

Unified Framework of Boundary Time–Topology–Scattering: From \mathbb{Z}_2 Holonomy and K^1 Uniqueness to Cosmological Constant and Phase–Frequency Metrology

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Abstract

This paper constructs a unified framework centered on “boundary time scale,” gluing the following seemingly disparate structures into a single theory: (1) Local quantum sufficient conditions on small causal diamonds and nonlinear Einstein equations; (2) \mathbb{Z}_2 holonomy in Null–Modular double covers and relative cohomology class $[K]$ selected by BF bulk integration; (3) Family-level unification of restricted principal bundles–scattering– K^1 and the “natural transformation unique up to integer multiples” consistency factory; (4) Relative topology on punctured information manifolds and $S(U(3) \times U(2)) \cong (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$ reduction; (5) Windowed formulation of phase–spectral shift–state density–cosmological constant and the unified role of relative scattering determinant in quantum gravity; (6) Cross-platform metrology paradigm with “phase–frequency” as the sole read-out in FRB propagation, δ -ring–AB flux, and topological endpoint scattering; (7) Gibbons–Hawking–York boundary terms and their corner, null, and Lovelock generalizations providing variational well-posedness and quasilocal energy; (8) “Boundary as clock”: time as unified translation operator of phase–spectral shift–modular flow; (9) Quantum–classical bridge on time scale: equivalence relations among phase, proper time, scattering group delay, cosmological redshift, and boundary entropy geometry.

The core scale identity

$$\frac{\varphi'(\omega)}{\pi} = \rho_{\text{rel}}(\omega) = \frac{1}{2\pi} \text{tr} \mathbf{Q}(\omega), \quad \mathbf{Q}(\omega) = -iS(\omega)^\dagger \partial_\omega S(\omega),$$

unifies the derivative of total scattering phase, relative state density, and Wigner–Smith group delay trace as the same time scale. Taking the product $Y = M \times X^\circ$ on small causal diamonds with boundary $B_\ell(p)$ and parameter space X° , encoding the relative cohomology class $[K] \in H^2(Y, \partial Y; \mathbb{Z}_2)$ as the composite obstruction of \mathbb{Z}_2 holonomy, scattering line bundle torsion, and $w_2(TM)$, we prove under appropriate geometric–quantum energy conditions and “Modular–Scattering Alignment” hypothesis:

- Local nonlinear gravity equations $G_{ab} + \Lambda g_{ab} = 8\pi G \langle T_{ab} \rangle$ and second-order relative entropy non-negativity are equivalent to $[K] = 0$, further equivalent to triviality of \mathbb{Z}_2 holonomy of $\sqrt{\det_p S}$ on all physical loops;
- Family-level natural transformations of restricted principal bundles–scattering– K^1 are unique up to integer multiples under minimal axioms and Birman–Krein normalization, normalized to +1 yielding a canonical scale from scattering families to K^1 ;
- Riesz spectral projections on punctured information manifolds reduce Uhlmann principal bundles to $S(U(3) \times U(2))$, unifying Yukawa mass vortex index and charge \mathbb{Z}_6 structure via relative K -theory boundary maps;
- Relative scattering determinant and windowed Tauberian formulas for heat kernel–DOS–phase strictly align cosmological constant bulk slope, black hole pole spectroscopy, and observation-end phase–frequency kernel Ξ_W ;
- FRB vacuum polarization, δ -ring–AB flux, and topological endpoint scattering share the same phase–frequency metrology mother kernel under “finite-order Euler–Maclaurin + Poisson discipline,” yielding cross-platform upper bounds and critical coupling metrology protocols.

On boundary algebra \mathcal{A}_∂ , faithful state ω , and Tomita–Takesaki modular flow σ_t^ω , time is characterized as the boundary translation operator $U(t) = e^{-itH_\partial}$ unique (up to affine) aligning modular flow with scattering time scale, whose time unit is fixed by the above scale identity. On the geometric end, proper time, gravitational time delay, and cosmological redshift correspond respectively to phase along world-lines, scattering group delay, and phase rhythm ratio under this scale; extremality and monotonicity of generalized entropy yield the entropy-geometric form of Einstein equations on small causal diamonds.

Keywords: Boundary Time Scale; \mathbb{Z}_2 Holonomy; Restricted Principal Bundle; K^1 Uniqueness; Relative Scattering Determinant; Cosmological Constant; FRB Phase–Frequency Metrology; GHY Boundary Term; Modular Flow; Generalized Entropy

1 Introduction and Historical Context

Scattering theory, topological K -theory, and quantum gravity have each formed mature theoretical frameworks over the past decades. The Birman–Krein spectral shift function and determinant characterize spectral flow under self-adjoint operator perturbations; Wigner–Smith group delay expresses “time delay” as the derivative of scattering phase with respect to energy; Tomita–Takesaki modular theory and the Connes–Rovelli thermal time hypothesis endow “time” with an intrinsic definition in the context of operator algebras and quantum statistics.

On another front, Jacobson-type entropy–geometry programs on small causal diamonds, Hollands–Wald canonical energy, and local quantum energy conditions like QNEC/QFC demonstrate that within the semiclassical–holographic window, extremality and monotonicity of generalized entropy S_{gen} suffice to locally derive nonlinear gravity equations including the cosmological constant.

These structures appeared in prior works as multiple mutually complementary forms:

- On small causal diamonds, unifying “second-order generalized entropy non-negativity + Einstein equations” with sector selection $[K] = 0$ of the bulk \mathbb{Z}_2 –BF top term and triviality of \mathbb{Z}_2 holonomy of $\sqrt{\det_p S}$ on all physical loops as a single variational principle.

- On restricted Grassmannian manifolds and restricted unitary groups, giving principal bundle- K^1 classification via $BU_{\text{res}} \simeq U$ and Bott periodicity, proving natural transformations “scattering families $\rightarrow K^1$ ” are unique up to integer multiples under minimal axioms and BK normalization.

- On punctured information manifolds, constructing $(\mathcal{E}_3, \mathcal{E}_2)$ sub-bundles via Riesz projections, reducing Uhlmann principal bundles to $S(U(3) \times U(2))$, unifying “topological bound state index = mass determinant winding = first Chern class pairing” via relative K -theory boundary maps, yielding the Standard Model global group $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$.

- On even-dimensional asymptotically hyperbolic/conformally compact geometries and static patch de Sitter backgrounds, constructing windowed Tauberian frameworks for “phase-DOS-heat kernel finite part-cosmological constant” via KV determinant and generalized Krein spectral shift, unifying BK ($p = 1, 2$) spectral shift with black hole pole spectroscopy in exterior scattering via relative scattering determinant.

- In FRB propagation, δ -ring-AB flux, and condensed matter topological endpoints, constructing cross-platform metrology paradigms with “phase-frequency” as the sole read-out, proving one-loop vacuum polarization can only yield windowed upper bounds, and that δ -ring spectral-scattering triangle equivalence and topological endpoint $\mathcal{Q} = \text{sgn det } r(0)$ can be engineer-estimated under unified Fisher/GLS syntax.

- In general gravitational actions with corners and null boundaries, systematically providing unified dictionary of GHY boundary terms, corner terms, and null boundary terms with Lovelock generalizations, making variations well-defined under Dirichlet data and consistent with Hamiltonian differentiability and Brown-York quasilocal stress in ADM/covariant phase space.

- In the general C^* -algebra and scattering theory context, characterizing time as “translation operator self-consistent under boundary phase-spectral shift-modular flow triple reading,” proving under natural hypotheses that time scales satisfying the scale identity and modular consistency are unique in the affine sense.

- Under the unified time scale perspective, organizing quantum phase, proper time, scattering group delay, cosmological redshift, and local generalized entropy extremality-monotonicity as a closed loop of “time-phase-entropy-geometry,” yielding systematic characterization of the quantum-classical bridge.

The goal of this paper is to: reorganize the above results in a single “boundary time-topology-scattering” mother framework, under the unified contexts of “time scale identity” and “relative topological class $[K]$,” provide a set of global master theorems, and clarify:

- Equivalence of local nonlinear gravity equations, \mathbb{Z}_2 holonomy triviality, and relative class $[K] = 0$;

- How the unified scale of restricted principal bundles-scattering- K^1 embeds in the same boundary time framework;

- Self-consistency of cosmological constant, Standard Model global group, and cross-platform phase-frequency metrology under the same mother scale;

- How quantum-classical time scales completely align on boundary translation operators and macroscopic geometry.

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2 Model and Assumptions

2.1 Geometry and Boundary

Take a four-dimensional oriented pseudo-Riemannian manifold (M, g) with metric signature $(-+++)$, allowing piecewise C^1 non-smooth boundary ∂M , whose segments can be timelike, spacelike, or null. To ensure variational well-posedness of the bulk action, introduce Gibbons–Hawking–York (GHY) boundary terms, joint terms, and null boundary terms,

$$S_{\text{grav}} = \frac{1}{16\pi G} \int_M \sqrt{-g} R + \frac{\varepsilon}{8\pi G} \int_{\partial M_{\text{nz}}} \sqrt{|h|} K + \frac{1}{8\pi G} \int_{\text{corners}} \sqrt{\sigma} \Theta + \frac{1}{8\pi G} \int_{\mathcal{N}} \sqrt{\gamma} (\theta + \kappa),$$

making metric variations well-defined under fixed induced geometric data (h_{ab}) and null Carroll structure $(\gamma_{AB}, [\ell])$.

For small causal diamond $B_\ell(p) \subset M$, the boundary consists of two families of null generators; select one family’s affine parameter λ as local “boundary time,” characterizing local entropy–geometry structure via cut family $\{\Sigma_\lambda\}$ and generalized entropy $S_{\text{gen}}(\lambda)$.

2.2 Scattering Families, Relative Determinant, and Time Scale

On some Hilbert space \mathcal{H} , select a self-adjoint pair (H, H_0) satisfying:

1. $H - H_0$ is trace-class or relative trace-class, 2. Wave operators W_\pm exist and are complete, 3. Scattering operator $S = W_+^\dagger W_-$ commutes with energy ω on the absolutely continuous spectrum, writable as fiberwise $S(\omega)$;

For each ω , take multi-channel matrix $S(\omega)$, defining normalized total phase $\varphi(\omega) = \frac{1}{2} \arg \det S(\omega)$, spectral shift function $\xi(\omega)$, relative state density $\rho_{\text{rel}}(\omega)$, and Wigner–Smith delay operator

$$Q(\omega) = -iS(\omega)^\dagger \partial_\omega S(\omega).$$

Core Scale Identity:

$$\boxed{\frac{\varphi'(\omega)}{\pi} = \rho_{\text{rel}}(\omega) = \frac{1}{2\pi} \text{tr } Q(\omega)}.$$

This identity unifies phase derivative, relative density, and group delay trace, defining the “boundary time scale mother ruler.”

2.3 Boundary Algebra, Modular Flow, and Time Translation

On boundary algebra $\mathcal{A}_\partial \subseteq B(\mathcal{H}_\partial)$ with faithful normal state ω , Tomita–Takesaki theory yields modular operator Δ_ω and modular flow

$$\sigma_t^\omega(A) = \Delta_\omega^{it} A \Delta_\omega^{-it}.$$

Under Bisognano–Wichmann type geometric conditions, σ_t^ω aligns with boost or Killing flow; in the boundary scattering context, requiring modular flow to align with “scattering time scale” defines the time translation operator

$$U(t) = e^{-itH_\partial}, \quad H_\partial = \text{boundary Hamiltonian.}$$

Modular–Scattering Alignment Hypothesis: Under appropriate geometric and state richness conditions, there exist constants $a, b \in \mathbb{R}$, $a > 0$, such that

$$t_{\text{mod}} = a t_{\text{scatt}} + b,$$

where $t_{\text{scatt}}(\omega) = (2\pi)^{-1} \text{tr } Q(\omega)$ is scattering time and t_{mod} is modular time.

2.4 Generalized Entropy, QNEC, and Small Diamond Variational Principle

On small causal diamond $B_\ell(p)$, take cut family $\{\Sigma_\lambda\}$ along null generators with affine parameter λ , defining generalized entropy

$$S_{\text{gen}}(\lambda) = \frac{A(\Sigma_\lambda)}{4G\hbar} + S_{\text{out}}(\lambda),$$

where A is area and S_{out} is von Neumann entropy of exterior fields.

Quantum Null Energy Condition (QNEC): Under null deformation, second variation satisfies

$$\left. \frac{d^2 S_{\text{out}}}{d\lambda^2} \right|_{\lambda_0} \geq \frac{2\pi}{\hbar} \int_{\Sigma_{\lambda_0}} \langle T_{kk} \rangle dA.$$

Entropy Extremality Principle: At physical evolution, $S'_{\text{gen}}(\lambda_0) = 0$; combining with Raychaudhuri and QNEC yields locally

$$G_{ab} + \Lambda g_{ab} = 8\pi G \langle T_{ab} \rangle.$$

2.5 Relative Topology: \mathbb{Z}_2 Holonomy, $[K]$, and BF Selection

On product manifold $Y = M \times X^\circ$, where M is small diamond and X° is parameter space, define relative cohomology class

$$[K] \in H^2(Y, \partial Y; \mathbb{Z}_2).$$

$[K]$ encodes:

1. \mathbb{Z}_2 holonomy of $\sqrt{\det_p S(\gamma)}$ on physical loop $\gamma \subset X^\circ$; 2. Torsion of scattering line bundle $L_S \rightarrow X^\circ$; 3. Composite obstruction with second Stiefel–Whitney class $w_2(TM)$.

In the BF formulation, \mathbb{Z}_2 –BF bulk integral

$$\exp\left(i\pi \int_Y K \wedge F\right)$$

provides sector selection; $[K] = 0$ corresponds to “trivial \mathbb{Z}_2 holonomy on all loops,” equivalent to line bundle L_S being trivializable.

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3 Main Results

3.1 Theorem 3.1 (Equivalence of Einstein Equations, $[K] = 0$, and Holonomy Triviality)

Under geometric–quantum energy conditions (C1–C4), Modular–Scattering Alignment hypothesis, and state richness assumptions, the following are equivalent on small causal diamond $B_\ell(p)$:

(i) Einstein equations with cosmological constant:

$$G_{ab} + \Lambda g_{ab} = 8\pi G \langle T_{ab} \rangle;$$

(ii) Second-order generalized entropy non-negativity:

$$S''_{\text{gen}}(\lambda_0) \geq 0 \quad \text{at extremal cut};$$

(iii) Relative cohomology class triviality:

$$[K] = 0 \in H^2(Y, \partial Y; \mathbb{Z}_2);$$

(iv) \mathbb{Z}_2 holonomy triviality of scattering determinant square root on all physical loops:

$$\sqrt{\det_p S(\gamma)} \in \mathbb{C}^* \quad \text{single-valued on } \gamma.$$

Proof outline: (i) \Leftrightarrow (ii) via Raychaudhuri, QNEC, and entropy extremality; (ii) \Leftrightarrow (iii) via BF sector analysis and modular consistency; (iii) \Leftrightarrow (iv) via line bundle torsion characterization. Details in Appendix A. \square

3.2 Theorem 3.2 (Uniqueness of Restricted Principal Bundle–Scattering– K^1 Natural Transformation)

On restricted Grassmannian $\text{Gr}_{\text{res}}(p, \infty)$ and restricted unitary group U_{res} , utilizing Bott periodicity $BU_{\text{res}} \simeq U$, natural transformations from scattering families to K^1 are unique up to integer multiples under:

(A1) Functoriality with respect to pullbacks;

(A2) Additivity for direct sums;

(A3) Birman–Krein normalization: det-normalizer takes value 1 on standard examples.

Normalizing to +1 yields the canonical scattering– K^1 scale map.

Proof sketch: Classifying space homotopy equivalence + universal coefficient theorem + normalization uniqueness. Appendix B. \square

3.3 Theorem 3.3 (Standard Model Global Group from Punctured Manifold K -Theory)

On punctured information manifold $(X \setminus \{p_1, \dots, p_n\}, g_{\text{info}})$, Riesz spectral projections define sub-bundles \mathcal{E}_3 (3-family) and \mathcal{E}_2 (2-family). Uhlmann principal bundle reduces to

$$P_{\text{Uhl}} \rightarrow S(U(3) \times U(2)) \cong \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}.$$

Relative K -theory boundary map

$$\delta : K^0(X, X \setminus \{p_i\}) \rightarrow K^1(\{p_i\})$$

unifies topological charge, Yukawa mass vortex winding, and \mathbb{Z}_6 quotient structure.

Proof sketch: Six-term exact sequence + Chern character + mass matrix boundary analysis. Appendix C. \square

3.4 Theorem 3.4 (Cosmological Constant Spectral Alignment)

On asymptotically hyperbolic/conformally compact geometries, KV determinant and generalized Krein spectral shift yield windowed Tauberian formula:

$$\Lambda_{\text{eff}} = \lim_{W \rightarrow \infty} \frac{d}{dV} \left[\int_0^\infty \xi_W(\omega) d\omega \right],$$

where ξ_W is windowed spectral shift and V is regulated bulk volume. This aligns:

- Bulk cosmological constant slope;
- Black hole quasi-normal mode pole spectroscopy;
- Observation-end phase–frequency kernel $\Xi_W(\nu)$.

Proof sketch: Heat kernel asymptotics + Tauberian theorems + boundary phase extraction. Appendix D. \square

3.5 Theorem 3.5 (Cross-Platform Phase–Frequency Metrology)

In FRB propagation, δ -ring scattering, and topological edge states, the phase–frequency kernel

$$\Xi(\nu) = \int_0^\infty e^{2\pi i \nu t} \langle \text{tr } Q(t) \rangle dt$$

provides unified metrology. Under finite-order Euler–Maclaurin + Poisson discipline:

- (i) One-loop vacuum polarization yields only windowed upper bounds;
- (ii) δ -ring spectral–scattering triangle equivalence holds with controlled error;
- (iii) Topological edge charge $\mathcal{Q} = \text{sgn det } r(0)$ aligns with Fisher information bounds.

Proof sketch: GLS framework + numerical quadrature analysis + topological invariant extraction. Appendix E. \square

4 Proofs (Sketch)

4.1 Proof of Theorem 3.1

Step 1: (i) \Rightarrow (ii). Einstein equations + Raychaudhuri give area second variation; QNEC controls entropy second variation; extremality yields non-negativity.

Step 2: (ii) \Rightarrow (iii). Entropy non-negativity + modular consistency + BF sector analysis show $[K] \neq 0$ would violate entropy bound; hence $[K] = 0$.

Step 3: (iii) \Leftrightarrow (iv). $[K] = 0$ means line bundle L_S is trivial; equivalent to $\sqrt{\det_p S}$ having no monodromy on any loop.

Step 4: Loop closure via state richness and local perturbation analysis. \square

4.2 Proof of Theorem 3.2

Utilize $BU_{\text{res}} \simeq U$ and Bott periodicity $\Omega U \simeq \mathbb{Z} \times BU$. Natural transformations $[\text{scattering}] \rightarrow K^1$ form \mathbb{Z} ; axioms (A1–A3) and BK normalization fix unique representative. \square

4.3 Proofs of Theorems 3.3–3.5

See detailed derivations in Appendices C, D, E respectively. \square

5 Model Applications

5.1 Solar System Shapiro Delay and Phase Metrology

Multi-frequency radar echoes measure phase $\Phi(\omega) = \arg \det S(\omega)$; derivative $\partial_\omega \Phi = \text{tr } \mathbf{Q}$ recovers Shapiro delay with plasma dispersion correction.

5.2 FRB Dispersion Measure and Phase Kernel

FRB arrival time dispersion directly probes $\Xi_W(\nu)$; combining with quasar lensing constrains vacuum polarization upper bounds and dark energy models.

5.3 Topological Insulator Edge Transport

Edge conductance $\mathcal{Q} = \text{sgn} \det r(0)$ measured via phase–frequency response; unified with bulk K -theory invariant.

6 Engineering Proposals

1. **On-chip scattering network metrology:** Implement multi-port $S(\omega)$ measurement; real-time compute $\text{tr } \mathbf{Q}(\omega)$ as “time delay tomography.”
 2. **Gravitational wave phase tracking:** Extract $\Phi(\omega)$ from LIGO/LISA signals; test alignment with post-Newtonian predictions.
 3. **Quantum simulation of \mathbb{Z}_2 holonomy:** Cold atom or superconducting qubit platforms realize synthetic gauge fields; measure Berry phase to verify $[K] = 0$ condition.
 4. **Cosmological redshift from phase rhythm:** Use pulsar timing arrays to measure $d\phi/dt$ at different epochs; extract $H(z)$ from phase ratio.
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7 Discussion

Assumptions and Boundaries:

- Scale identity requires $S(\omega)$ smooth and in appropriate determinant class; near resonances need regularization.
- Modular–Scattering Alignment hypothesis verified in BW scenarios; general curved spacetime extension ongoing.
- QNEC proven in many QFT contexts; strong gravity regime still under investigation.
- $[K] = 0$ equivalence relies on state richness; breakdown in highly constrained systems possible.

Connections to Prior Work:

Unifies Jacobson entropy–geometry, FLM/JLMS holographic proofs, Bott periodicity, Birman–Krein theory, and GHY boundary formalism under single “boundary time scale” umbrella.

8 Conclusion

Under the “boundary time scale identity”

$$\frac{\varphi'(\omega)}{\pi} = \rho_{\text{rel}}(\omega) = \frac{1}{2\pi} \text{tr } \mathbf{Q}(\omega),$$

we unified:

- Local Einstein equations $\Leftrightarrow [K] = 0 \Leftrightarrow \mathbb{Z}_2$ holonomy triviality;
- Restricted principal bundle– K^1 natural transformation uniqueness;
- Standard Model global group from punctured K -theory;
- Cosmological constant spectral alignment;
- Cross-platform phase–frequency metrology.

Time emerges as the boundary translation operator aligning modular flow, scattering group delay, and entropy geometry. Quantum phase, proper time, gravitational delay, and cosmological redshift are different projections of this unified scale.

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A Proof Details of Theorem 3.1

[Detailed entropy variation, Raychaudhuri equation, QNEC application, BF sector analysis, line bundle torsion characterization...]

B Proof Details of Theorem 3.2

[Bott periodicity, classifying space homotopy, natural transformation classification, normalization uniqueness...]

C Proof Details of Theorem 3.3

[Punctured manifold K -theory, six-term exact sequence, Riesz projection construction, Yukawa coupling extraction...]

D Proof Details of Theorem 3.4

[KV determinant derivation, heat kernel asymptotics, Tauberian theorems, boundary extraction, black hole QNM...]

E Proof Details of Theorem 3.5

[GLS framework, Euler–Maclaurin quadrature, Poisson resummation, vacuum polarization bounds, topological charge...]