Isometries of the Cube

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Introduction

This paper is divided into two parts.

Part 1 is a short, sweet, and direct construction of all isometries $\operatorname{Isom}(\mathcal{C})$ of the cube as signed permuation matrices over $\mathbb{R}^3$.

Part 2 shows that all of $\operatorname{Isom}(\mathcal{C})$ can be generated from a rotation (leaving no vertex fixed), a reflection, and a "spin", or a rotation about an axis passing through two vertices on opposite corners (leaving them fixed).

Part 3 describes the representation of $\operatorname{Isom}(\mathcal{C})$ over the adjacency matrix of the cube, and gives an isomorphism between this representation and the $\mathbb{R}^3$ representation. Following this, *tilings* of isometries in the adjacency representation are constructed that offer a somewhat interesting visualization. Two classes of examples of these tilings are provided in the Appendix.

Part 1

Constuction

Let the cube $\mathcal{C}$ be represented by its eight vertices $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_8 \in \mathbb{R}^3$ where

$$\begin{aligned} &\mathbf{y}_1 = (\ \ 1,\ \ 1,\ \ 1), \quad \mathbf{y}_2 = (\ \ 1,-1,\ \ 1), \quad \mathbf{y}_3 = (-1,-1,\ \ 1), \quad \mathbf{y}_4 = (-1,\ \ 1,\ \ 1), \\ &\mathbf{y}_5 = (-1,\ \ 1,-1), \quad \mathbf{y}_6 = (-1,-1,-1), \quad \mathbf{y}_7 = (\ \ 1,-1,-1), \quad \mathbf{y}_8 = (\ \ 1,\ \ 1,-1). \end{aligned} \tag{1.1.1}$$

We can easily determine the order $|\operatorname{Isom}(\mathcal{C})|$, since it must preserve which vertices neighbor which. First, choose one of the eight vertices. Then, there are exactly three other vertices that it must be connected to, which we may choose from and place beside it. Finally, there are two vertices that may be placed beside this, since it is already beside the originally chosen vertex. The number of isometries must then be

$$|\operatorname{Isom}(\mathcal{C})| = 8 \cdot 3 \cdot 2 = 48.$$

Each face of $\mathcal{C}$ may be represented by a standard basis vector or its negative

$$\pm\hat{\mathbf{x}}_1 = (\pm 1, 0, 0), \quad \pm\hat{\mathbf{x}}_2 = (0, \pm 1, 0), \quad \pm\hat{\mathbf{x}}_3 = (0, 0, \pm 1).$$

In particular, it will be useful in Section 3.2 to know that

$$\hat{\mathbf{x}}_1 = \frac{1}{4}(\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_7 + \mathbf{y}_8), \quad \hat{\mathbf{x}}_2 = \frac{1}{4}(\mathbf{y}_1 + \mathbf{y}_4 + \mathbf{y}_5 + \mathbf{y}_8), \quad \hat{\mathbf{x}}_3 = \frac{1}{4}(\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 + \mathbf{y}_4). \tag{1.1.2}$$

Any $M \in \mathrm{Isom}(\mathcal{C})$ must map faces to faces, and so must map $\pm\hat{\mathbf{x}}_j \mapsto \pm\hat{\mathbf{x}}_k$ for some $j, k = 1, 2, 3$. M is also invertible since $\mathbf{v} \mapsto M\mathbf{v}$ is one-to-one by definition of $\mathrm{Isom}(\mathcal{C})$, and so its' columns must be linearly independent. This forces the columns of M to take the form

$$M = (\pm\hat{\mathbf{x}}_a, \pm\hat{\mathbf{x}}_b, \pm\hat{\mathbf{x}}_c) \tag{1.1.3}$$

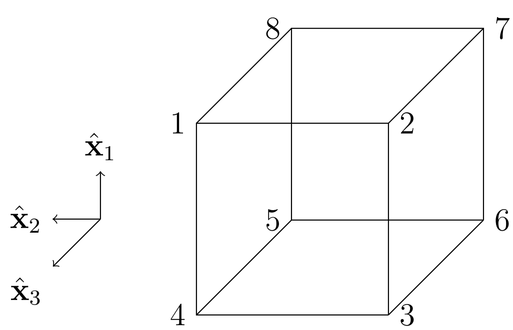
where (a, b, c) is a permutation of $(1, 2, 3)$. There are $3!$ permutations and 2^3 choices of sign, making

$$3! \cdot 2^3 = 48$$

such matrices, so this necessarily characterizes all $M \in \mathrm{Isom}(\mathcal{C})$.

Part 2
Generators

Number the vertices of $\mathcal{C}$ in accordance with Eq. 1.1.1:



All isometries of $\mathcal{C}$ may be formed from three generators, here written as permutations of the vertices:

$$R = (1234)(8765), \quad r = (12)(34)(56)(78), \quad s = (135)(268). \tag{2.0.1}$$

R is the rotation of the x_3-face; r is the reflection in the x_1x_3-plane; and s is the rotation ("spin") that holds 4 and 7 fixed. It is easy to see, from visualization or computation, that the orders of these elements are

$$|\langle R \rangle| = 4, \quad |\langle r_1 \rangle| = |\langle r_2 \rangle| = 2, \quad |\langle s \rangle| = 3.$$

To show that these are indeed generators, it will be useful to consider the reflections r_{12} and r_{23} about the respective x_1x_2-, x_2x_3-planes (noting that $r = r_{13}$), as well as the rotations R_1, R_2, R_3 about the respective x_1-, x_2-, x_3-axes (noting that $R = R_3^{-1}$). We see that

$$\begin{aligned} srs^{-1} &= (36)(54)(18)(72) = r_{12}, \quad & RrR^{-1} &= (23)(41)(85)(67) = r_{23}, \\ sRs^{-1} &= (3654)(2781) = R_1, \quad & s^2Rs^{-2} &= (5814)(6723) = R_2^{-1}. \end{aligned} \tag{2.0.2}$$

So these elements (and their inverses) are in $\langle R, r, s\rangle$. Consider now the $\mathbb{R}^3$ representations of $r_{12}, r_{23}, r_{13}, R_1, R_2, R_3$, which are easily obtainable by considering the transformation of the faces of $\mathcal{C}$:

$$\begin{aligned} r_{12} &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad r_{23} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_{13} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R_1 &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad R_2 \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_3 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

In particular, it is easiest to study the combinations

$$r_{12} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad r_{23} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_{13} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.0.3a}$$

$$r_{12}R_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad r_{23}R_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad rR_3 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.0.3b}$$

The matrices of $r_{12}R_1, r_{23}R_2, r_{13}R_3$ are precisely the standard permutation matrices of (23), (123), (12), respectively, and generate all six possible permutation matrices:

$$(1), \quad (12), \quad (23), \quad \underbrace{(13)}_{=(12)(23)}, \quad (123), \quad \underbrace{(321)}_{=(12)(123)}.$$

It was shown in Part 1 that all elements of $M \in \mathrm{Isom}(\mathcal{C})$ are of the form (Eq. 1.1.3)

$$M = (\pm\hat{\mathbf{x}}_a, \pm\hat{\mathbf{x}}_b, \pm\hat{\mathbf{x}}_c)$$

for some permutation (a, b, c) of $(1, 2, 3)$ and choice of signs, and that each choice of permutation and sign corresponds to a unique element of $\mathrm{Isom}(\mathcal{C})$. The elements Eq. 2.0.3a allow us to choose the sign, and the elements Eq. 2.0.3b allow us to choose the permutation. Given a choice of sign $\iota = (\iota_1, \iota_2, \iota_3)$ and permutaton $\sigma \in S_3$, we can now build every element $M_\sigma^\iota \in \mathrm{Isom}(\mathcal{C})$ as

$$M_\sigma^\iota = m(\sigma) r_{23}^{n(\iota_1)} r_{13}^{n(\iota_2)} r_{12}^{n(\iota_3)} = r_{23}^{n(\iota_{\sigma(1)})} r_{13}^{n(\iota_{\sigma(2)})} r_{12}^{n(\iota_{\sigma(3)})} m(\sigma), \tag{2.0.4}$$

where

$$n(x) = \frac{1-x}{2},$$

$$\begin{aligned} m((1)) &= 1, & m((12)) &= r_{13}R_3, & m((23)) &= r_{12}R_1, \\ m((13)) &= r_{12}R_1r_{13}R_3, & m((123)) &= r_{23}R_2, & m((321)) &= r_{13}R_3r_{23}R_2. \end{aligned}$$

It is thus that $\langle R, r, s\rangle = \mathrm{Isom}(\mathcal{C})$.

Aside

The second equality in Eq. 2.0.4 comes from the fact that right multiplication by e.g. r_{23} sets the sign of the first *column* of $m(\sigma)$, and left multiplication sets the sign of the first *row*. Each row and column of $m(\sigma)$ contains exactly one 1, and the map that links corresponding rows and columns is exactly σ.

It is clear that $\langle r_{23}, r_{13}, r_{12}\rangle$ and the set of permutation matrices $P_3 = \{m(\sigma) \mid \sigma \in S_3\} \cong S_3$ are trivially intersecting subgroups of $\mathrm{Isom}(\mathcal{C})$. This second equality then shows us that $\langle r_{23}, r_{13}, r_{12}\rangle$ is a normal subgroup of $\mathrm{Isom}(\mathcal{C})$, and it follows that

$$\mathrm{Isom}(\mathcal{C}) \big/ \langle r_{23}, r_{13}, r_{12}\rangle \cong S_3.$$

Of course, this is not surprising since we've already shown that $\mathrm{Isom}(\mathcal{C})$ is the set of signed permutation matrices. Note however that this does not mean that $\mathrm{Isom}(\mathcal{C})$ is a product group:

$$\mathrm{Isom}(\mathcal{C}) \ncong C_2^3 \times S_3,$$

where $C_2^3 \cong \langle r_{23}, r_{13}, r_{12}\rangle$ is the product of three order-two cyclic groups. While Eq. 2.0.4 does tell us that $\langle r_{23}, r_{13}, r_{12}\rangle$ is normal, it also explicitly shows that it does not commute with P_3.

Part 3
Adjacency Representation

3.1 Description

We form the adjacency matrix Z by writing $Z_{ij} = 1$ when vertices i and j share an edge:

$$Z = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

For any permutation $\sigma \in S_8$, we may form a permutation matrix $I_\sigma \in P_8 \subset \mathbb{R}^{8\times 8}$ that interchanges the rows of another matrix $X \in \mathbb{R}^{8\times 8}$ via matrix multiplication $I_\sigma X$. For example,

$$I_{(12)} = \left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{$\mathbf{0}$} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{1} \end{array}\right), \quad I_{(1234)} = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ \hline \multicolumn{4}{c|}{\mathbf{0}} & \mathbf{1} \end{array}\right),$$

where $\mathbf{0}$ and $\mathbf{1}$ stand for appropriate zero and identity matrices. In general, the components of I_σ may be written

$$[I_\sigma]_{ij} = \delta_{\sigma(i),j}, \quad \text{for } i, j = 1, \ldots, 8, \tag{3.1.1}$$

where δ is the Kronecker delta. These matrices are necessarily orthogonal, with $I_\sigma I_\sigma^\top = \mathbf{1}$. They also allow us to permute columns by

$$\left(I_\sigma X^\top\right)^\top = X I_\sigma^\top.$$

Permuting vertices of $\mathcal{C}$ corresponds to permuting the corresponding rows and columns; we define the action of σ over X by

$$\sigma * X = I_\sigma X I_\sigma^\top = I_\sigma X I_\sigma^{-1}.$$

With this action, P_8 is in fact a linear representation of S_8:

$$\begin{gathered} I_{\sigma_1}(I_{\sigma_2} A I_{\sigma_2}^\top) I_{\sigma_1}^\top = I_{\sigma_1\sigma_2} A I_{\sigma_1\sigma_2}^\top, \\ I_\sigma(\alpha A + \beta B) I_\sigma^\top = \alpha(I_\sigma A I_\sigma^\top) + \beta(I_\sigma B I_\sigma^\top), \end{gathered}$$

for $\alpha, \beta \in \mathbb{R}$ and $A, B \in \mathbb{R}^{8\times 8}$. Futhermore, the representation is faithful, or

$$\mathbf{1} X \mathbf{1}^\top = X, \tag{3.1.2}$$

which implies it is one-to-one. The isometries of $\mathcal{C}$ are exactly those automorphisms $X \to \sigma * X$ with Z as a fixed point, or

$$\mathrm{Isom}(\mathcal{C}) \cong \left\{ I \in P_8 \;\middle|\; I_\sigma Z I_\sigma^\top = Z \right\}.$$

In this representation, the generators (2.0.1) take the form

$$I_R = \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{pmatrix}, \quad I_r = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{pmatrix},$$

$$I_s = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix}.$$

(The ones have been bolded for visibility.)

3.2 Isomorphism between $\mathbb{R}^3$ Representation

Recall the construction in Eq. 1.1.2 of the standard basis vectors $\{\hat{\mathbf{x}}_j\}_{j=1}^3$ from the vertex vectors $\{\mathbf{y}_k\}_{k=1}^8$:

$$\hat{\mathbf{x}}_1 = \frac{1}{4}(\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_7 + \mathbf{y}_8), \quad \hat{\mathbf{x}}_2 = \frac{1}{4}(\mathbf{y}_1 + \mathbf{y}_4 + \mathbf{y}_5 + \mathbf{y}_8), \quad \hat{\mathbf{x}}_3 = \frac{1}{4}(\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 + \mathbf{y}_4).$$

Define the columns of the matrix $Y \in \mathbb{R}^{3\times 8}$ as $\{\mathbf{y}_k\}_{k=1}^8$:

$$Y_{ak} = [\mathbf{y}_k]_a, \quad a = 1, 2, 3,\ k = 1, \ldots, 8.$$

For an $M \in \text{Isom}(\mathcal{C})$, since adjacency representation is faithful (Eq. 3.1.2) there is a unique $I \in P_8$ corresponding to M. The matrix M is understood most easily as a mapping of the faces of $\mathcal{C}$; I is understood as a mapping of the vertices. Multiplying by Y by I on the right permutes the columns of Y according to the permutation that I represents. Constructing the faces of $\mathcal{C}$ from YI then gives us the columns of M:

$$M_{a,1} = \frac{1}{4}\Big([YI]_{a,1} + [YI]_{a,2} + [YI]_{a,7} + [YI]_{a,8}\Big), \quad M_{a,2} = \frac{1}{4}\Big([YI]_{a,1} + [YI]_{a,4} + [YI]_{a,5} + [YI]_{a,8}\Big),$$
$$M_{a,3} = \frac{1}{4}\Big([YI]_{a,1} + [YI]_{a,2} + [YI]_{a,3} + [YI]_{a,4}\Big).$$

We can write this in a nicer form by noting that by defintion $[YI]_{a,j} = \sum_{k=1}^{8} Y_{ak} I_{kj}$, so

$$[YI]_{a,1} + [YI]_{a,2} + [YI]_{a,7} + [YI]_{a,8} = \sum_{k=1}^{8} Y_{ak}(I_{k,1} + I_{k,2} + I_{k,7} + I_{k,8}) = [YJ(I)]_{a,1}$$
$$[YI]_{a,1} + [YI]_{a,4} + [YI]_{a,5} + [YI]_{a,8} = \sum_{k=1}^{8} Y_{ak}(I_{k,1} + I_{k,4} + I_{k,5} + I_{k,8}) = [YJ(I)]_{a,2}$$
$$[YI]_{a,1} + [YI]_{a,2} + [YI]_{a,3} + [YI]_{a,4} = \sum_{k=1}^{8} Y_{ak}(I_{k,1} + I_{k,2} + I_{k,3} + I_{k,4}) = [YJ(I)]_{a,3},$$

where we've defined $J(I) \in \mathbb{R}^{8\times 3}$ as

$$J_{k,1}(I) = I_{k,1} + I_{k,2} + I_{k,7} + I_{k,8}, \quad J_{k,2}(I) = I_{k,1} + I_{k,4} + I_{k,5} + I_{k,8},$$
$$J_{k,3}(I) = I_{k,1} + I_{k,2} + I_{k,3} + I_{k,4}$$

for $k = 1, \ldots, 8$. It follows that $M = \frac{1}{4} Y J(I)$.

The inverse map is constructed simply by acting on the vertices $\{y_k\}_{k=1}^{8}$ with M, which must permute them by some permutation $\sigma \in S_8$:

$$My_k = y_{\sigma(k)}, \quad k = 1, \ldots, 8,$$

The corresponding matrix in the adjacenecy representation is just I_σ from Eq. 3.1.1.

3.3 Tilings

We may generate interesting visual representations of the elements of $\mathrm{Isom}(\mathcal{C})$ by considering tilings of the adjacency representation matrices in the following way: for $I \in P_8$ corresponding to an elements of $\mathrm{Isom}(\mathcal{C})$, construct the matrices A, B, C as

$$A_{jk} = I_{j,(9-k)}, \quad B_{jk} = I_{(9-j),k}, \quad C_{jk} = I_{(9-j),(9-k)}, \quad j, k = 1, \ldots, 8,$$

which are the reversals of the columns, rows, and columns and rows of I, respectively. Now form the matrix t and the infinite *tiling matrix* $T(I)$ of I by

$$t = \left(\begin{array}{c|c} I & A \\ \hline B & C \end{array}\right), \quad T_I = \begin{pmatrix} \ddots & \vdots & \ddots & \vdots & \\ \cdots & t & \cdots & t & \cdots \\ \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & t & \cdots & t & \cdots \\ & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

I present the tilings this way because this is how I first approached them. We can now generalize by realizing that for the matrix $F \in P_8$ where

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that

$$A = IF^{\top}, \quad B = FI, \quad C = FIF^{\top}.$$

Since $F^2 = \mathbf{1}$, we can view $T(I) = T_F(I)$ as successive conjugations of I by F along the 8×8 diagonal and counter-diagonal, along with the intermediate products FI and $IF^{\top}$:

$$T_F(I) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & FIF^{-1} & FI & FIF & \cdots \\ \cdots & IF^{-1} & I & IF & \cdots \\ \cdots & F^{-1}IF^{-1} & F^{-1}I & F^{-1}IF & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

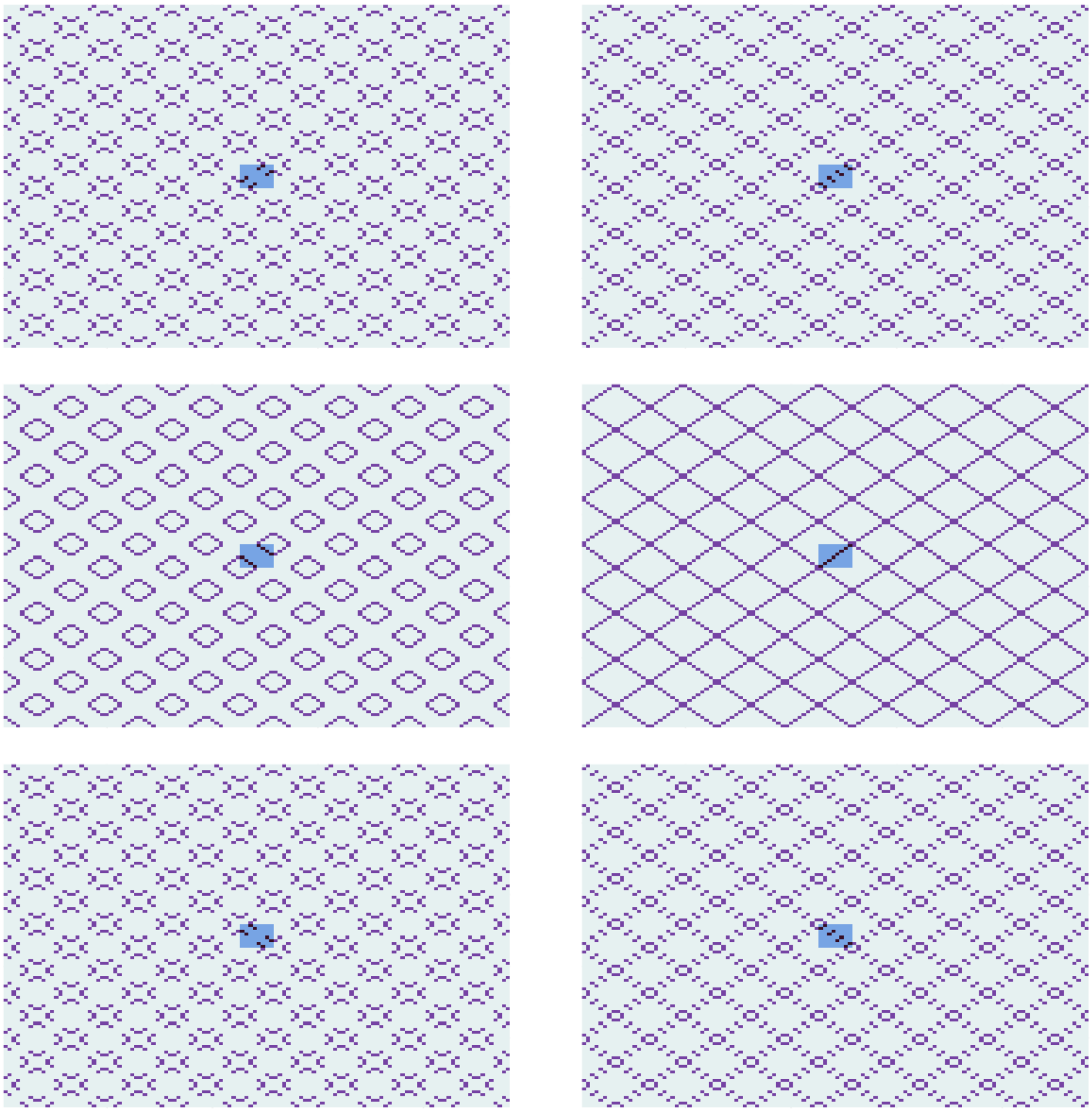
Note that for this specific case $F = F^{\top} = F^{-1}$, but in general for any $K \in P_8$ we have $K^{\top} = K^{-1}$. F is an isometry, specifically $I_{(87654321)}$, so there are at most four isometries $\{I, IF, FI, FIF\}$ that generate the same tiling $T_F(I)$.

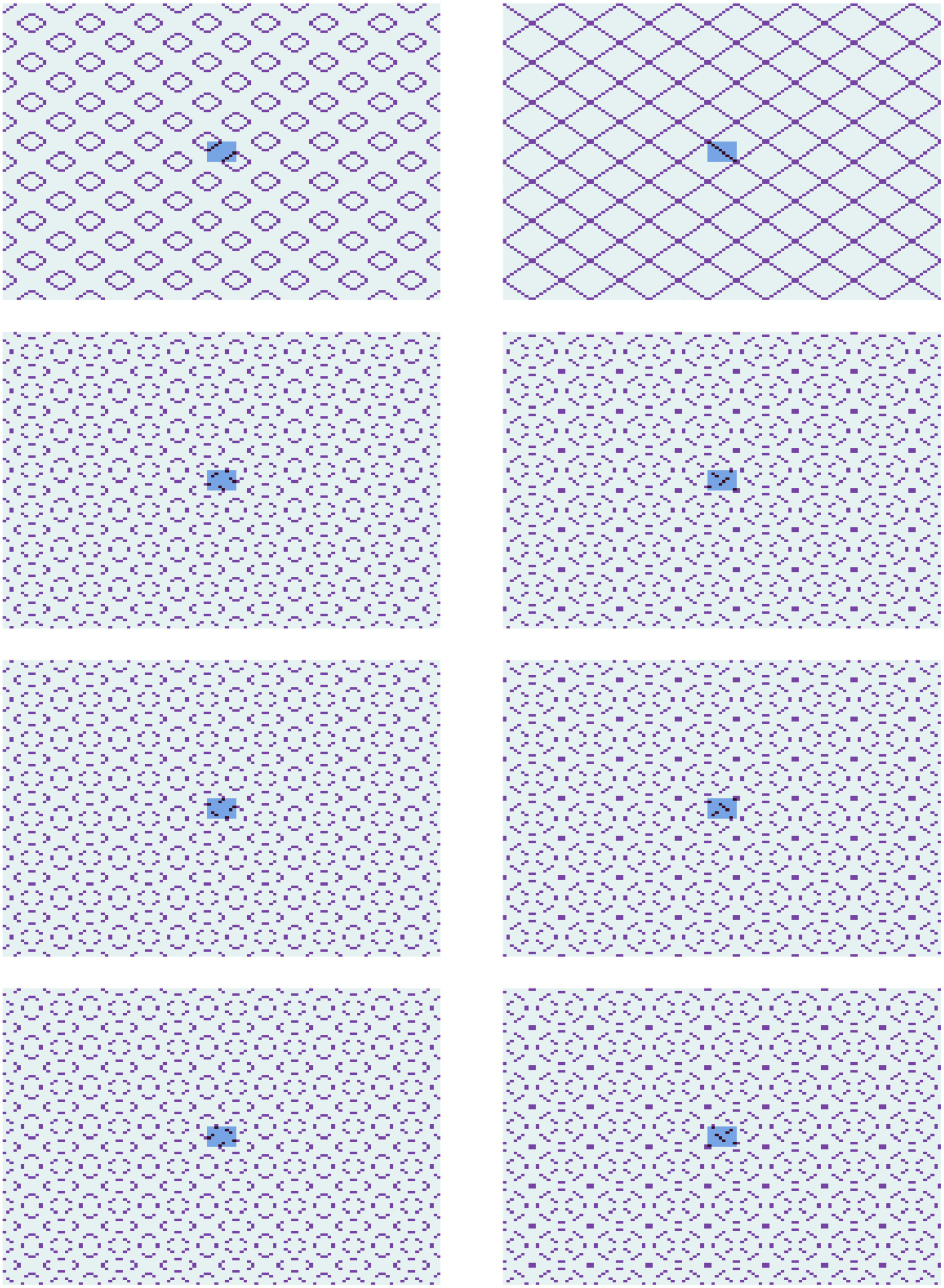
If we consider an arbitrary isometry $K \in P_8$ and generate $T_K(I)$, then if K is of order a, $T_K(I)$ repeats every a columns and a rows over from I. There are then at most a^2 isometries that generate the same tiling based around I; this number is lowered if some power of K commutes with I. Most interesting is to note that the tiling $T_K(\mathbf{1})$, the tiling of the identity by K, is a pictorial representation of the cyclic group $\langle K \rangle$ generated by K; we can easily identify the order of K by noting where the diagonal identity conjugation line repeats. By this method, I've found that the highest order of an element of $\mathrm{Isom}(\mathcal{C})$ is six.

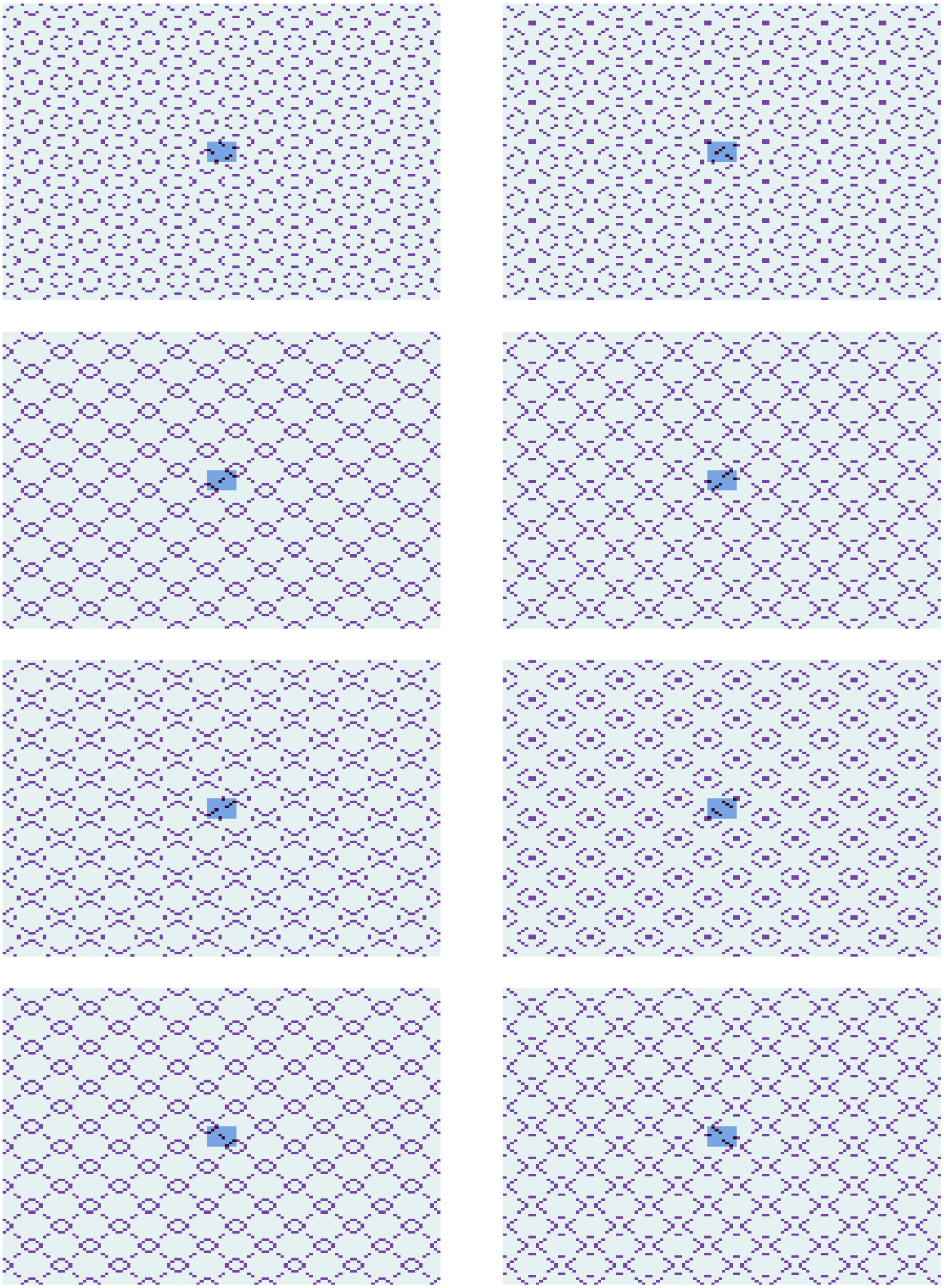
$T_F(\mathrm{Isom}(\mathcal{C}))$ and $T_{\mathrm{Isom}(\mathcal{C})}(\mathbf{1})$ are provided in Sections 1 and 2 in the Appendix. The tilings were generated by a Julia program I wrote, which you can find online at https://github.com/loppy1243/Cube.

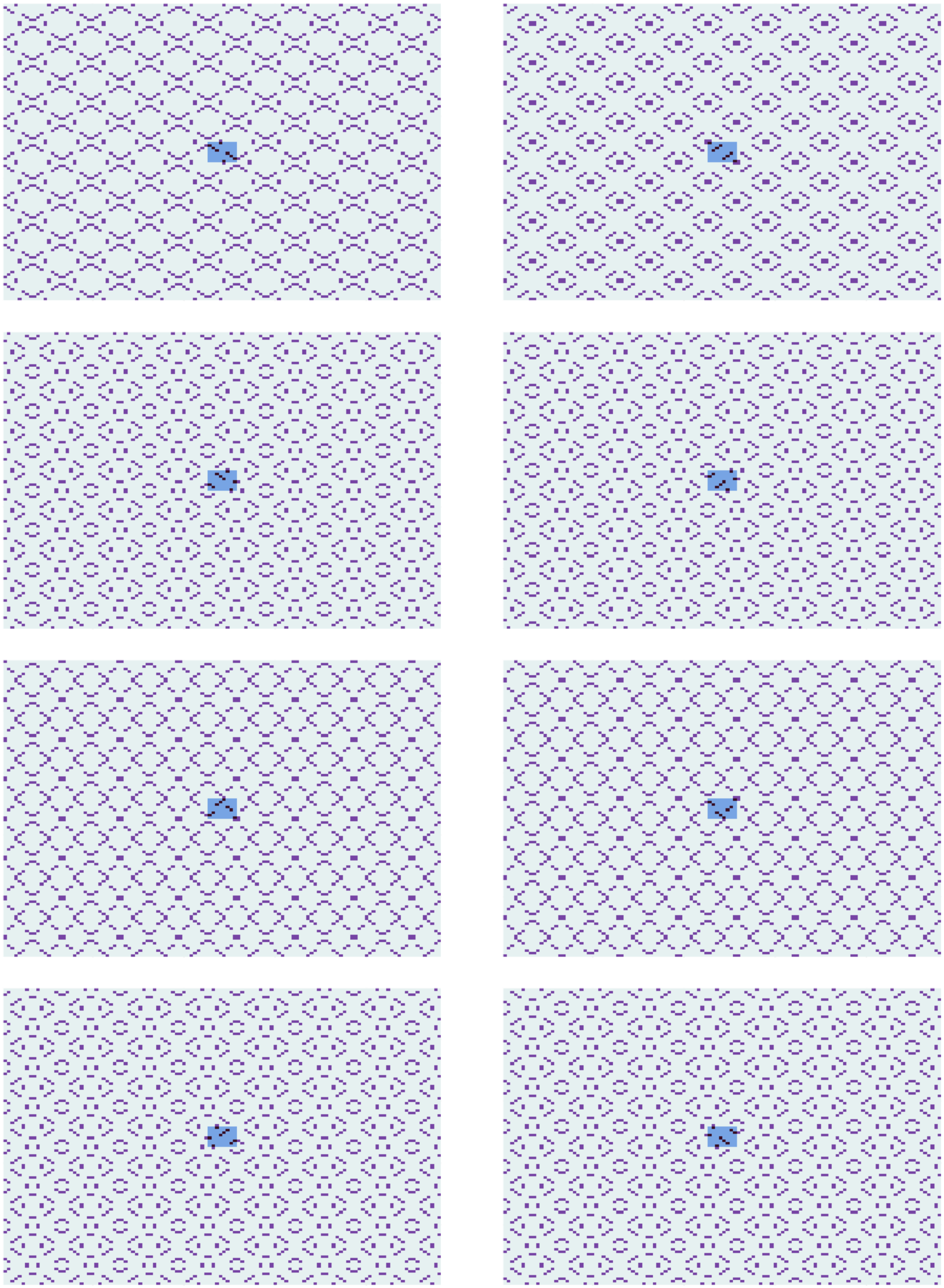
Appendix

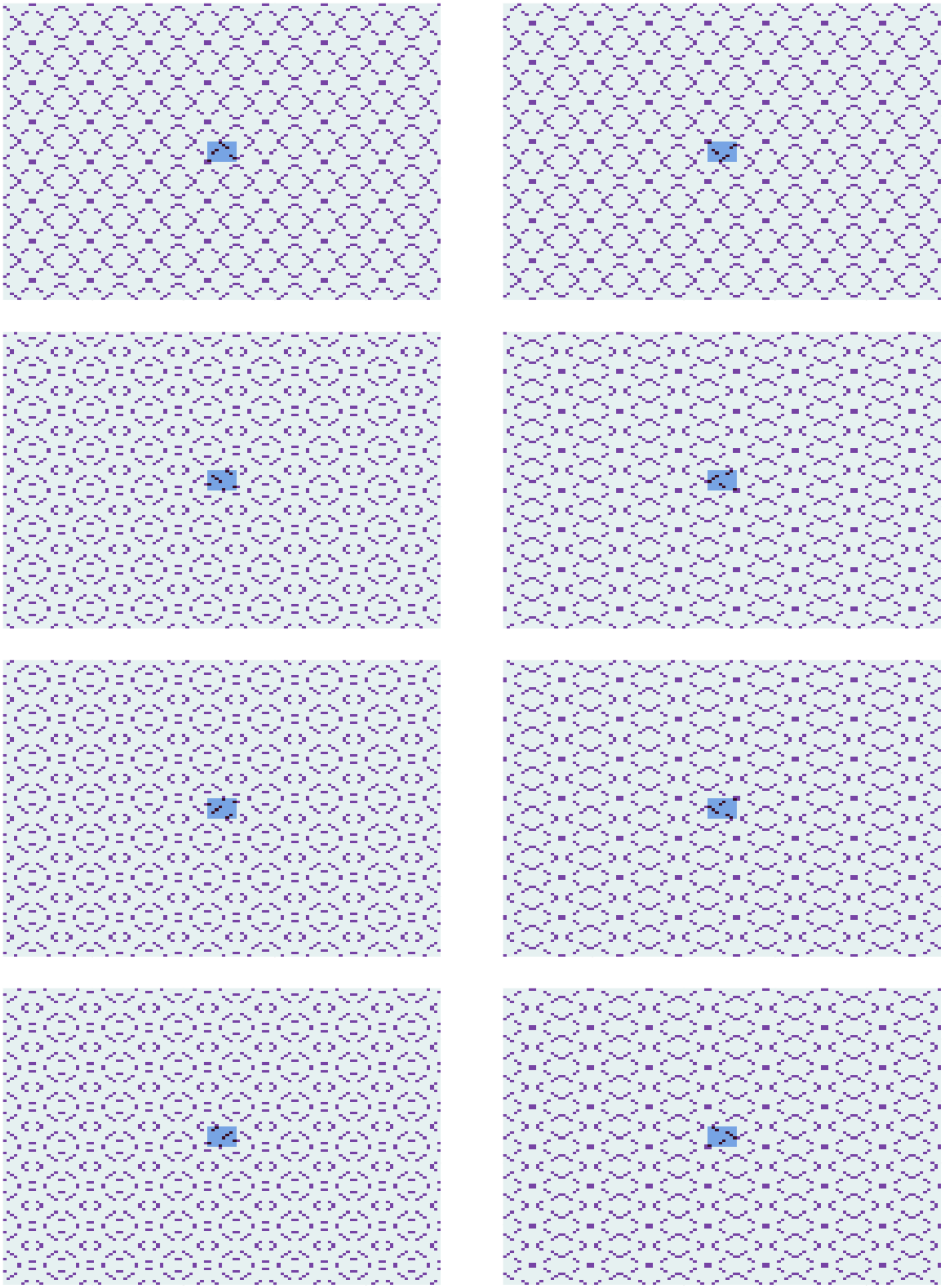
1 Tilings of $\mathrm{Isom}(\mathcal{C})$ by F $(T_F(\mathrm{Isom}(\mathcal{C})))$

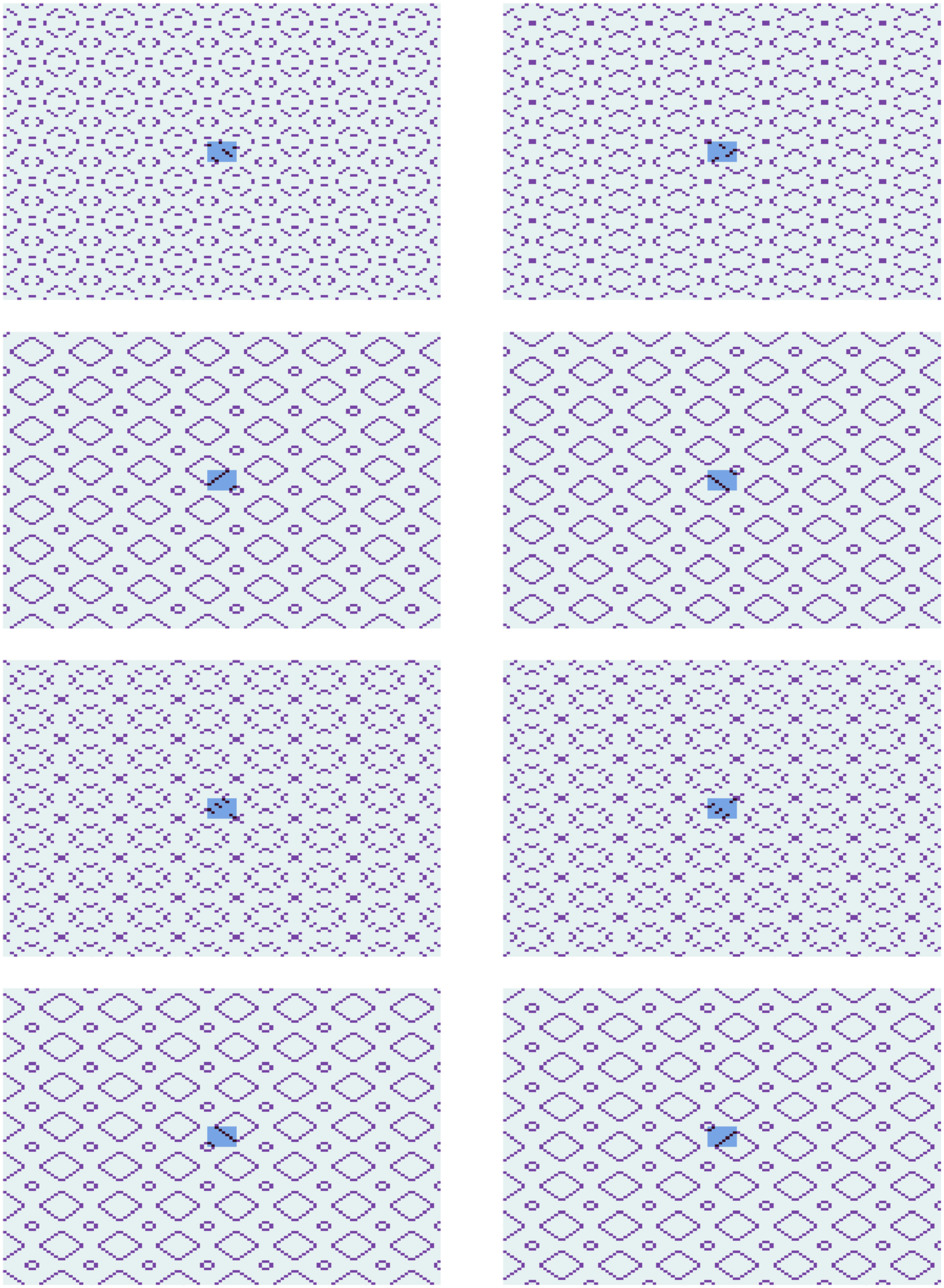


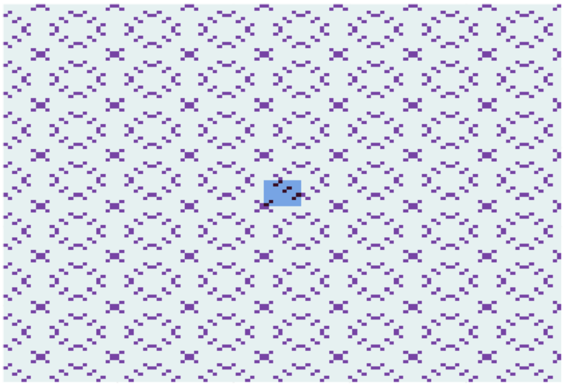


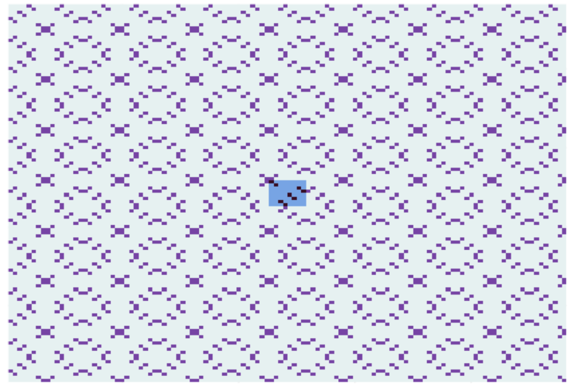












2 Tilings of the Identity by $\mathrm{Isom}(\mathcal{C})$ ($T_{\mathrm{Isom}(\mathcal{C})}(\mathbf{1})$)