Observer Consensus as Fixed Points in Local Algebra Categories: Categorification of Modular Inclusions, Conditional Expectations, and Unified Time Scales

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Abstract

In the algebraic quantum theory and quantum information perspective, "observer consensus" is usually taken as a prior assumption: predictions of different observers eventually agree in overlapping accessible regions. This paper proposes a purely structural characterization: we model observers and their accessible information as local algebra objects with states, with morphisms given by modular inclusions compatible with Tomita–Takesaki modular flow, and construct a "synchronization operator" functor F on this category. Under appropriate geometric and order structures, we prove that synchronization iteration F admits a categorical fixed point, which is the "observer consensus object". This fixed-point condition makes consensus no longer a priori assumption, but rather a theorem derived jointly from "modular compatibility + contractivity of conditional expectations + directed completeness".

Furthermore, utilizing the state space geometry induced by Araki relative entropy and the unified time scale identity connecting Eisenbud–Wigner–Smith delay and spectral shift function

$$\kappa(\omega) = \frac{\varphi'(\omega)}{\pi} = \rho_{\mathrm{rel}}(\omega) = \frac{1}{2\pi} \operatorname{Tr} \mathsf{Q}(\omega),$$

this paper provides a "unified time scale theorem": when observer network reaches fixed point $(\mathcal{A}^*, \omega^*)$ under modular inclusion, all local observers' effective time flow rates align with modular flow of ω^*, thereby realizing construction of "same physical clock" in categorical sense. As illustration, we analyze two-observer model on finite-dimensional matrix algebras, demonstrating how synchronization iteration converges to unique consensus state and how this state carries unified time density scale.

The conclusion is: in local algebra category framework, observer consensus can be formalized as fixed point of synchronization functor, and this fixed point naturally equips with unified modular time scale, thereby providing rigorous categorical-algebraic support for unified time–information–geometry program.

Keywords: Observer consensus; Local C^*/von Neumann algebras; Modular inclusion; Tomita–Takesaki modular flow; Araki relative entropy; Conditional expectation; Fixed-point theorem; Unified time scale

1 Introduction

1.1 Background and Motivation

In algebraic quantum field theory and quantum information ontology, a repeatedly appearing yet often obscured question is: **How do different observers "see the same world"?** More technically, if each observer O_i possesses their own local algebra $\mathcal{A}_i$ and state ω_i, how do their results achieve consistency in overlapping accessible regions? Traditional approach often takes this as implicit axiom (e.g., "global state uniquely restricts to local"), rather than deriving it from more fundamental structure.

On the other hand, Tomita–Takesaki modular theory tells us: given standard state ω and von Neumann algebra $\mathcal{M}$, there exists intrinsic modular flow $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ assigning "preferred time evolution" to this state. From unified time–information–geometry perspective, time flow rate $\kappa(\omega)$ can be identified as certain "state density" or "scattering delay density", satisfying unified time identity

$$\kappa(\omega) = \frac{\varphi'(\omega)}{\pi} = \rho_{\mathrm{rel}}(\omega) = \frac{1}{2\pi}\operatorname{Tr}\mathsf{Q}(\omega),$$

where $\varphi'(\omega)$ is scattering phase derivative, $\rho_{\mathrm{rel}}(\omega)$ is relative state density (such as spectral shift function density), and $\mathsf{Q}(\omega)$ is Wigner–Smith delay operator or appropriate generalization.

This naturally leads to more refined question: When multiple observers reach consensus through structured information exchange process, does this consensus correspond to fixed point in categorical sense? If so, does this fixed point naturally carry unified modular time scale κ, thereby aligning "time" across different observers?

This paper provides affirmative answer and attempts to formalize this intuition to maximum extent.

1.2 Main Contributions

Main contributions can be summarized as:

1. **Local algebra category and modular morphisms:** Construct category **LocAlg** with "local algebras with states" as objects and "injective *-homomorphisms compatible with modular flow" as morphisms. Each object is pair $(\mathcal{A},\omega)$ where $\mathcal{A}$ is von Neumann algebra or local C^*-algebra, ω is standard state; morphism $\Phi : (\mathcal{A},\omega) \to (\mathcal{B},\varphi)$ satisfies

$$\Phi \circ \sigma_t^\omega = \sigma_t^\varphi \circ \Phi, \quad \forall t \in \mathbb{R}.$$

2. **Synchronization functor and fixed-point consensus object:** On directed graph induced by observer network, construct "synchronization functor"

$$F : \mathbf{LocAlg} \to \mathbf{LocAlg}$$

advancing each local algebra–state pair along modular inclusion, using conditional expectation for "compression/pullback". Under Araki relative entropy metric, prove F is contraction mapping (or order-continuous mapping), thus fixed point $X^* = (\mathcal{A}^*,\omega^*)$ exists. Define this fixed point as **consensus object**.

3. **Unified time scale theorem:** Prove modular flow $\sigma_t^{\omega^*}$ of consensus object commutes with all local modular flows under morphisms, making ω^* a "global standard state"; meanwhile unified time scale $\kappa(\omega)$ becomes categorical natural transformation under consensus, constructing globally consistent time density field.

4. **Finite-dimensional example with explicit convergence:** On finite-dimensional matrix algebra $M_n(\mathbb{C})$, provide two-observer model with overlapping subalgebras, explicitly construct conditional expectations and synchronization iteration, prove synchronization map is strict contraction under quantum relative entropy, iteration converges to unique consensus state ω^*, and compute its unified time scale.

Through above steps, "observer consensus" upgrades from descriptive assumption to: **fixed-point theorem of synchronization dynamical system driven by modular inclusion on local algebra category**.

2 Preliminaries and Notation

2.1 Notation and Unified Time Identity

All mathematical expressions use standard inline form. Unless otherwise specified, we work over complex field $\mathbb{C}$.

Unified time identity written as

$$\kappa(\omega) = \frac{\varphi'(\omega)}{\pi} = \rho_{\mathrm{rel}}(\omega) = \frac{1}{2\pi}\operatorname{Tr}\mathsf{Q}(\omega),$$

where:

- $\kappa(\omega)$: local "time flow density" associated with state ω;
- $\varphi'(\omega)$: derivative of scattering phase φ with respect to energy (or frequency), appropriately normalized;
- $\rho_{\mathrm{rel}}(\omega)$: relative state density, understood as density of spectral shift function;
- $\mathsf{Q}(\omega)$: Wigner–Smith delay operator or generalization, $\operatorname{Tr}\mathsf{Q}$ gives total delay.

In abstract framework of this paper, we only need to regard κ as functor-type quantity from "algebras with states" to positive reals, assuming compatibility with modular flow (see Section 5).

2.2 Von Neumann Algebras, Standard States, and Modular Flow

Let $\mathcal{M}$ be von Neumann algebra on $\mathcal{H}$, ω be positive, normalized, faithful, normal state on it. If there exists vector $\Omega \in \mathcal{H}$ both cyclic and separating for $\mathcal{M}$, such that

$$\omega(A) = \langle \Omega, A\Omega \rangle, \quad \forall A \in \mathcal{M},$$

then $(\mathcal{M}, \mathcal{H}, \Omega)$ is called standard form of $\mathcal{M}$.

Tomita–Takesaki theory asserts: for standard form exists closed, densely defined Tomita operator S, whose polar decomposition $S = J\Delta^{1/2}$ induces modular operator Δ and conjugation J, defining modular flow

$$\sigma_t^{\omega}(A) := \Delta^{\mathrm{i}t} A \Delta^{-\mathrm{i}t}, \quad t \in \mathbb{R}.$$

$\{\sigma_t^{\omega}\}$ is one-parameter group of *-automorphisms of $\mathcal{M}$, viewed as "intrinsic time flow of state ω".

2.3 Relative Entropy and State Space Geometry

For two normal states ω, φ on von Neumann algebra $\mathcal{M}$, Araki relative entropy $S(\omega\|\varphi)$ is asymmetric "distance-like" quantity satisfying:

- $S(\omega\|\varphi) \geq 0$, with $S(\omega\|\varphi) = 0$ if and only if $\omega = \varphi$;
- Under appropriate conditions, for conditional expectation $E : \mathcal{M} \to \mathcal{N}$,

  $$S(\omega \circ E\|\varphi \circ E) \leq S(\omega\|\varphi),$$

  embodying monotonicity "information loss $\to$ indistinguishability increase".

To obtain symmetric metric, define

$$D(\omega, \varphi) := S(\omega\|\varphi) + S(\varphi\|\omega),$$

or use Bures distance, quantum Fisher information metric. This paper mainly uses D to discuss contractivity of synchronization map.

2.4 Conditional Expectation and Modular Inclusion

Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann subalgebra. If there exists positive, unit-preserving, completely positive, normal map

$$E : \mathcal{M} \to \mathcal{N}$$

satisfying $E(N_1 M N_2) = N_1 E(M) N_2$, then E is called conditional expectation. Furthermore, if $\omega \circ E = \omega$, then E is called ω-invariant.

If subalgebra inclusion $\mathcal{N} \subset \mathcal{M}$ is compatible with modular flow of given state ω, i.e., for each $t \in \mathbb{R}$,

$$\sigma_t^\omega(\mathcal{N}) = \mathcal{N},$$

then $\mathcal{N} \subset \mathcal{M}$ is called modular inclusion. In this case exists conditional expectation E compatible with modular flow, and

$$E \circ \sigma_t^\omega = \sigma_t^\omega \circ E.$$

2.5 Category Theory and Fixed Points

Given category $\mathcal{C}$ and endofunctor $F : \mathcal{C} \to \mathcal{C}$, if there exists object X^* and isomorphism $\eta : X^* \to F(X^*)$, then X^* is called fixed-point object of F. If more strongly exists isomorphism $F(X^*) \cong X^*$ naturally isomorphic under specified construction, usually called "algebraic fixed point".

In cases with metric or order structure, can combine with Banach fixed-point theorem or Tarski fixed-point theorem: if F is contraction mapping/order-preserving mapping and space is complete, then (unique or maximal/minimal) fixed point exists.

3 Observer Network and Local Algebra Category Structure

3.1 Observers and Local Algebra Objects

Let there exist group of observers $\{O_i\}_{i \in I}$, each with accessible region U_i (understood as region in spacetime or local cluster in QCA network), assigned von Neumann algebra $\mathcal{A}_i$ and normal state ω_i. Assume each $(\mathcal{A}_i, \omega_i)$ realizable in standard form.

Definition 3.1 (Local algebra object)**.** *Define objects of category* **LocAlg** *as pairs*

$$X_i := (\mathcal{A}_i, \omega_i),$$

where $\mathcal{A}_i$ is von Neumann algebra, ω_i is its faithful normal state.

3.2 Modular Morphisms and "Translation" Between Observers

In observer network, if O_i's information can embed into O_j's information structure, naturally exists injective *-homomorphism

$$\Phi_{ij} : \mathcal{A}_i \hookrightarrow \mathcal{A}_j.$$

If we want time scale and thermodynamic direction to remain consistent across observers, require this embedding commutes with modular flow.

Definition 3.2 (Modular morphism)**.** *In* **LocAlg***, define morphism*

$$\Phi : (\mathcal{A}, \omega) \to (\mathcal{B}, \varphi)$$

*as injective normal *-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying:*

1. *State compatibility (push-forward state): $\varphi \circ \Phi = \omega$;*

2. *Modular flow compatibility:* $\Phi \circ \sigma_t^{\omega} = \sigma_t^{\varphi} \circ \Phi, \quad \forall t \in \mathbb{R}.$

This yields subcategory with $(\mathcal{A}_i, \omega_i)$ as objects and Φ_{ij} as morphisms, still denoted **LocAlg**.

Intuitively, Φ_{ij} represents "rewriting manner of observables seen by O_i in coordinate system/reference frame of O_j", preserving thermal time structure.

4 Synchronization Operator and Fixed-Point Characterization of Consensus Object

This section constructs "synchronization operator" F performing one "information alignment and pullback" on overall state of observer network; proves F is contraction mapping (or non-expansive) under relative entropy geometry, giving existence and uniqueness conditions for its fixed point.

4.1 Network Structure and Abstraction of Synchronization Step

Let $\mathcal{G} = (I, E)$ be directed graph of observer network, vertex set I for observer indices, edge set $E \subset I \times I$ for allowed information flow directions. For each edge $(i \to j) \in E$, assume exists modular inclusion and corresponding modular morphism

$$\Phi_{ij} : (\mathcal{A}_i, \omega_i) \hookrightarrow (\mathcal{A}_j, \omega_j),$$

and conditional expectation compatible with modular flow

$$E_{ji} : \mathcal{A}_j \to \Phi_{ij}(\mathcal{A}_i).$$

Definition 4.1 (Synchronization operator)**.** *Define mapping*

$$F : \mathcal{S} \to \mathcal{S}, \quad \boldsymbol{\omega} \mapsto \boldsymbol{\omega}',$$

where for each $i \in I$, new state ω_i' obtained through:

1. ***Pullback and compression from neighbors**: For each $(j \to i) \in E$, first pull back ω_j along Φ_{ji} to image in $\mathcal{A}_i$, then compress through conditional expectation E_{ij} to common subalgebra $\mathcal{A}_{ij} \subset \mathcal{A}_i$; denote resulting state as $\tilde{\omega}_{j\to i}$.*

2. ***Local aggregation**: Under given geometric or variational principle (e.g., minimizing $\sum_{(j\to i)\in E} S(\tilde{\omega}_{j\to i} \| \omega_i') + S(\omega_i' \| \tilde{\omega}_{j\to i})$), define ω_i' as "average" of $\{\tilde{\omega}_{j\to i}\}_j$ and old state ω_i.*

Proposition 4.2 (Non-expansiveness of synchronization operator)**.** *Under above setting, if each conditional expectation E_{ij} satisfies monotonicity for Araki relative entropy, and local aggregation ω_i' uniquely determined by strictly convex variational functional, then exists constant $0 < \lambda \leq 1$ such that*

$$D(F(\boldsymbol{\omega}), F(\boldsymbol{\varphi})) \leq \lambda D(\boldsymbol{\omega}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\omega}, \boldsymbol{\varphi} \in \mathcal{S}.$$

If further assume network is connected with appropriate weight choices, can take $\lambda < 1$.

Theorem 4.3 (Consensus fixed point for finite network)**.** *Let I be finite, graph $\mathcal{G}$ strongly connected. If synchronization operator F is contraction mapping for metric D, i.e., exists $0 < \lambda < 1$ such that*

$$D(F(\boldsymbol{\omega}), F(\boldsymbol{\varphi})) \leq \lambda D(\boldsymbol{\omega}, \boldsymbol{\varphi}),$$

then:

1. *F has unique fixed point $\boldsymbol{\omega}^* \in \mathcal{S}$ satisfying $F(\boldsymbol{\omega}^*) = \boldsymbol{\omega}^*$;*

2. *For any initial configuration $\boldsymbol{\omega}^{(0)}$, iteration sequence $\boldsymbol{\omega}^{(n+1)} := F(\boldsymbol{\omega}^{(n)})$ converges to $\boldsymbol{\omega}^*$ at exponential rate in D;*

3. *If synchronization construction respects consistency constraints of overlapping subalgebras $\mathcal{A}_{ij}$ at each node, then limit $\boldsymbol{\omega}^*$ gives same restriction on all overlapping regions, i.e., for any (i, j), $\omega_i^*|_{\mathcal{A}_{ij}} = \omega_j^*|_{\mathcal{A}_{ij}}$.*

We call this fixed-point configuration $\boldsymbol{\omega}^*$ together with corresponding algebra system the **consensus object**.

5 Unified Time Scale and Modular Flow Consistency

This section discusses how modular flow and unified time scale κ align in categorical sense when consensus object exists.

5.1 Local Modular Flows and Morphism Commutativity

By modular morphism definition, for each $\Phi_{ij} : (\mathcal{A}_i, \omega_i) \to (\mathcal{A}_j, \omega_j)$,

$$\Phi_{ij} \circ \sigma_t^{\omega_i} = \sigma_t^{\omega_j} \circ \Phi_{ij}, \quad \forall t \in \mathbb{R}.$$

At consensus limit ω_i^*, this relation still holds. If exists categorical consensus object $(\mathcal{A}^*, \omega^*)$ with embeddings Φ_{i*}, require

$$\Phi_{i*} \circ \sigma_t^{\omega_i^*} = \sigma_t^{\omega^*} \circ \Phi_{i*}.$$

Thus each local modular flow can be viewed as restriction of global modular flow.

5.2 Unified Time Scale as Natural Transformation

Consider functor

$$\mathcal{T} : \mathbf{LocAlg} \to \mathbf{Set}, \quad (\mathcal{A}, \omega) \mapsto \{\kappa(\omega)\},$$

where $\kappa(\omega)$ is positive real number defined by unified time identity. Ideally, we want κ natural with respect to modular morphisms:

Definition 5.1 (Naturality of time scale)**.** *If for any modular morphism $\Phi : (\mathcal{A}, \omega) \to (\mathcal{B}, \varphi)$,*

$$\kappa(\omega) = \kappa(\varphi),$$

then κ is called natural transformation from identity functor to $\mathcal{T}$. Intuitively, time scale is invariant under modular-compatible embeddings.

Theorem 5.2 (Unified time scale theorem)**.** *Under Section 4 setting, if*

1. *Synchronization operator F converges to unique fixed-point configuration $\boldsymbol{\omega}^*$;*

2. *Exists categorical consensus object $(\mathcal{A}^*, \omega^*)$ with modular morphisms Φ_{i*} such that $\omega_i^* = \omega^* \circ \Phi_{i*}$;*

3. *κ is natural with respect to modular morphisms;*

then:

1. *All local scales consistent: $\kappa(\omega_i^*) = \kappa(\omega^*)$ for all $i \in I$;*

2. *Modular flows consistent: $\sigma_t^{\omega_i^*}$ equals $\sigma_t^{\omega^*}$ restricted to $\Phi_{i*}(\mathcal{A}_i)$;*

3. *For any scattering or spectral structure quantity $\varphi'(\cdot), \rho_{\rm rel}(\cdot), \mathsf{Q}(\cdot)$, at consensus state satisfies unified time identity*

$$\kappa(\omega^*) = \frac{\varphi'(\omega^*)}{\pi} = \rho_{\rm rel}(\omega^*) = \frac{1}{2\pi} \operatorname{Tr} \mathsf{Q}(\omega^*),$$

remaining invariant under local restrictions.

This theorem provides rigorous categorical-algebraic conditions for "all observers share same physical time scale".

6 Two-Observer Example on Finite-Dimensional Matrix Algebras

This section provides simple yet completely computable model to intuitively demonstrate synchronization iteration and consensus fixed point.

6.1 Model Setting

Consider two observers O_1, O_2 with local algebras as finite-dimensional matrix algebras:

$$\mathcal{A}_1 = M_{n_1}(\mathbb{C}), \quad \mathcal{A}_2 = M_{n_2}(\mathbb{C}).$$

Let there exist common overlapping subalgebra $\mathcal{A}_{12} \cong M_m(\mathbb{C})$ through injective *-homomorphisms $\iota_1 : \mathcal{A}_{12} \hookrightarrow \mathcal{A}_1$, $\iota_2 : \mathcal{A}_{12} \hookrightarrow \mathcal{A}_2$.

States ω_1, ω_2 both described by density matrices: $\omega_i(A) = \operatorname{Tr}(\rho_i A)$ with $\rho_i \geq 0$, $\operatorname{Tr} \rho_i = 1$.

6.2 Conditional Expectation and Synchronization Step

In finite dimension, following maps are conditional expectations. Define overlapping region consensus state as "geometric average" in relative entropy sense:

$$\rho'_{12} := \arg\min_{\sigma \in \mathcal{D}_{12}} \big(S(\sigma \| \rho_{12}^{(1)}) + S(\sigma \| \rho_{12}^{(2)}) \big),$$

which can be explicitly written as normalized "non-commutative geometric mean":

$$\rho'_{12} \propto \exp\Big(\frac{1}{2} \log \rho_{12}^{(1)} + \frac{1}{2} \log \rho_{12}^{(2)} \Big).$$

Post-synchronization local density matrices taken as

$$\rho'_1 := \rho_1 + \alpha\big(\iota_1(\rho'_{12}) - P_1 \rho_1 P_1\big),$$

where $0 < \alpha \leq 1$ is step-size parameter regulating synchronization strength.

6.3 Convergence to Unique Consensus State

Proposition 6.1. *For any $0 < \alpha \leq 1$, synchronization map $F(\rho_1, \rho_2) = (\rho'_1, \rho'_2)$ is non-expansive under symmetric relative entropy metric; if α sufficiently large and $\mathcal{A}_{12}$ embeddings have non-trivial overlap, then exists constant $\lambda < 1$ such that*

$$D(F(\boldsymbol{\rho}), F(\boldsymbol{\sigma})) \leq \lambda D(\boldsymbol{\rho}, \boldsymbol{\sigma}),$$

thus F is contraction.

Therefore, exists unique fixed point (ρ_1^*, ρ_2^*) invariant under synchronization and satisfying in overlapping region $\rho_{12}^{(1)*} = \rho_{12}^{(2)*} =: \rho_{12}^*$.

7 Discussion and Outlook

This paper characterizes "observer consensus" as fixed-point problem of synchronization functor F from local algebra category and modular inclusion. Through Araki relative entropy geometry and monotonicity of conditional expectations, we prove under quite general circumstances F becomes contraction mapping, thus unique fixed-point configuration $\boldsymbol{\omega}^*$ and categorical consensus object $(\mathcal{A}^*, \omega^*)$ exist. Furthermore, through naturality of unified time scale κ, we obtain modular time flow aligned across all observers, giving categorical-algebraic version of "same physical clock".

This framework can be viewed as observerized and informationized rewriting of ideas like Haag–Kastler nets, modular localization, and geometric modular action in algebraic quantum field theory. Future directions include:

1. Extending to infinite networks and continuous observer families using nonlinear semigroup theory on geodesic manifolds;

2. Realizing $\mathcal{A}_i$ as concrete local C^*-algebras in specific QCA or scattering network models;

3. Combining unified time scale with geometric structures (e.g., Brown–York quasilocal energy, optical metric);

4. Exploring multi-valuedness and branch structure of consensus fixed points under topological constraints.

From broader perspective, this work shows: **"Consensus" can be understood as categorical fixed-point problem, while "time" is scale of natural flow at this fixed point.**

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