Proof Translations in Classical and Intuitionistic Logic

Ian Ribeiro de Faria Leite

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Introduction

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  Translating Tableaux Proofs into Sequent Proofs in classical and Intuitionistic Logic.

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  Translating Tableaux Proofs ? into Sequent Proofs ? in classical and Intuitionistic Logic?

Outline

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  - Intuitionistic Logic
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    - Kripke Frames
    - Forcing
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  - Sequent proofs
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Classical Logic

Is $\exists x(P(x) \to (\forall y)P(y))$ classically valid ?

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$\langle \{c_1\}, \{P(c_1)\} \rangle$

Here we are denoting $\langle$ Domain,All true atomic sentences $\rangle$:

Classical Logic

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$\langle\{c_1, c_2\}, \{\}\rangle\checkmark$

$\langle\{c_1, c_2\}, \{P(c_1)\}\rangle\checkmark$

$\langle\{c_1, c_2\}, \{P(c_1)P(c_2)\}\rangle\checkmark$

...

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Intuitionistic Logic

Is $\exists x(P(x) \rightarrow (\forall y)P(y))$
intuitionistically valid ?

Intuitionistic Logic

Is $\exists x(P(x) \to (\forall y)P(y))$ intuitionistically valid ?

$$\begin{array}{c} \langle\{c_1\},\{P(c_1)\}\rangle \text{X} \\ | \\ \langle\{c_1,c_2\},\{P(c_1),P(c_2)\}\rangle \\ | \\ \langle\{c_1,c_2,c_3\},\{P(c_1),P(c_2)\}\rangle \end{array}$$

Here we are denoting $\langle$ Domain,All true atomic sentences $\rangle$:

Intuitionistic Logic/Structures

- A **structure for a Language** $\mathcal{L}$ (an $\mathcal{L}$-structure) consists of:
  - A domain. (ex: $\mathbb{Z}$)
  - Assignments for constant symbols to elements of the domain (ex: "1" $\mapsto$ 1)
  - Functions and predicates corresponding to function and predicate symbols. (ex: P(x) $\mapsto$ is odd)

Intuitionistic Logic/Kripke Frames

- A **Kripke Frame** of a Language $\mathcal{L}$, $\mathcal{C} = (R, \{C(p)\}_{p \in R})$ consists of
  a partially ordered set R, and an $\mathcal{L}$-structure $C(p)$ for all p's in R. Also:
  - all interpretations are preserved
  - if $p \leq q$, then C(q) extends C(p):
    all sentences that are true in C(p) are true in C(q)
    and the domain of C(p) is included in the domain of C(q)

Intuitionistic Logic/Kripke Frames

- When a sentence ϕ of a language $\mathcal{L}$ is **forced** by a structure $C(p)$ of a frame $\mathcal{C}$, we denote:
  $p \vDash_{\mathcal{C}} \phi$
- Forcing is defined by induction:
  - $p \vDash_{\mathcal{C}} \phi \Leftrightarrow \phi$ is true in $C(p)$ (if ϕ is an atomic sentence)
  - $p \vDash_{\mathcal{C}} (\phi \to \psi) \Leftrightarrow$ for all $q \geq p$, if $q \vDash_{\mathcal{C}} \phi$, then $q \vDash_{\mathcal{C}} \psi$
  - $p \vDash_{\mathcal{C}} \neg\phi \Leftrightarrow$ for all $q \geq p$, q does not force ϕ
  - $p \vDash_{\mathcal{C}} (\forall x)\phi(x) \Leftrightarrow$ for all $q \geq p$ and d in $\mathcal{L}_{C(q)}$, $q \vDash_{\mathcal{C}} \phi(d)$
  - $p \vDash_{\mathcal{C}} (\exists x)\phi(x) \Leftrightarrow$ exists a d in $\mathcal{L}_{C(q)}$, such that $p \vDash_{\mathcal{C}} \phi(d)$
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does $\emptyset \vDash_{\mathcal{C}} \phi$?

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does $\emptyset \vDash_{\mathcal{C}} \exists x(P(x) \to (\forall y)P(y))$?

$$\begin{tikzcd}
\langle\{c_1\},\{P(c_1)\}\rangle \arrow[d, no head] \\
\langle\{c_1,c_2\},\{P(c_1),P(c_2)\}\rangle \arrow[d, no head] \\
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\end{array}$$

The tableaux method

- In order to prove that a sentence ϕ is Intuitionistically valid, we can assume that it is not ($F\emptyset \vDash_{\mathcal{C}} \phi$), and systematically search for the frame counter-example $\mathcal{C}$.

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| \\
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$$T00 \vDash_{\mathcal{C}} \neg\alpha$$

$$F00 \vDash_{\mathcal{C}} \neg\alpha$$

X

Sequent Calculus

A Sequent is an expression of the form:

$$\Gamma \vdash \Delta$$

where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \dots\}$ and $\Delta = \{\Delta_1, \Delta_2, \Delta_3, \dots\}$ are finite sets of formulas.
A sequent is intuitionistically valid if for all Kripke frames, if Γ are forced, then at least one of Δ is forced.

Classical Sequent Calculus

$$\frac{}{\Gamma, \alpha \vdash \alpha, \Delta}\ \text{Ax}\ \alpha$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$$

$$\frac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\text{L}\ \neg\alpha \qquad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta}\ \neg\text{R}\ \neg\alpha$$

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\text{L}\ \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\text{R}\ \alpha \wedge \beta$$

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$$\frac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x\alpha(x) \vdash \Delta}\ \forall\ \text{L}\ \forall x\alpha(x) \qquad \frac{\Gamma \vdash \alpha(y), \Delta}{\Gamma \vdash \forall x\alpha(x), \Delta}\ \forall\ \text{R}\ \forall x\alpha(x)$$

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$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\mathrm{L}\ \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\mathrm{R}\ \alpha \wedge \beta$$

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Intuitionistic Sequent Calculus

$$\frac{}{\Gamma, \alpha \vdash \alpha, \Delta} \text{ Ax } \alpha$$

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$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \wedge\text{L } \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta} \wedge\text{R } \alpha \wedge \beta$$

$$\frac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta} \vee\text{ L } \alpha \vee \beta \qquad \frac{\Gamma \vdash \alpha, \beta\Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \vee\text{ R } \alpha \vee \beta$$

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta} \to\text{L } \alpha \to \beta \qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta} \to\text{R } \alpha \to \beta$$

$$\frac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x\alpha(x) \vdash \Delta} \forall\text{ L } \forall x\alpha(x) \qquad \frac{\Gamma \vdash \alpha(y)}{\Gamma \vdash \forall x\alpha(x)} \forall\text{ R } \forall x\alpha(x)$$

$$\frac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x\alpha(x) \vdash \Delta} \exists\text{ L } \exists x\alpha(x) \qquad \frac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x\alpha(x), \Delta} \exists\text{ R } \exists x\alpha(x)$$

$$\frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$$

Intuitionistic Sequent Calculus

$$\frac{}{\boldsymbol{\Gamma, \alpha \vdash \alpha, \Delta}}\ \text{Ax}\ \boldsymbol{\alpha}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \alpha}$$

$$\frac{\Gamma, \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg\text{L}\ \neg\alpha \qquad \frac{\boldsymbol{\Gamma, \alpha \vdash}}{\boldsymbol{\Gamma \vdash \neg\alpha}}\ \neg\text{R}\ \boldsymbol{\neg\alpha}$$

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge\text{L}\ \alpha \wedge \beta \qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}\ \wedge\text{R}\ \alpha \wedge \beta$$

$$\frac{\Gamma, \alpha \vdash \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}\ \vee\ \text{L}\ \alpha \vee \beta \qquad \frac{\boldsymbol{\Gamma \vdash \alpha, \beta, \Delta}}{\boldsymbol{\Gamma \vdash \alpha \vee \beta, \Delta}}\ \vee\ \text{R}\ \boldsymbol{\alpha \vee \beta}$$

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \to \beta \vdash \Delta}\ \to\text{L}\ \alpha \to \beta \qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}\ \to\text{R}\ \alpha \to \beta$$

$$\frac{\Gamma, \alpha(t) \vdash \Delta}{\Gamma, \forall x \alpha(x) \vdash \Delta}\ \forall\ \text{L}\ \forall x \alpha(x) \qquad \frac{\Gamma \vdash \alpha(y)}{\Gamma \vdash \forall x \alpha(x)}\ \forall\ \text{R}\ \forall x \alpha(x)$$

$$\frac{\Gamma, \alpha(y) \vdash \Delta}{\Gamma, \exists x \alpha(x) \vdash \Delta}\ \exists\ \text{L}\ \exists x \alpha(x) \qquad \frac{\Gamma \vdash \alpha(t), \Delta}{\Gamma \vdash \exists x \alpha(x), \Delta}\ \exists\ \text{R}\ \exists x \alpha(x)$$

$$\frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta}$$

Outline

- ☑ Context
  - ☑ Introduction
  - ☑ Classical Logic
  - ☑ Intuitionistic Logic
    - ☑ Structures
    - ☑ Kripke Frames
    - ☑ Forcing
  - ☑ Tableaux proofs
  - ☑ Sequent proofs
- ☐ Steps taken
  - ☐ Cumulative Tableaux
    - ☐ Inference Function f
  - ☐ Intuitionistic Translation
    - ☐ Thinning
    - ☐ Non-branching tree properties
  - ☐ Classical translation
    - ☐ Translating Nodes
    - ☐ Examples

Cumulative Tableaux

$$
\begin{tikzcd}
F_{\emptyset}\alpha \to \neg\neg\alpha \arrow[d] \\
T_0\alpha \arrow[d] \\
F_0\neg\neg\alpha \arrow[d] \\
T_{00}\neg\alpha \arrow[d] \\
F_{00}\alpha \arrow[d] \\
T_{00}\neg\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_{\emptyset}\alpha \to \neg\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha,\, T_{00}\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha,\, T_{00}\neg\alpha,\, F_{00}\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha,\, T_{00}\neg\alpha,\, F_{00}\alpha\, T_{00}\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_{\emptyset}\alpha \to \neg\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha,\, T_{00}\neg\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha,\, T_{00}\neg\alpha,\, F_{00}\alpha \arrow[d] \\
F_{\emptyset}\alpha \to \neg\neg\alpha,\, T_0\alpha,\, F_0\neg\neg\alpha,\, T_{00}\neg\alpha,\, F_{00}\alpha\, T_{00}\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}
$$

Cumulative Tableaux

$$\begin{tikzcd}
F\alpha \to \neg\neg\alpha \arrow[d] \\
T\alpha \arrow[d] \\
F\neg\neg\alpha \arrow[d] \\
T\neg\alpha \arrow[d] \\
F\alpha \arrow[d] \\
T\neg\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}$$

$$\begin{tikzcd}
F\alpha \to \neg\neg\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha, T\neg\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha, T\neg\alpha, F\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha, T\neg\alpha, F\alpha T\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}$$

$$\begin{tikzcd}
F\alpha \to \neg\neg\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha, T\neg\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha, T\neg\alpha, F\alpha \arrow[d] \\
F\alpha \to \neg\neg\alpha, T\alpha, F\neg\neg\alpha, T\neg\alpha, F\alpha T\alpha \arrow[d] \\
\mathsf{X}
\end{tikzcd}$$

Cumulative Tableaux/Inference function F

- $f(T_p, L) = [L - \sigma : \sigma]$
- $f(T_p \neg\alpha, L) = [\sigma : L - \sigma : F_{p'}\alpha]$
  for a minimal $p' \geq p$ present in $\sigma : L$.
- $f(F_p \neg\alpha, L) = [L - \sigma : \sigma : T'_p\alpha]$
  for a new $p' \geq p$
- $f(T_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : T_p\alpha : T_p\beta]$.
- $f(F_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : F_p\alpha], [\sigma : L - \sigma : F_p\beta]$.
- $f(T_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : T_p\alpha], [\sigma : L - \sigma : T_p\beta]$.
- $f(F_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : F_p\alpha : F_p\beta]$.
- $f(T_p(\alpha \rightarrow \beta), L) = [\sigma : L - \sigma : F_{p'}\alpha], [L - \sigma : \sigma : T_{p'}\beta]$
  for a new $p' \geq p$
- $f(F_p(\alpha \rightarrow \beta), L) = [\sigma : L - \sigma : T_{p'}\alpha : F_{p'}\beta]$
  for a minimal $p' \geq p$ present in $\sigma : L$.

Cumulative Tableaux/Inference function F

- $f(T_p, L) = [L - \sigma : \sigma]$
- $f(T_p \neg\alpha, L) = [\sigma : L - \sigma : F_{p'}\alpha]$
  for a minimal $p' \geq p$ present in $\sigma : L$.
- $\mathbf{f(F_p \neg\alpha, L) = [\sigma : L - \sigma : T'_p\alpha]}$
  for a new $\mathbf{p' \geq p}$
- $f(T_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : T_p\alpha : T_p\beta]$.
- $f(F_p(\alpha \wedge \beta), L) = [\sigma : L - \sigma : F_p\alpha], [\sigma : L - \sigma : F_p\beta]$.
- $f(T_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : T_p\alpha], [\sigma : L - \sigma : T_p\beta]$.
- $\mathbf{f(F_p(\alpha \vee \beta), L) = [\sigma : L - \sigma : F_p\alpha : F_p\beta]}$.
- $f(T_p(\alpha \to \beta), L) = [\sigma : L - \sigma : F_{p'}\alpha], [L - \sigma : \sigma : T_{p'}\beta]$
  for a new $p' \geq p$
- $f(F_p(\alpha \to \beta), L) = [\sigma : L - \sigma : T_{p'}\alpha : F_{p'}\beta]$
  for a minimal $p' \geq p$ present in $\sigma : L$.

Classical Translation/Thinning

$$\mathbf{F}_{\emptyset}\mathbf{x} \to \neg\neg\mathbf{x}$$

$$\downarrow$$

$$\mathbf{F}_{\mathbf{0}}\neg\neg\mathbf{x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T}_{\mathbf{0}}\mathbf{x}$$

$$\downarrow$$

$$\mathbf{T}_{\mathbf{00}}\neg\mathbf{x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T}_{\mathbf{0}}\mathbf{x}, F_{0}\neg\neg x$$

$$\downarrow$$

$$\mathbf{F}_{\mathbf{00}}\mathbf{x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T}_{\mathbf{0}}\mathbf{x}, F_{0}\neg\neg x, T_{00}\neg x$$

$$\downarrow$$

$$\mathbf{F}_{\mathbf{00}}\mathbf{x}, F_{\emptyset}x \to \neg\neg x, \mathbf{T}_{\mathbf{00}}\mathbf{x}, T_{0}x, F_{0}\neg\neg x, T_{00}\neg x$$

$$\downarrow$$

$$\times$$

Classical Translation/Translating Nodes

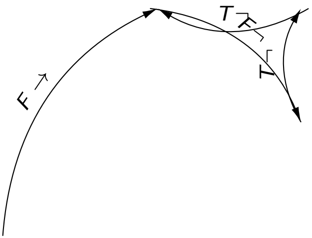
$p = q$ $p > q$ $p \lessgtr\!\!\!\!/\ q$

$F_p\ T_q$

T_p $T_p\ T_q$

F_p $F_p\ F_q$

$T_p\ F_q$



Classical Translation/Translating Nodes

Definition

Node Translating Function $\mathcal{T}$

Given a signed sentence list L with sentences $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ and $\{F_{q_1}\delta_1, F_{q_2}\delta_2 ...\}$ and a frame $w \in \{p_1, p_2, ...\} \cup \{q_1, q_2, ...\}$ then:

$\mathcal{T}(L, w) = \Gamma_1, \Gamma_2, ... \vdash \Delta_1, \Delta_2, ...$, where:

$\{T_{p'_1}\Gamma_1, T_{p'_2}\Gamma_2, ...\}$ are the elements of $\{T_{p_1}\gamma_1, T_{p_2}\gamma_2, ...\}$ such that $p' \leq w$ and $\{F_{q'_1}\Delta_1, F_{q'_2}\Delta_2, ...\}$ are the elements of $\{F_{q_1}\delta_1, F_{q_2}\delta_2, ...\}$ such that $q' \geq w$

For the classical case: $\mathcal{T}(L) = \mathcal{T}(L, \emptyset)$