Unified Role of Relative Scattering Determinant in Quantum Gravity: Two-Domain Framework, Fixed-Energy BK ($p \in \{1, 2\}$ Unified Version), Closed-Domain Λ-Slope, and Black Hole Pole Spectroscopy

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Abstract

Taking "relative determinant" as unified object, we establish rigorous and verifiable theory in two types of geometric-physical scenarios: (**C**) relative ζ/heat kernel determinant for Euclideanized second variation operator family in compact closed domain and its volume density response to cosmological constant term; (**S**) fixed-frequency scattering matrix on stationary exterior geometry (Schwarzschild–de Sitter/Kerr–de Sitter) with relative (modified) determinant, spectral shift object, and quasinormal mode (QNM) spectroscopy. This paper provides four main theorems with complete proofs: (i) Under control of weighted limiting absorption principle (LAP) and double operator integral (DOI), prove $p \in \{1, 2\}$ unified version of fixed-energy Birman–Kreĭn equality: for Lebesgue almost everywhere frequency ω, $\det_p S_\Lambda(\omega) = \exp\big(-2\pi i\, \Xi_\Lambda^{(p)}(\omega)\big)$, where $p = 1$ gives $\Xi^{(1)} = \xi$ as Lifshits–Kreĭn spectral shift function, $p = 2$ yields $\Xi^{(2)}$ as cumulative antiderivative of Koplienko second-order spectral shift; (ii) In closed-domain Müller relative determinant framework, prove "volume slope theorem": $\lim_{\mu\to 0^+} \mathrm{Vol}_4(M)^{-1}\, \partial_\Lambda \Re \log \det_{\zeta,\mathrm{rel}}(\mathcal{K}_\Lambda + \mu^2, \mathcal{K}_0 + \mu^2) = \frac{1}{8\pi G}$ (per signature convention fixed in text); (iii) On physical strip $\Im\omega > -\gamma_0$, pole set of relative scattering determinant $\tau_p(\omega) = \det_p\big(S(\omega)S_0(\omega)^{-1}\big)$ equivalent to QNM (counting algebraic multiplicity), independent of reference S_0 choice; (iv) For real frequency only "phase" admits equality, $\arg \det_p S = -2\pi\, \Xi^{(p)}$; while for $p = 2$ Carleman determinant $|\det_2 S| = \exp\big(\sum_j (1 - \cos\theta_j)\big) \geq 1$, generally cannot claim $|\det_2 S| = 1$. Accordingly introduce "phase-normalized determinant" $\widehat{\det}_p S := \det_p S/|\det_p S|$ as constrained object for frequency-domain "globally meromorphic fitting", providing principal angle upper bound for Fisher information. Paper concludes with parameters and acceptance standards for three reproducible experimental pipelines: closed-domain rel-zeta, exterior-domain meromorph-fit, channel–pseudo-unitary verification.

1 Introduction

Closed-domain relative ζ/heat kernel determinant and exterior-domain relative scattering determinant share essential structure—"relative phase". On closed-domain side, this phase recovers on-shell action's volume density response to cosmological constant Λ via logarithmic derivative; on exterior-domain side, it's controlled by BK/LK (and second-order Koplienko version) spectral shift function to scattering matrix phase on energy fibers. This paper unifies two domains into closed loop of "verifiable hypothesis $\Rightarrow$ theorem $\Rightarrow$ detailed proof":

1. Achieve fixed-energy implementation under operator-Lipschitz and DOI techniques, with weighted LAP dominated convergence;

2. Implement regularization independence and item-wise cancellation of corner/boundary/ghost under Müller relative determinant;

3. Unify relative scattering determinant poles as QNM under analytic Fredholm framework, proving reference independence;

4. Separate "block-level modulus conservation" from "global Carleman modulus non-constant identity" under pseudo-unitary (J-unitary) framework, imposing real-axis modulus constraint via "phase-normalized determinant".

All mathematical expressions in text presented inline with $\cdot$ form, avoiding ambiguity from display/environment switching.

2 Setting, Notation, and Verifiable Hypotheses

2.1 Spectrum, Ideals, and Modified Determinants

Take separable Hilbert space $\mathcal{H}$. Denote self-adjoint operator pair (H_Λ, H_0), difference $V = H_\Lambda - H_0$. Schatten ideal $\mathfrak{S}_p$ standard definition. For $K \in \mathfrak{S}_1$ take Fredholm determinant $\det(I+K)$; for $K \in \mathfrak{S}_2$ take Carleman determinant $\det_2(I+K) = \det\big((I+K)\exp(-K)\big)$. If U unitary with $U - I \in \mathfrak{S}_2$, spectral angles $\{\theta_j\} \in \ell^2$ satisfy $|\det_2(U)| = \exp\big(\sum_j (1-\cos\theta_j)\big) \geq 1$, $\arg\det_2(U) = \sum_j(\theta_j - \sin\theta_j)$.

2.2 Spectral Shift Objects and DOI

First-order spectral shift ξ and second-order spectral shift measure η respectively satisfy $\mathrm{Tr}(f(H_\Lambda) - f(H_0)) = \int f'(E)\,\xi(E)\,dE$, $\mathrm{Tr}(f(H_\Lambda) - f(H_0) - f'(H_0)V) = \int f''(E)\,d\eta(E)$, function class taking operator-Lipschitz/appropriate Besov intersection. Cumulative antiderivative $\Xi^{(2)}(E) = \eta((-\infty, E))$, normalized $\Xi^{(2)}(-\infty) = 0$. Double operator integral representation $f(H_\Lambda) - f(H_0) = \iint \Phi_f(\lambda,\mu)\,dE_\Lambda(\lambda)\,V\,dE_0(\mu)$, where $\Phi_f(\lambda,\mu) = (f(\lambda) - f(\mu))/(\lambda - \mu)$ has Schur/Haagerup bound.

2.3 Weighted LAP and Energy Fiberization

There exist $s > \frac{1}{2}$, energy window I, constant C_I such that $|\langle x\rangle^{-s}(H_\# - \lambda \mp i0)^{-1}\langle x\rangle^{-s}| \leq C_I$ holds for $\lambda \in I$, $\# \in \{\Lambda, 0\}$. Stationary exterior region (SdS/KdS) stationary under time Killing field with frequency ω, partial wave decomposition yields channel matrix $S_{\ell m}(\omega)$.

2.4 Closed-Domain Relative ζ-Determinant and Volume Slope

Euclideanized second variation operator family $\mathcal{K}_\Lambda$ with reference $\mathcal{K}_0$ matching principal symbol, boundary conditions and Faddeev–Popov ghost pairing consistent, zero modes/threshold resonances removed via deprojection. Difference heat kernel $K_{\mathrm{rel}}(t) = \mathrm{Tr}\big(e^{-t(\mathcal{K}_\Lambda + \mu^2)} - e^{-t(\mathcal{K}_0+\mu^2)}\big)$ has short-time expansion, define $\log\det_{\zeta,\mathrm{rel}}(\mathcal{K}_\Lambda + \mu^2, \mathcal{K}_0 + \mu^2) = -\int_0^\infty t^{-1}K_{\mathrm{rel}}(t)\,dt$. Metric signature and action convention fixed as $\partial_\Lambda S_{\text{on-shell}} = (8\pi G)^{-1}\mathrm{Vol}_4(M)$.

2.5 Exterior-Domain Reference and Pseudo-Unitary

On strip $\Im\omega \in (-\gamma_0, 0]$ choose reference scattering matrix $S_0(\omega)$, require analyticity on same sheet without zero/poles. Each channel constructs energy flux quadratic form η via Jost–Wronskian normalization making $S_{\ell m}^\dagger \eta S_{\ell m} = \eta$.

2.6 Verifiable Hypotheses (Assumption Box)

(H-AC): Wave operators exist and complete, AC part admits energy fiberization;

(H-LAP): Weighted LAP (parameter $s > 1/2$, constant C_I);

(H-LK/DOI): Poisson smoothing $f_\varepsilon \in \mathrm{OL}$, $|f_\varepsilon|_{\mathrm{OL}} \le C/\varepsilon$, DOI kernel has uniform Schur/Haagerup bound;

(H-S$_p$): For a.e. $\omega \in I$, $\chi_{(-\infty,\omega]}(H_\Lambda) - \chi_{(-\infty,\omega]}(H_0) \in \mathfrak{S}_p$ and $S_\Lambda(\omega)S_0(\omega)^{-1} - I \in \mathfrak{S}_p$ (typical $p = 2$);

(H-relDet): Principal symbol consistent, boundary/ghost matching, no zero modes or dereso'd, difference heat kernel has short-time expansion;

(H-Ref): Reference S_0 analytic on strip without zero/poles;

(H-Can): Channel energy flux gauge fixed, block-level pseudo-unitary holds.

3 Main Theorems and Conclusions

Theorem 1 (3.1: Fixed-Energy BK: $p \in \{1, 2\}$ Unified Version)**.** *Under* (H-AC), (H-LAP), (H-LK/DOI), (H-S$_p$), *for Lebesgue almost everywhere* $\omega \in I$*: when* $p = 1$, $\det S_\Lambda(\omega) = \exp\big(-2\pi i\,\xi_\Lambda(\omega)\big)$*; when* $p = 2$, $\det_2 S_\Lambda(\omega) = \exp\big(-2\pi i\,\Xi^{(2)}_\Lambda(\omega)\big)$*. Thus* $\arg\det_p S_\Lambda(\omega) = -2\pi\,\Xi^{(p)}_\Lambda(\omega)$*.*

Theorem 2 (3.2: Closed-Domain "Volume Slope")**.** *Under* (H-relDet) *and deresonance projection,* $\lim_{\mu\to 0^+} \mathrm{Vol}_4(M)^{-1}\,\partial_\Lambda \Re \log \det_{\zeta,\mathrm{rel}}(\mathcal{K}_\Lambda + \mu^2, \mathcal{K}_0 + \mu^2) = \frac{1}{8\pi G}$ *(per text signature convention).*

Theorem 3 (3.3: τ_p Poles = QNM, Reference Independent)**.** *Let* $\tau_p(\omega) = \det_p\big(S(\omega)S_0(\omega)^{-1}\big)$*. Under* (H-Ref)*, on strip* $\Im\omega \in (-\gamma_0, 0]$*, pole set of* τ_p *coincides with* S *poles (QNM) counting algebraic multiplicity. If changing reference to* $\widetilde{S}_0$ *still satisfying* (H-Ref)*, then* $\tau_p/\widetilde{\tau}_p$ *is analytic outer function without zeros/poles on strip, leaving pole set unchanged.*

Theorem 4 (3.4: Real-Frequency Phase and Modulus; Phase-Normalized Determinant)**.** *Block-level: If* $S^\dagger_{\ell m}(\omega)\eta S_{\ell m}(\omega) = \eta$*, then* $|\det S_{\ell m}(\omega)| = 1$*. Global: generally only* $\arg\det_p S(\omega) = -2\pi\,\Xi^{(p)}(\omega)$ *holds. When* $S(\omega)$ *unitary with* $S(\omega) - I \in \mathfrak{S}_2$, $|\det_2 S(\omega)| = \exp\big(\sum_j (1 - \cos\theta_j(\omega))\big) \ge 1$*. Define* $\widehat{\det}_p S(\omega) = \det_p S(\omega)/|\det_p S(\omega)|$, $\widehat{\tau}_p(\omega) = \tau_p(\omega)/|\tau_p(\omega)|$ *as real-axis "modulus equals 1" constrained objects.*

4 Proof of Theorem 3.1 (DOI–LAP Dominated Convergence to Fixed Energy)

Proof strategy overview: Approximate step function via Poisson smoothing f_ε, apply DOI expression with weighted LAP establishing uniform $\mathfrak{S}_p$ domination inequality, then exchange limit $\varepsilon \downarrow 0$ at Lebesgue points of spectral shift object, finally identify scattering phase via AC fiberization and exponentiate to determinant equality. Difference between $p = 1$ and $p = 2$ carried by first/second-order trace formulas.

Step 1 (Poisson smoothing and DOI kernel bound): Take $f_\varepsilon(\lambda) = \frac{1}{2} + \frac{1}{\pi}\arctan((\omega - \lambda)/\varepsilon)$. Then $f_\varepsilon \in \mathrm{OL}$ with $|f_\varepsilon|_{\mathrm{OL}} \le C/\varepsilon$. DOI expression $f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0) = \iint \Phi_{f_\varepsilon}(\lambda, \mu)\, dE_\Lambda(\lambda)\, V\, dE_0(\mu)$, where $|\Phi_{f_\varepsilon}|_{\mathrm{Schur}} \le C/\varepsilon$.

Step 2 (Weighted LAP and Schatten domination): Write weighted projection boundary value resolvent form via Stone formula, apply (H-LAP) yielding $|\langle x\rangle^{-s} R_\#(\omega \pm i0)\langle x\rangle^{-s}| \le C_I$. By Birman–Solomyak type estimate obtain $|f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0)|_{\mathfrak{S}_p} \le C_I(C/\varepsilon)\, M_p(I)$, where $M_p(I) = \sup_{\lambda\in I} |\langle x\rangle^{-s}(R_\Lambda(\lambda \pm i0) - R_0(\lambda \pm i0))\langle x\rangle^{-s}|_{\mathfrak{S}_p}$ bounded.

Step 3 ($p = 1$**: spectral shift and BK**): First-order trace formula yields $\mathrm{Tr}(f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0)) = \int f_\varepsilon'(E)\,\xi(E)\,dE$. Taking $\varepsilon \downarrow 0$ with dominated convergence yields $\mathrm{Tr}(\chi_{(-\infty,\omega]}(H_\Lambda) - \chi_{(-\infty,\omega]}(H_0)) = \xi(\omega)$ at Lebesgue points of ω. AC fiberization with stationary scattering shows $\det S(\omega) = \exp(-2\pi i\,\xi(\omega))$.

Step 4 ($p = 2$**: Koplienko phase**): Second-order trace formula yields $\mathrm{Tr}\big(f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0) - f_\varepsilon'(H_0)V\big) = \int f_\varepsilon''(E)\,d\eta(E)$. Integrate right side twice by parts, taking $\varepsilon \downarrow 0$ yields $\Xi^{(2)}(\omega) = \eta((-\infty,\omega))$. Fixed-energy implementation same as above, thus $\det_2 S(\omega) = \exp(-2\pi i\,\Xi^{(2)}(\omega))$. QED.

5 Proof of Theorem 3.2 (Relative Heat Kernel Item-Wise Cancellation, Tauberian Exchange, and Signature Convention)

Step 1 (Logarithmic derivative heat kernel representation): $\partial_\Lambda \log\det_{\zeta,\mathrm{rel}} = -\int_0^\infty t^{-1}\,\partial_\Lambda K_{\mathrm{rel}}(t)\,dt$, where $K_{\mathrm{rel}}(t) = \mathrm{Tr}(e^{-t(\mathcal{K}_\Lambda+\mu^2)} - e^{-t(\mathcal{K}_0+\mu^2)})$.

Step 2 (Short-time expansion and item-wise cancellation): Under principal symbol consistency, boundary/ghost pairing consistency, multiplicative anomaly vanishing, $K_{\mathrm{rel}}(t) \sim \sum_{k\ge 0} a_k^{\mathrm{rel}}\,t^{(k-d)/2}$ ($d = 4$), local coefficients (including GHY, corners, ghost pairing) cancel item-wise except volume term a_0^{rel}, i.e., $a_{k>0}^{\mathrm{rel}} = 0$.

Step 3 (Tauberian exchange and volume slope): Introduce small mass $\mu > 0$ controlling large t part, split $\int_0^\infty = \int_0^{t_0} + \int_{t_0}^\infty$. Former dominated by a_0^{rel}, latter under deresonance projection has uniform bound. Exchanging $\mu \downarrow 0$ with volume density limit yields $\mathrm{Vol}_4^{-1}\,\partial_\Lambda \Re \log\det_{\zeta,\mathrm{rel}} = \partial_\Lambda c_0$. By text convention $\partial_\Lambda S_{\text{on-shell}} = (8\pi G)^{-1}\mathrm{Vol}_4$, alignment yields $\partial_\Lambda c_0 = \frac{1}{8\pi G}$. QED.

6 Proof of Theorem 3.3 (Analytic Fredholm and Reference Independence)

Step 1 (Analytic Fredholm): On strip $\Im\omega \in (-\gamma_0, 0]$, write $S(\omega) = I + K(\omega)$ where $K(\omega)$ is $\mathfrak{S}_p$-valued meromorphic family. Determinant $\mathcal{D}_p(\omega) = \det_p(I + K(\omega))$ meromorphic, its zero order equals kernel dimension (algebraic multiplicity) of $I + K(\omega)$.

Step 2 (Relativization and pole counting): Define $\tau_p(\omega) = \det_p(S(\omega)S_0(\omega)^{-1})$. If S_0 analytic nonzero on strip, τ_p shares poles and orders with S, poles being QNM.

Step 3 (Reference independence): If choosing another $\widetilde{S}_0$ also satisfying condition, then $\tau_p/\widetilde{\tau}_p = \det_p(S_0\widetilde{S}_0^{-1})$ is analytic outer function without zeros/poles, leaving pole set unchanged. QED.

7 Proof of Theorem 3.4 (Block-Level Pseudo-Unitary and Global Carleman Modulus)

Block level: By $S_{\ell m}^\dagger \eta S_{\ell m} = \eta$ and $\det(\eta^{-1}S_{\ell m}^\dagger \eta S_{\ell m}) = 1$ obtain $|\det S_{\ell m}| = 1$.

Global phase: By Theorem 3.1 obtain $\arg\det_p S(\omega) = -2\pi\,\Xi^{(p)}(\omega)$.

Global modulus ($p = 2$): If $S(\omega)$ unitary with $S(\omega) - I \in \mathfrak{S}_2$, spectral angles $\{\theta_j(\omega)\} \in \ell^2$ yield $|\det_2 S(\omega)| = \exp\big(\sum_j (1 - \cos\theta_j(\omega))\big) \ge 1$. In general J-unitary case modulus non-constant, thus phase-normalization $\widehat{\det}_p$ natural object for real-axis modulus constraint. QED.

8 Globally Meromorphic Fitting and Fisher Projection Geometry (For Data-Side Implementation)

On strip $\Im\omega \in [-\gamma_0, 0]$ parametrize $\log \widehat{\tau}_p(\omega) = \sum_{j=1}^{J} \log \frac{\omega-\omega_j}{\omega-\overline{\omega_j}} + iQ(\omega)$, where ω_j are lower half-plane poles, Q low-order entire function taking purely imaginary values on real axis. Enforce conjugate pairing and "phase-normalized modulus equals 1", suppress false poles via strip cross-validation.

Proposition 5 (8.1: Fisher Principal Angle Upper Bound)**.** *For whitened observation* $y_k = \Im \log \widehat{\tau}_p(\omega_k) + \epsilon_k$*, Jacobian* J *with constraint submanifold tangent space projection* $P_{\mathcal{M}}$ *yields restricted Fisher* $F_{\mathcal{M}} = (P_{\mathcal{M}}J)^\top (P_{\mathcal{M}}J)$*. If* ϑ *is maximum principal angle between* $\operatorname{range}(J)$ *and* $\operatorname{range}(P_{\mathcal{M}})$*, then variance reduction factor* $\mathcal{R} \le 1/|\sin\vartheta|$*. Proof in Appendix F.*

9 Reproducible Experimental Protocols (P1–P3)

P1 | rel-zeta (closed domain): Grid step h (three levels), heat kernel window $t \in [t_{\min}, t_{\max}]$ ($t_{\min} \sim c\,h^2$), extrapolation order $N \in \{2, 3\}$, small mass μ (three to five logarithmic points). Target quantity $\mathrm{Vol}_4^{-1}\,\partial_\Lambda \Re \log \det_{\zeta,\mathrm{rel}}$. Acceptance: slope error $< 1\%$; drift under different corner triangulations/gauges $< 0.5\%$.

P2 | meromorph-fit (exterior domain): Fit $\log \widehat{\tau}_p$ recovering $\{\omega_j\}$. Priors: pairwise symmetric, strip analytic, real-axis modulus constraint (on $\widehat{\tau}_p$), and $\Re \log \det_2(\omega) \ge 0$ (if using $p = 2$). Acceptance: CRLB improvement over mode-by-mode $\ge 1.3\times$; false alarm rate $\le 5\%$; cross-strip consistent.

P3 | bh-channels (pseudo-unitary and BK phase): Jost–Wronskian normalization constructs η, compute $|S_{\ell m}^\dagger \eta S_{\ell m} - \eta|$ and phase closure $\arg \widehat{\det}_p S + 2\pi \Xi^{(p)}$. Acceptance: pseudo-unitary residual $< 10^{-12}$, phase closure $< 10^{-3}$ radians; converges as $a + b/\ell_{\max}$ with $\ell_{\max}$.

10 Discussion and Outlook

Under explicitly verifiable analytic hypotheses, this paper completes four main conclusions: $p \in \{1, 2\}$ unified version of fixed-energy BK, closed-domain volume slope, reference independence of relative scattering determinant poles = QNM, and real-frequency phase–modulus decomposition, providing reproducible experimental pipelines. Limitations: LAP constant may deteriorate under strong trapping or extreme spin; non-local boundaries and singular geometry require separate verification of multiplicative anomaly; statistical side needs robust regularization against model bias. Future work includes: extending modulus–phase formula for $\det_p$ under Kreĭn spaces; seamlessly incorporating BK version for differential forms/electromagnetic fields; testing stability of "reference independent" poles using multi-station strip data.

A Complete Derivation of DOI–LAP Dominated Convergence

A.1 Kernel Bound and Weight Insertion

Take $f_\varepsilon(\lambda) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{\omega-\lambda}{\varepsilon}$. $f_\varepsilon \in \mathrm{OL}$, $|f_\varepsilon|_{\mathrm{OL}} \le C/\varepsilon$. DOI expression yields $f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0) = \iint \Phi_{f_\varepsilon}(\lambda,\mu)\, dE_\Lambda(\lambda)\, V\, dE_0(\mu)$, where Φ_{f_ε} satisfies $\sup_\lambda \int |\Phi_{f_\varepsilon}(\lambda,\mu)|\, d\mu \le C/\varepsilon$, $\sup_\mu \int |\Phi_{f_\varepsilon}(\lambda,\mu)|\, d\lambda \le C/\varepsilon$.

Insert $\langle x\rangle^{\pm s}$ obtaining $f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0) = \iint (\langle x\rangle^{-s} dE_\Lambda(\lambda))\, (\langle x\rangle^{s} V \langle x\rangle^{s})\, (dE_0(\mu)\langle x\rangle^{-s})\, \Phi_{f_\varepsilon}(\lambda,\mu)$.

A.2 Schatten Domination Inequality

By Haagerup/Schur bound with Hölder inequality (on $\mathfrak{S}_p$), $|f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0)|_{\mathfrak{S}_p} \le |\Phi_{f_\varepsilon}|_{\mathrm{Schur}} \cdot \sup_{\lambda\in I} |\langle x\rangle^{-s} E'_\Lambda(\lambda)\langle x\rangle^{-s}| \cdot |\langle x\rangle^s V \langle x\rangle^s|_{\mathfrak{S}_p} \cdot \sup_{\mu\in I} |\langle x\rangle^{-s} E'_0(\mu)\langle x\rangle^{-s}|$.

Use Stone formula $E'_{\#}(\lambda) = \pi^{-1}\Im R_{\#}(\lambda + i0)$ with (H-LAP) obtaining $|f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0)|_{\mathfrak{S}_p} \le C_I\,\varepsilon^{-1}\,|\langle x\rangle^s V \langle x\rangle^s|_{\mathfrak{S}_p}$.

In scattering setting better use difference resolvent form $|\langle x\rangle^{-s}(R_\Lambda(\lambda \pm i0) - R_0(\lambda \pm i0))\langle x\rangle^{-s}|_{\mathfrak{S}_p}$ controlling $|\langle x\rangle^s V \langle x\rangle^s|_{\mathfrak{S}_p}$, thus obtaining unified domination $|f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0)|_{\mathfrak{S}_p} \le C_I\,\varepsilon^{-1}\,M_p(I)$.

A.3 Lebesgue Points and Limit Exchange

Let ω be Lebesgue point of spectral shift object. By above obtain family of ε-uniformly integrable dominating function $M(\omega) = \sup_{\varepsilon<\varepsilon_0} |f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0)|_{\mathfrak{S}_p} \in L^1_{\mathrm{loc}}(I)$. Thus limit of DOI-trace as $\varepsilon \downarrow 0$ commutes with local integration over ω, obtaining fixed-energy version of first/second-order trace formula, completing Theorem 3.1 proof.

B Relative Heat Kernel Item-Wise Cancellation and Λ-Slope Refinement

B.1 Short-Time Expansion and Local Coefficients

For Laplace-type operator $\mathcal{K}_{\#}$ (including gauge-ghost pairing) have $\mathrm{Tr}(e^{-t\mathcal{K}_{\#}}) \sim \sum_{j\ge 0} a_j(\mathcal{K}_{\#})\,t^{(j-d)/2}$. Under principal symbol consistency with boundary/ghost matching, relative difference $a^{\mathrm{rel}}_{j>0} = a_j(\mathcal{K}_\Lambda) - a_j(\mathcal{K}_0) = 0$; corner and boundary term coefficients also cancel in "relative difference" (Wodzicki residue zero ensures multiplicative anomaly absent).

B.2 Logarithmic Derivative and Volume Term

Relative ζ written as $\zeta_{\mathrm{rel}}(s;\mu) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} e^{-t\mu^2}\,K_{\mathrm{rel}}(t)\,dt$. Differentiating with respect to Λ, only $a_0^{\mathrm{rel}} = \mathrm{Vol}_4(M)c_0$ contributes, yielding $\partial_\Lambda \zeta'_{\mathrm{rel}}(0;\mu) = -\partial_\Lambda a_0^{\mathrm{rel}} \int_0^\infty t^{-1} e^{-t\mu^2}\,dt + \text{finite terms}$. Volume density with $\mu \downarrow 0$ exchange, by text convention $\partial_\Lambda c_0 = \frac{1}{8\pi G}$, obtaining Theorem 3.2.

B.3 Tauberian Exchange Error Estimate

Take $t_0 = \mu^{-2\alpha}$ ($\alpha \in (0,1)$), $\int_{t_0}^\infty t^{-1} e^{-t\mu^2}\,K_{\mathrm{rel}}(t)\,dt$ controlled by spectral gap and deresonance projection as $\mathcal{O}(\mu^{2(1-\alpha)})$, while $(0, t_0)$ segment error after higher-order coefficient cancellation becomes $\mathcal{O}(t_0^{1/2}) = \mathcal{O}(\mu^{-\alpha})$ coefficient nullification term, overall can take α making total error $o(1)$.

C Reference Independence of Relative Scattering Determinant and Pole Counting

C.1 Analytic Fredholm Tools

Write $S(\omega) = I + K(\omega)$, $K(\omega)$ being $\mathfrak{S}_p$-valued meromorphic family. Then $\mathcal{D}_p(\omega) = \det_p(I + K(\omega))$ meromorphic with zero order equal to $\dim\ker(I + K(\omega))$.

C.2 Relativization and Pole Transfer

Assume S_0 analytic without zeros/poles on strip, define $\tau_p(\omega) = \det_p(S(\omega)S_0(\omega)^{-1})=\det_p(I + K(\omega)) \cdot \det_p(S_0(\omega)^{-1})$. Latter factor analytic nonzero, thus τ_p poles synchronize with $\mathcal{D}_p$, order being QNM algebraic multiplicity.

C.3 Reference Independent Outer Function Factor

If changing reference to $\widetilde{S}_0$, then $\tau_p/\widetilde{\tau}_p = \det_p(S_0\widetilde{S}_0^{-1})$ is analytic without zeros/poles (outer function), leaving pole set and multiplicity unchanged.

D Pseudo-Unitary: Channel Construction and Global Carleman Modulus

D.1 Energy Flux Quadratic Form and J-Unitary

Take Jost solutions $u_{\text{in/out}}$ at both ends of radial equation with Wronskian normalization, making energy flux $\mathcal{F} = \Im(\overline{u}\,\partial_r u)$ consistent at both ends. Accordingly define channel quadratic form $\eta = \mathrm{diag}(1, -1)$ making $S_{\ell m}^{\dagger}\eta S_{\ell m} = \eta$.

D.2 Block-Level Determinant Unit Modulus and Global Phase

Finite-dimensional block directly yields $|\det S_{\ell m}| = 1$. Global direct sum under $\mathfrak{S}_2$ setting only preserves phase equality; if globally unitary with $S - I \in \mathfrak{S}_2$, spectral angle expansion yields $|\det_2 S| = \exp(\sum(1 - \cos\theta_j)) \geq 1$.

E Koplienko Phase Fixed-Energy Construction ($p = 2$)

E.1 Second-Order Trace Formula and DOI

For f_ε have $\mathrm{Tr}\big(f_\varepsilon(H_\Lambda) - f_\varepsilon(H_0) - f'_\varepsilon(H_0)V\big) = \int f''_\varepsilon(E)\,d\eta(E)$. Integrate right side twice by parts yielding $-\int f'_\varepsilon(E)\,d\Xi^{(2)}(E)$.

E.2 Dominated Convergence and Lebesgue Points

By Appendix A's $\mathfrak{S}_2$ domination with $|f'_\varepsilon|_{L^1} \leq C$ obtain integrable domination, as $\varepsilon \downarrow 0$, f'_ε converges to δ_ω (weak sense) recovering $\Xi^{(2)}(\omega)$ at Lebesgue points.

E.3 Scattering Phase and Determinant

Fixed-energy implementation with AC fiberization identifies $\Xi^{(2)}(\omega)$ as scattering phase second-order spectral shift antiderivative, exponentiating yields $\det_2 S(\omega) = \exp(-2\pi i\,\Xi^{(2)}(\omega))$.

F Proof of Fisher Projection Geometry Upper Bound

F.1 Model and Projection

Whitened observation $y = J\theta + \epsilon$, hard constraint $C(\theta) = 0$ defining differentiable submanifold $\mathcal{M}$ with tangent space projection $P_{\mathcal{M}}$ satisfying $P_{\mathcal{M}}^2 = P_{\mathcal{M}}$.

F.2 Principal Angle and Spectral Bound

Let ϑ be maximum principal angle between range(J) and range($P_{\mathcal{M}}$), have $|P_{\mathcal{M}}v| \geq |\sin\vartheta|\,|v|$ for all $v \in$ range(J). Thus $v^\top F_{\mathcal{M}} v = |P_{\mathcal{M}} J v|^2 \geq \sin^2\vartheta\,|Jv|^2 = v^\top(\sin^2\vartheta\, F)v$, yielding $F_{\mathcal{M}} \succeq \sin^2\vartheta\, F$. Taking maximum eigenvalue yields $\mathrm{Tr}(F_{\mathcal{M}}^{-1}) \leq \mathrm{Tr}(F^{-1})/\sin^2\vartheta$, i.e., variance reduction factor $\mathcal{R} \leq 1/|\sin\vartheta|$. QED.

G Reproducible Experiment Parameters and Error Budget (Brief Table)

G.1 P1 (closed domain): $h \in \{h_0,\, h_0/2,\, h_0/4\}$; $t_{\min} \sim c\,h^2$, $t_{\max}$ satisfying semiclassical window; μ taking $\{\mu_0,\, \mu_0/3,\, \mu_0/9,\, \mu_0/27\}$. Extrapolation using bilinear (for (μ, h)) and Richardson (for t-window) hybrid. Tolerance: slope $< 1\%$; drift under different corner triangulations/gauges $< 0.5\%$.

G.2 P2 (exterior domain): Strip $\Im\omega \in [-\gamma_0, 0]$ uniform sampling; pole number J jointly determined by AIC/BIC and strip cross-validation; penalty term constrains $Q(\omega)$ degree and real-axis purely imaginary condition; prior $\Re\log\det_2 \geq 0$ only as soft regularization. Tolerance: CRLB improvement ≥ 1.3, false alarm $\leq 5\%$.

G.3 P3 (channels): Extrapolation radius, matching radius, integration step calibrated via grid search; Wronskian normalization difference $< 10^{-12}$; $|S_{\ell m}^\dagger \eta S_{\ell m} - \eta|_\infty < 10^{-12}$; phase closure $< 10^{-3}$ radians; converges as $a + b/\ell_{\max}$ with $\ell_{\max}$.

End of Main Text and Appendices