Plane Geometry from the Ground Up

Nicholas Todoroff

December 1, 2021

I. Wandering around the Plane

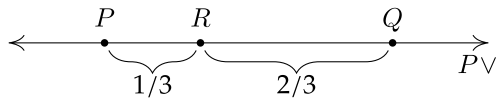
Most are familiar with plane geometry beginning with *points*, little indivisible pin-pricks which make up the plane. From this point-of-view, *lines* are something which are made up of points. We will begin our journey by throwing away this conception: we will think of lines as basic entities on *equal footing* with points. This is our first step towards teasing out the *duality* between lines and points: just as a line is made up of points, a point can be thought of as made up of all the lines which pass through it.

Practically all of this material has its start in the work of Charles Gunn; his Ph.D. thesis [1] is a good starting point.

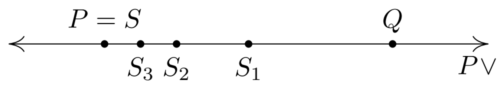
0.1 Combinations of Points

Any two points P and Q have a unique line which passes through them; we will call this line $P\vee Q$, read "P join Q". How can we describe the points which make up $P\vee Q$ knowing P and Q? Think about *ratios* of distances: we can think of the point in $P\vee Q$ between P and Q which is, say, twice as close to P as it is to Q, a ratio of $2{:}1$. (Why we use *closeness* rather than *farness* will be explained in a moment.) A ratio is not uniquely represented though; we could also have called it $10{:}5$, or $1{:}1/2$, or many other things. To make this ratio unique we need to *normalize* it. For a ratio $\alpha{:}\beta$ we will do this by requiring that $\alpha+\beta=1$, and represent the point R that has this ratio *symbolically* by $R=\alpha P+\beta Q$. We will call this symbolic combination of points an *affine combination*. In our example, since for $2{:}1$ we have $2+1=3$, we need to divide by 3 to get $2/3{:}1/3$; now $2/3+1/3=1$ but we still have the same ratio. The point given by this ratio is then the affine combination $R=\frac{2}{3}P+\frac{1}{3}Q$.

Figure 1: Point Addition



(**a**) The affine combination $R=\frac{2}{3}P+\frac{1}{3}Q$ on the line $P\vee Q$.



(**b**) The affine combination $S=0P+1Q$, with S_1,S_2,S_3 from (1) approaching it.

In Figure 1a we can see what this looks like. But why $\frac{2}{3}P+\frac{1}{3}Q$ when in the picture $\frac{1}{3}$ is associated with P and $\frac{2}{3}$ with Q? It's because affine combinations tell you *how close* to each point you are; since $2/3>1/3$, this affine combination is closer to P than Q. Indeed, it must be this way: consider the affine combination $1P+0Q$. We would *like* this to be the same as P since we would like to be able to do something like $1P+0Q=P+0=P$. Similarly, think of a sequence like

$$S_1=\frac{1}{2}P+\frac{1}{2}Q,\quad S_2=\frac{3}{4}P+\frac{1}{4}Q,\quad S_3=\frac{7}{8}P+\frac{1}{8}Q,\quad \ldots,\quad S=1P+0Q, \tag{1}$$

demonstrated in Figure 1b. The coefficient of Q is halved each time, getting closer and closer to 0, while the coefficient of P is getting closer and closer to 1. So in order to allow our little algebraic manipulation $1P+0Q=P$, the points in (1) must be getting closer and closer to P. It stands to reason that the coefficient of a point in an affine combination should tell us how close the combination is to that point.

It's important to note that our points P, Q, R and the like are *not* numbers. The use of "addition" and "multiplication" here is not the familiar addition and multiplication of numbers; an affine combination like $\frac{2}{5}P+\frac{3}{5}Q$ is just *notation* which describes a point between P and Q. This notation is meant to be suggestive of what properties these affine combinations will have as we develop our formalism further.

0.2 Combinations of Lines

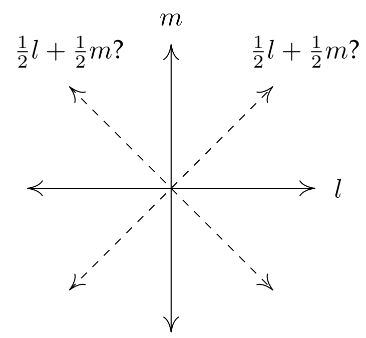
Just like two points P and Q have a line passing through them, two lines l and m have a point where they intersect; we call this $l\wedge m$, read "l meet m". Two points P and Q add together to make a point on $P\vee Q$, and *dually* two lines should add together to make a line through $l\wedge m$. However, this appears to be trickier than adding points together. We would like to say, for example, that $\frac{1}{2}l+\frac{1}{2}m$ is the line through $l\wedge m$ halfway inbetween l and m. As you can see in Figure 2a, this completely breaks down for two perpendicular lines: not only are there multiple choices we could make, there is also no way to choose one over the other. What we need is to introduce the notion of *orientation*.

Intuitively, a line splits the plane into two halves. We may *orient* a line by naming one half "positive" (+) and the other "negative" (−). The names "positive" and "negative" are just convenient labels; we could just have well used "cat" and "dog", or anything else, but "positive" and "negative" will turn out to be useful. Any two oriented lines define three types of regions:

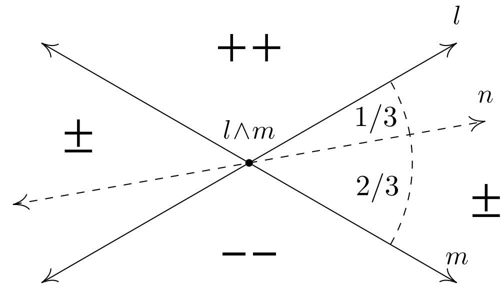
a $(++)$ region that both lines make positive; a $(--)$ region that both lines make negative; and what we'll call a $(\pm)$ region that one line makes positive and the other makes negative. Refer to Figure 2b.

Using this notion of orientation, we can now define an affine combination of lines unambiguously. Just like with points we use a ratio $\alpha{:}\beta$ (with $\alpha+\beta = 1$) for oriented lines l and m telling us *how close* the affine combination $n = \alpha l + \beta m$ is to l and m. To be clear: α (compared to β) tells us how close n is to l, and β (compared to α) tells us how close n is to m. Finally, we have the additional caveat that n will be in the $(\pm)$ region of l and m and have orientation consistent with the $(++)$ and $(--)$ regions.

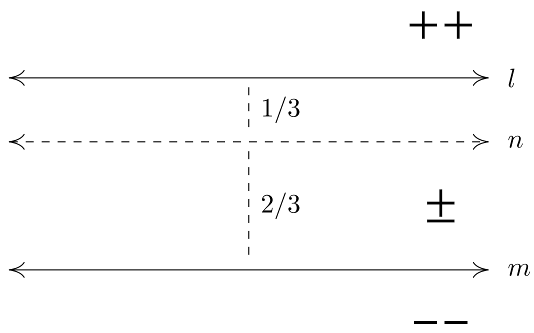
Figure 2: Line Addition



(a) Perpendicular lines l and m. Which of the dashed lines should be $\frac{1}{2}l + \frac{1}{2}m$?



(b) The affine combination $n = \frac{2}{3}l + \frac{1}{3}m$, with all the lines meeting at the point $l \wedge m$. The regions defined by the orientation of the lines are marked. Note that n is in the $(\pm)$ region of l and m, and also that l's positive side is towards the top left, m's positive side is towards the top right, and so n's positive side is up towards the $(++)$ region.



(c) The same affine combination but with l and m parallel.

References

[1] Charles Gunn. "Geometry, Kinematics, and Rigid Body Mechanics in Cayley-Klein Geometries". In: (Dec. 19, 2011). DOI: 10.14279/depositonce-3058. URL: https://depositonce.tu-berlin.de/handle/11303/3355 (visited on 05/25/2021).