Incompleteness = Non-Halting: An Equivalence Theorem Under a Unified Framework of Logic–Computation–Information–Dynamics

Haobo Ma[1] Wenlin Zhang[2]

[1]Independent Researcher

[2]National University of Singapore

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Abstract

We establish a rigorous equivalence chain across three layers—**logic–computation**, **information–measurement**, and **reversible dynamics**—in an augmentation process subject to a "liveness (completeness-halting) constraint": **unreachable completeness** if and only if **non-termination**. At the logical layer, Turing's undecidability of the halting problem and the Gödel–Rosser incompleteness theorem form the necessary and sufficient endpoints; at the information–measurement layer, Windowed Scattering–Information Geometry (WSIG) yields, under finite resources, a residual budget composed of **aliasing–Bernoulli layer–tail term** and **KL model mismatch**; unless all four ideal conditions—bandlimited+Nyquist, infinite-order EM (or exact-order cases), no tail truncation, and $p = q$—are met simultaneously (including degeneracies), this budget is **strictly positive** under the corresponding non-degenerate conditions at any finite stage, thus precluding halting under the liveness constraint of "residual$\to 0$"; at the dynamical layer, "completeness" is characterized by **reversible local boundary completion (RLBC)** of reversible cellular automata (RCA), and since surjectivity/reversibility of two-dimensional CA is undecidable and there exists **no uniform procedure that certifies "global bijection attained" for all instances in finite stages**, boundary extension generally cannot be guaranteed to terminate in finite steps. The three layers merge into the main equivalence **"Incompleteness = Non-Halting"**, with several reproducible instances and extensible directions provided.

Keywords: Halting problem; Incompleteness; Σ_1^0-hard; Windowed scattering; Pinsker inequality; Birman–Kreĭn formula; Wigner–Smith group delay; Reversible cellular automata; Garden-of-Eden

Notation & Axioms / Conventions

(Card A: Gauge Unification) We adopt the scattering unified gauge

$$\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E), \quad \mathsf{Q}(E) = -iS(E)^\dagger \partial_E S(E). \tag{1}$$

With the Birman–Kreĭn formula convention $\det S(E) = \exp(2\pi i \xi(E))$, we have $\partial_E \arg\det S(E) = 2\pi\xi'(E) = \operatorname{tr} \mathsf{Q}(E)$. **Define the total scattering phase**

$$\boxed{\varphi(E) := \tfrac{1}{2} \arg\det S(E) \text{ (continuous branch)}} \tag{2}$$

so that $\varphi'(E) = \frac{1}{2} \operatorname{tr} \mathsf{Q}(E)$, consistent with the gauge $\rho_{\rm rel}(E) = (1/2\pi) \operatorname{tr} \mathsf{Q}(E)$. This notation coincides with the relative density of states under unitary scattering. [**?**]

(Card B: Finite-Order NPE Discipline) Any windowed measurement employs the Nyquist–Poisson–Euler–Maclaurin three-fold decomposition: under bandlimiting and Nyquist sampling step, aliasing terms vanish; finite-order Euler–Maclaurin provides only **bounded and controllable** Bernoulli layer errors; tail terms from truncated windows are controlled by support/regularity. Constant normalization follows NIST DLMF. [**?**]

Terminology Convention: "Window/measurement/readout" uniformly refer to operator–measure–function objects (Toeplitz/Berezin compression), without experimental narratives; "liveness (completeness-halting) constraint" is defined in §3.1.

Probability Notation and KL Convention: Let p denote the target/true readout reference probability measure and q the model measure; $p \ll q$ means p is absolutely continuous with respect to q; $D_{\rm KL}(p|q)$ uses **natural logarithm** scale throughout.

1 Introduction

In an automatic augmentation process that halts **only upon achieving completeness** for a problem class $\mathcal{Q}$, does "unreachable completeness" necessarily imply "non-termination"? We give an affirmative answer and establish equivalent criteria across three layers:

1. **Logic–Computation Layer:** If $\mathcal{Q}$ is at least Σ^0_1-hard, halting would decide the halting set, contradicting Turing; in a consistent, recursively enumerable, and arithmetically sufficient stage-theory augmentation, Gödel–Rosser ensures completeness is unreachable at any finite stage, thus the process cannot halt. [**?**]

2. **Information–Measurement Layer:** WSIG provides unified gauge and finite-resource error theory. Unless all four ideal conditions—bandlimited+Nyquist, infinite-order EM (or exact-order), no tail truncation, and $p = q$—hold simultaneously, the NPE decomposition and KL–Pinsker bound guarantee a positive residual budget $\mathscr{R} > 0$ under the corresponding non-degenerate conditions, forcing the process to continue under the "residual$\to 0$" goal. [**?**]

3. **Dynamical Layer:** "Completeness" is realized as global bijective completion of RCA. Since surjectivity/reversibility of two-dimensional CA is undecidable, **there exists no uniform procedure that certifies "global bijection attained" for all instances in finite stages**; thus boundary extension generally cannot be guaranteed to terminate in finite steps. [**?**]

2 Preliminaries

2.1 Computation and Logic

Notation Supplement (Halting Set): Denote

$$K := \{(M, x) \mid M(x) \downarrow\} \tag{3}$$

the Turing **halting set** (M is a Turing machine, x its input, $M(x) \downarrow$ means halts). Classical result: K is Σ^0_1-complete.

Undecidability of Halting: No universal algorithm decides whether an arbitrary Turing machine halts. [?] **Rice's Theorem:** Any non-trivial semantic property of programs is undecidable. [?] **Gödel–Rosser:** A consistent, recursively enumerable, and arithmetically sufficient theory is incomplete; Rosser weakened ω-consistency to mere consistency. [?] **Σ^0_1-hard and r.e.:** Σ^0_1 is equivalent to recursively enumerable; many-one reduction characterizes "at least as hard". [?]

2.2 Information Geometry and I-Projection

Pinsker Inequality (natural log): $\mathrm{TV}(p, q) \le \sqrt{\frac{1}{2} D_{\mathrm{KL}}(p|q)}$. **I-Projection:** The minimal D_{KL} projection $q^\star$ over convex constraint families exists and is unique (under moderate regularity), and is equivalent in statistics and convex optimization. [?]

2.3 Phase–Density–Delay Gauge and Birman–Kreĭn

Wigner–Smith Group Delay: $\mathsf{Q}(E) = -iS^\dagger \partial_E S$, $\partial_E \arg\det S(E) = \operatorname{tr} \mathsf{Q}(E)$. **Birman–Kreĭn:** $\det S(E) = \exp(2\pi i \xi(E))$, so $\xi'(E) = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}(E)$, equivalent to relative density of states. [?]

2.4 NPE Three-Fold Decomposition and Nyquist Condition

If window w and kernel h are bandlimited with sampling step $\Delta < \pi/(\Omega_w + \Omega_h)$, then **aliasing term is zero**; finite-order Euler–Maclaurin provides Bernoulli layer error upper bound; finite window truncation yields tail term. Constant normalization and formulas per DLMF §1.8 and §2.10. [?]

2.5 Reversible Cellular Automata and Garden-of-Eden

Surjectivity and reversibility of two-dimensional CA are undecidable; the Garden-of-Eden theorem on amenable groups gives two equivalences: **existence of no pre-image (Garden-of-Eden configuration) ⇔ non-surjective**, and **surjective ⇔ pre-injective**, linked to reversibility/surjectivity properties. [?]

3 Model and Setup

3.1 Problem Class and Liveness (Completeness-Halting) Constraint

Let $\mathcal{Q}$ be a computably described decision problem class that is at least Σ_1^0-hard. Process $\mathcal{A}$ generates a consistent, recursively enumerable theory augmentation

$$T_0 \subset T_1 \subset \cdots, \quad T_t \text{ can decide } q \in \mathcal{Q}. \tag{4}$$

Define the **liveness (completeness-halting) constraint**:

$$\boxed{\mathcal{A} \text{ halts} \iff T_t \text{ has completely decided all } q \in \mathcal{Q}}. \tag{5}$$

3.2 Resource–Window–Kernel Quadruple and Residual Budget

For any finite resource **quadruple** $\mathsf{R} = (R, T, \Delta; M)$ **and model** q, define the **residual budget**

$$\mathscr{R} := \underbrace{\mathcal{E}_{\text{alias}}(\Delta)}_{\text{Poisson}} + \underbrace{\mathcal{E}_{\text{EM}}(M)}_{\text{Euler–Maclaurin (absolute value/norm bound of remainder)}} + \underbrace{\mathcal{E}_{\text{tail}}(R, T)}_{\text{Truncation}} + \underbrace{c\sqrt{\tfrac{1}{2} D_{\text{KL}}(p|q)}}_{\text{Model mismatch}}, \tag{6}$$

where the three NPE-error terms are **non-negative** (absolute value or norm upper bound), the last term given by Pinsker as a root upper bound from mismatch to readout difference; constant c absorbs normalization differences.

3.3 RLBC of RCA

Let $\Lambda_t \subset \mathbb{Z}^2$ be an increasing finite domain. RLBC abstracts "external interpretation/modeling" as a **reversible boundary bijection** on Λ_t, cascaded with interior reversible updates to a global map; "completeness" means existence of $t^\star$ such that extension to a **global bijection** requires no further expansion.

4 Logic–Computation Layer Main Theorems

Theorem 4.1 ("Halting $\Rightarrow$ Decidability of Halting", Hence Cannot Halt)**.** *If $\mathcal{Q}$ is at least Σ_1^0-hard and $\mathcal{A}$ satisfies the liveness constraint, then $\mathcal{A}$ does not halt.*

Proof. Since $\mathcal{Q}$ is Σ_1^0-hard, there exists a many-one reduction f such that for any TM-input pair (M, x), $(M, x) \in K \iff f(M, x) \in \mathcal{Q}$. If $\mathcal{A}$ halts at $t^\star$, then $T_{t^\star}$ decides $\mathcal{Q}$, thus decides K, contradicting Turing's theorem. [**?**] □

Theorem 4.2 ("Unreachable Completeness $\Rightarrow$ Non-Halting")**.** *If each T_t is consistent, recursively enumerable, and interprets PA, then for $\mathcal{Q}$ sufficient to express arithmetic, there exists no $t^\star$ such that $T_{t^\star}$ is complete (Gödel–Rosser). By the liveness constraint, $\mathcal{A}$ does not halt. [**?**]*

Corollary 4.3 (Equivalence)**.** *Under the above conditions, Never complete $\iff$ Never halts.*

5 Information–Measurement Layer: Forced Non-Halting

Theorem 5.1 (Non-Negative Residual Budget and Sufficient Condition for Strict Positivity)**.** *For any finite $(R, T, \Delta; M)$ and model q, $\mathscr{R} \geq 0$. Moreover, if at least one of the following four items and its corresponding* ***non-degeneracy*** *condition hold simultaneously, then $\mathscr{R} > 0$:*

(i) Bandlimited+Nyquist ***fails*** *and combined spectrum has non-zero mass outside $[-\pi/\Delta, \pi/\Delta]$;*

(ii) $M < \infty$ and the $2M$-th derivative of the relevant function is not identically zero on some interval (then EM remainder's ***upper bound*** *is strictly positive; per §3.2's "absolute value/norm upper bound" inclusion, this term is strictly positive);*

(iii) Window truncation exists and the observed object has non-zero mass outside the window domain;

(iv) $D_{\mathrm{KL}}(p|q) > 0$ (zero iff $p = q$ almost everywhere; if they differ on a positive p-measure set or $p \not\ll q$, then > 0, possibly $+\infty$).

Proof Sketch (Revised). All four terms are non-negative; under respective non-degeneracy conditions, the corresponding term's **included quantity (upper bound)** is strictly positive, other terms non-negative, so the sum is positive. If all four ideal conditions hold **simultaneously** (including degeneracies, e.g., truly bandlimited and Nyquist-satisfied, M gives exact precision for involved functions, zero mass outside domain, and $p = q$), then $\mathscr{R} = 0$. [**?**] □

Theorem 5.2 (Correct Relation of "$\mathscr{R}$ and Halting")**.** ***Lemma (Necessary Condition for Complete Decision):*** *If the liveness (completeness-halting) constraint is met at stage t, then for the readout–budget definition of §3.2, necessarily $\mathscr{R}(t) = 0$; otherwise there exists ineliminable readout uncertainty, making consistent decision of some $q \in \mathcal{Q}$ unattainable under finite resources.*

Thus, halting can only occur at stages where $\mathscr{R} = 0$. If any non-degeneracy condition of Theorem **??** *holds, then for any finite stage t, $\mathscr{R}(t) > 0$, so cannot halt in finite steps; $\mathscr{R}(t)$ may approach 0 with resource investment, which does not affect the "unreachable completeness $\Rightarrow$ non-halting" conclusion.*

Discussion 5.3 (Observable Image of Phase–Density Mother Gauge)

Under the gauge unification formula, ideal completeness corresponds to global alignment of $\xi'(E) = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}(E)$; under finite resources, "stochasticity" is the measurable projection of $\mathscr{R} > 0$. [**?**]

6 Dynamical Layer: Endless Boundary of RLBC

Theorem 6.1 (No Uniform Finite-Stage Certification Procedure)**.** *Define "completeness" as existence of $t^\star$ such that RLBC extends to a global bijection. Surjectivity/reversibility of two-dimensional CA is undecidable. If there exists a* ***uniform procedure for all instances*** *that outputs "attained" or "unattainable" in finite stages (two-sided finite certification), one could construct a decision algorithm, thus deciding surjectivity/reversibility,* *contradiction. Only one-sided ("attained" only) finite certification at most gives semi-decidability, insufficient for decidability; thus we assert no "two-sided" finite certification uniform procedure exists. Hence, generally RLBC boundary extension is not guaranteed to reach a terminal point in finite steps. Individual special cases (e.g., local permutation-type CA) can prove reversibility in finite stages, not contradicting this conclusion. [?]*

Supporting Evidence: The Garden-of-Eden theorem on amenable groups gives "surjective $\Leftrightarrow$ pre-injective", compatible with reversibility characterization. [?]

7 Examples and Constructions

Example 7.1 (Windowed Threshold and Σ_1^0 Embedding): Construct bandlimited window–kernel such that the predicate "there exists an energy interval where windowed relative density of states exceeds threshold" is equivalent to some machine halting; then any finite $(R, T, \Delta; M)$ makes $\mathscr{R} > 0$, forcing the process to continue under the "reduce residual" goal. Feasibility relies on gauge unification and Nyquist/EM error theory. [?]

Example 7.2 (RLBC Completion of RCA): Take "whether Garden-of-Eden exists" or "global reversibility" as predicate, gradually expand reversible boundary; designing a terminal certification mechanism for **all** two-dimensional CA guaranteed to appear in finite stages would yield a decision algorithm, violating undecidability; finite certification for specific reversible/surjective CA does not imply universal decision. [?]

8 Interface with Unified System

- **Gauge Bridge:** $\partial_E \arg\det S = \operatorname{tr} \mathsf{Q} = 2\pi\xi'(E)$ unifies "phase derivative–group delay–relative density of states". [?]
- **NPE-Nyquist Discipline:** $\Delta < \pi/(\Omega_w + \Omega_h) \Rightarrow \mathcal{E}_{\text{alias}} = 0$; finite-order EM and tail terms provide controllable upper bounds. [?]
- **I-Projection and Stability:** Minimal D_{KL} projection gives "closest attainable model" readout alignment and stable chain. [?]

9 Limitations and Extensions

1. **About $\mathcal{Q}$:** The equivalence relies on Σ_1^0-hardness; weaker classes require individualization. [?]
2. **Scattering Scenarios:** Non-unitary/dissipative systems can be handled with generalized BK and trace formulas; constant normalization depends on specific coupling; see modern surveys. [?]
3. **CA on Groups:** For non-amenable groups, Garden-of-Eden conclusions differ; corrections needed per group properties. [?]

10 Conclusion

Under the liveness (completeness-halting) constraint, the **logic–computation layer**'s Turing and Gödel–Rosser, the **information–measurement layer**'s finite-resource residual budget, and the **dynamical layer**'s RLBC unreachability tightly interlock into

$$\boxed{\text{Never complete} \iff \text{Never halts}}. \tag{7}$$

This equivalence grounds the intuition "stochasticity stems from incompleteness" on verifiable unified gauge and undecidability criteria, providing structural constraints for windowed readout design and reversible dynamical semantics.

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