Proof Translations in Classical and Intuitionistic Logic

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Introduction

- **The Subject:**

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  Translating Tableaux Proofs into Sequent Proofs in classical and Intuitionistic Logic.

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  Translating Tableaux Proofs ? into Sequent Proofs ? in classical and Intuitionistic Logic?

Outline

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Classical Logic

Is $\exists x(P(x) \to (\forall y)P(y))$ classically valid ?

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$\langle\{c_1\},\{P(c_1)\}\rangle$

Here we are denoting $\langle$ Domain,All true atomic sentences $\rangle$

Classical Logic

Is $\exists x(P(x) \rightarrow (\forall y)P(y))$ classically valid ?

$\langle\{c_1\},\{P(c_1)\}\rangle\checkmark$

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Classical Logic

Is $\exists x(P(x) \rightarrow (\forall y)P(y))$ classically valid ?

$$\langle \{c_1\}, \{\} \rangle \checkmark$$

$$\langle \{c_1\}, \{P(c_1)\} \rangle \checkmark$$

$$\langle \{c_1, c_2\}, \{\} \rangle \checkmark$$

$$\langle \{c_1, c_2\}, \{P(c_1)\} \rangle \checkmark$$

$$\langle \{c_1, c_2\}, \{P(c_1), P(c_2)\} \rangle \checkmark$$

$$\ldots$$

Here we are denoting $\langle$ Domain,All true atomic sentences $\rangle$

Intuitionistic Logic

Is $\exists x(P(x) \rightarrow (\forall y)P(y))$
intuitionistically valid ?

Intuitionistic Logic

Is $\exists x(P(x) \to (\forall y)P(y))$ intuitionistically valid ?

$$\begin{array}{c} \langle\{c_1\}, \{P(c_1)\}\rangle \text{X} \\ | \\ \langle\{c_1, c_2\}, \{P(c_1), P(c_2)\}\rangle \\ | \\ \langle\{c_1, c_2, c_3\}, \{P(c_1), P(c_2)\}\rangle \end{array}$$

Here we are denoting $\langle$ Domain,All true atomic sentences $\rangle$:

Intuitionistic Logic/Structures

- A **structure for a Language** $\mathcal{L}$ (an $\mathcal{L}$-structure) consists of:
  - A domain. (ex: $\mathbb{Z}$)
  - Assignments for constant symbols to elements of the domain (ex: "1" $\mapsto 1$)
  - Functions and predicates corresponding to function and predicate symbols. (ex: P(x) $\mapsto$ is odd)

Intuitionistic Logic/Kripke Frames

- A **Kripke Frame** of a Language $\mathcal{L}$, $\mathcal{C} = (R, \{C(p)\}_{p \in R})$ consists of
  a partially ordered set R, and an $\mathcal{L}$-structure $C(p)$ for all p's in R. Also:
  - all interpretations are preserved
  - if $p \leq q$, then C(q) extends C(p):
    all sentences that are true in C(p) are true in C(q)
    and the domain of C(p) is included in the domain of C(q)

Intuitionistic Logic/Kripke Frames

- When a sentence ϕ of a language $\mathcal{L}$ is **forced** by a structure $C(p)$ of a frame $\mathcal{C}$, we denote:
  $p \vDash_{\mathcal{C}} \phi$
- Forcing is defined by induction:
  - $p \vDash_{\mathcal{C}} \phi \Leftrightarrow \phi$ is true in $C(p)$ (if ϕ is an atomic sentence)
  - $p \vDash_{\mathcal{C}} (\phi \rightarrow \psi) \Leftrightarrow$ for all $q \geq p$, if $q \vDash_{\mathcal{C}} \phi$, then $q \vDash_{\mathcal{C}} \psi$
  - $p \vDash_{\mathcal{C}} \neg\phi \Leftrightarrow$ for all $q \geq p$, q does not force ϕ
  - $p \vDash_{\mathcal{C}} (\forall x)\phi(x) \Leftrightarrow$ for all $q \geq p$ and d in $\mathcal{L}_{C(q)}$, $q \vDash_{\mathcal{C}} \phi(d)$
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does $\emptyset \vDash_{\mathcal{C}} \phi$?

$$\begin{tikzcd}
\langle \{c_1\}, \{\} \rangle \arrow[d, dash] \\
\langle \{c_1\}, \{\phi\} \rangle
\end{tikzcd}$$

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does $\emptyset \vDash_{\mathcal{C}} \neg\phi$?

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does $\emptyset \models_{\mathcal{C}} \exists x(P(x) \rightarrow (\forall y)P(y))$?

$$\begin{array}{c} \langle\{c_1\},\{P(c_1)\}\rangle \\ | \\ \langle\{c_1,c_2\},\{P(c_1),P(c_2)\}\rangle \\ | \\ \langle\{c_1,c_2,c_3\},\{P(c_1),P(c_2)\}\rangle \end{array}$$

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does $\emptyset \Vdash_{\mathcal{C}} \forall x(P(x) \rightarrow P(x))$?

$$\langle \{c_1\}, \{P(c_1)\} \rangle$$

$$\langle \{c_1, c_2\}, \{P(c_1), P(c_2)\} \rangle$$

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The tableaux method

- In order to prove that a sentence ϕ is Intuitionistically valid, we can assume that it is not $(F\emptyset \vDash_{\mathcal{C}} \phi)$, and systematically search for the frame counter-example $\mathcal{C}$.

The tableaux method

- In order to prove that a formula ϕ is Intuitionistically valid, we can assume that it is not ($F\emptyset \vDash_{\mathcal{C}} \phi$), and systematically search for the frame counter-example $\mathcal{C}$.

$$F\emptyset \vDash_{\mathcal{C}} \alpha \to \neg\neg\alpha$$

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$$\begin{array}{c} F_{\emptyset}\alpha \to \neg\neg\alpha \\ \downarrow \\ T_0\alpha \\ \downarrow \\ F_0\neg\neg\alpha \end{array}$$

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$$
\begin{array}{c}
F_{\emptyset}\alpha \to \neg\neg\alpha \\
\downarrow \\
T_0\alpha \\
\downarrow \\
F_0\neg\neg\alpha \\
\downarrow \\
T_{00}\neg\alpha \\
\downarrow \\
F_{00}\alpha \\
\downarrow \\
T_{00}\alpha \\
\downarrow \\
\mathrm{X}
\end{array}
$$

Sequent Calculus

A Sequent is an expression of the form:

$$\Gamma \vdash \Delta$$

where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \dots\}$ and $\Delta = \{\Delta_1, \Delta_2, \Delta_3, \dots\}$ are finite sets of formulas.
A sequent is intuitionistically valid if for all Kripke frames, if Γ are forced, then at least one of Δ is forced.

Classical Sequent Calculus

$$\frac{}{\Gamma, \alpha \vdash \alpha, \Delta} \text{ Ax } \alpha$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$$

$$\frac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta} \ \neg\text{L } \neg\alpha \qquad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta} \ \neg\text{R } \neg\alpha$$

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \ \wedge\text{L } \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta} \ \wedge\text{R } \alpha \wedge \beta$$

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$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta} \ \to\text{L } \alpha \to \beta \qquad \frac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta} \ \to\text{R } \alpha \to \beta$$

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$$\frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$$

Classical Sequent Calculus

$$\frac{}{\Gamma, \alpha \vdash \alpha, \Delta}\ \text{Ax}\ \alpha$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$$

$$\frac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\text{L}\ \neg\alpha \qquad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta}\ \neg\text{R}\ \neg\alpha$$

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\text{L}\ \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\text{R}\ \alpha \wedge \beta$$

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Intuitionistic Sequent Calculus

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$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta}\ \to\text{L}\ \alpha \to \beta \qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}\ \to\text{R}\ \alpha \to \beta$$

$$\frac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x\alpha(x) \vdash \Delta}\ \forall\ \text{L}\ \forall x\alpha(x) \qquad \frac{\Gamma \vdash \alpha(y)}{\Gamma \vdash \forall x\alpha(x)}\ \forall\ \text{R}\ \forall x\alpha(x)$$

$$\frac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x\alpha(x) \vdash \Delta}\ \exists\ \text{L}\ \exists x\alpha(x) \qquad \frac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x\alpha(x), \Delta}\ \exists\ \text{R}\ \exists x\alpha(x)$$

$$\frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$$

Intuitionistic Sequent Calculus

$$\frac{}{\Gamma, \alpha \vdash \alpha, \Delta}\ \text{Ax}\ \alpha$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$$

$$\frac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\text{L}\ \neg\alpha \qquad \frac{\Gamma, \alpha \vdash}{\Gamma \vdash \neg\alpha}\ \neg\text{R}\ \neg\alpha$$

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\text{L}\ \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\text{R}\ \alpha \wedge \beta$$

$$\frac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}\ \vee\text{L}\ \alpha \vee \beta \qquad \frac{\Gamma \vdash \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta}\ \vee\text{R}\ \alpha \vee \beta$$

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta}\ \to\text{L}\ \alpha \to \beta \qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}\ \to\text{R}\ \alpha \to \beta$$

$$\frac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x \alpha(x) \vdash \Delta}\ \forall\text{L}\ \forall x \alpha(x) \qquad \frac{\Gamma \vdash \alpha(y)}{\Gamma \vdash \forall x \alpha(x)}\ \forall\text{R}\ \forall x \alpha(x)$$

$$\frac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x \alpha(x) \vdash \Delta}\ \exists\text{L}\ \exists x \alpha(x) \qquad \frac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x \alpha(x), \Delta}\ \exists\text{R}\ \exists x \alpha(x)$$

$$\frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$$

Outline

- ☑ Context
  - ☑ Introduction
  - ☑ Classical Logic
  - ☑ Intuitionistic Logic
    - ☑ Structures
    - ☑ Kripke Frames
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  - ☑ Sequent proofs
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    - ☐ Thinning
    - ☐ Non-branching tree properties
  - ☐ Classical translation
    - ☐ Translating Nodes
    - ☐ Examples

Cumulative Tableaux

$$
\begin{tikzcd}
F_{\emptyset}\alpha \to \neg\neg\alpha \arrow[d] \\
T_0\alpha \arrow[d] \\
F_0\neg\neg\alpha \arrow[d] \\
T_{00}\neg\alpha \arrow[d] \\
F_{00}\alpha \arrow[d] \\
T_{00}\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_{\emptyset}\alpha \to \neg\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_0\alpha,\ F_0\neg\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_0\alpha,\ F_0\neg\neg\alpha,\ T_{00}\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_0\alpha,\ F_0\neg\neg\alpha,\ T_{00}\neg\alpha,\ F_{00}\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_0\alpha,\ F_0\neg\neg\alpha,\ T_{00}\neg\alpha,\ F_{00}\alpha,\ T_{00}\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}
$$

Cumulative Tableaux

$$
\begin{array}{c}
F_{\emptyset}\alpha \to \neg\neg\alpha \\
\downarrow \\
T_{\emptyset}\alpha \\
\downarrow \\
F_{\emptyset}\neg\neg\alpha \\
\downarrow \\
T_{\emptyset}\neg\alpha \\
\downarrow \\
F_{\emptyset}\alpha \\
\downarrow \\
T_{\emptyset}\alpha \\
\downarrow \\
\mathsf{X}
\end{array}
$$

$$
\begin{array}{c}
F_{\emptyset}\alpha \to \neg\neg\alpha \\
\downarrow \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_{\emptyset}\alpha,\ F_{\emptyset}\neg\neg\alpha \\
\downarrow \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_{\emptyset}\alpha,\ F_{\emptyset}\neg\neg\alpha,\ T_{\emptyset}\neg\alpha \\
\downarrow \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_{\emptyset}\alpha,\ F_{\emptyset}\neg\neg\alpha,\ T_{\emptyset}\neg\alpha,\ F_{\emptyset}\alpha \\
\downarrow \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\ T_{\emptyset}\alpha,\ F_{\emptyset}\neg\neg\alpha,\ T_{\emptyset}\neg\alpha,\ F_{\emptyset}\alpha,\ T_{\emptyset}\alpha \\
\downarrow \\
\mathsf{X}
\end{array}
$$

Cumulative Tableaux/Inference function F

- $f(T_p, L) = [L - \sigma : \sigma]$
- $f(T_p \neg\alpha, L) = [\sigma : L - \sigma : F_{p'}\alpha]$
  for a minimal $p' \geq p$ present in $\sigma : L$.
- $f(F_p \neg\alpha, L) = [L - \sigma : \sigma : T'_p\alpha]$
  for a new $p' \geq p$
- $f(T_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : T_p\alpha : T_p\beta]$.
- $f(F_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : F_p\alpha], [\sigma : L - \sigma : F_p\beta]$.
- $f(T_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : T_p\alpha], [\sigma : L - \sigma : T_p\beta]$.
- $f(F_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : F_p\alpha : F_p\beta]$.
- $f(T_p(\alpha \to \beta), L) = [\sigma : L - \sigma : F_{p'}\alpha], [L - \sigma : \sigma : T_{p'}\beta]$
  for a new $p' \geq p$
- $f(F_p(\alpha \to \beta), L) = [\sigma : L - \sigma : T_{p'}\alpha : F_{p'}\beta]$
  for a minimal $p' \geq p$ present in $\sigma : L$.

Cumulative Tableaux/Inference function F

- $f(T_p, L) = [L - \sigma : \sigma]$
- $f(T_p \neg\alpha, L) = [\sigma : L - \sigma : F_{p'}\alpha]$
  for a minimal $p' \geq p$ present in $\sigma : L$.
- $\mathbf{f(F_p \neg\alpha, L) = [\sigma : L - \sigma : T'_p\alpha]}$
  for a new $\mathbf{p' \geq p}$
- $f(T_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : T_p\alpha : T_p\beta]$.
- $f(F_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : F_p\alpha], [\sigma : L - \sigma : F_p\beta]$.
- $f(T_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : T_p\alpha], [\sigma : L - \sigma : T_p\beta]$.
- $\mathbf{f(F_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : F_p\alpha : F_p\beta]}$.
- $f(T_p(\alpha \to \beta), L) = [\sigma : L - \sigma : F_{p'}\alpha], [L - \sigma : \sigma : T_{p'}\beta]$
  for a new $p' \geq p$
- $f(F_p(\alpha \to \beta), L) = [\sigma : L - \sigma : T_{p'}\alpha : F_{p'}\beta]$
  for a minimal $p' \geq p$ present in $\sigma : L$.

Classical Translation/Thinning

$$\mathbf{F_{\emptyset}x} \to \neg\neg\mathbf{x}$$

$$\downarrow$$

$$\mathbf{F_0}\neg\neg\mathbf{x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T_0x}$$

$$\downarrow$$

$$\mathbf{T_{00}}\neg\mathbf{x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T_0x}, F_0\neg\neg x$$

$$\downarrow$$

$$\mathbf{F_{00}x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T_0x}, F_0\neg\neg x, T_{00}\neg x$$

$$\downarrow$$

$$\mathbf{F_{00}x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T_{00}x}, T_0x, F_0\neg\neg x, T_{00}\neg x$$

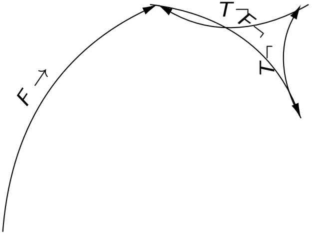
$$\downarrow$$

Classical Translation/Translating Nodes

$p = q$ $\quad$ $p > q$ $\quad$ $p \not\lessgtr q$

$F_p\ T_q$

T_p



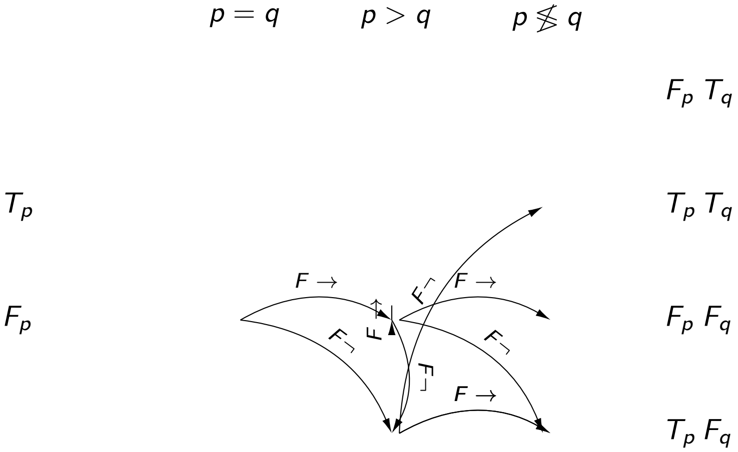
$T_p\ T_q$

F_p

$F_p\ F_q$

$T_p\ F_q$

Classical Translation/Translating Nodes



Definition

Node Translating Function $\mathcal{T}$

Given a signed sentence list L with sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ and $\{F_{q_1}\delta_1, F_{q_2}\delta_2...\}$ and a frame $w \in \{p_1, p_2, ...\} \cup \{q_1, q_2, ...\}$ then:

$\mathcal{T}(L, w) = \Gamma_1, \Gamma_2, ... \vdash \Delta_1, \Delta_2, ...$, where:

$\{T_{p'_1}\Gamma_1, T_{p'_2}\Gamma_2, ...\}$ are the elements of $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ such that $p' \leq w$ and $\{F_{q'_1}\Delta_1, F_{q'_2}\Delta_2, ...\}$ are the elements of $\{F_{q_1}\delta_1, F_{q_2}\delta_2, ...\}$ such that $q' \geq w$

For the classical case: $\mathcal{T}(L) = \mathcal{T}(L, \emptyset)$