(a)

$$A = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}$$

positive definite를 확인하기위해서, leading principal submatrices가 positive determinant를 갖는지 확인하면 됩니다.

$$\det(5) = 5 > 0, \quad \det\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = 25 - 9 = 16 > 0$$

$$\det(A) = \sum_{j=1}^{3} a_{1j} A_{1j} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 5A_{11} + 3A_{12}$$

$$= 5M_{11} - 3M_{12} = 5(25+1) - 3(15) = 85 > 0.$$

therefore A is positive definite and also tridiagonal.

(b) Let's jacobi's method.

$$A = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & -1 & 5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 30 \\ 40 \\ -35 \end{pmatrix}$$

$$\text{Let } A = D - L - U = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(D - L - U)\mathbb{x} = \mathbb{b}$$

$$D\mathbb{x} = (L+U)\mathbb{x} + \mathbb{b}$$

$$\mathbb{x} = \underbrace{D^{-1}(L+U)}_{T_j} + \mathbb{x} + \underbrace{D^{-1}\mathbb{b}}_{C_j}$$

To find D^{-1}, using $DD^{-1} = E$.

$$5ED^{-1} = E$$

$$5D^{-1} = E.$$

$$D^{-1} = \frac{1}{5}E.$$

$$\therefore T_j = \frac{1}{5}E(L+U) = \frac{1}{5}\begin{pmatrix} 0 & -3 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{5} & 0 \\ -\frac{3}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & 0 \end{pmatrix}$$

$$C_j = \frac{1}{5}E \cdot \mathbb{b} = \begin{pmatrix} 6 \\ 8 \\ -7 \end{pmatrix}$$

(c) $E_1: \quad 5x_1 + 3x_2 = 30$ $\qquad x_1 = -\frac{3}{5}x_2 + 6$

$E_2: \quad 3x_1 + 5x_2 - x_3 = 40$ $\quad \Rightarrow \quad x_2 = -\frac{3}{5}x_1 + \frac{1}{5}x_3 + 8$

$E_3: \quad -x_2 + 5x_3 = -35$ $\qquad x_3 = \frac{1}{5}x_2 - 7$

(also using $T_j x + c_j$)

Let $x^{(0)} = (0,0,0)$

$x_1^{(1)} = 6 \qquad x_1^{(2)} = -\frac{24}{5} + 6 = \frac{6}{5} \qquad x_1^{(3)} = -\frac{9}{5} + 6 = \frac{21}{5}$

$x_2^{(1)} = 8 \qquad x_2^{(2)} = -\frac{3}{5}\cdot 6 + \frac{1}{5}(-7) + 8 = 3 \qquad x_2^{(3)} = -\frac{3}{5}\cdot\frac{6}{5} + \frac{1}{5}\left(-\frac{27}{5}\right) + 8 = -1$

$x_3^{(1)} = -7 \qquad x_3^{(2)} = \frac{8}{5} - 7 = -\frac{27}{5} \qquad x_3^{(3)} = \frac{1}{5}\cdot 3 - 7 = -\frac{32}{5}$

$x^{(1)} = (6, 8, -7)^t \qquad x^{(2)} = \left(\frac{6}{5}, 3, -\frac{27}{5}\right)^t \qquad x^{(3)} = \left(\frac{21}{5}, -1, -\frac{32}{5}\right)^t$

(d) by Theorem, If A is tridiagonal and positive definite,

$$\text{optimal } \omega = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}}$$

To find $\rho(T_j)$, find $\det(T_j - \lambda I)$

$$\det(T_j-\lambda I) = \det\begin{pmatrix} -\lambda & -\frac{3}{5} & 0 \\ -\frac{3}{5} & -\lambda & \frac{1}{5} \\ 0 & \frac{1}{5} & -\lambda \end{pmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= -\lambda \cdot \det\begin{pmatrix} -\lambda & \frac{1}{5} \\ \frac{1}{5} & -\lambda \end{pmatrix} + \frac{3}{5} \cdot \det\begin{vmatrix} -\frac{3}{5} & \frac{1}{5} \\ 0 & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2 - \frac{1}{25}\right) + \frac{3}{5}\left(\frac{3}{5}\lambda\right)$$

$$= \lambda\left(\frac{9}{25} - \lambda^2 + \frac{1}{25}\right) = -\lambda\left(\lambda^2 - \frac{2}{5}\right) \qquad \lambda = 0 \text{ or } \pm\sqrt{\frac{2}{5}}$$

$\therefore \rho(T_j) = \sqrt{\frac{2}{5}}$, and the optimal $\omega = \dfrac{2}{1+\sqrt{1-\frac{2}{5}}} = \dfrac{2}{1+\sqrt{\frac{3}{5}}} \doteq 1.12701$

$\doteq 1.13$

$(\max|\lambda|)$