EE636: Matrix Computations Lecture 3

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Recap

\bullet How a computer solves

$$Ax = b$$
, $\min ||Ax - b||$, $Ax = \lambda x$, $Av = \sigma u$

where $\sigma \geqslant 0, \{u_1, u_2, \dots, u_m\} \subseteq \mathbb{R}^m, \{v_1, v_2, \dots, v_n\} \subseteq \mathbb{R}^n$ are orthonormal.

- The effect of round off error.
- When a solution can be trusted.
- FLOPS to measure computation time.
- Counted the FLOPS for basic matrix multiplication: 2mn for a matrix of size $m \times n$ and a vector of size n.
- How to solve linear systems of equations: triangular systems.
- The (almost) general case: Gaussian elimination.
- Gaussian elimination (without pivotting) is possible if and only if the leading principal minors of A are non-zero.
- Gaussian elimination without pivotting results in the LU decomposition of A: that is A = LU, where U is an upper triangular matrix and L is a unity lower triangular matrix.
- ullet The L and U in the LU decomposition is unique.

FLOPs count of LU decomposition

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & \overline{a_{32}} & \overline{a_{33}} & \cdots & \overline{a_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

- · (n-1) FLOPS for div? to get miss. · (n-1)2+(n-1)] to augmented · (n-1)2+(n-1)] to ELOPS for mult?.
- (p-1) FLOPS for add?

FLOPs count of LU decomposition

$$2(n-1)^{2} + 2(n-1) + (n-1)$$

$$(n-1)[2(n-1) + 3]$$

$$= (n-1)[2n+1] \approx 2n^{2} \approx$$

$$= \left(\frac{2n^2 - n - 1}{n}\right)$$
 @ the end of step 1.

Total PLOPS = 2 1 + 201-1) + 2(m)

A= A

Definition (symmetric positive definite matrices)

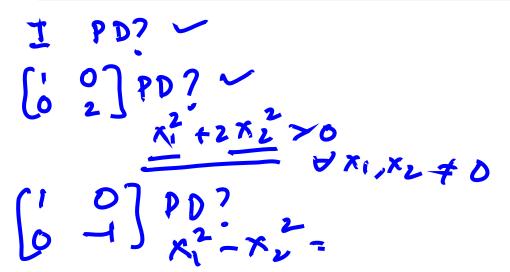
A symmetric matrix $A = A^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ is said to be positive definite (PD) if $x^{\mathrm{T}}Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

• Geometry of quadratic forms.

Theorem

 $A = A^{\mathrm{T}}$ is PD if and only if every eigenvalue of A is positive.





A=AT =)
$$\exists Q \in \mathbb{R}^{n \times n}$$
, $Q^{\dagger}Q = QQ^{\dagger} = 1$,
 $S^{\dagger}AQ = \begin{bmatrix} n_1 & 0 \\ 0 & n_n \end{bmatrix}$ SPECTRAL
THEOREM.
 $A = Q \wedge Q^{\dagger} \qquad Y = Q^{\dagger}X$
 $X^{\dagger}AX = X^{\dagger}Q \wedge Q^{\dagger}X = Y^{\dagger}AY$
 $X^{\dagger}AX = X^{\dagger}Q \wedge Q^{\dagger}X = Y^{\dagger}AY$
 $X^{\dagger}AX = X^{\dagger}Q \wedge Q^{\dagger}X = Y^{\dagger}AY$

$$Y^{T} \wedge Y = y^{2} \wedge 1 + \cdots + y^{2} \wedge n \neq 0$$

$$(a) \wedge 1 \neq 0 \cdot PSD$$

$$(b) \wedge 1 \neq 0 \cdot PD$$

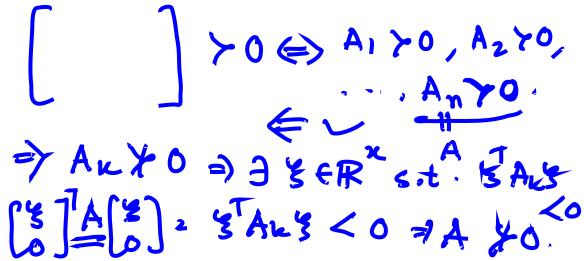
$$(c) \wedge 1 \neq 0 \cdot PD$$

$$(d) \wedge 1 \neq 0 \cdot PD$$

$$(d)$$

Theorem

 $A = A^{\mathrm{T}}$ is PD if and only if every leading principal submatrix of of A is PD.



Lemma

If $A = A^{\mathrm{T}}$ is PD then det A > 0.

The Schur complement

The Schur complement

Theorem

Let $A = A^{T}$ be PD, then A admits an LU factorization.

LU decomposition of symmetric PD matrices: Cholesky factorization

LU decomposition of symmetric PD matrices: Cholesky factorization

LU decomposition of symmetric PD matrices: Cholesky factorization

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Thank you