

21MAB201T : TRANSFORMS AND BOUNDARY VALUE PROBLEMS

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UNIT - V 21MAB201T : Z - Transforms

CONTENTS:

- Z - Transforms - Elementary properties of Z - Transforms
- Inverse Z - Transform (using partial fraction and residues)
- Initial value theorem and final value theorem
- Convolution theorem
- Formation of difference equations
- Solution of difference equations using Z - Transform

Z - TRANSFORM

INTRODUCTION

- The Z - transform plays a significant role in the Modern communication theory especially in signal processing.
- In signal processing, the Z - transform converts a discrete time - domain signal, which is a sequence of real numbers, into a complex frequency - domain representation.
- The Z - transform and advanced Z - transform were introduced (under the Z - transform name) by E. I. Jury in 1958 in Sampled - Data Control Systems (John Wiley & Sons). The idea contained within the Z - transform was previously known as the "generating function method".
- There are two mainly known Z - transforms, namely, unilateral and bilateral.
- The (unilateral) Z - transform is to discrete - time signals what the one - sided Laplace transform is to continuous - time signals.

Z - TRANSFORM

INTRODUCTION

- In other words, just as Laplace transforms are used to solve differential equations, Z - transforms are used to solve difference equations.
- The Z - transform is an essential part of a structured control system design.
- One of the very nice things about the Z - transform is the ease with which we can write a software from it.
- Using Z - transform, almost by inspection, theoretically correct coding from a transfer function can be done.

Z - TRANSFORM

DEFINITION : Unilateral Z - transform

Let $\{f(n)\}$ be any sequence defined for $n = 0, 1, 2, \dots$.

Then the Z - transform of the given sequence $\{f(n)\}$ is defined as

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

where z is an arbitrary complex variable.

The right hand side of the above equation is a function of z and hence it is denoted by $Z[f(n)] = F(z)$.

This transform is known as **Unilateral Z - transform** (or) **One - Sided Z - transform** of the given sequence $\{f(n)\}$.

Z - TRANSFORM

DEFINITION : Unilateral Z - transform for discrete values of t

Let $\{f(t)\}$ be a sequence defined for discrete values of $t = 0, T, 2T, 3T, \dots$.
i.e., $\{f(t)\}$ is defined for discrete values of t , where $t = nT$,
 $n = 0, 1, 2, 3, \dots$ and T being the sampling period,

Then the Z - transform of the given sequence $\{f(t)\}$ is defined as

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

i.e., If the sequence is given as $\{f(t)\}$, then replace t by nT and find the Z - Transform of $\{f(nT)\}$.

Z - TRANSFORM

DEFINITION : Bilateral Z - transform

Let $\{f(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \dots$.

Then the Z - transform of the given sequence $\{f(n)\}$ is defined as

$$Z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

This Z - transform is called as **Bilateral Z - transform** (or) **two - sided Z - transform** of the given sequence $\{f(n)\}$.

NOTE :

- ① The domain for the sequence in unilateral Z-transform is $n = \{0, 1, 2, \dots\}$ whereas for Bilateral Z-transform is $n = \{0, \pm 1, \pm 2, \dots\}$.
- ② Throughout this chapter we consider only the unilateral Z - transform.

Basic Formulae :

- ① $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$ where $|x| < 1$
- ② $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$ where $|x| < 1$
- ③ $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ where $|x| < 1$
- ④ $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$ where $|x| < 1$
- ⑤ $-\log(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$
- ⑥ $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Z - Transform of some elementary sequences

I. Find the Z - Transform of the following sequences :

1). 1 2). a^n 3). n 4). $\frac{1}{n}$ 5). $\frac{1}{n+1}$

6). $\frac{1}{n-1}$ 7). $\frac{1}{n!}$ 8). $\frac{1}{(n+1)!}$ 9). $\frac{1}{n+2}$

(10). Unit Impulse Sequence. (11). Unit Step Sequence.

(1). To Find : $Z[1]$

Solution : We know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\therefore Z[1] = \sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{z}\right)^n$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \quad \text{where } \left|\frac{1}{z}\right| < 1$$

$$= \left(1 - \frac{1}{z}\right)^{-1}, \quad 1 < |z|$$

$$= \left(\frac{z-1}{z}\right)^{-1}, \quad |z| > 1$$

$$\boxed{Z[1] = \frac{z}{z-1}}, \quad |z| > 1$$

(2). To Find : $Z[a^n]$

Solution : We know that

$$\begin{aligned} Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\ \therefore Z[a^n] &= \sum_{n=0}^{\infty} a^n \cdot \left(\frac{1}{z}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \\ &= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \quad \text{where } \left|\frac{a}{z}\right| < 1 \\ &= \left(1 - \frac{a}{z}\right)^{-1}, \quad |a| < |z| \\ &= \left(\frac{z-a}{z}\right)^{-1}, \quad |z| > |a| \end{aligned}$$

$$\boxed{Z[a^n] = \frac{z}{z-a}}, \quad |z| > |a|$$

(3). To Find : $Z[n]$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z[n] &= \sum_{n=0}^{\infty} n. \left(\frac{1}{z}\right)^n \\ &= 0 + 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \dots \\ &= \frac{1}{z} \left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \dots \right] \\ &= \frac{1}{z} \left(1 - \frac{1}{z} \right)^{-2} \\ &= \frac{1}{z} \left(\frac{z-1}{z} \right)^{-2} = \frac{1}{z} \left(\frac{z}{z-1} \right)^2 \end{aligned}$$

$$\boxed{Z[n] = \frac{z}{(z-1)^2}, \quad |z| > 1}$$

(4). To Find : $Z\left[\frac{1}{n}\right]$, $n > 0$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z\left[\frac{1}{n}\right] &= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) z^{-n} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \frac{1}{z^n} \\ &= \frac{1}{1} \left(\frac{1}{z}\right) + \frac{1}{2} \left(\frac{1}{z}\right)^2 + \frac{1}{3} \left(\frac{1}{z}\right)^3 + \dots \\ &= \frac{\left(\frac{1}{z}\right)}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \dots \\ &= -\log\left(1 - \frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z-1}{z}\right)^{-1} \end{aligned}$$

$$\boxed{Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)} //$$

(5). To Find : $Z\left[\frac{1}{n+1}\right]$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z\left[\frac{1}{n+1}\right] &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right) \left(\frac{1}{z}\right)^n \\ &= 1 + \frac{1}{2} \left(\frac{1}{z}\right) + \frac{1}{3} \left(\frac{1}{z}\right)^2 + \frac{1}{4} \left(\frac{1}{z}\right)^3 \dots \\ &= z \left[\left(\frac{1}{z}\right) + \frac{1}{2} \left(\frac{1}{z}\right)^2 + \frac{1}{3} \left(\frac{1}{z}\right)^3 + \frac{1}{4} \left(\frac{1}{z}\right)^4 \dots \right] \\ &= -z \log \left(1 - \frac{1}{z}\right) = -z \log \left(\frac{z-1}{z}\right) = z \log \left(\frac{z-1}{z}\right)^{-1} \end{aligned}$$

$$\boxed{Z\left[\frac{1}{n+1}\right] = z \log \left(\frac{z}{z-1}\right), \quad |z| > 1 \quad //}$$

(6). To Find : $Z\left[\frac{1}{n-1}\right]$, $n > 1$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z\left[\frac{1}{n-1}\right] &= \sum_{n=2}^{\infty} \left(\frac{1}{n-1}\right) z^{-n} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1}\right) \left(\frac{1}{z}\right)^n \\ &= \frac{1}{1} \left(\frac{1}{z}\right)^2 + \frac{1}{2} \left(\frac{1}{z}\right)^3 + \frac{1}{3} \left(\frac{1}{z}\right)^4 + \dots \\ &= \frac{1}{z} \left[\left(\frac{1}{z}\right) + \frac{1}{2} \left(\frac{1}{z}\right)^2 + \frac{1}{3} \left(\frac{1}{z}\right)^3 + \dots \right] \\ &= -\frac{1}{z} \log \left(1 - \frac{1}{z}\right) = -\log \left(\frac{z-1}{z}\right) = \frac{1}{z} \log \left(\frac{z-1}{z}\right)^{-1} \end{aligned}$$

$$\boxed{Z\left[\frac{1}{n-1}\right] = \frac{1}{z} \log \left(\frac{z}{z-1}\right)} , \quad |z| > 1 \quad //$$

(7). To Find : $Z\left[\frac{1}{n!}\right]$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z\left[\frac{1}{n!}\right] &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n \\ &= 1 + \frac{1}{1!} \left(\frac{1}{z}\right) + \frac{1}{2!} \left(\frac{1}{z}\right)^2 + \frac{1}{3!} \left(\frac{1}{z}\right)^3 + \frac{1}{4!} \left(\frac{1}{z}\right)^4 + \dots \\ &= 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^4}{4!} + \dots \end{aligned}$$

$$\boxed{Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}} \quad //}$$

(8). To Find : $Z\left[\frac{1}{(n+1)!}\right]$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z\left[\frac{1}{(n+1)!}\right] &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{z}\right)^n \\ &= \frac{1}{1!} + \frac{1}{2!} \left(\frac{1}{z}\right) + \frac{1}{3!} \left(\frac{1}{z}\right)^2 + \frac{1}{4!} \left(\frac{1}{z}\right)^3 + \dots \\ &= z \left[\frac{1}{1!} \left(\frac{1}{z}\right) + \frac{1}{2!} \left(\frac{1}{z}\right)^2 + \frac{1}{3!} \left(\frac{1}{z}\right)^3 + \frac{1}{4!} \left(\frac{1}{z}\right)^4 + \dots \right] \end{aligned}$$

$$\boxed{Z\left[\frac{1}{(n+1)!}\right] = z \left[e^{\frac{1}{z}} - 1\right]} //$$

(9). To Find : $Z\left[\frac{1}{n+2}\right]$

Solution : We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z\left[\frac{1}{n+2}\right] &= \sum_{n=0}^{\infty} \frac{1}{(n+2)} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(n+2)} \left(\frac{1}{z}\right)^n \\ &= \frac{1}{2} + \frac{1}{3} \left(\frac{1}{z}\right) + \frac{1}{4} \left(\frac{1}{z}\right)^2 + \dots \\ &= z^2 \left(\frac{1}{2} \left(\frac{1}{z}\right)^2 + \frac{1}{3} \left(\frac{1}{z}\right)^3 + \frac{1}{4} \left(\frac{1}{z}\right)^4 + \dots \right) \\ &= z^2 \left(-\log \left(1 - \frac{1}{z} \right) - \frac{1}{z} \right) = z^2 \left(-\log \left(\frac{z-1}{z} \right) - \frac{1}{z} \right) \\ &= z^2 \left(\log \left(\frac{z}{z-1} \right) - \frac{1}{z} \right) \end{aligned}$$

$$\boxed{Z\left[\frac{1}{n+2}\right] = z^2 \log \left(\frac{z}{z-1} \right) - z} //$$

(10). To Find : The Z - Transform of Unit Impulse Sequence

Solution : The Unit Impulse Sequence is defined by

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$\therefore Z[f(n)] = \sum_{n=0}^{\infty} f(n) \cdot z^{-n}$$

$$\begin{aligned} Z[\delta(n)] &= \sum_{n=0}^{\infty} \delta(n) \cdot z^{-n} \\ &= \sum_{n=0}^{\infty} \delta(n) \cdot \left(\frac{1}{z}\right)^n \end{aligned}$$

$$= \delta(0) \cdot 1 + \delta(1) \cdot \left(\frac{1}{z}\right) + \delta(2) \cdot \left(\frac{1}{z}\right)^2 + \delta(3) \cdot \left(\frac{1}{z}\right)^3 + \dots$$

$$= 1 + 0 + 0 + 0 + \dots$$

$$\boxed{Z[\delta(n)] = 1} \quad //$$

(11). To Find : The Z - Transform of Unit Step Sequence

Solution : The Unit Step Sequence is defined by

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) \cdot z^{-n}$$

$$\begin{aligned} Z[u(n)] &= \sum_{n=0}^{\infty} u(n) \cdot \left(\frac{1}{z}\right)^n \\ &= u(0) \cdot 1 + u(1) \cdot \left(\frac{1}{z}\right) + u(2) \cdot \left(\frac{1}{z}\right)^2 + u(3) \cdot \left(\frac{1}{z}\right)^3 + \dots \\ &= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \\ &= \left(1 - \frac{1}{z}\right)^{-1} = \left(\frac{z-1}{z}\right)^{-1} \end{aligned}$$

$$Z[u(n)] = \frac{z}{z-1} \quad //$$

Properties of Z - Transform

(1). Linearity Property

$$Z [af(n) \pm bg(n)] = aZ[f(n)] \pm bZ[g(n)]$$

(2). Differentiation in z - Domain :

If $Z[f(n)] = F(z)$, Then

$$Z[nf(n)] = -z \frac{d}{dz} [F(z)]$$

(3). First Shifting [Frequency Shifting] Property (or) Damping Rule :

If $Z[f(n)] = F(z)$, Then

$$Z[a^n f(n)] = F\left(\frac{z}{a}\right)$$

Properties of Z - Transform

(4). Second Shifting [Time Shifting] Property :

If $Z[f(n)] = F(z)$, Then

$$\textcircled{i} \quad Z[f(n+1)] = zF(z) - zf(0)$$

$$\textcircled{ii} \quad Z[f(n+2)] = z^2F(z) - z^2f(0) - zf(1)$$

$$\textcircled{iii} \quad Z[f(n+3)] = z^3F(z) - z^3f(0) - z^2f(1) - zf(2)$$

(2). Differentiation in z - Domain :

Statement :

If $Z[f(n)] = F(z)$, Then $Z[nf(n)] = -z \frac{d}{dz} [F(z)]$

Proof : We know that

$$F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\text{i.e., } F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\frac{d}{dz} [F(z)] = \sum_{n=0}^{\infty} f(n) (-n \cdot z^{-n-1})$$

$$\frac{d}{dz} [F(z)] = - \sum_{n=0}^{\infty} n f(n) z^{-n} \cdot z^{-1}$$

$$\frac{d}{dz} [F(z)] = -\frac{1}{z} \sum_{n=0}^{\infty} n f(n) z^{-n}$$

$$-z \frac{d}{dz} [F(z)] = \sum_{n=0}^{\infty} n f(n) z^{-n}$$

$$-z \frac{d}{dz} [F(z)] = Z[nf(n)]$$

$$\text{i.e., } \boxed{Z[nf(n)] = -z \frac{d}{dz} [F(z)]}$$

(3). First Shifting [Frequency Shifting] Property (or) Damping Rule :

Statement : If $Z[f(n)] = F(z)$, Then $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\begin{aligned} Z[a^n f(n)] &= \sum_{n=0}^{\infty} a^n f(n) z^{-n} \\ &= \sum_{n=0}^{\infty} f(n) \frac{a^n}{z^n} \\ &= \sum_{n=0}^{\infty} f(n) \left(\frac{a}{z}\right)^n \\ &= \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n} \end{aligned}$$

$$\boxed{Z[a^n f(n)] = F\left(\frac{z}{a}\right)} \quad //$$

(4). Second Shifting [Time Shifting] Property :

(i) **Statement :** If $Z[f(n)] = F(z)$, Then $Z[f(n+1)] = zF(z) - zf(0)$.

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\therefore Z[f(n+1)] = \sum_{n=0}^{\infty} f(n+1) z^{-n}$$

Put $m = n+1 \implies n = m-1$, Also $n = 0 \implies m = 1$

$$\begin{aligned} Z[f(n+1)] &= \sum_{m=1}^{\infty} f(m) z^{-(m-1)} = z \sum_{m=1}^{\infty} f(m) z^{-m} \\ &= z \left(\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) \right) \\ &= z(F(z) - f(0)) \end{aligned}$$

$$\boxed{Z[f(n+1)] = zF(z) - zf(0)} \quad //$$

(4). Second Shifting [Time Shifting] Property :

(ii) **Statement :** If $Z[f(n)] = F(z)$, Then

$$Z[f(n+2)] = z^2 F(z) - z^2 f(0) - zf(1)$$

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\therefore Z[f(n+2)] = \sum_{n=0}^{\infty} f(n+2) z^{-n}$$

Put $m = n+2 \implies n = m-2$, Also $n = 0 \implies m = 2$

$$\begin{aligned} Z[f(n+2)] &= \sum_{m=2}^{\infty} f(m) z^{-(m-2)} = z^2 \sum_{m=2}^{\infty} f(m) z^{-m} \\ &= z^2 \left(\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) z^{-0} - f(1) z^{-1} \right) \\ &= z^2 (F(z) - f(0) - f(1) z^{-1}) \end{aligned}$$

$$\boxed{Z[f(n+2)] = z^2 F(z) - z^2 f(0) - zf(1)} \quad //$$

(4). Second Shifting [Time Shifting] Property :

(iii) **Statement :** If $Z[f(n)] = F(z)$, Then

$$Z[f(n+3)] = z^3 F(z) - z^3 f(0) - z^2 f(1) - z f(2)$$

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$\therefore Z[f(n+3)] = \sum_{n=0}^{\infty} f(n+3) z^{-n}$$

Put $m = n+3 \Rightarrow n = m-3$, Also $n = 0 \Rightarrow m = 3$

$$\begin{aligned} Z[f(n+3)] &= \sum_{m=3}^{\infty} f(m) z^{-(m-3)} = z^3 \sum_{m=3}^{\infty} f(m) z^{-m} \\ &= z^3 \left(\sum_{m=0}^{\infty} f(m) z^{-m} - f(0) z^{-0} - f(1) z^{-1} - f(2) z^{-2} \right) \\ &= z^3 (F(z) - f(0) - f(1) z^{-1} - f(2) z^{-2}) \end{aligned}$$

$Z[f(n+3)] = z^3 F(z) - z^3 f(0) - z^2 f(1) - z f(2)$ //

Initial Value Theorem

Statement : Initial Value Theorem

If $Z[f(n)] = F(z)$, Then $f(0) = \lim_{z \rightarrow \infty} F(z)$.

Proof : We know that $F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$F(z) = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$$

$$F(z) = \frac{f(0)}{z^0} + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots$$

Taking limit $z \rightarrow \infty$ on both the sides , We get

$$\lim_{z \rightarrow \infty} F(z) = \lim_{z \rightarrow \infty} \left[\frac{f(0)}{z^0} + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots \right]$$

$$\lim_{z \rightarrow \infty} F(z) = f(0)$$

$$\text{i.e., } \boxed{f(0) = \lim_{z \rightarrow \infty} F(z)} \quad //$$

Final Value Theorem

Statement : Final Value Theorem

If $Z[f(n)] = F(z)$, Then $\lim_{n \rightarrow \infty} f(n) = \lim_{z \rightarrow 1} (z-1)F(z)$.

Proof : We know that

$$(i) \quad F(z) = Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$(ii) \quad Z[f(n+1)] = zF(z) - zf(0)$$

$$\text{Now } Z[f(n+1) - f(n)] = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$Z[f(n+1)] - Z[f(n)] = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$zF(z) - F(z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$(z-1)F(z) = \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

Taking limit $z \rightarrow 1$ in both the sides , We get

$$\lim_{z \rightarrow 1} [(z-1)F(z) - zf(0)] = \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$\begin{aligned} \lim_{z \rightarrow 1} (z-1)F(z) - f(0) &= \sum_{n=0}^{\infty} [f(n+1) - f(n)] \\ &= \lim_{n \rightarrow \infty} \sum_{m=0}^n [f(m+1) - f(m)] \\ &= \lim_{n \rightarrow \infty} \{[f(1) - f(0)] + [f(2) - f(1)] + \cdots + \\ &\quad [f(n) - f(n-1)] + [f(n+1) - f(n)]\} \\ &= \lim_{n \rightarrow \infty} [f(n+1) - f(0)] \end{aligned}$$

$$\lim_{z \rightarrow 1} (z-1)F(z) - f(0) = \lim_{n \rightarrow \infty} f(n) - f(0)$$

$$\lim_{z \rightarrow 1} (z-1)F(z) = \lim_{n \rightarrow \infty} f(n)$$

$$i.e., \boxed{\lim_{n \rightarrow \infty} f(n) = \lim_{z \rightarrow 1} (z-1)F(z)} \quad //$$

Problems on Z - Transform

Find the Z - Transform of the following :

$$(1). (-3)^n + \frac{1}{3^n} - 7 \quad (2). 4 \cdot 3^n + 2(-1)^n \quad (3). e^{an} \quad (4). e^{-an}$$

$$(5). e^{at} \quad (6). e^{-at} + t \quad (7). a^n \cosh n\theta \quad (8). a^n \sinh n\theta$$

$$(9). r^n \cos n\theta \quad (10). r^n \sin n\theta \quad (11). \cos n\theta \quad (12). \sin n\theta$$

$$(13). \cos\left(\frac{n\pi}{2}\right) \quad (14). \sin\left(\frac{n\pi}{2}\right) \quad (15). \cos^2\left(\frac{n\pi}{2}\right)$$

$$(16). \sin^2\left(\frac{n\pi}{2}\right) \quad (17). \cos^3\left(\frac{n\pi}{6}\right) \quad (18). \sin^3\left(\frac{n\pi}{6}\right)$$

$$(19). \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) \quad (20). \sin\left(\frac{n\pi}{2} + a\right) //$$

Homework Problems : (16) , (17) & (20)

Problems on Z - Transform

Find the Z - Transform of the following :

$$(21). \frac{1}{n(n+1)}$$

$$(22). \frac{n-2}{n(n+1)}$$

$$(23). \frac{1}{(n+1)(n+2)}$$

$$(24). \frac{2n+3}{(n+1)(n+2)}$$

$$(25). \frac{2n^2+4n+3}{n(n+1)(n+2)} //$$

Homework Problems : (22) , (23) & (25)

(1). To Find : $Z \left[(-3)^n + \frac{1}{3^n} - 7 \right]$

Solution : We know that $Z[a^n] = \frac{z}{z-a}$

$$\begin{aligned} \therefore Z \left[(-3)^n + \frac{1}{3^n} - 7 \right] &= Z[(-3)^n] + Z \left[\left(\frac{1}{3} \right)^n \right] - 7Z[1] \\ &= \frac{z}{z-(-3)} + \frac{z}{z-\frac{1}{3}} - 7 \frac{z}{z-1} \\ Z \left[(-3)^n + \frac{1}{3^n} - 7 \right] &= \frac{z}{z+3} + \frac{3z}{3z-1} - \frac{7z}{z-1} \quad // \end{aligned}$$

(2). To Find : $Z[43^n + 2(-1)^n]$

Solution : We know that $Z[a^n] = \frac{z}{z-a}$

$$\begin{aligned} \therefore Z[43^n + 2(-1)^n] &= 4Z[3^n] + 2Z[(-1)^n] \\ &= 4 \frac{z}{z-3} + 2 \frac{z}{z+1} \quad // \end{aligned}$$

(3). To Find : $Z[e^{an}]$

Solution : We know that $Z[a^n] = \frac{z}{z-a}$

$$\begin{aligned} Z[e^{an}] &= Z[(e^a)^n] \\ &= \frac{z}{z-e^a} \quad // \end{aligned}$$

(4). To Find : $Z[e^{-an}]$

Solution : We know that $Z[a^n] = \frac{z}{z-a}$

$$\begin{aligned} Z[e^{-an}] &= Z[(e^{-a})^n] \\ &= \frac{z}{z-e^{-a}} \quad // \end{aligned}$$

(5). To Find : $Z[e^{at}]$

Solution : We know that $Z[a^n] = \frac{z}{z-a}$ and $Z[f(t)] = Z[f(nT)]$

$$\begin{aligned}Z[e^{at}] &= Z[e^{anT}] \\&= Z[(e^{aT})^n] \\Z[e^{at}] &= \frac{z}{z-e^{aT}} \quad //\end{aligned}$$

(6). To Find : $Z[e^{-at} + t]$

Solution : We know that $Z[a^n] = \frac{z}{z-a}$ and $Z[f(t)] = Z[f(nT)]$

$$\begin{aligned}Z[e^{-at} + t] &= Z[e^{-anT} + nT] = Z[(e^{-aT})^n + T.n] \\&= Z[(e^{-aT})^n] + TZ[n] \\Z[e^{-at} + t] &= \frac{z}{z-e^{-aT}} + T \frac{z}{(z-1)^2} \quad //\end{aligned}$$

(7). To Find : $Z [a^n \cosh n\theta]$

Solution : We know that (i) $Z [a^n] = \frac{z}{z-a}$ and (ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

$$\begin{aligned} Z [a^n \cosh n\theta] &= Z \left[a^n \left(\frac{e^{n\theta} + e^{-n\theta}}{2} \right) \right] \\ &= \frac{1}{2} Z [a^n (e^{n\theta} + e^{-n\theta})] \\ &= \frac{1}{2} Z [a^n e^{n\theta} + a^n e^{-n\theta}] \\ &= \frac{1}{2} Z [(ae^\theta)^n + (ae^{-\theta})^n] \\ &= \frac{1}{2} \left\{ Z [(ae^\theta)^n] + Z [(ae^{-\theta})^n] \right\} \\ &= \frac{1}{2} \left\{ \frac{z}{z - ae^\theta} + \frac{z}{z - ae^{-\theta}} \right\} \\ &= \frac{z}{2} \left\{ \frac{1}{z - ae^\theta} + \frac{1}{z - ae^{-\theta}} \right\} \end{aligned}$$

$$\begin{aligned}
 Z[a^n \cosh n\theta] &= \frac{z}{2} \left\{ \frac{z - ae^{-\theta} + z - ae^{\theta}}{z^2 - aze^{\theta} - aze^{-\theta} + a^2} \right\} \\
 &= \frac{z}{2} \left\{ \frac{2z - a(e^{\theta} + e^{-\theta})}{z^2 - az(e^{\theta} + e^{-\theta}) + a^2} \right\} \\
 &= \frac{z}{2} \left\{ \frac{2z - 2a \cosh \theta}{z^2 - 2az \cosh \theta + a^2} \right\}
 \end{aligned}$$

$$\boxed{Z[a^n \cosh n\theta] = \frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}} \quad //$$

(8). To Find : $Z[a^n \sinh n\theta]$

Solution : We know that (i) $Z[a^n] = \frac{z}{z - a}$ and (ii) $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\begin{aligned}
 Z[a^n \sinh n\theta] &= Z\left[a^n \left(\frac{e^{n\theta} - e^{-n\theta}}{2}\right)\right] = \frac{1}{2} Z[a^n (e^{n\theta} - e^{-n\theta})] \\
 &= \frac{1}{2} Z[a^n e^{n\theta} - a^n e^{-n\theta}]
 \end{aligned}$$

$$\begin{aligned}
 Z[a^n \sinh n\theta] &= \frac{1}{2} Z[(ae^\theta)^n - (ae^{-\theta})^n] \\
 &= \frac{1}{2} \left\{ Z[(ae^\theta)^n] - Z[(ae^{-\theta})^n] \right\} \\
 &= \frac{1}{2} \left\{ \frac{z}{z - ae^\theta} - \frac{z}{z - ae^{-\theta}} \right\} \\
 &= \frac{z}{2} \left\{ \frac{1}{z - ae^\theta} - \frac{1}{z - ae^{-\theta}} \right\} \\
 &= \frac{z}{2} \left\{ \frac{z - ae^{-\theta} - z + ae^\theta}{z^2 - aze^\theta - aze^{-\theta} + a^2} \right\} \\
 &= \frac{z}{2} \left\{ \frac{a(e^\theta - e^{-\theta})}{z^2 - az(e^\theta + e^{-\theta}) + a^2} \right\} \\
 &= \frac{z}{2} \left\{ \frac{2a \sinh \theta}{z^2 - 2az \cosh \theta + a^2} \right\}
 \end{aligned}$$

$Z[a^n \sinh \theta] = \frac{az \sinh \theta}{z^2 - 2az \cosh \theta + a^2}$	//
--	----

(9). To Find : $Z[r^n \cos n\theta]$ and also (10). To Find : $Z[r^n \sin n\theta]$

Solution : We know that (i) $Z[a^n] = \frac{z}{z-a}$ and (ii) $e^{in\theta} = \cos n\theta + i \sin n\theta$

Put $a = re^{i\theta}$ in (i)

$$Z[(re^{i\theta})^n] = \frac{z}{z - re^{i\theta}}$$

$$Z[r^n e^{in\theta}] = \frac{z}{z - re^{i\theta}}$$

$$Z[r^n (\cos n\theta + i \sin n\theta)] = \frac{z}{z - r(\cos \theta + i \sin \theta)}$$

$$Z[r^n \cos n\theta + ir^n \sin n\theta] = \frac{z}{z - r \cos \theta - ir \sin \theta}$$

$$Z[r^n \cos n\theta] + iZ[r^n \sin n\theta] = \frac{z}{(z - r \cos \theta) - ir \sin \theta}$$

$$= \frac{z}{(z - r \cos \theta) - ir \sin \theta} \times \frac{(z - r \cos \theta) + ir \sin \theta}{(z - r \cos \theta) + ir \sin \theta}$$

$$= \frac{z(z - r \cos \theta) + irz \sin \theta}{(z - r \cos \theta)^2 + (r \sin \theta)^2}$$

$$\begin{aligned}
 Z[r^n \cos n\theta] + iZ[r^n \sin n\theta] &= \frac{z(z - r \cos \theta) + i r z \sin \theta}{z^2 - 2zr \cos \theta + r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\
 &= \frac{z(z - r \cos \theta) + i r z \sin \theta}{z^2 - 2zr \cos \theta + r^2} \\
 Z[r^n \cos n\theta] + iZ[r^n \sin n\theta] &= \frac{z(z - r \cos \theta)}{z^2 - 2zr \cos \theta + r^2} + i \frac{r z \sin \theta}{z^2 - 2zr \cos \theta + r^2}
 \end{aligned}$$

Comparing the real and imaginary parts on both the sides , We get

$$Z[r^n \cos n\theta] = \frac{z(z - r \cos \theta)}{z^2 - 2zr \cos \theta + r^2}$$

&

$$Z[r^n \sin n\theta] = \frac{r z \sin \theta}{z^2 - 2zr \cos \theta + r^2} //$$

(11). To Find : $Z [\cos n\theta]$

Solution : We know that

$$Z [a^n \cos n\theta] = \frac{z (z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2} \quad (1)$$

Put $a = 1$ in (1), We get

$$Z [\cos n\theta] = \frac{z (z - \cos \theta)}{z^2 - 2z \cos \theta + 1} //$$

(12). To Find : $Z [\sin n\theta]$

Solution : We know that

$$Z [a^n \sin n\theta] = \frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2} \quad (1)$$

Put $a = 1$ in (1), We get

$$Z [\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} //$$

(13). To Find : $Z \left[\cos \left(\frac{n\pi}{2} \right) \right]$

Solution : We know that $Z [a^n \cos n\theta] = \frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$ (1)

Put $a = 1$ and $\theta = \frac{\pi}{2}$ in (1), We get

$$Z \left[\cos \left(\frac{n\pi}{2} \right) \right] = \frac{z(z - \cos \frac{\pi}{2})}{z^2 - 2z \cos \frac{n\pi}{2} + 1} = \frac{z^2}{z^2 + 1} //$$

(14). To Find : $Z \left[\sin \left(\frac{n\pi}{2} \right) \right]$

Solution : We know that $Z [a^n \sin n\theta] = \frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}$ (1)

Put $a = 1$ and $\theta = \frac{\pi}{2}$ in (1), We get

$$Z \left[\sin \left(\frac{n\pi}{2} \right) \right] = \frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} = \frac{z}{z^2 + 1} //$$

(15). To Find : $Z \left[\cos^2 \left(\frac{n\pi}{2} \right) \right]$

Solution : We know that $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

$$\begin{aligned}\therefore \cos^2 \left(\frac{n\pi}{2} \right) &= \frac{1 + \cos 2 \left(\frac{n\pi}{2} \right)}{2} = \frac{1 + \cos n\pi}{2} = \frac{1 + (-1)^n}{2} \\ Z \left[\cos^2 \left(\frac{n\pi}{2} \right) \right] &= Z \left[\frac{1 + (-1)^n}{2} \right] = \frac{1}{2} Z [1 + (-1)^n] \\ &= \frac{1}{2} \{ Z[1] + Z[(-1)^n] \} \\ &= \frac{1}{2} \left\{ \frac{z}{z-1} + \frac{z}{z+1} \right\} \\ &= \frac{z}{2} \left\{ \frac{1}{z-1} + \frac{1}{z+1} \right\} \\ &= \frac{z}{2} \left\{ \frac{z+1+z-1}{z^2-1} \right\} = \frac{z}{2} \left\{ \frac{2z}{z^2-1} \right\}\end{aligned}$$

$$\boxed{Z \left[\cos^2 \left(\frac{n\pi}{2} \right) \right] = \frac{z^2}{z^2-1} \quad //}$$

(18). To Find : $Z \left[\sin^3 \left(\frac{n\pi}{6} \right) \right]$

Solution : We know that $\sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$

Also
$$Z [\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$\therefore \sin^3 \left(\frac{n\pi}{6} \right) = \frac{1}{4} \left[3 \sin \left(\frac{n\pi}{6} \right) - \sin 3 \left(\frac{n\pi}{6} \right) \right]$$

$$\sin^3 \left(\frac{n\pi}{6} \right) = \frac{3}{4} \sin \left(\frac{n\pi}{6} \right) - \frac{1}{4} \sin \left(\frac{n\pi}{2} \right)$$

$$\Rightarrow Z \left[\sin^3 \left(\frac{n\pi}{6} \right) \right] = \frac{3}{4} Z \left[\sin \left(\frac{n\pi}{6} \right) \right] - \frac{1}{4} Z \left[\sin \left(\frac{n\pi}{2} \right) \right]$$

$$= \frac{3}{4} \frac{z \sin \left(\frac{\pi}{6} \right)}{z^2 - 2z \cos \left(\frac{\pi}{6} \right) + 1} - \frac{1}{4} \frac{z}{z^2 + 1}$$

$$= \frac{3}{4} \frac{z^{\frac{1}{2}}}{z^2 - 2z \frac{\sqrt{3}}{2} + 1} - \frac{1}{4} \frac{z}{z^2 + 1}$$

$$\boxed{Z \left[\sin^3 \left(\frac{n\pi}{6} \right) \right] = \frac{3}{8} \frac{z}{z^2 - \sqrt{3}z + 1} - \frac{1}{4} \frac{z}{z^2 + 1}} \quad //$$

(19). To Find : $Z \left[\cos \left(\frac{n\pi}{2} + \frac{\pi}{4} \right) \right]$

Solution : WKT
$$Z \left[\cos \left(\frac{n\pi}{2} \right) \right] = \frac{z^2}{z^2 + 1} \quad \& \quad Z \left[\sin \left(\frac{n\pi}{2} \right) \right] = \frac{z}{z^2 + 1}$$

$$\text{Now } \cos \left(\frac{n\pi}{2} + \frac{\pi}{4} \right) = \cos \left(\frac{n\pi}{2} \right) \cos \left(\frac{\pi}{4} \right) - \sin \left(\frac{n\pi}{2} \right) \sin \left(\frac{\pi}{4} \right)$$

$$\cos \left(\frac{n\pi}{2} + \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} \left[\cos \left(\frac{n\pi}{2} \right) - \sin \left(\frac{n\pi}{2} \right) \right]$$

$$\begin{aligned} \Rightarrow Z \left[\cos \left(\frac{n\pi}{2} + \frac{\pi}{4} \right) \right] &= \frac{1}{\sqrt{2}} Z \left[\cos \left(\frac{n\pi}{2} \right) - \sin \left(\frac{n\pi}{2} \right) \right] \\ &= \frac{1}{\sqrt{2}} \left\{ Z \left[\cos \left(\frac{n\pi}{2} \right) \right] - Z \left[\sin \left(\frac{n\pi}{2} \right) \right] \right\} \\ &= \frac{1}{\sqrt{2}} \left\{ \frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1} \right\} = \frac{1}{\sqrt{2}} \left[\frac{z^2 - z}{z^2 + 1} \right] \end{aligned}$$

$$Z \left[\cos \left(\frac{n\pi}{2} + \frac{\pi}{4} \right) \right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)} \quad //$$

(21). To Find : $Z \left[\frac{1}{n(n+1)} \right]$

Solution : Let $f(n) = \frac{1}{n(n+1)}$. Using the method of partial fraction

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$1 = A(n+1) + Bn$$

$$\text{Put } n = 0 \text{ we get } 1 = A + 0 \implies A = 1$$

$$\text{Put } n = -1 \text{ we get } 1 = 0 - B \implies B = -1$$

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\begin{aligned} Z \left[\frac{1}{n(n+1)} \right] &= Z \left[\frac{1}{n} \right] - Z \left[\frac{1}{n+1} \right] \\ &= \log \left(\frac{z}{z-1} \right) - z \log \left(\frac{z}{z-1} \right) \end{aligned}$$

$$\boxed{Z \left[\frac{1}{n(n+1)} \right] = (1-z) \log \left(\frac{z}{z-1} \right)} \quad //$$

(24). To Find : $Z \left[\frac{2n+3}{(n+1)(n+2)} \right]$

Solution : Let $f(n) = \frac{2n+3}{(n+1)(n+2)}$. Using the method of partial fraction

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{(n+1)} + \frac{B}{(n+2)}$$

$$2n+3 = A(n+2) + B(n+1)$$

Put $n = -1$ we get $-2+3 = A+0 \implies A = 1$

Put $n = -2$ we get $-4+3 = 0-B \implies B = 1$

$$\therefore f(n) = \frac{2n+3}{(n+1)(n+2)} = \frac{1}{(n+1)} + \frac{1}{(n+2)}$$

$$\begin{aligned} Z[f(n)] &= Z \left[\frac{2n+3}{(n+1)(n+2)} \right] = Z \left[\frac{1}{n+1} \right] + Z \left[\frac{1}{n+2} \right] \\ &= z \log \left(\frac{z}{z-1} \right) + z^2 \log \left(\frac{z}{z-1} \right) - z \end{aligned}$$

$$\boxed{Z \left[\frac{2n+3}{(n+1)(n+2)} \right] = z(z+1) \log \left(\frac{z}{z-1} \right) - z} \quad //$$

Problems on Z - Transform Using Properties

Find the Z - Transform of the following :

(1). n^2 (2). n^3 (3). nC_2 (4). $n(n-1)$

(5). $n(n-1)(n-2)$ (6). $(n+1)(n+2)$

(7). $a^n n$ (8). $a^n n^2$ (9). $a^n n^3$ (10). $\frac{a^n}{n!}$

(11). $\frac{a^n}{n}$ (12). $a^n \sin n\theta$ (13). $a^n \cos n\theta$

(14). $a^n \cos\left(\frac{n\pi}{2}\right)$ (15). $a^n \sin\left(\frac{n\pi}{2}\right)$

(16). $a^n \delta(n-k)$ (17). $a^n u(n)$ //

Homework Problems : (4) , (6) , (8) , (9) , (11) , (12) , (13)

(1). To Find : $Z[n^2]$

Solution : WKT $Z[nf(n)] = -z \frac{d}{dz} (Z[f(n)])$ & $Z[n] = \frac{z}{(z-1)^2}$

$$\begin{aligned} \text{Now } Z[n^2] &= Z[n.n] = -z \frac{d}{dz} (Z[n]) = -z \frac{d}{dz} \left(\frac{z}{(z-1)^2} \right) \\ &= -z \left(\frac{(z-1)^2 \cdot 1 - z \cdot 2(z-1)}{(z-1)^4} \right) \\ &= -z \left(\frac{z^2 - 2z + 1 - 2z^2 + 2z}{(z-1)^4} \right) \\ &= z \left(\frac{z^2 - 1}{(z-1)^4} \right) = \frac{z(z+1)(z-1)}{(z-1)^4} \end{aligned}$$

$$\boxed{Z[n^2] = \frac{z(z+1)}{(z-1)^3}} //$$

(2). To Find : $Z [n^3]$

Solution : We know that

$$(i) Z [nf(n)] = -z \frac{d}{dz} (Z [f(n)]) \quad \&$$

$$(ii) Z [n^2] = \frac{z(z+1)}{(z-1)^3}$$

$$\begin{aligned} \text{Now } Z [n^3] &= Z [n.n^2] = -z \frac{d}{dz} (Z [n^2]) \\ &= -z \frac{d}{dz} \left(\frac{z(z+1)}{(z-1)^3} \right) \\ &= -z \frac{d}{dz} \left(\frac{(z^2+z)}{(z-1)^3} \right) \end{aligned}$$

After the differentiation using $d \left(\frac{u}{v} \right)$, We get

$$\boxed{Z [n^3] = \frac{z(z^2 + 4z + 1)}{(z-1)^4}} \quad //$$

(3). To Find : $Z[nC_2]$

Solution : We know that

$$(i) Z[n] = \frac{z}{(z-1)^2} \quad \& \quad (ii) Z[n^2] = \frac{z(z+1)}{(z-1)^3}$$

$$nC_2 = \frac{n(n-1)}{1 \times 2} = \frac{1}{2}(n^2 - n)$$

$$\begin{aligned} Z[nC_2] &= Z\left[\frac{1}{2}(n^2 - n)\right] \\ &= \frac{1}{2}\{Z[n^2] - Z[n]\} \\ &= \frac{1}{2}\left\{\frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}\right\} \\ &= \frac{z}{2}\left\{\frac{(z+1) - (z-1)}{(z-1)^3}\right\} \end{aligned}$$

$$\boxed{Z[nC_2] = \frac{z}{(z-1)^3}} \quad //$$

(5). To Find : $Z[n(n-1)(n-2)]$

Solution : We know that (i) $Z[n] = \frac{z}{(z-1)^2}$ & (ii) $Z[n^2] = \frac{z(z+1)}{(z-1)^3}$

$$(iii) Z[n^3] = \frac{z(z^2 + 4z + 1)}{(z-1)^4}.$$

$$\begin{aligned} \text{Now } Z[n(n-1)(n-2)] &= Z[n^3 - 3n^2 + 2n] \\ &= Z[n^3] - 3Z[n^2] + 2Z[n] \\ &= \frac{z(z^2 + 4z + 1)}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2} \\ &= \frac{z(z^2 + 4z + 1) - 3z(z+1)(z-1) + 2z(z-1)^2}{(z-1)^4} \\ &= \frac{z^3 + 4z^2 + z - 3z^3 + 3z + 2z^3 - 4z^2 + 2z}{(z-1)^4} \end{aligned}$$

$$Z[n(n-1)(n-2)] = \frac{6z}{(z-1)^4} //$$

(7). To Find : $Z[a^n n]$

Solution : We know that (i) $Z[a^n f(n)] = (Z[f(n)])_{z \rightarrow \frac{z}{a}}$ & (ii) $Z[n] = \frac{z}{(z-1)^2}$

$$\text{Now } Z[a^n n] = (Z[n])_{z \rightarrow \frac{z}{a}} = \left(\frac{z}{(z-1)^2} \right)_{z \rightarrow \frac{z}{a}} = \frac{z}{a\left(\frac{z}{a} - 1\right)^2}$$

$$\boxed{Z[a^n n] = \frac{az}{(z-a)^2}} \quad //$$

(10). To Find : $Z\left[\frac{a^n}{n!}\right]$

Solution : We know that (i) $Z[a^n f(n)] = (Z[f(n)])_{z \rightarrow \frac{z}{a}}$ & (ii) $Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}$

$$\boxed{Z\left[\frac{a^n}{n!}\right] = \left(Z\left[\frac{1}{n!}\right]\right)_{z \rightarrow \frac{z}{a}} = \left(e^{\frac{1}{z}}\right)_{z \rightarrow \frac{z}{a}} = e^{\frac{a}{z}}} \quad //$$

$$(14). \text{ To Find : } Z \left[a^n \cos \left(\frac{n\pi}{2} \right) \right]$$

Solution : We know that (i) $Z[a^n f(n)] = (Z[f(n)])_{z \rightarrow \frac{z}{a}}$ &

$$(ii) Z \left[\cos \left(\frac{n\pi}{2} \right) \right] = \frac{z^2}{z^2 + 1}$$

$$\begin{aligned} Z \left[a^n \cos \left(\frac{n\pi}{2} \right) \right] &= \left(Z \left[\cos \left(\frac{n\pi}{2} \right) \right] \right)_{z \rightarrow \frac{z}{a}} = \left(\frac{z^2}{z^2 + 1} \right)_{z \rightarrow \frac{z}{a}} \\ &= \frac{\left(\frac{z}{a} \right)^2}{\left(\frac{z}{a} \right)^2 + 1} \\ &= \frac{z^2}{a^2 \left(\frac{z^2 + a^2}{a^2} \right)} \end{aligned}$$

$$\boxed{Z \left[a^n \cos \left(\frac{n\pi}{2} \right) \right] = \frac{z^2}{z^2 + a^2}} //$$

(15). To Find : $Z \left[a^n \sin \left(\frac{n\pi}{2} \right) \right]$

Solution : We know that (i) $Z [a^n f(n)] = (Z [f(n)])_{z \rightarrow \frac{z}{a}}$ &

(ii) $Z \left[\sin \left(\frac{n\pi}{2} \right) \right] = \frac{z}{z^2 + 1}$

$$\begin{aligned} Z \left[a^n \sin \left(\frac{n\pi}{2} \right) \right] &= \left(Z \left[\sin \left(\frac{n\pi}{2} \right) \right] \right)_{z \rightarrow \frac{z}{a}} = \left(\frac{z}{z^2 + 1} \right)_{z \rightarrow \frac{z}{a}} \\ &= \frac{\left(\frac{z}{a} \right)}{\left(\frac{z}{a} \right)^2 + 1} \\ &= \frac{z}{a \left(\frac{z^2 + a^2}{a^2} \right)} \end{aligned}$$

$$\boxed{Z \left[a^n \sin \left(\frac{n\pi}{2} \right) \right] = \frac{az}{z^2 + a^2}} //$$

(16). To Find : $Z[a^n \delta(n-k)]$

Solution : We know that (i) $\delta(n-k) = \begin{cases} 1 & \text{if } n=k \\ 0 & \text{if } n \neq k \end{cases}$ & (ii) $Z[\delta(n-k)] = \frac{1}{z^k}$

$$\begin{aligned} Z[a^n \delta(n-k)] &= (Z[\delta(n-k)])_{z \rightarrow \frac{z}{a}} = \left(\frac{1}{z^k} \right)_{z \rightarrow \frac{z}{a}} \\ &= (z^{-k})_{z \rightarrow \frac{z}{a}} = \left(\frac{z}{a} \right)^{-k} = \left(\frac{a}{z} \right)^k // \end{aligned}$$

(17). To Find : $Z[a^n u(n)]$

Solution : We know that (i) $Z[a^n f(n)] = (Z[f(n)])_{z \rightarrow \frac{z}{a}}$ & (ii) $Z[u(n)] = \frac{z}{z-1}$

$$\begin{aligned} Z[a^n u(n)] &= (Z[u(n)])_{z \rightarrow \frac{z}{a}} = \left(\frac{z}{z-1} \right)_{z \rightarrow \frac{z}{a}} \\ &= \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{a\left(\frac{z-a}{a}\right)} = \frac{z}{z-a} // \end{aligned}$$

UNIT - V Z - TRANSFORMS AND DIFFERENCE EQUATIONS

CONTENTS:

- Z - Transforms - Elementary properties of Z - Transforms
- Inverse Z - Transform (using partial fraction and residues)
- Initial value theorem and final value theorem
- Convolution theorem
- Formation of difference equations
- Solution of difference equations using Z - Transform

INVERSE Z - TRANSFORM

Inverse Z - Transform : Definition

If $Z[f(n)] = F(z)$, Then $f(n)$ is called the inverse Z - Transform of $F(z)$. It is denoted by $Z^{-1}[F(z)] = f(n)$

Some Important Inverse Z - Transform Formulae

$$(1). Z^{-1} \left[\frac{z}{z-1} \right] = 1$$

$$(2). Z^{-1} \left[\frac{z}{z-a} \right] = a^n$$

$$(3). Z^{-1} \left[\frac{z}{(z-1)^2} \right] = n$$

$$(4). Z^{-1} \left[\frac{z}{(z-a)^2} \right] = na^{n-1}$$

$$(5). Z^{-1} \left[\frac{z}{(z-a)^3} \right] = \frac{n(n-1)a^{n-2}}{2!}$$

$$(6). Z^{-1} \left[\frac{z^2}{z^2 + a^2} \right] = a^n \cos \left(\frac{n\pi}{2} \right)$$

$$(7). Z^{-1} \left[\frac{z}{z^2 + a^2} \right] = a^{n-1} \sin \left(\frac{n\pi}{2} \right)$$

Problems on Inverse Z - Transform by Partial Fraction Method

Find the Inverse Z - Transform of the following using partial fraction method:

$$(1). Z^{-1} \left[\frac{10z}{z^2 - 3z + 2} \right]$$

$$(2). Z^{-1} \left[\frac{z^3}{(z-1)^2(z-2)} \right]$$

$$(3). Z^{-1} \left[\frac{z^2}{(z+2)(z^2+4)} \right]$$

$$(4). Z^{-1} \left[\frac{4z^2 - 2z}{(z-1)(z-2)^2} \right]$$

$$(5). Z^{-1} \left[\frac{z}{(z-1)(z-2)} \right]$$

$$(6). Z^{-1} \left[\frac{z^2 + z}{(z-1)(z^2+1)} \right]$$

Homework Problems : (4) , (5) & (6)

(1). To Find : $Z^{-1} \left[\frac{10z}{z^2 - 3z + 2} \right]$

Solution : Let

$$F(z) = \frac{10z}{z^2 - 3z + 2} = \frac{10z}{(z-1)(z-2)}$$

$$\boxed{\frac{F(z)}{z} = \frac{10}{(z-1)(z-2)}}$$

Use the method of partial fraction , We get

$$\frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\boxed{10 = A(z-2) + B(z-1)}$$

Put $z = 1$, We get $10 = -A + B(0) \implies \boxed{A = -10}$

Put $z = 2$, We get $10 = 0 + B \implies \boxed{B = 10}$

$$\begin{aligned}
 \therefore \frac{10}{(z-1)(z-2)} &= \frac{-10}{z-1} + \frac{10}{z-2} \\
 \frac{10}{(z-1)(z-2)} &= 10 \left(\frac{1}{z-2} - \frac{1}{z-1} \right) \\
 \frac{F(z)}{z} &= 10 \left(\frac{1}{z-2} - \frac{1}{z-1} \right) \\
 F(z) &= 10 \left(\frac{z}{z-2} - \frac{z}{z-1} \right) \\
 Z^{-1}[F(z)] &= 10Z^{-1} \left(\frac{z}{z-2} - \frac{z}{z-1} \right) \\
 f(n) &= 10 \left\{ Z^{-1} \left(\frac{z}{z-2} \right) - Z^{-1} \left(\frac{z}{z-1} \right) \right\} \\
 \boxed{f(n) = 10 \{2^n - 1\}} & \quad //
 \end{aligned}$$

(2). To Find : $Z^{-1} \left[\frac{z^3}{(z-1)^2(z-2)} \right]$

Solution : Let

$$F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\boxed{\frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)}}$$

Use the method of partial fraction , We get

$$\frac{z^2}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$$

$$\boxed{z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2}$$

Put $z = 1$, We get $1 = A(0) + B(-1) + C(0) \implies \boxed{B = -1}$

Put $z = 2$, We get $4 = A(0) + B(0) + C(1) \implies \boxed{C = 4}$

Comparing the coefficients of z^2 on both the sides , We get

$$1 = A + C \implies A = 1 - C = 1 - 4 \implies \boxed{A = -3}$$

$$\therefore \frac{z^2}{(z-1)^2(z-2)} = \frac{-3}{(z-1)} + \frac{-1}{(z-1)^2} + \frac{4}{(z-2)}$$

$$\frac{F(z)}{z} = \frac{-3}{(z-1)} + \frac{-1}{(z-1)^2} + \frac{4}{(z-2)}$$

$$F(z) = -3 \frac{z}{(z-1)} - \frac{z}{(z-1)^2} + 4 \frac{z}{(z-2)}$$

$$Z^{-1}[F(z)] = -3Z^{-1}\left(\frac{z}{(z-1)}\right) - Z^{-1}\left(\frac{z}{(z-1)^2}\right) + 4Z^{-1}\left(\frac{z}{(z-2)}\right)$$

$$f(n) = -3(1^n) - n + 4(2)^n$$

$$\boxed{f(n) = 4(2^n) - n - 3} \quad //$$

(3). To Find : $Z^{-1} \left[\frac{z^2}{(z+2)(z^2+4)} \right]$

Solution : Let $F(z) = \frac{z^2}{(z+2)(z^2+4)}$

$$\boxed{\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)}}$$

Use the method of partial fraction , We get

$$\frac{z}{(z+2)(z^2+4)} = \frac{A}{(z+2)} + \frac{Bz+C}{(z^2+4)}$$

$$\boxed{z = A(z^2+4) + [Bz+C](z+2)}$$

Put $z = -2$, We get $-2 = A(4+4) + [Bz+C](0) \Rightarrow \boxed{A = \frac{-1}{4}}$

Put $z = 0$, We get $0 = 4A + 2C \Rightarrow 0 = -1 + 2C \Rightarrow \boxed{C = \frac{1}{2}}$

Comparing the coefficients of z^2 on both the sides , We get

$$0 = A + B \implies B = -A \implies \boxed{B = -\frac{1}{4}}$$

$$\therefore \frac{z}{(z+2)(z^2+4)} = \frac{-\frac{1}{4}}{(z+2)} + \frac{\frac{1}{4}z + \frac{1}{2}}{(z^2+4)}$$

$$\frac{F(z)}{z} = -\frac{1}{4} \frac{1}{(z+2)} + \frac{1}{4} \frac{z}{(z^2+4)} + \frac{1}{2} \frac{1}{(z^2+4)}$$

$$F(z) = -\frac{1}{4} \frac{z}{(z+2)} + \frac{1}{4} \frac{z^2}{(z^2+4)} + \frac{1}{2} \frac{z}{(z^2+4)}$$

$$Z^{-1}[F(z)] = -\frac{1}{4} Z^{-1} \left(\frac{z}{z+2} \right) + \frac{1}{4} Z^{-1} \left(\frac{z^2}{z^2+4} \right) + \frac{1}{2} Z^{-1} \left(\frac{z}{z^2+4} \right)$$

$$f(n) = -\frac{1}{4}(-2)^n + \frac{1}{4}2^n \cos\left(\frac{n\pi}{2}\right) + \frac{1}{2}2^{n-1} \sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{f(n) = -\frac{1}{4}(-2)^n + \frac{1}{4}2^n \cos\left(\frac{n\pi}{2}\right) + \frac{1}{4}2^n \sin\left(\frac{n\pi}{2}\right)} \quad (or)$$

$$\boxed{f(n) = \frac{2^n}{4} \left[\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) - (-1)^n \right]} \quad //$$

Inverse Z - Transform by Residue Method

I.Z.T by Residue Method : Formula

$$Z^{-1} [F(z)] = \text{Sum of the residues of } z^{n-1}F(z) \text{ at its poles.}$$

Residue Formula :

The residue of $z^{n-1}F(z)$ at a pole $z = a$ of order m is given by

$$\text{Res} [z^{n-1}F(z), z = a] = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m \cdot z^{n-1}F(z)]$$

At pole of order 1 [i.e., at simple pole] $z = a$, The residue is given by the formula

$$\text{Res} [z^{n-1}F(z), z = a] = \lim_{z \rightarrow a} (z-a) \cdot z^{n-1}F(z)$$

Problems on Inverse Z - Transform by Residue Method

Find the Inverse Z - Transform of the following using residue method :

$$\begin{array}{ll} (1). Z^{-1} \left[\frac{z}{z^2 + 3z + 2} \right] & (2). Z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] \\ (3). Z^{-1} \left[\frac{z^3}{(z-1)^2(z-2)} \right] & (4). Z^{-1} \left[\frac{z}{z^2 - 3z + 2} \right] \\ (5). Z^{-1} \left[\frac{4z^2 - 2z}{(z-1)(z-2)^2} \right] & (6). Z^{-1} \left[\frac{z^2 - 3z}{(z-5)(z+2)} \right] \end{array}$$

Homework Problems : (4) , (5) & (6)

(1). To Find : $Z^{-1} \left[\frac{z}{z^2 + 3z + 2} \right]$

Solution : Let $F(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)}$

$$\Rightarrow \boxed{z^{n-1}F(z) = \frac{z^n}{(z+1)(z+2)}}$$

To Find : The poles of $z^{n-1}F(z)$

Assume that $(z+1)(z+2) = 0$

$\Rightarrow z = -1$ is a simple pole and $z = -2$ is also a simple pole

To Find : The Residues of $z^{n-1}F(z)$

$$\text{Res} [z^{n-1}F(z), z = -1] = \lim_{z \rightarrow -1} (z+1) z^{n-1}F(z)$$

$$\begin{aligned} \text{Res}(-1) &= \lim_{z \rightarrow -1} (z+1) \frac{z^n}{(z+1)(z+2)} \\ &= \lim_{z \rightarrow -1} \frac{z^n}{(z+2)} = \frac{(-1)^n}{(-1+2)} \end{aligned}$$

$$\boxed{\text{Res}(-1) = (-1)^n}$$

$$\text{Res} [z^{n-1} F(z), z = -2] = \lim_{z \rightarrow -2} (z+2) z^{n-1} F(z)$$

$$\begin{aligned} \text{Res}(-2) &= \lim_{z \rightarrow -2} (z+2) \frac{z^n}{(z+1)(z+2)} \\ &= \lim_{z \rightarrow -2} \frac{z^n}{(z+1)} \\ &= \frac{(-2)^n}{(-2+1)} \end{aligned}$$

$$\boxed{\text{Res}(-2) = -(-2)^n}$$

$$\begin{aligned} \therefore Z^{-1} \left[\frac{z}{(z+1)(z+2)} \right] &= Z^{-1} [F(z)] \\ &= \text{Sum of the residues of } z^{n-1} F(z) \end{aligned}$$

$$f(n) = \text{Res}(-1) + \text{Res}(-2)$$

$$\boxed{f(n) = (-1)^n - (-2)^n} \quad //$$

(2). To Find : $Z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right]$

Solution : Let $F(z) = \frac{z(z+1)}{(z-1)^3}$

$$\Rightarrow \boxed{z^{n-1}F(z) = \frac{z^n(z+1)}{(z-1)^3}}$$

To Find : The poles of $z^{n-1}F(z)$

Assume that $(z-1)^3 = 0 \Rightarrow z = 1$ is a pole of order $m = 3$

To Find : The Residues of $z^{n-1}F(z)$

$$\text{Res} [z^{n-1}F(z), z=1] = \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} [(z-1)^3 z^{n-1}F(z)]$$

$$\text{Res}(1) = \frac{1}{2} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left[(z-1)^3 \frac{z^n(z+1)}{(z-1)^3} \right]$$

$$= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} [z^{n+1} + z^n]$$

$$\begin{aligned}
 \text{Res}(1) &= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d}{dz} [(n+1)z^n + nz^{n-1}] \\
 &= \frac{1}{2} \lim_{z \rightarrow 1} [n(n+1)z^{n-1} + n(n-1)z^{n-2}] \\
 &= \frac{1}{2} [n(n+1) + n(n-1)] \\
 &= \frac{1}{2} [n^2 + n + n^2 - n] \\
 &= \frac{1}{2} [2n^2]
 \end{aligned}$$

$$\boxed{\text{Res}(1) = n^2}$$

$$\begin{aligned}
 \therefore Z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] &= Z^{-1} [F(z)] \\
 &= \text{Sum of the residues of } z^{n-1} F(z) \\
 f(n) &= \text{Res}(1)
 \end{aligned}$$

$$\boxed{f(n) = n^2} \quad //$$

(3). To Find : $Z^{-1} \left[\frac{z^3}{(z-1)^2(z-2)} \right]$

Solution : Let $F(z) = \frac{z^3}{(z-1)^2(z-2)}$

$$\Rightarrow z^{n-1}F(z) = \frac{z^{n-1}z^3}{(z-1)^2(z-2)}$$

$$\Rightarrow \boxed{z^{n-1}F(z) = \frac{z^{n+2}}{(z-1)^2(z-2)}}$$

To Find : The poles of $z^{n-1}F(z)$

Assume that $(z-1)^2(z-2) = 0$

$\Rightarrow z = 1$ is a pole of order 2 and

$z = 2$ is a pole of order 1

To Find : The Residues of $z^{n-1}F(z)$

$$\text{Res} [z^{n-1}F(z), z=1] = \frac{1}{1!} \lim_{z \rightarrow 1} \frac{d}{dz} [(z-1)^2 z^{n-1}F(z)]$$

$$\text{Res}(1) = \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 \frac{z^{n+2}}{(z-1)^2(z-2)} \right]$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{z^{n+2}}{(z-2)} \right]$$

$$= \lim_{z \rightarrow 1} \left[\frac{(z-2)(n+2)z^{n+1} - z^{n+2}(1)}{(z-2)^2} \right]$$

$$= \left[\frac{(-1)(n+2) - 1}{(1)^2} \right]$$

$$\boxed{\text{Res}(1) = -(n+3)}$$

$$\text{Res} [z^{n-1} F(z), z=2] = \lim_{z \rightarrow 2} (z-2) z^{n-1} F(z)$$

$$\begin{aligned} \text{Res}(2) &= \lim_{z \rightarrow 2} (z-2) \frac{z^{n+2}}{(z-1)^2 (z-2)} \\ &= \lim_{z \rightarrow 2} \left[\frac{z^{n+2}}{(z-1)^2} \right] \\ &= \frac{2^{n+2}}{(2-1)^2} \end{aligned}$$

$$\boxed{\text{Res}(2) = 4(2)^n}$$

$$\therefore Z^{-1} \left[\frac{z^3}{(z-1)^2 (z-2)} \right] = Z^{-1} [F(z)]$$

$$= \text{Sum of the residues of } z^{n-1} F(z)$$

$$f(n) = \text{Res}(1) + \text{Res}(2)$$

$$\boxed{f(n) = -(n+3) + 4(2)^n} \quad //$$

Inverse Z - Transform by Convolution Method

Convolution Product : Definition

Let $f(n)$ and $g(n)$ be two sequences, then the convolution product of $f(n)$ and $g(n)$ is defined by

$$f(n) * g(n) = \sum_{k=0}^n f(k) g(n-k)$$

Convolution Theorem : Statement

If $Z[f(n)] = F(z)$ and $Z[g(n)] = G(z)$, then

$$Z[f(n) * g(n)] = Z[f(n)] Z[g(n)] \quad (OR)$$

$$Z[f(n) * g(n)] = F(z) G(z)$$

NOTE : $Z^{-1}[F(z) G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

Problems on Inverse Z - Transform by Residue Method

Find the Inverse Z - Transform of the following using convolution method :

$$(1). Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$$

$$(2). Z^{-1} \left[\frac{8z^2}{(2z-1)(4z-1)} \right]$$

$$(3). Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right]$$

$$(4). Z^{-1} \left[\frac{z^2}{\left(z - \frac{1}{2}\right) \left(z - \frac{1}{4}\right)} \right]$$

$$(5). Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right]$$

$$(6). Z^{-1} \left[\frac{z^2}{(z-3)(z-4)} \right]$$

$$(7). Z^{-1} \left[\frac{z^2}{(z-a)^2} \right]$$

$$(8). Z^{-1} \left[\frac{z^2}{(z+a)^2} \right]$$

Homework Problems : (2) , (4) , (6)

(1). To Find : $Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$

Solution : We know that

(i) $Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^n f(k) g(n-k)$

$$\begin{aligned} Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] &= Z^{-1} \left[\frac{z}{(z-a)} \cdot \frac{z}{(z-b)} \right] \\ &= Z^{-1} [F(z) \cdot G(z)] \\ &= Z^{-1} [F(z)] * Z^{-1} [G(z)] \\ &= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-b} \right] \\ &= a^n * b^n \\ &= f(n) * g(n) \\ &= \sum_{k=0}^n f(k) g(n-k) \end{aligned}$$

$$\begin{aligned}
 Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] &= \sum_{k=0}^n a^k b^{n-k} = \sum_{k=0}^n a^k b^n b^{-k} \\
 &= b^n \sum_{k=0}^n \left(\frac{a}{b} \right)^k = b^n \sum_{k=0}^n r^k \\
 &= b^n (1 + r + r^2 + \cdots + r^n) \\
 &= b^n \left(\frac{1 - r^{n+1}}{1 - r} \right) \\
 &= b^n \left(\frac{1 - \left(\frac{a}{b} \right)^{n+1}}{1 - \frac{a}{b}} \right) \\
 &= b^n \left(\frac{b}{b^{n+1}} \times \frac{b^{n+1} - a^{n+1}}{b - a} \right) \\
 &= \frac{b^{n+1} - a^{n+1}}{b - a}
 \end{aligned}$$

$$\boxed{Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = \frac{a^{n+1} - b^{n+1}}{a - b}} //$$

(3). To Find : $Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right]$

Solution : We know that (i) $Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^n f(k) g(n-k)$

$$\begin{aligned} Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] &= Z^{-1} \left[\frac{8z^2}{2 \left(z - \frac{1}{2}\right) 4 \left(z + \frac{1}{4}\right)} \right] \\ &= Z^{-1} \left[\frac{z^2}{\left(z - \frac{1}{2}\right) \left(z + \frac{1}{4}\right)} \right] \\ &= Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] \end{aligned}$$

Where $a = \frac{1}{2}$ and $b = -\frac{1}{4}$

$$= Z^{-1} \left[\frac{z}{(z-a)} \cdot \frac{z}{(z-b)} \right]$$

$$\begin{aligned}
Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] &= Z^{-1} [F(z) \cdot G(z)] \\
&= Z^{-1} [F(z)] * Z^{-1} [G(z)] \\
&= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-b} \right] \\
&= a^n * b^n = f(n) * g(n) \\
&= \sum_{k=0}^n f(k) g(n-k) \\
&= \sum_{k=0}^n a^k b^{n-k} \\
&= \sum_{k=0}^n a^k b^n b^{-k} \\
&= b^n \sum_{k=0}^n \left(\frac{a}{b} \right)^k = b^n \sum_{k=0}^n r^k \\
&= b^n (1 + r + r^2 + \dots + r^n)
\end{aligned}$$

$$\begin{aligned}
 Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] &= b^n \left(\frac{1-r^{n+1}}{1-r} \right) \\
 &= b^n \left(\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}} \right) \\
 &= b^n \left(\frac{b}{b^{n+1}} \times \frac{b^{n+1} - a^{n+1}}{b-a} \right) \\
 &= \frac{b^{n+1} - a^{n+1}}{b-a} \\
 &= \frac{a^{n+1} - b^{n+1}}{a-b}
 \end{aligned}$$

Put $a = \frac{1}{2}$ and $b = \frac{-1}{4}$, We get

$$= \frac{\left(\frac{1}{2}\right)^{n+1} - \left(\frac{-1}{4}\right)^{n+1}}{\frac{1}{2} + \frac{1}{4}}$$

$$\boxed{Z^{-1} \left[\frac{8z^2}{(2z-1)(4z+1)} \right] = \frac{4}{3} \left[\left(\frac{1}{2}\right)^{n+1} - \left(\frac{-1}{4}\right)^{n+1} \right]} //$$

(5). To Find : $Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right]$

Solution : We know that

(i) $Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^n f(k) g(n-k)$

$$Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] = Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$$

Where $a = 1$ and $b = 3$

$$\begin{aligned} &= Z^{-1} \left[\frac{z}{(z-a)} \cdot \frac{z}{(z-b)} \right] \\ &= Z^{-1} [F(z) \cdot G(z)] \\ &= Z^{-1} [F(z)] * Z^{-1} [G(z)] \\ &= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-b} \right] \end{aligned}$$

$$\begin{aligned}
Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] &= a^n * b^n \\
&= f(n) * g(n) \\
&= \sum_{k=0}^n f(k) g(n-k) \\
&= \sum_{k=0}^n a^k b^{n-k} = \sum_{k=0}^n a^k b^n b^{-k} \\
&= b^n \sum_{k=0}^n \left(\frac{a}{b} \right)^k \\
&= b^n \sum_{k=0}^n r^k \\
&= b^n (1 + r + r^2 + \dots + r^n) \\
&= b^n \left(\frac{1 - r^{n+1}}{1 - r} \right)
\end{aligned}$$

$$\begin{aligned}
 Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] &= b^n \left(\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}} \right) \\
 &= b^n \left(\frac{b}{b^{n+1}} \times \frac{b^{n+1} - a^{n+1}}{b - a} \right) \\
 &= \frac{b^{n+1} - a^{n+1}}{b - a} \\
 &= \frac{a^{n+1} - b^{n+1}}{a - b}
 \end{aligned}$$

Put $a = 1$ and $b = 3$

$$= \frac{1 - 3^{n+1}}{1 - 3}$$

$$\boxed{Z^{-1} \left[\frac{z^2}{(z-1)(z-3)} \right] = \frac{1}{2} [3^{n+1} - 1]} \quad //$$

(7). To Find : $Z^{-1} \left[\frac{z^2}{(z-a)^2} \right]$

Solution : We know that

(i) $Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^n f(k) g(n-k)$

$$\begin{aligned} Z^{-1} \left[\frac{z^2}{(z-a)^2} \right] &= Z^{-1} \left[\frac{z}{(z-a)} \cdot \frac{z}{(z-a)} \right] \\ &= Z^{-1} [F(z) \cdot G(z)] \\ &= Z^{-1} [F(z)] * Z^{-1} [G(z)] \\ &= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-a} \right] \\ &= a^n * a^n \\ &= f(n) * g(n) \end{aligned}$$

$$\begin{aligned}
 Z^{-1} \left[\frac{z^2}{(z-a)^2} \right] &= \sum_{k=0}^n f(k) g(n-k) \\
 &= \sum_{k=0}^n a^k a^{n-k} \\
 &= \sum_{k=0}^n a^{k+n-k} \\
 &= \sum_{k=0}^n a^n \\
 &= \underbrace{\left(a^n + a^n + \cdots + a^n \right)}_{(n+1) \text{ times}}
 \end{aligned}$$

$$\boxed{Z^{-1} \left[\frac{z^2}{(z-a)^2} \right] = (n+1) a^n} \quad //$$

(8). To Find : $Z^{-1} \left[\frac{z^2}{(z+a)^2} \right]$

Solution : We know that

(i) $Z^{-1} [F(z) \cdot G(z)] = Z^{-1} [F(z)] * Z^{-1} [G(z)]$

(ii) $f(n) * g(n) = \sum_{k=0}^n f(k) g(n-k)$

$$\begin{aligned} Z^{-1} \left[\frac{z^2}{(z+a)^2} \right] &= Z^{-1} \left[\frac{z}{(z+a)} \cdot \frac{z}{(z+a)} \right] \\ &= Z^{-1} [F(z) \cdot G(z)] \\ &= Z^{-1} [F(z)] * Z^{-1} [G(z)] \\ &= Z^{-1} \left[\frac{z}{z+a} \right] * Z^{-1} \left[\frac{z}{z+a} \right] \\ &= (-a)^n * (-a)^n \\ &= f(n) * g(n) \end{aligned}$$

$$\begin{aligned}
 Z^{-1} \left[\frac{z^2}{(z+a)^2} \right] &= \sum_{k=0}^n f(k) g(n-k) \\
 &= \sum_{k=0}^n (-a)^k (-a)^{n-k} \\
 &= \sum_{k=0}^n (-a)^{k+n-k} \\
 &= \sum_{k=0}^n (-a)^n \\
 &= \left(\underbrace{(-a)^n + (-a)^n + \cdots + (-a)^n}_{(n+1) \text{ times}} \right)
 \end{aligned}$$

$Z^{-1} \left[\frac{z^2}{(z+a)^2} \right] = (n+1)(-a)^n \quad //$
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UNIT - V Z - TRANSFORMS AND DIFFERENCE EQUATIONS

CONTENTS:

- Z - Transforms - Elementary properties of Z - Transforms
- Inverse Z - Transform (using partial fraction and residues)
- Initial value theorem and final value theorem
- Convolution theorem
- Formation of difference equations
- Solution of difference equations using Z - Transform

Formation of Difference Equations

Form the difference equations from the following by eliminating the arbitrary constants given :

$$(1). u_n = A2^n + Bn$$

$$(2). y(n) = a2^n + b(-2)^n$$

$$(3). u_n = (A + Bn)(-3)^n$$

$$(4). y_n = an + bn^2$$

$$(5). x(n) = (An + B)2^n$$

$$(6). u_n = a2^{n+1}$$

Homework : (4) & (5)

(1). Form the difference equations from $u_n = A2^n + Bn$ by eliminating the arbitrary constants A and B .

Solution :

$$u_n = A2^n + Bn$$

$$u_{n+1} = A2^{n+1} + B(n+1)$$

$$u_{n+2} = A2^{n+2} + B(n+2)$$

$$\begin{vmatrix} u_n & 2^n & n \\ u_{n+1} & 2^{n+1} & n+1 \\ u_{n+2} & 2^{n+2} & n+2 \end{vmatrix} = 0 \implies 2^n \begin{vmatrix} u_n & 1 & n \\ u_{n+1} & 2 & n+1 \\ u_{n+2} & 4 & n+2 \end{vmatrix} = 0$$
$$\implies \begin{vmatrix} u_n & 1 & n \\ u_{n+1} & 2 & n+1 \\ u_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$$u_n [2(n+2) - 4(n+1)] - u_{n+1} [n+2 - 4n] + u_{n+2} [n+1 - 2n] = 0$$

$$(1-n)u_{n+2} - (2-3n)u_{n+1} - 2nu_n = 0$$

This is the required difference equation of second order. //

(2). Form the difference equations from $y(n) = a2^n + b(-2)^n$ by eliminating the arbitrary constants a and b .

Solution : Given that $y(n) = a2^n + b(-2)^n$

$$\Rightarrow y(n+1) = a2^{n+1} + b(-2)^{n+1} \quad \& \quad y(n+2) = a2^{n+2} + b(-2)^{n+2}$$

$$\begin{vmatrix} y(n) & 2^n & (-2)^n \\ y(n+1) & 2^{n+1} & (-2)^{n+1} \\ y(n+2) & 2^{n+2} & (-2)^{n+2} \end{vmatrix} = 0$$

$$2^n(-2)^n \begin{vmatrix} y(n) & 1 & 1 \\ y(n+1) & 2 & -2 \\ y(n+2) & 4 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} y(n) & 1 & 1 \\ y(n+1) & 2 & -2 \\ y(n+2) & 4 & 4 \end{vmatrix} = 0$$

$$y(n)[8+8] - y(n+1)[4-4] + y(n+2)[-2-2] = 0$$

$$-4y(n+2) + 16y(n) = 0$$

$$\boxed{y(n+2) - 4y(n) = 0} \quad //$$

(3). Form the difference equations from $u_n = (A + Bn)(-3)^n$ by eliminating the arbitrary constants A and B .

Solution : Given that $u_n = (A + Bn)(-3)^n$

$$u_n = A(-3)^n + Bn(-3)^n$$

$$u_{n+1} = A(-3)^{n+1} + B(n+1)(-3)^n$$

$$u_{n+2} = A(-3)^{n+1} + B(n+2)(-3)^n$$

Consider the determinant

$$\begin{vmatrix} u_n & (-3)^n & n(-3)^n \\ u_{n+1} & (-3)^{n+1} & (n+1)(-3)^{n+1} \\ u_{n+2} & (-3)^{n+2} & (n+2)(-3)^{n+2} \end{vmatrix} = 0$$
$$(-3)^n(-3)^n \begin{vmatrix} u_n & 1 & n \\ u_{n+1} & -3 & -3(n+1) \\ u_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

$$\begin{vmatrix} u_n & 1 & n \\ u_{n+1} & -3 & -3(n+1) \\ u_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

Expand the determinant along the first column , We get

$$\left. \begin{aligned} &u_n [-27(n+2) + 27(n+1)] - u_{n+1} [9(n+2) - 9n] \\ &+ u_{n+2} [-3(n+1) + 3n] \end{aligned} \right\} = 0$$

$$\left. \begin{aligned} &u_n [-27n - 54 + 27n + 27] - u_{n+1} [9n + 18 - 9n] \\ &+ u_{n+2} [-3n - 3 + 3n] \end{aligned} \right\} = 0$$

$$-3u_{n+2} + 18u_{n+1} - 27u_n = 0$$

$$\boxed{u_{n+2} - 6u_{n+1} + 9u_n = 0}$$

This is our required difference equation of second order.

(3). Form the difference equations from $u_n = a2^{n+1}$ by eliminating the arbitrary constants a

Solution : Given that

$$\begin{aligned}u_n &= a2^{n+1} \\u_{n+1} &= a2^{n+2} \\&= a2^{n+1} \times 2 \\u_{n+1} &= 2u_n\end{aligned}$$

$$\boxed{u_{n+1} - 2u_n = 0}$$

This is our required difference equation of first order.

Solution of Difference Equations using Z - Transform

Solve the following difference equations :

1). $u_{n+2} + 3u_{n+1} + 2u_n = 0$ given that $u_0 = 1$, $u_1 = 2$

2). $y_{n+2} - 4y_n = 0$ given that $y_0 = 0$, $y_1 = 2$ [Homework]

3). $y(n+3) - 3y(n+1) + 2y(n) = 0$ given $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

4). $y_{n+2} - 4y_{n+1} + 4y_n = 0$ given $y_0 = 1$, $y_1 = 0$ [Homework]

5). $y_{n+2} + y_n = 2$ given that $y_0 = y_1 = 0$ [Homework]

6). $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$

7). $u_{n+2} + 4u_{n+1} + 3u_n = 2^n$ given $u_0 = 0$, $u_1 = 1$ [Homework]

8). $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ given $y_0 = 0$, $y_1 = 1$

9). $u_{n+2} - 5u_{n+1} + 6u_n = 4^n$ given $u_0 = 0$, $u_1 = 1$ [Homework]

10). $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ given $y_0 = 3$, $y_1 = -5$

(1). Solve $u_{n+2} + 3u_{n+1} + 2u_n = 0$ given that $u_0 = 1$, $u_1 = 2$

Solution: Given difference equation is

$$u_{n+2} + 3u_{n+1} + 2u_n = 0 \quad (1)$$

Also given that $u_0 = 1$, $u_1 = 2$

Taking Z - Transform on both the sides of (1), We get

$$Z[u_{n+2} + 3u_{n+1} + 2u_n] = Z[0]$$

$$Z[u_{n+2}] + 3Z[u_{n+1}] + 2Z[u_n] = Z[0]$$

$$[z^2 U(z) - z^2 u(0) - zu(1)] + 3[zU(z) - zu(0)] + 2U(z) = 0$$

$$(z^2 + 3z + 2) U(z) - z^2(1) - z(2) - 3z(1) = 0$$

$$\begin{aligned}(z^2 + 3z + 2) U(z) &= z^2 + 5z \\ (z + 1)(z + 2) U(z) &= z(z + 5)\end{aligned}$$

$$\boxed{U(z) = \frac{z(z + 5)}{(z + 1)(z + 2)}} \quad (2)$$

To Find : $Z^{-1}[U(z)] = Z^{-1}\left[\frac{z(z + 5)}{(z + 1)(z + 2)}\right]$

$$(2) \implies \frac{U(z)}{z} = \frac{z + 5}{(z + 1)(z + 2)}$$

Using the method of partial fraction , We get

$$\begin{aligned}\frac{z + 5}{(z + 1)(z + 2)} &= \frac{A}{z + 1} + \frac{B}{z + 2} \\ z + 5 &= A(z + 2) + B(z + 1)\end{aligned}$$

Put $z = -1$, We get $4 = A(1) + B(0) \implies \boxed{A = 4}$

Put $z = -2$, We get $3 = A(0) + B(-1) \implies \boxed{B = -3}$

$$\begin{aligned}\frac{z+5}{(z+1)(z+2)} &= \frac{4}{z+1} - \frac{3}{z+2} \\ \frac{U(z)}{z} &= \frac{4}{z+1} - \frac{3}{z+2} \\ U(z) &= 4\frac{z}{z+1} - 3\frac{z}{z+2}\end{aligned}$$

Taking inverse Z - transform on both the sides, We get

$$\begin{aligned}Z^{-1}[U(z)] &= 4Z^{-1}\left[\frac{z}{z+1}\right] - 3Z^{-1}\left[\frac{z}{z+2}\right] \\ \boxed{u(n)} &= 4(-1)^n - 3(-2)^n \quad //\end{aligned}$$

(3). Solve $y(n+3) - 3y(n+1) + 2y(n) = 0$ given $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

Solution: Given difference equation is

$$y(n+3) - 3y(n+1) + 2y(n) = 0 \quad (1)$$

Also given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned} Z[y(n+3) - 3y(n+1) + 2y(n)] &= Z[0] \\ Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] &= Z[0] \end{aligned}$$

$$\left. \begin{aligned} [z^3 Y(z) - z^3 y(0) - z^2 y(1) - z y(2)] \\ - 3[zY(z) - zy(0)] + 2Y(z) \end{aligned} \right\} = 0$$

$$(z^3 - 3z + 2) Y(z) - 4z^3 - 8z + 12z = 0$$

$$(z^3 - 3z + 2) Y(z) = 4z^3 - 4z$$

$$(z-1)^2 (z+2) Y(z) = 4z(z^2 - 1)$$

$$Y(z) = \frac{4z(z^2 - 1)}{(z-1)^2 (z+2)}$$

$$Y(z) = \frac{4z(z+1)(z-1)}{(z-1)^2(z+2)}$$

$$\boxed{Y(z) = \frac{4z(z+1)}{(z-1)(z+2)}} \quad (2)$$

To Find : $Z^{-1}[Y(z)] = Z^{-1}\left[\frac{4z(z+1)}{(z-1)(z+2)}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)}$$

Using the method of partial fraction , We get

$$\frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$4(z+1) = A(z+2) + B(z-1)$$

Put $z = 1$, We get $8 = A(3) + B(0) \implies \boxed{A = \frac{8}{3}}$

Put $z = -2$, We get $-4 = A(0) + B(-3) \implies \boxed{B = \frac{4}{3}}$

$$\begin{aligned}\frac{4(z+1)}{(z-1)(z+2)} &= \frac{8}{3} \frac{1}{z-1} + \frac{4}{3} \frac{1}{z+2} \\ \frac{Y(z)}{z} &= \frac{8}{3} \frac{1}{z-1} + \frac{4}{3} \frac{1}{z+2} \\ Y(z) &= \frac{8}{3} \frac{z}{z-1} + \frac{4}{3} \frac{z}{z+2}\end{aligned}$$

Taking inverse Z - transform on both the sides, We get

$$\begin{aligned}Z^{-1}[Y(z)] &= \frac{8}{3} Z^{-1} \left[\frac{z}{z-1} \right] + \frac{4}{3} Z^{-1} \left[\frac{z}{z+2} \right] \\ y(n) &= \frac{8}{3} (1) + \frac{4}{3} (-2)^n \\ \boxed{y(n) &= \frac{4}{3} [2 + (-2)^n]} \quad //\end{aligned}$$

(6). Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$

Solution: Given difference equation is

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n \quad (1)$$

Also given that $y_0 = 0$ & $y_1 = 0$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned} Z[y_{n+2} + 6y_{n+1} + 9y_n] &= Z[2^n] \\ Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] &= Z[2^n] \\ \left. \begin{aligned} [z^2 Y(z) - z^2 y(0) - zy(1)] \\ 6[zY(z) - zy(0)] + 9Y(z) \end{aligned} \right\} &= \frac{z}{z-2} \\ (z^2 + 6z + 9) Y(z) - 0 - 0 - 0 &= \frac{z}{z-2} \\ (z+3)^2 Y(z) &= \frac{z}{z-2} \\ \boxed{Y(z) = \frac{z}{(z-2)(z+3)^2}} & \quad (2) \end{aligned}$$

To Find : $Z^{-1}[Y(z)] = Z^{-1}\left[\frac{z}{(z-2)(z+3)^2}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2}$$

Using the method of partial fraction , We get

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}$$
$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z = 2$, We get $1 = A(25) + B(0) + C(0) \implies A = \frac{1}{25}$

Put $z = -3$, We get $1 = A(0) + B(0) + C(-5) \implies C = \frac{-1}{5}$

Comparing the coefficients of z^2 on both the sides , We get

$$0 = A + B \implies B = -A \implies B = \frac{-1}{25}$$

$$\begin{aligned}\frac{1}{(z-2)(z+3)^2} &= \frac{1}{25} \frac{1}{(z-2)} - \frac{1}{25} \frac{1}{(z+3)} - \frac{1}{5} \frac{1}{(z+3)^2} \\ \frac{Y(z)}{z} &= \frac{1}{25} \frac{1}{(z-2)} - \frac{1}{25} \frac{1}{(z+3)} - \frac{1}{5} \frac{1}{(z+3)^2} \\ Y(z) &= \frac{1}{25} \frac{z}{(z-2)} - \frac{1}{25} \frac{z}{(z+3)} - \frac{1}{5} \frac{z}{(z+3)^2}\end{aligned}$$

Taking inverse Z - transform on both the sides, We get

$$\begin{aligned}Z^{-1}[Y(z)] &= \frac{1}{25} Z^{-1} \left[\frac{z}{(z-2)} \right] - \frac{1}{25} Z^{-1} \left[\frac{z}{(z+3)} \right] - \frac{1}{5} Z^{-1} \left[\frac{z}{(z+3)^2} \right] \\ y(n) &= \frac{1}{25} 2^n - \frac{1}{25} (-3)^n - \frac{1}{5} n (-3)^{n-1} \\ \boxed{y(n) &= \frac{1}{25} 2^n - \frac{1}{25} (-3)^n + \frac{1}{15} n (-3)^n} \quad //\end{aligned}$$

(8). Solve $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ given $y_0 = 0, y_1 = 1$

Solution: Given difference equation is

$$y_{n+2} + 4y_{n+1} + 3y_n = 3^n \quad (1)$$

Also given that $y_0 = 0, y_1 = 1$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned} Z[y_{n+2} + 4y_{n+1} + 3y_n] &= Z[3^n] \\ Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] &= Z[3^n] \\ \left. \begin{aligned} [z^2 Y(z) - z^2 y(0) - zy(1)] \\ + 4[zY(z) - zy(0)] + 3Y(z) \end{aligned} \right\} &= \frac{z}{z-3} \\ (z^2 + 4z + 3) Y(z) - z &= \frac{z}{z-3} \\ (z+1)(z+3) Y(z) &= \frac{z}{z-3} + z \\ (z+1)(z+3) Y(z) &= \frac{z + z(z-3)}{z-3} \end{aligned}$$

$$Y(z) = \frac{z + z^2 - 3z}{(z+1)(z+3)(z-3)}$$

$$Y(z) = \frac{z^2 - 2z}{(z+1)(z+3)(z-3)}$$

$$Y(z) = \frac{z(z-2)}{(z+1)(z+3)(z-3)}$$

$$\boxed{Y(z) = \frac{z}{(z+1)(z+3)(z-3)}} \quad (2)$$

To Find : $Z^{-1}[Y(z)] = Z^{-1}\left[\frac{z(z-2)}{(z+1)(z+3)(z-3)}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)}$$

Using the method of partial fraction , We get

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)$$

Put $z = -1$, We get $-3 = A(2)(-4) + B(0) + C(0) \implies A = \frac{3}{8}$

Put $z = -3$, We get $-5 = A(0) + B(-2)(-6) + C(0) \implies B = \frac{-5}{12}$

Put $z = 3$, We get $1 = A(0) + B(0) + C(4)(6) \implies C = \frac{1}{24}$

$$\begin{aligned}\frac{z-2}{(z+1)(z+3)(z-3)} &= \frac{3}{8} \frac{1}{z+1} - \frac{5}{12} \frac{1}{z+3} + \frac{1}{24} \frac{1}{z-3} \\ \frac{Y(z)}{z} &= \frac{3}{8} \frac{1}{z+1} - \frac{5}{12} \frac{1}{z+3} + \frac{1}{24} \frac{1}{z-3} \\ Y(z) &= \frac{3}{8} \frac{z}{z+1} - \frac{5}{12} \frac{z}{z+3} + \frac{1}{24} \frac{z}{z-3}\end{aligned}$$

Taking inverse Z - transform on both the sides, We get

$$\begin{aligned}Z^{-1}[Y(z)] &= \frac{3}{8}Z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12}Z^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{24}Z^{-1}\left[\frac{z}{z-3}\right] \\ y(n) &= \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}3^n \quad //\end{aligned}$$

(10). Solve $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ given $y_0 = 3, y_1 = -5$

Solution: Given difference equation is

$$y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8 \quad (1)$$

Also given that $y_0 = 3, y_1 = -5$

Taking Z - Transform on both the sides of (1), We get

$$\begin{aligned} Z[y_{n+2} + 4y_{n+1} - 5y_n] &= Z[24n - 8] \\ Z[y_{n+2}] + 4Z[y_{n+1}] - 5Z[y_n] &= 24Z[n] - 8Z[1] \\ \left. \begin{aligned} [z^2 Y(z) - z^2 y(0) - zy(1)] + \\ 4[zY(z) - zy(0)] - 5Y(z) \end{aligned} \right\} &= 24 \frac{z}{(z-1)^2} - 8 \frac{z}{z-1} \\ (z^2 + 4z - 5) Y(z) - 3z^2 + 5z - 12z &= \frac{24z - 8z(z-1)}{(z-1)^2} \\ (z+5)(z-1) Y(z) &= \frac{24z - 8z^2 + 8z}{(z-1)^2} + 3z^2 + 7z \end{aligned}$$

$$\begin{aligned}
 (z+5)(z-1)Y(z) &= \frac{32z - 8z^2 + (3z^2 + 7z)(z^2 - 2z + 1)}{(z-1)^2} \\
 Y(z) &= \frac{32z - 8z^2 + 3z^4 - 6z^3 + 3z^2 + 7z^3 - 14z^2 + 7z}{(z+5)(z-1)^3} \\
 Y(z) &= \frac{3z^4 + z^3 - 19z^2 + 39z}{(z+5)(z-1)^3} \\
 \boxed{Y(z) &= \frac{z[3z^3 + z^2 - 19z + 39]}{(z+5)(z-1)^3}} \quad (2)
 \end{aligned}$$

To Find : $Z^{-1}[Y(z)] = Z^{-1}\left[\frac{z[3z^3 + z^2 - 19z + 39]}{(z+5)(z-1)^3}\right]$

$$(2) \implies \frac{Y(z)}{z} = \frac{3z^3 + z^2 - 19z + 39}{(z+5)(z-1)^3}$$

Using the method of partial fraction , We get

$$\frac{3z^3 + z^2 - 19z + 39}{(z+5)(z-1)^3} = \frac{A}{(z+5)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$
$$3z^3 + z^2 - 19z + 39 = \begin{cases} A(z-1)^3 + B(z+5)(z-1)^2 + \\ C(z+5)(z-1) + D(z+5) \end{cases}$$

Put $z = -5$, We get

$$-375 + 25 + 95 + 39 = A(-6)^3 \implies -216 = -216A \implies \boxed{A = 1}$$

Put $z = 1$, We get

$$3 + 1 - 19 + 39 = D(6) \implies 24 = 6D \implies \boxed{D = 4}$$

Equating the coefficients of z^3 on both the sides , We get

$$3 = A + B \implies B = 3 - A = 3 - 1 \implies \boxed{B = 2}$$

Put $z = 0$, We get

$$39 = -A + 5B - 5C + 5D \implies -39 = -1 + 10 - 5C + 20$$
$$\implies 39 = 29 - 5C$$
$$\implies \boxed{C = -2}$$

$$\frac{3z^3 + z^2 - 19z + 39}{(z+5)(z-1)^3} = \frac{1}{(z+5)} + \frac{2}{(z-1)} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

$$\frac{Y(z)}{z} = \frac{1}{(z+5)} + \frac{2}{(z-1)} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

$$Y(z) = \frac{z}{(z+5)} + 2\frac{z}{(z-1)} - 2\frac{z}{(z-1)^2} + 4\frac{z}{(z-1)^3}$$

Taking inverse Z - transform on both the sides, We get

$$Z^{-1}[Y(z)] = \begin{cases} Z^{-1}\left[\frac{z}{z+5}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right] \\ -2Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{(z-1)^3}\right] \end{cases}$$

$$y(n) = (-5)^n + 2 - 2n + 4\frac{n(n-1)}{2}$$

$$y(n) = (-5)^n + 2 - 2n + 2n^2 - 2n$$

$$\boxed{y(n) = (-5)^n + 2n^2 - 4n + 2} \quad //$$