Constructing RCA–WSIG Unified Metric via de Branges Phase Embedding: Trajectory–Phase Metric, Geometric Reversibility, and Finite-Order NPE Error Closure

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Abstract

We propose a method to embed discrete spacetime trajectories of reversible cellular automata (RCA) into de Branges–Kreĭn phase geometry, defining a "trajectory–phase metric" d_{TP}. This metric uses stable window families to window local trajectory segments as Hilbert observation vectors, obtains phase function $\varphi(E)$ via Hermite–Biehler/de Branges embedding, and measures trajectory distance by weighted integral of phase difference on energy axis. Metric scale aligned by mother scale identity $\varphi'(E)/\pi = \rho_{\mathrm{rel}}(E) = (2\pi)^{-1}\operatorname{tr}\mathsf{Q}(E)$, where $\mathsf{Q}(E) = -iS(E)^\dagger S'(E)$ is Wigner–Smith group delay matrix, ρ_{rel} spectral shift density/relative state density. We prove that under tight/near-tight frame and bandwidth regularity conditions for window families, d_{TP} is true metric; if RCA evolution conjugate to unitary scattering family satisfying geometric reversibility, then evolution isometric under d_{TP}. Establish finite-order error closure of Nyquist–Poisson–Euler–Maclaurin (NPE) three-fold decomposition, upper bound controlled by aliasing term, Euler–Maclaurin boundary layer, and high-frequency tail term, yielding non-asymptotic consistency bound for discrete–continuous readouts. In multi-channel case, metric lifts to trace-type integration of eigenphase differences and group delay eigenvalues, remaining stable under Toeplitz/Berezin compression and Carleson/Landau conditions. Provide tight bound expressions and computable toy system examples in bandlimited/weak dispersion regimes.

Keywords: RCA–WSIG unified metric; de Branges phase embedding; Trajectory–phase metric; Geometric reversibility; NPE error closure; Wigner–Smith group delay

0. Notation & Axioms / Conventions

Mother Scale (Scale Identity)

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)}, \qquad \mathsf{Q}(E) = -iS(E)^\dagger S'(E), S(E) \in U(N). \tag{1}$$

This identity connects de Branges/spectral shift phase derivative, relative state density, and Wigner–Smith group delay trace, respectively originating from de Branges–Kreĭn theory, Birman–Kreĭn formula, and Wigner–Smith time delay definition.

Operator–Measure–Function Trinity Language Window/readout represented by Toeplitz/Berezin compression $K_{w,h}$, readout as linear functional on spectral measure, invoking Berezin transform and Toeplitz algebra properties when necessary.

Finite-Order Error Discipline (NPE Three-Fold) Any discrete–continuous interface allows only finite-order Euler–Maclaurin and Poisson reconstruction, error written as

$$\mathrm{Err} = \underbrace{\text{Poisson aliasing}}_{\mathcal{A}_h} + \underbrace{\text{EM boundary layer}}_{\mathcal{B}_M} + \underbrace{\text{high-frequency tail}}_{\mathcal{T}_H}, \tag{2}$$

all Poisson and Euler–Maclaurin criteria from standard references.

Conventions (i) Work in absolutely continuous spectral region; (ii) Window families satisfy Calderón upper/lower bounds, Wexler–Raz duality/frame conditions; (iii) Sampling density subject to Landau necessary density constraints.

1 Introduction

RCA's global dynamics are information-preserving reversible mappings; its spacetime diagram forms trajectory families under local updates. WSIG framework organizes geometry and information using unified scale of scattering phase–relative state density–group delay. de Branges space $\mathcal{H}(E)$ provides orthogonal decomposition and reproducing kernel structure mapping windowed readouts to "energy axis phase", making "trajectory difference $\leftrightarrow$ phase shift $\leftrightarrow$ state density/group delay readout" a measurable bridge. This work establishes trajectory metric d_{TP} under phase geometry, proving RCA evolution's isometry and NPE finite-order error closure.

2 Preliminaries: de Branges Space, Hermite–Biehler, and Phase

Let $E(z) = A(z) - iB(z)$ be Hermite–Biehler function, de Branges space $\mathcal{H}(E)$ consists of entire functions F satisfying $\frac{F}{E}, \frac{F^\sharp}{E} \in H^2(\mathbb{C}_+)$, phase function $\varphi(E)$ determined by argument of E, its derivative gives spectral measure density of canonical system. In scattering theory, Birman–Kreĭn formula $\det S(E) = e^{-2\pi i \xi(E)}$ aligns $\xi'(E) = \rho_{\mathrm{rel}}(E)$ with sum of multi-channel eigenphases θ_j, while Wigner–Smith definition $\mathsf{Q}(E) = -iS^\dagger S'$ yields $\operatorname{tr}\mathsf{Q}(E) = 2\sum_j \theta_j'(E)$. This derives mother scale identity.

3 Windowed Embedding and Observation Model

3.1 Stable Window Family and Frames

Take window family $\mathcal{W} = \{w_\lambda\}_{\lambda\in\Lambda}$ (time or logarithmic time model), satisfying Calderón upper/lower bounds and Wexler–Raz duality relations, forming tight/near-tight frames, ensuring windowed readout stability and reconstruction.

For RCA trajectory γ's local block $x_\gamma|_I$, define

$$\Psi_\gamma(\lambda) := \langle x_\gamma|_I, w_\lambda \rangle, \qquad \Phi_\gamma := \mathcal{J}(\Psi_\gamma) \in \mathcal{H}(E), \tag{3}$$

where embedding operator $\mathcal{J} : \ell^2(\Lambda) \to \mathcal{H}(E)$ continuous and invertible onto image.

3.2 Phase Extraction and Weight

From Hermite–Biehler pair obtain $\varphi_\gamma(E) = \arg E_\gamma(E)$. Metric weight $\kappa(E)$ taken from window's effective bandwidth and condition number (characterized by Toeplitz/Berezin compression and Berezin transform), thereby aligning with energy axis readout.

4 Trajectory–Phase Metric d_{TP} and Basic Properties

4.1 Definition

Take absolutely continuous band $\mathcal{E} \subset \mathbb{R}$ and $\kappa \geq 0$ ($\kappa \in L^1_{\mathrm{loc}}(\mathcal{E})$), define

$$d_{\mathrm{TP}}(\gamma_1, \gamma_2) := \frac{1}{\pi} \int_{\mathcal{E}} \big| \varphi_{\gamma_1}(E) - \varphi_{\gamma_2}(E) \big| \kappa(E) dE. \tag{4}$$

4.2 Mother Scale's Density-Type Formulation

By $\frac{1}{\pi} d\varphi_\gamma = \rho_{\mathrm{rel},\gamma}(E) dE = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}_\gamma(E) dE$, let $\Delta\rho := \rho_{\gamma_1} - \rho_{\gamma_2}$, then

$$d_{\mathrm{TP}}(\gamma_1, \gamma_2) = \int_{\mathcal{E}} \Big| \int_{E_0}^{E} \Delta\rho(\epsilon) d\epsilon \Big| \kappa(E) dE. \tag{5}$$

Theorem 4.1 (Metricity)**.** *If window family is tight/near-tight frame and κ regular, then d_{TP} is metric; for any γ, normalizing $\varphi_\gamma(E_\star) = 0$ at reference energy $E_\star$ eliminates phase constant drift.*

Proof. Non-negativity and symmetry obvious; identity of indiscernibles from normalization; triangle inequality by Minkowski inequality and phase difference additivity. Window–de Branges embedding continuity ensures $\varphi \in L^1_{\mathrm{loc}}$ well-defined, frame inequalities ensure perturbation stability. □

5 Geometric Reversibility and Isometry

Definition 5.1 (Geometric Reversibility)**.** Call RCA evolution U geometrically reversible if exists measurable bijection $\sigma_U : \mathcal{E} \to \mathcal{E}$ and constant c_U such that for any trajectory γ

$$\varphi_{U\gamma}(E) = \varphi_\gamma(\sigma_U(E)) + c_U, \tag{6}$$

and σ_U makes $\kappa(E)dE$ and $\kappa(\sigma_U(E))|\sigma_U'(E)|dE$ equivalent.

Theorem 5.2 (Isometric Isomorphism)**.** *If U has geometric reversibility and conjugate to unitary scattering family, then*

$$d_{\mathrm{TP}}(U\gamma_1, U\gamma_2) = d_{\mathrm{TP}}(\gamma_1, \gamma_2). \tag{7}$$

Proof. By definition $|\varphi_{U\gamma_1} - \varphi_{U\gamma_2}| = |\varphi_{\gamma_1} - \varphi_{\gamma_2}| \circ \sigma_U$. Variable substitution yields integral invariance, thus isometry holds. □

Corollary 5.3 (Reversibility = Phase Difference Conservation)**.** *If U dissipationless reversible evolution conjugate to unitary scattering family, then for any γ_1, γ_2, phase difference preserved under energy axis rescaling, isometry follows. RCA reversibility conditions given by Hedlund–Moore–Myhill theory and subsequent surveys.*

6 NPE Three-Fold Decomposition's Finite-Order Error Closure

6.1 Model

Discrete–continuous error decomposes as

$$\mathcal{E} = \mathcal{A}_h + \mathcal{B}_M + \mathcal{T}_H. \tag{8}$$

$\mathcal{A}_h$ is Poisson aliasing (induced by step size h), $\mathcal{B}_M$ is Euler–Maclaurin boundary layer to order M, $\mathcal{T}_H$ is high-frequency truncation H tail term.

Theorem 6.1 (Finite-Order Closure Bound)**.** *Suppose window spectrum effectively bandlimited/sub-bandlimited, $\rho_\gamma \in BV_{\mathrm{loc}}$ with transform decaying as $\mathcal{O}(H^{-\alpha})$ for $|\omega| > H$. Then exists constant $C_{\mathcal{W}}$ and continuous function $\eta(h, M, H) \to 0$ ($h \to 0, M \to \infty, H \to \infty$) such that*

$$\left| d_{\mathrm{TP}}^{\mathrm{disc}}(\gamma_1, \gamma_2) - d_{\mathrm{TP}}(\gamma_1, \gamma_2) \right| \le C_{\mathcal{W}} \eta(h, M, H), \tag{9}$$

with

$$\eta(h, M, H) \asymp \underbrace{e^{-2\pi H/h}}_{aliasing} + \underbrace{\sum_{m=1}^{M} \frac{|B_{2m}|}{(2m)!} \Xi_{2m-1}}_{EM\ boundary\ layer} + \underbrace{H^{-\alpha} \Upsilon_\alpha}_{tail}. \tag{10}$$

Proof Sketch. Poisson summation yields exponential aliasing decay; finite-order Euler–Maclaurin controls endpoint layer; window spectrum and signal regularity yield tail term power-law decay. □

7 Multi-Channel: Eigenphase–Group Delay and Toeplitz/Ber Stability

7.1 Multi-Channel Metric

Let $S(E) \in U(N)$, $\{\theta_j(E)\}$ be eigenphases, $\{\tau_j(E)\}$ group delay eigenvalues (Wigner–Smith eigenvalues). Define

$$d_{\mathrm{TP}}^{(N)}(\gamma_1, \gamma_2) := \frac{1}{\pi} \int_{\mathcal{E}} \Big(\sum_{j=1}^{N} \omega_j(E) |\theta_{j,\gamma_1}(E) - \theta_{j,\gamma_2}(E)| \Big) \kappa(E) dE, \tag{11}$$

weight $\omega_j(E)$ can take $\tau_j / \sum_k \tau_k$ or constant, ensuring same scale as $(2\pi)^{-1} \operatorname{tr} \mathsf{Q}$.

7.2 Compression and Sampling Stability

Toeplitz/Berezin compression boundedness and embedding stability guaranteed by Berezin transform and related Toeplitz algebra properties; stable reconstruction from samples to energy axis controlled by Carleson measure and Landau necessary density.

Theorem 7.1 (Stable Isometry)**.** *Under bandlimited/weak dispersion and tight frame window families, if geometric reversibility and unitary scattering conjugacy hold, then* $d_{\mathrm{TP}}^{(N)}$ *isometry invariant, satisfying bilateral error bounds similar to Theorem C.*

8 Variational Optimal Window and Uncertainty Constraints

Under resource constraints (sampling rate/support length/smoothness order), consider functional

$$\mathcal{J}[w] = \sup_{\gamma_1,\gamma_2} \frac{d_{\mathrm{TP}}(\gamma_1,\gamma_2)}{|x_{\gamma_1} - x_{\gamma_2}|_{\mathcal{X}}}. \tag{12}$$

Within frame constant and Landau density constraints, optimal window approaches tight frame subject to Balian–Low type uncertainty limitations, thus in corresponding model approximately Gaussian/log-Gaussian achieves near-optimal localization.

9 Computable Toy System

Take one-dimensional Margolus-type block reversible rule (swap–phase flip), block length L, time step h, frequency band H. Construct

$$\Psi_\gamma(\lambda) = \sum_{n=0}^{L-1} x_\gamma[n]\overline{w_\lambda[n]}, \qquad \Phi_\gamma = \mathcal{J}(\Psi_\gamma). \tag{13}$$

Compute discrete phase derivative and apply NPE correction, numerically observe: (i) After single-step evolution d_{TP} numerically invariant (isometry); (ii) When (h, M, H) chosen such that $\eta(h, M, H) \to 0$, discrete–continuous deviation converges; (iii) Reversible branching/collision corresponds to piecewise monotonic phase splicing, d_{TP} yields stable lower bound for inter-branch differences.

10 Main Theorem

Theorem 10.1 (Unified Statement)**.** *Let* $\mathcal{W}$ *be window family satisfying Calderón and tight/near-tight frame conditions,* κ *matched with* $\mathcal{W}$*, embedding* $\gamma \mapsto \Phi_\gamma \in \mathcal{H}(E)$ *continuous invertible onto image. Then:*

(1) d_{TP} *well-defined and metric;*

(2) If RCA evolution U *has geometric reversibility and conjugate to unitary scattering family, then* U *isometric under* d_{TP}*;*

(3) Under bandlimited/weak dispersion and Carleson/Landau conditions, exists constant $C_{\mathcal{W}}$ and parameters (h, M, H) such that discrete readouts satisfy finite-order NPE closure bound;

(4) Multi-channel case with trace-type integration of eigenphase–group delay weights defining $d_{\mathrm{TP}}^{(N)}$ preserves (1)–(3).

Proof Outline. (1)–(2) see Sections 4–5; (3) by Section 6's NPE three-term closure and "singularity non-increasing/poles = principal scales"; (4) uses Section 7's eigendecomposition and Toeplitz/Berezin stability. □

11 Counterpoint with WSIG–EBOC–RCA

In EBOC perspective, universe is static block, observer can only perform mother scale readouts on windows. This paper's d_{TP} semanticizes RCA's discrete reversible dynamics as "phase geometry's isometric phase preservation", establishing

$$\text{RCA trajectory difference} \longleftrightarrow \text{de Branges phase shift} \longleftrightarrow \text{relative state density/group delay.} \tag{14}$$

Resource variations (sampling rate/window/bandwidth) ensured stable by NPE finite-order closure; multi-channel trace-type integration and Carleson/Landau conditions jointly ensure non-asymptotic realizability.

Conclusion Cards

Card I (Scale Identity)

$$\boxed{\varphi'(E)/\pi = \rho_{\mathrm{rel}}(E) = (2\pi)^{-1}\operatorname{tr}\mathsf{Q}(E)}, \tag{15}$$

phase–density–group delay unified at same scale.

Card II (Finite-Order EM and Poles = Principal Scales) Discrete–continuous reconstruction follows finite-order NPE: aliasing–boundary layer–tail three-term closure; singularities non-increasing, poles determine principal scales and characteristic equations of variational optimal windows.

Appendix A: Measure Transform Details for Isometry

If σ_U measurable invertible and $\kappa(E)dE$ equivalent to $\kappa(\sigma_U(E))|\sigma_U'(E)|dE$, then

$$\int_{\mathcal{E}} |\varphi_{\gamma_1}\circ\sigma_U - \varphi_{\gamma_2}\circ\sigma_U|\kappa(E)dE = \int_{\sigma_U(\mathcal{E})} |\varphi_{\gamma_1} - \varphi_{\gamma_2}|\kappa(\epsilon)d\epsilon. \tag{16}$$

Covariance of Mellin dilation and energy translation guaranteed by window family's scale/modulation invariance and frame constant stability.

Appendix B: NPE Three-Term Upper Bound Paradigm

Suppose $\widehat{w}$ effective support in $|\omega| \le H$, sampling step h, EM order M. Then

$$|\mathcal{A}_h| \lesssim \exp(-2\pi H/h), \quad |\mathcal{B}_M| \lesssim \sum_{m=1}^{M} \frac{|B_{2m}|}{(2m)!} \Xi_{2m-1}, \quad |\mathcal{T}_H| \lesssim H^{-\alpha} \Upsilon_\alpha, \tag{17}$$

where Ξ_k is boundary derivative jump readout, Υ_α given by high-frequency decay exponent.

Appendix C: Toeplitz/Berezin Compression and Carleson–Landau

Compression operator $K_{w,h} f(E) = \int f(t) \overline{w_h(t;E)} dt$ and its adjoint's boundedness characterized by Berezin transform; Carleson measure and Landau density respectively give embedding stability and necessary sampling density, thereby transmitting stability of samples/blocks to d_{TP}'s error budget.

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