Self-Referential Scattering and the Birth of Fermions: Riccati Square Roots, Spinor Double Cover, and a $\mathbb{Z}_2$ Exchange Phase

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November 17, 2025

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Abstract

We define a unified $\mathbb{Z}_2$ invariant $\nu_{\sqrt{S}}$ on the discriminant-free parameter space $X^\circ \subset X$. It is the monodromy of the square-root pullback $\mathfrak{s}^*(p)$ of the fixed-energy scattering phase exponential and equals, modulo two, the winding of $\mathfrak{s}$, the unitary spectral flow, the mod-2 linking with the discriminant, and the exchange parity of hyperbolic fixed points in a self-consistent loop.

Formally, on parameter space X° after removing discriminant, consider phase exponential map $\mathfrak{s} : X^\circ \to U(1)$ of fixed-energy scattering. Along square cover $p : U(1) \to U(1)$, $p(z) = z^2$, the pullback

$$P_{\sqrt{\mathfrak{s}}} = \mathfrak{s}^*(p) = \{(x, \sigma) \in X^\circ \times U(1) :\ \sigma^2 = \mathfrak{s}(x)\} \to X^\circ$$

defines square-root cover of scattering. Holonomy characterized by overall phase one-form

$$\alpha = \frac{1}{2\mathrm{i}}\,(\det S)^{-1}\mathrm{d}(\det S)$$

yields

$$\nu_{\sqrt{S}}(\gamma) = \exp\Big(i \oint_\gamma \alpha\Big) = (-1)^{\deg(\det S|_\gamma)} \in \{\pm 1\},$$

a natural $\mathbb{Z}_2$ invariant. Spectral-theoretically, under short-range conditions, combining Birman–Kreĭn formula with spectral flow yields mod-2 Levinson relation. Functional-analytically, under boundary triple and Nevanlinna–Möbius structure, we make rigorous self-referential closed loop giving existence theorem and two fixed-point exchange in hyperbolic region. Using one-dimensional δ-potential and Aharonov–Bohm model as examples, we give explicit winding number calculations. For topological superconductor endpoint scattering, we distinguish Altland–Zirnbauer symmetry classes: Class D's $\operatorname{sgn}\det r(0)$ and Class DIII's $\operatorname{sgn}\operatorname{Pf} r(0)$ each equivalent to branch sign of $\sqrt{\det r(0)}$. This framework directly applicable to Fermi/Bose statistics in $d \geq 3$; in $d = 2$ gives $\mathbb{Z}_2$ projection of anyon $U(1)$ statistics.

Experimentally readable mod-2 indicator: In gate-tunable Josephson junctions [1, 2], when Andreev channel number $\lesssim 4$, perform single 2π scan of superconducting phase difference ϕ at zero energy bias; the dimensionless $\mathbb{Z}_2$ index $G_{\mathbb{Z}_2} \equiv \nu_{\sqrt{S}} \in \{+1, -1\}$ flips at each Majorana crossing event, realizing single-shot $\mathbb{Z}_2$ magnetometer.

Convention: In single-channel case $\mathfrak{s} = S$; in multichannel or partial-wave settings $\mathfrak{s} = \det_p S$.

Keywords: scattering phase square-root cover; $\mathbb{Z}_2$ holonomy; covering lift; spectral shift; Birman–Kreĭn; Riccati; boundary triple; Pfaffian index; Aharonov–Bohm scattering

Contents

0 Notation, Assumptions, Objects and Core Physical Picture

0.1 Core Idea and Physical Picture

Convention. We write $\mathfrak{s} := S$ in the single-channel case. In multichannel or partial-wave settings we set $\mathfrak{s} := \det_p S$ and drop the subscript when no confusion arises.

Core idea of this work: unify three seemingly different negative-sign sources using unified $\mathbb{Z}_2$ holonomy index. Background on scattering theory can be found in standard references [3, 4]

$$\nu_{\sqrt{S}}(\gamma) = \exp\Big(i \oint_\gamma \frac{1}{2\mathrm{i}} (\det S)^{-1} \mathrm{d}(\det S)\Big) :$$

negative sign from exchanging two fermions, negative sign from rotating spinor by 2π, and branch-switching negative sign of scattering semi-phase. Physical picture as follows.

1. **Branch of scattering semi-phase** For single-channel scattering, $S = e^{2i\delta}$. If adiabatically evolving along external parameter loop γ, δ may return to initial value plus integer multiple of π. Viewing $e^{i\delta}$ as "square root" of S, after one cycle square root's sign may flip, precisely physical meaning of $\nu_{\sqrt{S}}$.
2. **Exchange and spinors** In $d \geq 3$, two-particle exchange path in unordered pair configuration space homotopic to π rotation of relative coordinate; its lift on rotation group corresponds to non-trivial class of $\pi_1(\mathrm{SO}(d)) \cong \mathbb{Z}_2$ (represented by 2π rotation). Spinor field takes negative under 2π rotation; sending this non-trivial class via scattering map into loop on $U(1)$, winding number parity consistent with spinor negative sign, precisely given by $\nu_{\sqrt{S}}$.
3. **Spectral flow and bound states** When external parameter circling causes bound state to cross eigenphase reference point, integer spectral flow changes by 1, thus $\nu_{\sqrt{S}}$ flips. Equivalently, if loop transversely crosses discriminant of "generating or annihilating upper-half-plane Jost zero" once, $\nu_{\sqrt{S}}$ also flips.
4. **Fixed-point exchange of self-referential closed loop** In cases modeled by transport or scattering self-consistency equations, system boundary conditions and response form closed loop via Möbius self-map. Hyperbolic parameter region has two boundary fixed-point branches; crossing discriminant hypersurface once exchanges these two branches once. This exchange parity equivalent to $\nu_{\sqrt{S}}$.

Thus, whether observing from configuration space topology, spinor double cover, scattering analytic structure or self-consistent dynamics, appearing is same $\mathbb{Z}_2$ holonomy. This index has observational accessibility: can extract S's phase continuation from interference measurement, or obtain from phase spectral flow and bound state counting.

General notation hint: This work uses symbol L in two different contexts. In §3, $L = \psi'/\psi$ is the Riccati variable. In §7, L is a boundary parameter taking values on the extended real line. The two usages are separate within their respective sections and do not mix. When mentioning "self-referential closed loop $L = \Phi_{\tau,E}(L)$," it always refers to the boundary parameter from §7.

0.2 Parameter Space and Discriminant

Let X be piecewise smooth manifold, take discriminant

$D = \{$upper-half-plane Jost zero generation or annihilation, zero-energy threshold anomaly, embedded eige

denote $X^\circ = X \setminus D$. On X° scattering data continuous or analytic in parameters.

Proposition 0.1 (Existence and naturality of w_D)**.** *Let X be second countable piecewise C^1 oriented manifold, $D \subset X$ be closed tameable codimension-one stratified submanifold. Let $X^\circ := X \setminus D$. Then exists unique $w_D \in H^1(X^\circ; \mathbb{Z}_2)$ (linking class) such that pairing with any sufficiently small normal positive loop is* 1*. This class is natural under embeddings and C^0-stable under small perturbations of D.*

Proposition 0.2 (Stratified transversality case)**.** *Let X, D be as above, $N(D)$ be compact tubular neighborhood of D. For any closed path's avoidance version $\gamma_\varepsilon \subset X^\circ$, when $\gamma_\varepsilon \pitchfork \partial N(D)$, have*

$$\langle w_D, [\gamma_\varepsilon] \rangle = \big([\gamma_\varepsilon] \cdot [\partial N(D)]\big) \bmod 2.$$

This class is invariant under operations adjusting (γ, D) to stratified transversality via small C^1 perturbations.

Proofs: By Alexander duality and Poincaré–Lefschetz duality. See Appendix D.

0.3 Connection and Winding Number

Unique convention (Throughout): In this work, the notation "$\sqrt{S}$" **always** refers to the **map-level square-root covering** of $\mathfrak{s} : X^\circ \to U(1)$ (i.e., the monodromy of principal $\mathbb{Z}_2$-bundle $P_{\sqrt{\mathfrak{s}}} = \mathfrak{s}^*(p)$), **not** a matrix square root. Phrases like "branch sign of $\sqrt{\det r(0)}$" always mean the **sign of the covering monodromy branch**, not taking matrix square roots of $\det r(0)$. See §2 (covering–lift criterion) and §9 (D/DIII indices).

$$\alpha = \frac{1}{2\mathrm{i}}\,(\det S)^{-1}\mathrm{d}(\det S), \qquad \nu_{\sqrt{S}}(\gamma) = \exp\Big(i \oint_\gamma \alpha\Big) = (-1)^{\deg(\det S|_\gamma)}.$$

$$\deg(\det S|_\gamma) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma (\det S)^{-1}\mathrm{d}(\det S) \in \mathbb{Z},$$

Note: For multichannel/partial wave cases, the above $\det S$ should be understood as $\det / \det_p S$ as needed; for single channel it reduces to scalar $S = e^{2i\delta}$.

Closed path orientation adopts mathematical positive convention.

Definition 0.3 (Argument version of winding number and integral equivalence)**.** Let $\mathfrak{s} : X^\circ \to U(1)$ be continuous along closed path $\gamma_\varepsilon \subset X^\circ$. Take any continuous argument choice $\operatorname{Arg}\mathfrak{s} \in \mathbb{R}/2\pi\mathbb{Z}$, define

$$\deg(\mathfrak{s}|_\gamma) := \frac{1}{2\pi}\,\Delta_{\gamma_\varepsilon}\big(\operatorname{Arg}\mathfrak{s}\big) \in \mathbb{Z}.$$

If further $\mathfrak{s}$ along γ_ε is piecewise C^1 or of bounded variation, then

$$\frac{1}{2\pi\mathrm{i}} \oint_{\gamma_\varepsilon} \mathfrak{s}^{-1}\mathrm{d}\mathfrak{s} = \deg(\mathfrak{s}|_\gamma), \qquad \oint_{\gamma_\varepsilon} \mathrm{d}\xi_p = -\deg(\mathfrak{s}|_\gamma),$$

where $\oint \mathrm{d}\xi_p$ is interpreted in Lebesgue–Stieltjes sense. The two definitions are equivalent.

Warning (comparison scope of spectral loops vs parameter loops): Any integer-level equalities in this work concern **external parameter closed paths γ only**. Spectral parameter loop C is used only for analytic structure integer accounting of $S(k)$. The two types compare **only at $\mathbb{Z}_2$ level**. The equivalence chain in Main Theorem 1.1 is always understood as statements for γ.

Hierarchy and sign convention: This work at **integer level** only uses

$$\mathrm{Sf}(\gamma) = \deg(\det S|_\gamma) \in \mathbb{Z},$$

see §4; while main Theorem 1.1 connecting N_b, I_2 holds only at $\mathbb{Z}_2$ level:

$$(-1)^{\deg(\det S|_\gamma)} = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{N_b(\gamma)} = (-1)^{I_2(\gamma,D)}.$$

For general closed paths, the **integer sign** of $N_b(\gamma)$ depends on **avoidance manner** of crossing D and parameter orientation, thus no integer identity with deg is established; only its parity (N_b mod 2) is an invariant. Following discussion unfolds according to this convention.

Declaration (space and invariant hierarchy): All winding numbers $\deg(\det S|_\gamma)$, spectral flow $\mathrm{Sf}(\gamma)$, bound state counting $N_b(\gamma)$, and intersection number $I_2(\gamma, D)$ in this work take the **same parameter-energy closed path** $\gamma \subset X^\circ$ as argument, and comparison proceeds only at $\mathbb{Z}_2$ level.

Remark 0.4 (Global: Spectral loop vs Parameter loop)**.** The $\deg(S|_C) = -\sum_j m_j$ in §3 is a **momentum** (k) **plane spectral loop** C analytic counting, used for spectral structure analysis of $S = f(-k)/f(k)$; it does **not** identify with parameter loop γ at integer level. This work does **not** claim integer equalities like $\deg(\det S|_\gamma) = \deg(S|_C)$ or $\deg_\lambda = \deg_k$. Main Theorem 1.1 only asserts mod-2 equivalences:

$$(-1)^{\deg(\det S|_\gamma)} = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{N_b(\gamma)} = (-1)^{I_2(\gamma,D)}.$$

All integer-level statements concern **parameter loops γ only**. The two types of loops compare **only at $\mathbb{Z}_2$ level** via intersection criterion. Other mentions of this distinction refer back to this remark.

0.4 Short-Range and Spectral Assumptions

Potential V belongs to short-range class [4, 3]: in $d = 1$ (and with certain additional conditions for some $d = 2$ cases) ensures $S(E, \lambda) - \mathbf{1}$ is trace-class; for more general $d \geq 2$ cases typically only obtain $S(E, \lambda) - \mathbf{1}$ belongs to suitable Schatten class [5], thus need to use modified Fredholm determinant $\det_p$ and its continuous branch to define spectral shift. Below for brevity denote uniformly as "$\det / \det_p$". Other assumptions remain unchanged: $(E, \lambda) \mapsto S$ piecewise C^1 along closed path γ, and γ avoids thresholds and embedded eigenvalues; if cannot completely avoid thresholds, describe via mod-2 intersection number. Single channel $S = e^{2i\delta}$; multichannel/partial waves use $\det / \det_p S$ as overall phase exponential.

(A3‴) (Regularity and integral interpretation): On γ_ε can take continuous branch of spectral shift ξ_p, satisfying along γ_ε being piecewise C^1 or of bounded variation. Thus

$$\oint_{\gamma_\varepsilon} d\xi_p$$

is defined in Lebesgue–Stieltjes sense, and equals the negative of total variation of $\operatorname{Arg} \mathfrak{s}$ divided by 2π. In particular $(-1)^{\deg(\mathfrak{s}|_\gamma)} = \exp\left(-i\pi \oint_{\gamma_\varepsilon} d\xi_p\right)$.

0.5 Birman–Kreĭn and Spectral Shift

On absolutely continuous spectral energy segment

$$\det S(E,\lambda) = e^{-2\pi i\,\xi(E,\lambda)}, \qquad 2\delta(E,\lambda) \equiv -2\pi\,\xi(E,\lambda) \pmod{2\pi}.$$

When γ simultaneously changes energy and external parameter, $\oint_\gamma d\xi$ taken from continuous branch of (modified) Fredholm determinant; reverse orientation gives $\oint_\gamma d\xi \mapsto -\oint_\gamma d\xi$, not changing parity.

0.6 Dimension–Decay–Determinant and Regularization

Dim. d	Short-range assumption	Determinant and ξ	Remarks
1	$V \in L^1 \cap L^2$	Classical $\det S$ valid	Threshold anomaly controllable
2	$V = O(\langle x\rangle^{-1-\varepsilon})$	Need $\det_2$ or partial-wave cutoff	AB flux separable
≥ 3	$V \in L^{d/2+\varepsilon}$ etc.	Often need modified $\det_p$ and continuation	See Yafaev, Pushnitski et al.

0.7 Main Claims: What is New

This work establishes cross-disciplinary connections and provides first rigorous treatments in the following aspects:

1. **Self-consistent Möbius fixed-point exchange:** First rigorous equivalence between exchange parity of boundary fixed points in self-referential closed loop $L = \Phi_{\tau,E}(L)$ and $\nu_{\sqrt{S}}(\gamma)$, with sign formula for $\partial_L \arg\det S$ under Herglotz monotonicity (Theorem 7.3, Lemma 7.4).
2. **Mod-2 Levinson for general Schatten class:** Extension of Birman–Kreĭn–spectral flow equality from trace-class to general Schatten $\mathfrak{S}_p$ ($p \geq 2$), establishing branch independence of continuous spectral shift ξ_p and mod-2 stability (Theorem 1.1, Theorem 4.3).
3. **Unified interpretation of topological superconductor indices:** Rigorous demonstration that Class D's $\operatorname{sgn}\det r(0)$ and Class DIII's $\operatorname{sgn}\operatorname{Pf} r(0)$ both equal branch sign of covering monodromy $\sqrt{\det r(0)}$ (not matrix square root), connecting endpoint scattering to map-level square-root topology (Lemma 9.1).
4. **Stratified discriminant and avoidance independence:** Systematic construction of linking class $w_D \in H^1(X^\circ;\mathbb{Z}_2)$ for codimension-one stratified discriminant, with rigorous transversality criterion and proof that all $\mathbb{Z}_2$ conclusions are insensitive to avoidance choice (Propositions 0.1–5.1, Theorem 5.2).

All four equivalences in Main Theorem 1.1 concern **parameter-space closed paths** γ **only**; comparisons with spectral loops hold **only at** $\mathbb{Z}_2$ **level**. Integer-level claims are explicitly restricted to trace-class settings or specific examples.

1 Main Results (Four Equivalent Links)

Theorem 1.1 (Unified equivalence; integer=trace-class, mod 2=general Schatten)**.** *Under Section 0 short-range and regularity settings, assume the following two conditions hold:*

Assumption A (Schatten-continuity and spectral shift continuation): *There exists $p \geq 2$ and a continuous branch of regularized spectral shift ξ_p such that $(E, \lambda) \mapsto S(E, \lambda) - \mathbf{1} \in \mathfrak{S}_p$ is continuous along closed path γ (avoiding discriminant as in §0.2a to get $\gamma_\varepsilon \subset X^\circ$), and define $\mathfrak{s} := e^{-2\pi i \xi_p} \in U(1)$. Literature: Yafaev [4], §8–§9; Pushnitski [6]; Behrndt–Hassi–de Snoo [7], §10.*

Assumption D (Transverse-codimension-one regularity): *Discriminant $D \subset X$ is a codimension-one piecewise C^1 closed submanifold, corresponding to events of Jost zero generation/annihilation, threshold anomaly, embedded eigenvalue, or channel opening/closing (§0.2).*

Then for any parameter closed path $\gamma \subset X$ (if $\gamma \cap D \neq \varnothing$, take avoidance path $\gamma_\varepsilon \subset X^\circ$ as in §0.2a), have

$$\nu_{\sqrt{S}}(\gamma) = \exp\Big(i \oint_\gamma \tfrac{1}{2\mathrm{i}} (\det S)^{-1} \mathrm{d}(\det S)\Big) = (-1)^{\deg(\det S|_\gamma)} = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{N_b(\gamma)} = (-1)^{I_2(\gamma, D)}.$$

*where for **multichannel/partial wave cases** unified substitution uses overall phase exponential $\det S$ (when necessary use modified determinant $\det_p S$); for **single channel** have $\det S = S$. Notation "$\sqrt{S}$" refers to **map-level square-root covering** of $\mathfrak{s} : X^\circ \to U(1)$ (monodromy of principal $\mathbb{Z}_2$-bundle $P_{\sqrt{\mathfrak{s}}} = \mathfrak{s}^*(p)$), **not** matrix square root.*

- ***Integer level (trace-class version):*** *If $S - \mathbf{1} \in \mathfrak{S}_1$ is continuous along γ_ε, then $\mathfrak{s} = \det S$ and*
$$\mathrm{Sf}(\gamma) = \deg(\det S|_\gamma) = -\oint_{\gamma_\varepsilon} d\xi \in \mathbb{Z}.$$
- ***Mod 2 level (general Schatten version):*** *Under Assumption A ($p \geq 2$), only assert*
$$\exp\Big(-i\pi \oint_{\gamma_\varepsilon} d\xi_p\Big) = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{N_b(\gamma)} = (-1)^{I_2(\gamma, D)}.$$

Important convention: *All above equivalences concern **parameter-space closed path γ only**. At integer level this work does not compare spectral parameter loop $C \subset k$-plane (for Jost zero counting) with parameter loop γ; the two types compare **only at $\mathbb{Z}_2$ level** via intersection criterion $I_2(\gamma, D)$ (see Remark 0.4 and §3 note).*

*Define **bound state parity index***

$$N_b(\gamma) := I_2(\gamma, D) = \langle w_D, [\gamma] \rangle \in \mathbb{Z}_2.$$

*If **additionally** there exists piecewise C^1 2-chain transverse to D with $\partial\Sigma = \gamma$, then have degenerate equivalence*

$$I_2(\gamma, D) = \#(\Sigma \cap D) \bmod 2.$$

$\mathrm{Sf}(\gamma)$ *is spectral flow of eigenphase with respect to reference phase.*

Proof (mod 2): By Birman–Kreĭn formula, on absolutely continuous spectral energy segment exists continuous spectral shift ξ such that $\det S = e^{-2\pi i \xi}$. Taking continuous branch along closed path γ, have $\deg(\det S|_\gamma) = -\oint_\gamma \mathrm{d}\xi = \mathrm{Sf}(\gamma)$. Thus $(-1)^{\deg(\det S|_\gamma)} = \exp(-i\pi \oint_\gamma \mathrm{d}\xi) = (-1)^{\mathrm{Sf}(\gamma)}$. Let $D \subset X$ be discriminant. By §5's definition, $I_2(\gamma, D) = \langle w_D, [\gamma] \rangle$ is defined for any closed path $\gamma \subset X^\circ$. If exists piecewise C^1 2-chain Σ transverse to D with $\partial\Sigma = \gamma$, then each intersection point corresponds to exactly one eigenphase first-order crossing at reference phase, spectral flow jumps by ± 1, and $I_2(\gamma, D) = \#(\Sigma \cap D) \bmod 2$; thus $(-1)^{\mathrm{Sf}(\gamma)} = (-1)^{I_2(\gamma, D)}$. By definition $N_b(\gamma) := I_2(\gamma, D)$, combining above

$$(-1)^{\deg(\det S|_\gamma)} = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{N_b(\gamma)} = (-1)^{I_2(\gamma, D)}. \quad \square$$

Lemma 1.2 (BK to spectral flow mod 2)**.** *Under Section 0 short-range and regularity assumptions, along closed path γ*

$$\deg(\det S|_\gamma) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\det S)^{-1} \mathrm{d}(\det S) = -\oint_\gamma \mathrm{d}\xi, \qquad \mathrm{Sf}(\gamma) = \deg(\det S|_\gamma) \in \mathbb{Z},$$

$$\exp\Big(-i\pi \oint_\gamma \mathrm{d}\xi\Big) = (-1)^{\mathrm{Sf}(\gamma)}.$$

Continuous branch and reverse orientation only change integral sign, not affecting parity.

Lemma 1.3 (Intersection number to bound state parity)**.** *When γ transverse to D, each transverse point corresponds to first-order bifurcation of phase and spectral flow ± 1, thus*

$$(-1)^{\mathrm{Sf}(\gamma)} = (-1)^{I_2(\gamma, D)}.$$

Proofs see Appendices C and D.

2 Covering–Lift and Flat Line Bundle

2.1 Covering–Lift and Principal $\mathbb{Z}_2$-Bundle

Terminology convention: In this work, "holonomy" exclusively refers to the single-valued-lifting sign of the $\mathbb{Z}_2$ principal bundle, taking values $\{\pm 1\}$; "holonomy index" refers to the exponential path function written as $\nu_{\sqrt{S}}(\gamma) = \exp(i \oint \alpha)$. The two are related by $\mathrm{Hol} = \exp(i\pi \deg)$ and equivalent.

Since $U(1) = K(\mathbb{Z}, 1)$, have $[X^\circ, U(1)] \cong H^1(X^\circ; \mathbb{Z})$. Square cover $p : z \mapsto z^2$ corresponds to multiplication by two on fundamental group and first cohomology. For any closed path γ

$$\nu_{\sqrt{S}}(\gamma) = \exp\Big(i \oint_\gamma \tfrac{1}{2\mathrm{i}} (\det S)^{-1} \mathrm{d}(\det S)\Big) = (-1)^{\deg(\det S|_\gamma)}.$$

Definition 2.1 (Cover's single-valued lift and so-called holonomy)**.** The "holonomy" of principal $\mathbb{Z}_2$-bundle $P_{\sqrt{\mathfrak{s}}} = \mathfrak{s}^*(p)$ refers to single-valued-ness of the cover's lift along closed paths. Namely, take $\tilde{\gamma}$ to be the lift of γ_ε in $P_{\sqrt{\mathfrak{s}}}$; if starting sheet labeled $+1$, then ending sheet labeled ± 1. Denote this sign as $\mathrm{Hol}_{P_{\sqrt{\mathfrak{s}}}}(\gamma_\varepsilon) \in \{\pm 1\}$. Then

$$\mathrm{Hol}_{P_{\sqrt{\mathfrak{s}}}}(\gamma_\varepsilon) = (-1)^{\deg(\mathfrak{s}|_\gamma)}.$$

Here no differential-geometric connection form is introduced on the $\mathbb{Z}_2$-bundle; the equality is purely topological.

Theorem 2.2 (Covering–lift criterion)**.** *Exists continuous $s : X^\circ \to U(1)$ such that $s^2 = S$ if and only if $[S] \in 2H^1(X^\circ; \mathbb{Z})$. Corresponding principal $\mathbb{Z}_2$-bundle $P_{\sqrt{S}} = S^*(p)$ holonomy equals $\nu_{\sqrt{S}}$.*

2.2 Flat Line Bundle, Bockstein and Two Types of Lifting Problems

This work involves two independent types of lifting/square-root problems:

(A) Map level (square root of function): Given $S : X^\circ \to U(1)$, lift $s : X^\circ \to U(1)$ of square cover $p : z \mapsto z^2$ such that $s^2 = S$ exists if and only if $[S] \in 2H^1(X^\circ; \mathbb{Z})$ (since $U(1) \simeq K(\mathbb{Z}, 1)$ and $p_* = \times 2$). Its $\mathbb{Z}_2$ obstruction given by holonomy of principal $\mathbb{Z}_2$-bundle $P_{\sqrt{S}} = S^*(p)$, i.e., $\nu_{\sqrt{S}}$.

(B) Line bundle level (square root of bundle): For any $U(1)$-principal bundle/complex line bundle $\mathcal{L}$, necessary and sufficient condition for existence of $\mathcal{M}$ such that $\mathcal{M}^{\otimes 2} \cong \mathcal{L}$ is $c_1(\mathcal{L}) \in 2H^2(X^\circ;\mathbb{Z})$. Here c_1 arises from connecting homomorphism of exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}^\infty(\mathbb{R}) \xrightarrow{\exp(2\pi i\,\cdot)} \mathcal{C}^\infty(U(1)) \longrightarrow 0$$

given by

$$\delta:\ H^1\left(X^\circ;\mathcal{C}^\infty(U(1))\right) \xrightarrow{\simeq} H^2\left(X^\circ;\mathbb{Z}\right)$$

satisfying $\delta([\mathcal{L}]) = c_1(\mathcal{L})$.

These two target different objects and **generally not mutually deducible**. Directly related to this work's $\mathbb{Z}_2$ index is $P_{\sqrt{S}}$; its associated flat complex line bundle $\mathcal{L}_{\sqrt{S}}$ via $\{\pm 1\} \hookrightarrow U(1)$ has c_1 being 2-torsion (given by Bockstein of multiplication by two short exact sequence), can reflect obstruction of $\nu_{\sqrt{S}}$ at torsion and mod-2 level, but not equivalent to lifting condition (A).

Two criteria (parallel statement):

- **Map level** ($U(1) = K(\mathbb{Z},1)$): $\exists\, s : X^\circ \to U(1)$ s.t. $s^2 = S \iff [S] \in 2H^1(X^\circ;\mathbb{Z})$.
- **Line bundle level** (exponential sequence and Bockstein): For any complex line bundle $\mathcal{L}$, $\exists\, \mathcal{M},\ \mathcal{M}^{\otimes 2} \cong \mathcal{L} \iff c_1(\mathcal{L}) \in 2H^2(X^\circ;\mathbb{Z})$.

These two target different objects and **generally not mutually deducible**; this work's $\nu_{\sqrt{S}}$ equivalent to (A), not equivalent to c_1 parity of any given $\mathcal{L}$.

Supplementary (de Rham perspective): First Chern class of a flat line bundle is a torsion element. Its de Rham representative is the zero form. Therefore, the information of $\nu_{\sqrt{S}}$ in this work resides entirely at the torsion and $\mathbb{Z}_2$ levels and is not reflected in curvature forms.

3 Riccati Variable, Weyl–Titchmarsh and Jost Structure

Let $L = \psi'/\psi$, then

$$L' + L^2 = V - E.$$

Weyl–Titchmarsh m-function and abstract Weyl function $M(z)$ belong to Herglotz or Nevanlinna class. In one-dimensional solvable models [8, 9], choose Jost function f such that

$$S(k) = \frac{f(-k)}{f(k)} = e^{2i\delta(k)}.$$

If C is small positive loop in k-plane enclosing only upper-half-plane zeros k_j (counting multiplicity m_j), then

$$\frac{1}{2\pi \mathrm{i}} \oint_C S^{-1}\mathrm{d}S = \frac{1}{2\pi \mathrm{i}} \oint_C \Big(-\frac{f'(-k)}{f(-k)} - \frac{f'(k)}{f(k)}\Big)\,\mathrm{d}k = -\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(k)}{f(k)}\,\mathrm{d}k = -\sum_j m_j.$$

Thus $\nu_{\sqrt{S}}(C) = (-1)^{\sum_j m_j}$. If simultaneously enclosing $\pm k_j$, two terms cancel and winding number zero.

Note (spectral loop vs parameter loop): The above takes small positive loop C in k-plane, enclosing only upper-half-plane zeros $\{k_j\}$. It gives **spectral parameter** integer counting $\deg(S|_C) = -\sum_j m_j$, used for analyzing analytic structure of $S = f(-k)/f(k)$. It is **not** a closed path γ in external parameter space, thus does not define $N_b(\gamma)$. When comparison with N_b is needed, should first select closed path γ avoiding D in parameter space and apply Sf = deg from §4 and $\mathbb{Z}_2$ equivalence from §5, only retaining parity information.

Terminology clarification (bound state parity): The "bound state parity" here refers to the **parity of transverse events of eigenphase crossing reference phase** 0 **or** π **on the unit circle** (i.e., mod-2 of phase spectral flow crossing count), not equivalent to bound energy levels crossing on the energy axis for the Hamiltonian; choosing reference $\theta = 0$ or π are $\mathbb{Z}_2$-equivalent. This terminology follows scattering theory tradition; its essence is the $\mathbb{Z}_2$ index of unitary matrix spectral flow.

4 Birman–Kreĭn, Spectral Shift and Mod-2 Levinson

Theorem 4.1 ($\det_p$ continuous branch and spectral flow equality)**.** *Let along closed path $\gamma \subset X^\circ$ one of the following holds:*

(i) $S(E,\lambda) - \mathbf{1} \in \mathfrak{S}_1$ and $(E,\lambda) \mapsto S$ piecewise C^1; or

(ii) There exists $p \geq 2$ such that $S(E,\lambda) - \mathbf{1} \in \mathfrak{S}_p$, and take continuous branch of modified determinant $\det_p$ with corresponding spectral shift ξ_p.

Then along γ exists continuous branch of $\log \det_p S$, and

$$\det_p S = e^{-2\pi i\,\xi_p}, \qquad \mathrm{Sf}(\gamma) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (\det_p S)^{-1} \mathrm{d}(\det_p S) = -\oint_\gamma \mathrm{d}\xi_p \in \mathbb{Z}.$$

Hint: Follows from analyticity of modified Fredholm determinant and continuous choice of spectral shift function (see [6, 10, 5]). For general Schatten class $\mathfrak{S}_p$ ($p \geq 2$), see [11, 12, 13].

Above notation consistent with "$\det / \det_p$" in §0.4.

Theorem 4.2 (Birman–Kreĭn and spectral shift)**.** *On absolutely continuous spectral energy segment under condition $S - \mathbf{1}$ trace-class (multichannel take modified Fredholm determinant), exists continuous spectral shift ξ such that*

$$\det S(E,\lambda) = e^{-2\pi i\,\xi(E,\lambda)}.$$

Thus along closed path γ

$$\mathrm{Sf}(\gamma) = \deg(\det S|_\gamma) = -\oint_\gamma \mathrm{d}\xi \in \mathbb{Z}.$$

When γ simultaneously changes energy and external parameter, ξ taken from continuous branch of modified Fredholm determinant; reverse orientation only changes sign of $\oint_\gamma \mathrm{d}\xi$ without changing parity. By Theorem 4.1 have $\mathrm{Sf}(\gamma) = \deg(\det S|_\gamma)$.

Theorem 4.3 (Mod-2 Levinson)**.**

$$\nu_{\sqrt{S}}(\gamma) = \exp\Big(-i\pi \oint_\gamma \mathrm{d}\xi\Big) = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{N_b(\gamma)}.$$

Here $N_b(\gamma)$ refers to parity of signed counting of threshold events along γ, not difference of total bound state numbers at endpoints. When γ cannot completely avoid D, only $(-1)^{\mathrm{Sf}(\gamma)}$ is invariant; integer $\mathrm{Sf}(\gamma)$ and $\deg(\det S|_\gamma)$ sign depends on avoidance direction.

5 Discriminant and Mod-2 Intersection Number

Let

$D = \{$Jost upper-half-plane zero generation or annihilation, threshold anomaly, embedded eigenvalue, chan

Under generic position D is codimension-one piecewise smooth submanifold. Following conclusions universally hold at $\mathbb{Z}_2$ level; even if closed path must cross D, its $\nu_{\sqrt{S}}(\gamma)$ and $I_2(\gamma, D)$ remain well-defined and equivalent to each other.

Proposition 5.1 (Avoidance and orientation $\mathbb{Z}_2$ invariance, general outline)**.** *Under Assumption A and Assumption D, let closed path $\gamma \subset X$ have finitely many transverse crossings with discriminant D. Then:*

(i) *Avoidance choice only affects the sign of integer winding* $\deg(\mathfrak{s}|_\gamma)$*, not its parity;*

(ii) *Reversing orientation only changes the sign of* $\deg(\mathfrak{s}|_\gamma)$*, not its parity;*

(iii) *Therefore, all $\mathbb{Z}_2$ conclusions ($\nu_{\sqrt{S}}(\gamma)$, $(-1)^{\mathrm{Sf}(\gamma)}$, $(-1)^{I_2(\gamma,D)}$) are* ***insensitive*** *to avoidance choice and orientation reversal.*

Proof: Follows from continuity of spectral shift branch selection and transversality of crossing (see Appendices C and D). □

Definition (mod-2 intersection/linking number, universal version for closed paths): Let $D \subset X$ be codimension-one piecewise smooth closed submanifold, denote $X^\circ = X \setminus D$. Let $w_D \in H^1(X^\circ; \mathbb{Z}_2)$ be $\mathbb{Z}_2$ cohomology class induced by D in complement space (equivalently, monodromy class of cutting double cover on X°). For any closed path $\gamma \subset X^\circ$, define

$$I_2(\gamma, D) := \langle w_D, [\gamma] \rangle \in \mathbb{Z}_2.$$

This definition meaningful for any closed path, related to homotopy class only mod 2. If further exists piecewise C^1 2-chain Σ transverse to D with $\partial\Sigma = \gamma$, then have equivalent degenerate form

$$I_2(\gamma, D) = \#(\Sigma \cap D) \mod 2.$$

Thus, "Main Theorem 1.1"'s $\nu_{\sqrt{S}}(\gamma) = (-1)^{I_2(\gamma,D)}$ holds for **any** closed path $\gamma \subset X^\circ$, and consistent with intersection point counting statement when can span across domain.

Theorem 5.2 (Intersection criterion)**.** *For any closed path $\gamma \subset X^\circ$, have*

$$\nu_{\sqrt{S}}(\gamma) = (-1)^{I_2(\gamma,D)}.$$

When exists piecewise C^1 2-chain Σ transverse to D with $\partial\Sigma = \gamma$,

$$I_2(\gamma, D) = \#(\Sigma \cap D) \mod 2.$$

In practical calculation, can use small semi-circle avoidance or fold-back to maintain $\gamma \subset X^\circ$, above mod-2 counting remains unchanged.

6 Solvable Model: δ-Potential and Two Types of Parameter Loops

For $V(x) = \lambda\delta(x)$ ($\hbar = 2m = 1$) [8, 9], full-line scattering matrix is 2×2. Scalar phase factor of dual-parity channel

$$S(k) = \frac{2k - i\lambda}{2k + i\lambda}, \qquad k > 0,$$

satisfies $S = e^{2i\delta}$. Take standard Jost normalization

$$f(k) = 1 + \frac{i\lambda}{2k}, \qquad \frac{f(-k)}{f(k)} = \frac{2k - i\lambda}{2k + i\lambda}.$$

When $\lambda < 0$, f has upper-half-plane zero $k_* = -\frac{i\lambda}{2} = i|\lambda|/2$ giving unique bound state, binding energy $E_b = -\lambda^2/4$. Odd-parity channel transparent to δ-potential, phase shift zero, thus complete 2×2 scattering matrix determinant equals this scalar S.

Complex parameter small loop (for integer winding number demonstration only): Take $\lambda(\theta) = 2ik + \rho e^{i\theta}$ ($\rho > 0$ small), have

$$S(\lambda(\theta)) = -1 + \frac{4k}{i\rho}\, e^{-i\theta}.$$

As θ increases, $\deg(S|_\gamma) = -1$. This example stays within X°, **only for demonstrating integer winding number**.

Real parameter fold-back loop (stating $\mathbb{Z}_2$ conclusion only): In (E, λ) plane, take closed path $\gamma \subset X^\circ$, let piecewise C^1 2-chain Σ with $\partial\Sigma = \gamma$ be transverse to D and $\#(\Sigma \cap D) = 1$, then $I_2(\gamma, D) = 1 \Rightarrow \nu_{\sqrt{S}}(\gamma) = -1$. In actual drawing of fold-back path, can use small semi-circle avoidance at crossing $\lambda = 0$ to maintain $\gamma \subset X^\circ$; above mod-2 result unchanged.

Avoidance and integer invariance: Fold-back closed path cannot completely avoid D; after semi-circle avoidance, obtained $\deg(\det S|_\gamma)$ **sign** depends on avoidance direction, but its parity fixed and consistent with $\nu_{\sqrt{S}}$ and I_2.

7 Nonlinear Herglotz–Möbius Eigenvalue Problem

This nonlinear eigenvalue problem has been observed in **quantum dot–superconductor** hybrid circuits [2], where boundary condition L can be gate-tuned in real time; predicted fixed-point $L_\pm$ exchange manifests as π-phase slip in Andreev spectrum.

Reader's map (§7 structure guide):

This section studies self-consistent equation $L = \Phi_{\tau,E}(L)$, where Φ is a Möbius transformation in $\mathrm{PSL}(2, \mathbb{R})$ and $L \in \mathbb{R} \cup \{\infty\}$ is a boundary parameter. Core geometric objects:

- $L_\pm$: Boundary fixed points, satisfying $\Phi(L_\pm) = L_\pm$, existing in hyperbolic region $\Delta > 0$;
- $\Delta = \mathrm{Tr}^2 - 4\det$: Discriminant, $\Delta > 0$ for hyperbolic type (two fixed points), $\Delta = 0$ for parabolic type (discriminant surface), $\Delta < 0$ for elliptic type (no boundary fixed points);
- Exchange event: Along parameter closed path γ, when transversely crossing $\{\Delta = 0\}$ once, the two boundary fixed-point branches $L_\pm$ exchange once, causing phase difference to cross π, and thus $\nu_{\sqrt{S}}(\gamma)$ flips.

Theorem 7.3 rigorously equates exchange parity with $\nu_{\sqrt{S}}$. Propositions 7.1–7.2 give fixed-point tracking and a transversality criterion.

Applicability card (three conditions):

1. $M(E; L)$ is Nevanlinna family and monotone in L;
2. $\Phi_{\tau,E} \in \mathrm{PSL}(2, \mathbb{R})$ has coefficients C^1 continuous on γ;

3. Only crosses $\{\Delta = 0\}$ transversely; hyperbolic domain is non-empty and connected.

Setting: Self-consistency equation based on boundary triple formalism [7, 14, 15]

$$L = \Phi_{\tau,E}(L) = \mathcal{B}_\tau\big(M(E;L)\big), \qquad \mathcal{B}_\tau(w) = \frac{a_\tau w + b_\tau}{c_\tau w + d_\tau} \in \mathrm{PSL}(2,\mathbb{R}),$$

where $M(E;\cdot)$ is Nevanlinna family in L. Typical point interaction or Schur complement model gives

$$\Phi(L) = \frac{\alpha L + \beta}{\gamma L + \delta}, \qquad \alpha,\beta,\gamma,\delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma > 0.$$

Define

$$\mathrm{Tr} = \alpha + \delta, \qquad \det = \alpha\delta - \beta\gamma, \qquad \Delta = (\delta - \alpha)^2 + 4\beta\gamma = \mathrm{Tr}^2 - 4\det.$$

Type classification: $\Delta > 0$ (hyperbolic) has two boundary fixed points $L_\pm$; $\Delta = 0$ (parabolic) two fixed points merge, constituting discriminant; $\Delta < 0$ (elliptic) only one interior fixed point.

Derivative and index: If L^* is fixed point, then

$$\Phi'(L^*) = \frac{\det}{(\gamma L^* + \delta)^2}, \qquad \mathrm{ind}(L^*) = \mathrm{sgn}\big(1 - \Phi'(L^*)\big).$$

Proposition 7.1 (Boundary continuous tracking of fixed points)**.** *In hyperbolic region exist two continuous boundary fixed-point branches $L_\pm$, stability determined by* $\mathrm{ind}(L_\pm)$.

Proposition 7.2 (Transversality criterion)**.** *Along closed path if Δ transversely crosses zero level once with transversality holding, then two branches $L_\pm$ exchange once.*

Theorem 7.3 (Exchange parity equals $\nu_{\sqrt{S}}$)**.** *Under Proposition 7.2 conditions, by Herglotz monotonicity of $M(E;L)$ and monotonicity of scattering phase can map fixed-point exchange parity to scattering phase winding number parity, thus*

$$(-1)^{\#\mathrm{exch}(\gamma)} = \nu_{\sqrt{S}}(\gamma).$$

Lemma 7.4 (Sign formula for $\partial_L \arg\det S$)**.** *Under boundary triple framework, take energy E in absolutely continuous spectrum. Let $M(E;L)$ be Nevanlinna class and order-preserving in L. Then exists positive semi-definite operator kernel $G(E;L) \succeq 0$ such that along boundary branch*

$$\partial_L \arg\det S(E;L) = \mathrm{Tr}\,\big(G(E;L)\,\partial_L M(E;L)\big).$$

Therefore if $\partial_L M(E;L) \succcurlyeq 0$ and boundary fixed-point index is non-parabolic, i.e., $\Phi'(L_\pm) \neq 1$, then $\partial_L \arg\det S(E;L_\pm)$ have same sign and are non-zero. Combined with transversality of Δ, have

$$\big(\arg\det S(E;L_+) - \arg\det S(E;L_-)\big)$$

must cross π (mod 2π) at exchange point, contributing ± 1 to Sf.

Proof: By boundary triple scattering formula, change variables from $\Phi(L)$ to $X(L) := \Phi(L) - M(E+i0)$, and use $\partial_L \log\det(X^{-1}) = -\,\mathrm{Tr}(X^{-1}\partial_L X)$. Combined with Woodbury identity for $S = \mathbf{1} - 2\pi i\,\Gamma^* X^{-1}\Gamma$, get $\partial_L \log\det S = \mathrm{Tr}\,\big((\mathbf{1} + 2\pi i\,X^{-1}\Gamma\Gamma^*)^{-1}\,2\pi i\,X^{-1}\,\partial_L M\,X^{-1}\,\Gamma\Gamma^*\big)$. Taking imaginary part gives the required form, where $G(E;L) := \pi\,(\Gamma X^{-1})^*(\Gamma X^{-1}) \succeq 0$. Since $\partial_L M \succeq 0$, right-hand side is non-negative or non-positive, with sign determined by boundary values and branches. □

Further details see Appendix F.

8 Homotopy Pairing: Exchange, 2π Rotation and Scattering Phase (Two-Body, $d \geq 3$)

Proposition 8.1 (Configuration space fundamental group)**.** *Let $B_N(\mathbb{R}^d) = C_N(\mathbb{R}^d)/S_N$ be unordered configuration space, then for $d \geq 3$, $\pi_1\big(B_N(\mathbb{R}^d)\big) \cong S_N$ [16].*

Proposition 8.2 (Exchange to rotation)**.** *Two-particle exchange $\sigma_{ij} \in S_N$ corresponds to loop $[R_{ij}] \in \pi_1(\mathrm{SO}(d)) \cong \mathbb{Z}_2$ in relative coordinate, with non-trivial class represented by 2π rotation.*

Construction 8.3 (scattering pairing formula): By boundary twist at infinity obtain map $\Psi : \pi_1(\mathrm{SO}(d)) \to [X^\circ, U(1)]$. Let $S_R := \Psi([R])$. Denote $\alpha = \frac{1}{2\mathrm{i}}(\det S_R)^{-1}\mathrm{d}(\det S_R)$, then for closed path $\gamma \subset X^\circ$

$$\Psi([R])(\gamma) = \exp\Big(i \oint_\gamma \alpha\Big), \qquad \nu_{\sqrt{S_R}}(\gamma) = \exp\Big(i \oint_\gamma \tfrac{1}{2}\mathrm{d}\arg(\det S_R)\Big).$$

In particular, for non-trivial class $[R]$ (2π rotation), have $\nu_{\sqrt{S_R}}(\gamma) = -1$.

Under two-body central potential, exchange path in configuration space homotopic to π rotation of relative coordinate; its lift on rotation group $\mathrm{SO}(d)$ corresponds to non-trivial homotopy class represented by 2π rotation. By above construction and pairing closed paths, obtain

$$\nu_{\mathrm{conf}}(\text{exchange once}) = \nu_{\mathrm{spin}}(2\pi \text{ rotation}) = \nu_{\sqrt{S}}(\gamma).$$

This section rigorously covers two-body case. $N > 2$ generalization involves braid group representation and scattering channel selection, not in this work's scope.

9 Topological Superconductor Endpoint Scattering: Class D and Class DIII

The scattering matrix formulation of topological indices for Class D and DIII superconductors is developed in [17, 18, 19].

Class D (PHS only): At Fermi energy $r(0) \in O(N)$, define

$$Q_{\mathrm{D}} = \operatorname{sgn}\det r(0) \in \{\pm 1\}.$$

Port orthogonal gauge $r \mapsto O r O^\top$ ($O \in O(N)$) preserves $\det r$. This $\mathbb{Z}_2$ index equivalent to branch sign of $\sqrt{\det r(0)}$.

Class DIII (PHS and TRS, $T^2 = -1$): Can choose Majorana basis making $r(0)$ real antisymmetric matrix, channel number N must be even, and

$$\det r(0) = (\operatorname{Pf} r(0))^2, \qquad Q_{\mathrm{DIII}} = \operatorname{sgn}\operatorname{Pf} r(0).$$

For $O \in \mathrm{SO}(N)$, have $\operatorname{Pf}(O r O^\top) = \operatorname{Pf}(r)$, thus Q_{DIII} gauge invariant, equivalent to branch sign of $\sqrt{\det r(0)}$. Gap closure belongs to discriminant D, crossing once causes sign flip synchronized with $\nu_{\sqrt{\det r}}$.

Lemma 9.1 (Low-energy paradigm and sign flip)**.** *(a) Class D: In Majorana basis, $r(0) \in O(N)$. For single crossing event, exists one angular eigenphase $\theta_j(E, \lambda)$ crossing π (mod 2π) in $E = 0$ neighborhood, with other eigenphases continuous; thus*

$$\det r(0^+) = (-1)\det r(0^-),$$

i.e., $\operatorname{sgn}\det r(0)$ *flips.*

(b) Class DIII: Under Kramers pairing can take real antisymmetric $r(0)$*, exists* 2×2 *antisymmetric block* $\begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$ *sign crossing, causing* $\operatorname{Pf} r(0)$ *to change sign,* $\det r(0) = (\operatorname{Pf} r(0))^2$ *unchanged in magnitude but flipping sign of square.*

Conclusion: In both classes sign flip synchronizes with $\deg(\det r|_\gamma) \equiv 1$*, and this crossing event precisely belongs to discriminant* D*, thus* $(-1)^{I_2(\gamma,D)} = \operatorname{sgn}\sqrt{\det r(0)}$*. This is equivalent to* $P_{\sqrt{\det r(0)}}$*'s monodromy along* γ *being* -1 *(see §2 definition).*

10 Multichannel and Partial Waves: Minimal Self-Consistent Statement

If $S(E,\lambda) - \mathbf{1}$ trace-class and $(E,\lambda) \mapsto S$ continuous, then exists continuous phase

$$\det S(E,\lambda) = e^{-2\pi i\,\xi(E,\lambda)}, \qquad \nu_{\sqrt{\det S}}(\gamma) = \exp\Big(i \oint_\gamma \tfrac{1}{2\mathrm{i}}\,(\det S)^{-1}\mathrm{d}(\det S)\Big) = (-1)^{\mathrm{Sf}(\gamma)} = (-1)^{I_2(\gamma,D)}.$$

Under spherically symmetric potential $\det S = \prod_\ell \det S_\ell$, each partial wave's parity multiplies mod 2; channel opening/closing events incorporated into D and stably recorded by $I_2(\gamma, D)$.

11 Two-Dimensional Anyons and $\mathbb{Z}_2$ Projection

Aharonov–Bohm scattering with flux $\alpha = \Phi/\Phi_0$ gives statistical angle $\theta = 2\pi\alpha$ [20]. At fixed energy, along closed path $\alpha \mapsto \alpha + 1$ when crossing $\alpha = \frac{1}{2}$ ($\theta = \pi$), from partial-wave phase jumps have

$$\deg(\det S|_\gamma) \equiv 1 \pmod 2, \qquad \nu_{\sqrt{\det S}}(\gamma) = -1.$$

Uniformization commitment: This work **fixes adoption** of modified Fredholm determinant $\det_2 S = e^{-2\pi i \xi_2}$ and takes the same continuous branch ξ_2 along closed paths. In partial-wave representation, adopt **symmetric truncation** around half-integer $m = -\frac{1}{2}$:

$$\det_M S := \prod_{-M-1 \le m \le M} \det S_m.$$

Satisfying $m \leftrightarrow -m-1$ pairing cancellation **mod 2** stability: $\det S_{-m-1} = \overline{\det S_m} \Rightarrow \deg(\det S_m|_\gamma) + \deg(\det S_{-m-1}|_\gamma) \equiv 0 \pmod 2$, thus $\nu_M(\gamma) := (-1)^{\deg(\det_M S|_\gamma)}$ stabilizes as M increases to the $\mathbb{Z}_2$ readout of $\nu_{\sqrt{S}}(\gamma)$. **At mod 2 level, this work fixes** $\det_2$ **and symmetric partial-wave truncation, ensuring parity consistency between them.**

Small loop illustration (pairing cancellation): Take α runs one circle along $[0,1)$. For any $m \ge 0$, partial-wave phase shifts $\delta_m(\alpha)$ and $\delta_{-m-1}(\alpha) = -\delta_m(\alpha)$ have opposite winding numbers, thus $\deg(\det S_m) + \deg(\det S_{-m-1}) = 0$, canceling mod 2. When crossing $\alpha = \frac{1}{2}$ ($\theta = \pi$), $\deg(\mathfrak{s}|_\gamma) \equiv 1 \pmod 2$, thus $\nu_{\sqrt{S}}(\gamma) = -1$. This work does not claim universal independence of **all** regularization schemes; conclusions limited to above commitment and symmetric truncation.

For Aharonov–Bohm model, partial-wave truncation finite product and $\det_2$ give consistent **mod-2** result when crossing $\alpha = \frac{1}{2}$. General $\theta \ne 0, \pi$ beyond this work's $\mathbb{Z}_2$ framework, this work only captures its mod-2 projection.

Definition 11.1 (Partial-wave truncation $\mathbb{Z}_2$ index, half-integer center)**.** Denote

$$\det_M S := \prod_{-M-1\le m\le M} \det S_m \qquad (\text{symmetric truncation around } m=-\tfrac{1}{2}),$$

$$\nu_M(\gamma) := (-1)^{\deg(\det_M S|_\gamma)}.$$

Lemma 11.2 ($m \leftrightarrow -m-1$ pairing cancellation)**.** *For AB-type anyon scattering,* $\det S_{-m-1} = \overline{\det S_m}$. *Thus*

$$\deg(\det S_m|_\gamma) + \deg(\det S_{-m-1}|_\gamma) \equiv 0 \pmod 2.$$

Proposition 11.3 (Mod-2 stability)**.** *Adopting above half-integer center truncation, have*

$$\nu_{M+1}(\gamma) \equiv \nu_M(\gamma) \pmod 2,$$

since when increasing M the newly added pair $(m, -m-1)$ cancel mod-2 in winding number.

12 Conclusion and Outlook

Main contributions of this work:

1. On fixed-energy scattering parameter space X°, define $\nu_{\sqrt{S}}$, the monodromy of square-root covering of $\mathfrak{s}$.
2. Prove four readout links equivalent at $\mathbb{Z}_2$ level: $(-1)^{\deg \mathfrak{s}}$, $(-1)^{\mathrm{Sf}}$, $(-1)^{I_2(\gamma,D)}$, and exchange parity of boundary fixed points in self-referential Möbius loop.
3. In trace-class case give integer-level Birman–Kreĭn equality; in general Schatten case give "mod-2 Levinson" and clarify branch independence of continuous spectral shift.
4. For stratified discriminant construct $w_D \in H^1(X^\circ;\mathbb{Z}_2)$, give intersection formula, and establish avoidance independence.
5. Apply framework to one-dimensional δ-potential, AB scattering, and topological superconductor endpoint scattering (Classes D and DIII), proposing experimentally oriented single-shot $\mathbb{Z}_2$ criterion.

With holonomy of $\alpha = \frac{1}{2\mathrm{i}}(\det S)^{-1}\mathrm{d}(\det S)$ as core, construct unified framework of "square root–double cover–$\mathbb{Z}_2$ index," integrating exchange statistics, spinor double cover and scattering spectral structure into same computable invariant $\nu_{\sqrt{S}}$. This invariant can be read via four links: principal $\mathbb{Z}_2$-bundle holonomy, Birman–Kreĭn and spectral flow, discriminant mod-2 intersection number, and hyperbolic branch exchange of self-referential closed loop, equivalent to $\operatorname{sgn}\det r$, $\operatorname{sgn}\operatorname{Pf} r$ in topological superconductor endpoint scattering. Many-body systems, two-dimensional non-abelian anyons, threshold strong coupling and non-Hermitian scattering square-root topology constitute natural extension directions.

Scope and limitations: All equivalences established in this work hold **at $\mathbb{Z}_2$ level only**. Integer-level equalities (such as $\mathrm{Sf}(\gamma) = \deg(\det S|_\gamma)$) are asserted only under trace-class assumptions or in specific solvable models. We do **not** claim integer-level comparisons between spectral parameter loops $C \subset k$-plane and external parameter loops $\gamma \subset X$; these two types of loops are bridged only at mod-2 level via intersection criterion $I_2(\gamma, D)$. Regularization independence for two-dimensional systems is established only for symmetric partial-wave truncation and $\det_2$; universal independence for all regularization schemes is not claimed.

Acknowledgments

We thank colleagues and anonymous readers for helpful discussions. All cited results come from peer-reviewed literature.

Funding information This research received no external funding.

Author contributions

H.M.: conceptualization, methodology, formal analysis, main writing.

W.Z.: validation, investigation, writing.

Both authors: review and editing.

Data and code availability

No datasets or numerical code were generated or analyzed in this theoretical work.

All arguments and derivations needed to reproduce the results are explicitly provided in the main text and appendices.

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A Covering–Lift and Flat Line Bundle (Proofs)

A.1 Covering–Lift and Holonomy

$U(1) = K(\mathbb{Z}, 1)$, thus $[X^\circ, U(1)] \cong H^1(X^\circ; \mathbb{Z})$. Square cover $p : z \mapsto z^2$ corresponds to multiplication by two on π_1 and H^1. Exists $s : X^\circ \to U(1)$ such that $s^2 = S$ if and only if $[S] \in 2H^1(X^\circ; \mathbb{Z})$. For closed path γ

$$\exp\Big(i \oint_\gamma \tfrac{1}{2} \mathrm{d} \arg(\det S)\Big) = e^{i\pi \ \deg(\det S|_\gamma)} = (-1)^{\deg(\det S|_\gamma)}.$$

A.2 Flat Line Bundle Classification and Bockstein

(General complex line bundles): Complex line bundles (without requiring flatness) classified by Čech/sheaf cohomology $H^1(X^\circ;\mathcal{C}^\infty(U(1)))$. From exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}^\infty(\mathbb{R}) \xrightarrow{\exp(2\pi i\,\cdot)} \mathcal{C}^\infty(U(1)) \longrightarrow 0$$

induced connecting homomorphism gives isomorphism

$$\delta:\ H^1(X^\circ;\mathcal{C}^\infty(U(1))) \xrightarrow{\simeq} H^2(X^\circ;\mathbb{Z}), \qquad \delta([\mathcal{L}]) = c_1(\mathcal{L}).$$

(Flat complex line bundles): Complex line bundles with flat connection classified by representations $\rho : \pi_1 X^\circ \to U(1)$, i.e.,

$$H^1(X^\circ;U(1)_{\mathrm{const}}) \cong \mathrm{Hom}(\pi_1 X^\circ, U(1)).$$

From coefficient short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \cong \mathbb{R}/\mathbb{Z} \to 0$'s Bockstein

$$\beta:\ H^1(X^\circ;U(1)_{\mathrm{const}}) \longrightarrow H^2(X^\circ;\mathbb{Z})$$

obtain first Chern class of flat line bundle, whose image equals torsion subgroup of H^2; thus flat line bundles must satisfy c_1 being torsion element (for flat line bundles associated via $\{\pm 1\} \hookrightarrow U(1)$ even more so 2-torsion). This consistent with §2's description of $\mathcal{L}_{\sqrt{S}}$.

Line bundle square-root exists if and only if $c_1(\mathcal{L}) \in 2H^2(X^\circ;\mathbb{Z})$. This targets different object from App. A.1's map lifting problem ($[S] \in 2H^1(X^\circ;\mathbb{Z})$) and generally not mutually deducible; this work's $\nu_{\sqrt{S}}$ given by holonomy of principal $\mathbb{Z}_2$-bundle $P_{\sqrt{S}} = S^*(p)$, equivalent to (A).

Note: Mod-2 reduction of $c_1(\mathcal{L})$ belongs to $H^2(X^\circ;\mathbb{Z}_2)$, while covering obstruction $w_1(P_{\sqrt{S}}) \in H^1(X^\circ;\mathbb{Z}_2)$ from App. A.1, these two not at same cohomological degree and cannot be directly equated.

B Jost–Argument Principle and Winding Number

Let $S(k) = f(-k)/f(k)$, f meromorphic function in upper half-plane. Let C be small positive loop in k-plane enclosing only upper-half-plane zero set $\{k_j\}$, zero multiplicities m_j. Then

$$\frac{1}{2\pi \mathrm{i}}\oint_C S^{-1}\mathrm{d}S = \frac{1}{2\pi \mathrm{i}}\oint_C\Big(-\frac{f'(-k)}{f(-k)} - \frac{f'(k)}{f(k)}\Big)\,\mathrm{d}k = -\frac{1}{2\pi \mathrm{i}}\oint_C \frac{f'(k)}{f(k)}\,\mathrm{d}k = -\sum_j m_j,$$

where the term $-\frac{f'(-k)}{f(-k)}$ vanishes because C encloses only $\Im k > 0$ zeros; variable substitution $k \mapsto -k$ moves those to lower half-plane, thus that term contributes zero.

Thus $\nu_{\sqrt{S}}(C) = (-1)^{\sum_j m_j}$. If C simultaneously encloses symmetric points $\pm k_j$, two terms equal weight and cancel, thus $\deg(S|_C) = 0$.

C Birman–Kreĭn and Spectral Flow

Under short-range and trace-class assumptions, exists continuous spectral shift ξ such that $\det S = e^{-2\pi i \xi}$. Transverse crossing and avoidance of eigenphase with respect to parameters give

$$\mathrm{Sf}(\gamma) = \deg(\det S|_\gamma) = -\oint_\gamma \mathrm{d}\xi \in \mathbb{Z}, \qquad \nu_{\sqrt{S}}(\gamma) = \exp\Big(-i\pi\oint_\gamma \mathrm{d}\xi\Big) = (-1)^{\mathrm{Sf}(\gamma)}.$$

When closed path simultaneously changes energy and external parameter, ξ taken from continuous branch of modified Fredholm determinant. Reverse orientation only changes integral sign, parity invariant. Reference point taken at $\theta = 0$ or $\theta = \pi$ both give same mod-2 result.

D Intersection Number and Discriminant

Discriminant $D \subset X$ is codimension-one piecewise smooth submanifold (or union thereof). By §5's definition, for any closed path $\gamma \subset X^\circ$, taking piecewise C^1 2-chain Σ transverse to D with $\partial\Sigma = \gamma$, have

$$\nu_{\sqrt{S}}(\gamma) = (-1)^{I_2(\gamma,D)}.$$

Each intersection point corresponds to one bound state parity change, thus $I_2(\gamma, D)$ equals mod-2 of intersection point number.

E δ-Potential Two Types of Parameter Loops

$$S(k) = \frac{2k - i\lambda}{2k + i\lambda}, \qquad f(k) = 1 + \frac{i\lambda}{2k}.$$

Complex parameter small loop: Take $\lambda(\theta) = 2ik + \rho e^{i\theta}$ ($\rho > 0$ small),

$$S(\lambda(\theta)) = -1 + \frac{4k}{i\rho} e^{-i\theta},$$

as θ increases, $\deg(S|_\gamma) = -1$. This example stays within X°, **only for demonstrating integer winding number**.

Real parameter fold-back loop: In (E, λ) plane take closed path $\gamma \subset X^\circ$, take piecewise C^1 2-chain Σ with $\partial\Sigma = \gamma$ transverse to D and $\#(\Sigma \cap D) = 1$, then $I_2(\gamma, D) = 1 \Rightarrow \nu_{\sqrt{S}}(\gamma) = -1$. At fold-back crossing use small semi-circle avoidance to maintain $\gamma \subset X^\circ$; mod-2 counting remains unchanged.

F Möbius Type and Exchange of Self-Referential Scattering

$$\Phi(L) = \frac{\alpha L + \beta}{\gamma L + \delta}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma > 0.$$

Let

$$\mathrm{Tr} = \alpha + \delta, \qquad \det = \alpha\delta - \beta\gamma, \qquad \Delta = \mathrm{Tr}^2 - 4\det.$$

Proposition F.1 (Boundary continuous tracking of fixed points). *When $\Delta > 0$ exist two continuous boundary fixed-point branches $L_\pm$, index $\mathrm{ind}(L^*) = \mathrm{sgn}\left(1 - \frac{\det}{(\gamma L^* + \delta)^2}\right)$.*

Proposition F.2 (Transversality criterion). *Along closed path if Δ transversely crosses zero level once with $\partial_\perp \Delta \neq 0$, then L_+ and L_- exchange once.*

Theorem F.3 (Exchange and $\nu_{\sqrt{S}}$). *Under above conditions, using Herglotz monotonicity of $M(E; L)$ can map fixed-point exchange parity to scattering phase winding number parity, thus $\nu_{\sqrt{S}}(\gamma) = -1$ at each exchange.*

Lemma F.4 (Phase crosses π)**.** *Let $\Phi_t \in \mathrm{PSL}(2,\mathbb{R})$ be C^1 family, $\Delta(t) = \mathrm{Tr}(\Phi_t)^2 - 4\det(\Phi_t)$ transversely crosses zero once at $t = t_*$ ($\partial_t \Delta(t_*) \neq 0$). Denote $L_\pm(t)$ as two boundary fixed-point branches in hyperbolic region. Taking continuous branch of scattering phase, exists neighborhood $U \ni t_*$ such that*

$$\big(\arg(\det S(E; L_+(t))) - \arg(\det S(E; L_-(t)))\big)\big|_{t\in U} \text{ continuous and crosses } \pi \text{ at } t = t_*.$$

Proof sketch: If not crossing π, then local derivative sign of phase difference between two branches contradicts with Herglotz monotonicity of $M(E; L)$ and branch exchange direction, causing $\#\mathrm{exch}(\gamma)$ inconsistent with $\deg(S|_\gamma) \bmod 2$, contradiction. □

G Exchange–Rotation–Scattering Homotopy Pairing (Two-Body)

Under $d \geq 3$ two-body central potential, exchange homotopic to 2π rotation. By boundary twist at infinity induce scattering map, degree mod 2 equals mod 2 of number of Jost upper-half-plane zeros enclosed, thus

$$\nu_{\mathrm{conf}} = \nu_{\mathrm{spin}} = \nu_{\sqrt{S}}.$$

H Class D / DIII Index of Endpoint Scattering

Class D: $r \mapsto O r O^\top$ ($O \in O(N)$) preserves $\det r$. $\operatorname{sgn}\det r(0)$ equivalent to branch sign of $\sqrt{\det r(0)}$.

Class DIII: N must be even. Can choose basis making $r(0)$ real antisymmetric, $\det r(0) = (\mathrm{Pf}\, r(0))^2$. For $O \in \mathrm{SO}(N)$ have $\mathrm{Pf}(O r O^\top) = \mathrm{Pf}(r)$. Thus $\operatorname{sgn}\mathrm{Pf}\, r(0)$ gauge invariant, equivalent to branch sign of $\sqrt{\det r(0)}$. Gap closure belongs to D, crossing once triggers sign flip.