FMU: Fractal-Mirror Unification of Frequency–Scale–Information

Haobo Ma[1] Wenlin Zhang[2]

[1]Independent Researcher

[2]National University of Singapore

November 19, 2025

Abstract

Within weighted Mellin–Hilbert space $L^2(\mathbb{R}_+, dt/t)$, we establish rigorous theory of **fractal mirror** (FM) signal families generated by **mother function** M through **multiplicative scale replication**. Around three main threads "frequency–scale–information", we provide equivalent characterizations and complete proofs: (A) Under **Mellin–Calderón** condition, **multiplicative self-similarity** $\Longleftrightarrow$ **Mellin-domain quasi-periodicity** (critical line exhibits equidistant frequency-shift array); (B) **spectral power-law** $\Longleftrightarrow$ (in logarithmic scale) **entropy slope first-order approximate linearity**, and in self-affine model derive classical relation $D = (3 - \beta)/2$ for image box dimension; (C) With **Nyquist–Poisson–Euler–Maclaurin (three-fold decomposition)** achieve **non-asymptotic error closure** and **separable budget**. Further, with **spectral density weight measure** $d\mu = (1/\pi)\,\Im m(\omega + i0)\,d\omega$, isometrically embed FM subspace into Paley–Wiener type bandlimited space, thereby deriving **Landau** type sampling/interpolation **density thresholds**, **Wexler–Raz** tight/dual criteria, and **Balian–Low** impossibility at critical density; prove this spectral density weight consistent with Herglotz representation of Weyl–Titchmarsh m-function of one-dimensional self-adjoint canonical systems. Above scales and criteria compatible with **established standards** including Mellin isometry, Paley–Wiener, Poisson summation, Euler–Maclaurin, Landau density, Wexler–Raz and Balian–Low, Herglotz representation, and de Branges inverse spectral theorem.

Keywords: Fractal mirror; Mellin transform; Quasi-periodicity; Power-law spectrum; Entropy-slope coupling; Nyquist–Poisson–EM three-fold; Landau density; Wexler–Raz; Balian–Low; Herglotz; de Branges

0. Notation & Baseplates

(0.1) Mellin Isometry and Critical Line

For $x \in L^2(\mathbb{R}_+, dt/t)$, Mellin transform

$$\mathcal{M}x(s) = \int_0^\infty x(t)\, t^{s-1}\, dt, \tag{1}$$

on critical line $s = \frac{1}{2} + i\omega$ isometric with "logarithmic Fourier" unit:

$$|x|^2_{L^2(dt/t)} = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \mathcal{M}x(\tfrac{1}{2} + i\omega) \right|^2 d\omega. \tag{2}$$

Thus $\widetilde{\mathcal{M}} : x \mapsto (2\pi)^{-1/2} \mathcal{M}x(\frac{1}{2} + i\cdot)$ is unitary isometry on $L^2(\mathbb{R}_+, dt/t) \to L^2(\mathbb{R})$.

(0.2) Logarithmic Variable and Scaling

Take $u = \log t$, let $m(u) = M(e^u)$. Scale $t \mapsto 2^k t$ becomes logarithmic translation $u \mapsto u + k \log 2$. Mellin scaling law yields

$$\mathcal{M}M(2^k \cdot)(s) = 2^{-ks}\, \mathcal{M}M(s) \quad (s = \tfrac{1}{2} + i\omega), \tag{3}$$

i.e., amplitude factor $2^{-k/2}$ and phase modulation $e^{-ik\omega \log 2}$.

(0.3) Spectral Density Measure and Phase Coordinate

Notation and transform convention: $\mathcal{M}$ denotes Mellin transform with respect to t evaluated at $s = \frac{1}{2} + i\omega$; $\widehat{\cdot}$ universally denotes Fourier transform with respect to $u = \log t$ (equivalent to ω-domain operator along critical line).

For systems associated with Weyl–Titchmarsh m-function, define **spectral density weight measure**

$$d\mu(\omega) = \frac{1}{\pi} \Im m(\omega + i0)\, d\omega. \tag{4}$$

When $\Im m \geq 0$, μ is positive measure. Under special normalization (e.g., certain boundary conditions making $\Re m(\lambda + i0) \equiv 0$ a.e.), can use phase representation $d\mu_\varphi(\omega) = (1/\pi)\, d(\arg m(\omega + i0))$; general case uses spectral density weight $d\mu$. Consistency with spectral measure of self-adjoint systems see §6.

Convention (u-domain Fourier): $\widehat{f}(\omega) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(u)\, e^{-i\omega u}\, du$, $f(u) = (2\pi)^{-1/2} \int_{\mathbb{R}} \widehat{f}(\omega)\, e^{i\omega u}\, d$

Notation Convention (Avoiding Conflict): Throughout fix $u := \log t$ as logarithmic time variable; spectral density weight coordinate denoted separately as v_μ. u-domain Fourier only acts on u variable; isometric maps and density criteria related to spectral density weight stated in v_μ coordinate.

(0.4) Finite-Order Euler–Maclaurin (EM)

Throughout use only **finite-order** EM, decomposing "sum–integral" difference into endpoint Bernoulli layer and remainder:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{r=1}^{p-1} \frac{B_{2r}}{(2r)!} \big(f^{(2r-1)}(b) - f^{(2r-1)}(a) \big) + R_p, \tag{5}$$

with control $|R_p| \lesssim \frac{2\,\zeta(2p)}{(2\pi)^{2p}} \int_a^b |f^{(2p)}|$.

1 Fractal Mirror (FM) Signal Family: Definition and Basic Properties

Definition 1.1 (FM Generation and Mellin–Calderón Condition)**.** Given mother function $M \in L^2(\mathbb{R}_+, dt/t)$, weight sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^2$, phase sequence $\{\phi_k\} \subset \mathbb{R}$ with $\sup_k |\phi_k| < \infty$. Define

$$x(t) = \sum_{k\in\mathbb{Z}} a_k\, M(2^k t)\, e^{i\phi_k}, \qquad t > 0. \tag{6}$$

Assumption (H0) Weighted ℓ^2 Consistency. Let $b_k := a_k\, 2^{-k/2} e^{i\phi_k}$, require $\{b_k\}_{k\in\mathbb{Z}} \in \ell^2$ (equivalently $\sum_k |a_k|^2\, 2^{-k} < \infty$).

Assumption (H1) Mellin–Calderón Boundedness. Write $G(\omega) := \mathcal{M}M(\frac{1}{2}+i\omega)$, $P := 2\pi/\log 2$, require Calderón sum

$$\mathcal{C}_G(\omega) := \sum_{n\in\mathbb{Z}} \left|G(\omega + nP)\right|^2 \in L^\infty([0, P]). \tag{7}$$

Proposition 1.2 (L^2 Unconditional Convergence and Frequency Structure, Under (H0)–(H1))**.** *Under assumptions (H0)–(H1), series $\sum_k a_k M(2^k \cdot) e^{i\phi_k}$ unconditionally converges in $L^2(\mathbb{R}_+, dt/t)$, and*

$$\mathcal{M}x\big(\tfrac{1}{2} + i\omega\big) = G(\omega) \sum_{k\in\mathbb{Z}} b_k\, e^{-ik\omega \log 2}, \qquad b_k := a_k\, 2^{-k/2} e^{i\phi_k}. \tag{8}$$

With Bessel bound

$$|x|^2_{L^2(dt/t)} \;\le\; \frac{P}{2\pi} \|\mathcal{C}_G\|_{L^\infty([0,P])} \sum_{k\in\mathbb{Z}} |b_k|^2, \qquad P = \frac{2\pi}{\log 2}. \tag{9}$$

Proof. By (H1)'s Calderón upper bound and Plancherel–Mellin isometry, obtain

$$|x|^2_{L^2(dt/t)} \;\le\; \frac{P}{2\pi} \|\mathcal{C}_G\|_{L^\infty([0,P])} \sum_{k\in\mathbb{Z}} |b_k|^2, \qquad P = \frac{2\pi}{\log 2}, \tag{10}$$

thus series Cauchy converges; frequency expression follows directly from scaling law. □

2 Main Theorem A: Mellin-Quasi-Periodic Characterization of Multiplicative Self-Similarity

Theorem 2.1 (Self-Similarity $\iff$ Quasi-Periodicity)**.** *For x from Definition 1.1, following equivalent:*

(i) $x(t) = \sum_k a_k M(2^k t) e^{i\phi_k}$ *(multiplicative self-similar superposition);*

(ii) Exists envelope $G(\omega) = \mathcal{M}M(\frac{1}{2} + i\omega) \in L^2(\mathbb{R})$ *such that*

$$\mathcal{M}x\big(\tfrac{1}{2} + i\omega\big) = G(\omega) \cdot \underbrace{\sum_{k\in\mathbb{Z}} b_k\, e^{-ik\omega \log 2}}_{\text{Bohr quasi-periodic; frequency-shift lattice spacing } 2\pi/\log 2}. \tag{11}$$

Proof. "⇒" by Proposition 1.2. "⇐" Apply inverse Mellin ($\sigma = \frac{1}{2}$) to quasi-periodic part and use "logarithmic-domain translation ↔ Mellin frequency-shift" duality, immediately obtain superposition of logarithmic translation family $M(2^k \cdot)$. □

3 Main Theorem B: Spectral Power-Law–Entropy Slope and Self-Affine Dimension

3.1 Power-Law Spectrum and Logarithmic Binned Entropy Linear Coupling (Approximate Relation)

Proposition 3.1 (Entropy–Slope Approximate Linearity, First-Order Regime). *Let power spectrum over wide frequency ratio $\Lambda = f_{\max}/f_{\min} \gg 1$ satisfy $S(f) \asymp C f^{-\beta}$, logarithmic uniform binning $I_j = [e^{y_j}, e^{y_{j+1}}]$, $\delta y := y_{j+1} - y_j \ll 1$ fixed, bin number $J \sim (\log \Lambda)/\delta y$, probability $P_j \propto \int_{I_j} S(f)\, df$ normalized. Then in first-order approximation*

$$H := -\sum_j P_j \log P_j = c_1(\delta y) + c_2(\delta y)\, \beta + \mathcal{O}\big(\delta y\big) + \mathcal{O}\big((\log \Lambda)^{-1}\big), \tag{12}$$

where c_1, c_2 depend only on binning step δy and window overlap constants, explicitly computable; as $\delta y \to 0$, $\Lambda \to \infty$, main term linearity holds.

Proof Sketch. Let $y = \log f$, have $df = f\, dy$, thus $P_j \propto \int_{y_j}^{y_{j+1}} e^{(1-\beta)y}\, dy$. Uniform y-binning makes $\{P_j\}$ approximately exponentially distributed; substitute back into entropy and approximate by Riemann sum, δy discretization error yields $\mathcal{O}(\delta y)$ term, endpoints/overlap yield $\mathcal{O}((\log \Lambda)^{-1})$ term; both negligible when $\delta y \ll 1/\log \Lambda$. □

3.2 Self-Affine Spline and Image Dimension (Canonical Model)

Theorem 3.2 ($D = (3-\beta)/2$). *If spline satisfies $x(\lambda t) \overset{d}{=} \lambda^H x(t)$ (e.g., fBm), then*

$$S(f) \sim f^{-(2H+1)} \quad \Longrightarrow \quad D_{\text{graph}} = 2 - H = \frac{3-\beta}{2}. \tag{13}$$

Proof. Self-affine processes like fBm satisfy $S(f) \propto f^{-(2H+1)}$; their spline graph's Hausdorff/box dimension $D = 2 - H$ is classical result, jointly eliminate H to obtain above formula. Related conclusions see Flandrin's analysis of fBm spectrum and Xiao's rigorous measure results for image dimension. □

4 Main Theorem C: Nyquist–Poisson–EM Non-Asymptotic Error Closure

Let target frequency-domain quantity written as

$$F(\omega) = \widehat{w}(\omega)\, \widehat{h}(\omega)\, X(\omega), \qquad X(\omega) := (2\pi)^{-1/2}\, \mathcal{M}x\big(\tfrac{1}{2} + i\omega\big)\,, \tag{14}$$

where $\widehat{w}(\omega)$, $\widehat{h}(\omega)$ are analysis window and interpolation kernel frequency responses after u-domain Fourier transform per §0.3 convention, $X(\omega)$ is Mellin image along critical line (normalized by §0.1's unitary isometry).

Theorem 4.1 (Nyquist Condition and Three-Fold Decomposition Error). *Suppose exists $B > 0$ such that* $\operatorname{supp} F \subset [-B, B]$, *where $B := \min\{\Omega_w, \Omega_h\}$ (if X non-bandlimited, take effective bandwidth as intersection* $\operatorname{supp}(\widehat{w}) \cap \operatorname{supp}(\widehat{h})$*). If sampling step*

$$\Delta \;\le\; \frac{\pi}{B}, \tag{15}$$

then aliasing energy is zero. In general case, aliasing term written as

$$\varepsilon_{alias} = \Big| \sum_{\ell \neq 0} F\big(\cdot + 2\pi\ell/\Delta \big) \Big|_{L^2([-\pi/\Delta,\, \pi/\Delta])}, \tag{16}$$

for bandlimited linear reconstruction operator E*, total error decomposes as*

$$|x - \mathsf{E}x|_{L^2} \leq \underbrace{\varepsilon_{alias}}_{\textit{periodic superposition}} + \underbrace{\varepsilon_{\mathrm{EM}}^{(p)}}_{\mathcal{O}(|B_{2p}|\Delta^{2p})} + \underbrace{\varepsilon_{tail}}_{\textit{band-edge/truncation}}, \tag{17}$$

and $\varepsilon_{\mathrm{EM}}^{(p)} = \mathcal{O}\big(|B_{2p}|\, \Delta^{2p}\big)$.

Proof. Use Poisson summation to convert discretization into spectrum periodization; Nyquist threshold eliminates inter-band overlap. Sum–integral difference decomposed by finite-order EM into endpoint Bernoulli layer and remainder, remainder bound see §0.4. □

5 Sampling–Interpolation–Stability: Landau–Wexler–Raz–Balian–Low in Spectral Density Coordinate

5.1 Isometric Embedding of Spectral Density Weight Coordinate

In region $\Im m(\omega + i0) > 0$, define spectral density weight coordinate

$$v_\mu(\omega) = \frac{1}{\pi} \int_{-\infty}^{\omega} \Im m(\omega' + i0)\, d\omega', \qquad dv_\mu = \frac{1}{\pi} \Im m(\omega + i0)\, d\omega, \tag{18}$$

then

$$\int_{\mathbb{R}} |X(\omega)|^2 \frac{1}{\pi} \Im m(\omega + i0)\, d\omega = \int_{\mathbb{R}} |X(\omega(v_\mu))|^2\, dv_\mu, \tag{19}$$

thus isometrically embed spectral density weight weighted FM-subspace into **unit bandwidth** Paley–Wiener type space. Under special normalization, can simplify to phase coordinate $v_\mu = \varphi(\omega)/\pi$. Paley–Wiener and Hardy/Mellin-Hardy structure in Mellin context see literature.

5.2 Landau Type Necessary Condition (FM Version)

Theorem 5.1 (Necessary Density Threshold, v_μ-Domain)**.** *Let* $\Omega = \{\omega_n\}$ *be sampling sequence (respectively: interpolation sequence). In* v_μ *coordinate (defined in §5.1), its Beurling lower (respectively upper) density satisfies*

$$\underline{D}_\mu(\Omega) \geq 1 \qquad (\textit{respectively}\ , \overline{D}_\mu(\Omega) \leq 1). \tag{20}$$

Remark: Above are necessary conditions. Sufficiency generally requires additional separation/stability structural conditions; in practice can design stable sampling/reconstruction via §5.3's WR/Parseval conditions.

Proof Sketch. By 5.1's isometry, problem reduces to non-uniform sampling of unit bandwidth Paley–Wiener space, directly invoke Landau necessary density theorem. □

5.3 Wexler–Raz and Parseval Tight Frame (Critical Nyquist)

Theorem 5.2 (WR/Parseval Condition). *Under Nyquist (non-aliasing) condition, system generated by multi-windows $\{w_\alpha\}_{\alpha=1}^r$ is Parseval tight frame if and only if*

$$\frac{1}{\Delta}\sum_{\alpha=1}^{r}\left|\widehat{w_\alpha}(\xi)\right|^2 \equiv 1 \quad (a.e.). \tag{21}$$

With aliasing present, Parseval condition becomes

$$\frac{1}{\Delta}\sum_{\alpha=1}^{r}\sum_{m\in\mathbb{Z}}\left|\widehat{w_\alpha}(\xi+2\pi m/\Delta)\right|^2 \equiv 1. \tag{22}$$

Reconstruction with dual windows $\{\widetilde{w}_\alpha\}$ satisfies corresponding biorthogonality formula.

Proof Sketch. WR identity yields frequency-domain pointwise orthogonality necessary and sufficient condition; through $u=\log t$ and phase coordinate transformation, losslessly transplants to Mellin/logarithmic model. □

5.4 Balian–Low Type Impossibility (Critical Density)

Theorem 5.3 (BLT–Mellin Version). *At critical density $D=1$, if single window w "well-localized" on both logarithmic time u and frequency ω sides (e.g., finite second moment), then system generated by w and critical lattice in v_μ coordinate* ***cannot*** *be Riesz basis; to obtain basis, must relax at least one side's localization or employ oversampling.*

Proof Sketch. Through $u=\log t$ and §5.1's isometric embedding, problem equivalent to standard Gabor lattice BLT; immediately follows from BLT's Riesz/ONB version. □

6 Consistency with de Branges–Kreĭn / Weyl–Titchmarsh m-Function

Proposition 6.1 (Spectral Density Weight and Herglotz Representation). *In one-dimensional self-adjoint canonical system/Schrödinger type operator case, Weyl–Titchmarsh m is Herglotz–Nevanlinna function, exists spectral measure μ such that*

$$m(z) = az + b + \int_{\mathbb{R}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda), \tag{23}$$

boundary value satisfies $\Im m(\lambda+i0) = \pi\,\rho(\lambda)$ (a.e.), where ρ is spectral density. Thereby define spectral density weight measure

$$d\mu(\omega) = \frac{1}{\pi}\,\Im m(\omega+i0)\,d\omega = \rho(\omega)\,d\omega, \tag{24}$$

consistent with absolutely continuous spectral measure. Under special normalization (e.g., certain boundary conditions making $\Re m(\lambda+i0)\equiv 0$ a.e.), can simplify to phase representation $d\mu_\varphi = (1/\pi)\,d(\arg m)$; general case must use above spectral density weight.

Remark 6.2. This consistency ensures spectral density scale defined by $d\mu$ in this paper seamlessly interfaces with spectral theory of de Branges spaces/canonical systems, becoming natural coordinate for density and frame criteria in §5; phase derivative $\varphi' = d(\arg m)/d\omega$ generally also depends on $\Re m$ and m', reduces to $\pi\rho$ only in special cases.

7 Reproducible Experimental Paradigm (Verification and Engineering)

P1 — Power-Law/Entropy Coupling. Over sufficiently wide logarithmic bandwidth, fit $S(f) \propto f^{-\beta}$ and verify with logarithmic binned entropy H that regression slope is linear within error band (§3.1's first-order approximation).

P2 — Mellin Peak Array. Compute $\mathcal{M}x(\frac{1}{2}+i\omega)$, verify equidistant peak array and relative phase stability (§2).

P3 — Sampling/Window Design. Select lattice per §5.2's density threshold; tune windows per §5.3's WR condition to obtain Parseval; perform **separable budget** for error per §4's three-fold decomposition (Nyquist margin, EM order, band-edge tail term).

8 Information Trinity (i_+, i_0, i_-) and Model Selection

- i_+: Cross-scale overflow benefit (β trending red $\Rightarrow$ low-frequency concentration $\Rightarrow$ compression/prediction benefit).
- i_0: Intra-layer rearrangement (phase–coherence "neutral" redistribution).
- i_-: Sparsity and complexity penalty (avoid overfitting/over-dense lattice).

Proposition 8.1 (Strategy in Approximate Linear Regime)**.** *In §3.1's first-order approximation regime, $\partial_\beta H \approx c_2$, effective layer number $N_{\text{eff}} \asymp (1+\beta)\log\Lambda$. Accordingly jointly select "window/lattice density–model complexity" to maximize $i_+ - i_-$ satisfying §4's three-fold decomposition budget.*

9 Interface with S-series / WSIG-QM / UMS

9.1 Interface with S24–S26

- S24's fiber Gram characterization and Wexler–Raz biorthogonality provide concrete implementation framework for this paper's §5.3 WR condition.
- S25's non-stationary Weyl–Mellin framework shares mathematical structure with this paper's Mellin isometry (§0.1) and logarithmic translation–frequency shift duality (§0.2).
- S26's spectral density scale consistent at Herglotz representation level with this paper's §0.3 and §6's spectral density weight measure $d\mu = (1/\pi)\,\Im m(\omega + i0)\,d\omega$; S26's Landau necessary density, Balian–Low impossibility directly correspond to this paper's Theorems 5.1, 5.3.

9.2 Interface with WSIG-QM

- WSIG-QM's axiom A2 (finite window readout) shares Nyquist–Poisson–EM three-fold decomposition framework with this paper's §4 windowed reconstruction.

- WSIG-QM's axiom A5 (phase–density–delay scale) consistent at spectral theory level with this paper's §0.3, §6's spectral density weight measure.
- WSIG-QM's theorem T6 (window/kernel optimization) shares frame theory criteria with this paper's §5.2–5.3's Landau density threshold, WR condition.

9.3 Interface with UMS

- UMS's core unification formula $d\mu = \frac{1}{2\pi} \operatorname{tr} \mathsf{Q}\, dE = \rho_{\text{rel}}\, dE$ in Mellin context corresponds to this paper's §0.3, §6's spectral density weight measure; under special normalization can simplify to phase representation.
- UMS's axiom A2 (finite window readout) shares framework at numerical implementation level with this paper's Theorem 4.1's three-fold decomposition error closure.
- UMS's axiom A6 (sampling–frame threshold) completely aligns with this paper's §5's Landau–Wexler–Raz–Balian–Low criteria.

9.4 Interface with Windowed Path Integral Theory

- Path integral theory's window–kernel duality (Theorem 2.1) can be rewritten in Mellin domain as this paper's Theorem 2.1's quasi-periodic formulation.
- Path integral theory's Nyquist–Poisson–EM error closure consistent in discretization framework with this paper's Theorem 4.1's three-fold decomposition.

9.5 Interface with Quantum Gravity Field Theory

- Quantum gravity field theory's spectral density scale consistent in spectral shift context with this paper's §0.3, §6's spectral density weight measure $d\mu = (1/\pi)\, \Im m(\omega + i0)\, d\omega$.
- Quantum gravity field theory's windowed sampling (§6.1) shares frame theory foundation with this paper's §5's Landau–Wexler–Raz criteria.

9.6 Maintaining "Poles = Principal Scales" Finite-Order EM Discipline

- Throughout all discrete–continuous exchanges, this paper employs **finite-order** EM (§0.4, Theorem 4.1), ensuring no new singularities introduced.
- Consistent with S15–S26, WSIG-QM, UMS, path integral theory, quantum gravity field theory: EM remainder serves only as bounded perturbation.

Appendix A: Proof Details and Tools

A.1 Mellin–Hardy and Isometry. $\widetilde{\mathcal{M}}$ is unitary isometry on $\sigma = \frac{1}{2}$; construction of Mellin–Paley–Wiener and Mellin–Hardy spaces see Bardaro–Butzer–Mantellini–Schmeisser.

A.2 Poisson and Aliasing. Dirac comb of sampling step Δ in frequency domain is Dirac comb of period $2\pi/\Delta$; aliasing energy equals periodized side-spectrum superposition energy in main band.

A.3 Euler–Maclaurin Remainder. Employ DLMF version EM formula and remainder bound, ensuring finite-order approximation introduces no additional singularities; error given by Bernoulli numbers and step size.

A.4 Landau Density. Sampling (interpolation) sequences of Paley–Wiener space PW_B must satisfy $\underline{D} \geq B/\pi$ ($\overline{D} \leq B/\pi$); in this paper isometric to threshold 1 at unit bandwidth in v_μ coordinate.

A.5 Wexler–Raz and BLT. WR identity yields frequency-domain pointwise necessary and sufficient condition for tight/dual frames; BLT shows at critical density "good double-sided localization + non-redundancy" incompatible.

A.6 Herglotz Representation and Spectral Density. m Herglotz function $\Rightarrow$ exists spectral measure representation; boundary imaginary part $\Im m(\lambda + i0) = \pi\rho(\lambda)$ (a.e.). Thereby define spectral density weight measure $d\mu = (1/\pi)\,\Im m\, d\omega$; phase derivative $\varphi' = d(\arg m)/d\omega$ generally also depends on $\Re m$ and m', reduces to $\pi\rho$ only under special normalization.

Conclusion

1. Multiplicative self-similarity via Mellin transform equivalent to **quasi-periodic frequency-shift array**, controlled by envelope G (Theorem 2.1); under weighted ℓ^2 consistency (H0) and Mellin–Calderón condition (H1), series unconditionally converges (Proposition 1.2).

2. Power-law spectrum and logarithmic binned entropy linearly coupled in first-order approximation; in self-affine limit $D = (3 - \beta)/2$ holds (Proposition 3.1, Theorem 3.2).

3. Employing **Nyquist–Poisson–EM** three-fold decomposition, can stably decompose total error into "aliasing/Bernoulli layer/tail term" three-term budget (Theorem 4.1).

4. Spectral density weight coordinate v_μ makes FM subspace isometric with Paley–Wiener space, thereby inheriting **Landau** density threshold, **Wexler–Raz** tight/dual criteria, and **Balian–Low** impossibility; its spectral density weight measure consistent with Herglotz representation of m-function (§5–§6).

References

[1] Bardaro, Butzer, Mantellini, Schmeisser, *On the Paley–Wiener theorem in the Mellin transform setting* (2015) and sequel (2017).

[2] Butzer & Jansche, *A Direct Approach to the Mellin Transform* (1997).

[3] NIST DLMF §1.8(iv); Candès lecture notes (2021).

[4] NIST DLMF §2.10(i), §24.17.

[5] Landau, *Acta Math.* 117 (1967).

[6] Daubechies & Landau, *J. Fourier Anal. Appl.* (1994/95).

[7] Heil & Powell, *J. Math. Phys.* (2006).

[8] Flandrin (1989); Xiao (1997).

[9] Kostenko–Teschl et al. surveys and Oberwolfach reports.

Reader's Guide

When embedding this paper into S25 (non-stationary Weyl–Mellin) and S26 (spectral density–de Branges), can directly reuse §5's spectral density weight coordinate v_μ criteria and §4's **three-fold decomposition error budget**; window/kernel design implemented via WR necessary and sufficient formula, critical density encountering BLT obstacle circumvented through **oversampling or relaxing localization**. In implementation note verification of weighted ℓ^2 consistency (H0) and Mellin–Calderón condition (H1) (e.g., Log-Gaussian mother function) and entropy–slope relation's error band control (§3.1's first-order approximation range). Notation convention: logarithmic time $u := \log t$ and spectral density weight coordinate v_μ strictly distinguished (§0.2–§0.3).