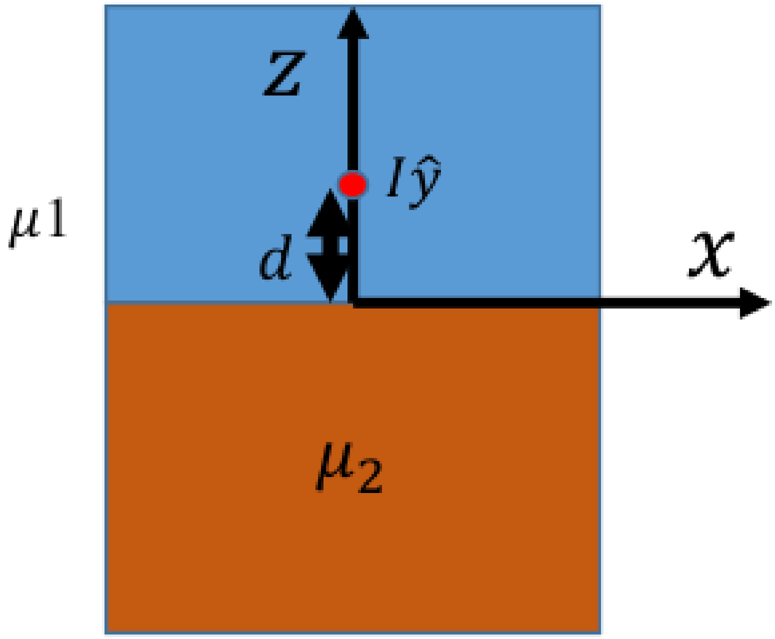
Question 1

Two semi-infinite large magnetic materials with permeability μ_1 and μ_2 meet at z = 0 plane. Inside one of the material there is a current I running in the y direction, at a distance d from the interface. Calculate the magnetic field in each of the materials.



The free current inside the blue material will generate a magnetic field. Without taking into account the magnetic material, the magnetic field $B_1 = \frac{\mu_0 I}{2\Pi r}$. Taking into account the magnetic materials ... the brown material can be replaced by an imaginary current I_2 a distance 2d below I, that effects the magnetic field in the blue material. $B_1 = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I_2}{2\pi r_2}$. However, this still doesn't take into account the magnetic material this field is propagating through. To account for this, the 'H' field needs to be accounted for. $H_1 = \frac{B_1}{\mu_1} = \frac{\mu_0 I}{2\mu_1 \pi r} + \frac{\mu_0 I_2}{2\mu_1 \pi r_2}$. In the brown material... the magnetic field seen there will be based on the current in the blue region and the blue region itself can be replaced by an imaginary current I_1 in the same place. Thus the magnetic field is $B_2 = \frac{\mu_0 (I+I_1)}{2\pi r}$ and $H_2 = \frac{\mu_0 (I+I_1)}{2\mu_2 \pi r}$.

$$B_1 = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I_2}{2\pi r_2}$$
$$H_1 = \frac{\mu_0 I}{2\mu_1 \pi r} + \frac{\mu_0 I_2}{2\mu_1 \pi r_2}$$
$$B_2 = \frac{\mu_0 (I + I_1)}{2\pi r}$$
$$H_2 = \frac{\mu_0 (I + I_1)}{2\mu_2 \pi r}$$

Now, the imaginary currents I_1, I_2 need to be solved for. This can be done taking into account the boundary conditions between the materials.

$$B^{\perp}_{above} = B^{\perp}_{below}$$
$$H^{\parallel}_{above} - H^{\parallel}_{below} = K_f \times \hat{n}$$

Takind into account that there was no free surface current at the interface($K_f = 0$):

$$H^{\parallel}_{above} = H^{\parallel}_{below}$$

If I assume the currents I and I_2 are all running the same direction, the magnetic fields will all wrap in the same direction around each current. At the interface, finding the perpendicular component of the B field will add to each other. Introducing some random angle θ which is the angle between the interface and the field at any point along the interface.

$$B^{\perp}_1 = \frac{\mu_0 I}{2\pi r} sin(\theta) + \frac{\mu_0 I_2}{2\pi r_2} sin(\theta)$$
$$B^{\perp}_2 = \frac{\mu_0 (I + I_1)}{2\pi r} sin(\theta)$$

(Note along the interface, $r = r_2$)

$$\frac{\mu_0 I}{2\pi r} sin(\theta) + \frac{\mu_0 I_2}{2\pi r} sin(\theta) = \frac{\mu_0 (I + I_1)}{2\pi r} sin(\theta)$$

This simplifys down to:

$$I_1 = I_2$$

Similar arguments with H:

$$H_1^{\|} = \frac{\mu_0 I}{2\mu_1 \pi r} cos(\theta) - \frac{\mu_0 I_2}{2\mu_1 \pi r_2} cos(\theta)$$
$$H_2^{\|} = \frac{\mu_0 (I + I_1)}{2\mu_2 \pi r} cos(\theta)$$
$$\frac{\mu_0 I}{2\mu_1 \pi r} cos(\theta) - \frac{\mu_0 I_2}{2\mu_1 \pi r_2} cos(\theta) = \frac{\mu_0 (I + I_1)}{2\mu_2 \pi r} cos(\theta)$$
$$\frac{I - I_2}{\mu 1} = \frac{I + I_1}{\mu_2}$$

Replacing I_2 by I_1:

$$\mu_2 I - \mu_2 I_1 = \mu_1 I + \mu_1 I_1$$
$$I_1(\mu_1 + \mu_2) = I(\mu_2 - \mu_1)$$
$$I_1 = I_2 = I\frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}$$

Plugging currents into the magnetic fields:

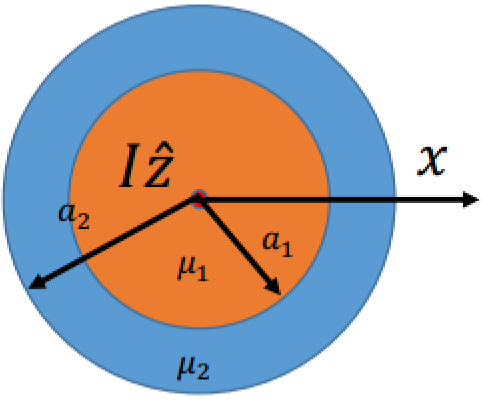
$$B_1 = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0(\mu_2 - \mu_1) I}{2\pi r_2(\mu_1 + \mu_2)}$$
$$B_2 = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0(\mu_2 - \mu_1) I}{2\pi r(\mu_1 + \mu_2)}$$

The direction of the magnetic current is perpendicular to r orr_2 respectively.

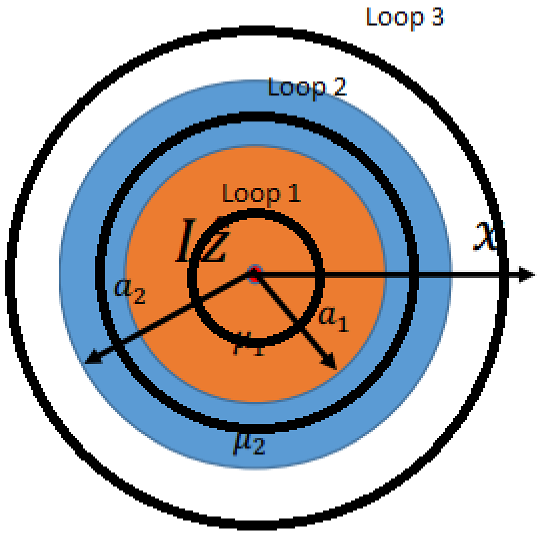
$$\mathbf{B_1} = \frac{\mu_0 I}{2\pi r}(cos(\phi_1)\hat{x} + sin(\phi_1)\hat{z}) + \frac{\mu_0(\mu_2 - \mu_1) I}{2\pi r_2(\mu_1 + \mu_2)}(cos(\phi_2)\hat{x} + sin(\phi_2)\hat{z})$$
$$\mathbf{B_2} = \left(\frac{\mu_0 I}{2\pi r} + \frac{\mu_0(\mu_2 - \mu_1) I}{2\pi r(\mu_1 + \mu_2)}\right)(cos(\phi_1)\hat{x} + sin(\phi_1)\hat{z})$$

Question 2

Two coaxial linear magnetic materials enclose a current running along their axis. Calculate the bound currents (both volume and surface) and calculate the magnetic field produced by the bound currents.



Starting with Maxwell's equations, and using ampere's loops at various locations we can solve for the H field at each loop. There are three loops I will concern myself with. Loop 1 will be in the μ_1 material, 2 will be in the μ_2 material, and loop 3 will be outside the cable.



Starting with Loop 3... The curl of $\vec{H}$ is equal to the free current $\vec{J}_f + \frac{\partial D}{\partial t}$, $\vec{J}_f$ is equal to the free current which was placed into the system by us. (Nothing is changing with respect to time so $\frac{\partial D}{\partial t} = 0$). In integral form:

$$\int \vec{H} \cdot \vec{dl} = I_f$$

No matter how I walk around this loop, the problem looks the same, and thus, $\vec{H}$ should not change with $\vec{dl}$ and can be taken out of the integral:

$$H \int dl = I_f$$

The integral now is just equal to $2\pi r$ where $r > a_2$. Thus:

$$H = \frac{I_f}{2\pi r}$$

This expression for H works for all three loops chosen, just the specific r will be restricted to certain values. (loop 2: $a_1 < r < a_2$ and loop 1: $0 < r < a_1$.)

Refocusing on loop 3. The H field allows us to find the B field in this area.

$$\vec{B} = \mu \vec{H}$$

In this region, there is no magnetic material, $\mu = \mu_0$. Thus:

$$B = \frac{\mu_0 I_f}{2\pi r}$$

Another of Maxwell's equations states that in general the curl of $\vec{B}$ is equal to μ_0 times the total current $\vec{J_T}$ plus $\mu_0 \epsilon_0$ times $\frac{\partial \vec{E}}{\partial t}$. (Static system) Integral form:

$$\int \vec{B} \cdot \vec{dl} = \mu_0 I_T$$

Again due to symetry, B can be pulled out of the integral and the integral is again equal to $2\pi r$. Since this is the same loop that was used for this region's H field:

$$\begin{aligned} B(2\pi r) &= \mu_0 I_T \\ \mu_0 I_f &= \mu_0 I_T \\ I_f &= I_T = I_f + I_{b3} \end{aligned}$$

This means the total bound current in this loop is 0. $I_{b3} = I_f - I_f = 0$
Solving for another loop should clear up the picture a bit. Are there bound currents that just cancel outside? Or is there truely no bound current in the system. Lets solve for loop 2 and see what it tells us.

Loop 2:

The H field was already found:

$$H = \frac{I_f}{2\pi r}, \qquad a_1 < r < a_2$$

Solving for B, μ in this region is μ_2:

$$B = \frac{\mu_2 I_f}{2\pi r}$$

Finding I_T:

$$\begin{aligned} B(2\pi r) &= \mu_0 I_T \\ \mu_2 I_f &= \mu_0 I_T \\ \frac{\mu_2}{\mu_0} I_f &= I_T = I_f + I_{b2} \end{aligned}$$

Solving for I_{b2}:

$$I_{b2} = \left(\frac{\mu_2}{\mu_0} - 1\right) I_f$$

Noting that $\frac{\mu}{\mu_0} - 1 = \chi_m$:

$$I_{b2} = \chi_{m2} I_f$$

This is only equal to 0 when no magnetic material is present ($\mu = \mu_0$), and thus there is some sort of bound charge going on inside loop 2. This leads me to believe that the bound charge being 0 in region 3 means that between loop 2 and a_2 there is a bound charge that exists whose magnitude is negative I_{b2} This leads me to believe it is a surface current flowing along the exterior surface of the coaxial. (Bound currents outside are surface and inside are volume)

Before solving for the surface and volume bound currents, loop 1 is going to be analyzed so the complete picture is clear.

Loop 1:

H:

$$H = \frac{I_f}{2\pi r}, \qquad 0 < r < a_1$$

B, μ in this region is μ_1:

$$B = \frac{\mu_1 I_f}{2\pi r}$$

Finding I_T:

$$B(2\pi r) = \mu_0 I_T$$
$$\mu_1 I_f = \mu_0 I_T$$
$$\frac{\mu_1}{\mu_0} I_f = I_T = I_f + I_{b1}$$

Solving for I_{b1}:

$$I_{b1} = \left(\frac{\mu_1}{\mu_0} - 1\right) I_f$$

:

$$I_{b1} = \chi_{m1} I_f$$

Again, this shows that there is a bound current inside loop 1, in this case, very near the free current. Inside loop 1, the only bound current that could exist is a volume current, and thus $\vec{J_{b1}} = \chi_{m1} I \hat{z}$. Using this value, the magnetization $\vec{M}$ in region 1 can be found. The curl of $\vec{M}$ is equal to $\vec{J_b}$. Finding M_1 using the integral version and the same loop 1 (M_1 is constant along the loop.):

$$M_1 \int dl = \chi_{m1} I_f$$
$$M_1 = \chi_{m1} \frac{I_f}{2\pi r}$$

The direction of the magnetization will be $\pm\hat{\phi}$, for the sake of these calculations, I am going to assume positive phi and let the current and chi values sort it out when actual numbers are applied.

At the intersection between materials μ_1 and μ_2, I assume there is going to be surface currents. The surface currents $\vec{\mathbf{K_{b1}}}$ and $\vec{\mathbf{K_{b2i}}}$ can be found by taking the magnetization and crossing with $\hat{n}$. ($\vec{\mathbf{M_1}}$ for $\vec{\mathbf{K_{b1}}}$ and $\vec{\mathbf{M_2}}$ for $\vec{\mathbf{K_{b2i}}}$)

$$\vec{\mathbf{M_1}} \times \hat{n} = \vec{\mathbf{K_{b1}}}$$
$$\hat{n} = \hat{r}$$
$$\vec{\mathbf{K_{b1}}} = \chi_{m1} \frac{I_f}{2\pi a_1} \hat{\phi} \times \hat{r}$$
$$= -\chi_{m1} \frac{I_f}{2\pi a_1} \hat{z}$$

When integrated around the circumference of the inner magnetic material, the total surface current of material 1 is $\chi_{m1} I_f$ flowing in a direction negative to the volume current found near the free current inside. Thus the total current from the bound currents in this material will add up to 0.

Repeating these steps near the same boundary but for the second magnetic material will result in a similar answer that should be equal to the bound current observed when analyzing loop 2.

$$\begin{aligned}
\vec{\mathbf{M_2}} \times \hat{n} &= \vec{\mathbf{K_{b2i}}} \\
\hat{n} &= -\hat{r} \\
\vec{\mathbf{K_{b2i}}} &= \chi_{m2} \frac{I_f}{2\pi a_1} \hat{\phi} \times -\hat{r} \\
&= \chi_{m2} \frac{I_f}{2\pi a_1} \hat{z}
\end{aligned}$$

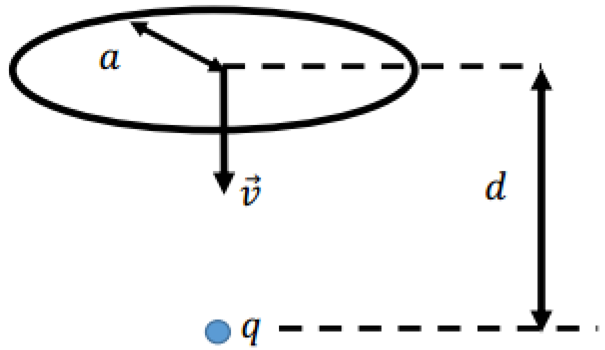
Integrated around the circumference of the inner magnetic material again, the total surface current from the second material does indeed add up to what was found earlier: $I_{b2} = \chi_{m2} I_f$ and knowing that outside the material, no bound currents are observed, the bound current on the outer surface of material 2 must be equal and opposite, or $\vec{\mathbf{K_{b2o}}} = -\chi_{m2} \frac{I_f}{2\pi a_2} \hat{z}$

To sum up all the answers for problem 2:

$$\begin{aligned}
\vec{\mathbf{J_b}} &= \frac{\delta(r) I}{2\pi r} \hat{z} \\
\vec{\mathbf{K_{b1}}} &= -\chi_{m1} \frac{I}{2\pi a_1} \hat{z} \\
\vec{\mathbf{K_{b2i}}} &= \chi_{m2} \frac{I}{2\pi a_1} \hat{z} \\
\vec{\mathbf{K_{b2o}}} &= -\chi_{m2} \frac{I}{2\pi a_2} \hat{z} \\
\vec{\mathbf{H}} &= \frac{I_f}{2\pi r} \hat{\phi}, \ r \neq 0 \\
\vec{\mathbf{B_1}} &= \frac{\mu_1 I}{2\pi r} \hat{\phi}, \ 0 < r < a_1 \\
\vec{\mathbf{B_2}} &= \frac{\mu_2 I}{2\pi r} \hat{\phi}, \ a_1 < r < a_2 \\
\vec{\mathbf{B_3}} &= \frac{\mu_0 I}{2\pi r} \hat{\phi}, \ r > a_2
\end{aligned}$$

Question 3

A ring of current of radius a is moving toward a magnetic monopole at a constant velocity v, calculate the total EMF potential in the circuit.



The idea of a magnetic monopole ires me to no end, however, I imagine it would react somewhat like a point current, and we can use hybrid equations from the point charge using magnetic conventions. Here goes:

The EMF potential in the circuit is equal to the negative change in flux with respect to time.

$$U_{EMF} = -\frac{\partial \Phi}{\partial t}$$

The flux of the B field through the loop generated by this circuit is:

$$\Phi = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

dA in this integral is going to be cylindrical and is equal to:

$$d\vec{\mathbf{A}} = sd\phi ds\hat{z}$$

The B field may look something like this:

$$\vec{\mathbf{B}} = \frac{\mu_0 q}{4\pi r^2}\hat{r}$$

This r-hat is in spherical coordinates, to be usefull it needs to be converted to cylindrical:

$$\hat{r} = sin\theta cos\phi\hat{x} + sin\theta sin\phi\hat{y} + cos\theta\hat{z}$$

At this point, nothing more needs to be done, as the only component needed is z-hat and z-hat for cartesian and cylindrical are identical. So, the component of the B field we need is:

$$\vec{\mathbf{B}} = \frac{\mu_0 q}{4\pi(s^2+z^2)} cos\left(tan^{-1}\left(\frac{s}{z}\right)\right)\hat{z}$$

Dotting B with dA gives:

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{s\mu_0 q ds d\phi}{4\pi(s^2+z^2)} cos\left(tan^{-1}\left(\frac{s}{z}\right)\right)$$

Integrating the d phi portion just gives 2 pi:

$$\Phi = \int \frac{s\mu_0 q ds}{2(s^2+z^2)} cos\left(tan^{-1}\left(\frac{s}{z}\right)\right)$$

This integral is particularly annoying to me and thus I used Mathematica to solve it:

$$\Phi = -\frac{\mu_0 q}{2\sqrt{1+\frac{s^2}{z^2}}}$$

At this point, the derivative of the flux with respect to time needs to be found:

$$\frac{d\Phi}{dt} = \frac{d\Phi}{dz} \cdot \frac{dz}{dt}$$

The derivative of z with respect to t is simply v. Using mathematica again to find the derivative of Phi with respect to z:

$$\frac{d\Phi}{dz} = -\frac{\mu_0 q s^2}{2\left(1+\frac{s^2}{z^2}\right)^{\frac{3}{2}} z^3}$$

Replacing z with d, the EMF potential at this point in time is:

$$U_{EMF} = \frac{\mu_0 q s^2}{2\left(1+\frac{s^2}{d^2}\right)^{\frac{3}{2}} d^3} v$$