Translating Tableaux Proofs into Sequent Proofs in Classical and Intuitionistic Logic

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Abstract. This work aims to discuss the translation between Tableaux proofs and sequent calculus proofs in first-order intuitionistic predicate logic and in first-order classical predicate logic. It begins with an overview of the definitions and clarification of the notation. It then shows a translation process in classical logic and its OCAML implementation restricted to propositional logic. Finally, a potential extension towards translation in intuitionistic logic is discussed.

Keywords: Tableaux proof · sequent calculus · intuitionistic logic.

1 Introduction

1.1 Notation

In this work, sentences will implicitly refer to first-order predicate logic sentences; for intuitionistic logic, their meaning will come from Kripke's semantics [3] . The notation for structures and frames will be heavily based on [1]. In order to make this document slightly more self-reliant and to clarify the notation, we will briefly explain:

1.2 Definitions

Definition 1. ***A Structure of a Language*** *consists of a domain and:*

- *An assignment from the constant symbols of the language to the domain.*
- *An assignment from the predicate symbols of the language to predicates in the domain.*

Structures represent possible worlds or possible states of knowledge inside a frame:

Definition 2. ***A Kripke Frame*** *of a Language $\mathcal{L}$, $\mathcal{C} = (R, \{C(p)\}_{p \in R})$ consists of a partially ordered set R, and an $\mathcal{L}$-structure $C(p)$ for all p's in R. Furthermore, in a Kripke Frame, if $p \leq q$, then $C(q)$ extends $C(p)$:*

- *All sentences that are true in $C(p)$ are true in $C(q)$.*
- *The domain of $C(p)$ is included in the domain of $C(q)$.*
- *The assignments in $C(p)$ are the same as in $C(q)$ for the domain in common.*

Particularly, in order to simplify the notation, R will always be the set of sequences of integers, and $p \leq q$ if it exists an l such that $q = p||l$. Also, from now on, the constant elements of a language will always be in the ordered set $\{c_0, c_1, c_2...\}$

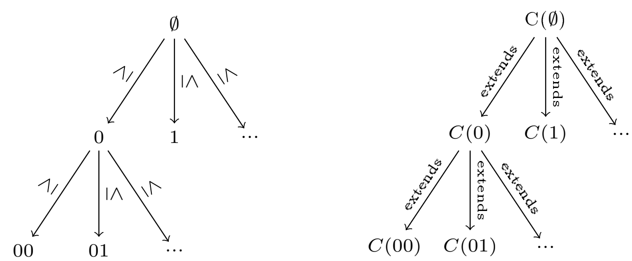

Fig. 1. R and a Kripke frame.

Definition 3. ***Forcing***. *When a sentence ϕ of a language $\mathcal{L}$ is* ***forced*** *by a structure $C(p)$ of a $\mathcal{L}$-frame $\mathcal{C}$, we denote: $p \Vdash_{\mathcal{C}} \phi$*
Forcing is defined by induction: [1]

- $p \Vdash_{\mathcal{C}} \phi \Leftrightarrow \phi$ *is true in $C(p)$ (if ϕ is an atomic sentence).*
- $p \Vdash_{\mathcal{C}} (\phi \to \psi) \Leftrightarrow$ *for all $q \geq p$, if $q \Vdash_{\mathcal{C}} \phi$, then $q \Vdash_{\mathcal{C}} \psi$.*
- $p \Vdash_{\mathcal{C}} \neg\phi \Leftrightarrow$ *for all $q \geq p$, q does not force ϕ.*
- $p \Vdash_{\mathcal{C}} (\forall x)\phi(x) \Leftrightarrow$ *for all $q \geq p$ and d in $\mathcal{L}_{C(q)}$, $q \Vdash_{\mathcal{C}} \phi(d)$.*
- $p \Vdash_{\mathcal{C}} (\exists x)\phi(x) \Leftrightarrow$ *there exists a d in $\mathcal{L}_{C(q)}$, such that $p \Vdash_{\mathcal{C}} \phi(d)$.*
- $p \Vdash_{\mathcal{C}} (\phi \wedge \psi) \Leftrightarrow p \Vdash_{\mathcal{C}} \phi$ *and* $p \Vdash_{\mathcal{C}} \psi$.
- $p \Vdash_{\mathcal{C}} (\phi \vee \psi) \Leftrightarrow p \Vdash_{\mathcal{C}} \phi$ *or* $p \Vdash_{\mathcal{C}} \psi$.

Definition 4. ***Intuitionistic Validity***. *A sentence of a language $\mathcal{L}$ is Intuitionistically valid if it is forced in all structures of all Kripke frames of $\mathcal{L}$.*

In classical logic, this definition simplifies to the one of forcing, and it's simplified again by the fact that p = q; in fact, here we will define classical validity as: [1]

Definition 5. ***Classical Validity***. *A sentence of a language $\mathcal{L}$ is classically valid if it is forced by all single-structure Kripke frames of $\mathcal{L}$.*

2 The Intuitionistic Tableaux Method

Considerations Here we first define a slightly different version of the destructive tableaux proof tree described by [1], were each node is a truth assertion. This different version will allow for a more implementation-oriented approach and the translation later on.

The correspondence of the destructive tableaux proof tree described in [1] to our new one is shown in Figure 2. Generally speaking, a node in the usual definition is replaced by a sequence of all nodes in the path that goes from the root to it. Afterwards, some nodes are removed from the newly formed tableaux by adjoining its son(s) and its parent. A node should be removed if its corresponding node in the original tableaux was not a leaf of the atomic tableaux [1] that introduced it.

$$
\begin{tikzcd}
F_\emptyset \alpha \to \neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha \arrow[dd] \\
T_0\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha \arrow[d] & \\
F_0\neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha, F_0\neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha, F_0\neg\neg\alpha \arrow[d] \\
T_{00}\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha, F_0\neg\neg\alpha, T_{00}\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha, F_0\neg\neg\alpha, T_{00}\neg\alpha \arrow[d] \\
F_{00}\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha, F_0\neg\neg\alpha, T_{00}\neg\alpha, F_{00}\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0\alpha, F_0\neg\neg\alpha, T_{00}\neg\alpha, F_{00}\alpha \arrow[d] \\
\mathrm{X} & \mathrm{X} & \mathrm{X}
\end{tikzcd}
$$

Fig. 2. Example of a destructive tableaux proof tree from [1], the intermediary structure and the non-destructive tableaux proof tree.

In this new presentation, one can see each node of the tree as an assumption of the existence of a frame that respects a list of constraints, and each edge as an implication between assumptions.

The Intuitionistic Tableaux Method The tableaux stands on some definitions, they will be justified briefly:

Definition 6. ***A Signed Sentence*** *is a forcing assertion inside of a tableaux proof. It looks like* $T_q\phi$ *or* $F_p\phi$
A Signed Sentence List *is a list forcing assertions inside of a tableau proof.*

We say that a list of forcing assertions having sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ and $\{F_{q_1}\delta_1, F_{q_2}\delta_2...\}$ is "intuitionistically valid" (Question 2: maybe use a term != valid?)if there exists a frame $\mathcal{C}$ for which $\mathcal{C}(p_1) \Vdash \gamma_1$ and $\mathcal{C}(p_2) \Vdash \gamma_2$ and ... $\mathcal{C}(q_1) \nVdash \delta_1$ and $\mathcal{C}(q_2) \nVdash \delta_2$

A Signed sentence list can be seen as an existence assumption that may or may not be intuitionistically valid. We can infer other assumptions that are, by definition, consequences of a given assumption. The function f, defined bellow, is one of the ways we can do that:

Definition 7. *The function f takes a signed sentence σ and a signed sentence list L and returns one or two signed sentence lists.*
$f(\sigma, L)$ is defined as follows:
(here we denote $l||l' = l_1, l_2..., l_{|l|}, l'_1, l'_2.., l'_{|l'|}$)
if $\sigma \in L$:

- $f(T_p\neg\alpha, L) = [L - \sigma||\sigma||F_{p'}\alpha]$
  for a minimal $p' \geq p$ present in $h||L$.
- $f(F_p\neg\alpha, L) = [L - \sigma||\sigma||T'_p\alpha]$
  for a new $p' \geq p$
- $f(T_p(\alpha \wedge \beta), L) = [L - \sigma||\sigma||T_p\alpha||T_p\beta]$.
- $f(F_p(\alpha \wedge \beta), L) = [L - \sigma||\sigma||F_p\alpha], [L - \sigma||h||F_p\beta]$.
- $f(T_p(\alpha \vee \beta), L) = [L - \sigma||\sigma||T_p\alpha], [L - \sigma||h||T_p\beta]$.
- $f(F_p(\alpha \vee \beta), L) = [L - \sigma||\sigma||F_p\alpha||F_p\beta]$.
- $f(T_p(\alpha \rightarrow \beta), L) = [L - \sigma||\sigma||F_{p'}\alpha], [L - \sigma||h||T_{p'}\beta]$
  for a new $p' \geq p$
- $f(F_p(\alpha \rightarrow \beta), L) = [L - \sigma||\sigma||T_{p'}\alpha||F_{p'}\beta]$
  for a minimal $p' \geq p$ present in $h||L$.
- $f(T_p(\forall x)\phi(x), L) = [L - \sigma||\sigma||T_p\phi(c_i)]$
  for the first constant c_i for which $T_p\phi(c_i)$ is not in L.
- $f(F_{p'}(\forall x)\phi(x), L) = [L - \sigma||\sigma||F_p\phi(c_i)]$
  for the first constant c_i not present in $h||L$ and a new $p' \geq p$
- $f(T_{p'}(\exists x)\phi(x), L) = [L - \sigma||\sigma||T_p\phi(c_i)]$
  for the first constant c_i not present in $h||L$ and a new $p' \geq p$.
- $f(F_p(\exists x)\phi(x), L) = [L - \sigma||\sigma||F_p\phi(c_i)]$
  for the first constant c_i such that $F_p\phi(c_i)$ is not in L and .

[TODO revise: a stronger criterium for contradiction makes some of these useless. in any case, this "simetric" form is more easy to implement]
if $\sigma \notin L$:

- $f(\sigma, L) = L$

Although not justified here, the reordering of the terms plays an important role in the implementation and in the completeness of the systematic tableaux: each signed sentence will be σ infinitely many times.

On the other hand, the soundness of the tableaux method depends on the fact that f is 'well-behaved":

Theorem 1. *Given a signed sentence list x, if x is intuitionistically valid then $f(x)$ is intuitionistically valid*

Proof. [1] proves this by using the definition of forcing. The choices of p' and c_i garantes the completnees of the tableaux.

Case: if $L \ni F_p(\forall x)\phi(x)$ is intuitionistically valid, then there exists frame $\mathcal{C}$ that "respects" it. We take a frame $\mathcal{C}'$ that is exactly like $\mathcal{C}$ with adition of a structure $p' \geq p$ such that $\mathcal{C}'(p') \nVdash \phi(c_i)$ for a new c_i. We now that $\mathcal{C}'$ exists by the defintion of forcing for $\forall$. Also $\mathcal{C}'$ is a frame that "respects" $f(\forall x\phi(x), L) = [L - \sigma||\sigma||T_p\phi(c_i)]$

By choosing c_i differently, we might arrive at at a intuitionistic invalid [Q irrespectable?] tableaux.

Definition 8. ***The tableaux Development of a List of Sentences*** *is defined inductivelly:*

- *A tree with the single node L is a tableaux development of L.*
- *If τ is a tableaux development of L, then $\hookleftarrow(\sigma, \tau)$ is a tableaux development of ϕ. Where:*

  $\hookleftarrow(\sigma, \tau) = \tau$ *with* $f(\sigma, l)$*added to all leaves* l *that contain* σ

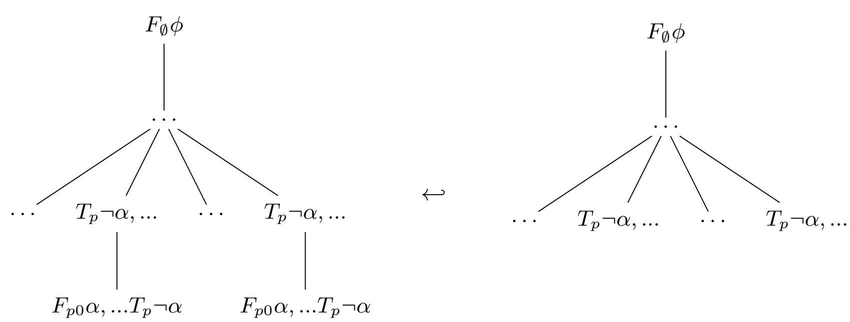

Fig. 3. Example of $\hookleftarrow(T\neg\alpha, \tau)$ and τ.

Theorem 2. *If $F_\emptyset\phi$ is intuitionistically valid then one of the leaves of $\hookleftarrow(\sigma_n, \hookleftarrow(\sigma_{n-1}(.....(\hookleftarrow(\sigma_1, F_\emptyset\phi)...))))$ is intuitionistically valid.*

Proof. The proof goes by induction:

The base case is true by definition: either the root $F_\emptyset\phi$ is a intuitionistically valid leaf or the premisse is false. Next we assume $\tau' = \hookleftarrow(\sigma, \tau)$. There are two cases to consider:

- If there does not exist a frame that does not force ϕ, $F\phi$ is not intuitionistically valid and the theorem holds for τ'.

- If there exists a frame that does not force ϕ:
  By the induction hypothesis, there exists a intuitionistically valid leaf σ in τ.
  - If σ is a leaf in τ, then $F_{\emptyset}\phi$ the theorem holds for τ'.
  - If σ is not a leaf in τ, then: One or two nodes were added to σ in τ' by the definition of $\hookleftarrow$. By the theorem 1, one of the added nodes is also intuitionistically valid. Consequently, the theorem holds for τ'.

From this, we can conclude that: If all leaves of $\hookleftarrow (\sigma_n, \hookleftarrow (\sigma_{n-1}(.....(\hookleftarrow (\sigma_1, F\phi)...))))$ are contradictory, then $F\phi$ is not intuitionistically valid, by consequence ϕ is intuitionistically valid.

Completeness For purposes of implementation, we define an $\hookleftarrow_c$ that can be applied systematically:

Definition 9. *$\hookleftarrow_c (\tau) = \tau$ with $f(h,l)$ added to all leaves l that contain σ. h is the first signed sentence of the shalowest (and after that leftmost) non-contradictory leaf.*

we can use $\hookleftarrow_c$ instead of $\hookleftarrow$ to define a systematic tableaux
[TODO]

3 The Classical Tableaux Method

The classical tableaux method can be seen as the intuitionistic tableaux method restricted to one-framed structures. The adapted definitions are:

Definition 10. ***A Signed Sentence*** *is a forcing assertion inside of a tableaux proof. It looks like $T\phi$ or $F\phi$*
A Signed Sentence List *is a list forcing assertions inside of a tableau proof. We say that a list of forcing assertions having sentences $\{T\gamma_1, T\gamma_2, ...\}$ and $\{F\delta_1, F\delta_2...\}$ is "classically valid" (Question 2: maybe use a term != valid?)if there exists a single-structured frame $\mathcal{C}$ such that $\mathcal{C}(\emptyset) \Vdash \gamma_1$ and $\mathcal{C}(\emptyset) \Vdash \gamma_2$ and ... $\mathcal{C}(\emptyset) \nVdash \delta_1$ and $\mathcal{C}(\emptyset) \nVdash \delta_2$*

Definition 11. *The function f takes a signed sentence σ and a signed sentence list L and returns one or two signed sentence lists.*
$f(\sigma, L)$ is defined as follows:
(here we denote $l||l' = l_1, l_2..., l_{|l|}, l'_1, l'_2.., l'_{|l'|}$)
if $\sigma \in L$:

- $f(T\neg\alpha, L) = [L - \sigma||\sigma||F\alpha]$.
- $f(F\neg\alpha, L) = [L - \sigma||\sigma||T\alpha]$.
- $f(T(\alpha \wedge \beta), L) = [L - \sigma||\sigma||T\alpha||T\beta]$.
- $f(F(\alpha \wedge \beta), L) = [L - \sigma||\sigma||F\alpha], [L||\sigma||F\beta]$.
- $f(T(\alpha \vee \beta), L) = [L - \sigma||\sigma||T\alpha], [L||\sigma||T\beta]$.
- $f(F(\alpha \vee \beta), L) = [L - \sigma||\sigma||F\alpha||F\beta]$.

- $f(T(\alpha \to \beta), L) = [L - \sigma||\sigma||F\alpha], [L||\sigma||T\beta]$.
- $f(F(\alpha \to \beta), L) = [L - \sigma||\sigma||T\alpha||F\beta]$.
- $f(T(\forall x)\phi(x), L) = [L - \sigma||\sigma||T\phi(c_i)]$
  for the first constant c_i for which $T\phi(c_i)$ is not in L.
- $f(F(\forall x)\phi(x), L) = [L - \sigma||\sigma||F\phi(c_i)]$
  for the first constant c_i not present in L.
- $f(T(\exists x)\phi(x), L) = [L - \sigma||\sigma||T\phi(c_i)]$
  for the first constant c_i not present in L.
- $f(F(\exists x)\phi(x), L) = [L - \sigma||\sigma||F\phi(c_i)]$
  for the first constant c_i for which $F\phi(c_i)$ is not in L.

[TODO revise]
if $\sigma \notin L$:

- $f(\sigma, L) = L$

4 Sequent Calculus

Here we use the multi-conclusion sequent calculus defined in [2].

Definition 12. ***A Sequent*** *is an expression of the form*

$$\Gamma \vdash \Delta$$

where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, ...\}$ and $\Delta = \{\Delta_1, \Delta_2, \Delta_3 ...\}$ are finite sets of formulas. Γ is called the antecedent, and Δ is called the succedent.

Γ represents multiple necessary hypothesis, while Δ represents multiple possible conclusions.

Definition 13. ***Intuitionistical Validity of a Sequent*** *A sequent is intuitionistically valid if for all kripke frames, if Γ are forced, then at least one of Δ is forced.*

The more awkward definition:

" A sequent is classically valid if there does not exist a single-sentenced frame that forces all sentences in Γ and does not force any sentence in Δ. "

when compared with the definition of list of signed sentences, hints us towards an equivalence for single framed structures. This will not be discussed in depth.

The meaning of a sequent calculus rule is [Question : is "meaning" too strong? -> maybe change to "a sequent calculus rule can be interpreted as"] the implication: if the top sequent is valid, then the bottom sequent is valid.

Syntactically, each of the rules represent vertices that can be added to a sequent proof tree development to obtain another sequent proof tree development

4.1 Classical Sequent Calculus

Definition 14. *A single sequent is a classical sequent proof tree development Any of the rules of the table 1 applied to a classical sequent proof tree development is a sequent proof tree development*
A classical sequent proof tree is a classical sequent proof tree development with axioms on all leaves.

Rules for classical multi-consequence sequent calculus are given in 1

Table 1. Rules for L's classical multi-consequence sequent calculus.

Rule	Inference
Axiom	$\dfrac{}{\alpha \vdash \alpha}$
Weakening	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \quad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$
Negation	$\dfrac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\mathrm{R}\ \neg\alpha \quad \dfrac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta}\ \neg\mathrm{L}\ \neg\alpha$
Conjunction	$\dfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\mathrm{L}\ \alpha \wedge \beta \quad \dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\mathrm{R}\ \alpha \wedge \beta$
Disjunction	$\dfrac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}\ \vee\ \mathrm{L}\ \alpha \vee \beta \quad \dfrac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\ \mathrm{R}\ \alpha \vee \beta \quad \dfrac{\Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\ \mathrm{R}\ \alpha \vee \beta$
Implication	$\dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta}\ \to \mathrm{L}\ \alpha \to \beta \quad \dfrac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta}\ \to \mathrm{R}\ \alpha \to \beta$
Quantifiers	$\dfrac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x\alpha(x) \vdash \Delta}\ \forall\ \mathrm{L}\ \forall x\alpha(x) \quad \dfrac{\Gamma \vdash \alpha(y), \Delta}{\Gamma \vdash \forall x\alpha(x), \Delta}\ \forall\ \mathrm{R}\ \forall x\alpha(x)$ $\dfrac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x\alpha(x) \vdash \Delta}\ \exists\ \mathrm{L}\ \exists x\alpha(x) \quad \dfrac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x\alpha(x), \Delta}\ \exists\ \mathrm{R}\ \exists x\alpha(x)$
Contraction	$\dfrac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \quad \dfrac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$

4.2 Intuitionistic Sequent calculus

Definition 15. *A single sequent is a intuitionistic sequent proof tree development*

Any of the rules of the table 1 aplied to a intuitionistic sequent proof tree development is a sequent proof tree development
A intuitionistic squent proof tree is a intuitionistic sequent proof tree development with axioms on all leafs.

Rules for L's intuitionistic multi-consequence sequent calculus are given in 2

Table 2. Rules for L's classical multi-consequence sequent calculus.

Rule	**Inference**
Axiom	$\dfrac{}{\alpha \vdash \alpha}$
Weakening	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$ $\quad$ $\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$
Negation	$\dfrac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\mathrm{L}\ \neg\alpha$ $\quad$ $\dfrac{\Gamma, \alpha \vdash}{\Gamma \vdash \neg\alpha}\ \neg\mathrm{R}\ \neg\alpha$
Conjunction	$\dfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\mathrm{L}\ \alpha \wedge \beta$ $\quad$ $\dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\mathrm{R}\ \alpha \wedge \beta$
Disjunction	$\dfrac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}\ \vee\ \mathrm{L}\ \alpha \vee \beta$ $\quad$ $\dfrac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\ \mathrm{R}\ \alpha \vee \beta$ $\quad$ $\dfrac{\Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\ \mathrm{R}\ \alpha \vee \beta$
Implication	$\dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \rightarrow \beta \vdash \Delta} \rightarrow \mathrm{L}\ \alpha \rightarrow \beta$ $\quad$ $\dfrac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \rightarrow \beta} \rightarrow \mathrm{R}\ \alpha \rightarrow \beta$
Quantifiers	$\dfrac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x\alpha(x) \vdash \Delta}\ \forall\ \mathrm{L}\ \forall x\alpha(x)$ $\quad$ $\dfrac{\Gamma \vdash \alpha(y)}{\Gamma \vdash \forall x\alpha(x)}\ \forall\ \mathrm{R}\ \forall x\alpha(x)$ $\dfrac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x\alpha(x) \vdash \Delta}\ \exists\ \mathrm{L}\ \exists x\alpha(x)$ $\quad$ $\dfrac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x\alpha(x), \Delta}\ \exists\ \mathrm{R}\ \exists x\alpha(x)$
Contraction	$\dfrac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$ $\quad$ $\dfrac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$

5 Translation

Definition 16. ***Node Translating Function*** $\mathcal{T}$
Given a signed sentence list L with sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ *and* $\{F_{q_1}\delta_1, F_{q_2}\delta_2...\}$ *and a world* $w \in \{p_1, p_2, ...\} \cup \{q_1, q_2, ...\}$ *then:*

$\mathcal{T}(L,w) = \Gamma_1, \Gamma_2, ... \vdash \Delta_1, \Delta_2, ...$, were:
$\{T_{p'_1}\Gamma_1, T_{p'_2}\Gamma_2, ...\}$ are the elements of $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ such that $p' \geq w$ and $\{F_{q'_1}\Delta_1, F_{q'_2}\Delta_2, ...\}$ are the elements of $\{F_{q_1}\delta_1, F_{q_2}\delta_2, ...\}$ such that $q' \leq w$

For the classical case: $\mathcal{T}(L) = \mathcal{T}(L, \emptyset)$

Theorem 3. *Given signed sentence list L and a w, if L is intuitionistically valid then the sequent $\mathcal{T}(L,w)$ is not intuitionistically valid.*

Proof. If L has the sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ and $\{F_{q_1}\delta_1, F_{q_2}\delta_2 ...\}$,it exists a frame $\mathcal{C}$ such that:

$\mathcal{C}(p_1) \Vdash \gamma_1$ and $\mathcal{C}(p_2) \Vdash \gamma_2$ and ... $\mathcal{C}(q_1) \nVdash \delta_1$ and $\mathcal{C}(q_2) \nVdash \delta_2$
which implicitly means, by the definition of extension:

($\mathcal{C}(p) \Vdash \gamma_1$ for all $p \geq p_1$) and ($\mathcal{C}(p) \Vdash \gamma_2$ for all $p \geq p_2$) and ... ($\mathcal{C}(p) \nVdash \delta_1$ for all $p \leq q_1$) and ($\mathcal{C}(p) \nVdash \delta_2$ for all $p \leq q_2$)...

Take the structure $\mathcal{C}(w)$ inside of $\mathcal{C}$. We can infer:

$\mathcal{C}(w) \Vdash \Gamma_1$ and $\mathcal{C}(w) \Vdash \Gamma_2$ and ... $\mathcal{C}(w) \nVdash \Delta_1$ and $\mathcal{C}(w) \nVdash \Delta_2$....

$\mathcal{C}(w)$ is thus a counterexample proving the non-validity of $\Gamma_1, \Gamma_2, ... \vdash \Delta_1, \Delta_2, ...$

5.1 Classical Translation

Theorem 4. *Given a classical signed sentence list l with $\sigma \in l$, then $\dfrac{\mathcal{T}(f(\sigma, l))}{\mathcal{T}(l)}$ is a classically valid rule.*

Proof. Here will show the implicit contradiction rule being used:
if $L - \sigma$ has sentences $\{T\Gamma_1, T\Gamma_2, ...\}$ and $\{F\Delta_1, F\Delta_2 ...\}$, then:

– $$\frac{\mathcal{T}(f(T\neg\alpha, l))}{\mathcal{T}(L)} = \frac{\mathcal{T}((L - T\neg\alpha)||T\neg\alpha||F\alpha)}{\mathcal{T}(L)} = \frac{\Gamma, \neg\alpha \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta} = \cfrac{\cfrac{\Gamma, \neg\alpha \vdash \alpha, \Delta}{\Gamma, \neg\alpha, \neg\alpha \vdash \Delta}\,\neg\mathrm{R}}{\Gamma, \neg\alpha \vdash \Delta}$$

– $$\frac{\mathcal{T}(f(F\neg\alpha, l))}{\mathcal{T}(L)} = \frac{\mathcal{T}((L - F\neg\alpha)||F\neg\alpha||T\alpha)}{\mathcal{T}(L)} = \frac{\Gamma, \neg\alpha \vdash \neg\alpha, \Delta}{\Gamma \vdash, \neg\alpha\Delta} = \cfrac{\cfrac{\Gamma, \alpha \vdash \neg\alpha, \Delta}{\Gamma \vdash \neg\alpha, \neg\alpha\Delta}\,\neg\mathrm{L}}{\Gamma, \vdash \neg\alpha\Delta}$$

– $$\frac{\mathcal{T}(f(T(\alpha \wedge \beta), l))}{\mathcal{T}(L)} = \frac{\mathcal{T}((L - T(\alpha \wedge \beta))||\alpha \wedge \beta||T\alpha||T\beta)}{\mathcal{T}(L)} = \frac{\Gamma, \alpha, \beta, \alpha \wedge \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} = \cfrac{\cfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha, \beta, \alpha \wedge \beta \vdash \Delta}\,\wedge\mathrm{L}}{\Gamma, \alpha \wedge \beta \vdash \Delta}$$

– $$\frac{\mathcal{T}(f(F(\alpha \wedge \beta), l))}{\mathcal{T}(L)} = \frac{\mathcal{T}(L - \sigma||\sigma||F\alpha) \qquad \mathcal{T}(L - \sigma||\sigma||F\beta)}{\mathcal{T}(L)} = \frac{\Gamma, \alpha, \alpha \wedge \beta \vdash \Delta \qquad \Gamma, \beta, \alpha \wedge \beta \vdash \Delta}{\Gamma, \vdash \alpha \wedge \beta, \Delta} = \cfrac{\cfrac{\Gamma, \alpha, \alpha \wedge \beta \vdash \Delta \qquad \Gamma, \beta, \alpha \wedge \beta \vdash \Delta}{\Gamma \vdash \alpha \wedge \beta, \alpha \wedge \beta, \Delta}\,\wedge\mathrm{R}}{\Gamma, \vdash \alpha \wedge \beta, \Delta}$$

- $\dfrac{\mathcal{T}(f(T(\alpha \vee \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||T\alpha) \qquad \mathcal{T}(L-\sigma||\sigma||T\beta)}{\mathcal{T}(L)}$
$= \dfrac{\Gamma, \alpha, \alpha \vee \beta \vdash \Delta \qquad \Gamma, \beta, \alpha \vee \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta} = \dfrac{\dfrac{\Gamma, \alpha \vdash \alpha \vee \beta, \Delta \qquad \Gamma, \beta, \alpha \vee \beta \vdash \Delta}{\Gamma \vdash \alpha \vee \beta, \alpha \vee \beta, \Delta}\ \vee\mathrm{L}}{\Gamma \vdash \alpha \vee \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(F(\alpha \vee \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}((L-F(\alpha \vee \beta))||F\alpha||F\beta)}{\mathcal{T}(L)}$
$= \dfrac{\Gamma \vdash \alpha \vee \beta, \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} = \dfrac{\dfrac{\Gamma \vdash \alpha \vee \beta, \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \alpha \vee \beta, \alpha, \beta, \Delta}\ \wedge\mathrm{R}}{\Gamma \vdash \alpha \vee \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(T(\alpha \to \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\alpha) \qquad \mathcal{T}(L-\sigma||\sigma||T\beta)}{\mathcal{T}(L)}$
$= \dfrac{\Gamma, \alpha, \alpha \to \beta \vdash \Delta \qquad \Gamma, \alpha \to \beta \vdash \beta, \Delta}{\Gamma, \alpha \to \beta \vdash \Delta} = \dfrac{\dfrac{\Gamma, \alpha, \alpha \to \beta \vdash \Delta \qquad \Gamma, \alpha \to \beta \vdash \beta, \Delta}{\Gamma, \alpha \to \beta, \alpha \to \beta \vdash \Delta}\ \to\mathrm{L}}{\Gamma, \alpha \to \beta \vdash \Delta}$
- $\dfrac{\mathcal{T}(f(F(\alpha \to \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}((L-F(\alpha \to \beta))||F(\alpha \to \beta)||T\alpha||F\beta)}{\mathcal{T}(L)}$
$= \dfrac{\Gamma, \alpha \vdash \beta, \alpha \to \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta} = \dfrac{\dfrac{\Gamma, \alpha \vdash \beta, \alpha \to \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \alpha \to \beta, \Delta}\ \to\mathrm{R}}{\Gamma \vdash \alpha \to \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(T(\forall x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||T\phi(c_i))}{\mathcal{T}(L)}$
$= \dfrac{\Gamma, \forall x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \forall x\, \phi(x) \vdash \Delta} = \dfrac{\dfrac{\Gamma, \forall x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \forall x\, \phi(x), \forall x\, \phi(x) \vdash \Delta}\ \forall\mathrm{L}}{\Gamma, \forall x\, \phi(x) \vdash \Delta}$
and c_i does not occour in Γ or Δ, as it is a new one.
- $\dfrac{\mathcal{T}(f(F(\forall x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\phi(c_i))}{\mathcal{T}(L)}$
$= \dfrac{\Gamma \vdash \phi(c_i), \forall x\, \phi(x), \Delta}{\Gamma \vdash \forall x\, \phi(x), \Delta} = \dfrac{\dfrac{\Gamma \vdash \phi(c_i), \forall x\, \phi(x), \Delta}{\Gamma \vdash \forall x\, \phi(x), \forall x\, \phi(x), \Delta}\ \forall\mathrm{R}}{\Gamma \vdash \forall x\, \phi(x), \Delta}$
- $\dfrac{\mathcal{T}(f(T(\exists x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||T\phi(c_i))}{\mathcal{T}(L)}$
$= \dfrac{\Gamma, \exists x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \exists x\, \phi(x) \vdash \Delta} = \dfrac{\dfrac{\Gamma, \exists x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \exists x\, \phi(x), \exists x\, \phi(x) \vdash \Delta}\ \exists\mathrm{L}}{\Gamma, \exists x\, \phi(x) \vdash \Delta}$
and c_i does not occour in Γ or Δ, as it is a new one.
- $\dfrac{\mathcal{T}(f(F(\exists x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\phi(c_i))}{\mathcal{T}(L)}$
$= \dfrac{\Gamma \vdash \phi(c_i), \exists x\, \phi(x), \Delta}{\Gamma \vdash \exists x\, \phi(x), \Delta} = \dfrac{\dfrac{\Gamma \vdash \phi(c_i), \exists x\, \phi(x), \Delta}{\Gamma \vdash \exists x\, \phi(x), \exists x\, \phi(x), \Delta}\ \exists\mathrm{R}}{\Gamma \vdash \exists x\, \phi(x), \Delta}$

Definition 17. ***Tree Translating Function*** *Given a tableaux proof tree development and its root r τ, we define $\mathcal{T}_p(\tau)$:*

- *if $\tau = r$ and there are $T\sigma$ and $F\sigma$ on r:*
$\mathcal{T}_p(\tau) = \dfrac{}{\mathcal{T}(r)}\ Ax\ \sigma$

- *if $\tau = r$ and there is no σ such that $T\sigma$ and $F\sigma$ are on r:*
  $\mathcal{T}_p(\tau) = \mathcal{T}(\tau)$
- *if r has a single child r_0 with a corresponding subtree τ_0:*
  by definition, it exits a sentence σ such that $f(\sigma, r) = f_{Rule}(\sigma, r) = r_0$, and so
  $\mathcal{T}_p(\tau) = \dfrac{\mathcal{T}_p(\tau_0)}{\mathcal{T}(r)}$ *Rule on σ.*
- *if r has two children r_1 and r_2 with corresponding subtrees τ_1 and τ_2:*
  by definition there exists a sentence σ such that $f(\sigma, r) = f_{Rule}(\sigma, r) = \{r_1, r_2\}$, and so:
  $\mathcal{T}_p(\tau) = \dfrac{\mathcal{T}_p(\tau_1) \qquad \mathcal{T}_p(\tau_2)}{\mathcal{T}(r)}$ *Rule on σ*

Theorem 5. *given a classical tableaux development τ, $\mathcal{T}_p(\tau)$ is a valid sequent proof*

Proof. The proof goes by induction on the size of τ:

- if $\tau =$ r then there are $T\sigma$ and $F\sigma$ on r, $\mathcal{T}_p(\tau) = \dfrac{}{\mathcal{T}(r)}$ Ax σ is a valid sequent proof.
- if r has a single child r_0 with a corresponding subtree τ_0: $\mathcal{T}_p(\tau) = \dfrac{\mathcal{T}_p(\tau_0)}{\mathcal{T}(r)}$ Rule on σ is a valid sequent proof since $\mathcal{T}_p(\tau_0)$ is valid by induction hypothesis and the rule is valid by the last theorem.
- if r has two children r_1 and r_2 with corresponding subtrees τ_1 and τ_2: $\mathcal{T}_p(\tau) = \dfrac{\mathcal{T}_p(\tau_1) \qquad \mathcal{T}_p(\tau_2)}{\mathcal{T}(r)}$ Rule on σ is a valid sequent proof since $\mathcal{T}_p(\tau_1)$ and $\mathcal{T}_p(\tau_2)$ are valid by induction hypothesis and the rule is valid by the last theorem.

5.2 Intuitionistic Translation

When looking at the tableaux proof of $\neg(A \wedge \neg A) \wedge (\neg X \wedge \neg Y)$ and trying to chose proper w's for Translating like we did for classical logic we find a problem with the $\neg R$ rule: "useless" terms from our acumulative tableaux method can "overcrowd" the right side of the sequent equation. One naive solution would be to remove all useless terms: (Another way of seeing is that sequent calculus has a Weakening rule that we are not using yet)

Definition 18. ***Tree Thinning Function*** *Given a tableaux proof tree development and its root r, we define $\mathcal{F}(\tau)$:*

- *If $\tau = r$ and there are $T_f\sigma$ and $F_f\sigma$ on r:*
  $\mathcal{F}(\tau) = [T_f\sigma, F_f\sigma]$
- *If r has a single child $f(\sigma, r)$:*
  with a corresponding subtree τ_0 and r_0' is the root of $\mathcal{F}(\tau_0)$:
  $(f(\sigma, r) - r) \cap r_0'$ are the elements generated by the inference f that are necessary to prove the non-existance of a frame that respects $f(\sigma, r)$. In this case:

  - *If* $f(\sigma, r) - r) \cap r'_0 \neq \emptyset$
    $\mathcal{F}(\tau) =$ *a tree with root* $(r'_0 - f(\sigma,\sigma))||\sigma$ *connected to the child* $r'_0||(f(\sigma,\sigma) - \sigma)$*, that has as a child the subtree* $\mathcal{F}(\tau_0)$
  - *If* $f(\sigma, r) - r) \cap r'_0 = \emptyset$
    $\mathcal{F}(\tau) = \mathcal{F}(\tau_0)$
- *if* r *has the children* $f(\sigma, r)[1]$ *and* $f(\sigma, r)[2]$ *:*
  with the corresponding subtrees τ_1 *and* τ_2*,* r'_1 *is the root of* $\mathcal{F}(\tau_1)$ *and* r'_2 *is the root of* $\mathcal{F}(\tau_2)$*, then* $(f(\sigma, r) - r) \cap (r'_1 \cup r'_2))$ *also defines if* f *was "usefull":*
  - *If* $(f(\sigma, r) - r) \cap (r'_1 \cup r'_2) \neq \emptyset$*:*
    $\mathcal{F}(\tau) =$ *a tree with root* $r'_0||\sigma$ *connected to the children* $r'_1||(f(\sigma,\sigma)[1] - r)$ $r'_2||(f(\sigma,\sigma)[2] - r)$*, each having* $\mathcal{F}(\tau_1)$ *and* $\mathcal{F}(\tau_2)$*, respectivelly, bellow them.*
  - *If* $(f(\sigma, r) - r) \cap (\mathcal{F}(r'_1) \cup \mathcal{F}(r'_2)) = \emptyset$*:*
    $\mathcal{F}(\tau) = \mathcal{F}(\tau_1) = \mathcal{F}(\tau_1)$

Theorem 6. *(proof/search for a counter ex) If* τ *does not branch, the root* r' *of* $\mathcal{F}(\tau)$ *has not more than 2 signed sentences* F

Proof. The proof goes by induction on the size of τ:

- If $\tau =$ r, $\mathcal{F}(\tau) = [T_f\sigma, F_f\sigma]$. The theorem holds
- If r has a single child $f(\sigma, r)$:
  with a corresponding subtree τ_0 and r'_0 is the root of $\mathcal{F}(\tau_0)$:
  - If $f(\sigma, r) - r) \cap r'_0 \neq \emptyset$:
    the root of $\mathcal{F}(\tau)$ is $(r'_0 - f(\sigma,\sigma))||\sigma$ r'_0 has at most one F, so the only way for $(r'_0 - f(\sigma,\sigma))||\sigma$ to have more than 1 signed sentence of type is :
    if the other signed sentence in r' is of type F (we denote F_qB)
    and $f = f_{F\neg}(or f_{F\rightarrow})$ that is,
    $\sigma = F_p\neg\alpha$. This would mean that $f(\sigma,\sigma) = T_{p'}\alpha$ for a new p'. By definition, p' can not be $\leq q$
  - If $f(\sigma, r) - r) \cap r'_0 = \emptyset$
    $\mathcal{F}(\tau) = \mathcal{F}(\tau_0)$

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