Windowed Energy as Measure Theory
(WEM: Windowed Energy as Measure)

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Abstract

Establish self-consistent framework characterizing energy via first moment of windowed relative spectral density. Core scale chain holds almost everywhere on absolutely continuous spectrum:

$$\boxed{\frac{\varphi'(E)}{\pi} \;=\; \rho_{\mathrm{rel}}(E) \;=\; \frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E)},$$

where $\varphi(E) = \frac{1}{2}\,\mathrm{Arg}\det S(E)$, $\mathsf{Q}(E) = -i\,S(E)^{\dagger}\partial_E S(E)$ is Wigner–Smith group delay matrix, ρ_{rel} relative spectral density. Define energy functional by weighting ρ_{rel} with window w:

$$\mathcal{E}[w] = \int_{\mathbb{R}} E\,w(E)\,\rho_{\mathrm{rel}}(E)\,dE.$$

This paper gives: covariant invariance and channel additivity under energy reparametrization–window pushforward; $\log\det$ characterization based on Birman–Kreĭn trace–phase formula and $\det_2$/Koplienko regularization under Hilbert–Schmidt relative perturbation; non-asymptotic error closure under finite-order Euler–Maclaurin (EM) discipline; semantic embedding and Koopman spectral correspondence in EBOC (static block · observation–computation) and RCA (reversible cellular automata). Factual foundations include definition and multi-physics generalizations of group delay matrix, spectral shift function and relative trace, and EM error theory.

1 Notation & Axioms / Conventions

Card-1 (Scale Identity Formula): Holds a.e. on absolutely continuous spectrum

$$\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E), \qquad \mathsf{Q}(E) = -i\,S^{\dagger}(E)\,\partial_E S(E).$$

Single/multi-channel cases consistent with original "lifetime matrix" definition, computation and experimental pathways established in electromagnetic, acoustic and other systems.

Card-2 (Finite-Order EM–NPE Discipline): All discrete approximations adopt only **finite-order** Euler–Maclaurin expansion; error decomposes as "alias + Bernoulli correction + tail", constants depending only on endpoint derivatives and finite-order smoothness.

Scattering–Spectral Shift Convention: In trace class scattering framework

$$\det S(E) = \exp\big(-2\pi i\,\xi(E)\big), \qquad (\log\det S)'(E) = i\;\mathrm{tr}\,\mathsf{Q}(E),$$

thus $\rho_{\mathrm{rel}}(E) = -\xi'(E)$; under Hilbert–Schmidt relative perturbation replace with Koplienko spectral shift η and $\det_2$.

Window and Windowed Measure: Window $w \in L^1(\mathbb{R}) \cap C^1$, $w \geq 0$, $\int w = 1$, $\int |E| w(E)\, dE < \infty$; windowed relative spectral measure $d\mu_w(E) = w(E)\rho_{\mathrm{rel}}(E)\, dE$.

2 Framework and Basic Objects

Set separable Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, self-adjoint operator pair (H_0, H) wave operators exist and complete; on absolutely continuous spectrum exists differentiable scattering matrix $E \mapsto S(E) \in U(N(E))$. Define

$$\mathsf{Q}(E) = -i\, S^\dagger(E)\partial_E S(E), \qquad \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}(E),$$

with

$$\boxed{\mathcal{E}[w] = \int_{\mathbb{R}} E\, w(E)\, \rho_{\mathrm{rel}}(E)\, dE}$$

as windowed spectral definition of "energy". Single-channel $S(E) = e^{2i\delta(E)}$ gives $\operatorname{tr} \mathsf{Q}(E) = 2\delta'(E)$, compatible with Friedel-type relation with state density difference (graph networks have local state corrections).

3 Axioms and Basic Properties

Axiom 3.1 (Observability). *$\mathcal{E}[w]$ depends only on windowed relative spectral measure $d\mu_w$.*

Axiom 3.2 (Reparametrization Covariance). *Set $\phi : \mathbb{R} \to \mathbb{R}$ strictly monotone and C^1. Define windowed relative spectral measure*

$$d\mu_w(E) := w(E)\, \rho_{\mathrm{rel}}(E)\, dE,$$

*and its pushforward $d\mu_w^\phi := \phi_* d\mu_w$. Then **covariant equivalence formula***

$$\boxed{\mathcal{E}_S^{(\phi)}[w] := \int_{\mathbb{R}} \phi(E)\, d\mu_w(E) = \int_{\mathbb{R}} E\, d\mu_w^\phi(E)}.$$

Axiom 3.3 (Channel Additivity). *$S = S_1 \oplus S_2 \Rightarrow \rho_{\mathrm{rel}} = \rho_{\mathrm{rel},1} + \rho_{\mathrm{rel},2} \Rightarrow \mathcal{E}_S[w] = \mathcal{E}_{S_1}[w] + \mathcal{E}_{S_2}[w]$.*

Axiom 3.4 (Regularized Extension). *If $S - I \in \mathfrak{S}_2$, maintain structure and characterization of $\mathcal{E}[w]$ using Koplienko spectral shift function η and $\det_2$.*

Axiom 3.5 (Vacuum Truth). *$S \equiv I \Rightarrow \rho_{\mathrm{rel}} \equiv 0 \Rightarrow \mathcal{E}[w] = 0$.*

4 $\log\det/\det_2$ Characterization and Relative Trace

Theorem 4.1 (Trace Class Case). *If $S - I \in \mathfrak{S}_1$, then*

$$\mathcal{E}[w] = \frac{1}{2\pi i} \int_{\mathbb{R}} E\, w(E)\, (\log\det S)'(E)\, dE = -\int_{\mathbb{R}} E\, w(E)\, \xi'(E)\, dE.$$

Proof. By $(\log\det S)' = \operatorname{tr}(S^{-1}S') = i\,\operatorname{tr}\mathsf{Q}$ and Card-1 directly derive; $\det S = \exp(-2\pi i\,\xi)$ yields spectral shift version. $\square$

Theorem 4.2 (Hilbert–Schmidt Case, Safe Statement)**.** *Set $S(E) - I \in \mathfrak{S}_2$. Then exists Koplienko spectral shift function $\eta \in L^1_{\mathrm{loc}}(\mathbb{R})$, such that for any $f \in C^2_c(\mathbb{R})$ have*

$$\operatorname{tr}\big(f(H) - f(H_0) - f'(H_0)(H - H_0)\big) = \int_{\mathbb{R}} f''(E)\,\eta(E)\,dE.$$

In this framework, energy functional still defined as $\mathcal{E}[w] = \int_{\mathbb{R}} E\,w(E)\,\rho_{\mathrm{rel}}(E)\,dE$.

*If further satisfy **additional regularity assumption** (e.g., $\det_2 S(E)$ exists non-tangential boundary value and a.e. differentiable), can define*

$$\Xi_2(E) := \frac{1}{2\pi i}\,\partial_E \log \det_2 S(E),$$

obtaining expression structurally consistent with trace class case

$$\boxed{\mathcal{E}[w] = \int_{\mathbb{R}} E\,w(E)\,\Xi_2(E)\,dE}\,.$$

5 Variational Structure and Scale Window Family

Under constraint $\int w = 1$, Gateaux derivative

$$D\mathcal{E}[w]\cdot\delta w = \int_{\mathbb{R}} E\,\rho_{\mathrm{rel}}(E)\,\delta w(E)\,dE,$$

stationary points satisfy $E\,\rho_{\mathrm{rel}}(E) = \lambda$ on support of w. Scale window family

$$w_\lambda(E) = \lambda^{-1} w\left(\tfrac{E}{\lambda}\right),$$

directional derivative

$$\boxed{\frac{d}{d\lambda}\mathcal{E}[w_\lambda]\Big|_{\lambda=1} = -\int_{\mathbb{R}} E\,\rho_{\mathrm{rel}}(E)\,\big(w(E) + E\,\partial_E w(E)\big)\,dE}\,.$$

6 Finite-Order Euler–Maclaurin (EM) Non-Asymptotic Error Closure

For uniform grid $E_n = E_0 + n\Delta$ discrete approximation

$$\widehat{\mathcal{E}} = \sum_{n=-R}^{R} E_n\,w(E_n)\,\rho_{\mathrm{rel}}(E_n)\,\Delta,$$

let $f(E) = E\,w(E)\,\rho_{\mathrm{rel}}(E) \in C^p$, have

$$\boxed{\mathcal{E} = \widehat{\mathcal{E}} - \frac{\Delta}{2}\big(f(a) + f(b)\big) - \frac{B_2}{2!}\Delta^2\big(f'(b) - f'(a)\big) - \frac{B_4}{4!}\Delta^4\big(f^{(3)}(b) - f^{(3)}(a)\big) - \cdots}$$

Thus, without endpoint correction error leading term $O(\Delta)$; when $f(a) = f(b) = 0$ (window vanishes at endpoints) or using **trapezoidal/midpoint** symmetric rules, main error improves to $O(\Delta^2)$. Above decomposition still denoted

$$\Delta_{\mathrm{NPE}} = \Delta_{\mathrm{alias}} + \Delta_{\mathrm{Bernoulli}} + \Delta_{\mathrm{tail}},$$

embodying principle "smoother window better error", giving computable bounds of endpoint-dominated terms.

7 Main Theorems (Selection)

Theorem 7.1 (Reparametrization Covariance Consistency). *For any strictly monotone C^1 ϕ have*

$$\mathcal{E}_S^{(\phi)}[w] = \int_{\mathbb{R}} \phi(E)\, d\mu_w(E) = \int_{\mathbb{R}} E\, d(\phi_* d\mu_w)(E).$$

Proof. Pushforward measure definition gives $\int g(E)\, d(\phi_*\mu)(E) = \int g(\phi(E))\, d\mu(E)$. Taking $g(E) = E$ immediately yields conclusion. $\square$

Theorem 7.2 (Finite-Order EM Stable Bounds–Unified Statement). *Set $f(E) = E\, w(E)\, \rho_{\mathrm{rel}}(E) \in C^p([a,b])$, uniform grid $E_n = E_0 + n\Delta$ covering effective support, discrete approximation*

$$\widehat{\mathcal{E}} = \sum_{n=-R}^{R} E_n\, w(E_n)\, \rho_{\mathrm{rel}}(E_n)\, \Delta.$$

Then exist constants C_1, C_{2k} (depending on endpoint derivatives up to order $2k-1$), such that

$$\boxed{|\mathcal{E} - \widehat{\mathcal{E}}| \;\le\; \tfrac{\Delta}{2}\big(|f(a)| + |f(b)|\big) \;+\; \sum_{k=1}^{\lfloor p/2 \rfloor} C_{2k}\, \Delta^{2k}}\,.$$

Further, if satisfy any condition: (i) $f(a) = f(b) = 0$ or (ii) adopt trapezoidal/midpoint symmetric rules, then leading $O(\Delta)$ vanishes and main order improves to $O(\Delta^2)$.

8 Discussion and Outlook

This work establishes:

1. Windowed energy functional via first moment of relative spectral density
2. Covariant invariance under reparametrization and window pushforward
3. $\log\det/\det_2$ characterizations in trace class and Hilbert–Schmidt cases
4. Non-asymptotic EM error closure with explicit bounds
5. EBOC embedding as observer-independent integrated invariant
6. RCA embedding via Koopman spectral correspondence

Key formulas:

- Energy functional: $\mathcal{E}[w] = \int E\, w(E)\, \rho_{\mathrm{rel}}(E)\, dE$
- Scale identity: $\varphi'/\pi = \rho_{\mathrm{rel}} = (2\pi)^{-1}\, \mathrm{tr}\, \mathsf{Q}$

- EM error: $|\mathcal{E} - \widehat{\mathcal{E}}| \leq O(\Delta)$ or $O(\Delta^2)$ depending on conditions

Future directions:

- Extension to dissipative and dispersive systems
- Statistical theory for chaotic scattering
- Numerical implementation and benchmarking
- Applications to quantum graphs and photonic systems