

Unified Framework and Engineering Pathways for Time Crystals: From Floquet Phases to Open System Limits, Quasiperiodic Drives, and Topological Protection

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Abstract

Construct unified time crystal theory penetrating closed and open quantum systems, periodic and quasiperiodic drives, and topological constraints. Using group representation and non-equilibrium statistics as foundation, propose operational order parameters, temporal correlations, and spectral criteria; at mathematical level via high-frequency Floquet–Magnus expansion, Lieb–Robinson bounds, and spectral pairing structure, prove rigidity and robustness of prethermal discrete time crystals on exponentially long timescales; under disordered many-body localization (MBL) and Lindblad open systems respectively provide existence and stability theorems, extending to multi-frequency quasiperiodically driven "temporal quasicrystals"; further demonstrate topological order protection mechanism for discrete time crystals, constructing topological time crystals using surface code logical operators as order parameters; finally toward experiments, propose engineering schemes and measurement protocols for superconducting qubit arrays, Rydberg gases, trapped ions, and liquid crystal/phonon–polariton platforms. This paper's theorems and schemes systematically absorb recent key advances: equilibrium time crystal no-go theorems, rigorous definition and first experimental observation of Floquet discrete time crystals, prethermal exponential lifetime upper bounds, latest realizations of dissipative and topological time crystals, and experimental evidence for continuous/spatiotemporal time crystals and multi-frequency temporal quasicrystals.

Keywords: Discrete time crystal; spontaneous breaking of time translation symmetry; Floquet prethermalization; many-body localization; Lindblad open systems; temporal quasicrystal; topological time crystal; surface code logical operators; Rydberg gas; superconducting qubits

1 Introduction & Historical Context

Time crystal idea originates from fundamental question "can time translation symmetry spontaneously break". Although initial continuous time crystal proposals attracted wide attention, rigorous no-go theorems exclude such phases in ground states or canonical ensembles for broad classes with short-range interactions, shifting focus toward non-equilibrium driven systems.

In periodically driven many-body quantum systems, Else–Bauer–Nayak provided rigorous definition of discrete time crystal (DTC): system under fundamental period T Floquet symmetry spontaneously selects longer subharmonic period mT , exhibiting "rigid" against perturbations subharmonic response, accompanied by long-range temporal correlations and characteristic spectral lines. Subsequent theoretical work revealed unified framework for rigidity, criticality, and realizability, providing clear blueprint for experiments on different platforms.

Experimentally, two milestone works in 2017 respectively observed discrete time crystalline order in trapped ions and room-temperature diamond spin systems, establishing DTC observability and robustness; these two results simultaneously published in same issue of *Nature*. Subsequently, superconducting quantum processors confirmed "eigenstate time crystalline order" at spectral-dynamical level, emphasizing eigenstate ordering structure under Floquet-MBL/prethermal background. Open system direction, strongly interacting Rydberg gases observed dissipative time crystals (both continuous and discrete types), demonstrating controllable realization and spectral gap protection of open system limit cycles. At topological level, two-dimensional superconducting qubit arrays realized "long-lived prethermal topological time crystal", whose subharmonic response significantly exhibited only by non-local logical operators, accompanied by non-zero topological entanglement entropy. More macroscopic spatiotemporal symmetry breaking also developing: continuous media (liquid crystals, superfluid ^3He , etc.) based on nonlinear-dissipative-pumping coupling observed continuous/spatiotemporal time crystals and controllable coupling with mechanical modes; in 2025 realized discrete "temporal quasicrystal" under multi-frequency drive.

Above advances jointly point toward unified question: **How to define, determine, and engineer time crystal existence and robustness under general conditions?** Below provides systematic answer.

2 Model & Assumptions

2.1 Local Quantum Model (Closed System) Consider spin/boson/fermion system on d -dimensional lattice Λ , local interactions with finite range or fast decay. Periodic drive

$$H(t) = H_0 + \sum_{\alpha=1}^r V_{\alpha}(t), \quad V_{\alpha}(t+T) = V_{\alpha}(t), \quad |V_{\alpha}| \leq J.$$

Floquet unitary $F = \mathcal{T}e^{-i\int_0^T H(t)dt}$ generates discrete time translation \mathbb{Z} . High-frequency limit ($\omega = 2\pi/T \gg J$) defines prethermal region; disorder case H_0 supports MBL.

2.2 Lindblad Open System Model Density matrix ρ satisfies

$$\dot{\rho} = \mathcal{L}_t(\rho) = -i[H(t), \rho] + \sum_{\mu} \left(L_{\mu}\rho L_{\mu}^{\dagger} - \frac{1}{2}\{L_{\mu}^{\dagger}L_{\mu}, \rho\} \right), \quad \mathcal{L}_{t+T} = \mathcal{L}_t,$$

single-period quantum channel $\mathcal{E} = \mathcal{T} \exp\left(\int_0^T \mathcal{L}_t dt\right)$ describes Poincaré section dynamics.

2.3 Symmetry and Order Parameters If system possesses finite internal symmetry group G and discrete time translation \mathbb{Z} , steady-state/eigenstate symmetry can spontaneously break to subgroup $H \subset G \times \mathbb{Z}$. Take local observable O odd under G transformation, define temporal order parameter

$$\mathcal{C}_O(n) = \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \langle O_x(nT) O_x(0) \rangle,$$

exhibiting strict m -periodic spacing in n with non-trivial long-time limit, characterizing m -subharmonic DTC order. This definition equivalent to representation theory perspective.

3 Main Results (Theorems and Statements)

Theorem 1 (1: Existence and Rigidity of Prethermal Discrete Time Crystal). *Let $H(t)$ be local periodically driven system, piecewise drive composed of near- π global symmetric "pulses" exists,*

making Floquet unitary writable as

$$F = U_* e^{-iH_* T} X U_*^\dagger + \Delta,$$

where $X^2 = \mathbb{I}$ is \mathbb{Z}_2 internal symmetry generator, H_* commutes with X and quasilocal, $|\Delta| \leq C e^{-c\omega/J}$. Then any local operator O odd under X transformation exhibits stable $2T$ subharmonic locking, maintaining coherence for polynomial time $t \lesssim \tau_* \sim e^{c\omega/J}$; for small perturbation δH have $\text{dist}(C_O, C_O^{(0)}) \leq K|\delta H|$. Proof based on quasiconserved quantities from high-frequency Floquet–Magnus truncation and exponential slow heating theorem.

Theorem 2 (2: Spectral Pairing and Eigenstate Order in MBL–DTC). *On one-dimensional strongly disordered localized chain, Floquet operator with near- π global symmetric kick admits quasilocal unitary U making*

$$F \simeq \tilde{X} e^{-iH_{\text{MBL}} T},$$

where \tilde{X} is quasilocal \mathbb{Z}_2 symmetry, H_{MBL} diagonalized by set of l-bits. Spectrum exhibits π pairing (eigenstates differ by π), typical eigenstates have parity degeneracy, inducing state-independent $2T$ subharmonic response (eigenstate order).

Theorem 3 (3: Sufficient Condition for Open System Dissipative Time Crystal). *For periodic Lindblad semigroup \mathcal{L}_t single-period channel \mathcal{E} , if all eigenvalues within spectral radius satisfy $|\lambda_j| < 1$, while unique modulus group consists of m phases $\{e^{2\pi i k/m}\}_{k=0}^{m-1}$ with corresponding Jordan blocks non-splittable, then almost all initial states converge long-time to period- mT limit cycle attractor family, manifesting as m -subharmonic dissipative time crystal; limit cycle structurally stable against small perturbations under Liouvillian spectral gap. This criterion consistent with Perron–Frobenius type results for quantum channels.*

Theorem 4 (4: Multi-Frequency Drive and Temporal Quasicrystal). *For quasiperiodic drive with k mutually irrational frequencies $\{\omega_i\}$, if $\min_i \omega_i \gg J$ enters prethermal region, quasilocal unitary U and effective Hamiltonian H_* exist such that on \mathbb{Z}^k temporal lattice*

$$\mathcal{U}(\vec{n}) = U e^{-iH_* \sum_i n_i T_i} \mathbf{g}(\vec{n}) U^\dagger + O\left(e^{-c \min_i \omega_i / J}\right),$$

where \mathbf{g} is finite index subgroup representation of \mathbb{Z}^k . This implements spontaneous breaking for \mathbb{Z}^k , forming "temporal quasicrystal" with multi-subharmonic spectral lines.

Theorem 5 (5: Topological Time Crystal: Non-Local Order Parameter and Protection). *Under two-dimensional stabilizer code (surface code) drive implementing logical π flip and Hamiltonian engineering, Floquet unitary in code subspace equivalent to*

$$F_{\text{logical}} \approx \overline{X}_L e^{-iH_*^{\text{top}} T},$$

where \overline{X}_L is logical non-local symmetry. Any local operator insignificantly exhibits subharmonic order, while non-local logical closed string/membrane operators exhibit rigid mT subharmonic response, quantitatively supported by topological entanglement entropy. This mechanism experimentally verified on programmable superconducting arrays.

Proposition 6 (6: Continuous/Macroscopic Spatiotemporal Time Crystal). *In nonlinear–dissipative–pumping coupled continuous media (e.g., nematic liquid crystals) and superfluid ^3He systems, Hopf–Turing cooperative instability forms stable limit cycle–stripes simultaneously breaking space and time translation (spatiotemporal crystal), with controllable coupling to mechanical modes.*

4 Proofs

4.1 Exponentially Long Time Bound in Prethermal Region (Theorem 1)

For local driven system adopt Floquet–Magnus expansion $F = \exp\{-iT \sum_{n \geq 0} \Omega_n\}$, truncate at optimal order $n_* \sim \omega/J$ obtaining quasilocal effective Hamiltonian $H_* = \sum_{n \leq n_*} \Omega_n$. Rigorous results provide energy absorption and truncation error exponentially small within time $\tau_* \sim \exp(c\omega/J)$, i.e.

$$|F - e^{-iH_*T}| \leq Ce^{-c\omega/J}, \quad \left| \frac{d}{dt} \langle H_* \rangle \right| \leq C' e^{-c\omega/J}.$$

Introduce near- π symmetric kick U_X (global \mathbb{Z}_2 flip) in piecewise drive, combined with X and H_* near-commutativity and existence of quasilocal unitary transformation U_* , obtain structural decomposition and subharmonic locking; error controlled for $t \leq \tau_*$, deriving exponential longevity and rigidity.

4.2 MBL π Spectral Pairing and Eigenstate Order (Theorem 2)

Under strong disorder quasilocal unitary U exists making UH_0U^\dagger diagonalized by l -bits. Near- π kick in U representation produces quasilocal \tilde{X} , commuting with H_{MBL} . On spectrum each state $|\psi\rangle$ with $\tilde{X}|\psi\rangle$ constitutes π paired subspace, inducing state-independent $2T$ subharmonic response. This "eigenstate order" consistent with processor experiment spectral–dynamical consistency.

4.3 Open System Limit Cycle Spectral Condition (Theorem 3)

Decompose single-period CPTP channel \mathcal{E} spectrum into generalized eigenmodes. If spectral radius periphery contains only modulus 1 pure phase eigenvalues of m phases, with spectral gap elsewhere, then \mathcal{E}^n iteration projects arbitrary initial state to m -cycle attractor cluster, convergence rate given by Liouvillian spectral gap. This result belongs to positive operator–quantum channel Perron–Frobenius theory (Evans–Høegh-Krohn et al.), compatible with recent algebraic characterization of limit cycles/synchronization.

4.4 Multi-Frequency Temporal Quasicrystal Group Representation and Prethermal Protection (Theorem 4)

Time translation group for quasiperiodic drive is \mathbb{Z}^k . Under joint high-frequency limit, construct finite index subgroup representation $\mathfrak{g} \subset \mathbb{Z}^k$, making effective evolution on lattice decompose as product of $e^{-iH_* \sum_i n_i T_i}$ and $\mathfrak{g}(\vec{n})$; latter's finite image induces incommensurate multi-subharmonic peaks, forming "temporal quasicrystal", consistent with general theory of "multiple time–translation symmetry protection".

4.5 Topological Time Crystal Non-Local Order and Protection (Theorem 5)

Under periodic engineering of surface code Hamiltonian, Floquet unitary in code subspace manifests as superposition of logical π flip and effective Hamiltonian. Non-local logical operators (closed strings/membranes) constitute order parameters, whose subharmonic locking insensitive to local noise; topological entanglement entropy provides independent evidence. Superconducting array experiments observe subharmonic peaks significant only for logical operators with non-zero topological entanglement entropy.

5 Model Apply (Representative Platforms and Observables)

5.1 Superconducting Qubit Array (Topological DTC)

Device: two-dimensional square array, surface code stabilizers (A_s, B_p) + periodic "logical π kick".
Observables: autocorrelation and Fourier spectrum of non-local \bar{Z}_L, \bar{X}_L ; topological entanglement entropy γ . Expected signal: frequency domain exhibits $\omega/2$ subharmonic peak, significant only in logical channel; $\gamma > 0$ appears cooperatively with logical subharmonic response.

5.2 Rydberg Atom Gas (Dissipative Time Crystal)

Device: room-temperature vapor cell, continuous optical pumping and decoherence channel; Observables: fluorescence/transmission intensity autocorrelation, population polarization Poincaré map trajectory; Expected signal: stable limit cycle and parameter phase diagram crossing (disorder–limit cycle–multistability), convergence rate controlled by Liouvillian spectral gap.

5.3 Trapped Ions (Prethermal DTC)

Device: long-range interaction chain, high-frequency drive suppresses heating; Observables: single-body/many-body spin correlation functions and Fourier spectra; Expected signal: $2T$ subharmonic peak with lifetime exponentially growing with ω , discernible dependence on initial state energy density.

5.4 Liquid Crystal and Superfluid ^3He (Continuous/Spatiotemporal Time Crystal)

Device: photoinduced drive of nematic liquid crystal; magnon condensation and free surface coupling in ^3He ; Observables: spatiotemporal patterns coexisting stripes–oscillations; coherent frequency and phase locking; Expected signal: Hopf–Turing cooperative mode and tunable mechanical coupling.

5.5 Multi-Frequency Drive Temporal Quasicrystal

Device: quantum simulator simultaneously driven by multicolor microwave/laser; Observables: incommensurate subharmonic peak family and multiple "rigidities"; Expected signal: peak positions locked only determined by finite image of \mathbb{Z}^k representation, not drifting with small perturbations.

6 Engineering Proposals (Realizability and Error Budget)

6.1 Prethermal DTC Pulse Design and Frequency Window

Adopt piecewise drive $F = e^{-iH_Z\tau_z}e^{-iH_X\tau_x}$, implementing near- π flip at τ_x ; choose ω satisfying $e^{-c\omega/J} \lesssim \varepsilon/10$ ensuring $\tau_* \sim e^{c\omega/J}$ covers $10^2 \sim 10^3$ periods. Gate noise $\varepsilon \ll e^{-c\omega/J}$ maintains subharmonic locking; decoherence time $T_2 \gg \tau_*$.

6.2 Topological Time Crystal Logical Gate–Error Correction Coexistence

Embed logical π flip into surface code periodic sequence, ensuring stabilizer measurement and phase accumulation close loop within τ_* ; non-local readout reduces local noise sensitivity. Reference recent device metrics (logical order significant only in non-local channel observing non-zero topological entanglement entropy).

6.3 Open System CTC Gain–Decoherence Ratio

Open limit cycle via controllable pumping–decoherence ratio G/κ , spectral gap Δ_{Liouv} sets recovery time $\tau_r \sim 1/\Delta_{\text{Liouv}}$ and perturbation robustness; Rydberg platform provides room-temperature feasible parameter domain.

6.4 Multi-Frequency Temporal Quasicrystal Drive Arrangement

In \mathbb{Z}^k framework design drive phase and frequency "coprime", avoiding accidental integer period recurrence; experimental spectral lines use incommensurate peak positions as fingerprint.

7 Discussion (Boundaries, Risks, and Related Work)

Boundaries and risks: (i) Outside high-frequency window heating and chaos may extinguish prethermal DTC; (ii) MBL stability limited in high dimensions and long-range interactions; (iii) open system engineering noise and non-Markovianity may induce phase wandering and multistable competition.

Alignment with existing work: This paper's Theorems 1–2 compatible with Else–Bauer–Nayak definition, absorbing Yao et al. framework on rigidity and criticality; exponential longevity consistent with Mori–Kuwahara–Saito and Abanin–De Roeck–Ho–Huveeneers prethermal upper bounds; Theorem 3 consistent with Rydberg dissipative time crystal observations and positive operator spectral theory; Theorem 5's logical–non-local order matches superconducting array "topological time crystal" data; macroscopic continuous/spatiotemporal time crystals and He-3 coupling results provide cross-scale phenomenology.

Methodological parallel (\mathbb{Z}_2 holonomy and " π " fingerprint): DTC π frequency–spectral pairing and subharmonic rigidity characterizable via \mathbb{Z}_2 quantized fingerprint. Related \mathbb{Z}_2 holonomy/square root determinant–holonomy ideas transplantable as criterion in relative cohomology and modular connection contexts, identifying spectral transition and non-trivial phase winding of " π " flip type.

8 Conclusion

Established unified time crystal theory and engineering pathways: constructing common structure among closed system prethermalization, MBL, open system limit cycles, and topological protection; extending to multi-frequency temporal quasicrystals and cross-scale continuous/spatiotemporal time crystals; proposing reproducible experimental protocols supporting multiple platforms. This framework integrates "group–representation–spectrum" with "prethermalization/localization/dissipation–protection" dynamical principles, providing solid foundation toward robust time-frequency devices, controllable synthesis of non-equilibrium phases, and topological–logical storage.

Acknowledgements, Code Availability

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A Prethermal DTC Construction and Exponential Lifetime

A.1 Prethermal Upper Bound and Effective Hamiltonian

Under $\omega \gg J$ condition Floquet–Magnus expansion

$$\Omega_0 = \frac{1}{T} \int_0^T H(t) dt, \quad \Omega_1 = \frac{1}{2Ti} \iint_{0 < t_1 < t_2 < T} [H(t_2), H(t_1)] dt_1 dt_2, \dots$$

optimal order $n_* \sim \alpha\omega/J$ truncation defines H_* , with

$$|F - e^{-iH_*T}| \leq Ce^{-c\omega/J}, \quad \left| \frac{d}{dt} \langle H_* \rangle \right| \leq C'e^{-c\omega/J},$$

constants depending only on locality (implemented via Lieb–Robinson bound).

A.2 Near- π Kick and \mathbb{Z}_2 Internal Symmetry

Let

$$U_X = \exp\left(-i\frac{\pi + \epsilon}{2} \sum_i \sigma_i^x\right) = X \exp\left(-i\frac{\epsilon}{2} \sum_i \sigma_i^x\right),$$

with H_0 commuting with X to exponential accuracy. Then

$$F = U_X e^{-iH_0T} \approx X e^{-iH_*T} + O(\epsilon) + O(e^{-c\omega/J}),$$

combined with quasilocal unitary U_* yields Theorem 1's structural decomposition. Subharmonic locking from $XOX = -O$ odd transformation property and error suppression.

B MBL–DTC Spectral Pairing and Eigenstate Order

B.1 l -bit Diagonalization and Quasilocal \tilde{X}

Unitary U exists making $UH_0U^\dagger = f(\{\tau_i^z\})$; near- π kick in this representation becomes $\tilde{X} = UXU^\dagger \simeq \prod_i \tilde{\sigma}_i^x$, commuting with H_{MBL} .

B.2 π Spectral Pairing

If $F \simeq \tilde{X}e^{-iH_{\text{MBL}}T}$, then for eigenstate $|\psi\rangle$

$$F|\psi\rangle = e^{-iET} \tilde{X}|\psi\rangle, \quad F(\tilde{X}|\psi\rangle) = e^{-i(ET+\pi)}|\psi\rangle,$$

thus eigenstates differ by π . Arbitrary initial state expanded in paired subspace yields state-independent $2T$ subharmonic response, consistent with superconducting processor spectral–dynamical observations.

C Open System Dissipative Time Crystal Spectral Criterion

C.1 CPTP Mapping Perron–Frobenius Structure

Assume \mathcal{E} peripheral spectrum contains only m eigenphases $\{e^{2\pi ik/m}\}$, remaining spectrum strictly contracting. By positive operator spectral theory (Evans–Høegh-Krohn): m mutually disjoint attractor components exist, \mathcal{E}^n converges arbitrary initial state to period- m limit cycle; spectral gap Δ_{Liouv} controls convergence rate and perturbation resistance.

C.2 Interfacing with Algebraic Synchronization Theory

For time-independent Liouvillian, can characterize purely imaginary eigenvalues and persistent oscillation modes via algebraic symmetry–Lie algebra structure, delimiting conditions for stable/metastable synchronization and multi-frequency commensurability; this picture consistent with above peripheral spectrum group structure.

D Multi-Frequency Drive and Temporal Quasicrystal Group Representation Proof

D.1 Time Translation Group and Finite Image

Quasiperiodic drive makes time translation group \mathbb{Z}^k . Under high-frequency limit construct effective evolution

$$\mathcal{U}(\vec{n}) = U e^{-iH_\star \sum_i n_i T_i} \mathbf{g}(\vec{n}) U^\dagger + O(e^{-c \min_i \omega_i/J}),$$

where $\mathbf{g} : \mathbb{Z}^k \rightarrow G_f$ is quotient representation to finite group G_f . If $\text{Im } \mathbf{g}$ non-trivial, then $\mathcal{C}_O(\vec{n})$ exhibits peaks on multiple incommensurate subharmonic lines, defining temporal quasicrystal order.

E Topological Time Crystal Logical Order and Entanglement Entropy

E.1 Code Subspace Floquet Unitary

Under surface code Hamiltonian H_{SC} and periodic sequence, $F_{\text{logical}} \simeq \overline{X}_L e^{-iH_\star^{\text{top}} T}$. Non-local logical operators exhibit period-doubled rigid response.

E.2 Order Parameter and Topological Entanglement Entropy

Define autocorrelation $\mathcal{C}_{W_C}(n)$ of non-local loop operator W_C and topological term γ of subsystem entropy $S(A)$. Experiments measure non-zero γ jointly with W_C subharmonic locking establishing topological time crystal order.

F Experimental Error Budget and Calibration

F.1 Prethermal Window Engineering

Given hardware noise amplitude ε , select ω making $e^{-c\omega/J} \lesssim \varepsilon/10$, with pulse area error $|\epsilon| \lesssim \varepsilon$; ensure $T_2 \gg \tau_* \sim e^{c\omega/J}$.

F.2 Open System Limit Cycle Noise Shaping

Realize single limit cycle phase region via G/κ and detuning scan measuring \mathcal{E} peripheral eigenvalue family; Δ_{Liouv} and coherence-decoherence ratio determine stability.