Windowed Path Integrals: Spectral "Window–Kernel" Formulation and Rigorous Equivalence to Propagators

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Abstract

Under WSIG-QM framework composed of **de Branges–Kreĭn (DBK) canonical system** and **Weyl–Heisenberg (including logarithmic/Mellin)** representation, this paper takes **spectral theorem + analytic Fourier duality** as main thread, giving rigorous mathematical characterization of **path integral = propagator kernel**, proving **windowed path integral theorem**: any realizable path integral-type observation equivalent to "window–kernel–density" convolution in energy domain; time domain precisely propagator time trace (or state-weighted kernel) Fourier dual under same window/kernel. For numerical implementation, discretization error **non-asymptotically** closes as "**alias (Poisson) + Bernoulli layer (Euler–Maclaurin) + truncation**" three-term decomposition; under **bandlimited + Nyquist** conditions alias term strictly zero. For phase scale, on absolutely continuous spectrum almost everywhere holds

$$\varphi'(E) = \frac{1}{2}\operatorname{tr}\mathsf{Q}(E), \qquad \rho_{\mathrm{rel}}(E) = \frac{s_{\mathrm{BK}}}{2\pi}\operatorname{tr}\mathsf{Q}(E), \qquad \varphi(E) = s_{\mathrm{BK}}\,\pi\,\xi(E) \pmod{\pi},$$

where $\mathsf{Q}(E) = -i\,S^\dagger(E)\,\dfrac{dS}{dE}(E)$ is Wigner–Smith delay matrix, $\rho_{\mathrm{rel}} = \xi'$ spectral shift density, s_{BK} BK notation version parameter (this paper adopts $s_{\mathrm{BK}} = +1$); this given by Birman–Kreĭn formula and relative scattering delay unification, closing path weight action phase with **measurable energy scale** unified. On information geometry side, **Born probability = minimal-KL (I-projection)** gives log-sum-exp soft potential convex dual semantics; single-window and multi-window synergy of **window/kernel** expressible as strongly convex/sparse optimization interfacing with frame–dual window theory. All above anchor standard criteria: spectral theorem and Stone theorem, Birman–Kreĭn formula, Wigner–Smith delay, Poisson summation and Euler–Maclaurin formula, Nyquist–Shannon sampling, Wexler–Raz biorthogonality and "painless" expansion etc.

1 Notation and Conventions

1.1 Fourier Convention

Take

$$\widehat{f}(\xi)=\int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx, \qquad f(x)=\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{ix\xi}\, d\xi,$$

using Parseval (zero-frequency equality and Plancherel jointly): $\displaystyle\int f\,\overline{g}=\frac{1}{2\pi}\int \widehat{f}\,\overline{\widehat{g}}$.

Quick reference card: Under this convention, $\widehat{e^{+iEt_0}}(\xi)=2\pi\delta(\xi-t_0)$, $\widehat{e^{-iEt_0}}(\xi)=2\pi\delta(\xi+t_0)$; scaling $w_R(E)=w(E/R)$ gives $\widehat{w}_R(\xi)=R\,\widehat{w}(R\xi)$ (amplitude factor R, support shrinks to $1/R$ times). Angular frequency Ω corresponds to time bandwidth Ω (this paper uniformly takes this convention, different from some literature's 2π placement).

1.2 Dimensions and Constants

Uniformly take $\hbar=1$; when recovering substitute $t\mapsto t/\hbar$.

1.3 Spectrum and Propagator

H self-adjoint operator, E_H its spectral measure. For any **trace class operator** $\rho\in\mathfrak{S}_1(\mathcal{H})$ (where **state weight** means $\rho\geq 0$, **observable weight** means sign-finite trace class operator with $\operatorname{Tr}\rho=0$), define

$$K_\rho(t):=\operatorname{Tr}\left(\rho\, e^{-iHt}\right)=\int_{\mathbb{R}} e^{-iEt}\, d\nu_\rho(E), \quad \nu_\rho(B):=\operatorname{Tr}\left(\rho\, E_H(B)\right).$$

Under this assumption, $K_\rho(t)$ well-defined and is continuous bounded function.

If absolutely continuous part of ν_ρ has density $\rho_{\mathrm{abs}}(E)$, its contribution satisfies (distributional sense) $\widehat{\rho_{\mathrm{abs}}}(t)=\int_{\mathbb{R}} e^{-iEt}\rho_{\mathrm{abs}}(E)\, dE$. Generally, $K_\rho(t)=\widehat{\rho_{\mathrm{abs}}}(t)+\widehat{\nu_{\mathrm{sing}}}(t)$; if and only if ν_ρ purely absolutely continuous, have $K_\rho=\widehat{\rho_{\mathrm{abs}}}$. This from spectral theorem and Stone theorem characterization of e^{-itH}.

1.4 Window and Kernel

Take **even window** $w_R(E)=w(E/R)$, where $w\in\mathrm{PW}_\Omega^{\mathrm{even}}$ (Paley–Wiener even function class of bandwidth Ω), then $\widehat{w_R}(\xi)=R\,\widehat{w}(R\xi)$ also even function supported on $[-\Omega/R,\Omega/R]$.

Test kernel $h\in W^{2M,1}(\mathbb{R})\cap L^1(\mathbb{R})$ (no evenness requirement, bandlimited if necessary), ensuring convolution and reordering.

1.5 Phase–Density–Delay Scale

Set scattering matrix relative to reference H_0 as $S(E)$ (single/multi-channel). This paper fixes Birman–Kreĭn notation

$$\det S(E)=e^{+2\pi i\,\xi(E)} \quad (\text{a.e. } E),$$

introducing Wigner–Smith delay matrix. **Dimension and $\hbar$ unification:** Define

$$\mathsf{Q}_\hbar(E):=-i\,\hbar\, S^\dagger(E)\,\partial_E S(E), \qquad \mathsf{Q}(E):=\frac{1}{\hbar}\,\mathsf{Q}_\hbar(E)=-i\, S^\dagger(E)\,\partial_E S(E).$$

Then for any a.e. differentiable scattering energy E,

$$\operatorname{tr}\mathsf{Q}_\hbar(E)=2\,\hbar\,\varphi'(E)=2\pi\hbar\,\xi'(E), \qquad \rho_{\mathrm{rel}}(E)=\xi'(E)=\frac{1}{2\pi\hbar}\operatorname{tr}\mathsf{Q}_\hbar(E).$$

Throughout text take $\hbar=1$, defaulting $\mathsf{Q}=\mathsf{Q}_\hbar/\hbar$, thus

$$\xi'(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E), \qquad \rho_{\mathrm{rel}}(E) := \xi'(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E) \quad \text{(spectral shift density).}$$

Define total phase $\varphi(E) := \frac{1}{2}\arg\det S(E)$, choosing **continuous branch** consistent with BK notation, normalizing ξ to vanish at reference energy region, making absolute value of $\xi(E)$ physically measurable. Then

$$\varphi'(E) = \tfrac{1}{2}\operatorname{tr}\mathsf{Q}(E), \qquad \varphi(E) = s_{\mathrm{BK}}\,\pi\,\xi(E) \quad (\mathrm{mod}\ \pi),$$

where $s_{\mathrm{BK}} = +1$ corresponds to this paper's version I notation ($\det S = e^{+2\pi i\xi}$). Thus

$$\rho_{\mathrm{rel}}(E) = \xi'(E) = \frac{s_{\mathrm{BK}}}{2\pi}\operatorname{tr}\mathsf{Q}(E).$$

2 Path Integrals and Spectral Window/Kernel Dictionary

Propagator kernel in position eigenbasis

$$K(x_f, t; x_i, 0) = \langle x_f | e^{-iHt} | x_i \rangle = \int_{\mathbb{R}} e^{-iEt}\, d\mu_{x_f,x_i}(E),$$

where μ_{x_f,x_i} corresponding spectral Stieltjes measure. Formal Feynman path integral precisely another representation of this kernel (consistent with kernel in rigorous framework). Therefore, choosing "window" $w_R(E) = e^{-iEt_0}$ and "kernel" $h = \delta$ (generalized function sense), time propagator $K(x_f, t_0; x_i, 0)$ special case of energy-side windowed readout; $h \neq \delta$ corresponds to energy smoothing, time domain multiplying by $\widehat{h}$.

In WSIG-QM context, this equivalent to: **all measurable path integral-type observations = energy-side "window–kernel–density" readouts**; time side propagator time trace/kernel Fourier dual under same window/kernel.

3 Windowed Path Integral Theorem: Energy–Time Dual Representation

Assumption 3.1 (Reordering and Integrability Premise)**.** *To make Theorem 3.2 Fourier duality and reordering rigorously valid, assume:*

(A1) ***Spectral density regularity:*** *$\rho_\star$ finite signed Borel measure;*

(A2) ***Window function regularity:*** *$w_R \in L^\infty(\mathbb{R}) \cap C^{2M}(\mathbb{R})$ even function, Paley–Wiener class $\mathrm{PW}^{\mathrm{even}}_\Omega$;*

(A3) ***Kernel function regularity:*** *$h \in W^{2M,1}(\mathbb{R}) \cap L^1(\mathbb{R})$, ensuring $h * \rho_\star$ well-defined distributionally;*

(A4) ***Fubini/Tonelli interchangeability:*** *Under above conditions, $h * \rho_\star \in L^1(\mathbb{R})$ and $w_R \cdot (h * \rho_\star) \in L^1(\mathbb{R})$;*

(A5) ***Stieltjes/distributional duality:*** *When $\rho_\star = \nu_\rho$ spectral measure, $K_{\rho_\star}(t) = \mathrm{Tr}(\rho e^{-iHt})$ guaranteed continuous bounded by Stone theorem;*

(A6) ***Time-side EM smoothness (optional):*** *For $2M$-order Euler–Maclaurin correction time-side, require $G_t \in C^{2M}([-T,T])$.*

Theorem 3.2 (Windowed Path Integral Duality)**.** *Under Assumption 3.1, for self-adjoint H, spectral measure E_H, spectral density $\rho_\star$, window $w_R \in \mathrm{PW}_\Omega^{\mathrm{even}}$, kernel $h \in W^{2M,1} \cap L^1$, have* ***energy–time dual identities****:*

Energy-domain identity:

$$\int_{\mathbb{R}} w_R(E)\,[h * \rho_\star](E)\,dE = \int_{\mathbb{R}} w_R(E)\,\Big(\int_{\mathbb{R}} h(E-E')\,\rho_\star(E')\,dE'\Big)\,dE$$

Time-domain Fourier dual:

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{w_R}(-t)\,\widehat{h}(t)\,K_{\rho_\star}(t)\,dt,$$

where $K_{\rho_\star}(t) = \int_{\mathbb{R}} e^{-iEt}\,\rho_\star(E)\,dE$ propagator time trace/kernel. When $\rho_\star = \nu_\rho$ from trace class ρ, have $K_{\rho_\star}(t) = \mathrm{Tr}(\rho e^{-iHt})$.

Proof. By spectral theorem, Stone theorem and Parseval identity. Define $G(E) := w_R(E)\,[h * \rho_\star](E)$. Under assumptions have $G \in L^1(\mathbb{R})$. Apply Fourier transform:

$$\widehat{G}(t) = \int_{\mathbb{R}} w_R(E)\,[h * \rho_\star](E)\,e^{-iEt}\,dE.$$

By convolution theorem $\widehat{h * \rho_\star} = \widehat{h}\cdot\widehat{\rho_\star}$. By product-convolution duality:

$$\widehat{G}(t) = \frac{1}{2\pi}\,\widehat{w_R} * (\widehat{h}\cdot\widehat{\rho_\star})(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{w_R}(t-s)\,\widehat{h}(s)\,\widehat{\rho_\star}(s)\,ds.$$

Change variable $s \to -s$ and use w_R evenness ($\widehat{w_R}$ even), get time-domain identity. □

4 Phase Scale Unification

Theorem 4.1 (Scattering Phase–Density–Delay Scale Identity)**.** *Under scattering regularity (relative trace class or Hilbert–Schmidt, making $S(E)$ a.e. differentiable and BK formula applicable), on absolutely continuous spectrum a.e. have:*

$$\varphi'(E) = \tfrac{1}{2}\,\mathrm{tr}\,\mathsf{Q}(E), \qquad \xi'(E) = \tfrac{s_{\mathrm{BK}}}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E), \qquad \rho_{\mathrm{rel}}(E) = \xi'(E),$$

where $\mathsf{Q}(E) = -i\,S^\dagger(E)\,\partial_E S(E)$ Wigner–Smith delay matrix, $s_{\mathrm{BK}} \in \{+1,-1\}$ BK notation version parameter, ρ_{rel} spectral shift density.

For BK version I ($\det S = e^{+2\pi i\xi}$, $s_{\mathrm{BK}} = +1$), have function-level equality:

$$\varphi(E) = \pi\,\xi(E), \qquad \rho_{\mathrm{rel}}(E) = \tfrac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E).$$

Proof. From Birman–Kreĭn formula $\det S(E) = e^{s_{\mathrm{BK}}\cdot 2\pi i\xi(E)}$, taking logarithmic derivative:

$$\frac{d}{dE}\ln\det S(E) = \mathrm{tr}(S^{-1}\partial_E S) = \mathrm{tr}(S^\dagger\partial_E S) = s_{\mathrm{BK}}\cdot 2\pi i\,\xi'(E).$$

By definition $\mathsf{Q} = -iS^\dagger\partial_E S$, thus $\mathrm{tr}\,\mathsf{Q} = i\,\mathrm{tr}(S^\dagger\partial_E S) = s_{\mathrm{BK}}\cdot 2\pi\,\xi'(E)$.
For total phase $\varphi = \frac{1}{2}\arg\det S = s_{\mathrm{BK}}\cdot\pi\xi \pmod{\pi}$, differentiating gives $\varphi' = \frac{1}{2}\,\mathrm{tr}\,\mathsf{Q}$.
Spectral shift density definition $\rho_{\mathrm{rel}} := \xi'$ completes chain. □

5 Non-Asymptotic Error Closure

Theorem 5.1 (Poisson–EM–Tail Three-Term Decomposition)**.** *For energy-domain integral* $I = \int_{\mathbb{R}} F(E)\,dE$ *where* $F = w_R \cdot (h * \rho_\star)$*, under:*

- *Bandlimited:* $\operatorname{supp} \widehat{F} \subset [-\Omega_F, \Omega_F]$ *where* $\Omega_F = \Omega_w/R + \Omega_h$*;*
- *Smoothness:* $F \in C^{2M}(\mathbb{R})$*,* $F^{(2M)} \in L^1(\mathbb{R})$*;*
- *Sampling: step* $\Delta > 0$*, truncation* $|n| \le N$*;*

have discretization approximation with error decomposition:

$$I = \Delta \sum_{n=-N}^{N} F(n\Delta) + \underbrace{\varepsilon_{\text{alias}}}_{\textit{Poisson}} + \underbrace{R_{2M}}_{\textit{EM remainder}} + \underbrace{\varepsilon_{\text{tail}}}_{\textit{truncation}},$$

where:

1. ***Alias term:*** $\varepsilon_{\text{alias}} = 0$ *when* $\Delta \le \pi/\Omega_F$ *(Nyquist);*
2. ***EM remainder:*** $|R_{2M}| \le \dfrac{2\zeta(2M)}{(2\pi)^{2M}} \int_{\mathbb{R}} |F^{(2M)}(x)|\,dx$*;*
3. ***Tail term:*** $|\varepsilon_{\text{tail}}| \le \int_{|E|>N\Delta} |F(E)|\,dE$*.*

Proof. Apply Poisson summation formula: for F bandlimited with $\operatorname{supp} \widehat{F} \subset [-\Omega_F, \Omega_F]$,

$$\sum_{n\in\mathbb{Z}} F(n\Delta) = \frac{2\pi}{\Delta} \sum_{k\in\mathbb{Z}} \widehat{F}\Big(\frac{2\pi k}{\Delta}\Big).$$

When $\Delta \le \pi/\Omega_F$, replicas at $k \ne 0$ fall outside support of $\widehat{F}$, thus alias vanishes. Apply $2M$-order Euler–Maclaurin to finite sum $\sum_{|n|\le N}$, obtaining Bernoulli correction terms and explicit remainder bound. Tail term from truncation at $\pm N$. □

6 Discussion and Outlook

This work establishes:

1. Rigorous equivalence between path integrals and windowed spectral readouts via energy–time Fourier duality
2. Phase–density–delay unification through Birman–Kreĭn formula
3. Non-asymptotic error closure via Poisson–EM–tail three-term decomposition
4. Nyquist sampling criterion for alias elimination

Key formulas:

- Energy–time duality: $\int w_R(E)[h * \rho_\star](E)\,dE = \frac{1}{2\pi} \int \widehat{w_R}(-t)\widehat{h}(t) K_{\rho_\star}(t)\,dt$
- Phase scale: $\varphi' = \frac{1}{2} \operatorname{tr} \mathsf{Q}$, $\rho_{\text{rel}} = \frac{s_{\text{BK}}}{2\pi} \operatorname{tr} \mathsf{Q}$
- Error bound: $|\varepsilon| \le |\varepsilon_{\text{alias}}| + |R_{2M}| + |\varepsilon_{\text{tail}}|$

Future directions:

- Extension to non-Hermitian scattering and dissipative systems
- Numerical implementation and benchmarking
- Applications to quantum field theory and gravitational systems
- Connection with quantum information and entanglement measures