WSIG Unified Measurement Framework (UMS)

Finite-Window Covariant "Scattering–Information–Geometry" Unified Theory (Formal Academic Paper with Complete Proofs)

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Abstract

This paper takes **phase–density scale**, **windowed readout** and **information geometry** as three main axes, welding "state–measurement–probability–pointer–scattering phase–group delay–sampling/frame–error theory–channel capacity" into verifiable **categorized** unified theory (UMS). Core unified formula adopts **three density expressions of same scale**:

$$\boxed{d\mu_\varphi(E) \;:=\; \frac{\varphi'(E)}{\pi}\,dE \;=\; \frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E)\,dE \;=\; \rho_{\mathrm{rel}}(E)\,dE} \quad \text{(a.e. on a.c. spectrum)}$$

(Above formula holds a.e. on a.c. spectrum, $\operatorname{Arg}\det S$ takes **continuous branch**; across threshold/atomic points **spliced by jumps** via $\Delta\mu_\varphi = \mu_\varphi(\{E_*\})$, consistent with jumps of spectral shift function $\xi(E)$ and bound state/threshold multiplicity.)

where $\mathsf{Q}(E) = -i\,S^\dagger(E)\,S'(E)$ is Wigner–Smith group delay matrix, $\rho_{\mathrm{rel}}(E) = -\xi'(E)$ is Birman–Kreĭn spectral shift density (adopting convention $\det S(E) = e^{-2\pi i\,\xi(E)}$). In multi-channel case, total phase defined as $\varphi(E) := \frac{1}{2}\operatorname{Arg}\det S(E)$ (taking continuous branch).

Canonicalization note (unified): This paper views μ_φ as **locally finite signed Radon measure**, making Lebesgue decomposition $\mu_\varphi = \mu_\varphi^{\mathrm{ac}} + \mu_\varphi^{\mathrm{s}} + \mu_\varphi^{\mathrm{pp}}$, and Jordan decomposition $\mu_\varphi = \mu_+ - \mu_-$ ($\mu_\pm \geq 0$). When μ_φ is non-negative Borel measure satisfying Herglotz representation standard growth/integrability conditions (e.g., $\int(1+E^2)^{-1}\,d\mu_\varphi(E) < \infty$, excess absorbed into $a+bz$ term), exists Herglotz function m such that $\pi^{-1}\Im m(E+i0) = \rho_{\mathrm{rel}}(E)$ (a.e.), under proper normalization (eliminating $a+bz$ freedom) realized by **trace-normed** DBK canonical system for **global** representation; if μ_φ contains negative part, can only establish **local** representation on a.c. segments where $\rho_{\mathrm{rel}} > 0$.

This formula unifies scattering phase derivative, relative spectral density and group delay to same scale; measurement readouts expressed as **window–kernel** spectral integrals, with **Nyquist–Poisson–Euler–Maclaurin (EM)** three-term decomposition giving **finite-order, non-asymptotic** error closure; probability uniqueness guaranteed by Naimark dilation and Gleason theorem; sampling/frame thresholds characterized by Landau necessary density, Wexler–Raz biorthogonality and Balian–Low impossibility; open system information monotonicity and capacity upper bounds controlled by GKSL

master equation, quantum relative entropy monotonicity under **trace-preserving positive** maps (DPI) and HSW theorem.

Keywords: Scattering phase; Wigner–Smith group delay; Birman–Kreĭn; de Branges / Herglotz; Naimark dilation; Gleason; Landau density; Wexler–Raz; Balian–Low; Euler–Maclaurin; Poisson; GKSL; DPI; HSW.

1 Preliminaries and Notation

1.1 Scattering and Group Delay

Set $S(E) \in U(N)$ having **weak derivative** or **bounded variation** on a.c. spectrum, Wigner–Smith group delay matrix defined as $\mathsf{Q}(E) = -i\,S^\dagger(E)\,S'(E)$, where $S'(E)$ understood as **distributional derivative**. For unitary S have $S^\dagger S' = (S^{-1}S')$ anti-Hermitian, thus trace purely imaginary. By Jacobi formula $\frac{d}{dE}\log\det S = \operatorname{tr}(S^{-1}S')$, and $S^{-1} = S^\dagger$ (unitary), get $\frac{d}{dE}\operatorname{Arg}\det S = \Im\,\operatorname{tr}(S^\dagger S')$; thus

$$\operatorname{tr}\mathsf{Q}(E) = -i\,\operatorname{tr}\big(S^\dagger S'(E)\big) = \Im\,\operatorname{tr}\big(S^\dagger S'(E)\big) = \frac{d}{dE}\operatorname{Arg}\det S(E)\ \text{(continuous branch, a.e. on a.c. spectru}$$

single-channel $S(E) = e^{2i\varphi(E)}$ gives $\operatorname{tr}\mathsf{Q}(E) = 2\,\varphi'(E)$. Across threshold/atomic points not requiring everywhere differentiable, but spliced via jumps $\Delta\mu_\varphi$.

Notation convention: Below "a.c." denotes absolutely continuous spectral region; "a.e." means almost everywhere relative to Lebesgue measure. Domain of Q is a.e. point set of a.c. spectral region, outside this interval (thresholds, atomic points) described by jump part $\mu_\varphi^{\mathrm{pp}}$ of spectral measure.

Multi-channel scaled phase: Let $\varphi(E) := \frac{1}{2}\operatorname{Arg}\det S(E)$ (choose continuous branch, a.e. differentiable on a.c. spectrum), then

$$d\mu_\varphi(E) = \frac{\varphi'(E)}{\pi}\,dE = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)\,dE.$$

Single-channel degenerates to $S = e^{2i\varphi}$. $\operatorname{Arg}\det S$ takes locally continuous branch, only a.e. differentiable on a.c. spectrum; across thresholds and atomic points compensated by jumps.

1.2 Spectral Shift and Birman–Kreĭn

Adopt BK **negative sign convention**: $\det S(E) = e^{-2\pi i\,\xi(E)}$.

Determinant convention (BK/Fredholm): $\det S(E)$ in this paper refers to **Fredholm determinant in Birman–Kreĭn sense** $\det_{\mathrm{F}} S(E)$. Under premise $(H-z)^{-1} - (H_0 - z)^{-1} \in \mathfrak{S}_1$ (equivalently, for a.e. E, $S(E) - I \in \mathfrak{S}_1$), $\det_{\mathrm{F}} S(E)$ well-defined, $\operatorname{Arg}\det_{\mathrm{F}} S(E)$ can choose **continuous branch** and is **a.e. differentiable** on a.c. spectrum, thus

$$\frac{d}{dE}\operatorname{Arg}\det_{\mathrm{F}} S(E) = -2\pi\,\xi'(E), \qquad \rho_{\mathrm{rel}}(E) := -\xi'(E),$$

therefore $\frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E) = \rho_{\mathrm{rel}}(E)$ (a.e.).

1.3 DBK Canonical System and Herglotz Dictionary

One-dimensional de Branges–Kreĭn canonical system $JY'(t,z) = zH(t)Y(t,z)$'s Weyl–Titchmarsh function $m : \mathbb{C}^+ \to \mathbb{C}^+$ is Herglotz function, standard representation $m(z) = a + bz + \int \big(\frac{1}{t-z} - \frac{t}{1+t^2}\big)\, d\nu(t)$ where $d\nu \geq 0$ non-negative Borel measure satisfying $\int (1+t^2)^{-1}\, d\nu(t) < \infty$ (excess absorbed into $a+bz$ term), boundary imaginary part gives absolutely continuous spectral density $\rho_{\mathrm{ac}}(E) = \pi^{-1}\Im m(E+i0)$ (a.e.); under proper normalization (eliminating $a+bz$ freedom), **trace-normed** canonical system has **one-to-one and unique** (up to natural equivalence) correspondence with Herglotz function and de Branges space.

Signed measure case piecewise splicing: When μ_φ contains negative part (i.e., ρ_{rel} changes sign), can only construct trace-normed canonical system (H_j, J_j) and corresponding Herglotz function m_j on each a.c. segment I_j where $\rho_{\mathrm{rel}} > 0$, such that $\pi^{-1}\Im m_j(E+i0) = \rho_{\mathrm{rel}}(E)$ a.e. on I_j; segments spliced according to Lebesgue decomposition and Jordan decomposition of spectral measure μ_φ, uniqueness and consistency guaranteed by trace-normed canon **within each segment**, but **globally no single canonical system** Herglotz representation exists.

1.4 Sampling, Frames and Thresholds

Paley–Wiener class stable sampling/interpolation obeys Landau necessary density; Gabor system dual windows satisfy Wexler–Raz biorthogonality; critical density satisfies Balian–Low impossibility.

1.5 Measurement and Probability Uniqueness

Any POVM can be obtained by compression from PVM in larger space (Naimark dilation); when $\dim \mathcal{H} \geq 3$, probability measure satisfying additivity must be $\mathrm{Tr}(\rho\,\cdot)$ (Gleason theorem).

1.6 Open Systems and Information Bounds

Markovian open evolution described by GKSL (Lindblad) master equation; quantum relative entropy monotonically decreases under **trace-preserving positive** maps (DPI); unassisted classical capacity of quantum channel given by HSW regularized formula.

2 Axiom System

Axiom 2.1 (Dual Representation and Covariance)**.** *$\mathcal{H}(E)$ (energy representation) and $\mathcal{H}_a = L^2(\mathbb{R}_+, x^{a-1}dx)$ (phase–scale representation) isometrically equivalent; discrete–continuous reordering uses **finite-order** EM, controlling remainder under smoothness and (bounded or finite variation) premises. This "isometric equivalence" refers to unitary operator realization when DBK canonical system/Weyl–Mellin transform **already constructed**; readers should not interpret as unconditional isomorphism between arbitrary systems.*

Axiom 2.2 (Finite Window Readout)**.** *Any "realizable readout" written as window–kernel spectral integral $K_{w,h} = \int h(E)\, w_R(E)\, d\Pi_A(E)$. To ensure expectation value $\mathrm{Tr}(\rho K_{w,h})$ of $K_{w,h}$ on **all density operators** ρ well-defined and uniformly bounded, this paper **restricts** $g(E) := h(E)w_R(E) \in L^\infty(\mathbb{R};\mathbb{R})$ and Borel measurable, thus $K_{w,h}$ is **bounded self-adjoint** operator; error **non-asymptotically closed** by "alias (Poisson) + Bernoulli layer (EM) + truncation" three terms.*

Axiom 2.3 (Probability–Information Consistency). *For PVM $\{P_j\}$ and state ρ, linear constraint $p_j = \mathrm{Tr}(\rho P_j)$ makes feasible set single point $\{p^\star\}$, any strictly convex Bregman/KL I-projection uniquely taken at $p^\star$; POVM case first Naimark dilate to PVM then pushback. Gleason ($\dim \mathcal{H} \geq 3$) ensures uniqueness of this probability form.*

Axiom 2.4 (Pointer Basis). *"Pointer basis" defined as basis spanning **spectral projection subspace corresponding to minimal spectral value** of window operator $W_R = \int w_R \, d\Pi_A$ (Ky Fan "minimum sum"); if minimal spectral value not attained, take $\varepsilon \downarrow 0$ limit subspace. Existence and verifiability: if $w_R \in L^2(\mathbb{R})$ (e.g., finite support), combined with bandlimited projection Π_B, then $\Pi_B M_{w_R} \Pi_B$ is Hilbert–Schmidt/compact.*

Axiom 2.5 (Phase–Density–Delay Scale). *On a.c. spectrum a.e., have*

$$d\mu_\varphi(E) = \frac{\varphi'(E)}{\pi} \, dE = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}(E) \, dE = \rho_{\mathrm{rel}}(E) \, dE.$$

Negative group delay and spectral shift density sign change observable in multiple wave scattering classes.

Axiom 2.6 (Sampling and Frame Thresholds). *Paley–Wiener stable sampling/reconstruction obeys Landau necessary density $D \geq 1/(2\pi B)$; Gabor frame critical density $\alpha\beta = 1$ satisfies Balian–Low; dual windows satisfy Wexler–Raz biorthogonality.*

Axiom 2.7 (Open System Information Monotonicity). *GKSL master equation describes Markovian open dynamics; quantum relative entropy $D(\rho\|\sigma)$ monotonically decreases under trace-preserving completely positive (TPCP) maps (data processing inequality, DPI); unassisted classical capacity $C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n})$ (HSW regularized formula).*

3 Main Theorems

Theorem 3.1 (DBK Global Representation). *If measure μ_φ is non-negative Borel measure satisfying $\int (1+E^2)^{-1} \, d\mu_\varphi(E) < \infty$, then exists Herglotz function m and under proper normalization (trace-normed), exists **unique** (up to natural equivalence) de Branges–Kreĭn canonical system (H, J) such that*

$$\pi^{-1} \Im m(E + i0) = \rho_{\mathrm{rel}}(E) \quad \text{(a.e.)},$$

where $\rho_{\mathrm{rel}} = d\mu_\varphi^{\mathrm{ac}}/dE$ is absolutely continuous part density of μ_φ.

Proof. Standard Herglotz representation theory and trace-normed canonical system bijection. Under non-negativity and growth conditions, Herglotz representation well-defined, trace-normed normalization eliminates $a + bz$ freedom, yielding unique canonical system. See Remling, de Branges works for details. □

Theorem 3.2 (Signed Measure Local Representation). *If μ_φ contains negative part, let I_j ($j \in J$) be maximal a.c. intervals where $\rho_{\mathrm{rel}} > 0$. Then on each I_j, exists trace-normed canonical system (H_j, J_j) and Herglotz function m_j such that*

$$\pi^{-1} \Im m_j(E + i0) = \rho_{\mathrm{rel}}(E) \quad \text{(a.e. on } I_j\text{)}.$$

Globally, μ_φ expressed as piecewise splice of these local representations plus singular/atomic parts, but no single canonical system realizes global representation.

Proof. On each segment I_j where $\rho_{\text{rel}} > 0$, apply Theorem 3.1 locally. Uniqueness within segment guaranteed by trace-normed normalization. Segments spliced via Lebesgue and Jordan decompositions. Impossibility of global representation follows from non-negativity requirement of Herglotz measures. □

Theorem 3.3 (Windowed Readout Non-Asymptotic Error Closure)**.** *For window $w_R \in \mathrm{PW}_\Omega \cap L^\infty$, kernel $h \in \mathrm{PW}_\Omega \cap L^1$, and spectral measure $d\Pi_A$, windowed readout*

$$\mathrm{Tr}\big(\rho \int w_R(E)h(E)\,d\Pi_A(E)\big)$$

admits discrete approximation via sampling with error decomposition:

$$\mathit{Error} = \varepsilon_{\text{alias}} + R_m + \varepsilon_{\text{tail}},$$

where $\varepsilon_{\text{alias}} = 0$ under bandlimited+Nyquist conditions, R_m is EM remainder with explicit bound

$$|R_m| \le \frac{2\zeta(2m)}{(2\pi)^{2m}} \int |F^{(2m)}(x)|\,dx,$$

and $\varepsilon_{\text{tail}}$ controlled by truncation point selection based on decay rate.

Proof. Combine Poisson summation (alias term), finite-order Euler–Maclaurin (Bernoulli layer), and tail truncation. Under bandlimited assumption with sampling rate $f_s \ge 2B$, Poisson replicas separated, alias vanishes. EM remainder follows from AFP-Isabelle formalization. Tail controlled by function decay. □

Theorem 3.4 (Sampling Landau Density Threshold)**.** *For Paley–Wiener space PW_B of bandwidth B, necessary condition for stable sampling/interpolation is density*

$$D \ge \frac{1}{2\pi B}.$$

Equivalently, sampling period $T \le \pi/B$ (angular frequency) or $T \le 1/(2f_B)$ (Hertz, $f_B = B/(2\pi)$).

Proof. Landau 1967 classical result. Follows from uncertainty principle and Fourier analysis on bandlimited functions. □

Theorem 3.5 (Balian–Low Impossibility at Critical Density)**.** *For Gabor frame with time-frequency lattice (α, β) satisfying $\alpha\beta = 1$ (critical density), if frame is Riesz basis, then window g satisfies*

$$\int t^2|g(t)|^2\,dt \cdot \int \omega^2|\widehat{g}(\omega)|^2\,d\omega = \infty.$$

Cannot have both time and frequency good localization simultaneously at critical density.

Proof. Standard Balian–Low theorem. Follows from uncertainty principle and critical density constraint. See Daubechies 1992 for complete proof. □

Theorem 3.6 (Open System Capacity HSW Bound)**.** *For quantum channel $\mathcal{N}$ with ensemble $\{p_i, \rho_i\}$, Holevo information*

$$\chi(\{p_i, \rho_i\}) = S\Big(\sum_i p_i \mathcal{N}(\rho_i)\Big) - \sum_i p_i S\big(\mathcal{N}(\rho_i)\big),$$

and unassisted classical capacity

$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n}) = \lim_{n\to\infty} \frac{1}{n} \max_{\{p_i,\rho_i\}} \chi(\mathcal{N}^{\otimes n}, \{p_i, \rho_i\}).$$

Proof. Holevo–Schumacher–Westmoreland theorem. Follows from quantum relative entropy monotonicity (DPI) and coding theorem arguments. See Holevo 1998, Schumacher–Westmoreland 1997. □

4 Discussion and Outlook

This work establishes unified framework connecting:

- Scattering theory (phase, delay, spectral shift)
- Information geometry (KL projection, Fisher metric)
- Measurement theory (windows, frames, sampling)
- Error analysis (Poisson–EM–tail decomposition)
- Open systems (GKSL, DPI, capacity bounds)

Key achievements:

1. Unified scale formula $d\mu_\varphi = \frac{\varphi'}{\pi} dE = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}\, dE = \rho_{\mathrm{rel}}\, dE$
2. DBK global/local representation dichotomy for signed measures
3. Non-asymptotic error closure via Nyquist–Poisson–EM
4. Landau–Balian–Low sampling/frame thresholds
5. HSW capacity bound for open systems

Future directions:

- Extension to non-Hermitian/dissipative scattering
- Categorical formulation of measurement framework
- Numerical implementation and experimental validation
- Connections to quantum gravity and holography