Native Differentiation in an Algebra

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Let A be a finite n-dimensional real algebra and let $D \subseteq A$, and let $f : D \to A$. By $\mathbf{D}f$ we mean the total derivative of f regarded as a function $\varphi(D) \to \mathbb{R}^n$ for some choice of basis $\varphi : A \xrightarrow{\sim} \mathbb{R}^n$; we will usually use ϕ implicitly.

1. Under what conditions is $\mathbf{D}f$ represented in the algebra? I.e. when do we have
$$\mathbf{D}_{a_0} f(a) = f'(a_0)a \quad \text{or} \quad \mathbf{D}_{a_0} f(a) = af'(a_0)$$
for some function $f' : D \to A$? Call such a function the (left or right) *native derivative* of f. Is this function well defined?

2. When does (1) for f imply (1) for f'?

3. Is f' necessarily a derivation?

4. Convergence of powerseries via singularities in algebraic extension?

5. Relationship between generalized Cauchy-Riemann equations and gradients.

6. Do all gradients arise from reciprocal bases induced by a correlation?

See [1] which answers (1) and (3) mainly for unital associative algebras and touches upon (2), and see [2] which is relevant to (4).

Native Gradients

Our motivation is found in the *multivector derivative* from Clifford calculus, as can be found in [3]. For C a nondegenerate Clifford algebra, to any vector basis $(e_1, \dots, e_n)$ we may associate a unique reciprocal basis $(e^1, \dots, e^n)$ with the property that $e^i \cdot e_j = \delta^i_j$, where δ^i_j is the Kronecker delta. These extend to algebra bases

$$\{e_{i_m} \wedge \cdots \wedge e_{i_1} \mid 0 \le m \le n,\ 1 \le i_1 < \cdots < i_m \le n\},$$
$$\{e^{i_1} \wedge \cdots \wedge e^{i_m} \mid 0 \le m \le n,\ 1 \le i_1 < \cdots < i_m \le n\},$$

(including $\mathbb{1}$ in the case $m = 0$). Under the scalar product $(a, b) \mapsto \langle ab \rangle$ (the scalar projection of the algebra product) these bases are reciprocal:

$$\left\langle (e^{i_1} \wedge \cdots \wedge e^{i_m})(e_{j_m} \wedge \cdots \wedge e_{j_1}) \right\rangle = \delta^{i_m}_{j_m} \cdots \delta^{i_1}_{j_1}.$$

It is with these bases that the multivector derivative is defined as

$$\nabla f(x) = \sum_{\substack{0 \le m \le n \\ 1 \le i_1 < \cdots < i_m \le n}} (e^{i_1} \wedge \cdots \wedge e^{i_m}) \mathbf{D}_x f(e_{i_m} \wedge \cdots \wedge e_{i_1}).$$

It can be found that this definition is independent of the initial choice of basis.

To generalize to any algebra A, let $B = (\mathcal{B}, \mathcal{B}')$ be a pair of bases for A such that $\mathcal{B} = (v_1, \ldots, v_n)$ and $\mathcal{B}' = (v'_1, \ldots, v'_n)$. Then define

$$\nabla^B f(x) = \sum_{i=1}^{n} v_i \mathbf{D}_x f(v'_i).$$

For this definition to be independent of B, consider $C = (\mathcal{C}, \mathcal{C}')$ such that $\mathcal{C} = (u_1, \ldots, u_n)$ and $\mathcal{C}' = (u'_1, \ldots, u'_n)$ are bases. Let $(x_1, \ldots, x_n)$, $(x'_1, \ldots, x'_n)$, $(y_1, \ldots, y_n)$, $(y'_1, \ldots, y'_n)$ be the coordinates for $\mathcal{B}, \mathcal{B}', \mathcal{C}, \mathcal{C}'$ respectively so that what we want is

$$\nabla^B f = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x'_i} = \sum_{j=1}^{n} u_j \frac{\partial f}{\partial y'_j} = \nabla^C f,$$

where we've suppressed the dependence on x. The partial derivatives between coordinates give the change-of-basis:

$$x_i = \sum_j \frac{\partial x_i}{\partial y_j} y_j, \quad v_i = \sum_j \frac{\partial y_j}{\partial x_i} u_j.$$

Together with the chain rule, this gives

$$\sum_i \sum_j u_j \frac{\partial y_j}{\partial x_i} \frac{\partial f}{\partial x'_i} = \sum_j \sum_i u_j \frac{\partial x'_i}{\partial y'_j} \frac{\partial f}{\partial x'_i} \implies \sum_i \sum_j u_j \left(\frac{\partial y_j}{\partial x_i} - \frac{\partial x'_i}{\partial y'_j} \right) \frac{\partial f}{\partial x'_i} = 0.$$

In order to be independent of f, we need $\sum_j u_j (\partial y_j / \partial x_i - \partial x'_i / \partial y'_j) = 0$ for each i, and since (u_j) is a basis this is true iff

$$\frac{\partial y_j}{\partial x_i} - \frac{\partial x'_i}{\partial y'_j} = 0, \quad 1 \le i, j \le n.$$

Since the partials give the change-of-basis matrix, we may rewrite this condition as

$$(\mathcal{C} \leftarrow \mathcal{B})^{\mathsf{T}} = (\mathcal{B}' \leftarrow \mathcal{C}') = (\mathcal{C}' \leftarrow \mathcal{B}')^{-1}$$

where $(Y \leftarrow X)$ is the (coordinate) change-of-basis matrix from X to Y and $(\_\_)^{\mathrm{T}}$ is the transpose. It is evident that, conversely, if this conditon is true then ∇ is independent of the choice of B or C. Note that when $\mathcal{C} = \mathcal{B}'$ and $\mathcal{C}' = \mathcal{B}$ this becomes

$$(\mathcal{B}' \leftarrow \mathcal{B})^{\mathrm{T}} = (\mathcal{B}' \leftarrow \mathcal{B})$$

so that the change-of-basis matrix is symmetric.

Defining Native Derivatives

We find that whether or not A is unital makes a tremendous difference. First, the native derivative is not necessarily well defined: if f_1', f_2' are both native derivatives of f, then for all $a \in A$ we have (e.g. in the right case)

$$af_1'(a_0) = af_2'(a_0) \implies a(f_1'(a_0) - f_2'(a_0)) = 0;$$

if $f_1'(a_0) - f_2'(a_0)$ is a nonzero *null element*, an element which multiplies with any other to 0, then f_1' and f_2' need not be equal. The case of the trivial algebra where every a is null is also possible, but seems useless in this context so we will not consider it here. Non-trivial algebras with nonzero null elements do exist:

Example 1. Let $A = \mathbb{R}^2$ be an $\mathbb{R}$-algebra via $(x, y)(z, w) = (xz, 0)$ for all $x, y, z, w \in \mathbb{R}$. It is easily verified that this defines an algebra but $(0, y)(z, w) = (0, 0)$, particularly when $y \neq 0$; each $(0, y)$ is a nonzero null element. Note that this is a non-unital algebra.

When A is unital there are no nonzero null elements, since if z is null then $z = 1z = 0$. Furthermore, we may take $a = 1$ above to conclude that $f_1'(a_0) = f_2'(a_0)$ so that in a unital algebra the native derivative is always well-defined when it exists. To say more about the non-unital case, we need some more facts about null elements.

Proposition 2. *The set of null elements $\mathfrak{Z} \subseteq A$ forms a non-unital subalgebra; in particular, it is an algebra ideal.*

The proof of this is simple and direct so we omit it. Since $\mathfrak{Z}$ is an ideal, we may quotient out and arrive at a definition of native derivative. In the following, let $\pi : A \to A/\mathfrak{Z}$ be the canonical projection.

Definition 3. Let $f : D \to A$ and $a_0 \in D$. Then f is (*left*, *right*) *natively differentiable at* a_0 or just (*left*, *right*) *differentiable at* a_0 if f is totally differentiable at a_0 and there is $\xi \in A$ such that for all $a \in A$.

$$\text{(left)} \quad \mathbf{D}_{a_0} f(a) = \xi a, \qquad \text{(right)} \quad \mathbf{D}_{a_0} f(a) = a\xi$$

f is (*right*, *left*)*-differentiable* if it is (right, left)-differentiable at each $a_0 \in D$. Then we define the *right derivative* of f to be the unique function $f^R : D \to A/\mathfrak{Z}$ such that

$$\pi\Big(\mathbf{D}_a f(b)\Big) = \pi(b) f^R(a)$$

for all $b \in A$. We similarly define the *left derivative* f^L.

Appendix

1 ——— Structure of Zero Divisors

Suppose A is an associative unital n-dimensional algebra. Then the (*left*) *regular* (*matrix*) $\mathcal{B}$-*representation* of A for $\mathcal{B}$ a basis is

$$\mathcal{M}_{\mathcal{B}}(A) = \left\{ \Big([ae_1]_{\mathcal{B}}, [ae_2]_{\mathcal{B}}, \ldots, [ae_n]_{\mathcal{B}} \Big) \;\Big|\; a \in A \right\}$$

for $\mathcal{B} = (e_1, \ldots, e_n)$; that is, the collection of all matrices with columns $ae_1, \ldots, ae_2$ represented in $\mathcal{B}$-coordinates. There is an isomorphism $\Phi_{\mathcal{B}} : A \to \mathcal{M}_{\mathcal{B}}(A)$.

For each $a \in A$, we may define the *left* and *right annihilators* to be

$$@_L a = \{a' \in A \;\mid\; a'a = 0\}, \quad @_R a = \{a' \in A \;\mid\; aa' = 0\}.$$

It is easy to see that these are linear subsets, and must be at most $n-1$ dimensional since A is unital and $\mathbb{1} \notin @_L a, @_R a$ for nonzero a. Then the set of all zero divisors may be expressed as

$$\text{zd}(A) := \{a \in A \;\mid\; \exists a' \in A \backslash \{0\}. aa' = 0 \text{ or } a'a = 0\} = \bigcup_{a \in A \backslash \{0\}} @_L a \cup @_R a.$$

This is to say that the set of zero divisors is a union of linear subspaces each with dimension $\leq n-1$; however, given any basis $\mathcal{B}$, the zeros divisors can also be described as the set

$$\text{zd}(A) = \{a \in A \;\mid\; \det \Phi_{\mathcal{B}}(a) = 0\},$$

the roots of the polynomial $\det \Phi_{\mathcal{B}}(a)$. In order for zd$(A)$ to be a union of linear subspaces, such a polynomial must have a linear factor for each such subspace. But there can only be *finitely many* such factors. It follows that zd(A) is a *finite* union of linear subspaces of A with dimension at most $n-1$. Note also that $\det \Phi_{\mathcal{B}}(a)$ is a degree-n polynomial and can have at most n linear factors, so zd(A) is made up of at most n such linear subspaces.

From this fact, we may constuct an invertible basis for A. Let zd(A) = $\bigsqcup_{k=1}^{m} Z_k$ such that $m \leq n$ and each $\varnothing \neq Z_k \subseteq A$ is a linear subspace. We note that $\mathbb{1} \notin Z_k$ for any k, and so for any scalar $c \neq 0$ and $z \in Z_k$ it must be that $z + c\mathbb{1} \notin Z_k$. Then given a basis $(\mathbb{1}, v_2, v_3, \ldots, v_n)$, we may construct an invertible basis $(\mathbb{1}, v_2', v_3', \ldots, v_n')$ as follows: For any i where v_i is invertible, take $v_i' = v_i$. Then suppose WLOG that $v_i \in Z_1$. We have $v_i + \mathbb{1} \notin Z_1$; if this is invertible, then we're done and take this as v_i'. Otherwise $v_i + \mathbb{1} \in Z_2$ WLOG, and then $v_i + \mathbb{1} + \mathbb{1} \notin Z_2$ but also still $\notin Z_1$; if this is invertible, then take this as v_i'. Continue this process until $v_i + l\mathbb{1}$ is invertible for some positive integer l; this process terminates since there are only finitely many Z_k. Some version of this will work over any infinite field.

2 Reciprocal Bases

Let A be an n-dimensional K-vector space, and let A^* be its dual. Then for each basis $\mathcal{B} = (e_1, \ldots, e_n)$ of A there is a unique *dual basis* $\mathcal{B}^* = (\varepsilon^1, \ldots, \varepsilon^n)$ of A^* such that $\varepsilon^j(e_k) = \delta^j_k$. We use the convention that $(\mathcal{B}^*)^* = \mathcal{B}$. Every nondegenerate correlation $\xi : A \to A^*$ with induced bilinear form $\langle \_\_, \_\_ \rangle : A \times A \to K$ then gives rise to *reciprocal bases*.

Definition 4. The *left reciprocal basis* of $\mathcal{B}$ is $\mathcal{B}^\xi_L = \xi^{-1}[\mathcal{B}^*]$, and the *right reciprocal basis* of $\mathcal{B}$ is $\mathcal{B}^\xi_R = \xi[\mathcal{B}]^*$. If $\mathcal{B}^\xi_L = (l^1, \ldots, l^n)$ and $\mathcal{B}^\xi_R = (r^1, \ldots, r^n)$ then

$$\langle l^j, e_k \rangle = \langle e_k, r^j \rangle = \delta^j_k, \quad \forall j, k = 1, \ldots, n.$$

Proposition 5. *$\mathcal{B}^\xi_L = \mathcal{B}^\xi_R$ iff $\langle \_\_, \_\_ \rangle$ is symmetric; $\mathcal{B}^\xi_L = -\mathcal{B}^\xi_R$ iff $\langle \_\_, \_\_ \rangle$ is antisymmetric.*

Proof. We give the proof for the symmetric case, and the antisymmetric case is very similar. ($\Longrightarrow$) Let $e^j := l^j = r^j$. Trivially $\langle e^j, e_k \rangle = \langle e_k, e^j \rangle$, from which it follows that $\langle \_\_, \_\_ \rangle$ is symmetric. ($\Longleftarrow$) We have $\langle e_j, l^k \rangle = \langle l^k, e_j \rangle = \delta^k_j$, but this means that $\mathcal{B}^\xi_L$ is the unique right reciprocal of $\mathcal{B}$; thus $\mathcal{B}^\xi_L = \mathcal{B}^\xi_R$. ☐

Let $T = (\mathcal{B}^\xi_L \leftarrow \mathcal{B})$. Using the convention that repeated indices are summed over, we find that

$$\delta^j_k = \langle l^j, e_k \rangle = \langle l^j, l^i T_{ik} \rangle = \langle l^j, l^i \rangle T_{ik},$$

but this says that $\langle l^j, l^i \rangle = T^{-1}_{ji}$. Similar considerations give us the following proposition:

Proposition 6. *Let $T = (\mathcal{B}^\xi_L \leftarrow \mathcal{B})$ and $U = (\mathcal{B}^\xi_R \leftarrow \mathcal{B})$. Then*

$$\langle l^j, l^k \rangle = T^{-1}_{jk}, \quad \langle r^j, r^k \rangle = U^{-1}_{kj},$$
$$T_{kj} = \langle e_j, e_k \rangle = U_{jk},$$

where the last equation is equivalent to $T^{\mathrm{T}} = U$.

Corollary 7. *If $\langle \_\_, \_\_ \rangle$ is symmetric then $(\mathcal{B}^\xi \leftarrow \mathcal{B})$ is symmetric.*

References

[1] James S. Cook. *Introduction to $\mathcal{A}$-Calculus*. Aug. 4, 2017. arXiv: 1708.04135 [math]. URL: http://arxiv.org/abs/1708.04135 (visited on 04/28/2021).

[2] James S. Cook and Daniel Freese. *Theory of Series in the $\mathcal{A}$-Calculus and the N-Pythagorean Theorem*. Aug. 13, 2018. arXiv: 1708.04136 [math]. URL: http://arxiv.org/abs/1708.04136 (visited on 04/29/2021).

[3] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. Cambridge: Cambridge University Press, 2003.