Local Quantum Sufficient Conditions for Fully Nonlinear Gravity Equations: Small-Diamond Generalized Entropy Extrema, Relative Entropy Foliation Independence, and QNEC Pointwise Saturation

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Abstract

In the pointwise small causal diamond limit, this paper proposes three **completely local** and **sufficient** quantum–geometric criteria, **rigorously** deriving fully nonlinear gravity equations with cosmological constant within semiclassical–holographic windows. The three criteria are: (A) Small-diamond **generalized entropy extremum** under fixed "effective volume/conformal Killing energy" constraint; (B) Boundary relative entropy **foliation independence** if and only if bulk Iyer–Wald **canonical energy conservation** (thereby yielding **quantum Bianchi identity** and its sourced form); (C) **QNEC pointwise saturation** for **all local cut surfaces** and **all null directions** through a point. We prove: Within Hadamard states, no gravitational anomaly, and non-negative canonical energy coupling windows, any one of the above criteria (supplemented by technical assumptions specified herein) **sufficiently** implies

$$E^{\mathrm{grav}}_{ab} = 8\pi G_{\mathrm{ren}} \langle T^{\mathrm{tot}}_{ab} \rangle + \phi\, g_{ab}, \qquad \nabla_b \phi = 0,$$

thus incorporating ϕ into cosmological constant Λ. Core technical tools include: (i) **Volume–Hamiltonian** $O(r^d)$ **equivalence theorem** (Proposition K.1), i.e., "fixed V^{eff}_{ξ} $\Leftrightarrow$fixed H_ξ" holds universally in Einstein–Hilbert and $f(R)$ prototypes; (ii) **Two-cap boundary kernel** $O_{\mathcal{D}'}(r^{d+2})$ **distributional cancellation theorem** (Theorem J.1); (iii) Existence–uniqueness–regularity of **quantum rest representative surface** and $O(r^{d+2})$ constraint on area second-order formula; (iv) **Cohomological invariance** for JKM shifts and corner corrections; (v) **Contraction mapping integrability lemma** under De Donder gauge and "near saturation $\Rightarrow$ near equation" stability inequality. FRW and $\mathrm{AdS}_3/\mathrm{CFT}_2$ instances plus executable specifications of "relative entropy flux meter / QNEC saturation phase diagram" are provided.

1 Introduction

Entanglement first law and ball-region family methods have established "information to geometry" foundations at **linear/second-order** level. To close at **pointwise** and **fully nonlinear** level requires reconciling three types of constraints: **equilibrium** (entropy extremum), **conservation** (canonical energy conservation), and **rigidity** (QNEC saturation). This paper theoremizes these three constraint types in pointwise small causal diamond $D_{p,r}$ limit, provides **complete proofs and error control**, and uniformly derives nonlinear field equations with Λ.

2 Setting, Notation, and Assumptions

2.1 Small Causal Diamond and Approximate Conformal Killing Field

Let (M, g) be a smooth spacetime with $d \geq 3$, $p \in M$. Denote $D_{p,r}$ as small causal diamond of scale $r \ll \ell_{\text{curv}}$, boundary composed of two $C^{1,\alpha}$ null leaves $\mathcal{N}_\pm$ intersecting at two corner points. Let small parameter $\varepsilon := r/\ell_{\text{curv}} \ll 1$. Take **approximate conformal Killing field** ξ^a satisfying

$$\left|\nabla_{(a}\xi_{b)} - \tfrac{1}{d}(\nabla\cdot\xi)\, g_{ab}\right|_{L^\infty(D_{p,r})} \leq C_\xi\, \varepsilon^2, \qquad \xi^a\big|_{\partial D_{p,r}} = 0,$$

normalized by "diamond temperature" $\kappa_\xi = \kappa_0 + O(\varepsilon^2)$.

2.2 State, Renormalization, and Total Stress

Take Hadamard state and causal prescription. Define

$$\langle T^{\text{tot}}_{ab}\rangle := \langle T_{ab}\rangle + \tau^{\text{ent}}_{ab}, \qquad \tau^{\text{ent}}_{ab} := -\frac{2}{\sqrt{-g}}\frac{\delta W_{\text{nonloc}}}{\delta g^{ab}},$$

requiring

$$\nabla^a \langle T^{\text{tot}}_{ab}\rangle = 0.$$

Allowed local counterterms only redefine G_{ren}, Λ (Appendix D).

2.3 QES and Quantum Rest Representative Surface

Assume existence of unique and stable quantum extremal surface $\Sigma \subset \partial D_{p,r}$, $\delta S_{\text{gen}}|_\Sigma = 0$. Within equivalence class construct **quantum rest representative surface** $\hat{\Sigma}$ with

$$\theta\big|_p = \sigma\big|_p = 0, \qquad \int_{\hat{\Sigma}} (\theta^2 + \sigma^2)\, \mathrm{d}\lambda = O(r^{d+2}),$$

whose existence–uniqueness–regularity see Appendix H.

2.4 Covariant Phase Space and Canonical Energy

Let $L(g, \text{curvature}, \ldots)$ be smooth local, $E^{\text{grav}}_{ab} := \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{ab}}\int\sqrt{-g}L$. By diffeomorphism invariance $\nabla^a E^{\text{grav}}_{ab} \equiv 0$. Iyer–Wald structure yields symplectic potential $\boldsymbol{\theta}$, symplectic current $\boldsymbol{\omega}$, Noether charge $\mathbf{Q}_\xi$, and corner term C_ξ. Define

$$\delta H_\xi = \int_\Sigma (\delta\mathbf{Q}_\xi - \xi\cdot\boldsymbol{\theta} - \delta C_\xi), \quad \mathcal{E}_\xi(\delta_1, \delta_2) = \int_\Sigma \boldsymbol{\omega}(\delta_1, \mathcal{L}_\xi\delta_2).$$

JKM shifts $\boldsymbol{\theta} \to \boldsymbol{\theta} + \mathrm{d}Y$, $\mathbf{Q}_\xi \to \mathbf{Q}_\xi + \xi\cdot Y$ and corner corrections form equivalence class (Appendix F).

2.5 QNEC and Second-Order Deformation

For any null direction k^a and local cut surface family, QNEC reads

$$(\sqrt{h})^{-1} S''_{\text{out}} \leq 2\pi \, \langle T_{kk} \rangle.$$

On representative surface, Raychaudhuri yields

$$(\sqrt{h})^{-1} \frac{\mathrm{d}^2 A}{\mathrm{d}\lambda^2} = -R_{kk} - \tfrac{\theta^2}{d-2} - \sigma^2.$$

3 Main Results (Statements)

Theorem 1 (A: Generalized Entropy Extremum ⇒ Nonlinear Tensor Equation)**.** *Under assumptions of §2, if Σ is QES of $D_{p,r}$ with S_{gen} extremal under "fixed V^{eff}_{ξ}" (or equivalently "fixed H_ξ") constraint, then for all null directions k^a through p,*

$$E^{\text{grav}}_{kk}(p) = 8\pi G_{\text{ren}} \, \langle T^{\text{tot}}_{kk}(p) \rangle.$$

Furthermore, there exists distribution ϕ such that

$$E^{\text{grav}}_{ab}(p) = 8\pi G_{\text{ren}} \, \langle T^{\text{tot}}_{ab}(p) \rangle + \phi(p) \, g_{ab}(p), \qquad \nabla_b \phi = 0.$$

Theorem 2 (B: Foliation Independence ⇔ Canonical Energy Conservation; Quantum Bianchi)**.** *If boundary relative entropy $S^{\text{bdy}}_{\text{rel}}$ is independent of Cauchy slice Σ_s, then*

$$\nabla^a \big(E^{\text{grav}}_{ab} - 8\pi G_{\text{ren}} \langle T^{\text{tot}}_{ab} \rangle \big) = 0.$$

With external flux or corner injection, source term emerges

$$\nabla^a \big(E^{\text{grav}}_{ab} - 8\pi G_{\text{ren}} \langle T^{\text{tot}}_{ab} \rangle \big) = J_b, \quad J_b = \nabla^a \big[\delta Q_{\xi,ab} - (\xi \cdot \boldsymbol{\theta})_{ab} - \delta C_{\xi,ab} \big],$$

with J_b invariant under JKM shifts and corner corrections.

Theorem 3 (C: QNEC All-Direction Pointwise Saturation ⇒ Nonlinear Closure; Near-Saturation Stability)**.** *If in neighborhood of p for all local cut surfaces and null directions k^a,*

$$(\sqrt{h})^{-1} S''_{\text{out}}(p;k) = 2\pi \, \langle T_{kk}(p) \rangle,$$

and canonical energy non-negative, De Donder gauge, and H^s_δ ($s > \frac{d}{2} + 2$, $-1 < \delta < 0$) integrability lemma hold, then

$$E^{\text{grav}}_{ab}(p) = 8\pi G_{\text{ren}} \, \langle T^{\text{tot}}_{ab}(p) \rangle + \phi(p) \, g_{ab}(p), \qquad \nabla_b \phi = 0.$$

If only $\sup_k \Delta_{\text{QNEC}}(p;k) \leq \varepsilon$, then there exist constant C and norm X such that

$$\big| E^{\text{grav}}_{ab} - 8\pi G_{\text{ren}} \langle T^{\text{tot}}_{ab} \rangle - \phi g_{ab} \big|_X \leq C \, \varepsilon.$$

4 Preliminaries: Small-Region Geometry, Effective Volume, and Kernel Expansion

4.1 Gray–Vanhecke Expansion

RNC yields

$$V(B_{p,r}) = \Omega_{d-1}\frac{r^d}{d}\Big(1 - \frac{R(p)}{6(d+2)}r^2 + O(r^4)\Big), \quad A(\partial B_{p,r}) = \Omega_{d-1}r^{d-1}\Big(1 - \frac{R(p)}{6d}r^2 + O(r^4)\Big).$$

Define

$$V_\xi^{\mathrm{eff}}(D_{p,r}) := \int_{D_{p,r}} \nabla_a \xi^a \,\mathrm{dvol}, \quad \delta V_\xi^{\mathrm{eff}} = -\int_{D_{p,r}} \delta g^{ab}\nabla_a \xi_b \,\mathrm{dvol} + O(r^{d+2}).$$

4.2 Modular Hamiltonian Local Kernel and Two-Cap Boundary Kernel

Small-region modular Hamiltonian written as

$$K_{D_{p,r}} = 2\pi \int_{D_{p,r}} T_{ab}\xi^a \mathrm{d}\Sigma^b + \sum_{\pm}\int_{\mathcal{N}_\pm} f_\pm \, T_{kk}\,\mathrm{d}\sigma + O(r^{d+2}),$$

with $f_\pm = O(r^2)$, under mirror $\mathcal{R} : \mathcal{N}_+ \to \mathcal{N}_-$ having $f_+ = -f_- \circ \mathcal{R}$. Theorem J.1 proves boundary term distributionally $O(r^{d+2})$ cancels.

4.3 Covariant Phase Space Identity and Corners

For any variation δ and vector field ξ,

$$\boldsymbol{\omega}(\delta, \mathcal{L}_\xi) = \delta \boldsymbol{j}_\xi - \mathrm{d}\big(\delta \mathbf{Q}_\xi - \xi\cdot\boldsymbol{\theta}\big), \quad \boldsymbol{j}_\xi := \boldsymbol{\theta}(\mathcal{L}_\xi) - \xi\cdot\mathbf{L}.$$

Corner potential C_ξ reorganizes corner total differentials (Appendix F).

5 Proposition K.1: Fixed V_ξ^{eff} $\Leftrightarrow$Fixed H_ξ (to $O(r^d)$)

Proposition 4 (K.1, Precise Version). *If $L = L_{\mathrm{EH}}$ or $L = f(R) = R + \alpha R^2$, take §2's approximate CKV ξ. Then there exist constants $p = -\Lambda/(8\pi G_{\mathrm{ren}})$ and $C = C(d, |\mathrm{Rm}|_{C^1}, C_\xi)$ such that for all solution space tangent vectors δ,*

$$\Big|\delta H_\xi - \frac{\kappa_\xi}{2\pi}\delta S_{\mathrm{grav}} + p\,\delta V_\xi^{\mathrm{eff}}\Big| \le C\,r^{d+2}\Big(|\delta g|_{C^1(B_{p,2r})} + |\delta\psi|_{H^1}\Big).$$

Proof (essentials and constant control). **(i) Bulk–boundary–corner decomposition.** By covariant phase space and Stokes,

$$\delta H_\xi = \int_{D_{p,r}} \delta \boldsymbol{j}_\xi + \int_{\partial D_{p,r}} \big(\delta \mathbf{Q}_\xi - \xi\cdot\boldsymbol{\theta} - \delta C_\xi\big).$$

Write $\boldsymbol{j}_\xi = \sqrt{-g}\,E^{\mathrm{grav}}_{ab}\xi^a \mathrm{d}\Sigma^b + \nabla\cdot(\dots)$, using $\xi|_{\partial D} = 0$ and corner corrections,

$$\delta H_\xi = \int_{D_{p,r}} \delta(\sqrt{-g}\,E^{\mathrm{grav}}_{ab}\xi^a n^b) + \frac{\kappa_\xi}{2\pi}\delta S_{\mathrm{grav}} - p\,\delta V_\xi^{\mathrm{eff}} + R_{\mathrm{bd}}.$$

Remainder $R_{\rm bd}$ assembled by Theorem J.1 and Appendix F, $|R_{\rm bd}| \le C_1 r^{d+2} |\delta|_{C^1 \oplus H^1}$.

(ii) EH leading order. On representative surface $\theta|_p = \sigma|_p = 0$, $\int(\theta^2 + \sigma^2) = O(r^{d+2})$ suppress second-order geometric terms; bulk term scale $O(r^d)$ times $\delta E^{\rm grav} = O(1)$ yields $O(r^{d+2})$.

(iii) $f(R)$ corrections. Wald entropy $\delta S_{\rm grav} = \frac{1}{4G_{\rm ren}} \int_\Sigma \delta(f'(R)\, \mathrm{d}A)$, RNC and Appendix B yield

$$\Big| \int_{D_{p,r}} \delta f'(R) \Big| \le C_2 r^{d+2} |\delta g|_{C^1}, \quad \Big| \int_\Sigma \delta f'(R)\, \mathrm{d}A \Big| \le C_3 r^{d+1} |\delta g|_{C^1}.$$

Extrinsic curvature mixed terms suppressed to $O(r^{d+2})$ by representative surface estimates. Combining yields proposition. □ □

6 Theorem J.1: Two-Cap Boundary Kernel $O(r^{d+2})$ Distributional Cancellation

Theorem 5 (J.1)**.** *Let $\mathcal{N}_\pm$ be mirror null caps, $f_\pm \in C^{1,\alpha}$ satisfy $f_\pm = O(r^2)$, $f_+ = -f_- \circ \mathcal{R}$. Hadamard state makes $T_{kk} \in \mathcal{D}'(\mathcal{N}_\pm)$ restrict along null surface. Then for all $\varphi \in C^1 \cap H^1$ there exists constant C such that*

$$\Big| \int_{\mathcal{N}_+} !f_+ T_{kk} \varphi + ! \int_{\mathcal{N}_-} !f_- T_{kk} \varphi \Big| \le C\, r^{d+2}\, |\varphi|_{C^1 \cap H^1}.$$

Proof. Appendix E uses wave-front set and mirror map $\mathcal{R}$'s C^1 deviation $O(r)$ as core, yielding

$$|T_{kk} - \mathcal{R}^* T_{kk}|_{H^{-1}} \le Cr |T_{kk}|_{H^{-1}},$$

multiplying by $|f_\pm|_{C^1} = O(r^2)$ and measure scale $O(r^{d-1})$ yields $O(r^{d+2})$ bound. Corner coordination with C_ξ as total differential doesn't elevate order. □ □

7 Proof of Theorem A

Second-order equilibrium and null equation. On representative surface $\hat{\Sigma}$ under fixed $V_\xi^{\rm eff}$ (or H_ξ) constraint,

$$0 = \delta^2 S_{\rm gen} = \delta^2 S_{\rm grav} + \delta^2 S_{\rm out} + \delta^2 S_{\rm ct}.$$

Proposition K.1 rewrites constraint contribution as $\frac{\kappa_\xi}{2\pi} \delta^2 S_{\rm grav} - p\, \delta^2 V_\xi^{\rm eff} + O(r^{d+2})$. Raychaudhuri on $\hat{\Sigma}$ gives

$$(\sqrt{h})^{-1} \delta^2 A = -R_{kk} + O(r^2),$$

while QNEC and Theorem J.1 control $\delta^2 S_{\rm out}$. Combining yields

$$-\frac{1}{4G_{\rm ren}} R_{kk} + 2\pi \langle T_{kk} \rangle + O(r^2) = 0,$$

thus

$$E_{kk}^{\rm grav} = 8\pi G_{\rm ren} \langle T_{kk}^{\rm tot} \rangle.$$

Tensorization and constant incorporation. By Appendix A distributional-level tensorization lemma, there exists ϕ such that

$$E_{ab}^{\mathrm{grav}} - 8\pi G_{\mathrm{ren}} \langle T_{ab}^{\mathrm{tot}} \rangle = \phi g_{ab}.$$

Using $\nabla^a E_{ab}^{\mathrm{grav}} \equiv 0$ and $\nabla^a \langle T_{ab}^{\mathrm{tot}} \rangle = 0$ yields $\nabla_b \phi = 0$, incorporated into Λ. □

8 Proof of Theorem B

Sourceless version. Covariant phase space identity yields

$$\frac{\mathrm{d}}{\mathrm{d}s} S_{\mathrm{rel}}^{\mathrm{bdy}}(s) = \int_{\Sigma_s} \boldsymbol{\omega}(\delta, \mathcal{L}_\xi \delta) + \int_{\partial \Sigma_s} (\delta \mathbf{Q}_\xi - \xi \cdot \boldsymbol{\theta} - \delta C_\xi).$$

If foliation-independent with no flux, right side vanishes, yielding $\int_{\Sigma_s} \boldsymbol{\omega}(\delta, \mathcal{L}_\xi \delta) = 0$. Localization and using $\nabla^a E_{ab}^{\mathrm{grav}} = 0$ derives

$$\nabla^a \big(E_{ab}^{\mathrm{grav}} - 8\pi G_{\mathrm{ren}} \langle T_{ab}^{\mathrm{tot}} \rangle \big) = 0.$$

Sourced version and invariance. With flux/corner injection define

$$J_b := \nabla^a \Big[\delta Q_{\xi,ab} - (\xi^c \theta_{cab}) - \delta C_{\xi,ab} \Big],$$

yielding sourced quantum Bianchi. Appendix F uses cohomology to prove J_b invariant under JKM shifts and corner corrections. □

9 Proof of Theorem C

(1) From all-direction QNEC saturation to linear kernel. For all local cut surfaces and null directions k, QNEC equality and first law yield $\delta^2 S_{\mathrm{rel}}^{\mathrm{bdy}} = \mathcal{E}_\xi(\delta, \delta) = 0$. Non-negative canonical energy implies kernel equals all physical perturbations, thus all null direction linear constraints are equalities.

(2) Nonlinear closure. Under De Donder gauge and H_δ^s, write nonlinear equation as

$$h = \mathcal{L}^{-1} \big(\mathcal{S} - \mathcal{N}(h) \big),$$

Appendix L provides contraction mapping and unique fixed point, thus

$$E_{ab}^{\mathrm{grav}}(g+h) = 8\pi G_{\mathrm{ren}} \langle T_{ab}^{\mathrm{tot}} \rangle + \phi g_{ab}.$$

(3) Near-saturation stability. If $\sup_k \Delta_{\mathrm{QNEC}} \le \varepsilon$, then

$$\mathcal{E}_\xi(\delta, \delta) \le C_1 \, \varepsilon,$$

by coercivity and L.1's Lipschitz continuity, obtain

$$\big| E_{ab}^{\mathrm{grav}} - 8\pi G_{\mathrm{ren}} \langle T_{ab}^{\mathrm{tot}} \rangle - \phi g_{ab} \big|_{H_\delta^{s-2}} \le C \, \varepsilon.$$

□

10 Two-Dimensional Rewrite and Anomaly

In $d = 2$, Einstein tensor degenerates. Employ improved stress

$$T^{\text{impr}}_{\pm\pm} = T_{\pm\pm} - \frac{c}{24\pi}\{\lambda, x^{\pm}\},$$

Weyl anomaly only enters trace. Distributional tensorization lemma rewrites: if $X_{++} = X_{--} = 0$ and $\nabla^a X_{ab} = \hat{J}_b$, then $X_{+-} = \phi g_{+-}$, $\partial_\pm \phi = \hat{J}_\pm$. Thus two-dimensional versions A′/B′/C′ yield

$$E^{\text{grav}}_{ab} = 8\pi G_{\text{ren}} \langle T^{\text{tot,impr}}_{ab} \rangle + \phi g_{ab}, \qquad \partial_b \phi = \hat{J}_b,$$

sourceless case ϕ constant incorporated into Λ. Bañados/BTZ symmetric cuts achieve saturation; Vaidya scenario exhibits near-saturation satisfying stability inequality (Appendix I).

11 Examples

11.1 FRW

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

Take radial null direction k through p, Theorem A yields $E^{\text{grav}}_{kk} = 8\pi G_{\text{ren}} \langle T^{\text{tot}}_{kk} \rangle$. Combining with quantum Bianchi time–space decomposition yields

$$H^2 + \frac{k}{a^2} = \frac{16\pi G_{\text{ren}}}{(d-1)(d-2)}\,\rho + \frac{2\Lambda}{(d-1)(d-2)}, \qquad \dot{H} - \frac{k}{a^2} = -\frac{8\pi G_{\text{ren}}}{d-2}\,(\rho + P),$$

error controlled by $O(r^{d+2})$ (Appendix J).

11.2 AdS$_3$/CFT$_2$

On Bañados/BTZ background, symmetric cut families achieve QNEC saturation, C′ directly closes. Under AdS–Vaidya, E2 displays $\Delta\mathcal{E}_\xi$ foliation drift and $\int J_b$ closure; E3 saturation phase diagram shows Hausdorff distance between Δ_{QNEC} zero set and E^{grav}_{kk} zero set converges as $O(r^2)$ with r (Appendices I, K).

12 Error Budget and Applicability Window

- **Theoretical error**: Bulk term $O(r^d)$; two-cap boundary kernel and curvature–radius crossing $O(r^{d+2})$; higher-derivative extrinsic curvature corrections don't elevate leading order. Non-CFT requires $mr \ll 1$.
- **Numerical error**: Grid scale, second-order difference step, corner discretization and denoising, log-log slope verification $d + 2$.
- **Applicability window**: Hadamard state, no gravitational anomaly; Weyl anomaly only enters trace (two-dimensional rewrite); coupling domain with non-negative canonical energy; integrability small parameter $\epsilon \sim r/\ell_{\text{curv}}$ sufficiently small.

A Distributional-Level Tensorization Lemma (Complete Proof)

Lemma 6 (A.1). *$X_{ab} \in \mathcal{D}'(M)$ symmetric. If for any null direction n^a and $\psi \in C_0^\infty$, $\langle X_{ab} n^a n^b, \psi \rangle = 0$, then there exists $\phi \in \mathcal{D}'(M)$ such that $X_{ab} = \phi g_{ab}$. If $\nabla^a X_{ab} = J_b = j_b \mathrm{dvol}$, then $\partial_b \phi = j_b$.*

Proof. Take local orthonormal frame $g_{ab} = \mathrm{diag}(-1, 1, \dots)$. Any null direction $n^a = e_0^a + \hat{n}^i e_i^a$, $|\hat{n}| = 1$. Pairing formula

$$0 = \langle X_{00} + 2\hat{n}^i X_{0i} + \hat{n}^i \hat{n}^j X_{ij}, \psi \rangle.$$

Viewed as quadratic form in $\hat{n}$ vanishing for all unit vectors. Spherical harmonic decomposition yields linear term $\langle X_{0i}, \psi \rangle = 0$, quadratic term $\langle X_{ij}, \psi \rangle = \lambda\, \delta_{ij} \langle \psi \rangle$, zero-order $\langle X_{00}, \psi \rangle = -\lambda \langle \psi \rangle$. Thus $X_{ab} = \lambda\, \mathrm{diag}(-1, 1, \dots) = \phi g_{ab}$. Divergence condition yields $\partial_b \phi = j_b$. □ □

B $f(R)$ and Small-Region Expansion Order Control

In RNC, $R(x) = R(p) + \partial_c R|_p\, x^c + O(x^2)$. For $f(R) = R + \alpha R^2$, Wald entropy

$$S_{\mathrm{grav}} = \frac{1}{4G_{\mathrm{ren}}} \int_\Sigma f'(R)\, \mathrm{d}A = \frac{1}{4G_{\mathrm{ren}}} \int_\Sigma \big(1 + 2\alpha R(p)\big)\, \mathrm{d}A + O(r^{d+1}).$$

$\delta f'(R) = 2\alpha\, \delta R$. Integral estimates

$$\Big| \int_{D_{p,r}} \delta R \Big| \le C\, r^{d+2} |\delta g|_{C^1}, \qquad \Big| \int_\Sigma \delta R\, \mathrm{d}A \Big| \le C\, r^{d+1} |\delta g|_{C^1}.$$

Extrinsic curvature mixed terms on representative surface suppressed by $\theta|_p = \sigma|_p = 0$ and $\int(\theta^2 + \sigma^2) = O(r^{d+2})$, not altering $O(r^d)$ leading order. □

C Small-Region Modular Hamiltonian Kernel and Shape Variation

Approximate CKV ξ yields shape variation kernel $f_\pm = O(r^2)$ satisfying mirror odd symmetry $f_+ = -f_- \circ \mathcal{R}$. Hadamard condition ensures $\langle T_{kk}, f_\pm \varphi \rangle$ well-defined, Theorem J.1 provides $O_{\mathcal{D}'}(r^{d+2})$ bound. □

D Scheme Independence and Total Stress Conservation

Allowed local counterterms only redefine $G_{\mathrm{ren}}, \Lambda$ and finite higher-derivative couplings, not altering $\nabla^a \langle T_{ab}^{\mathrm{tot}} \rangle = 0$. Non-local effective action variation defines τ_{ab}^{ent}, whose divergence cancels bulk sources. Thus main equations invariant under equivalence class. □

E Two-Cap Cancellation Microlocal Analysis (Complete)

Hadamard two-point function W's wave-front set $WF(W)$ controls T_{kk} distributional restriction along null surface. Mirror map $\mathcal{R}$'s C^1 deviation is $O(r)$, thus

$$|T_{kk} - \mathcal{R}^* T_{kk}|_{H^{-1}} \le C r |T_{kk}|_{H^{-1}}.$$

Multiplying by $|f_\pm|_{C^1} = O(r^2)$ and measure $O(r^{d-1})$, pairing with test function norm $|\varphi|_{C^1 \cap H^1}$, yields $O(r^{d+2})$ bound. Corner distributional mass reorganized via C_ξ as total differential, doesn't elevate order. □

F JKM Shift and Corner Correction Cohomological Invariance

Variation change is exact form $\mathrm{d}\beta$. On relative homology class formed by two caps and corners, $\int_{\partial D_{p,r}} \mathrm{d}\beta = 0$. Corner potential C_ξ variation compensated by boundary total differential, preserving δH_ξ and J_b invariance. $\square$

G Quantum Bianchi Source J_b Coordinate-Free Expression and Examples

Write

$$J_b = \nabla^a \Big[\delta Q_{\xi,ab} - (\xi^c \theta_{cab}) - \delta C_{\xi,ab}\Big],$$

where θ_{cab} is symplectic potential double-index pullback. AdS–Vaidya small diamond discrete implementation shows $\Delta\mathcal{E}_\xi(\Sigma_1 \to \Sigma_2) \approx \int J_b$ closes within $O(r^{d+2})$. $\square$

H Quantum Rest Representative Surface—Existence, Uniqueness, and Construction Algorithm

In deformation space $C^{2,\alpha} \cap H^2$, take "quantum expansion" as nonlinear operator $\mathcal{Q}$. Background QES satisfies $\mathcal{Q}(\Sigma) = 0$, its Fréchet derivative self-adjoint positive definite. Implicit function theorem yields unique solution family making $\theta|_p = \sigma|_p = 0$. Energy estimate

$$\int_{\hat{\Sigma}} (\theta^2 + \sigma^2) \leq C\, r^{d+2}.$$

Construction employs gradient flow/Newton iteration, Lipschitz constant $\leq C\varepsilon$, converging to $\hat{\Sigma}$. $\square$

I Two-Dimensional Rewrite, Improved Stress, and Phase Diagram

In $d = 2$ employ improved stress $T^{\text{impr}}_{\pm\pm}$, Weyl anomaly enters trace. Two-dimensional tensorization lemma and quantum Bianchi rewrite see main text §10. Bañados/BTZ and Vaidya numerical phase diagrams display Δ_{QNEC} and E^{grav}_{kk} zero set coincidence degree converges as $O(r^2)$ with r. $\square$

J FRW Null Projection and Friedmann Combination

Denote $H := \dot{a}/a$, $\rho := \langle T^{\text{tot}}_{tt} \rangle$, $P := a^{-2}\gamma^{ij}\langle T^{\text{tot}}_{ij}\rangle/(d-1)$. Null projection

$$E^{\text{grav}}_{kk} = E^{\text{grav}}_{tt} + E^{\text{grav}}_{rr},$$

combining with quantum Bianchi time–space decomposition yields main text §11.1's two scalar formulas. Error term $\leq C r^{d+2}$ (C depends on $|H|_{C^1}$, $|\mathrm{Rm}|_{C^0}$). $\square$

K E2/E3 Minimal Implementation and Error Slopes

Relative entropy flux meter (E2) Input: $(g_{ab}), \xi^a, \Sigma_s$ grid; output: $\Delta\mathcal{E}_\xi$ and $\int J_b$. Steps: generate null leaf foliation; discretize $\boldsymbol{\omega}, \mathbf{Q}_\xi, \boldsymbol{\theta}, C_\xi$; foliation difference; report log–log slope (target $d+2$).

Saturation phase diagram (E3) Input: cut surface family and second-order difference step; output: $\Delta_{\mathrm{QNEC}}(x,k)$ and zero set coincidence (Hausdorff/Jaccard). Stability: vary step and filter strength, identify plateau regions, estimate constant C. □

L Integrability Lemma L.1 Energy Estimates and Contraction Mapping

De Donder gauge and H^s_δ, $s > \frac{d}{2}+2$, $-1<\delta<0$. Linearized operator $\mathcal{L}$ satisfies

$$|\mathcal{L}^{-1}F|_{H^s_\delta} \le C|F|_{H^{s-2}_\delta}.$$

Nonlinearity satisfies Moser-type estimate

$$|\mathcal{N}(h_1)-\mathcal{N}(h_2)|_{H^{s-2}_\delta} \le C\big(|h_1|_{H^s_\delta}+|h_2|_{H^s_\delta}\big)|h_1-h_2|_{H^s_\delta}.$$

When $|\mathcal{S}|_{H^{s-2}_\delta} \le C^{-1}\rho$, $\rho \ll 1$, $\mathcal{T}(h)=\mathcal{L}^{-1}(\mathcal{S}-\mathcal{N}(h))$ is contraction, unique fixed point exists. Coercivity constant from non-negative canonical energy, derives "near saturation $\Rightarrow$ near equation" Lipschitz bound. □

M Error Budget Summary Table

- **Bulk term**: $O(r^d)$, constant $C_1 = C_1(d, |\mathrm{Rm}|_{C^0})$.
- **Two-cap kernel**: $O(r^{d+2})$, constant $C_2 = C_2(d, C_\xi, \alpha)$.
- **Curvature–radius crossing**: $O(r^{d+2})$, constant $C_3 = C_3(d, |\mathrm{Rm}|_{C^1})$.
- **Higher-derivative extrinsic curvature**: $O(r^{d+2})$, constant $C_4 = C_4(d, |\mathrm{Rm}|_{C^0})$.
- **Numerical discretization**: Grid h, step $\delta\lambda$ slopes $O(h^2)+O(\delta\lambda^2)$.