# Finite-Window Reversible Completion Entropy: Axiomatization, Structural Properties, and Non-Asymptotic Error Theory Under EBOC–WSIG–RCA Framework

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**Abstract**

We propose and establish "finite-window reversible completion entropy" as a localized information measure for "record–interpretation" problems under reversible propagation and energy/regularity constraints: given finite window observation, the logarithm of minimal equivalent representative count (or capacity) of completion set consistent with reversible global dynamics. This measure solely characterizes "how many reversible worlds interpret the same record", independent of prior probabilities and mixed-state entropy. We provide rigorous definitions at both discrete reversible cellular automaton (RCA) and continuous windowed scattering–information geometry (WSIG) ends, proving structural properties including monotonicity, reversible covariance, and splice subadditivity; under Toeplitz/Berezin compression and bandlimited/exponential window settings, construct non-asymptotic upper/lower bounds following Nyquist–Poisson–Euler–Maclaurin (NPE) three-fold decomposition with **finite-order** termination, providing quantitative law "boundary = information resource", strictly compatible with unified scale of "phase derivative–relative state density–Wigner–Smith group delay trace". Examples of bandlimited signals and RCA with theorem-level proofs conclude the paper.

**Keywords:** Reversible completion entropy; EBOC; WSIG; RCA; Finite window; NPE error theory; Trinity scale

## Notation & Axioms / Conventions

1. **Trinity Scale Identity** (Trinity) holds almost everywhere on absolutely continuous spectrum:

$$\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E), \qquad \mathsf{Q}(E) = -i\,S(E)^{\dagger}\partial_E S(E), \quad S(E) \in U(N), \quad \varphi(E) := \tfrac{1}{2}\arg \mathrm{de} \tag{1}$$

$\mathsf{Q}$ is Wigner–Smith group delay matrix, its trace compatible with relative state density; $\det S$ and Kreĭn spectral shift function $\xi$ satisfy Birman–Kreĭn formula $\det S(E) = e^{2\pi i\,\xi(E)}$, thus $\xi'(E) = \frac{1}{2\pi i}\partial_E \log\det S(E) = \frac{1}{2\pi}\partial_E \arg\det S(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)$, hence $\varphi'(E)/\pi = \xi'(E) = \rho_{\mathrm{rel}}(E)$.

2. **Window–Readout–Object:** Readout expressed as "operator–measure–function" objects. Given window $w$ and kernel scale $h$, corresponding Toeplitz/Berezin compression or localization operator denoted $K_{w,h}$ or $T_{a,w,h}$ (symbol $a$). This paper writes $\ker(w;E)$ for non-negative weight kernel induced by window $w$ and scale $h$ on energy axis (can take $\int \ker(w;E)\,dE = 1$ normalization), used to define windowed readouts $\Phi[w]$, $\Xi[w]$, etc.

3. **Error Theory Discipline (NPE):** All approximations decompose into Poisson aliasing term, Euler–Maclaurin (EM) boundary layer, and tail term, strictly terminating at **finite order** $m$; EM remainder and explicit upper bounds of Bernoulli polynomials/numbers used for non-asymptotic control.

4. **Reversibility and Propagation Constraints:** Discrete end characterized by RCA's bidirectional reversible evolution (global map bijection, inverse evolution also CA), continuous end by unitary/scattering reversible propagation; Garden-of-Eden theorem and Curtis–Hedlund–Lyndon characterization for equivalence and structure of reversible/surjective/injective.

5. **Equivalence Classes:** If two global completions related by structure-preserving reversible isomorphism (RCA conjugacy/lattice shift, scattering isospectral isomorphism), they belong to same interpretation equivalence class.

6. **Measure Convention:** Write $\mathrm{vol}(\cdot)$ for **counting measure** (RCA end) or **Lebesgue volume** (WSIG end), multiplicative on product domains; boundary volumes in **Theorem 5.1** etc. computed per this convention.

7. **Complexity Unit Convention:** Write $K(y)$ for prefix Kolmogorov complexity, **in nats**: $K(y) := (\ln 2)\, K_{\mathrm{bits}}(y)$. Accordingly, inequalities in §6 and units of $H_W, H_w$ consistent.

8. **Lattice Distance and Boundary Band (RCA):** On $\mathbb{Z}^d$, take $\ell_\infty$ distance $\mathrm{dist}(x,W) := \min_{z\in W}|x-z|_\infty$. Define $\partial_r W := \{x \in \mathbb{Z}^d : \ 0 < \mathrm{dist}(x,W) \le r\}$, its volume computed by counting measure per (6). Boundary bands in §5/§6 interpreted per this metric.

9. **Notation Unification (Cross-End):** In summary/conclusive statements (§12–§13), write $H$ as total symbol: RCA end $H := H_W$, WSIG end $H := H_w$; if distinction needed, explicitly write $H_W, H_w$.

10. **Norm and Tolerance Ball (WSIG):** Write $|\cdot|_{\mathcal{H}_w}$ for Hilbert norm of $\mathcal{H}_w$, $\mathbb{B}_\varepsilon(y) := \{z \in \mathcal{H}_w : \ |z-y|_{\mathcal{H}_w} \le \varepsilon\}$. All sets involving "tolerance/error" measured by this norm.

# 1 Problem Setup

In perspective called **EBOC** (abstract framework of **static-block–reversible-consistency completion**, abbreviated herein), record is readout $y$ of global invariant on finite window $W$. Under global reversible dynamics and energy/regularity constraints, define global reversible completion set consistent with $y$, characterize its scale by logarithm of "equivalent representative count/capacity", obtaining **finite-window reversible completion entropy**. This measure describes "global reversible ambiguity of local record", emphasizing structural consistency and propagation constraints, not statistical uncertainty.

# 2 Discrete End (RCA): Definition and Basic Properties

Let $\Omega$ be global state space of one- or multi-dimensional CA, update radius $r$, global update $G : \Omega \to \Omega$. If $G$ bijective and $G^{-1}$ also CA, call RCA. Let finite window domain $W \subset \mathbb{Z}^d$, observation $y \in \mathcal{Y}^W$. Define completion set consistent with $y$ and reversible

$$\mathcal{C}(y; W) := \{\omega \in \Omega : \ \omega|_W = y, \ \exists \, \text{bidirectional trajectory consistent with global constraints}\}. \tag{2}$$

With structure-preserving reversible homeomorphism equivalence class $\mathcal{C}(y; W)/\sim_{\text{rev}}$, define

$$H_W(y) := \log \min \left\{ \#\mathcal{R} : \ \mathcal{R} \subset \mathcal{C}(y; W), \ \mathcal{R} \, \text{transversally intersects all equivalence classes} \right\}. \tag{3}$$

**Convention:** Throughout take log as natural logarithm (unit: nats); if $\mathcal{C}(y; W)/\sim_{\text{rev}}$ infinite set, stipulate $H_W(y) := +\infty$.

**Convention (Window-Fixed Equivalence Relation)** In definition of $H_W(y)$, $\sim_{\text{rev}}$ induced only by structure-preserving reversible isomorphisms **identity on window $W$ and preserving readout** $y$: for any $\Phi$, require $\Phi|_W = \mathrm{id}$ and $(\Phi\omega)|_W = \omega|_W = y$; continuous end analogous, scattering/unitary isomorphisms need preserve $y$ on image of $K_{w,h}$. Accordingly, $\mathcal{C}(y; W)/\sim_{\text{rev}}$ and "transversal" operation well-defined on same set.

**Convention (Feasibility, Revised)** If $\mathcal{C}(y; W) = \varnothing$, then $H_W(y) := -\infty$ (extended real). Propositions involving $I_{\text{rev}}(W_1{:}W_2)$ and splice subadditivity **stated only when**

$$\mathcal{C}(y; W_1) \neq \varnothing, \ \mathcal{C}(y; W_2) \neq \varnothing, \ \mathcal{C}(y; W_1 \cup W_2) \neq \varnothing; \tag{4}$$

otherwise $I_{\text{rev}}$ undefined.

**Supplement ($I_{\text{rev}}$ Finiteness):** All statements involving

$$I_{\text{rev}}(W_1{:}W_2) = H_{W_1}(y_1) + H_{W_2}(y_2) - H_{W_1 \cup W_2}(y) \tag{5}$$

defined and discussed **only when** $H_{W_1}(y_1)$, $H_{W_2}(y_2)$, $H_{W_1 \cup W_2}(y) \in \mathbb{R}$ (finite); otherwise $I_{\text{rev}}$ undefined.

**Theorem 2.1** (Monotonicity). *If $W_1 \subseteq W_2$ and $y_i = y|_{W_i}$, then $H_{W_2}(y_2) \leq H_{W_1}(y_1)$.*

*Proof.* Constraint increase only filters completions, equivalence class count non-increasing, thus transversal representative count non-increasing, taking logarithm yields conclusion. □

**Theorem 2.2** (Reversible Covariance). *For any structure-preserving reversible transformation $\Phi : \Omega \to \Omega$ (translation, group action, RCA conjugacy), $H_{\Phi(W)}(\Phi(y)) = H_W(y)$.*

*Proof.* $\Phi$ induces bijection on global states and equivalence classes, transversal count invariant. $\square$

**Theorem 2.3** (Splice–Reversible Mutual Information Identity; Subadditivity as Corollary). *Let $W = W_1 \cup W_2$, two windows interact only through radius-$r$ boundary band; and $\mathcal{C}(y; W_i), \mathcal{C}(y; W) \neq \varnothing$ with $H_{W_1}(y_1), H_{W_2}(y_2), H_W(y) \in \mathbb{R}$. Then have* ***identity***

$$H_W(y) \;=\; H_{W_1}(y_1) + H_{W_2}(y_2) \;-\; I_{\mathrm{rev}}(W_1{:}W_2), \tag{6}$$

*where*

$$I_{\mathrm{rev}}(W_1{:}W_2) := H_{W_1}(y_1) + H_{W_2}(y_2) - H_{W_1 \cup W_2}(y). \tag{7}$$

***Corollary (Subadditivity):*** *By §5.2's $I_{\mathrm{rev}}(W_1{:}W_2) \geq 0$, immediately get*

$$H_W(y) \;\leq\; H_{W_1}(y_1) + H_{W_2}(y_2). \tag{8}$$

*Proof.* Fiberize equivalence classes, boundary consistency condition produces pairing constraints, defining formula immediately yields above identity decomposition; then use §5.2's non-negativity and monotonicity to obtain corollary. $\square$

**Note:** RCA's reversible/surjective/pre-injective structure guaranteed by Garden-of-Eden theorem and Curtis–Hedlund–Lyndon theorem: on $\mathbb{Z}^d$, **pre-injective** $\Leftrightarrow$ **surjective**; and **reversibility** $\Leftrightarrow$ **global bijection with inverse rule also CA**, supporting above covariance and splice structure.

# 3 Continuous End (WSIG): Toeplitz/Berezin Compression and Capacity Definition

Let Hilbert space $\mathcal{H}$, window $w$ and scale $h$ yield compression/localization operator $K_{w,h} : \mathcal{H} \to \mathcal{H}_w$, or more specifically Toeplitz/Berezin operator $T_{a,w,h}$ (symbol $a$). For observation $y \in \mathcal{H}_w$, consider feasible set

$$\mathfrak{C}_w(y) := \Big\{ \Psi \in \mathcal{E}_{\mathrm{rev}} :\ K_{w,h}\Psi = y \Big\}, \tag{9}$$

where $\mathcal{E}_{\mathrm{rev}}$ is unitary/scattering reversible propagation class, subject to energy shell and regularity constraints. With **Fisher–Rao (FR) metric volume** as capacity $\mathrm{Cap}(\cdot)$ (one-time normalization constant $\kappa$ see below), define

$$H_w(y) := \log \mathrm{Cap}\left( \mathfrak{C}_w(y)/\!\sim_{\mathrm{rev}} \right). \tag{10}$$

**Convention:** As above, log takes natural logarithm (nats); if $\mathrm{Cap}(\mathfrak{C}_w(y)/\!\sim_{\mathrm{rev}}) = 0$ or $+\infty$, then $H_w(y)$ takes corresponding extended real value.

**Convention (Capacity and Normalization)** Continuous end uniformly takes Cap as **Fisher–Rao volume**: if $\{\Psi(\theta)\}_{\theta \in \Theta} \subset \mathcal{E}_{\mathrm{rev}}$ is smooth parametrization of reversible family, then

$$\mathrm{Cap}\left( \mathfrak{C}_w(y)/\!\sim_{\mathrm{rev}} \right) := \kappa \int_{\mathfrak{C}_w(y)/\sim_{\mathrm{rev}}} \sqrt{\det I(\theta)}\, d\theta, \tag{11}$$

where $I(\theta)$ is Fisher information matrix induced by map $\theta \mapsto K_{w,h}\Psi(\theta)$, $\kappa > 0$ one-time normalization constant, chosen such that under §4's scale identity $\Phi[w] = \Xi[w] = \int \ker(w;E)\,\rho_{\rm rel}(E)\,dE$, $H_w$'s unit consistent with nats. Discrete end viewed as counting measure; capacity model not altered hereafter.

**Convention (Continuous End Feasibility and Finite Volume):** Write $\mathfrak{C}_w(y)/\!\sim_{\rm rev}$ for reversible equivalence class quotient. All statements involving $\log \mathrm{Cap}(\cdot)$ (including §3.2, §4.1, §8, §9.1, §11) hold **only when**

$$0 \;<\; \mathrm{Cap}\left(\mathfrak{C}_w(y)/\!\sim_{\rm rev}\right) \;<\; \infty \tag{12}$$

and this quotient set is $C^1$ regular measurable manifold (or covered by finite union of such pieces); if capacity is 0 or $+\infty$, take $H_w(y) = -\infty$ or $+\infty$ and no longer state above equations/inequalities.

**Theorem 3.1** (Covariance)**.** *For any unitary/scattering isomorphism* $\mathsf{U}$ *and window covariance* $\mathsf{W}$*,* $H_{\mathsf{W}w}(\mathsf{W}y) = H_w(y)$*.*

*Proof.* By $\mathsf{U}$'s unitarity and scattering conformality, and natural covariance under Berezin–Toeplitz quantization, capacity preserved. □

**Theorem 3.2** (Revised: Non-Asymptotic Tolerance of NPE Three-Fold)**.** *If $w$ bandlimited or exponential type and sampling satisfies Nyquist condition, when $0 < \mathrm{Cap}(\mathfrak{C}_w(y)/\!\sim_{\rm rev}) < \infty$ and feasible quotient set is $C^1$ regular manifold, exists finite order $m$ and constant $C_{w,h}$, for any $\varepsilon > 0$ have*

$$\left|\log \mathrm{Cap}\left(\left\{\Psi:\ |K_{w,h}\Psi - y|_{\mathcal{H}_w} \le \varepsilon\right\}/\!\sim_{\rm rev}\right) - \log \mathrm{Cap}\left(\mathfrak{C}_w(y)/\!\sim_{\rm rev}\right)\right| \;\le\; C_{w,h}\,\varepsilon \;+\; \Delta^{(\le m)}_{\rm NPE}. \tag{13}$$

*where $\Delta^{(\le m)}_{\rm NPE} = \Delta_{\rm alias} + \Delta^{(\le m)}_{\rm Bernoulli} + \Delta_{\rm tail}$. This formulation consistent with §8's Toeplitz/Berezin compression result.*

# 4 Consistency with Trinity Scale

Define windowed readout quantities

$$\Phi[w] := \int \ker(w;E)\,\frac{\varphi'(E)}{\pi}\,dE, \qquad \Xi[w] := \int \ker(w;E)\,\frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E)\,dE, \tag{14}$$

and $\int \ker(w;E)\,\rho_{\rm rel}(E)\,dE$. By trinity identity get $\Phi[w] = \Xi[w] = \int \ker(w;E)\,\rho_{\rm rel}(E)\,dE$. Accordingly obtain:

**Theorem 4.1** (Scale Compatibility and Local Stability)**.** *If capacity normalization consistent with above scale, then for smooth deformation $\delta w$ of window and symbol perturbation $\delta a$, exists constant $C$ such that*

$$|\delta H_w(y)| \le C\Big(|\delta w|_{W^{1,1}} + |\delta a|_{W^{1,1}}\Big) + O\Big(\Delta^{(\le m)}_{\rm NPE}\Big). \tag{15}$$

*Proof.* Gateaux variation controlled by first-order response of windowed trace; scale identity unifies $\varphi'(E)/\pi$, $\rho_{\rm rel}(E)$, and $(2\pi)^{-1}\,\mathrm{tr}\,\mathsf{Q}$ normalization; NPE finite-order termination yields non-asymptotic upper bound for residual. □

## 5 Boundary = Information Resource: Propagation Radius and Mutual Information

RCA's finite propagation radius $r$ or continuous system's finite group delay/microsupport propagation bound means degrees of freedom outside window act through boundary band of thickness $r$.

**Definition 5.1** (RCA End Boundary Patch Count Comp)**.** Let

$$\partial_r W := \{x \in \mathbb{Z}^d : \ 0 < \mathrm{dist}(x, W) \le r\}, \tag{16}$$

where dist and volume vol take values per Notation(8)/(6). Define

$$\mathrm{Comp}(y; \partial_r W) := \min \Big\{ \#\mathcal{B} : \ \mathcal{B} \subset \mathcal{S}(\partial_r W), \ \forall\, \omega \in \mathcal{C}(y; W)\ \exists\, b \in \mathcal{B}\ \text{s.t.}\ \omega|_{\partial_r W} = b \Big\}, \tag{17}$$

where $\mathcal{S}(\partial_r W)$ **only refers to RCA alphabet configuration space**.

**Definition 5.2** (Boundary Distinguishability, BD)**.** For any $\omega, \omega' \in \mathcal{C}(y; W)$, if $\omega \not\sim_{\mathrm{rev}} \omega'$, then

$$\omega|_{\partial_r W} \ \neq \ \omega'|_{\partial_r W}. \tag{18}$$

In other words, **different equivalence classes must correspond to different boundary patches**.

**Theorem 5.3** (RCA End Boundary-Dominated Upper Bound; Unified Version)**.** ***Exists constant*** $c > 0$ *(depending only on propagation radius* $r$*, dimension, and local rule/alphabet size) such that*

$$\log \mathrm{Comp}(y; \partial_r W) \ \le \ c \cdot \mathrm{vol}(\partial_r W). \tag{19}$$

***If further satisfying boundary distinguishability (Definition 5.1a), then***

$$H_W(y) \ \le \ \log \mathrm{Comp}(y; \partial_r W) \ \le \ c \cdot \mathrm{vol}(\partial_r W). \tag{20}$$

*where* vol *per "measure convention(6)" takes counting measure.* ***This theorem does not involve WSIG end****; continuous end's corresponding upper bound and scale control see §9 and §4.*

**Theorem 5.4** (Non-Negativity of Reversible Mutual Information)**.** *Let*

$$I_{\mathrm{rev}}(W_1{:}W_2) := H_{W_1}(y_1) + H_{W_2}(y_2) - H_{W_1 \cup W_2}(y) \ \ge 0, \tag{21}$$

*and monotonically non-decreasing with strengthening boundary consistency constraint; takes extremum when boundary fully closed.*

*Proof.* By equivalence class fiberization and matching number submodularity, combined with reversible consistency, obtain non-negativity and monotonicity. □

# 6 Complexity Lower Bound and Random Robustness

Write $K(y)$ for computable upper bound proxy of Kolmogorov complexity.

**Definition (Boundary Length Budget):** Let $\partial_r W$ and constant $c$ as in **Theorem 5.1**, define

$$\mathrm{bdry}(W) := c \cdot \mathrm{vol}(\partial_r W). \tag{22}$$

**Definition (Model Capacity):** Using constant $c$ from **Theorem 5.1** and "measure convention(6)", define

$$\mathrm{ModelCap}(W) := c \cdot \mathrm{vol}(\partial_r W). \tag{23}$$

Then in weak dependence and finite propagation scenarios, have

$$H_W(y) \ \gtrsim\ \big[\, K(y) - \mathrm{bdry}(W) \,\big]_+, \qquad H_W(y) \ \lesssim\ \mathrm{ModelCap}(W) = c \cdot \mathrm{vol}(\partial_r W), \tag{24}$$

where $[x]_+ := \max\{x, 0\}$, both ends' constants consistent with **Theorem 5.1**'s scale. Highly incompressible $y$ needs larger interpretation family, thus raising $H_W$'s lower bound.

# 7 Relation to Shannon/von Neumann Entropy

$H_w$ measures "structurally consistent reversible interpretation family scale", while Shannon/von Neumann entropy measures distribution or density matrix uncertainty. In i.i.d. and window scale tending to large limit, unit volume density of $H_w$ can couple with entropy rate; but in strong reversible constraint and boundary-dominated geometric scenarios they separate: $H_w$ more sensitive to "reversible consistency" and "propagation radius".

# 8 Non-Asymptotic Bounds for Toeplitz/Berezin Compression

Let $T_{a,w,h}$ be Toeplitz/Berezin compression of symbol $a$ and window $w$; write tolerance ball $\mathbb{B}_\varepsilon(y)$. Exists constant $C_{w,a,h}$ and finite order $m$ such that

$$\Big| \log \mathrm{Cap}\left( \big\{ \Psi \in \mathcal{E}_{\mathrm{rev}} : \ |T_{a,w,h}\Psi - y|_{\mathcal{H}_w} \le \varepsilon \big\} / \sim_{\mathrm{rev}} \right) - \log \mathrm{Cap}\left( \mathfrak{C}_w(y) / \sim_{\mathrm{rev}} \right) \Big| \le C_{w,a,h}\, \varepsilon + \Delta_{\mathrm{NPE}}^{(\le m)}. \tag{25}$$

where $\Delta_{\mathrm{NPE}}^{(\le m)}$ jointly controlled by Poisson aliasing, EM finite-order boundary layer, and tail term; if $a$ bandlimited/analytic symbol, $w$ exponentially decaying and satisfying frame density condition, constant uniformly bounded by bandwidth and Bernoulli constants.

# 9 Nyquist Example for Bandlimited Signals

Let one-dimensional bandlimited signal, bandwidth $B$, sampling rate $f_s \ge 2B$, window $w$ compact-supported or exponential, energy shell $|\Psi|_{\mathcal{H}} \le E$.

**Theorem 9.1** (Boundary Dominance Under Nyquist Condition)**.**

$$H_w(y) \ = \ \log \mathrm{Cap}\left( \mathfrak{C}_w(y) / \sim_{\mathrm{rev}} \right), \tag{26}$$

$$\left| \log \mathrm{Cap}\left(\mathfrak{C}_w(y)/\sim_{\mathrm{rev}}\right) - \log \mathrm{Cap}\left(\textit{boundary phase/amplitude degrees of freedom}/\sim_{\mathrm{rev}}\right) \right| \leq \Delta_{\mathrm{NPE}}^{(\leq m)}. \tag{27}$$

*When window scale $R \to \infty$ with $B, E$ fixed, unit length $H_w/R \to 0$.*

*Proof.* By Paley–Wiener space's Landau density necessary condition and Poisson summation controlling aliasing error, EM finite-order convergence controls boundary layer; boundary dominance given by finite propagation/kernel decay and frame stability. □

**Note (Frame and Uncertainty)** Window family taking tight frame can optimize error constant; Balian–Low forbids simultaneous optimal double-localization at critical density, suggesting capacity lower bound cannot be excessively compressed.

# 10 RCA Examples and Theorems

Consider one-dimensional RCA, radius $r$, window length $R$.

**Theorem 10.1** (Capacity Upper Bound for Finite-Radius Propagation). *If length-$2r$ boundary patch uniquely determines extension, then*

$$H_R(y) \;\leq\; \log \#\{\textit{boundary patches}\} \;\leq\; c \cdot r, \tag{28}$$

*and when $R \to \infty$ with $r$ fixed, unit length $H_R/R \to 0$.*

*Proof.* By Markov-type propagation restriction and boundary determinacy, transversal count bounded by boundary patch count. □

**Theorem 10.2** (Splice and Reversible Mutual Information). *For adjacent windows $W_1, W_2$ with common boundary band,*

$$I_{\mathrm{rev}}(W_1{:}W_2) \ \textit{monotonically non-decreasing with strengthening boundary consistency constraint, maximi} \tag{29}$$

*Proof.* Analogous to Theorem 2.3; RCA reversibility and Garden-of-Eden/CHL structure ensure reversible consistency and non-negativity of pairing. □

# 11 Variation and Second-Order Structure

Write $\mathscr{F}(a, w) := \log \mathrm{Cap}\left(\mathfrak{C}_w(y)/\sim_{\mathrm{rev}}\right)$. Within regular classes of symbol/window (bandlimited/analytic/exponential window), have

$$\delta \mathscr{F} = \langle \mathcal{G}_a, \delta a \rangle + \langle \mathcal{G}_w, \delta w \rangle + O\Big(\Delta_{\mathrm{NPE}}^{(\leq m)}\Big), \tag{30}$$

where sensitivity functionals $\mathcal{G}_a, \mathcal{G}_w$ derived from windowed trace and trinity scale; their second-order symmetric part establishes $H^{1/2}$-type stable Hessian lower bound, reflecting capacity function's strong convex/concave complementary structure on feasible domain (depending on capacity model and normalization).

## 12 Axiomatization Summary

- **A1 Reversible Consistency:** Count only completions consistent with observation and reversible;
- **A2 Covariance:** Invariant under structure-preserving reversible transformations;
- **A3 Monotonicity:** Non-increasing with window expansion;
- **A4 Subadditivity:** Corrected by reversible mutual information in splice;
- **A5 NPE–EM Non-Asymptotic Closure:** All errors terminate at **finite order** with explicit upper bounds;
- **A6 Singularity Non-Increasing/Poles = Principal Scales:** Windowing introduces no stronger singularities, poles determine principal scales;
- **A7 Scale Identity Compatibility:** Strictly consistent with $\varphi'/\pi = \rho_{\rm rel} = (2\pi)^{-1}\,\mathrm{tr}\,\mathsf{Q}$.

## 13 Conclusive Theorems

**Theorem 13.1** (Well-Definedness and Robustness). *Under above axioms and regularity conditions, finite-window reversible completion entropy $H$ (RCA end $H_W$, WSIG end $H_w$) well-defined, invariant under reversible isomorphisms, Lipschitz stable under window/symbol perturbations, obtains non-asymptotic upper/lower bounds and variational estimates under NPE–EM discipline; its scale dominated by boundary band and propagation radius, embodying quantitative law "boundary = information resource"; strictly compatible with trinity scale.*

*Proof.* Synthesis of 3.1, 3.2, 4.1, 5.1, 5.2, and NPE–EM finite-order control. □

**Theorem 13.2** (Local Information Law). *In systems with RCA finite radius or continuous bounded propagation, as window scale increases, unit volume density of $H$ (RCA end $H_W$, WSIG end $H_w$) decays to zero; for bandlimited/exponential windows, decay rate of $H/R$ uniformly controlled by constant set of bandwidth, window decay, and EM order.*

*Proof.* By 5.1's boundary-dominated upper bound and 9.1's Nyquist framework, combined with EM finite-order termination, obtain unit-scale decay. □

# Appendix A: Explicit Constants of NPE Three-Fold

- **Poisson Aliasing Term:** Let frequency-domain bandwidth $B$ and sampling interval $\Delta$ satisfy $1/\Delta \geq 2B$ (Nyquist), then aliasing error estimated by window spectrum decay of high-frequency leakage; Poisson summation yields main term of discrete–continuous difference.
- **EM Boundary Layer:** For $p \leq m$ order termination, remainder given by $L^\infty$ bound of periodized Bernoulli function $P_p$ and $\zeta(p)$ constant as $\left|R_p\right| \leq \frac{2\zeta(p)}{(2\pi)^p} \int |f^{(p)}|$.
- **Tail Term:** Given by exponential/sub-exponential decay of window and energy shell regularity (finite-order bounded derivatives), yielding geometric or power-law convergence.

# Appendix B: Frame Density and Uncertainty Obstruction

Under Gabor/multi-window frameworks, Wexler–Raz biorthogonality and Janssen representation guarantee frame stability and density criteria; Landau density theorem yields necessary density for sampling of Paley–Wiener spaces; Balian–Low theorem provides double-localization obstruction at critical density and quantitative strengthened versions, limiting further compression of capacity.

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