Data Structures in Mathematics

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October 4, 2024

Lecture 1: What exactly is a set?

Tue 28 May 23:25

Definition 1 (Set). A number of things of the same kind that belong or are used together. [Merriam-Webster]

A group or collection of things that belong together or resemble one another or are usually found together [Oxford]

Definition 2 (Thesaurus Set). Collection, group, arrangement, clump, multitude, batch, cluster, formation, body, host, bunch, lineup, bundle, lot, display, menagerie, throng, assortment, compendium, aggregate, ensemble, assemblage, class, clique, progression, ...

Whatever a 'set' turns out to be, we would like the following examples.

Example. $A = \{1, 2, 3\} = \{2, 1, 3\} = \{3, 1, 2\}$
$B = \{47\}$
$C = \{\{5, 11\}, \{1\}, 6\}$
We want a 'set' to be <u>unordered</u>, and to contain only <u>one</u> of a given element.

Example. $D = \{1, 3, 1, 2\} \longleftarrow$ Not a set!
We also want the objects in the 'set' to be well-defined 'mathematical objects'. So closely related to the question of what is a 'set' is the question: What is a 'mathematical object'? Are any of the following sets?

Example. $F = \{\text{Fred}\}$, $G = \left\{\frac{1}{2} + \frac{2}{3}\right\}$, $H = \left\{{}^{2}3\right\}$.
Are we going to consider 'sets' with an 'infinite number' of elements?

Example. $X = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ or $Y = \{\text{all possible equations}\}$
How to tell if two sets A and B are equal?

Example. $A = \{3, 17, 18, 21, 5, 4, 1, 5, R, 13, 11, 8, 9, 51, 14, 28\}$
$B = \{51, 4, R, 8, 3, 1, 21, R, 9, 41, 17, 47, 28, 13\}$
Is $A = B$? ■
Implicit in the above question: how can we tell if mathematical objects are the same?

Example. $C = \{3 \times 7, \text{ the smallest prime } > 18, 17 = 13\}$

Example. $D = \{19, 13 = 17, 22 - 1\}$
Is $C = D$?
And what about 'sets' whose 'elements' cannot be specified but only 'described'?

$$E = \{\text{the smallest prime} > 20^{20^{20^{20^{20}}}}, \text{ the smallest equation written down, the One Piece wa}\}$$

Is E a legitimate 'set'? ■

How about self-referential sets?

Example. $F = \{F\}$ i.e. $F = \{F, \{F\}\}$ i.e. $F = \{F, \{F, \{F\}\}\}$ and taking the limit $F = \{F, \{F, \{F, \{F, \{F, \dots\}\}\}\}\}$

Does this make sense? ■
Or empty sets: Should we allow a set with 0 elements? i.e. $G = \{\}$

Sets with negative number of elements?
$\{1, 2\} \cup \{3\} = \{1, 2, 3\}$??
$\{1, 2\} \cup \{2\}$ A negative number of 2's?
$\{\text{Fred}\} \cup \{\text{Fred}\}$ A negative Fred?

Choice versus algorithmic construction of sets [L. Wittgenstein]

Choice: $C = \left\{3, \frac{1}{34}, \text{electric fan}, T, \text{car}, 1337, \dots\right\}$

Algorithmic: 1) Insert 1
2) Insert what you've built so far, and repeat.
i.e. $A = \{1, \{1\}, \{1, \{1\}\}, \{1, \{1, \{1, \{1\}\}\}\}, \dots\}$

Whenever you are dealing with infinite processes the distinction between choice and algorithm is never far away.
How about choice/algorithm question applied to nesting rather than listing issues?

Example. $R = \{a, b\}$ where $a = \{c, d\}$, $b = \{e, f\}$, $c = \{x, y, z\}$, $d = \{f\}$
$e = \{1, 2, r\}$, $f = \{k, l, m\}$, … ■
What about sets given by conditions which we can't (easily check)?

Example. $P = \{n \mid n \text{ is an odd perfect number}\}$
Is P a valid set? ■

Georg Cantor [1870's]

> "Eine Menge, ist die Zusommenfasung bestimmter, wohlunterschedener Objekte unserer Anschaung oder unseres Dekens-welche Elemente der Menge genannt werden."

Translation: "A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought-which are called elements of the set."

Lecture 2: Sets and other data structures in mathematics

Wed 29 May 21:18

Set theory is an attempt to organize mathematical objects. This is parallel to attempts to organize physical or real-life objects. There are obvious connections to computer science.

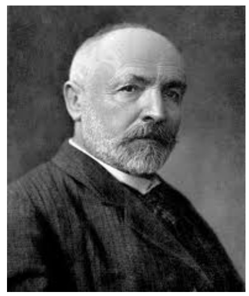

Figure 1: Georg Cantor 1845-1918

Definition 3 (Generic set)**.** A set is unordered and without repetitions.

There are four types of data structures:

	Unordered	**Ordered**
Without Repetition	set $\{1\,2\,3\}$	ordered set $\{1,2,3\}$
With Repetition	multiset $[1\,1\,4\,2]$	list $[1,1,4,2]$

Notice the different notation where no commas suggest no order and braces suggest repetitions.

{ } – Without repetitions
[ ] – With repetitions
Space delimiter – Unordered
Comma delimiter – Ordered

$$\underset{\text{set}}{A = \{1\;2\;3\}}$$
$$\underset{\text{oset}}{H = \{2,7\}}$$
$$\underset{\text{mset}}{D = [1\;5\;5\;4]}$$
$$\underset{\text{list}}{F = [7,5,4]}$$

Data structures in real life

People in a company: set

Names in a company: multiset (mset)

Words in a dictionary: ordered set (oset)

Letters in a word: list

Webpages: set

Images found on the internet: multiset (mset)

Primes from 2 to 19: ordered set (oset)

Bus numbers that arrive at a stop: list

While the kinds of elements in real life that appear in data structures are highly varied, we will start out with very focused examples.

$$\begin{aligned} \mathbf{Nat}: &\quad 1, 2, 3, 4, \ldots \\ \mathbf{Nat}^+: &\quad 0, 1, 2, 3, \ldots \end{aligned}$$

For a precise theory, we are going to agree that the Hindu-Arabic form of a number $n \in Nat$ or $k \in Nat^+$ is the only one allowed!

Example. $n = 13$✓
$n = 12 + 1$×
$n = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 4$×
$n =$ smallest prime > 12×
$n =$ Mary's lucky number×

Example. $k = 57109$✓, $k = 57.109$×, $k = 57.109 \cdot 0$×
Or first (cooked up) datastructure is not going to be a set.

Definition 4. k-list: For $n, k \in$ Nat we define a k-list from n to be an expression of the form $L = [m_1, m_2, \ldots, m_k]$ where each $m_i \in$ Nat and $1 \leq m_i \leq n$.

This defines a new type object:

$$\underline{List(n, k)}.$$

Example. i) $[3, 2, 4] \in \text{List}(4, 3)$ ii) $[13] \in \text{List}(15, 1)$
iii) $[1, 5, 5, 1] \in \text{List}(5, 4)$ iiii) $[2, 2, 2, 2] \in \text{List}(2, 4)$

The objects (natural numbers) $m_1, \ldots, m_k$ in the k-list L=$[m_1, m_2, \ldots, m_k]$ are called the elements of L. They are considred as distinct elements, even if as numbers they are the same/equal.

Example. The 4-list $L = [3, 1, 3, 2]$ has four elements, namely 3,1,3 and 2. The size of the k-list $L = [m_1, m_2, \ldots, m_k]$ is

$$|L| = k.$$

The list the most fundamental data structure. An ordered set is a variant, where the elements are required to be distinct.

Definition 5 (k-ordered set)**.** For $n, k \in Nat$, a k-ordered set from n is an expression of the form

$$O = m_1, m_2, \ldots, m_k$$

where each $m \in$ Nat, $1 \leq m_i \leq n$, and the m_i are all distinct, ie. $m_i \neq mj$ if $i \neq j$.

Example. O $= \{1, 3, 5\} \in$ Oset$(5, 3)$ with $|O| =$ k $= 3$.
Another variant of a list is a multiset, where order is no longer important.

Definition 6 (k-multiset)**.** For $n, k \in$ Nat, a k-multiset from n is an expression of the form

$$M = [m_1, m_2, \ldots, m_n]$$

where each $m_i \in$ Nat, $1 \leq m_i \leq n$, and the order of the elements m_i is unimportant.

Example. $M = [1\;3\;3\;1\;15] = [15\;1\;3\;1\;3] \in$ Mset$(15, 5)$ with $|M| = 5$.

Definition 7 (k-set)**.** For $n, k \in$ Nat we define a k-set from n to be an expression of the form

$$S = \{m_1, m_2, \ldots, m_k\}$$

where each $m_1 \in$ Nat, $1 \leq m \leq n$, and the m_i are all distinct, and their order is unimportant.

Example. $S = \{13\;7\;2\} = \{2\;7\;13\} \in \{13\;3\}$ with $|S| = 3$.