WSIG–EBOC Unified Theory of **Spacetime/Time/Space**

Haobo Ma[1] Wenlin Zhang[2]

[1]Independent Researcher

[2]National University of Singapore

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Abstract

Within the WSIG (Windowed Scattering & Information Geometry) and EBOC (Eternal-Block Observer-Computing) frameworks, we establish an operational spacetime theory based on **windowed scattering**: taking the **phase derivative–spectral shift density–Wigner–Smith group delay** triple equivalence as the metrological primitive; using the **Kramers–Kronig causality–analyticity** (restricted to stable LTI) and the light cone support of wave equation **retarded Green's function** (time-domain support, applicable to LTV), we provide **an upper bound in terms of front-propagating optical metric: causal front does not exceed** c; and the equality holds **if and only if the front-detectability condition (§5)** is satisfied. Using the first-detection time of **threshold mutual information** to establish the **information light cone bound**: in vacuum or static media (LTI),

$$c_{\text{info}} := \lim_{\delta\downarrow 0}\sup \frac{D}{T_\delta} \ \leq\ c,$$

with equality $c_{\text{info}} = c$ **if and only if** the **front-detectability** (§5) holds; **where** $D = D_{\text{front}}$ (link normalization, $t_h = 0$) or $D = D_{\text{sys}} := D_{\text{front}} + c\,t_h$ (system normalization, $t_h > 0$). For LTV, only $T_\delta \geq t_{\min}$ is obtained. We provide **operational definitions** for **time** and **space**, and characterize **spacetime** as a four-tuple $(\mathcal{E}, \preceq, g, \mu_\varphi)$: where $\preceq$ is induced by light cone support, g is the (media-inclusive) optical Lorentzian metric, $\mu_\varphi = (\varphi'/\pi)\,dx$ is the **phase density measure** given by de Branges kernel diagonal. The core theorems prove the four-way equivalence of "**phase slope = group delay = spectral shift density = SI realization**", and provide **non-asymptotic** detection bounds under the **Nyquist–Poisson–Euler–Maclaurin (NPE)** error ledger. The theory is compatible with the discrete light cone of the CHL theorem for **reversible cellular automata (RCA)**, and establishes isomorphisms with density thresholds (Landau, Wexler–Raz, Balian–Low) for sampling/interpolation/frames.

Keywords: Wigner-Smith group delay; Birman-Kreĭn spectral shift; retarded Green's function; optical metric; information light cone; NPE error ledger; de Branges phase density; reversible cellular automata; Landau density; Balian-Low theorem

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Contents

1 Notation and Preliminaries

- Scattering matrix $S(E) \in U(N)$. **Wigner–Smith delay matrix** defined as $Q(E) := -\,i\,S(E)^\dagger \frac{dS}{dE}(E)$, with $\operatorname{tr} Q(E)$ having dimension 1/energy; physical **group delay** is $\tau_{\mathrm{WS}}(E) := \hbar \operatorname{tr} Q(E)$, with dimension of time. This definition traces back to Smith's characterization of the "lifetime matrix". Standard literature defines Wigner–Smith delay as "derivative of phase with respect to energy" corresponding to $\operatorname{tr} Q$, with $\tau = \hbar\,\mathrm{tr}Q$ having time dimension.

- **Birman–Kreĭn (BK) formula**: If $\det S(E) = e^{-2\pi i\,\xi(E)}$, then $\operatorname{tr} Q(E) = -2\pi\,\xi'(E)$.

- **KK–causality (LTI restricted)**: For **stable LTI** systems (impulse response $h_{\mathrm{sys}}(t)$), strict causality $\Rightarrow$ frequency response analytic in upper half-plane; and under **additional conditions** ($h_{\mathrm{sys}} \in L^1(\mathbb{R})$, $H(\omega)$ polynomially bounded growth, no upper half-plane poles, etc.), real and imaginary parts satisfy Kramers–Kronig dispersion, and strict causality can be recovered (analyticity $\Rightarrow$ causality). **LTV/nonstationary** cases do not apply this frequency-domain equivalence; this paper uses only the time-domain support formulation of G_{ret} (see §5).

- **Retarded Green's function**: In 3D **vacuum unbounded domain**, the solution to the scalar wave equation $\Box G_{\mathrm{ret}} = \delta$ is $G_{\mathrm{ret}}(t, \mathbf{r}) = \delta(t - |\mathbf{r}|/c)/(4\pi|\mathbf{r}|)$, with support exactly at $t = r/c$; **Maxwell**'s time-domain **dyadic** kernel is obtained from this scalar kernel via tensor–differential operators (containing δ and its derivatives), with support likewise **only on the light cone**. **In bounded domains/media/dispersion**, **precursors/tails** generally appear; see **§0 "front decomposition (definition)"**.

- **Optical metric (Gordon)**: Front time **satisfies**

$$t_{\min} \ge \frac{d_{\mathrm{front}}}{c},$$

  where d_{front} is defined by the **3D Fermat metric** of high-frequency limit refractive index n_∞. The equality $t_{\min} = d_{\mathrm{front}}/c$ holds **if and only if the front-detectability condition (§5)** is satisfied; 3D vacuum pure propagation is a special case. Many passive media satisfy "vacuumization" at high frequencies $n(\mathbf{x}, \omega) \to 1$ (thus $n_\infty = 1$), giving $d_{\mathrm{front}} = |A - B|$; in this case, **in experimental coordinates** one can unconditionally only assert $t_{\min} \ge |A - B|/c$, and **only when** the above **equality condition** holds does one **recover** "front velocity = c". (Group velocity can differ from c—Sommerfeld–Brillouin precursor). For modern reviews and extensions, see Leonhardt–Philbin and subsequent work.

- **Fast/slow light and information velocity**: Experimental and information-theoretic definitions show that **detectable information velocity does not exceed** c.

- **NPE error ledger**: Nyquist sampling theorem and aliasing condition (Shannon/Nyquist); Poisson summation formula (NIST DLMF §1.8(iv)); Euler–Maclaurin bounded remainder (DLMF §2.10).

- **de Branges kernel diagonal**: For the space $\mathcal{H}(\mathcal{E})$ of a Hermite–Biehler function $\mathcal{E}$, we have $K(x,x) = \frac{1}{\pi}\,\varphi'(x)\,|\mathcal{E}(x)|^2$, where φ is the phase function. **(Notation unification)** This paper uses E to denote energy (also as real frequency variable); in de Branges sections, x denotes the same real axis variable and is identified with E, written as $x \equiv E$. Thus $\mu_\varphi = (\varphi'(E)/\pi)\,dE$, consistent with the normalization $\frac{1}{2\pi}\int w_R(E)\,dE = 1$ in §2.

- **Weyl–Heisenberg representation and uniqueness**: Stone–von Neumann theorem and foundations of coherent/time-frequency frameworks, see Folland.

- **Windowed convolution and averaging notation**: Take energy window $w_R \in L^1(\mathbb{R})$ and **unit integral kernel** $\eta \in L^1(\mathbb{R})$. Normalization

$$\frac{1}{2\pi}\int_{\mathbb{R}} w_R(E)\,dE = 1, \qquad \int_{\mathbb{R}} \eta(\nu)\,d\nu = 1.$$

  Define convolution and windowed average

$$[\eta \star f](E) := \int_{\mathbb{R}} \eta(\nu)\,f(E-\nu)\,d\nu, \qquad \langle f \rangle_{w,\eta} := \frac{1}{2\pi}\int_{\mathbb{R}} w_R(E)\,[\eta \star f](E)\,dE.$$

  For vacuum pure delay link $S_L(E) = e^{iEL/(\hbar c)}$, from $\operatorname{tr} Q_L \equiv L/(\hbar c)$ and the above normalization, we obtain $\hbar\langle \operatorname{tr} Q_L\rangle_{w,\eta} = L/c$ (see §4). **(Symbol clarification)** This paper uses η for **unit integral kernel** (for windowed readout), $h_{\mathrm{sys}}(t)$ for **system impulse response** (LTI filter), with $t_h := \inf\{t : h_{\mathrm{sys}}(t) \neq 0\}$ as its onset delay.

- **Distributions and generalized function notation**: $\delta(\cdot)$ denotes the Dirac δ distribution; $\Theta(t)$ is the Heaviside step function, $\Theta(t) = 0$ $(t < 0)$, $\Theta(t) = 1$ $(t > 0)$ (taking $\Theta(0) = \frac{1}{2}$ does not affect results).

- **Front decomposition (definition)**: In vacuum or static media (LTI) and satisfying front assumption (F), the retarded Green's function in the **distributional sense** can be written as

$$G_{\mathrm{ret}}(t;x,y) = \sum_{k=0}^{m} K_k(x,y)\,\delta^{(k)}\!\Big(t - \tfrac{d_{\mathrm{front}}(x,y)}{c}\Big) + \Theta\Big(t - \tfrac{d_{\mathrm{front}}(x,y)}{c}\Big)\,g(t;x,y),$$

  where $m < \infty$, $K_k(x,y)$ are amplitude coefficients of front singularities, g is the **tail kernel** (locally integrable). We say "**front-detectability** (at the distributional level)" if and only if **there exists some** $k \geq 0$ **such that** $K_k(x,y) \neq 0$; Maxwell case allows $k \geq 1$.

- **Scattering background and energy representation**: Let $\mathcal{H}$ be a Hilbert space, H_0, H be self-adjoint operators (free/full Hamiltonian). Assume wave operators $W_\pm := s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ exist and are complete, then scattering operator $\mathsf{S} := W_+^* W_-$ fiberizes in energy representation to $S(E) \in U(N)$. Wigner–Smith delay matrix is defined as $Q(E) := -\,i\,S(E)^\dagger \partial_E S(E)$, consistent with notation in §0.

- **Trace notation**: This paper uses tr for finite-dimensional matrix trace (e.g. $S(E) \in U(N)$), reserving Tr for trace-class operators on Hilbert spaces (e.g. T_{w_φ} in §6).

2 Axioms (WSIG–EBOC Spacetime)

Let

$$\mathbb{S} = \big(\mathcal{E},\ \preceq,\ g,\ \mu_\varphi;\ H, H_0, S(\cdot);\ \mathcal{W}, \mathcal{H}\big).$$

Hierarchy clarification: This paper calls $(\mathcal{E}, \preceq, g, \mu_\varphi)$ the **spacetime four-tuple** (ontological layer), while $(H, H_0, S(\cdot); \mathcal{W}, \mathcal{H})$ are **realization data** (realization layer, used to give windowed scattering readouts and proof chain); these two must not be confused.

We call $\mathbb{S}$ a **WSIG–EBOC spacetime** if:

(A1) Events and causal order: $\mathcal{E}$ is the event set; if the wave equation's $G_{\rm ret}$ **has nonzero support on the pair** (e_1, e_2) (i.e. $G_{\rm ret}(e_2, e_1) \neq 0$), then set $e_1 \preceq e_2$. For discrete systems, take RCA neighborhood propagation bound (see §9).

(A2) Representation and observability: $\mathcal{H}$ is a Hilbert space; (U_τ, V_σ) is a projective unitary representation of Weyl–Heisenberg, supporting windowed readouts.

(A3) Phase–density–delay dictionary: There exists window $\mathcal{W}$ such that windowed readouts of $\partial_E \arg\det S = \operatorname{tr} Q = -2\pi\, \xi'(E)$ hold.

(A4) Causality–analyticity consistency: In the **stable LTI** case, strict causality $\Rightarrow$ frequency response analytic in upper half-plane; and under **additional conditions** ($h_{\rm sys} \in L^1(\mathbb{R})$, $H(\omega)$ polynomially bounded growth, no upper half-plane poles, etc.), real and imaginary parts satisfy Kramers–Kronig dispersion, and strict causality can be recovered (analyticity $\Rightarrow$ causality). **LTV/nonstationary** cases do not use this frequency-domain equivalence, only the time-domain support formulation of $G_{\rm ret}(t, \tau)$ (see §5). **In time-invariant/static media**, we **only assert**

$$t_{\min} \geq \frac{D_{\rm front}}{c}, \qquad D_{\rm front} = \begin{cases} L, & \text{vacuum}, \\ d_{\rm front}(x, y), & \text{media/inhomogeneous/bounded domain}. \end{cases}$$

The equality $t_{\min} = d_{\rm front}/c$ holds **if and only if front-detectability (§5)** is satisfied; 3D vacuum pure propagation is a special case. If cascaded with a strictly causal LTI filter (impulse response $h_{\rm sys}(t)$, onset delay $t_h := \inf\{t :\ h_{\rm sys}(t) \neq 0\}$), then

$$t_{\min}\ \geq\ \frac{D_{\rm front}}{c} + t_h,$$

and **only when front-detectability (§5) is satisfied and there is no systematic cancellation at the front**, do we have

$$t_{\min} = \frac{D_{\rm front}}{c} + t_h.$$

(A5) Front–information consistency: In **vacuum or static media (LTI)**, the first-detection time T_δ of threshold mutual information satisfies

$$c_{\rm info} := \limsup_{\delta \downarrow 0} \frac{D}{T_\delta}\ \leq\ c,$$

where D is the **front optical path** in the normalization used. The equality $c_{\rm info} = c$ holds if and only if **front-detectability** is satisfied; here **"front-detectability"** means: **according to §0 front decomposition** $G_{\rm ret}(t; x, y) = \sum_{k=0}^{m} K_k(x, y)\, \delta^{(k)}\big(t - \frac{D}{c}\big) + \Theta(\cdot)\, g$, **there exists some** $k \geq 0$ **such that** $K_k(x, y) \neq 0$, or the measurement chain contains a **Dirac pass-through component** (impulse response contains $\delta(t)$), or there exist $t_n \downarrow D/c$ and unit-energy short pulse ψ such that $\int G_{\rm ret}(t_n,\tau;x,y)\psi(\tau)\,d\tau \neq 0$. **(i) Link normalization** ($t_h = 0$): $D = D_{\rm front}$. **(ii) System normalization** ($t_h > 0$): $D = D_{\rm sys} := D_{\rm front} + c\,t_h$. **(LTV supplement)**: For time-varying systems, only assert $T_\delta \geq t_{\min}$ (defined by time-domain support of $G_{\rm ret}(t,\tau)$).

(A6) Sampling–error closure: Readout error is given by NPE three-term decomposition with non-asymptotic upper bounds.

(A7) SI alignment: c takes SI fixed value; "length from delay" and "time from length" are mutually inverse realizations (§4, §12).

(A8) Phase density geometry: de Branges kernel diagonal $K(x,x) = \frac{1}{\pi}\,\varphi'(x)\,|\mathcal{E}(x)|^2$ induces $\mu_\varphi := (\varphi'/\pi)\,dx$ as windowed calibration.

3 Operational Definitions: Time and Space

Definition 2.1 (Time) Take window w_R and unit integral kernel η. For link scattering $S(E)$, define **windowed group delay readout**

$$\mathsf{T}[w_R,\eta] := \hbar\,\frac{1}{2\pi}\int_{\mathbb{R}} w_R(E)\,\big[\eta \star \operatorname{tr} Q\big](E)\,dE,$$

where $Q := -iS^\dagger \frac{dS}{dE}$, and $\operatorname{tr} Q$ is equivalent to $\partial_E \arg\det S$ and $-2\pi\xi'(E)$.

Normalization: Convention $(2\pi)^{-1}\int_{\mathbb{R}} w_R(E)\,dE = 1$ and $\int_{\mathbb{R}}\eta(E)\,dE = 1$ (unit integral kernel), giving $\mathsf{T} = L/c$ on vacuum pure delay link. Verification: for $S_L(E) = e^{iEL/(\hbar c)}$ we have constant $\operatorname{tr} Q = L/(\hbar c)$, substituting into the above and using the normalization immediately yields $\mathsf{T} = L/c$; consistent with Theorem 4.1.

For a vacuum link of length L, the defined **time coordinate difference** satisfies $\Delta t = L/c$.

Definition 2.2 (Space) In vacuum, define spatial distance by **radar distance**: $d(A,B) := \frac{c}{2}\,\mathsf{T}_{\rm roundtrip}(A \to B \to A)$.

In media/inhomogeneous/bounded domains, take high-frequency limit refractive index $n_\infty(\mathbf{x}) = \lim_{\omega\to\infty} n(\mathbf{x},\omega)$. Define 3D **Fermat (optical) metric**

$$ds_{\rm front} = n_\infty(\mathbf{x})\,|d\mathbf{x}|,$$

and accordingly define **front optical path**

$$d_{\rm front}(A,B) := \inf_{\gamma: A\to B}\int_\gamma n_\infty(\mathbf{x})\,ds,$$

then **earliest reachable time** satisfies

$$t_{\min}(A,B) \geq \frac{d_{\rm front}(A,B)}{c}.$$

Equality holds if and only if front-detectability (§5) is satisfied. For isotropic case with $n_\infty \equiv 1$, $d_{\rm front} = |A - B|$. The above is equivalent to the null geodesic description of 4D Gordon optical metric in the high-frequency limit, but for timing purposes the 3D optical path formulation is more direct.

Group refractive index: For isotropic passive media, define

$$n_g(\mathbf{x},\omega) := n(\mathbf{x},\omega) + \omega\,\partial_\omega n(\mathbf{x},\omega) = \frac{c}{v_g(\mathbf{x},\omega)},$$

where v_g is the group velocity. Thus group time $t_g = \int_\gamma n_g\, ds/c$, phase time $t_\phi = \int_\gamma n\, ds/c$. **In general, there is no guarantee of the ordering between t_g or t_ϕ and $t_{\min}$** (in anomalous dispersion and gain/loss situations, one can have $t_g < t_{\min}$ or $t_\phi < t_{\min}$). **What is universal and consistent with causality is only**

$$t_{\min} \ge \frac{d_{\text{front}}}{c}, \qquad T_{\text{info}} \ge t_{\min},$$

and **only when front-detectability (§5) is satisfied** do we have $t_{\min} = d_{\text{front}}/c$, where T_{info} is the first arrival time of any **detectable information** (see §5).

Round-trip time:

$$\mathsf{T}_{\text{roundtrip}}(A \to B \to A;\, w_R, \eta) := \mathsf{T}[w_R, \eta;\, A \to B] + \mathsf{T}[w_R, \eta;\, B \to A].$$

If link and readout protocol are reciprocal (bidirectional symmetric), then $d(A, B) = \frac{c}{2}\, \mathsf{T}_{\text{roundtrip}}(A \to B \to A)$.

Front assumption (F): Medium is passive linear, isotropic, and high-frequency refractive index has finite limit

$$n_\infty(\mathbf{x}) := \lim_{\omega \to \infty} n(\mathbf{x}, \omega) \in [1, \infty).$$

Accordingly define front optical metric g^{front} and front optical path d_{front}. If only upper/lower bounds for n_∞ can be given $1 \le \underline{n}_\infty(\mathbf{x}) \le n_\infty(\mathbf{x}) \le \overline{n}_\infty(\mathbf{x}) < \infty$, then corresponding front optical paths satisfy $\underline{d}_{\text{front}} \le d_{\text{front}} \le \overline{d}_{\text{front}}$. **In general, one can only unconditionally assert**

$$\boxed{t_{\min}(A, B) \;\ge\; \frac{\underline{d}_{\text{front}}(A, B)}{c}},$$

while

$$t_{\min}(A, B) \;\le\; \frac{\overline{d}_{\text{front}}(A, B)}{c}$$

holds only under additional conditions: $t_{\min} \le \overline{d}_{\text{front}}/c$ **only when front-detectability (§5)** holds; if cascaded with a strictly causal LTI filter and onset delay t_h has an upper bound, then

$$t_{\min} \;\ge\; \frac{d_{\text{front}}}{c} + t_h,$$

and **only when front-detectability (§5) is satisfied and there is no systematic cancellation at the front** do we have $t_{\min} = \frac{d_{\text{front}}}{c} + t_h \le \frac{\overline{d}_{\text{front}}}{c} + t_h$.

Definition 2.3 (Simultaneity slice) Select a reference worldline and round-trip protocol, let $\Sigma_t := \{e \in \mathcal{E} :\ \mathsf{T}_{\text{roundtrip}} = 2t\}$, whose three-dimensional metric is induced by radar distance or g^{front}.

4 Structured Definition of Spacetime

Definition 3.1 (WSIG–EBOC Spacetime) If there exists windowed scattering readout such that:

(1) **Metric–readout consistency**: On vacuum link L we have $\mathsf{T} = L/c$; media **front optical path** is consistent with front, i.e. d_{front} is the **geodesic optical path minimum** of **3D Fermat (optical) metric** $ds_{\text{front}} = n_\infty |\mathrm{d}x|$;

(2) **Triple dictionary**: $\partial_E \arg\det S = \operatorname{tr} Q = -2\pi\xi'(E)$;

(3) **NPE detectability**: Error aliasing/Poisson/EM remainder have global upper bounds;

(4) **Information–causality consistency (vacuum or static media [LTI])**: First-detection time T_δ of mutual information satisfies

$$c_{\text{info}} := \lim_{\delta\downarrow 0} \sup \frac{D_{\text{front}}}{T_\delta} \ \le\ c,$$

with equality $c_{\text{info}} = c$ **only when front-detectability** is satisfied (see A5, §5). **(Cascade filter supplement)**: If measurement chain contains strictly causal LTI filter with onset delay $t_h > 0$, then all D_{front} above **are replaced by** $D_{\text{sys}} := D_{\text{front}} + ct_h$. **(LTV supplement)**: For time-varying systems, only assert $T_\delta \ge t_{\min}$ (defined by time-domain support of $G_{\text{ret}}(t,\tau)$).

Then the four-tuple $(\mathcal{E}, \preceq, g, \mu_\varphi)$ is a **spacetime**.

5 Main Equivalence Theorem (Phase–Delay–Spectral Shift–SI)

Theorem 4.1 (Four-way equivalence) Let vacuum link L have scattering $S_L(E) = \exp\big(iEL/(\hbar c)\big)$. Then

$$\mathsf{T}[w_R, \eta; L] = \hbar \left\langle \partial_E \arg\det S_L \right\rangle_{w,\eta} = \hbar \left\langle \operatorname{tr} Q_L \right\rangle_{w,\eta} = -\,\hbar\, 2\pi \left\langle \xi'(E) \right\rangle_{w,\eta} = \frac{L}{c},$$

and in the Nyquist bandwidth limit, the obtained $c = \lim L/\mathsf{T}$ is independent of window/kernel and agrees with SI value.

Proof. **Assumption (single-mode pure delay link)**: Take single-channel $S_L(E) = \exp\big(iEL/(\hbar c)\big)$. Window $w_R \in L^1(\mathbb{R})$, kernel $\eta \in L^1(\mathbb{R})$ satisfy normalization

$$\frac{1}{2\pi}\int_{\mathbb{R}} w_R(E)\, dE = 1, \qquad \int_{\mathbb{R}} \eta(E)\, dE = 1.$$

Lemma 1 (logarithmic derivative = Wigner–Smith): For differentiable unitary matrix (scalar here) S,

$$\partial_E \arg\det S(E) = \operatorname{Im}\, \partial_E \log\det S(E) = \operatorname{Im}\, \operatorname{tr}\big(S^\dagger(E)\, \partial_E S(E)\big) = -\,i\ \operatorname{tr}\big(S^\dagger(E)\, \partial_E S(E)\big) = \operatorname{tr} Q(E),$$

where $Q(E) := -\,i\, S^\dagger(E)\, \partial_E S(E)$. For scalar S_L, $\operatorname{tr} Q_L(E) = -iS_L^*(E)\partial_E S_L(E) = L/(\hbar c)$ (constant).

Lemma 2 (Birman–Kreĭn): If $\det S(E) = e^{-2\pi i\, \xi(E)}$, then

$$\operatorname{tr} Q(E) = -2\pi\, \xi'(E).$$

Thus for S_L we get $\xi'(E) = -\frac{1}{2\pi}\frac{L}{\hbar c}$.

Main proof: Define windowed readout

$$\mathsf{T}[w_R, \eta; L] = \hbar\, \frac{1}{2\pi} \int_{\mathbb{R}} w_R(E)\, \big[\eta \star \operatorname{tr} Q_L\big](E)\, dE.$$

Since $\operatorname{tr} Q_L \equiv L/(\hbar c)$ is constant, convolution and integration commute and using both normalizations,

$$\mathsf{T}[w_R, \eta; L] = \hbar \cdot \frac{1}{2\pi} \cdot \frac{L}{\hbar c} \int w_R(E)\, dE \cdot \int \eta(\nu)\, d\nu = \frac{L}{c}.$$

Combining Lemmas 1–2, given the integrand is constant,

$$\mathsf{T} = \hbar \left\langle \partial_E \arg\det S_L \right\rangle_{w,\eta} = \hbar \left\langle \operatorname{tr} Q_L \right\rangle_{w,\eta} = -\,\hbar \cdot 2\pi \left\langle \xi'(E) \right\rangle_{w,\eta} = \frac{L}{c}.$$

Error closure note (NPE): Nyquist/Poisson/Euler–Maclaurin three terms are each 0 in the "constant integrand" case: alias term is 0, Poisson periodization term is 0, EM remainder is 0 because higher derivatives are 0. Thus limit is independent of window/kernel and agrees with SI value. □

6 Causal Front and Information Light Cone

Theorem 5.1 (Causal front does not exceed c; equality condition) For linear strictly causal, **time-invariant (static media)** channel, the earliest nonzero response time satisfies

$$t_{\min} \geq \frac{D_{\text{front}}}{c}, \qquad D_{\text{front}} = \begin{cases} L, & \text{vacuum}, \\ d_{\text{front}}(x, y), & \text{media/inhomogeneous/bounded domain.} \end{cases}$$

For **3D vacuum free space pure propagation**, $t_{\min} = L/c$; for general media/bounded domains, **if front-detectability (§5)** holds, also $t_{\min} = d_{\text{front}}/c$; if cascaded with strictly causal LTI filter, $t_{\min} \geq D_{\text{front}}/c + t_h$ (equality requires front-detectability and no cancellation).

(LTV remark): For linear **time-varying** systems, front is formulated only via **time-domain support** of $G_{\text{ret}}(t, \tau)$; this paper does not represent time-varying media front by static d_{front}.

Proof. Assume system is linear, strictly causal, time-invariant (static media). By causality,

$$G_{\text{ret}}(t, \tau; x, y) = 0 \quad \text{when} \quad t < \tau.$$

Let source signal x be supported on $t \geq 0$, then

$$y(t) = \int_{\mathbb{R}} G_{\text{ret}}(t, \tau; x, y)\, x(\tau)\, d\tau, \qquad y(t) = 0 \ (t < 0).$$

Vacuum, uniform, lossless 3D scalar wave equation:

$$G_{\text{ret}}(t, r) = \frac{\delta(t - r/c)}{4\pi r},$$

support only at $t = r/c$, so for link length L we have $t_{\min}(L) = L/c$. **Maxwell** case uses dyadic kernel applying tensor–differential operators (containing δ and its derivatives) to the above, support likewise only on light cone, conclusion unchanged.

General media/inhomogeneous/bounded domain (infimum calibration): Under **front assumption (F)**, $n_\infty(\mathbf{x})$ exists and is finite, from high-frequency geometric optics front propagation bound

$$\operatorname{supp}_t G_{\rm ret}^{\rm med}(t,\tau;x,y) \ \subseteq\ [\tau + d_{\rm front}(x,y)/c,\ \infty).$$

Thus when $t < d_{\rm front}(x,y)/c$ response is zero. **If front-detectability (§5)** holds, then at $t = d_{\rm front}(x,y)/c$ there exists distributional nonzero response (front singularity $\sum_k K_k \delta^{(k)}$ or Dirac pass-through component), giving

$$t_{\min} = \tfrac{d_{\rm front}(x,y)}{c};$$

otherwise only

$$\boxed{t_{\min}\ \geq\ \tfrac{d_{\rm front}(x,y)}{c}},$$

typically strictly greater. If cascaded with strictly causal LTI filter with onset delay $t_h > 0$,

$$t_{\min}\ \geq\ \tfrac{d_{\rm front}(x,y)}{c} + t_h,$$

and **only when front-detectability (§5) is satisfied and there is no systematic cancellation at the front** can equality be taken.

Thus in vacuum case, when $t < L/c$, for any $\tau \geq 0$, $t-\tau < L/c \Rightarrow G_{\rm ret}(t-\tau, L) = 0$, so $y(t) = 0$. **Take unit-energy short pulse family** $x_\eta = \psi_\eta$ ($\psi_\eta \to \delta$ approximate identity kernel), then

$$y_\eta(t) = \big(G_{\rm ret} * \psi_\eta\big)(t) = \frac{\psi_\eta(t - L/c)}{4\pi L},$$

thus $\lim_{\eta\downarrow 0} y_\eta(t) = \delta(t - L/c)/(4\pi L)$, earliest nonzero response still at $t = L/c$. This argument is consistent with "detectable readout" calibration. □

Theorem 5.2 (Information light cone; bound and equality condition) In vacuum or static media (LTI), first-detection time T_δ of threshold mutual information satisfies

$$c_{\rm info} := \lim_{\delta\downarrow 0}\sup \frac{D}{T_\delta}\ \leq\ c,$$

where D is the **front optical path** in the normalization used. **Equality** $c_{\rm info} = c$ **if and only if front-detectability** holds.

(i) Link normalization ($t_h = 0$): $D = D_{\rm front}$; if **front-detectability** (same definition as A5) holds, then $c_{\rm info} = c$.

(ii) System normalization (cascade strictly causal LTI, $t_h > 0$): $D = D_{\rm sys} := D_{\rm front} + c\,t_h$; if **front-detectability** holds, then $c_{\rm info} = c$.

(LTV supplement): For linear time-varying systems, only assert $T_\delta \geq t_{\min}$.

Proof. Let input process X be supported on $t \geq 0$, noise N independent of X, receiver observes

$$Y_t := \{Y(s) : 0 \leq s \leq t\}, \qquad Y(t) = \int_{\mathbb{R}} G_{\rm ret}(t,\tau;x,y)\,X(\tau)\,d\tau + N(t).$$

(1) Zero mutual information (before front): For case (i), if in **vacuum** $t < L/c$ (or in **media/inhomogeneous/bounded domain** $t < d_{\rm front}(x,y)/c$), then $Y(s) = N(s)$, so $I(X;Y_t) = 0$. For case (ii), front changes to $D_{\rm sys}/c$.

(2) After front: If **front-detectability** (case i) or system contains cascade filter (case ii) holds, for any small $\varepsilon > 0$, let

$$t_\varepsilon := \tfrac{D_{\text{front}}}{c} + \varepsilon \quad \text{(case i)}, \qquad t_\varepsilon := \tfrac{D_{\text{sys}}}{c} + \varepsilon \quad \text{(case ii)},$$

take scalar test $X_\alpha(\tau) = \alpha\,\psi(\tau)$ (ψ unit-energy short-time pulse),

$$Z_\varepsilon := \int_0^{t_\varepsilon} G_{\text{ret}}(t_\varepsilon, \tau; x, y)\,\psi(\tau)\,d\tau \neq 0.$$

Introduce equal-probability symbol $S \in \{\pm 1\}$, set $X_S(\tau) := S\,\alpha\,\psi(\tau)$, then $Y(t_\varepsilon) = S\,\alpha\,Z_\varepsilon + N(t_\varepsilon)$. Let $N(t_\varepsilon) \sim \mathcal{N}(0, \sigma^2)$, denote $X := S\,\alpha$, take $\text{SNR} := \alpha^2|Z_\varepsilon|^2/\sigma^2$. By I–MMSE small SNR slope

$$I\big(X; Y(t_\varepsilon)\big) = \tfrac{1}{2}\,\text{SNR} + o(\text{SNR}) = \frac{\alpha^2|Z_\varepsilon|^2}{2\sigma^2} + o(\alpha^2).$$

Since $Y(t_\varepsilon)$ is a measurable function of $Y_{[0,t_\varepsilon]}$, $I\big(X; Y_{[0,t_\varepsilon]}\big) \geq I\big(X; Y(t_\varepsilon)\big)$. Given any small $\delta > 0$, take

$$\alpha = \alpha(\delta) := \frac{\sqrt{2\,\sigma^2\,\delta}}{|Z_\varepsilon|},$$

so $\text{SNR} = \alpha^2|Z_\varepsilon|^2/\sigma^2 = 2\delta$ (still in small SNR regime), thus

$$I\big(X; Y_{[0,t_\varepsilon]}\big) \;\geq\; I\big(X; Y(t_\varepsilon)\big) \;\geq\; \delta.$$

Hence $T_\delta \leq t_\varepsilon$. And when $\delta \downarrow 0$ we have $\alpha(\delta) \downarrow 0$, consistent with "small signal limit" assumption. Therefore $\forall \varepsilon > 0,\ \exists \delta(\varepsilon) \downarrow 0 : T_\delta \leq t_\varepsilon$, giving case (i) $c_{\text{info}} = c$, case (ii) likewise $c_{\text{info}} = c$. □

7 Phase Density Geometry and Trace Formula

Proposition 6.1 On de Branges space $\mathcal{H}(\mathcal{E})$, almost everywhere $K(x,x) = \frac{1}{\pi}\,\varphi'(x)\,|\mathcal{E}(x)|^2$. Thus **phase density** $\rho(x) := \varphi'(x)/\pi$ gives a natural measure. Let reproducing kernel orthogonal projection Π, and take $T_{w_\varphi} := M_{\sqrt{w_\varphi}}\Pi M_{\sqrt{w_\varphi}}$. **If** $w_\varphi \geq 0$ and $w_\varphi \in L^1 \cap L^\infty(d\mu_\varphi)$ (or understood via monotone truncation limit $w_\varphi^{(n)} \uparrow w_\varphi$), **then** T_{w_φ} is trace class, and

$$\text{Tr}(T_{w_\varphi}) = \int_{\mathbb{R}} w_\varphi(E)\,\rho(E)\,dE, \qquad \rho(E) = \frac{\varphi'(E)}{\pi}.$$

Here integration variable E is consistent with energy variable in §2. This identity is the realization of "phase–density–delay" in RKHS.

8 Sampling/Interpolation/Frames: Density Thresholds and Obstructions

- **Landau necessary density** (Paley–Wiener special case): Sampling/interpolation sequence must satisfy endpoint density threshold.
- **Wexler–Raz biorthogonal relation**: Characterizes dual window–lattice parameter relations for Gabor frames and generalizations.
- **Balian–Low obstruction**: At critical density, single window cannot be simultaneously time–frequency tightly localized; can circumvent via frames/multiple windows.

9 NPE Error Ledger (Non-asymptotic Bounds)

Error decomposition and tail term definition: Convention: non-ideal terms in time readout decompose to $\varepsilon_{\text{alias}}$ (aliasing/undersampling), ε_{EM} (Euler–Maclaurin finite-order remainder), and $\varepsilon_{\text{tail}}$ (finite support or non-compact support tail of window/kernel). Take $R > 0$ so main support of w_R is contained in $[-R, R]$. Define

$$\varepsilon_{\text{tail}} := \hbar \, \frac{1}{2\pi} \int_{|E|>R} w_R(E) \, [\eta \star \mathrm{tr} Q](E) \, dE.$$

If $\int_{|E|>R} |w_R(E)| \, dE \leq \epsilon$, $\eta \in L^1$, denote $\Omega_E := \mathrm{supp}(w_R) \oplus \mathrm{supp}(\eta)$, and $\mathrm{tr}\, Q \in L^\infty(\Omega_E)$, then

$$|\varepsilon_{\text{tail}}| \leq \frac{\hbar}{2\pi} \, \epsilon \, \|\eta\|_{L^1} \operatorname*{ess\,sup}_{E \in \Omega_E} |\mathrm{tr}\, Q(E)|.$$

10 Discrete Spacetime and CHL Light Cone

Let lattice spacing be a, discrete time step Δt. Radius r RCA influence domain at t steps is $\pm rt$, equivalent "discrete light cone", with discrete "speed of light" defined as

$$c_{\text{disc}} = \frac{r\, a}{\Delta t}.$$

Curtis–Hedlund–Lyndon theorem characterizes "continuous and commutes with shift $\Leftrightarrow$ sliding block code", and guarantees when reversible the inverse evolution is also CA, achieving reversible causal propagation.

11 Compatibility with Relativity/Field Theory

- **Lorentz covariance**: Light cone support of G_{ret} is equivalent to Minkowski light cone.
- **Microcausality/no superluminality**: In fast/slow light systems, "information velocity $\leq c$" is consistent with spacelike commut activity of quantum fields.
- **Media geometry**: Gordon optical metric absorbs refraction/flow velocity into metric, thus **in front optical metric** by definition always have "front $= c$". **In experimental coordinates** only guarantee $t_{\min} \geq d_{\text{front}}/c$; and **only when front-detectability (§5)** holds (including $\delta/\delta^{(k)}$ front or Dirac pass-through, or short-pulse limit detectable), can one recover "front $= c$". If moreover $n_\infty \equiv 1$, then $d_{\text{front}} = |A - B|$ (whether equality holds still depends on above condition).

12 Conclusive Definitions (Summary)

- **Time**: Under given window–kernel and readout protocol, time is **windowed group delay coordinate**, i.e. $t \equiv \hbar \int \frac{w_R}{2\pi} \, [\eta \star \mathrm{tr}\, Q] \, dE$.
- **Space**: Via isochronous round-trip readout selected **simultaneity slice** Σ_t and its three-dimensional metric. In vacuum defined by radar distance; for media case use **geodesic optical path minimum** of **3D Fermat metric** $ds_{\text{front}} = n_\infty(\mathbf{x})|\mathrm{d}x|$

induced by **front optical metric** (high-frequency limit $n_\infty = \lim_{\omega\to\infty} n(\mathbf{x},\omega)$ determines) to define d_{front}, earliest reachable time satisfies $t_{\min} \geq d_{\text{front}}/c$; **only when front-detectability (§5)** holds, do we have $t_{\min} = d_{\text{front}}/c$. **In front optical metric** wavefront velocity is by definition c; **in experimental coordinates**: if $n_\infty \equiv 1$, then $d_{\text{front}} = |A - B|$ and unconditionally only $t_{\min} \geq |A-B|/c$; **only under** front-detectability (§5) (or external strictly causal LTI filter with $t_h > 0$ replacing with D_{sys}) can one **recover** "front velocity = c" conclusion.

- **Spacetime**: Four-tuple $(\mathcal{E}, \preceq, g, \mu_\varphi)$, where $\preceq$ comes from G_{ret} light cone support, g is (optical) Lorentzian metric, $\mu_\varphi = (\varphi'/\pi)\,dx$ is phase density calibration; detectable and calibratable under NPE ledger, mutually inverse realization with SI.

13 Implementation Protocol (Brief)

1. Select geometrically known vacuum link L;
2. Broadband excitation, measure $\hat{\tau} = \mathsf{T}[w_R, \eta; L]$;
3. Check Nyquist (bandlimited/anti-aliasing), estimate Poisson/EM remainder;
4. Take $\hat{c} = L/\hat{\tau}$, cross-calibrate with frequency chain/interferometer length chain closed loop.

14 Proof Thread and External Index

- $\partial_E \arg\det S = \operatorname{tr} Q$ and BK identity: arXiv:1006.0639
- KK–causality equivalence (stable LTI restricted) and front light cone: Phys. Rev. 104, 1760
- Information light cone and threshold mutual information: Nature 425, 695
- NPE three-term decomposition non-asymptotic bounds: Shannon (1949)
- de Branges kernel diagonal and phase density: standard de Branges space theory
- Weyl–Heisenberg/Stone–von Neumann: Folland (1989)
- CHL theorem and discrete light cone: Math. Systems Theory 3 (1969)

15 References (Bibliographic Information, by Topic)

Scattering and group delay: F. T. Smith, "Lifetime Matrix in Collision Theory," *Phys. Rev.* 118 (1960) 349–356; A. Pushnitski, "The Birman–Krein formula…" (2010, arXiv:1006.0639); M. S. Birman & D. R. Yafaev, "The spectral shift function…" *Alg. Anal.* 4 (1992) 1–20.

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Weyl–Heisenberg and uniqueness: G. B. Folland, *Harmonic Analysis in Phase Space*, Princeton (1989).

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Conclusion (Theorem)

Theorem (Unified Calibration Theorem): The c defined by **Nyquist limit of windowed group delay** on vacuum link is **uniquely** determined, and is equivalent pairwise to

(A) phase slope/spectral shift density, (B) causal front (KK & light cone), (C) information light cone (threshold mutual information), (D) SI realization.

Therefore: **time** is **windowed group delay coordinate**; **space** is **isochronous round-trip slice and its metric**; **spacetime** is the measurable structure of **causal order + (optical) metric + phase density measure**. The above equivalence and detectability are supported by proofs and literature in §4–§8.

Proof. Let vacuum link L. By Theorem 4.1, windowed group delay readout gives $\mathsf{T} = L/c$. By Theorem 5.1, earliest nonzero response time $t_{\min}(L) = L/c$, so "phase slope/group delay" agrees with "causal front". By Theorem 5.2, vacuum link because $G_{\mathrm{ret}}(t,r) = \delta(t - r/c)/(4\pi r)$ contains front singularity (δ distribution) satisfies **front-detectability** condition, so threshold mutual information first-detection time $T_\delta(L) \to L/c$ ($\delta \downarrow 0$), thus "information light cone" agrees with front. In SI, c is a fixed constant, length–time mutually inverse realization (§2, §12 round-trip/radar protocol) gives $\hat{c} = L/\hat{\tau}$ closed-loop consistency with frequency chain/interferometer length chain. Therefore the four (A) phase slope/spectral shift density, (B) causal front, (C) information light cone, (D) SI realization, are pairwise equivalent. □