$$\frac{I_{max}\{\theta,\varepsilon\} - I_{min}\{\theta,\varepsilon\}}{I_{max}\{\theta,\varepsilon\} + I_{min}(\theta,\varepsilon)} = \sqrt{1 - \frac{4\det\hat{J}}{(\mathrm{Tr}(\hat{J}))^2}}$$

Examples

(a) Unpolarized light : $I(\theta,\varepsilon) = \text{constant for all } \theta, \varepsilon$

$$\Rightarrow J_{xx} = J_{yy} \,;\; J_{xy} = 0$$

$$\hat{J} = \frac{1}{2} I_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Completely polarized light

Monochromatic $\Rightarrow$ a_1, a_2, ϕ_1, ϕ_2 are time independent

$$\hat{J} = \begin{bmatrix} a_1^2 & a_1 a_2 e^{+i\delta} \\ a_1 a_2 e^{-i\delta} & a_2^2 \end{bmatrix} \qquad \delta = \phi_1 - \phi_2$$

$$\det(\hat{J}) = 0$$

Complex degree of coherence

$$j_{xy} = \frac{J_{xy}}{\sqrt{J_{xx}}\sqrt{J_{yy}}} = e^{i\delta} \Rightarrow |j_{xy}| = 1$$

– Special case: Linear polarization $\Rightarrow$ $\delta = m\pi$

$$\hat{J} \rightarrow \begin{bmatrix} a_1^2 & (-1)^m a_1 a_2 \\ (-1)^m a_1 a_2 & a_2^2 \end{bmatrix}$$

x- polarized light : $\hat{J} = I_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

y- polarized light : $\hat{J} = I_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

+45° - linearly polarized light : $\frac{1}{2} I_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

− 45° - linearly pol. light : $\frac{1}{2} I_0 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

- Circular polarization : $\delta = (2m \pm 1)\pi/2$

$$\hat{J} = \frac{1}{2} I_0 \begin{pmatrix} 1 & \pm i \\ \mp i & 1 \end{pmatrix}$$

Extension to non-monochromatic light

↳ a_1, a_2, ϕ_1, ϕ_2 are all time dependent

↳ $\frac{a_2(t)}{a_1(t)} = \underset{\text{Const}}{q}, \quad \delta = \phi_1(t) - \phi_2(t) = \underset{\text{Const}}{\chi}$ *

$$J_{xx} = \langle a_1^2 \rangle, \quad J_{xy} = q \langle a_1^2 \rangle e^{i\chi}$$

$$J_{yx} = q \langle a_1^2 \rangle e^{-i\chi}, \quad J_{yy} = \langle a_2^2 \rangle = q^2 \langle a_1^2 \rangle$$

$$\det(\hat{J}) = 0$$

$$E_x = \sqrt{\langle a_1^2 \rangle}\, e^{i(\alpha - 2\pi\bar{\nu}t)}, \quad E_y = \sqrt{\langle a_2^2 \rangle}\, e^{i(-\chi + \alpha - 2\pi\bar{\nu}t)}$$

NOTION OF A DEGREE OF POLARIZATION $\boxed{\text{INDEPENDENT}}$

* <u>Question 1</u>: What is the coherency matrix when several light waves are superposed?

$$E_x = \sum_{n=1}^{N} E_x^{(n)} \quad , \quad E_y = \sum_{n=1}^{N} E_y^{(n)}$$

$$J_{kl} = \langle E_k E_l^* \rangle = \sum_{n=1}^{N} \sum_{m=1}^{N} \langle E_k^{(n)} (E_l^{(m)})^* \rangle$$

$$= \sum_n \langle E_k^{(n)} (E_l^{(n)})^* \rangle + \sum_{n \neq m} \underbrace{\langle E_k^{(n)} E_l^{(m)*} \rangle}_{\text{zero}}$$

$$J_{kl} = \sum_n J_{kl}^{(n)} \qquad \square$$

<u>Question 2</u>

Any quasi-monochromatic wave can be regarded as the sum of a completely unpolarized light and a completely polarized light

↳ Is such a representation unique?

<u>Proof</u>:

$$\underline{J} = J^{(1)} + J^{(2)}$$

$$J^{(1)} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \qquad J^{(2)} = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix}$$

with $A \geq 0, \quad B \geq 0, \quad C \geq 0, \quad BC - DD^* = 0$

$$\left. \begin{array}{ll} J_{xx} = A + B \quad , & J_{yy} = A + C \\ J_{xy} = D \quad , & J_{yx} = D^* \end{array} \right\} \text{ --- } (1)$$

From determinant condition:

$$(J_{xx} - A)\,(J_{yy} - A) \;-\; J_{xy} J_{yx} = 0$$

$$A^2 - (J_{xx} + J_{yy})\,A + \det(J) = 0$$

$$A = \frac{1}{2}(J_{xx} + J_{yy}) \pm \frac{1}{2}\sqrt{\underline{(J_{xx} + J_{yy})^2 - 4\det} \hat{J}} \quad - (2)$$

$$\begin{aligned} \det(J) &= J_{xx} J_{yy} - J_{xy} J_{yx} \\ &\leq J_{xx} J_{yy} \\ &\leq \frac{1}{4}(J_{xx} + J_{yy})^2 \end{aligned}$$

Both roots are real and non-negative $\Leftarrow$

Which root is physical?

$$A = \frac{1}{2}(J_{xx} + J_{yy}) \pm \frac{1}{2}\left((J_{xx} + J_{yy})^2 - 4\det J\right)^{1/2}$$

$$B = \frac{1}{2}(J_{xx} - J_{yy}) \mp \frac{1}{2}\left((J_{xx} + J_{yy})^2 - 4\det J\right)^{1/2}$$

$$D = J_{xy}, \quad D^* = J_{yx}$$

$$C = \underline{\frac{1}{2}(J_{yy} - J_{xx})} \mp \frac{1}{2}\left((J_{xx} + J_{yy})^2 - 4\det(J)\right)^{1/2}$$

B & C must be +ve

Check:
$$\begin{aligned} &\left[(J_{xx} + J_{yy})^2 - 4(J_{xx} J_{yy} - J_{xy} J_{yx})\right]^{1/2} \\ &= \left[(J_{xx} - J_{yy})^2 + 4 J_{xy} J_{yx}\right]^{1/2} \\ &\geq |J_{xx} - J_{yy}| \end{aligned}$$

$\Rightarrow$ −ve sign in Eq. 2 is physical

$\Rightarrow$ Thus, we have a unique decomposition:

$$I_{tot} = Tr(\hat{J}) = J_{xx} + J_{yy}$$

$$I_{pol} = Tr(J^{(2)}) = B + C$$

$$= \left[(J_{xx} + J_{yy})^2 - 4 \det(\hat{J}) \right]^{1/2}$$

Degree of polarization: $\mathcal{P} = \dfrac{I_{pol}}{I_{tot}} = \sqrt{1 - \dfrac{4 \det(\hat{J})}{(Tr(\hat{J}))^2}}$

$$0 \leq \mathcal{P} \leq 1$$

SANITY CHECK: unpolarized light: $\mathcal{P} = 0$

$\Rightarrow (J_{xx} + J_{yy})^2 - 4 (J_{xx} J_{yy} - J_{xy} J_{yx}) = 0$

$\Rightarrow (J_{xx} - J_{yy})^2 + 4 J_{xy} J_{yx} = 0$

$\Rightarrow J_{xx} = J_{yy}, \quad J_{xy} = J_{yx} = 0$

SANITY CHECK: completely polarised light: $\mathcal{P} = 1$

$\det(\hat{J}) = 0 \Rightarrow |j_{xy}| = 1$

Ending of Wave Optics section of PH 202.

What have we become?
-Connoisseurs of Light

Maxwell's Equations
Stokes and Poincare
Brewster
Fabry-Perot
Huygen-Fresnel
Green's identities
Kirchhoff Fresnel
Fourier Theory
Fraunhofer
Babinet
Statistics of Light
Van-Cittert Zernike
Michelson
Degree of polarization