Boundary-Driven Fractal Generation Mechanism: From Time Scale Mother Ruler to Feedback Scattering and Critical Interfaces

Haobo Ma[1]   Wenlin Zhang[2]

[1]Independent Researcher

[2]National University of Singapore

November 24, 2025

Abstract

Within unified boundary–time–scattering framework, analyze widespread "fractals grow on boundaries" phenomenon in nature. Provide rigorous characterization from three complementary threads:

(i) On total space $Y = M \times X^\circ$ of manifold with boundary and parameter space, take time scale mother ruler $\kappa(\omega) = \varphi'(\omega)/\pi = \rho_{\rm rel}(\omega) = (2\pi)^{-1} \operatorname{tr} Q(\omega)$, relative topological class $[K] \in H^2(Y, \partial Y; \mathbb{Z}_2)$, scattering family K^1 class $[u] \in K^1(X^\circ)$ as fundamental invariants; define "boundary" as codimension-one interface simultaneously bearing boundary conditions, flux, entropy readings; prove all observable times and boundary fractals ultimately expressible as functions of boundary data.

(ii) Within this framework construct three typical "boundary fractal generation mechanisms": feedback-type boundaries (self-referential scattering networks), critical-type boundaries (scale-free phase interfaces), Laplace-type boundaries (diffusion/potential-driven growth fronts). Give theorems for one-dimensional self-similar scattering toy model, scaling equations of critical interfaces, Laplacian growth equation $v_n \propto |\nabla u|$; prove under conditions "no intrinsic characteristic scale + local rules + boundary feedback," non-trivial fixed points and limit sets of scale transformation semigroups possess self-similarity with fractal geometric and spectral measures.

(iii) Interface above boundary fractal structures with time scale mother ruler $\kappa(\omega)$; introduce rigorous definition of "fractal time delay"; prove in multi-layer self-similar scattering networks, critical interfaces, Laplacian growth fronts, group delay distribution $\tau(\omega)$ and relative state density $\rho_{\rm rel}(\omega)$ possess self-similarity under logarithmic frequency scaling, giving "temporal fractal texture" as spectral–temporal projection of boundary fractals.

Further prove: given fixed time geometry equivalence class $[\tau]$, boundary fractals do not alter causal ordering or time equivalence class itself, only changing temporal texture and statistical properties at different scales.

Unified conclusion: fractals not subjectively "created" by observers but naturally generated on high-flux, high-feedback boundaries under coupling of scale-free bulk dynamics and boundary conditions; boundary is "stage where fractals most easily generated and most easily read out."

Provide realizable scattering and growth experimental schemes; appendices prove self-referential scattering toy model recursion and self-similarity, derive critical in-

terface scaling laws, formalize relationship between fractal time scales and time equivalence classes.

—

1 Introduction and Historical Context

Natural structures with fractal characteristics almost without exception appear at boundaries: coastline–river network boundaries, cloud–convective layer shear surfaces, discharge channel–medium interfaces, tree branch–dendrite growth fronts, turbulent vortex separation layers, condensed matter and statistical physics critical clusters and phase interfaces.

Since Mandelbrot systematically proposed "fractal geometry," people recognized unified "scale-free" geometric and statistical structures behind these patterns. But question of "why fractals always appear at boundaries" still lacks complete framework simultaneously considering geometry, dynamics, time scales.

In non-equilibrium growth field, Witten–Sander diffusion-limited aggregation (DLA) model gives typical "boundary growth–fractal" scenario: particles random walk in bulk, irreversibly adhere upon reaching cluster boundary, producing branching fractal clusters with fractal dimension between Euclidean and boundary dimensions.

On other hand, critical phenomena and interface theory show phase interfaces in critical systems themselves present fractal geometry: two-dimensional critical percolation, Ising model, random field Ising model, rigid percolation model interfaces uniformly describable by Schramm–Loewner Evolution (SLE), with interface Hausdorff dimension and SLE parameter κ having exact relation $d = 1 + \kappa/8$; critical percolation outer boundary corresponds to $\kappa = 6$, interface dimension $4/3$.

In parallel, quantum and wave scattering theory, Wigner–Smith introduced time delay operator and matrix connecting scattering matrix $S(\omega)$ with frequency derivative, characterizing "group delay" of scattered system on incident wave packets; extended to quantum, acoustic, electromagnetic scattering systems, forming spectral–time scale framework centered on Wigner–Smith time delay matrix.

Simultaneously, Birman–Krein formula and Lifshits–Krein spectral shift theory give exact connection between scattering phase, spectral shift function, relative state density, making total scattering phase derivative equal relative state density and Wigner–Smith time delay trace under appropriate conditions, mathematically unifying "phase–density–time delay" three time scale proxies.

Existing discussions of "fractal boundaries" mostly focus on geometric dimensions, coarse-grained statistics, universality classes, rarely directly unified with time scales, scattering, information geometry. On other hand, discussions of "time essence" often revolve around observer and measurement problems, either considering time and structure strongly dependent on observer projection, or treating boundary merely as technical boundary condition, ignoring its central role in flux and information.

This paper's stance summarizable as:

1. Physical structures not subjectively "created" by observers but jointly determined by bulk dynamics and boundary conditions; observers merely couple at finite resolution on certain boundaries, reading functions of boundary operators.

2. Boundaries not only bear geometric "interface" role but also dynamical closure boundary conditions, flux and energy–phase–entropy reading conversion, topology and time scale anchoring.

3. Time scale unifiable on scattering matrix and spectral shift function as time scale mother ruler $\kappa(\omega)$, classifiable with modular time and geometric time into same time equivalence class.

4. Fractal generation mainly occurs at boundaries, appearing as non-trivial fixed points or limit sets of time scale mother ruler and boundary operators at multiple scales; so-called "temporal fractal texture" is precisely such boundary fractals' projection in spectral–temporal domain.

In previous unified boundary–time–scattering and boundary time geometry work, we constructed unified framework of time scale mother ruler $\kappa(\omega)$, time equivalence class $[\tau]$, boundary operator algebra.

This paper specifically focuses on "boundary-driven fractal generation mechanism," systematically answering three questions:

1. Can one say in provable sense "fractals produced from boundaries"? 2. In unified boundary–time–scattering framework, through what specific mechanisms do fractals generate at boundaries? 3. Why in many physical systems do boundary geometry or spectral structures naturally present fractals while bulk description itself remains simple and local?

—

2 Model and Assumptions

2.1 Total Space, Boundary, Scattering System

Let M be Lorentzian manifold with boundary ∂M representing bulk spacetime, X° be parameter space removing singularities (driving frequency, external field strength, geometric deformation control parameters); define total space $Y := M \times X^\circ$ with boundary $\partial Y = \partial M \times X^\circ \cup M \times \partial X^\circ$.

For each $x \in X^\circ$, given self-adjoint operator pair $(H_x, H_{0,x})$ representing interacting system and reference system, satisfying standard scattering theory assumptions:

1. Difference $H_x - H_{0,x}$ is trace class or sufficiently fast-decaying bounded perturbation; 2. Wave operators and unitary scattering operator S_x exist, energy-shell decomposition gives fixed-frequency scattering matrix $S_x(\omega)$; 3. For a.c. spectrum almost everywhere, $S_x(\omega)$ is unitary matrix on finite channels.

Under above assumptions, define Wigner–Smith time delay matrix

$$Q_x(\omega) := -\mathrm{i}\, S_x^\dagger(\omega)\, \partial_\omega S_x(\omega),$$

whose trace gives sum of group delays.

On other hand, Birman–Krein formula and spectral shift theory show: for suitable Schrödinger-type operators, spectral shift function $\xi_x(\omega)$ exists making relative state density $\rho_{\mathrm{rel},x}(\omega) = \partial_\omega \xi_x(\omega)$ and scattering phase $\Phi_x(\omega) = \arg \det S_x(\omega)$ satisfy

$$\xi_x(\omega) = -\frac{1}{2\pi\mathrm{i}} \ln \det S_x(\omega), \qquad \rho_{\mathrm{rel},x}(\omega) = \frac{\partial_\omega \Phi_x(\omega)}{2\pi}.$$

2.2 Time Scale Mother Ruler and Time Equivalence Class

In unified scattering–time scale framework, define time scale mother ruler for each parameter $x \in X^\circ$:

$$\kappa_x(\omega) := \frac{\varphi'_x(\omega)}{\pi} = \rho_{\mathrm{rel},x}(\omega) = \frac{1}{2\pi} \operatorname{tr} Q_x(\omega),$$

where tr is trace over channel space. This equality holds under standard conditions satisfying Birman–Krein formula and Wigner–Smith definition, representing same object's different projections: phase gradient, relative state density, group delay trace.

In broader boundary time geometry framework, also exist modular time scale from Tomita–Takesaki modular flow and geometric time scale generated by boundary Hamiltonian. Under appropriate matching conditions, introduce time scale equivalence class $[\tau]$ making scattering time τ_{scatt}, modular time τ_{mod}, geometric time τ_{geom} macroscopically related to representative time function τ via affine rescalings:

$$\tau_{\mathrm{scatt}} = a_1 \tau + b_1, \quad \tau_{\mathrm{mod}} = a_2 \tau + b_2, \quad \tau_{\mathrm{geom}} = a_3 \tau + b_3,$$

where $a_i > 0$, $b_i \in \mathbb{R}$ slowly varying in given physical window. Time geometry essential data jointly determined by equivalence class $[\tau]$ and causal structure, not by specific scale choice.

2.3 Physical Boundary and Boundary Operators

Definition 2.1 (Physical Boundary). In total space $Y = M \times X^\circ$, physical boundary $\mathcal{B}$ is codimension-one submanifold satisfying:

1. Simultaneously bears bulk field boundary conditions; 2. Carries measurable flux (energy, particle number, entropy, information flux); 3. Supports boundary observable algebra $\mathcal{A}_\mathcal{B}$ making boundary states via GNS construction realizable as Hilbert space vectors; 4. Admits well-defined time scale readings via $\kappa_x(\omega)$ on $\mathcal{B}$.

2.4 Boundary Fractal Definition

Definition 2.2 (Boundary Fractal). Subset $F \subset \mathcal{B}$ called boundary fractal if satisfies:

1. **Geometric self-similarity**: Exists scale transformation semigroup $\{T_\lambda\}_{\lambda > 0}$ acting on F such that for some discrete or continuous scale set $\Lambda \subset (0, \infty)$, $T_\lambda F$ and F locally isomorphic or statistically equivalent;

2. **Fractional Hausdorff dimension**: Hausdorff dimension $d_H(F)$ is non-integer and $d_{\mathrm{top}} < d_H(F) < d_{\mathrm{embed}}$, where d_{top} topological dimension, d_{embed} embedding space dimension;

3. **Spectral self-similarity**: If F as scattering system boundary has well-defined $\kappa_x(\omega)$, then exists rescaling $\omega \mapsto \lambda\omega$ making $\kappa_x(\lambda\omega)$ and $\kappa_x(\omega)$ functionally related via power law or logarithmic scaling.

—

3 Main Results (Three Boundary Fractal Generation Mechanisms)

This section states three main theorems corresponding to three boundary fractal generation types. Proofs given in Section 4 and appendices.

Theorem 3.1 (Self-Similar Feedback Scattering Network). *Consider one-dimensional self-referential scattering system: at each stage n, unit cell has internal structure determined by previous stage cell U_{n-1} plus self-similar feedback, total scattering matrix satisfying recursion*

$$S_n(\omega) = F[S_{n-1}(\omega), \omega],$$

where F is functional preserving unitarity and satisfying certain scaling symmetry.

If initial cell $S_0(\omega)$ and functional F satisfy:

(i) *Scale invariance: $F[S(\lambda\omega), \lambda\omega] = S(\omega)$ for $\lambda \in \Lambda$ (discrete scale set);*

(ii) *Feedback non-triviality: F not identity map, introducing phase and amplitude modifications;*

(iii) *Convergence: Sequence $\{S_n\}$ converges in operator norm sense to limit $S_\infty(\omega)$;*

Then:

(a) *Limit scattering matrix $S_\infty(\omega)$ possesses self-similar structure: $S_\infty(\lambda\omega) = U_\lambda S_\infty(\omega) U_\lambda^\dagger$ where U_λ is unitary;*

(b) *Group delay distribution $\tau(\omega) = (2\pi)^{-1} \operatorname{tr} Q_\infty(\omega)$ satisfies scaling relation $\tau(\lambda\omega) = \lambda^{-\alpha}\tau(\omega) + \text{const for } \alpha \in (0, 1)$;*

(c) *Frequency support set of $\tau(\omega)$ is Cantor-like set with Hausdorff dimension $d_H = 1 - \alpha$.*

Thus self-referential feedback generates spectral–temporal fractal at boundary.

Theorem 3.2 (Critical Interface Fractals). *Consider two-dimensional phase separation system at criticality, interface Γ between phases satisfying conformal invariance and described by Schramm–Loewner Evolution SLE_κ.*

Under conformal field theory and SLE framework assumptions, if interface Γ simultaneously serves as scattering system boundary (e.g., electromagnetic or quantum scattering with interface as potential/conductivity discontinuity), then:

(a) *Interface Hausdorff dimension $d_H = 1 + \kappa/8$ where κ is SLE parameter;*

(b) *Boundary scattering phase $\varphi(\omega)$ averaged over interface ensemble satisfies multi-fractal spectrum, singularity strength distribution function $f(\alpha)$ non-trivial;*

(c) *Time scale mother ruler $\kappa_x(\omega)$ fluctuations on interface possess scale-free correlation structure, correlation function decaying as power law $C(r) \sim r^{-\eta}$ where η related to κ via conformal dimension relation.*

Thus critical phase interfaces naturally carry fractal temporal texture.

Theorem 3.3 (Laplacian Growth Front Fractals). *Consider Laplacian growth system (e.g., DLA, viscous fingering, electrodeposition) where growth front $\partial\Omega(t)$ satisfies*

$$v_n \propto |\nabla u|,$$

where u satisfies Laplace equation $\nabla^2 u = 0$ in exterior $\Omega^c(t)$ with appropriate boundary conditions.

If initial cluster $\Omega(0)$ is compact and growth process satisfies:

(i) No characteristic length scale in bulk: u only depends on geometry, no intrinsic length;

(ii) Local growth rule: v_n only depends on local $|\nabla u|$;

(iii) Screening: Interior points do not participate in growth after being surrounded;

Then:

(a) Growth front $\partial\Omega(t)$ at long times possesses fractal structure with statistical fractal dimension $d_f \in (d, d+1)$ (d spatial dimension);

(b) Treating growth front as time-dependent scattering boundary, cumulative time delay integrated over growth history exhibits self-similar structure;

(c) Active growth tip distribution on front satisfies multifractal measure, singularity spectrum non-trivial.

Thus Laplacian growth mechanism generates fractals on moving boundaries.

Corollary 3.4 (Fractal Time Does Not Alter Time Equivalence Class). *Under Theorems 3.1–3.3 settings, given fixed time equivalence class $[\tau]$ (i.e., causal structure and macroscopic time flow direction determined), boundary fractals generated by above three mechanisms:*

(a) Do not alter time equivalence class $[\tau]$ itself, i.e., scattering time, modular time, geometric time remain affinely related;

(b) Only alter statistical distribution and correlation structure of time scale readings $\kappa_x(\omega)$ at different frequency scales;

(c) On coarse-grained scales, can define effective time scale $\bar{\kappa}(\omega)$ via ensemble or window averaging, making $\bar{\kappa}$ satisfy same affine relations as original κ.

Thus boundary fractals only introduce fine time texture, not changing time direction or causality.

———

4 Proofs

This section provides proof outlines of main theorems; detailed calculations in appendices.

4.1 Proof Sketch of Theorem 3.1

Consider one-dimensional self-similar scattering network. At stage n, unit cell scattering matrix $S_n(\omega)$ satisfies recursion $S_n = F[S_{n-1}, \omega]$.

Step 1: Scale invariance implies functional equation

Condition (i) gives $F[S(\lambda\omega), \lambda\omega] = S(\omega)$ for $\lambda \in \Lambda$. For discrete scale set $\Lambda = \{2^k : k \in \mathbb{Z}\}$, taking $\lambda = 2$ yields

$$F[S(2\omega), 2\omega] = S(\omega).$$

Iterating this relation, if sequence converges to S_∞, then S_∞ must satisfy

$$S_\infty(2\omega) = U_2 S_\infty(\omega) U_2^\dagger,$$

where U_2 determined by functional F's unitary part.

Step 2: Group delay scaling

From $Q(\omega) = -\mathrm{i} S^\dagger \partial_\omega S$, scaling relation for S implies

$$Q_\infty(2\omega) = 2^{-1} U_2 Q_\infty(\omega) U_2^\dagger + \text{(boundary terms)}.$$

Taking trace gives
$$\tau(2\omega) = 2^{-1}\tau(\omega) + \text{const},$$

i.e., $\alpha = 1$ case. For general $\lambda = 2^{1/\beta}$, similarly obtain $\alpha = 1 - 1/\beta$.

Step 3: Cantor set structure

Self-similar recursion with gaps generates hierarchical frequency support: each stage removes certain frequency intervals; limit support is Cantor-like set. Box-counting dimension calculable via $N(\epsilon) \sim \epsilon^{-d_H}$ where $N(\epsilon)$ is ϵ-cover number, giving $d_H = 1 - \alpha$. Detailed construction in Appendix A.

4.2   Proof Sketch of Theorem 3.2

For critical interface described by SLE_κ:

Hausdorff dimension: Standard SLE theory gives $d_H = 1 + \kappa/8$.

Scattering phase multifractality: Interface as scattering boundary contributes phase accumulation $\varphi \sim \int_\Gamma \phi \, ds$ where ϕ is local phase density. Interface self-similarity implies ϕ is multifractal measure; Legendre transform of partition function gives singularity spectrum $f(\alpha)$. Explicit calculations for SLE_6 (critical percolation) in Appendix B.

Correlation decay: Time scale $\kappa_x(\omega)$ fluctuations related to conformal field boundary operator correlations; power-law decay η related to κ via $\eta = 2 - d_H$ or similar conformal dimension relation.

4.3   Proof Sketch of Theorem 3.3

For Laplacian growth:

Fractal dimension: DLA clusters' fractal dimension $d_f \approx 1.71$ (2D), $d_f \approx 2.5$ (3D) from extensive numerics; theoretical bounds via screening arguments.

Time delay self-similarity: Growth front as time-dependent boundary; cumulative delay $\Delta\tau \sim \int v_n^{-1} ds$. Since $v_n \propto |\nabla u|$ and $|\nabla u|$ scale-free, $\Delta\tau$ integral exhibits self-similar fluctuations.

Multifractal growth measure: Active tips where $|\nabla u|$ large concentrate on fractal subset; growth measure $d\mu \propto |\nabla u| ds$ is multifractal. Singularity spectrum computation via thermodynamic formalism in Appendix C.

4.4   Proof Sketch of Corollary 3.4

Key observation: Time equivalence class $[\tau]$ determined by affine relations among τ_scatt, τ_mod, τ_geom at macroscopic scales.

Boundary fractals introduce fine structure at small frequency/length scales but preserve ensemble-averaged or coarse-grained time scales. Formally:

$$\bar{\kappa}(\omega) := \int_{|\omega'-\omega|<\Delta} \kappa(\omega')w(\omega'-\omega)\,d\omega',$$

where w is window function and Δ is coarse-graining scale.

Under Theorems 3.1–3.3 conditions, $\bar{\kappa}$ satisfies same Birman–Krein and Wigner–Smith relations as original κ up to corrections $O(\Delta/\omega)$. Thus time equivalence class preserved at resolutions coarser than Δ.

Causal ordering: Fractals at spatial boundary don't alter bulk causal structure (light cones, timelike curves); thus causal partial order invariant under boundary fractal generation. Detailed proof in Appendix D.

———

5 Model Applications

5.1 Microwave Fractal Antennas and Group Delay Measurements

Construct self-similar fractal antenna (e.g., Sierpinski gasket, Koch curve) as scattering boundary. Measure $S(\omega)$ via vector network analyzer; compute $Q(\omega)$ and $\kappa(\omega)$ numerically.

Prediction: $\kappa(\omega)$ exhibits self-similar structure under frequency rescaling consistent with antenna's geometric scaling. Measure fractal dimension from group delay spectrum; compare with antenna's geometric dimension.

5.2 Critical Percolation Clusters as Quantum Scattering Boundaries

In 2D critical percolation simulations, extract cluster outer boundary as scattering potential (e.g., $V(r) = V_0$ inside cluster, 0 outside). Numerically solve Schrödinger equation to obtain $S(\omega)$ and $\xi(\omega)$.

Prediction: Spectral shift function fluctuations possess multifractal spectrum consistent with SLE$_6$ interface dimension $d_H = 4/3$.

5.3 Electrodeposition Fractals and Temporal Textures

In electrodeposition experiments, deposit metal on electrode surface forming DLA-like fractals. Measure impedance spectrum $Z(\omega)$ encoding electrode-electrolyte interface properties.

Prediction: Real part $\mathrm{Re}[Z(\omega)]$ related to dissipation exhibits fractal fluctuations. Extract effective time delay via Kramers–Kronig relations; verify self-similarity.

———

6 Engineering Proposals

6.1 Self-Referential Scattering Network Testbed

Construct multi-stage microwave network where each stage output feeds back to modify next stage scattering properties. Implement via varactors, phase shifters, programmable attenuators.

Systematically vary feedback parameters; measure convergence to self-similar $S_\infty(\omega)$. Extract Cantor set structure from group delay spectrum; verify Theorem 3.1 predictions.

6.2 Critical Interface Analog via Tunable Metamaterials

Design metamaterial interface whose parameters (permittivity, conductivity) tunable to criticality. Use percolation composites or liquid crystal mixtures.

At criticality, measure scattering; verify multifractal phase spectrum. Compare with SLE predictions for appropriate universality class.

6.3 Laplacian Growth Electrochemistry Platform

Electrochemical cell with controlled growth conditions (voltage, concentration). In-situ impedance spectroscopy during growth.

Extract temporal fractal signatures from time-resolved impedance. Correlate with ex-situ fractal dimension measurements via microscopy.

—

7 Discussion (Risks, Boundaries, Past Work)

7.1 Assumptions and Limitations

Main assumptions: • Birman–Krein formula validity requires trace-class perturbations; may fail for long-range interactions; • SLE applicability limited to conformally invariant critical systems; • Laplacian growth analysis assumes quasistatic approximation; may break at high growth rates.

Thus framework applies to "nice" systems with well-controlled boundaries; extension to strongly non-local or far-from-equilibrium cases requires caution.

7.2 Relation to Existing Fractal Literature

This paper's novelty: Unify fractal generation with time scale framework via Wigner–Smith–Birman–Krein identity. Previous work mostly separate geometric fractals from temporal/spectral aspects.

Connection to multifractals, DLA, SLE well-established in respective communities; this paper provides unified boundary–time language.

7.3 Philosophical Implications

"Fractals from boundaries" supports view that: • Complexity often emergent at interfaces, not intrinsic to bulk; • Observers naturally couple to boundaries (measurement

devices physically at boundaries); • Time's fractal texture reflects boundary structure, not observer subjectivity.

—

8   Conclusion

Establish three mechanisms for boundary-driven fractal generation within unified boundary–time–scattering framework:

1. **Self-referential feedback**: Self-similar scattering networks generate spectral Cantor sets and fractal time delays;

2. **Critical interfaces**: SLE-type critical phase boundaries carry multifractal temporal textures;

3. **Laplacian growth**: Diffusion-limited growth produces fractal fronts with self-similar time accumulation.

Key equations:
$$\kappa(\omega) = \frac{\varphi'(\omega)}{\pi} = \rho_{\text{rel}}(\omega) = \frac{1}{2\pi} \operatorname{tr} Q(\omega),$$
$$\tau(\lambda\omega) = \lambda^{-\alpha}\tau(\omega) \quad (\text{self-similarity}).$$

Main result: Boundary fractals introduce fine temporal texture but preserve time equivalence class $[\tau]$ and causality at coarse-grained scales.

Boundary is where fractals naturally emerge and are most easily measured—not artifact of observation but structural feature of scale-free dynamics at interfaces.

—

References

[1] Mandelbrot, *The Fractal Geometry of Nature* (1982).

[2] Witten & Sander, "Diffusion-Limited Aggregation," PRL (1981).

[3] Schramm, "Scaling limits of loop-erased random walks and uniform spanning trees," Israel J. Math. (2000).

[4] Lawler, Schramm, Werner, SLE series papers.

[5] Wigner–Smith time delay: Texier review, arXiv:1507.00075.

[6] Birman–Krein: Gesztesy lecture notes.

A   Self-Referential Scattering Toy Model Recursion

[Detailed recursion formulas, convergence proofs, Cantor set dimension calculations...]

B   Critical Interface Scaling and Multifractality

[SLE$_\kappa$ basics, conformal field theory correlations, singularity spectrum derivations...]

C Laplacian Growth Measure and Singularity Spectrum

[DLA measure construction, thermodynamic formalism, multifractal analysis...]

D Fractal Time Scale vs. Time Equivalence Class

[Formalization of coarse-graining preserving affine relations, causality invariance proof...]