

# Geometrization of Causal Structures: Spacetime as Minimal Lossless Compression of Causal Constraints

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November 24, 2025

## Abstract

Starting from first principles, this paper treats “causal structure” as the most primitive and economical information object in the physical world, and proposes a unified perspective that “spacetime geometry = minimal lossless compression of causal constraints”. Specifically, we characterize causal reachability by partial order relations on event sets, recover conformal structure via Alexandrov topology and time orientation, and supplement the absolute scale of the metric with volume calibration and statistical information. We prove that under quite general conditions: given causal structure and volume measure, the conformal class of spacetime metric can be uniquely reconstructed. Subsequently, we introduce a “description length–curvature” variational principle, interpreting curvature as the “redundancy density” of correlations among causal constraints that cannot be eliminated, and propose the functional

$$\mathcal{F}[g] = \mathcal{C}(\text{Reach}(g)) + \lambda \int_M |\text{Riem}(g)|^2 d\text{Vol}_g$$

as an abstract model for the trade-off between geometric reconstruction and causal compression. At the level of linear quantum field theory and information geometry, we discuss how microcausality, modular flow, and relative entropy monotonicity define the “information length” of distinguishable paths under given causal cone structure, thereby geometrizing the relationships among causality, geometry, and information. Through examples of Minkowski spacetime, FRW universe, and finite causal set embeddings, we demonstrate how this perspective unifies the intuition that “flatness = no redundancy” and “curvature = constraint correlations cannot be flattened”. The appendices provide theorem statements and proof sketches for how causal structure determines conformal class, technical details of Alexandrov topology and strong causality, and formal derivation of variational equations for description complexity functionals.

**Keywords:** Causal Structure, Geometrization, Information Compression, Spacetime Emergence, Conformal Class, Alexandrov Topology, Description Complexity

**MSC 2020:** 83C45, 94A29, 06A06, 53C50

## Contents

# 1 Introduction

## 1.1 Problem Background and Basic Position

In general relativity, spacetime is characterized as a differentiable manifold  $(M, g)$  equipped with a Lorentz metric  $g$ . The metric  $g$  plays two roles simultaneously:

1. Determining light cones and causal reachability (causal structure);
2. Determining quantitative calibration of time and space (lengths and volumes).

However, numerous theorems show that under appropriate causal conditions, merely knowing “which events can causally influence which events” as a reachability relation is sufficient to largely recover the conformal structure of spacetime, i.e., the conformal class  $[g]$  of  $g$ . This suggests a highly compression-oriented viewpoint:

If we only care about “what influences causality allows”, then complete metric information is far more than necessary; causal structure itself is a more primitive and economical information object, while geometry is a kind of “encoding” built upon it.

This paper asks three questions from this perspective:

1. If only causal structure is given, to what extent can geometry be recovered?
2. Can geometry be viewed as a kind of “minimal lossless compression” of causal constraints?
3. Can curvature, a traditional geometric quantity, be interpreted as the “redundancy density of correlations among causal constraints”?

## 1.2 Structural Overview of This Paper’s Contributions

The main contributions of this paper can be summarized as follows:

1. **Axiomatic Causal Structure and Geometric Reconstruction:** Based on partially ordered sets and Alexandrov topology, we introduce the concept of “causal space” and provide theorem statements for correspondence between causal homeomorphisms and conformal homeomorphisms: under strong causality and moderate regularity assumptions, causal structure uniquely determines the conformal class.
2. **Compression Perspective and Variational Principle:** Define description complexity function  $\mathcal{C}$  of causal reachability graphs, propose the functional

$$\mathcal{F}[g] = \mathcal{C}(\text{Reach}(g)) + \lambda \int_M |\text{Riem}(g)|^2 d\text{Vol}_g$$

,

interpreting curvature as “accounting for correlations among causal constraints that cannot be simultaneously flattened”.

3. **Embedding of Information Geometry and Quantum Field Theory:** Under given causal structure and boundary algebra, construct “information metric” in the interior of causal cones using microcausality, modular flow, relative entropy, and Fisher information, connecting the length of “distinguishable paths” with causal constraints.
4. **Case Analysis and Finite Models:** Demonstrate how the above structures are concretely realized in Minkowski, FRW, and discrete causal set cases, along with intuitive interpretations.

The appendices provide formally precise statements and proof sketches for involved theorems, with supplements on technical details of topology and measure.

## 2 Causal Structure and Partial Order Models

### 2.1 Partial Order Characterization of Causal Structure

Let  $M$  be a four-dimensional differentiable manifold equipped with a Lorentz metric  $g$ . For any point  $p \in M$ , define the future and past of timelike samples:

- $J^+(p)$ : The set of points reachable from  $p$  via non-spacelike causal curves (including null and timelike);
- $J^-(p)$ : Similarly defined as the set of points that can causally reach  $p$ .

Define binary relation  $\leq$  as follows: for  $p, q \in M$ , write  $p \leq q$  if  $q \in J^+(p)$ . Under appropriate causal conditions (such as no closed causal curves),  $\leq$  constitutes a partial order relation on  $M$ .

**Definition 2.1** (Causal Preorder and Causal Set). Let  $X$  be a set and  $\preceq$  a reflexive, transitive relation on  $X$ . Call  $(X, \preceq)$  a causal preordered set. If additionally  $\preceq$  has antisymmetry, call it a causally partially ordered set or causal set.

For spacetime  $(M, g)$ ,  $(M, \leq)$  is a causally partially ordered set. We often use notation  $p \ll q$  to denote  $q \in I^+(p)$ , where  $I^+(p)$  is the strictly timelike future.

### 2.2 Alexandrov Topology and Causal Open Sets

A natural topology can be generated from causal structure.

**Definition 2.2** (Alexandrov Basis). For  $p, q \in M$  with  $p \ll q$ , define the bicone open set

$$A(p, q) := I^+(p) \cap I^-(q).$$

Call  $\mathcal{B} := \{A(p, q) : p, q \in M, p \ll q\}$  the Alexandrov basis. The topology generated by  $\mathcal{B}$  is called the Alexandrov topology.

If  $(M, g)$  is strongly causal, then the Alexandrov topology coincides with the original differentiable manifold topology. Thus under the strong causality assumption, **topological structure can be reconstructed from causal structure alone**.

### 3 Causal Structure, Time Orientation, and Conformal Class

#### 3.1 Conformal Structure and Light Cones

The conformal class of a Lorentz metric  $g$  is defined as

$$[g] := \{\Omega^2 g : \Omega : M \rightarrow (0, \infty) \text{ smooth}\}.$$

Conformally equivalent metrics have the same null vector cones, i.e., the same light cone structure and causal relations. Therefore:

- Causal structure  $(M, \leq)$  depends only on the conformal class  $[g]$ ;
- If two Lorentz metrics  $g, \tilde{g}$  induce the same causal structure, then under appropriate conditions they are conformally equivalent.

**Axiom 3.1** (Time Orientation). Assume spacetime  $(M, g)$  is time-orientable, i.e., there exists a globally consistent choice of “future” direction, allowing timelike vector fields to be globally oriented. Time orientation and causal structure together determine information about “which side is the future”.

Under this assumption, causal structure contains information about:

1. Which points can causally influence which points;
2. For each causal curve, which direction is “future”.

#### 3.2 Causal Homeomorphism and Conformal Homeomorphism

**Definition 3.2** (Causal Homeomorphism). Let  $(M, [g]), (\tilde{M}, [\tilde{g}])$  be two spacetimes with causal relations denoted  $\leq, \tilde{\leq}$  respectively. If there exists a bijection  $\Phi : M \rightarrow \tilde{M}$  such that for any  $p, q \in M$ ,

$$p \leq q \iff \Phi(p) \tilde{\leq} \Phi(q),$$

then call  $\Phi$  a causal homeomorphism.

Under strong causality, local compactness, and appropriate regularity assumptions, the following fundamental theorem can be stated (formalized version and proof sketch given in the appendix):

**Theorem 3.3** (Causal Structure Determines Conformal Class, Theorem Form). *Let  $(M, g), (\tilde{M}, \tilde{g})$  be strongly causal, locally compact spacetimes, assuming no “pathological” causal boundaries. If there exists a causal homeomorphism  $\Phi : M \rightarrow \tilde{M}$ , then  $\Phi$  is a conformal homeomorphism, i.e., there exists a smooth function  $\Omega : \tilde{M} \rightarrow (0, \infty)$  such that*

$$\Phi^* \tilde{g} = \Omega^2 g.$$

*Therefore, under these assumptions, causal structure + time orientation suffices to determine the conformal class.*

## 4 Volume Calibration and Absolute Metric Reconstruction

### 4.1 Causality + Volume Measure

The conformal class only determines the structure of “light cones” and “null geodesics”, not yet determining the absolute length scale. For this, introduce volume calibration:

**Axiom 4.1** (Volume Calibration). On  $(M, [g])$ , given a Borel measure  $\mu$  compatible with the volume form  $dVol_g$  of some representative metric  $g$ , i.e., there exists a positive smooth function  $\rho : M \rightarrow (0, \infty)$  such that for all measurable sets  $A \subset M$ ,

$$\mu(A) = \int_A \rho dVol_g.$$

Intuitively,  $\mu$  captures information about “event density” or “volume calibration”.

Under appropriate regularity conditions, causal structure + volume measure can recover the absolute scale of the metric: by comparing how the volume of Alexandrov sets  $A(p, q)$  varies with  $p, q$ , one can inversely determine the assignment of conformal factor  $\Omega$ , thus recovering the specific metric.

### 4.2 Step-by-Step Narrative of Geometric Reconstruction

In summary, geometric reconstruction can be viewed as three steps:

1. **From Partial Order to Topology:** Causal partial order  $(M, \leq)$  generates Alexandrov topology, which under strong causality equals the manifold topology;
2. **From Causality to Conformal Class:** Causal structure and time orientation under strong causality and regularity uniquely determine conformal class  $[g]$ ;
3. **From Volume to Metric:** Introduce volume measure  $\mu$ ; by comparing volumes of Alexandrov sets with conformal structure, recover absolute scaling factor to obtain specific  $g$ .

Therefore, under appropriate conditions, the following data can be viewed as equivalent geometric “encodings”:

$$(M, g) \iff (M, \leq, \mu).$$

In the right-hand data,  $\leq$  and  $\mu$  are closely related to “reachable/unreachable” and “volume calibration”, having more “information compression” implications.

## 5 Geometry as Minimal Lossless Compression of Causal Constraints

### 5.1 Causal Reachability Graph and Description Complexity

After discretizing spacetime, events can be viewed as vertices and causal relations as edges of a directed acyclic graph, obtaining a causal reachability graph  $\mathcal{G} = (V, E)$ . Even in the continuous case, causal structure can be approximated as such graphs through some sampling or coarse-graining.

**Definition 5.1** (Description Complexity). Let  $\mathcal{G} = (V, E)$  be a finite directed acyclic graph,  $\mathcal{D}$  a class of description languages (such as adjacency matrix encoding, adjacency list, hierarchical decomposition, etc.). Define the minimal description complexity under precision  $\epsilon$  as

$$\mathcal{C}_\epsilon(\mathcal{G}) := \min \{\text{encoding length}(\text{Code}) : \text{Code} \in \mathcal{D}, \text{reconstruction error} \leq \epsilon\}.$$

In appropriate limits, a continuous version can be defined denoted  $\mathcal{C}(\text{Reach}(g))$ , where  $\text{Reach}(g)$  represents the causal reachability structure induced by metric  $g$ .

Intuitively,  $\mathcal{C}(\text{Reach}(g))$  measures “how much information is needed at minimum to record all causal constraints”.

## 5.2 Curvature as Redundancy Density

Geometrically, local flatness means that in sufficiently small neighborhoods, coordinate systems can be found making the metric approximately Minkowski, with light cone structure “as straight as possible”. If we view local causal structures in each local neighborhood as “local constraints”, then when these constraints can be compatibly assembled into a globally flat structure, curvature is zero; when this compatibility fails, curvature records this “closed-loop deviation that cannot be eliminated” through the Riemann tensor  $\text{Riem}$ .

Therefore, the following interpretation can be proposed:

Curvature can be viewed as the “redundancy density of correlations among causal constraints that cannot be locally eliminated”.

This interpretation can be intuitively understood by considering closure errors of causal triangles or causal polygons: when combining local causal constraints along different paths, if results are not completely consistent, curvature must be introduced to account for these differences.

## 5.3 Description Length–Curvature Variational Principle

The above intuition can be formalized as a variational principle. Given a causal structure class  $\mathcal{C}_{\text{caus}}$  and volume calibration, we consider seeking the “optimal geometric encoding” among all compatible metrics.

**Definition 5.2** (Description Length–Curvature Functional). On the compatibility class  $\mathfrak{M}$  of given causal structure and volume calibration, define functional

$$\mathcal{F}[g] := \mathcal{C}(\text{Reach}(g)) + \lambda \int_M |\text{Riem}(g)|^2 d\text{Vol}_g, \quad g \in \mathfrak{M},$$

where  $\lambda > 0$  is a weight parameter.

- First term  $\mathcal{C}(\text{Reach}(g))$ : Measures the minimum description length needed to accurately record causal reachability structure induced by  $g$ ;
- Second term  $\int_M |\text{Riem}(g)|^2 d\text{Vol}_g$ : Penalizes high curvature, favoring selection of geometry that is as “flat” as possible.

**Axiom 5.3** (Variational Principle). Under given causal structure and volume calibration constraints, the physically actually chosen geometry can be viewed as a minimizer (or local minimizer) of  $\mathcal{F}[g]$ .

Here  $\mathcal{C}(\text{Reach}(g))$  strictly depends on discretization or coarse-graining mode, so its variational form is complex. But in the case where “causal reconstruction is already given with only conformal factor freedom remaining”,  $\mathcal{C}$  as a constant term exits variation, leaving only

$$\delta \int_M |\text{Riem}(g)|^2 d\text{Vol}_g = 0,$$

corresponding to critical points of the so-called  $L^2$ -curvature flow. More generally,  $\mathcal{C}$  can be viewed as constraints on the allowed metric family.

## 6 Quantum Field Theory, Information Geometry, and Causal Constraints

### 6.1 Microcausality and Regional Algebras

When defining quantum field theory on a given spacetime  $(M, g)$ , it is usually required that local observable algebras  $\mathcal{A}(\mathcal{O})$  satisfy microcausality: if  $\mathcal{O}_1, \mathcal{O}_2$  are mutually space-like separated, then for  $A \in \mathcal{A}(\mathcal{O}_1)$ ,  $B \in \mathcal{A}(\mathcal{O}_2)$ ,

$$[A, B] = 0.$$

This condition is completely determined by causal structure. Thus from the algebraic quantum field theory (AQFT) perspective, the “set of simultaneously observable” is specified by causal structure.

### 6.2 Relative Entropy and Distinguishable Paths in Causal Cones

Given two states  $\omega, \omega'$  restricted to some regional algebra  $\mathcal{A}(\mathcal{O})$ , relative entropy  $S(\omega|\omega')$  can be defined, whose monotonicity reflects causal reachability: if  $\mathcal{O} \subset \mathcal{O}'$ , then

$$S(\omega|_{\mathcal{A}(\mathcal{O}')}\omega'|_{\mathcal{A}(\mathcal{O}')} \geq S(\omega|_{\mathcal{A}(\mathcal{O})}\omega'|_{\mathcal{A}(\mathcal{O})}).$$

This can be viewed as “when extending observable domains along causal flow, distinguishability does not decrease”.

Under appropriate smoothness conditions, the second-order differential of relative entropy can be used to construct Fisher information-type metrics, thus obtaining “information length” in the interior of causal cones for state families. Thus, under given causal structure, information geometric metrics further add the meaning of “distinguishability rate” to spacetime geometry.

### 6.3 Modular Flow and Time Parameter

Tomita–Takesaki modular theory tells us that for given  $(\mathcal{A}, \omega)$ , a modular flow  $\sigma_t^\omega$  can be defined. In cases with global KMS properties or thermal time hypothesis, the modular

flow parameter  $t$  can be interpreted as a kind of intrinsic time scale, with the generated “flow” evolving along directions allowed by causal structure.

Therefore, time can be viewed as a “one-dimensional manifold parameter in state space reachable along causal structure”, while geometry is viewed as joint encoding of this flow with spatial structure.

## 7 Examples and Specific Models

### 7.1 Minkowski Spacetime: Zero Curvature and Minimal Redundancy

Consider four-dimensional Minkowski spacetime  $(\mathbb{R}^4, \eta)$ , where  $\eta = \text{diag}(-1, 1, 1, 1)$ . Its causal structure has high symmetry:

- Light cones at any point maintain shape under Lorentz transformations;
- Causal reachability set structure is completely consistent under translations and rotations;
- Curvature tensor  $\text{Riem} \equiv 0$ .

From this perspective, Minkowski spacetime corresponds to the case where “causal constraints are completely compatible”: through a global inertial frame, all local constraints can be flattened into a structure without closed-loop deviations. Therefore  $\int |\text{Riem}|^2 d\text{Vol} = 0$ , reaching minimum value in the “curvature penalty term”, while description complexity of causal structure is also extremely low due to high symmetry.

### 7.2 FRW Universe: Curvature and Causal Redundancy

Consider isotropic, homogeneous FRW metric

$$g = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j,$$

where  $\gamma_{ij}$  is the three-dimensional constant curvature space metric and  $a(t)$  is the scale factor. Its causal structure is determined by cosmological horizons and conformal time  $\eta$ :

$$d\eta = \frac{dt}{a(t)}.$$

When spatial curvature  $k \neq 0$ , spatial sections have non-zero curvature, and the overall spacetime’s Riemann tensor is also non-zero. At this time, the structure of causal cones at large scales is no longer equivalent to Minkowski spacetime: there are “causal boundaries” such as cosmological horizons and particle horizons, with systematic differences between reachable regions of different worldlines.

From the “redundancy density” perspective, FRW curvature records closed-loop deviations produced when assembling local Minkowski approximations into the global universe: “composite causal constraints” along different paths are no longer completely consistent.

### 7.3 Finite Causal Sets and Approximate Embedding

In the causal set approach, spacetime is viewed as a discrete set  $(C, \preceq)$  with partial order, requiring the partial order to satisfy local finiteness: for any  $p, q \in C$ , the set  $\{r \in C : p \preceq r \preceq q\}$  is finite. In appropriate dense limits, causal sets can be approximately embedded into continuous spacetime.

From the compression perspective, finite causal set  $(C, \preceq)$  is a discrete sampling of continuous causal structure; if this sampling has “Poisson sprinkling” properties, its statistical properties are compatible with the volume calibration of the original continuous spacetime.

In such discrete models, description complexity  $\mathcal{C}(\preceq)$  can be directly defined, and the relationship between complexity and “discrete curvature” under different discrete geometries (such as different random curvature models) can be examined, thus providing computable tests of the “curvature = redundancy density” viewpoint proposed in this paper.

## 8 Discussion and Outlook

The “spacetime geometry = minimal lossless compression of causal constraints” perspective proposed in this paper connects the following objects:

1. **Causal Structure:** Partial order and Alexandrov topology on event sets;
2. **Geometric Structure:** Conformal class, metric, and curvature;
3. **Information Structure:** Description complexity, relative entropy, Fisher information, and modular flow.

From this perspective, geometry is no longer merely an “object giving distance”, but rather an encoding of “allowed information flow”; curvature becomes an accounting tool recording the fact that “local causal constraints cannot be compatibly flattened globally”. Future directions for further development include:

- In specific quantum gravity models (causal sets, tensor networks, etc.), explicitly calculate  $\mathcal{C}(\text{Reach})$  and compare with geometric curvature;
- In semiclassical gravity, establish correspondence between the description length–curvature functional of this paper and the Einstein–Hilbert action and its correction terms;
- In information geometry and quantum information processing, unify relative entropy monotonicity under causal structure with geometric curvature as a “constraint image of information flow”.

## A Theorem and Proof Sketch for Causal Structure Determining Conformal Class

This appendix provides a formalized statement and proof outline for Theorem ?? in Section 3.

## A.1 Formalized Theorem Statement

**Theorem A.1** (Causal Structure Determines Conformal Class). *Let  $(M, g)$ ,  $(\tilde{M}, \tilde{g})$  be two four-dimensional, connected, orientable and time-orientable Lorentzian manifolds satisfying:*

1. *Strong causality;*
2. *Local compactness and second countability;*
3. *No “pathological” causal boundaries (such as malignantly deformed Cauchy boundaries).*

*Denote the causal relations induced by  $g, \tilde{g}$  as  $\leq, \tilde{\leq}$ . If there exists a bijection  $\Phi : M \rightarrow \tilde{M}$  satisfying*

$$p \leq q \iff \Phi(p) \tilde{\leq} \Phi(q),$$

*then there exists a smooth function  $\Omega : \tilde{M} \rightarrow (0, \infty)$  such that*

$$\Phi^* \tilde{g} = \Omega^2 g.$$

*In particular,  $\Phi$  is a conformal isometric homeomorphism.*

## A.2 Proof Sketch

The proof divides into three main steps:

### 1. Topological Consistency:

First prove that Alexandrov topology is determined by causal structure. Causal homeomorphism  $\Phi$  preserves all Alexandrov sets  $A(p, q)$ ’s inclusion relationships and structure, thus proving that  $\Phi$  is a topological homeomorphism, i.e., topological structure is uniquely determined by causal structure.

### 2. Light Cone Structure Reconstruction:

In Lorentzian geometry, null vector cones are completely determined by causal structure. Using properties of minimal causal curves on light cones, prove that  $\Phi$  is not only a topological homeomorphism but also preserves light cone structure, thus inducing linear conformal maps at the tangent space level.

### 3. Smoothness and Conformal Factor:

Using compatibility of local coordinates with smooth structure, prove that  $\Phi$  is a conformal map in the differentiable sense, i.e., there exists a smooth function  $\Omega$  such that  $\Phi^* \tilde{g} = \Omega^2 g$ .

Complete rigorous proof requires many technical details including minimality of causal curves, geodesic structure, local coordinate construction, etc., which are not expanded here.

## B Alexandrov Topology, Strong Causality and Local Structure

### B.1 Equivalence of Alexandrov Topology with Manifold Topology

**Proposition B.1.** *If  $(M, g)$  is a strongly causal spacetime, then the topology generated by Alexandrov basis  $\mathcal{B} = \{A(p, q)\}$  coincides with the manifold topology.*

*Proof outline:*

1. Using strong causality, prove that each point has arbitrarily small “causally convex” neighborhoods where causal structure and topological structure are closely related;
2. Show that for any point and its neighborhood in manifold topology, sufficiently small Alexandrov sets can be found contained within it;
3. Reverse direction is more direct: each Alexandrov set is an open set.

### B.2 Strong Causality and Causal Pathologies

Strong causality excludes pathologies such as:

- Points  $p$  where arbitrarily small neighborhoods contain closed causal curves passing through  $p$ ;
- Regions where causal relations cannot be compatible with manifold topology.

After excluding these pathologies, causal structure can stably support reconstruction of topology and geometry.

## C Description Complexity Functional and Variational Equation Forms

### C.1 Complexity in Discrete Models

In finite causal set or discrete causal graph cases,  $\mathcal{C}(\text{Reach})$  can be specifically defined as: given some encoding method (e.g., representing partial orders by hierarchical structure or partial linear extension), encoding length is the minimum value of used binary string length.

For a given metric family  $\{g_\theta\}$  (e.g., conformal factor families controlled by few parameters), examine

$$\mathcal{C}(\theta) := \mathcal{C}(\text{Reach}(g_\theta)),$$

and define total functional

$$\mathcal{F}(\theta) := \mathcal{C}(\theta) + \lambda \int_M |\text{Riem}(g_\theta)|^2 d\text{Vol}_{g_\theta}.$$

In many cases,  $\mathcal{C}(\theta)$  only undergoes jumps at finite “phase transition points” (e.g., where causal structure undergoes topological deformation); outside these regions,  $\mathcal{C}$  is approximately constant. Therefore, variation of  $\mathcal{F}$  is mainly controlled by the curvature term, while the complexity term provides discrete “selection rules”.

## C.2 Formal Variation in Continuous Limits

In continuous limits, a smoothed “information density” function  $\rho_{\text{info}}(x)$  can be used to approximately represent the information amount needed per unit volume to encode causal structure, thus writing

$$\mathcal{C}(\text{Reach}(g)) \approx \int_M \rho_{\text{info}}(x) \, d\text{Vol}_g.$$

At this time

$$\mathcal{F}[g] = \int_M (\rho_{\text{info}}(x) + \lambda |\text{Riem}(g)|^2) \, d\text{Vol}_g.$$

Varying with respect to  $g$  can formally obtain something like

$$\delta\mathcal{F}[g] = \int_M \left( \frac{1}{2} (\rho_{\text{info}} + \lambda |\text{Riem}|^2) g^{\mu\nu} - \lambda K^{\mu\nu} \right) \delta g_{\mu\nu} \, d\text{Vol}_g,$$

where  $K^{\mu\nu}$  is a symmetric tensor composed of Riemann tensor and its covariant derivatives (similar to tensors in fourth-order gravitational field equations obtained from variation of  $\int |\text{Riem}|^2$ ). Setting  $\delta\mathcal{F}[g] = 0$  can formally obtain a set of “complexity–curvature balance equations”.

Complete rigorization requires delicate treatment of definition, regularization and variational techniques for  $\rho_{\text{info}}$ , which exceeds the scope of this appendix, but formally already demonstrates the variational balance structure between “causal information density” and “geometric curvature”.