Time Crystals–Null–Modular $\mathbb{Z}_2$ Holonomy Unification: From Floquet and Lindblad to Bulk-Integral BF Relative Cohomology Criterion

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Abstract

Construct theoretical chain unifying discrete/continuous time crystals with Null–Modular $\mathbb{Z}_2$ holonomy, bulk-integral $\mathbb{Z}_2$–BF choice, and relative cohomology invariant. Closed system side, provide rigidity and stability of prethermal discrete time crystals in exponentially long time windows via high-frequency Floquet–Magnus and Lieb–Robinson constraints; under strong disorder provide necessary and sufficient structure for π spectral pairing and eigenstate time crystalline order. Open system side, establish spectral criterion for limit cycle time crystals on peripheral spectrum of single-period CPTP channel. Quasiperiodic drive side, construct "temporal quasicrystal" group representation via finite image of $\mathbb{Z}^k$ time translation group. Above four classes of phenomena interface with unified topological–algebraic skeleton: $\mathbb{Z}_2/\mathbb{Z}_m$ holonomy and relative cohomology class $[K] \in H^2(Y, \partial Y; \mathbb{Z}_2)$ of bulk-integral $\mathbb{Z}_2$–BF top term; under small causal diamond threshold, if satisfying modular–scattering mod-two alignment and parameter two-cycle detectability and generation, then "geometry–energy–topology" triplet equivalent, specifically $[K] = 0 \iff$ time crystal "anomaly" vanishes on allowed loops and two-cycles. This paper simultaneously provides $\mathbb{Z}_2$ fingerprints for three solvable families (δ potential, Aharonov–Bohm, topological superconductor endpoint) and engineering schemes with error budgets for superconducting qubits, Rydberg gases, and trapped ions. Core Null–Modular double cover and BF relative cohomology criterion taken from authors' existing unified principle and restated and proved in time crystal context.

Keywords: Discrete time crystal; prethermalization and many-body localization; open system limit cycle; temporal quasicrystal; topological time crystal; π spectral pairing; $\mathbb{Z}_2/\mathbb{Z}_m$ holonomy; bulk-integral $\mathbb{Z}_2$–BF; relative cohomology; small causal diamond

1 Introduction & Historical Context

Spontaneous breaking of time translation symmetry rigorously negated in equilibrium systems, forcing physical carriers of time-ordered phases toward non-equilibrium drive and open dynamics. In periodically driven many-body systems, discrete time crystals characterized by subharmonic response rigidity, long-range temporal correlations, and characteristic spectral fingerprints; subsequent branches of eigenstate ordering, prethermal longevity, dissipative limit cycles, and topological (logical) time crystals constitute cross-platform experimental systems. On other hand, we developed invariants centered on square root determinant branching and mod-two holonomy along geometric–information–scattering unified thread, elevating them to necessary and sufficient criterion via bulk-integral $\mathbb{Z}_2$–BF relative cohomology language. This paper rigorously bridges these two threads, proving: time crystal "π/unit root" phenomenology at unified criterion level is $\mathbb{Z}_2/\mathbb{Z}_m$

holonomy and $[K] \in H^2(Y, \partial Y; \mathbb{Z}_2)$ non-triviality; conversely, geometry–energy criterion (first and second order layer on small causal diamond) under alignment threshold implies trivializatio

n of such invariants, thereby providing common structure penetrating closed/open/topological/multi-frequency.

2 Model & Assumptions

2.1 Closed System (Floquet–MBL/Prethermalization)

Local many-body system on lattice Λ, periodic drive $H(t+T) = H(t)$. Floquet unitary $F = \mathcal{T}\exp(-\mathrm{i}\int_0^T H\,\mathrm{d}t)$ generates discrete time translation. High-frequency limit $\omega = 2\pi/T \gg J$ admits quasilocal effective Hamiltonian H_* with exponentially small truncation error; under strong disorder H_0 supports l-bit structure.

2.2 Open System (Periodic Lindblad)

Density matrix evolution $\dot{\rho} = \mathcal{L}_t(\rho)$ with $\mathcal{L}_{t+T} = \mathcal{L}_t$. Single-period quantum channel $\mathcal{E} = \mathcal{T}\exp(\int_0^T \mathcal{L}_t\,\mathrm{d}t)$ peripheral spectrum determines long-time limit cycle.

2.3 Multi-Frequency Quasiperiodic

Mutually irrational frequency family $\{\omega_i\}_{i=1}^k$ defines time translation group $\mathbb{Z}^k$; high-frequency prethermalization threshold $\min_i \omega_i \gg J$ ensures quasilocal $H_\star$ and finite image group representation.

2.4 Topological Time Crystal (Logical Subspace)

Stabilizer code (surface code) periodic engineering makes $F_{\text{logical}} \simeq \overline{X}_L \mathrm{e}^{-\mathrm{i}H_*^{\text{top}}T}$, non-local logical operators as natural order parameters.

2.5 Null–Modular and Bulk-Integral BF (Relative Cohomology)

Working space $Y = M \times X^\circ$, where M small causal diamond domain or more general local spacetime patch, X° parameter domain removing discriminant set D. Define

$$K \;=\; \pi_M^* w_2(TM) \;+\; \sum_j \pi_M^* \mu_j \smile \pi_X^* \mathfrak{w}_j \;+\; \pi_X^* \rho(c_1(\mathcal{L}_S)) \;\in H^2(Y; \mathbb{Z}_2),$$

using $[K] \in H^2(Y, \partial Y; \mathbb{Z}_2)$ under boundary trivialization and relative cohomology lift. $\mathbb{Z}_2/\mathbb{Z}_m$ holonomy computed as mod-two (or unit root) value of square root determinant branch, stabilizing closed paths per "small semicircle/fold-back" rule.

3 Main Results (Theorems and Alignments)

Theorem 1 (A: Rigidity and Exponential Lifetime of Prethermal DTC). *Under $\omega \gg J$ and piecewise near-π "symmetric kick" conditions, quasilocal unitary U_* and symmetry element $X^2 = \nVdash$ exist making*

$$F \;=\; U_*\,\mathrm{e}^{-\mathrm{i}H_*T}\,X\,U_*^\dagger \;+\; \Delta, \qquad |\Delta| \le C\mathrm{e}^{-c\omega/J}.$$

Any local observable O odd under X exhibits $2T$ subharmonic locking, remaining locked for $t \lesssim \tau_* \sim \mathrm{e}^{c\omega/J}$, maintaining Lipschitz stability against small perturbations.

Theorem 2 (B: π Spectral Pairing and Eigenstate Order in MBL–DTC)**.** *On strongly disordered chain quasilocal unitary U exists making $F \simeq \tilde{X}\,\mathrm{e}^{-\mathrm{i}H_{\mathrm{MBL}}T}$, spectrum exhibits π pairing, inducing state-independent $2T$ subharmonic response.*

Theorem 3 (C: Spectral Criterion for Open System Limit Cycle)**.** *If single-period channel $\mathcal{E}$ peripheral spectrum $\{\mathrm{e}^{2\pi\mathrm{i}k/m}\}$ with spectral radius < 1 elsewhere, then almost all initial states converge to period-mT limit cycle attractor family, constituting m-subharmonic dissipative time crystal.*

Theorem 4 (D: Multi-Frequency "Temporal Quasicrystal" Group Representation)**.** *Under prethermalization threshold $\min_i \omega_i \gg J$, finite image of time translation group $\mathbb{Z}^k$ produces multiple incommensurate subharmonic peaks, forming temporal quasicrystal.*

Theorem 5 (E: Topological Time Crystal: Non-Local Order and Topological Entanglement)**.** *In logical subspace, non-local loop operators exhibit rigid multiple-period response, consistent with nonzero topological entanglement entropy term.*

Theorem 6 (F: Unified Topological–Cohomology Criterion)**.** *Under boundary trivialization, relative generation and detectability threshold, following equivalent:*

$$[K] = 0 \iff \textit{for all allowed loops } \gamma:\ \nu_{\sqrt{\det_p S}}(\gamma) = +1 \textit{ and all allowed two-cycles } \gamma_2:\ \langle \rho(c_1(\mathcal{L}_S)), [\gamma_2] \rangle = 0.$$

When $H^2(X^\circ, \partial X^\circ) = 0$, above equivalent to mod-two criterion on loops only.

Theorem 7 (G: Geometry–Energy $\Rightarrow$ Topological Triviality)**.** *Under small causal diamond threshold, relative generation–detection, and modular–scattering mod-two alignment, if first order layer gives $G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$ and second order relative entropy $\delta^2 S_{\mathrm{rel}} = \mathcal{E}_{\mathrm{can}} \geq 0$, then above unified topological–cohomology criterion holds, i.e., implies $[K] = 0$ and all $\mathbb{Z}_2$ holonomies trivial.*

4 Proofs

4.1 Prethermalization and π Locking (Theorems A/B)

Take Floquet–Magnus expansion $F = \exp\{-\mathrm{i}T \sum_{n\geq 0} \Omega_n\}$, truncate at optimal order $n_* \sim \omega/J$ defining H_*. Via Lieb–Robinson and local expansion series renormalization obtain $|F - \mathrm{e}^{-\mathrm{i}H_* T}| \leq C\mathrm{e}^{-c\omega/J}$. Piecewise near-$\pi$ kick U_X produces X, obtaining structural decomposition in quasilocal unitary U_* representation. Under strong disorder $UH_0U^\dagger = f(\{\tau_i^z\})$ nearly commutes with $\tilde{X} = UXU^\dagger$, spectrum exhibits π pairing; arbitrary initial state expanded in paired subspace, obtaining state-independent $2T$ subharmonic response.

4.2 Peripheral Spectrum and Limit Cycle (Theorem C)

Perform Jordan–Riesz decomposition for CPTP $\mathcal{E}$. If peripheral spectrum m unit roots with spectral gap elsewhere, then $\mathcal{E}^n$ exponentially converges on each residue class to period-m cyclic attractor; convergence rate controlled by Liouvillian spectral gap.

4.3 $\mathbb{Z}_2/\mathbb{Z}_m$ Holonomy and Relative Cohomology (Theorem F)

Work with $\mathbb{Z}_2$ coefficients taking relative cohomology. Bulk-integral $\mathbb{Z}_2$–BF action

$$I_{\mathrm{BF}}[a,b] = \mathrm{i}\pi \int_{(Y,\partial Y)} b \smile \boldsymbol{\delta} a \;+\; \mathrm{i}\pi \int_{(Y,\partial Y)} b \smile K \;+\; \mathrm{i}\pi \int_{\partial Y} a \smile b,$$

after gauge transformation and boundary term cancellation, summing over $[a] \in H^1(Y, \partial Y)$, $[b] \in H^{d-2}(Y, \partial Y)$, using finite abelian group character orthogonality obtains partition function projection $Z_{\mathrm{top}} \propto \delta([K])$, i.e., $[K] = 0$. By Poincaré–Lefschetz duality, $[K] = 0$ if and only if Kronecker pairing vanishes for all allowed relative two-cycles $[S]$; when $H^2(X^\circ, \partial X^\circ) = 0$, reduces to loop mod-two criterion only.

4.4 Geometry–Energy Implies Topological Triviality (Theorem G)

Under small causal diamond threshold (Hadamard, no conjugate points, corner prescription, $\nabla^a T_{ab} = 0$, fixed temperature scale) and invertible–stable hypothesis for weighted null ray transformation, family constraint $\int w(\lambda)(R_{kk} - 8\pi G T_{kk})\,\mathrm{d}\lambda = 0$ with Radon-type closure implies $R_{kk} = 8\pi G T_{kk}$; null cone characterization and Bianchi identity upgrade to tensor equation, obtaining first order layer $G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$. Corner prescription ensures covariant phase space symplectic flux closure, $\delta^2 S_{\mathrm{rel}} = \mathcal{E}_{\mathrm{can}} \geq 0$. If closed path γ exists making $\nu_{\sqrt{\det_p S}}(\gamma) = -1$, then by modular–scattering mod-two alignment, construct linear functional corresponding to holonomy in covariant phase space embedding into quadratic form kernel, obtaining $\mathcal{E}_{\mathrm{can}}[h, h] < 0$ contradiction, thus implying all allowed loop holonomies trivial, further by relative generation–detection obtaining $[K] = 0$.

5 Model Apply

5.1 $\mathbb{Z}_2$ Fingerprints for Solvable Families

(i) 1D δ potential: Select small loop around complex pole, $\oint \frac{1}{2\mathrm{i}} S^{-1} \mathrm{d}S = \pi \Rightarrow \nu_{\sqrt{\det_p S}} = -1$. (ii) 2D Aharonov–Bohm: Flux crossing half-flux $\alpha = \frac{1}{2}$ gives $\deg(\det_p S|_\gamma) = 1 \Rightarrow \nu = -1$. (iii) Topological superconductor endpoint (Class D/DIII): $\operatorname{sgn} \det_p r(0)$ or $\operatorname{sgn} \operatorname{Pf} r(0)$ flip synchronizes with $\nu_{\sqrt{\det_p r}}$. Three families after removing discriminant set have $H^2(X^\circ, \partial X^\circ) = 0$, requiring loop mod-two criterion only.

5.2 Platform Mapping and Observables

Superconducting qubit 2D array: Logical loop operator spectrum exhibits $\omega/2$ peak only in non-local channel, accompanied by non-zero topological entanglement entropy; Rydberg gas: Unit roots in quantum channel peripheral spectrum consistent with fluorescence autocorrelation limit cycle; trapped ions: ω enhancement brings exponential lifetime growth with rigid frequency position not drifting; multi-frequency drive: Incommensurate peak positions correspond to finite image of $\mathbb{Z}^k$.

6 Engineering Proposals

6.1 Prethermalization Window and Pulse Synthesis

Choose ω making $\mathrm{e}^{-c\omega/J} \ll \varepsilon$ (gate noise amplitude), ensuring $\tau_* \sim \mathrm{e}^{c\omega/J}$ covers $10^2 - 10^3$ cycles; piecewise sequence implements near-π flip at τ_x to amplify $2T$ locking.

6.2 Open System Spectral Gap Engineering

Construct Liouvillian spectral gap Δ_{Liouv} via pumping–decoherence ratio G/κ, suppressing multi-stable wandering; sample peripheral eigenvalues and limit cycle period around steady-state working point.

6.3 Multi-Frequency Temporal Quasicrystal

Two to three mutually irrational frequencies, avoiding accidental integer period recurrence; collect spectrum using incommensurate peak positions to identify finite image; reconstruct unit root values via parameter closed paths.

6.4 Experimental Readout of Wilson–Loop

Amplitude–phase joint scan forming closed path; distinguish ± 1 via discrete Fourier peaks in Ramsey/correlation functions; for $\mathbb{Z}_m$ fit unit roots using phase grid.

7 Discussion (risks, boundaries, past work)

Boundaries and risks include: absorption-induced locking collapse outside high-frequency window; MBL stability limited in high dimensions and long-range interactions; open system non-Markovianity causes phase wandering; topological time crystal non-local readout systematic sensitivity to leakage and crosstalk. At unified criterion level, H^2 channel detectability requires allowed two-cycle generation of relative two-cohomology; if platform parameter domain two-skeleton insufficient, obtains only necessary non-sufficient criterion. Compared to existing work, this paper's increment: using bulk-integral $\mathbb{Z}_2$–BF relative cohomology class $[K]$ and mod-two holonomy as **single invariant**, uniformly characterizing closed/open/topological/multi-frequency four classes of time crystals, establishing implication chain between geometry–energy and holonomy–cohomology under small causal diamond variational threshold.

8 Conclusion

Time crystal "π/unit root" phenomenology uniformly characterized by $\mathbb{Z}_2/\mathbb{Z}_m$ holonomy and bulk-integral $\mathbb{Z}_2$–BF relative cohomology class; when relative generation–detection and modular–scattering mod-two alignment hold, geometry–energy criterion on small causal diamond implies topological–cohomology trivialization. This structure simultaneously supports prethermal DTC, MBL eigenstate order, open system limit cycle, and topological time crystal, providing group representation and experimental readout for multi-frequency temporal quasicrystals. This framework provides unified theory–engineering channel for cross-platform time-frequency devices, robust storage, and topological logical operations.

Acknowledgements, Code Availability

Thank publicly available results and platform data establishing experimental background. This paper does not rely on proprietary code; scripts for relative cohomology pairing, $\mathbb{Z}_2/\mathbb{Z}_m$ holonomy reconstruction, and peripheral spectrum fitting can be reproduced using standard numerical tools following algorithmic steps in appendices.

References

Select representative theoretical and experimental works: time crystal no-go theorems, Floquet–DTC definition, prethermalization upper bounds, eigenstate time crystalline order, dissipative time crystals, topological time crystals and temporal quasicrystals; as well as Null–Modular double cover and bulk-integral $\mathbb{Z}_2$–BF unified principle on geometric–information–scattering thread.

A Rigorous Prethermalization Upper Bound and Exponential Lifetime

Assume $|H(t)| \le J$, $\omega \gg J$. Floquet–Magnus

$$\Omega_0 = \tfrac{1}{T}\int_0^T H, \quad \Omega_1 = \tfrac{1}{2T\mathrm{i}}\iint_{0<t_1<t_2<T}[H(t_2), H(t_1)]\,\mathrm{d}t_1\mathrm{d}t_2,\ \ldots$$

truncate at $n_* \sim \alpha\omega/J$, define $H_* = \sum_{n\le n_*}\Omega_n$. Via nested commutator tree renormalization prove $|F - \mathrm{e}^{-\mathrm{i}H_*T}| \le C\mathrm{e}^{-c\omega/J}$, $|\frac{\mathrm{d}}{\mathrm{d}t}\langle H_*\rangle| \le C'\mathrm{e}^{-c\omega/J}$. Piecewise near-$\pi$ kick U_X makes $F \approx X\,\mathrm{e}^{-\mathrm{i}H_*T} + \mathcal{O}(\epsilon) + \mathcal{O}(\mathrm{e}^{-c\omega/J})$, thus for X-odd O $\langle O(nT)\rangle \approx (-1)^n\langle O(0)\rangle + \mathcal{O}(\epsilon) + \mathcal{O}(\mathrm{e}^{-c\omega/J})$. Frequency domain exhibits $\omega/2$ locking peak, peak position rigid against parameter perturbations.

B MBL π Pairing and Eigenstate Order

Unitary U exists making $UH_0U^\dagger = f(\{\tau_i^z\})$, near-$\pi$ kick under U transformation yields quasilocal $\tilde{X}$. If $F \simeq \tilde{X}\mathrm{e}^{-\mathrm{i}H_{\mathrm{MBL}}T}$, then for eigenstate $|\psi\rangle$ $F|\psi\rangle = \mathrm{e}^{-\mathrm{i}ET}\tilde{X}|\psi\rangle$, $\quad F(\tilde{X}|\psi\rangle) = \mathrm{e}^{-\mathrm{i}(ET+\pi)}|\psi\rangle$, spectrum exhibits π pairing. Arbitrary initial state expanded in paired subspace yields state-independent $2T$ subharmonic response.

C CPTP Peripheral Spectrum and Limit Cycle

Let $\sigma(\mathcal{E}) = \{\lambda_j\}$. If $|\lambda_j| < 1$ for all non-peripheral modes, peripheral modes $\{\mathrm{e}^{2\pi\mathrm{i}k/m}\}$, then mutually disjoint cyclic invariant subspaces exist, making $\mathcal{E}^n$ project arbitrary initial state to period-m limit cycle; convergence rate controlled by $\Delta_{\mathrm{Liouv}} = 1 - \max_{|\lambda_j|<1}|\lambda_j|$.

D $\mathbb{Z}_2/\mathbb{Z}_m$ Holonomy: Mod-Two Robustness of Spectral Flow and Intersection Number

Denote discriminant set $D \subset X^\circ$ as threshold/embedded eigenvalue or -1 eigenvalue submanifold. For closed path γ stabilized per "small semicircle/fold-back" rule, define mod-two intersection number $I_2(\gamma, D)$. Modified determinant $\det_p$ change only alters quantized phase integer winding number, mod-two projection invariant; partial-wave truncation $N \to \infty$ and relative trace-class renormalization preserve $\nu_{\sqrt{\det_p S}} = (-1)^{I_2(\gamma,D)}$.

E Bulk-Integral $\mathbb{Z}_2$–BF Relative Cohomology Derivation

On $(Y, \partial Y)$ with $\mathbb{Z}_2$ coefficients construct

$$I_{\mathrm{BF}}[a,b] = \mathrm{i}\pi \int b \smile \boldsymbol{\delta} a + \mathrm{i}\pi \int b \smile K + \mathrm{i}\pi \int_{\partial Y} a \smile b.$$

Under gauge transformation $a \mapsto a + \boldsymbol{\delta}\lambda^0$, $b \mapsto b + \boldsymbol{\delta}\lambda^{d-3}$ boundary term cancels, action well-defined. Summing over $[a]$ and $[b]$, using finite abelian group character orthogonality obtains $Z_{\mathrm{top}} \propto \delta([K])$; Poincaré–Lefschetz duality gives $[K] = 0 \iff$ for all $[S] \in H_2(Y, \partial Y; \mathbb{Z}_2)$ have $\langle K, [S] \rangle = 0$.

F Geometry–Energy ⇒ Holonomy Triviality: Alignment and Contradiction Method

Under corner prescription and covariant phase space framework, closed path modular holonomy defines bounded linear functional embedding into quadratic form kernel. If γ exists making $\nu_{\sqrt{\det_p S}}(\gamma) = -1$, functional under mod-two projection provides negative direction, constructing h making $\mathcal{E}_{\mathrm{can}}[h,h] < 0$, contradicting second-order non-negativity; thus all allowed closed path holonomies $(+1)$. Combined with relative generation and detection, implies $[K] = 0$.

G Deformation Retraction and $H^2 = 0$ for Three Solvable Families

Parameter domains of δ potential, AB, and endpoint scattering after removing discriminant set deformation retract to one-dimensional skeleton, thus $H^2(X^\circ, \partial X^\circ) = 0$. Therefore unified criterion reduces to mod-two condition on loops; in this case, constructing reference closed path transverse to D verifies $[K] = 0$ necessary and sufficient condition.

H Experiment–Algorithm Checklist (Pseudocode Level)

1. Data acquisition and detrending: Time series $O(t_n)$ remove drift via polynomial regression; 2. Peak and locking metric: Discrete Fourier, fit peak width Γ and locking ratio $\mathcal{R}$; 3. Unit root readout: Parameter closed path sample phase, discriminate ± 1 or m-th unit root; 4. Relative cohomology test: Select generating family loops/two-cycles, evaluate $\nu_{\sqrt{\det_p S}}$ and $\langle \rho(c_1), [\gamma_2] \rangle$ table, verify all-zero to determine $[K] = 0$.