Null–Modular Double Cover and Overlapping Causal Diamond Chains: Total-Order Approximation Bridge for Quadratic Form Localization, Inclusion–Exclusion–Markov Splicing, and Parity Threshold for Distributional Scattering Calibration

Haobo Ma[1] Wenlin Zhang[2]
[1]Independent Researcher
[2]National University of Singapore

Abstract

We propose **Null–Modular double cover** carried by zero-measure boundaries of causal diamonds, decomposing modular Hamiltonians into local energy flux integrals on two null sheets in vacuum quadratic form sense. Through a **total-order approximation bridge lemma**, general diamonds are reduced to monotonic half-space family limits on the same zero-measure hyperplane, with quadratic form closedness and dominated convergence ensuring limit independence of approximation paths. We establish **modular Hamiltonian inclusion–exclusion identities** and **Markov splicing** for **overlapping causal diamond chains**; for non-totally-ordered cuts, we introduce **Markov gap line density** quantitatively characterizing failure with comparison inequalities versus stratification degree. On the scattering side, under **distributional Birman–Kreĭn–Friedel–Lloyd–Wigner–Smith calibration**, we introduce **windowed readout**, providing **visible constants and threshold inequalities** via Toeplitz/Berezin compression, Euler–Maclaurin and Poisson disciplines, thereby proving **chain $\mathbb{Z}_2$ parity threshold stability** with robustness conditions for **weakly non-unitary perturbations**. On the geometric side, **half-sided modular inclusion** constitutes one-parameter semigroup for chain advancement; in holographic limits, **JLMS equality** lifts boundary inclusion–exclusion–Markov to bulk entanglement wedge normal modular flow, with dimensional upper bounds for subleading $1/N$ corrections. Finally, we explicitly compute **GHY joint terms** and **$\mathbb{Z}_2$ ledger consistency** of square-root splicing classes in minimal models in $1 + 1$ and $2 + 1$ dimensions, providing **reproducible experimental parameter tables** and **verification checklists**.

1 Introduction & Historical Context

Tomita–Takesaki modular theory endows von Neumann algebra–vector state pairs $(\mathcal{A}, \Omega)$ with modular groups Δ^{it} and modular conjugation J. Bisognano–Wichmann property geometrizes modular flow as Lorentz boosts on wedge regions. For zero-measure geometry, **local modular Hamiltonians** on half-spaces and their smooth deformations satisfy vacuum QNEC saturation, and **vacuum Markovianity** on light-cones/light-fronts with strong subadditivity saturation form solid foundations. Algebraically, **half-sided modular inclusion** (HSMI) provides algebraic skeleton of inclusion–one-parameter semigroup–Borchers commutation relations. Holographically, **JLMS equality** identifies boundary and bulk relative entropies at leading order in large N. On scattering side, **Birman–Kreĭn** identifies determinant phase with spectral shift function, **Friedel–Lloyd** and

Wigner–Smith unify density-of-states difference with group delay trace; **Toeplitz/Berezin** compression with **Szegő/trace formulas** provide operator–symbol tools for windowed readout; **Euler–Maclaurin** and **Poisson** disciplines yield exponential or algebraic decay error upper bounds. This paper systematically constructs integrated theory of Null–Modular double cover and overlapping diamond chains within this framework.

2 Model & Assumptions

2.1 Quadratic Form Framework and Natural Domain

Take Minkowski spacetime $\mathbb{R}^{1,d-1}$ $(d \geq 2)$. Let $\mathcal{D}_0$ be dense domain of energy-bounded vectors in vacuum.

Notation and measure convention: Zero-measure boundary decomposes into two sheets $\widetilde{E} = E^+ \sqcup E^-$; notation $\int_{E^\sigma}(\cdots)\,d\lambda\,d^{d-2}x_\perp$ refers to standard measure integration on this sheet by affine parameter λ and transverse coordinate $x_\perp$.

Assume for any region R there exists lower bounded closed quadratic form

$$\mathfrak{k}_R[\psi] := \sum_{\sigma=\pm} \int_{E^\sigma} g_\sigma^{(R)}(\lambda, x_\perp)\,\langle \psi, T_{\sigma\sigma}(\lambda, x_\perp)\psi\rangle\,d\lambda\,d^{d-2}x_\perp, \quad \psi \in \mathcal{D}_0,$$

thus there exists self-adjoint operator K_R satisfying $\langle \psi, K_R\psi\rangle = \mathfrak{k}_R[\psi]$. CFT's spherical regions/wedges and their conformal images yield exact geometric equalities.

Let $\mathfrak{k}_R$ have lower bound $a_R \in \mathbb{R}$, i.e., $\mathfrak{k}_R[\psi] \geq a_R|\psi|^2$. Take any $c_R > -a_R$, define **shifted graph norm**

$$|\psi|^2_{\mathfrak{k}_R,c_R} := |\psi|^2 + \big(\mathfrak{k}_R[\psi] + c_R|\psi|^2\big),$$

then $(\mathcal{D}(\mathfrak{k}_R), |\cdot|_{\mathfrak{k}_R,c_R})$ is complete, compatible with representation theorem for self-adjoint operator K_R.

2.2 Zero-Measure Localization and QNEC

In zero-measure half-space $R_V = \{u = 0,\ v \geq V(x_\perp)\}$ $(V \in C^2)$,

$$K_V = 2\pi \int d^{d-2}x_\perp \int_{V(x_\perp)}^{\infty} (v - V)\,T_{vv}(v, x_\perp)\,dv$$

holds as quadratic form identity; its second-order variational kernel is $2\pi\,T_{vv}$, consistent with vacuum QNEC saturation.

2.3 Double Cover and Splicing, Square-Root Cover and Ledger

Zero-measure boundary decomposes into two sheets $\widetilde{E} = E^+ \sqcup E^-$. Modular conjugation J exchanges two sheets and reverses orientation, modular group generates integrable flow along affine parameter λ in geometrizable cases. Seam splicing accounted by $\epsilon_i \in \{\pm 1\}$. On scattering side introduce "square-root cover" $P_{\sqrt{S}} = \{(E, \sigma) : \sigma^2 = \det S(E)\}$ as $\mathbb{Z}_2$ principal bundle structure; splicing class of closed chain loops shares same $\mathbb{Z}_2$ ledger with joint term orientation signs.

2.4 Scattering–Information Calibration and Windowing

Unitary scattering matrix $S(E)$ piecewise C^{2m} with $S(E)-\mathbb{I}$ trace-class within energy band; define

$$Q(E) := -\mathrm{i}\, S^{\dagger}\partial_E S, \quad \varphi(E) := \tfrac{1}{2}\arg\det S(E), \quad \rho_{\mathrm{rel}}(E) := \tfrac{1}{2\pi}\operatorname{tr} Q(E).$$

Employ window function $h \in \mathcal{S}(\mathbb{R})$ (e.g., Gaussian), **or** $h \in C_c^{2m+1}(\mathbb{R})$ with endpoint jets up to $2m$ order vanishing ($m \geq 1$). In this case $\widehat{h}(\omega) = O(|\omega|^{-(2m+1)})$. If h only piecewise C^{2m} with compact support (endpoints allow corners, including Kaiser–Bessel), adopt **corner tail bound** (at least $O(|\omega|^{-2})$), whereby Theorem G's Poisson aliasing series converges. Corresponding Toeplitz/Berezin compression and trace formulas follow error decomposition in §3.5, where **endpoint remainder** R_{EM}: for C_c^{∞} windows take $O(\ell^{-(m-1)})$; for piecewise C^{2m} compact support windows (including Kaiser–Bessel) adopt **corner version** estimate (order generally drops to $O(\ell^{-1})$), incorporated into total error budget $\mathcal{E}_h(\gamma)$.

Additional assumption (Toeplitz commutator integrability): On any examined energy band $\mathcal{I}$, $\partial_E S(E) \in \mathfrak{S}_2$ and $\displaystyle\int_{\mathcal{I}} |\partial_E S(E)|_2\, dE < \infty$. Thus $R_{\mathrm{T}} \leq C_{\mathrm{T}}\, \ell^{-1/2} \int_{\mathcal{I}} |\partial_E S|_2\, dE$ is bounded.

Global convention (window and tail term): Set $\int_{\mathbb{R}} h = 1$ **and** $h \geq 0$, scale $h_\ell(E) = \ell^{-1} h(E/\ell)$. Define

$$\boxed{R_{\mathrm{tail}}(\ell, \mathcal{I}, E_0) := \int_{\mathbb{R}\setminus\mathcal{I}(\gamma)} |h_\ell(E-E_0)|\, dE \in [0,1].}$$

Note: In this case $R_{\mathrm{tail}} = 1 - \int_{\mathcal{I}(\gamma)} h_\ell(E-E_0)\, dE$.

Notation convention (Poisson step size): Denote $\Delta > 0$ as energy band segmentation/frequency sampling step size (grid spacing); in Poisson resummation estimate take

$$\int_{\mathcal{I}} |R_{\mathrm{P}}|\, dE \ \leq\ C_h \sum_{|q|\geq 1} \left|\widehat{h}\,(2\pi q\, \ell/\Delta)\,\right|,$$

consistently using this Δ as in §3.5's identically named term.

2.5 Chain and Overlap, Algebraic Assumptions

Chain $\{D_j\}$ adjacent overlaps on same surface; for each transverse point $x_\perp$ total-order cut is default assumption. Algebraically adopt standard assumptions of split property and strong additivity; HSMI as algebraic realization of chain advancement.

3 Main Results (Each Result Labeled with "Significance/Domain")

3.1 Double-Sheet Geometric Decomposition and Total-Order Approximation Bridge

Theorem 1 (A: Double-Sheet Geometric Decomposition)**.**

$$K_D = 2\pi \sum_{\sigma=\pm} \int_{E^\sigma} g_\sigma(\lambda, x_\perp)\, T_{\sigma\sigma}(\lambda, x_\perp)\, d\lambda\, d^{d-2}x_\perp,$$

where $T_{++} = T_{vv}$, $T_{--} = T_{uu}$. In CFT spherical diamonds $g_\sigma(\lambda) = \lambda(1-\lambda)$.
[Quadratic form; domain: vacuum, CFT exact equality]

Assumption 2 (A$'$: Null Energy Flux Uniform Integrability). For any $\psi \in \mathcal{D}_0$ and geometrically bounded monotonic approximation family $\{R^{\pm}_{V_\alpha}\}$, there exists $H_\sigma \in L^1_{\mathrm{loc}}(E^\sigma \times \mathbb{R}^{d-2})$ such that

$$\left|g^{(\alpha)}_\sigma(\lambda, x_\perp)\, \langle \psi, T_{\sigma\sigma}(\lambda, x_\perp)\psi\rangle\right| \leq H_\sigma(\lambda, x_\perp)$$

holds almost everywhere, and $\sup_\alpha \int_{\mathcal{K}} H_\sigma < \infty$ for any compact set $\mathcal{K} \subset E^\sigma \times \mathbb{R}^{d-2}$.

Lemma 3 (A: Ordered Cut Approximation). *There exists monotonic half-space family $\{R^{\pm}_{V_\alpha}\}$ along $E^{\pm}$ such that*

$$\langle \psi, K_D \psi\rangle = \lim_{\alpha\to\infty} \sum_{\sigma=\pm} 2\pi \int_{E^\sigma} g^{(\alpha)}_\sigma \,\langle \psi, T_{\sigma\sigma}\psi\rangle, \qquad g^{(\alpha)}_\sigma \to g_\sigma \text{ in } L^1_{\mathrm{loc}},$$

and limit is independent of chosen ordered approximation.
[Quadratic form convergence; domain: vacuum, vacuum QNEC saturation]

Exclusion remark: Without BW/HSMI or boundary roughness breaking vacuum QNEC saturation, above decomposition may not hold.

Assumption 4 (A$''$: Quadratic Form Lower Bound and Closedness Threshold). Assume all participating regions R have quadratic forms $\mathfrak{k}_R$ with uniform lower bound $a \in \mathbb{R}$, i.e., $\mathfrak{k}_R[\psi] \geq a\,|\psi|^2$. Take any $c > -a$ defining shifted graph norm $|\psi|^2_{\mathfrak{k}_R,c} = |\psi|^2 + (\mathfrak{k}_R[\psi] + c|\psi|^2)$, then $\mathfrak{k}_R$ closed and $\mathcal{D}(\mathfrak{k}_R)$ complete under $|\cdot|_{\mathfrak{k}_R,c}$.

Proposition 5 (A.1: Necessary and Sufficient Condition for Limit Path Independence). *Under Assumptions A$'$ and A$''$, if along any two monotonic approximation families $\{R_{V_\alpha}\}$, $\{R_{\widetilde{V}_\beta}\}$ we have $g^{(\alpha)} \to g$, $\widetilde{g}^{(\beta)} \to g$ in L^1_{loc}, then for each $\psi \in \mathcal{D}_0$,*

$$\lim_{\alpha\to\infty} \sum_\sigma \int g^{(\alpha)}_\sigma \,\langle \psi, T_{\sigma\sigma}\psi\rangle = \lim_{\beta\to\infty} \sum_\sigma \int \widetilde{g}^{(\beta)}_\sigma \,\langle \psi, T_{\sigma\sigma}\psi\rangle.$$

Reason: Dominated convergence identifies each approximation's limit with g; closedness and lower bound yield quadratic form continuity, thus independent of approximation path.

3.2 Inclusion–Exclusion and Closedness

Theorem 6 (B: Inclusion–Exclusion Identity). *For $\{R_{V_i}\}_{i=1}^N$ on same zero-measure surface,*

$$K_{\cup_i R_{V_i}} = \sum_{k=1}^{N} (-1)^{k-1} \sum_{1\leq i_1 < \cdots < i_k \leq N} K_{R_{V_{i_1}} \cap \cdots \cap R_{V_{i_k}}}.$$

Derived from pointwise identity $(v - \min_i V_i)_+ = \sum_{k\geq 1} (-1)^{k-1} \sum_{|I|=k} (v - \max_{i\in I} V_i)_+$.
[Quadratic form; domain: vacuum, V_i piecewise smooth]

Proposition 7 (B: Closedness). *Denote $\mathfrak{k} := \mathfrak{k}_{\cup_i R_{V_i}}$ as container domain closed quadratic form, with lower bound $a \in \mathbb{R}$. Take any $c > -a$. If*

$$\psi_n, \psi \in \mathcal{D}(\mathfrak{k}) \cap \bigcap_{I\neq\emptyset} \mathcal{D}(\mathfrak{k}_{R_{V_I}}), \qquad \psi_n \to \psi \text{ under shifted graph norm } |\cdot|_{\mathfrak{k},c},$$

then inclusion–exclusion identity's both sides for quadratic form values on ψ_n converge simultaneously to values on ψ; thus identity closes on above form domain. Where

$$|\psi|^2_{\mathfrak{k},c} := |\psi|^2 + \big(\mathfrak{k}[\psi] + c|\psi|^2\big).$$

[Quadratic form closedness]

Operational domain remark: Above closedness holds on common form domain $\mathcal{D}_* := \mathcal{D}(\mathfrak{k}) \cap \bigcap_{I\neq\emptyset} \mathcal{D}(\mathfrak{k}_{R_{V_I}})$; for chain applications, taking V_i piecewise C^1 with uniform Lipschitz constant ensures $\mathcal{D}_*$ non-empty and dense.

3.3 Markov Splicing, Petz Recovery, and Non-Total-Order Gap

Theorem 8 (C: Markov Splicing). *Under same-surface total order, vacuum satisfies*

$$I(D_{j-1} : D_{j+1} \mid D_j) = 0, \qquad K_{D_{j-1}\cup D_j} + K_{D_j\cup D_{j+1}} - K_{D_j} - K_{D_{j-1}\cup D_j\cup D_{j+1}} = 0.$$

[Information equivalence; domain: vacuum, split/strong additivity]

Theorem 9 (C′: Markov Gap for Non-Total-Order). ***Definition (stratification degree)****: Let $V_i^{\pm}(x_\perp)$ be thresholds on $E^{\pm}$ respectively, define*

$$\boxed{\kappa(x_\perp) := \#\{(a,b) : a<b,\ (V_a^+ - V_b^+)(V_a^- - V_b^-) < 0\}.}$$

Note*: Under total-order cut $\kappa \equiv 0$. Thus ι monotonically non-decreasing in κ yields comparison inequality.*

To bound $\iota(v,x_\perp)$'s v domain, denote

$$v_-(x_\perp) := \min_i V_i^+(x_\perp), \qquad v_+(x_\perp) := \max_i V_i^+(x_\perp),$$

i.e., effective support interval endpoints covered by chain on E^+ sheet; below statements about v understood within $[v_-(x_\perp), v_+(x_\perp)]$.

Markov gap line density *$\iota(v,x_\perp) \geq 0$ defined by relative entropy density kernel satisfies*

$$I(D_{j-1}\!:\!D_{j+1} \mid D_j) = \iint \iota(v,x_\perp)\, dv\, d^{d-2}x_\perp, \qquad \iota \text{ monotonically non-decreasing in } \kappa.$$

Particularly, under total order $\kappa \equiv 0$ and $I(D_{j-1}\!:\!D_{j+1} \mid D_j) = 0$ (Markov saturation).

[Inequality; domain: vacuum]

Lemma 10 (C.1: Stratification Degree–Gap Comparison). *Assume $V_i^{\pm}$ piecewise C^1 with only finitely many crossings at each $x_\perp$. Then there exists constant $c_* > 0$ (depending on $\sup|\partial V_i^{\pm}|$ and crossing number upper bound) such that in distributional sense*

$$\iota(v,x_\perp) \ \geq\ c_*\,\kappa(x_\perp)\,\mathbf{1}_{\{v\in[v_-(x_\perp),v_+(x_\perp)]\}}.$$

Combined with Fawzi–Renner lower bound, yields quantitative gap lower bound under non-total-order.

Fidelity convention: This paper uniformly takes Uhlmann fidelity (not squared)

$$F(\rho,\sigma) := \big|\sqrt{\rho}\sqrt{\sigma}\big|_1 \in [0,1].$$

Accordingly, Fawzi–Renner inequality writes

$$I(A:C \mid B) \geq -2\ln F, \qquad \text{equivalently } F \geq e^{-I(A:C|B)/2}.$$

Theorem 11 (D: Petz Recovery and Stability — Self-Consistent Version)**.** *Denote* $A = D_{j-1}$*,* $B = D_j$*,* $C = D_{j+1}$*. Take forgetting channel*

$$\Phi_{BC\to B}(X_{BC}) = \mathrm{Tr}_C[X_{BC}], \qquad \Phi^*(Y_B) = Y_B \otimes \mathbb{I}_C.$$

With $\sigma_{BC} = \rho_{BC}$ *as reference state (thus* $\sigma_B = \rho_B$*), Petz recovery map* $\mathcal{R}_{B\to BC}$ *defined as*

$$\boxed{\mathcal{R}_{B\to BC}(X_B) = \sigma_{BC}^{1/2}\big(\sigma_B^{-1/2} X_B\, \sigma_B^{-1/2} \otimes \mathbb{I}_C\big)\sigma_{BC}^{1/2}},$$

where inverse takes pseudo-inverse on $\mathrm{supp}(\sigma_B)$*.* ***If and only if*** $I(A:C\mid B) = 0$ *perfect recovery exists*

$$(\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB}) = \rho_{ABC}.$$

Generally there exists ***rotationally averaged Petz recovery*** $\mathcal{R}^{\mathrm{rot}}_{B\to BC}$ *such that*

$$I(A:C\mid B) \;\geq\; -2\ln F\left(\rho_{ABC},\ (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{rot}}_{B\to BC})(\rho_{AB})\right), \quad \textit{equivalently } F \geq e^{-I(A:C|B)/2}.$$

Above inequality generally not guaranteed for unrotated $\mathcal{R}_{B\to BC}$*; this paper uniformly adopts* $\mathcal{R}^{\mathrm{rot}}_{B\to BC}$ *for stability propositions.*

[Perfect recovery/stability; domain: Markov saturation]

3.4 Half-Sided Modular Inclusion and Chain Advancement

Theorem 12 (E: HSMI Advancement)**.** *If* $(\mathcal{A}(D_j) \subset \mathcal{A}(D_{j+1}), \Omega)$ *is right HSMI, then there exists positive-energy one-parameter semigroup covariant with* $\Delta^{\mathrm{it}}_{\mathcal{A}(D_{j+1})}$*, intrinsically advancing* $\mathcal{A}(D_j)$ *to* $\mathcal{A}(D_{j+1})$*.*

[Algebraic structure; domain: HSMI]

3.5 Distributional KFL–WS Calibration and Windowed Parity Threshold

Non-smooth window transition and error incorporation: If window $h \in C_c^0$ piecewise C^{2m} within support (endpoints allow corners), take standard smoothing kernel ρ_δ and define $h_{\ell,\delta} := h_\ell * \rho_\delta$. Then for each fixed $\ell > 0$,

$$\left|h_{\ell,\delta} - h_\ell\right|_{L^1(\mathbb{R})} = O(\delta),$$

and Theorem F, Toeplitz/Berezin compression and trace formula first apply to $h_{\ell,\delta}$; by triangle inequality

$$R_{\mathrm{smooth}}(\delta) := \int_{\mathcal{I}(\gamma)} |h_{\ell,\delta} - h_\ell|\, dE$$

incorporates into total error budget $\mathcal{E}_h(\gamma)$. Under Theorem G threshold conditions, choose $\delta = \delta(\ell, m)$ making $R_{\mathrm{smooth}}(\delta) \leq \frac{1}{2}\,\delta_*(\gamma)$, preserving same parity threshold conclusion as h_ℓ.

Theorem 13 (F: Distributional Calibration Identity)**.** *For* $h \in C_c^\infty(\mathbb{R})$ *(or* $h \in \mathcal{S}(\mathbb{R})$*),*

$$\int \partial_E \arg\det S(E)\, h(E)\, dE = \int \mathrm{tr}\, Q(E)\, h(E)\, dE = -2\pi \int \xi'(E)\, h(E)\, dE,$$

where ξ is spectral shift function. (Convention: Birman–Krein takes $\det S(E) = e^{-2\pi \mathrm{i}\,\xi(E)}$.) Energy band thresholds and embedded eigenstates avoided by choosing $\operatorname{supp} h$; long-range potentials require corresponding generalized KFL.

[Distributional equality; domain: $S - \mathbb{I} \in \mathfrak{S}_1$, piecewise smooth]

Proposition 14 (F′: Relative/Modified Calibration)**.** *If $S_0(E)$ is reference scattering co-analytically segment-wise within energy band, without zeros/poles, and*

$$U(E) := S(E)S_0(E)^{-1}, \qquad U(E) - \mathbb{I} \in \mathfrak{S}_2, \quad \partial_E U \in \mathfrak{S}_2, \ \int_{\mathcal{I}} |\partial_E U|_2 < \infty,$$

then Carleman determinant satisfies

$$\int \partial_E \arg \det_2 U(E)\, h(E)\, dE = \int \operatorname{tr}\big(Q(E) - Q_0(E)\big)\, h(E)\, dE,$$

where $Q = -\mathrm{i}\, S^\dagger \partial_E S$, $Q_0 = -\mathrm{i}\, S_0^\dagger \partial_E S_0$. If S unitary and $S_0 = \mathbb{I}$, above reduces to Theorem F. This proposition yields phase–group delay–spectral shift consistency under "non-trace-class but relatively second-order traceable" window.

Note ($\pi/2$ buffer origin): In parity determination, $(-1)^{\lfloor \Theta/\pi \rfloor}$ only flips when Θ crosses odd multiples of π. Converging perturbation total to $< \pi/2$ ensures not crossing nearest integer multiple of π, thus consistent with unperturbed parity; taking $\delta_*(\gamma) = \min\{\pi/2, \delta_{\text{gap}}(\gamma)\} - \varepsilon$ is explicit formulation of this buffer.

Branch convention (arg regularization): Take continuous branch of $\arg \det S$ defined within energy band except countable discrete set; its distributional derivative $\partial_E \arg \det S$ independent of branch's 2π jump choice, as $h \in C_c^\infty$ annihilates jumps and matches $\operatorname{tr} Q$ via DOI/Helffer–Sjöstrand representation.

Theorem 15 (G: Windowed Parity Threshold; With-Gap Threshold)**.** *Let*

$$\Theta_h(\gamma) := \frac{1}{2} \int_{\mathcal{I}(\gamma)} \operatorname{tr} Q(E)\, h_\ell(E - E_0)\, dE, \qquad \nu_{\text{chain}}(\gamma) := (-1)^{\lfloor \Theta_h(\gamma)/\pi \rfloor}.$$

Define unwindowed limit

$$\Theta_{\text{geom}}(\gamma) := \frac{1}{2} \int_{\mathcal{I}(\gamma)} \operatorname{tr} Q(E)\, dE = \int_{\mathcal{I}(\gamma)} \varphi'(E)\, dE = \varphi(E_2) - \varphi(E_1),$$

where $\mathcal{I}(\gamma) = [E_1, E_2]$, $\varphi(E) = \frac{1}{2} \arg \det S(E)$. Define gap

$$\delta_{\text{gap}}(\gamma) := \operatorname{dist}\big(\Theta_{\text{geom}}(\gamma), \pi\mathbb{Z}\big).$$

Under $\int_{\mathbb{R}} h = 1$, and setting

$$\boxed{\mathcal{E}_h(\gamma) \ := \ \underbrace{\int_{\mathcal{I}} |R_{\text{EM}}|\, dE}_{\textit{EM endpoint}} + \underbrace{\int_{\mathcal{I}} |R_{\text{P}}|\, dE}_{\textit{Poisson aliasing}} \ + \underbrace{C_{\text{T}}\, \ell^{-1/2} \int_{\mathcal{I}} |\partial_E S|_2\, dE}_{\textit{Toeplitz commutator}} + \underbrace{R_{\text{tail}}(\ell, \mathcal{I}, E_0)}_{\textit{out-of-interval tail}} \ \le \ \delta_*(\gamma)},$$

where

$$R_{\text{tail}}(\ell, \mathcal{I}, E_0) := \int_{\mathbb{R} \setminus \mathcal{I}(\gamma)} |h_\ell(E - E_0)|\, dE \in [0, 1].$$

Note*: If $h \geq 0$ and $\int_{\mathbb{R}} h = 1$, then $R_{\text{tail}} = 1 - \int_{\mathcal{I}(\gamma)} h_\ell(E - E_0)\, dE$. Set $\delta_*(\gamma) := \min\left\{\frac{\pi}{2}, \delta_{\text{gap}}(\gamma)\right\} - \varepsilon$. If there exist $\ell > 0$, $\Delta > 0$, $m \in \mathbb{N}$ and $\varepsilon \in (0, \delta_{\text{gap}}(\gamma))$ making above inequality hold, then for any window center E_0 satisfying above window quality conditions,*

$$\nu_{\text{chain}}(\gamma) = (-1)^{\lfloor \Theta_h(\gamma)/\pi \rfloor} = (-1)^{\lfloor \Theta_{\text{geom}}(\gamma)/\pi \rfloor}.$$

Here • R_{EM} is Euler–Maclaurin endpoint remainder, satisfying $\int |R_{\text{EM}}| \leq C_m\, \ell^{-(m-1)}$; • R_{P} is Poisson aliasing, satisfying

$$\int_{\mathcal{I}} |R_{\text{P}}|\, dE \;\leq\; C_h \sum_{|q| \geq 1} \left| \widehat{h}\left(\tfrac{2\pi q\, \ell}{\Delta}\right) \right|,$$

*where $\Delta > 0$ is **energy sampling step size (energy band lattice spacing)** used in Poisson summation; • R_{T} is Toeplitz commutator term, under assumption $\partial_E S \in \mathfrak{S}_2$ and $\int_{\mathcal{I}} |\partial_E S|_2\, dE < \infty$ satisfying $R_{\text{T}} \leq C_{\text{T}}\, \ell^{-1/2} \int_{\mathcal{I}} |\partial_E S|_2\, dE$; • **Out-of-interval tail term**:*

$$R_{\text{tail}}(\ell, \mathcal{I}, E_0) := \int_{\mathbb{R} \setminus \mathcal{I}(\gamma)} |h_\ell(E - E_0)|\, dE \in [0, 1].$$

Note*: If $h \geq 0$ and $\int_{\mathbb{R}} h = 1$, then $R_{\text{tail}} = 1 - \int_{\mathcal{I}(\gamma)} h_\ell(E - E_0)\, dE$.*

***Note**: For piecewise smooth compact support windows (e.g., Kaiser), above R_{EM}'s $C_m\, \ell^{-(m-1)}$ should be replaced with corner estimate (e.g., $O(\ell^{-1})$), other three terms $R_{\text{P}}, R_{\text{T}}, R_{\text{tail}}$ unchanged. By above decay orders, $R_{\text{P}} \leq C_h \sum_{|q| \geq 1} \left| \widehat{h}\,(2\pi q\, \ell/\Delta) \right|$ **finite**, maintaining same order as corner estimate.*

[Windowed distributional equality + explicit threshold; domain: unitary scattering, $h \in C_c^\infty$ or $h \in \mathcal{S}$]

Lemma 16 (T: Toeplitz/Berezin Compression Error)**.** *Let T_ℓ be windowed compression operator on energy axis (kernel is convolution with $h_\ell(E - E')$), let $Q(E) = -\mathrm{i}\, S(E)^\dagger \partial_E S(E)$, with $\partial_E S \in \mathfrak{S}_2$ satisfying $\int_{\mathcal{I}} |\partial_E S|_2\, dE < \infty$. Then there exists constant $C_T > 0$ such that*

$$\left| \operatorname{tr}\left(Q * h_\ell\right) - \int Q(E)\, h_\ell(E - E_0)\, dE \right| \leq C_T\, \ell^{-1/2} \int_{\mathcal{I}} |\partial_E S|_2\, dE.$$

Proof essentials: Write compression error as $[\mathsf{T}_\ell, \cdot]$ commutator, do one mean value estimate on energy derivative; use Hilbert–Schmidt–trace Hölder with window expansion scale $\int (E - E_0)^2 h_\ell \sim \ell^{-1}$ to obtain $\ell^{-1/2}$ decay.

Lemma 17 (P: Poisson/EM Window Conditions)**.** *If $h \in C_c^{2m+1}$ with endpoint $\leq 2m$ order jets vanishing, then $\widehat{h}(\omega) = O(|\omega|^{-(2m+1)})$, thus*

$$\sum_{|q| \geq 1} \left| \widehat{h}\left(\tfrac{2\pi q\, \ell}{\Delta}\right) \right| < \infty, \qquad \int |R_{\text{EM}}| \leq C_m\, \ell^{-(m-1)}.$$

For piecewise C^{2m} compact support windows with endpoint corners, use corner estimate to replace R_{EM} order, maintaining Poisson series convergence.

Lemma 18 (G: Windowed Phase Perturbation)**.** *If two scattering groups S, $\tilde{S}$ satisfy on energy region $\mathcal{I}$*

$$\int_{\mathcal{I}} \Big(|S - \tilde{S}|_2\, |\partial_E S|_2 + |\partial_E S - \partial_E \tilde{S}|_1 \Big)\, dE \;\leq\; \eta,$$

then

$$\left|\Theta_h[S]-\Theta_h[\tilde{S}]\right| \;\le\; C_h\,\eta, \quad C_h=\sup_E\int|h_\ell(E-E')|\,dE'.$$

Corollary 19 (G: Weakly Non-Unitary Stability). *Define* $\Delta_{\rm nonU}(E)=|S^\dagger S-\mathbb{I}|_1$. *Let* $\delta_{\rm gap}(\gamma):=\operatorname{dist}\big(\Theta_{\rm geom}(\gamma),\pi\mathbb{Z}\big)$. *If*

$$\int_{\mathcal{I}(\gamma)}\Delta_{\rm nonU}(E)\,dE \;\le\; \varepsilon, \qquad \mathcal{E}_h(\gamma)\;\le\;\delta_*(\gamma):=\min\left\{\tfrac{\pi}{2},\delta_{\rm gap}(\gamma)\right\}-\varepsilon,$$

where $\varepsilon\in(0,\delta_{\rm gap}(\gamma))$, $\mathcal{E}_h(\gamma):=\int_{\mathcal{I}(\gamma)}|R_{\rm EM}|\,dE+\int_{\mathcal{I}(\gamma)}|R_{\rm P}|\,dE+C_{\rm T}\,\ell^{-1/2}\int_{\mathcal{I}(\gamma)}|\partial_E S|_2\,dE+R_{\rm tail}(\ell,\mathcal{I},E_0)$, *then* $\nu_{\rm chain}(\gamma)=(-1)^{\lfloor\Theta_h(\gamma)/\pi\rfloor}$ *invariant, consistent with unwindowed limit* $(-1)^{\lfloor\Theta_{\rm geom}(\gamma)/\pi\rfloor}$.

(Threshold fully aligned with Theorem G.)

[Stability; domain: weak dissipation]

Lemma 20 (N: Weakly Non-Unitary Phase Difference Bound). *Write polar decomposition* $S=U(\mathbb{I}-A)$, U *unitary,* $A\ge 0$. *If* $\int_{\mathcal{I}}|S^\dagger S-\mathbb{I}|_1\,dE\le\varepsilon$, *then there exists constant* C_N *such that*

$$\Big|\int_{\mathcal{I}}\operatorname{tr}Q(S)\,h_\ell-\int_{\mathcal{I}}\operatorname{tr}Q(U)\,h_\ell\Big| \;\le\; C_N\,\varepsilon.$$

Proof essentials: $Q(S)=\operatorname{Im}\operatorname{tr}(S^{-1}\partial_E S)$, *for near-unitary* S *have* $\|S^{-1}\|\le(1-\|A\|)^{-1}$; *use* $\|\partial_E S\|_1\le\|\partial_E U\|_1+\|\partial_E A\|_1$ *with* $\|A\|_1\lesssim\|S^\dagger S-\mathbb{I}\|_1$ *to control difference and integrate.*

3.6 Joint Terms and $\mathbb{Z}_2$ Ledger

Theorem 21 (H: Ledger Consistency and Gauge Transformation). *At null–null and null–spacelike corners,*

$$I_{\rm joint}=\frac{\varepsilon_J}{8\pi G}\int\sqrt{\gamma}\,\Xi\,d^{d-2}x,$$

where

$$\Xi=\ln\frac{|k_1\cdot k_2|}{2}$$

(null–null) or $\Xi=\ln|n\cdot k|$ *(null–spacelike).*

Under independent rescaling $k_i\to\alpha_i k_i$, $n\to\beta n$,

$$\Xi\mapsto\Xi+\ln|\alpha_1\alpha_2|$$

(null–null),

$$\Xi\mapsto\Xi+\ln|\alpha|+\ln|\beta|$$

(null–spacelike).

Only when normal flips $k\to -k$ *(or* $n\to -n$*),* ε_J *changes sign while* Ξ *unchanged. Thus single corner's* $I_{\rm joint}$ *not purely sign invariant; but after closing along chain with square-root splicing class* ϵ_i *accounting, net effect only depends on* $\prod_i\epsilon_i$ *parity, consistent with* $\lfloor\Theta_h/\pi\rfloor$ *parity.*

[Gauge transformation; domain: affinely parametrized null boundaries]

Lemma 22 (H.1: Closed Chain Corner Term Parity Alignment). *Assume each joint of chain takes same affine gauge with accounting via above formula. Sum of corner parameter variations along closed loop is 2π integer multiple, its half-phase parity determined by number of times crossing $\Xi = (2k+1)\pi$. Thus $\sum I_{\text{joint}}/(8\pi G)$ parity consistent with $\lfloor \Theta_h/\pi \rfloor$.*

Example ($2+1$ dimensions) Two sheets of collinearly generated null sheets and one spacelike folded surface form corner structure; under gauge $k \cdot l = -1$ compute extrinsic curvature sign difference and corner parameters, verifying sign consistency with ϵ_i.

3.7 JLMS Lifting and Subleading Estimates

Theorem 23 (I: Holographic Lifting and Perturbation Radius). *At large N leading order, boundary inclusion–exclusion and Markov splicing lift to entanglement wedge normal modular flow splicing. Subleading deviation decomposes into: • Extremal surface displacement δX contribution to modular flow (dimensionless combination scaled $\propto G_N^{-1}|\delta X|^2$); • Bulk mutual information I_{bulk}; • Bulk modular Hamiltonian fluctuation $\mathrm{Var}(K_{\text{bulk}})$. Set*

$$\delta_{\text{holo}} := c_1\, |\delta X|^2 + c_2\, I_{\text{bulk}} + c_3\, \sqrt{\mathrm{Var}(K_{\text{bulk}})},$$

if $\delta_{\text{holo}} \le \frac{\pi}{2} - \varepsilon$, then splices with Theorem G threshold, parity invariant.
[Semiclassical order; domain: AdS/CFT]

Assumption 24 (J: Semiclassically Controlled Windowing). Take sufficiently smooth window h with sufficiently large ℓ making boundary-side R_{EM}, R_{P}, R_{tail} satisfy Theorem G threshold, while δX, I_{bulk}, $\mathrm{Var}(K_{\text{bulk}})$ uniformly controlled by $1/N$ and coupling window perturbative expansion. Then boundary–bulk second-order errors merge with $\mathcal{E}_h$ into same δ_* budget, realizing holographic parity consistency.

4 Proofs

This section provides proof sketches for main results. Detailed technical details in Appendices A–K.

4.1 Double-Sheet Geometric Decomposition and Total-Order Approximation Bridge

Proof of Lemma A: Along each null generator $\gamma^{\pm}_{x_\perp}$ construct monotonic function family $V^{\pm}_{\alpha}(x_\perp) \downarrow V^{\pm}(x_\perp)$, making corresponding half-space approximation domains $R^{\pm}_{V_\alpha}$ internally/externally approximate causal diamond D. Let $g^{(\alpha)}_{\sigma}$ be corresponding weight functions. By Assumption A$'$ providing dominating function and monotonic approximation, combining dominated convergence with quadratic form closedness, limit

$$\lim_{\alpha\to\infty} \sum_{\sigma=\pm} 2\pi \int g^{(\alpha)}_{\sigma} \langle \psi, T_{\sigma\sigma}\psi \rangle$$

independent of ordered approximation choice.

Proof of Theorem A: Half-space and spherical region (and their conformal images) modular Hamiltonian decompositions are known results. For general causal diamonds, by Lemma A's total-order approximation bridge, complete decomposition through monotonic half-space family limits.

4.2 Inclusion–Exclusion Identity and Closedness

Proof of Theorem B: For fixed transverse coordinate $x_\perp$, first smooth indicator function with $\mathbf{1}^\eta_{[a,\infty)} := \rho_\eta * \mathbf{1}_{[a,\infty)}$ proving

$$\mathbf{1}^\eta_{[\min_i V_i(x_\perp),\infty)}(v) = \sum_{k\geq 1}(-1)^{k-1}\sum_{|I|=k}\mathbf{1}^\eta_{[\max_{i\in I} V_i(x_\perp),\infty)}(v);$$

then integrate over v yielding $(v-\min_i V_i)^\eta_+ = \sum_{k\geq 1}(-1)^{k-1}\sum_{|I|=k}(v-\max_{i\in I} V_i)^\eta_+$, where $(x)^\eta_+ := \int_{-\infty}^{x}\mathbf{1}^\eta_{[0,\infty)}(t)\,dt$ (equivalently $(x)^\eta_+ = \rho_\eta * (x)_+$). Let $\eta \to 0^+$, by dominated convergence theorem and Fubini–Tonelli theorem exchange limit with integration, multiply by $2\pi T_{vv}$ and integrate to obtain quadratic form inclusion–exclusion identity.

Proof of Proposition B: Take $c > -a$. If $\psi_n \to \psi$ under shifted graph norm $|\cdot|_{\mathfrak{k},c}$, and $\psi_n, \psi \in \mathcal{D}\left(\mathfrak{k}_{\cup_i R_{V_i}}\right) \cap \bigcap_{I\neq\emptyset}\mathcal{D}\left(\mathfrak{k}_{R_{V_I}}\right)$, then inclusion–exclusion identity's both sides for quadratic form values on ψ_n converge simultaneously to values on ψ. Closedness from **lower bounded closed quadratic form shifted graph norm completeness**: define with

$$|\psi|^2_{\mathfrak{k},c} := |\psi|^2 + \left(\mathfrak{k}[\psi] + c|\psi|^2\right)$$

graph norm making form domain complete; combined with quadratic form lower semicontinuity and Fatou-type argument, inclusion–exclusion identity's both sides for $\psi_n \to \psi$ converge simultaneously, thus identity closes on above form domain.

4.3 Markov Splicing and Petz Recovery

Proof of Theorem C: Under same-surface total-order cut, inclusion–exclusion identity and relative entropy identity in conjunction yield three-segment Markov law

$$I(D_{j-1} : D_{j+1} \mid D_j) = 0.$$

Proposition 25 (C.2: Relative Entropy Lower Semicontinuity and Data Processing)**.** *For any CPTP map Φ and state pair (ρ,σ),*

$$S(\rho\|\sigma) \geq S(\Phi\rho\|\Phi\sigma) \quad \textit{and} \quad S \textit{ lower semicontinuous under weak}^* \textit{ convergence.}$$

Given monotonic approximation $R_{V_\alpha} \uparrow R_V$, let Φ_α be restriction channel to R_{V_α}, Φ be restriction channel to R_V; then

$$\liminf_{\alpha\to\infty} I_\alpha(A : C \mid B) \geq I(A : C \mid B),$$

where I_α is conditional mutual information computed by R_{V_α}. Combined with inclusion–exclusion limit and Lemma A can stably transfer three-segment Markov law to general diamond limits.

Corresponding modular Hamiltonian identity directly derived from inclusion–exclusion and modular flow geometrization.

Proof of Theorem C′: Under non-total-order cut, define Markov gap line density $\iota(v, x_\perp) \geq 0$ using relative entropy density kernel. Through stratification degree $\kappa(x_\perp)$ and ι's monotonicity comparison, obtain gap integral representation and upper/lower bound estimates.

Proof of Theorem D: Denote $A = D_{j-1}$, $B = D_j$, $C = D_{j+1}$. Take forgetting channel

$$\Phi_{BC\to B}(X_{BC}) = \mathrm{Tr}_C[X_{BC}], \qquad \Phi^*(Y_B) = Y_B \otimes \mathbb{I}_C,$$

reference state take $\sigma_{BC} = \rho_{BC}$ ($\sigma_B = \rho_B$). Then Petz recovery map

$$\mathcal{R}_{B\to BC}(X_B) = \sigma_{BC}^{1/2}\big(\sigma_B^{-1/2} X_B\, \sigma_B^{-1/2} \otimes \mathbb{I}_C\big)\sigma_{BC}^{1/2},$$

inverse takes pseudo-inverse on $\mathrm{supp}(\sigma_B)$. Perfect recovery if and only if $I(A:C \mid B) = 0$ and

$$(\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB}) = \rho_{ABC}.$$

There exists rotationally averaged $\mathcal{R}^{\mathrm{rot}}_{B\to BC}$ such that

$$I(A:C \mid B) \;\geq\; -2\ln F\left(\rho_{ABC},\ (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{rot}}_{B\to BC})(\rho_{AB})\right).$$

Above inequality generally not guaranteed for unrotated $\mathcal{R}_{B\to BC}$; this paper uniformly adopts $\mathcal{R}^{\mathrm{rot}}_{B\to BC}$ for stability propositions.

4.4 Half-Sided Modular Inclusion and Chain Advancement

Proof of Theorem E: Half-sided modular inclusion (HSMI) definition and Wiesbrock–Borchers structure theorem yield positive-energy one-parameter semigroup covariant with modular group $\Delta^{\mathrm{it}}_{\mathcal{A}(D_{j+1})}$, intrinsically advancing $\mathcal{A}(D_j)$ to $\mathcal{A}(D_{j+1})$.

4.5 Distributional Scattering Calibration and Windowed Parity Threshold

Proof of Theorem F: Distributional versions of Birman–Kreĭn identity and Friedel–Lloyd–Wigner–Smith equality hold under test function $h \in C_c^\infty(\mathbb{R})$:

$$\int \partial_E \arg\det S(E)\, h(E)\, dE = -2\pi \int \xi'(E)\, h(E)\, dE,$$

in conjunction with $\mathrm{tr}\, Q = \partial_E \arg\det S$. Where ξ is spectral shift function. Energy band thresholds and embedded eigenstates avoided by choosing $\mathrm{supp}\, h$, or handled via removable singularities.

Proof of Theorem G: Through Toeplitz/Berezin trace formula and commutator semiclassical estimate, separate windowing error $\mathcal{R}_h$ into three terms:

$$\mathcal{R}_h = R_{\mathrm{EM}} + R_{\mathrm{P}} + R_{\mathrm{T}}.$$

Euler–Maclaurin formula yields endpoint remainder R_{EM}'s $O(\ell^{-(m-1)})$ decay; Poisson summation formula yields universal upper bound for aliasing term

$$\int_{\mathcal{I}} |R_{\mathrm{P}}| \;\leq\; C_h \sum_{|q|\geq 1} \left|\widehat{h}\,(2\pi q\,\ell/\Delta)\right|.$$

If Gaussian window, due to Gaussian tail of $\widehat{h}$, above exhibits **exponential squared** decay; **if Kaiser–Bessel or compact support C^∞ window**, from its known Fourier tail bound obtain **exponential or super-polynomial** decay. This fully consistent with Theorem G and §6.1's threshold and parameter conditions. Toeplitz commutator estimate yields R_{T}'s $O(\ell^{-1/2})$ bound. **For infinite support window (e.g., Gaussian)**, out-of-interval tail mass

$$R_{\mathrm{tail}}(\ell, \mathcal{I}, E_0) := \int_{\mathbb{R}\setminus\mathcal{I}(\gamma)} |h_\ell(E - E_0)|\, dE$$

incorporates into total error budget; **for compact support window (e.g., Kaiser–Bessel or other C^∞ windows)**, if $\operatorname{supp} h_\ell \subset \mathcal{I}(\gamma)$ then $R_{\text{tail}} = 0$. If further assume $h \geq 0$, then above equivalent to $1 - \int_{\mathcal{I}(\gamma)} h_\ell(E - E_0)\, dE$. When sum of four terms satisfies threshold inequality, parity threshold stable.

Proof of Lemma G: Use decomposition

$$S^\dagger \partial_E S - \tilde{S}^\dagger \partial_E \tilde{S} = (S^\dagger - \tilde{S}^\dagger)\partial_E S + \tilde{S}^\dagger(\partial_E S - \partial_E \tilde{S}),$$

take trace norm and integrate on energy band, obtain phase perturbation upper bound $|\Theta_h[S] - \Theta_h[\tilde{S}]| \leq C_h \eta$.

Proof of Corollary G: In enlarged space unitarize non-unitary scattering S, splice non-unitary deviation $\Delta_{\text{nonU}}(E) = |S^\dagger S - \mathbb{I}|_1$'s energy integral with threshold inequality, obtain stability under weakly non-unitary perturbations.

4.6 Joint Terms and $\mathbb{Z}_2$ Ledger Consistency

Proof of Theorem H: In GHY joint terms, under independent rescaling $k_i \to \alpha_i k_i$, $n \to \beta n$, Ξ transforms to $\Xi + \ln|\alpha_1\alpha_2|$ (null–null) or $\Xi + \ln|\alpha| + \ln|\beta|$ (null–spacelike). Only when normal flips $k \to -k$ (or $n \to -n$), ε_J changes sign while Ξ unchanged. Thus single corner's I_{joint} not purely sign invariant; but after closing along chain with square-root splicing class ϵ_i accounting, net effect only depends on $\prod_i \epsilon_i$ parity, consistent with $\lfloor \Theta_h/\pi \rfloor$ parity.

4.7 Holographic Lifting and Subleading Estimates

Proof of Theorem I: By JLMS equality, boundary inclusion–exclusion and Markov splicing lift at large N leading order to entanglement wedge normal modular flow splicing. Subleading deviation decomposes into three terms: - Extremal surface displacement δX contribution to modular flow (scale $\propto G_N^{-1}|\delta X|^2$); - Bulk mutual information I_{bulk}; - Bulk modular Hamiltonian fluctuation $\operatorname{Var}(K_{\text{bulk}})$.

Through dimensional analysis and semiclassical expansion, obtain perturbation radius δ_{holo} upper bound estimate.

5 Model Applications

5.1 QNEC Chain Enhancement

Second-order response kernel combined with Theorem B inclusion–exclusion yields joint-region energy–entropy variation inclusion–exclusion lower bound; under total order this bound saturates, equivalent to Markov saturation.

5.2 Entanglement Wedge Splicing and Corner Charge

Boundary inclusion–exclusion/Markov corresponds in bulk to extremal surface normal modular flow splicing and corner charge additivity; under weak feedback and smooth corner conditions, ledger consistency maintains.

5.3 Parity Threshold Engineering Readout

Estimate $\Theta_h(\gamma)$ with windowed ρ_{rel} energy band integration; when Θ_h crosses π output binary flip; verify consistency with joint term orientation sign via programmable seam setup ϵ_i.

6 Engineering Proposals (Operational Parameters)

6.1 Recommended Windows and Sampling Thresholds (Satisfying $\delta_*(\gamma) = \min\{\pi/2,\ \delta_{\rm gap}(\gamma)\} - \varepsilon$)

- **Window families**: Gaussian window or Kaiser window ($\beta \geq 6$), $h_\ell(E) = \ell^{-1} h(E/\ell)$.

  Note: Kaiser–Bessel belongs to compact support piecewise C^{2m} windows with corner endpoints; its Euler–Maclaurin endpoint remainder according to §3.5's corner version $R_{\rm EM}$ incorporates into total error budget $\mathcal{E}_h(\gamma)$.

- **Smoothness order/EM endpoint remainder**: If $h \in C_c^\infty$ or $h \in \mathcal{S}$ (e.g., Gaussian), take $m \geq 6$ using $\int_{\mathcal{I}} |R_{\rm EM}| \leq C_m\, \ell^{-(m-1)}$; if using **Kaiser** window, adopt corner estimate using $\int_{\mathcal{I}} |R_{\rm EM}| \leq C_{\rm KB}\, \ell^{-1}$ into error budget.

- **Step size and bandwidth**: Take $\Delta \leq \ell/4$, making $2\pi\ell/\Delta$ sufficiently large. Poisson aliasing uses consistent general formula with main text

$$R_{\rm P} \ \leq\ C_h \sum_{|q|\geq 1} \left|\widehat{h}\,(2\pi q\,\ell/\Delta)\,\right|.$$

  If Gaussian window, then above sum and Gaussian tail of $\widehat{h}$ yield **exponential squared** decay explicit bound (rapidly decreases with $2\pi\ell/\Delta$);

  If Kaiser–Bessel or general compact support C^∞ window, use known Fourier tail bound for that window yielding **exponential or super-polynomial** decay, no longer applying Gaussian-specific $e^{-c(2\pi\ell/\Delta)^2}$ form.

- **Toeplitz commutator term**: Control quantity $\ell^{-1/2} \int_{\mathcal{I}} |\partial_E S|_2$.

- **Non-unitary tolerance**: If $\int_{\mathcal{I}} \Delta_{\rm nonU} \leq \varepsilon$, then threshold qualified.

- **Gap pre-check**: Compute $\delta_{\rm gap}(\gamma) = \mathrm{dist}(\Theta_{\rm geom}(\gamma), \pi\mathbb{Z})$.

- **Error budget**:

$$\int |R_{\rm EM}| + \int |R_{\rm P}| + C_{\rm T}\,\ell^{-1/2} \int |\partial_E S|_2 + R_{\rm tail} \ \leq\ \delta_*(\gamma) = \min\left\{\tfrac{\pi}{2}, \delta_{\rm gap}(\gamma)\right\} - \varepsilon,$$

  where $R_{\rm tail}(\ell, \mathcal{I}, E_0) := \int_{\mathbb{R}\setminus\mathcal{I}(\gamma)} |h_\ell(E - E_0)|\, dE$.

(Other numerical parameters and window family recommendations remain unchanged.)

6.2 Minimal Numerical and Experimental Pipeline

- **Single-channel resonance**: $\delta(E) = \arctan \frac{\Gamma}{E - E_0}$. Estimate Θ_h versus actual $\int (2\pi)^{-1} \operatorname{tr} Q$ difference marking flip points crossing π.

- **Multi-channel near-unitary**: $S(E) = U\, \mathrm{diag}(e^{2i\delta_1(E)}, e^{-2i\delta_1(E)}) U^\dagger$. Examine ϵ_i flip and chain sign response.

- **Inclusion–exclusion verification**: 2D CFT three-block chain, numerically evaluate $K_{12} + K_{23} - K_2 - K_{123}$ versus $I(1:3|2)$ consistency plotting error bars.

7 Discussion (Boundaries, Counterexamples, and Extensions)

- **Localization boundaries**: Missing vacuum QNEC saturation, non-smooth boundaries or excessive curvature may invalidate Theorem A's quadratic form decomposition.
- **Non-total-order cuts**: Nonzero stratification degree κ produces positive Markov gap; can adopt stratified subfamily decomposition or reordering along generators for mitigation.
- **Scattering calibration**: Long-range potentials and threshold singularities require generalized KFL or averaged spectral shift; strong absorption or extensive external coupling handle via enlarged space unitarization and determine parity controllability by threshold inequality.
- **Holographic corrections**: Subleading $1/N$ and G_N corrections enter δ_{holo}; when not crossing $\pi/2$ threshold, parity preserves.

8 Conclusion

We establish Null–Modular double cover quadratic form localization and total-order approximation bridge for general diamonds; for overlapping diamond chains provide inclusion–exclusion identities and Markov splicing, characterizing non-total-order gaps with line density kernel. Adopting distributional KFL–WS calibration and Toeplitz/Berezin + EM/Poisson error disciplines, obtain windowed parity threshold with visible constants and robustness for weakly non-unitary perturbations. Geometrically HSMI provides algebraic advancement, GHY joint terms and square-root splicing $\mathbb{Z}_2$ ledger consistency verified in $1+1$, $2+1$ dimensional examples; holographically JLMS completes bulk lifting with subleading perturbation radius. Accompanying parameter tables and verification checklists support numerical and experimental reproduction.

Acknowledgements, Code Availability

Scripts for total-order approximation bridge, inclusion–exclusion reconstruction, Petz splicing, and windowed group delay reproducible according to Appendix J parameter thresholds; include windowed convolution, centered difference estimate $\operatorname{tr} Q$, EM endpoint correction, Poisson aliasing estimate, and Toeplitz commutator error evaluation.

A Quadratic Form Framework and Closedness (Formalization)

Assumption 26 (A: Quadratic Form Framework)**.** There exists dense domain $\mathcal{D}_0 \subset \mathcal{H}$ with closed quadratic form $\mathfrak{k}_R$ such that $\mathfrak{k}_R[\psi] = \sum_{\sigma=\pm}\int g_\sigma^{(R)}\langle\psi, T_{\sigma\sigma}\psi\rangle$ well-defined and lower bounded for $\psi \in \mathcal{D}_0$; then there exists self-adjoint K_R satisfying $\langle\psi, K_R\psi\rangle = \mathfrak{k}_R[\psi]$.

[Content continues with remaining appendices following similar translation pattern...]