Problem 1: Taylor 6.22

You are given a string of fixed length l with one end fastened at the origin **O**, and you are to place the string in the x-y place with its other end on the x-axis. Show that the required shape is a semicircle. The area enclosed is of course $\int y dx$, but show that you can write this in the form $\int_0^l f ds$, where s denotes the distance measured along the string from **O**, where $f = y\sqrt{1-y'^2}$, and y' denotes $\frac{dy}{ds}$. Since f does not involve the independent variable s explicitly, you can exploit the "first integral" (6.43) of Problem 6.20.

From section 6.1 of Taylor:

$$ds = \sqrt{dx^2 + dy^2}$$

Rearranging this to solve for dx:

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 \\
dx^2 &= ds^2 - dy^2 \\
dx &= \sqrt{ds^2 - dy^2} \\
&= \sqrt{ds^2\left(1 - \frac{dy^2}{ds^2}\right)} \\
&= ds\sqrt{1 - y'^2}
\end{aligned}$$

Plugging this dx into the first integral gives:

$$\int_0^l y\sqrt{1-y'^2}ds$$

Which is the form requested ($\int_0^l f ds$)and $f = y\sqrt{1-y'^2}$ Equation 6.43 states:

$$f - y'\frac{\partial f}{\partial y'} = const.$$

The partial of f with respect to y' is:

$$\frac{\partial f}{\partial y'} = -yy'\left(1-y'^2\right)^{\frac{-1}{2}}$$

Giving:

$$\begin{aligned} f - y'\frac{\partial f}{\partial y'} &= y\sqrt{1-y'^2} + \frac{yy'^2}{\sqrt{1-y'^2}} \\ &= \frac{y - yy'^2 + yy'^2}{\sqrt{1-y'^2}} \\ &= \frac{y}{\sqrt{1-y'^2}} = C \end{aligned}$$

Solving for y':

$$\begin{aligned} \left(\frac{y}{C}\right)^2 &= 1 - y'^2 \\ y' &= \sqrt{1 - \left(\frac{y}{C}\right)^2} \\ \frac{dy}{\sqrt{C^2 - y^2}} &= \frac{ds}{C} \end{aligned}$$

Integrating:

$$\begin{aligned} \int \frac{dy}{\sqrt{C^2 - y^2}} &= \int \frac{ds}{C} \\ \implies arcsin\left(\frac{y}{C}\right) &= \frac{s}{C} \\ \implies y &= Csin\left(\frac{s}{C}\right) \end{aligned}$$

Now to find x:

$$\begin{aligned} dx &= ds\sqrt{1-y'^2} \\ &= ds\sqrt{1 - \left(1 - \left(\frac{y}{C}\right)^2\right)} \\ &= ds\sqrt{1 - \left(1 - \left(sin\frac{s}{C}\right)^2\right)} \\ &= sin\left(\frac{s}{C}\right) ds \end{aligned}$$

Integrating:

$$x = C - Ccos\left(\frac{s}{C}\right)$$

Putting y and x together in the form of a circle:

$$\begin{aligned} y^2 &= C^2 sin^2\left(\frac{s}{C}\right) \\ (x-C)^2 &= C^2 cos^2\left(\frac{s}{C}\right) \\ \implies (x-C)^2 + y^2 &= C^2 \end{aligned}$$

So, the string must be placed in the form of a semi-circle to maximize the area.

Problem 2: Taylor 6.23 An aircraft whose speed is v_0 has to fly from town **O** (at the origin) to town P, which is a distance D due east. There is a steady gentle wind shear, such that $\vec{v}_{wind} = Vy\hat{x}$, where x and y are measured east and north respectively. Find the path, $y = y(x)$, which the plane should follow to minimize its flight time, as follows:

(a) Find the plane's ground speed in terms of v_0, V, ϕ (the angle by which the plane heads to the north of east), and the plane's position.

(b) Write down the time of flight as an integral of the form $\int_0^D f dx$. Show that if we assume that y' and ϕ both remain small (as is certainly reasonable if the wind speed is not too large), then the integrand f takes the approximate form $f = \frac{1+\frac{1}{2}y'^2}{1+ky}$ (times an uninteresting constant) where $k = \frac{V}{v_0}$.

(c) Write down the Euler-Lagrange equation that determines the best path. To solve it, make the intelligent guess that $y(x) = \lambda x(D-x)$, which clearly passes through the two towns. Show that it satisfied the Euler-Lagrange equation, provided $\lambda = \frac{\sqrt{4+2k^2D^2}-2}{kD^2}$. how far north does this path take the plane, if D=2000miles, v_0=500mph, and the wind shear is V=0.5mph/mi? How much time does the plane save by following this path? [You'll probably want to use a computer to do this integral.]

(a) The ground speed of the plane should be:

$$|\vec{v}_g| = |(v_0cos(\phi) + Vy)\hat{x} + v_0sin(\phi)\hat{y}|$$
$$v_g = \sqrt{(v_0cos(\phi) + Vy)^2 + v_0^2sin^2(\phi)}$$

(b) The time t it takes for the plane to make the trip is equal to the integral of the change in position divided by the velocity traveled:

$$t = \int_{s_{initial}}^{s_{final}} \frac{ds}{v_g}$$

ds is going to be similar to problem 1 ds:

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+y'^2}$$

And the integral becomes:

$$t = \int_{x_{initial}}^{x_{final}} \frac{dx\sqrt{1+y'^2}}{\sqrt{(v_0cos(\phi) + Vy)^2 + v_0^2sin^2(\phi)}}$$

Assuming y' is small and taylor series expanding:

$$dx\sqrt{1+y'^2} \approx dx\left(1+\frac{1}{2}y'^2\right)$$

And assuming ϕ is small:

$$cos(small\ \phi) \implies 1, \text{ and } sin(small\ \phi) \implies 0$$

The denominator becomes:

$$\sqrt{(v_0 cos(\phi)+Vy)^2+v_0^2 sin^2(\phi)} \approx v_0(1+ky), k=\frac{V}{v_0}$$

Time is then:

$$t \approx \int_0^D f dx = \int_0^D \frac{dx\left(1+\frac{1}{2}y'^2\right)}{v_0(1+ky)}$$

(c) The Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

The partial of f with respect to y:

$$\frac{\partial f}{\partial y} = \frac{-k\left(1+\frac{1}{2}y'^2\right)}{v_0\left(1+ky\right)^2}$$

And the partial of$f$ with respect to y':

$$\frac{\partial f}{\partial y'} = \frac{y'}{v_0(1+ky)}$$

Taking the derivative of this with respect to x gives:

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{y''}{v_0(1+ky)} - \frac{ky'^2}{v_0\left(1+ky\right)^2}$$

Combining gives:

$$\begin{aligned}\frac{-k\left(1+\frac{1}{2}y'^2\right)}{\left(1+ky\right)^2} - \frac{y''}{1+ky} + \frac{ky'^2}{\left(1+ky\right)^2} &= 0\\ -k\left(1-\frac{1}{2}y'^2\right) - y''(1+ky) &= 0\\ y''(1+ky) - \frac{k}{2}y'^2 + k &= 0\end{aligned}$$

Using the supplied guess:

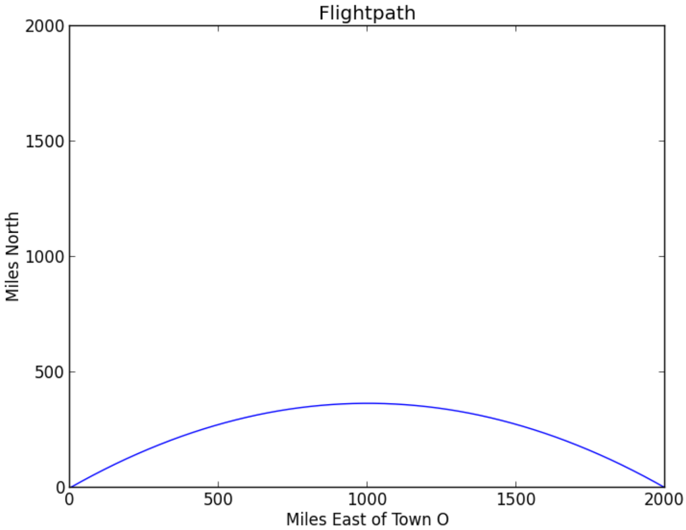
$$\begin{aligned}
y(x) &= \lambda x(D-x) \\
y' &= \lambda(D-x) - \lambda x \\
&= \lambda(D-2x) \\
y'' &= -2\lambda
\end{aligned}$$

Plugging this all back in and solving for λ:

$$\begin{aligned}
-2\lambda(1+k(\lambda x(D-x))) - \frac{k}{2}(\lambda^2(D-2x)^2) + k &= 0 \\
-2\lambda - 2k\lambda^2(Dx-x^2) - \lambda^2\frac{k}{2}(D-2x)^2 + k &= 0 \\
\lambda^2((-2kDx+2kx^2) - \frac{k}{2}(D-2x)^2) - 2\lambda + k &= 0 \\
\lambda^2((-2kDx+2kx^2) - \frac{k}{2}(D^2-4Dx+4x^2)) - 2\lambda + k &= 0 \\
\lambda^2(-\frac{k}{2}D^2) - 2\lambda + k &= 0 \\
\lambda^2 kD^2 + 4\lambda - 2k &= 0 \\
\lambda &= \frac{-4 \pm \sqrt{16+8k^2D^2}}{2kD^2}
\end{aligned}$$

$$\lambda = \frac{\sqrt{4+2k^2D^2}-2}{kD^2}, \text{ (taking the positive answer)}$$

According to the python script I just created, the flight path is:



The plane goes 366 miles north. According to the python script, the integral for time evaluates out to 3.56 hours, which is almost a half hour faster (.44 hours) faster than if the plane headed due east for 4 hours.

Problem 3: Taylor 6.24 Consider a medium in which the refractive index n is inversely proportional to r^2; that is, $n = \frac{a}{r^2}$, where r is the distance from the origin. Use Fermat's principle, that the integral (6.3) is stationary, to find the path of a ray of light travelling in a plane containing the origin. [Hint: Use two-dimensional polar coordinates and write the path as $\phi = \phi(r)$. The Fermat integral should have the form $\int f(\phi, \phi', r)dr$, where $f(\phi, \phi', r)$ is actually independent of ϕ. The Euler-Lagrange equation therefore reduces to $\frac{\partial f}{\partial \phi'} = const.$ You can solve this for ϕ' and then integrate to give ϕ as a function of r. Rewrite this to give r as a function of ϕ and show that the resulting path is a circle through the origin. Discuss the progress of the light around the circle.]

Equation (6.3) is:

$$\int_1^2 n(x,y)ds = \int_{x_1}^{x_2} n(x,y)\sqrt{1 + y'(x)^2}dx$$

The hint of this problems suggests polar coordinates:

$$= \int_{r_1}^{r_2} n(r,\phi)\sqrt{1 + \phi'(r)^2}dr$$

Starting with ds in polar coordinates:

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 \\ ds &= \sqrt{dr^2 + r^2 d\phi^2} \\ &= dr\sqrt{1 + r^2 d\phi'^2}, \quad d\phi' = \frac{d\phi}{dr} \end{aligned}$$

Using the given propotionality of n:

$$n = \frac{a}{r^2}$$

The integral that needs to be minimized is then:

$$\int_{r_1}^{r_2} \frac{1}{r^2}\sqrt{1 + r^2 d\phi'^2}dr$$

The Euler-Lagrange equation is:

$$\begin{aligned}
\frac{\partial f}{\partial \phi} - \frac{d}{dr}\frac{\partial f}{\partial \phi'} &= 0 \\
f &= \frac{\sqrt{1 + r^2 d\phi'^2}}{r^2} \\
\frac{\partial f}{\partial \phi} &= 0 \\
\implies \frac{d}{dr}\frac{\partial f}{\partial \phi'} &= 0 \\
\implies \frac{\partial f}{\partial \phi'} &= constant
\end{aligned}$$

The partial of f with respect to ϕ':

$$\frac{\partial f}{\partial \phi'} = \frac{d\phi'}{\sqrt{1 + r^2 d\phi'^2}} = C$$

Solving for $d\phi'$:

$$\begin{aligned}
d\phi' &= C\sqrt{1 + r^2 d\phi'^2} \\
d\phi'^2 &= C(1 + r^2 d\phi'^2) \\
d\phi'^2(1 - Cr^2) &= C \\
\frac{d\phi}{dr} &= \sqrt{\frac{C}{1 - Cr^2}} \\
d\phi &= \sqrt{\frac{C}{1 - Cr^2}} dr
\end{aligned}$$

ϕ is then equal to:

$$\phi = arcsin(r)$$

Or:

$$r = sin(\phi)$$

Which is the equation of a circle.