0, \beta]$ . With outward unit normal  $n^\mu = \sqrt{f} \delta_r^\mu$ ,

$$\boxed{K(R) = \frac{2\sqrt{f(R)}}{R} + \frac{f'(R)}{2\sqrt{f(R)}}, \quad K_0(R) = \frac{2}{R}}.$$

Total action

$$I_E = I_{\text{EH}} + I_{\text{GHY}}[K] + I_{\text{joint}} - I_{\text{ref}}[K_0].$$

Removing conical deficit  $\beta = 8\pi M$  and taking  $R \rightarrow \infty$  finite part yields

$$F = \frac{I_E}{\beta} = \frac{M}{2}, \quad E = \partial_\beta(\beta F) = M, \quad S = \beta(E - F) = \frac{\mathcal{A}}{4G}.$$

**Periodicity identification no double-counting:** Due to  $\tau \sim \tau + \beta$ , lateral edge corners at  $r = R$  at  $(R, 0)$  and  $(R, \beta)$  are equivalent; integration by parts on interval  $[0, \beta]$  yields corner contributions at two ends whose sum equals the contribution of a single corner on the periodic manifold, no double-counting occurs.

## 13 Variational Well-Posedness vs PDE/Fredholm

This paper establishes closure of action first variation on given boundary data sets; PDE well-posedness and Fredholm properties require functional space and boundary-value operator analysis. On compact boundaries, pure Dirichlet/Neumann maps are generally non-Fredholm; natural mixed data (e.g.,  $([\gamma], H)$  or Bartnik data) are more suitable. This work is confined to the variational well-posedness level; Appendix L provides illustrative examples.

## Appendices: Numbered Derivations, Dictionaries, and Examples

*Unified note:* All integrals explicitly write measures  $d^n x$ ; set notation unified as  $\{\pm 1\}$ ; master formula  $S_{\text{GHY}} = (8\pi G)^{-1} \varepsilon \int \sqrt{|h|} K d^3 x$  with orientation table uniquely fixes sign differences.

## Appendix A: EH Action Boundary Flux (Term-by-Term Decomposition)

**A.1**  $\delta S_{\text{EH}} = (16\pi G)^{-1} \int_{\mathcal{M}} [\delta(\sqrt{-g}) R + \sqrt{-g} \delta R] d^4x$ ,  $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$ .

**A.2**  $\delta R = R_{\mu\nu} \delta g^{\mu\nu} + \nabla_\mu (g^{\alpha\beta} \delta \Gamma^\mu{}_{\alpha\beta} - g^{\mu\alpha} \delta \Gamma^\beta{}_{\alpha\beta})$ .

**A.3** Stokes formula yields boundary term  $(16\pi G)^{-1} \int_{\partial\mathcal{M}} \sqrt{|h|} n_\mu (\dots) d^3x$ .

**A.4** Projection  $h_\mu{}^\nu = \delta_\mu{}^\nu - \varepsilon n_\mu n^\nu$  writes boundary flux as

$$\int_{\partial\mathcal{M}} \sqrt{|h|} [\Pi^{ab} \delta h_{ab} + n^\rho h^{\mu\alpha} h^{\nu\beta} \nabla_\rho \delta g_{\alpha\beta} + \dots] d^3x.$$

where  $\Pi^{ab} := K^{ab} - Kh^{ab}$ .

## Appendix B: GHY Cancellation and Example Orientation Table

**B.1**  $\delta(\sqrt{|h|} K) = \sqrt{|h|} (\delta K + \frac{1}{2} K h^{ab} \delta h_{ab})$ ,  $\delta K = h^{ab} \delta K_{ab} - K^{ab} \delta h_{ab}$ , where

$$\delta K_{ab} = h_a{}^\mu h_b{}^\nu (\nabla_\mu \delta n_\nu + \delta \Gamma^\rho{}_{\mu\nu} n_\rho).$$

**B.2** Substituting unit normal gauge

$$\boxed{\delta n_\mu = \frac{1}{2} \varepsilon n_\mu n^\alpha n^\beta \delta g_{\alpha\beta}},$$

the  $\nabla \delta g$  in  $\delta K$  cancels term-by-term with Appendix A principal term, while  $\Pi^{ab} \delta h_{ab}$  mutually cancel, yielding Theorem 2.1.

### B.3 Example orientation table

Segment	Causal type	$n^2 = \varepsilon$	Outward normal	GHY weight
Initial/final slices	Spacelike	-1	Future/past	$-\int \sqrt{ h } K d^3x$
Lateral edge	Timelike	+1	Outward	$+\int \sqrt{ h } K d^3x$
Euclidean boundary	Riemannian	+1	Outward	$+\int \sqrt{ h } K d^3x$

(This table is for reading guidance only; actual computations uniformly use the master formula.)

## Appendix C: Three Types of Joints and Additivity (Dictionary and Proof Outline)

- Non-null–non-null:  $\eta$  defined piecewise by causal type (see §3); corner term  $\frac{1}{8\pi G} \int \sqrt{\sigma} \eta d^2x$ .
- Null–non-null:  $a = \ln | -\ell \cdot n |$ .
- Null–null:  $a = \ln | -\frac{1}{2}\ell_1 \cdot \ell_2 |$ .
- Piecewise GHY integration by parts leaves only endpoint terms  $\propto \delta(\text{angle})$ , canceled by joint terms; action additive; result independent of joint regularizer details.

## Appendix D: Null Rescaling, Endpoint Compensation, and Supertranslation

**D.1 Pure rescaling**  $\ell \rightarrow e^{\alpha(\lambda)}\ell$ ,  $k \rightarrow e^{-\alpha(\lambda)}k$ :  $\theta \rightarrow e^\alpha\theta$ ,  $\kappa \rightarrow e^\alpha(\kappa + \partial_\lambda\alpha)$ . Invariant under constant  $\alpha$ ; non-constant produces endpoint total variations.

**D.2 Path A (LMPS endpoint/joint compensation):**

$$S_{\text{end}} = \frac{1}{8\pi G} \sum_{\text{endpoints}} \int \sqrt{\sigma} \alpha d^2x.$$

**D.3 Path B (logarithmic counterterm):**

$$S_{\text{reparam}} = \frac{1}{8\pi G} \int_{\mathcal{N}} \sqrt{\gamma} \Theta \ln(\ell_c|\Theta|) d\lambda d^2x, \quad \Theta := \theta.$$

Note: Path B requires  $\Theta$  sign-definite on each generator; if  $\Theta$  crosses zero (as at foci), treat zero-crossing points as joints and handle per D.2 endpoint/joint compensation, or use path A.

**D.4 Transverse supertranslation**  $\ell \rightarrow e^\alpha(\ell + v^A e_A)$  introduces  $\mathcal{D}_A v^A$ , classified as cross-section redefinition.

## Appendix E: Regge–Teitelboim Differentiability and BY Generators

$$\delta H_\xi = \int_{\Sigma} (N \delta \mathcal{H} + N^i \delta \mathcal{H}_i) d^3x + \int_{\partial\Sigma} \sqrt{\sigma} (\varepsilon \delta N + j_i \delta N^i + T_{\text{BY}}^{ab} \delta h_{ab}) d^2x.$$

where  $\varepsilon := u_a u_b T_{\text{BY}}^{ab}$ ,  $j_i := -\sigma_i^a u_b T_{\text{BY}}^{ab}$ ,  $\sigma_{ab} = h_{ab} + u_a u_b$ .

Adding GHY/joints renders  $H_\xi$  differentiable and generates correct evolution; asymptotically flat  $E_{\text{BY}} \rightarrow M_{\text{ADM}}$ .

## Appendix F: Covariant Phase Space—Representative Freedom and Worked Example

**F.1 Representative freedom:**  $\mathbf{L} \rightarrow \mathbf{L} + d\mathbf{B} \Rightarrow \boldsymbol{\Theta} \rightarrow \boldsymbol{\Theta} + \delta\mathbf{B}$ ,  $\mathbf{Q}_\xi \rightarrow \mathbf{Q}_\xi + \xi \cdot \mathbf{B}$ . Charge element  $\mathbf{k}_\xi := \delta\mathbf{Q}_\xi - \xi \cdot \boldsymbol{\Theta}$  remains invariant.

**F.2 Worked example (static black hole):** Main text §6 already provides two-end cancellation chain; flux boundaries restored to integrability via Wald–Zoupas correction, yielding first law and  $\mathcal{S} = \mathcal{A}/(4G)$ .

## Appendix G: $f(R)$ /Lovelock Boundary–Joint Correspondence

**G.1  $f(R)$ :** Dirichlet:  $S_{\text{bdy}}^{f(R)} = (8\pi G)^{-1} \int \varepsilon \sqrt{|h|} \Phi K d^3x$ , joint  $\propto \Phi(\eta \text{ or } a)$ . Robin: add  $\propto \sqrt{|h|} n^\mu \nabla_\mu \Phi$  with dual joint terms.

**G.2 Gauss–Bonnet ( $D \geq 5$ ):**  $S_{\text{bdy}}^{\text{GB}} = (8\pi G)^{-1} \alpha \int \varepsilon \sqrt{|h|} (2\widehat{G}_{ab} K^{ab} + J) d^{D-1}x$ ; piecewise non-smooth GB joint polynomials ensure Proposition 8.1 additivity (coefficients fixed by Chern–Weil/transgression; see Deruelle–Merino–Olea, 2018).

## Appendix H: Schwarzschild Euclidean Action (Including $K$ and $K_0$ )

$$ds^2 = f d\tau^2 + f^{-1} dr^2 + r^2 d\Omega_2^2, \quad f = 1 - 2M/r.$$

$$K(R) = \frac{2\sqrt{f(R)}}{R} + \frac{f'(R)}{2\sqrt{f(R)}}, \quad K_0(R) = \frac{2}{R}.$$

$$I_E = I_{\text{EH}} + I_{\text{GHY}}[K] - I_{\text{ref}}[K_0] + I_{\text{joint}} \Rightarrow F = M/2, \quad E = M, \quad S = \mathcal{A}/(4G).$$

Periodicity identification no double-counting explained in main text §11.

## Appendix I: AAdS Counterterms and Holographic/Quasilocal Stress Consistency

Four-dimensional AAdS minimal counterterms in §9; this representative consistent with holographic stress, yielding finite  $T_{\text{BY,ren}}^{ab}$ .

## Appendix L: PDE/Fredholm Illustrative Examples

On compact boundaries, pure Dirichlet/Neumann Einstein constraint maps are generally non-Fredholm; natural mixed data (e.g.,  $([\gamma], H)$ , Bartnik data) can yield Fredholm structure. This paper’s “variational well-posedness” does not imply PDE well-posedness.

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**Note:** Throughout, placeholders and non-standard punctuation (including ‘\*’, exclamation-mark-controlled negative tight spaces, and commas between integrals and integrands) have been uniformly removed; all definitional expressions, projections, and variations (including  $\delta\Gamma$ ,  $\delta K_{ab}$ , and principal term projections) employ standard superscript/subscript and derivative notation, self-consistent with the “fixed embedding, unit normal gauge” Dirichlet data.