UMMIC: Mother Map–Mellin–de Branges Unified Theory of "Information Conservation–Phase Density–Sampling Stability" (With Complete Proofs)

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Abstract

Starting from mother map kernels satisfying moderate axioms, under the Mellin isometry on $L^2(\mathbb{R}_+, dx/x)$ and the spectral dictionary of de Branges–Kreĭn canonical systems, we establish three parallel and closed main lines: (I) a **Noether-type flux continuity equation** (information conservation) with $\Lambda(s)$'s logarithmic potential as potential function; (II) **"phase density = spectral shift derivative = relative spectral density"** consistency (CCS) under self-adjoint scattering settings; (III) engineering-oriented **Nyquist–Poisson–Euler–Maclaurin (EM) three-fold decomposition** with **non-asymptotic** error closure. Accompanying this, under the parallel framework of positive-weight spectral density scale $d\mu(E) = \rho(E)\, dE$ and phase coordinate $v_\phi(E) = \delta(E)/\pi$, we introduce the relative spectral density $\rho_{\rm rel}(E) = \xi'(E) = -\frac{1}{\pi}\delta'(E)$ and its integral coordinate $v_{\rm rel}(E) = \xi(E)$, unifying engineering and spectral theory scales, and on this basis unify the proofs of **Landau** sampling/interpolation necessary density thresholds, **Wexler–Raz** tight/dual sufficiency conditions, and **Balian–Low** impossibility (Mellin/Weyl version), thereby welding "mother map–scattering–frame/sampling–error theory" into a non-asymptotic hard-core system.

Keywords: Mother map; Mellin transform; de Branges space; Information conservation; Phase density; Sampling theory; Wexler–Raz; Balian–Low; Nyquist–Poisson–Euler–Maclaurin

1 Axioms, Objects, and Notation

A0 (Mother Map and Mellin Embedding) Take mother kernel $k_{\mathcal{M}} \in L^2(\mathbb{R}_+, dx/x)$ with Mellin transform

$$Z_{\mathcal{M}}(s) = \int_0^\infty k_{\mathcal{M}}(x)\, x^{s-1}\, dx. \tag{1}$$

Along the critical line $s = \frac{1}{2} + i\omega$, there is an isometry

$$\int_0^\infty |k_{\mathcal{M}}(x)|^2 \frac{dx}{x} = \frac{1}{2\pi} \int_{\mathbb{R}} \left| Z_{\mathcal{M}}(\tfrac{1}{2} + i\omega)\right|^2 d\omega, \tag{2}$$

and scaling $k_{\mathcal{M}}(2^k\cdot)$ corresponds to frequency shift and amplitude factor $2^{-k/2}$. The above isometry is the standard statement of the Mellin–Plancherel theorem on $\frac{1}{2}+i\mathbb{R}$.

A1 (Completed Function and Mirror) There exists a normalization factor $\gamma_{\mathcal{M}}(s)$ such that $\Lambda_{\mathcal{M}}(s)=\gamma_{\mathcal{M}}(s)Z_{\mathcal{M}}(s)$ has well-defined phase $\varphi_{\mathcal{M}}(\omega)$ and modulus $R(\omega)$ on the critical line. Below we use only its boundary phase.

A2 (Spectral Density and Weyl–Titchmarsh Dictionary) Under self-adjoint canonical system/one-dimensional Schrödinger-type backgrounds, Weyl–Titchmarsh m is a Herglotz function, and the boundary imaginary part yields the absolutely continuous spectral density

$$\rho(E)=\frac{1}{\pi}\Im m(E+i0)\quad\text{(a.e.)},\tag{3}$$

accordingly defining the spectral density weight measure $d\mu(E)=\rho(E)\,dE$.

A3 (Density Coordinate, Phase Coordinate, and Isometry) Write $X(\omega)=(2\pi)^{-1/2}Z_{\mathcal{M}}(\frac{1}{2}+i\omega)$. Write $dv_\mu(x)=\rho(x)\,dx$; when $x=E$, $\rho(E)=\frac{1}{\pi}\Im m(E+i0)$ is the **absolute** spectral density (positive). Define phase coordinate $v_\phi(E)=\delta(E)/\pi$. The relative spectral density is defined as $\rho_{\rm rel}(E)=\xi'(E)=-\frac{1}{\pi}\delta'(E)$.

In the v_μ coordinate there is an isometry

$$\int_{\mathbb{R}}|X(\omega)|^2\,\rho(\omega)\,d\omega=\int_{\mathbb{R}}|X(\omega(v_\mu))|^2\,dv_\mu.\tag{4}$$

Here $d\mu=\rho\,dE$ always takes positive weight, no longer identified with δ'.

On each connected component of the absolutely continuous spectrum, v_μ is a non-decreasing function with a measurable right inverse for change of variables; accordingly $\int|X(\omega)|^2\rho(\omega)\,d\omega=\int|X(\omega(v_\mu))|^2\,dv_\mu$ holds (a.e.).

Energy formulation takes $v_\phi(E)=\delta(E)/\pi$; log-frequency formulation takes $v_\phi(\omega)=\varphi_{\mathcal{M}}(\omega)/\pi$. Below we default to E-variable, introducing relative density coordinate $v_{\rm rel}(E)=\xi(E)-\xi(E_\star)$ when necessary.

Phase Normalization (Anchor Point): Choose reference point $E_\star$ such that $\delta(E_\star)=0$ (equivalently $v_\phi(E_\star)=0$); for multi-channel, take $\delta=\frac{1}{2}\arg\det S$ with trace. This normalization does not affect $\underline{D}_\phi,\overline{D}_\phi$ and other translation-invariant quantities, but ensures uniqueness of v_ϕ and stability of change of variables.

Symbol Alignment: Write $\varphi_{\mathcal{M}}(\omega)=\arg\Lambda_{\mathcal{M}}(\frac{1}{2}+i\omega)$, scattering phase $\delta(E)=\frac{1}{2}\arg\det S(E)$. When embedding the mother map into the scattering model via the de Branges–Kreĭn interface, set $\delta\big(E(\omega)\big)=\varphi_{\mathcal{M}}(\omega)$. Absolute spectral density $\rho(E)\geq 0$ gives positive $d\mu(E)$; relative spectral density $\rho_{\rm rel}(E)=\xi'(E)=-\frac{1}{\pi}\delta'(E)$ can be positive or negative, given by spectral shift minus reference state density difference.

A4 (Finite-Order EM Discipline) Throughout we use only **finite-order** Euler–Maclaurin (endpoint Bernoulli layer + explicit remainder upper bound) in parallel with Poisson summation for error accounting, not introducing new singularities.

A5 (de Branges–Kreĭn Interface) When needed, invoke standard structures of de Branges spaces and canonical systems (kernel, ordering theorem, spectral measure and Hamiltonian duality).

A6 (Notation Convention) Write $A\lesssim B$ to mean there exists constant $C>0$ (independent of main variables, window scale R, sampling step Δ) such that $A\leq CB$; write $A\simeq B$ to mean both $A\lesssim B$ and $B\lesssim A$ hold.

2 Main Theorem I — Noether-Type Information Conservation (2D Flux Continuity Equation)

Theorem 2.1 (Flux Conservation and Point Source Counting)**.** *Let Λ be meromorphic in domain $\mathcal{D}$, $u(\sigma,\omega) = \log|\Lambda(\sigma + i\omega)|$, $\mathcal{J} = \partial_\omega u$, $\mathcal{H} = \partial_\sigma u$. Denote zero and pole sets as $\mathcal{Z}$, $\mathcal{P}$.*

(i) On $\mathcal{D} \setminus (\mathcal{Z} \cup \mathcal{P})$, $\partial_\omega \mathcal{J} + \partial_\sigma \mathcal{H} = \Delta u = 0$.

(ii) In distributional sense

$$\Delta u = 2\pi \sum_{z\in\mathcal{Z}} m_z\, \delta(\cdot - z) - 2\pi \sum_{p\in\mathcal{P}} n_p\, \delta(\cdot - p), \tag{5}$$

where m_z, n_p are multiplicities of zeros and poles.

(iii) Taking a rectangle R with the critical line as one side,

$$\int_{\partial R} (\mathcal{H}, \mathcal{J}) \cdot n\, ds = 2\pi\big(N_{\mathcal{Z}}(R) - N_{\mathcal{P}}(R)\big), \tag{6}$$

thus the interval integral along $\sigma = \frac{1}{2}$ equals "boundary normal flux + endpoint EM correction + point source counting" sum.

Proof. (i) Complex harmonicity: in source-free domain, $\log\Lambda$ is holomorphic, $u = \Re\log\Lambda$ is harmonic. (ii) Distributional source term: Laplacian of logarithmic singularity yields point mass. (iii) Green's identity: converts interior sources to boundary integral; line integral via finite-order EM gives endpoint correction; Poisson/EM tools for non-asymptotic closure of sum–integral difference. □

3 Main Theorem II — CCS Consistency: $-\frac{1}{\pi}\delta' = \xi' = \operatorname{tr}(\rho - \rho_0)$

Theorem 3.1 (Phase Density = Spectral Shift Derivative = Relative Spectral Density; Sign Unification)**.** *Let (H_0, H) be a self-adjoint scattering pair, $S(E)$ the scattering matrix (multi-channel with trace). Then almost everywhere*

$$-\frac{1}{\pi}\,\delta'(E) = \xi'(E) = \operatorname{tr}\big(\rho(E) - \rho_0(E)\big). \tag{7}$$

Proof. (a) Herglotz–Weyl identification: boundary imaginary part of m yields $\rho(E) = \frac{1}{\pi}\Im m(E + i0)$ (a.e.), similarly ρ_0. (b) Birman–Kreĭn and spectral shift: from $\det S(E) = e^{-2\pi i\xi(E)}$, $\xi'(E) = -\frac{1}{2\pi i}\partial_E \log\det S(E)$. Under trace-class assumptions, $\xi'(E) = \operatorname{tr}(\rho - \rho_0)(E)$. (c) Wigner–Smith delay: $Q(E) = -i\, S(E)^*\, \partial_E S(E)$, and $\partial_E \log\det S(E) = \operatorname{tr}(S^{-1}S') = i\ \operatorname{tr} Q(E)$, thus $\xi'(E) = -\frac{1}{2\pi}\operatorname{tr} Q(E)$. Single-channel $S = e^{2i\delta}$ gives $\operatorname{tr} Q(E) = 2\delta'(E)$, hence $-\frac{1}{\pi}\delta'(E) = \xi'(E)$. □

Remark 3.2. The above formula is the core of this paper's unified scale: scattering phase derivative, spectral shift function, and relative state density are equivalent under a sign-carrying relation; absolute spectral density $\rho(E) \geq 0$ gives positive weight measure, distinguished from relative density $\rho_{\text{rel}} = \xi' = -\frac{1}{\pi}\delta'$.

4 Main Theorem III — Non-Asymptotic Error Closure: Nyquist–Poisson–EM Three-Fold

Terminology and Scale (E-Domain) Take Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(E)\, e^{-i\xi E}\, dE, \qquad (h \star \rho)(E) = \int_{\mathbb{R}} h(E-t)\, \rho(t)\, dt.$$

Call g **bandlimited to** $[-B, B]$ if $\operatorname{supp}\widehat{g} \subset [-B, B]$; call **nearly bandlimited** to $[-B, B]$ if $\int_{|\xi|>B} |\widehat{g}(\xi)|^2\, d\xi$ is sufficiently small.

Below we stipulate $w_R(E) = w(E/R)$ and $E_n = E_0 + n\Delta$, where $w \in \mathcal{S}(\mathbb{R})$ is fixed.

Regularity Premise: Take $w \in \mathcal{S}(\mathbb{R})$; let kernel $h \in W^{2p,1}(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\widehat{h} \in L^1(\mathbb{R})$; absolute spectral density $\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$; accordingly $f(E) = w_R(E)\,(h \star \rho)(E) \in C^{2p}(\mathbb{R}) \cap L^1(\mathbb{R})$, with derivatives up to order $(2p-1)$ bounded and integrable at endpoints, $\widehat{h \star \rho} \in L^1(\mathbb{R})$. Under this premise, Theorem 3.1's EM remainder and aliasing upper bounds hold.

Theorem 4.1 (Three-Fold Decomposition Upper Bound; Symmetric Truncation Version)**.** *For any window w_R, kernel h, sampling step Δ, truncation radius $T > 0$, EM order p, there exists constant C such that*

$$\left| \int_{|E-E_0|\le T} w_R\big(h\star\rho\big)(E)\, dE - \Delta \sum_{|n-n_0|\le T/\Delta} w_R\big(h\star\rho\big)(E_n) \right| \le \varepsilon_{\mathrm{alias}} + \varepsilon_{\mathrm{EM}}^{(p)} + \varepsilon_{\mathrm{tail}}(T, R). \quad (8)$$

If $\widehat{h \star \rho}$ is bandlimited to $[-B, B]$ and $\Delta \le \pi/B$ then $\varepsilon_{\mathrm{alias}} = 0$; if only nearly bandlimited,

$$\varepsilon_{\mathrm{alias}} \le \frac{1}{\Delta} \sum_{m \ne 0} \int_{|\xi - 2\pi m/\Delta| > B} |\widehat{h \star \rho}(\xi)|\, d\xi.$$

$\varepsilon_{\mathrm{EM}}^{(p)}$ is given by finite-order EM's Bernoulli layer explicit upper bound; $\varepsilon_{\mathrm{tail}}(T, R)$ is caused only by $|E - E_0| > T$ and window w_R's decay, controllable by

$$\int_{|E-E_0|>T} |w_R(h \star \rho)(E)|\, dE + \Delta \sum_{|n-n_0|>T/\Delta} |w_R(h \star \rho)(E_n)|.$$

Proof. (i) Poisson summation: for equispaced sampling grid $E_n = E_0 + n\Delta$, discrete summation and Poisson summation formula yield periodized spectral superposition; under Nyquist, bands do not overlap, aliasing term is zero. (ii) Finite-order EM: sum–integral difference endpoint correction given by Bernoulli polynomial layer, remainder $R_p = \mathcal{O}(\Delta^{2p})$, constant depends only on p and several bounded derivatives. (iii) Symmetric truncation tail: constituted by integral and summation contributions from $|E - E_0| > T$ region; under symmetric window, w_R's decay and $h \star \rho$'s out-of-band energy upper bound give explicit control. Sum of three terms yields result. □

5 Sampling–Interpolation–Stability (Phase/Spectral Density Scale)

The following workspace is a reproducing kernel Hilbert space (such as de Branges space obtained via §A5 interface), so point evaluation functionals are continuous, $|f(E_n)|$ well-defined and satisfies standard kernel estimates.

Definition 5.1 (Density and Stability in Phase Coordinate v_ϕ)**.** Write sampling points E_n's phase coordinate $v_n = v_\phi(E_n) = \delta(E_n)/\pi$. For any $R > 0$ and $v \in \mathbb{R}$, let $I(v, R) = [v - R, v + R]$.

$$\underline{D}_\phi = \liminf_{R\to\infty} \inf_{v\in\mathbb{R}} \frac{\#\{n : v_n \in I(v, R)\}}{2R}, \quad \overline{D}_\phi = \limsup_{R\to\infty} \sup_{v\in\mathbb{R}} \frac{\#\{n : v_n \in I(v, R)\}}{2R}.$$

Call $\{E_n\}$ a **stable sampling sequence** if there exist constants $A, B > 0$ such that for every f in the workspace

$$A\|f\|^2_{L^2(d\mu)} \le \sum_n |f(E_n)|^2 \le B\|f\|^2_{L^2(d\mu)}.$$

Call $\{E_n\}$ an **interpolation sequence** if for any $\{c_n\} \in \ell^2$ there exists f with $f(E_n) = c_n$ and $\|f\|_{L^2(d\mu)} \lesssim \|\{c_n\}\|_{\ell^2}$.

Theorem 5.2 (Landau Necessary Density, Unit Bandwidth Scale)**.** *Via §A3's isometry, embed workspace into* $\mathrm{PW}_{1/2}$ *(Fourier support in* $[-1/2, 1/2]$*). If node set is a stable sampling sequence, then lower density* $\underline{D}_\phi \ge 1$*; if interpolation sequence, then upper density* $\overline{D}_\phi \le 1$*.*

Proof. After isometry, problem reduces to non-uniform sampling of $\mathrm{PW}_{1/2}$, threshold constant is 1. □

Theorem 5.3 (Parseval Tight Frame Necessary and Sufficient: Shift-Invariant vs. Gabor/WR)**.** ***(A) Shift-Invariant (Translation Only):*** *System* $\{(w_\alpha(E - n\Delta))_{\alpha,n}\}$ *is a Parseval tight frame if and only if*

$$\Delta^{-1} \sum_{\alpha=1}^{r} \sum_{m\in\mathbb{Z}} \left|\widehat{w_\alpha}\big(\xi + 2\pi m/\Delta\big)\right|^2 \equiv 1 \quad \textit{(a.e. } \xi\textit{)}.$$

If $\widehat{w_\alpha}$ *is bandlimited to* $[-B, B]$ *and* $\Delta \le \pi/B$ *(no aliasing), the above reduces to*

$$\Delta^{-1} \sum_{\alpha=1}^{r} \left|\widehat{w_\alpha}(\xi)\right|^2 \equiv 1.$$

(B) Gabor (Translation+Modulation, Wexler–Raz): *System* $\{e^{ik\Omega E}\, w_\alpha(E - n\Delta)\}_{\alpha,n,k}\}$ *is a Parseval tight frame if and only if (Wexler–Raz identity)*

$$\sum_{\alpha=1}^{r} \sum_{m,k\in\mathbb{Z}} \widehat{w_\alpha}\left(\xi + \tfrac{2\pi m}{\Delta}\right) \overline{\widehat{w_\alpha}\left(\xi + \tfrac{2\pi(m+\ell)}{\Delta}\right)}\, e^{ik\Omega\Delta\,\ell} = \Delta\Omega\, \delta_{\ell,0} \quad (\forall\, \ell \in \mathbb{Z}),$$

in particular at critical density $\Delta\Omega = 2\pi$ *with total fold to constant* 1 *reduces to Parseval condition.*

Proof. (A) Shift-invariant system's Parseval condition given by Calderón/Walnut representation; (B) Wexler–Raz identity gives frequency-domain pointwise orthogonality and Parseval necessary and sufficient; via $u = \log t$ and log-frequency variable transformation transplants to Mellin model. □

Theorem 5.4 (Balian–Low Impossibility: Mellin/Weyl Version)**.** *At critical density* $D = 1$ *with single window well-localized in both* u, ω *directions, the system generated by this window and critical lattice cannot be a Riesz basis; to obtain a basis, must relax at least one-side localization or adopt oversampling.*

Proof. Via §A3 isometry, reduce problem to standard Gabor lattice BLT (Riesz/ONB version), conclusion follows immediately. □

6 de Branges–Kreĭn Interface and "Phase Equidistant" Sampling

Definition 6.1 (Doubling Measure)**.** Write $\mu(I) = \int_I \rho(E)\, dE$. For any bounded open interval $I \subset \mathbb{R}$, denote $2I$ as the interval **with same center and doubled length**. If there exists constant $C_d \geq 1$ such that

$$\mu(2I) \leq C_d\, \mu(I) \quad \text{for all } I,$$

then call μ doubling. This, combined with reproducing-kernel diagonal estimates, allows matching local sampling spacing with ρ, thus supporting Proposition 5.1's stability conclusion.

Proposition 6.2 (Stable Frame with Phase Equidistance $\Delta\delta = \pi$)**.** *If spectral density/phase measure is "doubling", choose multi-windows with non-overlapping frequency bands such that Calderón sum is constant* 1*, and let sampling points satisfy*

$$\delta(x_{k+1}) - \delta(x_k) = \pi,$$

then obtain stable sampling frame; in strict equidistance case, Parseval tight frame. Proof relies on reproducing-kernel diagonal formula and scale consistency of relative density.

Proof Sketch. In de Branges space, kernel diagonal consistency with measure and canonical system's spectral correspondence give intrinsic relation between local sampling length and ρ; when matching kernel trace density with phase counting, use $\rho_{\text{rel}} = \xi' = -\frac{1}{\pi}\delta'$ to give sampling density scale; stability estimate's inner product and kernel diagonal always proceed under positive weight $d\mu = \rho\, dE$. □

7 Parallel and Inheritance with "Fractal Mirror (FMU)"

FMU has shown: multiplicatively self-similar signals exhibit "envelope × equidistant frequency shift array" in Mellin domain, with Bessel bound and unconditional convergence under weighted ℓ^2. UMMIC uses §A3's isometry to merge FMU's frequency–scale geometry with this paper's phase coordinate v_ϕ and density measure $d\mu$, making Landau/WR/BLT criteria and Nyquist–Poisson–EM three-fold decomposition close on the same coordinate, directly translatable to window/kernel design and error accounting.

8 Proof Tools and Minimal Sufficient Premises (Index Style)

- **Mellin isometry/scaling and log variable:** Isometry on $\frac{1}{2} + i\mathbb{R}$ and "scaling ↔ frequency shift" law.
- **Herglotz representation and spectral density:** $\rho = \frac{1}{\pi}\Im m$ (a.e.) and consistency of $d\mu = \rho\, dE$.
- **Poisson and Euler–Maclaurin:** For non-asymptotic accounting of sum–integral difference, aliasing and endpoint correction.

- **Birman–Kreĭn + Wigner–Smith:** $\det S = e^{-2\pi i\xi}$, $Q = -iS^*S'$, $\xi' = -\frac{1}{2\pi}\operatorname{tr} Q$.
- **Landau necessary density:** Sampling/interpolation threshold for Paley–Wiener spaces.
- **Wexler–Raz and BLT:** Tight/dual necessary and sufficient and critical density obstruction.
- **de Branges structure:** Dictionary of kernel, measure, and canonical system.

9 Verifiable Predictions and Engineering Interface (Minimal Experimental Template)

P1 — Flux Closure: In selected working band Ω, compute $\int_\Omega \partial_\omega \log|\Lambda(\frac{1}{2}+i\omega)|\,d\omega$, use Poisson+finite-order EM to give error three-fold account, verify constant-level closure.

P2 — Phase Scale Sampling Threshold: Evaluate $\underline{D}_\phi, \overline{D}_\phi$ in v_ϕ coordinate and observe threshold transition (Landau) from undersampling to reconstructible near critical.

P3 — WR-Parseval Design: Per WR condition solve windows jointly such that $\sum_\alpha |\widehat{w_\alpha}|^2$ (with folding) is constant 1, if aliasing use "total fold" formula.

P4 — Delay–Density Consistency: Numerically construct $S(E)$, compute $Q = -iS^*S'$ and phase of $\det S$, verify $\xi'(E) = -\frac{1}{2\pi}\operatorname{tr} Q(E) = -\frac{1}{\pi}\delta'(E)$ consistency with $\rho - \rho_0$.

10 Conclusion

This paper closes "mother map–Mellin–de Branges–scattering–frame/sampling–error theory" under the unified parallel framework of positive weight measure $d\mu = \rho\, dE$ and phase coordinate $v_\phi = \delta/\pi$ into a non-asymptotic and engineering-realizable theoretical framework: (I) $\nabla\cdot(\partial_\sigma u, \partial_\omega u)$ is zero in source-free domain, distributional source terms with zeros/poles yield flux counting identity, all boundary/endpoint costs packaged by **finite-order EM**; (II) $-\frac{1}{\pi}\delta' = \xi' = \operatorname{tr}(\rho-\rho_0)$ compresses scattering phase, spectral shift and state density to same scale; (III) **Nyquist–Poisson–EM** three-fold decomposition gives non-asymptotic error closure; (IV) **Landau/WR/BLT** on v_ϕ coordinate provide complete boundary for sampling–reconstruction–stability; (V) align term-by-term with de Branges structure, Weyl–Titchmarsh dictionary and FMU's frequency–scale geometry, thus landing directly on window/kernel design, spectral readout and delay measurement.

Appendix A: Common Criteria and Formulas (For Invocation)

A.1 Poisson Summation (Simple Type) $\displaystyle\sum_{n\in\mathbb{Z}} f(n\Delta) = \frac{1}{\Delta}\sum_{m\in\mathbb{Z}} \widehat{f}\Big(\frac{2\pi m}{\Delta}\Big)$. Under bandwidth limitation and $\Delta \le \pi/B$, aliasing shuts off.

A.2 Euler–Maclaurin (Finite-Order Version) $\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\big(f^{(2k-1)}(b) - f^{(2k-1)}(a)\big) + R_p$, with explicit upper bound for R_p.

A.3 Wigner–Smith Delay Matrix (Unified Notation) $Q(E) = -i\,S(E)^*\,\partial_E S(E)$, $\operatorname{tr} Q(E) = 2\delta'(E)$ (single-channel), and

$$\xi'(E) = -\frac{1}{2\pi}\operatorname{tr} Q(E).$$

A.4 Birman–Kreĭn Formula (Supplemented with Logarithmic Derivative) $\det S(E) = e^{-2\pi i \xi(E)}$, thus

$$\partial_E \log\det S(E) = -2\pi i\,\xi'(E), \qquad \xi'(E) = \operatorname{tr}\big(\rho - \rho_0\big)(E).$$

A.5 de Branges Space and Canonical System Correspondence of kernel and measure, subspace total order and canonical system's Hamiltonian dictionary.

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