Importance Sampling

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Outline

1. Recall: Monte Carlo integration
2. Importance Sampling
3. Examples of Importance Sampling

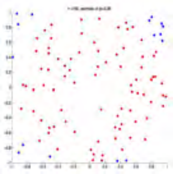
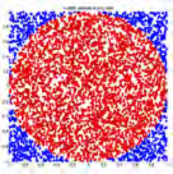
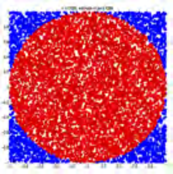
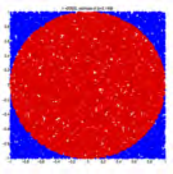
(a) Monte Carlo, Monaco

(b) Monte Carlo Casino

⋆ Some content and examples from Wasserman (2004)

Simple illustration: what is π?

$$\frac{Area_{\circ}}{Area_{\square}} = \frac{\pi r^2}{(2r)(2r)} = \frac{\pi}{4}$$

Monte Carlo Integration: motivation

$$I = \int_a^b h(y)dy$$

- Goal: evaluate this integral
- Sometimes we can find I (e.g. if $h(\cdot)$ is a function from Calc I)
- But sometimes we can't and need a way to approximate I. Monte Carlo methods are one (of many) approaches to do this.

The Law of Large Numbers

While nothing is more uncertain than the duration of a single life, nothing is more certain than the average duration of a thousand lives.

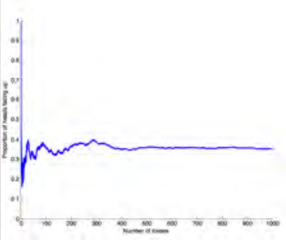
$\sim$ Elizur Wright

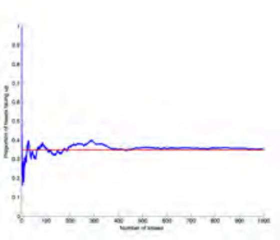
Figure: Elizur Wright (1804 - 1885), American mathematician, the "father of life insurance", "father of insurance regulation" (`http://en.wikipedia.org`)

Law of Large Numbers

- The Law of Large Numbers describes what happens when performing the same experiment many times.
- After many trials, the average of the results should be close to the expected value and will be more accurate with more trials.
- For Monte Carlo simulation, this means that we can learn properties of a random variable (mean, variance, etc.) simply by simulating it over many trials.

Suppose we want to estimate the probability, p, of a coin landing "heads up". How many times should we flip the coin?





Law of Large Numbers (LLN)

Given an independent and identically distributed sequence of random variables $Y_1, Y_2, \ldots, Y_n$ with $\bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i$ and $E(Y_i) = \mu$, then for every $\epsilon > 0$

$$P(|\bar{Y}_n - \mu| > \epsilon) \longrightarrow 0,$$

as $n \longrightarrow \infty$.

Monte Carlo Integration

General idea

Monte Carlo methods are a form of stochastic integration used to approximate expectations by invoking the law of large numbers.

$$I = \int_a^b h(y)dy = \int_a^b w(y)f(y)dy = E_f(w(Y))$$

where $f(y) = \frac{1}{b-a}$ and $w(y) = h(y) \cdot (b-a)$

- $f(y) = \frac{1}{b-a}$ is the pdf of a U(a,b) random variable
- By the LLN, if we take an iid sample of size N from $U(a,b)$, we can estimate I as

$$\hat{I} = N^{-1} \sum_{i=1}^{N} w(Y_i) \longrightarrow E(w(Y)) = I$$

Monte Carlo Integration: standard error

$$I = \int_a^b h(y)dy = \int_a^b w(y)f(y)dy = E_f(w(Y))$$

- Monte Carlo estimator: $\hat{I} = N^{-1}\sum_{i=1}^{N} w(Y_i)$
- Standard error of estimator: $\hat{SE} = \frac{s}{\sqrt{N}}$ where

$$s^2 = (N-1)^{-1}\sum_{i=1}^{N}\left(w(Y_i) - \hat{I}\right)^2$$

Monte Carlo Integration: Gaussian CDF example*

- Goal: estimate $F_Y(y) = P(Y \leq y) = E\left[I_{(-\infty,y)}(Y)\right]$ where $Y \sim N(0,1)$:

$$F(Y \leq y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \int_{-\infty}^{\infty} h(t) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

= E_{t ~ N(0,1)} (h(t))

where $h(t) = 1$ if $t < y$ and $h(t) = 0$ if $t \geq y$

sample from Gaussian

- Draw an iid sample $Y_1, \ldots, Y_N$ from a $N(0,1)$, then the estimator is

$$\hat{I} = N^{-1} \sum_{i=1}^{N} h(Y_i) = \frac{\#\text{ draws } < x}{N}$$

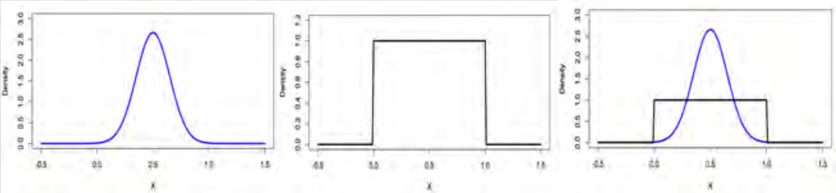
* Example 24.2 of Wasserman (2004)

Importance Sampling: motivation

i.e. non-uniform or non-gaussian say

- Standard Monte Carlo integration is great if you can sample from the *target* distribution (i.e. the desired distribution)
  $\longrightarrow$ But what if you can't sample from the target?

- Idea of importance sampling: draw the sample from a *proposal* distribution and re-weight the integral using *importance weights* so that the correct distribution is targeted



Monte Carlo Integration $\longrightarrow$ Importance Sampling

$$I = \int h(y)f(y)dy$$

- h is some function and f is the probability density function of Y
- When the density f is difficult to sample from, importance sampling can be used
- Rather than sampling from f, you specify a different probability density function, g, as the proposal distribution.

$$I = \int h(y)f(y)dy = \int h(y)\frac{f(y)}{g(y)}g(y)dy = \int \frac{h(y)f(y)}{g(y)}g(y)dy$$

Importance Sampling

$$I = E_f\left[h(Y)\right] = \int \frac{h(y)f(y)}{g(y)} g(y)dy = E_g\left[\frac{h(Y)f(Y)}{g(Y)}\right]$$

Hence, given an iid sample $Y_1, \ldots, Y_N$ from g, our estimator of I becomes

$$\hat{I} = N^{-1}\sum_{i=1}^{N} \frac{h(Y_i)f(Y_i)}{g(Y_i)} \longrightarrow E_g\left[\frac{h(Y)f(Y)}{g(Y)}\right] = I$$

Importance Sampling: selecting the proposal distribution

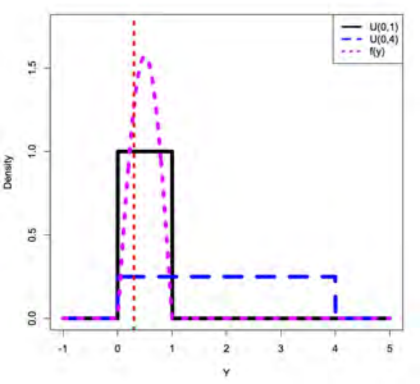
- The standard error of $\hat{I}$ could be infinite if $g(\cdot)$ is not selected appropriately $\longrightarrow$ g should have thicker tails than f (don't want ratio f/g to get large)

$$E_g\left[\left(\frac{h(Y)f(Y)}{g(Y)}\right)^2\right] = \int \left(\frac{h(y)f(y)}{g(y)}\right)^2 g(y)dy$$

- Select a g that has a similar shape to f, but with thicker tails
- Variance of $\hat{I}$ is minimized when $g(y) \propto |f(y)|$
- Want to be able to sample from $g(y)$ with ease

Importance sampling: Illustration

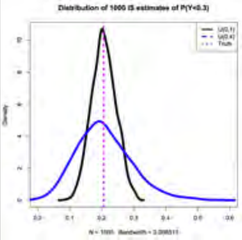
- Goal: estimate $P(Y < 0.3)$ where $Y \sim f$
- Try two proposal distributions: U(0,1) and U(0,4)



Importance sampling: Illustration, continued.

If take 1000 samples of size 100, and find the IS estimates, we get the following *estimated* expected values and variances.

	Expected Value	Variance
Truth	0.206	0
g_1: U(0,1)	0.206	0.0014
g_2: U(0,4)	0.211	0.0075



Monte Carlo Integration: Gaussian tail probability example*

- Goal: estimate $P(Y \geq 3)$ where $Y \sim N(0,1)$ (Truth is $\approx$ 0.001349)

$$P(Y > 3) = \int_3^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \int_{-\infty}^\infty h(t) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

where $h(t) = 1$ if $t > 3$ and $h(t) = 0$ if $t \leq 3$

- Draw an iid sample $Y_1, \ldots, Y_{100}$ from a $N(0,1)$, then the estimator is

$$\hat{I} = \frac{1}{100} \sum_{i=1}^{100} h(Y_i) = \frac{\#\text{ draws } > 3}{100}$$

* Example 24.6 of Wasserman (2004)

Gaussian tail probability example*, continued.

- Draw an iid sample $Y_1, \ldots, Y_{100}$ from a $N(0,1)$, then the estimator is

$$\hat{I} = \frac{1}{100}\sum_{i=1}^{100} h(Y_i)$$ g, f same, cancel out

- Draw an iid sample $Y_1, \ldots, Y_{100}$ from a $N(4,1)$, then the estimator is

$$\hat{I} = \frac{1}{100}\sum_{i=1}^{100} \frac{h(Y_i)f(Y_i)}{g(Y_i)}$$

where f is the density of a N(0,1) and g is the density of N(4,1)

* Example 24.6 of Wasserman (2004)

Gaussian tail probability example*, continued.

If take N samples of size 100, and find the MC and IS estimates, we get the following *estimated* expected values and variances.

$N = 10^5$

	Expected Value	Variance
Truth	0.00135	0
Monte Carlo	0.00136	1.3×10^{-5}
Importance Sampling	0.00135	9.5×10^{-8}

Extensions of Importance Sampling

- Sequential Importance Sampling
- Sequential Monte Carlo (Particle Filtering)
  $\longrightarrow$ See Doucet et al. (2001)
- Approximate Bayesian Computation $\longrightarrow$ See Turner and Zandt (2012) for a tutorial, and Cameron and Pettitt (2012); Weyant et al. (2013) for applications to astronomy

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