2. First-Order, Separable Differential Equations

Find the general solution of each of the followiing differential equations. Use Boas Chapter 8 Section 2 for review f an apprpriate technique.

(a)

$$
\begin{aligned}
y' &= \frac{x^2}{y} \\
\frac{dy}{dx} &= \frac{x^2}{y} \\
ydy &= x^2dx \\
\int ydy &= \int x^2dx \\
\frac{y^2}{2} &= \frac{x^3}{3} + C \\
y &= \pm\sqrt{\frac{2}{3}x^3 + C}
\end{aligned}
$$

(b)

$$
\begin{aligned}
xy' - xy &= y \\
x\left(\frac{dy}{dx} - y\right) &= y \\
\frac{1}{y}\left(\frac{dy}{dx} - y\right) &= \frac{1}{x} \\
\frac{dy}{y} - dx &= \frac{dx}{x} \\
\frac{dy}{y} &= \left(\frac{1}{x} + 1\right)dx \\
\int \frac{dy}{y} &= \int \left(\frac{1}{x} + 1\right)dx \\
ln(y) &= ln(x) + ln(c) + x \\
y &= cx + e^x
\end{aligned}
$$

(c)

$$
\begin{aligned}
xy' - xy &= x \\
\frac{dy}{dx} - y &= 1 \\
\frac{dy}{dx} &= y + 1 \\
\frac{dy}{y+1} &= dx \\
ln(y+1) &= x + c \\
y + 1 &= e^{x+c} \\
y &= e^x e^c - 1 \\
y &= ce^x - 1
\end{aligned}
$$

3. Quadratic Drag

An object is moving in one direction which is only subject to one force, quadratic drag, given by:

$$-cv^2$$

where v is the velocity of the object and c is a constant. Write the equation of motion for this object and solve for the velocity at any time t for an initial velocity of v_0.

Assuming c already has the mass of the object divided out, the acceleration of the object is $a = -cv^2$.

a is the time derivative of the velocity v and v is the time derivative of the position x thus:

$$
\begin{aligned}
v &= \frac{dx}{dt} \\
a &= \frac{dv}{dt} = \frac{d^2x}{dt^2}
\end{aligned}
$$

Solving for v:

$$\begin{aligned}
\frac{dv}{dt} &= -cv^2 \\
-\frac{dv}{v^2} &= cdt \\
\frac{1}{v} &= ct + C \\
v &= \frac{1}{ct + C}
\end{aligned}$$

Since $v(t = o) = v_0$, $\frac{1}{C} = v_0$

$$v = \frac{v_0}{cv_0 t + 1}$$

The above shows the equation for $v(t)$, seperating variables one more time will give the equation of motion.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{v_0}{cv_0 t + 1} \\
\int dx &= \int \frac{v_0 dt}{cv_0 t + 1} \\
x &= \frac{1}{c} ln(cv_0 t + 1) + C
\end{aligned}$$