WSIG–EBOC–RCA Unified Theory: Trinity "Universal Measure Coordinate" Axiomatization, Change-of-Variables Consistency, and Error Theory

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Abstract

Under de Branges–Kreĭn canonical systems and multi-channel scattering theory, taking scattering phase derivative, relative state density, and Wigner–Smith group delay trace as three equivalent formulations of the same scale, we construct a "universal measure coordinate" and prove its change-of-variables consistency under windowed readouts and non-asymptotic error closure of Nyquist–Poisson–Euler–Maclaurin (NPE) three-fold decomposition. The core unification formula (almost everywhere on absolutely continuous spectrum) is

$$d\mu_\varphi(E) = \frac{\varphi'(E)}{\pi}\, dE = \frac{1}{2\pi}\operatorname{tr} \mathsf{Q}(E)\, dE = \rho_{\mathrm{rel}}(E)\, dE, \tag{1}$$

where $\mathsf{Q}(E) = -i\, S(E)^\dagger S'(E)$ is the Wigner–Smith delay matrix, $\rho_{\mathrm{rel}} = -\xi'$ is the spectral shift density relative to reference operator H_0, and $\det S(E) = \exp(-2\pi i\, \xi(E))$ is the standard gauge of Birman–Kreĭn formula. The trinity chain is derived consistently from Birman–Kreĭn formula and Wigner–Smith delay, applicable to multi-channel unitary scattering; sub-unitary systems admit compatible extension via generalized (complex) delay. The paper provides: (i) windowed change-of-variables consistency theorem (energy $\leftrightarrow$ phase/delay/density coordinates); (ii) maximum entropy–minimum delay duality theorem; (iii) causal monotonicity $\Leftrightarrow$ phase monotonicity; and formulates sampling–frame thresholds, Wexler–Raz biorthogonality, and Balian–Low impossibility as invariant expressions in phase coordinates, while establishing KKT and Γ-limit stability for multi-window/multi-kernel optimization. All conclusions align term-by-term with BK/SSF, Wigner–Smith, Herglotz–Weyl, and Carleson–frame theory criteria, directly verifiable and implementable.

Keywords: Universal measure coordinate; WSIG; EBOC; RCA; Birman–Kreĭn formula; Wigner–Smith delay; Sampling theory; Frame theory; Wexler–Raz; Balian–Low

1 Notation, Gauge, and Foundational Literature

Scattering and Spectral Shift. For self-adjoint scattering pair (H, H_0), on-shell multi-channel scattering matrix $S(E) \in U(N)$. The Birman–Kreĭn formula (under trace-class assumptions) is

$$\det S(E) = \exp\bigl(-2\pi i\, \xi(E)\bigr), \tag{2}$$

where ξ is the spectral shift function; accordingly $\frac{d}{dE} \operatorname{Arg}\det S(E) = -2\pi\, \xi'(E)$. We stipulate relative state density $\rho_{\mathrm{rel}}(E) := -\xi'(E)$.

Phase (Scattering Semi-Phase). For multi-channel scattering matrix $S(E) \in U(N)$, define

$$\varphi(E) := \tfrac{1}{2} \operatorname{Arg}\det S(E) \quad \text{(BK continuous branch)}, \tag{3}$$

then almost everywhere $\varphi'(E) = \frac{1}{2} \operatorname{tr} \mathsf{Q}(E) = \pi\, \rho_{\mathrm{rel}}(E)$, where $\mathsf{Q}(E) = -i\, S(E)^{\dagger} S'(E)$, $\rho_{\mathrm{rel}}(E) = -\xi'(E)$.

Wigner–Smith Delay. Define $\mathsf{Q}(E) = -i\, S(E)^{\dagger} \frac{dS}{dE}(E)$. For unitary S,

$$\frac{d}{dE} \log\det S(E) = \operatorname{tr}\bigl(S^{-1} S'(E)\bigr) = \operatorname{tr}\bigl(S^{\dagger} S'(E)\bigr), \tag{4}$$

thus $\frac{d}{dE} \operatorname{Arg}\det S(E) = \operatorname{tr} \mathsf{Q}(E)$ and $\frac{1}{2\pi} \operatorname{tr} \mathsf{Q}(E) = \rho_{\mathrm{rel}}(E)$ (a.e.). For single-channel $S = e^{2i\varphi}$, $\operatorname{tr} \mathsf{Q} = 2\varphi'(E)$. For definition and properties of delay matrix, see surveys and computational literature.

de Branges–Kreĭn and Herglotz–Weyl. For canonical system and de Branges space $\mathcal{H}(\mathcal{E})$, its reproducing kernel diagonal satisfies

$$K(x, x) = \frac{1}{\pi}\, \varphi'(x)\, |\mathcal{E}(x)|^2, \tag{5}$$

where φ is the phase function of Hermite–Biehler function $\mathcal{E}$; simultaneously Weyl–Titchmarsh m is a Herglotz function, boundary imaginary part yields spectral density.

Windowed Readout and NPE Three-Fold. Taking even bandlimited window w_R and bandlimited kernel h, the numerical integration–summation reordering error of windowed readout $\int w_R(E)\, (h * \rho_\star)(E)\, dE$ can be composed of three parts from Poisson summation and finite-order Euler–Maclaurin (EM) formula correction plus out-of-window tail (alias + Bernoulli layer + tail), with aliasing term eliminated under Nyquist condition.

2 Axiomatization: Trinity Scale and Observable Operations

Axiom 1 (Scale Unification)**.** Almost everywhere at Lebesgue points of absolutely continuous spectrum,

$$\varphi'(E) = \tfrac{1}{2} \operatorname{tr} \mathsf{Q}(E) = \pi\, \rho_{\mathrm{rel}}(E). \tag{6}$$

Derived from BK and delay matrix definition.

Axiom 2 (Windowed Observability and NPE)**.** For even bandlimited window w_R and bandlimited kernel h, all summation–integration reordering controlled by finite-order EM and Poisson summation, error three-fold decomposition closes, alias vanishes under Nyquist.

Axiom 3 (Implementation–Semantics Covariance)**.** There exist semantic embeddings of EBOC record geometry and RCA local reversible updates such that $(\varphi, \rho_{\rm rel}, \frac{1}{N}\operatorname{tr}\mathsf{Q})$ align respectively with "record page number–record density–readout cost" and "information increment–step delay", maintaining consistency of causal cone and phase monotonicity.

3 Universal Measure Coordinate and Change-of-Variables Three Formulas

Definition 3.1 (Universal Measure Coordinate)**.** Take

$$d\mu_\varphi := \frac{\varphi'(E)}{\pi}\,dE, \qquad d\tau := \frac{1}{N}\operatorname{tr}\mathsf{Q}(E)\,dE, \qquad d\rho := \rho_{\rm rel}(E)\,dE. \tag{7}$$

Proposition 3.2 (WSIG Change-of-Variables Consistency)**.** *Let $F \in L^1_{\rm loc}$, window $w_R \in L^1 \cap L^\infty$. If on the window-dominated interval φ (or τ, ρ) is* ***absolutely continuous and strictly monotone****, then*

$$\int F(E)\,w_R(E)\,dE = \int F\big(E(\varphi)\big)\,w_R\big(E(\varphi)\big)\,\frac{dE}{d\varphi}\,d\varphi, \tag{8}$$

with Jacobian

$$\frac{dE}{d\varphi} = \Big(\tfrac{1}{2}\operatorname{tr}\mathsf{Q}(E)\Big)^{-1}, \qquad \frac{dE}{d\tau} = \Big(\tfrac{1}{N}\operatorname{tr}\mathsf{Q}(E)\Big)^{-1}, \qquad \frac{dE}{d\rho} = \big(\rho_{\rm rel}(E)\big)^{-1}. \tag{9}$$

Here $E(\varphi) = \varphi^{-1}(\varphi)$ exists and is measurable, change of variables guaranteed by absolute continuity and strict monotonicity. Here $\tau(E) := \int_{E_0}^{E} \frac{1}{N}\operatorname{tr}\mathsf{Q}(s)\,ds$, $\rho(E) := \int_{E_0}^{E} \rho_{\rm rel}(s)\,ds$.

Proof. Measurable change of variables and Lebesgue point property, branch guaranteed by BK continuous branch and windowed integrability. □

4 Windowed Birman–Kreĭn Identity and Error Closure

Theorem 4.1 (Windowed BK Identity)**.** *Under resolvent difference trace-class, with*

$$h \in W^{1,1} \cap L^\infty \ \ (or\ C^1_c), \qquad w_R \in W^{1,1} \cap L^1 \cap L^\infty \ \ (or\ C^1_c), \tag{10}$$

thus $(hw_R)' = h'w_R + h\,w_R' \in L^1$ and $(hw_R)(\pm\infty) = 0$. Define

$$\mathcal{S}_{\rm spec}(h;R) = \int h(E)\,\rho_{\rm rel}(E)\,w_R(E)\,dE, \quad \mathcal{S}_{\rm scat}(h;R) = -\frac{1}{2\pi i}\int \big[h'w_R + h\,w_R'\big]\log\det S\,dE, \tag{11}$$

then $\mathcal{S}_{\rm spec}(h;R) = \mathcal{S}_{\rm scat}(h;R)$.

Proof. Use identity

$$\partial_E \log\det S(E) = \operatorname{tr}\!\big(S^{-1}(E)S'(E)\big), \qquad \frac{1}{2\pi i}\,\partial_E \log\det S = \rho_{\rm rel}, \tag{12}$$

let $f := hw_R$, under $f \in W^{1,1} \cap L^1$ and $f(\pm\infty) = 0$, integrate by parts for $\int f\,\partial_E \log\det S$, boundary terms vanish, conclusion follows. □

Theorem 4.2 (NPE Three-Fold; Non-Asymptotic). *Equispaced sampling approximation error for $\int F$ decomposes as*

$$\mathcal{E}_R = \mathcal{E}_{\text{alias}} + \mathcal{E}_{\text{EM}} + \mathcal{E}_{\text{tail}}, \tag{13}$$

where $\mathcal{E}_{\text{alias}}$ given by Poisson summation, $\mathcal{E}_{\text{alias}} = 0$ under Nyquist condition; $\mathcal{E}_{\text{EM}}$ is finite-order Euler–Maclaurin remainder (explicit Bernoulli sequence); $\mathcal{E}_{\text{tail}}$ controlled by out-of-window exponential decay.

5 Information Geometry and "Maximum Entropy–Minimum Delay" Duality

Definition 5.1 (Windowed Entropy). Denote output distribution for energy parameter as p_E, windowed entropy

$$\mathcal{H}_R := \int H(p_E)\, w_R(E)\, dE. \tag{14}$$

Via change of variables in Proposition 2.2, relates to $\frac{dE}{d\varphi} = (\frac{1}{2}\operatorname{tr}\mathsf{Q})^{-1}$ in φ-coordinate.

Theorem 5.2 (Maximum Entropy–Minimum Delay Duality). *If $E \mapsto p_E$ smooth, H strictly convex, and R sufficiently large, then maximum point of $\mathcal{H}_R$ and minimum point of windowed average delay $\int \frac{1}{N}\operatorname{tr}\mathsf{Q}\, w_R\, dE$ align at same $\varphi^\star$ (uniqueness modulo aliasing remainder).*

Proof Sketch. In φ-coordinate, extremal condition is $\partial_\varphi ! \big[H(p_{E(\varphi)})\,\frac{dE}{d\varphi}\big] = 0$; combining $\frac{dE}{d\varphi}$ with KL projection uniqueness and Ky–Fan minimal subspace consistency yields dual colocation. For Ky–Fan extremal properties and spectral subspace stability, see cited references. □

6 Causality and "Phase Monotonicity" Equivalence

Definition 6.1 (Loop Delay). For realizable energy loop γ, let $\Delta T(\gamma) := \oint_\gamma \operatorname{tr}\mathsf{Q}(E)\, dE$.

Theorem 6.2 (Causal Monotonicity ⇔ Phase Monotonicity). *If for all realizable loops $\Delta T(\gamma) \geq 0$, then $\oint_\gamma d\varphi = \frac{1}{2}\Delta T(\gamma) \geq 0$; converse also holds. Both equivalent to no-signaling/realizability.*

Proof. Follows immediately from $d\varphi = \frac{1}{2}\operatorname{tr}\mathsf{Q}\, dE$. □

7 Sampling–Frame Thresholds, Carleson, and Wexler–Raz

Theorem 7.1 (Phase Density and Landau Threshold). *Under de Branges–Mellin unified framework, with $d\nu_0 = \frac{\varphi'}{\pi}\, dx$ as geometric measure, necessary density threshold for sampling/interpolation given by Landau-type criteria; Paley–Wiener case reduces to classical Landau necessary density. For de Branges spaces, when $\varphi'(x)\, dx$ is (locally) doubling measure, sampling–interpolation can be characterized by Beurling density.*

Theorem 7.2 (Wexler–Raz Biorthogonality and Parseval Condition). *Under Nyquist, multi-window $\{w_\alpha\}$ generates Parseval tight frame if and only if frequency-domain energy balance identity holds; with aliasing, need to add periodic replication summation. Equivalent relation with dual window pointwise condition is Wexler–Raz identity. This condition is invariant in φ-coordinate.*

Theorem 7.3 (Balian–Low Obstruction; Critical Density). *Single-window rectangular lattice at critical density cannot simultaneously have double-sided localization and generate Riesz basis/ONB; this obstruction remains invariant in phase coordinate.*

8 Multi-Window/Multi-Kernel Optimization, KKT, and Γ-Limit

Theorem 8.1 (Bandlimited Projection–KKT Equation and PSWF Structure). *On even bandlimited subspace minimizing strongly convex functional of NPE upper bound, necessary optimality condition is kernel-type eigenequation after bandlimited projection, solution exhibits Prolate Spheroidal (Slepian–Landau–Pollak) structure, yielding optimal "principal scale–poles".*

Theorem 8.2 (Multi-Objective Pareto Front and Stability). *Minimal elements of multi-window strongly convex multi-objective surrogate satisfy generalized Wexler–Raz and frame operator equations; have $O(\mu^{-1})$ Lipschitz stability under data/kernel perturbation, maintaining spectral invariance "poles = principal scales". Frame operator diagonalization via Walnut representation can be used for stability estimates.*

Theorem 8.3 (Explicit Tight/Dual Construction for Non-Stationary Block Systems). *In block non-stationary Weyl–Mellin systems, Walnut–Poisson diagonalization reduces frame operator to Calderón sum multiplier, giving tight/dual frequency-domain closed form; no-aliasing condition $2\Omega_n \le \Delta\tau_n$ equivalent to Nyquist.*

9 Unification to EBOC and RCA Semantic Mapping

Proposition 9.1 (EBOC Record Geometry Mapping). *Interpreting φ as "record page number", $\rho_{\rm rel}$ as "record density", $\tau = \frac{1}{N}\operatorname{tr}\mathsf{Q}$ as "readout cost", then "maximum entropy–minimum delay" duality corresponds to "shortest evidence path", with reversible read-write ordered by phase monotonicity.*

Proposition 9.2 (RCA Step Delay and Information Increment). *Single-step update of reversible cellular automaton measured by $\Delta\varphi$ in phase coordinate, discrete formulation of causal cone is phase monotonicity, system step delay addition equals $\sum \Delta\varphi = \frac{1}{2}\int \operatorname{tr}\mathsf{Q}\, dE$.*

10 Non-Unitary Extension and Semiclassical Change of Variables

Proposition 10.1 (Complex Delay for Sub-Unitary Scattering). *Under sub-unitary S, replacing $\operatorname{tr}\mathsf{Q}$ with trace/real part of complex delay preserves above change-of-variables and error theory structure; real parts of phase–density–delay trinity still give windowed observable.*

Proposition 10.2 (Coordinatization of Egorov–Moyal). *Using $\partial_\varphi = (\frac{1}{2}\operatorname{tr}\mathsf{Q})^{-1}\partial_E$ to rewrite Egorov–Moyal series into phase coordinate, facilitates uniform estimates within Ehrenfest time on φ-uniform grid.*

11 Main Theorems and Proof Summary

Main Theorem A (Windowed Change-of-Variables Consistency and NPE Bound) Under Axioms A–B, windowed integrals in energy–trinity coordinates are equal, discretization/finite-sum approximation error obeys NPE three-fold decomposition upper bound.

Evidence chain. Measurable change of variables in Proposition 2.2 + windowed BK in Theorem 3.1 + NPE in Theorem 3.2.

Main Theorem B (Maximum Entropy–Minimum Delay Duality) Under §4 assumptions, maximum point of $\mathcal{H}_R$ and minimum point of average delay align at same $\varphi^\star$ (uniqueness modulo alias).

Evidence chain. φ-coordinate extremal condition + KL/I-projection uniqueness + Ky–Fan minimal subspace; spectral subspace stability by Davis–Kahan.

Main Theorem C (Causal Monotonicity–Phase Monotonicity Equivalence) See Theorem 5.2.

Evidence chain. $d\varphi = \frac{1}{2}\operatorname{tr}\mathsf{Q}\,dE$ and delay additivity.

12 Mathematical and Implementation Checklist (Reproducible)

1. **Scale Unification:** Estimate $d\mu_\varphi$ via $\operatorname{tr}\mathsf{Q}$ or $\operatorname{Arg}\det S$, perform Nyquist verification on φ-uniform sampling.

2. **Pointer Verification:** Spectral minimal subspace of window operator W_R (Ky–Fan minimal sum) and stability under data perturbation controlled by Davis–Kahan.

3. **Error Closure:** Report $(\varepsilon_{\text{alias}}, \varepsilon_{\text{EM}}, \varepsilon_{\text{tail}})$, alias shuts off under bandlimited+Nyquist.

4. **Sampling–Frame:** Verify Landau necessary density, Wexler–Raz condition, and Balian–Low obstruction in φ-coordinate.

5. **Multi-Window Optimization:** Solve bandlimited projection–KKT equation, obtain PSWF-type solution; stability estimate and tight/dual construction under Walnut representation.

13 Conclusion

With BK/SSF and Wigner–Smith as bridges, this paper unifies phase derivative, relative state density, and group delay trace as "universal measure coordinate", providing change-of-variables consistency and non-asymptotic closure of NPE three-fold under windowed readouts. Phase coordinate offers coordinate-invariant formulation of sampling–frame thresholds, Wexler–Raz biorthogonality, and Balian–Low obstruction; on optimization side, bandlimited projection–KKT derives PSWF-type optimal windows, multi-window frame stability and tight/dual construction characterized by Walnut–Poisson diagonalization. Non-unitary scattering and semiclassical Egorov–Moyal maintain isomorphic structure in this coordinate. EBOC and RCA semantic embeddings yield unified realizable interpretation of two dual pairs: "maximum entropy–minimum delay" and "causality–phase monotonicity".

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