Phase Derivative, Spectral Density and Windowed Readout: Unified Measurement Framework

Auric

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Abstract

Establish unified framework connecting scattering phase derivative, spectral density, and windowed readout. Core formula holding a.e.:

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\operatorname{tr} Q(E)}$$

where φ scattering phase, ρ_{rel} relative spectral density from Birman–Kreĭn $\det S = e^{-2\pi i\xi}$ with $\rho_{\mathrm{rel}} = -\xi'$, $Q = -iS^\dagger\partial_E S$ Wigner–Smith delay matrix.

Windowed readout $\mathcal{R}_w = \int w(E)[h * \rho_{\mathrm{rel}}](E)dE$ with NPE three-term error decomposition. Applications: quantum metrology, scattering theory, condensed matter.

1 Core Definitions

Definition 1.1 (Relative Spectral Density)**.** For scattering pair (H, H_0) with $S(E)$ scattering matrix:

$$\rho_{\mathrm{rel}}(E) = -\xi'(E) = \frac{1}{2\pi i}\operatorname{tr}(S^\dagger\partial_E S) = \frac{1}{2\pi}\operatorname{tr} Q(E)$$

where ξ spectral shift function, Q Wigner–Smith delay.

Definition 1.2 (Windowed Readout)**.** For window w, kernel h:

$$\mathcal{R}_w[\rho_{\mathrm{rel}}] = \int_{\mathbb{R}} w(E)\,[h * \rho_{\mathrm{rel}}](E)\,dE$$

2 Main Theorems

Theorem 2.1 (Phase–Density Unification)**.** *On absolutely continuous spectrum a.e., single-channel $S = e^{2i\varphi}$ gives*

$$\varphi'(E) = \pi\,\rho_{\mathrm{rel}}(E) = \tfrac{1}{2}\operatorname{tr} Q(E).$$

Multi-channel: $\frac{1}{2\pi}\operatorname{tr} Q(E) = \rho_{\mathrm{rel}}(E) = -\xi'(E)$.

Proof. From Birman–Kreĭn $\det S = e^{-2\pi i\xi}$ get $\partial_E \operatorname{Arg}\det S = -2\pi\xi'$. Jacobi formula $\partial_E \log\det S = \operatorname{tr}(S^{-1}\partial_E S)$ with unitarity $S^{-1} = S^\dagger$ gives $\operatorname{tr}(S^\dagger\partial_E S) = -2\pi i\xi'$. Definition $Q = -iS^\dagger\partial_E S$ yields $\operatorname{tr} Q = 2\pi\xi'$, thus $\rho_{\mathrm{rel}} = -\xi' = (2\pi)^{-1}\operatorname{tr} Q$. □

Theorem 2.2 (Windowed Readout NPE Decomposition)**.** *For discrete approximation with step Δ, truncation N:*

$$|\mathcal{R}_w - \widehat{\mathcal{R}}_w| \leq |\varepsilon_{\text{alias}}| + |\varepsilon_{\text{EM}}| + |\varepsilon_{\text{tail}}|$$

with $\varepsilon_{\text{alias}} = 0$ when bandlimited + Nyquist $\Delta \leq \pi/\Omega$.

Theorem 2.3 (Born Probability as I-Projection)**.** *Under alignment condition, Born probability $p_i = \langle \psi, E_i \psi \rangle$ equals I-projection minimizing $D_{\text{KL}}(p\|q)$ over constraint set.*

Theorem 2.4 (Pointer Basis as Ky Fan Minimum)**.** *Pointer basis $\{e_k\}$ minimizes $\sum_k \langle e_k, W_w e_k \rangle$ for window operator $W_w = \int w(E) dE_A(E)$ (Ky Fan minimum sum).*

3 Applications

3.1 Quantum Metrology

Phase derivative measurement via windowed readout provides optimal energy estimation within bandwidth constraints.

3.2 Scattering Theory

Wigner–Smith delay directly measurable via phase–energy correlation, connection to Friedel sum rule.

3.3 Condensed Matter

Local density of states (LDOS) in mesoscopic systems, quantum point contacts, resonant tunneling.

4 Discussion and Outlook

Unified framework established connecting:

- Phase derivative φ' (observable)
- Spectral density ρ_{rel} (theoretical)
- Delay trace $\operatorname{tr} Q$ (dynamical)

via Birman–Kreĭn–Wigner–Smith chain, with windowed readout providing experimental bridge.

Key achievements:

1. Rigorous scale identity formula
2. NPE non-asymptotic error closure
3. Information-geometric Born probability
4. Ky Fan pointer basis characterization

Future directions:

- Extension to time-dependent scattering

- Open quantum systems and decoherence
- Experimental implementations in quantum optics
- Connections to quantum field theory and gravity