Translating from Tableaux Proofs into Sequent Proofs in Classical and Intuitionistic Logic

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Abstract. This work aims to discuss the translation between Tableaux proofs and sequent calculus proofs in both first-order intuitionistic predicate logic and in first-order classical predicate logic. It begins with an overview of the definitions in both intuitionistic and classical logic. It then shows a translation process in classical logic and its OCAML implementation restricted to propositional logic. Finally, a potential extension towards translation in intuitionistic logic is discussed.

Keywords: Tableaux proof · sequent calculus · intuitionistic logic.

1 Introduction

1.1 Notation

In this work, sentences will implicitly refer to first-order predicate logic sentences; for intuitionistic logic, their meaning will come from Kripke's semantics [3] . The notation for structures and frames will be heavily based on [1]. In order to make this document slightly more self-reliant and to clarify the notation, and we will briefly explain:

1.2 Definitions

Definition 1. ***A Structure of a Language*** *consists of a domain and:*

- *An assignment from the constant symbols of the language to the domain.*
- *An assignment from the predicate symbols of the language to predicates in the domain.*

Structures represent possible worlds or possible states of knowlege inside a frame:

Definition 2. ***A Kripke Frame*** *of a Language $\mathcal{L}$, $\mathcal{C} = (R, \{C(p)\}_{p\in R})$ consists of a partially ordered set R, and an $\mathcal{L}$-structure $C(p)$ for all p's in R. Furthermore, in a Kripke Frame, if $p \leq q$, then $C(q)$ extends $C(p)$:*

- *All sentences that are true in $C(p)$ are true in $C(q)$.*
- *The domain of $C(p)$ is included in the domain of $C(q)$.*
- *The assignments in $C(p)$ are the same as in $C(q)$ for the domain in common.*

Particularlly, in order to simplify the notation, R will always be the set of sequences of integers, and $p \leq q$ if it exists an l such that $q = p||l$. Also, Also, from now on, the constant elements of a language will allways be in the ordered set $\{c_0, c_1, c_2...\}$

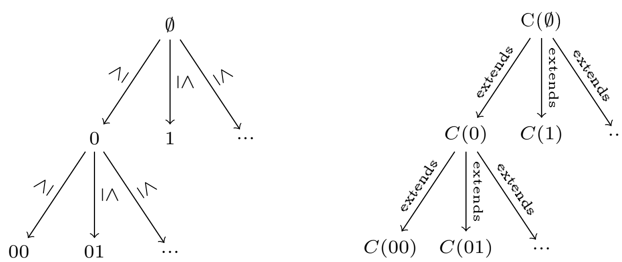

Fig. 1. R and a Kripke frame

Definition 3. ***Forcing***. *When a sentence ϕ of a language $\mathcal{L}$ is* ***forced*** *by a structure $C(p)$ of a frame $\mathcal{C}$, we denote: $p \Vdash_{\mathcal{C}} \phi$*
Forcing is defined by induction: [1]

- $p \Vdash_{\mathcal{C}} \phi \Leftrightarrow \phi$ *is true in $C(p)$ (if ϕ is an atomic sentence).*
- $p \Vdash_{\mathcal{C}} (\phi \rightarrow \psi) \Leftrightarrow$ *for all $q \geq p$, if $q \Vdash_{\mathcal{C}} \phi$, then $q \Vdash_{\mathcal{C}} \psi$.*
- $p \Vdash_{\mathcal{C}} \neg\phi \Leftrightarrow$ *for all $q \geq p$, q does not force ϕ.*
- $p \Vdash_{\mathcal{C}} (\forall x)\phi(x) \Leftrightarrow$ *for all $q \geq p$ and d in $\mathcal{L}_{C(q)}$, $q \Vdash_{\mathcal{C}} \phi(d)$.*
- $p \Vdash_{\mathcal{C}} (\exists x)\phi(x) \Leftrightarrow$ *there exists a d in $\mathcal{L}_{C(q)}$, such that $p \Vdash_{\mathcal{C}} \phi(d)$.*
- $p \Vdash_{\mathcal{C}} (\phi \wedge \psi) \Leftrightarrow p \Vdash_{\mathcal{C}} \phi$ *and* $p \Vdash_{\mathcal{C}} \psi$.
- $p \Vdash_{\mathcal{C}} (\phi \vee \psi) \Leftrightarrow p \Vdash_{\mathcal{C}} \phi$ *or* $p \Vdash_{\mathcal{C}} \psi$.

Definition 4. ***Intuitionistic Validity***. *A sentence of a language $\mathcal{L}$ is Intuitionistically valid if it is forced in all structures of all Kripke frames of $\mathcal{L}$.*

In classical logic, this definition simplifies to the one of forcing, and it's simplified again by the fact that p = q; in fact, here we will define classical validity as: [1]

Definition 5. ***Classical Validity***. *A sentence of a language $\mathcal{L}$ is classically valid if it is forced by all single-structure Kripke frames of that sentence's language.*

2 The intuitionistic tableaux method

Considerations Here we first define a slightly different version of the destructive tableaux proof tree described by [1], where each node is a truth assertion. Thi different version will allow for a more implementation-oriented approach and the translation later on.

The correspondence of the destructive tableaux proof tree described in [1] to our new one is shown in Figure 2. Generally speaking, a node in the usual definition is replaced by a sequence of all nodes in the path that goes from the root to it. Afterwards, some nodes are removed from the newly formed tableaux by adjoining its son(s) and its parent. A node should be removed if its corresponding node in the original tableaux was not a leaf of the atomic tableaux [1] that introduced it.

$$\begin{tikzcd}
F_\emptyset \alpha \to \neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha \arrow[dd] \\
T_0 \alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha \arrow[d] & \\
F_0 \neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha, F_0 \neg\neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha, F_0 \neg\neg\alpha \arrow[d] \\
T_{00} \neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha, F_0 \neg\neg\alpha, T_{00} \neg\alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha, F_0 \neg\neg\alpha, T_{00} \neg\alpha \arrow[d] \\
F_{00} \alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha, F_0 \neg\neg\alpha, T_{00} \neg\alpha, F_{00} \alpha \arrow[d] & F_\emptyset \alpha \to \neg\neg\alpha, T_0 \alpha, F_0 \neg\neg\alpha, T_{00} \neg\alpha, F_{00} \alpha \arrow[d] \\
\mathrm{X} & \mathrm{X} & \mathrm{X}
\end{tikzcd}$$

Fig. 2. Example of a destructive tableaux proof tree from [1], the intermediary structure and the non-destructive tableaux proof tree.

In this new presentation, one can see each node of the tree as an assumption of the existence of a frame that respects a list of constraints, and each edge as an implication between assumptions.

The Intuitionistic tableaux method The tableaux stands on some definitions, they will be justified briefly:

Definition 6. ***A Signed sentence*** *is a forcing assertion inside of a tableaux proof. It looks like* $T_q\phi$ *or* $F_p\phi$
A Signed sentence list *is a list forcing assertions inside of a tableau proof.*

We say that a list of forcing assertions having sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ *and* $\{F_{q_1}\delta_1, F_{q_2}\delta_2...\}$ *is "intuitionistically valid" (Question 2: maybe use a term != valid?)if there exists a frame* $\mathcal{C}$ *for which* $\mathcal{C}(p_1) \Vdash \gamma_1$ *and* $\mathcal{C}(p_2) \Vdash \gamma_2$ *and ...* $\mathcal{C}(q_1) \nVdash \delta_1$ *and* $\mathcal{C}(q_2) \nVdash \delta_2$

A Signed sentence list can be seen as an existance assumption that may or may not be intuitionistically valid. We can infer other assumptions that are, by definition, consequences of a given assumption. The function f, defined bellow, is one of the ways we can do that:

Definition 7. *The function* f *takes a signed sentence* σ *and a signed sentence list L and returns one or two signed sentence lists.*
$f(\sigma, L)$ *is defined as follows:*
(here we denote $l||l' = l_1, l_2..., l_{|l|}, l'_1, l'_2.., l'_{|l'|}$*)*
if $\sigma \in L$*:*

- $f(T_p\neg\alpha, L) = [L - \sigma||\sigma||F_{p'}\alpha]$
  for a $p' \geq p$ *not in* $h||L$.
- $f(F_p\neg\alpha, L) = [L - \sigma||\sigma||T\alpha]$
  for the first $p' \geq p$ *for such that* $F\neg\alpha$ *is not in* L.
- $f(T_p(\alpha \wedge \beta), L) = [L - \sigma||\sigma||T_p\alpha||T_p\beta]$.
- $f(F_p(\alpha \wedge \beta), L) = [L - \sigma||\sigma||F_p\alpha], [L||h||F_p\beta]$.
- $f(T_p(\alpha \vee \beta), L) = [L - \sigma||\sigma||T_p\alpha], [L||h||T_p\beta]$.
- $f(F_p(\alpha \vee \beta), L) = [L - \sigma||\sigma||F_p\alpha||F_p\beta]$.
- $f(T_p(\alpha \to \beta), L) = [L - \sigma||\sigma||F_{p'}\alpha], [L||h||T_{p'}\beta]$
  for the first $p' \geq p$ *such that* $F\neg\alpha$ *is not in* L.
- $f(F_p(\alpha \to \beta), L) = [L - \sigma||\sigma||T_{p'}\alpha||F_{p'}\beta]$
  for a $p' \geq p$ *not in* $h||L$.
- $f(T_p(\forall x)\phi(x), L) = [L - \sigma||\sigma||T_p\phi(c_i)]$
  for the first constant c_i *for which* $T_p\phi(c_i)$ *is not in* L.
- $f(F_p(\forall x)\phi(x), L) = [L - \sigma||\sigma||F_p\phi(c_i)]$
  for the first constant c_i *not present in* $h||L$.
- $f(T_p(\exists x)\phi(x), L) = [L - \sigma||\sigma||T_p\phi(c_i)]$
  for the first constant c_i *not present in* $h||L$.
- $f(F_p(\exists x)\phi(x), L) = [L - \sigma||\sigma||F_p\phi(c_i)]$
  for the first constant c_i *such that* $F_p\phi(c_i)$ *is not in* L.

[TODO revise]
if $\sigma \notin L$*:*

- $f(\sigma, L) = L$

Allthough not justified here, the reordering of the terms plays an important role in the implementation and in the completnees of the sistematic tableaux. On the other hand, the soundness of the tableaux method depends on the fact that f is "well behaved":

Theorem 1. *Given a signed sentence list x, if x is intuitionistically valid then f(x) is intuitionistically valid*

Proof. [1] proves this by using the definition of forcing. The choices of p' and c_i garantes the completnees of the tableaux. [TODO]

Definition 8. ***The Tableux development of a sentence*** *is defined inductivelly:*

- *A tree with the single node* $\{F\phi\}$ *is a tableaux development of* ϕ.
- *If* τ *is a tableaux development of* ϕ, *then* $\hookleftarrow(\sigma,\tau)$ *is a tableaux development of* ϕ. *Where:*

  $\hookleftarrow(\sigma,\tau) = \tau$ *with* $f(\sigma,l)$*added to all leaves* l *that contain* σ

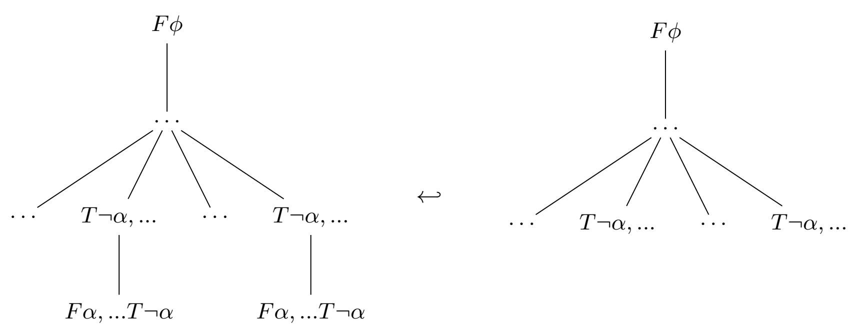

Fig. 3. Example of $\hookleftarrow(T\neg\alpha,\tau)$ and τ

Theorem 2. *If* $F\phi$ *is intuitionistically valid then one of the leaves of* $\hookleftarrow(\sigma_n,\hookleftarrow(\sigma_{n-1}(.....(\hookleftarrow(\sigma_1,)...))))$ *is intuitionistically valid.*

Proof. The proof goes by induction:

The base case is true by definition: either the root $F\phi$ is a intuitionistically valid leaf or the premisse is false. Next we assume $\tau' = \hookleftarrow(\sigma,\tau)$. There are two cases to consider:

- If there does not exist a frame that does not force ϕ, $F\phi$ is not intuitionistically valid and the theorem holds for τ'.
- If there exists a frame that does not force ϕ:
  By the induction hypothesis, there exists a intuitionistically valid leaf σ in τ.
  - If σ is a leaf in τ, then $F\phi$ the theorem holds for τ'.
  - If σ is not a leaf in τ, then: One or two nodes were added to σ in τ' by the definition of $\hookleftarrow$. By the theorem 1, one of the added nodes is also intuitionistically valid. Consequently, the theorem holds for τ'.

From this, we can conlude that: If all leaves of $\hookleftarrow(\sigma_n,\hookleftarrow(\sigma_{n-1}(.....(\hookleftarrow(\sigma_1,F\phi)...))))$ are contradictory, then $F\phi$ is not intuitionistically valid, by consequence ϕ is intuitionistically valid.

completnees For purposes of implementation, we define an $\hookleftarrow_c$ that can be applied systematically:

Definition 9. *$\hookleftarrow_c (\tau) = \tau$ with $f(h, l)$ added to all leaves l that contain σ. h is the first signed sentence of the shalowest non-contradictory leaf.*

we can use $\hookleftarrow_c$ instead of $\hookleftarrow$ to define a sistematic tableaux

[TODO]

3 The classical tableaux method

The classical tableaux method can be seen as a the intuitionistic tableaux method restricted to one-framed structures. The adapted definitions are:

Definition 10. ***A Signed sentence*** *is a forcing assertion inside of a tableaux proof. It looks like $T\phi$ or $F\phi$*
A Signed sentence list *is a list forcing assertions inside of a tableau proof. We say that a list of forcing assertions having sentences $\{T\gamma_1, T\gamma_2, ...\}$ and $\{F\delta_1, F\delta_2 ...\}$ is "classically valid" (Question 2: maybe use a term != valid?)if there exists a single-structured frame $\mathcal{C}$ such that $\mathcal{C}(\emptyset) \Vdash \gamma_1$ and $\mathcal{C}(\emptyset) \Vdash \gamma_2$ and ... $\mathcal{C}(\emptyset) \nVdash \delta_1$ and $\mathcal{C}(\emptyset) \nVdash \delta_2$*

Definition 11. *The function f takes a signed sentence σ and a signed sentence list L and returns one or two signed sentence lists.*
$f(\sigma, L)$ is defined as follows:
(here we denote $l||l' = l_1, l_2 ..., l_{|l|}, l'_1, l'_2 .., l'_{|l'|}$)
if $\sigma \in L$:

- $f(T\neg\alpha, L) = [L - \sigma||\sigma||F\alpha]$.
- $f(F\neg\alpha, L) = [L - \sigma||\sigma||T\alpha]$.
- $f(T(\alpha \wedge \beta), L) = [L - \sigma||\sigma||T\alpha||T\beta]$.
- $f(F(\alpha \wedge \beta), L) = [L - \sigma||\sigma||F\alpha], [L||\sigma||F\beta]$.
- $f(T(\alpha \vee \beta), L) = [L - \sigma||\sigma||T\alpha], [L||\sigma||T\beta]$.
- $f(F(\alpha \vee \beta), L) = [L - \sigma||\sigma||F\alpha||F\beta]$.
- $f(T(\alpha \to \beta), L) = [L - \sigma||\sigma||F\alpha], [L||\sigma||T\beta]$.
- $f(F(\alpha \to \beta), L) = [L - \sigma||\sigma||T\alpha||F\beta]$.
- $f(T(\forall x)\phi(x), L) = [L - \sigma||\sigma||T\phi(c_i)]$
  for the first constant c_i for which $T\phi(c_i)$ is not in L.
- $f(F(\forall x)\phi(x), L) = [L - \sigma||\sigma||F\phi(c_i)]$
  for the first constant c_i not present in L.
- $f(T(\exists x)\phi(x), L) = [L - \sigma||\sigma||T\phi(c_i)]$
  for the first constant c_i not present in L.
- $f(F(\exists x)\phi(x), L) = [L - \sigma||\sigma||F\phi(c_i)]$
  for the first constant c_i for which $F\phi(c_i)$ is not in L.

[TODO revise]
if $\sigma \notin L$:

- $f(\sigma, L) = L$

4 Sequent calculus

Here we use the multi-conclusion sequent calculus defined by [2].

Definition 12. ***Sequent*** *A sequent is an expression of the form*

$$\Gamma \vdash \Delta$$

where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, ...\}$ *and* $\Delta = \{\Delta_1, \Delta_2, \Delta_3 ...\}$ *are finite sets of formulas.* Γ *is called the antecedent, and* Δ *is called the succedent.*

Γ represents multiple necessary hipotesis, while Δ represents multiple possible conclusions.

Definition 13. *A sequent is intuitionistically valid if for all kripke frames, if* Γ *are forced, then at least one of* Δ *is forced.*

The more akward definition:

" A sequent is classically valid if , there does not exist a single-sentenced frame that forces all sentences in Γ and does not force any sentence in Δ. "

when compared with the definition of list of signed sentences, hints us towards a semantical equivalence for single framed structures, wich will not be discussed in depth.

The meaning of a sequent calculus rule is [Question : is "meaning" too strong? -> maybe change to "a sequent calculus rule can be interpreted as"] the implication: if the top sequent is valid, then the botton sequent is valid.

Sintatically [question sintaxically ?], each of the rules represent veriticies that can be added to a sequent proof tree development to obtain another sequent proof tree development

4.1 Classical Sequent calculus

Definition 14. *A single sequent is a classical sequent proof tree development*
Any of the rules of the table 1 aplied to a classical sequent proof tree development is a sequent proof tree development
A classical squent proof tree is a classical sequent proof tree development with axioms on all leafs.

Rules for classical multi-consequence sequent calculus are given in the table **??**

4.2 Intuitionistic Sequent calculus

Definition 15. *A single sequent is a intuitionistic sequent proof tree development*
Any of the rules of the table 1 aplied to a intuitionistic sequent proof tree development is a sequent proof tree development
A intuitionistic squent proof tree is a intuitionistic sequent proof tree development with axioms on all leafs.

Rules for L's intuitionistic multi-consequence sequent calculus are given in 3

Table 1. Cumulative Sequent Calculus Rules (Multi-Consequence)

Rule	**Inference**
Axiom	$\dfrac{}{\Gamma, \phi \vdash \phi, \Delta}$
Weakening	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \phi \vdash \Delta} \quad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \phi}$
Negation	$\dfrac{\Gamma, \neg\phi \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\,\neg\mathrm{R} \quad \dfrac{\Gamma, \phi \vdash \Delta, \neg\phi}{\Gamma \vdash \Delta, \neg\phi}\,\neg\mathrm{L}$
Conjunction	$\dfrac{\Gamma, \phi, \psi, \phi \wedge \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}\,\wedge\mathrm{L} \quad \dfrac{\Gamma \vdash \phi, \phi \wedge \psi, \Delta \qquad \Gamma \vdash \psi, \phi \wedge \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}\,\wedge\mathrm{R}$
Disjunction	$\dfrac{\Gamma, \phi \vee \psi, \phi \vdash \Delta \qquad \Gamma, \phi \vee \psi, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}\,\vee\mathrm{L} \quad \dfrac{\Gamma \vdash \phi \vee \psi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}\,\vee\mathrm{R}$
Implication	$\dfrac{\Gamma, \phi \to \psi \vdash \phi, \Delta \qquad \Gamma, \phi \to \psi, \psi \vdash \Delta}{\Gamma, \phi \to \psi \vdash \Delta}\,\to\mathrm{L} \quad \dfrac{\Gamma, \phi \vdash \psi, \phi \to \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta}\,\to\mathrm{R}$
Quantifiers	$\dfrac{\Gamma, \forall x\phi(x), \phi(t) \vdash \Delta}{\Gamma, \forall x\phi(x) \vdash \Delta}\,\forall\mathrm{L} \quad \dfrac{\Gamma \vdash \phi(y), \Delta}{\Gamma \vdash \forall x\phi(x), \Delta}\,\forall\mathrm{R}$ $\dfrac{\Gamma, \phi(y) \vdash \Delta}{\Gamma, \exists x\phi(x) \vdash \Delta}\,\exists\mathrm{L} \quad \dfrac{\Gamma \vdash \phi(t), \Delta}{\Gamma \vdash \exists x\phi(x), \Delta}\,\exists\mathrm{R}$

5 Translation

Definition 16. ***node translating function*** $\mathcal{T}$ *Given a signed sentence list L with sentences* $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ *and* $\{F_{q_1}\delta i, F_{q_2}\delta i+1...\}$ *and a* $w \in \{p_1, p_2, ...\} \cup \{q_1, q_2, ...\}$ *then:*
$\mathcal{T}(L, w) = \Gamma_1, \Gamma_2, ... \vdash \Delta_1, \Delta_2, ...,$ *were:*
$\{T_{q_1}\Gamma_1, T_{q_2}\Gamma_2, ...\}$ *are the elements of* $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ *such that* $p \geq w$ *and* $\{F_{q_1}\Delta_1, F_{q_2}\Delta_2, ...\}$ *are the elements of* $\{F_{q_1}\delta_1, F_{q_2}\delta_2, ...\}$ *such that* $q \leq w$

Theorem 3. *Given signed sentence list L and a* w*, if L is intuitionistically valid then the sequent* $\mathcal{T}(L, w)$ *is not intuitionistically valid.*

Proof. If L has the sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ and $\{F_{q_1}\delta i, F_{q_2}\delta i+1...\}$, that signifies an assumption of existance of a frame $\mathcal{C}$ such that:
$\mathcal{C}(p_1) \Vdash \gamma_1$ and $\mathcal{C}(p_2) \Vdash \gamma_2$ and ... $\mathcal{C}(q_1) \nvDash \delta_1$ and $\mathcal{C}(q_2) \nvDash \delta_2$
which implicitly means, by the definition of extension:
($\mathcal{C}(p) \Vdash \gamma_1$ for all $p \geq p_1$) and ($\mathcal{C}(p) \Vdash \gamma_2$ for all $p \geq p_2$) and ... ($\mathcal{C}(p) \nvDash \delta_1$ for all $p \leq q_1$) and ($\mathcal{C}(p) \nvDash \delta_2$ for all $p \leq q_2$)...

Table 2. Rules for L's classical multi-consequence sequent calculus

Rule	**Inference**
Axiom	$\dfrac{}{\alpha \vdash \alpha}$
Weakening	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$ $\quad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$
Negation	$\dfrac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\text{R}\ \neg\alpha$ $\quad \dfrac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta}\ \neg\text{L}\ \neg\alpha$
Conjunction	$\dfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\text{L}\ \alpha \wedge \beta$ $\quad \dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\text{R}\ \alpha \wedge \beta$
Disjunction	$\dfrac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}\ \vee\ \text{L}\ \alpha \vee \beta$ $\quad \dfrac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\ \text{R}\ \alpha \vee \beta$ $\quad \dfrac{\Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\ \text{R}\ \alpha \vee \beta$
Implication	$\dfrac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta} \to \text{L}\ \alpha \to \beta$ $\quad \dfrac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta} \to \text{R}\ \alpha \to \beta$
Quantifiers	$\dfrac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x\alpha(x) \vdash \Delta}\ \forall\ \text{L}\ \forall x\alpha(x)$ $\quad \dfrac{\Gamma \vdash \alpha(y), \Delta}{\Gamma \vdash \forall x\alpha(x), \Delta}\ \forall\ \text{R}\ \forall x\alpha(x)$ $\dfrac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x\alpha(x) \vdash \Delta}\ \exists\ \text{L}\ \exists x\alpha(x)$ $\quad \dfrac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x\alpha(x), \Delta}\ \exists\ \text{R}\ \exists x\alpha(x)$
Contraction	$\dfrac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$ $\quad \dfrac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$

Take the structure $\mathcal{C}(w)$ inside of $\mathcal{C}$. We can infer:
$\mathcal{C}(w) \Vdash \Gamma_1$ and $\mathcal{C}(w) \Vdash \Gamma_2$ and ... $\mathcal{C}(w) \nvDash \Delta_1$ and $\mathcal{C}(w) \nvDash \Delta_2$....
$\mathcal{C}(w)$ is a counterexample proving the non-validity of $\Gamma_1, \Gamma_2, ... \vdash \Delta_1, \Delta_2, ...$

5.1 Classical Translation

Theorem 4. *L is valid if and only if* $\mathcal{T}(L, \emptyset)$ *is not valid*

Proof. It goes directly from the definiton:

- If the signed sentence list is valid, then the sequent is invalid by corolary of [TODO]
- If the signed sentence list is not valid, it does not exist a counterexample for $\mathcal{T}$(L). (Note that we can not say the same for the intuitionistic case)

Table 3. Rules for L's intuitionistic multi-consequence sequent calculus

Rule	Inference
Axiom	$\frac{}{\phi \vdash \phi}$
Weakening	$\frac{\Gamma \vdash \Delta}{\Gamma, \phi \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \phi}$
Negation	$\frac{\Gamma, \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\ \neg\text{R}\ \neg\phi \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}\ \neg\text{L}\ \neg\phi$
Conjunction	$\frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}\ \wedge\text{L}\ \phi \wedge \psi \qquad \frac{\Gamma \vdash \phi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta}\ \wedge\text{R}\ \phi \wedge \psi$
Disjunction	$\frac{\Gamma, \phi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}\ \vee\ \text{L}\ \phi \vee \psi \qquad \frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}\ \vee\ \text{R}\ \phi \vee \psi \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}\ \vee\ \text{R}\ \phi \vee \psi$
Implication	$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \to \psi \vdash \Delta} \to \text{L}\ \phi \to \psi \qquad \frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta} \to \text{R}\ \phi \to \psi$
Quantifiers	$\frac{\Gamma, \phi(t) \vdash \Delta}{\Gamma, \forall x \phi(x) \vdash \Delta}\ \forall\ \text{L}\ \forall x \phi(x) \qquad \frac{\Gamma \vdash \phi(y), \Delta}{\Gamma \vdash \forall x \phi(x), \Delta}\ \forall\ \text{R}\ \forall x \phi(x)$ $\frac{\Gamma, \phi(y) \vdash \Delta}{\Gamma, \exists x \phi(x) \vdash \Delta}\ \exists\ \text{L}\ \exists x \phi(x) \qquad \frac{\Gamma \vdash \phi(t), \Delta}{\Gamma \vdash \exists x \phi(x), \Delta}\ \exists\ \text{R}\ \exists x \phi(x)$

Theorem 5. *Given a signed sentence list l with $\sigma \in l$, then $\frac{\mathcal{T}(f(\sigma, l))}{\mathcal{T}(l)}$ rule is a valid rule.*

Proof. Here will show the implicit quantifier rule being used:
if $L - \sigma$ has sentences $\{T\Gamma_1, T\Gamma_2, ...\}$ and $\{F\Delta_1, F\Delta_2...\}$, then:

- $\frac{\mathcal{T}(f(T\neg\alpha, l))}{\mathcal{T}(L)} = \frac{\mathcal{T}((L - T\neg\alpha) \| T\neg\alpha \| F\alpha)}{\mathcal{T}(L)} = \frac{\Gamma, \neg\alpha \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta} = \frac{\dfrac{\Gamma, \neg\alpha \vdash \alpha, \Delta}{\Gamma, \neg\alpha, \neg\alpha \vdash \Delta}\ \neg\text{R}}{\Gamma, \neg\alpha \vdash \Delta}$
- $\frac{\mathcal{T}(f(F\neg\alpha, l))}{\mathcal{T}(L)} = \frac{\mathcal{T}((L - F\neg\alpha) \| F\neg\alpha \| T\alpha)}{\mathcal{T}(L)} = \frac{\Gamma, \neg\alpha \vdash \neg\alpha, \Delta}{\Gamma \vdash, \neg\alpha\Delta} = \frac{\dfrac{\Gamma, \alpha \vdash \neg\alpha, \Delta}{\Gamma \vdash \neg\alpha, \neg\alpha\Delta}\ \neg\text{L}}{\Gamma, \vdash \neg\alpha\Delta}$
- $\frac{\mathcal{T}(f(T(\alpha \wedge \beta), l))}{\mathcal{T}(L)} = \frac{\mathcal{T}((L - T(\alpha \wedge \beta)) \| \alpha \wedge \beta \| T\alpha \| T\beta)}{\mathcal{T}(L)} = \frac{\Gamma, \alpha, \beta, \alpha \wedge \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}$
$= \frac{\dfrac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha, \beta, \alpha \wedge \beta \vdash \Delta}\ \wedge\text{L}}{\Gamma, \alpha \wedge \beta \vdash \Delta}$

- $\dfrac{\mathcal{T}(f(F(\alpha \wedge \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\alpha) \qquad \mathcal{T}(L-\sigma||\sigma||F\beta)}{\mathcal{T}(L)} = \dfrac{\Gamma, \alpha, \alpha \wedge \beta \vdash \Delta \qquad \Gamma, \beta, \alpha \wedge \beta \vdash \Delta}{\Gamma, \vdash \alpha \wedge \beta, \Delta}$
  $= \dfrac{\dfrac{\Gamma, \alpha, \alpha \wedge \beta \vdash \Delta \qquad \Gamma, \beta, \alpha \wedge \beta \vdash \Delta}{\Gamma \vdash \alpha \wedge \beta, \alpha \wedge \beta, \Delta} \wedge\mathrm{R}}{\Gamma, \vdash \alpha \wedge \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(T(\alpha \vee \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||T\alpha) \qquad \mathcal{T}(L-\sigma||\sigma||T\beta)}{\mathcal{T}(L)} = \dfrac{\Gamma, \alpha, \alpha \vee \beta \vdash \Delta \qquad \Gamma, \beta, \alpha \vee \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}$
  $= \dfrac{\dfrac{\Gamma, \alpha \vdash \alpha \vee \beta, \Delta \qquad \Gamma, \beta, \alpha \vee \beta \vdash \Delta}{\Gamma \vdash \alpha \vee \beta, \alpha \vee \beta, \Delta} \vee\mathrm{L}}{\Gamma \vdash \alpha \vee \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(F(\alpha \vee \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}((L-F(\alpha \vee \beta))||F\alpha||F\beta)}{\mathcal{T}(L)} = \dfrac{\Gamma \vdash \alpha \vee \beta, \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} = \dfrac{\dfrac{\Gamma \vdash \alpha \vee \beta, \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \alpha \vee \beta, \alpha, \beta, \Delta} \wedge\mathrm{R}}{\Gamma \vdash \alpha \vee \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(T(\alpha \to \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\alpha) \qquad \mathcal{T}(L-\sigma||\sigma||T\beta)}{\mathcal{T}(L)} = \dfrac{\Gamma, \alpha, \alpha \to \beta \vdash \Delta \qquad \Gamma, \alpha \to \beta \vdash \beta, \Delta}{\Gamma, \alpha \to \beta \vdash \Delta}$
  $= \dfrac{\dfrac{\Gamma, \alpha, \alpha \to \beta \vdash \Delta \qquad \Gamma, \alpha \to \beta \vdash \beta, \Delta}{\Gamma, \alpha \to \beta, \alpha \to \beta \vdash \Delta} \to\mathrm{L}}{\Gamma, \alpha \to \beta \vdash \Delta}$
- $\dfrac{\mathcal{T}(f(F(\alpha \to \beta), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}((L-F(\alpha \to \beta))||F(\alpha \to \beta)||T\alpha||F\beta)}{\mathcal{T}(L)} = \dfrac{\Gamma, \alpha \vdash \beta, \alpha \to \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta}$
  $= \dfrac{\dfrac{\Gamma, \alpha \vdash \beta, \alpha \to \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \alpha \to \beta, \Delta} \to\mathrm{R}}{\Gamma \vdash \alpha \to \beta, \Delta}$
- $\dfrac{\mathcal{T}(f(T(\forall x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||T\phi(c_i))}{\mathcal{T}(L)} = \dfrac{\Gamma, \forall x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \forall x\, \phi(x) \vdash \Delta} = \dfrac{\dfrac{\Gamma, \forall x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \forall x\, \phi(x), \forall x\, \phi(x) \vdash \Delta} \forall\mathrm{L}}{\Gamma, \forall x\, \phi(x) \vdash \Delta}$
  and c_i does not occour in Γ or Δ, as it is a new one.
- $\dfrac{\mathcal{T}(f(F(\forall x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\phi(c_i))}{\mathcal{T}(L)} = \dfrac{\Gamma \vdash \phi(c_i), \forall x\, \phi(x), \Delta}{\Gamma \vdash \forall x\, \phi(x), \Delta} = \dfrac{\dfrac{\Gamma \vdash \phi(c_i), \forall x\, \phi(x), \Delta}{\Gamma \vdash \forall x\, \phi(x), \forall x\, \phi(x), \Delta} \forall\mathrm{R}}{\Gamma \vdash \forall x\, \phi(x), \Delta}$
- $\dfrac{\mathcal{T}(f(T(\exists x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||T\phi(c_i))}{\mathcal{T}(L)} = \dfrac{\Gamma, \exists x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \exists x\, \phi(x) \vdash \Delta} = \dfrac{\dfrac{\Gamma, \exists x\, \phi(x), \phi(c_i) \vdash \Delta}{\Gamma, \exists x\, \phi(x), \exists x\, \phi(x) \vdash \Delta} \exists\mathrm{L}}{\Gamma, \exists x\, \phi(x) \vdash \Delta}$
  and c_i does not occour in Γ or Δ, as it is a new one.
- $\dfrac{\mathcal{T}(f(F(\exists x\, \phi(x)), l))}{\mathcal{T}(L)} = \dfrac{\mathcal{T}(L-\sigma||\sigma||F\phi(c_i))}{\mathcal{T}(L)} = \dfrac{\Gamma \vdash \phi(c_i), \exists x\, \phi(x), \Delta}{\Gamma \vdash \exists x\, \phi(x), \Delta} = \dfrac{\dfrac{\Gamma \vdash \phi(c_i), \exists x\, \phi(x), \Delta}{\Gamma \vdash \exists x\, \phi(x), \exists x\, \phi(x), \Delta} \exists\mathrm{R}}{\Gamma \vdash \exists x\, \phi(x), \Delta}$

Definition 17. ***Translation in classical logic*** *given a tableaux proof tree development and its root r τ, we define $\mathcal{T}_p(\tau)$:*

- *if $\tau = r$ and there are $T\sigma$ and $F\sigma$ on r:*
  $\mathcal{T}_p(\tau) = \dfrac{}{\mathcal{T}(r)}\ {\scriptstyle Ax\ \sigma}$
- *if $\tau = r$ and there is no σ such that $T\sigma$ and $F\sigma$ are on r:*
  $\mathcal{T}_p(\tau) = \mathcal{T}(\tau)$
- *if r has a single child r_0 with a corresponding subtree τ_0:*
  by definition the last theorem exits a sentence σ such that $f(\sigma, r) = f_{Rule}(\sigma, r) = r_0 = r_0$, and so $\mathcal{T}_p(\tau) = \dfrac{\mathcal{T}_p(\tau_0)}{\mathcal{T}(r)}\ {\scriptstyle Rule\ on\ \sigma}$

– *if r has two children r_1 and r_2 with corresponding subtrees τ_1 and τ_2: by the last theorem there exists a sentence σ such that $f(\sigma, r) = f_{Rule}(\sigma, r) = \{r_1, r_2\}$, and so* $\mathcal{T}_p(\tau) = \dfrac{\mathcal{T}_p(\tau_1) \qquad \mathcal{T}_p(\tau_2)}{\mathcal{T}(r)}\ {\scriptstyle Rule\ on\ \sigma}$

5.2 Intuitionistic Translation

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