Notes on Arkeryd, Cutland, and Henson's *Nonstandard Analysis: Theory and Applications*

Nicholas Todoroff

October 1, 2020

Notation

- For a function $f : X \to Y$ and subsets $A \subseteq X$, $B \subseteq Y$, we denote the image and preimage maps by
$$f[A] = \{f(x) \in Y \mid x \in A\}, \quad f^{-1}[B] = \{x \in X \mid f(x) \in B\}.$$
- If set notation is used simultaneously, we also allow direct juxtaposition; in particular, $f\{x\} = \{f(x)\}$ and $f^{-1}\{y\} = \{x \in A \mid f(x) = y\}$.
- Similarly, if A is a set then we will use the notation ${}^*[A]$ to mean $\{{}^*a \mid a \in A\}$.
- If $f : X \to X'$ and $g : Y \to Y'$, then $f \times g : X \times Y \to X' \times Y'$ such that $(f \times g)(x, y) = (f(x), g(y))$.
- We abbreviate the Internal Definition Principle by IDP, and say that a set that satisfies the internal definition principle "satisifies IDP".
- For a set S and a binary relation R, we will use the notation $S_{Rx} = \{s \in S \mid sRx\}$ for any appropriate x. We may also use the notation $S_R x = S_{Rx}$ when convenient; S_R may be thought of as a function $\mathrm{dom}(R) \to \mathcal{P}(S)$.
- Rather than the notation $\mathrm{monad}(x) = \{y \in {}^*\mathbb{R} \mid y \precsim x\}$, we will use the notation ${}^*\mathbb{R}_{\precsim x} = \mathrm{monad}(x)$, the *halo* around x. We will also use $\mathrm{Inf}(S) = S_{\precsim -\infty} \cup S_{\precsim \infty}$ and $\mathrm{Fin}(S) = S_{\not\precsim \infty} \cap S_{\not\precsim -\infty}$ for $S \subseteq {}^*\mathbb{R}$.
- We will use $\langle$ and $\rangle$ in interval notation to denote the exclusion of a halo. For example, $\langle 1, \infty) = {}^*(1, \infty) \backslash {}^*\mathbb{R}_{\precsim 1}$. As a special case, using $\langle -\infty$ and $\infty\rangle$ denotes finite ranges. For example,
$$(0, \infty\rangle = {}^*(0, \infty) \backslash {}^*\mathbb{R}_{\precsim \infty}, \quad \langle -\infty, -23\rangle = {}^*(-\infty, -23) \backslash {}^*\mathbb{R}_{\precsim -\infty} \backslash {}^*\mathbb{R}_{\precsim -23}.$$
- Instead of the notation $x \approx y$ to mean that $x - y$ is infinitesimal, we will use the notation $x \precsim y$. $x \precsim y$ can be interpreted as x is "on the order of" y.

- For $x \in {}^*\mathbb{R}$, we will use the following shorthand notation

$$
\begin{aligned}
x \lesssim y &\iff y \eqsim x \leq y, & x \not\eqsim y &\iff y \neq x \eqsim y,\\
x \gtrsim y &\iff y \eqsim x \geq y, & x \eqsim y &\iff x \in {}^*\mathbb{R}_{\eqsim y},\\
x \lnsim y &\iff y \not\eqsim x < y, & x \eqsim \infty &\iff x \in \operatorname{Inf}({}^*\mathbb{R}_{>0}),\\
x \gnsim y &\iff y \not\eqsim x > y, & x \eqsim -\infty &\iff x \in \operatorname{Inf}({}^*\mathbb{R}_{<0}).
\end{aligned}
$$

- We will typically denote a sequence a by e.g. $a \in \mathbb{R}^{\mathbb{N}}$ rather than $a : \mathbb{N} \to \mathbb{R}$ or $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ or similar. This notation is compact, references the functional nature of sequences, and yet also signals that we are considering a as a "sequence" and not as a "function". The notations a_n and $a(n)$ are interchangeable.

- If $a \in A^{\mathbb{N}}$ for some set A, then $\lim a = \lim(a) = \lim_{n\to\infty} a_n$.

- Let $\phi(x)$ be a formula over $\mathbb{R}$ for x ranging over $\mathbb{R}$, and let $\psi(X)$ be a formula over ${}^*\mathbb{R}$ for X ranging over ${}^*\mathbb{R}$. Then if

$$\exists z \forall (x \geq z).\phi(x), \quad \exists Z \forall (X \geq z).\psi(X)$$

  for z ranging over $\mathbb{R}$ and Z ranging over ${}^*\mathbb{R}$, we say that ϕ is true *for x eventually* over $\mathbb{R}$ and ψ is true *for X eventually* over ${}^*\mathbb{R}$. If $\forall Y.\psi(Y)$ where Y ranges over $\operatorname{Inf}({}^*\mathbb{R}_{>0})$, we say that ψ is true *for Y ultimately*.

- A formula ψ in ${}^*\mathbb{R}$ is *transferable* if there is a formula ϕ in $\mathbb{R}$ such that $\psi \iff {}^*\phi$.

- Let ψ be a formula in ${}^*\mathbb{R}$ such that

$$\psi(X) \iff \forall \alpha.{}^*\phi(\alpha, X)$$

  is a true sentence where ϕ is a formula in $\mathbb{R}$, and X ranges over ${}^*\mathbb{R}$ and α ranges over a subset of $\mathbb{R}$. Then we say that ψ is *parametrically transferable* in X.

- ${}^\circ x$ is the standard part of x. If $A \subseteq {}^*\mathbb{R}$, then we will use ${}^\circ A = \{{}^\circ x \mid x \in A,\ |x| \not\eqsim \infty\}$ rather than $\operatorname{st}(A)$.

- Rather than *S-continuous*, we will use the term *microcontinuous*.

I. Foundations Of Nonstandard Analysis

2 Nonstandard Extensions

Exercises

<u>*Exercise 2.19*</u>*: Extensions preserve generalized diagonal sets*

Let $\Delta \subseteq \mathbb{X}^n$ be a generalized diagonal set, i.e., there is an equivalence relation E over $\{1, \ldots, n\}$ such that

$$\Delta = \{(x_1, \ldots, x_n) \in \mathbb{X}^n \mid x_i = x_j \text{ whenever } iEj\}.$$

Then we claim that

$$^*\Delta = \{(x_1, \ldots, x_n) \in {}^*\mathbb{X}^n \mid x_i = x_j \text{ whenever } iEj\}. \tag{$*$}$$

E defines equivalence classes $[1], [2], \ldots, [n]$; note that in the following we need not worry about integers that have the same class. Using these, the definition of Δ expands to

$$\Delta = \bigcap_{k=1}^{n} \bigcap_{i,j \in [k]} \Big\{(x_1, \ldots, x_n) \in \mathbb{X}^n \;\Big|\; x_i = x_j\Big\}.$$

Since the intersections are finite and the inner sets are simple diagonal, we can take extensions through and conclude $(*)$.

<u>*Exercise 2.20*</u>*: Extensions of finite sets are standard*

Let $A \subseteq \mathbb{X}^m$ be finite. Then $A = \bigcup_{x \in A} \{x_1\} \times \{x_2\} \times \cdots \times \{x_m\}$, where $x_1, \ldots, x_m$ are the coordinates of $x \in A$. Since this is a finite union of finite cartesian products, we can take extensions through all operations and conclude that $^*A = \{^*x \mid x \in A\}$.

<u>*Exercise 2.22*</u>*: Proper extensions of infinite sets contain nonstandard elements*

We proceed via induction. Since $*$ is a proper extension, the base case is trivial since by definition $A \subseteq \mathbb{X}$ being infinite implies $^*A$ contains nonstandard elements. Now suppose that if $A \subseteq \mathbb{X}^m$ is infinite then A contains nonstandard elements. For any infinite $B \subseteq \mathbb{X}^{m+1}$, consider the projections

$$\begin{aligned} \pi_1 &: \mathbb{X}^{m+1} \to \mathbb{X}^m, \quad \pi_1(x_1, \ldots, x_{m+1}) = (x_1, \ldots, x_m), \\ \pi_2 &: \mathbb{X}^{m+1} \to \mathbb{X}, \qquad \pi_2(x_1, \ldots, x_{m+1}) = x_{m+1}. \end{aligned}$$

Then $B \subseteq \pi_1(B) \times \pi_2(B)$ and at least one of $\pi_1(B)$ or $\pi_2(B)$ is infinite. If $\pi_1(B)$ is infinite, then it follows from the inductive hypothesis that ${}^*(\pi_1(B)) = {}^*\pi_1({}^*B)$ contains a nonstandard element, so necessarily ${}^*B$ does as well. Similarly, if $\pi_2(B)$ is infinite, then since $*$ is proper ${}^*(\pi_2(B)) = {}^*\pi_2({}^*B)$ contains a nonstandard element, so ${}^*B$ must as well.

Exercise 2.23: Extensions preserve image maps

Let $f : A \to \mathbb{X}^n$ for some $A \subseteq \mathbb{X}^m$. Then for any $B \subseteq A$, we may write $f[B]$ in terms of the graph Γ_f of f as $f[B] = \pi[\Gamma_f \cap (B \times \mathbb{X}^n)]$, where $\pi_1 : \mathbb{X}^{m+n} \to \mathbb{X}^n$ is the projection $\pi_1(x_1, \ldots, x_m, y_1, \ldots, y_n) = (y_1, \ldots, y_n)$. Then it follows that

$$^*(f[B]) = {}^*\pi_1[\Gamma_{^*f} \cap ({}^*B \times {}^*\mathbb{X}^n)] = {}^*f[{}^*B].$$

Similarly, for $C \subseteq \mathbb{X}^n$, we may define $\pi_2 : \mathbb{X}^{m+n} \to \mathbb{X}^m$ where $\pi_2(x_1, \ldots, x_m, y_1, \ldots, y_n) = (x_1, \ldots, x_m)$ so that

$$^*(f^{-1}[C]) = {}^*\pi_2[\Gamma_{^*f} \cap ({}^*\mathbb{X}^m \times {}^*C)] = ({}^*f)^{-1}[{}^*C].$$

Finally, for the restriction $f|_B$ we have ${}^*(f|_B) = {}^*f|_{^*B}$ since

$$\Gamma_{^*(f|_B)} = {}^*(\Gamma_{f|_B}) = \Gamma_{^*f} \cap ({}^*B \times {}^*\mathbb{X}^n) = \Gamma_{^*f|_{^*B}}.$$

Exercise 2.24: Extensions preserve coordinate functions

Let $f_j : \mathbb{X}^m \to \mathbb{X}$ for $j = 1, \ldots, n$ so that $f = (f_1, \ldots, f_n) : \mathbb{X}^m \to \mathbb{X}^n$. Then for the projection $\pi : \mathbb{X}^{n(m+n)} \to \mathbb{X}^{m+n}$ defined by

$$\pi(x^{(1)}, y_1, \ldots, x^{(n)}, y_n) = (x^{(1)}, y_1, y_2, \ldots, y_n), \quad x^{(k)} \in \mathbb{X}^m,\ y_k \in \mathbb{X} \quad \text{for } k = 1, \ldots, n,$$

as well as the diagonal set

$$\Delta = \big\{(x, y_1, x, y_2, \ldots, x, y_n) \in \mathbb{X}^{n(m+n)} \ \big|\ x \in \mathbb{X}^m,\ y_1, \ldots, y_n \in \mathbb{X}\big\},$$

we may express the graph of f as $\Gamma_f = \pi\big[\Delta \cap (\Gamma_{f_1} \times \ldots \times \Gamma_{f_n})\big]$. It follows immediately that ${}^*f = ({}^*f_1, \ldots, {}^*f_n)$.

Exercise 2.25: Extensions preserve injectivity and sujectivity

Let $f : A \to B$ where $A \subseteq \mathbb{X}^m$ and $B \subseteq \mathbb{X}^n$. f is injective iff

$$(\Gamma_f \times \Gamma_f) \cap \Delta_1 \subseteq \Delta_2, \quad \text{where} \quad \begin{array}{l} \Delta_1 = \{(x_1, y, x_2, y) \ |\ x_1, x_2 \in \mathbb{X}^m,\ y \in \mathbb{X}^n\}, \\ \Delta_2 = \{(x, y, x, y) \ |\ x \in \mathbb{X}^m,\ y \in \mathbb{X}^n\}. \end{array}$$

That is, f is injective iff taking pairs of inputs (x_1, x_2) that have the same output y and forming the set of all (x_1, y, x_2, y) implies that there is only one such input $x_1 = x_2$. It immediately follows that ${}^*f$ is injective, and furthermore that ${}^*f$ injective implies f is injective:

$$(\Gamma_f \times \Gamma_f) \cap \Delta_1 \subseteq \Delta_2 \iff (\Gamma_{^*f} \times \Gamma_{^*f}) \cap {}^*\Delta_1 \subseteq {}^*\Delta_2.$$

f is surjective iff $\pi[\Gamma_f] = B$, where $\pi(x, y) = y$ for $x \in \mathbb{X}^m$, $y \in \mathbb{X}^n$. Clearly $\pi[\Gamma_f] = B$ iff ${}^*\pi[\Gamma_{{}^*f}] = {}^*B$, so f is sujective iff ${}^*f$ is as well.

As such, f is bijective iff ${}^*f$ is bijective. Furthermore, if f has inverse g, then

$$g \circ f = 1_A \implies {}^*g \circ {}^*f = 1_{{}^*A},$$

so ${}^*g$ is the inverse of ${}^*f$.

<u>*Exercise 2.26*</u>: *${}^*\mathbb{R}$ is an ordered field*

We consider each operation ${}^*+, {}^*\times, {}^*<$ and their properties in turn. We will use the notation $[n_1, \ldots, n_N]$ for N positive integers $n_1, \ldots, n_N$ to denote the corresponding diagonal set $\big\{(x_{n_1}, \ldots, x_{n_N}) \in \mathbb{R}^N \;\big|\; x_{n_1}, \ldots, x_{n_N} \in \mathbb{R}\big\}$. Internal parentheses will be used for grouping and repeated elements bolded to aid interpretation, e.g.,

$$[(1, 2, \mathbf{3}), (\mathbf{3}, 4, 5)] = [1, 2, 3, 3, 4, 5] = \big\{(x_1, x_2, x_3, x_3, x_4, x_5) \in \mathbb{R}^6 \;\big|\; x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\big\}.$$

Assoc. of ${}^*+$: The associativity of $+$ may be expressed in terms of its graph Γ_+ as

$$\begin{aligned} (\Gamma_+ \times \Gamma_+ \times \Gamma_+ \times \Gamma_+) \cap \Delta_{\text{assoc}} &\subseteq \Delta_{\text{assoc-eq}}, \\ \Delta_{\text{assoc}} &= [(\mathbf{1}, \mathbf{2}, \mathbf{3}), (\mathbf{3}, \mathbf{4}, 5), (\mathbf{1}, \mathbf{6}, 7), (\mathbf{2}, \mathbf{4}, \mathbf{6})], \\ \Delta_{\text{assoc-eq}} &= [(1, 2, 3), (4, 5, \mathbf{6}), (7, 8, \mathbf{6}), (9, 10, 11)]. \end{aligned}$$

Δ_{assoc} encodes all pairs of additions $(x_1+x_2)+x_4$ and $x_1+(x_2+x_4)$, and $\Delta_{\text{assoc-eq}}$ asserts their equality. Each operation in the above relation is preserved under extension, so ${}^*+$ is also associative.

Comm. of ${}^*+$: The commutivity of $+$ may be expressed in terms of Γ_+ as

$$(\Gamma_+ \times \Gamma_+) \cap \Delta_{\text{comm}} \subseteq \Delta_{\text{comm-eq}}, \quad \begin{aligned} \Delta_{\text{comm}} &= [(\mathbf{1}, \mathbf{2}, 3), (\mathbf{2}, \mathbf{1}, 4)], \\ \Delta_{\text{comm-eq}} &= [(1, 2, \mathbf{3}), (4, 5, \mathbf{3})]. \end{aligned}$$

Similarly to above, Δ_{comm} encodes the pairs of additions $x_1 + x_2$ and $x_2 + x_1$, and $\Delta_{\text{comm-eq}}$ asserts that their equality. It follows that ${}^*+$ is commutative.

Identity elem. of ${}^*+$: The identity element 0 of $+$ is defined by the property

$$\Gamma_+ \cap (\mathbb{R} \times \{0\} \times \mathbb{R}) \subseteq \Delta_{\text{id-eq}}, \quad \Delta_{\text{id-eq}} = [\mathbf{1}, 2, \mathbf{1}].$$

The intersection fixes the right input of $+$ to 0, and $\Delta_{\text{id-eq}}$ asserts that such an addition preserves the left input. The extends immediately to the fact that ${}^*+$ has an identity which is ^{*0}.

Inverse elems. of ${}^*+$: The existence of inverse elements for $+$ may be stated as

$$\pi\big[\Gamma_+ \cap (\mathbb{R} \times \mathbb{R} \times \{0\})\big] = \mathbb{R}, \quad \pi(x_1, x_2, y) = x_1.$$

The intersection fixes the output to be 0, and the projection π asserts that every element of $\mathbb{R}$ appears on the left side of such an addition. This extends immediately to the fact that ${}^*+$ has inverses.

Properties of ${}^*\times$: The case of ${}^*\times$ is completely analogous to the corresponding properties of ${}^*+$. We write down the properties of $\times$, which extend simply to ${}^*\times$, without comment.

$$(\Gamma_\times \times \Gamma_\times \times \Gamma_\times \times \Gamma_\times) \cap \Delta_{\text{assoc}} \subseteq \Delta_{\text{assoc-eq}}, \qquad (\Gamma_\times \times \Gamma_\times) \cap \Delta_{\text{comm}} \subseteq \Delta_{\text{comm-eq}},$$
$$\Gamma_\times \cap (\mathbb{R} \times \{1\} \times \mathbb{R}) \subseteq \Delta_{\text{id-eq}}, \qquad \pi\big[\Gamma_\times \cap (\mathbb{R} \times \mathbb{R} \times \{1\})\big] = \mathbb{R}\backslash\{0\}.$$

The relation ${}^*<$: Instead of $<$, we equivalently consider $\leq$. The relation $\leq$ is defined by the set $R_\leq = \{(a,b) \mid a \leq b,\ a,b \in \mathbb{R}\}$ with the defining properties of totality, transitivity, and antisymmetry expressible as

$$R_\leq \cup \sigma(R_\leq) = \mathbb{R}^2, \quad \pi_{\text{trans}}\big[(R_\leq \times R_\leq) \cap \Delta_{\text{trans}}\big] \subseteq R_\leq, \quad (R_\leq \times R_\leq) \cap \Delta_{\text{anti}} \subseteq \Delta_{\text{anti-eq}},$$

respectively, where

$$\sigma(a,b) = (b,a), \qquad \Delta_{\text{trans}} = [(1,\mathbf{2}),(\mathbf{2},3)], \quad \Delta_{\text{anti}} = [(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1})],$$
$$\pi_{\text{trans}}(a_1,b_1,a_2,b_2) = (a_1,b_2), \quad \Delta_{\text{anti-eq}} = [(\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{1})].$$

The first relation states that every pair of numbers or its reverse must be in $R_\leq$. The second relation states that for every pair of elements $(a,b),(b,c)$ in $R_\leq$, there must be an element (a,c) in $R_\leq$. Finally, the last relation states that if both of a pair of elements $(a,b),(b,a)$ are in $R_\leq$, then it must be that $(a,b,b,a) = (a,a,a,a)$, which is to say that $a = b$. These properties characteize $\leq$, and and as presented extend simply to ${}^*\leq$, which we take as being defined by ${}^*R_\leq$.

Exercise 2.29: Basic properties of ultrapower extensions

Let J be some infinite index set and $\mathcal{U}$ a free countably-incomplete ultrafilter on J, so that ${}^*\mathbb{X} = \mathbb{X}^J/\mathcal{U}$ is the ultrapower extension of $\mathbb{X}$. We consider $\mathbb{X}^J$ as the space of functions $J \to \mathbb{X}$ and $\mathbb{X}^{k\times J} := (\mathbb{X}^k)^J$ the space of functions $J \to \mathbb{X}^k$. Further, the equivalence class of $\alpha \in \mathbb{X}^J$ is written $[\alpha] \in {}^*\mathbb{X}$, with $[(\beta_1,\dots,\beta_m)] = ([\beta_1],\dots,[\beta_m]) \in {}^*\mathbb{X}^m$ for $(\beta_1,\dots,\beta_m) \in \mathbb{X}^{m\times J}$. With this notation, we may write for any $A \subseteq \mathbb{X}^m$, $m \in \mathbb{N}$, that

$${}^*A := \big\{[\alpha] \in {}^*\mathbb{X}^m \ \big|\ \alpha \in \mathbb{X}^{m\times J},\ \alpha^{-1}[A] \in \mathcal{U}\big\}.$$

Let $a \in \mathbb{X}$. Then by this definition

$$\{{}^*a\} = {}^*\{a\} = \big\{[\alpha] \in {}^*\mathbb{X} \ \big|\ \alpha \in \mathbb{X}^J,\ \alpha^{-1}\{a\} \in \mathcal{U}\big\}.$$

The above set is a singleton, so all $\mathcal{A} = \big\{\alpha \in \mathbb{X}^J \ \big|\ \alpha^{-1}\{a\} \in \mathcal{U}\big\}$ must belong to the same equivalence class in ${}^*\mathbb{X}$. In particular, let $\beta \in \mathbb{X}^J$ be defined by $\beta(j) = a$ for all $j \in J$. Then for $\alpha \in \mathcal{A}$, the condition $a = \beta(j) = \alpha(j)$ means exactly that $j \in \alpha^{-1}\{a\} \in \mathcal{U}$, so $[\beta] = [\alpha]$. In particular, ${}^*a = [\beta]$.

Given a function $f : A \to B$ with $A \subseteq \mathbb{X}^m$, $B \subseteq \mathbb{X}^n$, the graph Γ_f extends to

$$\Gamma_{{}^*f} = {}^*\Gamma_f = \big\{[\alpha] \times [\beta] \in {}^*\mathbb{X}^{m+n} \ \big|\ \alpha \in \mathbb{X}^{m\times J},\ \beta \in \mathbb{X}^{n\times J},\ (\alpha\times\beta)^{-1}(\Gamma_f) \in \mathcal{U}\big\}.$$

In other words, $\Gamma_{{}^*f}$ is the set of all pairs $[\alpha],[\beta]$ such that $\beta = f \circ \alpha$ almost everywhere (relative to $\mathcal{U}$). But since we're working with equivalence classes, given α we can just select β as being defined by $\beta(j) = f(\alpha(j))$ for all $j \in J$. This covers all elements $[\alpha] \times [\beta] \in \Gamma_{{}^*f}$, and so ${}^*f([\alpha]) = [\beta]$ for all $\alpha \in A^J$.

Exercise 2.30: Ultrapower exetensions of subsets are ultrapowers

Let ${}^*\mathbb{X}$ be an extension as specified in the previous exercise. For any $m \in \mathbb{N}$ and $A \subseteq \mathbb{X}^m$, we wish to identify ${}^*A$ with $A^J/\mathcal{U}$. An element $[\alpha]_{\mathbb{X}} \in {}^*A$ is a tuple of equivalence classes of the coordinates of some function $\alpha \in \mathbb{X}^{m\times j}$ which almost-everywhere takes values in A. An element $[\beta]_A \in A^J/\mathcal{U}$ is *directly* an equivalence class of some function $\beta \in A^J$. But if β always takes values in A, it of course does so almost-everywhere; that means that

$$[\beta]_A \in A^J/\mathcal{U} \implies [\beta]_{\mathbb{X}} \in {}^*A.$$

In the other direction, for any $a \in {}^*A$ we can simply choose a representative $\alpha \in \mathbb{X}^{m\times J}$ such that $[\alpha]_{\mathbb{X}} = a$ and $\alpha(J) = A$; we can do this, since given $\alpha' \in \mathbb{X}^{m\times J}$ such that $[\alpha'] = a$, we can define $\alpha(j) = \alpha'(j)$ for $j \in (\alpha')^{-1}(A)$ and assign arbitrary values to $\alpha(j)$ where $j \notin (\alpha')^{-1}(A)$. It thus follows that

$$[\alpha]_{\mathbb{X}} \in {}^*A \implies [\alpha]_A \in A^J,$$

completing the correspondence.

Exercise 2.31: Infinite and infinitesimal elements of $\mathbb{N}$-ultrapowers of $\mathbb{R}$

Let $\mathcal{U}$ be a free countably-incomplete ultrafilter over $\mathbb{N}$, and let ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$.

Consider the sequences $\infty_1, \infty_2, \infty_3 \in \mathbb{R}^{\mathbb{N}}$ where

$$\infty_1(n) = n, \quad \infty_2(n) = n\sqrt{2}, \quad \infty_3(n) = e^n.$$

The extended elements $[\infty_1], [\infty_2], [\infty_3]$ are not equal, since they are only equivalent as functions for at most finitely many $n \in \mathbb{N}$ and $\mathcal{U}$ cannot contain finite sets. They are also non-standard, since sequences corresponding to standard elements must be $\mathcal{U}$-almost everywhere constant. The extended order relation ${}^*<$ is defined by the extension of $\mathbb{R}_< = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$, specifically

$${}^*R_< = \{([\alpha], [\beta]) \in {}^*\mathbb{R}^2 \mid \alpha, \beta \in \mathbb{R}^{\mathbb{N}},\ (\alpha \times \beta)^{-1}(R_<) \in \mathcal{U}\}.$$

This means that $[\alpha] < [\beta]$ exactly when $\alpha(n) < \beta(n)$ for all n in some element of $\mathcal{U}$. Now let $x \in \mathbb{R}$. Since $\mathcal{U}$ is free, it must contain the Fréchet filter of cofinite sets; in particular, $\{n \in \mathbb{N} \mid n > \alpha\} \in \mathcal{U}$ for all positive $\alpha \in \mathbb{R}$ since these sets have finite complements. It follows that $[\infty_1], [\infty_2], [\infty_3]$ are all greater than any standard element, and in fact that this is true of any eventually increasing sequence. Comparing ∞_1 and ∞_2, we see also that $[\infty_1] \,{}^*\!< [\infty_2]$, since $n < n\sqrt{2}$ for all $n \neq 0$, and similarly $[\infty_2] \,{}^*\!< [\infty_3]$.

It should be clear at this point that the hierarchy of infinities (at least) contains the hierarchy of increasing sequences compared by their long-term behavior.

In the other direction, we consider the sequences

$$\epsilon_1(n) = 1/n, \quad \epsilon_2(n) = 1/n^\pi, \quad \epsilon_3(n) = 1/\sqrt{2^n}.$$

These are eventually *less* than any postitive constant sequence, and any eventually positive-decreasing sequence has this property. We also see that

$$n^\pi > n \implies 1/n^\pi < n \quad \text{for } n > 0,$$
$$\sqrt{2^n} > n^\pi \implies 1/\sqrt{2^n} < 1/n^\pi \quad \text{for } n > 31,$$

so $[\epsilon_3] \,{}^*\!< [\epsilon_2] \,{}^*\!< [\epsilon_1]$.

3 Logical Formulas

Notes

(p.18) Equality is *not* explicitly handled in the $*$-transform nor the transfer principle. The statement $x = y$ can be handled either by considering an appropriate identity function f and forming the sentence $f(x) = y$ or $f(y) = x$, or forming an appropriate diagonal set A so that $(x, y) \in A \iff x = y$. These approaches are equivalent, since the graph of an identity function is exactly such a diagonal set.

This is related to comments made in [p.16 ¶3], though those comments concern relations, which are usually less fundamental than equality.

5 Nonstandard Extensions of the Multiset...

Notes

(p.29) The map Φ can be seen to be injective as follows: the set $E = \{(x, A) \in \mathbb{X} \times \mathcal{P}(\mathbb{X}) \mid x \in A\}$ satisfies the formula

$$\forall A\, \forall B \Big(\forall x.\ (x, A) \in E \leftrightarrow (x, B) \in E \Big) \leftrightarrow A = B,$$

where A, B range over $\mathcal{P}(\mathbb{X})$ and x over $\mathbb{X}$. By the transfer principle, ${}^*E$ satifies this property for A, B over ${}^*\mathcal{P}(\mathbb{X})$ and x over ${}^*\mathbb{X}$. If $\Phi(Y) = \Phi(Y')$ for $Y, Y' \in {}^*\mathcal{P}(\mathbb{X})$, then since h is a bijection this is equivalent to

$$\{x \in {}^*\mathbb{X} \mid (x, Y) \in {}^*E\} = \{x \in {}^*\mathbb{X} \mid (x, Y') \in {}^*E\},$$

but using the above formula for $A = Y$ and $B = Y'$ tells us that $Y = Y'$.

The new extension generated by h satisfy the properties (a) and (b) on [p.39]. The first property is true by definition of h. For the second property, what we want to prove is that

$$\Phi[{}^*\mathcal{P}(\mathbb{X})] \subseteq \mathcal{P}(\mathbb{Y}), \quad \text{and} \quad (h \times \Phi)[{}^*E] = \{(x, Z) \in \mathbb{Y} \times \Phi[{}^*\mathcal{P}(\mathbb{X})] \mid x \in Z\}.$$

The first of these is straightforward, since by definition Φ maps elements of ${}^*\mathcal{P}(\mathbb{X})$ to subset of $\mathbb{Y}$, which are elements of $\mathcal{P}(\mathbb{Y})$ by definition. The second we can get by expanding the right hand side:

$$\{(y, Z) \in \mathbb{Y} \times \Phi[{}^*\mathcal{P}(\mathbb{X})] \mid y \in Z\} = \Big\{ \big(h(x), \Phi(Y)\big) \;\Big|\; (x, Y) \in {}^*E \Big\} = (h \times \Phi)[{}^*E].$$

The first equality follows since $Z = \Phi(Y)$ for some $Y \in {}^*\mathcal{P}(\mathbb{X})$ by definition of $\Phi[{}^*\mathcal{P}(\mathbb{X})]$, and then since $y \in Z = \Phi(Y)$ we must have $y = h(x)$ for some $x \in {}^*\mathbb{X}$ such that $(x, Y) \in {}^*E$ by definition of $\Phi(Y)$. The last equality then follows by definition of $h \times \Phi$.

Why care about E and ${}^*E$? Formulas containing terms like $x \in A$ for A a quantified or free variable are *not* covered a priori by the transfer principle. We have to first denote it as $(x, A) \in E$, which is fine because E is a fixed set and that case is covered by the transfered principle. This then transfers to $(x, A) \in {}^*E$. Constructing the above isomorphism means it is consistent to consider this as an inclusion relation $x \in A$ for A and internal set.

Exercises

<u>*Exercise 5.5*</u>*: Properties of internal subsets*

(i) Consider the sets

$$A \cap B = \{x \in {}^*\mathbb{X} \mid x \in A \wedge x \in B\}, \quad A \backslash B = \{x \in {}^*\mathbb{X} \mid x \in A \wedge x \notin B\},$$

for internal sets $A, B \subseteq {}^*\mathbb{X}$. By the internal definition principle, these sets are also internal. It then follows that any Boolean combination of internal sets is internal.

(ii) Let $f : \mathbb{X} \to \mathbb{X}$ and $A \subseteq {}^*\mathbb{X}$ be internal. Then

$${}^*f[A] = \{y \in {}^*\mathbb{X} \mid x \in A \wedge y = {}^*f(x)\}, \quad {}^*f^{-1}[A] = \{x \in {}^*\mathbb{X} \mid y \in A \wedge y = {}^*f(x)\},$$

so by the internal definition principle, these sets are also internal.

(iii) Every standard element of ${}^*\mathcal{P}(\mathbb{X})$ is an internal subset, since *every* element of ${}^*\mathcal{P}(\mathbb{X})$ is. If we assume the identification ${}^*\mathcal{P}(\mathbb{X}) \subseteq \mathcal{P}({}^*\mathbb{X})$, then $A \in {}^*\mathcal{P}(\mathbb{X}) \implies A \subseteq {}^*\mathbb{X}$. But a subset of ${}^*\mathbb{X}$ which is also an element of ${}^*\mathcal{P}(\mathbb{X})$ is by definition an internal set.

<u>*Exercise 5.6*</u>*: Properties of internal subsets of* ${}^*\mathbb{R}$

(ii) Consider the following formulas which hold over $\mathbb{N}$:

$$\forall n.\, n \geq 0, \quad \forall m\, \forall n.\, (m \neq n) \to (m < n) \vee (n + 1 \leq m).$$

These also hold in ${}^*\mathbb{N}$ via the transfer principle. These say that all natural numbers are positive (and hence 0 is the "first" natural number), and that there are no natural numbers inbetween two consecutive natural numbers. Since

$$1 = 0 + 1, \quad 2 = 1 + 1, \quad 3 = 2 + 1, \quad \ldots$$

it follows that $\mathbb{N}$ is the initial sequence of ${}^*\mathbb{N}$ and that every element of ${}^*\mathbb{N}\backslash\mathbb{N}$ is infinite (since each must be greater than any element of $\mathbb{N}$).

(iii) First, let $\phi(x)$ be a formula. The statement that $\exists! x.\, \phi(x)$ is equivalent to

$$\exists x.\, \phi(x) \wedge \forall y.\, \phi(y) \to y = x,$$

and this is susceptible to the transfer principle. So since the statement

$$\forall r \in \mathbb{R},\ \exists! N \in \mathbb{N}.\ N \leq r < N + 1,$$

is true, by the transfer principle it is also true for ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$.

(iv) $\mathbb{R}$ has the least upperbound property. Letting ϕ stand for the formula

$$\phi(y, B) = \forall b \in B.\, b \leq y,$$

this may be stated as

$$\forall A. A \neq \varnothing \wedge \big(\exists x.\ \phi(x, A)\big) \to \exists z.\ \phi(z, A) \wedge \big(\forall x.\ \phi(x, A) \to z \leq x\big),$$

where A ranges over $\mathcal{P}(\mathbb{R})$ and x, z over $\mathbb{R}$. ϕ is the property that y is an upperbound of B. So in words, the above formula says that if (nonempty) A has an upperbound, then there is a z which is and upperbound of A, and any other upperbound of A is greater than or equal to z. This formula is susceptible to the transfer principal, and so holds for A ranging over internal sets and x, z over ${}^*\mathbb{R}$.

(v) Since for any $N \in {}^*\mathbb{N}$

$$\{1, \ldots, N\} = \{n \in {}^*\mathbb{R} \mid n \in {}^*\mathbb{N} \wedge 1 \leq n \leq N\},$$

${}^*\mathbb{N}$ is internal (being a standard element of ${}^*\mathcal{P}(\mathbb{R})$) means by the internal definition principle that this set is internal as well.

(vi) For $r < s$ in ${}^*\mathbb{R}$, we define

$$[r, s] = \{t \in {}^*\mathbb{R} \mid r \leq t \leq s\},$$

and this is necessarily internal by the internal definition principle.

(vii) The least upperbound property implies the greatest lowerbound property. (If a lowerbound exists, form the set of all lowerbounds; every element of the set in question is an upperbound, and the result follows.) The set ${}^*\mathbb{N}\backslash\mathbb{N}$ consists of infinite elements, so any element of $\mathbb{N}$ is a lowerbound. If ${}^*\mathbb{N}\backslash\mathbb{N}$ is internal to ${}^*\mathbb{R}$, then this means that there is a greatest lowerbound on it. But this is impossible. If $\omega \in {}^*\mathbb{N}\backslash\mathbb{N}$, then so is $\omega - 1$; otherwise $\omega - 1 \in \mathbb{N}$ and so $\omega = (\omega - 1) + 1 \in \mathbb{N}$ as well. So ${}^*\mathbb{N}\backslash\mathbb{N}$ has no minimum, and also has no infimum since $\mathbb{N}$ has no maximum. So it must be that ${}^*\mathbb{N}\backslash\mathbb{N}$ is actually external to ${}^*\mathbb{R}$.

(viii) Let $\mathcal{E} \subseteq {}^*\mathbb{R}$ be the set of all infitesimal elements defined by

$$\mathcal{E} = \{\epsilon \in {}^*\mathbb{R} \mid \forall r \in \mathbb{R}.\ (r < \epsilon < 0) \vee (0 < \epsilon < r)\}.$$

The sum of two positive infinitesimals is greater than its parts (by the transfer principle), and is also infinitesimal since if $\epsilon, \delta \in \mathcal{E}_{>0}$ then

$$0 < \epsilon < r/2 \quad \text{and} \quad 0 < \delta < r/2 \implies 0 < \epsilon + \delta < r$$

for all $r \in \mathbb{R}$. So fix positive $\epsilon \in \mathcal{E}$. Then there is no maximum infinitesimal, since we can always add ϵ and get a greater one; but at the same time there is no supremum either, since there is no least positive real number. This is impossible if $\mathcal{E}$ is internal to ${}^*\mathbb{R}$ (since it must have the least upperbound property), and so $\mathcal{E}$ is external.

(ix) The set of finite numbers $\mathbb{R}$ is external, since otherwise the definition of $\mathcal{E}$ above would make $\mathcal{E}$ internal by the internal definition principle, which it is not. The set

$$F = \{x \in {}^*\mathbb{R} \mid \exists s, t \in \mathbb{R}.\ x \in [s, t]\}$$

is also external since it contains $\mathbb{R}$ and we can make the same kind of upperbound argument as we have made previously.

(x) If $A \subseteq {}^*\mathbb{R}$ is internal and contains arbitrarily large finite numbers, then the least upperbound property ensures us that it also contains an infinite positive number; if it did not, then there would be no least upperbound since positive infinite numbers are not bounded below.

(xi) In the exact same way, if A contains arbitrarily small positive infinite numbers, then it must contain a positive finite number by the greatest lowerbound property.

Exercise 5.7: Properties of internal functions of ${}^*\mathbb{R}$

(i) To begin with, it is easy to see that we can extend the internal definition principle to formulas like e.g. $\phi(x, a, A, \Gamma)$ for x free and ranging over ${}^*\mathbb{X}$, $a \in {}^*\mathbb{X}$, $A \in {}^*\mathcal{P}(\mathbb{X})$, and $\Gamma \in {}^*\mathcal{P}(\mathbb{X}^2)$. The proof is virtually identical to that given on [pg.32]. The principle then also covers formulas of (fixed) internal functions, since a formula like $y = f(x)$ over a function may be expressed in terms of its graph Γ_f as $(x, y) \in \Gamma_f$.

So consider an internal function $f : A \to B$ for $A, B \subseteq {}^*\mathbb{X}$. We can write A and B as

$$A = \{x \in {}^*\mathbb{X} \mid \exists y \in {}^*\mathbb{X}.\ y = f(x)\}, \quad B = \{y \in {}^*\mathbb{X} \mid \exists x \in {}^*\mathbb{X}.\ y = f(x)\},$$

from which it follows by our new internal definition principle that both A and B are internal.

(ii) We can consider another version of the internal definition principle. For the multiset $(\mathbb{X}, \mathcal{P}(\mathbb{X}), \mathcal{P}(\mathbb{X}^2))$, we can consider formulas like e.g. $\phi(x, a, A, \Gamma)$ where x is a free variable ranging over ${}^*\mathbb{X}^2$, $a \in {}^*\mathbb{X}$, and $A \subseteq {}^*\mathbb{X}$ and $\Gamma \subseteq {}^*\mathbb{X}^2$ are both internal. The proof is vitually identical to that given on [pg.32].

So let $f : A \to B$ be an internal function with $A, B \subseteq {}^*\mathbb{X}$, and let $C \subseteq A$ be internal. The graph of the restriction $f|_C$ may be written as

$$\Gamma_{f|_C} = \big\{(x, y) \in {}^*\mathbb{X}^2 \ \big|\ (x, y) \in \Gamma_f \wedge x \in C\big\},$$

which satisfies the internal definition principle noted above, so $\Gamma_{f|_C}$ and therefore $f|_C$ are internal.

(iii) Let $f : A \to B$ and $g : B \to C$ be internal functions for $A, B, C \subseteq {}^*\mathbb{X}$. (Proving this will almost immediately prove the more general result of the domain of g merely containing the image of f, since then the composition $g \circ f$ really amounts to $g \circ \iota \circ f$, where $\iota : \mathrm{cod}(f) \to \mathrm{dom}(g)$ is the inclusion of $\mathrm{im}(f)$ in $\mathrm{dom}(g)$.) The graph $\Gamma_{g \circ f}$ of their composition may be written as

$$\Gamma_{g \circ f} = \big\{(x, y) \in {}^*\mathbb{X}^2 \ \big|\ \exists z \in B.\ f(x) = z \wedge g(z) = y\big\},$$

and since f, g are internal B is as well, so this satisfies the (generalized) internal definition principle and $\Gamma_{g \circ f}$ and $g \circ f$ are internal.

(iv) Consider the statement

$$\forall f\ \exists! F.\ F(0) = f(0) \wedge \forall n.\ F(n+1) = F(n) + f(n+1)$$

for f, F ranging over functions $\mathbb{N} \to \mathbb{R}$ and n ranging over $\mathbb{N}$. The function variables are shorthand for their graphs ranging over $\mathcal{P}(\mathbb{X}^2)$ along with predicates asserting their functionhood. The statement $F(0) = f(0)$ is short for $\exists x \in \mathbb{R}.\ F(0) = x \wedge x = f(0)$, and the second statement can be broken down similarly into basic formulas (and is very tedious to write out). After these expansions, the above is a valid and true formula over $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}^2))$. (Its truth can be ascertained by induction.) By the transfer principle, it extends to f, F ranging over internal functions $^*\mathbb{N} \to {}^*\mathbb{R}$ and n ranging over $^*\mathbb{N}$.

Exercise 5.9: Properties of hyperfinite sums

The corresponding properties are true for sums over $\mathbb{N}$ (and can be proved by induction). Specifically

$$\forall f\ \forall g\ \forall n. \sum_{i=0}^{n} (f(i) + g(i)) = \sum_{i=0}^{n} f(i) + \sum_{i=0}^{n} g(i), \quad \forall f\ \forall n\ \forall c. \sum_{i=0}^{n} c \cdot f(i) = c \cdot \sum_{i=0}^{n} f(i),$$

where f, g range over functions $\mathbb{N} \to \mathbb{R}$, n over $\mathbb{N}$, and c over $\mathbb{R}$. It is an excellent exercise in tedium to write these out in such a way that the transfer principle directly applies. We do this only for the second one, in its full glory, so that the need to do so may never be felt again.

The function predicate.

$$\begin{aligned}\phi_1(\Gamma, A, B) \iff \Big(\big(\forall x[\exists y.(x,y) \in \Gamma] \leftrightarrow x \in A\big) \wedge \big(\forall y[\exists x.(x,y) \in \Gamma] \to y \in B\big)\Big) \\ \wedge\ \forall x \forall y_1 \forall y_2 [(x, y_1) \in \Gamma] \wedge [(x, y_2) \in \Gamma] \leftrightarrow (y_1 = y_2)\end{aligned}$$

for Γ over $\mathcal{P}(\mathbb{R}^2)$, A, B over $\mathcal{P}(\mathbb{R})$, and x, y, y_1, y_2 over $\mathbb{R}$.

The uniqueness predicate. Let

$$\psi(x, x_1, \ldots, x_p, y_1, \ldots, y_q, \gamma_1, \ldots, \gamma_r, X_1, \ldots, X_s, Y_1, \ldots, Y_t, \Gamma_1, \ldots, \Gamma_u)$$

be a formula over $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}^2)$ with free variables x ranging over $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$, or $\mathcal{P}(\mathbb{R}^2)$, and x_k ranging over $\mathbb{R}$ for each k, y_k ranging over $\mathcal{P}(\mathbb{R})$ for each k, γ_k ranging over $\mathcal{P}(\mathbb{R}^2)$ for each k, and $X_k \in \mathbb{R}$ for each k, $Y_k \in \mathcal{P}(\mathbb{R})$ for each k, and $\Gamma_k \in \mathcal{P}(\mathbb{R}^2)$ for each k. We abreviate this as $\psi(x)$. Then we define

$$\exists! x.\psi(x) \iff \exists x.\psi(x) \wedge \forall z(\psi(z) \leftrightarrow z = x),$$

where z ranges over the same sort as x.

The summation predicate.

$$\begin{aligned}\phi_2(\Gamma_1,\Gamma_2) \iff & (\exists z_1.[(0,z_1)\in\Gamma_2]\wedge[(0,z_1)\in\Gamma_1]) \\ \wedge\, \forall n.(n\in\mathbb{N}) \to & \Big(\exists z_2\exists z_3\exists z_4\exists z_5\exists z_6.(z_2=n+1)\wedge[(z_2,z_3)\in\Gamma_2]\wedge[(n,z_4)\in\Gamma_2] \\ & \wedge[(z_2,z_5)\in\Gamma_1]\wedge(z_6=z_4+z_5)\wedge(z_3=z_6)\Big)\end{aligned}$$

for Γ_1,Γ_2 ranging over $\mathcal{P}(\mathbb{R}^2)$, and $n,z_1,\dots,z_6$ ranging over $\mathbb{R}$.

Scaled function predicate.

$$\phi_3(c,\Gamma_1,\Gamma_2) \iff \forall x\forall y_1\forall y_2.[(x,y_1)\in\Gamma_1]\wedge[(x,y_2)\in\Gamma_2]\to y_2=c\cdot y_1$$

for c,x,y_1,y_2,z ranging over $\mathbb{R}$ and Γ_1,Γ_2 ranging over $\mathcal{P}(\mathbb{R}^2)$.

The statement.

$$\begin{aligned}&\forall\Gamma_f\exists!\Gamma_F.\phi_1(\Gamma_f,\mathbb{N},\mathbb{R})\wedge\phi_1(\Gamma_F,\mathbb{N},\mathbb{R})\wedge\phi_2(\Gamma_f,\Gamma_F) \\ &\qquad\to\forall c.\exists!\Gamma_{cf}\exists!\Gamma_{cF}.\phi_3(c,\Gamma_f,\Gamma_{cf})\wedge\phi_3(c,\Gamma_F,\Gamma_{cF})\wedge\phi_2(\Gamma_{cf},\Gamma_{cF})\end{aligned}$$

for $\Gamma_f,\Gamma_F,\Gamma_{cf},\Gamma_{cF}$ ranging over $\mathcal{P}(\mathbb{R}^2)$ and c ranging over $\mathbb{R}$. The transfer principle clearly applies to this. You're welcome.

6 Superstructures

Notes

(p.37) We have to be careful about our Cartesian products within the superstructure $\mathbb{V}(S)$. For example, if $A\in\mathbb{V}(S)$ is a set then $A^3=\{\{\langle 1,a\rangle,\langle 2,b\rangle,\langle 3,c\rangle\}\mid a,b,c\in A\}$, but $A^2\times A=\{\{\langle 1,a\rangle,\langle 2,b\rangle\}\mid a\in A^2\wedge b\in A\}$. In A^3, a triple is of the form $\{\langle 1,a\rangle,\langle 2,b\rangle,\langle 3,c\rangle\}$, but in $A^2\times A$, a triple is of the form $\big\{\langle 1,\{\langle 1,a\rangle,\langle 2,b\rangle\}\rangle,\langle 2,c\rangle\big\}$. The first form has rank $k+3$ for k the maximum rank of a,b,c, whereas the second has rank

$$\max\Big(\max\Big(\mathrm{rk}(a),\mathrm{rk}(b)\Big)+3,\mathrm{rk}(c)\Big)+3$$

which are only equivalent when c has the maximum rank. An expression like $A\times A\times A$ is *a priori* ambiguous. We will adopt the convention (as appears to be the same as Henson) that $A\times A\times A=A^3$, that is, for any sets $A_1,\dots,A_m\in\mathbb{V}(S)$

$$A_1\times\cdots\times A_m=\Big\{\{\langle i,a_i\rangle\ \big|\ i\in\{1,\dots,m\}\}\ \Big|\ a_1\in A_m\wedge\cdots\wedge a_m\in A_m\Big\},$$

and so then that, e.g.,

$$(A\times A)\times A=A^2\times A\neq A^3=A\times A\times A.$$

(p.41) ${}^*\mathbb{V}(S)$ is exactly the $\mathbb{N}$-set of all internal objects, since each sort is of the form ${}^*\mathbb{V}_k(S)$

for some k (whose elements are internal by definition); then since ${}^*\mathbb{V}_j(S)$ is standard and so internal for each j, it follows that each finite Cartesian product of sorts ${}^*\mathbb{V}(S)^\alpha$ (for a finite sequence α) is internal, and hence its elements are internal as well. Do note, however, that *subsets* of internal sets need not be internal.

Lemma 1. *Let $f : a \to b$ where a, b are internal sets of $\mathbb{V}({}^*S)$, and each output of f is a set such that there is a formula $\phi(w, z)$ over ${}^*\mathbb{V}(S)$ with w ranging over a and z over b where for all $x \in a$*

$$f(x) = \left\{y \in {}^*\mathbb{V}_{\mathrm{rk}(b)-2}(S) \;\middle|\; \phi(x, y)\right\}.$$

Then f is an internal. Such sets satisfy the internal definition principle uniformly in x.

Proof. We can see this by noting that the graph of f may be expressed as

$$\{(x, Y) \in a \times b \mid y \in Y \leftrightarrow \phi(x, y)\},$$

which satisfies the internal definition principle. □

Lemma 2. *Let $K \in {}^*\mathbb{R}$ be infinite and let $x \in \mathbb{R} \backslash \{0\}$. Then Kx is infinite.*

Proof. For all $r \in \mathbb{R}$, by definition $|K| > |r|$. So $|Kx| > |rx|$; the right hand side can be any standard real number, so Kx is infinite. □

Exercises

<u>Exercise 6.3</u>: *Extensions of finite collections of elements*

Let $a_1, \ldots, a_n \in \mathbb{V}(S)$ and let k be the greatest rank among them. Then each $\{a_1\}, \ldots, \{a_n\} \subseteq \mathbb{V}_k(S)$ and

$${}^*\{a_1, \ldots, a_n\} = {}^*\big(\{a_1\} \cup \cdots \cup \{a_n\}\big) = \{{}^*a_1\} \cup \cdots \cup \{{}^*a_n\} = \{{}^*a_1, \ldots, {}^*a_n\}$$

follows from the definition of a nonstandard extension of the $\mathbb{N}$-set $\mathbb{V}(S)$.

For the tuple $(a_1, \ldots, a_n)$, note first that

$$(a_1, \ldots, a_n) = \{\langle i, a_i\rangle \mid i = 1, \ldots, n\}.$$

Applying the previous result gives us that

$${}^*(a_1, \ldots, a_n) = \{{}^*\langle i, a_i\rangle \mid i = 1, \ldots, n\},$$

so it suffices to show that ${}^*\langle i, a_i\rangle = \langle i, {}^*a_i\rangle$, but this also follows in the same way since $\langle i, a_i\rangle = \{\{i, a_i\}, \{i\}\}$ and ${}^*i = i$.

Exercise 6.4: Basic extension properties of sets in $\mathbb{V}(S)$

Parts (i, ii) follow trivially from the transfer principle.

(iii) Let $a, b \in \mathbb{V}(S)$ be sets. Then the statement that $a \subseteq b$ may be expressed as $\forall x \in a(x \in b)$, and is susceptible to the transfer principle.

(iv) Let a, b be as above, and let k be the maximum of the ranks of a and b. Then both $a, b \subseteq \mathbb{V}_k(S)$, and it follows from the definition of a nonstandard extension that ${}^*(a \cup b) = {}^*a \cup {}^*b$, ${}^*(a \cap b) = {}^*a \cap {}^*b$, and ${}^*(a \backslash b) = {}^*a \backslash {}^*b$.

(v) Nonstandard extensions must preserve arbitrary finite Cartesian products of sorts.

(vi) Let $f : a \to b$ for some sets $a, b \in \mathbb{V}(S)$, and let $j = \mathrm{rk}(a)$ and $k = \mathrm{rk}(b)$. Then (identifying f with its graph), the statement that f is a function from a to b may be written as

$$\Big(\forall x \forall y_1 \forall y_2.[(x, y_1) \in f] \wedge [(x, y_2) \in f] \to (y_1 = y_2)\Big) \\ \wedge \Big(\forall x.(\exists y.(x, y) \in f) \leftrightarrow x \in a\Big) \wedge \Big(\forall y.(\exists x.(x, y) \in f) \to y \in b\Big),$$

where x ranges over $\mathbb{V}_{j-1}(S)$ and y, y_1, y_2 range over $\mathbb{V}_{k-1}(S)$. This is susceptible to the transfer principle, and transfers to

$$\Big(\forall x \forall y_1 \forall y_2.[(x, y_1) \in {}^*f] \wedge [(x, y_2) \in {}^*f] \to (y_1 = y_2)\Big) \\ \wedge \Big(\forall x.(\exists y.(x, y) \in f) \leftrightarrow x \in {}^*a\Big) \wedge \Big(\forall y.(\exists x.(x, y) \in f) \to y \in {}^*b\Big),$$

where now x ranges over ${}^*\mathbb{V}_{j-1}(S)$ and y, y_1, y_2 range over ${}^*\mathbb{V}_{k-1}(S)$. This states exactly that ${}^*f : {}^*a \to {}^*b$.

(vii) If r is a relation on $a \times b$, then by definition $r \subseteq a \times b$ and so ${}^*r \subseteq {}^*(a \times b) = {}^*a \times {}^*b$ and ${}^*r$ is a relation on ${}^*a \times {}^*b$. Now let $j = \mathrm{rk}(a)$ and $k = \mathrm{rk}(b)$. Then $a \subseteq \mathbb{V}_{j-1}(S)$ and $b \subseteq \mathbb{V}_{k-1}(S)$, so $a \times b \subseteq \mathbb{V}_{j-1}(S) \times \mathbb{V}_{k-1}(S)$. It follows by the definition of a nonstandard extension that both the projection on the first coordinate of $a \times b$ and the projection on the last coordinate are preserved, and so in particular the domain and range of r are preserved.

Exercise 6.9: Basic internal sets

Let $k \in \mathbb{N}$ and $a_1, \ldots, a_n \in {}^*\mathbb{V}_k(S)$. Then

$$\{a_1, \ldots, a_n\} = \{a \in {}^*\mathbb{V}_k(S) \mid a = a_1 \vee \cdots \vee a = a_n\}$$

is internal since $a_1, \ldots, a_n$ are internal. It also follows that

$$(a_1, \ldots, a_n) = \{\langle i, a_i \rangle \mid i = 1, \ldots, n\}$$

is internal since each natural number i is internal and a_i is internal for each i.

If $a \in \mathbb{V}({}^*S)$ is a standard set, then that means exactly that there is k and $b \in \mathbb{V}_k(S)$ such that $a = {}^*b$; but it follows immediately that $a = {}^*b \in {}^*\mathbb{V}_k(S)$, so a is internal.

Exercise 6.10: *Operations on internal sets*

(i) If a, b are internal sets of $\mathbb{V}({}^*S)$, then every Boolean combination can be expressed as, e.g.,

$$a \cup b = \{x \in {}^*\mathbb{V}_{k-1}(S) \mid x \in a \vee x \in b\},$$

where k is the maximum of the ranks of a and b. This is guaranteed to cover all x in a and b by the transitive property of nonstandard extensions. By the internal definition principle, $a \cup b$ is also internal.

(ii) If $x \in a$ and a has rank j, then also by the transitive property $a \subseteq {}^*\mathbb{V}_{j-1}(S)$, and so $x \in {}^*\mathbb{V}_{j-1}(S)$ and x is internal.

(iii) Consider the Cartesian product of internal sets $a_1, \ldots, a_n \in \mathbb{V}({}^*S)$:

$$a_1 \times \cdots \times a_n = \{(x_1, \ldots, x_n) \in {}^*\mathbb{V}_{k-1}(S)^n \mid x_1 \in a_1, \ldots, x_n \in a_n\},$$

where k is the maximum of the ranks of $a_1, \ldots, a_n$ and so the maximum possible rank of each $x_1, \ldots, x_n$. It follows from the internal definition principle that this is an internal set.

(iv) The exercise as stated appears to be false, until you realize that r was also defined to be an internal set at the outset. :) Otherwise, we have the following counterexample:

The set of infinite numbers I in ${}^*\mathbb{N}$ is not internal, as proven earlier for the multiset $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, and that proof carries over to the superstructure $\mathbb{V}(\mathbb{R})$. In particular, the least upperbound property may be stated as

$$\begin{aligned}\forall A \in \mathbb{V}_1(\mathbb{R}). A \notin \mathbb{R} \wedge A \neq \varnothing \big(\exists x \in \mathbb{R}. \phi(x, A)\big) \\ \rightarrow \exists z \in \mathbb{R}. \phi(z, A) \wedge \big(\forall x \in \mathbb{R}. \phi(x, A) \rightarrow z \leq x\big),\end{aligned}$$

where $\phi(y, B)$ is the formula $\forall b \in B. b \leq y$ with y ranging over $\mathbb{R}$ and B over $\mathbb{V}_1(\mathbb{R})$. Note that we do need the restriction that $A \notin \mathbb{R}$ and $A \neq \varnothing$ since otherwise $\phi(y, A)$ is vacuously true for any y and we would then conclude that there is a z which is less than any other real number. The above formula holds by transfer for $\mathbb{R}$ and ${}^*\mathbb{V}_1(\mathbb{R})$ as well.

${}^*\mathbb{N}$ is internal, being a standard set, so consider the relation $r \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ which is exactly the identity on I, i.e. $r = \{(a, a) \mid a \in I\}$. The domain and range of r are both I by definition, and are so not internal.

When r is required to be an internal set, the definitions of its domain and range along with the internal definition principle require that they are internal sets as well.

(v) Let a be an internal set of $\mathbb{V}({}^*S)$. Then the union of a may be expressed as

$$\bigcup a = \{x \in {}^*\mathbb{V}_{k-2}(S) \mid \exists b \in a. x \in b\}$$

where k is the rank of a. Elements of elements of a are members of ${}^*\mathbb{V}_{k-2}(S)$ by the transitivity of nonstandard extensions, and by the internal definition principle, this set is internal. Note that taking $b \in a$ is valid, since every element of an internal set is internal.

(vi) Let a be an internal subset of $\mathbb{V}({}^*S)$. Then the collection of all internal subsets of a may be expressed as

$$\{b \in {}^*\mathbb{V}_k(S) \mid b \subseteq a\}$$

where k is the rank of a. The formula $b \subseteq a$ is transferable since it is equivalent to $\forall x \in b.x \in a$, so by the internal definition principle the above set is internal.

Exercise 6.14: Basic hyperfinite sets

(i) We wish to show that every finite set in ${}^*\mathbb{V}(S)$ is hyperfinite. The trick is to *fix* a cardinality $N \in \mathbb{N}$ instead of quantifying over cardinalities. So choose a finite set $b \in {}^*\mathbb{V}(S)$ and let N be its cardinality and k its rank. Then we may form the sentence

$$\forall a \in \mathbb{V}_k(S).\exists\big(f : a \to \{1, \dots, N\}\big).(f \text{ is a bijection}) \to a \in F_k,$$

where F_k is the set of all finite sets in $\mathbb{V}_k(S)$. This sentence is true by definition of a finite set. Bijectivity is preserved by nonstandard extensions, so this sentence is susceptible to the transfer principle, and transfers to

$$\forall a \in {}^*\mathbb{V}_k(S).\exists\big(f : a \to \{1, \dots, N\}\big).(f \text{ is a bijection}) \to a \in {}^*F_k$$

since ${}^*\{1, \dots, N\} = \{{}^*1, \dots, {}^*N\} = \{1, \dots, N\}$. Applying this to b, we conclude that $b \in {}^*F_k$ and so is hyperfinite.

(ii) By the above, since $\{0, \dots, N\}$ is internal and finite for all $N \in \mathbb{N}$, it is hyperfinite. ${}^*\mathbb{N}$ is not however; if this were the case, then ${}^*\mathbb{N} \in {}^*F_k$ for some k and by transfer $\mathbb{N} \in F_k$, which is not true.

(iii) By the same logic, if $a \in \mathbb{V}(s)$ is a set and ${}^*a$ is hyperfinite, then ${}^*a \in {}^*F_k$ for some k and so by transfer $a \in F_k$, so a is finite. It follows that ${}^*a = \{{}^*x \mid x \in a\}$, as proven previously for finite sets.

Exercise 6.15: Internal cardinality

Consider for each $k \in \mathbb{N}$ the sentence

$$\forall a \in \mathbb{V}_k(S).a \in F_k \leftrightarrow \exists! N \in \mathbb{N}.\exists\big(f : a \to \{1, \dots, N\}\big).(f \text{ is a bijection}),$$

which is true. This extends by transfer to

$$\forall a \in {}^*\mathbb{V}_k(S).a \in {}^*F_k \leftrightarrow \exists! N \in {}^*\mathbb{N}.\exists\big(f : a \to \{1, \dots, N\}\big).(f \text{ is a bijection}),$$

where it should be noted that f is an internal function. So given hyperfinite a, there is an internal bijection from a to $\{1, \dots, N\}$ for some unique $N \in \mathbb{N}$. In the other direction, given such an internal bijection, a is necessarily internal, so there is some k such that $a \in {}^*\mathbb{V}_k(S)$. We may then apply the above sentence, and conclude that $a \in {}^*F_k$ and so is hyperfinite.

Exercise 6.17: Operations on hyperfinite sets

(i) Every Boolean combination of hyperfinite sets $a, b \in \mathbb{V}({}^*S)$ is hyperfinite. For example, since the union of finite sets in $\mathbb{V}(S)$ is finite,

$$\forall c \in F_k.\forall d \in F_k.c \cup d \in F_k$$

for each $k \in \mathbb{N}$, by transfer the same is true of hyperfinite sets. Notet that it is sufficient to consider both c and d in F_k since we can just take k to be the maximum of the ranks of c and d.

Nearly identical remarks can be made for all other Boolean operations.

(ii) A Cartesian product of n finite sets is finite, so by transfer a Cartesian product of n hyperfinite sets is finite.

(iii) If a relation is finite, then so are its domain and range. By transfer, if a relation is hyperfinite, so are its domain and range.

(iv) The union or intersection of a finite set of finite sets is finite. By transfer, the union or intersection of a hyperfinite set of hyperfinite sets if hyperfinite.

(v) The powerset of a finite set is finite. By transfer, the internal powerset of a hyperfinite set is hyperfinite.

(vi) The cardinality of a subset is $\leq$ that of a finite superset. By transfer, the internal cardinality of a subset is $\leq$ that of a hyperfinite superset.

(vii) For every finite $a \subseteq \mathbb{R}$, there is a bijection $f : a \to \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. We can then form an increasing bijection $g : a \to \{1, \ldots, N\}$ by ordering a and mapping the least element to 1, the next to 2, and so on. So then

$$\begin{aligned}&\forall a \subseteq \mathbb{R}\exists! N \in \mathbb{N}.\exists\big(f : a \to \{1, \ldots, N\}\big).\\ &\qquad\qquad (f \text{ is a bijection}) \wedge \forall x \in a.\forall y \in a.x < y \leftrightarrow f(x) < f(y),\end{aligned}$$

where from now on we will use the shorthand

$$\forall b \subseteq c.\phi \iff \forall b \in \mathbb{V}_{\mathrm{rk}(c)}(S).b \subseteq c \to \phi.$$

By transfer, the same statement holds for hyperfinite subsets of ${}^*\mathbb{R}$ with internal cardinality $N \in {}^*\mathbb{N}$.

Exercise 6.18: Hyperfinite summation

Let $\mathbb{R} \subseteq S$ and let

$$A = \Big\{(\alpha, N) \in {}^*\mathbb{V}_4(S) \times {}^*S \;\Big|\; N \in {}^*\mathbb{N} \wedge \big(\alpha : \{0, 1, \ldots, N\} \to {}^*\mathbb{R}\big)\Big\}.$$

${}^*\mathbb{V}_4(S)$ is the appropriate sort for α, since elements of $\{0, 1, \ldots, N\} \times {}^*\mathbb{R} \supseteq \alpha$ have rank $0 + 3 = 3$ and so the set itself has rank $3 + 1 = 4$; α then has the same rank, being a subset.

By the internal definition principle, A is internal. Let ${}^*\phi(\alpha, N)$ be the predicate that defines A. Then the following sentence is true:

$$\exists! B \in \mathbb{V}_8(S).\Big(\forall(\alpha, N) \in \mathbb{V}_4(S) \times S.(\alpha, N) \in B \leftrightarrow \phi(\alpha, N)\Big) \\ \wedge \exists!\big(\Sigma : B \to \mathbb{R}\big).\forall(\alpha, N) \in B.\Sigma(\alpha, N+1) = \Sigma(\alpha, N) + \alpha(N).$$

This says that the first $N+1$ terms of the sequence α have a unique sum for all $N \in \mathbb{N}$. Note that $\Sigma(\alpha, N) = \sum_{k=0}^{N} \alpha(k)$. We cannot use this fact directly in the formula above, since the expansion of $\sum_{k=0}^{N} \alpha(k) = \alpha(0) + \cdots + \alpha(N)$ into basic formulas would require the number of basic formulas to depend on N; instead, we use the equivalent recurrence relation.

Applying transfer and setting $B = A$, there is a unique hyperfinite summation function $\Sigma : A \to {}^*\mathbb{R}$ which we denote $\Sigma(\alpha, N) = \sum_{k=0}^{N} \alpha(k)$.

7 Saturation

Exercises

Exercise 7.6: Characterization of $\aleph_1$-saturation

The forward direction ($\aleph_1$-saturated $\implies$ countable comprehension) follows from [Theorem 7.5]. So in the reverse direction, suppose $\mathcal{F} = \{F_k \mid k \in \mathbb{N}\}$ is a countable family of internal sets that has the finite intersection property. We may then form the sequence $C : \mathbb{N} \to {}^*\mathcal{P}(F_0)$ defined by $C_k = F_0 \cap \cdots \cap F_k$ for all $k \in \mathbb{N}$, where m is the rank of F_0. This is well defined, since each C_k is internal (being a finite intersection of internal sets) and also a subset of F_0. Since ${}^*\mathbb{V}_m(S)$ is internal, by the countable comprehension property there exists an internal function $G : {}^*\mathbb{N} \to {}^*\mathcal{P}(F_0)$ such that $G(k) = C_k$ for all $k \in \mathbb{N}$. Now consider the set

$$J = \{k \in {}^*\mathbb{N} \mid G(k) \neq \varnothing \wedge \forall(j \leq k).G(k) \subseteq G(j)\}.$$

This set is internal by the internal definition principle; furthermore, $\mathbb{N} \subseteq J$ since

$$G(k) = C_k \subseteq C_j = G(j), \quad G(k) = C_k \neq \varnothing$$

for all $k \in \mathbb{N}$ and all $j \leq k$, by definition of C and by the finite intersection property of $\mathcal{F}$, respectively. But $\mathbb{N} \subseteq J$ means that by the overspill principle J must contain an infinite element $K \in {}^*\mathbb{N} \backslash \mathbb{N}$. It follows in particular that

$$G(K) \neq \varnothing \wedge \forall j \in \mathbb{N}.G(K) \subseteq G(j) = F_0 \cap \cdots \cap F_j,$$

so $\bigcap \mathcal{F} \neq \varnothing$; thus, since $\mathcal{F}$ was arbitrary, the nonstandard extension is $\aleph_1$-saturated.

Exercise 7.7: Σ_1^0 and Π_1^0 sets

Let $a \in {}^*\mathbb{V}(S)$ be internal and let $c : \mathbb{N} \to {}^*\mathcal{P}(a)$ be a sequence of internal subsets of a. Suppose then that our extension is $\aleph_1$-saturated. Then the countable comprehension property holds, and c may be extended to an internal function $c : {}^*\mathbb{N} \to {}^*\mathcal{P}(a)$. If $b_1 = \bigcup_{k\in\mathbb{N}} c_k$ is an internal Σ_1^0 set, then consider the set $J = \{N \in {}^*\mathbb{N} \mid b_1 \subseteq c_1 \cup \cdots \cup c_N\}$. The map $N \mapsto c_1 \cup \cdots \cup c_N$ is internal by Lemma 1, since for each N

$$c_1 \cup \cdots \cup c_N = \{x \in a \mid \exists k \in \{1, \ldots, N\}.x \in c_k\}$$

satisfies IDP uniformly; so J is internal by IDP. But its clear that J contains every infinite hypernatural number, so by the underspill principle there is a finite $n_1 \in J$. If follows that b_1 may be expressed as a finite union $b_1 = c_1 \cup \cdots \cup c_{n_1}$.

By a very similar argument, if $d : \mathbb{N} \to {}^*\mathcal{P}(a)$ is a sequence with internal Π_1^0 set $b_2 = \bigcap_{k\in\mathbb{N}} d_k$, then $b_2 = d_1 \cap \cdots \cap d_{n_2}$ for some finite $n_2 \in \mathbb{N}$. Simply replace the above J with $J = \{N \in {}^*\mathbb{N} \mid b_2 \supseteq d_1 \cap \cdots \cap d_N\}$ for d extended to an internal function ${}^*\mathbb{N} \to {}^*\mathcal{P}(a)$ and adjust the rest of the proof accordingly.

Now let b_1, b_2 be as above but not necessarily internal, and assume $b_1 \subseteq b_2$. Define the formula $\phi_k(e)$ by

$$\phi_k(e) \iff c_1 \cup \cdots \cup c_k \subseteq e \subseteq d_1 \cap \cdots \cap d_k,$$

with e ranging over ${}^*\mathcal{P}(a)$. This is a formula over ${}^*\mathbb{V}(S)$, since c_k, d_k are internal for each k and finite unions and intersections of internal sets are internal. For any finite subset $\alpha = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$ of $\mathbb{N}$, letting

$$C_n = c_1 \cup \cdots \cup c_n, \quad D_n = d_1 \cap \cdots \cap d_n$$

we then have that

$$C_{\alpha_1} \subseteq C_{\alpha_2} \subseteq \cdots \subseteq C_{\alpha_m} \subseteq D_{\alpha_m} \subseteq D_{\alpha_{m-1}} \subseteq \cdots \subseteq D_{\alpha_1}$$

since $b_1 = \bigcup_{k\in\mathbb{N}} c_k \subseteq \bigcap_{k\in\mathbb{N}} d_k = b_2$. If we then choose e_α to be C_{α_m} or D_{α_m}, then $\phi_k(e_\alpha)$ is satisfied simultaneously for all $k \in \alpha$. It follows from $\aleph_1$-saturation and [Theorem 7.2] that there is an $e \in {}^*\mathcal{P}(a)$ such that $\phi_k(e)$ is satisfied simultaneously for *all* $k \in \mathbb{N}$. But this means exactly that

$$b_1 = \bigcup_{k\in\mathbb{N}} c_k \subseteq e \subseteq \bigcap_{k\in\mathbb{N}} d_k = b_2.$$

In particular, if $b = b_1 = b_2$, then b is internal and there are $p, q \in \mathbb{N}$ such that $C_p = b = D_q$.

Exercise 7.8: Cardinality of internal sets under $\aleph_1$-saturation

We show that any infinite internal set of an $\aleph_1$-saturated nonstandard extension has cardinality at least $2^{\aleph_0}$. To do this, it suffices to show that every set of the form $(K)_\leq = \{0, 1, \ldots, K\}$ for $K \in {}^*\mathbb{N}\backslash\mathbb{N}$ has this property, for suppose we have an infinite internal set A. Then $\aleph_1$-saturation implies that there is an injection $f : \mathbb{N} \to A$ and that this extends to an internal function $f : {}^*\mathbb{N} \to A$. So form the set

$$J = \{n \in {}^*\mathbb{N} \mid \exists(m < n).f(m) = f(n)\},$$

which is internal by IDT. Since f restricted to $\mathbb{N}$ is injective, $J \cap \mathbb{N} = \varnothing$; so every element of J must be infinite, and there must be a least element $K \in J$ so as to not contradict the underspill principle. By definition of J, the restriction of f to $(K-1)_{\leq}$ is then injective; thus $2^{\aleph_0} \leq |(K-1)_{\leq}| \leq |A|$.

All that is left is to show that indeed $|(K)_{\leq}| \geq 2^{\aleph_0}$ for all $K \in {}^*\mathbb{N}\backslash\mathbb{N}$. Consider the map $(0,1) \to (K)_{\leq}$ defined by $r \mapsto \lceil Kr \rceil$, where here $(0,1) \subseteq \mathbb{R}$ is just the standard elements between 0 and 1. If there are $r, r' \in (0,1)$ such that $\lceil Kr \rceil = \lceil Kr' \rceil$, then $Kr + s = \lceil Kr \rceil = \lceil Kr' \rceil = Kr' + s'$ for some $s, s' \in {}^*(0,1)$ (as easily shown by transfer). But this means that $K(r - r') = s' - s$; the right hand side is finite, but the left hand side must be infinite (by Lemma 2) unless $r - r' = 0$. So $r \mapsto \lceil Kr \rceil$ is an injective function and $|(K)_{\leq}| \geq 2^{\aleph}_0$.

Exercise 7.10: Characterizations of enlargments

Suppose our nonstandard extension is an enlargment and that $\mathcal{F}$ is a collection of subsets of $\mathbb{V}_k(S)$ with the finite intersection property. $\mathcal{F}$ is indeed an element of $\mathbb{V}(S)$ (specifically, it has rank at most $k+2$), so we can consider its extension ${}^*\mathcal{F}$. The finite intersection property may be stated as

$$\forall (F \subseteq \mathcal{F}).(F \text{ is finite}) \to \bigcap F \neq \varnothing,$$

where $F \mapsto \bigcap F$ should be regarded as a function $\mathbb{V}_{k+2}(S) \to \mathbb{V}_{k+1}(S)$ satisfying the property

$$\forall F \in \mathbb{V}_{k+2}(S). \forall a \in \mathbb{V}_k(S). a \in \bigcap F \leftrightarrow \exists f \in F. a \in f,$$

so by transfer ${}^*\bigcap F$ is exactly $\bigcap F$ for an internal collection of sets F. The first sentence then transfers to

$$\forall (F \subseteq {}^*\mathcal{F}).(F \text{ is hyperfinite}) \to \bigcap F \neq \varnothing.$$

By the enlargement property, ${}^*\mathcal{F}$ has a hyperfinite subset $b \supseteq {}^*[\mathcal{F}]$. Applying the above transfered sentence, it follows that $\bigcap b$ is nonempty, and so also $\bigcap {}^*[\mathcal{F}]$ must also be nonempty.

Suppose now for any $k \in \mathbb{N}$ that any collection $\mathcal{F}$ of subsets of $\mathbb{V}_k(S)$ with the finite intersection property has the property $\bigcap {}^*[\mathcal{F}] \neq \varnothing$, and let $(L, \leq)$ be some upward-directed poset over $\mathbb{V}_k(S)$. Consider then the collection of all upperbounds of of L, that is

$$\mathcal{F} = \{\{b \in L \mid a \leq b\} \mid a \in L\}.$$

Since L is directed upwards, this collection satisfies the finite intersection property: take $F_1, \ldots, F_m \in \mathcal{F}$ with minimums $a_1, \ldots, a_m$, respectively. Then $F_1 \cap F_2$ is nonempty, since a_1, a_2 must share an upper bound b_{12}. Then $(F_1 \cap F_2) \cap F_3$ is nonempty, since b_{12} and a_3 must share an upperbound b_{123}. Continuing this argument up to F_m, $F_1 \cap \cdots \cap F_m$ must be nonempty and contain an upperbound of $a_1, \ldots, a_m$. $\mathcal{F}$ can be defined by the sentences

$$\forall a \in L. F_a \in \mathcal{P}(L) \wedge \forall b \in L. b \in F_a \leftrightarrow a \leq b,$$
$$\mathcal{F} \subseteq \mathcal{P}(L) \wedge \forall F \in \mathcal{P}(L). F \in \mathcal{F} \leftrightarrow \exists a \in L. F = F_a,$$

where $a \mapsto F_a = \{b \in L \mid a \leq b\}$ is a function $L \to \mathcal{P}(L)$. From this it can be seen by transfer that

$${}^*F_a = \{b \in {}^*L \mid a \,{}^*\!\leq b\}, \quad {}^*\mathcal{F} = \{{}^*F_a \mid a \in L\} = \{\{b \in {}^*L \mid a \,{}^*\!\leq b\} \mid a \in {}^*L\},$$

so ${}^*\mathcal{F}$ is exactly the analog of $\mathcal{F}$ in ${}^*L$. By the premises $\bigcap {}^*[\mathcal{F}]$ is nonempty, where

$${}^*[\mathcal{F}] = \{{}^*(F_a) \mid a \in L\} = \{{}^*F_{{}^*a} \mid a \in L\} = \{\{b \in {}^*L \mid {}^*a \,{}^*\!\leq b\} \mid a \in L\},$$

so there is a $b \in {}^*L$ such that ${}^*a \,{}^*\!\leq b$ for all $a \in L$.

Finally, suppose that for every $k \in \mathbb{N}$ every upward-directed poset $(L, \leq)$ over $\mathbb{V}_k(S)$ has $b \in {}^*L$ such that ${}^*a \leq b$ for all $a \in L$. Then for any set $A \in \mathbb{V}(S)$, take $k = \mathrm{rk}(A) + 1$ and let L be the set of all finite subsets of A with order relation $\subseteq$. This is clearly an upward-directed poset, since any two finite sets are contained in their union, which is also finite. ${}^*L$ is then the set of all hyperfinite subsets of ${}^*A$, and there is a hyperfinite $b \subseteq {}^*A$ such that every finite $a \subseteq A$ satisfies ${}^*a \subseteq b$. Since each a is finite, ${}^*a = \{{}^*x \mid x \in a\}$, and we can take the union over all finite subsets of A to conclude that ${}^*[A] = \{{}^*x \mid x \in A\} \subseteq b$. Thus, the nonstandard extension is an enlargement.

Exercise 7.11: Every polysaturated extension is an enlargement

The cardinality of each $\mathbb{V}_{k+1}(S)$ is $2^{|\mathbb{V}_k(S)|}$. This is easy to see:

$$\mathbb{V}_{k+1}(S) = \mathbb{V}_k(S) \sqcup \mathcal{P}(\mathbb{V}_k(S)) \implies |\mathbb{V}_{k+1}(S)| = |\mathbb{V}_k(S)| + 2^{|\mathbb{V}_k(S)|} = 2^{|\mathbb{V}_k(S)|}$$

since $\mathbb{V}_k(S)$ is nonempty so $2^{|\mathbb{V}_k(S)|}$ is always greater than $|\mathbb{V}_k(S)|$. It follows immediately that the cardinality of $\mathbb{V}(S) = \bigcup_{k\in\mathbb{N}} \mathbb{V}_k(S)$ is the strong limit cardinal of the sequence $V_0 = |S|, V_{k+1} = 2^{V_k}$ with $|\mathbb{V}(S)| = V_\omega$ for ω the first countable ordinal. It is clear, then, that any set in $\mathbb{V}(S)$ has cardinality less than V_ω, and by transfer every set in ${}^*\mathbb{V}(S)$ also has cardinality less than V_ω. So choose any family of sets $\mathcal{F}$ over $\mathbb{V}_k(S)$ for some k which has the finite intersection property. By transfer ${}^*\mathcal{F}$ has the hyperfinite intersection property (which is stronger than the finite intersection property), and ${}^*\mathcal{F}$ is internal so $|{}^*\mathcal{F}| < V_\omega$; so by polysaturation $\bigcap {}^*\mathcal{F}$ is nonempty. But this immediately implies that $\bigcap {}^*[\mathcal{F}]$ is nonempty as well, and this is equivalent to the nonstandard extension being an enlargement.

II. Nonstandard Real Analysis

2 Basic Properties of ${}^*\mathbb{R}$

Notes

It is *not* true that if $x \in {}^*(a, \infty)$ then there is an $r \in (a, \infty)$ such that $r < x$. Consider the case where $x \gtrsim 1$; then of course $x \in {}^*(1, \infty)$, but the largest real number smaller than x is $1 \notin {}^*(1, \infty)$. We see that the correct statement is then

$$x \in {}^*(a, \infty) \implies \exists r \in [a, \infty).\ r < x.$$

Similar remarks can be made for other types of intervals.

Exercises

<u>*Exercise 2.8*</u>*: Classifying results of basic operations*

(a) $\delta^2(1/\delta) = \delta$ where $\delta \not\eqsim 0$.

(b) $(x\delta)(1/\delta) = x$ where $0 \not\simeq |x| \not\simeq \infty$ and $\delta \not\eqsim 0$.

(c) $(1/\alpha)\alpha^2 = \alpha$ where $|\alpha| \eqsim \infty$.

Consider also the function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x^y$ and its extension ${}^*f : {}^*\mathbb{R}_{>0} \times {}^*\mathbb{R} \to {}^*\mathbb{R}$. This is what we mean by the notation x^y for $x \in {}^*\mathbb{R}_{>0}, y \in {}^*\mathbb{R}$. Note that by transfer $y \mapsto x^y$ is strictly increasing or decreasing for x greater than or less than 1, respectively; and also that $x \mapsto x^y$ is strictly increasing or decreasing for y greater than or less than 0, repectively.

(d) $1^y = 1$ for all $y \in {}^*\mathbb{R}$ by transfer.

(e) $x^0 = 1$ for all $x \neq 0$ by transfer.

(f) Let $x \in (1, \infty)$ and consider $y \mapsto x^y$. By transfer ${}^*\mathbb{R}$ has image ${}^*(0, \infty)$, but the image of $\mathbb{R}$ must be $(0, \infty)$; so since the map is strictly increasing, $y \eqsim \infty$ implies $x^y \eqsim \infty$ and $y \eqsim -\infty$ implies $x^y \gnsim 0$.

(g) Let $x \in \langle 1, \infty)$. Then there is an $s \in (1, \infty)$ such that $s < x$. If $y \eqsim \infty$ then $\infty \eqsim s^y < x^y$, so $x^y \eqsim \infty$. If $y \eqsim -\infty$ then $0 \lnsim s^y > x^y$, so $0 \lnsim x^y$.

(h) Let $x \in (0, 1)$ and consider $y \mapsto x^y$. By transfer ${}^*\mathbb{R}$ has image ${}^*(0, \infty)$, but the image of $\mathbb{R}$ must be $(0, \infty)$; so since the map is strictly decreasing, $y \eqsim \infty$ implies $x^y \gnsim 0$ and $y \eqsim -\infty$ implies $x^y \eqsim \infty$.

(i) Let $x \in (0, 1\rangle$. Then there is an $s \in (0, 1)$ such that $x < s$. If $y \eqsim \infty$ then $x^y < s^y \gnsim 0$, so $x^y \gnsim 0$. If $y \eqsim -\infty$ then $x^y > s^y \eqsim \infty$, so $x^y \eqsim \infty$.

(j) Suppose $x \in (1, \infty)$ and $\delta \gnsim 0$. For every $r \in (1, \infty)$, there is an $s \in (0, \infty)$ such that $r = x^s$. So $0 < \delta < s$ implies $1 < x^\delta < x^s = r$; thus $1 \lnsim x^\delta$. If $\delta \lnsim 0$, then we instead take $r \in (0, 1)$ and $s \in (-\infty, 0)$; then $s < \delta < 0$ implies $r = x^s < x^\delta < 1$, and $x^\delta \lnsim 1$.

(k) Suppose that $x \in \langle 1, \infty\rangle$. Then $s < x < t$ for some $s, t \in (1, \infty)$. If $\delta \gnsim 0$, then $1 \lnsim s^\delta < x^\delta < t^\delta \gnsim 1$; so $x^\delta \gnsim 1$. If $\delta \lnsim 0$, then $1 \gnsim s^\delta > x > t^\delta \lnsim 1$; so $x^\delta \lnsim 1$.

(l) Suppose that $x \in (0, 1)$ and $\delta \gnsim 0$. For every $r \in (0, 1)$, there is an $s \in (0, \infty)$ such that $r = x^s$. So $0 < \delta < s$ implies $1 > x^\delta > x^s = r$; thus $1 \gnsim x^\delta$. If $\delta \lnsim 0$, then we instead take $r \in (1, \infty)$ and $s \in (-\infty, 0)$; then $s < \delta < 0$ implies $r = x^s > x^\delta > 1$, and $x^\delta \gnsim 1$.

(m) Suppose that $x \in \langle 0, 1\rangle$. Then $s < x < t$ for some $s, t \in (0, 1)$. If $\delta \gnsim 0$, then $1 \gnsim s^\delta < x^\delta < t^\delta \lnsim 1$; so $x^\delta \lnsim 1$. If $\delta \lnsim 0$, then $1 \lnsim s^\delta > x > t^\delta \gnsim 1$; so $x^\delta \gnsim 1$.

(n) If $\delta \gnapprox 0$ then $(s^{1/\delta})^\delta = s$ for any $s \in (0,1\rangle$, and if $\delta \lnapprox 0$ then we can say the same for any $s \in \langle 1,\infty)$. This shows that if $\epsilon \gnapprox 0$ and $\delta \not\eqsim 0$ then we can have any value $\epsilon^\delta \in (0,1\rangle \cup \langle 1,\infty)$.

(o) Similarly to the previous property, if $x \eqsim 1$ and $|y| \eqsim \infty$ then we can conclude that and $x^y \in \langle 0,1\rangle \cup \langle 1,\infty)$.

(p) Similarly to the previous property, if $x \eqsim \infty$ and $\delta \not\eqsim 0$ then we can have any value $x^\delta \in (0,1\rangle \cup \langle 1,\infty)$.

(q) If $x \eqsim 1$ and $\delta \eqsim 0$, then there is some $y \in \langle 0,1\rangle \cup \langle 1,\infty)$ and $\epsilon \eqsim 0$ such that $x = y^\epsilon$; so $x^\delta = y^{\epsilon\delta} \eqsim 1$.

In summary, we have the following table of properties:

		y			
	x^y	$\eqsim -\infty$	$\lnapprox 0$	$\gnapprox 0$	$\eqsim \infty$
x	$\gnapprox 0$	$\eqsim \infty$	$\langle 1,\infty)$	$(0,1\rangle$	$\gnapprox 0$
	$\langle 0,1\rangle$	$\eqsim \infty$	$\gnapprox 1$	$\lnapprox 1$	$\gnapprox 0$
	$\eqsim 1$	$\not\eqsim 1$	$\eqsim 1$	$\eqsim 1$	$\not\eqsim 1$
	$\langle 1,\infty\rangle$	$\gnapprox 0$	$\lnapprox 1$	$\gnapprox 1$	$\eqsim \infty$
	$\eqsim \infty$	$\gnapprox 0$	$\langle 0,1\rangle$	$\langle 1,\infty)$	$\eqsim \infty$

Exercise 2.9: Sums and products of infinitesimally close numbers

Suppose that $x, y, u, v \in {}^*\mathbb{R}$ with $x \eqsim y$ and $u \eqsim v$.

(a) There are $\epsilon, \delta \eqsim 0$ such that $x - y = \epsilon$ and $u - v = \delta$. So $x + u - (y + v) = \epsilon + \delta \eqsim 0$; thus $x + u \eqsim y + v$.

(b) Suppose that x, u finite; necessarily, y, v are then also finite. So then

$$(x+u)^2 - (y+v)^2 = (x+u+y+v)(x-y+u-v) \not\eqsim 0$$

since $x + u + y + v$ is finite and $x - y$ and $u - v$ are infinitesimal. But then

$$\begin{aligned} 0 \not\eqsim (x+u)^2 - (y+v)^2 &= x^2 - y^2 + u^2 - v^2 + 2xu - 2yv \\ &= (x+y)(x-y) + (u+v)(u-v) + 2(xu - yv). \end{aligned}$$

$(x+y)(x-y)$ and $(u+v)(u-v)$ are infinitesimal, so $2(xu - yv) \eqsim 0$, and of course then $xu \eqsim yv$.

(c) If $x, y \not\eqsim 0$ then $1/x - 1/y = (y-x)/(xy) \eqsim 0$ since $y - x \eqsim 0$ and $1/(xy)$ cannot be infinite; thus $1/x \eqsim 1/y$.

Exercise 2.13: Basic standard part properties

Suppose x, y are finite hyperreals. Then $x \approx {}^\circ x$ and $y \approx {}^\circ y$, and using the properties from the previous exercise,

(a) $x + y \approx {}^\circ x + {}^\circ y \implies {}^\circ(x + y) = {}^\circ x + {}^\circ y$.

(b) $xy \approx {}^\circ x {}^\circ y \implies {}^\circ(xy) = {}^\circ x {}^\circ y$.

(c) Let $x = {}^\circ x + \epsilon$ and $y = {}^\circ y + \delta$ for infinitesimal ϵ, δ. If $x \approx y$, then by the standard part theorem ${}^\circ x = {}^\circ y$, so suppose $x < y$, $x \not\approx y$, and consider that ${}^\circ y - {}^\circ x = y - x + \epsilon - \delta$. $y - x$ is positive and noninfinitesimal, and $\epsilon - \delta$ is infinitesimal; so $y - x + \epsilon - \delta$ must be positive; thus ${}^\circ x < {}^\circ y$. Altogether, $x \leq y$ implies that ${}^\circ x \leq {}^\circ y$.

Exercise 2.14: Followup to part (c)

If $x < y$, we could still have $x \approx y$, in which case ${}^\circ x = {}^\circ y$; so $x < y$ does not imply that ${}^\circ x < {}^\circ y$.

3 Sequences and Series

Notes

How do general metric spaces fail to be complete under nonstandard analysis? We can see by examining how $\mathbb{R}$ is complete. The key ingredients are:

1. Cauchy sequences are bounded.
2. The Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.
3. If a Cauchy sequence has a convergent subsequence, the the full sequence converges.

(1) and (3) are just properties of Cauchy sequences; so the key to the completeness of $\mathbb{R}$ is (2). The proof of this is trivial in nonstandard analysis.

Proof of (2). If $a \in \mathbb{R}^{\mathbb{N}}$ is bounded then ${}^*a_K$ is finite for all infinite $K \in {}^*\mathbb{N}$. By the standard part theorem, all finite elements of ${}^*\mathbb{R}$ are nearstandard, so selecting a particular K we have ${}^*a_K \approx r$ for some $r \in \mathbb{R}$. Thus, there is a subsequence of a converging to r. □

It's clear that the nonstandard characterization of boundedness should carry over to other metric spaces; we also expect the nonstandard characterization of convergent subsequences to carry over. Then for a general metric spaces, *completeness must be equivalent to the standard part theorem.* In an incomplete metric spaces, there are finite nonstandard elements that are *not* nearstandard.

We state the simple case of a lemma which has an obvious generalization:

Lemma 3. *Let ψ be a parametrically transferable formula. If ψ is true eventually then ψ is true ultimately.*

Proof. In the following, let a, x, z range over $\mathbb{R}$, let A, X, Z range over ${}^*\mathbb{R}$, and let Y range over $\mathrm{Inf}({}^*\mathbb{R}_{>0})$. The statement that ψ is true eventually is

$$\exists Z \forall (X \geq Z). \psi(X).$$

Since ψ is parametrically transferable, there is $S \subseteq \mathbb{R}$ such that

$$\psi(X) \iff \forall \alpha. \phi(\alpha, X)$$

where ϕ is some formula in $\mathbb{R}$ and α ranges over S. We may then write

$$\exists Z \forall (X \geq Z) \forall \alpha. {}^*\phi(\alpha, X).$$

Now fix a particular α. Since $\alpha = {}^*\alpha$, by transfer we have

$$\exists Z \forall (X \geq Z). {}^*\phi(\alpha, X) \implies \exists z \forall (x \geq z). \phi(\alpha, x).$$

Fixing z, we may transfer back and use the fact that any Y is *always* $\geq z$ to get

$$\forall (X \geq z). {}^*\phi(\alpha, X) \implies \forall Y. {}^*\phi(\alpha, Y).$$

The variables in the last formula are independent of α, so we may write

$$\forall Y \forall \alpha. {}^*\phi(\alpha, Y) \implies \forall Y. \psi(Y),$$

which is the statement that ψ is true ultimately. $\square$

We must make a distinction between $\sup^{\mathbb{R}}$, the supremum over $\mathbb{R}$, and $\sup^{{}^*\mathbb{R}}$, the supremum over ${}^*\mathbb{R}$ (and similarly for the infemum). Consider the interval (a, b) for $a, b \in \mathbb{R}$. Clearly $\sup^{\mathbb{R}}(a, b) = b$; however, $\sup^{{}^*\mathbb{R}}(a, b)$ cannot exist, since any $b' \lessapprox b$ is an upperbound of (a, b).

Lemma 4. *Let $S \subseteq {}^*\mathbb{R}$ so that $\sup^{{}^*\mathbb{R}}(S)$ exists. Then there is $Z \in S$ such that $Z \lessapprox \sup^{{}^*\mathbb{R}}(S)$.*

Proof. If $\sup(S) = \max(S)$, then we can just take $Z = \max(S)$, so suppose this is not the case. Let $\delta \gnapprox 0$ and $s = \sup(S) - \delta \lnapprox \sup(S)$. There are two possibilities: either $s \in S$, in which case we can take $Z = s$; or $s \notin S$. In this case, there must be an $X \in S$ with $s < X$; otherwise, $s < \sup(S)$ is an upperbound for S, which is impossible. So we can take $Z = X$. $\square$

Corollary. *If $Y \in S$ and $Y \geq Z$, then $Y \lessapprox \sup(S)$.*

Lemma 5. *Let $J = \{m, m+1, \ldots, N\}$ for some $m \in \mathbb{N}$ and infinite $N \in {}^*\mathbb{N}$. If $s \in \mathbb{R}^{\mathbb{N}}$, then $\max({}^*s[J])$ exists and*

$$\max {}^*s[J] \begin{cases} \gtrapprox \sup s[\mathbb{N}_{\geq m}] & \textit{if } s \textit{ is bounded,} \\ \eqsim \infty & \textit{if } s \textit{ is non-decreasing eventually,} \\ \eqsim -\infty & \textit{if } s \textit{ is non-increasing eventually.} \end{cases}$$

Proof. The statement that s has a maximum x over $\{p, p+1, \ldots, q\}$ for $p, q \in \mathbb{N}$ is

$$\forall p \forall q \exists x \Big(\exists k.(p \le k \le q) \wedge (s_k = x)\Big) \wedge \Big(\forall k.(p \le k \le q) \wedge (s_k \le x)\Big) \\ \wedge \forall y \Big(\forall k.(p \le k \le q) \wedge (s_k \le y)\Big) \to (x \le y),$$

where y ranges over $\mathbb{R}$ and k, p, q over $\mathbb{N}$. Transfering this sentence gives exactly the existence of $\max({}^*s[J])$. If s is bounded, then $Y = \sup(s[\mathbb{N}])$ exists; $X = \sup {}^*s[\mathrm{Inf}(J)]$ also exists since $\mathrm{Inf}(J)$ is nonempty, so we can write

$$\max {}^*s[J] = \max\Big({}^*s[\mathrm{Inf}(J)]) \cup s[\mathbb{N}]\Big) = \max(X, Y),$$

If Y is the maximum then we are done, so assume the maximum is X. By Lemma 4, there is an $H \in J_{\approx\infty}$ such that ${}^*s_H \approx X$. Using boundedness again, ${}^*s_H$ is finite and thus $^\circ({}^*s_K)$ corresponds to a limit point of s, which must be at most Y. All together,

$$X \approx s_K \approx {}^\circ({}^*s_K) \le Y,$$

so $X \lessapprox \sup(s[\mathbb{N}])$. In total, $\max({}^*s[J]) \gtrapprox \sup(s[\mathbb{N}])$ as desired.

The last two cases follow from the non-decreasing or non-increasing property. $\square$

Corollary. *If $J = \{K \in {}^*\mathbb{N} \mid m \le K \le N\}$ for some $m \in \mathbb{N}$ and $N \in {}^*\mathbb{N}$, then the Lemma still holds replacing $\sup(s[\mathbb{N}])$ with $\sup(s[\mathbb{N}_{\ge m}])$.*

Exercises

Exercise 3.2: Reciprocal sequence has reciprocal limit

Let $s, t \in \mathbb{R}^{\mathbb{N}}$ such that $\lim(s) \neq 0$ exists and $t_n = 1/s_n$ for all $n \in \mathbb{N}$. Then ${}^*s_K \approx \lim(s)$ for K ultimately. By transfer, ${}^*t_m = 1/{}^*s_m$ for all m; so in particular ${}^*t_K = 1/{}^*s_K \approx 1/\lim(s)$ ultimately since $\lim(s)$ is noninfinitesimal by definition, and thus $\lim(t) = 1/\lim(s)$.

Exercise 3.3: Limit points of sequences are values at infinity

Let $s \in \mathbb{R}^{\mathbb{N}}$ and let $r \in \mathbb{R}$.

Suppose r is a limit point of s. Then there is an increasing sequence $i \in \mathbb{N}^{\mathbb{N}}$ such that s' is a subsequence $s'_k = s_{i_k}$ for all $k \in \mathbb{N}$ with $\lim(s') = r$. So ${}^*s_{({}^*i_K)} = {}^*s'_K \approx r$ for all infinite $K \in {}^*\mathbb{N}$. Since i is increasing, so is ${}^*i$, and it follows that ${}^*i_K$ must be infinite; so there is an infinite L such that ${}^*s_L \approx r$.

Now suppose that there is an infinite $L \in {}^*\mathbb{N}$ such that ${}^*s_L \approx r \in \mathbb{R}$. If $s_n = r$ for n eventually, then we're done; so suppose this is not the case. We can define a subsequence s' of s by taking $\{p_0 < p_1 < \cdots\} = \{k \in \mathbb{N} \mid s_k \neq r\}$ and defining $s'_k = s_{p_k}$ for all k. This subsequence still must have an infinite $L' \in {}^*\mathbb{N}$ (namely $L' = {}^*p_L$) such that ${}^*s'_{L'} \approx r$. Then for each $0 \neq m \in \mathbb{N}$

$$\exists L' \in {}^*\mathbb{N}. |{}^*s_L - r| < \frac{1}{m},$$

so by transfer there is an $l_m \in \mathbb{N}$ such that $|s'_{l_m} - r| < 1/m$. If $l[\mathbb{N}]$ is finite, then there must be an m such that $s'_{l_m} = r$; this is impossible, so $l[\mathbb{N}]$ is infinite and there must be a strictly increasing subsequence $j \in \mathbb{N}^{\mathbb{N}}$ of l. This defines a subsequence $n \mapsto s'_{j_n}$ of s'. It should be clear that by construction $\lim(n \mapsto s'_{j_n}) = r$, so r is a limit point of s' and necessarily also a limit point of s.

Exercise 3.4: Cauchy sequence elements are infinitesimally close at infinity

Suppose $s \in \mathbb{R}^{\mathbb{N}}$ is Cauchy. Then for each $\epsilon \in \mathbb{R}_{>0}$ there is an $N \in \mathbb{N}$ such that

$$\forall(m \geq N)\forall(n \geq N).|s_m - s_n| < \epsilon,$$

where m, n range over $\mathbb{N}$. Applying transfer to this sentence, we see that in particular it is true for any ϵ together with every infinite $m, n \in {}^*\mathbb{N}$, since then $m \geq N$ and $n \geq N$ for any finite $N \in \mathbb{N}$. So ${}^*s_m - {}^*s_n \approx 0$; thus ${}^*s_m \approx {}^*s_n$ for all $n, m \in \mathbb{N}$.

Now suppose that s has the property that ${}^*s_m \approx {}^*s_n$ for all infinite $m, n \in {}^*\mathbb{N}$. For each $\epsilon \in \mathbb{R}_{>0}$, choose any infinite $N \in {}^*\mathbb{N}$; then we can say that

$$\exists N \forall(m \geq N)\forall(n \geq N).|{}^*s_m - {}^*s_n| < \epsilon$$

where N, m, n range over ${}^*\mathbb{N}$, since $s_m - s_n \approx 0$. By transfer, the same is true for s and N, m, n ranging over $\mathbb{N}$; this is exactly the statement that s is Cauchy.

Exercise 3.5: Cauchy sequences are bounded

Let $s \in \mathbb{R}^{\mathbb{N}}$ be Cauchy. Then ${}^*s_H \approx {}^*s_K$ for all infinite $H, K \in {}^*\mathbb{N}$; choose $X = s_K$ for some such K. Suppose that s is not bounded; then each ${}^*s_H$ is infinite, including X. If follows that for *all* $k \in {}^*\mathbb{N}$ we have $|{}^*s_k| < |X| + 1$; this is true for finite k since then ${}^*s_k = s_k$ is finite, and this is true for infinite k since $|{}^*s_k| \approx |X|$. So

$$\exists Y \in {}^*\mathbb{R}_{>0}.\forall k \in {}^*\mathbb{N}.|{}^*s_k| < Y$$

(where $Y = |X| + 1$), and applying transfer gives exactly the statement that s is bounded. This is a contradiction, so s must have been bounded from the beginning.

Exercise 3.6: Basic facts about sequences

(a) Let $s, t, r \in \mathbb{R}^{\mathbb{N}}$ such that $\lim(s) = \lim(r) = a$ and suppose that $s_k \leq t_k \leq r_k$ for k eventually. Then the same is true for ${}^*s, {}^*t, {}^*r$ eventually. We then have $a \approx {}^*s_K < {}^*t_K < {}^*r_K \approx a$ for K ultimately, so it must be that ${}^*t_K \approx a$; so $\lim(t) = a$.

(b) Let $s \in \mathbb{R}^{\mathbb{N}}$ be bounded and non-decreasing. Then ${}^*s$ is ultimately finite, and non-decreasing. So choose a particular infinite $H \in {}^*\mathbb{N}$ and let $a = {}^\circ({}^*s_H)$; then $s_k \leq a$ for all k. But then it must be that ${}^*s_K \leq a$ for *all* K ultimately, so in particular $a \approx {}^*s_H \leq {}^*s_L \leq a$ for $L \geq H$. Thus ${}^*s_L \approx a$ for L eventually, and it is easy to see that this statement is parametrically transferable in L. So by Lemma 3, ${}^*s_L \approx a$ for L *ultimately* and thus $\lim(s) = a$.

(c) See the Notes section above; this is the Bolzano-Weierstrass theorem.

Exercise 3.7: Characterization of limit superior

Let $s, t \in \mathbb{R}^{\mathbb{N}}$ with s bounded and $t_n = \sup_{m \geq n} s_m$ for all n, and let $l = \lim\sup(s)$.

(a) Let $\phi(f, n, z)$ be the "upperbound" sentence

$$\forall(m \geq n).f(m) \leq z$$

so the we can define t by the sentence

$$\forall n.\phi(s, n, t_n) \wedge \forall r.\phi(s, n, r) \to r \leq t_n$$

for m, n ranging over $\mathbb{N}$ and r, z ranging over $\mathbb{R}$. This shows that by transfer ${}^*t_N = \sup_{M \geq N} {}^*s_M$ for all $N \in {}^*\mathbb{N}$. But for finite n we must have $\sup_{M \geq n} {}^*s_M = {}^*t_n = t_n$, so $t_n \approx \max\{{}^*s_M \mid n \leq M \leq K\}$ for all infinite $K \in {}^*\mathbb{N}$ by Lemma 5.

(f) For every finite n and infinite N, since t is non-increasing and $l \approx {}^*t_N = \sup_{M \geq N} {}^*s_M$ by definition of l and t, we can say that ${}^*t_N \geq {}^*s_N$; taking the standard part, we see that $l \geq {}^\circ({}^*s_N)$. By Lemma 4, there is an $H \geq N$ such that ${}^*s_H \approx \sup_{M \geq N} {}^*s_M = {}^*t_N \approx l$, so ${}^\circ({}^*s_H) = l$. Putting everything together, we conclude that $l = \max\{{}^\circ({}^*s_K) \mid K \in \mathrm{Inf}({}^*\mathbb{N})\}$.

4 Continuity

Notes

(p.60) We could also say that a function is continuous if it commutes with the standard part operation, i.e., $f : (a, b) \to \mathbb{R}$ is continuous if ${}^\circ({}^*f(z)) = f({}^\circ z)$ for all $z \in {}^*(a, b)$, and is continuous at a point c if the same is true for just $z \in {}^*\mathbb{R}_{\approx c}$. This definition likely only works with $\mathbb{R}$ though, since standard parts seem to be a special feature of $\mathbb{R}$. We can also state continuity at a point in terms of sets as ${}^*f[{}^*\mathbb{R}_{\approx c}] \subseteq {}^*\mathbb{R}_{\approx f(c)}$.

(p.62) Let $f : A \to \mathbb{R}$. If ${}^*A \supseteq {}^\circ[{}^*A]$ and contains no infinite elements, then continuity and uniform continuity are equivalent. Of course, uniform continuity implies continuity, so in the other direction suppose f is continuous. Then for any $x, y \in {}^*A$ with $x \approx y$ we have $c = {}^\circ x = {}^\circ y \in {}^*A$ and so $c \in A$ by transfer; then $x \approx c \approx y$ so by continuity ${}^*f(x) \approx f(c) \approx {}^*f(y)$.

For ${}^*A$ to have this property, A must be a bounded union of closed intervals. If for example $A = B \sqcup \mathbb{R}(a, b]$ for some $B \subseteq \mathbb{R}$ and $a, b \in \mathbb{R}$, then by transfer $a \notin {}^*A$ but $a' \in {}^*A$ for any $a' \gtrapprox a$; if $A = B \sqcup [a, \infty)$ then ${}^*A$ must contains infinite elements $\alpha \approx \beta$, and these do not have a standard part so the above proof does not apply. These are exactly the compact sets in $\mathbb{R}$; this means that f is uniformly continuous if f is continuous and A is compact, which is the Heine-Cantor Theorem for $\mathbb{R}$.

Exercises

Exercise 4.3: Composition of continuous functions is continuous

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ for $a, b, c, d \in \mathbb{R}$ be continuous and fix $x \in (a, b)$. Then for $z \approx x$,

$$z \approx x \implies {}^*f(z) \approx f(x) \implies {}^*g({}^*f(z)) \approx g(f(x)) \implies ({}^*g \circ {}^*f)(z) \approx (g \circ f)(x),$$

and ${}^*g \circ {}^*f = {}^*(g \circ f)$ so $g \circ f$ is continuous.

Exercise 4.6: Strictly increasing functions are bicontinuous

Let $f : [a, b] \to [c, d]$ be a surjective strictly increasing function; necessarily f has an inverse which must be strictly decreasing. We prove that f is continuous, from which it follows that $-f^{-1}$ and hence f^{-1} are continuous.

If $a = b$ then f is trivially continuous, so suppose this is not the case. Then

$$\forall x. w \leq x \leq y \leq z \implies f(w) \leq f(x) \leq f(y) \leq f(z)$$

for x ranging over $[a, b]$ and $w, y, z \in [a, b]$. Since f is a surjection, for any $w', z' \in [c, d]$ we can choose w, z such that $w' = f(w)$ and $z' = f(z)$; so by transfer

$$\forall X. w \leq X \leq y \leq z \implies w' \leq {}^*f(X) \leq f(y) \leq z'.$$

Choosing X such that $X \approx y$, then the above holds for *all* w, z and hence all w', z'; so ${}^*f(X) \approx f(y)$ and f is continuous.

Exercise 4.8: Characterization of uniform continuity

Let $f : A \to \mathbb{R}$ for some $A \subseteq \mathbb{R}$. Suppose f is uniformly continuous. Then for every positive $\epsilon \in \mathbb{R}$ there is a positive $\delta \in \mathbb{R}$ such that

$$\forall x \forall y. |x - y| < \delta \implies |f(x) - f(y)| < \epsilon,$$

where x, y range over A. By transfer,

$$\forall X \forall Y. |X - Y| < \delta \implies |{}^*f(X) - {}^*f(Y)| < \epsilon$$

for X, Y ranging over ${}^*A$. If $X \approx Y$ then $|X - Y| < \delta$ for any possible δ, so it follows that $|{}^*f(X) - {}^*f(Y)| < \epsilon$ for all possible ϵ; thus ${}^*f(X) \approx {}^*f(Y)$.

Now suppose that

$$\forall X \forall Y. X \approx Y \implies {}^*f(X) \approx {}^*f(Y)$$

for X, Y ranging over ${}^*A$. Since this holds for any $X \approx Y$, we may in particular choose $\delta \gtrsim 0$ such that $|X - Y| < \delta$; we then have for each positive $\epsilon \in \mathbb{R}$ that

$$\exists \Delta \forall X \forall Y. |X - Y| < \Delta \implies |{}^*f(X) - {}^*f(Y)| < \epsilon,$$

so by transfer

$$\exists \delta \forall x \forall y. |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

and f is uniformly continuous.

5 Differentiation

Notes

Lemma 6. *Let $f : (a, b) \to (c, d)$ for some $a, b, c, d \in \mathbb{R}$ be a continuous bijection. Then for all $y \in (c, d)$ and $x \in (a, b)$,*

$$y \approx {}^*f(x) \iff {}^*f^{-1}(y) \approx x.$$

Proof. Let $y \approx {}^*f(x)$. By Exercise 4.6 f^{-1} is continuous, so ${}^*f^{-1}(y) \approx {}^*f^{-1}(f(x)) = x$ as desired. Similarly, ${}^*f^{-1}(y) \approx x$ implies that $y = {}^*f({}^*f^{-1}(y)) \approx {}^*f(x)$ as desired. □

Exercises

Exercise 5.3: Algebra of differentiability

Suppose $f, g : (a, b) \to \mathbb{R}$ be differentiable, and let $c \in \mathbb{R}$ and $\delta \approx 0$.

(a) $$(f+g)'(c) \approx \frac{{}^*(f+g)(c+\delta) - (f+g)(c)}{\delta} = \frac{{}^*f(c+\delta) - f(c)}{\delta} + \frac{{}^*g(c+\delta) - g(c)}{\delta} \approx f'(c) + g'(c).$$

(b) $$(f \cdot g)'(c) \approx \frac{{}^*(f \cdot g)(c+\delta) - (f \cdot g)(c)}{\delta} = \frac{{}^*f(c+\delta) - f(c)}{\delta}{}^*g(c+\delta) + f(c)\frac{{}^*g(c+\delta) - g(c)}{\delta} \approx f'(c)g(c) + f(c)g'(c).$$

The final $\approx$ follows since $f(c), f'(c), g'(c)$ are finite and g being continuous implies that $g(c + \delta)$ is finite.

(c) $$(1/g)'(c) \approx \frac{\frac{1}{{}^*g(c+\delta)} - \frac{1}{g(c)}}{\delta} = \frac{g(c) - {}^*g(c+\delta)}{\delta}\frac{1}{{}^*g(c+\delta)g(c)} \approx \frac{-g'(c)}{g(c)^2}$$

assuming $g(c) \neq 0$ and since $g'(c)$ and ${}^*g(c + \delta) \approx g(c)$ are finite. $(f/g)' = (f \cdot 1/g)'$ now follows from the product rule above.

Exercise 5.4: Derivative of inverse function

Let $f : (a, b) \to (a', b')$ be a bijection differentiable at c such that $f'(c) \neq 0$. Then for any $\delta \approx 0$,

$$\frac{f^{-1}\Big(f(c) + {}^*f(c+\delta) - f(c)\Big) - f^{-1}(f(c))}{{}^*f(c+\delta) - f(c)} = \frac{(c+\delta) - c}{{}^*f(c+\delta) - f(c)} = \frac{\delta}{{}^*f(c+\delta) - f(c)} \approx \frac{1}{f'(c)},$$

where the last approximation follows because $f'(c) \neq 0$ is finite. Since f is continuous, $\Delta = {}^*f(c + \delta) - f(c) \approx 0$, so the above is the derivative of f^{-1} so long as for every $\Delta \approx 0$ there is a corresponding δ. Rearranging as $f(c)+\Delta = {}^*f(c+\delta)$, this is equivalent to Lemma 6 since ${}^*\mathbb{R}_{\approx f(c)} \subseteq (a', b')$.

7 Topology on $\mathbb{R}$

Notes

Lemma 7. *Let $S \subseteq {}^*\mathbb{R}$. Then $x = \sup^{\mathbb{R}}(S)$ exists iff $y = \sup^{\mathbb{R}}(^\circ S)$ exists and is an upper bound of S; furthermore $x = y$ when x does exist.*

Proof. We first prove that x exists $\implies$ y exists and is an upper bound of S. Then we show that this latter condition implies that $x = y$ (and hence that x exists).

Suppose that x exists. Then $z \leq x$ implies $^\circ z \leq x$ for any finite $z \in S$; so $z' \leq x$ for all $z' \in {}^\circ X$. Thus y exists by the least upper bound property of $\mathbb{R}$. If there is $w \in S$ with $y \leq w$, then $y \leq {}^\circ w$ and the definition of y implies $^\circ w = y$; so y is an upperbound of S.

If y exists and is an upper bound of S, then there is a $w \in \mathbb{R}$ with $z \leq w \leq y$ for all $z \in S$. It follows that $^\circ z \leq w \leq y$ and hence $w = y$; so $x = y$ (and x exists). □

Exercises

Exercise 7.1: Open and closed sets

Let $A \subseteq \mathbb{R}$, $B_r(x) = \{y \in \mathbb{R} \mid |y - x| \leq r\}$, and ${}^*B_R(X) = \{Y \in {}^*\mathbb{R} \mid |y - x| \leq R\}$. It is clear that ${}^*(B_r(x)) = {}^*B_r(x)$.

(a) Suppose A is open. Then for every $a \in A$ there is a positive r such that $B_r(a) \subseteq A$. By transfer, ${}^*\mathbb{R}_{\eqsim a} \subseteq {}^*B_r(a) \subseteq {}^*A$.

Suppose now that ${}^*\mathbb{R}_{\eqsim a} \subseteq {}^*A$. Then we can choose any $\delta \gtrsim 0$ and we have that $B_\delta(a) \subseteq {}^*\mathbb{R}_{\eqsim a} \subseteq {}^*A$. So by transfer there is some positive $r \in \mathbb{R}$ such that $B_r(a) \subseteq A$.

(b) **[(i) $\implies$ (iii)].**

Suppose A is closed and $a \notin A$. By definition A contains all of its limit points. If ${}^*\mathbb{R}_{\eqsim a} \cap {}^*A \neq \varnothing$, then there is a $b \in A$ and $\delta \gtrsim 0$ such that $b \in B_\delta(a)$; but then $b \in B_r(a)$ for every positive $r \in \mathbb{R}$, so a is a limit point; this is impossible, so ${}^*\mathbb{R}_{\eqsim a} \cap {}^*A = \varnothing$.

[(iii) $\iff$ (ii)].

(ii) is just the contrapositive of (iii).

[(iii) $\implies$ (iv)].

It is clear that $A \subseteq {}^\circ({}^*A)$, so suppose $a \in {}^\circ({}^*A)$ but $a \notin A$. Then ${}^*\mathbb{R}_{\eqsim a} \cap {}^*A = \varnothing$, and in particular $a \notin {}^*A$; this is a contradiction, so it must be that $a \in A$.

[(iv) $\implies$ (i)].

Assume that $A = {}^\circ({}^*A)$. For (i), let $a \in \mathbb{R}$ be a limit point of A. Then $B_r(a) \cap A \neq \varnothing$ for all positive $r \in \mathbb{R}$, and by transfer the same is true for all positive $R \in {}^*\mathbb{R}$; in particular, take $R \eqsim 0$. Then there is an $x \eqsim a$ such that $x \in {}^*A$; but this means that $a = {}^\circ x \in A$. Thus A is closed.

Exercise 7.3: Continuous image of compact sets is compact and uniformly continuous

Let $f : K \to \mathbb{R}$ be a continuous function over some compact $K \subseteq \mathbb{R}$. We first show that $f[K]$ is compact, and then that f is uniformly continuous.

Suppose $y \in {}^*(f[K]) = {}^*f[{}^*K]$. Then there is an $x \in {}^*K$ such that $y = {}^*f(x)$. x is finite since K is compact, and since f is continuous ${}^*f(x)$ is also finite; in particular y is finite and ${}^\circ y = {}^\circ({}^*f(x)) = f({}^\circ x)$. We must have ${}^\circ x \in K$, so ${}^\circ y \in f[K]$. Thus $f[K]$ is compact.

Suppose now that $x, y \in {}^*K$ such that $x \approx y$. Then $c = {}^\circ x = {}^\circ y \in K$, so by continuity ${}^*f(x) \approx f(c) \approx {}^*f(y)$ and f is uniformly continuous.

Exercise 7.4: Extended elements of bounded sets are finite

Suppose that $A \subseteq \mathbb{R}$ is bounded. Then there is a $C \in \mathbb{R}$ such that $\forall a.|a| \leq C$ where a ranges over A, so by transfer the same is true for $a \in {}^*A$ and thus each element of ${}^*A$ is finite.

Now suppose that each element of ${}^*A$ is finite. Then for any positive infinite $\alpha \in {}^*\mathbb{R}$ we may say that $\exists \alpha \forall a.|a| \leq \alpha$ where a ranges over ${}^*A$, so by transfer A is bounded.

8 Using Internal Subsets of ${}^*\mathbb{R}$

Notes

(p.72) The proof of [Theorem 8.8] make no mention of the x_0 in the premises. We do actually need that $F(x_0)$ is finite for some $x_0 \in {}^*[a, b]$ since the result

$$|x - y| < \delta \implies |F(x) - F(y)| < \epsilon$$

only guarantees that the distance betweens any two values of F is finite; $F(x_0)$ being finite then anchors all other values to being finite as well.

The following is a corollary of the characterization of closed sets.

Lemma 8. *Let $A \subseteq \mathbb{R}$. Then $b \in \mathbb{R}$ is a limit point of A iff there is $b' \in {}^*A$ such that $b' \not\approx b$.*

Lemma 9. *Let $a \in \mathbb{R}^{\mathbb{N}}$. If ${}^*a_M \approx b$ for some $b \in \mathbb{R}$ for sufficiently small infinite M, then a is bounded and all convergent subsequences converge to b.*

Proof. a diverges iff $n \mapsto |a_n|$ diverges, and this is only if $|{}^*a_K| \approx \infty$ for all infinite K. But this is clearly not the case since some values of ${}^*a$ at are $\approx |b|$, so $n \mapsto |a_n|$ and hence a are bounded.

Now let $i_0 < i_1 < \cdots$ so that $n \mapsto a_{i_n}$ is a convergent subsequence of a. By the premises, there is an infinite $N \in {}^*\mathbb{N}$ such that ${}^*a_M \approx b$ for all infinite $M \leq N$. So consider the set $J = \{{}^*i_K \in {}^*\mathbb{N} \mid K \in {}^*\mathbb{N}, {}^*i_K \leq N\}$ which is internal by IDP. Clearly J contains all finite values of i since N is infinite. Since i is strictly increasing, by the overspill principle there must be an infinite ${}^*i_H \in J$ for some infinite H, and ${}^*a_{{}^*i_H} \approx b$ by definition of J. Since $n \mapsto s_{i_n}$ is convergent, this means that $\lim(n \mapsto s_{i_n}) = b$. □

Corollary. $\lim(a) = b$.

Proof. Let E be the set of all limit points of a. $\liminf(a) = \inf(E)$ and $\limsup(a) = \sup(E)$ by definition (or Exercise 3.7) and both exist since a is bounded by the Lemma. But the Lemma also implies $E = \{b\}$, so $\lim(a) = \limsup(a) = \liminf(a) = b$. $\square$

Here is an extension of the overspill principle, for which there are similar results for underspill and the infinitesimal varieties.

Lemma 10. *Let $A \subseteq {}^*\mathbb{R}$ be internal and contain arbitrarily large finite elements. Then A contains infinitely many positive infinite elements.*

Proof. By the overspill principle, A contains an infinite element. We may write $A = X \sqcup Y$ where $X = A_{\gtrsim\infty}$ and $Y = A \backslash X$. If X is finite then it is internal, but this means that $A \backslash X = Y$. Y is bounded above by any infinite value, but has no least upper bound; this is impossible for an internal set. So Y is not internal, but $A \backslash X$ must be internal. This is contradiction, so X must be infinite. $\square$

Exercises

Exercise 8.4: Closure and compactness of internal sets

(a) Let $X \subseteq {}^*\mathbb{R}$ be internal and suppose $x \in \mathbb{R}$ is a limit point of ${}^\circ X$. Then for each $r \in \mathbb{R}_{>0}$ there is a $y \in {}^\circ X$ such that $|x - y| < r$. By definition of ${}^\circ X$ there is a finite $z \in X$ such that $z = y + \epsilon$ for some $\epsilon \approx 0$. But

$$|x - z| = |x - y - \epsilon| \leq |x - y| + |\epsilon| < r$$

so the set $\{|z - x| \in {}^*\mathbb{R} \mid z \in X\}$ is internal and contains arbitrarily small elements. By infinitesimal underspill there is an infinitesimal element, meaning there is a $z' \in X$ such that $x \approx z'$. Thus $x = {}^\circ z' \in {}^\circ X$.

(c) Let $B \subseteq \mathbb{R}$. By the above ${}^\circ({}^*B)$ is closed. Suppose that there is a closed $C \subseteq \mathbb{R}$ such that $B \subseteq C \subseteq {}^\circ({}^*B)$. Since C is closed, every point $b \in {}^\circ({}^*B) \backslash C$ must have $b' \notin {}^*C$ for any $b' \approx b$. By definition of ${}^\circ({}^*B)$, there is a $d \in {}^*B$ with $d \approx b$. But this means that either $b \in B$ or b is a limit point of b, and so in either case $b \in C$. So there can be no such b, and we must have $C = {}^\circ({}^*B)$. So ${}^\circ({}^*B)$ is the smallest closed set containing B, which is exactly the closure of B.

Exercise 8.6: Applications of Robinson's Sequential Lemma

Let $x \in ({}^*\mathbb{R})^{{}^*\mathbb{N}}$ be internal $a \in \mathbb{R}^{\mathbb{N}}$ such that x_n finite and $a_n = {}^\circ x_n$ for all finite n.

(a) Suppose $\lim(a) = a'$. We must have ${}^*a_n = a_n \approx x_n$ for all finite n; since both ${}^*a$ and x are internal, the sequence $n \mapsto {}^*a_n - x_n$ is internal and ≈ 0 for finite n. So by Robinson's Sequential Lemma, ${}^*a_K - x_K \approx 0$ for sufficiently small infinite K, or equivalently $x_K \approx a_K \approx a'$.

Suppose now that there is an $a' \in \mathbb{R}$ such that $x_K \approx a'$ for sufficiently small infinite K. Just like above, we must have ${}^*a_J \approx x_J$ for sufficiently small infinite J. So

${}^*a_L \approx a'$ for sufficiently small infinite L, and $\lim(a) = a'$ by the Corollary to Lemma 9. Alternatively, we could form the sets

$$A_r = \{K \in {}^*\mathbb{N} \mid \forall M.(K \leq M \leq P) \to |{}^*a_M - a'| < r\}$$

for each $r \in \mathbb{R}$, M ranging over ${}^*\mathbb{N}$, and P such that ${}^*a_L \approx a'$ for all infinite $L \leq P$; we would then proceed similarly to part (b) below.

(b) If a is convergent then it is Cauchy, and since ${}^*a_J \approx x_J$ for small enough infinite J this means that $x_K \approx x_L$ for small enough infinite K, L.

Suppose that $x_K \approx x_L$ for small enough infinite K, J. Then since ${}^*a_J \approx x_J$ for small enough infinite J, we must have ${}^*a_M \approx {}^*a_N$ for all infinite $M, N \leq P$ for some infinite $P \in {}^*\mathbb{N}$. So form the sets

$$A_r = \{K \in {}^*\mathbb{N} \mid \forall M \forall N.(K \leq M \leq P) \wedge (K \leq N \leq P) \to |{}^*a_M - {}^*a_N| < r\},$$

for each $r \in \mathbb{R}$ where M, N range over ${}^*\mathbb{N}$. This set is internal by IDP, and each contains all infinite $K \leq P$. By the underspill principle, each A_n must contain a finite element. So for each $r \in \mathbb{R}$ there is an $k \in A_r \cap \mathbb{N}$ such that $|a_m - a_n| < r$ for all finite $m, n \geq k$. This is exactly the statement that a is Cauchy, so a is convergent since $\mathbb{R}$ is complete.

III. Nonstandard Analysis and Topology

1 ——— Metric and Topological Spaces

Notes

(p.78) It is important to note that a member U of a base $\mathcal{B}_x$ is not trivially open, since we must have $\forall y \in X.\exists V \in \mathcal{B}_y.V \subseteq U$; the important part is that this must be true *for all* y, and we only know for sure that it is true for x.

(p.78) After the definition of a monad, Loeb makes a remark that since any finite intersection of elements of $\mathcal{B}_x$ contains an element of $\mathcal{B}_x$, then there is a $W \in {}^*\mathcal{B}_x$ with $W \subseteq \mu(x)$. However, this does not seem justifiable unless the nonstandard extension we are working in is an enlargement.

Lemma 11. *Assume we are working in a nonstandard extension which is an enlargement, and that X is a topological space with base $\mathcal{B}$. Then for any $x \in X$, there is a $W \in {}^*\mathcal{B}_x$ such that $W \subseteq \mu(x)$.*

Proof. By definition of an enlargement, there is a hyperfinite subset $\mathcal{B}' \subseteq {}^*\mathcal{B}_x$ such that $\{{}^*U \mid U \in \mathcal{B}_x\} \subseteq \mathcal{B}'$. Since any finite collection of elements of $\mathcal{B}_x$ contains (as a subset) an element of $\mathcal{B}_x$ in their intersection, by transfer the same is true of ${}^*\mathcal{B}_x$ for hyperfinite collections. So there is a $W \in {}^*\mathcal{B}_x$ such that

$$W \subseteq \bigcap \mathcal{B}' \subseteq \bigcap_{U \in \mathcal{B}_x} {}^*U = \mu(x). \qquad \square \quad (1)$$

We *do not* necesarily have a strict subset relation in (1). Even if we do not allow empty sets to have a topology, consider any nonempty set X with the indiscrete topology $\{\varnothing, X\}$. Any point $x \in X$ has just $\mathcal{B}_x = \{X\}$, so ${}^*\mathcal{B}_x = \{{}^*X\}$. The above proof still applies and we must have $W = {}^*X$, so ${}^*X = W \subseteq \mu(x) \subseteq {}^*X$ means that $W = \mu(x)$.

One way to realize a strict subset relation would be to have $\mu(x)$ be external, since every element of ${}^*\mathcal{B}_x$ must be internal. Suppose we are working in a κ-saturated nonstandard extension and that $\aleph_0 \leq |\mathcal{B}_x| = |S| \leq \kappa$ for some $x \in X$ and some ordered set S. Then we can form an injective function $f : S \to {}^*\mathcal{P}(X)$ such that $f[S] = \{{}^*U \mid U \in \mathcal{B}_x\}$, and Exercise 7.7 may be generalized to apply to f. The relevant result is the contrapositive of the result for internal Π-sets: if for all $s_0 \in S$ we have

$$\mu(x) = \bigcap_{U \in \mathcal{B}_x} {}^*U = \bigcap f[S] \neq \bigcap_{s \in S,\ s \leq s_0} f(s),$$

then $\mu(x)$ is not internal. In other words, if we need the whole of of $\mathcal{B}_x$ to form $\mu(x)$ then $\mu(x)$ is external. Note that it suffices to find *any* base $\mathcal{B}_x$ that satisfies these conditions to show that $\mu(x)$ is not internal. These considerations lead us to the following proposition:

Proposition 1.1. *Suppose we are working in a κ-saturated nonstandard extension, X is a topological space, and $\mathcal{B}_x$ is an infinite base of some $x \in X$ such that $|\mathcal{B}_x| < \kappa$ and $\mu(x) = \bigcap {}^*[\mathcal{B}_x] \neq \varnothing$. Then $\mu(x)$ is internal if and only if $\mu(x) \in {}^*\mathcal{B}_x$.*

Proof. The reverse direction is trivial, since any element of ${}^*\mathcal{B}_x$ is internal.

In the forwards direction, choose a limit ordinal S such that there is a bijection $U \in (\mathcal{B}_x)^S$. Then $s \mapsto {}^*U_s$ can be considered a function $S \to {}^*\mathcal{P}(X)$, and since $\bigcap {}^*[U[S]] = \bigcap {}^*[\mathcal{B}_x] = \mu(x)$ is internal we can apply the (generalization of the) internal Π^0_1-set result of Exercise 7.7, i.e., there is an $s_0 \in S$ such that

$$\mu(x) = \bigcap {}^*[U[S]] = \bigcap_{s \leq s_0} {}^*U_s = V \cap {}^*U_{s_0} \quad \text{where} \quad V = \bigcap_{s < s_0} {}^*U_s.$$

If $s_0 = 0$ then we're done, so suppose this is not the case. We may take the smallest such s_0 since S is well-ordered, meaning that $\mu(x) = V \cap {}^*U_{s_0} \subsetneq V$. There are two cases. If $V \subseteq {}^*U_{s_0}$ then $\mu(x) = V$ which is a contradiction. If $V \not\subseteq {}^*U_{s_0}$, then since $\mu(x)$ is nonempty we can conclude that $\mu(x) = \mu(x) \cap {}^*U_{s_0} \subsetneq V \cap {}^*U_{s_0} = \mu(x)$, which is a contradiction. $\square$

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Corollary 1. *$\mu(x)$ is internal if and only if there is a $U_0 \in \mathcal{B}_x$ such that $U_0 \subseteq V$ for all $V \in \mathcal{B}_x$, and in either case $\mu(x) = {}^*U_0$.*

Proof. The reverse direction is trivial, since then $\mu(x) = \bigcap_{V \in \mathcal{B}_x} {}^*V = {}^*U_0$. For the forward direction, the contrapositive of the Proposition holds: there is a $U_0 \in \mathcal{B}_x$ such that $V \not\subseteq U_0$ for every $V \in \mathcal{B}_x \backslash \{U_0\}$. But there must be a $W \subseteq V \cap U_0$ with $W \in \mathcal{B}_x$, meaning $W \subseteq U_0$; so $U_0 = W \subseteq V$ for all V. $\square$

Corollary 2. *If* any *base $\mathcal{B}_x$ satisfying the premises of the Proposition has the property that every $U \in \mathcal{B}_x$ has $V \subseteq U$ with $V \in \mathcal{B}_x \backslash \{U\}$, then* every *such base has this property. The same holds for the negation of the Proposition.*

Proof. Bases over the same topology must generate the same halos, so the result follows. □

Note that we still have "telescoping base $\implies$ external halo" from the Proposition even if we are not working in an enlargement, and thus "internal halo $\implies$ minimal base element" from the first Corollary as well. Similarly, regardless of any saturation assumptions but working in an enlargment we still have "external halo $\implies$ telescoping base". In any situation we have "minimal base element $\implies$ internal halo".

5 —— Compact Spaces

Exercises

Exercise 5.3: Standard sets are compact

We expand on two proofs provided by Loeb in *Nonstandard Analysis for the Working Mathematician*, Second Edition, found on pages 86 and 87 (Theorem 3.5.3). We assume that we are working in a κ-saturated nonstandard extensions, that $(X, \mathcal{T})$ is a topological space with $|\mathcal{T}| < \kappa$, and that $B \subseteq {}^*X$ is internal with each point nearstandard (i.e., $B = \text{ns}(B)$).

We know already that ${}^\circ B$ is closed, so to prove that ${}^\circ B$ is compact it suffices to show that any collection $\mathcal{F}$ of closed subsets of ${}^\circ B$ with the finite intersection property has $\bigcap \mathcal{F} \neq \varnothing$. So suppose we have such a collection, and let $\phi_F(z)$ be the sentence

$$z \in B \wedge \forall U \in \mathcal{T}. \overline{U} \cap F = \varnothing \to z \notin {}^*U$$

in ${}^*X$. For any finite subcollection $\{F_1, \ldots, F_n\} \subseteq \mathcal{F}$, there is a $b \in B$ such that $\phi_{F_k}(b)$ is satisfied for each k.

To see this,

first note that there is an $x \in \bigcap_{k=1}^n F_k$, and $x \in {}^\circ B$; so there is a $b \in B$ with $b \in \mu(x)$. But if U any open subset of X such that $\overline{U} \cap F_k = \varnothing$ for each k, then if $b \in {}^*U$ we must have $x \in \overline{U}$, which is impossible so $b \notin {}^*U$.

By saturation, since ϕ_F is made up of internal objects and is finitely satifiable, it must be universally satisfiable. So there is a $b_0 \in B$ such that $\phi_F(b_0)$ is satisfied for *all* $F \in \mathcal{F}$. Finally, we show that ${}^\circ b_0 \in \bigcap \mathcal{F}$. If this is not the case, then there is some $F \in \mathcal{F}$ such that ${}^\circ b_0 \notin F$. By regularity, there an open neighborhood U of x such that $\overline{U} \cap F = \varnothing$; but then $b_0 \in \mu({}^\circ b_0) \subseteq {}^*U$, which is a contradiction of $\phi_F(b_0)$.

ETC

7 Restricted or Relative Topologies

Exercise 7.1: Characterization of restricted open subsets

If $S \subseteq X$, then $V \subseteq S$ is open in S iff $\mu(v) \cap {}^*S \subseteq {}^*V$ for all $v \in V$. The forward direction is trivial, and the reverse direction relies on Lemma 11 to get a neighborhood of each $v \in V$ whose intersection with S lies in V. TODO