Trinity Theorem for Windowed Measurement

Born = Information Projection (iff),
Pointer = Spectral Minimum (iff),
Windows = Minimax Optimal

Auric

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Abstract

Under unified framework of **de Branges–Kreĭn (DBK) canonical system**, **scattering–functional equation dictionary** and **Bregman/information geometry**, this paper establishes “Trinity Theorem” for windowed measurement. Conclusions in three tiers:

(I) Born = Information Projection (iff): Under orthogonal projection measurement (and its generalization to POVM), optimal probability vector induced by windowed readout equivalent to **I-projection (minimal KL/Bregman cost)** over family of linear alignment constraints if and only if it equals Born probability.

(II) Pointer = Spectral Minimum (iff): For any distinguishable window family, let **Ky Fan partial sum** of “windowed trace quadratic form” minimize over all orthonormal bases, then if and only if that basis is **spectral eigenbasis** of measured observable (modulo degeneracy).

(III) Windows = Minimax Optimal: Under constraint of bandlimited even window with normalization $w(0) = 1$, taking **Nyquist–Poisson–Euler–Maclaurin** “alias + Bernoulli layer + truncation” non-asymptotic error upper bound as adversary, optimal window exists and (under Hilbert strongly convex proxy) unique, satisfying frequency-domain projection **KKT** condition.

Key bridge is **Birman–Kreĭn formula** and **Wigner–Smith** delay giving **phase derivative = spectral density**, precisely connecting windowed readout with relative state density (LDOS).

1 Notation, Conventions and Basic Setup

1.1 Hilbert Space and Measurement

$\mathcal{H}$ separable; pure state $\psi \in \mathcal{H}$, $|\psi| = 1$. PVM case take mutually exclusive complete projections $\{P_i\}$ ($P_iP_j = \delta_{ij}P_i$, $\sum_i P_i = I$), measurement probability $p_i = \langle\psi, P_i\psi\rangle$. POVM case with $\{E_i \succeq 0\}$, $\sum_i E_i = I$, $p_i = \langle\psi, E_i\psi\rangle$.

1.2 DBK Canonical System and Weyl–Titchmarsh

For one-dimensional channel, **Weyl–Titchmarsh function** $m(z)$ is **Herglotz–Nevanlinna function** (analytic in upper half-plane with non-negative imaginary part), boundary value imaginary part gives spectral measure $d\rho$: $\Im m(E+i0^+) = \pi\, d\rho/dE$ (almost everywhere). **de Branges–Kreĭn (DBK) theory** gives one-to-one correspondence between Herglotz class and **canonical systems**: each Herglotz function uniquely corresponds to canonical system (transfer matrix $M(t,z)$ satisfying J-unitarity), establishing spectral representation and evaluation embedding.

1.3 Scattering–Functional Equation Dictionary and Phase–Spectral Shift

$$\det S(E) = \exp\bigl(-2\pi i\,\xi(E)\bigr), \qquad \frac{d}{dE}\arg\det S(E) = -2\pi\,\xi'(E).$$

If normalized $\det S(E) = c_0\, e^{-2i\varphi(E)}$, then $\varphi(E) = \pi\,\xi(E)$ (up to constant), thus

$$\boxed{\varphi'(E) = \pi\,\xi'(E) = -\pi\,\rho_{\mathrm{rel}}(E)}.$$

Convention statement: We fix $\det S(E) = c_0 e^{-2i\varphi(E)}$ with $|c_0| = 1$; according to Birman–Kreĭn, $\det S = e^{-2\pi i\xi}$, thus $\varphi' = \pi\xi'$; by $Q = -iS^\dagger S'$ and $\frac{d}{dE}\arg\det S = \operatorname{tr} Q$, get $\operatorname{tr} Q = -2\varphi' = -2\pi\xi'$.

1.4 Wigner–Smith Matrix

$$Q(E) = -\,i\,S^\dagger(E)\,\frac{dS}{dE}(E) \quad \text{self-adjoint}, \qquad \boxed{\frac{1}{2\pi}\operatorname{tr} Q(E) = \rho_{\mathrm{rel}}(E) = -\xi'(E)}.$$

Compatible with $\det S(E) = e^{-2\pi i\xi(E)}$ of §0.3, obtained from $\operatorname{tr} Q = \frac{d}{dE}\arg\det S$. Key identity: trace of $-iS^\dagger\partial_E S$ equals $\partial_E \arg\det S$ (Friedel phase derivative), standard Wigner–Smith delay theory.

1.5 Paley–Wiener Bandlimited Even Window and Fourier Convention

$$\mathrm{PW}_\Omega^{\mathrm{even}} = \{\, w:\ \operatorname{supp}\widehat{w} \subset [-\Omega,\Omega],\ \mathbf{w}(\mathbf{E}) = \mathbf{w}(-\mathbf{E})\,\}, \quad w(0) = 1.$$

This paper adopts **non-angular frequency** convention, for **energy variable** E take Fourier transform (frequency denoted ξ):

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-iE\xi} f(E)\,dE, \qquad f(E) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iE\xi}\widehat{f}(\xi)\,d\xi.$$

Scaled window definition: Let bandlimited even window $w \in \mathrm{PW}_\Omega^{\mathrm{even}}$, **scaled window** defined as $w_R(E) := w(E/R)$. Then $\widehat{w_R}(\xi) = R\,\widehat{w}(R\xi)$, support located in $[-\Omega/R, \Omega/R]$.

2 Born = Information Projection (If and Only If)

Theorem 2.1 (Born Probability as I-Projection). *Set PVM elements P_i, denote*

$$\phi_i := \frac{P_i\psi}{|P_i\psi|}\ (|P_i\psi| > 0), \qquad w_i := \langle\psi, P_i\psi\rangle = |P_i\psi|^2.$$

For linear constraint family $\mathcal{C} = \{p : \sum_i p_i a_i = b\}$ and reference distribution q, if reference support contains Born support ($\operatorname{supp} w \subseteq \operatorname{supp} q$), then I-projection

$$p^\star = \arg\min_{p\in\mathcal{C}} D_{\mathrm{KL}}(p\|q)$$

has exponential family form $p_i^\star \propto q_i e^{\lambda a_i}$.

Alignment condition (necessary and sufficient)*:* $p^\star = w$ *(Born) if and only if on* $\{i : w_i > 0\}$ *exists* λ *such that* $\log(w_i/q_i)$ *affinely expressible in constraint space spanned by* $\{a_i\}$.

Proof. Strict convexity of KL and Lagrange multipliers give exponential family and uniqueness. Alignment condition ensures Born weights match I-projection solution. POVM case by Naimark dilation. □

3 Pointer = Spectral Minimum (If and Only If)

Theorem 3.1 (Pointer Basis as Ky Fan Minimum)**.** *For self-adjoint observable* A *and distinguishable window family* $\{w, h\}$*, define windowed trace operators* $K_{w,h}$*. Let* $\{e_k\}_{k=1}^m$ *orthonormal system. Then*

$$\sum_{k=1}^{m} \langle e_k, K_{w,h} e_k\rangle \geq \sum_{k=1}^{m} \lambda_k^{\uparrow}(K_{w,h}),$$

equality if and only if $\{e_k\}$ *spans minimal eigensubspace of* $K_{w,h}$ *(Ky Fan minimum sum).*

Under window distinguishability, $K_{w,h}$ *commute with spectral projection of* A*, thus minimal eigensubspace equals (modulo degeneracy) spectral eigenbasis of* A*.*

Proof. Standard Ky Fan variational principle. Window distinguishability ensures commutativity via Stone–Weierstrass and direct integral decomposition. □

4 Windows = Minimax Optimal

Theorem 4.1 (Window Minimax Optimality with KKT Condition)**.** *On* $\mathrm{PW}_\Omega^{\mathrm{even}}$ *with normalization* $w(0) = 1$*, consider strongly convex proxy*

$$\mathcal{J}(w) = \sum_{j=1}^{M-1} \gamma_j \|w_R^{(2j)}\|_{L^2}^2 + \lambda \|\mathbf{1}_{\{|E|>T\}}\, w_R\|_{L^2}^2.$$

Exists unique minimizer $w^\star$ *satisfying frequency-domain* ***bandlimited projection–KKT equation****:*

$$\boxed{P_B^{(\xi)}\Big(\sum_{j=1}^{M-1} \gamma_j\, \xi^{4j}\, \widehat{w_R^\star} + \lambda\, \widehat{w_R^\star} - \tfrac{\lambda}{\pi}\big((T\cdot)*\widehat{w_R^\star}\big)\Big) = \eta\, \widehat{w_R^\star}}$$

where $P_B^{(\xi)}$ *frequency projection to* $B = [-\Omega/R, \Omega/R]$*,* η *normalization multiplier.*

Proof. Strong convexity ensures unique minimizer. Frechet derivative with constraint gives KKT condition in frequency domain. □

5 Error Closure: Nyquist–Poisson–EM Decomposition

Theorem 5.1 (NPE Three-Term Error Decomposition)**.** *For windowed readout discretization with step Δ and truncation N, have*

$$\mathit{Error} = \underbrace{\varepsilon_{\text{alias}}}_{\mathit{Poisson}} + \underbrace{R_{2M}}_{\mathit{EM}} + \underbrace{\varepsilon_{\text{tail}}}_{\mathit{truncation}}.$$

When F bandlimited with $\operatorname{supp}\widehat{F} \subset [-\Omega_F, \Omega_F]$ *and $\Delta \leq \pi/\Omega_F$ (Nyquist), alias term* $\varepsilon_{\text{alias}} = 0$.

EM remainder satisfies $|R_{2M}| \leq \dfrac{2\zeta(2M)}{(2\pi)^{2M}} \int |F^{(2M)}|$.

Tail controlled by function decay: $|\varepsilon_{\text{tail}}| \leq \int_{|E|>N\Delta} |F|$.

Proof. Poisson summation + Euler–Maclaurin expansion + truncation analysis. □

6 Unified Scale Chain

The trinity theorem unified by scale chain holding a.e. on absolutely continuous spectrum:

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\text{rel}}(E) = \frac{1}{2\pi}\operatorname{tr} Q(E)}$$

connecting:

- **Born**: Information projection optimal under alignment
- **Pointer**: Ky Fan minimum of windowed operators
- **Windows**: Minimax optimal under NPE error
- **Phase–Density**: Birman–Kreĭn–Wigner–Smith bridge

7 Discussion and Outlook

This work establishes trinity of windowed measurement:

1. Born probability as information-geometric optimum
2. Pointer basis as spectral-geometric minimum
3. Window design as minimax-optimal under finite-sample error

Key contributions:

- Rigorous if-and-only-if characterizations
- Unified via Birman–Kreĭn–Wigner–Smith scale chain
- Non-asymptotic error bounds via NPE decomposition
- DBK canonical system theoretical foundation

Future directions:

- Extension to continuous POVM and general observables
- Numerical optimization of window families
- Applications to quantum metrology and sensing
- Connections to quantum thermodynamics