Finite-Window Energy and Spectral Conservation

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Abstract

Establish energy and spectral conservation laws under finite-window measurement. Core result: for windowed readout with window w and spectral density ρ, windowed energy $\mathcal{E}_w = \int E w(E) \rho(E) dE$ satisfies: (i) **Conservation**: under unitary evolution $\partial_t \mathcal{E}_w = 0$ when w time-independent; (ii) **Covariance**: gauge-invariant under phase transformations; (iii) **NPE closure**: discrete approximation with alias+EM+tail error decomposition.

1 Definitions

Windowed energy functional:

$$\mathcal{E}_w[\rho] = \int_{\mathbb{R}} E\, w(E)\, \rho(E)\, dE$$

where $w \geq 0$ normalized window, ρ spectral density.

Spectral measure: $d\mu_\rho(E) = \rho(E) dE$ for density, general $d\mu_\rho$ for measure.

2 Main Theorems

Theorem 2.1 (Windowed Energy Conservation). *For unitary evolution $\partial_t \rho = i[H, \rho]$ and time-independent window w, have $\partial_t \mathcal{E}_w = 0$.*

Theorem 2.2 (Gauge Covariance). *Under gauge transformation $\rho \mapsto U \rho U^\dagger$, $H \mapsto U H U^\dagger$ with U unitary, $\mathcal{E}_w$ invariant.*

Theorem 2.3 (NPE Error for Windowed Energy). *Discrete approximation $\widehat{\mathcal{E}}_w = \Delta \sum_n E_n w(E_n) \rho(E_n)$ satisfies*

$$|\mathcal{E}_w - \widehat{\mathcal{E}}_w| \leq |\varepsilon_{\text{alias}}| + |\varepsilon_{\text{EM}}| + |\varepsilon_{\text{tail}}|$$

with $\varepsilon_{\text{alias}} = 0$ under Nyquist.

3 Discussion

Finite-window framework provides:

- Energy conservation compatible with finite bandwidth

- Gauge-invariant formulation
- Non-asymptotic error control
- Connection to thermodynamics and resource theories