Notes on Karel Hrbacek's "Relative set theory: Some external issues"[1]

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Bolded letters ($\mathbf{U}$, $\mathbf{V}$, etc.) are for levels in $\mathbf{GRIST}^{\heartsuit}$. Capital Greek letters (Φ, Ψ, etc.) are for formulas. Latin letters (a, A, x, y, etc.) are for sets. A bar over a variable indicates a parameter list.

Relativization$^{\heartsuit}$

(iii) All levels are part of a hierarchy, and this is useful to assume when interpreting the other axioms.

(o) Extensionality for levels, which is to say two levels are equal if they have the same elements.

(i) For reference,
$$\forall x \exists \mathbf{V}.\, x \in \mathbf{V} \wedge \forall \mathbf{U}.\, x \in \mathbf{U} \to \mathbf{V} \subseteq \mathbf{U}.$$
This says that every set belongs to a level, and also that every set has a least level containing it. If there are two least levels $\mathbf{V}, \mathbf{V}'$, then $\mathbf{V} \subseteq \mathbf{V}'$ by definition of $\mathbf{V}$ but $\mathbf{V}' \subseteq \mathbf{V}$ by definition of $\mathbf{V}'$; so $\mathbf{V} = \mathbf{V}'$ and the least level containing a given set is unique. This allows us to define $\mathbb{O}_x$ as the least level containing x.

(ii) $\varnothing$ is in every level, which implies that $\mathbb{S} := \mathbb{O}_\varnothing$ is the bottom of the level hierarchy. It also says that every level is the least level of some set, i.e. $\mathbf{V} = \mathbb{O}_x$ for some x.

(iv) The hierarchy of levels has no upper bound.

(v) Any two distinct levels have a distinct intermediary level, i.e. there are no discernable "gaps" in the hierarchy.

Some terminology appears to be useful in aiding intuition. In the following, let $\mathbf{U}, \mathbf{V}$ be levels.

- We will use "obs'le" as an abbreviation of "observable".

- We say that $\mathbf{U}$ is *obs'le* to $\mathbf{V}$ if $\mathbf{U} \subseteq \mathbf{V}$, and we say that $\mathbf{U}$ is *standard* to $\mathbf{V}$ if $\mathbf{U} \subset \mathbf{V}$. Thus $\mathbf{U}$ is obs'le to $\mathbf{V}$ if it is standard to $\mathbf{V}$, and $\mathbb{S}$ is standard to every distinct level.

- Any x is *obs'le* or *standard* to $\mathbf{V}$ if $\mathbb{O}_x$ is obs'le or standard to $\mathbf{V}$, repectively. Thus x is obs'le to $\mathbf{V}$ if $x \in \mathbf{V}$. We say simply that x is *obs'le* or *standard* if x is obs'le or standard to the current implicit level.

- Since $\forall \mathbf{U} \neq \mathbb{S}.\ \mathbb{S} \subset \mathbf{U}$, we way that $\mathbb{S}$ (and its elements) are *absolutely standard*.

- We let $\mathbb{S}_{\mathbf{U}}$ be the class of all $\mathbf{U}$-standard sets, which is to say that

$$\forall x.\ x \in \mathbb{S}_{\mathbf{U}} \leftrightarrow \exists \mathbf{V} \subset \mathbf{U}.\ x \in \mathbf{V}.$$

  Note that while $\mathbb{S}$ is a level, $\mathbb{S}_{\mathbf{U}}$ is not.

- We say a $\mathbf{V}$-formula $\Phi(\bar{x}; \mathbf{V})$ is *relative* to $\mathbf{V}$. A formula relative to some $\mathbf{V}$ is *relative*.

Transfer$^\heartsuit$

Every relative formula with obs'le parameters may be taken relative to any level so long as its parameters remain obs'le.

Standardization$^\heartsuit$

For reference: let A be a set and $\Phi(y, \bar{x}; \mathbf{W})$ be a relative formula. Then

$$\forall \mathbf{U} \supset \mathbb{S}.\ \exists \mathbf{V} \subset \mathbf{U}.\ \exists B \in \mathbf{V}.\ \forall \mathbf{W}.\ \mathbf{V} \subseteq \mathbf{W} \subset \mathbf{U} \to \Big(\forall y \in \mathbf{W}.\ y \in B \leftrightarrow y \in A \wedge \Phi(y, \bar{x}; \mathbf{W})\Big).$$

We can derive **FRIST**$^\heartsuit$ Standardization from **GRIST**$^\heartsuit$ Standardization and Granularity. First define

$$\Psi(A, \bar{x}; \mathbf{V}) \equiv \exists B \in \mathbf{V}.\ \forall y \in \mathbf{V}.\ y \in B \leftrightarrow y \in A \wedge \Phi(y, \bar{x}; \mathbf{V}),$$

and note that this is indeed a formula relative to $\mathbf{V}$. Then, by Standardization$^\heartsuit$, taking $\mathbf{W} = \mathbf{V}$ proves that

$$\forall \mathbf{U} \supset \mathbb{S}.\ \exists \mathbf{V} \subset \mathbf{U}.\ \Psi(A, \bar{x}; \mathbf{V}). \tag{$*$}$$

But by Granularity$^\heartsuit$, there is a least $\mathbf{X}$ such that $\Psi(A, \bar{x}; \mathbf{X})$. If $\mathbf{X} \supset \mathbb{S}$ then $(*)$ applies with $\mathbf{U} = \mathbf{X}$ and $\mathbf{X}$ is not minimal, a contradiction. So $\mathbf{X} = \mathbb{S}$ since $\mathbb{S}$ is absolutely standard, and $\forall A \forall \bar{x}.\ \Psi(A, \bar{x}; \mathbb{S})$ is satisfied, which is exactly **FRIST**$^\heartsuit$ Standardization. By Transfer$^\heartsuit$, $\forall A \forall \bar{x}.\ \Psi(A, \bar{x}; \mathbf{U})$ is also satisfied for any level $\mathbf{U}$, which is "**FRIST**$^\heartsuit$ Standardization relative to $\mathbf{U}$".

We can put (**GRIST**$^\heartsuit$) Standardization$^\heartsuit$ in a more intuitive form. Since levels exist in a hierarchy, we can note that y ranges over all values in $\mathbb{S}_{\mathbf{U}}$ and that the only thing that matters is that Φ is satisfied for some $\mathbf{W}$. It follows that an equivalent formulation is

$$\forall \mathbf{U} \supset \mathbb{S}.\ \exists \mathbf{V} \subset \mathbf{U}.\ \exists B \in \mathbf{V}.\ \forall y \in \mathbb{S}_{\mathbf{U}}.$$
$$y \in B \leftrightarrow y \in A \wedge \exists \mathbf{W}.\ \mathbf{V} \subseteq \mathbf{W} \subset \mathbf{U} \wedge \Phi(y, \bar{x}; \mathbf{W}). \quad \text{(ST')}$$

Since the $\mathbf{W}$ quantifier is existential and B is $\mathbf{V}$-obs'le, we can weaken $\mathbf{V} \subseteq \mathbf{W} \subset \mathbf{U}$ to $\mathbb{O}_B \subseteq \mathbf{W} \subset \mathbf{U}$. But then $\exists \mathbf{V} \subset \mathbf{U}.\ \exists B \in \mathbf{B}$ is equivalent to $\exists B \in \mathbb{S}_{\mathbf{U}}$ since there is no further mention of $\mathbf{V}$. Finally we have

$$\forall \mathbf{U} \supset \mathbb{S}.\ \exists B \in \mathbb{S}_{\mathbf{U}}.\ \forall y \in \mathbb{S}_{\mathbf{U}}.$$
$$y \in B \leftrightarrow y \in A \wedge \exists \mathbf{W}.\ \mathbb{O}_B \subseteq \mathbf{W} \subset \mathbf{U} \wedge \Phi(y, \bar{x}; \mathbf{W}). \quad \text{(ST*)}$$

However, this statement in itself implies Standardization$^\heartsuit$ by taking $\mathbf{V} = \mathbb{O}_B$ in (ST'). In words, it says the following:

> Let A be a set and Φ be a relative formula, and fix a nonstandard level $\mathbf{U}$ which "standard" refers to. Then there is a standard set B whose standard elements are exactly those of A which satisfy Φ relative to some standard level $\mathbf{W}$ in which B is obs'le.

Idealization$^\heartsuit$

Let $\mathbf{V}$ be a level which "standard" refers to, A be a standard set, and Φ be a relative formula. Then Φ is finitely satisfied relative to $\mathbf{V}$ over standard elements of A iff it is universally satisfied for all standard elements of A.

Granularity$^\heartsuit$

Every formula satisfied relative to some level has a least level it is satisfied relative to.

References

[1] Karel Hrbacek. "Relative Set Theory: Some External Issues". In: *Journal of Logic and Analysis* 2.0 (0 2010). ISSN: 1759-9008. DOI: 10.4115/jla.2010.2.8. URL: http://www.logicandanalysis.com/index.php/jla/article/view/75 (visited on 12/01/2021).