概率 Prob.
考研14 P534

将一枚硬币独立地~~连续~~掷两次，引进事件：A_1 = {掷第一次出现正面}，A_2 = {掷第二次出现正面}，A_3 = {正反面各出现一次}，A_4 = {正面出现两次}。则：

(A). A_1, A_2, A_3 相互独立.

(B). A_2, A_3, A_4 相互独立.

(C). A_1, A_2, A_3 两两独立.

(D). A_2, A_3, A_4 两两独立.

Soln.

- 首先 A_1, A_2, A_3 相互独立, 指 $P(A_1A_2A_3)=0$

  两两独立. $P(A_1A_2)=0$, $P(A_2A_3)=0$ 就行.
  $P(A_1, A_3)=0$.

- 试验的样本空间有四个点. $\Omega=\{(正,正),(正,反),(反,正),(反,反)\}$

  样本

- $\underset{第一次正面}{A_1} \subset \underset{两次正面}{A_4}$ $\quad A_2 \subset A_4$, 且 A_3, A_4 不相容.

- 依古典概型. $P(A_1)=P(A_2)=P(A_3)=\frac{1}{2}$ $\quad P(A_4)=\frac{1}{4}$

- A_1, A_2 独立. obvious, 第一次和第二次出现正面没关系.

- A_1, A_2 与 A_3 亦独立, A_3: 正反面各出现一次 → 不一定第一次正面. 反之, 第一次正面 也不一定正反面各出现一次.

- 而 A_1, A_2, A_3 亦非相互独立.

$$\underbrace{P(A_1A_2)}_{两次正面} \overset{A_1 A_2 独立}{=} P(A_1)P(A_2).$$

$$P(\underbrace{A_1A_2}_{两次正面}\underbrace{A_3}_{一正一反})=0$$

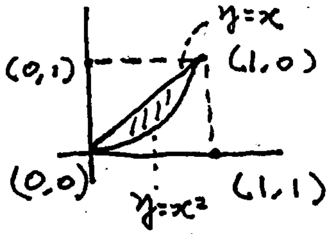
15考研全书
P535
概率

设平面区域D是由坐标为$(0,0)$，$(0,1)$，$(1,0)$，$(1,1)$的四个点围成之正方形，今向D内随机投入10个点，求这10个点中至少有2个点落于曲线$y=x^2$与直线$y=x$所围成区域D_1之概率.

Probability Problem

Soln:

- 投每一次个点视为一个随机试验。入 D_1 之概率。



Area of $D_1 = \int_0^1 (x-x^2)\,dx = \frac{1}{6}$

故每个点入 D_1 之 Prob: $P = \frac{1}{6}$

- 10次投点，意味着10次试验，

设 B_k = 10次投点中，有k个点落入 D_1 中。则 $B_k \sim B(\overset{n}{10}, \overset{p}{\tfrac{1}{6}}) =$

- {至少有两个点入 D_1} $\equiv \{B_2 \cup B_3 \cdots \cup B_{10}\}$

至少不好算，算至多。

{1 − 至多有一个点} $= \{1 - B_0 \cup B_1\}$

$$= 1 - C_{10}^{0}\left(\tfrac{1}{6}\right)^0\left(\tfrac{5}{6}\right)^{10} - C_{10}^{1}\left(\tfrac{1}{6}\right)^1\left(\tfrac{5}{6}\right)^{9}$$

$$\doteq 0.52.$$

甲乙二人轮流投篮，游戏规则规定甲先开始，且甲每轮只投一次，而乙每轮连续投两次，先投中者胜。设甲、乙每次投篮之命中率，分别为 p 与 0.5，则 $p=$ ___ 时，甲乙胜负概率相同。

概率
Prob
李KY95
P935

Soln:

什么时候甲会赢呢.

$$\begin{array}{ccccccc} \text{甲投进} & + & \underbrace{\text{甲不进}\cdot\text{乙不进}\cdot\text{乙不进}}\cdot\text{甲投进} & + & \underbrace{\text{甲不进}\cdot\text{乙不进}\cdot\text{乙不进}}\cdot & \underbrace{\text{甲不进}\cdot\text{乙不进}\cdot\text{乙不进}}\cdot & \text{甲投进} \\ P & & (1-P)(1-q)(1-q)\quad P & & (1-P)(1-q)^2 & (1-P)(1-q)^2 & P. \end{array}$$

$$P\{\text{甲会赢}\} = P + \underbrace{(1-P)(1-q)^2}_{\text{记做}\ a.} P + a^2 P.$$

$$= P(1+a+a^2+\cdots) = P\cdot\frac{1}{1-a}.$$

$$q=0.5 \Rightarrow a=\frac{1}{4}(1-P)$$

甲乙赢得概率同:

$$P\{\text{甲赢}\} = 0.5 \quad \text{即} \quad P\cdot\frac{1}{1-\frac{1}{4}(1-P)} = 0.5 \quad \Rightarrow P=\frac{3}{7}.$$

Math Prob 11

Let (X_1, X_2, X_3) be a three-dimensional random vector, where X_k, $k=1,2,3$ are independent geometric r.v.'s with parameter $1/3$

a) Calculate $P(X_1=1, X_2=2, X_3=3)$.

b) Calculate the probability that exactly two X_k's equals to 2.

What's the probability that it takes more than 5 flips to get a head? (Fair coin $p = \frac{1}{2}$)

$$\Pr(X>5) = 1-\Pr(X\le 5)$$
$$= 1-\left(1-\left(1-\tfrac{1}{2}\right)^{5}\right) = 1/32.$$

1) $$x = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$$

$$\frac{1}{2}x = \frac{1}{2^2} + \frac{2}{2^3} + \cdots$$

$$\downarrow$$

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$x = 2.$$

2) $$w = \frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \cdots$$

$$\frac{1}{2}w = \frac{1^2}{2^2} + \frac{2^2}{2^3} + \cdots$$

$$\downarrow$$

$$\frac{1}{2}w = \frac{1^2}{2} + \frac{2^2-1^2}{2^2} + \frac{3^2-2^2}{2^3} + \cdots$$

$$= \frac{1}{2} + \frac{(2+1)}{2^2} + \frac{(2+3)}{2^3} + \cdots + \frac{2N-1}{2^N}$$

$$= 2 \cdot \sum_{N=1}^{\infty} \frac{N}{2^N} - \sum_{N=1}^{\infty} \frac{1}{2^N}$$

$$= (2 \cdot 2 - 1) = 3$$

$$w = 6$$

→ 有个记忆之法.

$$\frac{1}{2} + \frac{1}{2^2} + \cdots = 1$$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{N}{2^N} = 2$$

即分子上的自然数之递增序列,在下面指数序列作用下,作用被削弱.

3). $$\lim_{N\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{N}{2^N} \cdot \frac{2^{N-1}}{N-1} = \frac{1}{2}$$

于是级数和收敛.

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots = \sum_{N=1}^{\infty} \frac{N}{2^N} \quad ?$$

$$\sum \frac{N^2}{2^N} \quad ?$$

$$\vdots$$

$$\sum \frac{N^5}{2^N} \quad ?$$

$$S = \sum_{1}^{\infty} \frac{N^m}{2^N} \quad \text{converge?}, \quad = ?$$

$$0 < m < \infty$$
$$\in \mathbb{N}$$

Plot $S \sim m$

I have a bag containing nine ordinary coins, and one double headed one; I remove a coin and flip it three times. It comes up heads each time. What's the probability that it's the double header?

根据 Bayers 公式：

A_1：拿出的硬币是 double header

A_2：—— single ——

B：连扔三次都是 Heads.

$$P(A_1 \mid B) = \frac{P(B \mid A_1) \cdot P(A_1)}{P(B)}$$

拿出的硬币也是 double header 的概率，based on 3次扔是 heads up.

- $P(B \mid A_1)$：如果是 double header，三次扔是 heads up 之 prob 为 1.

- $P(A_1)$：double header 的概率

- $P(B)$ 出现三次heads的概率，全概公式加分类讨论：

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2)$$
$$= 1 \cdot \frac{1}{10} + \frac{1}{8} \cdot \frac{9}{10} = \frac{17}{80}$$

于是

$$P(A_1 \mid B) = \frac{1 \cdot \frac{1}{10}}{\frac{17}{80}} = \frac{8}{17}.$$

Suppose we toss a fair coin, let N denote the number of tosses, until we get a head (including the final toss).

What's $\mathbb{E}(N)$ and $Var(N)$?

5:

硬币问题的本质，是全概公式 → 亦俗称之分类讨论，枚举. → 枚举不过是遍历.

第1次得到H	第2次	第3次
壹 H	TH	TTH
$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

a) $\bar{N} = \frac{1}{2}\cdot 1 + \frac{1}{2^2}\cdot 2 + \frac{1}{2^3}\cdot 3 + \cdots + \frac{1}{2^N}\cdot N + \cdots$

b) $\frac{1}{2}\bar{N} = \frac{1}{2^2}\cdot 1 + \frac{1}{2^3}\cdot 2 + \cdots$

a-b: $\frac{1}{2}\bar{N} = \frac{1}{2} + \frac{1}{2^2} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \quad \Rightarrow \bar{N} = 2$

$Var(N) = E(N^2) - [E(N)]^2$

$E(N^2) = \sum_{N=1}^{\infty} \frac{N^2}{2^N} = 6.$

$Var(N) = 6 - 2^2 = 2.$

Let's play a game. There're 4 sealed boxes. There's 100 pounds in one box and the others are empty. A player can pay X to open a box and take the content, as many times as they can. Assuming this is fair game, what is the value of X.

先算平均需n次，才能打开含100的箱子.

第一次中：$\frac{1}{4}$

第二次中：$\frac{3}{4}\cdot\frac{1}{3}$

第一次不中

余下的三个箱子，中有一个

第三次中：

$$\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{4}$$

第一次不中　第二次不中　第三次中

第四次中：$\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot 1$

$$\bar{n} = 1\cdot\frac{1}{4}+2\cdot\underbrace{\frac{3}{4}\cdot\frac{1}{3}}_{\frac{1}{4}}+3\cdot\frac{1}{4}+4\cdot\frac{1}{4}=2.5$$

$$x=\frac{100}{\bar{n}}=40.$$

- We play a game:

  I pick a number n from 1 to 100.

  If you guess correctly, I pay you \$$n$ and zero otherwise.

  How would you pay to play this game?

这个题的解答有些牵强.不过,如下:

抽的那个人当然选择抽的数,越小越好.但不能每次都是1.
但如若他每次都是1.那么被抓的概率加大.

于是乎,一个较好的 strategy 是, well,这里便是牵强的地方.

抽中 k 的概率 $\underbrace{\text{是}\ \frac{1}{k}}_{\text{正比于}}$.(为什么不是 $\frac{1}{k^2}$?)

于是抽中 k 的概率是:

$$P(k)=\frac{\frac{1}{k}}{\sum_{k=1}^{100}\frac{1}{k}} \qquad (\text{归一化})$$

$$\sum_{k=1}^{100}\frac{1}{k}=5.1 \text{ 左右}$$

精确值是一个狂复杂之分数.

$$P(k)\cdot\left(k\frac{1}{100}+0\cdot\frac{99}{100}\right)$$

抽中k之概率　猜中,于是得到k.　猜不中.

全概公式

$$X=\sum_{k=1}^{100}\frac{\frac{1}{k}}{\underbrace{\sum_{k=1}^{100}\frac{1}{k}}_{\text{记为常数}C}}\cdot\left(k\frac{1}{100}+0\right)=\frac{1}{C}\approx 0.19$$

$C\approx 5.1$

Joshi
P80
Probability

- Consider the following game. The player tosses a die once only. The payoff is 1 dollar for each dot on upturned face. Assuming a fair die, at what level should you set the ticket price of this game?

Soln:

很容易的一题.

色子随机抛. 6个面, 则每个面出现之概率为 $1/6$.

每朝上的面上的每一点 1 块. 于是 Expectation (or mean) 为

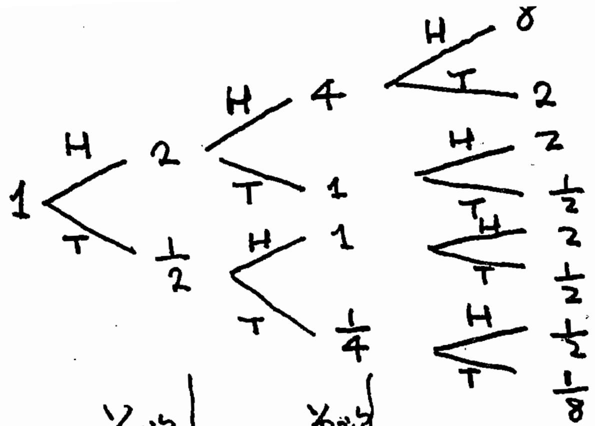
$$E(X) = \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{21}{6} = 3.5$$

平均收益为 3.5, 于是票价亦应该为 3.5.

Suppose you have a fair coin, and you start with a dollar, and if you toss H, your position doubles, if you toss T, your position halves. What's the expected value of the money you have if you toss infinitely?

S_1:



第1次 $\to$ $2\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{5}{4}$

第2次 $\to$ $4\cdot\frac{1}{4}+2\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4}=\frac{25}{16}=\left(\frac{5}{4}\right)^2$

第3次 $\to$ $\left(8+6+\frac{3}{2}+\frac{1}{8}\right)\cdot\frac{1}{8}$

$$=1+\frac{3}{4}+\frac{1}{64}+\frac{3}{16}$$

$$=\frac{64+48+1+12}{64}=\frac{125}{64}=\left(\frac{5}{4}\right)^3$$

于是，故可归纳得：第n次，是 $\left(\frac{5}{4}\right)^n$.

S_2.

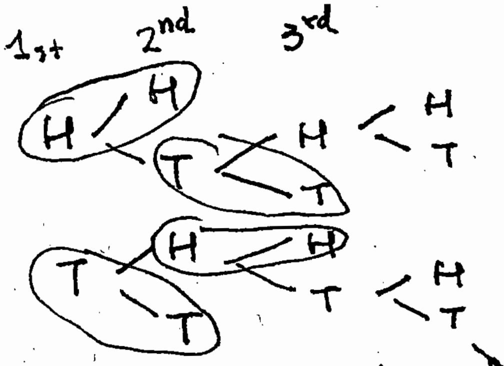
抛一次，从1变成$\frac{5}{4}$.

第二次，可视为从$\frac{5}{4}$的起始值开始抛，于是为$\left(\frac{5}{4}\right)\cdot\frac{5}{4}=\frac{25}{16}$

⋮

于是第n次为 $\left(\frac{5}{4}\right)^n$

We play a game, with a fair coin, the game stops when either two heads (H) or tail (T) appear consecutively. What's the expected time until the game stops?



$$2\cdot\left(\tfrac{1}{4}\times 2\right) + 3\times\left(\tfrac{1}{8}\right)\times 2 + 4\left(\tfrac{1}{16}\times 2\right)$$

$\bar{x} =$

$$x = \sum_{n=2}^{\infty} n\cdot\frac{1}{2^{n-1}} = \sum_{n=2}^{\infty}\left(\frac{n-1}{2^{n-1}} + \frac{1}{2^{n-1}}\right) = \sum_{n=1}^{\infty}\left(\frac{n}{2^{n}} + \frac{1}{2^{n}}\right) = 2+1=3.$$

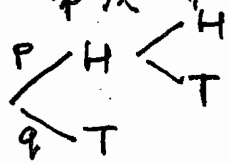
You toss a biased coin, what is the expected length of time until a head is tossed?

For two consecutive heads?

Soln:

1). 老办法, 二叉树

第一次 第二次



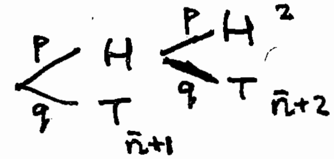
$$\bar{n} = 1\cdot p + 2\cdot qp + 3q^2p + \cdots$$

$$q\bar{n} = \qquad 1\cdot qp + 2q^2p + \cdots$$

$$(1-q)\bar{n} = p\frac{1}{1-q} = 1$$

$$\bar{n} = \frac{1}{p}.$$

2).



$$\bar{n} = p^2 2 + q(\bar{n}+1) + qp(\bar{n}+2)$$

$$\bar{n}(1-q-pq) = 2p^2+q+2pq$$

$$\bar{n} = \frac{2p^2+q+2pq}{1-q-pq}$$

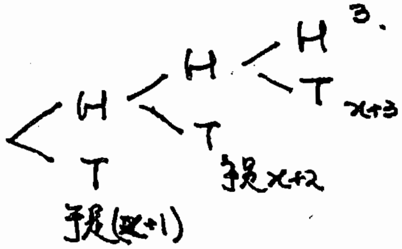
$$= \frac{2p^2+q+2pq}{p^2}$$

$$= 2 + \frac{q}{p^2} + \frac{2q}{p}.$$

For a fair coin, what is the expected number to get three heads in a row?

Soln: 老办法，一看到这种 consecutive 的题。

先画上二叉树，抛三次再说。



故 Expected value 为

$$x=\frac{1}{2}(x+1)+\frac{1}{2^2}(x+2)+\frac{1}{2^3}(x+3)+\frac{1}{2^3}\cdot 3$$

$$=\frac{7}{8}x+\left(\frac{1}{2}+\frac{1}{2}+\frac{3}{8}+\frac{3}{8}\right)$$

$$x=14.$$

平均需要抛 14 次才能得到 3 consecutive heads.

[illegible]：

n个 consecutive heads ?

$$x=\sum_{i=1}^{n}\frac{1}{2^i}(x+i)+\frac{1}{2^n}\cdot n$$

2. Suppose we play a game. I roll a die up to three times. Each time I roll, you can either take the number showing as dollars, or roll again. What's your expected winning.

Soln:

- 这个题首先要明确 strategy:

扔先头一个 dice, expected value: $\frac{1}{6}(1+2+3+4+5+6)=3.5$

故要是扔的 1, 2, 3, 则再扔一次;

扔的 4, 5, 6, 就认了.

- 于是

$A: \left.\begin{matrix}1\\2\\3\end{matrix}\right\}$

$B: \left.\begin{matrix}4\\5\\6\end{matrix}\right\}$

分成两组, $A=E(1,2,3)=2$

$B=5.$

出现概率均为1/2

第1次　第2次

$\frac{1}{2}$ A — 再扔 — $\frac{1}{2}$ A

　　　　　　　$\frac{1}{2}$ B

$\frac{1}{2}$ B

$$E = A\cdot\frac{1}{4} + B\cdot\frac{3}{4}$$
$$= 2\cdot\frac{1}{4} + 5\cdot\frac{3}{4}$$
$$= 4.25.$$

- 引申. 如果可以扔三次呢?

则 Based on 扔两次的均值 4.25,

扔到 1, 2, 3, 4, 会再扔. 5, 6, 则 take.

$\frac{2}{3}$ $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ — $\frac{2}{3}$ $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$

　　　　　　$\frac{1}{3}$ $\begin{pmatrix}5\\6\end{pmatrix}$

$\frac{1}{3}$ $\begin{pmatrix}5\\6\end{pmatrix}$

I take an ordinary-looking coin out of my pocket and flip it three times. Each time it is a head. What do you think is the probability that the next flip is also a head? What if I had flipped the coin 100 times and each flip was a head?

→ 若是3次均head，加上 ordinary looking，可是这很有可能还是一只 fair coin.
†只有3次
可是下次再扔 head 之几率为 1/2.

→ 100次均1/2. 基本不太可能是 fair，可是，认为 double-header，可是
下次再扔 Head 之 prob 为 1.

若无任何 further info. 可假设是 Double header 之概率为 1%. (P) 或设为 P

$$P\left(\begin{matrix}\text{Double}\\ \text{header}\end{matrix}\ \middle|\ 100\text{ Heads}\right)$$

$$= \frac{P(100\text{ Heads}\,/\,\text{Doube})\cdot P(\text{Double})}{P(100\text{ Heads}\,/\,\text{single})\,P(S) + P(100H/D)\,P(D)}$$

$$= \frac{1\cdot P}{\frac{1}{2^{100}}\cdot(1-P) + 1\cdot P} = \frac{P}{\frac{1}{2^{100}}(1-P) + P} \longrightarrow \text{无限接近于 } 1.$$

You throw a fair coin one million times.

What's the expected # of strings of 6 heads,

followed by 6 tails?

- 其实这是先从 中选出 1 个位置放. 12个为一组的 sequence. → 共有 1Million −11 个 vacency 可放.

(1M−11)

$$\underbrace{HHHHHH}_{6}\ \underbrace{TTTTTT}_{6}.$$

- 选出这个 seqence 的 slot 后, 再决定是 6H, 6T 的 seq.

$$\cancel{\frac{1}{1M-11}} \cdot \frac{1}{2^{12}} \times (1M-11).$$

- There are (1Million −11) possible slot for the sequency to occur, in each one of these slots, the possibility is 2^{-12}.

If X, Y ~~t~~totally independent.

$\mu_x \, \sigma_x$ $\quad$ $\mu_y \, \sigma_y$.

What's $E(X+Y)$ $\quad$ $Var(X+Y)$

$E(X-Y)$ $\quad$ $Var(X-Y)$

What if X, Y is not independent?

$E(X+Y) = \mu_X + \mu_Y$

$E(X-Y) = \mu_X - \mu_Y$ （期望的线性）

✓ $Var(X+Y) = \sigma_x^2 + \sigma_y^2$

$Var(X-Y) = \sigma_x^2 + \sigma_y^2$ 这个注意！ $Var(-Y) = Var(Y)$

此二式成立条件是 X, Y independent.

$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X, Y)$

$Var(X-Y) = Var(X) + Var(Y) - 2\,Cov(X, Y)$.

设随机变量 X 与 Y 的概率分布分别为

X	0	1
P	$\frac{1}{3}$	$\frac{2}{3}$

Y	-1	0	1
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

，且 $P\{X^2=Y^2\}=1$

(I) 求二维随机变量 (X,Y) 之概率分布.

(II) 求 $Z=XY$ 的概率分布.

(III) 求 X 与 Y 之相关系数.

首先要记得由两个一维构造二维，边缘分布概率，即为X与Y本身之概率。

Z(X,Y)	0	1	
−1	P_{11}	P_{12}	$\frac{1}{3}$
0	P_{21}	P_{22}	$\frac{1}{3}$
1	P_{31}		$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{2}{3}$	

P(X,Y)	Y: −1	0	1	
X: 0	0	b	0	$\frac{1}{3}$
1	a	0	c	$\frac{2}{3}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

三个0因为

$P(X^2=Y^2)=1$

X=0时，Y不可能取±1

叠加

下方的边缘概率！

于是 $a=b=c=\frac{1}{3}$，很容易得到

Z=XY 与Y独立同分布.

Z	−1	0	1
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

化作 $Cov(X,Y)=\underbrace{EXY}_{0}-\underbrace{EX\cdot EY}_{\frac{2}{3}\cdot 0}$

$$\rho_{XY}=corr(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}}=0$$

$$EX=0\cdot\frac{1}{3}+1\cdot\frac{2}{3}=\frac{2}{3}$$

可能之取值　取该值之概率

$$Var\,X=EX^2-EX\,EX=\frac{2}{3}-\frac{2}{3}\cdot\frac{2}{3}=\frac{2}{9}$$

$$0^2\cdot\frac{1}{3}+1^2\cdot\frac{2}{3}=\frac{2}{3}$$

另一种算法：

各个数据点离开mean的距离和，乘以概率。

$$Var\,X=\left(0-\frac{2}{3}\right)^2\cdot\frac{1}{3}+\left(1-\frac{2}{3}\right)^2\cdot\frac{2}{3}=\frac{6}{27}=\frac{2}{9}$$

Linear regression.

What is linear regression ?

In statistics, linear regression is an approach for modeling the relationship between a scalar dependent variable y and one or more explanatory variables denoted X. The case of one explanatory variable is called simple linear regression.

What is an Random variable (r.v.)

Math.

what's the difference between a r.v. & a common var like in $x+1=5$

代数系统中之变量.

$x+1=6$, x只能take一个确定值.

而 Random variable X, 可以take many values, with some probability.

比如:

$Y =$ sum of seven dice values (facing up).

于是Y可以取 7到 7×6 中的值 (with different p)

随机变量可以取多个值!!

$P(Y=42)$ 即Y取42的概率.

即 "sum of seven dice values" 取42

定义为Y后, 你不再需要写这么一长串了.

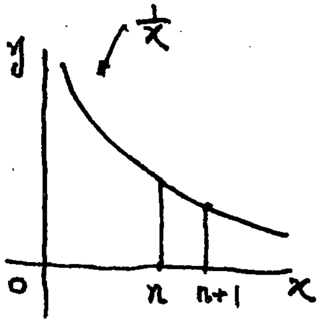
而是简单的写为$Y=42$即可, 带来很大的方便.

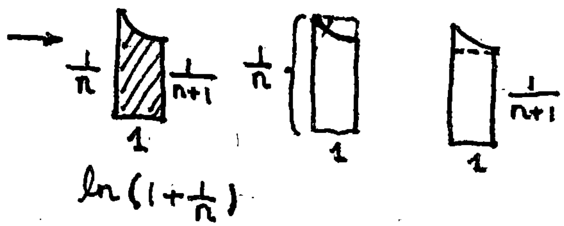
证明：1) $\forall n \in N$，有 $\frac{1}{n+1} < \ln(1+\frac{1}{n}) < \frac{1}{n}$ 成立．

2) 设 $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \quad (n = 1, 2, 3, \cdots)$

证明数列 $\{a_n\}$ 收敛．

1). 令 $f(x)=\frac{1}{x}$　　$\int_n^{n+1}\frac{1}{x}\,dx = \ln|x|\Big|_n^{n+1} = \ln\left(1+\frac{1}{n}\right)$

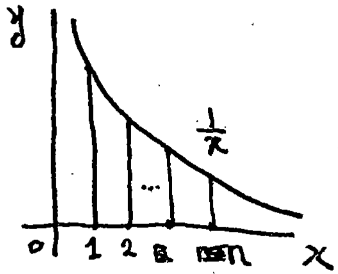


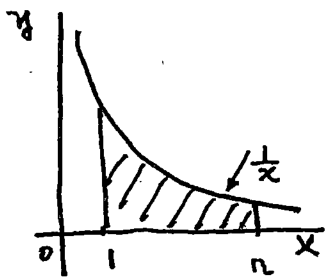


由几何图形可知

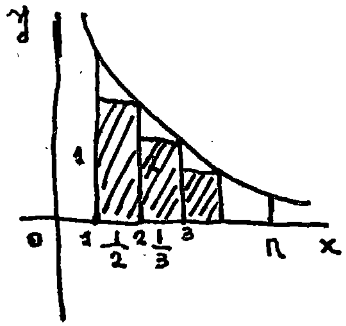
$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$

2). 证明 $a_n = \Sigma - \ln n$ 单调有界序列.

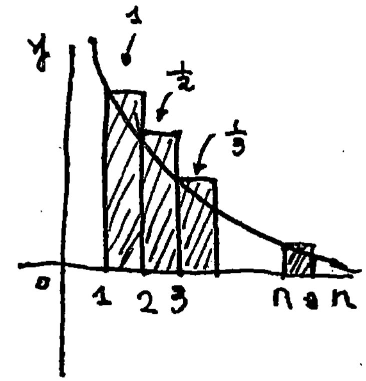




$\int_1^n \frac{1}{x}\,dx = \ln n$



$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} < \ln n$



$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > \ln n$

$$\overset{a_n}{\overset{\shortparallel}{}}$$

$$0 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n < 1$$

$$a_n - a_{n-1} = \frac{1}{n} - [\ln n - \ln(n-1)]$$

$$a_{n+1} - a_n = \frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right) < 0$$

于是 a_n 单调递减且有下界. a_n 收敛.

find the first four ~~power~~ items
of the power series of $\arctan(2x)$

直接求很麻烦. 但 recall $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

于是 $\frac{d}{dx}(\arctan 2x) = \frac{2}{1+4x^2} = 2\left[1 - 4x^2 + (4x^2)^2 - (4x^2)^3 + \dots\right]$

$= 2 - 8x^2 + 32x^4 - 128x^6 + \dots$

$\left(\frac{1}{1+x} = 1 - x + x^2 - x^3\right)$

$$\arctan 2x = \int 2 - 8x^2 + 32x^4 - 128x^6 + \dots$$

$$= \underbrace{2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \frac{128}{7}x^7}_{\text{于是为前4项.}} + C$$

$x=0$ 时

$\arctan 2x = 0$

$C = 0$

求积分：

$$\int \ln x \, dx$$

$$\int x^2 e^x \, dx$$

$$\int e^x \cos x \, dx$$

$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx.$$

何时用分部积分? Clue is, 两个 function, g, f, 求导后, 一个并未变复杂, 一个能简化.

比如 x^2, e^x

$$\frac{d}{dx}\downarrow \quad x^2 \to 2x \qquad e^x \to e^x$$

(简化, 并未变复杂)

$$\int \ln x\,dx = \ln x\cdot x - \int x\,d\ln x = \ln x\cdot x - \int x\cdot\frac{1}{x}\,dx = \ln x - x.$$

注: $d\ln x = \frac{1}{x}dx.$

$\int \frac{1}{x}dx = \ln|x|$

积分时要加绝对值 保证 $\ln x$ 有定义.

$$\int x^2 e^x dx = \int x^2\,de^x = x^2e^x - \int e^x\,2x\,dx$$

$$\int e^x x\,dx = \int x\,de^x = xe^x - \int e^x dx = xe^x - e^x.$$

$$= x^2e^x - 2xe^x + 2e^x + C.$$

$\int e^x\cos x\,dx$ (e^x, $\cos x$ 求导后, 二者都未简化, 也未变复杂. 于是分部积分可做. 谁进 d, 都可以)

$$= \int e^x\,d\sin x = e^x\sin x + \int -\sin x\, e^x dx$$

$$\int e^x\,d\cos x = e^x\cos x - \int \cos x\,e^x dx.$$

于是 $\int e^x\cos x\,dx = e^x\cdot\frac{1}{2}(\cos x+\sin x) + C.$

The average male drinks 2 L of water when active outdoors (with a standard deviation of .7L)

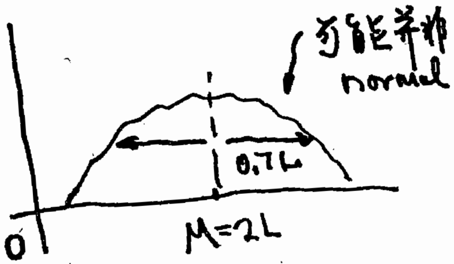
You're planning a full day nature trip for 50 men and will bring 110 L of water.

What's the probability that you will ran out?

key points 是. 不管最初是什么 distribution.
i.i.d. 叠加后. 均会 approach standard normal!

σ: standard deviation.
σ^2: variance (variance 是直接除以 n^2).

$\mu = 2L$, $\sigma = \frac{0.7L}{\sqrt{50}} = 0.099$

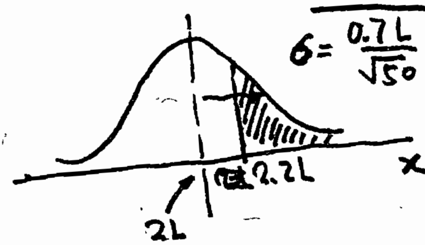


$P(\text{ran out}) = P(\text{use more than } 110L)$

$= P(\text{average use more than } 2.2L)$

$= \int_{2.2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$\sigma = \frac{0.7L}{\sqrt{50}}$



或者可以数一下 n个σ.

$\sigma = 0.099$

$\Delta x = 2.2 - 2L = 0.2$

$n = \frac{\Delta x}{\sigma} = \frac{0.2}{0.1} = 2$

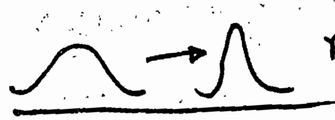
2σ

standard deviation

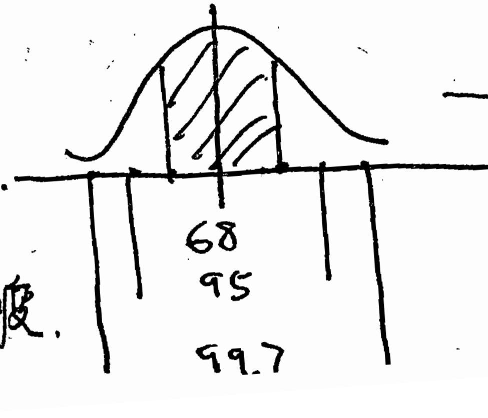
中心极限定理:
化腐朽为神奇的奥义.

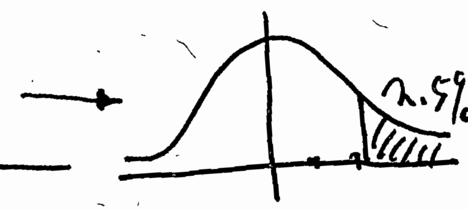
any distribution

i.i.d. 叠加 n越大越好.
n越大,越 → normal



n越大越瘦.





3.41 If S_t follows a log-normal Brownian motion, what process does the square of S_t follow?

若 S_t follows a log normal Brownian motion, 则它满足以下 stochastic differential eqn.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$ ，其中，$\mu > 0$, $\sigma > 0$, W_t : Brownian motion

Let $Y_t = S_t^2$

$$dY_t = d^2 S_t \overset{\text{ito's lemma}}{=} 2S_t dS_t + \frac{1}{2} 2 (dS_t)^2$$

$$(dS_t)^2 = \underbrace{\mu^2 S_t^2 dt^2 + 2\mu\sigma S_t^2 dW_t dt}_{\text{为}0} + \sigma^2 S_t^2 dt$$

$$= 2S_t^2 (\mu dt + \sigma dW_t) + \sigma^2 S_t^2 dt$$

$$= 2Y_t (\mu dt + \sigma dW_t) + \sigma^2 Y_t dt$$

$$= Y_t (2\mu + \sigma^2) dt + Y_t \cdot 2\sigma dW_t$$

于是 Y_t 仍然是一个 log-normal, with new parameter.

新的 drift 是 $2\mu + \sigma^2$

diffusion 是 2σ.

Suppose we are doing a random walk on the interval $[0, 1000]$, starting at 80. So with probability $\frac{1}{2}$, this number increases or decreases by one at each step. We stop when one of the boundries of the interval is reached. What is the probability that this boundary will be 0 ?

Crucial observation: our location is a martingale.

↳ Expected position at any time in the future is our current position.

Let X_t be the value at time t. It's a martingale.

Let p be the probability of hitting 1000. 则 hitting 0 之概率为 $1-p$.

于是 expected value at stopping is.

$$1000p + 0\cdot(1-p) = 80$$

于是 $p = \frac{8}{100}$

The probability to hit 0 is thus: $1-p = \frac{92}{100}$.

→ Jump obeys $\sim N(0,1)$

Everywhere continous, everywhere not differentiable.

→ The increments

$$W_{t_2} - W_{t_1},\ W_{t_3} - W_{t_2},\ \cdots,\ W_{t_m} - W_{t_{m-1}}$$

are independent, and the distribution of $W_{t_j} - W_{t_{j-1}}$ is given by

$$W_{t_j} - W_{t_{j-1}} \sim N(0,\ t_j - t_{j-1})$$

mean 相同 Variance 是 Δt.

If $M(x)$ is the cumulative Gaussian function and X is $N(0,1)$, then what is $\mathbb{E}[M(X)]$?

$$\int_{-\infty}^{x} e^{-x^2}\,dx$$

Explain change of measure. Give an example when change of measure is useful in Monte Carlo.

If X is $N(\mu, \sigma)$ and $\lambda > 0$, calculate the values of $\mathbb{E}(X^2)$ and $\mathbb{E}(\exp(\lambda X))$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$\mathbb{E}(X^2) = \sigma^2 + \mu^2$$

$$\mathbb{E}(\exp(\lambda X)) = \int_{-\infty}^{+\infty} \exp(\lambda x) \cdot \text{p.d.f.}\, dx$$

$$= \int_{-\infty}^{+\infty} \exp(\lambda x) \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{(x-\mu)^2}{2\sigma^2} dx$$

经2这里是2！

把 λx 乘进去，然后配方。

得.

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{x^2 - 2\mu x + \sigma^2 \lambda x + \mu^2}{2\sigma^2} dx$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x-(\mu+\sigma^2\lambda)}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} + \frac{(\mu+\sigma^2\lambda)^2}{2\sigma^2}\right)$$

$$= \exp\left(\mu\lambda + \frac{\sigma^2\lambda^2}{2}\right)$$

Prove the covariance matrix
is positive definite.

1. There're actually several forms of central Limit theorem.

→ Weekly convergance result in prob.

2. Central limit theorem involves the average of a sum of independent identically distributed R.V. with finite second moment.

→ As the # of R.V. increases, the average approaches a normal distribution with known parameters.

→ Remarkable point: this result is independent of the starting distribution of the R.V.

3. More formally, $X_1, X_2, \cdots, X_n$ be a seq of n independent and identically distributed R.V. with finite mean μ and variance σ^2. Let S_n denote the sum of R.V.

$S_n = X_1 + X_2 + \cdots + X_n$, R.V.

$$Y_n = \frac{S_n - \mu n}{\sigma \sqrt{n}}$$ converges (in distribution to the standard normal distribution)

Let X, Y be two independent
uniform R.V. on $[0, 1]$.
What's the p.d.f. of $X+Y$.

What is the distribution function and density function of the kth order statistics?

Suppose that three assets A, B, C are such that the correlation coefficient of A & B is 0.9, and the correlation of B & C is 0.8. Is it possible for A & C to have correlation coefficient 0.1 ?

写出 covariance matrix .

$$M=\begin{bmatrix} 1 & 0.9 & 0.8 \\ 0.9 & 1 & 0.1 \\ 0.8 & 0.1 & 1 \end{bmatrix}$$

$$|1| = 1 > 0$$

$$\begin{vmatrix} 1 & 0.9 \\ 0.9 & 1 \end{vmatrix} = 0.19 > 0$$

$$|M| = -0.316 < 0$$

非正定，于是 M can't be covariance.

A plane has one hundred seats and there are exactly one hundred people boarding. They line up at the gate in exactly the same order ~~of~~ as the order of the seats, on the plane, from 1 to 100. However, the first person is Grandma who doesn't see well, so instead of taking the first seat on the plane, she picks a random seat. Now the rule is, when any other person enters the plane, he or she will ~~w~~ try to sit at his own seat. ~~Eventually~~ If this is not possible, this person chooses one of the free seats randomly. Eventually, everyone boards the plane. What's the probabilities that the last person (number 100) will sit at his assigned seat #?

$\square 1 \ \underbrace{\square 2 \ \cdots \ \square 99}\ \square 100$

#1 乘客 take □1，于是大家全对。#100 亦会 take 第100号座位。

#1 take □100，于是 #100 never take his own seat.

#1 take □2 — □99 中的一个，设为 k。

□2 ~ □k-1 是对的，则不再考虑。

#k 面临的 seats，有 □1，□k+1 ··· □100。

同 #k
- □1 大家 (k+1 ··· 100) 全对，
- □100 #100 never takes his own seat
- □k+1 ~ □99

#1
- □1
- □k → #k
  - □1
  - □100
- □100

$\in$ □(2~99)

把整个概率空间分成两份，左：最后一个乘客坐自己座，右：没坐到自己座。

1：第一个乘客坐 □1，于是为左空间。
第 —— □100，右空间。
□k，继续往下看。$k \in [2, 99]$

2. #1 takes □k，于是k在 □1，□k+1，□n 中选择，依然是左右空间一样概率。

3. 最多递归至乘客 n-1 在 1 与 n 间选。

于是始终是 $\frac{1}{2}$。

通过以上推理发现，乘客100，只有两种可能，或者坐在 □1，或者 □100。于是概率为 1/2。

且达到两种状态 总之 symmetry 相同。

Given an example of a distribution
with inf variance.

Suppose the random variable X has a probability density function $f_X(x)$.

What density function does $g(X)$ have?

$$P(X \le x) = F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

Let $Y = g(X)$

$$\text{p.d.f. of } g(X) = \frac{d}{dx} P(Y \le x)$$

$$= \frac{d}{dx} P(g(X) \le x)$$

$$= \frac{d}{dx} P(X \le g^{-1}(x))$$

$$= \frac{d}{dx} F_X(g^{-1}(x))$$

$$= f_X(g^{-1}(x)) \cdot \frac{d}{dx} g^{-1}(x)$$

Suppose $x_1, x_2, \cdots, x_n$ are independent identically distributed from $[0, 1]$, and uniform on the interval. What's the expected value of the maximum?

What's the expected value of the difference between max & min?

Soln:

这一步很关键, 把 max 转化为取交.

$$\text{c.d.f. of } \max\{x_i\} = P\{\max\{x_i\} \le x\} = P\{\underset{\forall i}{x_i \le x}\} = \prod_{i \in (1,n)} \underbrace{P\{x_i \le x\}}$$

$P\{x_i \le x\} = \text{c.d.f. of } U[0,1]$

$$\frac{x-a}{b-a} = x.$$

$$= \begin{cases} 0 & x < 0 \\ x^n & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

$$\text{c.d.f.} = \int_{-\infty}^{x} \underset{\text{p.d.f.}}{p(y)}\, dy \longrightarrow \text{p.d.f.} = \frac{d}{dx}(\text{c.d.f.})$$

$$\text{p.d.f. of } \max\{x_i\} = \begin{cases} 0 & x < 0 \\ n x^{n-1} & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$$E\{\max\} = \int_{-\infty}^{+\infty} x \cdot p = \int_0^1 x \cdot x^{n-1} \cdot n \cdot dx = \frac{n}{n+1}$$

第二步先算 E of $\min\{x_i\}$.

还是构造概率.

$$P\{\underset{\forall i}{\min\{x_i\}} > x\} = \prod_{i=1}^{n} P\{x_i > x\} = \prod_{i=1}^{n}[1 - P\{x_i \le x\}] = (1-x)^n$$

利用 $x_i \overset{i.i.d}{\sim} U[0,1]$, 所以 $P\{x_i \le x\} = x$

$$= 1 - P\{\min\{x_i\} \le x\}$$

c.d.f. $F(m \le x) = 1 - (1-x)^n$

p.d.f. $p(\min\{x_i\}) = \frac{d}{dx} F = n(1-x)^{n-1}$, $x \in [0,1]$ $\longrightarrow$ $E m = \int_{-\infty}^{+\infty} x \cdot \text{pdf}\, dx = \int_0^1 x \cdot n(1-x)^{n-1} dx \overset{x = 1-t}{=\!=} n\int_0^1 (1-t) t^n\, dt = \frac{1}{n+1}$

$$\Rightarrow E(M - m) = \frac{n-1}{n+1}$$

What is the distribution function of a uniform

$U(a,b)$ R.V. ?

c.d.f.

p.d.f. density function
c.d.f. distribution function.

$U(a,b)$

p.d.f. $X = \begin{cases} 0 & x \le a,\ x > b. \\ \frac{1}{b-a} & a < x \le b \end{cases}$

$$F(x) = \int_{-\infty}^{x} \frac{1}{b-a}\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$$= \frac{1}{b-a}\, t \Big|_a^x = \frac{x-a}{b-a}$$

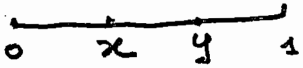
$$\underline{F(x) = \int_{-\infty}^{x} f(y)\,dy}$$

注意这个标记.
一般用大F.

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

What's the distribution function of a uniform $U(a,b)$ random variable?

Let $\{x, y\}$ be uniformly distributed on $[0, 1]$ and seperate $[0, 1]$ into 3 pieces, what is the probability that the 3 pieces can be constructed into a triangle?



不妨设 $y>x$

三段之长度: x,

$y-x$,

$1-y$

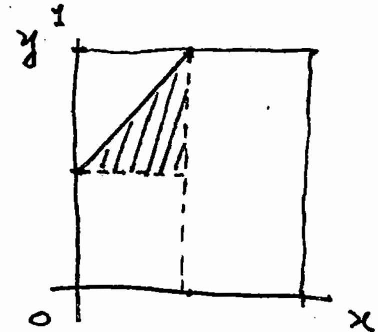
须有: $x+y-x \geq 1-y \Rightarrow y \geq \frac{1}{2}$

$x+1-y \geq y-x \Rightarrow \cancel{x-y \geq \frac{1}{2}}\ y-x \leq \frac{1}{2}$

$y-x+1-y \geq x \qquad x \leq \frac{1}{2}$.

$0 \leq y \leq 1$

$0 \leq x \leq 1$



Romeo & Juliet have agreed to meet for a date sometime between 9pm & 10pm. Each of them randomly picks a time within this interval and shows up and wait for the other for fifteen minutes. What's the probability that they will actually meet?

要诀，均匀分布，通常看能否化成几何解决.

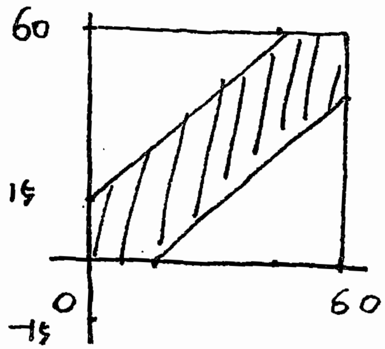
先转化为分钟，设 Romeo & Juliet 抵达时间分别为 x, y.

则有:

$$\left.\begin{array}{l} 0 \le x \le 60 \\ 0 \le y \le 60 \end{array}\right. \longrightarrow (\text{全集})$$

in order to meet,

$$|x-y| \le 15 \Rightarrow \left\{ |y-x| \le 15 \right. \longrightarrow x-15 \le y \le x+15$$



$$\frac{60^2-45^2}{60^2} = \frac{4^2-3^2}{4^2} = \frac{7}{16}.$$

You have been captured and blindfolded by pirates, then placed somewhere on a five-meter-long wooden plank. Disorientated, each step you take is one meter long but in a random direction — either towards the sharks waiting at one end or eventually freedom at the other. If x (integer) is the distance in meters you start from the safe end, determine the probability of your survival as a function of x.

$f = P(\text{safe end} \mid \text{start at } x)$

假若 $x=1$，则它有 $1/2$ 的概率，go to $f(0)$，$\frac{1}{2}$ goto $f(2)$

$$f(1) = \frac{1}{2} f(0) + \frac{1}{2} f(2) \qquad f(0) = 1$$

$$f(2) = \frac{1}{2} f(1) + \frac{1}{2} f(3)$$

$$f(3) = \frac{1}{2} f(2) + \frac{1}{2} f(4)$$

$$f(4) = \frac{1}{2} f(5) + \frac{1}{2} f(3)$$

$$f(5) = 0$$

$$\Rightarrow \quad f(4) = \cancel{\tfrac{5}{5}}\ \frac{1}{2} f(3)$$

$$f(3) = \frac{2}{3} f(2)$$

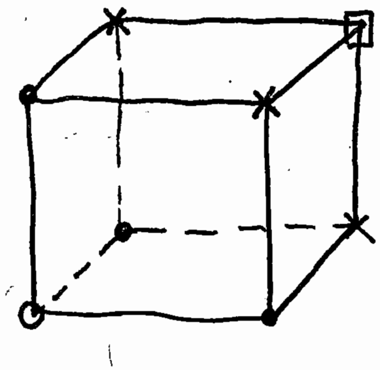
$$f(2) = \frac{3}{4} f(1)$$

$$f(1) = \frac{4}{5} f(0) = \frac{4}{5}$$

$$\Rightarrow \quad f(1) = \frac{4}{5},\ f(2) = \frac{3}{5},\ f(3) = \frac{2}{5},\ f(4) = \frac{1}{5}.$$

$$f(x) = \frac{5-x}{5}.$$

Suppose we have an ant travelling on edges of a cube, going from one vertex to another. The ant never stops, and it takes him one minutes to go along one edge. At every vertex, the ant randomly picks one of the three available edges and starts going along that edge. What's the expected number of minutes that it will take the ant to return to that same vertex.



还是正方体，但需不求.

设从距离为0的点，转回来后需要的时间 $f(0)$.

1 $f(1)$

2 $f(2)$

3 $f(3)$

$$
\begin{cases}
f(0) = f(1) + 1 \\
f(1) = \frac{1}{3}\cancel{1} + \frac{2}{3}\left(f(2)+1\right) \\
f(2) = \frac{2}{3}\left(f(1)+1\right) + \frac{1}{3}\left(f(3)+1\right) \\
f(3) = f(2) + 1
\end{cases}
$$

$\downarrow$ $\frac{1}{3}$ 的概率到起点.

$\frac{2}{3}$ 的 —— 到距离为2的点.

联立求解.

$f(1) = 7$

$f(0) = 8$.

A drawer contains 2 red socks and 2 black socks.
If you pull out 2 socks at random, what's the probability they match.

这题容易上当.

Soln 1) R R
B B

交叉, Red, black 各一只的情况.
其实有4种!

$$\frac{\#\{RR, BB\}}{\#\{RR, BB, RB, RB, BR, BR\}} = \frac{1}{3}$$

Soln 2) 先选一只, 比如是 Red. 则剩下3只中, 可以与之相配的概率为 $\frac{1}{3}$.

$$\frac{1}{2}\cdot\frac{1}{3}.$$

Black

$$\frac{1}{2}\cdot\frac{1}{3}$$

总概率为 $\underbrace{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}}_{\text{(全概公式)}} = \frac{1}{3}$

Soln 3). 总共的可能性, 4选2, 可能是 $C_4^2=6$

一样一只, 可能是 $C_2^1 \cdot C_2^1 = 4$ 可能配成对: $\frac{6-4}{6}=\frac{1}{3}$

红中取1 蓝中取1

3 What's the probability that the fourth business day of the month is Thursday?

此题的精要Tricky处在于，fourth business day！

周六，周日不算 weekday！
business day！

M Tu W Th
1 2 3 4

S M Tu W Th
1 2 3 4 5

仍为第fourth business day！

Sa Su M Tu W Th
1 2 3 4 5 6

故为 $\frac{3}{7}$.

→ A women has two babies. One of them is a girl. what's the probability that the other is a boy?

A family has two children whom you know at least one is a girl. What is the probality that both are girls.? What if you know that the eldest is a girl? What if you know that they have a girl called Clarissa?

elder younger :

B. B. w.p. 0.25

G. G. ——

B. G. ——

G. B. ——

At least, one is a girl: 0.75

Both are girls: 0.25

P(Both are girl | at least one is a girl)

$$= \frac{0.25}{0.75} = \frac{1}{3}.$$

(GG | GG, G.B. G.B.)

三种可能性中选一个.

If you know eldest is a girl (G.G , G.B)

Both are girls: G.G.

$\frac{1}{2}$.

Suppose you are throwing a dart at a circular board. What's your expected distance from the center?

Suppose you win a dollar, if you hit 10 times in a row inside a radius of $R/2$, where R is the radius of the board. You have to pay 10¢ for every try.

If you try 100 times, how much money would you have lost/made in expectation?

Does your answer change, if you are a professional and your probability of hitting inside $R/2$ is double of hitting outside of $R/2$?

→ 平均之 distance.

$$\bar{r} = \int_0^R \frac{2\pi r\,dr}{\pi R^2}\cdot r$$

↓

在 r 这一圈的概率 $\times\ r$

然后求平均.

$$\bar{r} = \frac{2}{3}R$$

- Try 100 次, have to pay $\frac{100\times 10¢}{100\ ¢/\$} = 10\$$.

hit 1 次 $R/2$, $P=\frac{1}{4}$

10次 in a row: $P=\left(\frac{1}{4}\right)^{10} = \frac{1}{2^{20}}$

Try 100 次, 91 个 slot, $91\cdot\frac{1}{2^{20}} = 0.86\times 10^{-4}$

very bad deal

- 中内圈 ($R/2$) 概率若为外圈两倍, 则

hit 10 times in a row: $P=\frac{2^{20}}{3^{20}}$

Try 100 次: $91\cdot\frac{2^{20}}{3^{20}} = 0.0274$

still bad deal.

Suppose $2N$ teams participate in a championship. Once a team loses, it is out. Suppose you know the ranking of teams in advance and you know that a team with higher rank will always win the game. In the beginning, all teams are randomly assigned to play with each other. What would be the probability that in the final, the best team are playing against the second best?

这题也 easy. 最好的肯定能进 Final.

所以让他随便选,选完了. second-best 入另一半即可.

N种可能. out of $(2N-1)$

于是 $P=\dfrac{N}{2N-1}$

记得把 best team 占的那个1减去.

If I draw two cards from an ordinary deck with replacement, what's the probability they are both aces? Without replacement?

Easy one.

With replacement. $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$

$\downarrow$ 52个slots中，4个可能性的Ace.

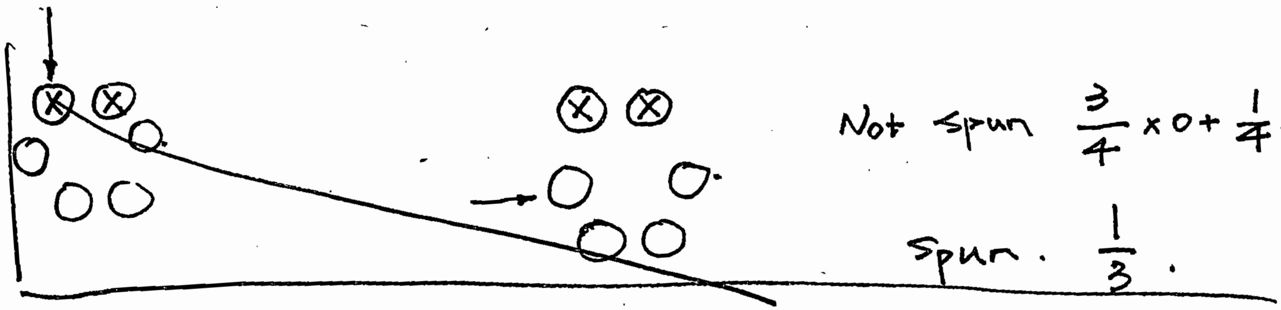
W/O replacement $\frac{4}{52} \times \frac{3}{51} = \frac{1}{13} \times \frac{1}{17}$

$\downarrow$

余下51张牌中，

3个是Ace.

Consider a deck of 52 cards, ordered such that $A > K > Q > J > \cdots > 2$. I pick one first, then you pick one. What's the probability that my card is larger than yours?



Not spun $\frac{3}{4}\times 0+\frac{1}{4}\times 1=\frac{1}{4}$

Spun. $\frac{1}{3}$.

Soln: 利用对称性 $P(\text{我的大}) = P(\text{我的小}) = p$

$$q(\text{Draw}) = \frac{3}{51} = 1-2p$$

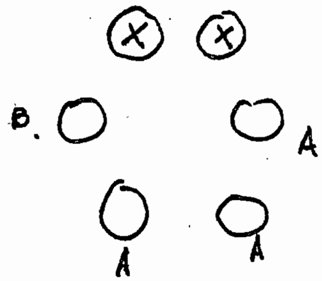
$$\Rightarrow p = \frac{8}{17}$$

你第一张任选.
第二张和第一张相等的概率是
从余下的51张中取到与第一张相同的3张.

You're playing Russian Roulette. There're precisely two bullets in neighbouring chambers of the six-shooter revolver. The barrel is spun. The trigger is pulled and the gun does not fire. You're next, do you spin again or pull the trigger?

(注: Russian Roulette you're fire yourself)

An easy question:



Not spin.

因为第一轮没开火，故可能处于空挡的4处。

只有B处，会fire yourself. $\frac{1}{4}$.

Spin 的话.

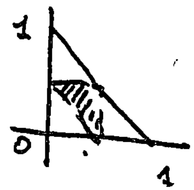
$\frac{2}{6} = \frac{1}{3}$ 概率 fire yourself.

Math.

Say I take a rubber band and randomly cut it into three pieces. What's the probability that one of the pieces has length greater than 1/2 of the original circumference of the rubber band.

$$x + y + z = 1$$

$$\Rightarrow \quad x + y < 1$$



$$x < \frac{1}{2}$$

$$y < \frac{1}{2}$$

$$z < \frac{1}{2} \quad \Rightarrow \quad x + y > \frac{1}{2}$$

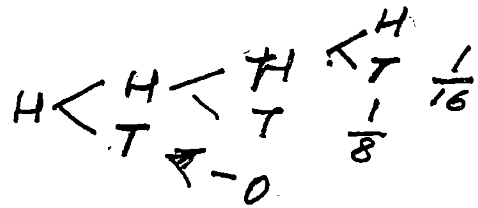
Math.

Flip a coin until either HHT or HTT appears.

Is one more likely to appear first?

If so, which one with what probability.

HHT

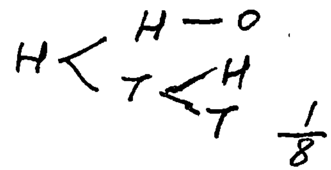


T – 0

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$$

$$\frac{\frac{1}{8}}{1-\frac{1}{2}}=\frac{1}{4}$$

~~H~~ HTT



T – 0

$$x=\frac{3}{8}+\frac{1}{4}(x+2)+\frac{1}{2}(x+1)$$

$$\frac{1}{4}x=\frac{3}{8}+\frac{1}{2}+\frac{1}{2}=\frac{11}{8}$$

$$x=\frac{11}{2}$$

$$p=\frac{1}{x}=\frac{2}{11}.$$

$98 - 72 = 26$

$67 + 31 = 98$

$+27$

$$\frac{21+22+24+31+36}{36} = \cancel{3 + \frac{13}{18}}$$

$$\frac{161}{36}$$

	1	2	3	4	5	6		
1	1	2	3	4	5	6	3.5	$\frac{21}{6}$
2	2	2	3	4	5	6	$\frac{22}{6}$	
3	3	3	3	4	5	6	$\frac{24}{6} = 4$	
4	4	4	4	4	5	6	$\frac{27}{6}$	
5	5	5	—	—	—	6	$\frac{31}{6}$	
6	6	—	—	—	—	6	6	

Math,

You have five coins. One is double-headed.

Pick one coin at random without looking and throw it 5 times.

Suppose the out are five heads, what is the probability that the coin picked is double-headed one?

$$P\left(\begin{matrix}\text{Double}\\ \text{headed}\end{matrix} \,\middle|\, 5\text{ heads}\right) = \frac{P(5\text{ heads})\cdot P\left(\begin{matrix}\text{two}\\ \text{headed}\end{matrix}\right)}{P(5\text{ heads})\cdot P\left(\begin{matrix}\text{One}\\ \text{headed}\end{matrix}\right) + P(5\text{ heads})\, P\left(\begin{matrix}\text{two}\\ \text{headed}\end{matrix}\right)}$$

$$= \frac{1\cdot\frac{1}{5}}{\frac{1}{32}\cdot\frac{4}{5} + 1\cdot\frac{1}{5}}$$

$$= \frac{8}{9}.$$

You're playing a game where the player gets to draw number 1-100 out of hat, replace and redraw as many times as they want, with their final number being how many dollars they win from the game. Each "redraw" cost extra $1.

How much would you charge someone to play this game?

假设最后一轮你拿到X，便停止.

也就是说 X要大于等于你之后的获利期望.

下一轮.

$$E = -1 + \frac{X-1}{100}E + \frac{\sum_{t=X}^{100} t}{100}$$

$$2(101-X)E = (101-X)(100+X) - 200$$

$$\rightarrow E = \frac{(101-X)(100+X)-200}{2(101-X)}$$

This is maximized by $101 - 10\sqrt{2} \doteq 86.86$.

Thus the best value for X is either 86 or 87.

These values give $E \doteq 86.33$ or $E \doteq 86.37$.

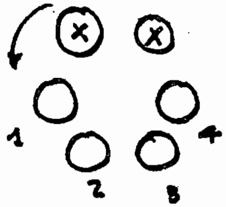
So $X = 87$ is better, and the value of the game is $1209/14$.

Russian roulette,

4 blank 2 bullets, all in a row.

If someone shoots a blank next to you,

would you take another shot or spin.



shoot blank,

1, 2, 3, 4 中选一个.

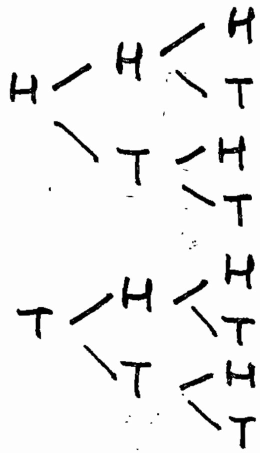
如果是1，再打会挂.

2,3,4 无事

不spin，$\frac{3}{4}$ 几率 survive.

spin，$\frac{2}{3}$ 几率 survive

故，不spin.



starts again, 作废

1

2

3

4

5

6

starts again, 作废

Math

翻A游戏，一副去掉大小王的扑克共有52张牌，庄家把牌一张一张翻开，每翻一张给你一块钱，一直到翻到一个A为止（这张也给你一块钱）。问，十块一次你玩不玩。

假设第X张，翻出A，

可见X是一个随机变量。

$$P(X=x) = \left(\frac{16}{17}\right)^{x-1}\left(\frac{1}{17}\right)^{x} \qquad x=1,2,\cdots,49$$

49,50,51,52

然后求期望。

$$EX=\sum_{x=1}^{49} xP(X=x)$$

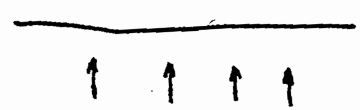
计算很麻烦！

亦可以这样想：

M张红牌，N张蓝牌，随机翻牌，第一次翻蓝牌的期望长度。

可以想成：

红牌M张



N张蓝牌插进去，求每一节之平均长度。

N张蓝牌将M张红牌分成N+1段，每段之平均长度是 $\frac{M}{N+1}$。

单指红牌。

每段最后加一个蓝牌。

平均长度加一。

故第一次翻出蓝牌之平均期望长度是：

$$1+\frac{M}{N+1}$$

回到原问题，期望长度是： $1+\frac{48}{5}=10.6$

一个骰子6面分别是1到6.

问你平均要投多少次,才能让每个数字都投过.

Math.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$$

$$=\frac{60+30+20+15+12+10}{60}=\frac{147}{60}$$

— 擲第一次可以得到一个数，第二次擲出不同数之概率是 $\frac{5}{6}$，第三次不同数之概率是 $4/6$.

故擲出第二个数之期望次数是 $\frac{6}{5}$　　其期望长度是 $\frac{6}{4}$.

類推可得：

$$1+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}=14.7.$$

Math.

Suppose you have a perfectly round disk.

You put three legs on this disk to form a table.

Suppose the legs are perfectly perpendicular to the disk and attached to the disk firmly.

What's the probability that the table will fall?

当三条腿在同一个半圆时，会倒

于是回到作废书问题，n 个点，随机投到圆周上，

问同时落到一个半圆上之概率？

一共 N 个点，固定一个点，从这点开始，

顺逆时针各有一个半圆，其他 N-1 个点

都落在同一个（比如顺时针）半圆上的概率是 $(\frac{1}{2})^{N-1}$.

总共 N 个点，所以总概率是

$$\frac{N}{2^{N-1}}$$

Math.

一个口袋有10个红色球，20个蓝色球，30个绿色球.
你随机地把球一个一个取出来.问红球最先被拿完的概率？

红球最先被拿完，亦即10个红色球被取出时，
口袋内至少还有一个蓝色球和一个绿色球的概率.

如果把这个排列反过来看，就是说看见红球前，
先看见蓝球和绿球. 共有两种情况.

红 蓝 绿
10 20 30

··· | |
蓝绿

··· | |
蓝绿

1. 先蓝后绿再红.

先蓝概率是 $\frac{1}{3}$，绿红两色中先绿再红之概率为 $\frac{3}{4}$.
(先蓝后其他蓝球不影响绿色的概率)
故第一种情况之概率是 $\frac{1}{3}\times\frac{3}{4}=\frac{1}{4}$

2. 绿蓝而后红.

先绿 $\frac{1}{2}$，蓝红两色中先蓝再红之概率是 $\frac{2}{3}$
第2种情况之概率是:

$$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$$

故总概率是 $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$.

仔细看一下上面之解法可知，
红蓝绿数量不紧. 比例是决定最后之结果.

Math.

We're playing a game, where we start at 0,
and flip a coin until someone wins.

Every head that is flipped adds +1 to the number,
and every tail adds -1.

If the number reaches -10, the interviewer wins
20, you win

What's the probability the interviewer wins?

The first one is 1/3, think of it like a random walk.

though you don't need

interviewer wins 之几率是 1/3.

Math.

Suppose you, player A, B are in a tennis tournament.

Player A has a better chance of beating you than player B does.

If you would like to maximize your chance of winning two games in a row, which sequence you prefer?

ABA or BAB ?

If you want to win two game,
you must win the middle game,
so you should play the weaker player in the middle: ABA.

Math

You have 4 coins, you throw them in the air, for every head that lends, you get 1 dollars.

How much would you pay to play this game?

$\searrow$ $\frac{1}{16} \times 0 + \frac{4}{16} \times (1) + \frac{6}{16} \times (2) + \frac{4}{16}(3) + \frac{1}{16}(4) = \$2$.

$\searrow$ 明显更优之解法!

binormial distribution (每个coin是head还是tail是独立的).

$$E(X) = np = 4 \times 0.5 = 2$$

MF.

When A predicts raining, the chance of raining is 60%.

When B — raining, ———————— is also 60%.

If both A and B ~~to~~predict to rain (assume they're independent)

What's the chance of raining?

P(A wrong)

两种情况:

- 预报下雨打没下
- 预报不下打下了.

若没有下雨, 则是 both A & B are wrong.

$P(\text{not rain}) = P(\text{both A \& B are wrong}) \underset{\text{A\&B independent}}{=\!=} P(\text{A wrong})\, P(\text{B wrong})$

$= 0.4 \times 0.4 = 0.16.$

$P(\text{rain}) = 1 - P(\text{not rain}) = 0.84.$

↑ 这是A, B完全 independent 之情况

↓ 若A与B正相关.

至少也要 0.6.

顶多是一个 range (0.6, 0.84)

$P(A \cup B) = 1 - (A \cap B) = 1 - (1-0.6) * (1-0.6) = 0.84.$

- 60%吧, confidence interval 变窄了而已.
  把a, b想成一个coin掷两遍好了.
- 这是一个典型之boost问题,
  multipredictor, how to sum up.
  其实过肯定大于 0.6.
- If A and B are two suckers, then nobody knows the chance of raining. If A and B are two gods, then 60% for sure. generally 60% +/- delta%.
  depends on the ability of the predic.

Find ~~positiver~~ positive integers x, y, z
and $x^x + y^y = z^z$.

MP.
Math problem.

- 列举一下 都比问别人快.

- 一道奥赛送分题，猜几个特殊解，而后证明大于那些数的都不行，便搞定了.

- $1^1 \quad 2^2 \quad 3^3 \quad 4^4 \quad 5^5 \quad 6^6$

  $1 \quad 4 \quad 27 \quad 256 \quad 3125 \quad 46656.$

明显无解!

假设 $z > x >= y > 0$

$$z^z >= (x+1)^{x+1} = \underbrace{(x+1)\cdot(x+1)\cdot\cdots\cdot(x+1)}_{x+1\text{个}}$$

$$= x^{x+1} + (x+1)\cdot x^x + \cdots > 2x^x >= x^x + y^y$$

P2.32. 某地区青年血压服从 $N(110, 12^2)$

任选一青年，测血压 X.

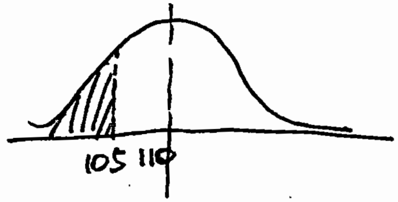
求 1). $P(X \le 105)$,

$P(100 \le X \le 120)$

2). 确定最小之 x，使

$P(X > x) \le 0.05$

1) $\dfrac{X-110}{\sqrt{122}} \sim N(0,1).$



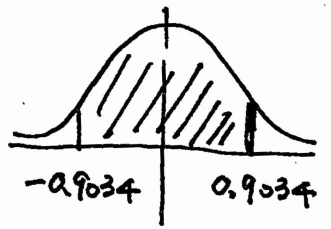
a. $\dfrac{105-110}{\sqrt{122}} = -0.453.$

normcdf$(-0.453) = 0.3254$

$P(X \leq 105) = 0.3254.$

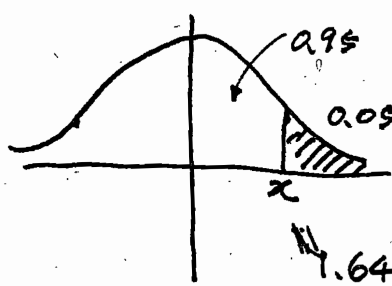
b. $\dfrac{100-110}{\sqrt{122}} = -0.9034.$

$\dfrac{120-110}{\sqrt{122}} = 0.9034$



$P(100 < X \leq 120) = 0.6347.$

2).



$\Downarrow$ 1.6449

$\dfrac{x-110}{\sqrt{122}} = 1.6449.$ $\qquad x = 128.169.$

$P_{2,30}$ $X \sim N(5,4)$

求 a 1) $P(X<a) = 0.90$

2) $P(|X-5|>a) = 0.01$.

$N(\mu, \sigma^2)$

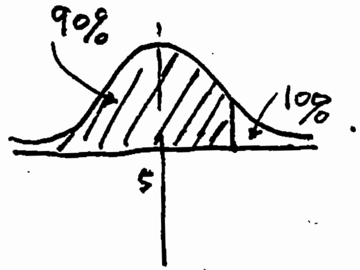
$\uparrow \quad \uparrow$

$5 \quad 4$

$\sigma = 2$.

$$P(X < a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\frac{-(x-\mu)^2}{2\sigma^2}\, dx.$$

1)



$X \sim N(\mu, \sigma^2)$

$\frac{X-\mu}{\sigma} \sim N(0,1) \qquad \frac{X-5}{2} \sim N(0,1)$

norminv(0.90) = 1.2816.

$N(0,1)$: $x = 1.2816$

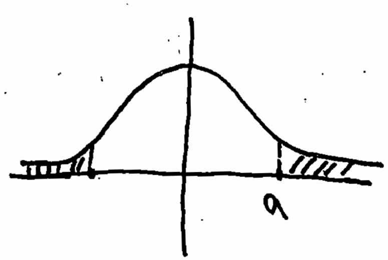
$\frac{X-5}{2} = 1.2816$

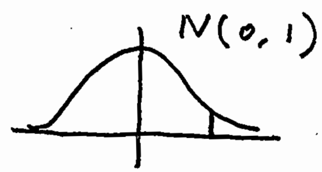
$X = 5 + 2.5632 = \underline{7.5632}$

思路是通过norminv
求标准正态之quantile
(即cdf的inv).

---

2). $P(|X-5| > a) = 0.01$





norminv(0.995) = 2.5758.

$\frac{X-5}{2} \sim N(0,1)$

$P\left(\frac{|X-5|}{2} > 2.5758\right) = 0.01$

$a = \underline{\cancel{4.5516}\ 5.14}$.

P2.24.

设某车间有同类型设备100台.

各台独立工作.

每台设备处于故障状态之概率为0.01.

又设每台设备的故障可由一名维修人员修理.

求此车间应配备多少维修人员.

首保证设备发生故障而不能及时修理的

概率小于0.01.

设KT 维修人员

X为设备处于故障之概率，X>k的概率为0.01

$$P(X>k)=\sum_{k+1}^{100} C_{100}^{k}\, 0.01^{k}\, 0.99^{100-k}$$

$$=1-\sum_{l=0}^{k} C_{100}^{l}\, 0.01^{l}\, 0.99^{100-l} = 0.01.$$

$l=0 \quad C_{100}^{0}\, 0.01^{0}\, 0.99^{100} = 0.3660$

$l=1 \quad C_{100}^{1}\, 0.01^{1}\, 0.99^{99} = 0.3697$

$l=2 \quad C_{100}^{2}\, 0.01^{2}\, 0.99^{98} = 0.1849$

$l=3 \quad C_{100}^{3}\, 0.01^{3}\, 0.99^{97} = 0.0610$

$l=4 \quad C_{100}^{4}\, 0.01^{4}\, 0.99^{96} = 0.0149.$

$k=4$个人.

$P(X>4) = 1-P(X=0,1,2,3,4) = 0.0035 < 0.01$

$P_{2.22}$.

设某类电子元件寿命的概率密度为：

$$f(x)=\begin{cases}\frac{a}{x^2} & x>100 \text{小时} \\ 0 & x\le 100\end{cases}$$

1). $a=?$

2). 一台设备中要使用三个这种元件。问在开始之150小时中，三个元件中至少有一个要替换之概率为？

3). 开始使用后三个元件全要替换之概率.

1). $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

$\int_{100}^{+\infty} \frac{a}{x^2}\,dx = 1$

$1 = a\cdot\left(-\frac{1}{x}\right)_{100}^{+\infty}$

$\frac{1}{a} = \frac{1}{100} - 0 \Rightarrow a = 100.$

单个是指数分布

三个是二项分布.

2). $P(X \leq 150) = \int_{-\infty}^{150} f(x)\,dx$

$= \int_{100}^{150} \frac{100}{x^2}\,dx$

$= 100 \left.\frac{1}{x}\right|_{150}^{100} = 100\left(\frac{1}{100} - \frac{1}{150}\right) = 1 - \frac{2}{3} = \frac{1}{3}$

P(至少有1个) $= 1 -$ P(一个也没有) $= 1 - \left(\frac{2}{3}\right)^3 = \frac{27-8}{27} = \frac{19}{27}.$

3) 全坏 $P(3个) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

P2.16.

一电话交换台每分钟收到呼唤之次数服从参数为4之泊松分布. 求

1) 每分钟恰有8次呼唤之概率.

2) 每分钟呼唤次数大于10之概率.

$$P(X=k)= e^{-\lambda}\frac{\lambda^k}{k!}$$

$\lambda=4,\ k=8$

$$P(X=8)= e^{-4}\frac{4^8}{8!} = 0{,}0298.$$

$$P(X>10)=1-P(X=1,2,3,\cdots,10)$$

P2.15 易

一繁忙汽车站，每天大量汽车通过，

设每辆车在一天某时段内事故率为 0.0001.

在某天该时段内有1000辆汽车通过.

问出事故次数不小于2之概率.

事故次数服从泊松分布.

$$P(x=k)= e^{-\lambda} \frac{\lambda^k}{k!}$$

1000辆车

$$\lambda = 0.0001 \times 1000 = 0.1$$

$$P(x=0)= e^{-0.1} \frac{0.1^0}{0!} = e^{-0.1}$$

$$P(x=1)= e^{-0.1} \frac{0.1^1}{1!} = 0.1\, e^{-0.1}$$

$$P(x\text{不小于}2) = 1- P(x\text{小于}2)$$

$$= 1 - P(x=0) - P(x=1)$$

$$= 1- e^{-0.1}(1+0.1)$$

$$= \cancel{0.4953}\ 0.0047.$$

書 P1.40

甲乙二人投篮，投中概率为 0.6，0.7. 今各投 3 次.

求

1) 二人投中概率相等之概率.

2) 甲比乙投中次数多之概率.

1. 相等 $= P(甲0\ 乙0) + P(甲1, 乙1) + P(甲2, 乙2) + P(甲3, 乙3)$

$= 0.4^3\, 0.3^3 + C_3^1\, 0.6\, 0.4^2 \cdot C_3^1\, 0.7 . 0.3^2 + C_3^2\, 0.6^2\, 0.4 \cdot C_3^2\, 0.7^2\, 0.3 + 0.6^3\, 0.7^3$

$= 0.3208.$

2. 甲 > 乙 $= P(甲1\ 乙0) + P(甲2, 乙0) + P(甲3, 乙0)$

$+ P(甲2, 乙1) + P(甲3, 乙1)$

$+ P(甲3, 乙2).$

西
P1.38

三人独立去破译一份密码，

各人能译出概率为 $\frac{1}{5}$，$\frac{1}{3}$，$\frac{1}{4}$.

问三人中至少一人能将此密码译出之概率.

$P(\text{至少一人}) = 1 - P(\text{一人没有})$.

$$= 1 - \frac{4}{5} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{5}.$$

蓄 P1.26.

甲乙两盒放螺栓.

甲 60长，40短

乙 20长，10短.

任取一盒，从中任取一栓.

发现是长螺栓.

求此栓来自甲盒概率.

$$P(\text{事前甲盒} \mid \text{长螺栓}) = \frac{P(\text{甲}\,\&\&\,\text{长})}{P(\text{长螺栓})} = \frac{\frac{1}{2} \times \frac{3}{5}}{\underbrace{\frac{1}{2}}_{\text{甲盒}} \times \underbrace{\frac{3}{5}}_{\text{甲盒中长栓}} + \underbrace{\frac{1}{2}}_{\text{乙}} \underbrace{\frac{2}{3}}_{\text{乙中长栓}}} = \frac{\frac{3}{10}}{\frac{3}{10} + \frac{1}{3}} = \frac{9}{19}$$

萬
1.25

已知男人中5%色盲.
女人中0.25%

今从男女人数相等之人群中随机选一人.
恰好是色盲. 问此人为男性之概率多少.

$$P(男 \mid 色盲) = \frac{P(男 \&\& 色盲)}{P(色盲)} = \frac{5\% \frac{1}{2}}{P(色盲 \mid 男) P(男) + P(色盲 \mid 女) P(女)}$$

$$= \frac{5\% \times \frac{1}{2}}{5\% \times \frac{1}{2} + 0.25\% \times \frac{1}{2}} = \frac{20}{21}$$

P书 1.24

甲乙两袋.

甲袋中n只白球，m只红球

乙袋中N只白球，M只红球

今先从甲袋中取一只球入乙.

再从乙中取一只.

问取到白球概率.

甲 → 乙

红 $\dfrac{m}{m+n}$ ⟶ 乙中 N 白, $M+1$ 红. $\dfrac{N}{N+M+1}$

白 $\dfrac{n}{m+n}$ ⟶ 乙中 $N+1$ 白, M 红 $\dfrac{N+1}{N+M+1}$.

$$P_{tot} = \frac{mN + n(N+1)}{(m+n)(M+N+1)}.$$

2. 袋中10球，9白1红.

10人依次于袋中各取一球.

每人取完后不再放回，

问第一人，第二人，直至最后一个取得红球各为多少.

均为 $1/10$.

与你取之顺序无关

P1.18 掷两骰子，已知两点之和为7.
求其中一颗为1点之概率.

$\underline{(1, 6)}$, (2, 5), (3, 4), (4, 3) (5, 2) $\underline{(6, 1)}$

$\frac{1}{3}$.

P1.17

毒 某油漆公司发出17桶漆，其中白10桶，黑4桶，红3桶。

搬运过程中所有标签脱落。

交货人随意将这些油漆发给顾客。

问一个定货4桶白，3黑，2红之顾客，

按所定颜色得货之概率为多少。

$$\frac{C_{10}^{4} \times C_{4}^{3} \times C_{3}^{2}}{C_{17}^{9}}$$

1.15

将3个球随机放入4个杯子.

求杯中球的最大数分别为1, 2, 3之概率.

答.

最大数为3. 三只投入一个杯子.

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{64}$$

第一只球
4杯子中选一个.

最大数为2.

$$C_3^2 \quad \cancel{2}\ 4 \quad 3 \quad = \cancel{2}^{3}\times 4\times 3.$$

哪两只球
在同一只杯.

两球在
哪只杯.

余球在
哪只杯.

总选择 4^3, $P=\frac{3\times 4\times 3}{64}=\frac{9}{16}$

最大数1.

大家都在不同杯.

$$\frac{4\times 3\times 2}{4\times 4\times 4}=\frac{3}{8}.$$

P1.12
离. 1500个产品中有400个次品，
1100个正品. 现任取200个,
1) 求恰有90个次品之概率
2) 求至少有2个次品概率.

任取一只，次品概率 $p = \frac{400}{15} = \frac{4}{15}$.

正品. $q = 1 - p = \frac{11}{15}$.

恰有90次品

$$C_{200}^{90} \; p^{90} q^{110} .$$

可能不对.

$$\checkmark \quad \frac{C_{400}^{90} \, C_{1100}^{110}}{C_{1500}^{200}}$$

至少有两个次品 = 1 − 0次品 − 1次品.

$$= 1 - q^{200} - C_{200}^{1} p^{1} q^{199} .$$

P1.10

房间中有10个人，分别佩戴1号到10号胸章。任选3人，记录其胸章，求

1) 最小号码为5的概率。

2). 最大号码为5的概率。

葛钊楼

3). 任取3个，无任何限制

$$C_{10}^{3} = \frac{10\times9\times8}{3\times2\times1} = \frac{720}{6} = 120.$$

(任意不是 $10\times9\times8$.

这样子里面有重复. 完 ~~全一次~~ A B C 是同一种.
B A C

1) 最小号码为5.

$$\cancel{C_5^1} \quad 1 \quad C_5^2 = \cancel{5}\times\frac{5\times4}{2} = \cancel{20}\ 10.$$

哪只为5   为5   6–10 中选出两只.

故最小号码为5 $P = \frac{10}{120} = \cancel{\frac{1}{4} = 25\%}\ \frac{1}{12}.$

6 7 8 9 10

2) 最大号码为5.

$$\cancel{C_3^1} \quad 1 \quad C_4^2 = \cancel{3}\times\frac{4\times3}{2} = \cancel{18}\ 6.$$

哪只为5   5   哪只   1–4 选出两只.

$\boxed{\text{不需要 顺序 does not matter}}$

$P = \frac{\cancel{18}\ 6}{120} = \cancel{\frac{3}{20}}\ \frac{1}{20} = \cancel{15\%}.$

1 2 3 4

Chapter 4 Probability Theory. soln.

- Coin toss game.

A toss n

B toss n.

Card game

Probability

A bigger x

equal $1-2x$ $\longrightarrow$ $P(\text{equal}) = \frac{1}{17}$

smaller x

两张一样.

A先随便选一张.

在剩余的51张中. B抽到与A同样的可能性是3张.

所以是

$$\frac{3}{51} = \frac{1}{17}$$

Prob. 1609

1. Let X, Y be r.v. with joint p.d.f. $f_{X,Y}(x,y)$, 1) show that pdf of X+Y

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\,dx$$

2) if X, Y are independent, write pdf of X+Y in terms of $\underline{f_X(x),\ f_Y(y)}$

marginal probability density functions.

Soln:

1) $F_Z(z) = P(Z \le z) = P(X+Y \le z)$

$$= \iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z-x} dy\, f(x,y)$$

= 二重积分 over region $x+y \le z$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{+\infty} dx \left[\frac{d}{dz} \underbrace{\int_{-\infty}^{z-x} dy\, f_{X,Y}(x,y)}_{\text{变限定积分}} \right]$$

$$= \int_{-\infty}^{+\infty} dx\, f_{X,Y}(x, z-x)$$

2). 若 X, Y 独立, 则 $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{+\infty} dx \left[\frac{d}{dz} \int_{-\infty}^{z-x} dy\, f_X(x)\, f_Y(y) \right]$$

独立于对 y 求积、对 z 求导

$$= \int_{-\infty}^{+\infty} f_X(x)\,dx \cdot \left[\frac{d}{dz} \int_{-\infty}^{z-x} dy\, f_Y(y) \right]$$

$$= \int_{-\infty}^{+\infty} dx\, f_X(x)\, f_Y(z-x)$$

$\downarrow$ 可作换元, 令 $u = x$ 都是 dummy var

$$= \int_{-\infty}^{+\infty} f_X(u)\, f_Y(z-u)\,du$$

$\longleftarrow$ 其实是 $f_X(x)$ $f_Y(y)$ 之卷积

$\xrightarrow{\text{根据 symmetry}}$ $f_Z(z) = \int_{-\infty}^{+\infty} f_Y(u)\, f_X(z-u)\,du$ as well.

Prob 1609.

2. Prove the following version of Chebyshev's inequality:

Let r.v. X be such that

$$E(|X-EX|^k) < \infty \text{ for some fixed } k \geq 1,$$

then it holds for all $t > 0$

$$P(|X-EX| \geq t) \leq \frac{E(|X-EX|^k)}{t^k}$$

Soln:

$$P(|X-\mu| \geq t) = \int_{|X-\mu| \geq t} dF_X(x) \leq \int_{|X-\mu| \geq t} \frac{|X-\mu|^k}{t^k} \, dF_X(x) \leq \int_{|X-\mu| \geq t} \frac{|X-\mu|^k}{t^k} \, dF_X(x) + \int_{|X-\mu| < t} \frac{|X-\mu|^k}{t^k} \, dF_X(x)$$

简便起见，记 $EX = \mu$

由 $|X-\mu| \geq t > 0$ 知 $\frac{|X-\mu|}{t}$ 是大于1之数.

$$\int \frac{|X-\mu|^k}{t^k} \, dF_X(x)$$

$$= t^k E(|X-\mu|^k)$$

Prob 1609.

3. X and Y are two indep. r.v., they have Poission dist.

with parameters $\lambda > 0$, and $\mu > 0$, respectively.

Show $X+Y$ is also distributed poission with parameter $\lambda+\mu$.

soln:

Let $Z=X+Y$, $Z=k$ 即 $X+Y=k$ 是以下事件之并

$$\{Z=k\} \equiv \left\{\begin{matrix} X=0 \\ \& \\ Y=k \end{matrix}\right\} \cup \left\{\begin{matrix} X=1 \\ \& \\ Y=k-1 \end{matrix}\right\} \cup \cdots \cup \left\{\begin{matrix} X=k \\ \& \\ Y=0 \end{matrix}\right\}$$

$$P\{Z=k\} = \sum_{i=0}^{k} P(X=i \;\&\&\; Y=k-i)$$

$\downarrow$ X, Y 独立

$$= \sum_{i=0}^{k} P(X=i)\cdot P(Y=k-i)$$

$$= \sum_{i=0}^{k} \frac{\lambda^i}{i!}e^{-\lambda} \cdot \frac{\mu^{k-i}}{(k-i)!}e^{-\mu}$$

X, Y 泊松. $P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}$

$$= \frac{e^{-(\lambda+\mu)}}{k!} \underbrace{\sum_{i=0}^{k} \frac{\lambda^i}{i!}\frac{\mu^{k-i}}{(k-i)!}\cdot k!}_{\equiv\ \underbrace{\sum_{i=0}^{k} C_k^i \lambda^i \mu^{k-i}}_{\equiv\ \text{二项定理}\ (\lambda+\mu)^k}}$$

$$= (\lambda+\mu)^k \frac{e^{-(\lambda+\mu)}}{k!}$$

即 $Z=(X+Y) \sim \text{Poission}(\lambda+\mu)$

1609.

4. Let X have normal dist. $N(\mu, \sigma^2)$, Let $Y = e^X$,

Compute the moment of $E(Y^k)$ for an arbitrary $k \in \mathbb{N}$.

Soln:

by definition, mgf

本题 $Y = e^X$

$$\phi(t) = \phi_X(t) = E(e^{tX}) = E(Y^t)$$

令 $t = k$, 且知 normal dist. 的 $\phi(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$

$$E(Y^k) = \phi(k) = \exp\left(\mu k + \frac{k^2\sigma^2}{2}\right).$$

Prob.

5.

$X_1, \cdots, X_n$ are i.i.d. r.v. with $E(X_i) = \mu$, $Var(X_i) = \sigma^2 < \infty$

Compute the expected value of the following r.v.

$$\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$\equiv \frac{1}{n} \sum_{i=1}^{n} X_i$$

Soln:

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X}_n)^2\right]$$

Expection 线性性质

$$=\frac{1}{n}\sum_{i=1}^{n}E\left(X_i-\overline{X}_n\right)^2$$

记 $Y_i = X_i - \overline{X}_n$

$$EY_i^2 = DY_i + (EY_i)^2$$

$$EY_i = E(X_i-\overline{X}) = EX_i - E(\overline{X}) = \mu - \frac{n\mu}{n} = 0$$

$$DY_i = D(X_i-\overline{X})$$

没有 x_i 了

$$= D\left[X_i - \frac{1}{n}x_i - \left(\frac{1}{n}x_1+\cdots+\frac{1}{n}x_n\right)\right]$$

$$= \left(\frac{n-1}{n}\right)^2 DX_i + \frac{1^2}{n^2}\left(DX_1+DX_2+\cdots DX_n\right)$$

缺 x_i

$$= \left(\frac{n-1}{n}\right)^2\sigma^2 + \frac{1}{n^2}(n-1)\sigma^2$$

$$= \frac{n\cdot(n-1)}{n^2}\sigma^2$$

$$= \frac{n-1}{n}\sigma^2$$

于是 $EY_i^2 = \frac{n-1}{n}\sigma^2$

Soln

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{n-1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2$$

Prof.
蔺令博
3-38.

一本500页的书中共有100个印刷错误.
每页错误个数近似服从Poission分布.
1) 随机取一页,求这一页上错误不少于2个之概率.
2) 随机取4页,求这一页上错误不少于5个之概率.

首先，两只泊松分布之叠加仍是泊松分布

$$X_1 \sim P(\lambda_1),\quad X_2 \sim P(\lambda_2)$$

$$X = X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$$

- $X \sim P(\lambda)$，则 $P(X=k) \xlongequal{\text{记为}} P_k = e^{-\lambda}\dfrac{\lambda^k}{k!}$

$$E(X) = \lambda,\quad D(X) = \lambda$$

- 设每页之错误分布 $X \sim P(\lambda)$

500页之错误分布 $Y \sim P(500\lambda)$　（500个"每页"叠加起来）

$$EY = 500\lambda = 100 \Rightarrow \lambda = 0.2.$$

$$X \sim P(0.2) \qquad P(X=k) = e^{-0.2}\frac{0.2^k}{k!}$$

- 随机取一页，其上错误不少于2个之概率。

$$P = 1 - P_0 - P_1$$

$$= 1 - e^{-0.2}\frac{0.2^0}{0!} - e^{-0.2}\frac{0.2^1}{1!} = 0.0175$$

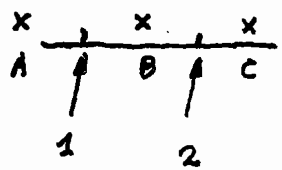
$$X \sim P(4\lambda) = P(0.8).$$

- 随机取4页，错误不少于5个之概率

$$P = 1 - \underbrace{P_0 - P_1 - P_2 - P_3 - P_4}_{\text{少于5个之概率}}$$

Prob
stat.

Jason throws two darts at a board, aiming for the center.
second lands farther from the center than the first.
If Jason throws a third dart, what's the probability
that the third is farther than the first?
Assume Jason's skillfulness is constant.



其实很简单.

1, 2 把 dartboard 分成 3段.

$$\frac{\#\{B, C\}}{\#\{A, B, C\}} = \frac{2}{3}.$$

How many people do we need in class to make the probability that two people have the same birthday more than $1/2$?

Prob
Stat.

P(at least two people have the same birthday)

= 1 - P (no people have the same birthday)

P(at least two) > $\frac{1}{2}$ $\Rightarrow$ P(no people have the same ..) < $\frac{1}{2}$

$$\Downarrow$$

$$\cancel{\frac{365}{365}}\ \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{365+1-n}{365} < \frac{1}{2}$$

$$\Downarrow$$

$n = 23.$

Prob & Stats.

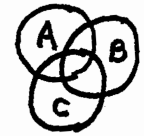
Six letters were to be placed in envelops for posting. Unfortunately, the letters were dropped before putting in their envelopes, and they were placed in at random.

- What's the probability that none is in the correct envelope?
- ———————— exactly one ————————.

Let A be the event that letter A is in its correct envelope, and similarly B is the event that B is in its ____________, then.

$P(A) = \frac{1}{5}$ (5个信封中选一个A信封)

$P(A \text{ and } B) = \underbrace{\frac{1}{5}}_{\text{A: 5选1}} \cdot \underbrace{\frac{1}{4}}_{\text{B: 余下4个中选1.}}$



$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - (P(A\cap B) + P(B\cap C) + P(C\cap A)) + P(A\cap B\cap C)$$

至少有1封对:

$$P(A \text{ or } B \text{ or } C \text{ or } D \text{ or } E)$$

$$= P(A) + P(B) + P(C) + P(D) + P(E)$$
$$- P(A \text{ and } B) - P(B \text{ and } C) - \cdots$$
$$+ P(A \text{ and } B \text{ and } C) + P(B \text{ and } C \text{ and } D) + \cdots$$
$$- P(A \text{ and } B \text{ and } C \text{ and } D) - P(\cdots) - \cdots$$
$$+ P(A \text{ and } B \text{ and } C \text{ and } D \text{ and } E)$$

$$= 5 \times \left(\frac{1}{5}\right) - C_5^2 \frac{1}{5} \times \frac{1}{4} + C_5^3 \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} - C_5^4 \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} + 1 \cdot \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1$$

$$= 1 - \frac{5 \times 4}{2} \times \frac{1}{5 \times 4} + \frac{5\cdot4\cdot3}{3\cdot2\cdot1}\,\frac{1}{5\cdot4\cdot3} - \frac{5\cdot4\cdot3\cdot2}{4\cdot3\cdot2\cdot1} \cdot \frac{1}{5\cdot4\cdot3\cdot2} + \frac{1}{5!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} = \frac{120-60+20-\overset{5}{\cancel{30}}+\overset{1}{\cancel{24}}}{120} = \frac{76}{120}.$$

$$P(\text{全不对}) = 1 - P(\text{至少一封对}) = 1 - \frac{76}{120} = \frac{44}{120} = \frac{11}{30}$$

& stat.

所以 using induction..

n封信全错:

$$\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\cdots+(-1)^n\frac{1}{n!}$$

$$e^{+x}=1+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

在 $x=0$ 处展开

$$e^{-1}=1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots(-1)^n\frac{1}{n!}=0.3678.$$

6封信全错:

$$\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}=\frac{53}{144}$$

6封信对1封:

imagine the letter laid out in a row, and the envelopes in a matching row.

$$\frac{\underset{\text{选出一封对的.}}{C_6^1}\ \underbrace{\text{5封全不对的}}_{\frac{11}{30}\times 5!}\text{ arrangement}}{\text{所有6封之可能排列 }6!}=\frac{6\times\frac{11}{30}\times 120}{720}=\frac{11}{30}$$

A deck of cards with numbers on. $1, 2, \cdots N$

Draw a card randomly from the deck,
keep it at hand, then draw another one;
if the new one is larger than the largest
number in hand, keep it, otherwise, discard it.
Stop when you have the card N.

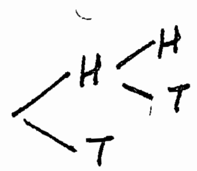
Question:
What is the expectation of numbers of
cards at hand.

We can use conditional probability on the first number drawn.

Let the result be $f(N)$

$$f(N) = \frac{1}{N} + \cdots + \frac{1}{1}$$

You have a fair coin, calculate the expected number N of throws that you needed to find two HH at the end of the sequence?



$$n = \frac{1}{2}(n+1) + \frac{1}{4}\cdot 2 + \frac{1}{4}(n+2)$$

$$n = 6.$$

Prob.

一个青蛙, equal chance to jump to right or left.

向右跳, 2分钟后青蛙将回到原地.

向左跳, 0.7的概率了4分钟后回到原地

0.3 —— 了 3分钟后 ~~回到原地~~ 跳出不回来.

假设x次跳出来.

$$x = \frac{1}{2}(x+2) + \frac{1}{2} \times 0.7 \times (x+4) + \frac{1}{2} \times 0.3 \times 3.$$

$$x = 19.$$

Chapter 4. Probability Theory, Questions.

- Coin toss game.

Two gamblers are playing a coin toss game. Gambler A has $(n+1)$ fair coins; B has n fair coins. What is the probability that A will have more heads than B if both flip all their coins.

- Card game.

A casino offers a simple card game. There are 52 cards in a deck with 4 cards for each value 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. Each time the cards are thoroughly shuffled (so each card has equal probability of being selected). You pick up a card from the deck and the dealer picks another one without replacement. If you have a larger number, you win; if the numbers are equal or yours is smaller, the house wins — as in all other casinos, the house always has better odds of winning. What is your probability of winning?

- Drunken passenger

A line of 100 airline passengers are waiting to board a plane. They each hold a ticket to one of 100 seats on the flights. For convenience, Let's say the n-th passenger in line has a ticket for the seat number n. Being drunk, the first person in line picks a random seat (equally likely for each seat). All of the other passengers are sober, and will go to their proper seats unless it's already occupied; In that case, they will randomly choose a free seat. You're person number 100. What is the probability that you end up in your seat (i.e., seat # 100)

- N points on a circle

Given N points drawn randomly on the circumference of a circle, what is the probability that they are all within a semicircle?

Stat.

药厂制剂车间用自动装瓶机封装药液，机器工作正常时，每瓶药液净重500克．某日随机取了10瓶，称得为：504，498，496，487，509，476，482，510，469，472．问此时机器是否正常工作．

- 500克是什么？总体均值.

假设是什么？总体均值等于500.

若不能否定，可理解为装瓶机工作正常.

假设检验分三步.

1. 提出原假设：总体期望 = 500.

2. 建立检验统计量：

$$\underbrace{\underbrace{\sqrt{n-1}}_{10个样本}\;\frac{\overset{样本均值}{\bar{y}}-\overset{500}{a_0}}{\underset{样本方差}{S}}}_{统计量\ T} \sim t(n-1)$$

即统计量T服从自由度为n-1的t-分布，且不含未知参数，它可做为判断H_0的检验统计量，这种检验方法，称为t检验法.

原理：

- 每一次抽样，可视为一个随机变量 R.V. 此R.V.与总体同分布.

- 10次抽样，可视为10个不同之R.V. → 独立同分布 i.i.d.

- 而 $\bar{y}$ 是把这10个随机变量相加再除以10求平均.

- $\bar{y}$ 本身也是随机变量

通过概率密度和差商之公式，若原总体符合正态，则 T 符合 t 分布.

→ ξ, η 相互独立之R.V.，

$\xi \sim N(0,1)$，$\eta \sim \chi^2(n)$

则 $t=\dfrac{\xi}{\sqrt{\frac{\eta}{n}}}$ ～ t分布（自由度为n）

$$f_{(t)}(x) = f_{\xi/\sqrt{\eta/n}}(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\cdot\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

Γ函数可视为阶乘之推广

1. The sample space of two children is given by

$$\Omega = \{(b,b), (b,g), (g,b), (g,g)\}$$

第一个是大孩子，第二个是小孩子.

Since Ms Jackson is invited, she has at least one son,

Let B be the event that at least one of the children is a boy, and A be the event that both children are boys.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{(b,b)\})}{P(\{(b,b),(g,b),(b,g)\})} = \frac{1}{3}$$

2. 另一个可能是男 $(\frac{1}{2})$

亦 —— 女 $(\frac{1}{2})$

故两个男孩之概率为 $\frac{1}{2}$.

Boys and girls:

1. A company is holding a dinner for working mothers with at least one son. Ms. Jackson, a mother with two children, is invited. What's the probability that both children are boys?

2. Your new collegue, Ms. Parker is known to have two children. If you see her walking with one of her children and that child is a boy, what is the probability that both children are boys?

一共52张牌中取5张，顺序 does not matter.

共有 C_{52}^5 种.

4张一样 $C_{13}^1 \cdot C_{48}^1$

13组 A ··· K

余下之48张.

$$P = \frac{13 \times 48}{C_{52}^5}$$

Full house.

$C_{13}^1 \quad C_{12}^1 \quad C_4^3 \quad C_4^2$

13组 取~~到~~ 1组3张

余下12组 中取 1组2张

$$P = \frac{13 \times 12 \times 4 \times 6}{C_{52}^5}$$

Two pairs.

余下一张

$\underbrace{C_{13}^1 \quad C_{12}^1 \quad C_4^2 \quad C_4^2}_{\text{两对}} \quad C_{44}^1$

$$P = \frac{13 \times 12 \times 6 \times 6 \times 44}{C_{52}^5}$$

Texas holdem.

得到4张一样的

full house (3张一样外加一对)

两对

的概率分别是多少?

我 $>$ 庄 $\quad p$

我 $=$ 庄 $\quad 1-2p = \frac{3}{51}$

我 $<$ 庄 $\quad p$.

$$1-2p = \frac{1}{17} \qquad p = \frac{8}{17}.$$

A casino offers a simple card game.

52 张牌，你和 dealer 各抽一张，无放回。

若你的牌大，你赢。

牌等或你的牌小，庄家赢。

问：你赌赢之概率多少。

soln:

A 和 B 都只扔 n 个时:

$A > B$ p $\xrightarrow{\text{不管怎样A依然大于B}}$ p.

$A = B$ $1-2p$ $\xrightarrow{\frac{1}{2}\text{概率A扔head,}}$ 从而 $A > B$ $\frac{1}{2}(1-2p)$

$A < B$ p $\xrightarrow{\text{不管怎样,}}$ A不大于B. (最多相等).   0

($A > B$ 之概率等于 $A < B$ by symmetry)

B扔n个. A扔n+1个.

故而都扔完后.

$A > B$ 之概率:

$p + \frac{1}{2}(1-2p) = \frac{1}{2}$.

Two gamblers are playing a coin toss game, Gambler A has $(n+1)$ fair coins; B has n fair coins. What's the probability that A will have more heads than B if both flip all their coins?

Prove S^2 is the unbiased estimator of σ^2

Stats

S^2: sample variance.

证明 $E(S^2) = E\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}\right) = \sigma^2$

Soln.

Let $X_1, X_2, \cdots, X_n$ be n independent observation from a population with mean μ and var σ^2.

- 则对于每个 X_i，有

$$E(X_i)=\mu, \quad Var(X_i)=\sigma^2$$

- $Var(X) = E(X^2)-(EX)^2$

  则 $E(X^2) = \sigma^2+\mu^2$

$Var(\bar{X}) = E(\bar{X}^2) - [E(\bar{X})]^2$

$\bar{X}$: Sample mean, 是个 R.V.

有：

$$E(\bar{X}^2) = \frac{\sigma^2}{n}+\mu^2$$

$$E\left[\sum(X_i-\bar{X})^2\right]$$

$$=E\left[\sum(X_i^2-2X_i\bar{X}+\bar{X}^2)\right]$$

$$=E\left(\sum X_i^2 - \sum 2X_i\bar{X} + \sum\bar{X}^2\right)$$

$\bar{X}$ 是常数，是 X_i 的估计　$\bar{X}=\frac{\sum X_i}{n}$

$$=E\left(\sum X_i^2 - 2\bar{X}\underbrace{\sum X_i}_{n\bar{X}} + n\bar{X}^2\right)$$

$$=E\left(\sum X_i^2 - n\bar{X}^2\right)$$

$$=\sum E(X_i^2) - nE(\bar{X}^2)$$

$$=\sum(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)$$

$$=n\sigma^2+n\mu^2-\sigma^2-n\mu^2=(n-1)\sigma^2$$

故

$$E(S^2)=\sigma^2$$

其中 $S^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}$，是 σ^2 之无偏估计.

Let play a game,

I give you a 12 sided die, and will pay you whatever the die lands on. If you're unhappy with the roll, you can choose to roll another two 6-sided dice, and I will pay you the sum of the two dice. How much are you willing to pay to play this game?

$$\frac{1+12}{2} = 6.5$$

7, 8, 9, 10, 11, 12

扔两只6面die；$\frac{1+6}{2} = 3.5$ $3.5 \times 2 = 7$.

于是低于7的都要重扔.

$$\underbrace{1\ 2\ 3\ 4\ 5\ 6\ 7}_{\text{重扔}}\quad \underbrace{8\ 9\ 10\ 11\ 12}_{\text{不扔}}$$

得 7

$$\frac{7 \times 7 + \frac{8+12}{2} \times 5}{12} = \frac{49+50}{12} = 8.25.$$

Chapter 4 Probability Theory, soln.

- Coin toss game.

    - A toss n & B toss n.

Heads of A	$>$	Heads of B	x
	$=$		$1-2x$
	$<$		x by symmetry.

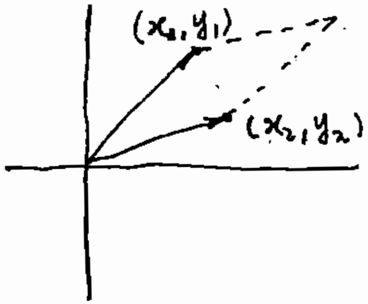
- 再掷一次. A 的 heads 比 B 多的概率:

$$x\cdot 1 + (1-2x)\cdot\frac{1}{2} + 0 = x + \frac{1}{2} - x = \frac{1}{2}.$$

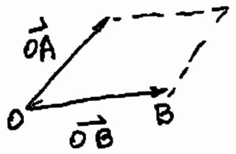
$x\cdot 1$: n次时比B多. n+1次不管怎么掷. 都还比B多

$(1-2x)\cdot\frac{1}{2}$: n次时与B相等. n+1次时, 若是head, 则比B多. 概率是$\frac{1}{2}$.

0: n次时比B少. 则不管怎么掷. 还是比B少.



如何迅速写出图示平行四边形之面积？

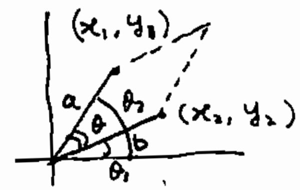


为一向量 OC

两个向量之叉积，大小为二者围成▱之面积，方向垂直于 AOB 平面.

在二维的情形下，

$$\left|\vec{OA}\times\vec{OB}\right| = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$



验证如下： $\text{Area} = a\cdot b\cdot \sin\theta.$

$$\sin\theta = \sin(\theta_2-\theta_1)$$
$$= \sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1$$
$$= \frac{y_2}{b}\frac{x_1}{a} - \frac{x_2}{b}\frac{y_1}{a}.$$

$$\text{Area} = y_2x_1 - x_2y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Math.

$\searrow \int_{-\infty}^{+\infty} e^{-x^2}\,dx.$

两种方法求解.

1. 令 $a=\int_{-\infty}^{+\infty} e^{-x^2}dx$

则 $a^2=\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2$ 换元

$=\int_{-\infty}^{+\infty} e^{-x^2}dx\cdot\int_{-\infty}^{+\infty} e^{-y^2}dy$ 累次积分化为重积分

$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy$

极坐标.

2. $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}=1$ (*)

$a=\int_{-\infty}^{+\infty} e^{-x^2}$

$\sqrt{\pi}$. 化[illegible].

令 $\mu=0$, $2\sigma^2=1$, $\sigma=\frac{1}{\sqrt{2}}$

*式化为

$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}=1$

于是 $a=\sqrt{\pi}$