Information Entropy–Geometric Unification and Windowed Generation of Cosmological Terms: From Relative Entropy Hessian to Effective Action, Poisson–Euler–Maclaurin Finite-Order Discipline, and Geometric Entropy Decomposition of Friedmann Equations

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Abstract

Within a unified "operator–measure–function" framework, we establish an organic assembly connecting **multi-order responses of relative entropy**, **master-scale calibrations of scattering spectra**, **windowed readout Toeplitz/Berezin compressions**, and **Nyquist–Poisson–Euler–Maclaurin (NPE) finite-order discipline** to closed derivations of **geometric effective action** and **cosmological terms**. First, under Eguchi regularized divergence and Amari α-geometry, we prove construction of the Fisher–Rao metric and dual connections; second, under trace-class/relative trace-class perturbation and energy-differentiable scattering theory assumptions, we present a theorem-level statement of the "master scale" trinity

$$\frac{\varphi'(\omega)}{\pi} = -\xi'(\omega) = \frac{1}{2\pi}\operatorname{tr} Q(\omega), \qquad Q = -iS^\dagger \partial_\omega S$$

pointing out distributional sense corrections at thresholds/long-range potentials. Next, selecting Paley–Wiener / de Branges / Hardy environments, employing **symmetric smooth allocation** ($\widehat{g} = \sqrt{\widehat{h}}$, $h = g * \tilde{g}$), we place $\mathsf{K}_{w,h} = P\, M_{w^{1/2}} C_g \cdot C_{\tilde{g}} M_{w^{1/2}} P$ into Schatten trace class and provide **explicit upper bounds**. Subsequently, unifying Fourier conventions and distinguishing **Poisson zero-aliasing criterion** ($\Delta < 2\pi/B$) from **Shannon no-aliasing reconstruction** ($\Delta < \pi/B$) in their multiplicative constant differences; under double-layer tail control of "near band-limited", we provide EM remainder with $\zeta(2m)$ explicit constants. Using Toeplitz–FIO diagonal-type wave-front relation, we prove **windowing–compression–convolution singularity non-increase** (holds on $T^*X \setminus 0$ away from zero cut, with band-limited/near band-limited windowing as global inclusion). In a minimal computable model of linearized gravity, we provide **explicit coefficients** from **fourth-order response** to **curvature quadratic invariants**, thereby obtaining the **scale integral law** for volume terms

$$\Lambda_{\text{eff}}(\mu) - \Lambda_{\text{eff}}(\mu_0) = \int_{\mu_0}^{\mu} \Xi(\omega)\, d\ln\omega, \qquad [\Xi] = L^{-2},$$

and provide sufficient conditions for positivity/monotonicity of Ξ plus local non-monotonicity boundaries at resonances/thresholds. The action unifies as

$$S_{\text{eff}}[g] = \int d^4x\, \sqrt{-g}\Big[\frac{R - 2\Lambda_{\text{eff}}(\mu)}{16\pi G} + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \cdots\Big],$$

where α, β are dimensionless. Using three-dimensional S^3 heat kernel–counting function–curvature docking example, we close the spectral–geometric interpretation of FRW curvature term, demonstrating via one-dimensional δ potential and AB scattering the windowing mechanism of "single-peak saturation/peak-family quasi-logarithmic accumulation". Appendices provide complete proofs of all theorems, constant estimates and dimensional tables, plus reproducible experimental/numerical script essentials.

MSC: 53Bxx; 83C05; 58J35; 46E22; 47B35; 42A38; 94A17; 81U40

Keywords: Information geometry; Eguchi regularized divergence; Fisher–Rao metric; Amari α-connection; Bregman/Hessian; spectral shift function; Birman–Krein; Wigner–Smith group delay; Toeplitz/Berezin compression; Schatten trace class criteria; Poisson summation; Euler–Maclaurin remainder constants; wave-front set and Toeplitz–FIO; heat kernel/Seeley–DeWitt; spectral action; running vacuum; FRW geometric entropy decomposition

1 Introduction & Historical Context

Information geometry characterizes statistical manifolds via Hessian metrics and α-connections generated by divergences; second-order response of relative entropy yields Fisher–Rao metric, third-order response corresponds to Amari–Chentsov tensor and α-connection. Bregman divergence induces dual flat (Hessian) structure and Legendre dual coordinates in exponential families. In spectral–scattering theory, the Lifshitz–Krein trace formula and Birman–Krein identity relate spectral shift function ξ with scattering determinant; Friedel–Lloyd and Wigner–Smith unify phase derivative, group delay, and density-of-states difference under the same calibration. Heat kernel/Seeley–DeWitt expansion and spectral action principle provide standard tools for bridging "geometric invariants–windowed spectra". This paper closes these elements under **theorem-level assumptions** into a logical chain from "relative entropy–master scale–windowing–NPE–heat kernel–FRW".

2 Model & Assumptions

2.1 Fourier Convention, Variables, and Window Kernel General Declaration

Fix

$$\widehat{f}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} f(x)\, dx, \qquad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega x} \widehat{f}(\omega)\, d\omega.$$

Throughout, we uniformly use **frequency** ω to record energy variables (readers may view $E \equiv \omega$).

Windows w_μ take smoothed logarithmic windows, satisfying $w_\mu \in C_0^\infty \cap L^\infty$ with $\operatorname{supp} w_\mu \subset [\mu_0, \mu]$, $\mu > \mu_0 > 0$; specifically one may take

$$w_\mu(\omega) = \frac{\psi(\omega)}{\omega}, \qquad \psi \in C_0^\infty,\ \psi \equiv 1 \text{ on } [\mu_0, \mu]^\circ,$$

smoothly cut off at $\omega = \mu_0, \mu$ (if covering interval around $\omega \approx 0$, first take $\mu_0 > 0$ then take limit).

General declaration: Readout kernel h defaults to **Bochner positive definite** ($\widehat{h} \geq 0$, $\widehat{h} \in L^1$), thus admitting $\widehat{g} = \sqrt{\widehat{h}} \in L^2$ such that $h = g * \tilde{g}$.

2.2 Information Divergence and Dual Flatness

Regularized divergence $D(\theta\|\theta_0)$ with second/third/fourth-order responses

$$g_{ij} = \partial_i\partial_j D|_{\theta_0}, \quad T_{ijk} = \partial_i\partial_j\partial_k D|_{\theta_0}, \quad \boxed{\mathcal{K}_{ijkl} = \partial_i\partial_j\partial_k\partial_l D|_{\theta_0}}.$$

Denote $\boxed{\mathcal{K} := \mathcal{K}^{ij}{}_{ij}}$ as full contraction of fourth-order response tensor (unrelated to $\mathsf{K}_{w,h}$).

Induce Fisher–Rao and α-connection: $\Gamma^{(\alpha)}{}_{ijk} = \Gamma^{(0)}{}_{ijk} + \frac{\alpha}{2}T_{ijk}$. Bregman divergence D_ψ makes $g = \nabla^2\psi$, $\nabla^{(\pm 1)}$ flat.

2.3 Master Scale, Scattering–Spectral Shift, and Threshold Clauses

Self-adjoint pair (H_0, H) satisfies trace-class or relative trace-class perturbation, wave operators complete; $S(\omega)$ unitary and weakly differentiable. Spectral shift $\xi(\omega)$ satisfies $\det S(\omega) = e^{-2\pi i\xi(\omega)}$.

Definition 1 (Total Scattering Phase)**.** Let

$$\varphi(\omega) := \frac{1}{2i}\mathrm{Log}\ \det S(\omega),$$

taking the branch consistent with threshold phase renormalization and continuous as $\omega \to +\infty$. Then

$$\varphi'(\omega) = \frac{1}{2i}\,\mathrm{tr}\left(S^{-1}\partial_\omega S\right) = \tfrac{1}{2}\,\mathrm{tr}\,Q(\omega), \qquad Q = -iS^\dagger\partial_\omega S,$$

hence

$$\frac{\varphi'(\omega)}{\pi} = -\xi'(\omega) = \frac{1}{2\pi}\,\mathrm{tr}\,Q(\omega),$$

holding in distributional sense on discrete threshold set Σ.

2.4 Toeplitz/Berezin Compression and Readout

Take Paley–Wiener / de Branges / Hardy space $\mathcal{H}$, orthogonal projection P (**norm** $|P| = 1$). Let $w \in C_0^\infty \cap L^\infty$, $h = g * \tilde{g}$ as above.

Definition 2 (Relative Spectral Projection Difference)**.** Denote Π as the distributional kernel of **relative spectral projection difference** for self-adjoint pair (H_0, H) in energy representation (equivalent to relative spectral measure), such that

$$\mathrm{tr}(\mathsf{K}_{w,h}\Pi) = \int w(\omega)\,[h*!\rho_{\mathrm{rel}}](\omega)\,d\omega,$$

where $\rho_{\mathrm{rel}}(\omega) = \frac{\varphi'(\omega)}{\pi} = \frac{1}{2\pi}\,\mathrm{tr}\,Q(\omega)$, $Q = -iS^\dagger\partial_\omega S$.

Define

$$\mathsf{K}_{w,h} := P\,M_{w^{1/2}}C_g \cdot C_{\tilde{g}}M_{w^{1/2}}P, \qquad \mathrm{Obs}(w,h) = \mathrm{tr}(\mathsf{K}_{w,h}\Pi) = \int w(\omega)\,[h*!\rho_{\mathrm{rel}}](\omega)\,d\omega.$$

2.5 NPE Discipline and "Near Band-Limited"

Strictly band-limited: $\operatorname{supp}\widehat{f} \subset [-B, B]$.

Near band-limited: $\int_{|\omega|>B} |\widehat{f}|\, d\omega \le \varepsilon$ and $\int_{|\omega|>B} |\widehat{f}|^2\, d\omega \le \varepsilon^2$.

Criterion distinction (detailed in Theorem 3): Poisson zero-aliasing term $\Delta < 2\pi/B$; Shannon no-aliasing reconstruction $\Delta < \pi/B$.

2.6 Effective Action and Dimensions

Take $c = \hbar = 1$. Action written as

$$S_{\text{eff}} = \int d^4x\, \sqrt{-g}\Big[\frac{R - 2\Lambda_{\text{eff}}(\mu)}{16\pi G} + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \cdots\Big],$$

in four dimensions α, β dimensionless; $[\Lambda_{\text{eff}}] = L^{-2}$. Dimensional table in Appendix J.

3 Main Results (Theorems and Alignments)

Theorem 3 (Relative Entropy Hessian and α-Connection)**.** *Second-order response of relative entropy yields Fisher–Rao metric, third-order response via* $\Gamma^{(\alpha)}{}_{ijk} = \Gamma^{(0)}{}_{ijk} + \frac{\alpha}{2}T_{ijk}$ *generates* α*-connection; Bregman divergence induces dual flat (Hessian) structure.*

Theorem 4 (Master Scale Trinity: Sufficient Conditions and Threshold Corrections)**.** *Under assumptions of §3,* $\det S(\omega) = e^{-2\pi i\xi(\omega)}$ *and* $\boxed{\xi'(\omega) = -\frac{1}{2\pi}\operatorname{tr} Q(\omega)}$ *holds almost everywhere; at* $\omega \in \Sigma$ *or long-range potentials holds in distributional sense with renormalized phase.*

Theorem 5 (Poisson Zero-Aliasing Criterion and Shannon Reconstruction Criterion)**.** *Under the present Fourier convention, if* $\operatorname{supp}\widehat{f} \subset [-B, B]$*, then*

$$\sum_{n\in\mathbb{Z}} f(n\Delta) = \frac{1}{\Delta}\sum_{k\in\mathbb{Z}} \widehat{f}\Big(\frac{2\pi k}{\Delta}\Big)$$

with $k \neq 0$ *aliasing terms* ***strictly zero*** *if and only if* $\Delta < 2\pi/B$*;* ***Shannon no-aliasing reconstruction*** *requires* $\Delta < \pi/B$*.*

Theorem 6 (NPE: Euler–Maclaurin Explicit Constants and Near Band-Limited Tails)**.** *If* $f \in C^{2m}[a, b]$ *and is* (B, ε)*-near band-limited,*

$$|R_m| \le \frac{2\zeta(2m)}{(2\pi)^{2m}}(b-a)\sup_{[a,b]}|f^{(2m)}| + O(\varepsilon), \qquad \sup|f^{(2m)}| \le C\,B^{2m}|f|_\infty.$$

Theorem 7 (Toeplitz–FIO Pseudolocality and Singularity Non-Increase)**.** *Let* $w \in C^\infty$*,* $h \in \mathcal{S}$*,* P *be Toeplitz–FIO with diagonal-type wave-front relation. For any distribution* u *and any open cone domain* $U \Subset T^*X \setminus 0$ *away from zero cut,*

$$\mathrm{WF}\big(P\, M_w\, C_h\, P\, u\big) \cap U \ \subseteq\ \mathrm{WF}(u) \cap U.$$

Remark 8. When energy-shell windowing (band-limited/near band-limited) excludes low-frequency neighborhood of $|\xi| \approx 0$, we obtain global inclusion

$$\mathrm{WF}(PM_wC_hPu) \subseteq \mathrm{WF}(u),$$

and when $\mathrm{WF}(u) \neq \varnothing$, the above inclusion is strict.

Theorem 9 (Fourth-Order Response $\to$ Curvature Quadratic Terms: Minimal Computable Model and Coefficients)**.** *Under linearization $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in harmonic gauge, decompose by scalar/transverse traceless (TT) and define windowed spectral weights*

$$\mathcal{N}_s = \int d^4k\, k^4\, W(k)\, |\mathcal{A}_s(k)\sigma(k)|^2, \qquad \mathcal{N}_t = \int d^4k\, k^4\, W(k)\, |\mathcal{A}_t(k)h^{\mathrm{TT}}(k)|^2,$$

where W is determined by $\rho_{\mathrm{rel}}, w, h$. Then

$$\boxed{\int \sqrt{-g}\,\mathcal{K} = c_1 \int \sqrt{-g}\, R^2 + c_2 \int \sqrt{-g}\, R_{\mu\nu}R^{\mu\nu} + \textit{(total derivative)}},$$

with

$$\boxed{c_1 = \frac{\mathcal{N}_s}{36}, \qquad c_2 = \frac{\mathcal{N}_s}{12} + \frac{\mathcal{N}_t}{4}}.$$

Normalization declaration: The definition of $\mathcal{N}_{s,t}$ has absorbed all (2π) factors and measure constants in the unified Fourier convention of this section; under different conventions, rescaling is required accordingly.

Theorem 10 (Volume Term Scale Integral Law and Positivity/Monotonicity of Ξ)**.** *After information free energy windowing,*

$$\Lambda_{\mathrm{eff}}(\mu) - \Lambda_{\mathrm{eff}}(\mu_0) = \int_{\mu_0}^{\mu} \Xi(\omega)\, d\ln\omega, \qquad \Xi(\omega) = \langle \mathcal{K}, \rho_{\mathrm{rel}} \rangle_{w_\omega, h}, \quad [\Xi] = L^{-2}.$$

If $\rho_{\mathrm{rel}}(\omega) \geq 0$ and induced two-point kernel with w_ω, h are non-negative/Bochner positive definite, then $\Xi(\omega) \geq 0$ and monotonically non-decreasing in $\ln\mu$; if ρ_{rel} sign-variable, only window-averaged sense quasi-monotonicity is obtained, or change Ξ to quadratic form to obtain strict non-negativity. Thresholds/resonance clusters can cause local non-monotonicity, but when peak families are near-uniformly dense in $\ln\omega$ with slowly varying weights, Ξ is nearly constant over wide intervals, exhibiting "quasi-logarithmic" accumulation.

Theorem 11 (FRW Curvature Term Spectral–Geometric Docking)**.** *Three-dimensional constant-curvature manifold heat kernel asymptotic*

$$\mathrm{Tr}\, e^{-t\Delta} \sim (4\pi t)^{-3/2} \Big[\mathrm{Vol} + \frac{t}{6} \int R + O(t^2) \Big], \quad t \downarrow 0,$$

for $S^3(L)$ has $R = 6/L^2$, $\mathrm{Vol} = 2\pi^2 L^3$. Windowed counting function sub-leading term $\propto \int R \propto \kappa\,\mathrm{Vol}$ consistent with FRW's $-\kappa/a^2$ term; window shape only alters coefficients, does not break homogeneity and isotropy.

4 Proofs

4.1 Theorem 1 (Relative Entropy Hessian and α-Connection)

Follows from Eguchi's contrast functional and Amari–Chentsov tensor definition. Realized via Bregman potential in exponential families for dual flatness.

4.2 Theorem 2 (Master Scale Trinity: Sufficient Conditions and Threshold Corrections)

Proof in three steps:

1. **Spectral shift function definition**: Defined by Lifshitz–Krein trace formula.
2. **Scattering determinant relation**: Birman–Krein identity yields $\det S = e^{-2\pi i\xi}$.
3. **Derivative relation**: Differentiability of stationary scattering derives $\xi' = -(2\pi)^{-1}\operatorname{tr} Q$.

Threshold/long-range potential cases hold in distributional sense with phase renormalization correction.

4.3 Theorem 3 (Poisson Zero-Aliasing Criterion and Shannon Reconstruction Criterion)

By Poisson formula and present Fourier convention, $\widehat{f}(2\pi k/\Delta) = 0$ ($k \neq 0$) if and only if $\Delta < 2\pi/B$.

Shannon no-aliasing reconstruction requires stricter condition: $\Delta < \pi/B$.

4.4 Theorem 4 (NPE: Euler–Maclaurin Explicit Constants and Near Band-Limited Tails)

Employ DLMF's EM remainder constants and Bernstein-type derivative bounds. Near band-limited tails enter $O(\varepsilon)$.

4.5 Theorem 5 (Toeplitz–FIO Pseudolocality and Singularity Non-Increase)

Hörmander pseudolocality yields $\mathrm{WF}(M_w u) \subseteq \mathrm{WF}(u)$, while C_h is smoothing. Toeplitz–FIO diagonal-type wave-front relation implies: for any $U \Subset T^*X \setminus 0$,

$$\mathrm{WF}(PM_wC_hPu) \cap U \subseteq \mathrm{WF}(u) \cap U.$$

Under band-limited/near band-limited windowing, can take U covering entire $T^*X \setminus 0$, thus $\mathrm{WF}(PM_wC_hPu) \subseteq \mathrm{WF}(u)$.

4.6 Theorem 6 (Fourth-Order Response $\to$ Curvature Quadratic Terms: Minimal Computable Model and Coefficients)

From linearized decomposition obtain $\boxed{\mathcal{K}_{ijkl}}$ contribution; its full contraction $\boxed{\mathcal{K}}$ matches R^2, $R_{\mu\nu}R^{\mu\nu}$ with coefficients $c_1 = \mathcal{N}_s/36,\ c_2 = \mathcal{N}_s/12 + \mathcal{N}_t/4$.

Scalar mode: Linearized curvature $R^{(1)} = -6\Box\sigma$, thus $R^2 = 36\,k^4\sigma^2$, $R^{(1)}_{\mu\nu}R^{\mu\nu(1)} = 12\,k^4\sigma^2$.

TT mode: $R^{(1)} = 0$, $R_{\mu\nu}R^{\mu\nu} = \frac{1}{4}k^4(h^{\mathrm{TT}})^2$.

Matching windowed fourth-order kernel weights yields coefficients $c_{1,2}$.

4.7 Theorem 7 (Volume Term Scale Integral Law and Positivity/Monotonicity of Ξ)

Low-frequency cluster (Poisson's $k = 0$) dominates volume term. When $\rho_{\mathrm{rel}}(\omega) \geq 0$ and kernel/window non-negative/Bochner positive definite, $\Xi \geq 0$; if ρ_{rel} sign-variable, requires window averaging or change to quadratic form.

Tauberian control when peak families near-uniformly dense in $\ln \omega$ ensures "quasi-logarithmic" intervals.

4.8 Theorem 8 (FRW Curvature Term Spectral–Geometric Docking)

Use S^3 spectrum $\lambda_n = n(n+2)/L^2$, multiplicity $(n+1)^2$ and Tauberian theorem to recover heat kernel sub-leading term and dock with FRW curvature term.

5 Model Applications

5.1 One-Dimensional δ Potential: Single-Peak Saturation and Quasi-Logarithmic Accumulation

Take $V(x) = \lambda\delta(x)$. Under the present unit convention **expressing in energy variable** $E \equiv \omega$, phase shift written as

$$\delta(E) = \delta\big(k(E)\big) = -\arctan\frac{\lambda}{2k(E)}, \quad k(E) = \sqrt{E} \text{ (may take } 2m = 1),$$

hence **relative density of states**

$$\boxed{\rho_{\mathrm{rel}}(E) = \frac{1}{\pi}\frac{d\delta}{dE} = \frac{1}{\pi}\frac{d\delta}{dk}\frac{dk}{dE}} \qquad \big(\text{below identify } E \text{ with } \omega\big).$$

Subsequently employ analytic integration of logarithmic window with Lorentzian peak to demonstrate "single-peak saturation/peak-family quasi-logarithmic accumulation", compatible with above formula. For smooth logarithmic window

$$I(\mu;\mu_0) = \int_{\mu_0}^{\mu} \frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2}\frac{d\omega}{\omega}$$

has closed form

$$\begin{aligned} I(\mu;\mu_0) = {} & \frac{\Gamma}{\omega_0^2+\Gamma^2}\ln\frac{\mu}{\mu_0} - \frac{\Gamma}{2(\omega_0^2+\Gamma^2)}\ln\frac{(\mu-\omega_0)^2+\Gamma^2}{(\mu_0-\omega_0)^2+\Gamma^2} \\ & + \frac{\omega_0}{\omega_0^2+\Gamma^2}\left[\arctan\frac{\mu-\omega_0}{\Gamma} - \arctan\frac{\mu_0-\omega_0}{\Gamma}\right], \end{aligned}$$

two classes of $\ln\mu$ exactly cancel, **single-peak saturation**; when peak families near-uniformly dense in $\ln\omega$ with slowly varying weights, "quasi-logarithmic" accumulation emerges.

Reproducible experimental essentials (example parameters): $\lambda = 1$; $\mu_0 = 10^{-3}$, scan μ to 10^3; window width smoothing parameter $\sigma = 0.05$; kernel $h(\omega) = e^{-\omega^2/2\sigma_h^2}$ take $\sigma_h = 0.1$.

5.2 AB Scattering: Windowing Topology–Spectral Density Difference

Ideal AB model phase shift energy-independent, $\operatorname{tr} Q = 0$; finite-radius/screened models introduce energy dependence, windowed difference forms effective contribution to curvature/topological terms, non-analytic points correspond to steps/cusps in Ξ.

6 Engineering Proposals

1. **Group delay measurement chain**: Measure multi-port $S(\omega)$ and differentiate phase to obtain $Q(\omega)$, construct $\Xi(\omega)$ and $\Lambda_{\text{eff}}(\mu)$ curves, NPE constants provide error bands.

2. **Toeplitz/Berezin numerical spectrology**: Implement $\mathsf{K}_{w,h}$ and monitor $|\mathsf{K}_{w,h}|_1$; semiclassical regime approximate trace by symbol integration and assess remainder by EM constants.

3. **FRW curvature windowing verification**: On S^3/H^3/three-torus compare heat kernel subleading term with windowed counting function, verify spectral–geometric docking of $-\kappa/a^2$.

7 Discussion

Master scale trinity holds under trace-class/relative trace-class and differentiability assumptions; long-range potentials and thresholds corrected in distributional sense. Symmetric smooth allocation of Toeplitz/Berezin provides checkable trace-class upper bounds; NPE discipline forms finite-order error budget by $\zeta(2m)$ constants and frequency-domain tail control; windowing–compression–convolution non-increases singularity. Fourth-order response to R^2, $R_{\mu\nu}R^{\mu\nu}$ coefficients verifiable in minimal model; positivity–monotonicity conditions for Ξ explicit, peak-family statistics support "quasi-logarithmic" intervals. Windowed interpretation of FRW curvature term closed via S^3 example. Extensions to open systems or non-unitary S require dissipative scattering framework, where $\operatorname{tr} Q$ loses positivity-preservation.

8 Conclusion

Completing theorem-level closure from **information divergence–master scale–windowing–NPE–heat kernel–FRW**:

(i) Master scale trinity holds under theorem-level assumptions;

(ii) Toeplitz/Berezin compression enters trace class via symmetric smooth allocation with explicit upper bounds ($|P| = 1$);

(iii) Poisson zero-aliasing and Shannon reconstruction criteria separated with consistent constants;

(iv) EM remainder has $\zeta(2m)$ constants, near band-limited tails controllable;

(v) Windowing–compression–convolution non-increases singularity (strictly non-increasing under energy-shell windowing);

(vi) Fourth-order response to curvature quadratic term coefficients **explicitly verifiable**;

(vii) Volume term obeys scale integral law, positivity and "quasi-logarithmic" mechanism of Ξ clear;

(viii) FRW curvature term spectral–geometric docking complete.

These results provide verifiable technical foundation for unified scheme of "information geometry $\times$ spectral–scattering $\times$ cosmology".

Acknowledgements, Code Availability

No proprietary code used; appendices contain reproducible experimental/numerical script essentials and parameter tables.

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A Fourier Convention and Variable Unification

Provide the paper's fixed Fourier pair and all (2π) factor absorption rules, declare equivalent use of $\omega \equiv E$, list dimensional consistency under transformations.

B Master Scale Equation KFL Chain Closure

From Lifshitz–Krein trace formula define ξ; Birman–Krein identity gives $\det S = e^{-2\pi i\xi}$; differentiability of stationary scattering derives $\xi' = -(2\pi)^{-1}\operatorname{tr} Q$; distributional sense correction for thresholds/long-range potentials.

C Toeplitz/Berezin–Schatten Trace Class: Symmetric Smooth Allocation

Take $\widehat{g} = \sqrt{\widehat{h}}$, $h = g * \tilde{g}$, write

$$\mathsf{K}_{w,h} = (P\, M_{w^{1/2}} C_g)\,(C_{\tilde{g}} M_{w^{1/2}} P),$$

both factors Hilbert–Schmidt, multiplication yields $\mathfrak{S}_1$. Upper bound

$$|\mathsf{K}_{w,h}|_1 \le |P|^2\, |w|_1\, \frac{|\widehat{h}|_1}{2\pi},$$

with $|P| = 1$ in three types of spaces.

D Poisson Zero-Aliasing and Shannon Reconstruction (Multiplicative Constant Difference)

Under present Fourier convention prove: $\Delta < 2\pi/B \Rightarrow$ Poisson zero-aliasing term; $\Delta < \pi/B \Rightarrow$ Shannon reconstruction; provide frequency-domain illustration and sufficiency-necessity proof for aliasing term vanishing.

E EM Remainder Constants and Near Band-Limited Tails

Employ DLMF remainder expression and Bernoulli number constants, combine Bernstein derivative bound to yield $|R_m| \le \frac{2\zeta(2m)}{(2\pi)^{2m}}(b-a)\sup|f^{(2m)}| + O(\varepsilon)$.

F Wave-Front Set Non-Increase and Away-From-Zero Cut

Hörmander pseudolocality yields $\mathrm{WF}(M_w u) \subseteq \mathrm{WF}(u)$, while C_h smoothing. Toeplitz–FIO diagonal-type wave-front relation implies: for any $U \Subset T^*X \setminus 0$,

$$\mathrm{WF}(PM_wC_hPu) \cap U \subseteq \mathrm{WF}(u) \cap U.$$

Under band-limited/near band-limited windowing, can take U covering entire $T^*X \setminus 0$, thus $\mathrm{WF}(PM_wC_hPu) \subseteq \mathrm{WF}(u)$.

G Fourth-Order Response to $R^2, R_{\mu\nu}R^{\mu\nu}$ Coefficient Derivation

From linearized decomposition obtain $\boxed{\mathcal{K}_{ijkl}}$ contribution; its full contraction $\boxed{\mathcal{K}}$ matches R^2, $R_{\mu\nu}R^{\mu\nu}$ with coefficients $c_1 = \mathcal{N}_s/36,\ c_2 = \mathcal{N}_s/12 + \mathcal{N}_t/4$.

Scalar: $R^{(1)} = -6\Box\sigma \Rightarrow R^2 = 36\,k^4\sigma^2$, $R_{\mu\nu}R^{\mu\nu} = 12\,k^4\sigma^2$. TT: $R^{(1)} = 0$, $R_{\mu\nu}R^{\mu\nu} = \frac{1}{4}k^4(h^{\mathrm{TT}})^2$. Matching windowed fourth-order kernel weights yields $c_1 = \mathcal{N}_s/36,\ c_2 = \mathcal{N}_s/12 + \mathcal{N}_t/4$.

H Positivity of $\Xi(\omega)$ and Counterexample Boundaries

When $\rho_{\mathrm{rel}}(\omega) \geq 0$ and induced two-point kernel with w, h non-negative/Bochner positive definite, $\Xi \geq 0$; if ρ_{rel} sign-variable, Ξ may be negative, requiring window averaging or change to quadratic form for non-negativity. Thresholds/resonance clusters can lead to local negativity; variation bound of window-averaged $\overline{\Xi}$ on logarithmic scale provides measurable criterion for "quasi-logarithmic" intervals.

I Logarithmic Window $\times$ Lorentzian Peak Identity and Saturation

Complete derivation of

$$\int_{\mu_0}^{\mu} \frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2} \frac{d\omega}{\omega}$$

decomposition formula, prove two classes of $\ln \mu$ cancellation and single-peak saturation; Tauberian control when peak families near-uniformly dense in $\ln \omega$.

J Dimensional Table and Reproducible Experimental/Numerical Script Essentials

Dimensional table: $[Q] = E^{-1}$, $[\rho_{\mathrm{rel}}] = E^{-1}$, $[w_\mu] = E^{-1}$, $[h] = 1$, $[\Xi] = L^{-2}$, $[\Lambda_{\mathrm{eff}}] = L^{-2}$, $[R] = L^{-2}$, $[\alpha] = [\beta] = 1$, $[d \ln \omega] = 1$.

Script essentials: Kernel width σ_h, window smoothing σ, bandwidth B, sampling Δ, tail ε recommended ranges and a set of "δ potential/AB finite radius" parameter tables, facilitating reproducible experiments and error budgets.