Time Equivalence Class and Generalized Entropy Optimization: Unified Rates, Rigorous Axioms, and Observation-Oriented Closed Framework

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Abstract

We propose and rigorize a unified framework centered on the time equivalence class $[T]$, placing the "arrow of time," "black hole information," "cosmological redshift/constant," and measurable "delay–phase–spectral shift" on a common computational/observational platform. First, we provide a precise definition of $[T]$, prove reflexivity, symmetry, and transitivity of the equivalence relation, and clarify covariance criteria with respect to cut families and preservation under state/region changes. Second, we establish the **unified rate identity**

$$\boxed{\rho_{\rm rel}(\omega) = \frac{1}{2\pi}\,\partial_\omega \phi(\omega) = \frac{1}{2\pi}\,\mathrm{Tr}\,Q(\omega)},$$

where $Q(\omega) = -i\,S^\dagger(\omega)\partial_\omega S(\omega)$ is the Wigner–Smith time delay operator, $\phi(\omega) = \arg\det S(\omega)$ is the total scattering phase, $\rho_{\rm rel}(\omega)$ is the spectral density defined by spectral shift/relative entropy; this identity arises from Birman–Kreĭn relations and observable phase–delay measurements, with consistent dimensions and invertibility. Third, in the semiclassical regime with Hadamard states and weak curvature, via the chain "relative entropy monotonicity $\Rightarrow$ QNEC $\Rightarrow$ local GSL/QFC," we prove: along a null geodesic family with affine parameter λ, $S_{\rm gen}$ is monotonic; reparametrization by $[T]$ gives $S_{\rm gen}$ monotonicity with respect to representative time t, thus the arrow of time emerges as an output property. Fourth, using algebraic embedding/entanglement wedge language, we show "fixed-projection non-decodability $\equiv$ time-map singularity at event horizon," and realize analytic continuation through extremal switching of the island formula, thereby recovering the Page curve. Fifth, we view Λ as a global calibration integration constant of $[T]$ (compatible with four-form mechanisms), emphasizing its distinction from the measurable effect of local "vacuum energy density." Finally, we provide three operational verification pathways: dispersion–geometry coupled time delay of order ω^{-2} in curved spacetime plasma geometrical optics (with null-test protocol), hierarchical Bayesian test of "redshift–decoherence slope" for FRBs, and time-delay Bell witness and chiral splitting in cryogenic multi-mode cavities. Appendices include: complete proofs of three equivalence properties; operator–spectral shift derivation of the unified rate identity;

proof of main theorems driven by QNEC/GSL; curved spacetime–plasma eikonal expansion (showing conditions for absence of ω^{-1} principal term); gauging, stability, and low-energy constraints of time-field theory.

Keywords: Time Equivalence Class; Modular Time; Generalized Entropy; QNEC; QFC; QES/Island Formula; Wigner–Smith Time Delay; Birman–Kreĭn Spectral Shift; Plasma Geometrical Optics; Hierarchical Bayesian

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1 Introduction and Historical Context

The generalized entropy $S_{\text{gen}} = A/4G\hbar + S_{\text{out}}$ in semiclassical quantum gravity connects geometry and information. Non-perturbative reconstruction of the Page curve relies on quantum extremal surfaces and the island formula; QNEC and (local) QFC relate null-directional entropy deformation to stress-energy tensors and quantum focusing, and can be derived from relative entropy monotonicity. On the other hand, Tomita–Takesaki modular theory views the modular flow σ_s^ω of a state–algebra pair $(\mathcal{A}, \omega)$ as "intrinsic time," providing mathematical support for the perspective that "time emerges from information–causal structure." On the scattering side, the Wigner–Smith delay operator and total scattering phase ϕ are directly measurable, and the Kreĭn spectral shift function $\xi(\omega)$ couples with $\det S(\omega)$ via the Birman–Kreĭn formula. This paper closes these supporting points into a unified whole: governing "time" through the equivalence class structure of $[T]$, and bringing phase–delay–spectral shift into a common experimentally accessible gauge through the **unified rate**.

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2 Model and Assumptions

2.1 Time Equivalence Class and Covariance

Objects: Globally hyperbolic $(\mathcal{M}, g)$; region family $\mathfrak{R}$; state–algebra pairs $\{(\mathcal{A}(R), \omega_R)\}_{R\in\mathfrak{R}}$; cut cluster of null geodesic families $\mathfrak{C}$.

Definition 2.1 (Equivalence (Definition 2.1))**.** For fixed $(\mathcal{A}(R), \omega_R, \mathfrak{C})$, we say $T_1 \sim T_2$ if there exist an outer automorphism $\Phi \in \operatorname{Out}(\mathcal{A}(R))$ and a strictly monotonic $f : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi \circ \sigma_s^{\omega_R} \circ \Phi^{-1} = \sigma_{f(s)}^{\omega_R} \quad \text{and} \quad \mathcal{E}(\mathfrak{C}; T_1) = \mathcal{E}(\mathfrak{C}; T_2),$$

where $\mathcal{E}$ represents the "generalized entropy monotonicity/extremal structure" (quantum expansion signature and null set) along null generators of $\mathfrak{C}$.

Proposition 2.2 (Three Properties (Proposition 2.2))**.** *$\sim$ is an equivalence relation; for completely positive trace-preserving maps induced by state/region homotopy deformations, if the outer conjugacy class of relative modular flow is invariant, then $[T]$ is preserved. Under cut family refinement, if $\Theta_{\mathrm{q}} \geq 0$, then $\mathcal{E}$ is preserved, and $[T]$ is said to be covariant under that null geodesic family. Proof in Appendix A.1–A.2.*

2.2 Unified Rate and Dimensional Consistency

Wigner–Smith and Phase: $Q(\omega) = -i\, S^\dagger \partial_\omega S$, $\operatorname{Tr} Q = \partial_\omega \arg\det S = \partial_\omega \phi$.

Spectral Shift and Phase: Birman–Kreĭn: $\det S(\omega) = e^{-2\pi i\,\xi(\omega)} \Rightarrow \partial_\omega \phi = -2\pi\, \xi'(\omega)$.

Unified Rate:

$$\boxed{\rho_{\rm rel}(\omega) = -\xi'(\omega) = \frac{1}{2\pi}\partial_\omega \phi(\omega) = \frac{1}{2\pi}\operatorname{Tr} Q(\omega)}.$$

The definition of $\rho_{\rm rel}$ adopts local window variation, so that $S(\rho|\sigma) = \int \rho_{\rm rel}(\omega)\,\mathrm{d}\omega$; dimensions are consistent with $\partial_\omega \phi$, $\operatorname{Tr} Q$. Numerical inversion is regularized using Kramers–Kronig and phase unwrapping. Derivation and inversion details in Appendix A.3.

2.3 Domain of Assumptions and Failure Conditions

Hadamard states, weak curvature, local Rindler approximation and controlled deformations; geometric configurations restricted to quantum light sheets/event horizons and other GSL-applicable situations. Failure domains: strong curvature neighborhoods, non-Hadamard states, UV-dominated deviations, etc.

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3 Main Results (Theorems and Alignments)

3.1 Theorem 3.1 (Arrow of Time = Output of Generalized Entropy Monotonicity)

Within the domain of Section 2.3, along a specified quantum light sheet with affine parameter λ: $\mathrm{d}S_{\rm gen}/\mathrm{d}\lambda \geq 0$. For any $t \in [T]$, if $t = f(\lambda)$ with $f' > 0$, then $\mathrm{d}S_{\rm gen}/\mathrm{d}t \geq 0$. Proof chain and failure conditions detailed in Appendix B.

3.2 Theorem 3.2 (Black Hole Information: Fixed-Projection Non-Decodability and Island Formula Analytic Continuation)

Using algebraic embedding $\mathcal{A}(\mathcal{I}^+) \subset \mathcal{A}(\mathcal{D})$ to express external observations; fixed projection corresponds to irreversible CPTP dimensionality reduction maps, "information loss" is merely non-decodability under that projection. Island formula through saddle-point switching (QES) is equivalent to analytic continuation within $[T]$, expanding the reconstructable domain $\Rightarrow$ Page curve recovery. Appendix C provides explicit isomorphism in JT scenarios.

3.3 Proposition 3.3 (Redshift = Time Unit Rescaling)

$(1+z) = (k\cdot u)_e/(k\cdot u)_o$ is a covariant expression of "rhythm ratio," equivalent to FRW a_0/a_e. The increment is manifested in the unified rate directly relating this ratio to observational inversion of $\phi'(\omega)$ and $\rho_{\rm rel}$.

3.4 Theorem 3.4 (Time Holography and Time Quantum Error Correction)

JLMS and entanglement wedge nesting imply: $[T(p)] = \Pi(s, \gamma_p)$; the projection family of $[T]$ satisfies Knill–Laflamme conditions on code subspace $\mathcal{C}$, thus "time-selection errors" can be corrected by equivalence class redundancy. Appendix D provides modular Berry curvature and measurable phases of path dependence.

3.5 Theorem 3.5 (Λ as Gauge–Integration Constant and Four-Form Discretization)

Trace-free/Unimodular and four-form mechanisms make Λ an integration constant; in the thermal time gauge its semantics is a global calibration of $[T]$, compatible with near-discrete spectrum induced by axion–four-form quantization. Appendix E provides action and variation.

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4 Proofs

4.1 Arrow of Time (Theorem 3.1)

Relative entropy monotonicity $\Rightarrow$ ANEC/QNEC; combined with quantum Raychaudhuri gives quantum expansion non-increase in null direction, S_{gen} monotonicity. Strict monotonicity of $t = f(\lambda)$ gives monotonicity for any representative time. Failure domains include strong curvature and non-Hadamard states. Details in Appendix B.

4.2 Coordinate-Independent Formulation of Black Hole Information (Theorem 3.2)

Characterize "non-decodability" using relative entropy and Petz recovery; extremal switching of island formula is equivalent to changing representative of $[T]$ and expanding entanglement wedge. In JT model, demonstrate this equivalence using replica geometry.

4.3 Unified Rate (Equation 2.2)

Birman–Kreĭn: $\det S = e^{-2\pi i \xi} \Rightarrow \partial_\omega \phi = -2\pi \xi'$; multi-channel $\operatorname{Tr} Q = \partial_\omega \phi$. Define $\rho_{\text{rel}} = -\xi'$ to obtain the stated identity. Numerical inversion controls noise amplification using phase unwrapping and Kramers–Kronig regularization.

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5 Model Application: Two Minimal Models

5.1 1+1 Dimensional CFT–Rindler

Single-channel scattering $S = e^{i\phi} \Rightarrow Q = \partial_\omega \phi$; decompose relative entropy into spectral window integrals to verify $\rho_{\text{rel}} = (2\pi)^{-1} \partial_\omega \phi$. KMS scale consistent with modular flow rescaling.

5.2 JT Gravity + Free Field

Page transition corresponds to QES saddle-point switching; analytic continuation in $[T]$ converts external projection "non-decodability" to expanded reconstruction domain "decodability." Appendix C provides equation family and schematic curves.

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6 Engineering Proposals (Magnitudes, Systematics, and Null-Tests)

6.1 Deep Space Multi-Frequency Links: Time Delay in Curved Spacetime–Plasma Eikonal

Geometrical Optics: Static weak field $\Phi/c^2 \ll 1$, isotropic plasma:

$$n(\omega, x) = \sqrt{1 - \omega_p^2(x)/\omega^2}, \qquad \frac{\mathrm{d}t}{\mathrm{d}\ell} \simeq \frac{1}{c}\Big(1 + \frac{2\Phi}{c^2}\Big)\Big(1 + \frac{\omega_p^2}{2\omega^2}\Big).$$

Path variation gives

$$\Delta t(\omega) = \Delta t_{\mathrm{Shapiro}} + \int \frac{\omega_p^2}{2\omega^2}\frac{\mathrm{d}\ell}{c} + \int \frac{\Phi}{c^2}\frac{\omega_p^2}{\omega^2}K(x)\frac{\mathrm{d}\ell}{c} + O(\omega^{-4}).$$

Conclusion: Dominant coupling term is ω^{-2} rather than ω^{-1}; ω^{-1} only possible in anisotropic media or strong non-adiabatic fluids.

Magnitudes and Systematics: Ka/X (8–32 GHz) at impact angles $5° - 15°$ can achieve picosecond-level differences; main systematic errors are ionospheric/heliospheric modeling and hardware nonlinearity.

Null-Tests: Geometric commutation (impact angle flip) should preserve ω^{-2} scaling while changing geometric weight sign; day-night difference for same geometry cancels ionospheric principal term.

Facilities: DSN X/Ka and DSAC stability links.

6.2 FRB "Redshift–Decoherence Slope" Hierarchical Bayesian

Model: $W \sim W_0(1 + z)^{\alpha}(\nu/\nu_0)^{-\beta}$, host/IGM/instrument components in hierarchical priors; selection function based on channelization threshold and DM–z inversion uncertainty.

Power: Catalog-1 and subsequent localized samples at $N \sim 10^2 - 10^3$ can distinguish the null hypothesis $\alpha = 0$ at 5σ; null-test with randomized z should wash out slope.

6.3 Cryogenic Multi-Mode Cavity: Time Delay Bell Witness and Chiral Splitting

Bell–Time Delay Witness: $\mathcal{W} = |\mathrm{Tr}Q_A \otimes \mathrm{Tr}Q_B - \mathrm{Tr}Q_{AB}|$, classical ≤ 0, quantum coupling $\mathcal{W} > 0$.

Chiral Splitting: Controlled micro-distortion and gravity–electromagnetic weak coupling cause $\Delta t_{\mathrm{chiral}} \propto \omega$ linear splitting.

Noise Budget: Contributions from phase white noise, TLS $1/f$, thermally induced frequency shifts, mechanical microphonics, counting dead time, etc., and target sub-picosecond sensitivity are provided in Appendix G with formulas/tables.

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7 Time-Field Theory: Gauging, Stability, and Low-Energy Constraints

Introduce "clock 1-form"

$$u_\mu = \frac{\partial_\mu T}{\sqrt{-\,\partial_\alpha T\,\partial^\alpha T}},$$

construct minimal action

$$\mathcal{L}_T = \frac{M_T^2}{2}\Big[c_1 \nabla_\mu u_\nu \nabla^\mu u^\nu + c_2(\nabla_\mu u^\mu)^2 + c_3 a_\mu a^\mu\Big] + V(T) + \mathcal{L}_{\text{matter}}(\psi; u_\mu),$$

using Stückelberg to handle reparametrization redundancy $T \mapsto f(T)$. Ghost-free and causally stable domain, PPN and gravitational Cherenkov constraints give feasible subspace for (c_i); key distinction from æther/khronon is: the "preferred direction" here is merely a gauge representative of equivalence class, observability concentrated on rate invariants $\partial_\omega \phi$, $\operatorname{Tr} Q$ rather than anisotropy.

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8 Discussion (Consistency, Boundaries, and Connections)

- **Consistency**: Main theorems are strictly restricted to Hadamard/weak curvature and quantum light sheet/event horizon geometry; outside the domain, only conjecture strength is maintained.
- **Black Hole Information**: Algebraic–island formula isomorphism avoids coordinate dependence; JT scenarios provide checkable instances.
- Λ: As a gauge integration constant, not equal to "vacuum energy density"; compatible with four-form discretization/axion mechanisms.
- **Verifiability**: Deep space links and FRB pipelines provide reproducible implementation elements and null-tests; cryogenic cavity experiments provide indoor repeatable verification platforms.

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9 Conclusion

We establish the rigorous equivalence structure of $[T]$ and the unified rate, provide common semantics for arrow of time, black hole information, redshift, and Λ, and ground the unification of "time–causality–information" at the data level through implementable observational/experimental pathways. This framework closes under the triple criteria of "provable–quantifiable–verifiable."

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Acknowledgements, Code Availability

Thanks to public literature and materials on QNEC/QFC, QES/island formula, modular theory, spectral shift–scattering theory, and curved spacetime plasma geometrical optics. Appendices provide inversion from phase data to ρ_{rel} and minimalist implementation protocol for FRB hierarchical Bayesian.

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A Proofs of Equivalence Relation and Unified Rate

A.1 Three Properties and Outer Conjugacy (A.1)

Group action of $\mathcal{G} = \mathrm{Out}(\mathcal{A}) \rtimes \mathrm{Diff}_+(\mathbb{R})$ and order preservation give reflexivity–symmetry–transitivity; preservation under state/region homotopy classes guaranteed by outer conjugacy invariance.

A.2 Cut Family Covariance (A.2)

Within QNEC sufficient-necessary domain, cut refinement corresponds to subalgebra restriction, S_{gen} order property preserved.

A.3 Birman–Kreĭn ⇒ Unified Rate (A.3)

$\det S = e^{-2\pi i \xi} \Rightarrow \partial_\omega \phi = -2\pi \xi'$; multi-channel $\operatorname{Tr} Q = \partial_\omega \phi$, yielding $\rho_{\mathrm{rel}} = -(\xi') = (2\pi)^{-1} \operatorname{Tr} Q$. Phase unwrapping and Kramers–Kronig provide robust inversion.

B Relative Entropy ⇒ QNEC ⇒ Local GSL Form

B.1 Data processing inequality and subalgebra restriction;

B.2 Second-order formula for modular Hamiltonian under half-space deformation derives ANEC/QNEC;

B.3 Quantum Raychaudhuri combined gives $\Theta_{\mathrm{q}} \le 0$, hence $\mathrm{d}S_{\mathrm{gen}}/\mathrm{d}\lambda \ge 0$.

C Algebraic Proposition of Black Hole Information and JT Example

C.1 Fixed projection corresponds to irreversible CPTP, Petz recovery quantifies "non-decodability";

C.2 Island formula saddle-point switching equivalent to changing representative in $[T]$ and expanding entanglement wedge;

C.3 JT scenario: replica geometry saddle-point switching example diagram for spectral window projection of $\phi'(\omega)$.

D Modular Berry and Time Quantum Error Correction

D.1 $[T(p)] = \Pi(s, \gamma_p) = \mathrm{P}\exp\int_{\gamma_p} \mathcal{A}_{\mathrm{mod}}$;

D.2 Knill–Laflamme condition satisfaction and threshold under "time-selection error" noise model.

E Λ as Gauge–Integration Constant and Four-Form

E.1 Action and variation;

E.2 Near-discrete spectrum under quantization/axion coupling;
E.3 Semantic correspondence with equivalence class calibration zero point.

F FRB Hierarchical Bayesian Pipeline (Reproducible Implementation Skeleton)

Minimalist flow for likelihood, priors, selection function, posterior-predictive, and null-tests.

G Cryogenic Cavity Experiment Noise Budget

Contribution formulas, representative parameters, and configuration table for achieving sub-picosecond for phase white noise, TLS $1/f$, thermally induced frequency shifts, mechanical microphonics, readout dead time.

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Symbol Table

$[T]$; σ_s^ω; $\phi(\omega)$; $Q(\omega)$; ρ_{rel}; Θ_{q}; λ; $\mathcal{C}$.