(1) **(Analyzing nested loops)** (10 points) Prove a Θ-bound on the running time of the following procedure as a function of n. In your analysis, do *not* use an exact formula for $\sum_i i$ or $\sum_i i^2$.

function $F(n)$ **begin**
  array $A[1{:}n, 1{:}n]$
  for $i := 1$ to n **do**
    $A[i,i] := 0$ $\to \Theta(1)$ $\Big\} \sum_{1\le i\le n}\Theta(1) = \Theta\Big(\sum_{1\le i\le n} 1\Big) = \Theta(n)$
  for $\ell := 2$ to n **do**
    for $i := 1$ to $n-\ell+1$ **do begin**
      $j := i+\ell-1$
      $A[i,j] := \infty$
      for $k := i$ to $j-1$ **do**
        $A[i,j] := \min\{A[i,j],\ A[i,k] + A[k+1,j] + ijk\}$ $\to \Theta(1)$ $\Big\} \sum_{i\le k\le j-1}\Theta(1) \to \Theta\sum_{i\le k\le \underbrace{i+\ell-1}_{j}-1} 1 = \Theta(i+\ell-2-i+1)$
    end
  return $A[1,n]$
end

$= \Theta(\ell-1) = \Theta(\ell)$

function $F(n)$ **begin**
  array $A[1{:}n, 1{:}n]$
  for $i := 1$ to n **do**
    $A[i,i] := 0$
  for $\ell := 2$ to n **do**
    for $i := 1$ to $n-\ell+1$ **do begin**
      $j := i+\ell-1$
      $A[i,j] := \infty$
      for $k := i$ to $j-1$ **do**
        $A[i,j] := \min\{A[i,j],\ A[i,k] + A[k+1,j] + ijk\}$
    end
  return $A[1,n]$
end

$\Theta(\ell) \Big\} \sum_{1\le i\le n-\ell+1}\Theta(\ell) = \Theta\sum_{1\le i\le n-\ell+1}\ell = \Theta\ell(n-\ell+1-1+1)$

$= \Theta\ell(n-\ell+1) = \Theta\ell(n-\ell)$

function $F(n)$ **begin**
  array $A[1{:}n, 1{:}n]$
  for $i := 1$ to n **do**
    $A[i,i] := 0$
  for $\ell := 2$ to n **do**
    for $i := 1$ to $n-\ell+1$ **do begin**
      $j := i+\ell-1$
      $A[i,j] := \infty$
      for $k := i$ to $j-1$ **do**
        $A[i,j] := \min\{A[i,j],\ A[i,k] + A[k+1,j] + ijk\}$
    end
  return $A[1,n]$
end

$\Theta\ell(n-\ell) \Big\} \sum_{2\le\ell\le n}\Theta\ell(n-\ell) = \Theta\sum_{2\le\ell\le n}\ell(n-\ell)$

$$\Theta\Big(\sum_{2\le\ell\le n}\ell n - \ell^2\Big) = \Theta\Big(\sum_{2\le\ell\le n}\ell n - \sum_{2\le\ell\le n}\ell^2\Big) = \Theta\Big(n\sum_{2\le\ell\le n}\ell - \frac{n^3}{3} + \cancel{\Theta(n^2)}\Big)$$

$$= \Theta\Big(n\Big(\frac{n^2}{2} + \cancel{\Theta(n)}\Big) - \frac{n^3}{3}\Big) \to \Theta\Big(\frac{n^3}{2} - \frac{n^3}{3}\Big) = \Theta\Big(\frac{n^3}{6}\Big) = \boxed{\Theta(n^3)}$$