## EE636: Matrix Computations Lecture 4

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## Recap

- How a computer solves

where  $\sigma \geqslant 0, \{u_1, u_2, \dots, u_m\} \subseteq \mathbb{R}^m, \{v_1, v_2, \dots, v_n\} \subseteq \mathbb{R}^n$  are orthonormal.

- The effect of round off error.
- When a solution can be trusted.
- FLOPS to measure computation time.
- Counted the FLOPS for basic matrix multiplication: 2mn for a matrix of size  $m \times n$  and a vector of size n.
- How to solve linear systems of equations: triangular systems.
- The (almost) general case: Gaussian elimination.
- Gaussian elimination (without pivotting) is possible if and only if the leading principal minors of A are non-zero.
- Gaussian elimination without pivotting results in the LU decomposition of A: that is A = LU, where U is an upper triangular matrix and L is a unity lower triangular matrix.
- The L and U in the LU decomposition is unique.
- LU requires  $(O)(n^3)$  FLOPs.
- Inversion of a matrix that admits LU requires  $(O)(n^3)$  FLOPs.
- Symmetric positive definite matrices.

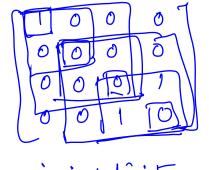
#### Theorem

 $A = A^{T}$  is PD if and only if every eigenvalue of A is positive.

PSD3 All leading principal minurs 
$$\geq 0$$

's necessary for PSD but NOT sufficient.

et  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 &$ 



is indefinite

All leading principal  $\hat{m}m > 0$ 

But Ais indifinition

All principal mime > 0 (=) APSD.

#### Theorem

 $A=A^{\mathrm{T}}$  is PD if and only if every leading principal submatrix of of A is PD.

$$(PD \Rightarrow dt > 0) \equiv (dt \not > 0 \Rightarrow \not > 0)$$

#### Lemma

If 
$$A = A^{\mathrm{T}}$$
 is PD then det  $A > 0$ .

$$\frac{\text{dit }Q = 1}{\text{dit }Q = 1}$$

## The Schur complement

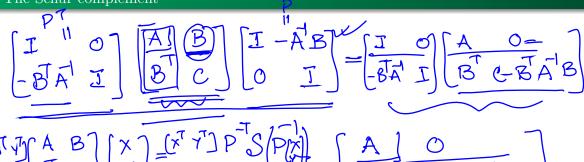
Suppose a symmetric matrix is given in block from as

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$
.

Further, suppose A is invertible. Then
M is insertible iff the Schur complement
Of A is invertible.

C-BTATB.

# The Schur complement



HW's Using Schur complement, and induction on the size of A probe lemma 1.

### Theorem

Let  $A = A^{T}$  be PD, then A admits an LU factorization.

## LU decomposition of symmetric PD matrices: Cholesky factorization

Suppose 
$$A = A^{T}$$
 (suppose  $A$  admits  $U$ )
$$A = LDU \Rightarrow A^{T} = U^{T}DL^{T}X$$

$$A = LDU$$

$$By uniquement of$$

By uniqueness of LD U decomposition.

$$A = LDLT$$

## LU decomposition of symmetric PD matrices: Cholesky factorization

$$X^TAX = X^T L D L^T X$$

# LU decomposition of symmetric PD matrices: Cholesky factorization

$$A = LDL^{T} = LVan Sann VD$$

$$F^{T} := JDL^{T}$$

$$\Rightarrow A = FF^{T} where F is Lower A^{T}$$

$$\in \mathbb{R}^{n \times n}$$

we assume that all leading principal minns 70, TST A is PD. Use induction the list of A o Base case & n=1 AER A>0=) Ais PD. Inductive etep; Suppose the nexult holds the for matrices of size (n-1) (or ws). TST the rusult must held for cizen.

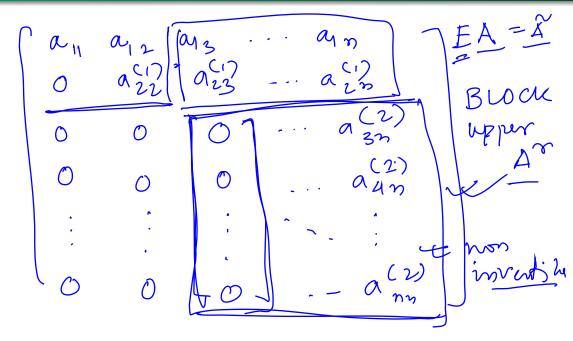
## Whiteboard

A= 
$$\begin{bmatrix} A_{11} \mid a_{12} \\ \overline{a_{12}} \mid a_{22} \end{bmatrix}$$
  $A_{11} \in \mathbb{R}^{(n-1)} \times (n-1)$   
 $A_{12} \in \mathbb{R}^{(n-1)}$ ,  $a_{22} \in \mathbb{R}$ .  
All hading principal minutes of  $A > 0 = 0$   
By inducting hypothesis =  $A_{11} \neq 0$   
 $A_{$ 

### Whiteboard

$$\frac{dut A > 0}{=} = \frac{1}{2} - \frac{1}{2$$

## LU decomposition with partial pivotting



### Permutation matrices

Orthogonal.

If A is invertible and a u.u. is zero =) 7 l = kf1, -, n s.t.  $a_0(u-1) \neq 0$ 

Permutaion involving only one swap is idempetent.

This idea of prisothing mins the LU

How the permutation matrix pervades through a chain of elementary row operation matrices

$$\pi_{35} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 & 0 & 0 \\ 0 & -m_{42} & 0 & 1 & 0 & 0 \\ 0 & -m_{52} & 0 & 0 & 1 & 0 \\ 0 & -m_{62} & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -m_{52} & 0 & 0 & 1 & 0 \\ 0 & -m_{42} & 0 & 1 & 0 & 0 \\ 0 & -m_{32} & 1 & 0 & 0 & 0 \\ 0 & -m_{62} & 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\pi_{35}$$

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$$\pi_{35}E_{2}E_{1}A = \pi_{35}A^{(2)}$$

$$\widehat{E}_{2}\pi_{35}E_{1}A = \widehat{A}^{(2)}$$

$$\widehat{E}_{2}\widehat{E}_{1}\pi_{35}A = \widehat{A}^{(2)}$$

Back to LU decomposition with partial pivotting

Thank you