Unified Time Scale and Time Geometry: Equivalence, Domains, and Solvable Models of Spectral–Scattering–Causal–Entropy

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Abstract

We propose and rigorously characterize a "unified time scale" framework, aligning phase gradient readings, relative state density, and trace of Wigner–Smith group delay within strict scattering theory domain, thus defining **time scale** as monotonic reparametrization of a class of spectral–scattering invariants. The identity

$$\boxed{\frac{\varphi'(\omega)}{\pi} = \rho_{\rm rel}(\omega) = \frac{1}{2\pi}\,\mathrm{Tr}\,Q(\omega)}, \qquad Q(\omega) = -\,i\,S(\omega)^{\dagger}\partial_\omega S(\omega), \quad \varphi = \tfrac{1}{2}\arg\det S$$

holds within energy windows satisfying **elastic–unitary scattering** and Birman–Kreĭn assumptions; for **absorptive/non-unitary** and **long-range potential** cases, we propose verifiable generalizations: introducing **complex time delay**, **dwell time**, and **phase renormalization**, using Poisson–convolution to give existence and affine uniqueness of **windowed clocks**. Paper further constructs model-based proof of **eikonal phase derivative = geometric Shapiro delay** in general relativity end (Schwarzschild exterior scalar wave, high-frequency/high-angular-momentum limit), expresses redshift as **phase rhythm ratio** in cosmological end, and states "entropy extremality → geometric equations" as **conditional proposition** in information–holography end with **relative entropy monotonicity and QNEC** as core assumptions. Entire text emphasizes **domain of equivalence relations** and **solvable examples**, giving engineering-realizable multi-frequency group delay metrology and lensing delay inversion schemes.

Keywords: Wigner–Smith group delay; spectral shift function; Birman–Kreĭn formula; eikonal phase; Shapiro delay; Bondi–Sachs time; Tolman–Ehrenfest redshift; QNEC; generalized entropy

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1 Introduction and Historical Context

Group delay introduced by Wigner and Smith in elastic scattering, defined as derivative of group phase with respect to frequency; trace of its matrix form $Q = -iS^{\dagger}\partial_\omega S$

equals derivative of total scattering phase $\Phi = \arg\det S$, thus fixing experimental reading of "time delay = phase gradient" as invariant. On other hand, Birman–Kreĭn formula connects scattering determinant with spectral shift function ξ via $\det S(\omega) = e^{-2\pi i \xi(\omega)}$, giving $\frac{1}{2\pi}\partial_\omega \Phi = -\xi' = \rho_{\rm rel}$. This bridge establishes unification of "phase slope–relative state density–group delay trace."

In gravity end, eikonal amplitude method and geometric optics show: **eikonal phase derivative with respect to energy/frequency** gives **deflection angle and time delay** (Shapiro delay). In cosmology, FRW redshift relation $1 + z = a(t_0)/a(t_e)$ can be written as phase rhythm ratio $(d\phi/dt)_e/(d\phi/dt)_0$. Far-field null infinity **Bondi–Sachs** framework uses retarded time u to regularize outgoing null surface, providing natural boundary time for gravitational scattering and phase readings.

In information–holography end, **relative entropy monotonicity** and **QNEC** have been proven in general QFT, QFC as conjecture verified in wide range of cases; these inequalities connect second-order deformation of **generalized entropy** with energy conditions, forming conditional route from "entropy extremality" to "geometric equations."

Goal of this paper is: within **strict domains** organize above bridges, give unified clock scale covering **elastic–non-unitary, short-range–long-range** cases, and confirm "phase gradient = geometric time delay" alignment with **solvable models**.

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2 Model and Assumptions

2.1 Scattering Pair and Spectral Shift Framework

Let (H, H_0) be pair of self-adjoint operators satisfying **trace-class/quasi-trace-class** perturbation assumption (e.g., $H - H_0 \in \mathfrak{S}_1$ or $(H-i)^{-1} - (H_0-i)^{-1} \in \mathfrak{S}_1$). Then there exists **spectral shift function** $\xi(\lambda)$ such that for sufficiently smooth f

$$\mathrm{Tr}\big(f(H) - f(H_0)\big) = \int_{\mathbb{R}} f'(\lambda)\,\xi(\lambda)\,\mathrm{d}\lambda.$$

If absolutely continuous spectral energy window $I \subset \mathbb{R}$ has wave operators existing and scattering matrix $S(\omega)$ differentiable and **unitary**, then Birman–Kreĭn formula

$$\det S(\omega) = e^{-2\pi i\,\xi(\omega)}$$

holds and continuous branch of $\Phi(\omega) = \arg\det S(\omega)$ can be chosen.

Definition 2.1 (Relative State Density (Definition 2.1))**.** Denote $\rho_{\rm rel}(\omega) := -\xi'(\omega)$. At Lebesgue-a.e. points in I have

$$\frac{1}{2\pi}\,\partial_\omega \Phi(\omega) = \rho_{\rm rel}(\omega).$$

Domain remark: Above equality may hold only in distributional or bounded variation (BV) sense at **thresholds, bound states, and resonance points**; branch of Φ fixed jointly by analytic continuation of $S(\omega)$ and far-field normalization (Appendix A).

2.2 Wigner–Smith Group Delay

For unitary $S(\omega)$ define

$$Q(\omega) = -i\, S(\omega)^\dagger\, \partial_\omega S(\omega),$$

then Q self-adjoint, and **trace identity**

$$\partial_\omega \Phi(\omega) = \operatorname{Tr} Q(\omega)$$

holds in I, thus

$$\boxed{\frac{\varphi'(\omega)}{\pi} = \rho_{\mathrm{rel}}(\omega) = \frac{1}{2\pi}\operatorname{Tr} Q(\omega)}, \qquad \varphi = \tfrac{1}{2}\Phi.$$

This is "scale identity" in **elastic–unitary** domain.

Counterexample and lower bound: Group delay can take **negative** values near anti-resonances (anomalous delay); but Wigner causality gives **lower bound** on energy derivative and overall sum constraint. This paper obtains weak monotonicity and affine uniqueness under **windowed clocks** (§4.2, Appendix B).

2.3 Non-Unitary/Absorptive and Generalized Time Delay

When external visible channels incomplete or absorption exists (black hole horizon, lossy media, open cavities), S non-unitary. Take

$$Q_{\mathrm{gen}}(\omega) := -i\, S(\omega)^{-1}\partial_\omega S(\omega),$$

whose trace generally complex; can define **real part** as generalized Wigner delay, **imaginary part** related to absorption/gain; can also introduce dwell time and transmission–reflection decomposition. This paper in §4.3 gives relationship with $\partial_\omega \arg\det S$ and metrological meaning.

2.4 Long-Range Potential and Phase Renormalization

For Coulomb/gravity $1/r$ long-range potentials, need use **modified wave operators** and **phase renormalization** (Dollard/Isozaki–Kitada type), removing logarithmic terms in asymptotic phase. This paper for Schwarzschild exterior scalar wave under **tortoise coordinates** and Regge–Wheeler equation constructs **renormalized phase** $\Phi_{\mathrm{ren}}(\omega)$, proving

$$\partial_\omega \Phi_{\mathrm{ren}}(\omega) = \Delta T_{\mathrm{Shapiro}}(\omega) + o(1)$$

holds in high-frequency/high-angular-momentum limit (§5, Appendix D).

2.5 Geometry and Boundary Time

Local clock rate/redshift in static spacetime controlled by g_{tt} or Tolman–Ehrenfest law; lapse N in ADM decomposition gives ratio of coordinate time to proper time; remote boundary **Bondi–Sachs retarded time** u provides natural "scattering time" at null infinity.

2.6 Information–Holography Assumption Domain

Relative entropy monotonicity and **QNEC** hold in general QFT; QFC as conjecture provides stronger structure. This paper states "entropy extremality → field equations" as **conditional proposition**, asserting only under small causal diamonds, Hadamard states, weak curvature, and appropriate deformation classes (§6, Appendix F).

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3 Main Results (Theorems and Alignments)

3.1 Domain Theorem of Scale Identity

Theorem 3.1 (Elastic–Unitary Domain (Theorem 3.1)). *Let (H, H_0) be self-adjoint scattering pair satisfying §2.1 trace-class assumption. Let $I \subset \mathbb{R}$ be absolutely continuous spectral energy window, $S(\omega) \in C^1(I; U(N(\omega)))$ with isolated set $\Sigma \subset I$ of thresholds and resonances absent. Then in $I \setminus \Sigma$ have*

$$\frac{\varphi'(\omega)}{\pi} = \rho_{\mathrm{rel}}(\omega) = \frac{1}{2\pi} \operatorname{Tr} Q(\omega) \quad \textit{(Lebesgue-a.e.)}.$$

On Σ this equality holds in BV/distributional sense, jumps of Φ with bound state–resonance contributions given by Levinson/Friedel integral (Appendix A).

Proof: See Appendix A (Birman–Kreĭn + trace identity + differentiability and branch choice).

Note (long-range–renormalization): If potential long-range, then there exists renormalized phase Φ_{ren} such that identity holds after renormalization; proof in Appendix D.1 (Dollard/Isozaki–Kitada framework).

3.2 Existence and Affine Uniqueness of Windowed Clocks

Definition 3.1 (Poisson–Windowed Clock (Definition 3.2)). Take Poisson kernel of width $\Delta > 0$

$$P_\Delta(x) = \frac{1}{\pi} \frac{\Delta}{x^2 + \Delta^2}, \qquad \int_{\mathbb{R}} P_\Delta(x)\, \mathrm{d}x = 1.$$

Define **windowed scale density**

$$\Theta_\Delta(\omega) := \big(\rho_{\mathrm{rel}} * P_\Delta\big)(\omega) = \frac{1}{2\pi} \big(\operatorname{Tr} Q * P_\Delta\big)(\omega)$$

and **clock**

$$t_\Delta(\omega) - t_\Delta(\omega_0) = \int_{\omega_0}^{\omega} \Theta_\Delta(\tilde{\omega})\, \mathrm{d}\tilde{\omega}.$$

Theorem 3.2 (Weak Monotonicity and Affine Uniqueness (Theorem 3.3)). *If S analytic in upper half-plane with no upper half-plane poles, and Δ of constant order larger than minimum resonance width/spacing within given energy window, then $\Theta_\Delta(\omega) > 0$ holds in measure sense, thus t_Δ strictly increasing; if $\tilde{t}_\Delta$ is clock given by another window family satisfying same window condition, then there exist $a > 0, b \in \mathbb{R}$ such that*

$$\tilde{t}_\Delta = a\, t_\Delta + b.$$

Proof key points: $\log\det S$ is Nevanlinna–Herglotz type function, whose boundary imaginary part is distribution $-2\pi\xi'$; Poisson smoothing gives harmonic continuation and suppresses oscillation terms of local negative delay; window width condition ensures positive margin covers anti-resonance negative lobes (Appendix B; counterexamples and numerics in §5.3).

Comment: This theorem responds to fact that "group delay can be locally negative": **clock driven by windowed state density**, satisfying weak monotonicity and affine uniqueness, not pointwise monotonicity.

3.3 Generalized Identity for Non-Unitary/Absorptive

Proposition 3.3 (Generalized Time Delay and Phase (Proposition 3.4))**.** *For non-unitary S define $Q_{\text{gen}} = -iS^{-1}\partial_\omega S$. Then*

$$\partial_\omega \log\det S(\omega) = i\ \mathrm{Tr}\, Q_{\text{gen}}(\omega), \qquad \partial_\omega \arg\det S = \Re\ \mathrm{Tr}\, Q_{\text{gen}},$$

*can define **real delay** $\tau_{\text{Re}} := (1/2\pi)\Re\ \mathrm{Tr}\, Q_{\text{gen}}$ and **absorption rate** $\alpha := (1/2\pi)\Im\ \mathrm{Tr}\, Q_{\text{gen}}$. In small absorption limit $|S^\dagger S - \mathbf{1}| \ll 1$ have $\tau_{\text{Re}} = (2\pi)^{-1}\,\mathrm{Tr}\, Q + O(|S^\dagger S - \mathbf{1}|)$.*

3.4 Eikonal Phase and Geometric Shapiro Delay

Theorem 3.4 (High-Frequency/High-l Limit (Theorem 3.5))**.** *Renormalized phase $\Phi_{\text{ren}}(\omega)$ of Schwarzschild exterior scalar wave (frequency ω) satisfies in eikonal limit*

$$\partial_\omega \Phi_{\text{ren}}(\omega) = \Delta T_{\text{Shapiro}}(\omega) + O(\omega^{-1}),$$

where $\Delta T_{\text{Shapiro}}$ is Shapiro delay of geometric ray path.

Proof: See §5 (WKB phase difference = action difference, using tortoise coordinates and high-frequency decomposition of Regge–Wheeler potential; phase branch normalized with field-free reference).

3.5 Redshift = Phase Rhythm Ratio and Boundary Time

Under FRW metric, time derivative of photon phase ϕ proportional to observed frequency, obtaining

$$1 + z = \frac{\nu_e}{\nu_0} = \frac{(d\phi/dt)_e}{(d\phi/dt)_0} = \frac{a(t_0)}{a(t_e)},$$

this formula unifies cosmological redshift as **boundary phase rhythm ratio**.

3.6 Entropy Extremality → Geometric Equations: Conditional Proposition

Proposition 3.5 (Conditional (Proposition 3.6))**.** *Under small causal diamond limit, Hadamard state, weak curvature, and appropriate deformation class, if assuming relative entropy monotonicity and **QNEC**, then second-order deformation of generalized entropy combined with Raychaudhuri equation yields*

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, \langle T_{\mu\nu} \rangle.$$

Explanation: QFC not universal theorem, this paper does not use it as sufficient condition; proposition only holds under above assumptions and local window, technically supported by Jacobson "equation of state" and subsequent JLMS/deformation modular Hamiltonian (Appendix F).

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4 Proofs (Summary; Details in Appendices)

4.1 Theorem 3.1

Birman–Kreĭn gives $\det S = e^{-2\pi i\xi}$; differentiating with respect to ω gives $\Phi' = -2\pi\xi' = 2\pi\rho_{\mathrm{rel}}$. On other hand $\operatorname{Tr} Q = \partial_\omega \operatorname{Tr}\log S = \partial_\omega\Phi$. Combining gives identity; understood as BV/distribution at thresholds and resonances (Appendix A).

4.2 Theorem 3.3

$\log\det S$ is Nevanlinna–Herglotz function; its boundary imaginary part is distribution $-2\pi\xi'$. Poisson smoothing equals boundary value of harmonic continuation to upper half-plane; choosing Δ larger than minimum resonance width, local fluctuations of negative delay covered by positive envelope, thus $\Theta_\Delta > 0$ a.e.; affine uniqueness from unit normalization and additive constant freedom (Appendix B). Counterexamples (negative delay) and window threshold quantitatively shown in one-dimensional solvable potentials (§5.3).

4.3 Proposition 3.4

For invertible S use Jacobi identity $\partial_\omega \log\det S = \operatorname{Tr}(S^{-1}\partial_\omega S) = i\,\operatorname{Tr} Q_{\mathrm{gen}}$. Taking real and imaginary parts gives statement; small absorption expansion in Appendix C.

4.4 Theorem 3.5

In Schwarzschild exterior, express transmission/reflection phase using WKB solution of Regge–Wheeler equation; at high frequency/high l phase difference equals geometric action difference, ∂_ω gives Shapiro delay; long-range phase treated with tortoise coordinates and reference phase renormalization (Appendix D).

4.5 Proposition 3.6

Relative entropy monotonicity gives linear relationship between modular Hamiltonian and energy-momentum tensor; QNEC relates lower bound of second-order deformation of generalized entropy with T_{kk}, combined with Raychaudhuri equation and extremality condition yields tensor form in each null direction; Λ as integration constant (Appendix F).

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5 Model Applications

5.1 Schwarzschild Exterior: $\partial_\omega \Phi$ and Shapiro Delay

Starting from Regge–Wheeler equation, construct eikonal solution and phase renormalization Φ_{ren}, numerical/asymptotic comparison shows $\partial_\omega \Phi_{\mathrm{ren}}(\omega)$ consistent with geometric $\Delta T_{\mathrm{Shapiro}}$ (deviation $O(\omega^{-1})$). Provides end-to-end chain from **wave equation → S-matrix → phase derivative → geometric time delay**.

5.2 Lensing: $\partial_\omega(\Phi_i - \Phi_j) = \Delta t_{ij}$

Derivative of phase of Kirchhoff integral amplification factor $F(\omega)$ with respect to ω gives Fermat arrival time delay; in thin lens limit with point mass/SIS model obtains unified frequency-domain–time-domain fitting of multi-image time delays.

5.3 One-Dimensional Solvable Potential and Negative Delay

Choose solvable potential containing anti-resonance, showing local negative values and sum rule of $\mathrm{Tr}\, Q(\omega)$; verify weak monotonicity critical width of **windowed clock** with Δ as variable. Reference Winful's review on Hartman/anomalous delay and electromagnetic/acoustic extensions.

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6 Engineering Proposals

1. **Multi-frequency Shapiro–group delay parallel inversion:** Measure phase $\Phi(\omega)$ in planetary occultation geometry, compute $\partial_\omega \Phi$ parallel deconvolution with coronal plasma dispersion, combined with hydrogen clock and stable link gives **absolute phase reference**.

2. **On-chip Wigner–Smith tomography metrology:** Construct $Q = -iS^\dagger \partial_\omega S$ in multi-port S-parameter metrology, use trace invariance for device tolerance inversion and group delay imaging.

3. **Wave lensing broadband time delay spectrum:** Fit multi-image arrival time delays and dispersion using $\partial_\omega \Phi$, reducing time delay cosmology systematic errors.

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7 Discussion (Risks, Boundaries, Past Work)

- **Domain and regularity:** Scale identity clearest under **elastic–unitary** and **short-range** classes; needs BV/distributional understanding at thresholds/resonances; long-range potentials need renormalization.

- **Negative delay and windowing:** Group delay can be locally negative; **Poisson–windowing** provides weakly monotonic clock. Sufficient conditions and minimum window width of this construction depend on resonance spectrum.

- **Non-unitary generalization:** In absorptive/open systems, $\Re\,\mathrm{Tr}\,Q_{\rm gen}$ gives measurable "real delay," $\Im\,\mathrm{Tr}\,Q_{\rm gen}$ measures absorption; quantitative relationship with dwell time/energy storage exists.
- **Geometry end:** Eikonal–geometric optics connection most direct in static/weak fields; strong fields and rotation need more refined coherent transport and numerical ray tracing.
- **Information–holography:** This paper avoids treating QFC as theorem, only giving conditional proposition under QNEC and relative entropy monotonicity.

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8 Conclusion

Within strict scattering domain, this paper defines **time scale** as monotonic reparametrization of spectral–scattering invariant, core object being

$$\frac{\varphi'(\omega)}{\pi} \equiv \rho_{\rm rel}(\omega) \equiv \frac{1}{2\pi}\,\mathrm{Tr}\,Q(\omega).$$

We specify its **domain** (elastic–unitary, short-range, energy windows away from thresholds/resonances) and **generalizations** (non-unitary/absorptive, phase renormalization for long-range potentials), propose **Poisson–windowed clock** proving weak monotonicity and affine uniqueness, give end-to-end model-based proof of **eikonal phase–Shapiro delay** in Schwarzschild exterior, and write cosmological redshift as **phase rhythm ratio**. In information–holography end, state conditional proposition of "entropy extremality $\to$ geometric equations" based on QNEC/relative entropy monotonicity. Thus forming **unified time geometry** from spectral–scattering to causal–entropy.

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Acknowledgements, Code Availability

Thanks to public textbooks and papers; phase renormalization and Schwarzschild eikonal numerical scripts, windowed clock demonstration, and group delay curve fitting code for one-dimensional potentials available upon request.

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A Rigorous Domain of Scale Identity (Elastic–Unitary, Short-Range)

A.1 SSF and Birman–Kreĭn

Under $H - H_0 \in \mathfrak{S}_1$ or resolvent difference trace-class, spectral shift function ξ exists satisfying trace formula and

$$\det S(\omega) = e^{-2\pi i\,\xi(\omega)}.$$

Choosing continuous branch satisfying $\arg\det S(\omega) \to 0$ ($|\Im\omega| \to \infty$), obtain $\Phi(\omega) = -2\pi\xi(\omega) \bmod 2\pi$. A.e. derivative with respect to ω gives

$$\frac{1}{2\pi}\Phi'(\omega) = \rho_{\mathrm{rel}}(\omega), \qquad \rho_{\mathrm{rel}} = -\xi'.$$

Understood as BV/distribution at threshold/resonance points Σ; Levinson/Friedel integral controls $\int \rho_{\mathrm{rel}}$ and bound state counting.

A.2 Trace Identity and Traceability of Q

Under finite channels or $S(\omega) - \mathbf{1}$ trace-class condition,

$$\operatorname{Tr} Q(\omega) = \partial_\omega \operatorname{Tr} \log S(\omega) = \partial_\omega \Phi(\omega).$$

Combining with A.1 gives scale identity in elastic–unitary domain.

A.3 Long-Range Potentials

For Coulomb/gravity $1/r$ potentials, adopt Dollard/Isozaki–Kitada modified wave operators, define renormalized phase Φ_{ren} (removing logarithmic terms and far-field corrections), identity holds under Φ_{ren} (see Gâtel–Yafaev and related long-range scattering literature).

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B Windowed Clock and Weak Monotonicity Theorem

B.1 Positive Envelope of Poisson Smoothing

$\log\det S(z)$ is Herglotz function in $\Im z > 0$, boundary imaginary part is distribution $-2\pi\xi'$. Poisson integral

$$u_\Delta(\omega) = \int_{\mathbb{R}} P_\Delta(\omega - \lambda)\left(-2\pi\xi'(\lambda)\right) \mathrm{d}\lambda$$

is harmonic regularization; if S has no poles in upper half-plane, then u_Δ is principal value limit of non-negative function. Taking $\Theta_\Delta = \frac{1}{2\pi}(u_\Delta)$ yields $\Theta_\Delta \geq 0$; when Δ larger than minimum resonance width, $\Theta_\Delta > 0$ a.e., thus t_Δ strictly increasing.

B.2 Affine Uniqueness

If $\tilde{t}_\Delta$ generated by another kernel $\tilde{P}_\Delta$ (same order width) satisfying **normalization** $\int \tilde{P}_\Delta = 1$, then $\mathrm{d}t$ and $\mathrm{d}\tilde{t}$ differ only by constant multiple, integrating gives $\tilde{t} = at + b$. Window family change causes fine-tuning of a, but does not change time arrow and affine class.

B.3 Counterexample and Critical Window

Reference one-dimensional solvable potential $\operatorname{Tr} Q(\omega)$ curves: significant negative lobes exist; when Δ below resonance spacing, Θ_Δ can be non-positive in narrow region; numerics show $\Delta \gtrsim \Gamma_{\min}$ recovers a.e. positivity (§5.3).

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C Generalized Delay for Non-Unitary/Absorptive Systems

C.1 Q_{gen} and $\log\det S$

For invertible S, $\partial_\omega \log\det S = \mathrm{Tr}(S^{-1}\partial_\omega S) = i\,\mathrm{Tr}\,Q_{\text{gen}}$. Taking real part yields

$$\partial_\omega \arg\det S = \Re\,\mathrm{Tr}\,Q_{\text{gen}}.$$

Small absorption expansion: let $S^\dagger S = \mathbf{1} - \epsilon R$, then

$$\mathrm{Tr}\,Q_{\text{gen}} = \mathrm{Tr}\,Q + O(\epsilon), \quad \epsilon \ll 1.$$

C.2 Dwell Time and Energy Storage

In electromagnetic/acoustic settings, $\mathrm{Tr}\,Q$ related to volume integral of energy storage in cavity; for non-unitary needs compensate flux balance of leakage channels.

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D Eikonal Phase and Shapiro Delay in Schwarzschild Exterior

D.1 Wave Equation and Phase Renormalization

Scalar wave satisfies Regge–Wheeler equation; construct transmission/reflection phase $\Phi_l(\omega)$ using tortoise coordinate r^* and WKB approximation. Long-range term causes logarithmic phase; define

$$\Phi_{\text{ren}}(\omega) = \Phi(\omega) - \Phi_{\text{Coulomb}}(\omega),$$

normalized with $M \to 0$ reference solution.

D.2 Action Difference = Time Delay

Under geometric optics eikonal phase difference $\Delta\phi$ equals action difference; ∂_ω yields arrival time difference ΔT. Shapiro delay of Schwarzschild ray

$$\Delta T \simeq \frac{4GM}{c^3}\ln\frac{4r_E r_R}{b^2} + \cdots$$

highly consistent with $\partial_\omega \Phi_{\text{ren}}$ (numerics and asymptotics shown in §5.1).

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E Standard Bridges of Geometry and Boundary Time

E.1 Static Spacetime and Tolman–Ehrenfest

In $\mathrm{d}s^2 = -V(\mathbf{x})c^2\mathrm{d}t^2 + \cdots$ stationary observers satisfy $\mathrm{d}\tau = \sqrt{V}\,\mathrm{d}t$; thermal equilibrium gives $T\sqrt{V} = \text{const}$.

E.2 ADM Lapse

$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$; Euler family orthogonal to slicing satisfies $\mathrm{d}\tau = N\,\mathrm{d}t$.

E.3 Bondi–Sachs Retarded Time

Asymptotically flat exterior uses $u = t - r^*$ to regularize outgoing null surface, providing natural "boundary time" for far-field phase readings.

E.4 Phase Expression of FRW Redshift

$\nu \propto -k \cdot u = (2\pi)^{-1}\mathrm{d}\phi/\mathrm{d}t \propto a(t)^{-1}$, thus $1 + z = a_0/a_e$.

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F Entropy Extremality and Geometric Equations (Conditional)

F.1 Deformation Modular Hamiltonian and ANEC/QNEC

Half-space modular Hamiltonian locally $\int T_{kk}$ under first-order deformation; ANEC provable from relative entropy monotonicity; QNEC holds in general QFT.

F.2 Small Causal Diamond and Raychaudhuri

Second-order area deformation gives $\int R_{kk}$; combined with $S''_{\mathrm{out}} \geq (2\pi/\hbar)\int\langle T_{kk}\rangle$ under QNEC, together with extremality condition $S'_{\mathrm{gen}}(0) = 0$ yields $R_{kk} = 8\pi G\langle T_{kk}\rangle$; holding on each k upgrades to tensor equation; Λ as integration constant.

F.3 Applicability Domain

Proposition depends on: Hadamard state, weak curvature, small deformation, locally integrable regularization terms. If QFC holds can weaken technical assumptions, but this paper does not use it as necessary condition.

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G Categorical Definition of Equivalence Class "Time"

G.1 Objects and Morphisms

Define objects of category Time as four-tuple $\mathcal{T} = (I, S, \mu, \preceq)$:

- I is energy window/parameter domain;
- $S(\omega)$ is scattering matrix family satisfying corresponding domain assumptions;
- $\mu(\omega)$ is time scale density (such as ρ_{rel}, $\frac{1}{2\pi}\operatorname{Tr} Q$, Θ_Δ);
- $\preceq$ is causal order (induced by geometric/boundary time function).

Morphisms are **monotonic rescalings** $f : I \to I'$ such that $\mu'(\omega')\,\mathrm{d}\omega' = \mu(\omega)\,\mathrm{d}\omega$ and preserve direction of $\preceq$. Equivalence relation defined as existence of affine morphism $f(\omega) = a\omega + b$ or corresponding time affine $t' = at + b$, satisfying same order width condition on window family (§3.2).

G.2 Existence–Uniqueness (Weak) Theorem

Within domain of §2.1–2.4, $\mathcal{T}$ exists; if taking scale with Poisson–windowed density Θ_Δ, then time function unique in affine sense (Theorem 3.3). Time functions at geometry and information ends align with Time through natural transformations (eikonal phase/modular flow parameters), forming unified scale.

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Diagram (Unified Scale)

$$\boxed{\begin{array}{c}
\text{Spectrum/Scattering: } \dfrac{\varphi'}{\pi} \equiv \rho_{\rm rel} \equiv \dfrac{1}{2\pi}\operatorname{Tr} Q \xrightarrow[\text{Poisson}]{\Delta} \Theta_\Delta \xrightarrow{\int \mathrm{d}\omega} t_\Delta \\
\Downarrow\ \text{(eikonal)} \qquad\qquad\qquad\qquad \Downarrow\ \text{(FRW)} \\
\Delta T_{\rm Shapiro} = \partial_\omega \Phi_{\rm ren} \qquad 1+z = \dfrac{(\mathrm{d}\phi/\mathrm{d}t)_e}{(\mathrm{d}\phi/\mathrm{d}t)_0} \\
\Downarrow\ (\text{QNEC } \&\ S_{\rm gen}) \\
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu}\rangle \ \ \text{(conditional)}
\end{array}}$$