Resource-Bounded Incompleteness Theory

Research Summary

Haobo Ma* Wenlin Zhang†

At a Glance

Resource-Bounded Incompleteness Theory (RBIT) gives a resource-parameterized version of Gödel's incompleteness, explaining how *finite* observers meet incompleteness in practice. It unifies logical proof complexity and statistical sample complexity in one framework, proving that (i) for every finite budget there are true but unreachable statements, and (ii) incompleteness persists under any computable axiom extensions—*without any complexity-separation assumptions*.

Core Innovation

RBIT bridges classical incompleteness (unbounded observers) and real systems (finite budgets) by making incompleteness *operational*:

- **Explicit, computable Gödel family:** a uniform, computable map $L \mapsto G_L$ with each G_L internal to arithmetic and of low logical complexity (Π_1 in PA; Δ_1 under a conservative Δ_0^E extension).
- **Minimal assumptions:** holds for any consistent r.e. theory that *extends* Q (Robinson); stronger bases (EA/PA) are optional.
- **Extension non-termination:** Rosser's refinement (under mere consistency) shows that adding any computable axiom fragment/sequence never terminates incompleteness.
- **Unified resource view:** proof-length thresholds and IPM-based indistinguishability share the same monotone resource geometry, with a clean split between information-theoretic indistinguishability (m, ε) and finite-sample accessibility N.

Main Theorems

Standing setting. Fix a consistent, recursively enumerable theory T extending Robinson's Q; $\mathrm{Proof}_T(x, y)$ denotes a fixed primitive-recursive proof-checking predicate for T.

Theorem 1: Resource-Bounded Gödel Family (one-way)

There exists a computable function f (uniform in L and effectively parameterized by the chosen T) such that for each budget L, $G_L = f(L)$ satisfies:

(1) $T \vdash G_L \leftrightarrow \forall x\,(\mathrm{Len}(x) \le L \to \neg\mathrm{Proof}_T(x, \ulcorner G_L \urcorner))$, equivalently $\forall x \le \mathrm{Bound}(L)\, \neg\mathrm{Proof}_T(x, \ulcorner G_L \urcorner)$.

*Independent Researcher. Email: aloning@gmail.com

†National University of Singapore. Email: e1327962@u.nus.edu

(2) **Arithmetic hierarchy:** in the *pure PA language*, G_L can be taken $\mathbf{\Pi_1}$; under a conservative definitional extension $\mathbf{\Delta_0^E}$ (adding length/exponentiation primitives), $G_L \in \mathbf{\Delta_1}$.

(3) If T is consistent, then $\mathbb{N} \models G_L$ and G_L has no T-proof of length $\leq L$.

Remark. This guarantees *no short proof*; it does *not* preclude a short refutation of G_L. Two-way undecidability at each stage follows via Rosser (Theorem 2).

Consequence: For *every* finite resource bound, there exist true statements beyond reach.

Theorem 2: Theory Extension Perpetuates Incompleteness

For any consistent chain $T_{t+1} = T_t + \Delta_t$ with computable axiom fragments Δ_t, each T_t has a sentence $G^{(t)}$ such that

$$T_t \nvdash G^{(t)} \quad \text{and} \quad T_t \nvdash \neg G^{(t)}.$$

Consequence: Adding computable axioms cannot terminate incompleteness—it reappears forever.

Theorem 3: Resolution Monotonicity

As resources increase:

- **Proof layer:** $\mathrm{Dec}_L(T) \subseteq \mathrm{Dec}_{L'}(T)$ for $L' \geq L$.
- **Statistical layer:** IPM-based indistinguishability is downward-closed under resource enhancement (larger m, larger N, smaller ε).

Consequence: More resources yield finer distinctions yet never eliminate the inaccessible region.

Unified Framework Structure

- **Semantic layer:** $\mathrm{Truth}(\varphi) \in \{\top, \bot\}$ (bivalence in the standard model).
- **Proof layer:** $\mathrm{ProvStatus}(\varphi) \in \{\mathrm{proved}, \mathrm{refuted}, \mathrm{undecided}\}$ with budget L.
- **Statistical layer:** $\mathrm{StatStatus}(\varphi) \in \{\mathrm{distinguishable}, \mathrm{indistinguishable}\}$ under (m, N, ε).

Resources: $R_{\log} = L \in \mathbb{N}$; $R_{\mathrm{stat}} = (m, N, \varepsilon)$ with $(m', N', \varepsilon') \geq (m, N, \varepsilon)$ meaning $m' \geq m$, $N' \geq N$, $\varepsilon' \leq \varepsilon$.

Technical Highlights

- **Base theory:** any consistent r.e. extension of $\mathbf{Q}$ suffices for concrete proof-instance verification; stronger bases such as $\mathbf{EA} = I\Delta_0 + \exp$ or $\mathbf{PA}$ also work.
- **Bounded-length internalization:** there is primitive-recursive $\mathrm{Bound}(L)$ with $\mathrm{Len}(x) \leq L \Rightarrow x \leq \mathrm{Bound}(L)$; hence the length bound is expressible by bounded quantifiers.
- **Hierarchy classification:** G_L is Π_1 in the PA language and Δ_1 under Δ_0^E.
- **Sample-complexity lens:** classical Chernoff/Hoeffding bounds are read as constraints in (m, N, ε).
- **Meta/object split:** semantic truth vs. syntactic provability are kept separate throughout.

Significance and Applications

Foundational Mathematics Quantifies the truth–provability gap under realistic constraints; gives constructive witnesses at each resource level; clarifies how extensions expand (but never complete) the knowable domain.

Computational Complexity Aligns proof complexity with sample complexity; exhibits parallel monotonicity patterns; suggests cross-domain resource conversion as a central open direction.

AI and Formal Verification Grounds cognitive limits of AI systems; informs resource-aware ATP/verification; supports reliability analyses under explicit budgets.

Philosophy of Mathematics Reconciles objective truth with finite accessibility; frames incompleteness as a structural feature of finitude rather than a defect.

Open Problems

1. **Quantitative conversion:** rigorous transformations between logical L and statistical (m, N, ε) resources.
2. **Growth-rate analysis:** characterize $\ell_T(G_L)$ (minimum proof length) as a function of L.
3. **Extended hierarchies:** systematic treatment of higher arithmetic levels under resource bounds.
4. **Practice bridge:** from abstract framework to engineering in AI/verification.

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