# EE636: Matrix Computations
# Lecture 3

Debasattam Pal

EE Department,
Indian Institute of Technology Bombay,
Mumbai, India.

IIT Bombay
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# Recap

- How a computer solves
$$Ax = b, \quad \min \|Ax - b\|, \quad Ax = \lambda x, \quad Av = \sigma u$$
where $\sigma \geqslant 0, \{u_1, u_2, \ldots, u_m\} \subseteq \mathbb{R}^m, \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{R}^n$ are orthonormal.
- The effect of round off error.
- When a solution can be trusted.
- FLOPS to measure computation time.
- Counted the FLOPS for basic matrix multiplication: $2mn$ for a matrix of size $m \times n$ and a vector of size $n$.
- How to solve linear systems of equations: triangular systems.
- The (almost) general case: Gaussian elimination.
- Gaussian elimination (without pivotting) is possible if and only if the "leading principal minors" of $A$ are non-zero.
- Gaussian elimination without pivotting results in the LU decomposition of $A$: that is $A = LU$, where $U$ is an upper triangular matrix and $L$ is a unity lower triangular matrix.
- The $L$ and $U$ in the LU decomposition is unique.

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{bmatrix} \right\} b = b^{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

- $(n-1)$ FLOPS for div^n to get $m_{ij}$ s.
- $(n-1)^2 + \boxed{(n-1)}$ → augmented FLOPS for mult^n.
- $(n-1)^2 + \boxed{(n-1)}$ FLOPS for addⁿ.

$$2(n-1)^2 + 2(n-1) + (n-1)$$

$$(n-1)\left[2(n-1) + 3\right]$$

$$= (n-1)\left[2n+1\right] \approx 2n^2$$

$$= \left(\underline{2n^2} - \underline{n - 1}\right)$$ @ the end of step 1.

$$\text{Total FLOPS} = 2n^2 + 2(n-1)^2 + 2(n-2)^2 + \cdots \approx O(n^3)$$

$A = A^T$

**Definition (symmetric positive definite matrices)**

A symmetric matrix $A = A^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ is said to be positive definite (PD) if $x^{\mathrm{T}} A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

- Geometry of quadratic forms.

$x \in \mathbb{R}^n$ "$x^T A x$" $\in \mathbb{R}$.

$x^T A x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$

$Ax = b$ PD.

$\{x \in \mathbb{R}^n \mid x^T A x = c\}$ = Level sets of $x^T A x$

**Theorem**

$A = A^{\mathrm{T}}$ is PD if and only if every eigenvalue of $A$ is positive. ✓

$I$ PD? ✓

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ PD? ✓

$$\underline{\underline{x_1^2} + \underline{2x_2^2}} > 0 \quad \forall x_1, x_2 \neq 0$$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ PD?

$$x_1^2 - x_2^2 =$$

$\underline{A = A^T} \Rightarrow \exists\, Q \in \mathbb{R}^{n \times n},\ Q^T Q = Q Q^T = I,$
s.t.

$$Q^T A Q = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

SPECTRAL THEOREM.

$\Lambda$

$A = Q \Lambda Q^T$

$\underline{\underline{y}} := Q^T \underline{\underline{x}}$

$$x^T A x = x^T Q \Lambda Q^T x = \underline{y^T \Lambda y}$$

$x^T A x \geq 0 \ \forall x \in \mathbb{R}^n \iff y^T \Lambda y \geq 0 \ \forall y \in \mathbb{R}^n$

$$y^T \Lambda y = y_1^2 \lambda_1 + \dots + y_n^2 \lambda_n \geq 0 \quad \forall y$$

$\Leftrightarrow$ $\underline{\lambda_i \geq 0}$. PSD

$\lambda_i > 0$, PD

$x_1^2 + y_2^2 = c$

GEOMETRY when A $\underline{\text{PSD}}$.

What if $A \neq A^T$.

**Theorem**

$A = A^{\mathrm{T}}$ is PD if and only if every leading principal submatrix of of $A$ is PD.

$$\begin{bmatrix} & & \end{bmatrix} \succ 0 \iff A_1 \succ 0,\ A_2 \succ 0,\ \ldots,\ A_n \succ 0.$$

$$\Leftarrow \checkmark \qquad \underline{\#}$$

$$\Rightarrow A_k \not\succ 0 \Rightarrow \exists\, \xi \in \mathbb{R}^{k} \text{ s.t. } \xi^T A_k \xi$$

$$\begin{bmatrix} \xi \\ 0 \end{bmatrix}^T A \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \xi^T A_k \xi < 0 \Rightarrow A \not\succ 0.$$

# Special case: symmetric positive definite matrices

**Lemma**

If $A = A^{\mathrm{T}}$ is PD then $\det A > 0$.

# The Schur complement

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## Special case: symmetric positive definite matrices

**Theorem**

Let $A = A^{\mathrm{T}}$ be PD, then $A$ admits an LU factorization.

# LU decomposition of symmetric PD matrices: Cholesky factorization

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# Whiteboard

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Thank you