(3) **(Understanding asymptotics)** (25 points)

(a) (15 points) Order the following functions according to their rate of growth. Specifically, group the functions into equivalence classes such that functions f and g are in the same class iff $f \in \Theta(g)$, and then order the equivalence classes from slowest to fastest growing.

For each successive pair of functions (f, g) in your order, state the relationship between f and g, namely either $f \in \Theta(g)$ or $f \in o(g)$.

$n!$	$\lfloor \ln n \rfloor!$	n^2	$(\ln n)^2$	$\ln(n!)$
2^{2^n}	$\ln \ln n$	$n^{\ln \ln n}$	$\sqrt{\ln n}$	$2^{\ln n}$
$(\ln n)^{\ln n}$	4^n	n	2^n	$n \ln n$

(Hint: First form a rough initial order of the functions, and then use insertion sort on this list to obtain the final order. To determine the relationship between two functions, take a limit of their ratio, or use the list of asymptotic properties given in class.)

Putting the functions into equivalence classes

[c^n, exponential] = $\{n!, 2^{2^n}, 4^n, 2^n\}$

[Polynomial] = $\{n^2, n^{\ln \ln n}, (\ln n)^{\ln n}\}$

[ln multiplied by linear] = $\{\lfloor \ln n \rfloor!, \ln(n!), n \ln n\}$

[linear] = $\{n\}$

[ln] = $\{(\ln n)^2, \ln \ln n, \sqrt{\ln n}, 2^{\ln n}\}$

2 $n!$
1 2^{2^n}
3 4^n
4 2^n

$n! = 2^{\log n!} \rightarrow 2^{\Theta(n \log n)}$

$2^{2^n} = 2^{(\lg 2) 2^n} = 2^{\Theta(2^n)}$

$4^n = 2^{(\log 4) n} = 2^{\Theta(n)}$

$2^n < 4^n$

since $\Theta(n \log n) = o(2^n)$ so $n! < 2^{2^n}$

since $\Theta(n) = o(n \log n)$ so $4^n < n!$

$2^n > n^{\ln \ln n}$

3 n^2
1 $n^{\ln \ln n}$
2 $(\ln n)^{\ln n}$

$$\lim_{n \to \infty} \frac{n^2}{n^{\ln \ln n}} \rightarrow \lim_{n \to \infty} \frac{\frac{d}{dn} n^2}{\frac{d}{dn} n^{\ln \ln n}} \rightarrow \lim_{n \to \infty} \frac{2n}{n^{\ln \ln n} \cdot \frac{d}{dn}\left(\ln(n) \cdot \ln(\ln(n))\right)}$$

$$\rightarrow \lim_{n \to \infty} \frac{2n}{n^{\ln(\ln(n))} \cdot \left(\left(\frac{1}{n} \cdot \ln(\ln(n))\right) + \left(\cancel{\ln(n)} \cdot \frac{1}{\cancel{\ln(n)}} \cdot \frac{1}{n}\right)\right)}$$

$$\rightarrow \lim_{n \to \infty} \frac{2n}{n^{\ln(\ln(n))}\left(\frac{\ln(\ln(n))}{n} + \frac{1}{n}\right)} = 0 \rightarrow n^2 \in o(n^{\ln \ln n})$$

$n^2 > \lfloor \ln(n) \rfloor!$

$$\lim_{n \to \infty} \frac{\frac{d}{dn} n^2}{\frac{d}{dn} (\ln(n))^{\ln(n)}} = \frac{2n}{(\ln(n))^{\ln(n)} \cdot \left(\frac{1}{n}\right)} \rightarrow \lim_{n \to \infty} \frac{\frac{d}{dn}(2n)}{\frac{d}{dn}(\ln(n))^{\ln(n)}\left(\frac{1}{n}\right)} \rightarrow 0$$

$\left((\ln(n))^{\ln(n)}\right) \approx n^{\ln(\ln(n))}$

$(2^{\lg \ln(n)})^{\ln(n)} = 2^{\Theta(\log \ln n \cdot \ln n)}$

$(2^{\log n})^{\ln \ln(n)} = 2^{\Theta(\log n \cdot \ln \ln n)}$ $\approx$

$\rightarrow n^2 \in o((\ln(n))^{\ln(n)})$

$((\ln n)^{\ln n})$ grows faster than

1 $\lfloor \ln(n) \rfloor!$
2 $\ln(n!)$
3 $n\ln(n)$

$n\ln(n) > n > 2^{\ln(n)}$

2 $(\ln(n))^2$
4 $\ln(\ln(n))$
3 $\sqrt{\ln(n)}$
1 $2^{\ln(n)}$

$\lfloor \ln(n) \rfloor! = \Theta\left(\sqrt{\ln n}\,(\ln n / e)^{\ln n}\right) = \Theta\left(\sqrt{\ln n} \cdot \frac{(\ln n)^{\ln n}}{n}\right)$ grows faster than $n\ln n$ as seen above

$\left.\begin{array}{l} \ln(n!) = \Theta(n \log n) \\ n\ln(n) = \Theta(n \ln n) \end{array}\right\} \approx$ from worksheet

$$\lim_{n\to\infty} \frac{(\ln(n))^2}{\ln(\ln(n))} = \lim_{n\to\infty} \frac{\frac{d}{dn}(\ln(n))^2}{\frac{d}{dn}\ln(\ln(n))} = \lim_{n\to\infty} \frac{2(\ln(n))\cdot 1/n}{\frac{1}{\ln(n)}\cdot\frac{1}{n}}$$

$$\to \lim_{n\to\infty} \frac{\frac{\ln(n)}{n}}{\frac{1}{n\ln(n)}} \qquad \frac{1}{n\ln(n)} < \frac{\ln(n)}{n}$$

$$\lim_{n\to\infty} \frac{\ln(\ln(n))}{(\ln(n))^{1/2}} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln(\ln(n))}{\frac{d}{dn}(\ln(n))^{1/2}} = \lim_{n\to\infty} \frac{\frac{1}{n\ln(n)}}{\frac{1}{2\sqrt{\ln(n)}}\cdot\frac{1}{n}}$$

$$\to \lim_{n\to\infty} \frac{\frac{1}{n\ln(n)}}{\frac{1}{n\sqrt{\ln(n)}}} \qquad \frac{1}{n\ln(n)} < \frac{1}{n\sqrt{\ln(n)}}$$

$$(\ln(n))^2 > (\ln(n))^{1/2}$$

$$2^{\ln(n)} > (\ln(n))^2$$

ascending order $= \ln(\ln(n)),\ \sqrt{\ln(n)},\ \ln^2 n,\ 2^{\ln(n)},\ n,\ n\ln(n),\ \ln(n!),\ \lfloor \ln(n) \rfloor!,\ n^2,\ (\ln(n))^{\ln(n)},$
$n^{\ln(\ln(n))},\ 2^n,\ 4^n,\ n!,\ 2^{2^n}$

(b) (10 points) Using the definition of Θ, prove the following.

Theorem *For any two functions $f(n)$ and $g(n)$ that are asymptotically non-negative,*

$$f(n) + g(n) = \Theta\big(\max\{f(n), g(n)\}\big).$$

In order to prove $f(n)+g(n)$ we must prove $f(n)+g(n) = \Omega(\max(f(n), g(n)))$
and $f(n)+g(n) = O(\max(f(n), g(n)))$

By definition $f(n)+g(n) \geq f(n)$ and $f(n)+g(n) \geq g(n)$ and in order to get the tightest lower bound we will pick the larger of $f(n)$ and $g(n)$
$\to f(n)+g(n) = \Omega(\max(f(n), g(n)))$

We also know that if we add two functions together they will be smaller than the largest function multiplied by 2

$\to \max + \min \leq 2\cdot\max$

$f(n) = \max \to f(n)+g(n) = 2\cdot f(n)$
$g(n) = \max \to f(n)+g(n) = 2\cdot g(n)$

Therefore, $f(n)+g(n) = O(2\max(f(n), g(n))) = O(\max(f(n), g(n)))$ dropping constants

To conclude we have a lower bound of $-\ f(n)+g(n) = \Omega(\max(f(n), g(n)))$
and an upper bound of $-\ f(n)+g(n) = O(\max(f(n), g(n)))$

Since both the lower bound and upper bound are equal we can show
$\to f(n)+g(n) = \Theta(\max(f(n), g(n)))$ //

(c) **(bonus)** (10 points) Find a function that grows faster than any polynomial, but slower than any exponential. More precisely, find a function $f(n)$ such that both $f \in \omega(n^a)$ for all a, and $f \in o(b^n)$ for all $b > 1$, where a and b are constants. Prove that your function satisfies these properties.

We want to find a function $g(n)$ that satisfies $f(n) \le g(n) \le b^{f(n)}$ where $f(n)$ is a polynomial function

Any exponential function will grow faster than a polynomial function so $g(n)$ will be in the form $b^{p(n)}$

In order to have $g(n)$ be smaller than $b^{f(n)}$ we must have $p(n) < f(n)$

If, at a minimum, $f(n) = n$ to satisfy being a polynomial then $p(n)$ can equal $\sqrt{n}$

Therefore $g(n) = b^{\sqrt{n}}$ where b is some constant

$$b^{\sqrt{n}} = (2^{\log b})^{\sqrt{n}} = 2^{\Theta(\sqrt{n}\log b)}$$
$$b^{n} = (2^{\log b})^{n} = 2^{\Theta(n \log b)}$$
$$\sqrt{n}\log b < n \log b$$

$$\lim_{n\to\infty} \frac{n}{b^{\sqrt{n}}} = \lim_{n\to\infty} \frac{\frac{d}{dn}\, n}{\frac{d}{dn}\, b^{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\ln(b)\cdot b^{\sqrt{n}}\cdot\frac{1}{2\sqrt{n}}}$$

$$\to \frac{1}{\ln(b)} \lim_{n\to\infty} \frac{1}{b^{\sqrt{n}}\cdot\frac{1}{2\sqrt{n}}} = \frac{1}{\ln(b)}\cdot 0 = 0 \quad \therefore\ n \in o(b^{\sqrt{n}})$$

$$\therefore \quad n \le b^{\sqrt{2}} \le b^{n} \;//$$