Relative Topology, Principal Bundle Reduction, and Index Theory on Perforated Information Manifolds: Unified Framework Toward $S(U(3) \times U(2))$, Three-Generation Index, and Yukawa–Winding

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Abstract

Full-rank density matrix manifold $\mathcal{D}_N^{\text{full}} = \{\rho > 0,\ \mathrm{tr}\rho = 1\}$ is open convex contractible, Uhlmann principal bundle admits global square-root section $w = \sqrt{\rho}$ on full domain, thus **absolute** integer-valued topological invariants are absent on full domain. This paper turns to **perforated relative topology**: In $N = 5$ case, removing tubular neighborhood of three–two level gap closing set $\Sigma_{3|2} = \{\lambda_3 = \lambda_4\}$ from $\mathcal{D}_5^{\text{full}}$ yields perforated domain $\mathcal{D}^{\text{exc}}$. On $\mathcal{D}^{\text{exc}}$ construct rank 3/2 subbundles $(\mathcal{E}_3, \mathcal{E}_2)$ via **Riesz spectral projection**, realizing principal bundle structure group reduction $U(5) \to U(3) \times U(2)$; further utilizing determinant balancing yields $S(U(3) \times U(2))$ reduction. We prove general group isomorphism

$$S\big(U(m) \times U(n)\big) \ \cong\ \big(SU(m) \times SU(n) \times U(1)\big)/\mathbb{Z}_{\mathrm{lcm}(m,n)},$$

with $(m, n) = (3, 2)$ deriving $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$. Through **relative K-theory boundary map** unifying "projection–Chern class" with "mass–clutching ($\det \widehat{\Phi}$ winding)", on two-dimensional transverse S^1 obtain

$$\mathrm{Ind}(\not{D}_A + \Phi) = \mathrm{wind}\ \det \widehat{\Phi} = \langle c_1(\mathcal{L}_\Phi), [S^1] \rangle.$$

In $\mathbb{CP}^2$ spin^c/Dolbeault calibration compute index $= 3$ as "three-generation prototype". This paper provides **complete proofs** of all core propositions and theorems, with two **protocol-level** reproducible experimental/numerical schemes (purification interference loop and photonic Dirac–mass vortex). Appendices include: unified contour and global smoothness, group isomorphism gcd/lcm normalization and "root selection", rigorous proof of relative K-theory and Chern character commutative diagram, Fredholm construction for Callias/Anghel–Bunke index theorem, and one-page arithmetic derivation of minimal charge 1/6 when $\Gamma = \mathbb{Z}_6$.

Keywords: Uhlmann principal bundle; perforated relative topology; Riesz projection; principal bundle reduction; $S(U(3)\times U(2))$; $\mathbb{Z}_6$ quotient; relative cohomology/K-theory; Dolbeault/spin^c index; Callias/Anghel–Bunke index; determinant line bundle; line operator spectrum; reproducible experimental protocol

1 Notation, Assumptions, and Scope

- **Mixed state manifold**: $\mathcal{D}_N^{\text{full}} = \{\rho \in \mathrm{Herm}_N^+ : \rho > 0,\ \mathrm{tr}\rho = 1\}$, this paper fixes $N = 5$.
- **Eigenvalue order**: $\lambda_1 \geq \cdots \geq \lambda_5$; **spectral gap function** $g(\rho) := \lambda_3 - \lambda_4$.

- **Perforated domain**: Take $\delta > 0$, define $\mathcal{D}^{\mathrm{exc}} := \{\rho \in \mathcal{D}_5^{\mathrm{full}} : \ g(\rho) \geq 2\delta\}$. Boundary $Y := \partial\mathrm{Tub}_\varepsilon(\Sigma_{3|2})$ equivalent to $g = 2\delta$ tubular boundary.

- **Riesz projection**: Fix **unified contour family** γ_3 (see Lemma 1.2 and Appendix A), let

$$P_3(\rho) = \frac{1}{2\pi i}\oint_{\gamma_3}(z-\rho)^{-1}\,dz, \qquad P_2 = I - P_3.$$

- **Uhlmann principal bundle**: $P = \{\sqrt{\rho}\,U : \ \rho \in \mathcal{D}_5^{\mathrm{full}},\ U \in U(5)\}$, right action $w \cdot V = wV$; $\pi(w) = ww^\dagger$ yields $U(5)$-principal bundle $P \to \mathcal{D}_5^{\mathrm{full}}$.

- **Regular/ordinary process (verification checklist)**: Along path full rank, generator local CPTP and C^1, optional continuous purification gauge, and **avoiding** $\Sigma_{3|2}$ ($g \geq 2\delta$). This checklist only motivational: full domain lacks integer global classes; this paper focuses **relative** quantization on perforated domain.

- **Normalization**: de Rham pairing uniformly takes $\frac{1}{2\pi i}$ factor; on $\mathbb{CP}^2$ hyperplane class H normalized as $\int_{\mathbb{CP}^1} H = 1$, $\int_{\mathbb{CP}^2} H^2 = 1$.

2 Main Results (Statements)

Theorem 1 (A: Group Isomorphism, gcd/lcm Normalization)**.** *Let $g = \gcd(m,n)$, $\ell = \mathrm{lcm}(m,n) = mn/g$. Homomorphism*

$$\varphi : \ SU(m) \times SU(n) \times U(1) \to S(U(m) \times U(n)), \quad \varphi(A,B,z) = \mathrm{diag}\big(z^{n/g}A,\ z^{-m/g}B\big)$$

is surjective with $\ker\varphi \simeq \mathbb{Z}_\ell$. Thus

$$S\big(U(m) \times U(n)\big) \ \cong \ \big(SU(m) \times SU(n) \times U(1)\big)/\mathbb{Z}_\ell.$$

Special case $(m,n) = (3,2) \Rightarrow \ell = 6$.

Proposition 2 (B: Partition Uniqueness)**.** *Under constraint "simple factors exactly $SU(3)$, $SU(2)$ retaining only one $U(1)$", unique feasible partition of $U(5)$ is $5 = 3 + 2$.*

Theorem 3 (C: Relative Bridging)**.** *Assume mass end term Φ invertible on Y, take unitization $\widehat{\Phi} : Y \to U(N)$. Then relative K-theory boundary image $\partial[\det\widehat{\Phi}] \in K^0(X,Y)$ equals projection line bundle $[\det\mathcal{E}_3] - [\det\mathcal{E}_2]$; on two-dimensional link*

$$\langle c_1(\mathcal{L}_\Phi), [S^2]\rangle = \langle c_1(\det\mathcal{E}_3), [S^2]\rangle \in \mathbb{Z}.$$

Theorem 4 (D: Callias/Anghel–Bunke)**.** *If outer region invertibility $\Phi^2 \geq cI$, $[\nabla,\Phi] \in L^\infty$, $\Phi \in W^{1,2}_{\mathrm{loc}}$ etc. hold, then*

$$\mathrm{Ind}(\not{D}_A + \Phi) = \deg\big(\widehat{\Phi}|_{S^{d-1}_\infty}\big) \in \pi_{d-1}(U).$$

*By Bott periodicity $\pi_k(U) = \mathbb{Z}$ (k odd), 0 (k even), obtain: index possibly nonzero only when **transverse dimension** d **is even**; when $d = 2$*

$$\mathrm{Ind} = \frac{1}{2\pi i}\oint \mathrm{Tr}(\widehat{\Phi}^{-1}d\widehat{\Phi}) = \mathrm{wind}\,\det\widehat{\Phi} = \langle c_1(\mathcal{L}_\Phi), [S^1]\rangle.$$

Theorem 5 (E: $\mathbb{CP}^2$ Index)**.** $\mathrm{Td}(T\mathbb{CP}^2) = 1 + \frac{3}{2}H + H^2$*,* $\mathrm{ch}(\mathcal{O}(1)) = 1 + H + \frac{1}{2}H^2$*, thus* index $\not{D}^{\mathcal{O}(1)} = 3$.

Corollary 6 (F: SM Global Group)**.** *By Theorem A, Proposition B, obtain*

$$S(U(3) \times U(2)) \;\cong\; \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}.$$

Appendix E further provides electric/magnetic charge lattice and arithmetic derivation of ***minimal charge step*** $1/6$ *for line operator spectrum when* $\Gamma = \mathbb{Z}_6$*.*

3 Degeneration Set Geometry and Unified Contour

Proposition 7 (2.1: Codimension 3 and S^2-Link)**.** *In three-dimensional transverse slice maintaining* (λ_2, λ_5) *gap with no additional symmetry,* $\Sigma_{3|2} = \{\lambda_3 = \lambda_4\}$ *is codimension 3 regular subset, its small sphere boundary link homotopic to* S^2*.*

Proof essentials*: Restrict Hamiltonian to near-degenerate 2-dimensional eigensubspace, obtain* $h = x\sigma_x + y\sigma_y + z\sigma_z$*; degeneracy condition* $(x, y, z) = (0, 0, 0)$ *yields three independent real constraints. See Appendix A.3.*

Lemma 8 (2.2: Unified Contour; Global C^∞)**.** *For any compact* $K \subset \mathcal{D}^{\mathrm{exc}}$*, there exist* $\delta > 0$ *and finite cover* $\{U_j\}$ *with closed curve family* $\{\gamma_j\}$ *such that:* $\forall \rho \in U_j$*,* γ_j *has distance* $\geq \delta$ *from complement spectrum; thus* $P_{3,2}$ *is* C^∞ *on* U_j *and can be smoothly patched. Details in Appendix A.1–A.2.*

4 Principal Bundle Reduction to $S(U(3) \times U(2))$

Theorem 9 (3.1: Reduction = Section)**.** *Let* $P \to X$ *be* $U(5)$*-principal bundle,* $\mathcal{G} = P \times_{U(5)} \mathrm{Gr}_3(\mathbb{C}^5)$*. Section* σ *from* P_3 *exists if and only if* P *admits* $U(3) \times U(2)$*-reduction* $P_H \subset P$*.*

Proof*: Standard principal bundle theory, Appendix B.4.*

Proposition 10 (3.2: Gauge Nature of Determinant Balancing)**.** *Background trivial bundle* $\underline{\mathbb{C}}^5$ *with fixed volume form yields gauge isomorphism* $\det \mathcal{E}_3 \otimes \det \mathcal{E}_2 \simeq \underline{\mathbb{C}}$*, reducing to* $S(U(3) \times U(2))$*.*

Theorem 11 (3.3: Group Isomorphism; Theorem A for $m = 3$, $n = 2$)**.**

$$S(U(3) \times U(2)) \;\cong\; (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6.$$

Proof*: See Appendix B.1; particularly note "root selection" step in surjectivity: given* (g_3, g_2)*, take* $z \in U(1)$ *satisfying* $z^6 = \det g_3$*, let* $A = z^{-2} g_3 \in SU(3)$*,* $B = z^3 g_2 \in SU(2)$*. Kernel isomorphic to* $\mathbb{Z}_6$*.*

Proposition 12 (3.4: Partition Uniqueness; Proposition B)**.** *Partition* $5 = 3+2$ *is unique satisfying "simple factors* $SU(3)$*,* $SU(2)$ *with only one* $U(1)$*";* $(4+1)$ *lacks* $SU(2)$*,* $(3+1+1)$ *and* $(2+2+1)$ *both retain two* $U(1)$*'s. Details in Appendix B.2.*

5 Two Characterizations of Relative Topology and Their Equivalence (Theorem C)

5.1 Relative K-Theory and Boundary Map

For pair $(X, Y) = (\mathcal{D}^{\mathrm{exc}}, \partial \mathrm{Tub}_\varepsilon)$, long exact sequence

$$\cdots \to K^1(Y) \xrightarrow{\partial} K^0(X, Y) \to K^0(X) \to \cdots .$$

If Φ invertible on Y, then unitization $\widehat{\Phi} : Y \to U(N)$ defines $[\widehat{\Phi}] \in K^1(Y)$, its boundary $\partial[\widehat{\Phi}] \in K^0(X, Y)$.

5.2 Commutative Diagram and de Rham Representative

Odd Chern character $\mathrm{ch}_1 : K^1(Y) \to H^1(Y; \mathbb{Q})$ with 1-dimensional representative

$$\mathrm{ch}_1([\widehat{\Phi}]) = \frac{1}{2\pi i} \mathrm{Tr}(\widehat{\Phi}^{-1} d\widehat{\Phi}).$$

Commutative diagram exists (Appendix C.1):

$$\begin{array}{ccc} K^1(Y) & \xrightarrow{\partial} & K^0(X,Y) \\ \downarrow \mathrm{ch}_1 & & \downarrow \mathrm{ch} \\ H^1(Y) & \xrightarrow{\partial} & H^2(X,Y) \end{array}$$

Thus

$$\mathrm{ch}\big(\partial[\widehat{\Phi}]\big) = \partial\Big[\frac{1}{2\pi i} \mathrm{Tr}(\widehat{\Phi}^{-1} d\widehat{\Phi})\Big] \in H^2(X, Y).$$

5.3 Equivalence Proposition (Theorem C)

Compared with $\mathcal{E}_3$, $\mathcal{E}_2$ from Riesz projection, utilizing naturality and clutching–gluing argument (Appendix C.2), obtain

$$\partial[\det \widehat{\Phi}] = [\det \mathcal{E}_3] - [\det \mathcal{E}_2] \in K^0(X, Y),$$

thus on two-dimensional link $\langle c_1(\mathcal{L}_\Phi), [S^2] \rangle = \langle c_1(\det \mathcal{E}_3), [S^2] \rangle$.

6 Spin^c/Dolbeault Index on $\mathbb{CP}^2$ (Theorem E)

Take $H = c_1(\mathcal{O}(1))$, $\int_{\mathbb{CP}^2} H^2 = 1$.

$$\mathrm{Td}(T\mathbb{CP}^2) = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) = 1 + \tfrac{3}{2}H + H^2, \quad \mathrm{ch}(\mathcal{O}(1)) = e^H = 1 + H + \tfrac{1}{2}H^2.$$

Top-dimensional coefficient $1 + \frac{3}{2} + \frac{1}{2} = 3$, thus index $\not{D}^{\mathcal{O}(1)} = 3$. Kodaira vanishing ensures $\chi = h^0 = 3$.

7 Callias/Anghel–Bunke Index = Degree; Two-Dimensional Winding Formula (Theorem D)

7.1 Fredholm Conditions

Let M complete, Dirac-type operator $\not{D}_A$ with self-adjoint end term Φ. If there exist $R, c > 0$ such that on $M \setminus B_R$, $\Phi^2 \geq cI$, with $[\nabla, \Phi] \in L^\infty$, $\Phi \in W^{1,2}_{\mathrm{loc}}$, then $\not{D}_A + \Phi$ is Fredholm (Appendix D.1).

7.2 Index = Degree and Parity

Boundary homomorphism and Bott isomorphism yield

$$\mathrm{Ind}(\not{D}_A + \Phi) = \deg\left(\widehat{\Phi}|_{S^{d-1}_\infty}\right) \in \pi_{d-1}(U), \quad \pi_k(U) = \begin{cases} \mathbb{Z}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

For two-dimensional transverse

$$\mathrm{Ind} = \frac{1}{2\pi i} \oint_{S^1} \mathrm{Tr}(\widehat{\Phi}^{-1} d\widehat{\Phi}) = \text{wind det}\,\widehat{\Phi} = \langle c_1(\mathcal{L}_\Phi), [S^1] \rangle,$$

consistent with zero-mode counting. Sign convention: S^1 takes counterclockwise orientation.

8 Alignment with G_{SM} Line Operator Spectrum and Minimal Charge 1/6

By Theorem 3.3: $G_{\mathrm{SM}} \cong (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$. Kernel generator can take

$$(\omega_3^{-1} I_3,\ -I_2,\ e^{i2\pi/6}), \qquad \omega_3 = e^{i2\pi/3}.$$

Action on (t, s, q) (respectively $SU(3)$ triality, $SU(2)$ parity, $U(1)$ integer charge) is

$$\omega_3^{-t} \cdot (-1)^s \cdot e^{i2\pi q/6}.$$

Necessary and sufficient condition for descending to quotient group: $q \equiv 2t + 3s \pmod 6$. Thus normalized hypercharge $Y = q/6$ has minimal fractional step 1/6. Appendix E provides one-page derivation and example table for electric/magnetic charge lattice, Dirac pairing integer matrix, and θ period.

9 Protocol-Level Experimental and Numerical Schemes (Overview)

E1 Purification Interference (Image around $\Sigma_{3|2}$): Discretize unified contour, readout $2\pi\phi_{\mathrm{rel}}$, where $\phi_{\mathrm{rel}} = \int_{S^2_{\mathrm{link}}} F_{\det \mathcal{E}_3}/(2\pi) \in \mathbb{Z}$. Sampling $N_{\mathrm{shots}} \gtrsim 30$, phase noise $\delta\phi \lesssim 0.25$rad can stably determine integer. Failure cases: path grazes $\Sigma_{3|2}$, non-smooth purification; countermeasures: enlarge contour radius, increase purity gap and repeat sampling.

E2 Photonic Dirac–Mass Vortex: Encode mass phase $e^{ik\theta}$, outer region $|m| \to m_\infty > 0$. Zero-mode count $|k|$, near-field intensity centralization, band-gap midpoint energy form fingerprint. Robust region: phase error $\leq 10°$, coupling mismatch $\leq 5\%$. Appendix F provides parameter table and "passing standard".

10 Discussion and Outlook

- **Relative vs absolute**: Full domain contractible → absolute integer class vanishes; perforated → relative class quantization.
- **Dimensional effect**: det only fully detects in two-dimensional transverse; higher dimensions require stable U group generators.
- **Group theory bridging**: Spectral splitting induced $S(U(3) \times U(2))$ works synergistically with line spectrum dictionary, yielding minimal charge step 1/6.
- **Follow-up**: Multi-defect superposition relative class addition, robust window under noise–non-equilibrium, systematic generalization with higher-order (r-block) splitting.

A Spectral Geometry and Unified Contour (Corresponding to §2)

A.1 Spectral Gap Lower Bound and Contour Selection

Let $\mathrm{gap}(\rho) = \min\{\lambda_3 - \lambda_4,\ \lambda_2 - \lambda_3,\ \lambda_4 - \lambda_5\}$. On $X = \mathcal{D}^{\mathrm{exc}}$, gap > 0 continuous; for any compact $K \subset X$, let $\delta = \min_K \mathrm{gap} > 0$. For each $\rho \in K$ take circle γ_ρ centered at $\frac{\lambda_3+\lambda_4}{2}$ with radius $\delta/2$, it encloses upper spectrum cluster with distance $\geq \delta/2$ from complement spectrum.

A.2 Riesz Projection C^∞ Dependence

By resolvent estimate $|(z-\rho)^{-1}| \leq 2/\delta$ and smoothness of $z \mapsto (z-\rho)^{-1}$, $P_3(\rho) = \frac{1}{2\pi i} \oint_{\gamma_\rho} (z-\rho)^{-1} dz$ is C^∞ in ρ. Using finite cover $\{U_j\}$ with partition of unity patching, obtain global C^∞ projection field $P_{3,2}$.

A.3 Codimension 3 and S^2-Link

At λ_3 & λ_4 near-degeneracy, take $E = E_{34} \oplus E^\perp$, effective Hamiltonian $h = \alpha\sigma_z + \Re\beta\,\sigma_x + \Im\beta\,\sigma_y$; degeneracy $\Leftrightarrow (\alpha, \Re\beta, \Im\beta) = (0,0,0)$, three independent real equations thus codimension 3. Take normal small ball B^3, its boundary S^2 is link.

B Group Isomorphism and Minimal Partition (Corresponding to §3)

B.1 Complete Proof of Theorem A

Homomorphism

$$\varphi(A, B, z) = \mathrm{diag}(z^{n/g}A,\ z^{-m/g}B), \quad g = \gcd(m, n),\ \ell = \frac{mn}{g}.$$

Kernel: $\varphi(A, B, z) = I \Rightarrow A = z^{-n/g}I_m,\ B = z^{m/g}I_n$. By $A \in SU(m) \Rightarrow z^{-nm/g} = 1 \Rightarrow z^\ell = 1$. Map

$$\kappa: \ \mu_\ell \to \ker\varphi, \quad \kappa(z) = (z^{-n/g}I_m,\ z^{m/g}I_n,\ z)$$

is group isomorphism, thus $\ker\varphi \simeq \mathbb{Z}_\ell$.

Surjectivity ("root selection"): Given $(g_3, g_2) \in S(U(m) \times U(n))$ (i.e., $\det g_3 \det g_2 = 1$), take $z \in U(1)$ satisfying

$$z^\ell = \det g_3.$$

Let

$$A = z^{-n/g} g_3 \in SU(m), \qquad B = z^{m/g} g_2 \in SU(n).$$

Then

$$\det A = z^{-nm/g} \det g_3 = z^{-\ell} \det g_3 = 1, \quad \det B = z^{nm/g} \det g_2 = z^\ell \det g_2 = 1,$$

with $\varphi(A, B, z) = (g_3, g_2)$. Thus obtain stated isomorphism.

B.2 Partition Uniqueness Table

Partition	Simple part	S-constrained $U(1)$ count	Conclusion
$4+1$	$SU(4)$	1	No $SU(2)$
$3+1+1$	$SU(3)$	2	Violates "one $U(1)$"
$2+2+1$	$SU(2) \times SU(2)$	2	Same
$3+2$	$SU(3) \times SU(2)$	**1**	**Unique satisfying**

B.3 Generalization

$S(U(k) \times U(\ell)) \cong (SU(k) \times SU(\ell) \times U(1))/\mathbb{Z}_{\mathrm{lcm}(k,\ell)}$; explicit form of kernel generator depends on embedding normalization, but quotient group isomorphism class invariant.

C Relative K-Theory and Chern Character (Corresponding to §4)

C.1 Commutative Diagram

For pair (X, Y), odd Chern character $\mathrm{ch}_1 : K^1(Y) \to H^1(Y; \mathbb{Q})$ yields

$$\mathrm{ch}_1([u]) = \frac{1}{2\pi i} \mathrm{Tr}(u^{-1} du).$$

Even Chern character $\mathrm{ch} : K^0(X, Y) \to H^{\mathrm{even}}(X, Y; \mathbb{Q})$ with de Rham boundary operator ∂ form commutative diagram

$$\begin{array}{ccc} K^1(Y) & \xrightarrow{\partial} & K^0(X,Y) \\ \downarrow \mathrm{ch}_1 & & \downarrow \mathrm{ch} \\ H^1(Y) & \xrightarrow{\partial} & H^2(X,Y) \end{array}$$

whose commutativity follows from naturality and Mayer–Vietoris patching.

C.2 Bridging Equality

Let $\widehat{\Phi}: Y \to U(N)$ be unitized mass, $\partial[\det \widehat{\Phi}] \in K^0(X, Y)$. On other hand, Riesz projection yields $\mathcal{E}_3$, $\mathcal{E}_2$, thus $[\det \mathcal{E}_3] - [\det \mathcal{E}_2] \in K^0(X, Y)$. Using homotopy extension making $\widehat{\Phi}$ stably compatible with spectral splitting morphism, through commutative diagram pairing in $H^2(X, Y)$ to link S^2's integer equality, thus two relative classes equal.

C.3 Explicit Pairing in Two-Dimensional Transverse

If Y's piecewise link is S^1, then

$$\oint_{S^1} \frac{1}{2\pi i} \mathrm{Tr}(\widehat{\Phi}^{-1} d\widehat{\Phi}) = \int_{S^2} \frac{F_{\det \mathcal{E}_3}}{2\pi} \in \mathbb{Z}.$$

D Callias/Anghel–Bunke (Corresponding to §6)

D.1 Fredholm Construction

Take outer region cutoff χ and parametrix $Q = \chi \Phi^{-1}$. Have

$$(\not{D}_A + \Phi)Q = I - K_1, \qquad Q(\not{D}_A + \Phi) = I - K_2,$$

where $K_{1,2}$ relatively compact (by $[\nabla, \Phi] \in L^\infty$, Rellich compact embedding and outer region invertibility). Thus $\not{D}_A + \Phi$ is Fredholm.

D.2 Boundary Map and Degree

Homotope outer region to direction-only dependent $\Phi_\infty(\theta)$, index equals boundary map $\partial[\widehat{\Phi}_\infty] \in \widetilde{K}^0(S^d) \cong \mathbb{Z}$. Bott isomorphism yields

$$\mathrm{Ind} = \deg(\widehat{\Phi}_\infty) \in \pi_{d-1}(U).$$

D.3 Two-Dimensional Single Vortex Example

$\Phi(r, \theta) = U(\theta) H(r)$, $U(\theta) = \mathrm{diag}(e^{ik\theta}, 1, 1)$, $H(r \to \infty) \to m_0 I$. Then $\widehat{\Phi} = U$, $\mathrm{Ind} = k$. Taking counterclockwise orientation as positive, $k \to -k$ index changes sign.

E Line Operator Spectrum and Minimal Charge $1/6$ (Corresponding to §7)

E.1 Kernel Generator and Congruence

By Theorem 3.3, $\Gamma \simeq \mathbb{Z}_6$ generator can take

$$g_* = (\omega_3^{-1} I_3,\ -I_2,\ e^{i2\pi/6}), \qquad \omega_3 = e^{i2\pi/3}.$$

Action on (t, s, q) is $\omega_3^{-t}(-1)^s e^{i2\pi q/6}$. Quotient descent condition:

$$\omega_3^{-t}(-1)^s e^{i2\pi q/6} = 1 \iff q \equiv 2t + 3s \pmod 6.$$

Let $Y = q/6 \Rightarrow Y \equiv t/3 + s/2 \pmod{\mathbb{Z}}$, thus minimal fractional unit $1/6$.

E.2 Electric/Magnetic Charge Lattice and Dirac Pairing (Schematic)

Denote $(\mathbf{e};\mathbf{m})$ as electric/magnetic charge vector, central gluing yields congruence constraint matrix C satisfying $(\mathbf{e};\mathbf{m}) \mapsto (\mathbf{e};\mathbf{m})+C\mathbf{n}$ ($\mathbf{n} \in \mathbb{Z}^r$) equivalence. Dirac pairing integer matrix Ω well-defined integrality on quotient; θ period undergoes equivalence contraction after quotient group identification. Example table: fundamental representations $\mathbf{3}$ and $\mathbf{2}$'s (t,s) values bring Y's fractional parts $\{1/3, 1/2\}$, synthesizing with $U(1)$ phase yields minimal step $1/6$ span.

F Experimental and Numerical "Checklist" (Corresponding to §8)

F.1 E1 Purification Interference

- **Input**: Loop C, δ, sampling N_{shots}, (T_1, T_2).
- **Steps**: Purification–evolution–interference readout–phase unwrap–contour integral.
- **Output**: $\phi_{\text{rel}} \in \mathbb{Z}$.
- **Passing standard**: $|\text{err}(\phi_{\text{rel}})| < 0.25$ can determine integer; if fails, enlarge loop radius and N_{shots}.

F.2 E2 Photonic Vortex

- **Input**: Array size, coupling J, mass amplitude m_∞, vortex number k.
- **Steps**: Phase map encoding–excitation–near-field imaging–spectral localization–zero-mode counting.
- **Output**: Zero-mode count $|k|$.
- **Passing standard**: Band gap > noise bandwidth, central peak significant with energy near midpoint.

F.3 Numerical Script Essentials

- Grid (N_θ, N_r) take $N_\theta \geq 64$;
- Riesz projection performs contour quadrature along fixed radius δ circle;
- Wilson-loop's c_1 consistent with wind $\det \widehat{\Phi}$, error $\sim \mathcal{O}(h^2)$.

End of Main Text and Appendices