

1 Theory

1.1 Definitions

Given a constraint system C where C is an ordered set of abstract constraints over some set of variables, $\{C_1, \dots, C_n\}$. The satisfiability problem is in conjunctive normal form (CNF): $C = \bigwedge_{i=1, \dots, n} C_i$; and each C_i is a disjunction of literals $C_i = l_{i1} \vee \dots \vee l_{ik_i}$ where each literal l_{ij} is either a Boolean variable x or its negation $\neg x$.

Satisfiability (SAT) : A CNF is satisfiable iff there exists an assignment of truth values to its variables such that the formula evaluates to true. If not, it is unsatisfiable (UNSAT).

Intuitively a constraint system contains the contracts that constrain component behavior and faults that are defined over these components. In the case of the nominal model, a constraint system is defined as follows:

Let F be the set of all faults defined in the model and G be the set of all component contracts (guarantees). $C = \{C_1, C_2, \dots, C_n\}$ where for $i \in \{1, \dots, n\}$, C_i has the following constraints for any $f_j \in F$ and $g_k \in G$ with regard to the top level property P :

$$C_i \in \begin{cases} f_j : \text{inactive} \\ g_k : \text{true} \\ P : \text{true} \end{cases}$$

Given a state space S , a transition system (I, T) consists of the initial state predicate $I : S \rightarrow \{0, 1\}$ and a transition step predicate $T : S \times S \rightarrow \{0, 1\}$. Reachability for (I, T) is defined as the smallest predicate $R : S \rightarrow \{0, 1\}$ which satisfies the following formulas:

$$\begin{aligned} \forall s. I(s) &\Rightarrow R(s) \\ \forall s, s'. R \wedge T(s, s') &\Rightarrow R(s') \end{aligned}$$

A safety property $\mathcal{P} : S \rightarrow \{0, 1\}$ is a state predicate. A safety property \mathcal{P} holds on a transition system (I, T) if it holds on all reachable states. More formally, $\forall s. R(s) \Rightarrow \mathcal{P}(s)$. When this is the case, we write $(I, T) \vdash \mathcal{P}$.

Given a transition system which satisfies a safety property P , it is possible to find which parts of the system are necessary for satisfying the safety property through the use of Ghassabani's *All Minimal Inductive Validity Cores* algorithm [2, 3]. This algorithm makes use of the collection of all minimal unsatisfiable subsets of a given transition system in terms of the negation of the top level property, i.e. the top level event. This is a minimal subset of the activation literals such that the constraint system C with the formula $\neg P$ is unsatisfiable when these activation literals are held true. Formally:

MUS : A Minimal Unsatisfiable Subset M of a constraint system C is : $\{M \subseteq C \mid M \text{ is UNSAT and } \forall c \in M: M \setminus \{c\} \text{ is SAT}\}$. This is the minimal explanation of the constraint systems infeasability.

A closely related set is a *minimal correction set* (MCS). The MCSs describe the minimal set of model elements for which if constraints are removed, the constraint system is satisfied. For C , this corresponds to which faults are not constrained to inactive (and are hence active) and violated contracts which lead to the violation of the safety property. In other words, the minimal set of active faults and/or violated properties that lead to the top level event.

MCS : A Minimal Correction Set M of a constraint system C is : $\{M \subseteq C \mid C \setminus M \text{ is SAT and } \forall S \subset M : C \setminus S \text{ is UNSAT}\}$. A MCS can be seen to “correct” the infeasability of the constraint system.

A duality exists between MUSs of a constraint system and MCSs as established by Reiter [6]. This duality is defined in terms of *hitting sets*. A hitting set of a collection of sets A is a set H such that every set in A is “hit” by H ; H contains at least one element from every set in A [4].

Hitting Set: Given a collection of sets K , a hitting set for K is a set $H \subseteq \cup_{S \in K} S$ such that $H \cap S \neq \emptyset$ for each $S \in K$. A hitting set for K is minimal if and only if no proper subset of it is a hitting set for K .

Utilizing this approach, we can easily collect the MCSs from the MUSs provided through the all MIVC algorithm and a hitting set algorithm by Murakami et. al. [1,5].

Cut sets and minimal cut sets provide important information about the vulnerabilities of a system. A *Minimal Cut Set* (MinCutSet) is a minimal collection of faults that lead to the violation of the safety property (or in other words, lead to the top level event). We define MinCutSet in terms of the constraint system in question as follows:

MinCutSet : Minimal Cut Set of a constraint system C with all faults in the system denoted as the set F is : $\{cut \subseteq F \mid C \setminus cut \text{ is SAT and } \forall c \in cut : C \setminus c \text{ is UNSAT}\}$.

When the MCS contains only faults, the MCS is equivalent to the MinCutSet as shown in the first part of the proof. When contracts exist in the MCS, a replacement can be made which transforms the MCS into the MinCutSet.

1.2 Transformation of MCS

Theorem: The MinCutSet can be generated by transformation of the MCS.

Proof: For faults in the model F and subcomponent contracts G :
 $MCS \cap F \neq \{\} \vee MCS \cap G \neq \{\}$.

Part 1: $MCS \cap G = \{\}$ (Leaf level of system)

(i) $MCS \subseteq \text{MinCutSet}$:

Let $M \in MCS$. Then $C \setminus M$ is SAT. Since $\exists g \in C$ for $g \in G$, thus $M \subseteq F$ and is a cut set.

By minimality of the MCS , M is a minimal cut set for $\neg P$.

(ii) $\text{MinCutSet} \subseteq MCS$:

Let $M \in \text{MinCutSet}$. Then all faults in M cause $\neg P$ to occur by definition. Thus, $C \setminus M$ is SAT.

By minimality of MinCutSet , M is also minimal and thus is a minimal correction set.

Part 2: $MCS \cap G \neq \{\}$ (Intermediate level of system)

Assume $\overline{C} = \{F, G, P\}$ with the constraints that all $f \in F$ are inactive and all $g \in G$ are valid with regard to top level property P , i.e. the nominal model proves.

Let $MCS = \{f_1, \dots, f_n, g_1, \dots, g_m\}$

For $g_1 \in MCS$, $\exists F_1 \subseteq F$ where F_1 is a minimal set of active faults that cause the violation of g_1 . Replace g_1 in MCS with F_1 . Then $MCS = \{f_1, \dots, f_n, F_1, g_2, \dots, g_m\}$.

Perform this replacement for all $g_i \in MCS$ until we reach $MCS = \{f_1, \dots, f_n, F_1, \dots, F_m\}$.

Since F_i is minimal, the MCS retains its minimality. Furthermore $MCS \subseteq F$ and $C \setminus MCS$ is SAT. Therefore the MCS is transformed into MinCutSet .

1.3 Replacements for Max N Faults Analysis: Algorithm and Theory

At the leaf level, only faults are contained in IVCs (and consequently in MCSs). Thus we store these cut sets in a lookup table for quick access throughout the algorithm. For this algorithm, we assume we are in an intermediate level of the analysis. Given a fault hypothesis of max N faults, we disregard any cut sets over cardinality N and collect the rest.

Assuming we have used the hitting set to generate MCS from the IVC, this is where the algorithm begins.

Let $MCS = F \cup G$ for faults $f \in F$ and contracts $g \in G$. If $|G| = 0$, then add MCS to contract lookup table (it is already a MinCutSet).

Assume $|G| = \alpha > 0$.

Let $Cut(g_j) = cut_1(g_j), \dots, cut_{\beta}(g_j)$ be all minimal cut sets for a contract $g_j \in G$ where $|Cut(g_j)| = \beta_j$.

Algorithm 1: Replacement

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1  $List(MCS) = MCS$  : initialize list of MCSs that contain contracts ;
2  $Cut(g_j)$  : all minimal cut sets with cardinality less than or equal to  $N$  of the
   contract  $g_j$  ;
3  $0 < j \leq \alpha$  ;
4  $0 < i \leq \gamma$  ;
5 for all  $MCS_i \in List(MCS)$  do
6   Remove  $MCS_i$  from  $List(MCS)$ ;
7   for all  $g_j \in MCS_i$  do
8     Remove  $g_j$  from  $MCS_i$  ;
9     for all  $cut_k(g_j) \in Cut(g_j)$  do
10      Add  $cut_k(g_j)$  to  $MCS_i$  ;
11      if  $|MCS_i| \leq N$  then
12        if  $\exists g_j \in MCS_i$  then
13          Add  $MCS_i$  to  $List(MCS)$ 
14        else
15          Add  $MCS_i$  to contract look up table (done)

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The number of replacements R that are made in this algorithm are given as the combination of minimal cut sets of the contracts within the MCS . The validity of this statement follows directly from the general multiplicative combinatorial principle. Therefore, the number of replacements R is given by:

$$R = \prod_{j=1}^{\alpha} |Cut(g_j)|$$

This is equivalent to the number of minimal cut sets generated which follows easily from the combinatorial calculation. It is also important to note that the cardinality of $List(MCS)$ is bounded. Every new MCS that is generated that still contains contracts is added to $List(MCS)$. Thus every contract up until the penultimate contract contributes to the cardinality of $List(MCS)$. The bound is given by:

$$|List(MCS)| \leq \prod_{j=1}^{\alpha-1} |Cut(g_j)|$$

Theorem 1. *The elimination of any $|MCS_i| \leq N$ will not eliminate any $|MinCutSet| \leq N$.*

Proof. If $\exists g_j \in MCS_i$, then $MCS_i = MinCutSet$ by proof previously given. Thus, this cut set is not required for consideration in the analysis since $|MinCutSet| > N$.

If $\exists g_j \in MCS_i$, then further replacement will need to be made. Thus in the next phase of the algorithm, we will have MCS_{i+1} . In this case, the size can only get larger and $|MCS_i| \leq |MCS_{i+1}|$ and we have not eliminated a cut set that must be considered for the analysis.

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