Proving Mathematical Statements

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- Proofs are used to establish whether a statement is true or false.
- Four proof methods are: direct proof, proof by contrapositive, proof by contradiction, and proof by induction.

Motivation

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- Helps to advance our understanding of mathematical concepts and their relationships
- Can lead to new discoveries and innovations that can have significant impacts in various fields

Direct Proof

A direct proof shows that an if-then statement is true by providing a logical chain of reasoning from the premises to the conclusion.

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4. State the conclusion and conclude the proof.

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2. By definition, an odd integer can be written as $2n + 1$, where n is an integer.
3. So, we can write $a = 2n_1 + 1$ and $b = 2n_2 + 1$ for some integers n_1 and n_2.

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4. Therefore, $a + b = 2n_1 + 1 + 2n_2 + 1 = 2(n_1 + n_2 + 1)$, which is even by definition.

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5. Hence, if a and b are odd integers, then $a + b$ is even.

Proof by Contrapositive

A proof by contrapositive is a proof that establishes the truth of an if-then statement by proving the truth of its contrapositive. The contrapositive of an if-then statement is formed by negating both the hypothesis and the conclusion and reversing the direction of the implication.

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3. Use direct proof to show that the negated hypothesis must also be true.

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5. Hence, if n is an odd integer, then n^2 is odd.
6. By proving the contrapositive, we have shown that if n^2 is even, then n must be even.

Proof by Contradiction

A proof by contradiction is a proof that establishes the truth of a statement by assuming its negation and then deriving a contradiction.

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3. Conclude that the original statement must be true.

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5. Let $p = 2k$ for some integer k.
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7. Since q^2 is even, q must also be even. [by Example 2]
8. But this contradicts our assumption that p and q have no common factors, since they are both even.
9. Therefore, our assumption that $\sqrt{2}$ is rational must be false, and $\sqrt{2}$ is irrational.

Proof by Induction

A proof by induction is a method of proof that establishes the truth of a statement for all natural numbers (or all integers greater than some fixed integer) by proving it for a base case and then showing that if the statement is true for some integer n, then it must also be true for $n + 1$.

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1. Prove the statement for a base case (usually $n = 1$ or $n = 0$).
2. Assume that the statement is true for $n = k$.
3. Show that it must also be true for $n = k + 1$.
4. Conclude that the statement is true for all natural numbers n.

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2. Assume that the statement is true for $n = k$, i.e., $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.
3. Consider the case when $n = k + 1$. We have:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

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3. Consider the case when $n=k+1$. We have:

$$1+2+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1) \quad =\frac{k(k+1)+2(k+1)}{2} \quad =\frac{(k+1)(k+2)}{2} \quad =\frac{(k+1)((k+1)+1)}{2}$$

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4. This shows that if the statement is true for $n = k$, then it must also be true for $n = k + 1$.
5. Therefore, by the principle of mathematical induction, the statement is true for all natural numbers n.

Thank You!