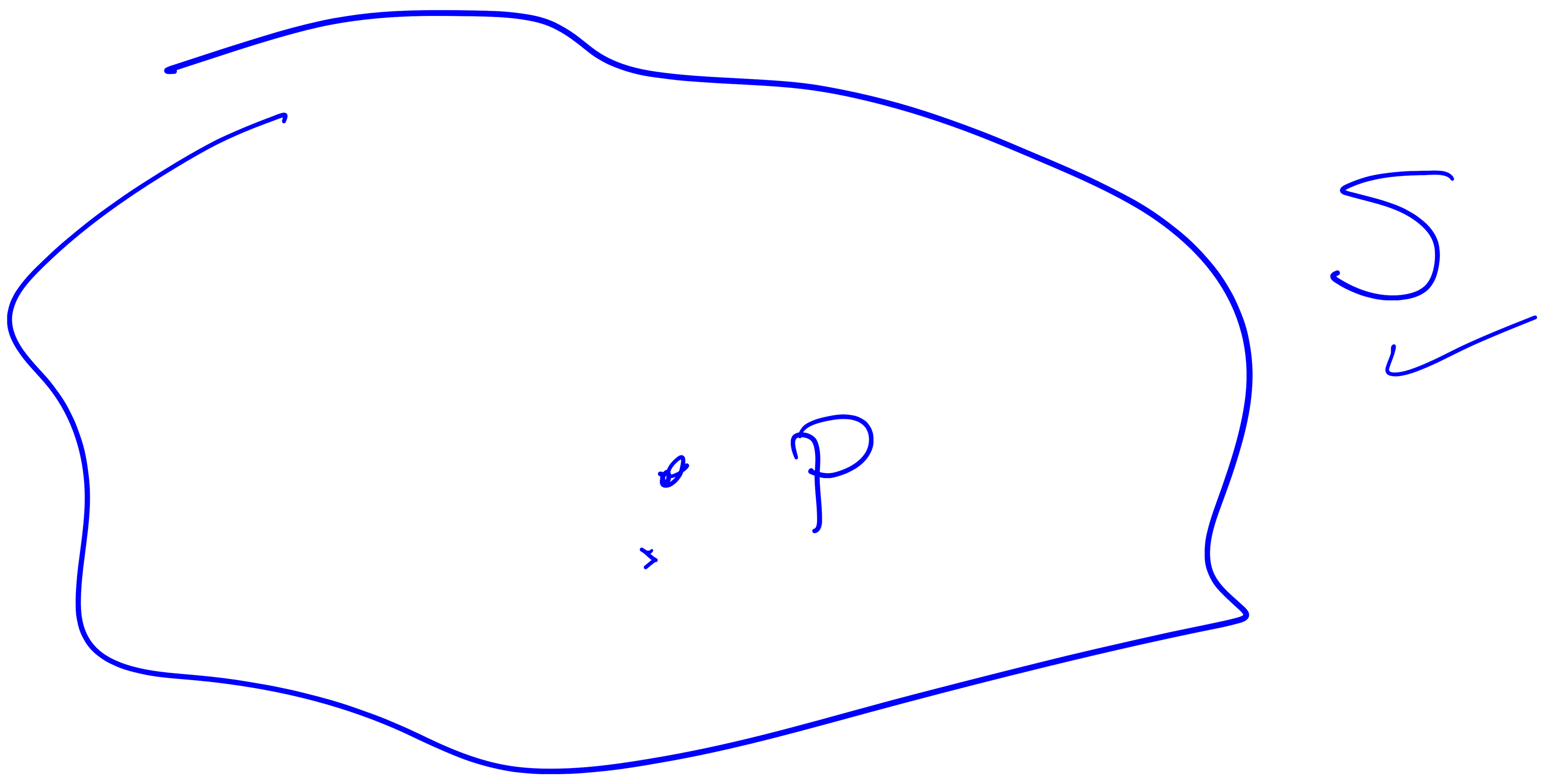
Kirchoff's theory of diffraction



(1) $V(x,y,z,t) = \underbrace{U(x,y,z)}\, e^{-i\omega t}$

$(\nabla^2 + k^2)\, U = 0$ ✓

(2) $\int_V \bar{\nabla}\cdot\bar{A}\, d^3\bar{r} = \oint_S \vec{A}\cdot\hat{n}\, dS$

$\vec{A} = U\nabla U'$

$\bar{\nabla}\cdot\bar{A} = \bar{\nabla}\cdot(U\nabla U') = U\nabla^2 U' + \bar{\nabla}U\cdot\nabla U'$

$$\int_V U\nabla^2 U' + \underline{\nabla U\cdot\nabla U'} = \oint_S U \underbrace{\frac{\partial U'}{\partial n}}\, dS \quad \text{— (1)}$$

$U \to U'$

$$\int_V U'\nabla^2 U + \underline{\nabla U\cdot\nabla U'} = \oint_S U' \frac{\partial U}{\partial n}\, dS \quad \text{— (2)}$$

$$\int_V \boxed{\left(U\nabla^2 U' - U'\nabla^2 U\right)} d^3\vec{r} = \int_S \left(U\frac{\partial U'}{\partial n} - U'\frac{\partial U}{\partial n}\right) dS$$

Extra condition

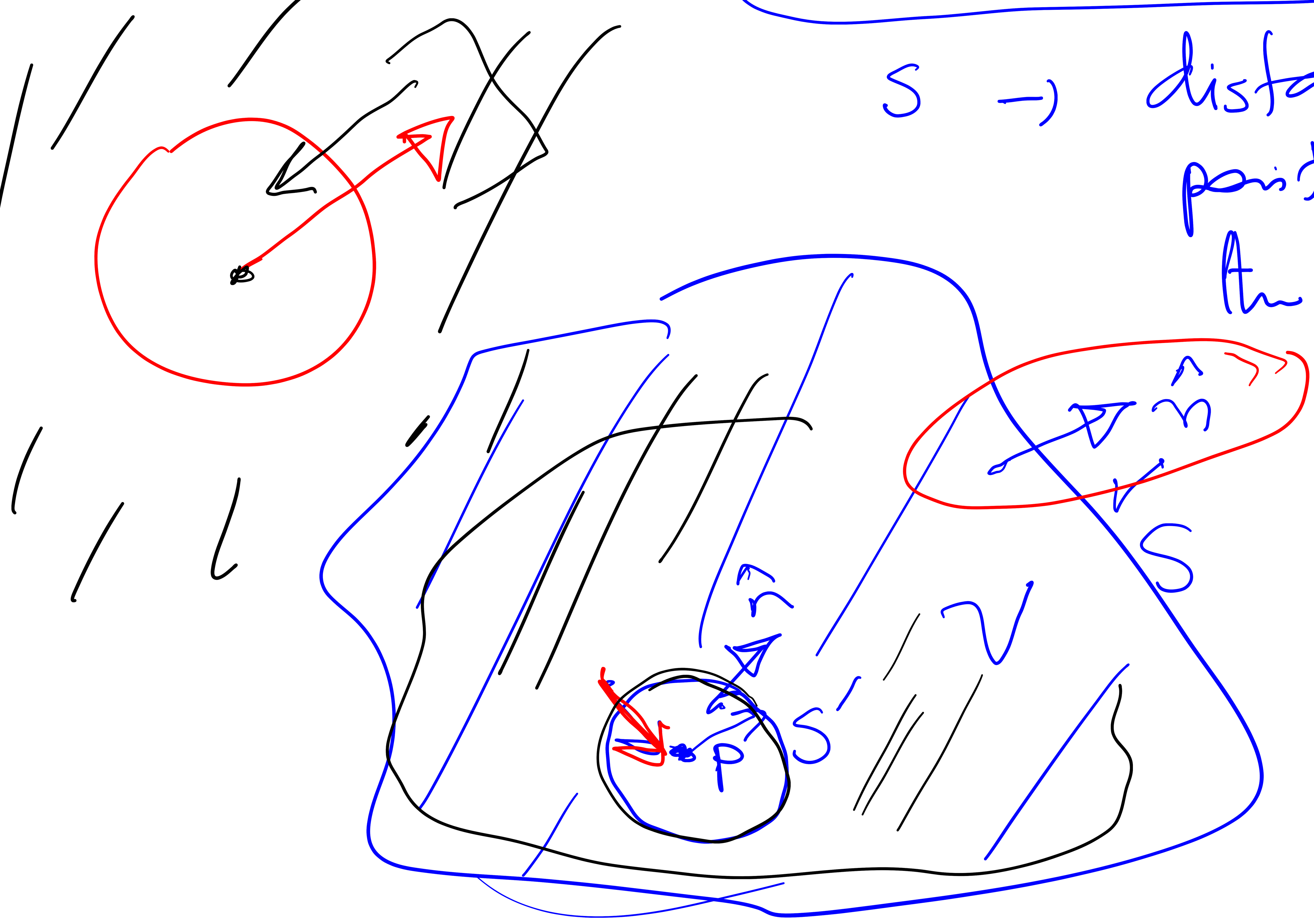
$(\nabla^2 + k^2) U = 0$, $\boxed{(\nabla^2 + k^2) U'} = 0$

LHS = 0

$$\int_S \left(U\frac{\partial U'}{\partial n} - U'\frac{\partial U}{\partial n}\right) dS = 0 \quad \text{— (3)}$$

Let's take $U' = e^{iks}/s$

$s \to$ distance between observation point and any point of the surface S



Excl. $s = 0$ singularity

e^{ikr}

$$\left(\int_S + \int_{S'}\right)\left(U\frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial U}{\partial n}\right) dS = 0$$

$$\underbrace{\int_S \left(U \frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial U}{\partial n}\right) dS}$$

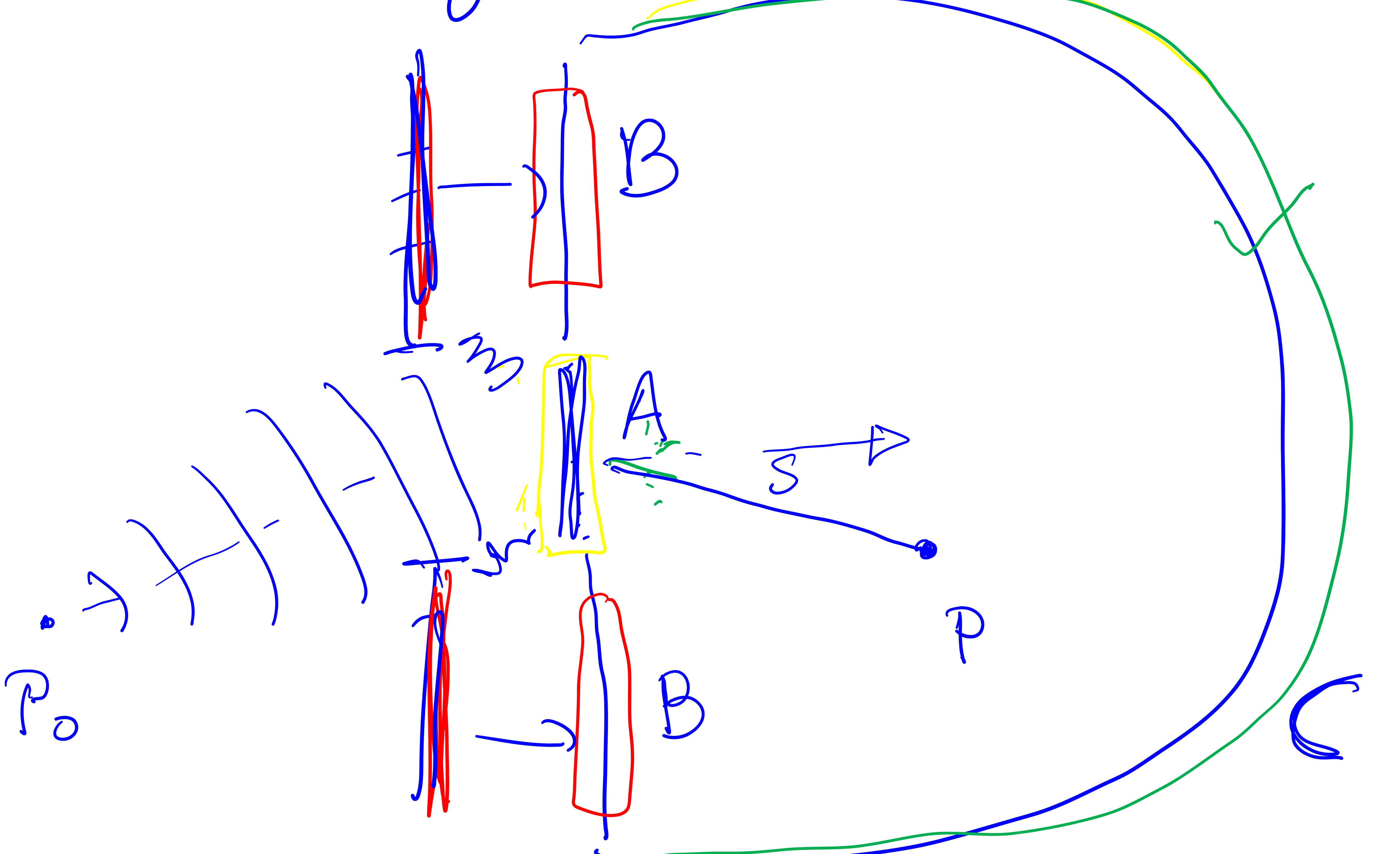
$$= \int_{S'} \left(U \frac{e^{iks}}{s}\left(ik - \frac{1}{s}\right) - \frac{e^{iks}}{s}\frac{\partial U}{\partial n}\right) dS'$$

$$= \int_{\Omega'} \left(U \frac{e^{ik\varepsilon}}{\varepsilon}\left(ik - \frac{1}{\varepsilon}\right) - \underbrace{\frac{e^{ik\varepsilon}}{\varepsilon}\frac{\partial U}{\partial n}}\right) \underbrace{\varepsilon^2}\, \underbrace{d\Omega'}$$

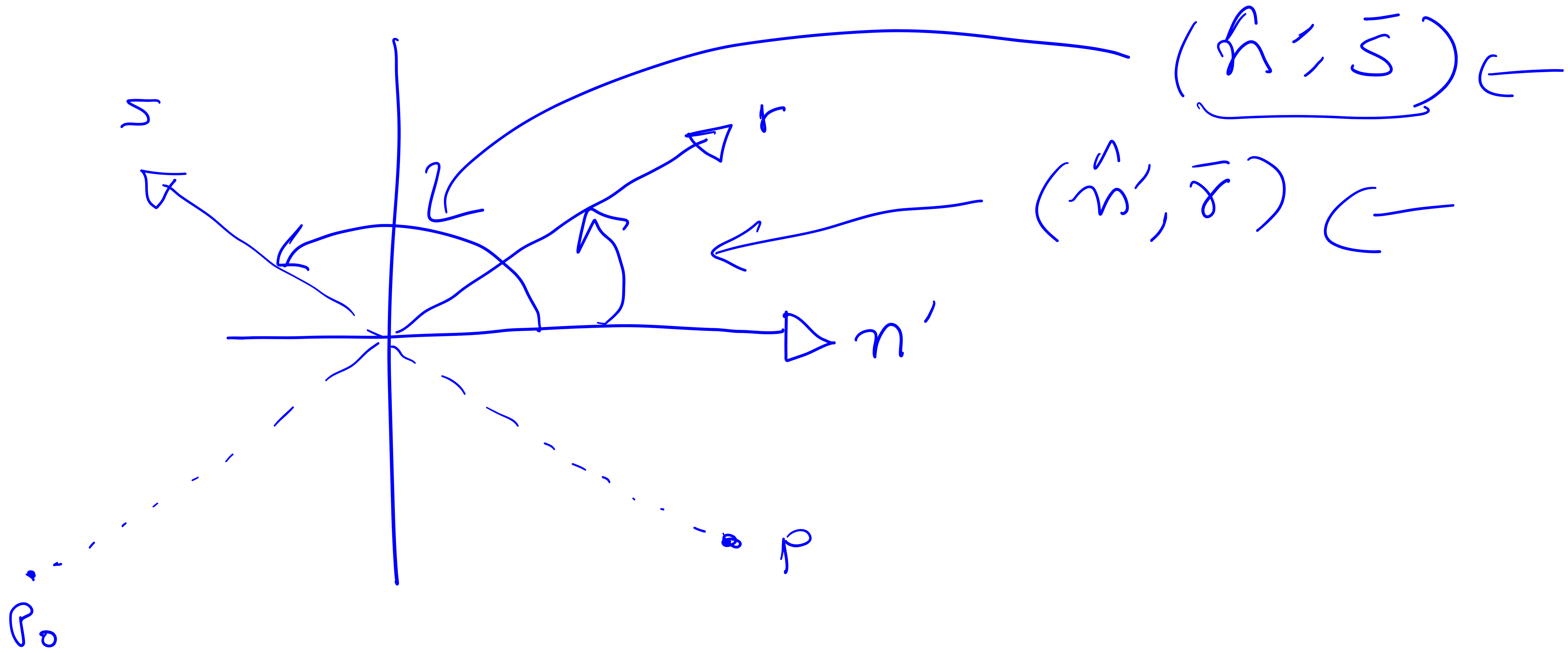
$$\varepsilon \to 0 \quad = \quad \underbrace{-4\pi\, U(P)}$$

$$\underbrace{U(P)} = -\frac{1}{4\pi} \iint_S \left(U \frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial U}{\partial n}\right) dS \qquad \square$$

Geometry



$$\left\{ \begin{array}{l} \text{opening} > \lambda \\ \text{opening} < AP_0,\ AP \end{array} \right.$$



Apply Kirchoff

$$U(P) = -\frac{1}{4\pi}\left(\int_A + \cancel{\int_B} + \cancel{\int_C}\right)\left[U \frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial U}{\partial n}\right] dS$$

"Kirchoff's assumptions"

- On A : $U = U^{(i)}$, $\frac{\partial U}{\partial n} = \frac{\partial U^{(i)}}{\partial n}$

$$U^{(i)} = A\frac{e^{ikr}}{r}, \qquad \frac{\partial U^{(i)}}{\partial n} = -\underbrace{A\frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right)}$$

- On B : $U = 0$, $\frac{\partial U}{\partial n} = 0$

(caveats)

- On C : Large R, $U, \frac{\partial U}{\partial n} \to 0$

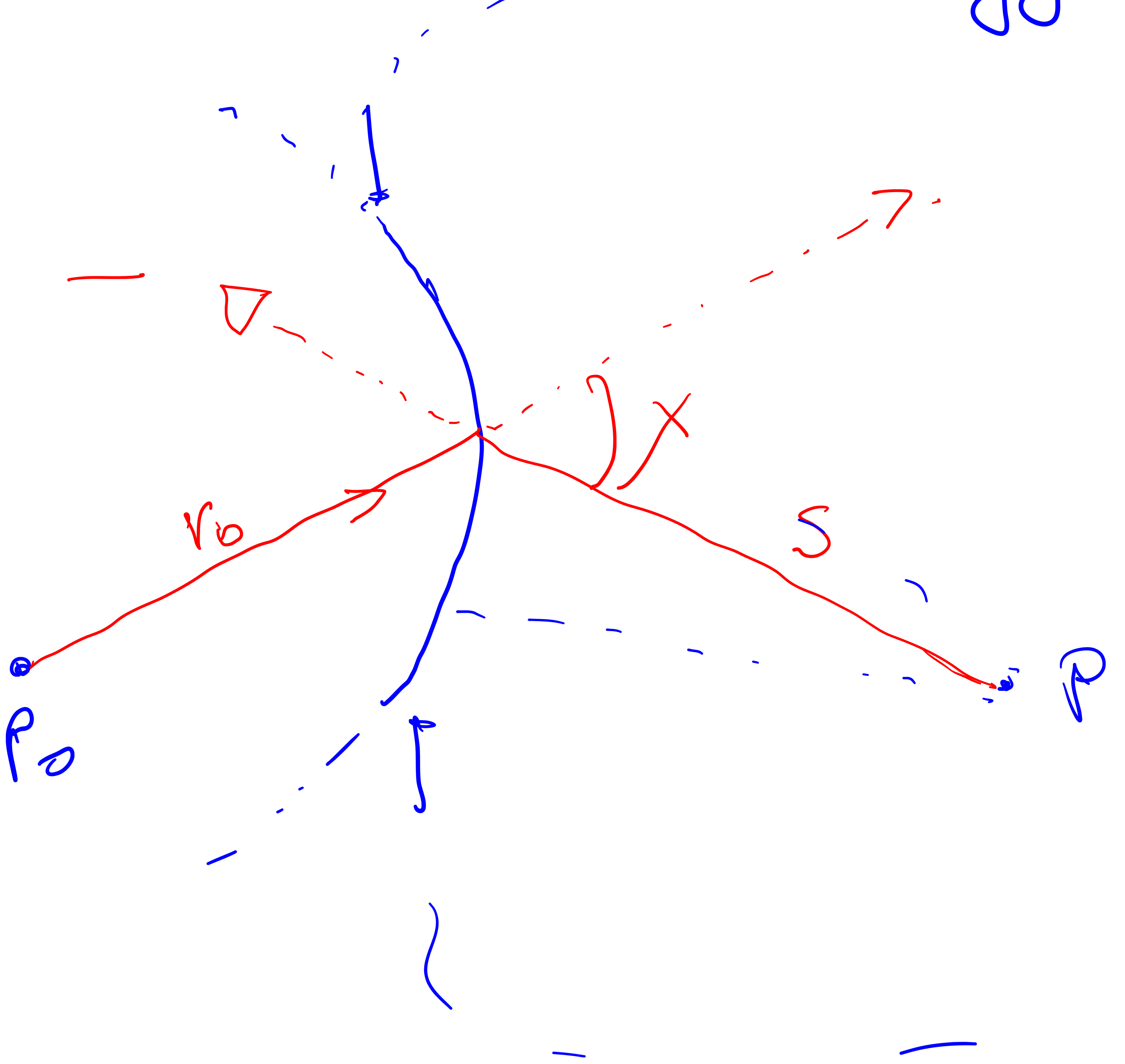
$$\Rightarrow U(P) = \frac{1}{4\pi} \iint_A \left(U \underbrace{\frac{\partial}{\partial n'} \left(\frac{e^{iks}}{s} \right)} - \frac{e^{iks}}{s} \underbrace{\frac{\partial U}{\partial n'}} \right) dS$$

$$= \frac{1}{4\pi} A \iint_A \frac{e^{ik(r+s)}}{rs} \left[\left(ik - \cancel{\frac{1}{s}} \right) \cos(n', s) - \left(ik - \cancel{\frac{1}{r}} \right) \cos(n', r) \right]$$

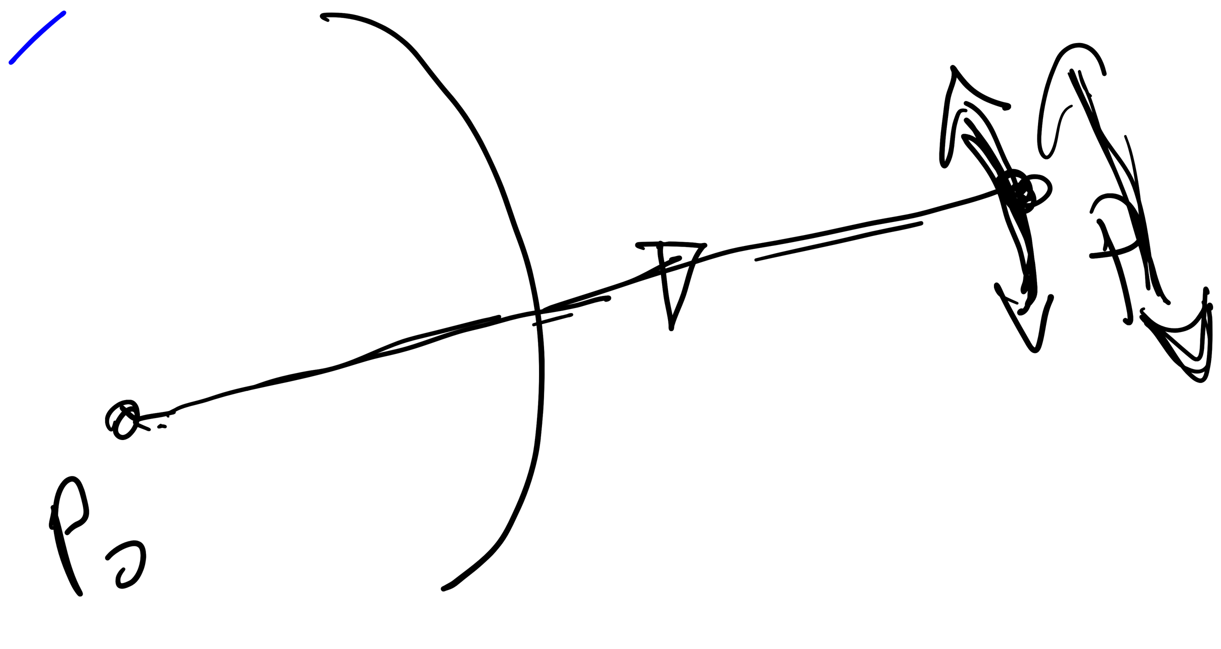
$r, s \rightarrow$ large compared to λ

$$U(P) = \frac{iA}{2\lambda} \iint_A \frac{e^{ik(r+s)}}{rs} \left[\cos(n', s) - \cos(n', r) \right] dS \quad \square$$

Connection to Huygen Fresnel theory



$$\underbrace{\chi = \pi - (r_0, s)}$$



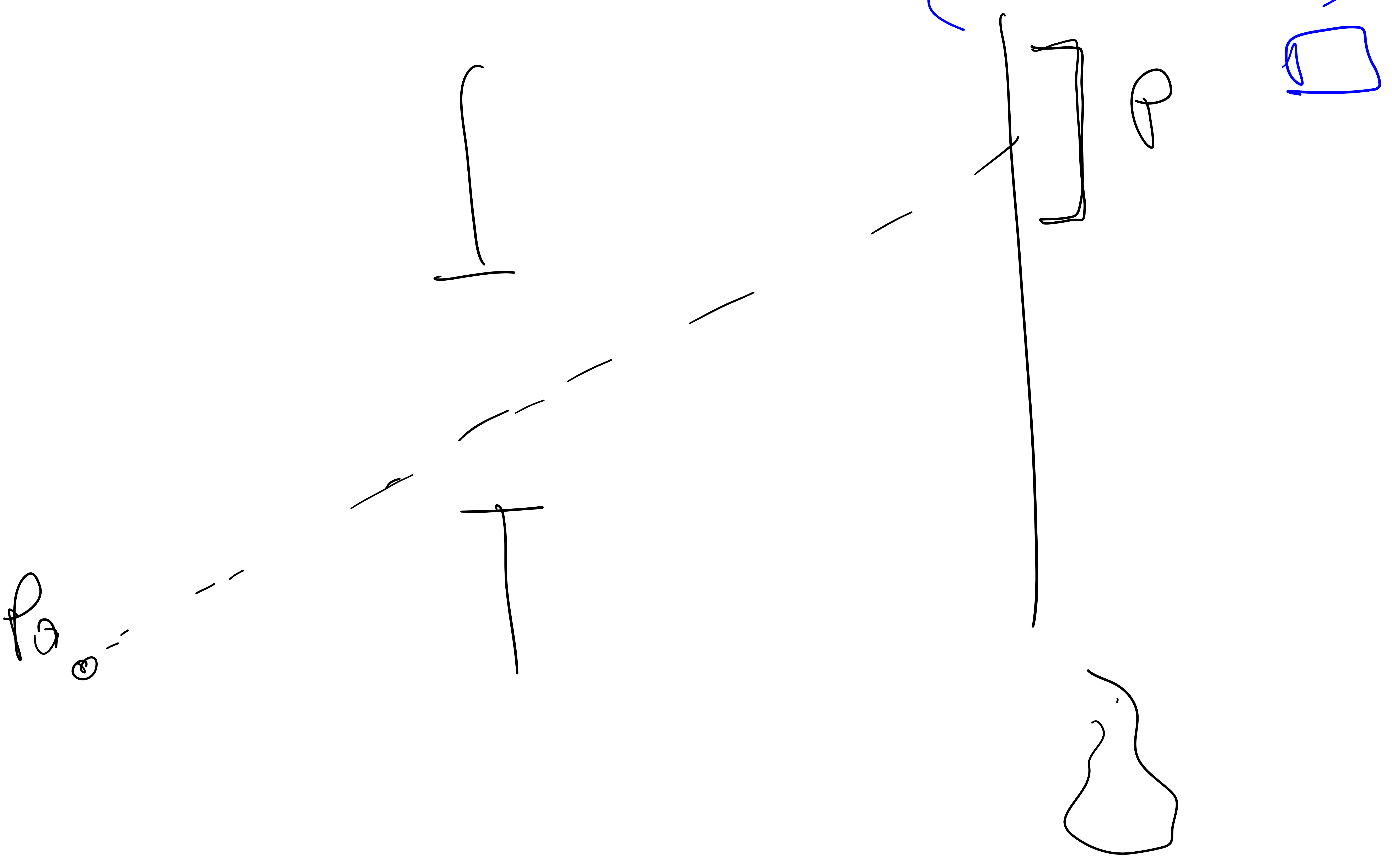
$$U(P) = \frac{-iA}{2\lambda} \frac{e^{ikr_0}}{r_0} \iint_A \frac{e^{iks}}{s} (1+\cos\chi)$$

Huygen - Fresnel

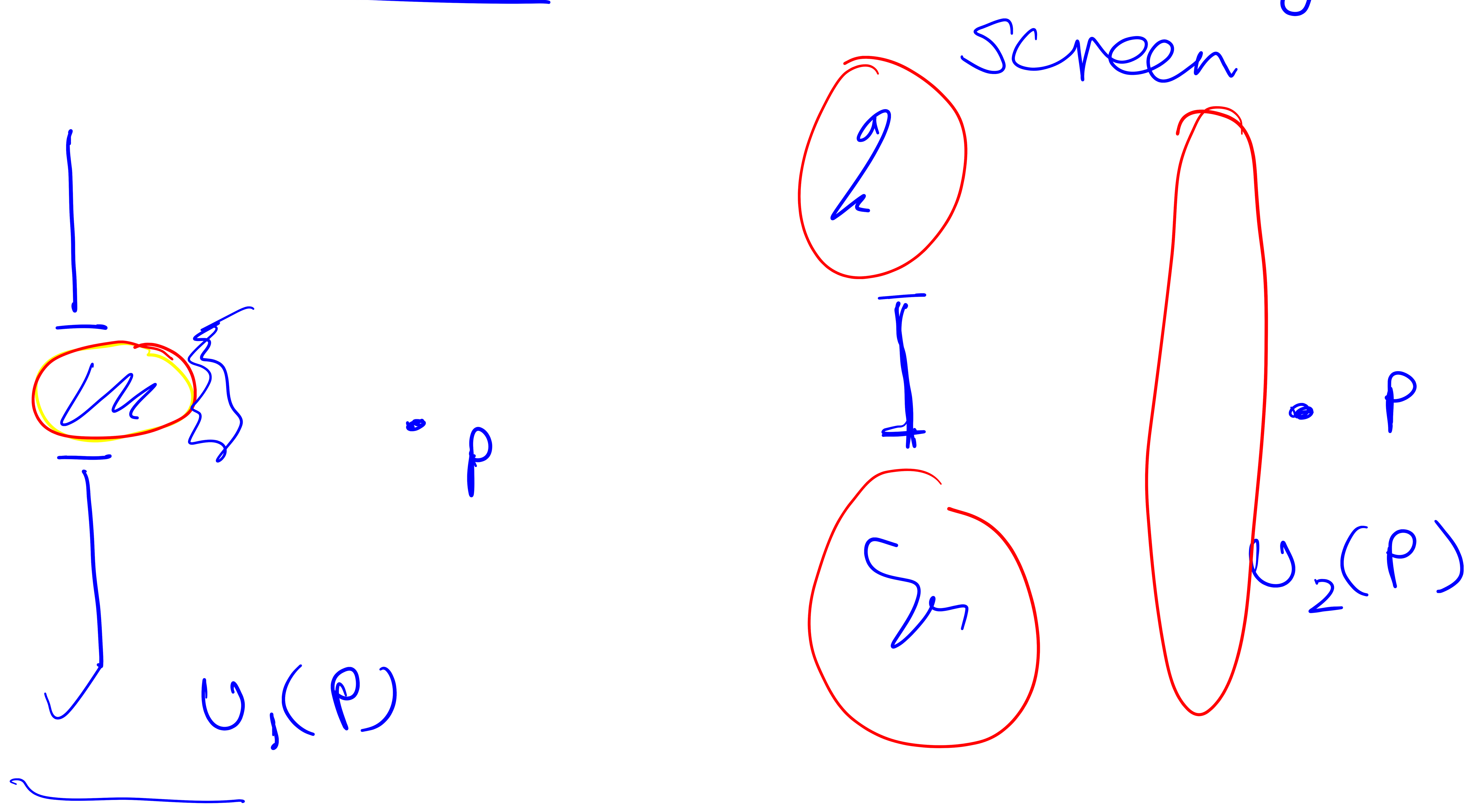
$$U(P) = \frac{A e^{ikr_0}}{r_0} \iint \frac{e^{iks}}{s} \underbrace{K(\chi)}\, dS$$

$$K(\chi) = \frac{-i}{2\lambda}(1+\cos\chi) \checkmark$$

(inclination)



Babinet principle : "Complementary"



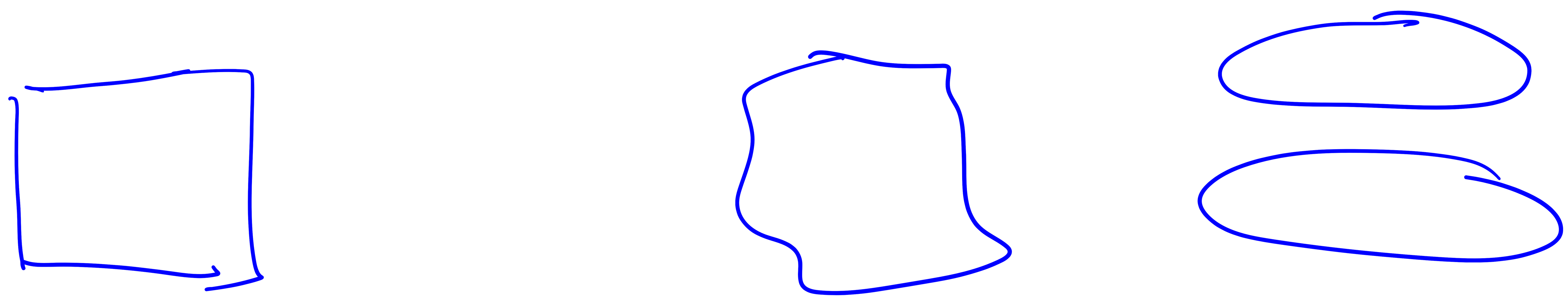
If $U(P)$ is the field at P when no screen is present,

$$U(P) = U_1(P) + U_2(P)$$

Babinet

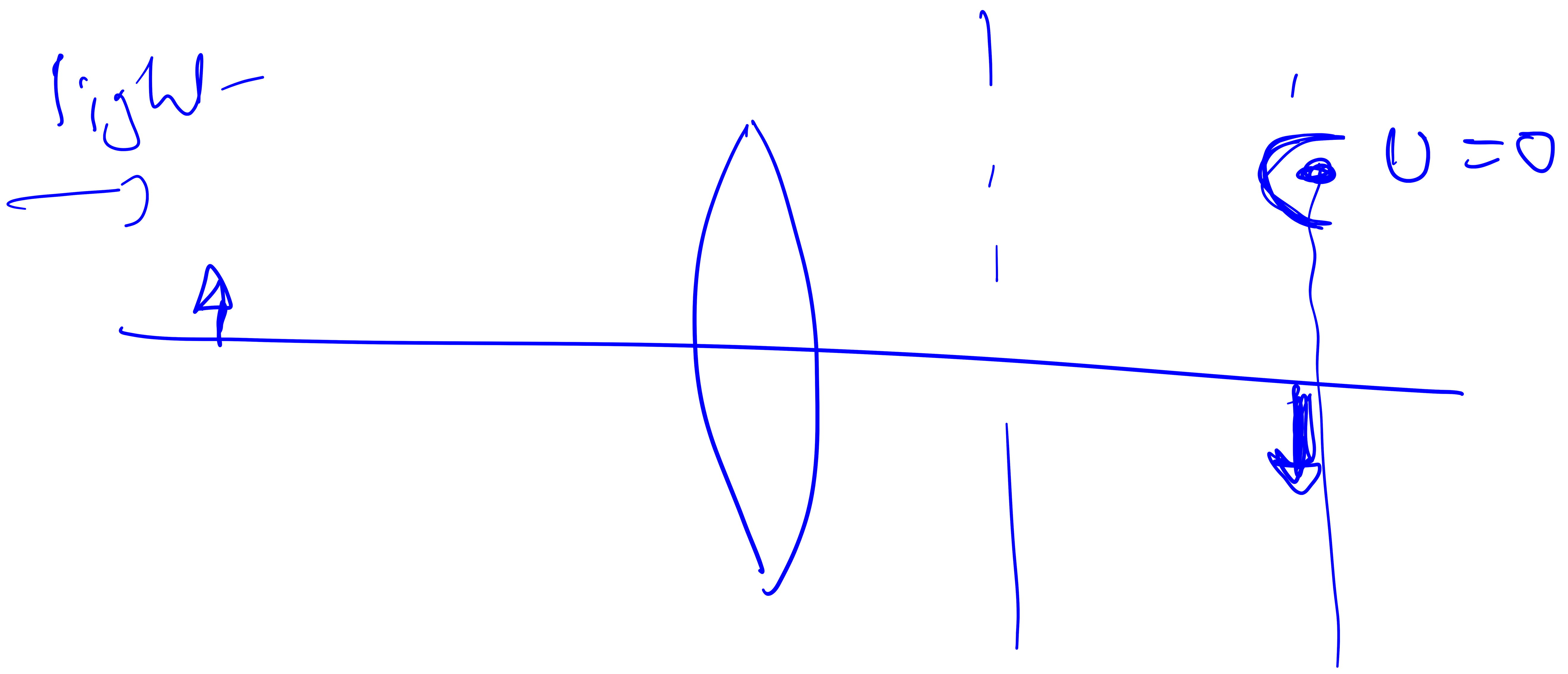
Implications

(1) If $U_1 = 0$; $U_2 = U$



(2) If $U = 0 \Rightarrow U_1 = -U_2$

Ex 1



$v_1 = -v_2$

Ex 2