Lagrangian Formalism for Electrodynamics

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The following report provides a comprehensive account of the Lagrangian formalism for Electrodynamics. We first introduce the Lagrangian formalism for the massless electromagnetic fields in section I and then move on to analyze the effects of photon mass in II. When a material enters its superconducting state, many of its electromagnetic properties change. In section III, we investigate one such phenomenon called the Meissner effect, in which the material behaves like a diamagnet. We then move to the study of symmetries of the Lagrangian and the construction of the Stress-Energy tensor in section IV. This is followed by a discussion on Noether's theorem in section V. Section VI deals with the solution of the wave equation in covariant form constructed from Lorentz invariant Green functions. We then study the application of the Lagrangian method to quantum electrodynamics and study the behaviour of the t'Hooft-Polyakov magnetic monopole, elaborated in sections VII - IX. In the final section, we construct the path integral for a spinning particle and establish its classical dynamics to be of a massless point particle on the unit 2-sphere coupled to a magnetic monopole field. Spin quantisation is shown to emerge as a consequence of global gauge invariance of the spin path integral.

I. LAGRANGIAN FOR THE ELECTROMAGNETIC FIELD

[1] The electromagnetic field is governed by Maxwell's equations. The four equations are as follows:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned} \tag{1}$$

These vector identities govern the behavior of the fields $\mathbf{E}$ and $\mathbf{B}$.
From here on out, we will be using the convention where $\epsilon_0 = 1$ and $c = 1$(c is the speed of light).
We observe that we have two quantities ρ and $\mathbf{j}$ which are the external 'entities' present in the equations. It can easily be shown that the two quantities transform according to Lorentz transformations. It will prove fruitful to define a 4-vector j^μ as

$$\begin{aligned}
j^0 &= \rho \\
j^i &= \mathbf{j}
\end{aligned} \tag{2}$$

We will also find it useful to have a 4-vector potential from which we obtain the fields (it will have its 0^{th} component as the scalar potential and the rest of the three representing the components of the vector potential). We will call that 4-vector potential as A_μ. To see its relation to the traditional electric and magnetic fields we know, it would be fruitful to carry out the analysis for a relativistic charged particle moving in the field. We will denote the charge on the particle by e and $d\tau$ as the differential of the proper time.

A. A Particle in an Electromagnetic Field

For a general particle moving, we have its action defined as

$$S = -\int m d\tau \tag{3}$$

We are integrating over τ to keep the action Lorentz invariant.[2] In addition to this, if the particle is moving in an electromagnetic field, an additional term is added as

$$S = -\left(\int m d\tau + \int eA_\mu dx^\mu\right) \tag{4}$$

This can be reduced to

$$S = -\int (m\sqrt{1-\dot{x}^2} + e(A_0 + \dot{x}^m A_m))dt \tag{5}$$

Here, $\dot{x}^2$ represents the sum of squares of $\dot{x}^1$,$\dot{x}^2$ and $\dot{x}^3$.[3]
We have obtained the Lagrangian for the particle as

$$\mathcal{L} = -(m\sqrt{1-\dot{x}^2} + e(A_0 + \dot{x}^m A_m)) \tag{6}$$

Here, we will apply our standard Euler-Lagrange equations for a single particle. We will obtain

$$\frac{d}{dt}\left(m\frac{dx^i}{d\tau}\right) = e\left(-\frac{\partial A_0}{\partial x^i} + \frac{\partial A_i}{\partial t}\right) + e\dot{x}^j\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) \tag{7}$$

This equation greatly resembles the Lorentz force law and we see that

$$\begin{aligned} E_i &= -\frac{\partial A_0}{\partial x^i} + \frac{\partial A_i}{\partial t} \\ (\mathbf{v}\times\mathbf{B})_i &= \dot{x}^j\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) \end{aligned} \tag{8}$$

We can show for the particle that

$$m\frac{d^2 x_\mu}{d\tau^2} = eF_{\mu\nu}\frac{dx^\nu}{d\tau} \tag{9}$$

Here, $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10}$$

It is evident that F is skew-symmetric, F is known as the electromagnetic field tensor. The components F_{0i} contain components of the electric field while the components F_{ij} contain the components of the magnetic field.

B. The Lagrangian of the Electromagnetic Field

We define the Lagrangian of the fields as

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{11}$$

For ease of caclulation, we will use natural units in which we get the lagrangian to be

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{12}$$

Before proceeding, we can observe certain properties of this Lagrangian. We can see that the Lagrangian upholds locality of the fields. Since we have constructed the Lagrangian as a scalar quantity, it is Lorentz invariant. We will be able to show the Gauge invariance of the Action from this Lagrangian as follows.[4]
First, note that $F_{\mu\nu}$ is gauge invariant.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{13}$$

Upon making the gauge transformation $A_\mu \to A_\mu + \partial_\mu \xi$[5], we see that

$$F'_{\mu\nu} = F_{\mu\nu} + \partial_\nu\partial_\mu\xi - \partial_\mu\partial_\nu\xi \tag{14}$$

Thus, $F_{\mu\nu}$ itself is gauge invariant and so, its contribution to the action is gauge invariant as well. Next, we need to investigate the gauge invariance of the other term. For this, we are concerned with the following integral

$$\int j^\mu \partial_\mu \xi d^4x$$

Using integration by parts, the 4 dimensional divergence theorem and the continuity equation, we can show that the integral is zero. For completeness, I will mention the continuity equation as

$$\partial_\mu j^\mu = 0 \tag{15}$$

Before moving on to the Euler-Lagrange Equations, we can quickly note that we have shown that the Lagrangian we chose is Lorentz Invariant, Gauge Invariant and preserves locality. We can note the Maxwell's equations in terms of the new quantities we defined are[6]:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0 \end{aligned} \tag{16}$$

The 4 equations we started out with have been encompassed in two equations in terms of the field tensor. We can see that the first of the two gives us the $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$ equations. The second gives us the $\nabla \times \mathbf{E}$ and $\nabla \cdot \mathbf{B}$ equations. Moving on to the Euler-Lagrange Equations for the field, we get

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\mu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\mu} \tag{17}$$

From this, we obtain

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\nu \\ &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu \end{aligned}$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -F^{\mu\nu}$$

Thus, from the Euler-Lagrange Equations, we get

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{18}$$

The second equation comes from the Bianchi Identity.

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{19}$$

This can be written more compactly as

$$\partial_\mu \star F^{\mu\nu} = 0 \tag{20}$$

Here, $\star F^{\mu\nu}$ is the Hodge dual of the field strength tensor defined as

$$\star F^{\mu\nu} = \frac{1}{2!} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{21}$$

Thus, we have demonstrated that Maxwells's equations pop out of the Lagrangian we constructed.

II. PROCA LAGRANGIAN - PHOTON MASS EFFECTS

A. Theoretical Treatment

An assumption that the photon is massless has been made throughout our calculations throughout the course (although it might not have been explicitly stated). We will now discuss the basis for such a claim.

A particle having mass manifests through a term of a very special kind in the Lagrangian. For particles originating from the gauge field, the term is a square of the gauge. Since we are working in a $U(1)$ gauge theory, the gauge field is just A_μ. Thus, the mass term looks like $\frac{1}{2}\frac{m_\gamma^2 c^2}{\hbar^2}A_\mu A^\mu$. For convenience, denote

$$\mu = \frac{m_\gamma^2 c^2}{\hbar^2} \tag{22}$$

Thus, our full Lagrangian with the mass term looks like

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}A_\mu A^\mu - J_\mu A^\mu \tag{23}$$

This Lagrangian is known as the **Proca-Lagrangian**. The homogenous Maxwell's equations remain unchanged as they stem from the antisymmetry of the Field tensor. Thus, we have (This is also called the Bianchi Identity for general gauge theories)

$$\partial_\mu \star F^{\mu\nu} = 0 \tag{24}$$

As for the equations of motion, we get an additional term giving us

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = \mu_0 J^\nu \tag{25}$$

With the Lorenz gauge (Now fixed owing to current conservation), we get

$$\Box A_\nu + \mu^2 A_\nu = \mu_0 J_\nu \tag{26}$$

For static fields, it reduces to a Helmholtz equation in A_ν

$$\nabla^2 A_\nu - \mu^2 A_\nu = \mu_0 J_\nu \tag{27}$$

The solution for the potential for a unit point charge at the origin (boundaries are at infinity) is

$$\Phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\frac{e^{-\mu r}}{r} \tag{28}$$

The dispersion relation in free space given by the equation is

$$\omega^2 = c^2k^2 + \mu^2c^2 \tag{29}$$

B. Experimental Methods

In a 1971 letter, P.A. Franken and G.W. Ampulski suggested a *table-top* experiment to determine the photon mass. In this subsection, we will go through the suggested experiment and also discuss the correctness of the results obtained. It is easy to prove that the free-space phase velocity of light v_ϕ is related to the angular frequency by

$$\left(\frac{v_\phi}{c}\right)^2 = \frac{\omega^2}{\omega^2 - \mu^2c^2} \tag{30}$$

Consider now that we have a resonant cavity whose fundamental frequency in the massless photon case is ω_0, and ω' in the massy case. The values of k for the various cavities depends only on the geometry of the cavity and is independent of μ. The relation between the phase velocity, angular frequency and the wavevector is fixed $v_\phi = \omega/k$. Therefore, we can write,

$$\frac{\omega'}{\omega_0} = \frac{v_\phi}{c} \tag{31}$$

Combining the results from equations (30) and (31), we get,

$$\omega'^2 = \omega_0^2 + \mu^2c^2 \tag{32}$$

This equation predicts that the lowest possible fundamental frequency for any resonant cavity can be μc. Albeit physically impossible, if we were to build larger and larger cavities the fundamental mode of the cavities will tend to μc. This will allow the determination of the mass of the photon. Since this is not practical, the use of RLC circuits is suggested. It is assumed that toroidal inductances, parallel plate capacitances are *folded* cavities, and therefore the previous discussion applies to them. Consider an RLC circuit now. The *ringing* or the damped natural frequency will be given by

$$\omega_0^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \tag{33}$$

ω_0 as defined above is the frequency that the circuit will oscillate at if it is not driven externally. R is an equivalent resistance that has been inputted to represent energy losses due to the non-ideality of the circuit. The quality factor Q of the circuit is given by $\omega_0 L/R$. If we have a high Q, the contribution of the latter term in equation (33) is very small. Let us denote the lowest mode for the massy case as ω' and assuming that the previously defined equations still stand, we get

$$\omega'^2 = \frac{1}{LC} - \frac{R^2}{4L^2} + \mu^2 c^2 \tag{34}$$

This again, ideally should allow us to determine the value of μ. In the actual experiment performed by Franken and Ampulski, they inserted charged capacitors into the RLC circuits and observed the frequency of the transient. Using the best available equipment, they put an upper bound of 10^{-49} g on the mass of a photon.
It should be noted that the experiment and the underlying mathematics rests on very shabby grounds. Franken and Ampulski agreed that there is no surety that the dispersion relationships assumed in equations (32) or (34) will be valid for a *warped* cavity like a RLC circuit, or even a rectangular cavity like that was assumed before. The experiment has a questionable interpretation because of the magnetic and dielectric materials, effects of which have been ignored in our analysis above. David Park and E.R. Williams in a letter later point out the various issues with the method proposed by Franken and Ampulski. We now discuss them.
Park and Williams consider a schematic cylinderical travelling-wave tube (TWT). What a TWT does not matter here and for the sake of brevity, we will not discuss it further. The only relevant point is that the TWT's walls are perfectly conducting along certain helical paths. Assume that the radius of the tube is a, the angle the conducting helices makes with the circumferential direction is ψ and in order to create a resonator, the tube is bended into a circle of circumference l. Under the assumption of a resistanceless helix, the tangential component of the electric field E on the surface of the TWT will vanish. Suppose

$$\begin{aligned} j_x &= -J\cos\psi\sin\phi f \\ j_y &= J\cos\psi\cos\phi f \\ j_z &= J\sin\psi f \end{aligned} \tag{35}$$

where $f = \delta(r-a)e^{\iota(kz-\omega t)}$ and ϕ is the azimuthal angle. One can solve and obtain the following

$$\begin{aligned} A_r &= 0 \\ A_\phi &= \frac{4a}{c}J\cos\psi I_1(\gamma r_<)K_1(\gamma r_>) \\ A_z &= \frac{4a}{c}J\sin\psi I_0(\gamma r_<)K_0(\gamma r_>) \\ V &= \frac{4ak}{\omega}J\sin\psi I_0(\gamma r_<)K_0(\gamma r_>) \end{aligned} \tag{36}$$

where $\gamma^2 = k^2 + \mu^2 - \omega^2/c^2$ and $r_>$ and $r_<$ respectively refer to the smaller and the larger value out of r and a. The vanishing of the tangential component of E at $r = a$ is given by $E_\phi \cos\psi + E_z \sin\psi = 0$. The results in the following dispersion relationship

$$\frac{c^2k^2}{\omega^2} = 1 + D\cot^2\psi, \quad D = \frac{I_1(\gamma a)K_1(\gamma a)}{I_0(\gamma a)K_0(\gamma a)} \tag{37}$$

Note that D is always positive. For small values of a, we get,

$$D \approx \frac{1}{2\ln(1.122/\gamma a)}$$

For the lowest possible mode of resonance, we will have $k = \frac{2\pi}{l}$. This gives,

$$\omega' = \frac{2\pi c}{l\sqrt{1 + D\cot^2\psi}} \tag{38}$$

It is clear here, that for any l, by varying ψ between 0 and $\pi/2$, we can make ω' as small as we want it to be. There is now lower limit set by μ. Assume now the case $\psi = \pi/4$. We approximate ω' to first order in μ^2

$$\omega'^2 = \omega_0^2\left(1 - \frac{D\mu^2}{\gamma^2}\right) \tag{39}$$

Under the conditions of the experiments assumed by Franken and Ampuslki, we have ω of the order of μc, and therfore we can approximate $\gamma^2 = (2\pi/l)^2$ to zeroth order in μ^2. Therefore, we get

$$\omega'^2 = \omega_0^2\left[1 - D\left(\frac{\mu l}{2\pi}\right)^2\right] \tag{40}$$

Note that as $\omega_0 \to 0$, ω' also vanishes, in agreement with equation (44). Although, the experiment performed in Fraken and Ampulski's utilzed a *RLC* circuit, Park and Williams believe that similar mathematical arguments can be utilized using electrostatics and electrodynamucs to arrive at a similar result. The analysis performed above shows that for $\mu > 0$, the fractional difference in the massless and massy case is of the order of $(\mu l)^2$ and not $(\mu c/\omega)^2$ as derived by Franklin and Ampulski. Therefore, it is at large apparatus sizes and not just low frequencies that the effects characteristic of the photon mass are observable, a conclusion that Franklin and Ampulski rightfully made when discussing the case of resonant cavities.

III. SUPERCONDUCTIVITY

Superconductivity is a phenomenon demonstrated by certain materials at cryogenic temperatures or very high pressures in which the electrical resistivity of the material vanishes. It was experimentally discovered by Kamerlingh Onnes in 1911. But it is not enough to model a superconductor as an ordinary electrical conductor with zero resistivity because superconductors show several novel magnetic properties. Several attempts have been made so far to provide a rigorous theoretical explanation for superconductivity. This section discusses a major milestone in the field: the London theory.

A. Meissner Effect

A superconducting material shows a temperature-dependent electromagnetic behavior. When the temperature is above T_C, the material has a finite resistance and behaves *normally.* Hence, this state is called the normal state. When the temperature drops below T_C, the material transitions into the *superconducting* state, i.e., the electrical resistance vanishes. Interestingly, when the aforementioned transition takes place, the magnetic field is expelled out of the material, i.e., the magnetic field is allowed to penetrate the sample only to a very extent given by the *London penetration depth* ($\mathcal{O}$(10 nm)). This effect is known as the Meissner effect, which is the consequence of an effective photon mass.

The London theory of superconductivity is a non-relativistic theory. It assumes non-relativistic charge transport, thus giving the following expression for current density $\mathbf{J}$:

$$\mathbf{J} = Qn_Q\mathbf{v} \tag{41}$$

Here, Q, n_Q, and $\mathbf{v}$ represent the charge possessed by the charge carriers, number density, and velocity, respectively. As discussed in the previous sections, in the presence of electromagnetic coupling, the canonical momentum $\mathbf{P}$ is given by

$$\mathbf{P} = \mathbf{p} + Q\frac{\mathbf{A}}{c}$$

where $\mathbf{p}$ is the kinetic momentum, and $\mathbf{A}$ represents the magnetic vector potential. Since we are in the classical, non-relativistic regime, the kinetic momentum can be written as $\mathbf{p} = m_Q\mathbf{v}$. This changes the equation for canonical momentum to the following

$$\mathbf{P} = m_Q\mathbf{v} + Q\frac{\mathbf{A}}{c} \tag{42}$$

The London brothers made the following assumption: *The superconducting state is a coherent state of charge carriers with zero canonical momentum.* It can be justified with a quantum mechanical treatment of superconductivity[78]. Periodic lattice potentials can be solved using Bloch's theorem. The statement of the theorem is: *If the potential is periodic, then the solution can be written as a phase multiplied by a periodic function with the same periodicity as the potential.* Mathematically,

$$\Psi_{\text{Bloch}} = e^{i\mathbf{k}.\mathbf{r}}u(\mathbf{r}) \tag{43}$$

where $u(\mathbf{r})$ is a periodic function with a period of say $\mathbf{R}$ which is the same as that of the potential. In the absence of an applied electromagnetic field, the expectation of the canonical momentum is 0, i.e.,

$$\langle\Psi|\mathbf{P}|\Psi\rangle = 0 \tag{44}$$

In the superconducting state, the wavefunction is *rigid*, i.e., the canonical momentum is zero even after the application of an electromagnetic field. It is well explained in the Bardeen-Cooper-Schrieffer (BCS) theory, in which the discretized Schrodinger equation for the superconducting lattice is solved. A detailed discussion of the BCS theory is beyond the scope of this project, but the process can be briefly chalked out as follows[9]:

(i) We first write down the effective second-quantized Wannier-transformed Hamiltonian using the spin-momentum coupled electron creation and annihilation operators.

(ii) To diagonalize the hamiltonian, we perform the *Bogoliubov* transformation, i.e., we introduce new quasi-fermionic creation and annihilation operators that are linear combinations of electronic creation and annihilation operators. These new quasiparticles are termed as *Bogoliubons.* The BCS ground state can be thought of as the vacuum in the Bogoliubon space.

(iii) When the effect of electromagnetic fields is investigated using perturbation theory, we obtain that in the superconducting state, the expectation of the canonical momentum is zero even after the application of the fields.

The current density in the superconducting state is given by

$$\mathbf{J} = Qn_Q(-\frac{Q}{m_Q}\frac{\mathbf{A}}{c}) = -\frac{Q^2}{m_Qc}n_Q\mathbf{A} \tag{45}$$

Next, we substitute the above expression for current density in the Lorenz gauge wave equation for the electromagnetic vector potential:

$$\begin{aligned}\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0(-\frac{Q^2}{m_Qc}n_Q\mathbf{A}) \\ &= \mu_0\frac{Q^2}{m_Qc}n_Q\mathbf{A}\end{aligned} \tag{46}$$

A little rearrangement shows that the above equation takes the *Proca* form as shown below:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu^2\mathbf{A} = 0 \tag{47}$$

where $\mu \equiv \sqrt{\mu_0\frac{Q^2}{m_Qc}n_Q}$. If we assume the static limit, i.e., $\partial_t^2\mathbf{A} = 0$, then we are left with the following second-order differential equation:

$$\nabla^2\mathbf{A} = \mu^2\mathbf{A} \tag{48}$$

The solutions are exponential $\sim e^{\pm\mu x}$. The *London penetration depth* is defined as

$$\lambda_L = \frac{1}{\mu} = \sqrt{\frac{m_Qc}{\mu_0Q^2n_Q}} \tag{49}$$

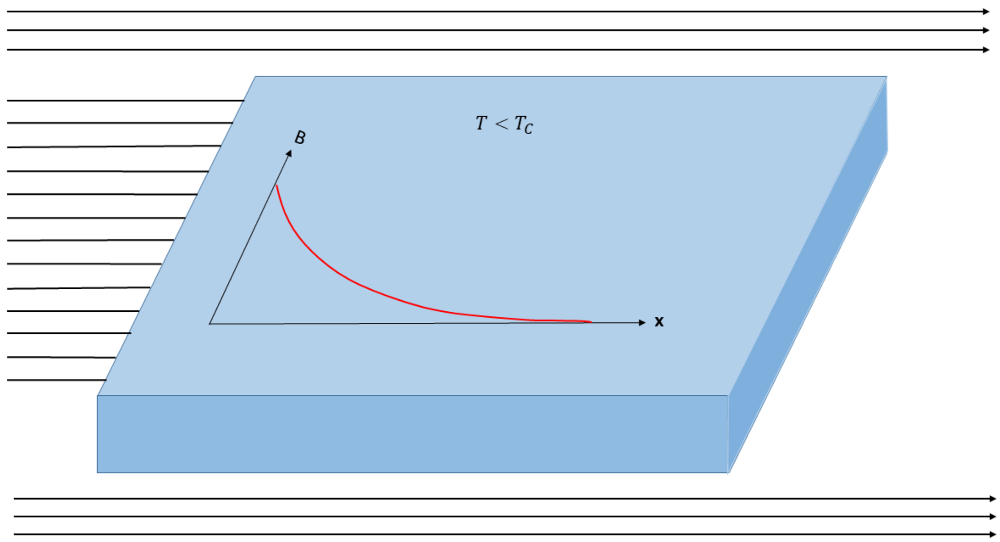

Figure 1. Meissner effect: At temperatures less than T_C, the magnetic field lines are expelled out

Clearly, beyond a distance of a few λ_L's, the potential is negligibly small. An order estimate of the London penetration depth will be done next. But before that, an important observation is that from the discussion on Proca Lagrangian, we know that the effective photon mass is $m_\gamma = \hbar\mu/c \implies$ the effective photon mass in this case is

$$m_\gamma = \frac{\hbar}{c\lambda_L} \tag{50}$$

BCS theory renders the following, which have also been corroborated experimentally: The conduction is due to electron-electron pairs loosely bound by phonons (Cooper pairs). Therefore,

(i) Charge of the charge carriers is twice the electronic charge $\implies Q = -2e$ (here, $e =$ electronic charge)

(ii) Mass of the charge carriers $m_Q = 2m_e$ (here, $m_e =$ mass of an electron)

(iii) The density of superconducting charge carriers n_Q is temperature dependent. At the ground state, $n_Q = n$, i.e., all the electrons participate in conduction, and the value gradually decreases to 0 as the temperature approaches the Curie temperature T_C.

To estimate the order of λ_L, we take the values of m_Q and Q as mentioned above, and $n_Q = \mathcal{O}(10^{28})m^{-3}$. This gives

$$\lambda_L \sim 40 \text{ nm} \tag{51}$$

Therefore, we observe that the extent to which the magnetic field can penetrate the material in the superconducting state is not more than a few nanometers.

B. Measurement of the London Penetration Depth

We saw in the previous subsection that the carrier density in the superconducting state is temperature-dependent. This brings a temperature dependence in λ_L, which can be measured by putting the superconducting specimen in series with a resonant circuit. Information about the temperature dependence of λ_L is encoded in the shift in the resonant frequency of the circuit. In the long wavelength limit, the surface impedance of the circuit is purely reactive,

$$Z_s = -i\frac{2\pi}{c\epsilon_0}\frac{\lambda_L}{\lambda} \tag{52}$$

An explicit calculation of the surface impedance, motivated by Problem 12.20 of the book Classical Electrodynamics by J.D. Jackson[1], is provided in the appendix.

IV. CANONICAL AND SYMMETRIC STRESS TENSORS

A. Generalization of the Hamiltonian: Canonical Stress Tensor

In classical particle mechanics, the transition to Hamiltonian formulation is made using the canonical transformation:

$$H = \sum_i p_i \dot{q}_i - L$$

In a local field theory, the total Lagrangian of the system, L, can be written as an integral over the Lagrangian density over space:

$$L = \int d^3\mathbf{x}\, \mathcal{L}(\phi_\nu, \partial_\mu \phi_\nu)$$

And similarly, for the Hamiltonian:

$$H = \int d^3\mathbf{x}\, \mathcal{H}(\phi_\mathbf{k}, \partial_\mu \phi_\mathbf{k})$$

For the electromagnetic free field, we have the Lagrange density $\mathcal{L}(A^\mu, \partial_\nu A^\mu) = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$, which is a function of the potentials and its derivatives. The canonical transformation is used to define the Hamiltonian density as before. The covariant generalization of the Hamiltonian is the canonical stress tensor:

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}_{EM}}{\partial(\partial_\alpha A^\lambda)} \partial^\beta A^\lambda - g^{\alpha\beta} \mathcal{L}_{EM} \tag{53}$$

$$\Longrightarrow T^{\alpha\beta} = -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \mathcal{L}_{EM}$$

Note: The sum over repeated indices is implied and $g^{\mu\nu}$ is the Minkowski metric with negative signature. To get a more physical picture of the components of the tensor, we use $\mathcal{L} = \left(\epsilon_0 \mathbf{E}^2 - \frac{1}{\mu_0}\mathbf{B}^2\right)/2$ to get (using $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{B} - c^{-2}\partial_t \mathbf{E} = 0$):

$$\begin{aligned}
T^{00} &= \frac{1}{2}\left(\epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{\mu_0}\right) + \epsilon_0 \nabla \cdot (\Phi \mathbf{E}) \\
T^{0i} &= \frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})_i + \frac{1}{\mu_0 c} \nabla \cdot (A_i \mathbf{E}) \\
T^{i0} &= \frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})_i + \frac{1}{\mu_0 c}\left[(\nabla \times \Phi \mathbf{B})_i - \partial_{x_0}(\Phi E_i)\right]
\end{aligned}$$

Due to the finite velocity of propagation, one can always assume the fields to be localized in a finite region of space. This allows us to set the all-space integral of the divergence terms in the above equations to zero. Thus, the integrals of T^{00} and T^{0i} over all 3-space at a fixed time in some inertial frame are:

$$\begin{aligned}
\int T^{00} d^3x &= \frac{1}{2} \int \left(\epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{\mu_0}\right) d^3x = \int u\, d^3x = U_{field} \\
\int T^{0i} d^3x &= \frac{1}{c} \int \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})_i\, d^3x = \frac{1}{c} \int s_i\, d^3x = c P^i_{field}
\end{aligned}$$

Thus, although T^{00} and T^{0i} differ from the energy density u and the momentum density s_i (i^{th} component of the Poynting vector) by divergence terms, integration over all space in an inertial frame can be interpreted as the total energy and c times the total momentum of the EM fields in that frame. The above generalizations of the field energy and momentum hint at a generalization of the conservation equation:

$$\partial_t u + \nabla \cdot \mathbf{s} = 0$$

The differential conservation statement is:

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{54}$$

The proof goes as follows:

$$
\begin{aligned}
\partial_\alpha T^{\alpha\beta} &= \sum_\mu \partial_\alpha \left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\mu)} \cdot \partial^\beta A^\mu\right] - \partial^\beta \mathcal{L} \\
&= \sum_\mu \left[\partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\mu)} \cdot \partial^\beta A^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\mu)} \partial_\alpha \partial^\beta A^\mu\right] - \partial^\beta \mathcal{L}
\end{aligned}
$$

Euler Lagrange equation of motion

$$
\partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\mu)} = \frac{\partial\mathcal{L}}{\partial A^\mu} \tag{55}
$$

Thus, we have:

$$
\begin{aligned}
\partial_\alpha T^{\alpha\beta} &= \sum_\mu \left[\frac{\partial\mathcal{L}}{\partial A^\mu} \cdot \partial^\beta A^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\mu)} \partial^\beta (\partial_\alpha A^\mu)\right] - \partial^\beta \mathcal{L} \\
&= \partial^\beta \mathcal{L}(A^\mu, \partial_\alpha A^\mu) - \partial^\beta \mathcal{L} = 0
\end{aligned}
$$

Using the identifications of the integrals derived for localized fields:

$$
\frac{d}{dt} E_{field} = 0\,, \quad \frac{d}{dt} \mathbf{P}_{field} = 0 \tag{56}
$$

The above results are valid for an observer at rest in the frame in which the fields are specified. The integral laws are not manifestly covariant. To avoid having electromagnetic energy and momentum defined separately in each inertial frame, we need to construct explicitly covariant integral expressions for the electromagnetic energy and momentum.

B. Symmetric Stress Tensor

The canonical stress tensor considered so far has some drawbacks. One is the lack of T^{0i} and T^{i0} symmetry. The field angular momentum

$$
\mathbf{L}_{field} = \int \mathbf{x} \times \mathbf{P}_{field}\, d^3x = \frac{1}{\mu_0 c^2} \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3x
$$

has a covariant generalization in terms of a rank-3 tensor:

$$
M^{\alpha\beta\gamma} = T^{\alpha\beta} x^\gamma - T^{\alpha\gamma} x^\beta
$$

The conservation of angular momentum then translates to the vanishing of the following 4-divergence:

$$
\begin{aligned}
&\partial_\alpha M^{\alpha\beta\gamma} = 0 \\
\Longrightarrow\ &\cancelto{0}{(\partial_\alpha T^{\alpha\beta})} x^\gamma + T^{\gamma\beta} - \cancelto{0}{(\partial_\alpha T^{\alpha\gamma})} x^\beta - T^{\beta\gamma} = 0
\end{aligned}
$$

Thus, the conservation of angular momentum demands the symmetry of the field strength tensor. Two other drawbacks of the $T^{\alpha\beta}$ tensor defined so far are that it explicitly involves the potentials and thus is not gauge invariant, and its trace $T_\alpha^{\ \alpha}$ is non-zero, as required for mass-less photons. To construct the symmetric, traceless, and gauge-invariant stress tensor, we proceed as follows:

$$
\begin{aligned}
T^{\alpha\beta} &= -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \mathcal{L}_{EM} = -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \\
&= \frac{1}{\mu_0} \left[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right] - \frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\lambda A^\beta \\
&= \frac{1}{\mu_0} \left[F^{\alpha\mu} g_{\mu\nu} F^{\nu\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right] - \frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\lambda A^\beta
\end{aligned}
$$

The term in the square brackets is symmetric in α and β, and is gauge invariant. The second term can be written as:

$$\begin{aligned} T_D^{\alpha\beta} &\equiv -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\lambda A^\beta = \frac{1}{\mu_0} F^{\lambda\alpha} \partial_\lambda A^\beta \\ &= \frac{1}{\mu_0}(F^{\lambda\alpha}\partial_\lambda A^\beta + A^\beta \partial_\lambda F^{\lambda\alpha}) = \frac{1}{\mu_0}\partial_\lambda (F^{\lambda\alpha} A^\beta) \end{aligned}$$

The tensor $T_D^{\alpha\beta}$ is a 4-divergence, and thus, its all space integral for local fields satisfies:

$$\int T_D^{0\beta} d^3x = 0$$

The 4-divergence of $T_D^{\alpha\beta}$ is also zero. $\partial_\alpha T_D^{\alpha\beta} = 0$ is easily verifiable, following the same procedure as before. Thus, the conservation laws will hold for $T^{\alpha\beta} - T_D^{\alpha\beta}$ if they do for $T^{\alpha\beta}$.

We can therefore define the symmetric, traceless, and gauge-invariant stress tensor as:

$$\Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right] \tag{57}$$

Explicit calculations give the following components:

$$\Theta^{00} = \frac{1}{2}\left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right) = u \tag{58}$$

$$\Theta^{0i} = \frac{1}{\mu_0 c}(\mathbf{E}\times\mathbf{B})_i = \frac{s_i}{c} \tag{59}$$

$$\Theta^{ij} = -\frac{1}{2}\left[\epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\right] \tag{60}$$

The differential conservation law becomes:

$$\partial_\alpha \Theta^{\alpha\beta} = 0 \tag{61}$$

The $\beta = 0$ component is, in fact, Poynting's theorem: $0 = \partial_\alpha \Theta^{\alpha 0} = \frac{1}{c}\left(\partial_t u + \nabla \cdot \mathbf{s}\right)$. The space-space components Θ^{ij} of the symmetrized stress tensor (58) are exactly what we had worked out in class, just using a different approach. Here we started with the Lagrangian to obtain the covariant generalization of the Hamiltonian and then worked things out based on symmetry arguments. In class, we started with Maxwell's equations, which are just the equations of motion for the Lagrangian.

C. Conservation Laws for Electromagnetic Fields Interacting with Charged Particles

In the presence of sources, i.e., charge and current densities, the symmetric stress-energy tensor retains its form, but the coupling to the source current $[j^\mu \equiv (c\rho, j^1, j^2, j^3) = (c\rho, \mathbf{j})]$ makes its divergence non-zero. The divergence calculation goes as:

$$\begin{aligned} \partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{\mu_0}\left[\partial^\mu F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4}\partial^\beta F_{\mu\nu} F^{\mu\nu}\right] \\ &= \frac{1}{\mu_0}\left[(\partial^\mu F_{\mu\lambda}) F^{\lambda\beta} + F_{\mu\lambda}(\partial^\mu F^{\lambda\beta}) + \frac{1}{2} F_{\mu\nu}\partial^\beta F^{\mu\nu}\right] \end{aligned}$$

Using the inhomogenous Maxwell's equations and rearranging:

$$\begin{aligned} \partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c} F^{\beta\lambda} j_\lambda &= \frac{1}{2\mu_0} F_{\mu\nu}(\partial^\mu F^{\lambda\beta} + \partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda}) \\ &= \frac{1}{2\mu_0} F_{\mu\nu}(\partial^\mu F^{\lambda\beta} + \partial^\lambda F^{\mu\beta}) \\ &= 0 \;\; \text{(contraction of a symmetric and an antisymmetric tensor)} \end{aligned}$$

In the second step, we use the homogeneous Maxwell's equation: $\partial^\mu F^\lambda\beta + \partial^\beta F^\mu\lambda + \partial^\lambda F^\beta\mu = 0$. Thus, the divergence of the stress tensor becomes:

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c} F^{\beta\lambda} j_\lambda \tag{62}$$

The space and time components become the equations for the conservation of energy and momentum:

$$\frac{1}{c}\left(\partial_t u + \nabla \cdot \mathbf{s}\right) = -\frac{1}{c}\mathbf{j}\cdot\mathbf{E} \tag{63}$$

$$\frac{1}{c^2}\frac{\partial s_i}{\partial t} + \partial_j \Theta_{ij} = -\left[\rho E_i + \frac{1}{c}(\mathbf{j}\times\mathbf{B})_i\right] \tag{64}$$

The right-hand for the above equations together are the components of the Lorentz force density 4-vector:

$$f^\beta \equiv \frac{1}{c} F^{\beta\mu} j_\mu = \left(\frac{1}{c}\mathbf{j}\cdot\mathbf{E}\ ,\ \rho\mathbf{E} + \frac{1}{c}(\mathbf{j}\times\mathbf{B})\right) \tag{65}$$

Using the Lorentz force equation, the integral of f^β can be identified to be the rate of change of energy and momenta of the charged particles (which are assumed to be the sources of the fields):

$$\int f^\beta d^3x = \frac{d}{dt} P^\beta_{particles} \tag{66}$$

Thus, the integral (over all space) form of the conservation equation can be written as a conservation law for the total energy-momentum content of the field + particles system:

$$\int d^3x (\partial_\alpha \Theta^{\alpha\beta} + f^\beta) = \frac{d}{dt}(P^\beta_{field} + P^\beta_{particles}) = 0 \tag{67}$$

V. NOETHER'S THEOREM

An extremely important and profound relationship exists between symmetries and conservation laws in classical and quantum mechanics and classical and quantum field theory.

Noether's Theorem: For every continuous transformation (of the coordinates and/or field functions) that leaves the action invariant (i.e., $\delta S = 0$), there corresponds is a conserved current (which could be a function of fields and their derivatives and possibly also the coordinates).

So far, we have looked particularly at the Lagrangian for Electromagnetism. We now look at a more general approach to deriving conservation laws based on various symmetries of a given Lagrangian.
In a local field theory, the total Lagrangian of the system, L, can be written as an integral over the Lagrangian density over space: $L = \int d^3x\ \mathcal{L}(\phi_i, \partial_\mu \phi_i)$
Let's first consider the simple case of only one scalar field. Noether's theorem considers continuous transforms of the field $\phi(x)$:

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon\Delta\phi(x)$$

Here ϵ is an infinitesimal parameter, and $\Delta\phi$ is some deformation of the field configuration. The above transformation is a symmetry if it leaves the equations of motion invariant, which in turn means that the action is left invariant. More generally, the action can change by a surface term since this would not affect the derivation of the Euler-Lagrange equations of motion. Thus, under the transformation, the Lagrangian density must also be invariant upto a 4-divergence (leading to the surface term), with Λ^μ not determined by the EOMs:

$$\begin{aligned} \mathcal{L}(x) &\to \mathcal{L}(x) + \epsilon\partial_\mu \Lambda^\mu \\ \implies \epsilon\Delta\mathcal{L} &= \epsilon\partial_\mu\Lambda^\mu \end{aligned}$$

The change in the Lagrangian density due to the change in the field ϕ is:

$$\begin{aligned} \epsilon\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}(\epsilon\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\epsilon\Delta\phi) \\ &= \epsilon\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \epsilon\left[\cancelto{0}{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)}\right]\Delta\phi \end{aligned}$$

The second term vanishes (Euler Lagrange equations of motion). Equating the remaining term to the total divergence term ($\partial_\mu \Lambda^\mu$) obtained above:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \Lambda^\mu \right) \equiv \partial_\mu j^\mu = 0 \tag{68}$$

Thus, each continuous symmetry of the Lagrangian gives rise to a conserved current j^μ. Like the Electromagnetic field, if the symmetry involves more than one field $\phi_i(x)$, with $i = 1, 2..$, the conserved current is:

$$\{j^0, \mathbf{j}\} \equiv j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta\phi_i - \Lambda^\mu \tag{69}$$

Now, the integral over all space of the 4-divergence of the current is (κ is a constant):

$$\begin{aligned} &\int \partial_\mu j^\mu \; d^3x = \kappa \\ \implies Q \equiv &\int j^0 \; d^3x = \cancelto{0}{\int \nabla \cdot \mathbf{j} \, d^3x} + \kappa \\ &\implies \frac{dQ}{dt} = 0 \end{aligned}$$

Thus, the conservation law implies that the "charge" Q is conserved. In general, the conservation law (68) will correspond to a vanishing 4-divergence of a higher rank tensor. Let's work out an example to understand the same.

Example: Consider a translation of both the source and the electromagnetic fields in the direction of an arbitrary 4-vector n^ρ, coupled with a gauge transformation by a function χ. The variations of the fields are then given by:

$$\begin{aligned} A_\mu &\to A_\mu + \epsilon \partial_\rho A_\mu n^\rho + \partial_\mu \chi \\ \implies \Delta A_\mu &= \partial_\rho A_\mu n^\rho + \partial_\mu \chi \end{aligned}$$

To make the variation of the 4-potential gauge invariant, we choose $\chi = -A_\rho n^\rho$. This gives:

$$\Delta A_\mu = \partial_\rho A_\mu n^\rho - \partial_\mu A_\rho n^\rho = F_{\rho\mu} n^\rho$$

Now, we expect the above transformation to be a symmetry of the Lagrangian, meaning that the transformation leaves the Lagrangian invariant. We evaluate the change in the Lagrangian density under the infinitesimal continuous transform:

$$\begin{aligned} \Delta\mathcal{L} &= 0 \\ \Delta\mathcal{L} &= -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \Delta A^\alpha \right) \\ &= -\frac{1}{\mu_0} \partial_\mu \left(F^{\mu\alpha} \Delta A_\alpha \right) \\ &= -\frac{1}{\mu_0} \partial_\mu \left(F^{\mu\alpha} F_{\beta\alpha} n^\beta \right) = \frac{1}{\mu_0} \partial_\mu \left(F^{\mu\alpha} F_{\alpha\beta} n^\beta \right) \\ &= \frac{1}{\mu_0} \partial_\mu (g^{\mu\nu} F_{\nu\alpha} F^{\alpha\beta}) n_\beta \\ &= \left[\partial_\mu \Theta^{\mu\beta} - \frac{1}{\mu_0} \partial_\mu \left(\frac{1}{4} g^{\mu\beta} F_{\sigma\tau} F^{\sigma\tau} \right) \right] n_\beta \end{aligned} \tag{70}$$

Note that n_β is an arbitrary constant vector along which we translated. Thus, the terms in the square bracket must go to zero independently. $g^{\mu\beta}$ is a constant matrix and can be pulled out of the derivative. Now, we simplify the second term separately:

$$\begin{aligned} \partial_\mu (F_{\sigma\tau} F^{\sigma\tau}) &= 2(\partial_\mu F_{\sigma\tau}) F^{\sigma\tau} \\ &= -2(\partial_\sigma F_{\tau\mu} + \partial_\tau F_{\mu\sigma}) F^{\sigma\tau} \\ &= -2[\partial_\sigma (F_{\tau\mu} F^{\sigma\tau}) - F_{\tau\mu} (\partial_\sigma F^{\sigma\tau}) + \partial_\tau (F_{\mu\sigma} F^{\sigma\tau}) - F_{\mu\sigma} (\partial_\tau F^{\sigma\tau})] \\ &= -\frac{2\mu_0}{c} [\cancel{\partial_\sigma (F_{\tau\mu} F^{\sigma\tau})} - F_{\tau\mu} (j^\tau) + \cancel{\partial_\tau (F_{\mu\sigma} F^{\sigma\tau})} - F_{\mu\sigma} (-j^\sigma)] \\ &= -\frac{4\mu_0}{c} j^\nu F_{\mu\nu} = -4\mu_0 f_\mu \end{aligned} \tag{71}$$

Here, we used the Bianchi identity followed by the antisymmetry of the $F_{\mu\tau}$ to cancel to terms. Then, used the expression for Maxwell's source equations. Substituting back into the equation $\Delta\mathcal{L} = 0$ to get the conservation law:

$$\partial_\mu \Theta^{\mu\beta} + \frac{1}{c} g^{\mu\beta} f_\mu = \partial_\mu \Theta^{\mu\beta} + f^\beta = 0 \tag{72}$$

We have arrived at the energy-momentum equations (62) derived before.

VI. SOLUTION OF THE WAVE EQUATION IN COVARIANT FORM AND INVARIANT GREEN FUNCTIONS

The inhomogeneous Maxwell's equations, when recast in terms of the electromagnetic field strength tensor $F^{\alpha\beta}$, arising from an external source $J^\alpha(x)$, are written as:

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

In terms of the four-potential A^α, the above equation becomes

$$\Box A^\beta - \partial^\beta(\partial_\alpha A^\alpha) = \mu_0 J^\beta$$

where the d'Alembertian operator $\Box \equiv \partial^\alpha \partial_\alpha$. Using the Lorenz gauge $\partial_\alpha A^\alpha = 0$, the components of the four-potential are solutions of the four-dimensional wave equation

$$\Box A^\beta = \mu_0 J^\beta \tag{73}$$

The solution to the above equation is accomplished by finding a Green's function $D(x, x')$ for the equation

$$\Box_x D(x, x') = \delta^4(x - x') \tag{74}$$

where $\delta^4(x - x') = \delta(x_0 - x_0^{'})\delta(\mathbf{x} - \mathbf{x}^{'})$ is a four-dimensional delta dunction. In the absence of boundary surfaces, the Green's function depends only on the 4-vector difference $z^\alpha = x^\alpha - x^{'\alpha}$. Thus $D(x, x') = D(x - x') = D(z)$ and equation 74 becomes

$$\Box_z D(z) = \delta^4(z)$$

We now move from real space wave to wave number space using Fourier integrals. The Fourier transform $\tilde{D}(k)$ of the Green's function is

$$D(z) = \frac{1}{(2\pi)^4} \int d^4k \tilde{D}(k) e^{-ik\cdot z}$$

where $k \cdot z = k_0 z_0 - \mathbf{k} \cdot \mathbf{z}$ (the 4-vector dot product). The delta function representation is

$$\delta^4(z) = \frac{1}{(2\pi)^4} \int d^4k e^{-ik\cdot z}$$

Thus one finds the k-space Green's function to be

$$\tilde{D}(k) = -\frac{1}{k \cdot k}$$

The Green's function is, therefore

$$D(z) = -\frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot z}}{k \cdot k}$$

The above integrand has singularities that need to be handled appropriately. We proceed by performing the integration over dk_0 first,

$$D(z) = -\frac{1}{(2\pi)^4} \int d^3k\, e^{i\mathbf{k}\cdot\mathbf{z}} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}$$

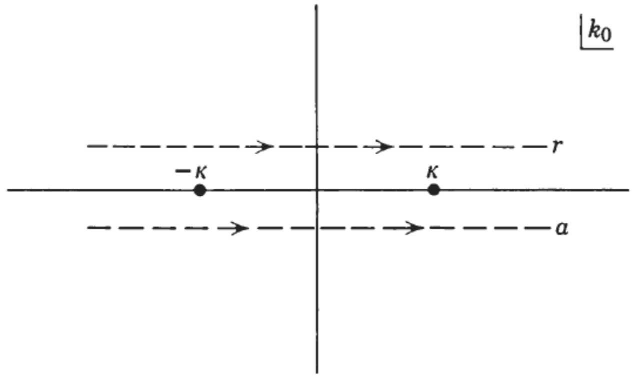

Figure 2. The contours r and a along with the singularities at κ and $-\kappa$ in the complex plane

where $\kappa = |\mathbf{k}|$. The integral is given meaning by considering k_0 as a complex variable and treating the integral as a contour integral in the k_0 plane. The integrand is two simple poles at $k_0 = \pm\kappa$. Green functions with differing behaviour are obtained by choosing different contours of integration relative to the poles. These are labelled r and a in figure 2. These contours may be closed at infinity with a semicircle in the upper/lower half plane, depending on the sign of z_0. For $z_0 > 0$, the exponential $e^{-ik_0 z_0}$ increases without limit in the upper half plane. Hence one must close the contour in the lower half plane. The opposite holds for $z_0 < 0$.

Consider the contour r. If $z_0 < 0$, the integral vanishes since the contour encircles no singularities. For $z_0 > 0$, the integral over k_0 is

$$\oint_r dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} = -2\pi i Res\left(\frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}\right) = -\frac{2\pi}{\kappa} sin(\kappa z_0)$$

The Green function is then

$$D_r(z) = \frac{\theta(z_0)}{(2\pi)^3} \int d^3k e^{i\mathbf{k}\cdot\mathbf{z}} \frac{sin(\kappa z_0)}{\kappa}$$

Integrating the angular part leads to

$$D_r(z) = \frac{\theta(z_0)}{2\pi^2 R} \int_0^\infty d\kappa sin(\kappa R) sin(\kappa z_0)$$

where $R = |\mathbf{z}| = |\mathbf{x} - \mathbf{x}'|$ is the spatial distance between x^α and $x^{'\alpha}$. Expressing the sines in terms of exponentials, we get

$$D_r(z) = \frac{\theta(z_0)}{8\pi^2 R} \int_{-\infty}^\infty d\kappa [e^{i(z_0 - R)\kappa} - e^{i(z_0 + R)\kappa}]$$

In terms of delta functions, the Green's function for the contour r is

$$D_r(z) = D_r(x - x') = \frac{\theta(x_0 - x_0')}{4\pi R} \delta(x_0 - x_0' - R)$$

Note that the 2nd term in the integral is always 0 since both R and z_0 are positive. This Green function is called the “retarded ”or “causal ”Green function since the source-point time x_0' is always earlier than the observation time x_0.

With the choice of contour a, a similar analysis leads to the “advanced ”Green function

$$D_a(x - x') = \frac{\theta[-(x_0 - x_0')]}{4\pi R} \delta(x_0 - x_0^{'} + R)$$

These Green functions can be written in more covariant form using the identity

$$\begin{aligned} \delta[(x - x')^2] &= \delta[(x_0 - x_0')^2 - |\mathbf{x} - \mathbf{x}'|^2] \\ &= \delta[(x_0 - x_0' - R)(x_0 - x_0' + R)] \\ &= \frac{[\delta(x_0 - x_0' - R) + \delta(x_0 - x_0' + R)]}{2R} \end{aligned}$$

Since the theta terms select one or the other of the two terms, we finally end with the following covariant expressions for the Green functions

$$D_r(x-x') = \frac{1}{2\pi}\theta(x_0 - x_0')\delta[(x-x')^2] \tag{75}$$

$$D_a(x-x') = \frac{1}{2\pi}\theta(x_0' - x_0)\delta[(x-x')^2] \tag{76}$$

It can be shown that the above expressions for Green functions are Lorentz invariant. The solution of the wave equation (73) can be written in terms of the Green functions:

$$A^\alpha(x) = A^\alpha_{in}(x) + \mu_0 \int d^4x' D_r(x-x')J^\alpha(x') \tag{77}$$

or

$$A^\alpha(x) = A^\alpha_{out}(x) + \mu_0 \int d^4x' D_a(x-x')J^\alpha(x') \tag{78}$$

where A^α_{in} and A^α_{out} are solutions of the homogeneous wave equation.

VII. QUANTUM ELECTRODYNAMICS [10]

Quantum Electrodynamics describes the interactions of Fermions (electrons, muons etc.) with the electromagnetic field.
[11]We can construct the Lagrangian for QED by looking at each term individually. Since we need a 'kinetic' term for the particle, we get a term of the form

$$\mathcal{L}_1 = i\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{79}$$

The mass term for the particles needs to be incorporated as

$$\mathcal{L}_2 = m\bar{\psi}\psi \tag{80}$$

Now, we have the term for the field strength

$$\mathcal{L}_3 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{81}$$

Lastly, we have the interaction term, which looks (the choice looks rather artificial at first glance) like

$$\mathcal{L}_4 = -e\bar{\psi}\gamma^\mu\psi A_\mu \tag{82}$$

This gives us the final Lagrangian as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial^\mu\psi + m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{\psi}\gamma^\mu\psi A_\mu \tag{83}$$

A more elegant way of arriving at this Lagrangian will be given in the next section.

VIII. CONSTRUCTION OF LAGRANGIANS

In the previous sections, we directly stated the Lagrangian be it for classical electrodynamics or QED. However, there is a more systematic way to construct the Lagrangian for a theory based on the symmetry group of the theory.
A symmetry group for a theory is a Lie group describing the continuous symmetry of the system. The symmetry group for Electrodynamics is $U(1)$ while that for QCD is $SU(3)$. Let us give a more general treatment to Lagrangian construction.
Assume the symmetry group of the theory is the Lie Group L. Denote the gauge fields corresponding to the generators of this algebra as A^a_μ where a indexes the gauge field (each gauge field is a 4 vector). Let f^{abc} denote the structure constants of the Lie Algebra corresponding to the Lie group.
The first element we need to look at is the so-called Covariant Derivative. This is a special extension of the derivative

to incorporate the same transformation properties under the symmetry group as the 'particle fields'. This will have the form

$$D_\mu = \partial_\mu - igA_\mu^a T^a \tag{84}$$

Here, D denotes the covariant derivative and T^a are the generators of the algebra.g is the coupling strength. The next crucial component is the field strength component. We define the field strength tensor as

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \tag{85}$$

The 'gauge fixing' term in the Lagrangian is, thus,

$$-\frac{1}{4}tr(G_{\mu\nu}G^{\mu\nu}) = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} \tag{86}$$

The mass term remains as it is. The mass term is quadratic in the fields and is constructed from them in such a way so as to make the Lagrangian a Lorentz Scalar. In all of this, we haven't yet considered sources. We will have a source coupled to each gauge field. Thus, our Lagrangian simply for the gauge fields will look like

$$\mathcal{L} = -\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu)}) - J_{mu}^a A^{a\mu} \tag{87}$$

The complete Lagrangian will include the kinetic and interaction terms for the particle fields as well as any potential function for the particle fields. We can now obtain the equations of motion

$$D_\mu \star F^{\mu\nu} = 0 \tag{88}$$

Here, $\star F^{\mu\nu}$ is the dual field strength tensor defined as

$$\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{89}$$

The equation of motion here is just the Bianchi Identity. The other equation of motion is

$$\partial_\mu F^{a\mu\nu} + gf^{abc}A_\mu^b F^{b\mu\nu} = -J^{a\nu} \tag{90}$$

Let us now use this technique to construct Lagrangians for QED. The symmetry group, in this case, is $U(1)$. The structure constants for the algebra of $U(1)$ are all 0 since it is abelian. Since we have only one generator for the algebra, we get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{91}$$

The covariant derivative becomes

$$D_\mu = \partial_\mu - ieA_\mu \tag{92}$$

We thus get the full Lagrangian for the gauge field to be

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \tag{93}$$

This is exactly what we had considered earlier! Thus, we can see that this method gives a systematic procedure to construct a **quadratic** Lagrangian given the gauge group. Another example (perhaps less relevant to electrodynamics) is that of the Georgi-Glashow for the electroweak theory. The symmetry group in concern for the bosonic part of the theory is $SO(3)$ while that for the full theory is $SU(2) \times U(1)$. The Lagrangian for the bosonic part of the model is

$$\mathcal{L} = \frac{-1}{4}\vec{G}^{\mu\nu} \cdot \vec{G}_{\mu\nu} + \frac{1}{2}D^\mu\vec{\phi} \cdot D_\mu\vec{\phi} - V(\phi) \tag{94}$$

Where

$$\vec{G}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - e\vec{W}_\mu \times \vec{W}_\nu \tag{95}$$

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} - e\vec{W}_\mu \times \vec{\phi} \tag{96}$$

$$V(\vec{\phi}) = \frac{1}{4}\lambda(\phi^2 - a^2)^2 \tag{97}$$

Here, V is the Higgs potential and $\vec{W}_\mu$ are the gauge fields. We can see that this conforms with our technique of constructing the Lagrangian ($f^{abc} = \epsilon^{abc}$)

IX. A DESCRIPTION OF A MAGNETIC MONOPOLE [12]

The Lagrangian for the Bosonic part of the Georgi-Glashow model can be used to obtain an entity that shows the properties of a magnetic monopole. The complete Georgi-Glashow model describes the properties of electroweak interactions. The bosonic part of this model consists of an $SO(3)$ Yang-Mills field theory coupled to a Higgs field in the adjoint representation. The Lagrangian for the bosonic part of the model is

$$\mathcal{L} = \frac{-1}{4}\vec{G}^{\mu\nu} \cdot \vec{G}_{\mu\nu} + \frac{1}{2}D^{\mu}\vec{\phi} \cdot D_{\mu}\vec{\phi} - V(\phi) \tag{98}$$

Here,

$$\vec{G}_{\mu\nu} = \partial_{\mu}\vec{W}_{\nu} - \partial_{\nu}\vec{W}_{\mu} - e\vec{W}_{\mu} \times \vec{W}_{\nu} \tag{99}$$

$$D_{\mu}\vec{\phi} = \partial_{\mu}\vec{\phi} - e\vec{W}_{\mu} \times \vec{\phi} \tag{100}$$

$$V(\vec{\phi}) = \frac{1}{4}\lambda(\phi^2 - a^2)^2 \tag{101}$$

Here, $\vec{G}_{\mu\nu}$ is the Gauge field strength. $\vec{\phi}$ is a vector in the 3D adjoint representation of $SO(3)$ and $V(\vec{\phi})$ is the Higgs potential (depending on the Higgs field $\vec{\phi}$).
With these expressions involved in the Lagrangian, we can see that it is invariant under

$$\vec{\phi} \mapsto \vec{\phi}' = g\vec{\phi} \tag{102}$$

$$\vec{W}_{\mu} \mapsto \vec{W}'_{\mu} = g\vec{W}_{\mu}g^{-1} + \frac{1}{e}\partial_{\mu}gg^{-1} \tag{103}$$

Where $g = g(x)$ is a 3×3 orthogonal matrix[13] with a unit determinant.
Further, we obtain the equations of motion as

$$D_{\nu}\vec{G}^{\mu\nu} = -e\vec{\phi} \times D^{\mu}\vec{\phi} \tag{104}$$

$$D^{\mu}D_{\mu}\vec{\phi} = -\lambda(\phi^2 - a^2)\vec{\phi} \tag{105}$$

$$D^{\star}_{\mu}\vec{G}^{\mu\nu} = 0 \tag{106}$$

The last of these equations is just the Bianchi Identity, while the first two are the equations of motion obtained from the Lagrangian.
If we consider the momenta conjugate to the gauge fields, we obtain

$$\vec{E}^{i} = -\vec{G}^{0i} \tag{107}$$

$$\vec{\Pi} = D_0\vec{\phi} \tag{108}$$

We can then define

$$\vec{G}_{ij} = -\epsilon_{ijk}\vec{B}^{k} \tag{109}$$

Now, the Hamiltonian density can be defined as

$$\mathcal{H} = \frac{1}{2}\vec{E}_i \cdot \vec{E}_i + \frac{1}{2}\vec{\Pi} \cdot \vec{\Pi} + \frac{1}{2}\vec{B}_i \cdot \vec{B}_i + \frac{1}{2}D_i\vec{\phi} \cdot D_i\vec{\phi} + V(\vec{\phi}) \tag{110}$$

This Hamiltonian is positive definite and gauge invariant under the transformation mentioned earlier.
For a vacuum configuration, $\mathcal{H}=0$. This gives us

$$\vec{G}_{\mu\nu}=0 \tag{111}$$

$$D^{\mu}\vec{\phi}=0 \tag{112}$$

$$V(\vec{\phi})=0 \tag{113}$$

The last of these gives us

$$\phi^2=a^2 \tag{114}$$

This reduces the symmetry group from $SO(3)$ to $SO(2)$ or $U(1)$. This exhibits spontaneous symmetry breaking in the model.
We can now look at the spectrum of the model.
Let

$$\vec{\phi}=\vec{a}+\vec{\varphi} \tag{115}$$

We have the Lagrangian density to be

$$\mathcal{L}=-\frac{1}{4}\vec{G}^{\mu\nu}\cdot\vec{G}_{\mu\nu}+\frac{1}{2}D^{\mu}\cdot\vec{\phi}D_{\mu}\vec{\phi}-V(\vec{\phi}) \tag{116}$$

Where

$$D_{\mu}\vec{\phi}=\partial_{\mu}\vec{\phi}-e\vec{W}_{\mu}\times\vec{\phi} \tag{117}$$

Substituting the expression for $\vec{\phi}$ in this, we have

$$\begin{aligned}\mathcal{L}&=-\frac{1}{4}\vec{G}^{\mu\nu}\cdot\vec{G}_{\mu\nu}+\frac{1}{2}(\partial^{\mu}(\vec{a}+\vec{\varphi})-e\vec{W}^{\mu}\times(\vec{a}+\vec{\varphi}))\cdot(\partial_{\mu}(\vec{a}+\vec{\varphi})-e\vec{W}_{\mu}\times(\vec{a}+\vec{\varphi}))-\frac{1}{4}\lambda((\vec{a}+\vec{\varphi})^2-\vec{a}^2)^2\\&=-\frac{1}{4}\vec{G}^{\mu\nu}\cdot\vec{G}_{\mu\nu}+\frac{1}{2}(\partial^{\mu}\vec{\varphi}-e\vec{W}^{\mu}\times\vec{\varphi}-e\vec{W}^{\mu}\times\vec{a})\cdot(\partial_{\mu}\vec{\varphi}-e\vec{W}_{\mu}\times\vec{\varphi}-e\vec{W}_{\mu}\times\vec{a})-\frac{1}{4}\lambda(2\vec{a}\cdot\vec{\varphi}+\varphi^2)\\&=-\frac{1}{4}\vec{G}^{\mu\nu}\cdot\vec{G}_{\mu\nu}+\frac{1}{2}D_{\mu}\vec{\varphi}\cdot D^{\mu}\vec{\varphi}+\frac{1}{2}(-e\vec{W}_{\mu}\times\vec{a})\cdot(-e\vec{W}_{\mu}\vec{a})+\frac{1}{2}D^{\mu}\vec{\varphi}\cdot(e\vec{W}_{\mu}\times\vec{a})+\frac{1}{2}D_{\mu}\vec{\varphi}\cdot(-e\vec{W}^{\mu}\times\vec{a})\\&\quad+\frac{1}{4}\lambda(2\vec{a}\cdot\vec{\varphi}+\varphi^2)^2\end{aligned} \tag{118}$$

From this expression, we can look at the quadratic terms to read off the mass of the Higgs Boson as well as the W^+ and W^- as:

$$\mathcal{L}=\cdots+\frac{1}{2}\left(\frac{M_H}{\hbar}\right)^2\psi^2+\frac{1}{2}\left(\frac{M_W}{\hbar}\right)^2W_{\mu}^{+}W^{\mu-}+\dots \tag{119}$$

Where we have $\psi=\frac{1}{a}\vec{a}\cdot\vec{\varphi}$. We can thus read off the masses as

$$\begin{aligned}M_H&=a\sqrt{2\lambda}\hbar\\M_W&=ae\hbar\end{aligned} \tag{120}$$

A. Finite Energy Solutions

We are interested in solutions where $\int\mathcal{H}d^3x$ exists. This means that the fields approach vacuum at infinity. This also means that the Higgs field approaches the Higgs vacuum at infinity.
We have the Higgs potential

$$V:\mathbb{R}^3\to\mathbb{R} \tag{121}$$

Let us consider the points in $\mathbb{R}^3$ at whom $V = 0$. Denote this space by $\mathcal{M}_0 \subset \mathbb{R}^3$. In this model, we have $\mathcal{M}_0$ to be a sphere of radius a.We will consider a map from the sphere at infinity to $\mathcal{M}_0$

$$\vec{\phi}_\infty(\hat{r}) = \lim_{r\to\infty} \vec{\phi}(\vec{r}) \qquad \in \mathcal{M}_0 \tag{122}$$

This is effectively a map from S^2 to S^2. Thus, there exists a surjection between the set of possible $\vec{\phi}_\infty$ and $\pi_2(S^2)$. To put it differently, the group of all equivalence classes of $\vec{\phi}_\infty$ with composition is isomorphic to $\pi_2(S^2)$
It is already known that

$$\pi_2(S^2) = \mathbb{Z}$$

Let us denote the equivalence class to which this map belongs to by the degree of the map. By definition, the degree of the map is undeterred through continuous deformation (in this case, we are interested in time evolution). Thus, if the map starts out non-trivial (degree 0), then it remains non-trivial i.e. it will be stable.
We will now look at the *'t Hooft-Polyakov ansatz* for the monopole as follows:

$$\begin{aligned}\vec{\phi}(\vec{r}) &= \frac{\vec{r}}{er^2}H(aer)\\ W_b^i &= -\epsilon_{bij}\frac{r^j}{er^2}(1-K(aer))\\ W_b^0 &= 0\end{aligned} \tag{123}$$

When we plug this ansatz into the expression for $\mathcal{H}$, we will get

$$\mathcal{H} = \frac{4\pi a}{e\xi^2}\left(\xi^2\frac{dH}{d\xi} + \frac{1}{2}(\xi\frac{dH}{d\xi} - H)^2 + \frac{1}{2}(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\right) \tag{124}$$

Here, $\xi = aer$ Note that due to the ansatz, the terms involving $\vec{E}^i$ and $\vec{\Pi}$ disappear from the Hamiltonian (They become 0). The first two terms and the fourth term come from $D_i\vec{\phi}\cdot D_i\vec{\phi}$. The third term comes from $\vec{B}^i\cdot\vec{B}_i$. The last term is the Higgs potential.
We are concerned with the existence of $\int \mathcal{H}d^3x$. Thus, we need to investigate conditions for convergence of

$$E = \frac{4\pi a}{e}\int_0^\infty \frac{d\xi}{\xi^2}\left(\xi^2\frac{dH}{d\xi} + \frac{1}{2}(\xi\frac{dH}{d\xi} - H)^2 + \frac{1}{2}(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\right) \tag{125}$$

Let us look at the condition that the integrand must vanish at $\xi \to \infty$:
Looking at the the last term, we have the condition that

$$\frac{H}{\xi} \to 1 \qquad \xi \to \infty \tag{126}$$

Looking at the fourth term, we get

$$K \to 0 \qquad \xi \to \infty \tag{127}$$

Next, we look at the integrand as $\xi \to 0$:
We will obtain

$$\begin{aligned}K - 1 &< \mathcal{O}(\xi) \qquad \xi \to 0\\ \frac{H}{\xi} &< \mathcal{O}(\xi) \qquad \xi \to 0\end{aligned} \tag{128}$$

This basically tells us that $K \to 1$ and $H \to 0$.
We can now investigate the map $\vec{\phi}_\infty$. We obtain

$$\begin{aligned}\vec{\phi}_\infty(\hat{r}) &= \lim_{r\to\infty}\vec{\phi}(\vec{r})\\ &= \lim_{r\to\infty}\frac{\hat{r}H(aer)}{er}\\ &= a\hat{r}\end{aligned} \tag{129}$$

This is just the identity map on $\mathcal{M}_0$. Thus, we can see that the degree of $\vec{\phi}_\infty$ is 1. This tells us that it is a non-trivial stable solution.
Directly plugging the ansatz into the equations of motion gives us

$$\begin{aligned} \xi^2 \frac{d^2 H}{d\xi^2} &= KH^2 + K(K^2-1) \\ \xi^2 \frac{d^2 K}{d\xi^2} &= 2K^2 + \frac{\lambda}{e^2} H(H^2 - \xi^2) \end{aligned} \tag{130}$$

If we look at the limit of this expression as $\xi \to \infty$, we will get

$$\begin{aligned} \frac{d^2 K}{d\xi^2} &= K \\ \frac{d^2 \tilde{H}}{d\xi^2} &= \frac{2\lambda}{e^2} \tilde{H} \qquad \tilde{H} = H - \xi \end{aligned} \tag{131}$$

Upon plugging in the boundary conditions for H and K, we get

$$\begin{aligned} K \quad &\alpha \quad \exp\left(\frac{-M_W r}{\hbar}\right) \\ \tilde{H} \quad &\alpha \quad \exp\left(\frac{-M_H r}{\hbar}\right) \end{aligned} \tag{132}$$

We can investigate the asymptotic electromagnetic field by studying

$$F_{\mu\nu} = \frac{1}{a} \vec{\phi} \cdot \vec{G}_{\mu\nu} \tag{133}$$

Since we are looking at the static solution, we get the electric field components to be 0. Thus, it suffices to investigate

$$F_{ij} = \frac{1}{a} \vec{\phi} \cdot \vec{G}_{ij} \tag{134}$$

Plug in $H(aer) = aer$ and $K = 0$. Thus,

$$\vec{\phi} = a\hat{r} \tag{135}$$

We can also obtain

$$\vec{G}_{ij} = \hat{r} \epsilon_{ijk} \frac{r^k}{er^3} \tag{136}$$

From this, we can directly obtain

$$F_{ij} = \epsilon_{ijk} \frac{r^k}{er^3} \tag{137}$$

Thus, we get the magnetic field to be

$$\vec{B} = -\frac{1}{e} \frac{\vec{r}}{r^3} \tag{138}$$

It is worth investigating this expression in detail[14]. The first thing we see is that the magnetic charge is $\frac{4\pi}{e}$. The reasoning behind this can be given by looking at the way $U(1)$ is embedded in $SO(3)$ in this theory. The electric charge is an eigenvalue of the T_3 isospin generator. This gives us a minimum electric charge in the theory as $\frac{e}{2}$. Thus, we see that the quantization is identical to that in the case of the Dirac monopole, just with different minimum electric charge. One more thing worth mentioning is the nature of the monopole. The Dirac monopole is effectively a singularity in the field. Here, we obtained a similar form of the magnetic field only when we considered $r \to \infty$. However, all fields involved are smooth. Thus, the origin of the magnetic monopole is purely topological and it arises naturally.

B. The topological origin of the magnetic monopole

As we look at points far away from the origin, we have

$$
\begin{aligned}
D_\mu \vec{\phi} &= 0 \\
\vec{\phi}\cdot\vec{\phi} &= a^2
\end{aligned}
\tag{139}
$$

This is just the vacuum configuration of the Higgs field. The second equation is important to state independently. The first equation implies that $\vec{\phi}\cdot\vec{\phi}$ is a constant. What the second equation tells us is that it takes a value so as to minimize the potential.
The field behaves this way far away from the 'core' of the monopole. It is thus reasonable to extend our consideration of 'core' monopoles and the regions far away from them to systems that are not necessarily stable. We assume that the Higgs field assumes its vacuum configuration everywhere except finitely many disconnected regions. In simple terms, we are considering a 'dilute gas' of monopoles.
From $D_\mu\vec{\phi} = 0$, we obtain

$$
\vec{W}_\mu \times \vec{\phi} = \frac{1}{e}\partial_\mu\vec{\phi} \tag{140}
$$

Simplifying this gives us

$$
W_\mu = \frac{1}{a^2 e}\vec{\phi}\times\partial_\mu\vec{\phi} + \frac{1}{a^2}(\vec{\phi}\cdot\vec{W}_\mu)\vec{\phi} \tag{141}
$$

We will denote

$$
A_\mu = \hat{\phi}\cdot\vec{W}_\mu \tag{142}
$$

Thus, we have

$$
W_\mu = \frac{1}{a^2 e}\vec{\phi}\times\partial_\mu\vec{\phi} + \frac{1}{a}A_\mu\vec{\phi} \tag{143}
$$

We can now investigate the nature of the field strength tensor in the Higgs vacuum.

$$
\vec{G}_{\mu\nu} = \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu - e\vec{W}_\mu\times\vec{W}_\nu \tag{144}
$$

We can look at this term by term:

$$
\begin{aligned}
\partial_\mu\vec{W}_\nu &= \frac{1}{a^2 e}\partial_\mu(\vec{\phi}\times\partial_\nu\vec{\phi}) + \frac{1}{a}\partial_\mu(\vec{\phi}A_\nu) \\
&= \frac{1}{a^2 e}[\vec{\phi}\times\partial_\mu\partial_\nu\vec{\phi} + \partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}] + \frac{1}{a}[\partial_\mu A_\nu + \vec{\phi}\partial_\mu A_\nu]
\end{aligned}
\tag{145}
$$

Thus, we will get

$$
\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu = \frac{2}{a^2 e}\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi} + \frac{1}{a}[\partial_\mu\vec{\phi}A_\nu - \partial_\nu\vec{\phi}A_\mu] + \frac{1}{a}\vec{\phi}[\partial_\mu A_\nu - \partial_\nu A_\mu] \tag{146}
$$

We also have

$$
\begin{aligned}
\vec{W}_\mu\times\vec{W}_\nu &= [\frac{1}{a^2 e}\vec{\phi}\times\partial_\mu\vec{\phi} + \frac{1}{a^2}A_\mu\vec{\phi}]\times[\frac{1}{a^2 e}\vec{\phi}\times\partial_\nu\vec{\phi} + \frac{1}{a^2}A_\nu\vec{\phi}] \\
&= \frac{1}{a^4 e}[\vec{\phi}\cdot(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})]\vec{\phi} + \frac{1}{a^3 e}[a^2\partial_\mu\vec{\phi}A_\nu - a^2\partial_\nu\vec{\phi}A_\mu] - \frac{1}{a^3 e}[\vec{\phi}A_\nu(\partial_\mu\vec{\phi}\cdot\vec{\phi}) - \vec{\phi}A_\mu(\partial_\nu\vec{\phi}\cdot\vec{\phi})]
\end{aligned}
\tag{147}
$$

Thus, upon cancelling a few terms, we get

$$
\vec{G}_{\mu\nu} = \frac{1}{a^4 e}\vec{\phi}[\vec{\phi}\cdot(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})] + \frac{1}{a}\vec{\phi}[\partial_\mu A_\nu - \partial_\nu A_\mu] \tag{148}
$$

We can write this as

$$
\vec{G}_{\mu\nu} = \frac{1}{a}\vec{\phi}F_{\mu\nu} \tag{149}
$$

where

$$F_{\mu\nu} = \frac{1}{a^3 e}\vec{\phi}\cdot(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}) + [\partial_\mu A_\nu - \partial_\nu A_\mu] \tag{150}$$

Upon inserting this form for $\vec{G}$ into its equations of motion (as well as the Bianchi identity) along with the conditions on $\vec{\phi}$ due to it being a vacuum, we get

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= 0\\ \partial^\star_\mu F^{\mu\nu} &= 0\end{aligned} \tag{151}$$

These are Maxwell's equations!
However, we notice that the form of the 'field strength tensor' is not quite what we have classically. We have an extra term coming from derivatives of the Higgs field. This is the topologically non-trivial part which gives us the monopole like nature.
We can investigate further by looking at the surface integral on a surface around a monopole.

$$g_\Sigma = \int_\Sigma \vec{B}\cdot d\vec{S} \tag{152}$$

Thus, we get

$$g_\Sigma = \frac{1}{2a^2 e}\int_\Sigma \epsilon_{ijk}\vec{\phi}\cdot(\partial_i\vec{\phi}\times\partial_j\vec{\phi})dS_i \tag{153}$$

From this, we can see that only the tangential components of $\partial_i\phi$ will contribute. We can also see that the integral only depends on the nature of $\vec{\phi}$ on Σ as well as the homotopy of the map $\phi:\Sigma\to\mathcal{M}_0$. Furthermore, we can see that g is invariant under infinitesmal deformations of the Higgs field which preserve the vacuum. Consider variations of the Higgs field which preserve the vacuum. These will obey

$$\begin{aligned}D_\mu\delta\vec{\phi} &= 0\\ \vec{\phi}\cdot\delta\vec{\phi} &= 0\end{aligned} \tag{154}$$

This gives us

$$\delta(\epsilon_{ijk}\vec{\phi}\cdot(\partial_j\vec{\phi}\times\partial_k\vec{\phi})) = 3\epsilon_{ijk}\delta\vec{\phi}\cdot(\partial_j\vec{\phi}\times\partial_k\vec{\phi}) + 2\epsilon_{ijk}\partial_j(\vec{\phi}\cdot(\delta\vec{\phi}\times\partial_k\vec{\phi})) \tag{155}$$

Now, the second term from this will integrate out to 0. We can thus focus our effort on the first term. From the vacuum condition $D_\mu\vec{\phi}=0$, we get that $\vec{\phi}\cdot\partial_j\vec{\phi}$ is 0 for all $j=1,2,3$. Thus must have $\vec{\phi}$ and $\partial_i\vec{\phi}\times\partial_j\vec{\phi}$ to be in the same direction. Thus, we must have

$$\delta\vec{\phi}\cdot(\partial_i\vec{\phi}\times\partial_j\vec{\phi}) = 0 \tag{156}$$

Thus, we will get

$$\delta g_\Sigma = 0 \tag{157}$$

This holds under several different homotopies. The one we will look at is the continuous deformation of Σ.
Σ can be thought of as the surface enclosing the magnetic monopole. Now, consider two such local monopoles, g_1 and g_2. Assume that Σ encloses both. We can construct two subsurfaces enclosing only one monopole each. Thus, we can see that the $g_\Sigma = g_1 + g_2$. We obtain that magnetic charges are additive.
Let us consider the integral of the Jacobian of the map $\vec{\phi}:\Sigma\to\mathcal{M}_0$. This is

$$N_\Sigma = \frac{1}{8a^2\pi}\int_\Sigma \epsilon_{ijk}\vec{\phi}\cdot(\partial_j\vec{\phi}\times\partial_k\vec{\phi})dS_i \tag{158}$$

This quantity can be seen to be related to g_Σ as

$$g_\Sigma = -\frac{4\pi}{e}N_\Sigma \tag{159}$$

Since the Jacobian of the map considered will be its degree, N_Σ needs to be an integer. This gives us the new 'Dirac Quantization condition' as

$$eg_\Sigma = -4\pi N_\Sigma \tag{160}$$

X. EMERGENT ELECTROMAGNETIC FIELD AND PATH INTEGRAL QUANTISATION OF A SPINNING PARTICLE [15] [16] [17]

In this section we develop a path integral for spinning particles. The classical action for a spin-half particle is constructed and shown to be that of a massless point particle on a sphere coupled to a monopole field. In addition we elucidate the quantisation of spin purely on the basis of geometrical considerations. This also leads to a justification of why a classical formalism of spin is outside the scope of traditional treatments of classical mechanics.

The quantum mechanics of a spin-(1/2) particle is a standard example taught in introductory courses in quantum physics. The Hilbert space characterized by the particle's spin degree of freedom is 2 dimensional, making it one of the simplest systems which encapsulate nontrivial quantum behavior. However, for a long time, the spin problem could not be incorporated into the path integral formalism. The difficulty of constructing a path integral for spin is attributed to its exponentiated classical action term. The formulation of such a term is unfamiliar as the "*classical mechanics of spin*" is a subject that is eluded even in advanced courses of quantum mechanics.

Before formulating the spin path integral, it is instructive to have a brief glimpse into the representation theory of $SU(2)$, the special unitary group in two dimensions. It is the set of all 2×2 complex matrices g satisfying $g^\dagger g = \mathbb{1}_2$ with det $g = 1$, where $\mathbb{1}_2$ is the identity matrix in two dimensions. The group has three independent real parameters, or equivalently its Lie algebra $su(2)$ is described by three generators $\hat{S}^i,\ i = 1, 2, 3$. The generators satisfy the algebra $[\hat{S}^i, \hat{S}^j] = i\epsilon_{ijk}\hat{S}^k$, where ϵ_{ijk} is the fully antisymmetric tensor. Each group element can be expressed in terms of exponentiated generators. For example, in terms of Euler angles ϕ, θ, ψ, the group elements are represented as

$$SU(2) : g(\phi, \theta, \psi) = e^{-i\phi\hat{S}^3} e^{-i\theta\hat{S}^2} e^{-i\psi\hat{S}^3} \mid \phi, \psi \in [0, 2\pi], \theta \in [0, \pi] \tag{161}$$

The Hilbert spaces $\mathcal{H}_S$ of quantum spin are irreducible representation spaces of $SU(2)$. The representation matrices of $SU(2)$ (denoted by g) act on these spaces. Within each $\mathcal{H}_S$, there is a distinguished state of highest 'weight' $|\uparrow\rangle$, defined as the eigenstate of $\hat{S}^3$ with maximum eigenvalue S. Physically $|\uparrow\rangle$ corresponds to the spin state polarised in the z direction. Since the representations of $SU(2)$ are irreducible, each state of the Hilbert space $\mathcal{H}_S$ can be obtained by applying Euler-angle parameterized elements of the representation to the maximum weight state.

Being a compact Lie group, it makes sense to integrate over the group manifold of $SU(2)$ with an appropriate integration measure. This is necessary as we make use of integrals of the form $\int_{SU(2)} dg\, f(g)$, where f is a function of g and dg is the integration measure over the group manifold.

A. Description of the Path Integral

Consider a particle of spin S subject to the Hamiltonian

$$\hat{H} = \mathbf{B}\cdot\hat{\mathbf{S}}$$

where $\mathbf{B}$ is the applied magnetic field and $\hat{\mathbf{S}} = \{\hat{S}^1, \hat{S}^2, \hat{S}^3\}$ is the vector of generators in the spin S representation. We calculate the imaginary time path integral representation of the particle, expressed by the partition function $\mathcal{Z} \equiv tr e^{-\beta\hat{H}}$. With the focus being the analysis of the action term on the path integral and not its explicit construction, we reserve the path integral derivation for the appendix. The path integral is compactly represented as

$$\mathcal{Z} = \int Dg \exp\left[-\int_0^\beta d\tau \left(-\langle\partial_\tau g|g\rangle + \langle g|\,\mathbf{B}\cdot\hat{\mathbf{S}}\,|g\rangle\right)\right] \tag{162}$$

where Dg is the integration measure over all paths $g(\tau)$ on the group manifold and τ is the (imaginary) time. The states $|g\rangle$ in the Euler-angle representation are

$$|\tilde{g}(\phi, \theta, \psi)\rangle \equiv e^{-i\phi\hat{S}^3} e^{-i\theta\hat{S}^2} e^{-i\psi\hat{S}^3}\,|\uparrow\rangle$$

These are referred to as *spin coherent states* in the literature. We make some simplifications at this point - note that that maximum weight state $|uparrow\rangle$ by definition satisfies $\hat{S}^3\,|\uparrow\rangle = S\,|\uparrow\rangle$. Thus, $|\tilde{g}(\phi, \theta, \psi)\rangle \equiv e^{-i\phi\hat{S}^3} e^{-i\theta\hat{S}^2}\,|\uparrow\rangle e^{-i\psi S}$. The angle ψ enters the coherent state merely as a phase factor. The other two angles θ and ϕ are true rotation angles acting through the generators $\hat{S}^3, \hat{S}^2$. Now θ and ϕ define a standard representation of the 2-sphere $\mathbb{S}^2$. The connection between the states $|\tilde{g}(\phi, \theta, \psi)\rangle$ covering the entire Hilbert space H_S and its parameters θ, ϕ spanning $\mathbb{S}^2$ is further elucidated when we compute the expectation values

$$n_i \equiv \langle\tilde{g}(\phi, \theta, \psi)|\,\hat{S}^i|\tilde{g}(\phi, \theta, \psi)\rangle, \quad i \in \{1, 2, 3\} \tag{163}$$

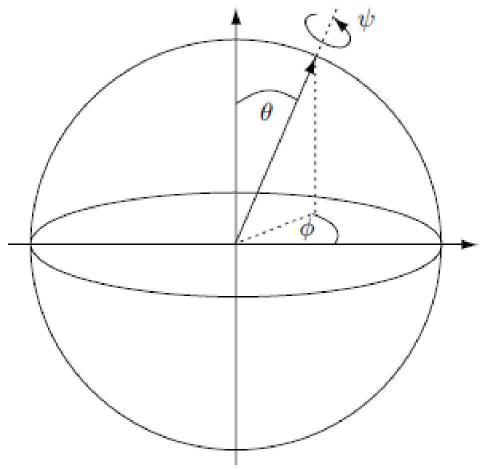

Figure 3. The unit sphere $\mathbb{S}^2$ paramterised by θ and ϕ. The spin coherent states are uniquely paramterised by θ and ϕ, upto a phase ψ

Using the Euler-angle representation for the states and the identity,

$$e^{-i\phi\hat{S}^i}\hat{S}^j e^{i\phi\hat{S}^i} = \hat{S}^j cos\phi + \epsilon_{ijk}\hat{S}^k sin\phi \tag{164}$$

leads to the expression for $\mathbf{n}$:

$$\mathbf{n} = S(sin\theta cos\phi, sin\theta sin\phi, cos\theta) \tag{165}$$

Thus, $\mathbf{n}$ is proportional to a unit vector spanning the unit 2-sphere. An important observation is that the $\mathbf{B}$ dependent part of the action is $U(1)$ gauge invariant, as

$$S_B[\phi,\theta] \equiv \int_0^\beta d\tau \left\langle \tilde{g}\right| \mathbf{B}\cdot\hat{\mathbf{S}} \left|\tilde{g}\right\rangle = \int_0^\beta d\tau \left\langle g\right| \mathbf{B}\cdot\hat{\mathbf{S}} \left|g\right\rangle = S\int_0^\beta d\tau \mathbf{n}\cdot\mathbf{B} = SB\int_0^\beta d\tau cos\theta \tag{166}$$

where the gauge-independent part $|g(\theta,\phi)\rangle \equiv e^{-i\phi\hat{S}^3}e^{-i\theta\hat{S}^2}\left|\uparrow\right\rangle$. Substituting this representation in the first term of the action in equation 162,

$$S_{top}[\phi,\theta] \equiv -\int_0^\beta d\tau \left\langle \partial_\tau\tilde{g}|\tilde{g}\right\rangle = -\int_0^\beta d\tau \left\langle \partial_\tau e^{-iS\phi}g \middle| ge^{-iS\phi}\right\rangle = -\int_0^\beta d\tau(\langle\partial_\tau g|g\rangle - iS\partial_\tau\phi\,\langle g|g\rangle) = -\int_0^\beta d\tau\,\langle\partial_\tau g|g\rangle$$

where the last equality holds because $\langle g|g\rangle = 1$ and ϕ is periodic in β. Thus the path integral does not depend on the gauge phase, and it effectively extends over paths on the 2-sphere.
Using the identity in equation 164, the expression for $S_{top}[\theta,\phi]$ is further simplified as:

$$S_{top}[\theta,\phi] = -\int_0^\beta d\tau\,\langle\partial_\tau g|g\rangle = -iS\int_0^\beta d\tau\partial_\tau\phi cos\theta = iS\int_0^\beta d\tau\partial_\tau\phi(1-cos\theta)$$

Along with the $\mathbf{B}$ dependent part, the total action for the spinning particle is written as

$$S[\theta,\phi] = S_B[\theta,\phi] + S_{top}[\theta,\phi] = S\int_0^\beta d\tau[Bcos\theta + i(1-cos\theta)\partial_\tau\phi] \tag{167}$$

B. Analysis of the action and Spin Quantisation

We rewrite the 2nd term of the action equation 167 in a more illuminating way. Note that $\dot{\mathbf{n}} = \dot{\theta}\hat{e}_\theta + \dot{\phi}sin\theta\hat{e}_\phi$, where $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ are the basis vectors for the spherical coordinate system. Rewriting $S_{top}[\theta,\phi]$ as

$$S_{top}[\theta,\phi] = iS\int_0^\beta d\tau\dot{\mathbf{n}}\cdot\mathbf{A} = iS\oint_\gamma d\mathbf{n}\cdot\mathbf{A} \tag{168}$$

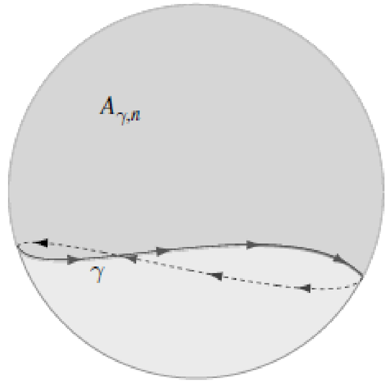

Figure 4. A closed path γ on $\mathbb{S}^2$ bounding the region $A_{\gamma,n}$ which covers the North pole

where

$$\mathbf{A} = \frac{1 - cos\theta}{sin\theta}\hat{e}_\phi \tag{169}$$

First, note that equation 168 is not a coordinate invariant formulation for S_{top}, for reasons deeply rooted in the topology of $\mathbb{S}^2$. Secondly, the expression for S_{top} can be read off as the (imaginary time) action of a *massless particle of charge S moving under the influence of a vector potential* $\mathbf{A}$. Using the curl in spherical coordinates, we obtain $\mathbf{B}_m = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{e}_r$ everywhere, apart from the negative z axis where $\mathbf{B}_m$ has a singularity, i.e the particle moves under a radial magnetic field of constant strength unity. Note that this field is not the applied magnetic field $\mathbf{B}_m$ and arises solely due to the geometry of the configuration space, which is the 2-sphere and is manifested in the expression for the action $S_{top}[\theta, \phi]$. For this reason, we refer to $\mathbf{B}_m$ as the **emergent** magnetic field arising from a magnetic monopole of strength 4π centred at the origin of the sphere. To explore the consequences of the magnetic monopole, we first use Stokes' theorem to write

$$S_{top}[\mathbf{n}] = iS \oint_\gamma d\mathbf{n} \cdot \mathbf{A} = iS \int_{A_{\gamma,n}} dS \cdot (\nabla \times \mathbf{A}) = iS \int_{A_{\gamma,n}} dS \cdot \mathbf{e}_r = iSA_{\gamma,n} \tag{170}$$

where $A_{\gamma,n}$ is the region of $\mathbb{S}^2$ which has the curve γ as its boundary and contains the North Pole. The integral produces the area of this surface which is denoted again by $A_{\gamma,n}$ Note that we cannot choose the South Pole since B is singular there and Stokes' theorem isn't valid. However the designated preference of the choice of the North pole leads to some symmetry breaking not present in the original problem.

Indeed one could have defined the action $S_{top}[\theta, \phi] = iS \oint d\mathbf{n} \cdot \mathbf{A}'$ where $\mathbf{A}' = -\frac{1+cos\theta}{sin\theta}\hat{e}_\phi = \mathbf{A} - 2\nabla\phi$ differs from $\mathbf{A}$ only by a gauge transformation. The new vector potential $\mathbf{A}'$ is non singular in the Southern hemisphere, and the application of Stokes' theorem leads to

$$S_{top}[\mathbf{n}] = iS \int_{A_{\gamma,s}} dS \cdot \mathbf{B}_m = -iSA_{\gamma,s} \tag{171}$$

Here $A_{\gamma,s}$ is the area of the surface bounded by γ but covering the south pole. Note the minus sign due to the orientation of the surface $A_{\gamma,s}$. This leads to the implication that the result obtained for S_{top} depends on the gauge choice for the monopole vector potential! The difference between the Northern and Southern gauge choices for the action is

$$iS \int_{A_{\gamma,n}} dS \cdot \mathbf{B}_m + iS \int_{A_{\gamma,s}} dS \cdot \mathbf{B}_m = iS \int_{\mathbb{S}^2} dS \cdot \mathbf{e}_r = 4\pi iS \tag{172}$$

Physically, we expect the path integral to be independent of this gauge choice, and should not depend on the choice of coordinate systems. Recall that physical quantities are determined by the *exponentiated* action $\exp(-S[\mathbf{n}])$ and not the action itself. Thus the gauge invariance is enforced only if the factor $\exp(4\pi iS) = 1$, which is possible if S is either **integer** or **half-integer**.

The quantisation of spin thus emerges as a consequence of global gauge invariance of the spin path integral. Summarising, we have found that the classical dynamics of a spin is that of a massless point particle on the 2-sphere coupled to a monopole field $\mathbf{B}_m$. The vector potential of the field isn't globally continuous on the sphere. In general, the phase space $\mathbb{S}^2$ cannot be represented in terms of a global system of coordinates and momentum, which places it outside the scope of traditional treatments of classical mechanics.

Appendix A

Here we show an explicit calculation of the surface impedance of a superconductor. Consider a uniform superconductor with London penetration depth λ_L, and it fills the half-space $x > 0$. The vector potential is tangential, and for the region $x < 0$, it is given by

$$A_y = (ae^{ikx} + be^{-ikx})e^{-i\omega t}$$

The vector potential inside the surface can be easily found by using equation (37). Since the time dependence is harmonic, we can write

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\omega^2 \mathbf{A} \tag{A1}$$

Thus, we can rewrite the massive Proca equation as

$$\nabla^2 \mathbf{A}` + (\frac{\omega^2}{c^2} - \mu^2)\mathbf{A} = 0 \tag{A2}$$

Defining $k^2 = (\omega/c)^2 - \mu^2 \implies$ in the long wavelength regime (low frequency) $|\mathbf{k}| = i\sqrt{\mu^2 - (\omega/c)^2}$. Since the vector potential outside the superconducting region propagates along x direction, it is natural to expect a solution of the form (exploiting the fact that due to the symmetry about the $x-$direction, $A_z = 0$ because it is zero at the boundary)

$$A_y = \left(A_1 e^{-\sqrt{\mu^2-(\omega/c)^2}x} + A_2 e^{\sqrt{\mu^2-(\omega/c)^2}x}\right) e^{-i\omega t} \tag{A3}$$

The second term blows up, but the potential must damp down as $x \to \infty$. Thus, $A_2 = 0 \implies$

$$A_y = \left(A_1 e^{-\sqrt{\mu^2-(\omega/c)^2}x}\right) e^{-i\omega t} \tag{A4}$$

Continuity of the vector potential at $x = 0$ gives

$$A_1 = a + b \tag{A5}$$

We now calculate the fields at the boundary. The electric field can be computed as follows

$$\begin{aligned}\mathbf{E}\big|_{x=0} &= -\frac{\partial \mathbf{A}}{\partial t}\big|_{x=0} \\ &= i\omega\,(a+b)\,e^{-\sqrt{\mu^2-(\omega/c)^2}x}e^{-i\omega t}\hat{\mathbf{y}}\big|_{x=0} \\ &= i\omega\,(a+b)\,e^{-i\omega t}\hat{\mathbf{y}}\end{aligned} \tag{A6}$$

The magnetic field comes out to be

$$\begin{aligned}\mathbf{B}\big|_{x=0} &= \nabla \times \mathbf{A}\big|_{x=0} \\ &= -\sqrt{\mu^2 - (\omega/c)^2}\,(a+b)\,e^{-i\omega t}\hat{\mathbf{z}}\end{aligned} \tag{A7}$$

The surface impedance can be calculated from the ratio of the tangential components of $\mathbf{E}$ and $\mathbf{B}$.

$$Z_s = Z_0 \frac{E_t}{B_t} \tag{A8}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$, and E_t and B_t represent the tangential components of the electric and magnetic fields, respectively.

$$\begin{aligned}Z_s &= Z_0 \frac{i\omega\,(a+b)\,e^{-i\omega t}}{-\sqrt{\mu^2 - (\omega/c)^2}\,(a+b)\,e^{-i\omega t}} \\ &= -Z_0 \frac{i\omega}{\sqrt{\mu^2 - (\omega/c)^2}}\end{aligned} \tag{A9}$$

Substituting $\omega = 2\pi c/\lambda$ and $\mu = 1/\lambda_L$ in equation (A9), we get

$$\begin{aligned} Z_s &= -i\frac{2\pi c Z_0}{\lambda}\left(\frac{1}{\lambda_L^2} - \left(\frac{2\pi}{\lambda}\right)^2\right)^{-\frac{1}{2}} \\ &= -i2\pi c Z_0\frac{\lambda_L}{\lambda}\left(1 - \left(\frac{2\pi\lambda_L}{\lambda}\right)^2\right)^{-\frac{1}{2}} \end{aligned} \tag{A10}$$

Since we are in the long wavelength regime, $\lambda_L << \lambda$. Thus, we obtain the surface impedance Z_s to the first order in λ_L/λ as

$$Z_s = -i2\pi c Z_0\frac{\lambda_L}{\lambda} \tag{A11}$$

From the above discussion, it is clear that the surface impedance is purely reactive, as promised in section III.

Appendix B: Construction of the Path Integral for spinning particles using Spin Coherent states

To express the partition function $\mathcal{Z} = \text{tr}\, e^{-\beta\hat{H}}$, we divide the (imaginary) time β into N equal intervals $\epsilon = \beta/N$. Therefore $\mathcal{Z} = \text{tr}(e^{-\epsilon\hat{H}})^N$. Next, we insert a suitable chosen resolution of identity between each of the terms $e^{-\epsilon\hat{H}}$. A representation that leads directly to the final form of the path integral is specified by

$$\text{id} = C\int dg\, |g\rangle\langle g| \tag{B1}$$

where "id" represents the unit operator on $\mathcal{H}_S$. Here C is some constant, and $|g\rangle \equiv g\,|\rangle$ is the state obtained by the action of the representation matrix g act on the maximum weight state $|\uparrow\rangle$. For integration over the group manifold, the integration measure dg is chosen as the *Haar* measure, which is invariant under left and right multiplication of g by fixed group elements:

$$\forall h \in SU(2):\ \int dg\, f(gh) = \int dg\, f(hg) = \int dg\, f(g) \tag{B2}$$

Substituting the resolution of identity into the time-sliced partition function and observing that

$$\begin{aligned} \langle g_{i+1}|\, e^{-\epsilon\mathbf{B}\cdot\hat{\mathbf{S}}}\, |g_i\rangle &\simeq \langle g_{i+1}|g_i\rangle - \epsilon\,\langle g_{i+1}|\,\mathbf{B}\cdot\hat{\mathbf{S}}\,|g_i\rangle = 1 - \langle g_i|g_i\rangle + \langle g_{i+1}|g_i\rangle - \epsilon\,\langle g_{i+1}|\,\mathbf{B}\cdot\hat{\mathbf{S}}\,|g_i\rangle \\ &\simeq \exp\Big(\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle - \epsilon\,\langle g_{i+1}|\,\mathbf{B}\cdot\hat{\mathbf{S}}\,|g_i\rangle\Big) \end{aligned}$$

for large N. The partition function is, therefore

$$\mathcal{Z} = \lim_{N\to\infty}\int_{g_N=g_0}\prod_{i=0}^{N} dg_i \exp\left[-\epsilon\sum_{i=0}^{N-1}\left(-\frac{\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle}{\epsilon} + \langle g_{i+1}|\,\mathbf{B}\cdot\hat{\mathbf{S}}\,|g_i\rangle\right)\right]$$

By taking the limit $N \to \infty$, this can be cast in the path integral form compactly as,

$$\mathcal{Z} = \int Dg\, \exp\left[-\int_0^\beta d\tau\,\Big(-\langle\partial_\tau g|g\rangle + \langle g|\,\mathbf{B}\cdot\hat{\mathbf{S}}\,|g\rangle\Big)\right] \tag{B3}$$

where the $\mathcal{H}_S$-valued function $|g(\tau)\rangle$ is the continuum limit of $|g_i\rangle$.

Appendix C: Tutorial Problem

Q: Recall that in the class, we arrived at the laws of conservation of linear and angular momenta by starting with the following equations:

$$\frac{d\mathbf{P}_m}{dt} = \int (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\, d^3\mathbf{x}$$
$$\frac{d\mathbf{L}}{dt} = \int (\mathbf{x}\times(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}))\, d^3\mathbf{x}$$

where $\mathbf{P}_m$ is the mechanical momentum and $\mathbf{L}$ is the angular momentum. After a little manipulation, we obtained the Maxwell stress tensor $T^{\mu\nu}$ and the angular momentum flux tensor $M^{\alpha\beta\gamma}$. In section V of this report, we showed how the conservation laws emerge out of Noether's theorem. Derive the law of conservation of angular momentum using **Noether's Theorem**. You may refer to the example solved in the section on Noether's theorem for reference. *Hint*: consider the symmetry of the Lagrangian under an infinitesimal Lorentz transformation along with a Gauge transformation.

Solution: The Lagrange density for EM fields is: $\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - j^\mu A^\mu$
The coordinates x^μ change as follows under an infinitesimal *continuous* Lorentz transformation:

$$x'^\mu = \Lambda^\mu_{\ \nu}x^\nu \simeq (\delta^\mu_{\ \nu} + \omega^\mu_{\ \nu})x^\nu \implies \Delta x^\mu = \omega^\mu_{\ \nu}x^\nu \tag{C1}$$

Here, ω^μ_ν is the matrix of infinitesimal Lorentz transformation parameters. This is easy to see by considering a special case of an infinitesimal boost (rotation) along (about) a particular axis. The matrix of the infinitesimal parameters satisfies: $\omega^\mu_{\ \sigma}\partial_\mu x^\rho = \omega^\mu_{\ \sigma}\delta^\sigma_{\ \mu} = \omega^\mu_{\ \mu} = 0$.
Now, the transformed 4-vector potential components are:

$$\Delta A_\mu = \partial_\rho A_\mu \omega^\rho_{\ \sigma}x^\sigma + \omega^\nu_{\ \mu}A_\nu + \partial_\mu\chi \tag{C2}$$

Here, we choose $\chi = -A_\rho\omega^\rho_{\ \sigma}x^\sigma$ to ensure Gauge invariance for ΔA_μ which becomes:

$$\Delta A_\mu = (\partial_\rho A_\mu - \partial_\mu A_\rho)\omega^\rho_{\ \sigma}x^\sigma = F_{\rho\mu}g^{\rho\kappa}\omega_{\kappa\sigma}x^\sigma \tag{C3}$$

Analogous to the set of equations (70), we are now ready to compute the change in the Lagrange density under the given transformation:

$$\begin{aligned}\Delta\mathcal{L} &= 0\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\Delta A_\nu\right)\\ &= -\frac{1}{\mu_0}\partial_\mu\left(F^{\mu\alpha}\Delta A_\alpha\right)\\ &= \frac{1}{\mu_0}\partial_\mu\left(F^{\alpha\mu}F_{\rho\alpha}g^{\rho\kappa}\omega_{\kappa\sigma}x^\sigma\right)\end{aligned} \tag{C4}$$

Here we have used the antisymmetry of $F^{\alpha\mu}$. Now, $\omega_{\kappa\sigma}$ are arbitrary constant parameters and can be pulled out of the derivative. Also, as per equation (57) we have:

$$\begin{aligned}\Theta^{\kappa\mu} &= \frac{1}{\mu_0}g^{\kappa\rho}F_{\rho\alpha}F^{\alpha\mu} + \frac{1}{4\mu_0}g^{\kappa\mu}F^{\delta\tau}F_{\delta\tau}\\ \implies \Theta^{\kappa\mu} - \frac{1}{4\mu_0}g^{\kappa\mu}F^{\delta\tau}F_{\delta\tau} &= \frac{1}{\mu_0}g^{\kappa\rho}F_{\rho\alpha}F^{\alpha\mu}\end{aligned} \tag{C5}$$

Thus, we in (C4) have:

$$\partial_\mu\left[\left(\Theta^{\kappa\mu} - \frac{1}{4\mu_0}g^{\kappa\mu}F^{\delta\tau}F_{\delta\tau}\right)x^\sigma\right]\omega_{\kappa\sigma} = 0 \tag{C6}$$

Swap dummy indices and use the antisymmetry of $\omega_{\kappa\sigma}$:

$$-\partial_\mu\left[\left(\Theta^{\sigma\mu} - \frac{1}{4\mu_0}g^{\sigma\mu}F^{\delta\tau}F_{\delta\tau}\right)x^\kappa\right]\omega_{\kappa\sigma} = 0 \tag{C7}$$

Add (C6) and (C7):

$$
\begin{aligned}
0 &= \partial_\mu \left[\Theta^{\kappa\mu} x^\sigma - \Theta^{\sigma\mu} x^\kappa - \frac{1}{4\mu_0} (g^{\kappa\mu} x^\sigma - g^{\sigma\mu} x^\kappa) F^{\delta\tau} F_{\delta\tau} \right] \omega_{\kappa\sigma} \\
&= \partial_\mu (M^{\sigma\kappa\mu}) - \frac{1}{4\mu_0} (g^{\kappa\mu} x^\sigma - g^{\sigma\mu} x^\kappa) \partial_\mu (F^{\delta\tau} F_{\delta\tau}) \\
&= \partial_\mu (M^{\sigma\kappa\mu}) - \frac{1}{4\mu_0} (g^{\kappa\mu} x^\sigma - g^{\sigma\mu} x^\kappa) \partial_\mu (F^{\delta\tau} F_{\delta\tau}) \\
&= \partial_\mu (M^{\sigma\kappa\mu}) + x^\sigma j^\kappa - x^\kappa f^\sigma
\end{aligned}
\tag{C8}
$$

Here, we use the expression derived in (71). Also, $\partial_\mu (g^{\kappa\mu} x^\sigma - g^{\sigma\mu} x^\kappa) = 0$ is easy to verify. Here, $f^\kappa = \frac{1}{c} F^{\kappa\nu} j_\nu$ as defined before and $M^{\sigma\kappa\mu} = \Theta^{\kappa\mu} x^\sigma - \Theta^{\sigma\mu} x^\kappa$. Thus, we have arrived at the law of conservation of angular momentum.

CONTRIBUTION REPORT

- **Anurag**: Section I, VII, VIII, IX
- **Devashish**: Section IV, V
- **Mehul**: Section II
- **Sagnik**: Section VI, X, Appendix B
- **Shashwat**: Section III, Appendix A

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[2] In general, we impose 3 conditions while dealing with fields theories: The Action must be Lorentz invariant, The Lagrangian must be Gauge invariant and the Lagrangian must be local.

[3] A Greek index can take values from 0 to 3 while a Latin index can take values from 1 to 3.

[4] If the action is gauge invariant, the field equations will be gauge invariant as well.

[5] ξ is a scalar.

[6] These are mentioned so we can verify whether the Lagrangian gives us the correct field equations.

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