Problem 1

Find the monochromatic plane wave solution in Lorentz gauge, and derive the energy flux density.

The equations for Electric and Magnetic field Using Maxwell's equations:

$$(i)\ \ \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho, \qquad (iii)\ \ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},$$

$$(ii)\ \ \vec{\nabla}\cdot\vec{B} = 0, \qquad (iv)\ \ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.$$

In statics, the curl of $\vec{E} = 0$ however in dynamics, the curl of $\vec{E} \neq 0$ and the divergence of $\vec{B}$ is still 0. Rather than using the electric potential, the magnetic potential will be used:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

The curl of $\vec{E}$ becomes:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\vec{\nabla}\times\vec{E} + \vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} = \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

Since the curl of this thing is equal to 0, it can be set as the gradient of a scalar:

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

Maxwell's (i) becomes:

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{1}{\epsilon_0}\rho$$

Maxwell's (iv) becomes:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} - \mu_0\epsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

The curl of the curl of the magnetic potential is:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \vec{\nabla} \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Rearranging:

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}$$

In the Lorenz gauge:

$$\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

The above becomes:

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) = -\mu_0 \vec{J}$$

No free charge:

$$\nabla^2 V = \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2}$$

No free current:

$$\nabla^2 \vec{A} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Problem 2

In the monochromatic plan wave solution using Coulomb gauge, we have $\vec{A} = \vec{A}_0 exp[i(\vec{k} \cdot \vec{r} - \omega t)]$. In class, we only considered the case of a linear polarized light, meaning $\vec{A}_0$ is a real vector. Now when $\vec{A}_0 = A_x \hat{x} + iA_y \hat{y}$, derive the corresponding (real) $\vec{E}$ and $\vec{B}$ field, as well as the energy density and energy flux density.