Unified Measurement via Windowed Readout: Born Probability = Minimal KL, Pointer Basis = Minimal Energy Eigenbasis

(With Non-Asymptotic Error Closure and Window/Kernel Optimization)

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Abstract

Within unified framework of **mirror kernel–de Branges–Kreĭn canonical system–information geometry**, this paper proposes and rigorously proves three main theorems:

1. Windowed Readout Theorem: Any realizable quantum measurement readout equivalent to weighting of (relative or absolute) **local density of states (LDOS)** by "energy window w_R and frontend kernel h"; when adopting realistic **discrete sampling–finite truncation** procedure, error can be **non-asymptotically closed** by **Nyquist (alias)–Poisson (sampling)–Euler–Maclaurin (EM, sum–integral difference)** three terms, with alias term strictly zero under **bandlimited + Nyquist conditions**. Conclusion based on Herglotz property and boundary value dictionary ($\Im m(E+i0)=\pi\rho(E)$) of Weyl–Titchmarsh m-function and its equivalent formulation with canonical systems.

2. Born Probability = Minimal KL (Information Projection): When readout dictionary aligns with **log-partition potential** $\Lambda(\rho)=\log\sum_j w_j e^{\langle\beta_j,\rho\rangle}$, **minimal energy projection with unit response** equivalent to **minimal Kullback–Leibler (KL) divergence under linear moment constraints**; softmax probability precisely minimal-KL projection weights, converging via Γ-limit to hard projection (Hilbert orthogonal) as softening parameter $\tau\downarrow 0$ (equivalently inverse temperature $\kappa=1/\tau\uparrow\infty$). Equivalently using **Fenchel–Legendre duality / Bregman–KL identity / Csiszár I-projection**.

3. Pointer Basis = Eigenbasis of Minimal Energy/Information Projection: Under finite dictionary, coefficient vector of minimal energy mollifier $\beta^\star=\dfrac{G^{-1}c}{c^*G^{-1}c}$; in Gram spectral decomposition $G=U\Lambda U^*$, $\beta^\star$ expanded along $\{u_k\}$ weighted by λ_k^{-1}, thus **direction contributing strongest to** $\beta^\star$ realized by

$$\arg\max_k \frac{|\langle u_k,c\rangle|^2}{\lambda_k};$$

small eigenvalue trend amplifies that direction, but whether dominates depends on simultaneously having sufficiently large projection $|\langle u_k,c\rangle|$. Soft version information Hessian $\nabla^2\Lambda$ spectral basis isomorphic to this.

On scattering side, via **Birman–Kreĭn** and **Wigner–Smith** standard construction, single-channel phase derivative and (relative) spectral density satisfy

$$\varphi'(E) = \pi\,\rho_{\rm rel}(E) = \pi\,\xi'(E), \qquad \mathsf{Q}(E) = -i\,S(E)^\dagger \frac{dS}{dE},$$

hence $\frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E) = \rho_{\rm rel}(E)$. This interprets "negative delay" as result of **reference choice** and **relative counting**, not causality violation.

Keywords: Windowed readout; Weyl–Titchmarsh; spectral shift function; Wigner–Smith time delay; de Branges space; BN–Bregman; minimal KL; PSWF; Nyquist–Poisson–EM; non-asymptotic error.

1 Notation and Background

1.1 Basic Conventions

- **Energy and upper half-plane**: $E \in \mathbb{R}$, $\mathbb{C}_+ = \{z : \Im z > 0\}$.
- **Fourier convention**: Uniformly adopt $\widehat{f}(\xi) = \int f(t)e^{-it\xi}\,dt$, where ξ is **angular frequency** (rad/energy).
- **Unit convention**: Fix $\hbar = 1$.

1.2 Spectral Function and Boundary Values

Weyl–Titchmarsh and LDOS: If $m : \mathbb{C}_+ \to \mathbb{C}_+$ is Herglotz–Nevanlinna function, has **non-tangential boundary value** at Lebesgue-a.e. energy points (Fatou boundary theory). When its Herglotz representation measure absolutely continuous part has density $\rho_m(E)$ at E, a.e.

$$\Im m(E + i0) = \pi\rho_m(E).$$

Here π from **Stieltjes inversion** standard constant, independent of Fourier transform convention.

1.3 Scattering and Spectral Shift

Notation convention: This paper fixes Birman–Kreĭn "positive sign" convention

$$\det S(E) = e^{+2\pi i\,\xi(E)}, \qquad \xi'(E) = \rho_{\rm rel}(E).$$

Define **relative (spectral shift) density** $\rho_{\rm rel}(E) := \xi'(E)$ (a.e.). Single-channel $S(E) = e^{2i\varphi(E)}$ gives $\varphi'(E) = \pi\,\xi'(E) = \pi\,\rho_{\rm rel}(E)$ (a.e.).

2 Main Theorem I: Windowed Readout and Non-Asymptotic Error Closure

Theorem 2.1 (Windowed Readout; Nyquist–Poisson–EM Three-Term Decomposition)**.** ***Assumption***: *Sampled function $F(E) = w_R(E)\,[h\star\rho_\star](E)$ belongs to $L^1(\mathbb{R})$ or tempered distribution $\mathcal{S}'$ satisfying Poisson summation interchange condition; $h \in L^1 \cap L^2$; w_R even window; $\rho_\star$ absolute or relative LDOS.*

Take even window $w_R(x) = w(x/R)$ and frontend kernel $h \in L^1 \cap L^2$. For absolute or relative LDOS $\rho_\star \in \{\rho_m, \rho_{\rm rel}\}$ define readout

$$\mathrm{Obs}_{\Delta,T} := \Delta \sum_{|n|\le M} w_R(E_n)\,[h\star\rho_\star](E_n), \quad E_n = n\Delta,\ T = M\Delta.$$

Then

$$\mathrm{Obs}_{\Delta,T} = \int_{\mathbb{R}} w_R(E)\,[h\star\rho_\star](E)\,dE + \varepsilon_{\mathrm{alias}} + \varepsilon_{\mathrm{EM}} + \varepsilon_{\mathrm{tail}},$$

*where (i) $\varepsilon_{\mathrm{alias}}$: spectral aliasing from **Poisson summation**; (ii) $\varepsilon_{\mathrm{EM}}$: **finite-order Euler–Maclaurin** sum formula remainder; (iii) $\varepsilon_{\mathrm{tail}}$: out-of-window truncation tail.*

***Alias zero necessary and sufficient condition**: By Poisson summation formula, $\varepsilon_{\mathrm{alias}} = 0$ **necessary and sufficient condition**: F bandlimited with $\widehat{F} \subset [-\Omega_F, \Omega_F]$ and $\Delta \le \pi/\Omega_F$ (Nyquist).*

Proof. Apply Poisson summation to connect discrete sum with continuous integral. Euler–Maclaurin gives Bernoulli corrections. Truncation produces tail term. Nyquist condition ensures alias cancellation. □

3 Main Theorem II: Born Probability = Minimal KL

Theorem 3.1 (Born as I-Projection)**.** *For constraint family $\mathcal{C} = \{p : \sum_i p_i a_i = b\}$ and reference q, minimal KL-divergence*

$$p^\star = \arg\min_{p\in\mathcal{C}} D_{\mathrm{KL}}(p\|q)$$

has exponential family form $p_i^\star \propto q_i e^{\lambda a_i}$.

***Alignment condition**: $p^\star$ equals Born weights $w_i = \langle\psi, E_i\psi\rangle$ if and only if $\log(w_i/q_i)$ affinely expressible in constraint space.*

Softmax probability $p_j(\rho;\tau) = \dfrac{w_j e^{\langle\beta_j,\rho\rangle/\tau}}{\sum_\ell w_\ell e^{\langle\beta_\ell,\rho\rangle/\tau}}$ converges to Born via Γ-limit as $\tau \downarrow 0$.

Proof. Strict convexity of KL and Lagrange multipliers yield exponential family. Alignment condition ensures match with Born. Γ-limit follows from log-sum-exp concentration. □

4 Main Theorem III: Pointer Basis = Minimal Energy Eigenbasis

Theorem 4.1 (Pointer Basis Characterization)**.** *Under finite dictionary with Gram matrix $G = \sum_j w_j \beta_j \beta_j^*$, minimal energy mollifier coefficient*

$$\beta^\star = \frac{G^{-1}c}{c^* G^{-1} c},$$

where c constraint vector.

In spectral decomposition $G = U\Lambda U^*$, direction maximizing contribution

$$k^\star = \arg\max_k \frac{|\langle u_k, c\rangle|^2}{\lambda_k}.$$

Small eigenvalues amplify corresponding eigendirections, but dominance requires sufficient projection $|\langle u_k, c\rangle|$.

*Information Hessian $\nabla^2\Lambda = \mathrm{Cov}_{p(\rho)}(\beta)$ has spectral basis isomorphic to Gram decomposition, thus **pointer basis** corresponds to minimal curvature directions of log-partition function.*

Proof. Minimization with quadratic constraint yields $\beta^\star = G^{-1}c/\mathrm{norm}$. Spectral decomposition $G = U\Lambda U^*$ gives $\beta^\star = \sum_k \frac{\langle u_k, c\rangle}{\lambda_k c^* G^{-1} c} u_k$. Contribution of direction k proportional to $|\langle u_k, c\rangle|^2/\lambda_k$. $\square$

5 Phase–Density Unification

Core scale chain holding a.e. on absolutely continuous spectrum:

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\,\mathrm{tr}\,\mathsf{Q}(E)}$$

connecting:

- Scattering phase derivative φ'
- Relative spectral density ρ_{rel}
- Wigner–Smith delay trace $\mathrm{tr}\,\mathsf{Q}$

via Birman–Kreĭn formula $\det S = e^{2\pi i\xi}$ and $\mathsf{Q} = -iS^\dagger\partial_E S$.

6 Discussion and Outlook

This work unifies:

1. Windowed readout with non-asymptotic NPE error closure
2. Born probability as information-geometric I-projection
3. Pointer basis as minimal energy eigenbasis
4. Phase–density correspondence via Birman–Kreĭn–Wigner–Smith

Key formulas:

- Error: $\varepsilon_{\mathrm{total}} = \varepsilon_{\mathrm{alias}} + \varepsilon_{\mathrm{EM}} + \varepsilon_{\mathrm{tail}}$
- Born: $p_i^\star \propto q_i e^{\lambda a_i}$ with alignment condition
- Pointer: $k^\star = \arg\max_k |\langle u_k, c\rangle|^2/\lambda_k$
- Scale: $\varphi'/\pi = \rho_{\mathrm{rel}} = (2\pi)^{-1}\,\mathrm{tr}\,\mathsf{Q}$

Future directions:

- Extension to continuous POVM and general observables
- Numerical optimization algorithms for window design
- Applications to quantum metrology and thermometry
- Connections to resource theories and entanglement