EE636: Matrix Computations
Lecture 2

Hello everyone!
We'll start shortly.

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Recap

$\mathbb{R}$

- How a computer solves
$$Ax = b, \quad \min \|Ax - b\|, \quad Ax = \lambda x, \quad Av = \sigma u$$
where $\sigma \geqslant 0, \{u_1, u_2, \ldots, u_m\} \subseteq \mathbb{R}^m, \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{R}^n$ are orthonormal.
- The effect of round off error.
- When a solution can be trusted.
- FLOPS to measure computation time.
- Counted the FLOPS for basic matrix multiplication: $2mn$ for a matrix of size $m \times n$ and a vector of size n.
- How to solve linear systems of equations: triangular systems.
- The (almost) general case: Gaussian elimination.

$$E_{n-1}E_{n-2}\cdots E_2 E_1 A = \widetilde{A} \text{ (upper triangular)}$$

E_1

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix}$$

$$m_{ij} := \frac{a_{ij}^{(j-1)}}{a_{jj}^{(j-1)}}$$

$\neq 0$

$a_{jj}^{(j-1)}$

Solving a linear system of equations: when is it possible to carry out Gaussian elimination?

$$\forall j \; \underline{a_{jj}^{(j-1)} \neq 0} \Leftarrow A_1, A_2, \ldots, A_n \text{ are}$$

$$\overset{?}{\Rightarrow} \text{ all invertible. } \checkmark$$

$$A_k := A(1:k, 1:k) \quad (\text{HW})$$

$$A_1 := A(1:1, 1:1) = a_{11}$$

$$\underline{A_1} \text{ is INVERTIBLE} \Rightarrow a_{11} \neq 0$$

$$a_{11}^{(0)}$$

Solving a linear system of equations: when is it possible to carry out Gaussian elimination?

$$\begin{bmatrix} I_{j-1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{j+1,j} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n,j} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \widetilde{A}_{11}^{(j-1)} & \widetilde{A}_{12}^{(j-1)} & \widetilde{A}_{13}^{(j-1)} & \cdots & \widetilde{A}_{1n}^{(j-1)} \\ 0 & a_{j,j}^{(j-1)} & a_{j,j+1}^{(j-1)} & \cdots & a_{j,n}^{(j-1)} \\ 0 & a_{j+1,j}^{(j-1)} & a_{j+1,j+1}^{(j-1)} & \cdots & a_{j+1,n}^{(j-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n,j}^{(j-1)} & a_{n,j+1}^{(j-1)} & \cdots & a_{n,n}^{(j-1)} \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{11}^{(j-1)} & \widetilde{A}_{12}^{(j-1)} & \widetilde{A}_{13}^{(j-1)} & \cdots & \widetilde{A}_{1n}^{(j-1)} \\ 0 & a_{j,j}^{(j-1)} & a_{j,j+1}^{(j-1)} & \cdots & a_{j,n}^{(j-1)} \\ 0 & 0 & a_{j+1,j+1}^{(j)} & \cdots & a_{j+1,n}^{(j)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n,j+1}^{(j)} & \cdots & a_{n,n}^{(j)} \end{bmatrix}$$

det = 1

$\det A_{j+1} = \not= 0$

$\det \widetilde{A}_{11}^{(j-1)} \cdot a_{j,j}^{(j-1)} \cdot a_{j+1,j+1}^{(j)} = \not= 0$

$\not= 0 \qquad \not= 0 \qquad \not= 0$

$$\underline{Ax = b} \Rightarrow MAx = Mb$$

$$\Rightarrow \tilde{A}x = \tilde{b}$$

$$\underbrace{E_{n-1} \cdot E_{n-2} \cdot \cdots \cdot E_2 \cdot E_1}_{M} \cdot A = \tilde{A}$$

M is invertible

$$(Ax = b) \Leftrightarrow \tilde{A}x = \tilde{b}$$

BACKWARD SUBN.

$$[A \mid b] \rightarrow [\tilde{A} \mid \tilde{b}] \quad [\tilde{A}\ \tilde{b}] = \underline{M}\,\underline{[(A)\,b]}$$

AUGMENTED

$$A \rightarrow \underline{M} \qquad Ax = \underline{b}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}}_{} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}}_{} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ (\underline{m_{21} m_{32}} - \underline{m_{31}}) & -m_{32} & 1 \end{bmatrix}$$

$\underline{\underline{M}}$

$$\tilde{b} = Mb = \begin{bmatrix} b_1^{(0)} \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix} \begin{array}{l} \Rightarrow b_1^{(0)} = b_1 \\ \Rightarrow y_2 = b_2 - m_{21} y_1 \\ \Rightarrow y_3 = b_3 - m_{31} y_1 - m_{32} y_2 \end{array}$$

$$y := $$

$$y_j = b_j - m_{j1} y_1 - m_{j2} y_2 - \dots - m_{j,j-1} y_{j-1}$$

$$\underline{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = b$$

$$\underset{=}{\tilde{A}}\, x = y$$

$$\tilde{A} =: U$$

$$L U \underset{=}{x} = L y = \underset{=}{b}$$

$$Ax = b$$

$$Ax - LUx = 0$$

$$\underline{\underline{M}}A = U$$

$$\Rightarrow \boxed{A = [L]U}$$

LU-decomposition

$$L = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & 0 \\ m_{31} & m_{32} & \ddots & \\ \vdots & \vdots & & \\ m_{n1} & m_{n2} & \cdots & 1 \end{bmatrix} = A$$

$$= M^{-1}$$

$A = LU$

$A_k = L(1:k, 1:k)\, U(1:k, 1:k)$

unit lower Δ^r

THM: Suppose A admits LU decomposition

upper Δ^r

$\Rightarrow L, U$ are unique

(*) **HW:** $a_{j,j}^{(j-1)} \neq 0 \;\forall j \Rightarrow$

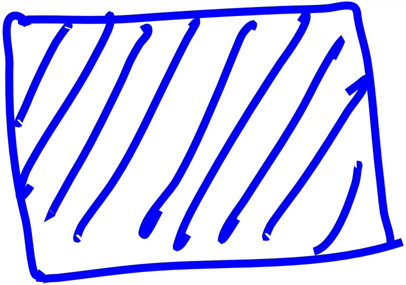
$A_1, A_2, \ldots, A_n$ are invertible.

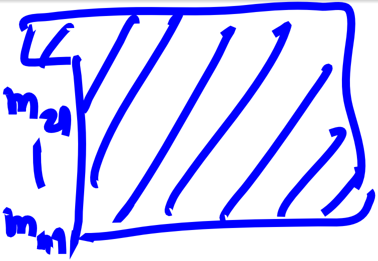
"Submit by Monday"

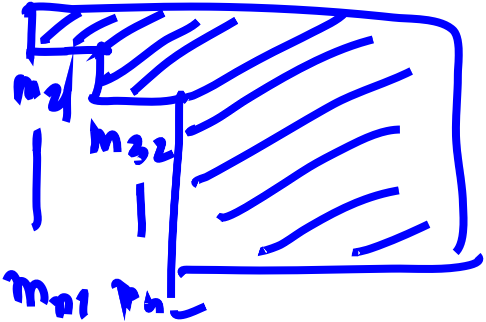
(*) **HW:** Write a pseudocode for LU decomposition.

Whiteboard

$A =$







LU

$A = LU$

$$\underline{U(2,1:n) + \frac{a_{21}}{a_{11}} U(1,1:n)} = A(2,1:n)$$

$$\begin{bmatrix} 1 & 0 & 0 & - & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 & - & 0 \\ | & & & & \\ \frac{a_{n1}}{a_{11}} & \boxed{} \; x & & & \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & - & u_{1n} \\ 0 & u_{22} & u_{23} & - & u_{2n} \\ 0 & 0 & u_{33} & - & u_{3n} \\ | & | & | & \searrow & | \\ 0 & 0 & 0 & - & u_{nn} \end{bmatrix} \; x$$

$$U(1,1:n) = A(1,1:n)$$

$$\cancel{a_{11}} L(1:n,1) = A(1:n,1)/a_{11}$$

$R_1(U) \to C_1(L) \to R_2(U) \to C_2(L)$

$\to \cdots R_n(U) \to C_n(L)$

LU decompⁿ is <u>unique</u>.

Positive definite matrices.

$A = A^T$ is said to be PD if

$x^T A x \geq 0 \ \forall x \in \mathbb{R}^n$

$= 0 \iff x = 0.$

$A = L L^T$ CHOLESKY.

Whiteboard

Thank you