

Chapter 11

White Dwarfs and Neutron Stars

Red giants will eventually consume all their accessible nuclear fuel.

- After ejection of the envelope, the cores of these stars shrink to the very hot, very dense objects that we call **white dwarfs**.
- An even more dense object termed a neutron star can be left behind after the evolution of more massive stars terminates in a core-collapse supernova explosion.

Technically, white dwarfs and neutron stars are stellar corpses, not stars, but it is common to refer to them loosely as stars.

11.1 Sirius B

The bright star Sirius, in Canis Major, is actually a double star.

- The brighter component is labeled Sirius A and the fainter companion star is known as Sirius B.
- Sirius B is an example of a white dwarf.
- Because of its proximity to Earth, Sirius B is not particularly dim (visual magnitude 8.5), but it is difficult to observe because it is so close to Sirius A.
- Sirius B is clearly not a normal star; its spectrum and luminosity indicate that it is hot (about 25,000 K surface temperature) but very small.
- This spectrum contains pressure-broadened hydrogen lines, implying a surface environment with much higher density than that of a normal star.

- Assuming the spectrum of Sirius B to be blackbody and using the well-established distance to Sirius, we conclude from its luminosity that Sirius B has a radius of only about 5800 km.
- But Sirius is a visual binary with a very well studied orbit.
- Therefore, we may use Kepler's laws to infer that the mass of Sirius B is about $1.03 M_{\odot}$.
- We conclude that a white dwarf like Sirius B packs the mass of a star in an object the size of the Earth.

Sirius B is the nearest and brightest white dwarf and we shall often use it as illustration. But it is in some respects not so representative because its mass of about $1.03 M_{\odot}$ is much larger than the average mass of about $0.58 M_{\odot}$ observed for white dwarfs (Sirius B is in the 98th percentile of white dwarf masses).

11.2 Properties of White Dwarfs

The preceding discussion allows us to make some immediate estimates that will shed light on the nature of white dwarfs even before we carry out any detailed analysis.

11.2.1 Density and Gravity

- Since white dwarfs contain roughly the mass of the Sun in a sphere the size of the Earth, we expect that white dwarfs have densities in the vicinity of 10^6 g cm^{-3} .
- For Sirius B the average density calculated from the observed mass and radius is about $2.5 \times 10^6 \text{ g cm}^{-3}$.
- The gravitational acceleration and the escape velocity at the surface for Sirius B are

$$g = \frac{Gm}{R^2} \simeq 3.7 \times 10^8 \text{ cm s}^{-2} \quad \frac{v_{\text{esc}}}{c} = \sqrt{\frac{2Gm}{Rc^2}} \simeq 0.02,$$

respectively, indicating that

- the gravitational acceleration is almost 400,000 times larger than at the Earth's surface, but
- general relativity effects, while not completely negligible, are sufficiently small to be ignored in initial approximation.

11.2.2 Equation of State

- We conclude from the preceding that hydrostatic equilibrium under Newtonian gravitation is adequate as a first approximation for the structure of white dwarfs.
- What about the microphysics of the gas?
 - Can we apply a Maxwell–Boltzmann description, or will the quantum statistical properties of the gas play a crucial role?
 - Will electron velocities be describable classically or will velocities become relativistic?

Let us assume initially nonrelativistic velocities and that electrons are primarily responsible for the internal pressure of the white dwarf.

- For simplicity we shall also assume that the white dwarf is composed of a single kind of nucleus having atomic number Z , neutron number N , and atomic mass number $A = Z + N$.
- Then the average electron velocity is $\bar{v}_e = \bar{p}/m_e$ where \bar{p} is the average momentum and m_e is the electron mass.
- By the uncertainty principle, the average momentum may be estimated as

$$\bar{p} \simeq \Delta p \simeq \hbar/\Delta x \simeq \hbar n_e^{1/3},$$

where n_e is the electron number density.

- We may expect the gas to be completely ionized and the corresponding electron number density is

$$n_e = \left(\frac{\text{number } e^-}{\text{nucleon}} \right) \left(\frac{\text{number nucleons}}{\text{unit volume}} \right) = \left(\frac{Z}{A} \right) \left(\frac{\rho}{m_H} \right).$$

- Therefore, the average electron velocity may be approximated by

$$\frac{\bar{v}_e}{c} = \frac{\bar{p}}{m_e c} = \frac{\hbar n_e^{1/3}}{m_e c} = \frac{\hbar}{m_e c} \left(\frac{Z\rho}{Am_H} \right)^{1/3} \simeq 0.25,$$

where we assume that $A = 2Z$, as would be true for ^{12}C , ^{16}O , or ^4He , which are the primary constituents of most white dwarfs.

- We conclude on general grounds that electron velocities will become relativistic (significant fraction of c) for higher-density white dwarfs.

- The average spacing between electrons in the gas is

$$d \simeq n_e^{-1/3} \simeq 1.5 \times 10^{-10} \text{ cm},$$

using $Z/A = 0.5$ and the average density of Sirius B.

- The deBroglie wavelength of electrons in the gas is on average

$$\bar{\lambda}_e = \frac{h}{p} = \frac{h}{\bar{v}_e m_e} \simeq 9.6 \times 10^{-10} \text{ cm}.$$

- Because the separation of particles in the gas is comparable to their deBroglie wavelength, the electron gas will be degenerate, provided that the temperature is not too high.
- For a degenerate fermion gas the fermi energy is ($\hbar = c = 1$ units)

$$E_f = \sqrt{k_f^2 + m^2}.$$

The gas will remain degenerate as long as E_f is much larger than the characteristic energy kT of particles in the gas.

- Since from the preceding equation $E_f \geq m_e c^2 = 0.511 \text{ MeV}$, this implies that a temperature $T = E/k = 0.511 \text{ MeV}/k \simeq 6 \times 10^9 \text{ K}$ is required to break the degeneracy.
- Detailed calculations indicate that interior white dwarf temperatures are typically in the range 10^6 – 10^7 K , so we conclude that white dwarfs contain cold, degenerate gases of electrons.
- Therefore they may be approximated by polytropic equations of state having the form

$$P = K\rho^\gamma,$$

1. $\gamma = \frac{5}{3}$ for nonrelativistic degenerate electrons
2. $\gamma = \frac{4}{3}$ for ultrarelativistic degenerate electrons.

- While we expect the electrons to be degenerate and to become relativistic at higher densities, the ions are much more massive than the electrons.
- The ions are neither relativistic nor degenerate, and are well described by an ideal gas equation of state.
- Because the ions move slowly, they contribute little to the pressure.
- However, calculations indicate that most of the heat energy stored in the white dwarf is associated with motion of the ions.
- Finally, photons in the white dwarf constitute a relativistic gas approximated by a Stefan–Boltzmann equation of state,

$$P = \frac{1}{3} a T^4,$$

where a is the radiation density constant and T is the temperature.

Thus, the picture that emerges for a white dwarf is of a hot, dense object for which the mechanical properties (exemplified by the pressure, which is generated primarily by the degenerate electrons) are decoupled from the thermal properties (which are associated primarily with the ions at normal temperatures).

11.2.3 Ingredients of a White Dwarf Description

An initial description of a white dwarf may be afforded by a theory for which

1. The stable configurations correspond to hydrostatic equilibrium under Newtonian gravitation.
2. The ions carry most of the mass and store most of the thermal energy of the white dwarf, but the electrons are responsible for the bulk of the pressure.
3. The electron equation of state will be that of a cold degenerate gas, conveniently approximated in the polytropic form $P = K\rho^\gamma$ with $\gamma = \frac{5}{3}$ for nonrelativistic and $\gamma = \frac{4}{3}$ for relativistic electrons, respectively.
4. Ions are nonrelativistic and may be described by an ideal gas equation of state.
5. Photons are described by a Stefan–Boltzmann equation of state.
6. Because the degenerate electron gas is primarily responsible for the pressure but its equation of state does not depend on temperature, the thermal and mechanical properties of the white dwarf are decoupled.
7. As density increases the velocity of the electrons increases and special relativity becomes important, corresponding to a transition

$$P \simeq K\rho^{5/3} \longrightarrow P \simeq K'\rho^{4/3}.$$

in the electron equation of state.

Let us now turn to a theoretical description embodying these basic ideas in a relatively simple formulation.

11.3 Lane–Emden Equations

The equations of hydrostatic equilibrium may be combined to give the differential equation

$$\left. \begin{aligned} dm &= 4\pi r^2 \rho(r) dr \\ \frac{dP}{dr} &= -\frac{Gm(r)}{r^2} \rho \end{aligned} \right\} \rightarrow \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho.$$

Assuming a completely degenerate electron gas, we adopt a polytropic equation of state with

$$P = K\rho^\gamma = K\rho^{1+1/n} \quad \gamma \equiv 1 + \frac{1}{n}.$$

Introducing dimensionless variables ξ and θ through

$$\rho = \rho_c \theta^n \quad r = a\xi \quad a = \sqrt{\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}},$$

where $\rho_c \equiv \rho(r=0)$ is the central density, the differential equation embodying hydrostatic equilibrium for a polytropic equation of state may be expressed in terms of the new dependent variable $\theta(\xi)$ and independent variable ξ as,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

In terms of these new variables the boundary conditions are

$$\theta(0) = 1 \quad \theta'(0) \equiv \left. \frac{d\theta}{d\xi} \right|_{\xi=0} = 0,$$

- The first follows from the requirement that the correct central density $\rho_c = \rho(0)$ be reproduced.
- The second follows from requiring that the pressure gradient dP/dr vanish at the origin (necessary condition for hydrostatic equilibrium).

Then we may integrate

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

outward from the origin ($\xi = 0$) until the point $\xi = \xi_1$ where θ first vanishes, to define the surface of the star, since at this point $\rho = P = 0$ because

$$\rho = \rho_c \theta^n \quad P = K \rho^\gamma.$$

Solutions having this property generally exist for $n < 5$.

Table 11.1: Lane–Emden constants

n	γ	ξ_1	$\xi_1^2 \theta'(\xi_1) $
0	∞	2.4494	4.8988
0.5	3	2.7528	3.7871
1.0	2	3.14159	3.14159
1.5	5/3	3.65375	2.71406
2.0	3/2	4.35287	2.41105
2.5	1.4	5.35528	2.18720
3.0	4/3	6.89685	2.01824
4.0	5/4	14.97155	1.79723
4.5	1.22	31.83646	1.73780
5.0	1.2	∞	1.73205

- The equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n.$$

has analytical solutions for the special cases $n = 0, 1$, and 5 , but

- in the physically most interesting cases the equations must be integrated numerically to define the Lane–Emden constants ξ_1 and $\xi_1^2 |\theta'(\xi_1)|$ for given n .

These are tabulated for various values of n and γ in Table 11.1.

The transformation equations

$$\rho = \rho_c \theta^n \quad r = a\xi \quad a = \sqrt{\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}},$$

may then be used to express quantities of physical interest in terms of these constants for definite values of the polytropic index n . For example, the radius R is

$$R = a\xi_1 = \sqrt{\frac{(n+1)K}{4\pi G}} \rho_c^{(1-n)/2n} \xi_1,$$

and the mass M is given by (Exercise 11.4)

$$\begin{aligned} M &\equiv 4\pi a^3 \rho_c \left[-\xi^2 \frac{d\theta}{d\xi} \right]_{\xi=\xi_1} \\ &= 4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 |\theta'(\xi_1)|, \end{aligned}$$

Eliminating ρ_c between these two equations gives a general relationship between the mass and the radius,

$$M = 4\pi R^{(3-n)/(1-n)} \left(\frac{(n+1)K}{4\pi G} \right)^{n/(n-1)} \xi_1^{(3-n)/(n-1)} \xi_1^2 |\theta'(\xi_1)|.$$

for a solution with polytropic index n .

11.4 Polytropic Models of White Dwarfs

We expect that white dwarfs are approximately described by systems in hydrostatic equilibrium having degenerate electron equations of state.

- Thus, we may expect that solutions of the Lane–Emden equation with polytropic index $n = \frac{3}{2}$, corresponding to $\gamma = \frac{5}{3}$, are relevant for the structure of low-mass white dwarfs where electron velocities are nonrelativistic.
- Likewise, we may expect that in more massive white dwarfs the electrons become relativistic and the corresponding structure is related to a Lane–Emden solution with polytropic index $n = 3$, corresponding to $\gamma = \frac{4}{3}$.
- Between these extremes the electron equation of state must generally be described in numerical terms permitting an arbitrary level of degeneracy and degree of relativity.

11.4.1 Low-Mass White Dwarfs

Let us first consider a low-mass white dwarf.

- Assuming a $\gamma = \frac{5}{3}$ polytropic equation of state ($n = \frac{3}{2}$), the preceding equation

$$M = 4\pi R^{(3-n)/(1-n)} \left(\frac{(n+1)K}{4\pi G} \right)^{n/(n-1)} \xi_1^{(3-n)/(n-1)} \xi_1^2 |\theta'(\xi_1)|.$$

then gives

$$MR^3 = \text{constant},$$

since

$$R^{(3-n)/(1-n)} = R^{(3-3/2)/(1-3/2)} = R^{(3/2)/(-1/2)} = R^{-3}.$$

Thus the product of the mass and the volume of a white dwarf is constant.

We obtain the surprising result that, contrary to the behavior of normal stars, increasing the mass of a low-mass white dwarf causes its radius to **shrink**.

- This behavior is a direct consequence of a degenerate electron equation of state.

11.4.2 The Chandrasekhar Limit

If we continue to add mass to a white dwarf, the electrons will move faster by uncertainty principle arguments and eventually will become relativistic. If $\gamma = \frac{4}{3}$ (corresponding to $n = 3$),

$$M = 4\pi R^{(3-n)/(1-n)} \left(\frac{(n+1)K}{4\pi G} \right)^{n/(n-1)} \xi_1^{(3-n)/(n-1)} \xi_1^2 |\theta'(\xi_1)|.$$

then implies that

$$M = \text{constant} \times R^0 = \text{constant}$$

This even more surprising result defines the Chandrasekhar limiting mass, which implies that there is an upper limit for the mass of a white dwarf.

Inserting the constants, we find that for the radius of a high-mass white dwarf described by a relativistic, degenerate electron equation of state,

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{-1/3} \left(\frac{\mu_e}{2} \right)^{-2/3} \text{ km},$$

and for the Chandrasekhar mass,

$$M_0 = 1.457 \left(\frac{2}{\mu_e} \right)^2 M_\odot \simeq 1.4 M_\odot,$$

where the last estimate follows because generally $2/\mu_e$ is of order unity.

Thus the Chandrasekhar limiting mass is slightly composition dependent but implies an upper limit on the mass of a white dwarf of approximately $1.4 M_\odot$.

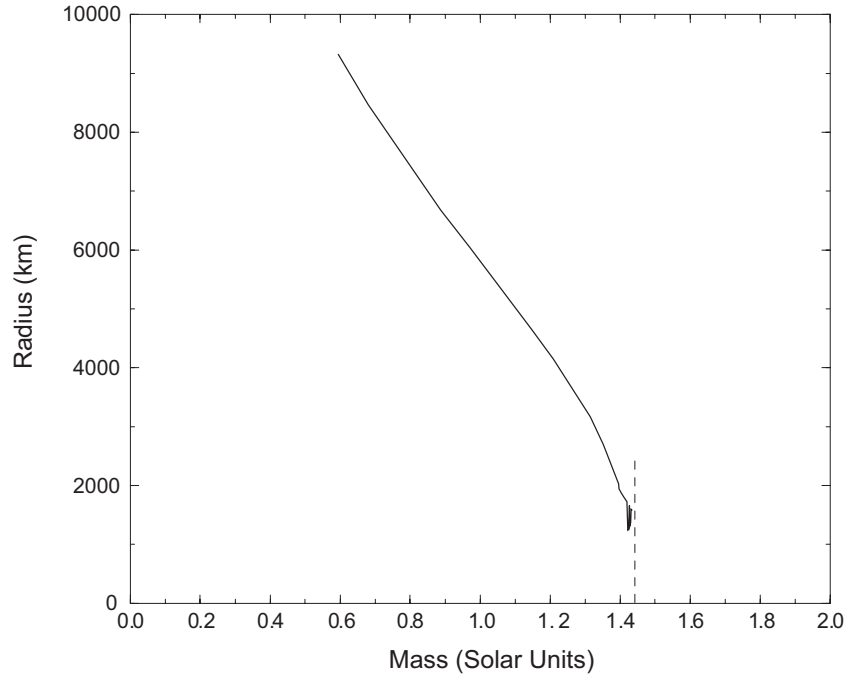
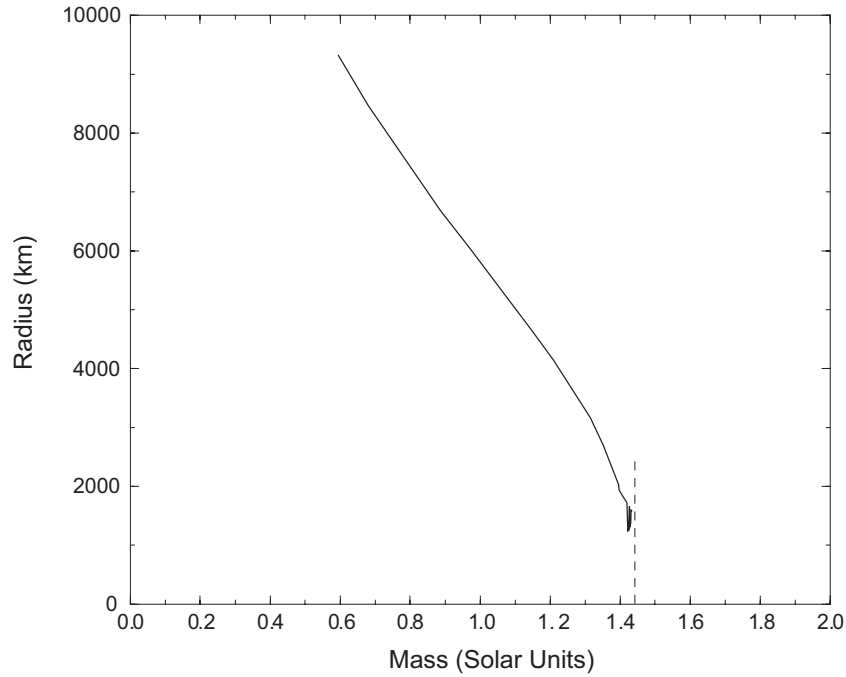


Figure 11.1: The dependence of radius on mass for a white dwarf. The limiting mass of 1.44 solar masses is indicated by the onset of numerical instability in the calculation. Calculated assuming $Y_e = 0.5$ and a central temperature $T_c = 5.0 \times 10^6$ K. (The electron fraction Y_e is the ratio of the number of electrons to the total number of nucleons. For symmetric matter $Z = N$, so for fully-ionized symmetric matter, $Y_e = \frac{1}{2}$.)

- In Fig. 11.1 the radius versus mass for white dwarfs in hydrostatic equilibrium is shown for a numerical simulation.
- This calculation uses a numerical equation of state that accounts fully for arbitrary degrees of electron degeneracy and arbitrary relativity for electrons.
- Thus, for electrons this equation of state approximates a $\gamma = \frac{5}{3}$ polytrope at low mass and a $\gamma = \frac{4}{3}$ polytrope at high mass, with a smooth transition in between.



- The ions of the white dwarf are assumed to obey an ideal gas equation of state and the photons are described by a Stefan–Boltzmann photon gas equation of state.
- We see clearly in the above figure the behavior implied by the preceding equations.

$$MR^3 = \text{constant},$$

- For lower masses the radius of the white dwarf decreases steadily with increase in mass, in accord with $MR^3 = \text{constant}$, but
- at high masses the curve turns over and approaches a vertical asymptote given by $M = M_0$, with the calculation becoming numerically unstable as the limiting mass is approached.

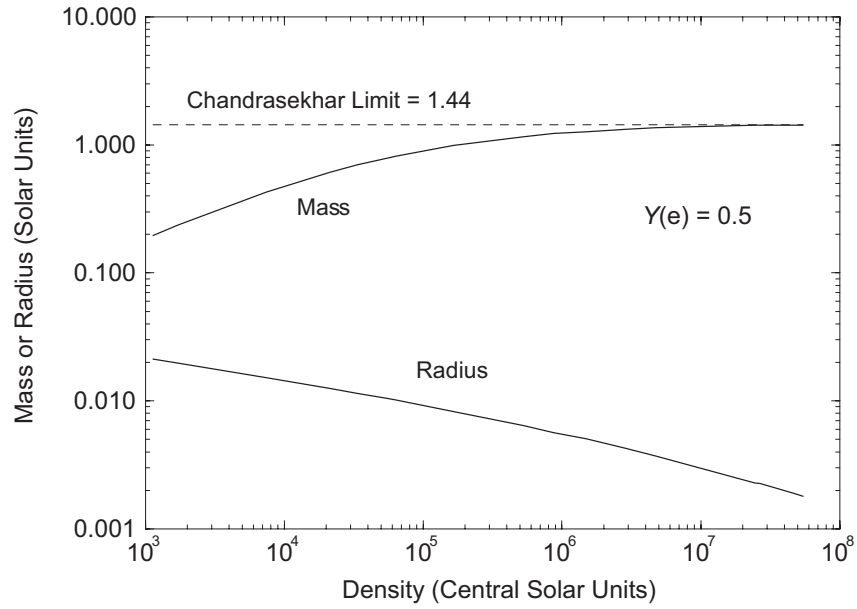


Figure 11.2: The variation of mass and radius for white dwarfs as a function of the central density in units of central solar densities.

- In Fig. 11.2 the variation of the mass and radius of white dwarfs as a function of the central density in central solar units is plotted for calculations similar to those described in Fig. 11.1.
- Note the steady trend to zero radius as the white dwarf approaches the limiting mass asymptotically.

11.4.3 Heuristic Derivation of the Chandrasekhar Limit

The Chandrasekhar limiting mass was obtained above as a consequence of the Lane–Emden equations, which embody polytropic equations of state and hydrostatic equilibrium. It will prove useful in understanding the limiting mass for white dwarfs to obtain the Chandrasekhar result in a somewhat more intuitive way.

- Assume a fully-ionized sphere of symmetric (equal numbers of protons and neutrons) matter containing N electrons.
- The mass of the sphere is then $M = 2m_p N$,
- the average spacing between electrons is $d \sim R/N^{1/3}$, and
- the average momentum of the electrons is (uncertainty principle)

$$p_f \sim \frac{\hbar}{d} \sim \frac{\hbar M^{1/3}}{R m_p^{1/3}}.$$

- By estimating the total energy of the degenerate electrons and balancing that against the gravitational energy of the protons (see Exercises 11.1–11.2), one obtains for energy balance in the non-relativistic and relativistic limits,

$$E = a \frac{M^{5/3}}{R^2} - b \frac{M^2}{R} \quad (\text{nonrelativistic})$$

$$E = c \frac{M^{4/3}}{R} - d \frac{M^2}{R} \quad (\text{relativistic})$$

where a , b , c , and d are positive constants.

- Notice that the two terms in the nonrelativistic case have different dependence on R .
- Thus, by setting $\partial E / \partial R = 0$, one finds an equilibrium configuration in the nonrelativistic case that generally satisfies $MR^3 = \text{constant}$.

- On the other hand, in the relativistic case

$$E = c \frac{M^{4/3}}{R} - d \frac{M^2}{R} \quad (\text{relativistic})$$

the two terms have the **same** dependence on R (a result that follows directly from the requirement $\gamma = \frac{4}{3}$).

- Thus, attempting to solve $\partial E / \partial R = 0$ for R corresponding to a gravitationally stable configuration leads to an **indeterminate result** (the resulting equation does not depend on R).
- The meaning of this result is clarified if we note that both terms in this equation vary as R^{-1} , but the first term depends on $M^{4/3}$ while the second varies as M^2 .
- The second term has a net negative sign and a stronger dependence on M than the first term, so **the total energy of the system becomes negative if the mass is made large enough**.
- But since the total energy scales as R^{-1} in this case, **once the total energy becomes negative the system can minimize its energy by shrinking to zero radius**:

For the relativistic degenerate gas there is a limiting mass beyond which the system is unstable against gravitational collapse.

- We may estimate this critical mass by equating the two terms in $E = cM^{4/3}/R - dM^2/R$, yielding (see Exercise 11.1)

$$M_0 = \left(\frac{\hbar c}{G m_p^{4/3}} \right)^{3/2} \simeq 1 M_\odot,$$

which is correct to order of magnitude.

11.4.4 The Adiabatic Index γ and Gravitational Stability

The preceding results are another variation of the theme introduced previously in conjunction with the collapse of protostars.

- There we found that an adiabatic index of $\gamma \leq \frac{4}{3}$ implies an instability against expansion or contraction.
- From the polytropic equation of state $P = K\rho^\gamma$,

$$\frac{dP}{d\rho} = K\gamma\rho^{\gamma-1} \rightarrow \frac{\rho}{P} \frac{dP}{d\rho} = \gamma \frac{\rho}{P} K\rho^{\gamma-1} = \gamma \frac{K\rho^\gamma}{P} = \gamma$$

so we may define γ for any equation of state $P(\rho)$ by

$$\gamma \equiv \frac{\rho}{P} \frac{dP}{d\rho} = \frac{d \ln P}{d \ln \rho}.$$

- Taking this logarithmic derivative as the definition of an effective adiabatic index γ_{eff} , we may expect that in any simulation of hydrostatic equilibrium,

$$\gamma_{\text{eff}} \equiv \frac{\rho}{P} \frac{dP}{d\rho} \simeq \frac{4}{3}$$

heralds the onset of a radial scaling instability.

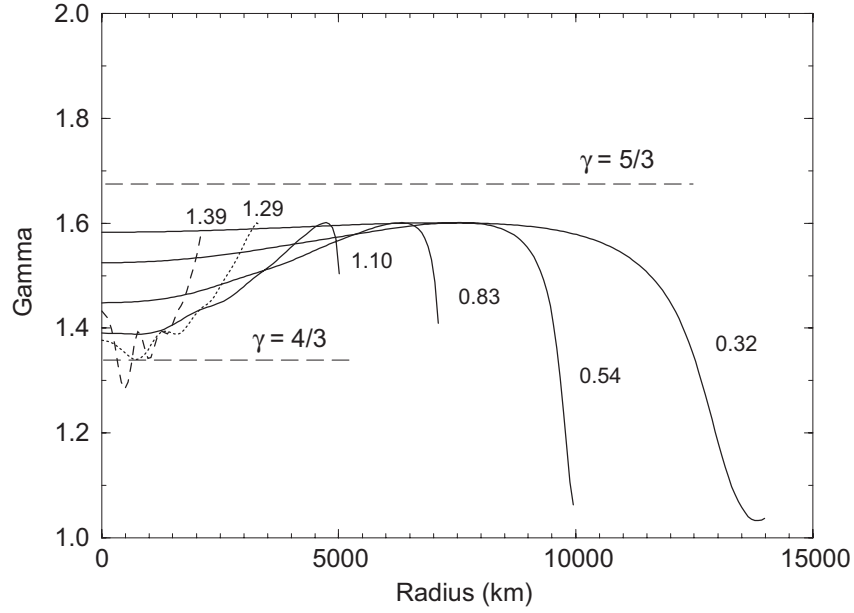
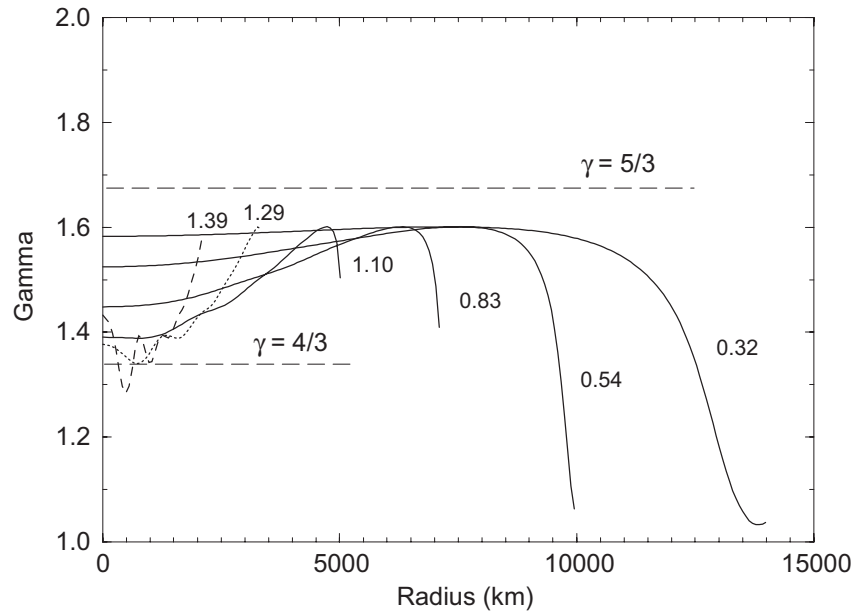


Figure 11.3: Values of the parameter $\gamma \equiv d \ln P / d \ln \rho$ at constant temperature for white dwarfs of various masses (solar units). The values of γ corresponding to nonrelativistic ($\gamma = 5/3$) and relativistic ($\gamma = 4/3$) polytropes are indicated. The calculation becomes unstable as the mass approaches the Chandrasekhar limiting mass which is 1.44 solar masses for this calculation (for which $Y_e = 0.5$). The central temperature is assumed to be 5×10^6 K in all calculations.

- In Fig. 11.3 the value of γ_{eff} as a function of radius is calculated numerically using

$$\gamma = \frac{\rho}{P} \frac{dP}{d\rho} = \frac{d \ln P}{d \ln \rho}.$$

for white dwarf solutions that have been obtained with an equation of state allowing arbitrary electron degeneracy and relativity



- For low-mass white dwarfs the effective value of γ is near the nonrelativistic expectation of $\frac{5}{3}$ for the entire interior.
- However, as the mass of the white dwarf is increased, the effective value of γ in the deep interior begins to drop.
- As the mass approaches the Chandrasekhar limit, γ_{eff} tends to $\frac{4}{3}$ and the numerical solution becomes very unstable.
- These numerical fluctuations reflect the incipient gravitational instability that in this case occurs at 1.44 solar masses.

The roles of relativity and quantum mechanics are central to the preceding results.

- Nonrelativistic degenerate matter has $\gamma \sim \frac{5}{3}$, which is gravitationally stable.
- But quantum mechanics (the uncertainty principle) requires the electrons to move faster as the density increases, implying that the velocities eventually become relativistic as the white dwarf mass increases.
- Relativistic degenerate matter has $\gamma \sim \frac{4}{3}$, which inherently is gravitationally unstable.
- Because the speed of the electrons is limited by the speed of light, there is a mass beyond which even the degeneracy pressure cannot prevent gravitational collapse of the system.

This critical point is the Chandrasekhar limiting mass.

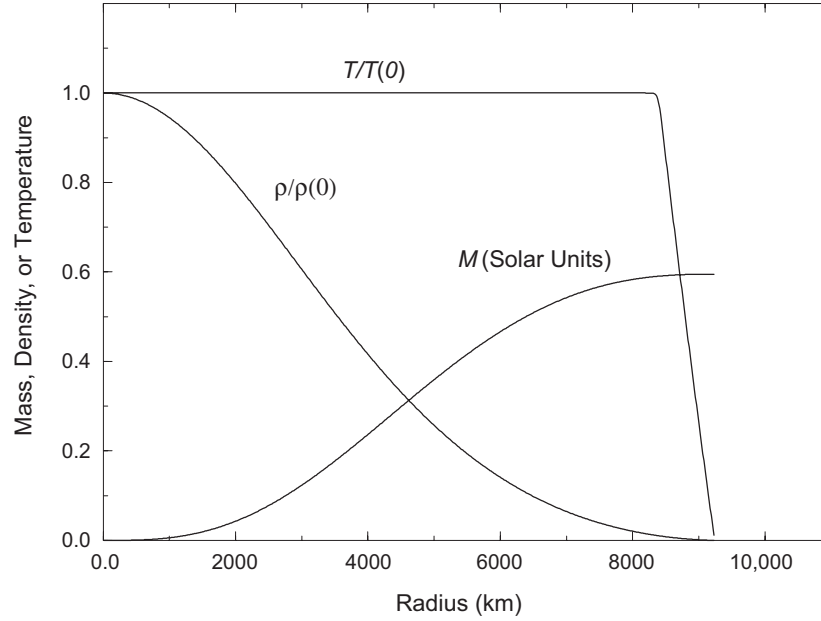


Figure 11.4: Behavior of density, enclosed mass, and temperature for a white dwarf. In this calculation the white dwarf has a central density of $2.9 \times 10^6 \text{ g cm}^{-3}$, a central temperature of $5.0 \times 10^6 \text{ K}$, a total mass of 0.595 solar masses, and a radius of 9234 km.

11.5 Internal Structure of White Dwarfs

- A numerical calculation of the internal structure of a white dwarf is illustrated in Fig. 11.4, which plots the density, enclosed mass, and temperature as a function of radius.
- The calculation corresponds to hydrostatic equilibrium with a realistic electron equation of state in which the electrons have arbitrary degeneracy and degree of relativity.
- The ions are assumed to obey an ideal gas equation of state and radiation to obey a Stefan–Boltzmann equation of state.

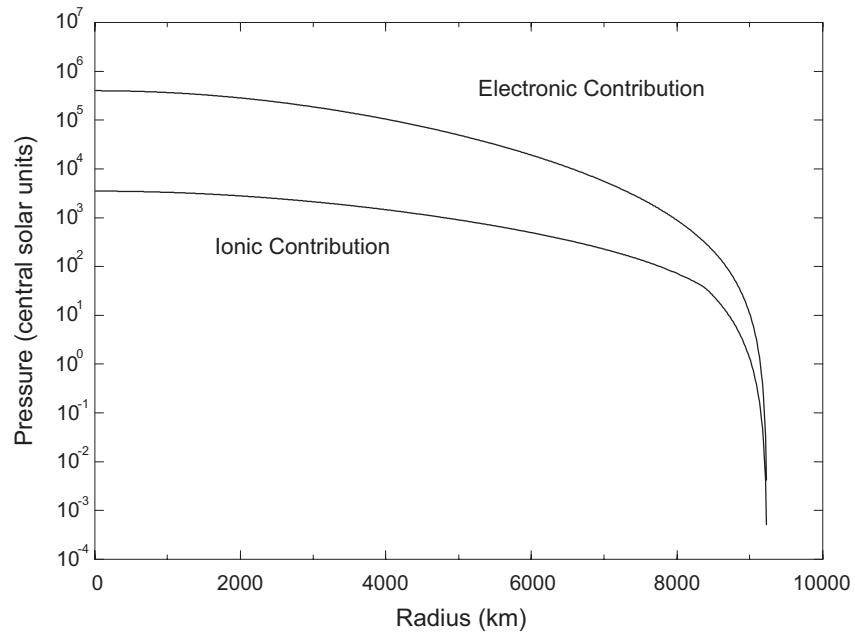
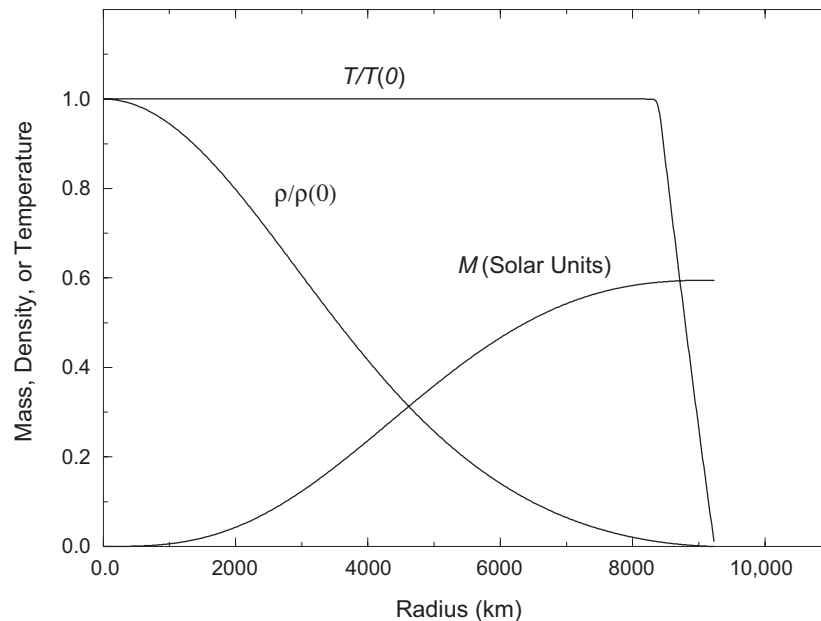


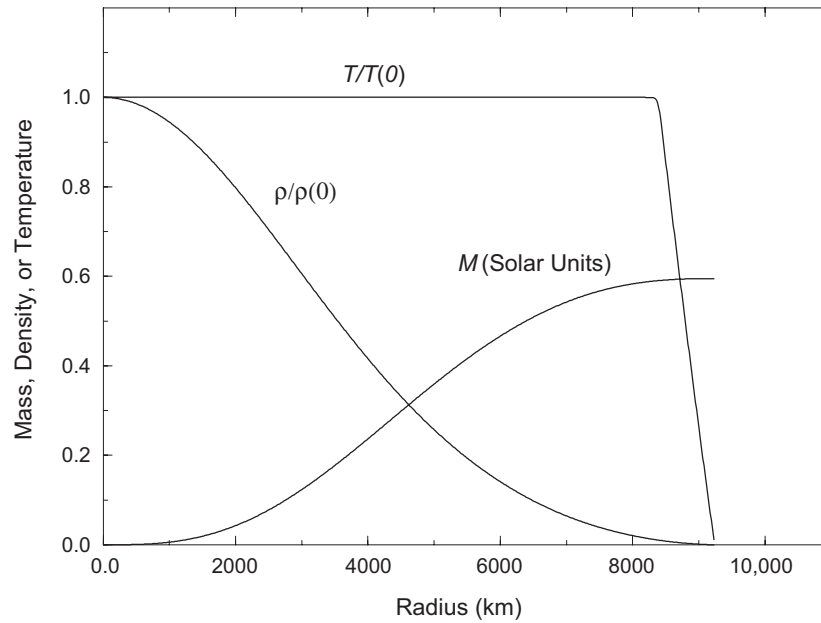
Figure 11.5: Relative contributions of the electronic pressure and ionic pressure for the calculation described in Fig. 11.4. The contribution to the pressure from radiation under these conditions is completely negligible relative to the electronic and ionic contributions. The electronic contribution is very nearly that expected for a fully degenerate gas.

Figure 11.5 illustrates the relative contribution of electrons and ions to the pressure in the preceding calculation, and provides strong justification for our earlier assumption that the pressure in white dwarfs is dominated by the contribution from degenerate electrons.



The internal temperature variation in the calculation shown above is determined as follows.

- Because degenerate matter is such a good conductor of thermal energy, the interior of a white dwarf cannot support a substantial temperature gradient and we assume all but a thin surface layer to be isothermal and strongly heat conducting.
- On the other hand, near the surface the density drops to zero and the nearly ideal gas expected there is a very good insulator.
- This suggests that a good model of how white dwarfs cool is one of a conducting sphere with no temperature gradient surrounded by a thin layer of normal gas with a gradient set by its transport properties (that is, by its opacity).
- This model is analogous mathematically to the cooling of a hot metal ball surrounded by a thin insulating jacket, since degenerate gases have many of the properties of metals.



In the “metal ball plus insulating blanket” model for the above figure,

- The interior is assumed fully conductive,
- The surface is assumed insulating with a radiative opacity given by the Kramers bound–free opacity, and
- the transition between the two is governed by the degeneracy parameter

$$\alpha = \frac{\mu - m_e c^2}{kT},$$

where μ is the chemical potential for the electrons.

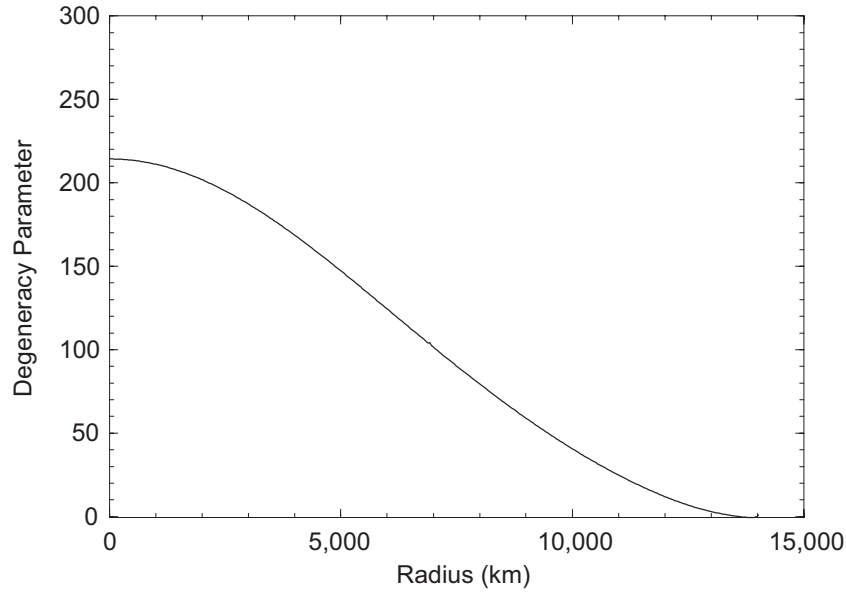


Figure 11.6: The degeneracy parameter $\alpha \equiv (\mu - m_e c^2)/kT$ as a function of radius in a white dwarf, where μ is the electron chemical potential and m_e the electron mass. The white dwarf has a calculated mass of 0.54 solar masses, a radius of 10,000 km, a central temperature of 5×10^6 K, and a central density of 2.135×10^6 g cm $^{-3}$. The figure was calculated with the equation of state used in Fig. 11.2.

- The variation of the degeneracy parameter

$$\alpha = \frac{\mu - m_e c^2}{kT},$$

with radius is illustrated in Fig. 11.6.

- In the interior α is large, indicating a highly degenerate gas of electrons.
- But very near the surface α falls to zero, implying that in a thin surface layer the electrons obey approximately an ideal gas law.

11.6 Cooling of White Dwarfs

Although white dwarfs have no internal heat source, they can remain luminous for long periods of time as the heat left over from their glory days slowly leaks away.

- The cooling curve for a white dwarf should then reflect both the internal structure and the age of the star.
- As we have seen, white dwarfs are well described by a spherical ball of electron degenerate matter surrounded by a very thin surface layer that obeys an ideal gas equation of state.

This can serve as a simple but quantitative model for cooling rates in white dwarfs.

11.7 Beyond White Dwarf Masses

The preceding discussion of limiting masses for white dwarfs assumes all pressure to derive from electrons.

- However, if the Chandrasekhar mass is exceeded and the system collapses, eventually a density will be reached where the nucleons (also fermions) will begin to produce a strong degeneracy pressure.
- Whether this nucleon degeneracy pressure can halt the collapse depends on the mass.
- Calculations indicate that for a mass less than about 2–3 solar masses (depending weakly on details such as the equation of state), the collapse converts essentially all protons into neutrons through the weak interactions, producing a neutron star.
- The degeneracy pressure of the neutrons halts the collapse at neutron-star densities and radii approximately 500 times smaller than for white dwarfs.
- Calculations, and general considerations for strong gravity, indicate that for masses greater than this even the neutron degeneracy pressure cannot overcome gravity and the system collapses to a black hole.
- These considerations also indicate that white dwarfs and neutron stars are the **only** possible stable configurations lying between normal stars and black holes.

Therefore, let us now consider neutron stars.

11.8 Basic Properties of Neutron Stars

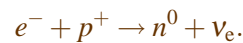
- Neutron stars were predicted in 1933 by Baade and Zwicky as a possible end result of what we would now call a core-collapse supernova.
- Oppenheimer and Volkov worked out equations describing their general structure and properties in 1939. (This will be covered in Astro 616, since it requires general relativity.)
- However, they were not taken very seriously until the discovery of radio pulsars in the 1960s pointed to rapidly rotating neutron stars as their most likely explanation.
- It is estimated that there are some 10^8 neutron stars in our galaxy.
- Of order 1000 of these have actually been observed by astronomers so far.

- Most neutron stars have been discovered as radio pulsars but the vast majority of the energy emitted by neutron stars is in very high-energy photons (X-rays and γ -rays), rather than radio waves.
- Typically only about 10^{-5} of their radiated energy is in the radio-frequency spectrum.
- Most neutron stars have masses of $1\text{--}2 M_{\odot}$ and diameters of $10\text{--}20$ km. Very loosely, a neutron star packs the mass of a normal star like the Sun into a volume of order 10 km in radius.
- From the density of a little over 1 g cm^{-3} and radius of about 7×10^5 km for the Sun, we may estimate immediately an average density of order $10^{14} \text{ g cm}^{-3}$ for neutron stars (it can actually be about an order of magnitude larger than that).
- Thus, they have enormous densities that are similar to those encountered in the nucleus of the atom.
- In fact, in certain ways (but not all), neutron stars are similar to giant atomic nuclei the size of a city.
- Their enormous densities imply strong gravitational fields and the possibility of significant general relativistic deviations from Newtonian gravity (to be discussed in Astro 616).

Box 11.2 Electron Capture and Neutronization

The formation of a neutron star results from a process called electron capture (a form of beta decay), which can follow the core collapse of a massive star late in its life to produce a supernova (see further Ch. 14).

- The process is also called **neutronization**, because its effect is to destroy protons and electrons and create neutrons. The basic reaction is



- It is slow under normal conditions (because it is mediated by the weak interaction), but very fast in the high density and temperature environment produced by core collapse in a massive star.
- In the supernova explosion the enormous amount of energy released gravitationally in the collapse of the core blows off the outer layers of the star and leaves behind an extremely dense, hot remnant.
- As the neutronization reaction proceeds, the neutrinos escape carrying off energy and leave behind the neutrons.
- Because neutrons carry no charge, there is no electrical repulsion as in normal matter and the core can collapse to very high density once it has become mostly neutrons.
- The structure of actual neutron stars is more complex than this, and they are not composed entirely of neutrons, but this simple picture captures the basic idea.

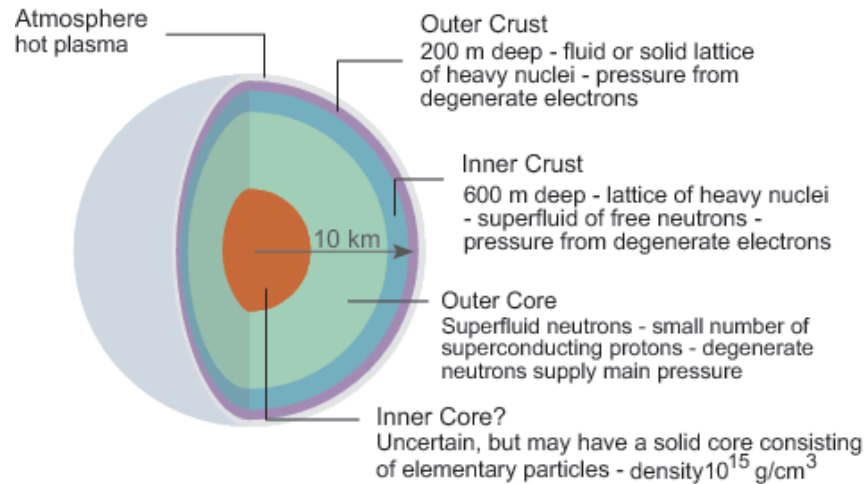
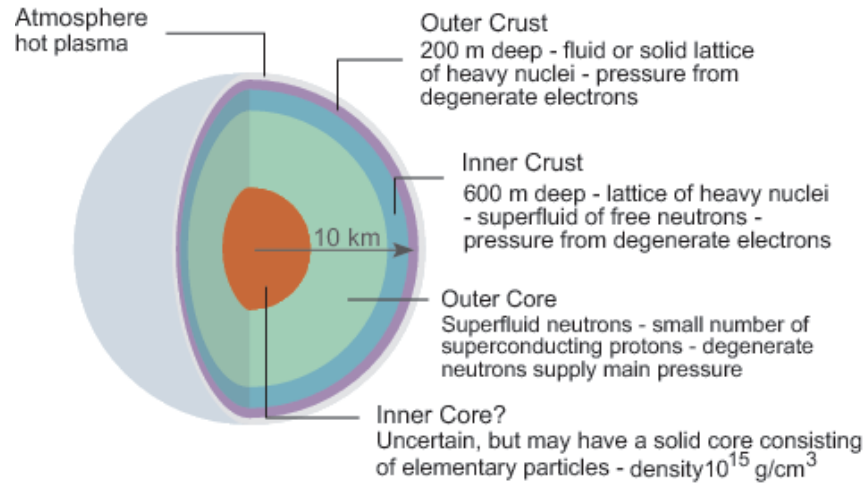


Figure 11.7: Internal structure of a typical neutron star.

Internally, we believe that a neutron star can be divided into the following general regions (see Fig. 11.7).

- The atmosphere is thin ($\sim 1 \text{ cm}$ thick) and consists of very hot, ionized gas.
- The outer crust is only about 200 meters thick and consists of a solid lattice or a dense liquid of heavy nuclei. The dominant pressure in this region is from electron degeneracy. The density is not high enough here to favor neutronization.
- The inner crust is from $\frac{1}{2}$ to 1 kilometer thick. The pressure is higher and the lattice of heavy nuclei is now permeated by free superfluid neutrons that begin to “drip” out of the nuclei. The pressure is still mostly from degenerate electrons.



- The outer core is composed primarily of superfluid neutrons and the neutrons supply most of the pressure through neutron degeneracy, though there are some free superconducting protons. This region is what gives the neutron star its name.
- The structure of the inner core is less certain than that of the outer portion of the star because we are less certain about how matter behaves under the intense pressure at the center (that is, the **equation of state** for matter under these conditions is not well understood).
- It might even consist of a solid core of particles more elementary than nucleons (pions, hyperons, quarks, ...).

Although much of a neutron star consists of closely packed neutrons and thus has some resemblance to a giant atomic nucleus, it is important to remember that it is gravity, not the nuclear force, that holds a neutron star together (see the following box).

Box 11.3 Neutron Stars Are Bound by Gravity

In some ways a neutron star is like 20-km diameter atomic nucleus, but there is one important difference:

A neutron star is bound by **gravity**, and the strength of that binding is such that the density of neutron stars is even greater than that of nuclear matter.

- How can the weakest force (gravity) produce an object more dense than atomic nuclei, which are held together by a diluted form of the strongest force?
- The answer: **range and sign of the forces involved**.
 - Gravity is weak, but long-ranged and attractive.
 - The strong nuclear force is short-ranged, acting only between nucleons that are near neighbors.
 - The normally attractive nuclear force becomes repulsive at very short distances. (A neutron star would **explode** if gravity were removed.)
- This is a kind of Tortoise and Hare fable:
 - Gravity is weak, but relentless and always attractive.
 - Thus, over large enough distances and long enough time, gravity—the plodding Tortoise of forces—always wins.

That is why the material in a neutron star can be compressed to such high density by the most feeble of the known forces.

11.9 Pulsars

In 1967 something remarkable was discovered in the sky: a star that appeared to be pulsing on and off with a period of about a second. Shortly, other such “pulsars” of even faster variation were discovered and the fastest now known (the **millisecond pulsars**) pulse on and off at nearly a thousand times a second.

Pulsars exhibit several common characteristics:

1. They have well-defined periods that challenge the accuracy of the best atomic clocks.
2. The measured periods range from about 4.3 seconds down to 1.6 milliseconds.

1.6 ms corresponds to approximately 640 revolutions per second, implying a 20-km wide object spinning as fast as a kitchen blender.

3. The period of a pulsar decreases slowly with time. The typical rate of decrease is a few billionths of a second each day, which implies that the frequency of pulsation will drop to zero after about 10 million years for typical pulsars.

What could cause this rather remarkable behavior? We shall now argue that only a neutron star can do the job.

11.9.1 The Pulsar Mechanism

The observational details for pulsars are inconsistent with an actual pulsation on that timescale for realistic objects but a rotating star could **appear to pulse** if it had some way to emit light in a beam that rotated with the source (just as a lighthouse appears to pulse as the beam sweeps over an observer).

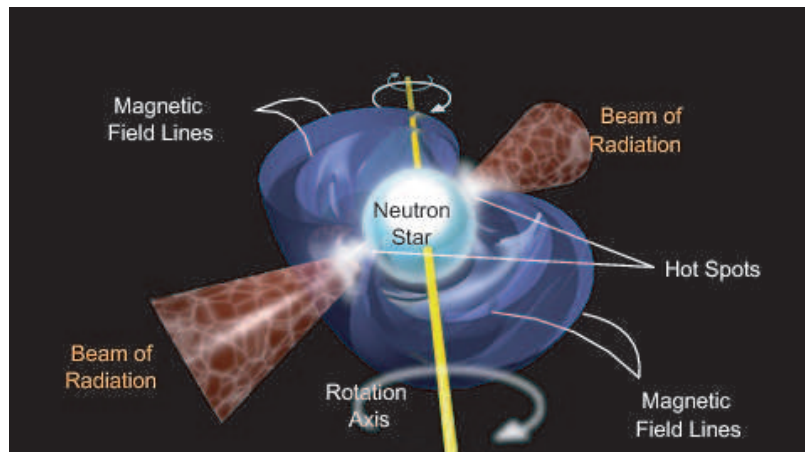
- What kind of object would be consistent with observed pulsar periods?
- Simple calculations show that only a very dense object could rotate fast enough and not fly apart because of the forces associated with the rapid rotation.
- A white dwarf is not dense enough. The minimum rotational period for a typical white dwarf would be several seconds; for shorter periods it would fly apart.
- But a neutron star is so dense that it could rotate more than a thousand times a second and still hold together.

This qualitative inference, augmented by much more detailed considerations, leads to the conclusion that the only plausible explanation for pulsars is that they are rapidly spinning neutron stars, with a mechanism to beam radiation in a kind of lighthouse effect.

Box 11.4 The Lighthouse Mechanism

A magnetic field varying in time produces an electrical field.

- Thus, the rapidly spinning magnetic field of the pulsar generates a very strong electrical field around the neutron star.
- This field accelerates electrons away from the surface at “hot spots” near the magnetic poles and these accelerated electrons produce radiation by the synchrotron effect.
- The synchrotron radiation is beamed strongly in the direction of electron motion.
- These beams rotate with the star, but the magnetic axis does not generally coincide with the rotation axis (recall Earth), so the beams rotate in a kind of corkscrew fashion:



- If these gyrating beams sweep over the Earth, they act similar to a lighthouse and we observe flashes of light.

Thus, the neutron star appears to be pulsing, even though it is neither pulsing nor is it really a star.

Table 11.2: Some typical magnetic field strengths

Object	Strength (gauss)
Earth's magnetic field	0.6
Simple bar magnet	100
Strongest sustained laboratory fields	4×10^5
Strongest pulsed laboratory fields	10^7
Maximum field for ordinary stars	10^6
Typical field for radio pulsar	10^{12}
Magnetars	10^{14} – 10^{15}

11.9.2 Magnetic Fields

Some pulsars contain the strongest magnetic fields known in our galaxy, and many of their basic properties are thought to derive from these fields.

- Some typical magnetic field strengths for various objects are listed in Table 11.2 (1 tesla = 10^4 gauss).
- From the table, the two classes of objects with the largest known magnetic fields are seen to be
 1. radio pulsars and
 2. magnetars (magnetars will be discussed below),
 both of which involve rotating neutron stars.

We infer that very strong magnetic fields are likely to be very common for neutron stars in general, although deducing that is more difficult if the neutron star is not observed as a pulsar or magnetar.

11.9.3 The Crab Pulsar

The first pulsar was found by Jocelyn Bell and Anthony Hewish at the Cambridge radio astronomy observatory in 1967. The most famous pulsar was discovered shortly after that.

- It lies in the Crab Nebula (M1), which is about 7000 light years away in the constellation Taurus.
- The Crab Pulsar rotates about 30 times a second, emitting a double pulse in each rotation in the radio through gamma-ray spectrum.
- In visible light, the Crab Pulsar appears to be a magnitude 16 star near the center of the nebula, but stroboscopic techniques reveal it to be pulsing.

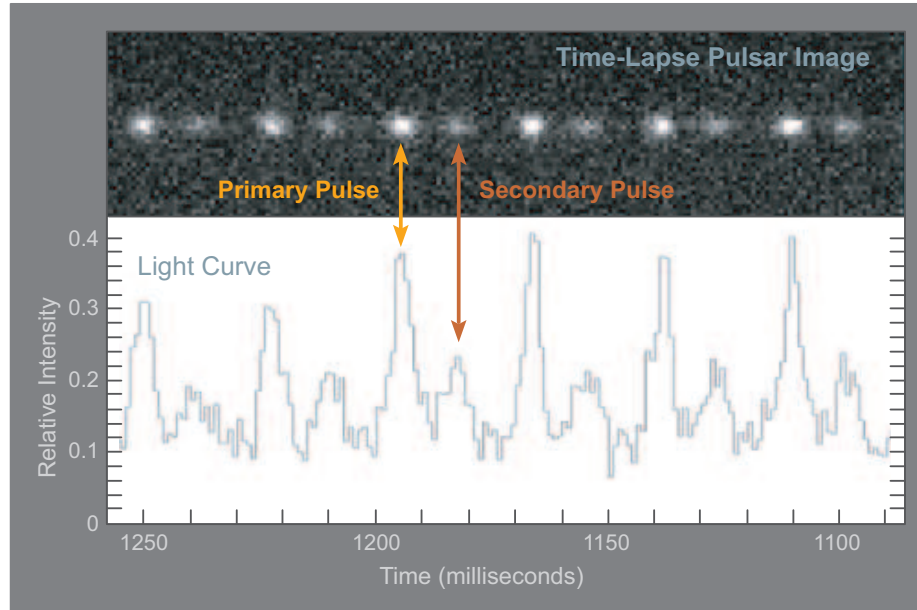
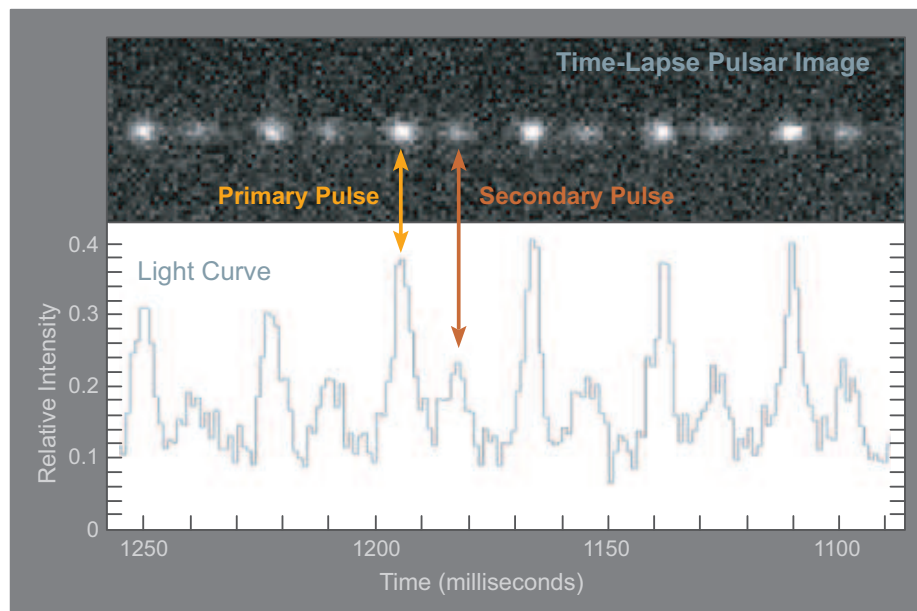


Figure 11.8: Light pulses from the Crab Pulsar. In this composite of European Southern Observatory data, the pulsar is shown in a time lapse image at the top and the light curve is displayed at the bottom on the same timescale.

Figure 11.8 shows the Crab Pulsar in action.

- The sequence is a composite of images taken through 3 different filters, all in the visible spectrum.
- Both the image sequence and the light curve show clearly the “double pulsing” of the Crab: in each cycle there is a strong primary pulse followed by a much weaker secondary pulse.
- The period (time between successive primary or secondary pulses) implies one primary and one secondary pulse about 30 times every second.



- This double pulsing effect can be explained by the lighthouse model if the geometry is such that the beam from one magnetic pole sweeps more directly over the Earth but the beam from the other pole does so only partially.

Although the Crab Pulsar emits visible light (and X-rays and gamma rays), most pulsars are detectable only by their radio frequency radiation. However, a few pulse strongly in other wavelength bands.

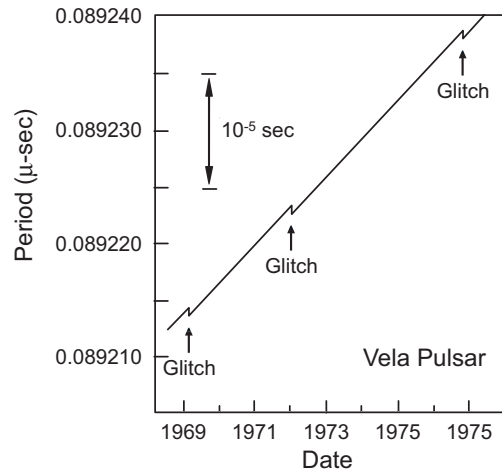


Figure 11.9: Glitches in the Vela Pulsar.

11.10 Pulsar Spindown and Glitches

In some pulsars “glitches” are observed where the spin rate suddenly jumps to a higher value (Fig. 11.9).

- The fractional change in period caused by a glitch is typically from 10^{-6} to 10^{-9} of the original period.
- Glitches indicate some internal rearrangement has altered the rotation rate by a small amount.
 - Proposal: “starquakes” in the dense crust cause the neutron star to contract slightly and thus to spin faster (angular momentum conservation).
 - Another: angular momentum stored in circulation of an internal superfluid liquid is suddenly transferred to the crust, altering the rotation rate.

11.11 Millisecond Pulsars

As a pulsar radiates away its energy, its spin rate decreases slowly. This change is small but can be measured very precisely.

- The rate of change in the rotational period for a radio pulsar is important because it can be used to estimate the strength of the magnetic field associated with the neutron star.
- Since pulsars are slowing down with time as they emit energy both in electromagnetic and gravitational waves, we may expect that the fastest pulsars are the youngest.
- For example, the Crab Pulsar is young (less than 1000 years), and pulses 30 times a second.
- However, this reasoning breaks down for the pulsars with millisecond periods.
- For many of these fast pulsars there is evidence that they are old, not young as we would expect for the fastest spin rates.
- This evidence consists primarily of the rate at which the pulsar spin is slowing, and where the millisecond pulsars are found.

- For example, the first millisecond pulsar discovered, PSR 1937 + 21, is very fast but it is spinning down very slowly.

This is an example of the standard pulsar naming system where the designation PSR indicates a pulsar, the first part of the number gives the approximate right ascension in hours and minutes, and the second part of the number gives the declination (with a plus or negative sign) in degrees.

- This slow spindown rate implies that it has a weak magnetic field and is old. (Older pulsars should have weaker fields and these should be less effective than younger, stronger fields in braking their motion.)
- Also, many of the millisecond pulsars that have been discovered are found in globular clusters, which contain an old population of stars.
- Therefore, they are not likely to be sites of recent supernova explosions that could have produced young pulsars since core collapse supernovae occur in very short-lived, massive stars.

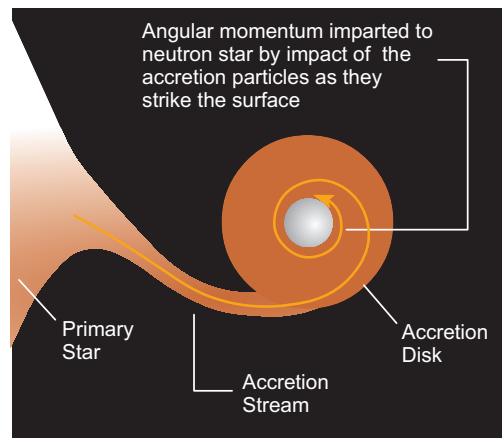


Figure 11.10: The spin-up mechanism for producing millisecond pulsars.

- The most plausible way of explaining the contradiction that the fastest pulsars seem very old is that **millisecond pulsars have been “spun up” since birth.**
- The proposed mechanism involves mass transfer in binary systems that adds angular momentum to the neutron star (Fig. 11.10).
- This accretion mechanism (**binary spinup**) transfers angular momentum from the orbital motion of the binary to rotation of the neutron star.
- Later, after the neutron star has been spun up to high rotational velocity, the primary star may become a supernova and disrupt the binary system.
- This leaves the rapidly spinning but old neutron star as a millisecond pulsar that defies the systematics expected from the evolution of isolated neutron stars.

11.12 Magnetars

Neutron stars have extremely strong magnetic fields. However, a new class of spinning neutron stars with abnormally large magnetic fields, even for a neutron star, have been discovered.

- These have been called **magnetars**.
- The magnetar SGR 1900+14 is estimated to have a magnetic field so strong (at least of order 10^{15} gauss) that if a magnet of comparable strength were placed halfway to the Moon, it could pull a metal pen out of your pocket on Earth!

SGR (soft gamma-ray repeater) indicates a magnetar. Like pulsars, the 1st part of the number gives the right ascension in hours and minutes, and the 2nd part of the number gives the declination (\pm) in degrees.

- In these rotating neutron stars it thought that the huge magnetic fields act as a kind of brake, slowing the rotation of the star.
- This slowing of the rotation disturbs the interior structure of the neutron star and “starquakes” in the star or magnetic field re-connection events release energy into the surrounding gases that cause emission of bursts of gamma rays.

Observationally, these are called **soft gamma ray repeaters (SGR)**:

- “Soft” means that the gamma rays are of low energy (in fact, they lie more in the X-ray portion of the spectrum);

- “Repeater” means that the bursts of gamma rays can repeat, unlike ordinary gamma ray bursts, which have not been observed to repeat.