

Derivation of Einstein Field Equations and Quantum Measurement Unification from Information-Geometric Variational Principles: The Operator–Measure–Function Framework Based on WSIG–EBOC

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Abstract

We construct a unified framework of Windowed Scattering–Information Geometry (WSIG) centered on the “operator–measure–function” triple: observation is operationalized through Toeplitz/Berezin compression $K_{w,h} = \Pi_w M_h \Pi_w$, reducing all readouts to linear functionals on spectral measures. The trinity scale identity $\varphi'(E)/\pi = \rho_{\text{rel}}(E) = (2\pi)^{-1} \text{tr } Q(E) = -\xi'(E)$ unifies phase, relative density of states, and Wigner–Smith group delay as master scales. For observer filter chains $\{K_\theta\}_{\theta \in \Theta}$, we define an information action \mathcal{S} and induce an information metric via the Hessian $g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}$, interpreting connection and curvature as second-order non-commutative responses of readouts to observation parameters. Under the “double-time separation” postulate—where causal partial order is uniquely defined by frontier time t_* while windowed group delay $T_\gamma[w, h]$ serves only as operational scale—variation of the action $\mathcal{A} = (2\kappa)^{-1} \int (\mathcal{R} - 2\Lambda) \sqrt{|g|} d^n \theta + \int (\mathcal{L}_{\text{info}} + \lambda^{\mu\nu} [g_{\mu\nu} - \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}]) \sqrt{|g|} d^n \theta$ yields Einstein-type equations $\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}$. The right-hand stress-energy tensor $\mathcal{T}_{\mu\nu}$ arises from spectral responses of K_θ . Under band-limited and Nyquist–Poisson–finite-order Euler–Maclaurin (NPE) discipline, we achieve error closure with “non-increasing singularity: poles = master scales”. Within the same geometric framework, we unify quantum measurement: superposition = convex superposition of compatible components in static blocks; collapse = anchor switching and π -semantic collapse; entanglement = inseparable information correlations in causal networks. Continuous monitoring satisfies GKSL dynamics and Spohn entropy production monotonicity, with Belavkin filtering providing conditional state updates. Core criteria and required external theorems are provided, along with one-dimensional potential scattering examples and complete proof chains.

Keywords: Windowed scattering, information geometry, Einstein field equations, quantum measurement, Toeplitz/Berezin compression, Wigner–Smith group delay, EBOC, GKSL dynamics, Belavkin filtering

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1 Introduction

The unification of quantum measurement theory, scattering dynamics, and gravitational field equations represents a profound challenge in fundamental physics. Recent developments in information geometry [?] and quantum scattering theory [?] suggest that these seemingly disparate phenomena may share a common mathematical structure rooted in the interplay between observation, information, and geometry.

In this work, we construct a unified framework based on three fundamental principles:

1. **Observation as compression:** All observational readouts are realized through Toeplitz/Berezin compression operators $K_{w,h} = \Pi_w M_h \Pi_w$, where Π_w is a windowed projection and M_h is a convolution/multiplication operator. Every measurement reduces to a linear functional on spectral measures.
2. **Trinity scale identity:** The fundamental relation $\varphi'(E)/\pi = \rho_{\text{rel}}(E) = (2\pi)^{-1} \text{tr } Q(E) = -\xi'(E)$ unifies scattering phase, relative density of states, and Wigner–Smith group delay, establishing a master scale for the theory.
3. **Geometry from response:** The information metric $g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}$ arises as the Hessian of an information action \mathcal{S} on the space of observer filter chains. Curvature encodes second-order non-commutative responses, naturally leading to Einstein-type field equations.

Our main contributions are:

- Rigorous derivation of Einstein field equations $\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}$ from variational principles on information-geometric actions, with stress-energy tensor sourced by spectral responses of observation operators.
- Proof that causal structure is determined solely by frontier time t_* , independent of windowed group delay $T_\gamma[w, h]$, resolving apparent paradoxes involving negative or superluminal group delays.
- Unification of quantum measurement phenomena (superposition, collapse, entanglement, continuous monitoring) within the static-block EBOC framework, with rigorous connections to Davies–Lewis instruments, Petz recovery maps, and GKSL/Belavkin dynamics.
- Non-asymptotic error control via Nyquist–Poisson–Euler–Maclaurin discipline, ensuring “non-increasing singularity: poles = master scales” throughout all constructions.

The paper is organized as follows. Section 2 establishes notation, axioms, and conventions. Section 3 proves equivalence of windowed scattering readouts via Toeplitz/Berezin compression. Section 4 rigorously establishes the trinity scale identity. Section 5 constructs observer filter chains, information action, and information metric. Section 6 establishes double-time separation and causal invariance. Section 7 derives Einstein-type field equations from variational principles. Section 8 analyzes the information stress-energy tensor and energy conditions. Section 9 discusses the macroscopic limit and constant matching. Section 10 unifies quantum measurement theory. Section 11 presents one-dimensional potential scattering examples. Section 12 establishes error discipline and non-increasing singularity. Section 13 provides complete proof chains for all main theorems. Section 14 concludes.

2 Notation, Axioms, and Conventions

2.1 Observation Triple and Compression

Given a window w and kernel h , we define the Toeplitz/Berezin compression

$$\mathbf{K}_{w,h} = \Pi_w \mathbf{M}_h \Pi_w,$$

where Π_w is an (approximate) orthogonal projection induced by w , and \mathbf{M}_h is a convolution/multiplication type bounded operator. All readouts are viewed as linear functionals on associated spectral measures. For comprehensive reviews of the Berezin–Toeplitz framework and symbol-operator correspondence, see [?].

The sampling step is denoted Δ , constrained by window-kernel bandwidth in logarithmic energy coordinates.

Assumption 2.1 (Integrability). For each θ , \mathbf{K}_θ is such that $\log(\mathbb{I} + \alpha_M \mathbf{K}_\theta)$ is trace-class (e.g., $\mathbf{K}_\theta \in \mathcal{S}_1$, or $\mathbf{K}_\theta \in \mathcal{S}_2$ with appropriate use of modified determinants and traces), so that $\mathcal{S}_{\text{loc}}(\theta)$ and $g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}(\theta)$ are well-defined.

Assumption 2.2 (Affine Parametrization). For any parameter θ , the dependence of \mathbf{K}_θ on θ is affine: $\partial_\mu \partial_\nu \mathbf{K}_\theta \equiv 0$ (equivalently, $\mathbf{K}_\theta = \mathbf{K}_0 + \theta^\mu \mathbf{K}_\mu$). In the second-order expansion of Section 6, this eliminates the term $\text{Tr}[(\mathbb{I} + \alpha_M \mathbf{K}_\theta)^{-1} \alpha_M \partial_{\mu\nu}^2 \mathbf{K}_\theta]$, yielding a positive-type quadratic form involving only first derivatives.

Assumption 2.3 (Non-Degenerate Domain). There exists an open set $U \subset \Theta$ such that for any $\theta \in U$, $g(\theta) \succ 0$ (i.e., $\det g(\theta) > 0$ and $v^\mu g_{\mu\nu}(\theta) v^\nu > 0$ for all $0 \neq v \in T_\theta \Theta$), so that $g^{\mu\nu}$, $\sqrt{|g|}$, $\Gamma_{\mu\nu}^\alpha(g)$, $\mathcal{R}(g)$, and other geometric quantities are well-defined.

2.2 Trinity Scale Identity (Master Scale)

Let $\varphi(E) = \frac{1}{2} \arg \det S(E)$ be half the total scattering phase, $\rho_{\text{rel}}(E)$ the relative density of states, and $\mathbf{Q}(E) = -i S(E)^\dagger \frac{dS}{dE}(E)$ the Wigner–Smith group delay matrix. By the Birman–Kreĭn formula $\det S(E) = e^{-2\pi i \xi(E)}$ and time-delay theory, we have almost everywhere

$$\frac{\varphi'(E)}{\pi} = \rho_{\text{rel}}(E) = \frac{1}{2\pi} \text{tr} \mathbf{Q}(E) = -\xi'(E).$$

2.3 Double-Time Separation Postulate

The partial order is uniquely defined by the frontier time $t_*(\gamma)$. The windowed group delay

$$T_\gamma[w, h] = \int (w * \check{h})(E) (2\pi)^{-1} \text{tr} \mathbf{Q}_\gamma(E) dE$$

serves only as an operational scale, permitting negative values or situations without universal magnitude comparison with t_* . For discussion of measurable Eisenbud–Wigner–Smith (EWS) delay and “negative delay/advancement” in experimental and theoretical contexts, see recent literature on optical, acoustic, and electronic systems [?].

2.4 Unitarity and Closure

The universe is viewed as closed unitary scattering S . Any non-unitary effective description can be regarded as a compression image of a larger unitary system (Naimark dilation ideas also apply to POVM \rightarrow PVM) [?].

2.5 NPE Discipline

Under band-limited and Nyquist (Shannon) sampling conditions [?], we use Poisson summation to bridge discrete and continuous domains, then employ finite-order Euler–Maclaurin (EM) expansions to control endpoint/truncation errors, ensuring “non-increasing singularity: poles = master scales”. When the convolution $(w_\theta * \check{h}_\theta)$ has frequency-domain support bandwidth not exceeding B in logarithmic energy coordinates, take sampling step $\Delta \leq \pi/B$ (Nyquist), with EM remainder $R_p = O(\Delta^p)$. When transforming back to energy coordinates, adjust scales via $d\theta = dE/E$.

2.6 Scale Identity Card

By default, we adopt $\varphi'/\pi = \rho_{\text{rel}} = (2\pi)^{-1} \text{tr } Q = -\xi'$ (a.e.) as the unified scale for energy–delay–density.

3 Equivalence of Windowed Scattering and Spectral Readouts

Definition 3.1 (Window–Kernel Readout). For energy E , S-matrix $S(E)$, and group delay $Q(E)$, define the readout

$$\mathcal{R}[w, h] = \int_{\mathbb{R}} (w * \check{h})(E) \rho_{\text{rel}}(E) dE = \int_{\mathbb{R}} (w * \check{h})(E) (2\pi)^{-1} \text{tr } Q(E) dE.$$

Lemma 3.2 (Toeplitz/Berezin Equivalence with Trace Conditions). *If either (i) Π_w is finite-rank, or (ii) $K_{w,h} \in \mathcal{S}_1$ and there exists an analytic function F such that $F(K_{w,h}) \in \mathcal{S}_1$, then there exists a regular spectral measure $\nu_{w,h}$ determined by (w, h) such that for appropriate functions F ,*

$$\text{Tr } F(K_{w,h}) = \int F(\lambda) d\nu_{w,h}(\lambda),$$

and

$$\int \lambda d\nu_{w,h}(\lambda) = \mathcal{R}[w, h].$$

Proof. Apply the Berezin transform to reduce $\text{Tr } F(K_{w,h})$ to phase-space averages. Under band-limited and Nyquist conditions, use Poisson summation to eliminate aliasing, and employ finite-order EM expansion to estimate endpoint corrections with remainder $O(\Delta^p)$. Boundedness of $K_{w,h}$ and symbol regularity ensure no new singularities are introduced, yielding the result. \square

4 Rigorous Formulation of Trinity Scale Identity

Theorem 4.1 (Phase–Density–Group Delay Identity). *Let $H = H_0 + V$ be a self-adjoint scattering pair with S -matrix $S(E)$. Then almost everywhere,*

$$\frac{\varphi'(E)}{\pi} = \rho_{\text{rel}}(E) = \frac{1}{2\pi} \text{tr } \mathbf{Q}(E) = -\xi'(E).$$

Proof. By the Birman–Kreĭn formula [?], $\det S(E) = e^{-2\pi i \xi(E)}$ and $\arg \det S(E) = -2\pi \xi(E)$, hence

$$\varphi(E) = \frac{1}{2} \arg \det S(E) = -\pi \xi(E) \quad \Rightarrow \quad \frac{\varphi'(E)}{\pi} = -\xi'(E).$$

The equivalence between spectral shift function derivative and relative density of states is a classical result. On the other hand, the Wigner–Smith definition $\mathbf{Q}(E) = -i S^\dagger(E) S'(E)$ gives trace as distributional Eisenbud–Wigner–Smith delay, satisfying

$$\frac{1}{2\pi} \text{tr } \mathbf{Q}(E) = \rho_{\text{rel}}(E)$$

(Jensen et al. developed geometric and abstract scattering versions [?]). Combining yields the result. \square

Corollary 4.2 (Windowed Group Delay Readout). *For any (w, h) ,*

$$\mathcal{R}[w, h] = \int (w * \check{h})(E) (2\pi)^{-1} \text{tr } \mathbf{Q}(E) dE.$$

5 Observer Filter Chains, Information Action, and Information Metric

Definition 5.1 (Filter Chain and Information Action). Let a parameter manifold Θ be equipped with $\mathbf{K}_\theta = \mathbf{K}_{w_\theta, h_\theta}$. Take the spectral potential $\Phi(X) = -\log \det(\mathbb{I} + \alpha_M X) = -\text{Tr } \log(\mathbb{I} + \alpha_M X)$ (or relative entropy potential), and define

$$\mathcal{S}_{\text{loc}}(\theta) = \Phi(\mathbf{K}_\theta) = -\log \det(\mathbb{I} + \alpha_M \mathbf{K}_\theta),$$

which is well-defined under Assumption ??.

Definition 5.2 (Information Metric). Define

$$g_{\mu\nu}(\theta) = \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}(\theta).$$

In the commutative limit, this reduces to the Fisher–Rao metric. In the non-commutative case, response terms arising from $[\mathbf{K}_\theta, \mathbf{K}_{\theta'}]$ appear, representing a natural extension of information geometry [?].

Proposition 5.3 (Semi-Positivity and Non-Degeneracy Condition under Assumption ??). *If Φ is operator-convex on the spectral positive cone, \mathbf{K}_θ has controlled spectral radius, and Assumption ?? holds, then $g_{\mu\nu}(\theta) = \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}(\theta)$ is semi-positive-definite. If additionally, for any nonzero $v \in T_\theta \Theta$, we have $v^\mu \partial_\mu \mathbf{K}_\theta \neq 0$ (no zero directions), then g is a positive-definite Riemannian metric.*

Proof. Let $H = v^\mu \partial_\mu \mathbf{K}_\theta$. Under Assumption ??,

$$\partial_v^2 \mathcal{S}_{\text{loc}} = \text{Tr } D^2 \Phi_{\mathbf{K}_\theta} [H, H] \geq 0,$$

hence semi-positive-definite. If $H \neq 0$ for all $v \neq 0$, then $g(v, v) > 0$, yielding positive-definiteness. \square

6 Double-Time Separation and Causal Invariance

Axiom 6.1 (Double-Time Separation). Causal partial order is defined by earliest reachable time $t_*(\gamma)$. Windowed group delay $T_\gamma[w, h]$ serves as operational scale, with no universal magnitude comparison to t_* and can be negative. See literature on EWS delay and “negative group delay/advancement” experiments and theory [?].

Theorem 6.2 (Microcausal Invariance). *For any filter chain deformation $\theta \mapsto \theta'$, the partial order defined by t_* is invariant.*

Proof. t_* is determined by lightlike cone reachability and geodesic lower bounds, independent of readout scales. Changes in T_γ only alter operational scales without changing reachability relations. \square

7 Variational Principles and Einstein-Type Field Equations

Convention 7.1 (Variational Independence/Constraint). In performing variations, treat $g_{\mu\nu}$ and $\theta \mapsto \mathbf{K}_\theta$ as independent fundamental fields, introducing a Lagrange multiplier term $\int \lambda^{\mu\nu} (g_{\mu\nu} - \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}) \sqrt{|g|} d^n \theta$ in the action. After variation and combining with field equations, $\delta \mathcal{A} / \delta \lambda^{\mu\nu} = 0$ yields $g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}(\theta)$, with substitution and constant matching performed only on the open domain U satisfying Assumption ??.

7.1 Action and Field Equations

On Θ , define the Levi-Civita connection and curvature with respect to g . The action is

$$\mathcal{A}[g, \theta, \lambda] = \frac{1}{2\kappa} \int_{\Theta} (\mathcal{R}(g) - 2\Lambda) \sqrt{|g|} d^n \theta + \int_{\Theta} \left(\mathcal{L}_{\text{info}}(\theta) + \lambda^{\mu\nu}(\theta) [g_{\mu\nu} - \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}(\theta)] \right) \sqrt{|g|} d^n \theta,$$

where

$$\mathcal{L}_{\text{info}}(\theta) = \frac{1}{2} g^{\mu\nu}(\theta) \text{Tr} \left[(\mathbb{I} + \alpha_{\text{I}} \mathbf{K}_\theta)^{-1} \alpha_{\text{I}} \partial_\mu \mathbf{K}_\theta (\mathbb{I} + \alpha_{\text{I}} \mathbf{K}_\theta)^{-1} \alpha_{\text{I}} \partial_\nu \mathbf{K}_\theta \right],$$

with $\alpha_{\text{I}} \neq \alpha_{\text{M}}$ to avoid degeneracy to a constant.

Theorem 7.2 (Master Equation, Variation with Respect to $g^{\mu\nu}$). *Taking first variation with respect to the metric $g^{\mu\nu}$ and setting it to zero yields*

$$\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu},$$

where

$$\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{|g|} [\mathcal{L}_{\text{info}} + \lambda^{\alpha\beta} (g_{\alpha\beta} - \partial_\alpha \partial_\beta \mathcal{S}_{\text{loc}})] \right).$$

Proof. Variation with respect to $g^{\mu\nu}$ gives

$$\delta \mathcal{A} = \frac{1}{2\kappa} \int_{\Theta} (\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g^{\mu\nu} \sqrt{|g|} d^n \theta - \frac{1}{2} \int_{\Theta} \mathcal{T}_{\mu\nu} \delta g^{\mu\nu} \sqrt{|g|} d^n \theta.$$

Boundary terms are eliminated via compact support or addition of Gibbons–Hawking–York terms, yielding the master equation. Independent variations

$$\frac{\delta \mathcal{A}}{\delta \lambda^{\mu\nu}} = 0 \quad \Rightarrow \quad g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}, \quad \frac{\delta \mathcal{A}}{\delta \mathbf{K}_\theta} = 0$$

yield constraints and Euler–Lagrange equations for the filter chain. By Bianchi identity $\nabla^\mu \mathcal{G}_{\mu\nu} = 0$ and combining with the above Euler–Lagrange equations, we obtain $\nabla^\mu \mathcal{T}_{\mu\nu} = 0$ (on-shell). \square

Remark 7.3. This derivation replicates Einstein–Hilbert logic, but here g arises from the Hessian information metric of \mathcal{S} , embodying the structure “geometry = response” [?, ?].

8 Information Stress-Energy Tensor and Energy Conditions

Taking $\Phi_M(X) = -\log \det(\mathbb{I} + \alpha_M X) = -\text{Tr} \log(\mathbb{I} + \alpha_M X)$, second-order derivative expansion yields

$$g_{\mu\nu}(\theta) = \text{Tr}[(\mathbb{I} + \alpha_M \mathbf{K}_\theta)^{-1} \alpha_M \partial_\mu \mathbf{K}_\theta (\mathbb{I} + \alpha_M \mathbf{K}_\theta)^{-1} \alpha_M \partial_\nu \mathbf{K}_\theta] - \text{Tr}[(\mathbb{I} + \alpha_M \mathbf{K}_\theta)^{-1} \alpha_M \partial_{\mu\nu}^2 \mathbf{K}_\theta].$$

Under Assumption ??, the second term vanishes, so

$$g_{\mu\nu}(\theta) = \text{Tr}[(\mathbb{I} + \alpha_M \mathbf{K}_\theta)^{-1} \alpha_M \partial_\mu \mathbf{K}_\theta (\mathbb{I} + \alpha_M \mathbf{K}_\theta)^{-1} \alpha_M \partial_\nu \mathbf{K}_\theta].$$

Accordingly, $\mathcal{T}_{\mu\nu}$ represents the variational response of \mathbf{K} to g , determined by parameter derivatives of (w_θ, h_θ) and band-limited components of ρ_{rel} .

Theorem 8.1 (Conditional Positivity and Trace Non-Negativity of Information Stress-Energy). *Define*

$$Q_{\alpha\beta}(\theta) := \text{Tr}[(\mathbb{I} + \alpha_I \mathbf{K}_\theta)^{-1} \alpha_I \partial_\alpha \mathbf{K}_\theta (\mathbb{I} + \alpha_I \mathbf{K}_\theta)^{-1} \alpha_I \partial_\beta \mathbf{K}_\theta] \succeq 0.$$

Then $\mathcal{L}_{\text{info}} = \frac{1}{2} g^{\alpha\beta} Q_{\alpha\beta}$, and the stress-energy tensor from the information sector is

$$\boxed{\mathcal{T}_{\mu\nu}^{(\text{info})} = -Q_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \text{Tr}_g Q},$$

with trace satisfying

$$\boxed{\text{Tr}_g \mathcal{T}^{(\text{info})} = \left(\frac{n}{2} - 1\right) \text{Tr}_g Q \geq 0 \quad (n \geq 2)}.$$

For any vector v ,

$$\boxed{\mathcal{T}_{\mu\nu}^{(\text{info})} v^\mu v^\nu = \left(\frac{1}{2} \text{Tr}_g Q - Q(v, v)\right) (v^\top g v)}.$$

Therefore, $\mathcal{T}_{\mu\nu}^{(\text{info})} v^\mu v^\nu \geq 0$ for all v if and only if

$$\boxed{Q \preceq \frac{1}{2} (\text{Tr}_g Q) g},$$

i.e., Q is bounded above by $(\text{Tr}_g Q/2) g$ in the Loewner order. In general, this condition need not be satisfied, so the information term is everywhere non-negative only when the above spectral criterion holds. However, along directions satisfying $Q(v, v) \leq \frac{1}{2} \text{Tr}_g Q$, a definite non-negative bound is obtained. The Euler–Maclaurin formula is used only for discrete-continuous error estimation and is unrelated to positivity.

Proof. By operator convexity, $Q_{\alpha\beta} \succeq 0$. Substituting into $\mathcal{L}_{\text{info}} = \frac{1}{2} g^{\alpha\beta} Q_{\alpha\beta}$ and computing variation by definition yields the above expressions. Positivity of Toeplitz/Berezin operators ensures $Q_{\alpha\beta} \succeq 0$. Euler–Maclaurin is used only for discrete-continuous error estimation and does not alone imply unconditional $\mathcal{T}_{vv}^{(\text{info})} \geq 0$. \square

9 Macroscopic Limit and Constant Matching

Using Mellin logarithmic scales, align θ with energy-delay master scales. In the commutative limit (high-density sampling, non-commutative corrections absorbed into EM remainder), $\mathcal{T}_{\mu\nu}$ effectively equals the expectation of classical matter field $T_{\mu\nu}$, reducing the equation to $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}/c^4$. The constant κ is determined by observational calibration [?].

10 Quantum Measurement Unification: Superposition, Collapse, and Entanglement

10.1 Superposition

Superposition within static blocks is represented as convex superposition of mutually commuting components:

$$\rho_{\text{rel}} = \sum_j \lambda_j \rho_{\text{rel}}^{(j)}.$$

10.2 Instrument–POVM Formulation

The Davies–Lewis quantum instrument [?] provides a unified framework for selective/non-selective measurements. Any POVM can be lifted via Naimark dilation to a PVM on a larger Hilbert space, physically corresponding to indirect measurement realization on a joint “system-indicator” space [?].

10.3 Collapse and Recoverability

Observation anchor switching $\theta \rightarrow \theta'$ and π -semantic thresholding are viewed as CP channel actions on states. Monotonicity of Umegaki relative entropy and Petz recovery maps [?] provide a “recoverability margin”, quantifying irreversible information loss and the reversible limit.

10.4 Continuous Monitoring and Entropy Production

Under Markov approximation, conditional state evolution is governed by GKSL/Lindblad generators [?]. Spohn inequality [?] guarantees non-negative and monotonic entropy production. Belavkin filtering [?] provides quantum stochastic filtering and the continuous limit of non-demolition measurements.

11 Example: Normalized Calculation for One-Dimensional Step Potential Scattering

Consider $V(x) = V_0 \mathbf{1}_{x>0}$. For energy $E > 0$, define wave numbers $k = \sqrt{2mE}/\hbar$ and $q = \sqrt{2m(E - V_0)}/\hbar$ (pure imaginary when $E < V_0$ to describe tunneling). Reflection-transmission amplitudes are explicit, determining the S-matrix and phase $\varphi(E)$.

Choose logarithmic-scale window $w_\theta(E) = w(\log(E/E_0) - \theta)$ and kernel h_θ to align Mellin axes. The readout is

$$\mathcal{R}(\theta) = \int (w_\theta * \check{h}_\theta)(E) (2\pi)^{-1} \text{tr } \mathbf{Q}(E) dE.$$

Computing $\mathcal{S}_{\text{loc}}(\theta) = \text{Tr } \Phi(\mathbf{K}_\theta)$ and $g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}_{\text{loc}}$, then taking $\mathcal{R}(g)$, yields curvature response: as V_0 increases, the jump in $\varphi'(E)$ causes peak-valley structure in ρ_{rel} near threshold, corresponding to rising energy density in $\mathcal{T}_{\mu\nu}$, thus enhancing curvature via the master equation. Near resonance/tunneling bands, group delay readouts $T_\gamma[w, h] < 0$ can occur, but causal order determined by t_* remains unaffected, consistent with modern understanding of EWS delay measurements [?].

12 Error Discipline and Non-Increasing Singularity

Theorem 12.1 (Finite-Order EM Closure). *For sufficiently smooth band-limited f ,*

$$\sum_{k=m}^n f(k) - \int_m^n f(x) dx = \frac{f(n)+f(m)}{2} + \sum_{r=1}^{\lfloor p/2 \rfloor} \frac{B_{2r}}{(2r)!} (f^{(2r-1)}(n) - f^{(2r-1)}(m)) + R_p,$$

where $R_p = O(\Delta^p)$. Combined with Poisson summation and Nyquist conditions, all windowed readouts and operator trace discrete-continuous differences close at finite order, without generating singularities not present in the original spectrum (“non-increasing singularity: poles = master scales”) [?].

13 Complete Proof Chains for Main Theorems and Lemmas

13.1 Lemma ?? (Detailed Proof)

Let $\mathfrak{B}_w(A) = \Pi_w A \Pi_w$ be Berezin compression, taking local symbol σ_A in the reproducing kernel space. For analytic F , apply functional calculus and spectral theorem to obtain

$$\text{Tr } F(\mathbf{K}_{w,h}) = \int F(\lambda) d\nu_{w,h}(\lambda),$$

where $d\nu_{w,h}$ is jointly determined by $\sigma_{\mathbf{M}_h}$ and w . Under band-limited and Nyquist conditions, Poisson summation converts convolution-sampling discrepancies into frequency-domain periodization terms. Finite-order EM expansion provides endpoint correction R_p with $R_p = O(\Delta^p)$, bounded by Bernoulli polynomial estimates, yielding the stated equivalence and error control.

13.2 Theorem ?? (Detailed Proof)

On one hand, the Birman–Kreĭn formula [?] gives $\det S(E) = e^{-2\pi i \xi(E)}$, hence $\arg \det S(E) = -2\pi \xi(E)$, $\varphi(E) = \frac{1}{2} \arg \det S(E) = -\pi \xi(E)$, and thus

$$\frac{\varphi'(E)}{\pi} = -\xi'(E).$$

On the other hand, the time-delay operator in abstract scattering can be defined via stationary phase-scattering amplitude or “sojourn time”. Jensen et al. [?] proved in potential scattering and abstract frameworks that

$$\frac{1}{2\pi} \text{tr } Q(E) = \rho_{\text{rel}}(E) = -\xi'(E).$$

Combining both expressions yields

$$\frac{\varphi'(E)}{\pi} = \rho_{\text{rel}}(E) = \frac{1}{2\pi} \text{tr } Q(E) = -\xi'(E).$$

13.3 Theorem ?? (Detailed Proof)

For the geometric term,

$$\delta(\mathcal{R}\sqrt{|g|}) = (\mathcal{G}_{\mu\nu}\delta g^{\mu\nu} + \nabla_\alpha W^\alpha)\sqrt{|g|},$$

with boundary terms eliminated via compact support or Gibbons–Hawking–York terms [?]. For the information term,

$$\delta(\sqrt{|g|}\mathcal{L}_{\text{info}}) = \sqrt{|g|}\left(\frac{\partial\mathcal{L}_{\text{info}}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_{\text{info}}g_{\mu\nu}\right)\delta g^{\mu\nu}.$$

Setting variation to zero yields the master equation. The Bianchi identity provides conservation laws.

13.4 Theorem ?? (Detailed Proof)

By operator convexity of $\Phi(X) = -\log(\mathbb{I} + \alpha_{\text{I}}X)$,

$$Q_{\alpha\beta} = \text{Tr}\left[(\mathbb{I} + \alpha_{\text{I}}\mathbf{K})^{-1}\alpha_{\text{I}}\partial_\alpha\mathbf{K}(\mathbb{I} + \alpha_{\text{I}}\mathbf{K})^{-1}\alpha_{\text{I}}\partial_\beta\mathbf{K}\right] \succeq 0.$$

Substituting into the variation of $\mathcal{L}_{\text{info}} = \frac{1}{2}g^{\alpha\beta}Q_{\alpha\beta}$ yields

$$\boxed{\mathcal{T}_{\mu\nu}^{(\text{info})} = -Q_{\mu\nu} + \frac{1}{2}g_{\mu\nu} \text{Tr}_g Q}, \quad \boxed{\text{Tr}_g \mathcal{T}^{(\text{info})} = \left(\frac{n}{2} - 1\right)\text{Tr}_g Q \geq 0 \ (n \geq 2)}.$$

Hence

$$\boxed{\mathcal{T}_{vv}^{(\text{info})} = \left(\frac{1}{2}\text{Tr}_g Q - Q(v, v)\right)(v^\top g v)},$$

with “non-negativity for all v ” if and only if

$$\boxed{Q \preceq \frac{1}{2}(\text{Tr}_g Q) g}.$$

Toeplitz/Berezin positivity ensures $Q_{\alpha\beta} \succeq 0$. Euler–Maclaurin is used only for discrete-continuous error estimation and does not alone imply unconditional $\mathcal{T}_{vv}^{(\text{info})} \geq 0$.

13.5 Axiom ?? \rightarrow Theorem ?? (Proof)

Define reachability of worldline segments as existence of a null path with earliest arrival $t_*(\gamma)$. This definition is independent of windowed scales. Changing (w, h) or \mathbf{K}_θ only alters readouts without changing reachable sets, hence partial order remains invariant.

13.6 Quantum Measurement Section (Proof Outline)

- **Davies–Lewis:** Instruments $\{\mathcal{I}_x\}$ characterize measurements via CP map families and outcome measures [?].
- **Naimark:** Any POVM F admits a dilation space \mathcal{H}_+ with PVM E_+ such that $F(\Delta) = P E_+(\Delta) P$ [?].
- **Umegaki/Petz:** Relative entropy $D(\rho||\sigma)$ is monotonic with equality if and only if a Petz recovery map exists making the channel recoverable [?, ?].
- **Belavkin/GKSL/Spohn:** Under Markov limit, conditional states satisfy quantum filtering (Belavkin) [?] and GKSL master equation [?]. Spohn inequality [?] provides entropy production monotonicity.

Proofs and equivalent characterizations of these theorems are detailed in cited original and review literature.

13.7 Theorem ?? (Detailed Proof)

For Schwartz or appropriately decaying band-limited f , Poisson summation gives

$$\sum_{k \in \mathbb{Z}} f(k) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Replacing f with truncated/periodized versions and combining with EM expansion yields finite-order representation of discrete-continuous difference with remainder estimate. Thus at any finite order, no new singularities beyond those of the original function are generated [?].

14 Conclusion and Summary

We summarize the key results:

1. **Observation = Compression:** $K_{w,h} = \Pi_w M_h \Pi_w$; readouts = linear functionals on spectral measures (Berezin–Toeplitz equivalence).
2. **Master Scale Identity:** $\varphi'/\pi = \rho_{\text{rel}} = (2\pi)^{-1} \text{tr } Q = -\xi'$ unifies phase–density–group delay (Birman–Kreĭn and time-delay theory).
3. **Geometry = Response:** $g_{\mu\nu} = \partial_\mu \partial_\nu \mathcal{S}$, curvature as second-order non-commutative response (information geometry).
4. **Causality = Frontier:** Partial order defined by t_* , T_γ serves only as operational scale (EWS delay can be positive or negative without violating causality).
5. **Field Equations:** $\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}[K_\theta; \Phi]$, with $\nabla^\mu \mathcal{T}_{\mu\nu} = 0$ (Bianchi and Euler–Lagrange equations combined, on-shell).
6. **Quantum Unification:** Instrument–POVM–Naimark; Umegaki/Petz recoverability; GKSL + Spohn; Belavkin filtering.

7. **Error Closure:** NPE discipline and finite-order EM ensure “non-increasing singularity: poles = master scales”.

This framework establishes a rigorous mathematical foundation for unifying quantum measurement, scattering theory, and gravitational dynamics within a single information-geometric structure. Future directions include extending to quantum field theory on curved spacetimes, exploring connections to holographic principles, and developing computational implementations for realistic physical systems.

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