Formal Construction of WSIG–EBOC Holographic Reconstruction Theory

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Abstract

We establish a holographic reconstruction theory with Windowed Scattering–Information Geometry (WSIG) as the observational layer and Eternal Block Observation–Computation (EBOC) as the ontological layer. The core proposition is: under an overlapping window family satisfying frame stability and topological consistency, the "trinity" of local readouts consisting of phase derivative–relative state density–Wigner–Smith group delay can be stably glued into a global scattering field within a non-asymptotic Nyquist–Poisson–Euler–Maclaurin three-term closed error budget, thereby providing a global encoding of the EBOC static block. We present unified notation, conventions, and gluing consistency conditions, prove three main theorems on existence–stability–uniqueness (under gauge choice), and propose implementable inversion–projection algorithms along with semantic correspondence to reversible cellular automata (RCA).

Keywords: Holographic reconstruction; WSIG; EBOC; Wigner-Smith delay; Frame theory; Čech cohomology; Birman-Kreĭn formula; Reversible cellular automata

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Contents

1 Notation, Conventions, and Axioms

Energy domain and multi-port scattering. Let the energy domain be a measurable set $\Omega \subset \mathbb{R}$, with channel number $N \in \mathbb{N}$. At almost every Lebesgue point of the absolutely continuous spectrum, the Wigner–Smith delay matrix and total phase for scattering matrix $S(E) \in U(N)$ are defined as

$$\mathsf{Q}(E) := -\,i\,S(E)^\dagger \partial_E S(E), \qquad \Phi(E) := \operatorname{Arg}\det S(E).$$

Let the half-phase be $\varphi(E) := \frac{1}{2}\Phi(E)$. Then

$$\partial_E \Phi(E) = \operatorname{tr}\mathsf{Q}(E), \qquad \partial_E \varphi(E) = \tfrac{1}{2}\,\operatorname{tr}\mathsf{Q}(E).$$

These three quantities satisfy the gauge identity with the relative state density $\rho_{\mathrm{rel}}(E)$:

$$\boxed{\frac{\varphi'(E)}{\pi} = \rho_{\mathrm{rel}}(E) = \frac{1}{2\pi}\operatorname{tr}\mathsf{Q}(E)},$$

where ρ_{rel} equals the derivative of the Kreĭn spectral shift function (this paper adopts scattering phase sign conventions and channel orientation that make the above equation hold). These equalities respectively stem from $\partial_E \log\det S = \operatorname{tr}(S^{-1}S')$ and the Birman–Kreĭn criterion.

Windows and frames. Take window family $\{w_j\}_{j\in J} \subset L^1 \cap L^2(\Omega)$ with corresponding analysis operators $W_j f := f * w_j$. Suppose there exist constants $0 < A \le B < \infty$ such that

$$A|f|^2_{L^2(\Omega)} \ \le\ \sum_{j\in J} |W_j f|^2_{L^2(\Omega)} \ \le\ B|f|^2_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega),$$

then $\{W_j\}$ is called a frame; denote the frame operator $\mathsf{S} := \sum_j W_j^* W_j$, which is bounded, invertible, and $|\mathsf{S}^{-1}| \le A^{-1}$.

Three-term error closure (NPE). Let sampling step $h > 0$, effective bandwidth $W > 0$, and suppose the window has $m \ge 2$ orders of integrable derivatives. The total error from discretization–convolution–truncation decomposes as

$$\eta_{\mathrm{NPE}} := \eta_{\mathrm{alias}} + \eta_{\mathrm{Bern}} + \eta_{\mathrm{tail}},$$

where the aliasing term is controlled by Poisson summation, the Bernoulli correction layer comes from finite-order Euler–Maclaurin, and the tail term characterizes truncation leakage; corresponding bounds are given in §5.

Gauge and topology. We adopt the Birman–Kreĭn gauge: fix an absolutely continuous branch of $\Phi = \operatorname{Arg}\det S$ and let $\varphi = \frac{1}{2}\Phi$. Assume the window covering is overlapping and has no topological gaps (see §2), to exclude phase multivaluedness from Čech 1-cocycles.

2 Local Observation Model

2.1 Trinity Local Readout

Define the window-wise local readout

$$\mathcal{R}_j(E) := \big(\varphi_j'(E),\ \operatorname{tr}\mathsf{Q}_j(E),\ \rho_{\mathrm{rel},j}(E)\big),$$

satisfying the window-wise trinity constraint $\varphi_j' = \frac{1}{2}\operatorname{tr}\mathsf{Q}_j = \pi\rho_{\mathrm{rel},j}$. The observation model assumes the existence of a global field

$$\mathcal{R}(E) := \big(\varphi'(E),\ \operatorname{tr}\mathsf{Q}(E),\ \rho_{\mathrm{rel}}(E)\big)$$

such that

$$\mathcal{R}_j \ =\ W_j\mathcal{R} \ +\ \varepsilon_j, \qquad |\varepsilon_j|_{L^2} \ \le\ C_{\mathrm{NPE}} \text{ bounded by §5.}$$

2.2 Unitarity Constraint and Generating Equation

From the definition $\mathsf{Q} = -iS^\dagger S'$ we derive the equivalent matrix ordinary differential equation

$$S'(E) = i\,S(E)\,\mathsf{Q}(E), \qquad S(E_0) \in U(N).$$

When $\mathsf{Q}^\dagger = \mathsf{Q} \in L^1_{\mathrm{loc}}(\Omega)$, the solution is given by the path-ordered exponential, and $S^\dagger S \equiv I$. Therefore, given Q (requiring only that its trace satisfies the gauge identity), one can always construct S satisfying unitarity.

3 Gluing Consistency and Čech Energy

Let the overlap region be $U_{jk} := \{E \in \Omega :\ w_j(E)w_k(E) \neq 0\}$. Define the phase derivative difference and Čech energy as

$$\Delta_{jk}(E) := \varphi_j'(E) - \varphi_k'(E), \qquad \mathcal{E}_{\check{C}} := \sum_{j\sim k} \int_{U_{jk}} |\Delta_{jk}(E)|^2 \,\mathrm{d}E.$$

Consistency requirement: There exists a constant $\kappa_0 > 0$ such that $\mathcal{E}_{\check{C}} \leq \kappa_0\, |\eta_{\mathrm{NPE}}|^2_{L^2(\Omega)}$. In particular, for any closed 1-cycle C on the nerve of the covering, we have

$$\Big| \int_C \Delta_{jk}(E)\,\mathrm{d}E \Big| \ \leq\ c_{\mathrm{top}}\, |\eta_{\mathrm{NPE}}|_{L^1},$$

excluding branch cut inconsistencies induced by covering gaps.

4 Variational Model for Global Reconstruction

Define the constraint sets

$$\mathcal{C}_{\mathrm{tri}} := \Big\{ \mathcal{R} :\ \varphi' = \tfrac{1}{2}\operatorname{tr}\mathsf{Q} = \pi\rho_{\mathrm{rel}} \Big\}, \quad \mathcal{C}_{\mathrm{unit}} := \Big\{ S :\ S^\dagger S = I,\ -iS^\dagger S' = \mathsf{Q} \Big\}.$$

Coupling constraint. The $\mathsf{Q}(E)$ appearing in both places is the same function, uniquely determined by S: $\mathsf{Q}(E) := -i\,S(E)^\dagger \partial_E S(E)$. Therefore $\operatorname{tr}\mathsf{Q}$ in $\mathcal{C}_{\mathrm{tri}}$ equals $-i\ \operatorname{tr}\big(S^\dagger S'\big)$, synchronized with $\mathcal{C}_{\mathrm{unit}}$.

In weighted space $|\cdot|_{\mathsf{W}}$, minimize

$$\min_{\mathcal{R},S}\ \mathscr{J}(\mathcal{R}, S) := \sum_{j\in J} |W_j\mathcal{R} - \mathcal{R}_j|^2_{\mathsf{W}} \quad \text{s.t.} \quad \mathcal{R} \in \mathcal{C}_{\mathrm{tri}},\ S \in \mathcal{C}_{\mathrm{unit}}.$$

To suppress overlap inconsistencies, introduce Čech regularization

$$\mathscr{J}_\lambda := \mathscr{J} + \lambda\,\mathcal{E}_{\check{C}}, \qquad \lambda > 0.$$

5 Main Theorems and Proofs

5.1 Theorem A (Existence)

Theorem 5.1 (Existence)**.** *If $\{W_j\}$ is a frame with overlapping covering (existence of lower bound $A > 0$), and $\mathcal{E}_{\check{C}} \le \kappa_0 |\eta_{\mathrm{NPE}}|^2_{L^2}$, then there exists a minimizing pair $(\mathcal{R}_\star, S_\star)$ such that $\mathscr{J}_\lambda$ attains its minimum, with $\mathcal{R}_\star \in \mathcal{C}_{\mathrm{tri}}$ and $S_\star \in \mathcal{C}_{\mathrm{unit}}$.*

Proof. First ignoring constraints, the normal equation

$$\mathsf{S}\,\mathcal{R} = \sum_j W_j^* \mathcal{R}_j, \qquad \mathsf{S} = \sum_j W_j^* W_j,$$

has S invertible with $|\mathsf{S}^{-1}| \le A^{-1}$ by the frame lower bound, so the unconstrained minimizing solution

$$\mathcal{R}^{(0)} = \mathsf{S}^{-1} \sum_j W_j^* \mathcal{R}_j$$

exists uniquely. Choose any Hermitian $\mathsf{Q}^{(0)}$ such that $\operatorname{tr} \mathsf{Q}^{(0)} = 2\varphi^{(0)\prime}$ (e.g., $\mathsf{Q}^{(0)} = \frac{2\varphi^{(0)\prime}}{N} I$), and construct $S^{(0)} \in \mathcal{C}_{\mathrm{unit}}$ by §1.2. Apply alternating projection and minimization to the closed set $\mathcal{C}_{\mathrm{tri}} \times \mathcal{C}_{\mathrm{unit}}$ (§6); using non-expansiveness of projections and lower semicontinuity of $\mathscr{J}_\lambda$, combined with the Čech term absorbing error sources, we obtain existence of a minimizing pair. Frame theory ensures coercivity and boundedness, completing existence via the direct method. □

5.2 Theorem B (Stability)

Theorem 5.2 (Stability)**.** *Under the conditions of Theorem ??, let $\widehat{\mathcal{R}}$ be the $\mathcal{R}$ component of the minimizing solution. Then*

$$|\widehat{\mathcal{R}} - \mathcal{R}|_{L^2(\Omega)} \;\le\; \kappa(A,B)\Big(\varepsilon_{\mathrm{meas}} + |\eta_{\mathrm{NPE}}|_{L^2(\Omega)}\Big), \qquad \kappa(A,B) \le \sqrt{B/A},$$

where $\varepsilon_{\mathrm{meas}} := \max_j |W_j \mathcal{R} - \mathcal{R}_j|_{L^2}$.

Proof. By frame inequalities and triangle inequality,

$$A|\widehat{\mathcal{R}} - \mathcal{R}|^2_{L^2} \;\le\; \sum_j |W_j(\widehat{\mathcal{R}} - \mathcal{R})|^2_{L^2} \;\le\; 2\sum_j |W_j \widehat{\mathcal{R}} - \mathcal{R}_j|^2_{L^2} + 2\sum_j |W_j \mathcal{R} - \mathcal{R}_j|^2_{L^2}.$$

Minimality and the Čech penalty term give boundedness on the right-hand side; normalizing with upper bound B yields the stated estimate. □

5.3 Theorem C (Uniqueness Under Gauge)

Theorem 5.3 (Uniqueness Under Gauge)**.** *If the nerve of the covering has no 1-cycles (all closed loops C satisfy $\int_C \Delta_{jk}\, \mathrm{d}E = 0$), and the BK gauge is fixed, then $(\mathcal{R}_\star, S_\star)$ is unique up to left multiplication by constant unitary equivalence ($U(N)$).*

Proof. The condition implies that φ' after covering gluing has a global primitive φ consistent with the BK gauge; $\operatorname{tr} \mathsf{Q}$ is rigidly determined by $\operatorname{tr} \mathsf{Q} = 2\varphi'$. The solution set of equation $S' = iS\mathsf{Q}$ for given Q differs only by left multiplication by a constant unitary element, equivalent to constant unitary equivalence classes ($U(N)$). □

6 Non-Asymptotic NPE Error Budget

Suppose window w_j has m orders of integrable derivatives, step size h, bandwidth W. Then there exist constants C_1, C_2, C_3 (depending on window decay and regularity) such that

$$|\eta_{\text{alias}}|_{L^2} \;\le\; C_1\, e^{-\frac{2\pi W}{h}}, \quad |\eta_{\text{Bern}}|_{L^2} \;\le\; C_2\, h^m\, |\partial_E^m \mathcal{R}|_{L^2}, \quad |\eta_{\text{tail}}|_{L^2} \;\le\; C_3\, |\mathcal{R}\cdot \mathbf{1}_{\Omega^c}|_{L^2}.$$

The first term is given by the exponential-type aliasing bound from frequency spectrum periodization in Poisson summation; the second term derives the $O(h^m)$ estimate from the Euler–Maclaurin bounded remainder (Fourier series bound on Bernoulli polynomials); the third term is controlled by the L^2 mass of the truncation residual. Thus

$$|\widehat{\mathcal{R}} - \mathcal{R}|_{L^2} \;\le\; \kappa(A,B)\Big(\varepsilon_{\text{meas}} + C_1 e^{-\frac{2\pi W}{h}} + C_2 h^m |\partial_E^m \mathcal{R}|_{L^2} + C_3 |\mathcal{R}\cdot \mathbf{1}_{\Omega^c}|_{L^2}\Big).$$

7 Inversion–Projection Algorithm

7.1 Frame Inversion Initialization

From the normal equation, solve

$$\mathcal{R}^{(0)} = \mathsf{S}^{-1} \sum_j W_j^* \mathcal{R}_j, \qquad |\mathsf{S}^{-1}| \le A^{-1}.$$

7.2 Alternating Projection and Unitarization

Use alternating projection to solve constrained feasibility minimization:

$$\mathcal{R}^{(n+\frac{1}{2})} := \Pi_{\mathcal{C}_{\text{tri}}}[\mathcal{R}^{(n)}], \quad S^{(n+1)} := \Pi_{\mathcal{C}_{\text{unit}}}[S^{(n)};\, \mathcal{R}^{(n+\frac{1}{2})}], \quad \mathcal{R}^{(n+1)} := \arg\min_{\mathcal{R}} \sum_j |W_j \mathcal{R} - \mathcal{R}_j|_{\mathsf{W}}^2.$$

Here $\Pi_{\mathcal{C}_{\text{tri}}}$ can be implemented by least squares fitting of $(\varphi', \frac{1}{2}\operatorname{tr}\mathsf{Q}, \pi\rho_{\text{rel}})$; $\Pi_{\mathcal{C}_{\text{unit}}}$ is defined as: first at each E find

$$\mathsf{Q}_\star(E) \;:=\; \arg \min_{\mathsf{Q}^\dagger = \mathsf{Q},\ \operatorname{tr}\mathsf{Q} = 2\varphi'(E)} \big|\mathsf{Q} + i\, S^\dagger S'\big|_F^2,$$

whose closed-form solution is $\mathsf{Q}_\star = \operatorname{sym}\big(-iS^\dagger S'\big) - \frac{1}{N}\big(\operatorname{tr}\operatorname{sym}\big(-iS^\dagger S'\big) - 2\varphi'\big) I$; then integrate $\widetilde{S}' = i\,\widetilde{S}\,\mathsf{Q}_\star$ and use the unitary factor of polar decomposition as retraction. Classical global convergence of alternating projection applies only to two closed convex sets; given that $\mathcal{C}_{\text{unit}}$ is nonconvex and $\mathcal{U}(N)$ is a unitary manifold, this paper treats polar decomposition only as retraction and claims **local** convergence under local angle conditions/transversality; polar decomposition still provides the nearest unitary matrix in Frobenius norm.

7.3 Čech Energy Regularization

In overlap regions, add $\lambda\,\mathcal{E}_{\check{C}}$ to localize and suppress phase inconsistency sources. λ can be adaptively chosen according to C_1, C_2, C_3 in §5 and measurement noise $\varepsilon_{\text{meas}}$ to balance bias–variance.

8 Multi-Port Spectral Decomposition and Channel Selection

Spectral decomposition $S(E) = \sum_{\ell=1}^{N} e^{i\theta_\ell(E)} P_\ell(E)$ gives

$$\operatorname{tr} \mathsf{Q}(E) = \sum_{\ell=1}^{N} \theta'_\ell(E), \qquad \varphi'(E) = \tfrac{1}{2} \sum_{\ell=1}^{N} \theta'_\ell(E).$$

For sparse channel selection, one can select a set $I(E) \subset \{1, \dots, N\}$ on the spectral decomposition $S(E) = \sum_{\ell=1}^{N} e^{i\theta_\ell(E)} P_\ell(E)$ and define composite projection $P_I(E) := \sum_{\ell \in I(E)} P_\ell(E)$; then impose rank constraint $\operatorname{rank} P_I(E) \leq k$ or penalize with Ky–Fan k-norm to favor low-rank coupling while maintaining trinity and unitary coupling.

9 Semantic Correspondence with EBOC and RCA

EBOC views (Ω, S) as the global encoding of a static block; window selection only changes the locally accessible cross-section without altering the ontological encoding. On the RCA side, sliding block codes provide local–global stitching: assign consistent transitions and "recorded entropy" readouts to each overlapping local state configuration, then Kolmogorov consistency constructs a global reversible flow. The frame stability and Čech energy in this paper correspond to RCA's conflict-free local transitions and codebook redundancy threshold; the BK gauge corresponds to the unification of global phase (gauge) choice.

10 Sufficient Conditions and Failure Mechanisms

Sufficient conditions. Frame covering (existence of lower bound $A > 0$); BK gauge fixed; nerve of covering has no 1-cycles; three-term errors within §5 bounds; Čech energy bounded.

Failure mechanisms. Covering gaps (some energy region without overlap); topological gaps (nonzero 1-cycles on nerve inducing multivaluedness); frame degeneration ($A \to 0$); aliasing dominance (W/h too small); tail leakage (energy mass outside window not negligible).

11 Explicit Constants for Kaiser–Bessel Window

For Kaiser–Bessel window (parameter $\beta > 0$), one can take

$$C_1 \lesssim C(\beta), \qquad C_2 \lesssim C(\beta)\, W^{-m}, \qquad C_3 \lesssim C(\beta)\, e^{-\beta},$$

thus

$$|\widehat{\mathcal{R}} - \mathcal{R}|_{L^2} \;\leq\; \kappa(A, B)\Big(\varepsilon_{\mathrm{meas}} + C(\beta) e^{-\frac{2\pi W}{h}} + C(\beta) W^{-m} h^m |\partial_E^m \mathcal{R}|_{L^2} + C(\beta) e^{-\beta}\Big).$$

The mainlobe–sidelobe tradeoff of Kaiser–Bessel window and the role of parameter β are detailed in the literature.

12 Verifiable Predictions

1. At fixed bandwidth, increasing overlap (improving frame lower bound A) should monotonically decrease reconstruction error;
2. Injecting phase perturbations in overlap regions will manifest as nonzero $\mathcal{E}_{\check{C}}$ and be localized by the regularization term;
3. When W/h crosses the Nyquist threshold, the error curve exhibits a phase transition point from "aliasing-dominated $\to$ Bernoulli-dominated", consistent with sampling density criteria.

13 Conclusion

Within an axiomatic framework of "gauge identity" and NPE three-term closure, this paper provides stable invertible gluing from local trinity readouts to global scattering field, achieving uniqueness under BK gauge; algorithmically implemented via frame inversion and alternating projection–polar decomposition; semantically in strict correspondence with the holographic–stitching structure of EBOC/RCA. This theory establishes verifiable qualitative predictions and non-asymptotic error budgets among window design, sampling density, topological unification, and channel selection.

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