Boundary as Unified Stage:
From Time Translation Operator, Null–Modular Double Cover
to GHY Boundary Term

Haobo Ma[1] Wenlin Zhang[2]

[1]Independent Researcher

[2]National University of Singapore

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Abstract

Construct unified framework with boundary as sole fundamental stage, gluing three mature but usually separate structures as three projections of same object:

(i) Based on Birman–Krein–Friedel–Wigner–Smith framework, "time translation operator" characterized by scattering phase, spectral shift function, Wigner–Smith time delay matrix;

(ii) Based on Tomita–Takesaki modular theory and recent results on causal diamond/null surface modular Hamiltonians, Markov property, QNEC, Null–Modular double cover and overlapping causal diamond chains;

(iii) Represented by Gibbons–Hawking–York (GHY) boundary term and its generalization with null sheets and joints, gravitational boundary action and Brown–York quasilocal energy.

Under clearly stated applicability domains and assumptions, give four main results:

(1) In scattering systems satisfying trace-class perturbation conditions, scattering half-phase derivative, Krein spectral shift density, Wigner–Smith delay matrix trace constitute same boundary time measure;

(2) In relativistic quantum field theory under standard assumptions, modular Hamiltonians on causal diamonds and null surfaces localizable as weighted integrals of stress–energy flow on Null–Modular double cover, satisfying Markov inclusion-exclusion law for overlapping diamond families;

(3) On piecewise spacetime boundaries (with null sheets and joints) in general relativity, after adding GHY-type boundary/joint terms, action variation for fixed induced geometry well-defined; Brown–York boundary stress tensor generates boundary time translation as third type of "boundary time";

(4) With appropriate matching maps, these three "boundary times" unifiable as different realizations of same one-parameter automorphism group, thus restating time, algebra, geometry simultaneously as different aspects of boundary data.

Provide unified scale examples in black hole thermodynamics, AdS/CFT, scattering network experiments; propose several engineering schemes testable on mesoscale experimental platforms.

1 Introduction and Historical Context

1.1 Paradigm Shift from Bulk to Boundary

Development of relativity, quantum field theory, quantum gravity shows clear "bulk-to-boundary" trend. Quantum scattering theory centered on S-matrix directly encodes dynamical information on spacetime asymptotic radiation boundary; Birman–Krein identity connects scattering determinant phase with spectral shift function, making "bulk spectral changes" readable from boundary scattering data.

In algebraic quantum field theory, Tomita–Takesaki modular theory shows: given local algebra and faithful state, modular flow σ_t^ω is canonical one-parameter automorphism group of algebra; geometric realization given clear characterization by Bisognano–Wichmann theorem, spherical region and null surface modular Hamiltonian local expressions.

In general relativity, Einstein–Hilbert bulk term alone cannot give well-defined variational principle for manifolds with boundary; must add Gibbons–Hawking–York boundary term; when null sheets and joints present, must introduce corresponding null boundary and corner terms to ensure action differentiability and Hamilton–Jacobi structure.

These advances jointly suggest: truly "computable" physical objects often concentrated on boundary, while bulk more like reconstruction or evolution result of boundary data.

1.2 Three Seemingly Different Theoretical Paradigms

Three specific paradigms this paper focuses on:

(1) Scattering end: Time as boundary translation scale

Let $(H_0, H_0 + V)$ be self-adjoint operator pair, V in appropriate trace class making wave operators and scattering matrix $S(\omega)$ well-defined satisfying Birman–Krein conditions. Then spectral shift function $\xi(\omega)$ exists satisfying

$$\det S(\omega) = \exp(-2\pi i \xi(\omega)),$$

whose derivative $\xi'(\omega)$ has exact trace formula relation with scattering phase derivative and Wigner–Smith time delay matrix $Q(\omega) = -iS(\omega)^\dagger \partial_\omega S(\omega)$ trace.

This makes $\varphi'(\omega)$, $\xi'(\omega)$, $\operatorname{tr} Q(\omega)$ jointly characterize "boundary time measure," viewing time as translation parameter generated by boundary spectral data.

(2) Algebraic–geometric end: Null–Modular double cover and overlapping causal diamond chains

In Minkowski or AdS boundary CFT, for vacuum state restricted to wedge region, spherical diamond, or null surface region, modular Hamiltonian writable as local integral of stress–energy tensor on boundary, forming infinite-dimensional Lie algebra and Markov property on null surfaces.

For causal diamond D, introduce Null–Modular double cover composed of future/past null boundaries (E^+, E^-); modular Hamiltonian K_D writable as

$$K_D = 2\pi \sum_{\sigma=\pm} \int_{E^\sigma} g_\sigma(\lambda, x_\perp) T_{\sigma\sigma}(\lambda, x_\perp) \, d\lambda \, d^{d-2}x,$$

where $T_{\sigma\sigma}$ are null direction components. Modular Hamiltonians of overlapping diamond chains satisfy inclusion-exclusion and Markov stitching properties.

(3) Gravity end: GHY boundary term, null boundaries, Brown–York quasilocal energy

On spacetime manifold $\mathcal{M}$ with boundary, after adding GHY boundary term

$$S = S_{\text{EH}} + S_{\text{GHY}}$$

variation for fixed induced metric well-defined; when boundary includes null sheets and joints, must supplement null boundary and corner terms.

Hamilton–Jacobi analysis for regions with boundary yields Brown–York boundary stress tensor

$$T_{\text{BY}}^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}},$$

whose time component gives quasilocal energy and Hamiltonian generating boundary time translation.

These three routes respectively highlight spectral–scattering, modular–algebraic, geometric–gravitational aspects, but all depend on boundary: scattering defined by asymptotic boundary, modular flow localized on region boundary, gravitational action differentiability determined by boundary terms.

1.3 Goals and Main Thread

Goal: Provide mathematically self-consistent framework with clear applicability domain, viewing above three paradigms as three projections of same boundary structure.

Core idea: 1. Take appropriate "boundary data triple"

$$(\partial\mathcal{M}, \mathcal{A}_\partial, \mu_\partial)$$

as fundamental object: $\partial\mathcal{M}$ geometric boundary, $\mathcal{A}_\partial$ observable algebra or scattering algebra on it, μ_∂ time scale measure from spectral shift or energy flow;

2. Restate time delay in scattering theory, modular flow in modular theory, Brown–York boundary Hamiltonian in gravity all as different representations of one-parameter automorphism group on this boundary structure;

3. Under strictly limited conditions, prove these representations mutually equivalent, compressing "time–algebra–geometry" three threads onto unified boundary stage.

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2 Model and Assumptions

Construct abstract model simultaneously accommodating scattering systems, algebraic QFT, gravitational boundaries; clarify applicability domains of all results.

2.1 Abstract Boundary Triple

Definition 2.1 (Boundary Triple)**.** Boundary triple is data

$$(\partial\mathcal{M}, \mathcal{A}_\partial, \omega_\partial)$$

where:

1. $\partial\mathcal{M}$ is piecewise smooth three-dimensional manifold, decomposable into timelike, spacelike, null sheets and their joints $\mathcal{C}$;

2. $\mathcal{A}_\partial$ is von Neumann algebra acting on Hilbert space $\mathcal{H}$, containing boundary observables (scattering channels, boundary fields, quasilocal energy operators, etc.);

3. ω_∂ is faithful normal state on $\mathcal{A}_\partial$; GNS triple denoted $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$.

Postulate 1 (Boundary Completeness)**.** *Physical content of bulk region $\mathcal{M}$ completely reconstructible from some boundary triple $(\partial\mathcal{M}, \mathcal{A}_\partial, \omega_\partial)$ (within given theory's applicability range); time evolution and response operators all determined by boundary one-parameter automorphism group and state evolution.*

This postulate has different concrete realizations in different contexts: wave operators and S-matrix in scattering theory, boundary CFT and bulk geometry in AdS/CFT, bulk solution reconstruction from boundary data in Hamilton–Jacobi perspective.

2.2 Scattering End Assumptions

Assumption 1 (S.1: Trace-class perturbation and BK conditions)**.** On Hilbert space $\mathcal{H}_{\text{scatt}}$, H_0 and $H = H_0 + V$ are self-adjoint operators, V in trace-class or stronger ideal making scattering matrix $S(\omega)$ exist for almost all energies ω with $S(\omega) - \mathbf{1} \in \mathfrak{S}_1$. Under these assumptions, Krein spectral shift function $\xi(\omega)$ and Birman–Krein identity

$$\det S(\omega) = \exp(-2\pi i \xi(\omega))$$

exist.

Assumption 2 (S.2: Time delay matrix and trace formula)**.** Wigner–Smith time delay operator defined as

$$Q(\omega) = -iS(\omega)^\dagger \partial_\omega S(\omega),$$

$Q(\omega)$ is trace-class operator satisfying

$$\operatorname{tr} Q(\omega) = 2\pi \xi'(\omega)$$

in appropriate sense.

Under these conditions, define "boundary time measure"

$$d\mu_\partial^{\text{scatt}}(\omega) := \frac{1}{2\pi} \operatorname{tr} Q(\omega)\, d\omega.$$

2.3 Modular Theory and Null–Modular Double Cover Assumptions

Assumption 3 (M.1: Standard modular structure). $\mathcal{A}_\partial \subset \mathcal{B}(\mathcal{H})$ is local algebra for causal region O, ω is vacuum or KMS state making Ω_ω cyclic and separating vector for $\mathcal{A}_\partial$, thus Tomita–Takesaki modular operator Δ and one-parameter modular flow

$$\sigma_t^\omega(A) = \Delta^{it} A \Delta^{-it}$$

exist.

Assumption 4 (M.2: Geometric modular flow). For wedge region, spherical causal diamond, or null surface region O in vacuum state, modular flow's geometric action is corresponding region's Lorentz transformation or conformal Killing flow; modular Hamiltonian $K_O = -\log \Delta$ writable as local integral of stress–energy tensor.

Assumption 5 (M.3: Markov property and inclusion-exclusion). For region families on null surface, modular Hamiltonians satisfy Markov property proved by Casini–Teste–Torroba: for nested or overlapping regions along null line, conditional mutual information saturates strong subadditivity, corresponding to modular Hamiltonian inclusion-exclusion identities.

Based on this, define Null–Modular double cover: for causal diamond D, boundary null hypersurfaces decompose as $E^+ \cup E^-$; define weighted energy flow integral representation modular Hamiltonians on both sheets.

2.4 Gravitational Boundary Assumptions

Assumption 6 (G.1: Action with boundaries/null boundaries). On four-dimensional Lorentzian manifold $\mathcal{M}$, gravitational action takes standard form

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda)\, d^4 x + S_\partial,$$

where

$$S_\partial = S_{\text{GHY}}^{\text{tl/sp}} + S_{\mathcal{N}}^{\text{null}} + S_{\mathcal{C}}^{\text{corner}}$$

are GHY boundary terms on timelike/spacelike sheets, improved terms on null sheets, corner terms at joints.

Assumption 7 (G.2: Variation well-definedness and Brown–York stress tensor). On variation family fixing boundary induced metric (and appropriate equivalent data for null boundaries), δS contains only bulk terms, yielding Einstein equations. For given timelike boundary three-sheet ^3B, action variation with respect to boundary metric defines Brown–York boundary stress tensor

$$T_{\text{BY}}^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}},$$

whose contraction with boundary Killing vector ξ^a gives quasilocal energy and Hamiltonian generating boundary time translation.

Under these assumptions, view Brown–York Hamiltonian as third type of "boundary time generator."

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3 Main Results (Theorems and Alignments)

State four main results establishing correspondences among three theoretical threads.

Theorem 3.1 (A: BK–Wigner–Smith–Time Scale Identity). *Under Assumptions S.1–S.2, define: • Total scattering phase $\Phi(\omega) = \arg \det S(\omega)$, half-phase $\varphi(\omega) = \frac{1}{2}\Phi(\omega)$; • Krein spectral shift function $\xi(\omega)$ satisfying $\det S(\omega) = \exp(-2\pi i \xi(\omega))$; • Wigner–Smith time delay operator $Q(\omega) = -iS(\omega)^\dagger \partial_\omega S(\omega)$.*

Then almost everywhere finite derivatives $\xi'(\omega)$ and $\varphi'(\omega)$ exist such that

$$\boxed{\frac{\varphi'(\omega)}{\pi} = \xi'(\omega) = \frac{1}{2\pi} \operatorname{tr} Q(\omega)}$$

holds almost everywhere.

Thus measure

$$d\mu_\partial^{\mathrm{scatt}}(\omega) := \frac{1}{2\pi} \operatorname{tr} Q(\omega)\, d\omega$$

equivalent to spectral shift measure $d\xi(\omega)$ and scattering phase scale $\pi^{-1} d\varphi(\omega)$, viewable as unified "boundary time scale."

Definition 3.2 (Null–Modular Double Cover). In d-dimensional Minkowski spacetime, consider causal diamond with vertices p, q:

$$D(p, q) = J^+(p) \cap J^-(q).$$

Boundary composed of future null hypersurface $\mathcal{N}^+$ and past null hypersurface $\mathcal{N}^-$. Define Null–Modular double cover

$$\widetilde{E}_D := E^+ \sqcup E^-,$$

where $E^\pm$ are two smooth leaves of $\mathcal{N}^\pm$ after removing joint points; introduce affine parameter λ and transverse coordinates $x_\perp$ on each leaf.

Theorem 3.3 (B: Modular Hamiltonian Null Measure Localization). *Under Assumptions M.1–M.2, for Minkowski vacuum state restricted to causal diamond D local algebra $\mathcal{A}(D)$, modular Hamiltonian writable as*

$$\boxed{K_D = 2\pi \sum_{\sigma=\pm} \int_{E^\sigma} g_\sigma(\lambda, x_\perp)\, T_{\sigma\sigma}(\lambda, x_\perp)\, d\lambda\, d^{d-2}x \,,}$$

where: 1. $T_{++} = T_{vv}$, $T_{--} = T_{uu}$ are stress–energy tensor components along two null directions; 2. Weight functions $g_\sigma(\lambda, x_\perp)$ determined solely by causal diamond geometric data, linearly degenerating at endpoints.

Moreover, for overlapping causal diamond family $\{D_j\}$ on same null surface, modular Hamiltonians satisfy inclusion-exclusion identity

$$K_{\cup_j D_j} = \sum_{k \geq 1} (-1)^{k-1} \sum_{j_1 < \cdots < j_k} K_{D_{j_1} \cap \cdots \cap D_{j_k}},$$

and Minkowski vacuum state restricted to these regions satisfies Markov property: for appropriately nested A, B, C, conditional mutual information

$$I(A : C \mid B) = 0$$

equivalent to above inclusion-exclusion identity.

Theorem B shows: Modular Hamiltonian completely localizable on null measure boundary; Null–Modular double cover provides purely boundary geometric realization of modular flow.

Theorem 3.4 (C: GHY–Brown–York Boundary Hamiltonian). *Under Assumptions G.1–G.2, consider spacetime region $\mathcal{M}$ with timelike/spacelike/null sheets and joints; total action*

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda)\, d^4x + S_{\text{GHY}}^{\text{tl/sp}} + S_{\mathcal{N}}^{\text{null}} + S_{\mathcal{C}}^{\text{corner}}$$

satisfies:

1. For all metric variations $\delta g_{\mu\nu}$ fixing induced geometry (and equivalent data on null boundaries) on boundary, total variation

$$\delta S = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu}\, d^4x,$$

yielding Einstein equations;

2. For given timelike boundary ^3B with timelike Killing vector ξ^a, Brown–York boundary stress tensor

$$T_{\text{BY}}^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}}$$

defines boundary Hamiltonian

$$H_{\partial}^{\text{grav}}[\xi] = \int_{B} \sqrt{\sigma}\, u_a\, T_{\text{BY}}^{ab}\, \xi_b\, d^2x$$

generating quasilocal time translation along ^3B time direction ξ^a; when ^3B extends to infinity, this Hamiltonian converges to ADM or Bondi mass.

Thus $H_{\partial}^{\text{grav}}$ viewable as third type of boundary time generator, at same level as Null–Modular and scattering end generators.

Definition 3.5 (Unified Boundary Time Generator). Given boundary triple $(\partial\mathcal{M}, \mathcal{A}_{\partial}, \omega_{\partial})$, call self-adjoint operator H_{∂} unified boundary time generator if:

1. In scattering representation, H_{∂}'s spectral decomposition measure for energy variable ω equivalent to $d\mu_{\partial}^{\text{scatt}}(\omega)$;

2. In algebraic representation, modular Hamiltonian satisfies $K_{\partial} = 2\pi\beta^{-1}H_{\partial}$ for some positive β, producing Tomita–Takesaki modular flow;

3. In geometric representation, Brown–York Hamiltonian writable as $H_{\partial}^{\text{grav}}[\xi] = \langle H_{\partial}, J[\xi] \rangle$ where $J[\xi]$ is boundary charge functional associated with Killing vector ξ.

Theorem 3.6 (D: Boundary Trinity Principle). *Assume following matching structure exists:*

1. Scattering system's incoming/outgoing channels embeddable into separable subalgebra of some QFT's boundary algebra $\mathcal{A}_{\partial}$, making scattering phase $\varphi(\omega)$ consistent with modular Hamiltonian's spectral phase in asymptotic regions;

2. QFT's stress–energy tensor expectation value connected to Brown–York boundary stress tensor via holographic dictionary or semiclassical Einstein equations;

3. Boundary Killing time and scattering energy normalization constants satisfy thermal time hypothesis normalization condition: modular flow parameter differs from physical time by constant scale.

Under these conditions, exists unique (up to global affine transformation) unified boundary time generator H_∂ making

Scattering time delay $\Longleftrightarrow$ Modular flow parameter $\Longleftrightarrow$ Brown–York boundary time

equivalent in common domain.

More specifically, positive constants c_1, c_2 exist making

$$\frac{\varphi'(\omega)}{\pi} = \xi'(\omega) = \frac{1}{2\pi} \operatorname{tr} Q(\omega),$$

$$K_D = 2\pi \int T_{\sigma\sigma} g_\sigma,$$

$$H_\partial^{\text{grav}}[\xi] = \int \sqrt{\sigma}\, u_a T_{\text{BY}}^{ab} \xi_b$$

satisfy

$$H_\partial = \int \omega \, d\mu_\partial^{\text{scatt}}(\omega) = c_1 K_D + c_2^{-1} H_\partial^{\text{grav}}$$

giving three realizations of same one-parameter group e^{-itH_∂} in different representations of same boundary Hilbert space.

Theorem D doesn't claim unified generator automatically constructible in arbitrary theories, but points out under above matching conditions, three threads naturally compress onto same boundary time object.

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4   Proofs

Provide proof skeletons of main theorems; finer technical details and special cases in appendices.

4.1   Proof Skeleton of Theorem A

Birman–Krein identity shows under Assumption S.1, spectral shift function $\xi(\omega)$ exists such that

$$\det S(\omega) = \exp(-2\pi i \xi(\omega)),$$

thus

$$\Phi(\omega) = \arg \det S(\omega) = -2\pi \xi(\omega).$$

Half-phase $\varphi(\omega) = -\pi \xi(\omega)$. Differentiating at smooth points:

$$\varphi'(\omega) = -\pi \xi'(\omega).$$

On other hand, Eisenbud–Wigner–Smith time delay theory shows: in potential scattering systems satisfying moderate decay and regularity conditions, Wigner–Smith time delay operator definable as

$$Q(\omega) = -i S(\omega)^\dagger \partial_\omega S(\omega),$$

proving its trace satisfies

$$\operatorname{tr} Q(\omega) = 2\pi \, \xi'(\omega).$$

Combining two equations gives

$$\frac{\varphi'(\omega)}{\pi} = \xi'(\omega) = \frac{1}{2\pi} \operatorname{tr} Q(\omega),$$

completing Theorem A proof. Rigorous treatment requires considering exceptional zero-measure sets on ω; see Appendix A.

[Proofs of Theorems B, C, D condensed for space...]

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5 Model Applications

Show unified boundary framework applicability in three typical physical scenarios: black hole thermodynamics, AdS/CFT, finite-scale scattering networks.

5.1 Black Hole Thermodynamics

For static black hole, event horizon is null boundary; surface gravity κ and Hawking temperature $T_{\mathrm{H}} = \kappa/(2\pi)$ jointly characterize thermal properties. QFT vacuum state outside horizon in wedge region's modular flow equivalent to Euclidean time translation around horizon, period being $\beta_{\mathrm{H}} = 1/T_{\mathrm{H}}$.

In unified boundary framework:

1. **Scattering end**: Consider fixed-energy scattering outside black hole; group delay $\operatorname{tr} Q(\omega)$ energy dependence encodes effective potential barrier and quasinormal mode structure near horizon;

2. **Null–Modular end**: Horizon itself viewable as part of Null–Modular double cover; modular Hamiltonian localized on null generator, proportional to T_{vv} integral;

3. **Gravity end**: For boundary two-sphere surrounding black hole, Brown–York quasilocal energy approaches ADM mass at infinity, forms Legendre structure with Bekenstein–Hawking entropy and temperature product near horizon.

Unified generator H_∂ gives compatible time translation and heat–geometry relations at three endpoints, restating black hole thermodynamics as purely boundary phenomenon.

5.2 AdS/CFT and Holographic Time Reconstruction

In AdS/CFT duality, boundary CFT time evolution generated by Hamiltonian H_{CFT}; modular Hamiltonian and generalized entropy extremal surfaces jointly determine bulk geometric response.

Unified boundary framework provides natural language: • **Scattering end**: Bulk high-energy process scattering delay corresponds to boundary CFT correlation function phase structure; • **Null–Modular end**: Boundary CFT spherical region modular flow corresponds via JLMS to bulk Killing flow and minimal surface; • **Gravity end**: These Killing flows generated by Brown–York boundary Hamiltonian, action readable from GHY term Hamilton–Jacobi variation.

Thus "holographic time reconstruction" understandable as unified generator H_∂'s two representations on CFT and AdS sides.

5.3 Finite-Scale Scattering Networks and Experimental Testability

In electromagnetic scattering and wave networks, Wigner–Smith time delay matrix $Q(\omega)$ already used to analyze complex medium group delay, mode focusing, topological phase.

Unified framework gives experimental route:

1. Measure $S(\omega)$ in microwave or optical networks; obtain $Q(\omega)$ and trace via numerical differentiation and spectral decomposition; construct boundary time measure $d\mu_\partial^{\mathrm{scatt}}$;

2. View network as discretized "null surface"; construct effective modular Hamiltonian for diamond-like regions via input/output port partitioning (approximable by second-order statistics or principal mode decomposition);

3. Construct geometric/topological analog "gravitational boundary" using effective medium or curved waveguides; define Brown–York-like quasilocal energy via energy storage and Q-factor changes.

If three readouts give consistent time scale under appropriate normalization, viewable as experimental fingerprint of unified boundary time generator H_∂.

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6 Engineering Proposals

For implementing unified boundary framework as testable engineering schemes, this section proposes three types of experimental routes implementable on mesoscale platforms.

6.1 Boundary Time Measure in Multi-Port Scattering Networks

In multi-port microwave scattering networks, can already high-precision measure frequency-dependent $S(\omega)$, construct Wigner–Smith time delay matrix $Q(\omega)$.

Engineering steps: 1. Design networks with different "interior domain" structures but same port geometry and external boundary; 2. Measure $\operatorname{tr} Q(\omega)$ for each network; obtain "effective residence time" scale via frequency integration; 3. Compare different structures' effects on $d\mu_\partial^{\mathrm{scatt}}$; test whether depends only on boundary conditions and spectral shift, robust to internal details.

Directly tests scattering end scale identity and boundary time's "outer boundary dominance."

6.2 Null Surface Simulation and Null–Modular Markov Property

In qubit chains or cold atom 1+1D systems, can construct approximate "null surface cuts": select spacetime region family whose future boundaries fall on approximate null lines; measure corresponding subsystem modular Hamiltonian approximations or conditional mutual information.

Engineering proposal: 1. In controllable many-body systems, implement time-dependent measurements or local couplings on spatial chain segments to effectively "simulate" modular Hamiltonians; 2. Construct overlapping segment triples (A, B, C); measure conditional mutual information $I(A : C \mid B)$; 3. Test whether $I(A : C \mid B)$ scaling approaches Markov saturation in "null surface approximation" limit, indirectly verifying Null–Modular inclusion-exclusion structure.

6.3 Quasilocal Energy and Geometric Boundary Simulation

In general relativity-like simulation platforms (e.g., superfluids, acoustic black holes, refractive index engineered optical media), can define effective metric bulk and boundary interfaces. Using energy conservation and reflection/transmission coefficients, can construct Brown–York-like quasilocal energy readouts.

Comparing these readouts with scattering end's $d\mu_{\partial}^{\text{scatt}}$ and Null–Modular structure can test whether "geometric boundary" and "spectral/modular boundary" unified scale experimentally distinguishable.

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7 Discussion (Risks, Boundaries, Past Work)

7.1 Applicability Domain and Potential Risks

Though unified boundary framework rigorous in intersection region of three mature theories, has clear applicability boundaries in several aspects:

1. **Scattering end limitations**: Theorem A largely depends on trace-class perturbations and good high-energy behavior; for long-range potentials, resonance accumulation, strong-coupling non-perturbative systems, spectral shift function and time delay trace formula applicability needs re-examination.

2. **Modular theory end limitations**: Null–Modular localization and Markov property currently rigorously proven mainly in conformal field theory and free field models; for general interacting QFT, especially non-local or non-Lorentz invariant theories, requires further generalization.

3. **Gravity end limitations**: GHY and Brown–York forms systematically analyzed in classical GR and some modified gravity, but boundary term definition at full quantum gravity or Feynman path integral level still has ambiguities, especially near topology changes or singularities.

Crossing these boundaries, existence and uniqueness of unified time generator H_{∂} requires additional assumptions or new mathematical tools.

7.2 Relation to Existing Work

This paper's framework viewable as "boundary reformulation" of several mature directions: • On scattering theory side, Birman–Krein, spectral shift function, Wigner–Smith time delay relations long used for spectral theory and conductivity properties research; • On modular theory side, spherical region and null surface modular Hamiltonians, ANEC/QNEC, Markov property become core topics combining relativistic QFT with quantum information; • On gravity side, GHY boundary term, Brown–York quasilocal energy, holographic entropy/relative entropy relations form foundation of modern gravity and holography.

This paper's contribution: View time scale unification as boundary spectral measure; view modular flow and Brown–York Hamiltonian as different representations of same generator; conceptually construct "boundary trinity" paradigm.

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8   Conclusion

Propose and construct unified framework with boundary as sole stage, viewing scattering time delay, Null–Modular modular flow, GHY–Brown–York gravitational boundary term as three representations of same boundary time generator H_∂.

Main conclusions:

1. In trace-class perturbation scattering theory, scattering half-phase derivative, Krein spectral shift density, Wigner–Smith delay matrix trace constitute unified boundary time measure;

2. In algebraic quantum field theory, modular Hamiltonians on causal diamonds and null surfaces completely localizable on Null–Modular double cover, satisfying Markov property and inclusion-exclusion law for overlapping diamond families;

3. In general relativity, action including GHY, null boundary, corner terms well-defined for variations fixing boundary geometry; Brown–York boundary Hamiltonian generates time translation as geometric boundary time;

4. With suitable matching maps, can construct unified H_∂ making scattering time delay, modular flow parameter, Brown–York boundary time equivalent on same spectral measure, restating "time–algebra–geometry" simultaneously as different aspects of boundary data.

This framework provides new angles for many open problems including black hole information, cosmological horizon time scales, mesoscale platform realizable boundary physics experiments.

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References

[1] Birman–Krein formula and spectral shift theory.

[2] Wigner–Smith time delay: various references.

[3] Tomita–Takesaki modular theory: Casini et al. papers.

[4] GHY boundary term: Gibbons, Hawking, York (1977).

[5] Brown–York quasilocal energy: Brown & York, PRD (1993).

[6] Null surface modular Hamiltonians: Casini, Teste, Torroba.

[7] Gravitational action with null boundaries: Lehner et al.

A   Spectral Shift, Phase, Time Delay Unification

[Detailed Birman–Krein formula derivation, Wigner–Smith trace formula...]

B   Null Surface Modular Hamiltonian and Markov Property

[Casini–Teste–Torroba results, inclusion-exclusion proofs...]

C GHY Boundary Term, Null Boundaries, Brown–York Energy

[Classical GHY variation calculation, null boundary and corner terms...]