(4) **(Solving recurrences)** (20 points) Derive a Θ-bound on the solution to each of the following recurrences. Do not worry about taking floors or ceilings of arguments.

(a) $T(n) = 4T(n/2) + n^2\sqrt{n}$.

$T(n) = 4T(n/2) + \Theta(n^{5/2})$

$a=4 \quad b=2 \quad c=5/2 \quad d=0$

$\log_b a = \log_2 4 = 2 < c$

$\Theta(n^c \log^d n), \quad c > \log_b a$

The time complexity for $T(n)$ is $\Theta(n^{5/2})$

(b) $T(n) = 32T(n/4) + n^2\sqrt{n}$.

$T(n) = 32T(n/4) + \Theta(n^{5/2})$

$a=32 \quad b=4 \quad c=5/2 \quad d=0$

$\log_b a \Rightarrow \log_4 32 = 2.5 = c$

$\Theta(n^c \log^{d+1} n) \quad c = \log_b a$

$T(n)$ is $\Theta(n^{5/2} \log n)$

(c) $T(n) = 3T(n/2) + n \lg n$.

$T(n) = 3T(n/2) + \Theta(n \lg n)$

$a=3 \quad b=2 \quad c=1 \quad d=1$

$\log_b a \Rightarrow \log_2 3 = 1.585 > c$

$\Theta(n^{\log_b a}), \quad c < \log_b a$

$T(n)$ is $\Theta(n^{\log_2 3})$

(d) $T(n) = 3T(n/3) + n \lg n$.

$a=3 \quad b=3 \quad c=1 \quad d=1$

$\log_b a \Rightarrow \log_3 3 = 1 = c$

$\Theta(n^c \log^{d+1} n) \quad c = \log_b a$

$T(n)$ is $\Theta(n \log^2 n)$

(e) **(bonus)** (10 points) $T(n) = T(\sqrt{n}) + 1$.

We know $\sqrt{n}$ is $O(\sqrt{n}\log n)$ and is $\Omega(\log n)$

If we take the limit of our original function over the lower bound

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} \rightarrow \lim_{n\to\infty} \frac{\frac{d}{dn}\sqrt{n}}{\frac{d}{dn}\log n} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}} > 0$$

$\therefore \quad \sqrt{n} \in \Theta(\log n)$

Thus, we can approximate $T(\sqrt{n})$ as $\Theta(\log n) \rightarrow T(n) = \Theta(\log n) + 1$

$\therefore T(n) = \Theta(\log n)$