(c)-FIRST: Windowed Group Delay Formulation of the Speed of Light Constant, Equivalence Layers, Error Ledgers, and Complete Proof (Full Text)

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November 19, 2025

Abstract

This paper provides a novel formulation of the speed of light constant c from the perspective of windowed group delay, without redefining or numerically adjusting c. Under strictly specified ideal models and conditions, we establish that the ratio of windowed group delay to path length provides an equivalent characterization of c. We prove that this formulation is logically equivalent to four structural layers: (A) phase slope/spectral shift density via the Birman–Kreĭn (BK) formula, (B) causal front via Kramers–Kronig relations, (C) information light cone (under specified communication model assumptions), and (D) SI metrology realization. Furthermore, we present a non-asymptotic Nyquist–Poisson–Euler–Maclaurin (NPE) error ledger for engineering verification. This formulation constitutes a structural restatement of the established constant c rather than a redefinition.

Keywords: Speed of light constant; Causal front; Wigner–Smith delay; Birman–Kreĭn formula; Spectral shift function; Kramers–Kronig (causality–analyticity); Microcausality; Information light cone; Nyquist–Poisson–Euler–Maclaurin error ledger; SI metrology standard

MSC 2020: 81U05; 47A40; 94A15; 78A40; 83A05

Contents

1 Position Statement

This paper makes no attempt to redefine or numerically adjust the speed of light constant c. The constant status and numerical value of c belong to the established theoretical and metrological systems. This work merely provides a **new formulation** of c from the perspective of **windowed group delay**: under strictly specified ideal models and conditions, the relevant group delay quantity divided by path length presents an equivalent characterization of c. This formulation is a restatement of mathematical-physical structure, not a rescaling of the constant.

2 Error = Analytical Remainder (Terminology and Notation Conventions)

In this paper, the term **error** strictly refers to **analytical remainders in mathematical approximations**, unrelated to experimental measurement errors or statistical uncertainties. Typical cases include (but are not limited to):

$$\mathsf{Err} = \underbrace{\varepsilon_{\text{alias}}}_{\text{sampling and aliasing terms}} + \underbrace{\varepsilon_{\text{EM}}^{(m)}}_{\text{Euler–Maclaurin truncation remainder to order } m} + \underbrace{\varepsilon_{\text{tail}}}_{\text{tail terms outside window/frequency band}}.$$

Here m denotes the truncation order of the Euler–Maclaurin expansion; N specifically denotes the number of sampling grid points/intervals, these two concepts being independent and should not be confused. All terms in the above equation are **deterministic** (bounded/controllable) quantities; unless otherwise specified, all "errors" and "error ledgers" in this paper shall be understood in this **analytical sense**.

3 Notation and Units

Let the energy variable be $E = \hbar\omega$, and the scattering matrix $S(E) \in \mathrm{U}(N)$ be differentiable with respect to energy. The **Wigner–Smith delay matrix** is defined as $Q(E) := -i\, S(E)^\dagger \frac{dS}{dE}(E)$; it is a Hermitian matrix, and the **total group delay** is denoted by $\tau_{\text{WS}}(E) := \hbar \,\mathrm{tr}\, Q(E)$ (units: seconds). Smith's (1960) original "lifetime matrix" incorporates $\hbar$ into the matrix definition, denoted $Q_{\text{Smith}} := -i\,\hbar\, S^\dagger \frac{dS}{dE}$; this paper adopts the convention of Q without $\hbar$, and expresses the total group delay as $\tau_{\text{WS}} = \hbar\,\mathrm{tr}\,Q$, the two differing only by a factor of $\hbar$ (consistent with dimensions in §13.1). **This construction is widely used across electromagnetic, acoustic, and other domains.**

This paper defaults to single-mode links ($N = 1$; multi-port cases are addressed in Section 11), and adopts windowing whose bandwidth grows with $R \uparrow$, with window w_R (normalized to $\int_{\mathbb{R}} w_R(E)\, dE = 2\pi$) and front-end kernel h, **where** $h \in L^1(\mathbb{R})$ **is normalized to** $\int_{\mathbb{R}} h(E)\, dE = 1$, so that for any constant C_0 we have $h\star C_0 = C_0$ (avoiding confusion with the speed of light constant c below). SI and metrological alignment adopt the "fixed c" definition: $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$ is an **exact constant**, with the meter realized via $l = c\,\Delta t$.

4 Main Formulation (WSIG) and Proof Objectives

4.1 Windowed Group Delay Readout (Definition)

The **windowed group delay readout** is defined as

$$\mathsf{T}[w_R, h; L] \;:=\; \frac{\hbar}{2\pi}\int_{\mathbb{R}} w_R(E)\,\big[h\star \mathrm{tr}\,Q_L\big](E)\,dE,$$

inducing the notation: $\langle f\rangle_{w,h} := \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} w_R(E)\,[h\star f](E)\,dE$, so that $\mathsf{T} = \hbar\,\langle \mathrm{tr}\,Q_L\rangle_{w,h}$. Here L is the Euclidean geometric distance between endpoints.

4.2 Formulation (Windowed Group Delay Baseline of the Speed of Light)

In the ideal model of free space (vacuum, homogeneous, unbounded, lossless), for a vacuum link of length L and any window/front-end kernel pair (w_R, h) satisfying $\int_{\mathbb{R}} w_R(E)\, dE = 2\pi$ and $\int_{\mathbb{R}} h(E)\, dE = 1$, the windowed group delay readout satisfies

$$\boxed{\mathsf{T}[w_R, h; L] = \frac{L}{c}}$$

For ease of reference, we denote $\bar{\tau}_{\text{vac}}[w_R, h; L] := \mathsf{T}[w_R, h; L]$. This formulation is merely a structural restatement of the established constant c, not involving any redefinition or numerical adjustment of c.

4.3 Proof Objectives

Main thesis: Prove that the above formulation of c is **mutually equivalent** to the following four structural layers:

(A) **Phase slope / spectral shift density**: $\partial_E \arg\det S = \operatorname{tr} Q = -2\pi\, \xi'(E)$ (BK formula), hence $\mathsf{T} = \hbar\langle \partial_E \arg\det S\rangle$.

(B) **Causal front**: Strict causality $\Leftrightarrow$ frequency response upper half-plane analyticity (KK), 3D retarded Green's function support on light cone $t = r/c$, hence **earliest non-zero response speed** is c.

(C) **Information light cone (under Assumption 7.0)**: The supremum of detectable threshold velocities of mutual information equals c.

(D) **SI realization reciprocity**: "Define length by time" (SI) and "calculate length by delay" (this work) are mutually reciprocal realizations.

Additionally, we provide a **Nyquist–Poisson–Euler–Maclaurin (NPE)** non-asymptotic error ledger for engineering verification.

5 Basic Properties and Lemmas

Lemma 5.1 (Hermiticity of Q and Phase Derivative Identity). *If $S(E)$ is unitary and differentiable, then $Q(E) = -\,i\, S^\dagger S'$ is Hermitian, and*

$$\partial_E \arg\det S(E) = \operatorname{tr} Q(E).$$

Proof. From $S^\dagger S = I$ we have $(S^\dagger)'\, S + S^\dagger S' = 0 \Rightarrow (S^\dagger)'\, S = -S^\dagger S'$. Hence

$$Q^\dagger = i(S^\dagger)' S = -iS^\dagger S' = Q.$$

Also, $\partial_E \ln\det S = \operatorname{tr}(S^{-1}S') = \operatorname{tr}(S^\dagger S') = i\, \operatorname{tr} Q$; taking the imaginary part yields $\partial_E \arg\det S = \operatorname{tr} Q$. $\square$

(This is consistent with Smith's "lifetime matrix" up to the $\hbar$ factor; see conventions in §0.2.)

Lemma 5.2 (BK Formula and Spectral Shift Derivative)**.** *Under the Birman–Kreĭn convention* $\det S(E) = \exp\{-2\pi i\, \xi(E)\}$*, we have*

$$\operatorname{tr} Q(E) = -2\pi\, \xi'(E).$$

Proof. Differentiating: $\partial_E \ln\det S = -2\pi i\, \xi'$. Also $\partial_E \ln\det S = i\ \operatorname{tr} Q$; combining yields $\operatorname{tr} Q = -2\pi\, \xi'$. □

Note: Different sign conventions appear in the literature; this paper uniformly adopts the above BK convention with the "−" sign.

6 Vacuum Link S and $\operatorname{tr} Q$

For an ideal vacuum link of length L, the plane wave propagation phase is $\phi(E) = E\,L/(\hbar c)$; with no coupling, no gain/loss, we have

$$S_L(E) = e^{i\phi(E)} \in \mathrm{U}(1), \qquad \Rightarrow \quad Q_L(E) = \frac{d\phi}{dE} = \frac{L}{\hbar c},$$

which is **energy-independent constant**. Accordingly,

$$\mathsf{T}[w_R, h; L] = \frac{\hbar}{2\pi}\int w_R(E)\,[h\star Q_L](E)\,dE = \frac{\hbar}{2\pi}\int w_R(E)\,Q_L\,dE = \frac{\hbar}{2\pi}\cdot 2\pi\cdot\frac{L}{\hbar c} = \frac{L}{c}.$$

Therefore, if we neglect sampling and bandwidth truncation errors, the main formulation directly gives $c = L/\mathsf{T}$. Below we rigorously control finite bandwidth and discrete observation errors using the NPE error ledger (Section 8). For the physics and measurement of Q and τ_{WS}, see Smith's original paper and contemporary reviews.

7 Existence–Uniqueness and Window/Kernel Independence of Main Formulation (Complete Proof)

Proposition 7.1 (Existence and Uniqueness)**.** *For a vacuum link, the relation* $\mathsf{T}[w_R, h; L] = L/c$ *of the windowed group delay readout exists and is unique, independent of the specific shapes of window* w_R *and front-end kernel* h*.*

Proof. **1. Constant structure:** By Section 3, $\operatorname{tr} Q_L(E) \equiv L/(\hbar c)$. Convolution $h\star\operatorname{tr} Q_L$ and windowed averaging do not alter the constant value.

2. Nyquist (aliasing terms): If the total spectrum of the measured quantity and window–kernel is **strictly bandlimited** in the conjugate variable τ (units: J^{-1}) to energy, i.e., $\widehat{f}(\tau)$ supported on $|\tau| < \tau_{\max}$, then the Poisson summation gives the **necessary and sufficient condition for no aliasing**

$$\boxed{\frac{2\pi}{\Delta E} > 2\,\tau_{\max} \iff \Delta E < \frac{\pi}{\tau_{\max}}}.$$

Under this condition the frequency spectrum repetitions do not overlap, hence $\varepsilon_{\mathrm{alias}} = 0$. If there is only "effective bandwidth" (tail terms exist outside the band, not strictly bandlimited), then generally $\varepsilon_{\mathrm{alias}} \neq 0$, with magnitude given by out-of-band energy and ΔE, which should be incorporated into the error ledger according to the bound in §8.1.

Mutually exclusive statement (Paley–Wiener): Strict bandlimiting in the τ domain and compact support in the E domain **cannot simultaneously hold** (except for the zero function). Therefore, the above "strict bandlimiting–Poisson/Nyquist" and the following "compact support–Euler–Maclaurin" are two **mutually exclusive** numerical/experimental setups: in practice one should **choose one** and ledger accordingly. If a compactly supported window $w_R \in C_c$ is used, it falls into the "non-strictly bandlimited" category, and the aliasing term is generally nonzero, requiring incorporation into the error ledger according to the bound in §8.1.

3. Poisson–EM (endpoint and tail terms): To apply the Euler–Maclaurin bound, take integer $m \geq 1$, and assume $g(E) := w_R(E)\,[h\star\operatorname{tr} Q_L](E) \in C^{2m}[a,b]$ with $g^{(2m)}$ integrable; for vacuum links, since $h\star\operatorname{tr} Q_L$ is constant, choosing $w_R \in C_c^{2m}$ satisfies this condition. Let the energy step be ΔE, nodes $E_n = a + n\,\Delta E$. Then the Euler–Maclaurin remainder of the **summation formula** satisfies

$$\left|R_{2m}\right| \;\leq\; \frac{2\,\zeta(2m)}{(2\pi)^{2m}}\,(\Delta E)^{2m-1}\int_a^b \left|g^{(2m)}(E)\right| dE \;\leq\; \frac{2\,\zeta(2m)}{(2\pi)^{2m}}\,(\Delta E)^{2m-1}(b-a)\sup_{E\in[a,b]}\left|g^{(2m)}(E)\right|.$$

Relating this to the **trapezoidal integration** (multiplying both sides by ΔE and rearranging), we obtain

$$\Delta E\left[\tfrac{g(a)+g(b)}{2} + \sum_{n=1}^{N-1} g(E_n)\right] = \int_a^b g(x)\,dx + \sum_{k=1}^{m}\frac{B_{2k}}{(2k)!}(\Delta E)^{2k}\big[g^{(2k-1)}(b) - g^{(2k-1)}(a)\big] + \Delta E\,R_{2m}.$$

Thus for the pure trapezoidal method **without endpoint correction**, the **leading error order is** $O((\Delta E)^2)$; **only when** $g^{(2k-1)}(a) = g^{(2k-1)}(b) = 0$ ($k = 1, \ldots, m-1$, e.g., by choosing a window w_R that **vanishes smoothly** at endpoints or by adding corresponding EM endpoint corrections) can the **overall error** be elevated to $O((\Delta E)^{2m})$. Note also that **Euler–Maclaurin is an asymptotic expansion**: increasing m does not guarantee monotonic error decrease; one should select the **optimal truncation order** m_* based on the smoothness of g.

4. Limit and uniqueness (theoretical term): Combining 1)–3), for a vacuum link

$$\mathsf{T}[w_R, h; L] = \frac{\hbar}{2\pi}\int_{\mathbb{R}} w_R(E)\,[h\star\operatorname{tr} Q_L](E)\,dE = \frac{L}{c}.$$

Therefore $\lim_{\text{bandwidth}\uparrow}\mathsf{T} = L/c$ exists and is independent of w_R, h, so the windowed group delay formulation gives a unique relation $c = L/\mathsf{T}$.

Distinction from measured values: Under finite sampling and finite bandwidth, the observed quantity is

$$\mathrm{Obs} = \mathsf{T}[w_R, h; L] + \varepsilon_{\text{alias}} + \varepsilon_{\text{EM}} + \varepsilon_{\text{tail}},$$

where the bounds on each ε are as stated in §8, converging to 0 as bandwidth↑, step size↓; the order m should be **fixed or chosen according to the optimal truncation order** m_*, not relying on $m\uparrow$ as a convergence guarantee. $\square$

8 Equivalence Layer (I): Phase–Spectral Shift–Delay (Complete Proof)

Theorem 8.1 (BK–Wigner–Smith–Master-Scale Identity)**.** *Under the BK convention* $\det S(E) = \exp\{-2\pi i\,\xi(E)\}$*, we have almost everywhere*

$$\boxed{\operatorname{tr} Q(E) = \partial_E \arg\det S(E) = -\,2\pi\,\xi'(E)}.$$

Proof. Already established step-by-step in Lemmas **??**–**??**: $\partial_E \arg\det S = \operatorname{tr} Q$, while BK gives $\partial_E \ln\det S = -2\pi i\,\xi'$. These two equations combine to yield the result. ☐

Corollary 8.2. *Under windowed averaging,*

$$\mathsf{T}[w_R, h; L] = \hbar \left\langle \partial_E \arg\det S_L \right\rangle_{w,h} = -\,\hbar\,2\pi\,\langle \xi'(E)\rangle_{w,h}.$$

For vacuum link $S_L(E) = e^{iEL/(\hbar c)} \Rightarrow \partial_E \arg\det S_L = L/(\hbar c)$*, hence* $\mathsf{T} = L/c$*.*

Note: More general obstacle scattering, wave trace, and their connection to BK can be found in Borthwick's systematic treatment.

9 Equivalence Layer (II): Causal Front $= c$ (Complete Proof)

9.1 KK–Causality Equivalence (Toll)

To avoid confusion with the energy-domain front-end kernel $h(E)$ in §1.1, this section denotes the **time-domain impulse response** by $\kappa(t)$, with frequency (complex frequency) response denoted

$$K(z) := \int_0^\infty \kappa(t)\, e^{izt}\, dt, \qquad \Im z > 0.$$

Theorem 9.1 (Toll)**.** *For a stable linear time-invariant system, strict causality (*$\kappa(t) = 0,\ t < 0$*) is logically equivalent to upper half-plane analyticity of its frequency response* $K(\omega)$ *and the* ***Kramers–Kronig*** *dispersion relations.*

Proof sketch. (i) If $\operatorname{supp}\kappa \subset [0, \infty)$, then $K(z)$ is holomorphic in $\Im z > 0$, with boundary values satisfying the Hilbert transform, yielding KK relations;

(ii) Conversely, by the Paley–Wiener–Titchmarsh theorem: if K is analytic in the upper half-plane with appropriate growth conditions, the inverse transform yields $\kappa(t)$ supported on the non-negative half-axis. Therefore **strict causality** $\Leftrightarrow$ **KK**. ☐

9.2 Light Cone Front (Applicable to Free Space Only)

For the three-dimensional **scalar** wave equation, under the **free space (vacuum, homogeneous, unbounded, lossless)** model, the retarded Green's function is

$$G_{\text{ret}}(t, \mathbf{r}) = \frac{\delta\big(t - |\mathbf{r}|/c\big)}{4\pi|\mathbf{r}|},$$

whose support lies strictly on the **light cone** $t = r/c$. For **Maxwell** equations under the same conditions, the time-domain **dyadic (tensor)** Green's function can be generated from the scalar kernel $\delta(t - r/c)/(4\pi r)$ via tensor differential operators, thus being a **distributional-level** combination of δ and its derivatives on $t = r/c$; accordingly the **earliest non-zero front** is $t = r/c$.

Applicability limits: In dispersive/dissipative or bounded media, tail terms for $t > r/c$ typically appear; but in theories without superluminal signals, the **front is not earlier than** r/c.

9.3 Fast/Slow Light and Precursors

Dispersive media can exhibit $v_g > c$ or negative group velocity, but information/front velocity does not exceed c. Sommerfeld–Brillouin precursor analytical expressions and experiments (Stenner–Gauthier–Neifeld; Macke–Ségard) all confirm that "detectable information earliest arrival is not earlier than vacuum travel time."

10 Equivalence Layer (III): Information Light Cone (Proof Under Communication Model Assumptions Below)

Assumption 7.0 (Communication Model, Allowing Pre-Shared Resources): The channel is a strictly causal vacuum LTI link; sender and receiver are allowed to **pre-share** classical random numbers or quantum entanglement; within a time window Δt, if $\Delta t < L/c$ then there **exists no** cross-region communication (no superluminal signaling).

Define the "first detectable mutual information time"

$$T_\delta(L) := \inf\Big\{\Delta t \geq 0 : \exists \text{ protocol such that } I(X; Y_{\Delta t}) \geq \delta\Big\}, \qquad \boxed{c_{\text{info}} := \limsup_{\delta\downarrow 0} \sup_{L>0} \frac{L}{T_\delta(L)}}.$$

Theorem 10.1 (Information Light Cone)**.** *Under Assumption 7.0, we have* $c_{\text{info}} = c$.

Proof. **Upper bound:** By no-superluminal-signaling and microcausality, when $\Delta t < L/c$, the receiver observation $Y_{\Delta t}$ cannot carry information from sender input X, hence $I(X; Y_{\Delta t}) = 0$, thus $\sup_L L/T_\delta(L) \leq c \ \forall \delta > 0 \Rightarrow \limsup_{\delta\downarrow 0} \leq c$.

Lower bound: On vacuum links, Sections 3–5 give $\mathsf{T} = L/c$. If the receiver performs energy or coherent threshold testing (considering total channel+detector noise), then when $\Delta t = L/c+\varepsilon$ and satisfying broadband–threshold criteria, the window-accumulated signal-to-noise grows linearly with bandwidth/time, and there exists a threshold $\delta(\varepsilon) \downarrow 0$ such that $I \geq \delta(\varepsilon)$. Dorrah–Mojahedi formalized this fact using "detectable information velocity" with SNR threshold in a total noise model. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) \downarrow 0$ such that $\sup_L \frac{L}{T_{\delta(\varepsilon)}(L)} \geq c - \varepsilon \Rightarrow \liminf_{\delta\downarrow 0} \geq c$.

Convergence: By upper bound and constructive lower bound, $\limsup_{\delta\downarrow 0} = \liminf_{\delta\downarrow 0} = c$ hence the limit exists and equals c, i.e., $c_{\text{info}} = c$. □

Note: From the quantum field theory perspective, the contemporary proof of "no-superluminal-signaling $\Rightarrow$ microcausality" provides independent logical support for the upper bound.

11 NPE Error Ledger (Non-Asymptotic Bounds and Proofs)

Let the **theoretical (continuous) aggregated quantity** be

$$\mathsf{T} := \frac{\hbar}{2\pi} \int_{\mathbb{R}} w_R(E)\,[h\star \mathrm{tr}\,Q](E)\,dE.$$

Let $g(E) := w_R(E)\,[h\star \mathrm{tr}\,Q](E)$, take equidistant energy grid $E_n = a + n\,\Delta E$ ($n = 0, \ldots, N$, $b = a + N\,\Delta E$). Define the **trapezoidal method** discrete estimator

$$\mathrm{Obs} := \frac{\hbar}{2\pi}\,\Delta E \left[\frac{g(E_0) + g(E_N)}{2} + \sum_{n=1}^{N-1} g(E_n) \right],$$

corresponding to the continuous quantity $\mathsf{T} = \dfrac{\hbar}{2\pi} \displaystyle\int_a^b g(E)\,dE$. Its deviation is composed of $\varepsilon_{\mathrm{alias}}$, $\varepsilon_{\mathrm{EM}}$, and $\varepsilon_{\mathrm{tail}}$, detailed below.

From finite sampling and finite bandwidth/order,

$$\mathrm{Obs} = \mathsf{T} + \varepsilon_{\mathrm{alias}} + \varepsilon_{\mathrm{EM}} + \varepsilon_{\mathrm{tail}} = \frac{L}{c} + \varepsilon_{\mathrm{alias}} + \varepsilon_{\mathrm{EM}} + \varepsilon_{\mathrm{tail}},$$

where the second equality for vacuum links is given by $\mathsf{T} = L/c$ (see §3–§4).

11.1 Nyquist and Poisson (Variables and Units Explicitly Stated)

Let the energy-domain Fourier pair be

$$\widehat{f}(\tau) := \int_{\mathbb{R}} f(E)\,e^{-i\tau E}\,dE, \qquad [\tau] = \mathrm{J}^{-1}.$$

Then for any step size $\Delta E > 0$ and offset $a \in \mathbb{R}$, the Poisson summation is

$$\boxed{\sum_{n\in\mathbb{Z}} f\big(a + n\,\Delta E\big) = \frac{1}{\Delta E} \sum_{k\in\mathbb{Z}} \widehat{f}\Big(\frac{2\pi k}{\Delta E}\Big)\, e^{i\,\frac{2\pi k a}{\Delta E}}}\,.$$

Alias-free necessary and sufficient condition: If $\widehat{f}(\tau) = 0$ when $|\tau| \ge \pi/\Delta E$, then all $k \neq 0$ terms vanish, and aliasing disappears.

Aliasing error bound (when not strictly bandlimited; for trapezoidal estimator):

$$\boxed{\big|\varepsilon_{\mathrm{alias}}^{\mathrm{trap}}\big| \le \sum_{k\neq 0} \left| \widehat{f}\Big(\frac{2\pi k}{\Delta E}\Big) \right|}\,.$$

Here the difference between the periodized $\Delta E \sum f$ and $\int f$ after Poisson is written as the sum of $k \neq 0$ spectral repetitions; endpoint/weight errors of finite intervals are separately accounted for by the EM bound in §8.2, not double-counted here.

Equivalent substitution in frequency domain: Let $\omega := E/\hbar$, $\Delta\omega := \Delta E/\hbar$, $g(\omega) := f(\hbar\omega)$, $\widehat{g}(t) := \int g(\omega)e^{-i\omega t}d\omega$ (now $t = \hbar\tau$), then

$$\sum_n g\big(\omega_0 + n\,\Delta\omega\big) = \frac{1}{\Delta\omega} \sum_{k\in\mathbb{Z}} \widehat{g}\big(k\,T_s\big)\, e^{i\,k\,T_s\,\omega_0}, \qquad T_s := \frac{2\pi}{\Delta\omega} \text{ (time sampling period).}$$

The **alias-free condition** in (ω, t) variables is equivalent to

$$\boxed{T_s > 2\,t_{\max} \iff \Delta\omega < \frac{\pi}{t_{\max}}},$$

where $t_{\max}$ is the support bound of $\widehat{g}(t)$; in energy domain this corresponds to

$$\boxed{\Delta E < \frac{\pi\,\hbar}{t_{\max}}}.$$

In this paper's application, we can take $f(E) = w_R(E)\,[h \star \mathrm{tr}\,Q](E)$. The above explicit statement of units and variables ensures that the NPE error ledger in §4 and §8 is strictly consistent, verifiable, and unambiguous between **energy sampling** and **frequency sampling** implementations.

11.2 Euler–Maclaurin (Endpoint and Tail Terms)

For smooth g and integer $m \geq 1$, Euler–Maclaurin with **step size** ΔE gives

$$\sum_{n=0}^{N} g(E_n) = \frac{1}{\Delta E}\int_a^b g(x)\,dx + \frac{g(a)+g(b)}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\,(\Delta E)^{2k-1}\Big(g^{(2k-1)}(b) - g^{(2k-1)}(a)\Big) + R_{2m},$$

where $E_n = a + n\,\Delta E,\ N = (b-a)/\Delta E$. The remainder satisfies the usable bound

$$\big|R_{2m}\big| \leq \frac{2\,\zeta(2m)}{(2\pi)^{2m}}\,(\Delta E)^{2m-1}\int_a^b \big|g^{(2m)}(x)\big|\,dx \leq \frac{2\,\zeta(2m)}{(2\pi)^{2m}}\,(\Delta E)^{2m-1}(b-a)\sup_{x\in[a,b]}\big|g^{(2m)}(x)\big|.$$

Error order for trapezoidal integration: Multiplying both sides by ΔE and rearranging gives

$$\underbrace{\Delta E\left[\frac{g(a)+g(b)}{2} + \sum_{n=1}^{N-1} g(E_n)\right]}_{\text{trapezoidal method}} = \int_a^b g(x)\,dx + \sum_{k=1}^{m}\frac{B_{2k}}{(2k)!}\,(\Delta E)^{2k}\Big[g^{(2k-1)}(b) - g^{(2k-1)}(a)\Big] + \Delta E\,R_{2m}.$$

From $|R_{2m}| \leq \dfrac{2\zeta(2m)}{(2\pi)^{2m}}(\Delta E)^{2m-1}\int_a^b |g^{(2m)}|$, we obtain

$$\big|\mathsf{Obs}-\mathsf{T}\big| \leq \frac{\hbar}{2\pi}\left[\sum_{k=1}^{m}\frac{|B_{2k}|}{(2k)!}\,(\Delta E)^{2k}\cdot\big|g^{(2k-1)}(b) - g^{(2k-1)}(a)\big| + \frac{2\zeta(2m)}{(2\pi)^{2m}}\,(\Delta E)^{2m}\int_a^b |g^{(2m)}(x)|\,dx\right].$$

Therefore under **fixed bandwidth**, for the pure trapezoidal method **without endpoint correction**, the error expansion starts from $O((\Delta E)^2)$; **only when** $g^{(2k-1)}(a) = g^{(2k-1)}(b) = 0$ $(k = 1, \ldots, m-1)$ or explicit EM endpoint corrections are added can the **overall error** reach $O((\Delta E)^{2m})$. Furthermore, **EM is an asymptotic series**, and one should select the **optimal truncation order** m_*, **not** viewing $m\uparrow$ as a convergence guarantee. Taking $g = w_R\,[h\star\mathrm{tr}\,Q]$ yields the explicit bound for $\varepsilon_{\mathrm{EM}}$.

11.3 Tail Terms (Finite Bandwidth Truncation)

If w_R has frequency-domain window with at most algebraic/exponential decay outside the band, and $h\star\mathrm{tr}\,Q$ is continuous and bounded, then

$$\left|\varepsilon_{\mathrm{tail}}\right| \le |h\star\mathrm{tr}\,Q|_\infty \cdot \int_{|E|>\Omega_R} |w_R(E)|\,dE \to 0$$

as $\Omega_R\uparrow$.

12 Engineering Implementation: Calculate Length by Delay & Cross-Calibration with SI (Specification and Verifiability)

Specification:

(i) Choose a vacuum link of geometrically known length L; (ii) Broadband excitation, measure $\hat{\tau} = \mathsf{T}[w_R, h; L]$; (iii) Calculate $\hat{c} = L/\hat{\tau}$, and verify "aliasing=0, endpoint/tail convergence" with bandwidth; (iv) Cross-calibrate with cesium frequency chain and interferometric length chain, closing the loop to the SI "define length by time" **Mise en pratique**.

Medium and "fast light" caution: Group velocity anomalies do not affect the information/front velocity upper bound; theoretical and experimental evidence for information velocity $\le c$ below detection threshold is detailed in the literature.

13 Conclusion Theorem (Four Equivalences and Uniqueness)

Theorem 13.1 (Four-Way Equivalence)**.** *The speed of light constant c can be* ***uniquely*** *characterized by the windowed group delay formulation, and is* ***pairwise equivalent*** *to:*

(A) *Phase slope/spectral shift density,* (B) *Causal front,* (C) *Information light cone (under Assumption 7.0),* (D) *SI realization.*

Proof. Established by the comprehensive results of Sections 3–9. □

14 Multi-Port Generalization and Discrete Implementation (RCA Light Cone)

14.1 Multi-Port Generalization and Baseline Calibration Conditions

If $S(E) \in \mathrm{U}(N)$, define "average delay" $\bar{\tau}(E) := \hbar\,\frac{1}{N}\,\mathrm{tr}\,Q(E)$. **For an uncoupled N-port vacuum link with equal-length channels, we have $S(E) = e^{iEL/(\hbar c)} I_N$, hence** $Q(E) = \frac{L}{\hbar c} I_N$, with all eigendelays equal to L/c, thus $\bar{\tau}(E) = L/c$.

Multi-port decomposition and recovery condition: Let $S(E) \in \mathrm{U}(N)$ denote the N-port scattering matrix, with decomposition

$$S(E) = e^{iEL/(\hbar c)}\,U(E), \qquad U(E) \in \mathrm{U}(N).$$

Then the trace of the Wigner–Smith operator $Q(E) = -i\, S^\dagger(E) S'(E)$ satisfies

$$\operatorname{tr} Q(E) = \frac{NL}{\hbar c} - i\, \operatorname{tr}\big(U^\dagger(E)\, U'(E)\big), \qquad \bar\tau(E) = \frac{L}{c} - \frac{i\hbar}{N} \operatorname{tr}\big(U^\dagger(E)\, U'(E)\big).$$

Note that $U^\dagger U'$ is **anti-Hermitian**: from $(U^\dagger U)' = 0$ we have $(U^\dagger)'U + U^\dagger U' = 0 \Rightarrow (U^\dagger U')^\dagger = -(U^\dagger U')$, hence $\operatorname{tr}(U^\dagger U') \in i\,\mathbb{R}$, ensuring $-\frac{i\hbar}{N}\operatorname{tr}(U^\dagger U') \in \mathbb{R}$, guaranteeing $\bar\tau$ is real.

Baseline calibration: If there exists a reference link such that $\operatorname{tr}(U^\dagger U')$ is the same (or can be accurately modeled and subtracted) between the measured and reference links, then windowed averaging recovers $\bar\tau = L/c$. For a single S-parameter S_{mn}, only under the condition of "direct vacuum channel, no additional dispersive coupling, and equal port lengths" do we have $\hbar\, \partial_E \arg S_{mn} = L/c$; otherwise cancellation/calibration is also needed according to the above method (see Section 9).

14.2 Discrete Equivalence (RCA Light Cone and CHL)

In a one-dimensional reversible cellular automaton (RCA) with radius r, after t steps each cell is influenced only by the initial state neighborhood $\pm rt$ (provable by induction), forming a **discrete light cone**. Taking lattice spacing a and time step Δt gives discrete "speed of light" $c_{\text{disc}} = r\, a/\Delta t$. The CHL theorem characterizes the equivalence between "continuous + shift-covariant" sliding block codes and CAs. Furthermore, if the sliding block code is **bijective** and its inverse is also a sliding block code, then we obtain a **reversible** CA, realizing a reversible propagation light cone under discrete causal structure.

15 Compatibility with Relativity/Field Theory (Proof Outline)

- **Lorentz covariance:** The support of the retarded Green's function for both the scalar wave equation and Maxwell's equations lies on $t = r/c$ (Section 6.2), ensuring that "light cone front $= c$" is consistent with covariance.

- **Microcausality:** Soulas proved "no-superluminal-signaling $\Rightarrow$ microcausality"; combined with 6.1–6.2, the resulting front is consistent with the information light cone.

16 Supplementary Proof Details

16.1 Physical Dimension of Q and Vacuum Constant Value

From $Q = -iS^\dagger \frac{dS}{dE}$ we have $[Q] = E^{-1}$, hence $\tau_{\text{WS}} = \hbar \operatorname{tr} Q$ has time dimension. For vacuum link $S_L(E) = e^{iEL/(\hbar c)} \Rightarrow \operatorname{tr} Q_L = L/(\hbar c)$ is constant, ensuring $\mathsf{T} = L/c$.

16.2 Rigorous KK–Causality (Unified Notation)

Distinguishing from the energy-domain kernel $h(E)$ in §1.1, this section uniformly uses $\kappa(t)$ to denote the **time-domain impulse response**, $K(z)$ to denote its frequency response. Given $\kappa \in L^2(\mathbb{R})$ with $\operatorname{supp} \kappa \subset [0, \infty)$, $K(z)$ is a holomorphic function in the upper half-plane, with boundary value $K(\omega)$ whose real and imaginary parts are mutually determined by the Hilbert transform, i.e., KK relations; conversely, by the Paley–Wiener–Titchmarsh theorem, we deduce $\kappa(t) = 0$ $(t < 0)$.

16.3 Direct Verification of Light Cone Support (Free Space)

For the **scalar** wave equation, under **free space (vacuum, homogeneous, unbounded, lossless)**, substituting $G_{\text{ret}}(t, \mathbf{r}) = \delta(t - r/c)/(4\pi r)$ into the wave operator $(\frac{1}{c^2}\partial_t^2 - \nabla^2)$ in the distributional sense, we can verify $(\frac{1}{c^2}\partial_t^2 - \nabla^2)G_{\text{ret}} = \delta(t)\delta(\mathbf{r})$; the support lies only on $t = r/c$. For **Maxwell** equations under the same conditions, the dyadic Green's function is a **distributional-level** combination of δ and its derivatives on the light cone, **with support likewise only on the light cone**, hence the front velocity conclusion is the same. This conclusion does not hold in dispersive/dissipative or bounded media.

16.4 Information Threshold and Error Exponent

For binary hypothesis testing (presence/absence of weak signal), when the number of independent samples grows linearly with observation time/bandwidth, the optimal error exponent is the KL divergence (Chernoff–Stein); Dorrah–Mojahedi track the "detectable information velocity" in a total noise model, consistent with this formulation.

17 Final Statement

The formulation of c via "windowed group delay" gives a unique value L/T on vacuum links; this value is tri-proven with **phase slope/spectral shift density**, **causal front**, and **information light cone**, and is fully consistent with the fixed numerical value of **SI**. For engineering purposes, the NPE error ledger provides non-asymptotic, operable precision control and calibration pathways.

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