Smallest Gaps Between Eigenvalues of Real Gaussian Matrices

Patrick Lopatto
Brown University

with Matthew Meeker

Slides: `https://lopat.to/bms.pdf`

April 10, 2024

Goals for This Talk

1. Explain a new result about the smallest eigenvalue gaps of (non-symmetric) real Gaussian matrices.
2. Advertise intriguing open problems about extremal eigenvalue gaps.

These are fundamental questions about the spectrum.

Despite significant advances in RMT, our understanding is very incomplete.

Question (Meta-Problem)

For a given random matrix ensemble, what is the distribution of the smallest gap between eigenvalues? The largest gap?

We will see more precise versions soon.

Can also ask about extreme values of characteristic polynomial (Fyodorov–Hiary–Keating conjecture), eigenvector entries, and so on.

Motivation: Analytic Number Theory

Gaps between zeros of $\zeta(s)$ resemble gaps between zeros of random matrix eigenvalues, even at the level of extremes. Denote zeros by

$$1/2 + \mathrm{i}\gamma_k, \qquad 0 < \gamma_1 < \gamma_2 \leq \dots .$$

Then

$$\tilde{\gamma}_i = \frac{\gamma_{i+1} - \gamma_i}{2\pi} \log\left(\frac{\gamma_i}{2\pi}\right)$$

is 1 on average.

RMT prediction supported by numerics:

$$\sup_{m \leq k \leq m+n} \tilde{\gamma}_k \approx \frac{\sqrt{32 \log n}}{2\pi}.$$

(Ben Arous–Bourgade, 2013)

Motivation: Numerical Analysis

Extremal gap questions appear in the analysis of numerical algorithms.

Example: Toda flow algorithm for diagonalizing a symmetric/Hermitian matrix. (Deift et al. 1983)

To understand its asymptotic complexity, one must understand the minimal eigenvalue gap of GOE/GUE, since this controls the speed of convergence.

(Ben Arous–Bourgade, 2013)

Definition

Gaussian Orthogonal Ensemble (GOE): An $n \times n$ symmetric matrix whose upper-triangular entries $\{w_{ij}\}_{j \geq i}$ are independent, centered Gaussians with

$$\operatorname{Var} w_{ij} = n^{-1} \text{ for } i \neq j, \qquad \operatorname{Var} w_{ii} = 2n^{-1}.$$

Definition

Gaussian Unitary Ensemble (GUE): Analogous, using complex Gaussians.

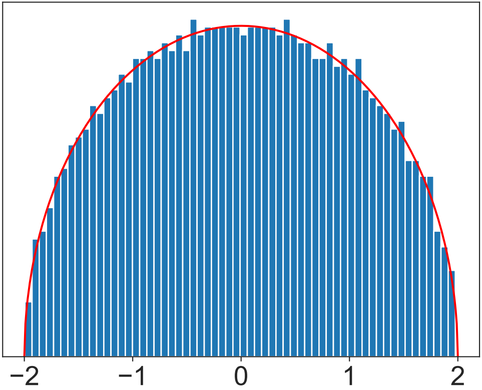
Note: This scaling places the asymptotic spectrum on $[-2, 2]$.

Semicircle Law

Theorem (Wigner '58)

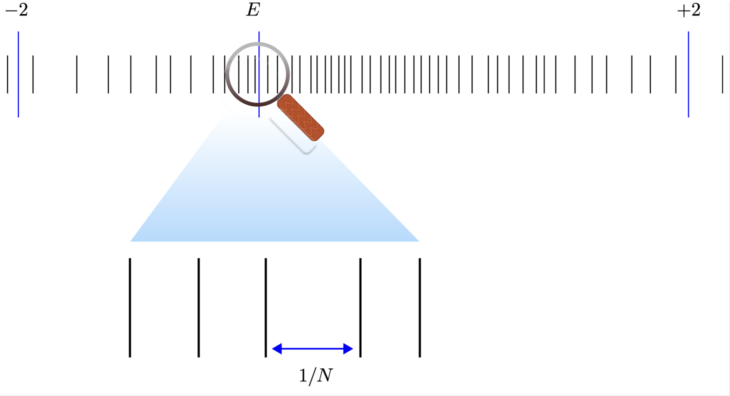
For $|E| \leq 2$, almost surely

$$\frac{1}{N}\left|\{i : \lambda_i \leq E\}\right| \to \frac{1}{2\pi}\int_{-2}^{E}\sqrt{4-x^2}\,dx.$$



Local Statistics

In the spectral bulk, the typical inter-particle distance is order n^{-1}.



History

Vinson (2001) determined the asymptotic distribution of the smallest GUE eigenvalue gap

$$\tau_1^{\mathrm{GUE}} = \min_{1 \le i \le n-1} \lambda_{i+1} - \lambda_i.$$

He showed that $n^{4/3}\tau_1^{\mathrm{GUE}}$ converges in law to a distribution with density

$$p(x) = 3x^2 e^{-x^3}.$$

Note τ_1^{GUE} is much smaller than the $O(n^{-1})$ typical spacing.

He also studied the circular unitary ensemble (CUE) and certain unitarily-invariant β-ensembles.

Ben Arous and Bourgade (2013) determined the k-th smallest and largest gaps of GUE (and CUE) for finite k.

The asymptotic density of $n^{4/3}\tau_k^{\mathrm{GUE}}$ is

$$p_k(x) = \frac{3}{(k-1)!} x^{3k-1} e^{-x^3}.$$

Defining the "largest gap" requires some care. Edge spacings are $O(n^{-2/3})$, so we restrict to the spectral bulk.

Fix $I = [-a, a] \subset (-2, 2)$ and define

$$\mathcal{T}^{\mathrm{GUE}} = \max_{i:\lambda_i \in I} \lambda_{i+1} - \lambda_i.$$

Then in any L^p norm,

$$\sqrt{4 - a^2} \cdot \frac{n\mathcal{T}^{\mathrm{GUE}}}{\sqrt{32 \log n}} \to 1.$$

CUE results are very similar (and suggest the zeta function prediction).

Higher-order asymptotics for largest gaps derived in (Feng–Wei, 2018). Gumbel distribution.

Heuristic for Largest Gap

Wigner Surmise: Based on exact 2×2 computation, we suspect that $n(\lambda_{i+1} - \lambda_i)$ should have density like

$$p(s) = \frac{32s^2}{\pi^2} e^{-4s^2/\pi}.$$

This has a Gaussian tail.

The maximum of k independent Gaussians with s.d. σ is order $\sigma\sqrt{\log k}$.

Here $k = O(n)$ and $\sigma = n^{-1}$, so we get $\sqrt{\log n}/n$.

For Poisson heuristic, see (Bourgade, 2022).

More History

Ben Arous and Bourgade relied on representation of GUE eigenvalues as a determinantal point process.

The GOE is Pfaffian, so new ideas are required.

Smallest gaps distribution for GOE identified in (Feng–Tian–Wei, 2019). τ_1^{GOE} has order $n^{-3/2}$.

(View as $n^{-(\beta+2)/(\beta+1)}$; see [Feng–Wei, 2020] for circular β case and integer β.)

Problem

Determine the distribution of the largest gap of GOE (in the bulk).

Non-Hermitian Models

Now we consider models with a 2-dimensional spectrum.

Definition (GinOE)

An $n \times n$ real matrix with entries g_{ij} such that

- g_{ij} are independent real Gaussian random variables.
- $\mathbb{E}[g_{ij}] = 0$, $\mathbb{E}[g_{ij}^2] = n^{-1}$.

Define GinUE similarly using complex Gaussians.

As $n \to \infty$, the spectrum of GinOE/GinUE becomes uniformly distributed on the unit disk in the complex plane.

Previous Result

Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of GinUE, and let $t_k^{(n)}$ be the k-th smallest element of $\{|\lambda_i - \lambda_j| : i < j\}$.

Set $\tau_k^{(n)} = (\pi/4)^{1/4} t_k^{(n)}$.

Theorem (Shi–Jiang, 2012)

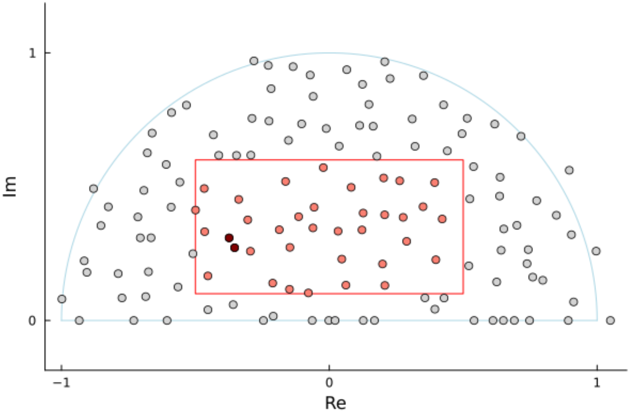
For all $k \in \mathbb{N}$, $n^{3/4}\tau_k^{(n)}$ converges in distribution to a limit with density

$$p(x) \propto x^{4k-1} e^{-x^4}.$$

This is smaller than the typical bulk spacing of $n^{-1/2}$ (and larger than independent point heuristic of n^{-1}).

Shi–Jiang rely on determinantal structure. What about the GinOE, which is Pfaffian?

GinOE Smallest Gaps



Eigenvalues come in conjugate pairs, with $O(\sqrt{n})$ on the real line.

Set

$$\mathbb{D}^+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\},$$

We restrict to domains at a positive distance from the boundary of $\mathbb{D}^+$.

Definition

A domain Ω is called *admissible* if $\overline{\Omega} \subset \mathbb{D}^+$.

Let

$$t_1^{(n)} = \min\{|\lambda_i - \lambda_j| : \lambda_i \in \Omega \wedge \lambda_j \in \Omega \wedge i < j\}$$

and define the rescaled gaps

$$\omega_\ell^{(n)} = n^{3/4} \left(\frac{\pi}{4|\Omega|}\right)^{1/4} t_\ell^{(n)}.$$

We start with the following result, which is a corollary of our main theorem.

Corollary (L.–Meeker)

For all $k \in \mathbb{N}$, $n^{3/4}\omega_k^{(n)}$ converges in distribution to a limit with density

$$p(x) \propto x^{4k-1} e^{-x^4}.$$

This is the same as the GinUE case — which is not surprising.

We also obtain the joint distribution of the smallest k gaps (as do Shi–Jiang).

Now we state our main theorem on Poisson convergence. Shi–Jiang also have a version, which requires some care in interpreting.

Definition

For $z_1, z_2 \in \mathbb{C}$, we say that $z_1 \prec z_2$ if $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$, or if $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$ and $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$.

Definition

Let Ω denote an admissible domain and set $\mathbb{R}^+ = [0, \infty)$. We define a point process $\chi_\Omega^{(n)}$ on $\mathbb{R}^+$ as follows. Define

$$i^* = \arg\min_{j \neq i} \left\{ |\lambda_j - \lambda_i| : \lambda_i \prec \lambda_j, \lambda_j \in \Omega \right\}$$

if the set of indices j such that $j \neq i$ and $\lambda_j \in \Omega$ is nonempty. Otherwise, we set $i^* = 0$. Then set

$$\chi_\Omega^{(n)} = \sum_{i : \lambda_i \in \Omega} \delta_{n^{3/4}|\lambda_{i^*} - \lambda_i|} \mathbb{1}_{\{i^* \neq 0\}}.$$

Theorem

Let Ω be an admissible domain. As $n \to \infty$, the processes $\chi_\Omega^{(n)}$ converge weakly to a Poisson point process χ_Ω on $\mathbb{R}^+$ with intensity

$$\mathbb{E}\big[\chi_\Omega(A)\big] = \frac{|\Omega|}{\pi} \int_A r^3 \, dr$$

for any bounded Borel set $A \subset \mathbb{R}^+$.

The joint distribution for the smallest k gaps then follows by routine manipulations.

The limiting intensity measure is identical to the one for the smallest gaps process of GinUE.

Proof Overview

There is well-developed theory for studying the smallest gaps of determinantal point processes (Soshnikov, 2005).

But the eigenvalues of the GinOE are not a determinantal point process.

Key Idea: In the spectral bulk, and away from the real line, the correlation functions are *approximately* determinantal. This allows us to adapt the Soshnikov strategy (with some extra technical work).

Correlation Functions

Let $\mathbb{C}^* = \mathbb{C} \backslash \mathbb{R}$ and $\mathcal{I}_k \subset \{1, \ldots, n\}^n$ be the set of pairwise distinct k-tuples of indices.

For $k, n \in \mathbb{N}$, there exists a function $\rho_k^{(n)} \colon (\mathbb{C}^*)^k \to \mathbb{R}^+$ such that

$$\int_{(\mathbb{C}^*)^k} f(z_1, \ldots, z_k) \rho_k^{(n)}(z_1, \ldots, z_k)\, dz_1 \ldots dz_k$$

$$= \mathbb{E}\left[\sum_{(i_1, \ldots, i_k) \in \mathcal{I}_k} f(w_{i_1}, \ldots, w_{i_k}) \right]$$

for every compactly supported, bounded $f \colon (\mathbb{C}^*)^k \to \mathbb{R}$.

Pfaffian

Definition

The Pfaffian of a $2n \times 2n$ skew-symmetric matrix $M = (M_{ij})_{i,j=1}^{2n}$ is defined by

$$\mathrm{Pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} M_{\sigma(2i-1),\sigma(2i)},$$

where S_{2n} is the symmetric group of degree $2n$.

We have $\mathrm{Pf}(M)^2 = \det(M)$.

Identification of correlation functions due to Forrester–Nagao, 2008; then Borodin–Sinclair, 2009, and others.

Lemma

The k-point complex–complex correlation functions of the n-dimensional real Ginibre ensemble G_n are given by

$$\rho_k(z_1, \ldots, z_k) = \mathrm{Pf}(K(z_i, z_j))_{1\leq i,j\leq k},$$

where $(K(z_i, z_j))_{1\leq i,j\leq k}$ is a $2k \times 2k$ matrix composed of the 2×2 blocks

$$K(z_i, z_j) = \begin{pmatrix} D_n\,(z_i, z_j) & S_n\,(z_i, z_j) \\ -S_n\,(z_j, z_i) & I_n\,(z_i, z_j) \end{pmatrix},$$

and D_n, I_n, and S_n are defined on the next slide.

$$S_n(z,w) = \frac{\mathrm{i}e^{-(1/2)(z-\bar{w})^2}}{\sqrt{2\pi}}(\bar{w}-z)G(z,w)s_n(z\bar{w}),$$

$$D_n(z,w) = \frac{e^{-(1/2)(z-w)^2}}{\sqrt{2\pi}}(w-z)G(z,w)s_n(zw),$$

$$I_n(z,w) = \frac{e^{-(1/2)(\bar{z}-\bar{w})^2}}{\sqrt{2\pi}}(\bar{z}-\bar{w})G(z,w)s_n(\bar{z}\bar{w}),$$

where $z, w \in \mathbb{C}^*$ and

$$G(z,w) = \sqrt{\operatorname{erfc}(\sqrt{2}\,\operatorname{Im}(z))\operatorname{erfc}(\sqrt{2}\,\operatorname{Im}(w))},$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt, \qquad s_n(z) = e^{-z}\sum_{j=0}^{n-1}\frac{z^j}{j!}.$$

Lemma

Let Ω be an admissible domain. Then

$$\sup_{z,w\in\Omega} \left|D_n(\sqrt{n}z, \sqrt{n}w)\right| \leq Ce^{-cn}, \qquad \sup_{z,w\in\Omega} \left|I_n(\sqrt{n}z, \sqrt{n}w)\right| \leq Ce^{-cn},$$
$$\sup_{z,w\in\Omega} \left|S_n(\sqrt{n}z, \sqrt{n}w)\right| \leq C.$$

From (Goel–L.–Xie, 2023); basic idea first noticed by (Kopel, 2015).

Can reduce “checkerboard Pfaffians” with blocks like

$$\begin{bmatrix} 0 & X \\ -X & 0 \end{bmatrix}$$

to determinants.

Lemma

Let $M = (M_{ij})_{i,j=1}^{2n}$ be a skew-symmetric $2n \times 2n$ matrix such that $M_{ij} = 0$ for every pair of indices (i,j) such that $i \equiv j \bmod 2$. Let $\tilde{M} = (\tilde{M})_{i,j=1}^{n}$ be the $n \times n$ matrix defined by $\tilde{M}_{ij} = M_{2i-1,2j}$. Then $\mathrm{Pf}(M) = \det(\tilde{M})$.

Use this lemma and the previous results to see that the GinOE is “approximately determinantal” on admissible domains, with kernel S_n.

Modified Point Process

Given a bounded Borel set $A \subset \mathbb{R}^+$, we define the corresponding set

$$B = \{z \in \mathbb{C} : |z| \in A, 0 \prec z\}.$$

We set

$$B_n = \{n^{-3/4} b : b \in B\},$$

and define

$$\xi^{(n)} = \sum_{i:\lambda_i \in \Omega} \delta_{\lambda_i}, \qquad \tilde{\xi}^{(n)} = \sum_{i:\lambda_i \in \Omega} \delta_{\lambda_i} \mathbb{1}_{\{\xi^{(n)}(\lambda_i + B_n) = 1\}},$$

where the set $\lambda_i + B_n$ has the usual definition as

$$\lambda_i + B_n = \{\lambda_i + z : z \in B_n\}.$$

It suffices to study $\tilde{\xi}^{(n)}$ in order to prove our main result.

Lemma

Let Ω be an admissible domain. For all bounded Borel sets $A \subset \mathbb{R}^+$, we have

$$\lim_{n\to\infty} \chi^{(n)}(A) - \tilde{\xi}^{(n)}(\Omega) = 0,$$

where the convergence is in distribution.

Heuristic: no three eigenvalues bunch together on the scale of the smallest gap.

Now we just need to show that $\tilde{\xi}^{(n)}(\Omega)$ is Poisson.

Conclusion

To show $\tilde{\xi}^{(n)}(\Omega)$ is Poisson, it suffices to compute its moments.

This is “just” a computation, since the correlation functions of $\tilde{\xi}^{(n)}(\Omega)$ have an explicit representation in terms of those for $\tilde{\xi}$ via an inclusion exclusion formula:

$$\tilde{\tau}_k(z_1, \dots, z_k) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{z_1+B_n} dx_1 \dots \int_{z_k+B_n} dx_k$$

$$\int_{((z_1+B_n)\cup\ldots\cup(z_k+B_n))^m} \tau_{2k+m}(z_1, x_1, \dots, z_k, x_k, y_1, \dots, y_m)\, dy_1 \dots dy_m.$$

A Similar Argument

For a domain Ω, define $f_\Omega : \mathbb{C} \to \mathbb{R}$ by $f_\Omega(z) = \mathbb{1}_\Omega(z)$. For the GinOE, define the (N-dependent) random variable

$$X_\Omega = \sum_{i=1}^{N} f_\Omega(\lambda_i) - \mathbb{E}\left[\sum_{i=1}^{N} f_\Omega(\lambda_i)\right].$$

Theorem (Goel–L.–Xie)

Let Ω be an admissible domain with Lipschitz boundary. Then we have the weak convergence

$$\lim_{n\to\infty} \frac{X_\Omega}{n^{1/4}} = \mathcal{N}\left(0, \frac{\ell(\partial\Omega)}{2\pi^{3/2}}\right).$$

Here $\ell(\partial\Omega)$ is the length of the boundary. Generalizes to fractal boundaries.

Variance of $O(n^{1/4})$ smaller than for sums of IID random variables.

Comments

The result for GinUE is classical; see the survey (Byun–Forrester, 2022).

Smooth test functions were done in (Kopel, 2005).

For determinantal point processes, one can use the method of Costin–Lebowitz (1995), based on a recursive cumulant expansion written in terms of the determinantal kernel. One shows that the higher cumulants vanish, which implies Gaussianity.

Comments

This reduction to an “approximately determinantal” process in the bulk of the GinOE is fairly generic; potentially other problems could be dealt with this way. (I welcome suggestions.)

One expects the behavior at the edge of the disk to be negligible for each of these two results. In principle, asymptotics are available to deal with these cases, though we did not work out the details.

Treating domains that cross the real axis seems harder.

Some Questions

Problem

Characterize the largest gap of GinOE/GinUE in a bulk domain.

Problem

What can be said about extremal gaps of other ensembles?

In the non-Hermitian case, high-probability lower bounds are available for i.i.d. matrices, but no other distributional results.

For symmetric/Hermitian ensembles, a bit more is known.

Universality

Definition (Wigner Matrix, Symmetric)

An $N \times N$ real matrix with entries H_{ij}, $1 \leq i, j \leq N$ such that:

- *H_{ij} are mutually independent for $i \leq j$,*
- *$H_{ij} = H_{ji}$ for $i \neq j$*
- *$\mathbb{E}[H_{ij}] = 0$ and $\mathbb{E}[H_{ij}^2] = N^{-1}$ for all i, j, all moments finite.*

Hermitian definition analogous.

Theorem (Landon–L.–Marcinek, Informal)

The largest gap of any Hermitian Wigner matrix (in the spectral bulk) is asymptotically the same as that of the GUE.

Automatically works for real symmetric Wigner once the GOE distribution is known.

Universality of the Smallest Gap?

Simultaneous work of Bourgade (2021) showed universality of the largest and smallest gaps for both symmetry classes under the assumption of smooth entry distributions.

Problem

Determine the smallest gap distribution for a real Bernoulli matrix.

Thank you!

Fyodorov–Hiary–Keating Conjecture

Conjecture

Let U_N be a $N \times N$ unitary matrix. The random variable X_N determined by

$$\max_{|z|=1} \log|\det(z - U_N)| = \log N - \frac{3}{4}\log\log N + X_N$$

converges as $N \to \infty$ to a limit X_∞ in distribution. This limit is the sum of two independent Gumbel random variables.

From physical reasoning. Based on analogy with the *circular logarithmic* model (N equidistant samples from 2-dim. GFF on circle), whose extrema are computed using the "freezing scenario."

The $3/4$ indicates long-range correlations (relative to $1/4$ for independent variables).

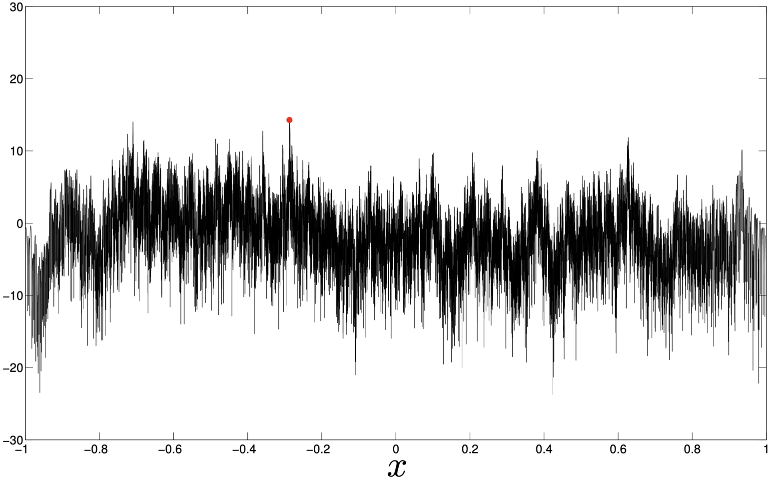


FIGURE 1.2. A plot of a single realization of $2\log\left(|p_N(x)|e^{-\mathbb{E}\log|p_N(x)|}\right)$ with $N = 3000$. The maximum value is marked with a red circle.

From (Fyodorov–Simm, 2016).

FHK for Wigner Matrices

Set $\det(E) = \prod_{i=1}^{N}(E - \lambda_i)$ and $I = [-2+\kappa, 2-\kappa]$.

Conjecture

For a Hermitian Wigner matrix,

$$\max_{E\in I} \left(\log|\det(E)| - \mathbb{E}\left[\log|\det(E)| \right] \right) = \log N - \frac{3}{4}\log\log N + Z_N,$$

where

$$\lim_{N\to\infty} Z_N \overset{(\mathrm{d})}{=} Z_\infty$$

for a random variable Z_∞ satisfying the tail decay asymptotic $c\, y e^{-2y} \leq \mathbb{P}(Z_\infty > y) \leq c^{-1} y e^{-2y}$ as $y \to \infty$, for some fixed $c > 0$.

Restrict to bulk of spectrum by analogy with unitary case.
(For real symmetric matrices, divide by $\sqrt{2}$.)