First principles approach to the Abraham–Minkowski controversy for the momentum of light in general linear non-dispersive media

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1 Introduction

There has always been extensive debate about the correct expression for energy-momentum tensor of light in material media. The most famous ones are the Abraham and Minkowski expressions, proposed at the beginning of the 20th century.

Minkowski proposed a non-symmetric energy momentum tensor, which can be derived from the Maxwell's macroscopic equations. The Abraham tensor is symmetric, but it cannot be derived from the first principles.

In the simplest case of a plane wave propagating in these media, the Minkowski and Abraham momentum densities read

$$\pi^M = \eta \frac{\mathcal{U}}{c} \hat{k} \qquad \pi^A = \frac{1}{\eta} \frac{\mathcal{U}}{c} \hat{k}$$

[4]

Here, $\mathcal{U}$ is the energy density of light in the medium, $\hat{k}$ is the propagation direction unit vector and c is the velocity of light in vacuum.

Qualitatively, these two expressions predict that the energy of a photon in any material media changes to $\frac{\eta \hbar w}{c}$ for the Minkowski tensor and $\frac{\hbar w}{\eta c}$ for the Abraham tensor, for a photon whose energy is $\frac{\hbar w}{c}$ in vacuum.

About 40 years ago, it was understood that neither of these forms is completely correct on its own.

The electromagnetic momentum of electromagnetic waves in linear media is always accompanied by a material momentum. Only the total energy momentum of the closed system has any physical meaning, and that the Abraham and Minkowski tensors just correspond to different separations of the same total tensor.

Both momenta can be measured, but in different situations. The Abraham momentum was identified as the 'kinetic' momentum of light in matter and the Minkowski momentum as the 'canonical' momentum because it is related to the generator of translations in the medium at rest. [2]
In this paper, we outline a field theoretic approach to obtaining both the Minkowski

and Abraham tensors, while seeing that they both describe the same physical situation.

2 Setting up the Lagrangian Formalism

In this section, we consider a general Lagrangian and derive conservation equations for the canonical stress energy tensor, the spin and orbital angular momentum tensors using various local symmetries. Once we have our conservation laws, we perform a "relocalization" (analogous to a choosing a gauge) such that the new tensors have very useful properties. The new stress energy tensor thus obtained is the so called Belinfante–Rosenfeld (BR) tensor. Finally, we explore the implications of the conservation equations under open and closed systems. We will use the notation of the paper: i, j refer to both spatial and temporal coordinates, while a, b refer to only spatial coordinates.

2.1 Balance Equations and relocalization

We begin with a lagrangian density of N fields $\mathcal{L}(\phi^A(x), \partial_i\phi^A(x))$. We note that for this general lagrangian, these fields may not be independent, and may have internal spaces of their own in which they interact (for example, a spin space). We further account for fields being external, so they may not satsify the Euler-Lagrange Equations. We impose local symmetries on this system, and each such symmetry gives rise to a conserved current via Noether's second theorem (see [1]).

Invariance of the lagrangian under space-time translations gives us a conservation equation for what is defined as the canonical stress tensor:

$$\Sigma_i^{\ j} = \frac{\partial \mathcal{L}}{\partial(\partial_j \phi^A)} \partial_i \phi^A - \delta_i^{\ j} \mathcal{L} \tag{1}$$

From invariance under Lorentz Transformations, we obtain a conservation equation for the spin angular momentum tensor (from boosts):

$$S_{kl}^{\ \ j} = \frac{\partial \mathcal{L}}{\partial(\partial_j \phi^A)} (s_{kl})^A_{\ B} \phi^B \tag{2}$$

and orbital angular momentum tensor (from spatial rotations):

$$L_{kl}^{\ \ j} = x_k \Sigma_l^{\ j} - x_l \Sigma_k^{\ j} \tag{3}$$

Here $(s_{kl})^A_B$ are Lorentz generators for the representation ϕ_A, ϕ_B belong to. It is intuitive to note that the spin density reduces to 0 if the fields have no microstructure, i.e., there is no internal mixing between the fields. Thus the spin density is a representation of the angular momentum generated from the microstructure of the fields alone.

Adding the spin and orbital angular momentum conservation equations, we obtain the total angular momentum conservation equation of the total angular momentum tensor:

$$J_{kl}^{\ \ j} = S_{kl}^{\ \ j} + L_{kl}^{\ \ j} \tag{4}$$

These four conservation equations are referred to as the balance equations:

$$\partial_j \Sigma_i^{\ j} = -\frac{\delta \mathcal{L}}{\delta \phi^A} \partial_i \phi^A \tag{5}$$

$$\partial_j S_{kl}^{\ \ j} = 2\Sigma_{[kl]} - \frac{\delta \mathcal{L}}{\delta \phi^A} (s_{kl})^A_{\ \ B} \phi^B \tag{6}$$

$$\partial_j L_{kl}^{\ \ j} = -2\Sigma_{[kl]} - \frac{\delta \mathcal{L}}{\delta \phi^A} (l_{kl})^A_{\ \ B} \phi^B \tag{7}$$

$$\partial_j J_{kl}^{\ \ j} = -\frac{\delta \mathcal{L}}{\delta \phi^A} (j_{kl})^A_{\ \ B} \phi^B \tag{8}$$

Where s_{kl}, l_{kl}, j_{kl} are the usual spin, orbital and total angular momentum generators. These terms are non-zero only if we allow mixing of the fields in an internal space.

Now, using a procedure called relocalization, one can transform these conserved currents such that the balance equations remain unchanged. The procedure is covered extensively in some of the references of the paper we are reviewing, but the final result is analogous to a gauge freedom. The general form of such a transformation:

$$\hat{\Sigma}_i{}^j = \Sigma_i{}^j - \partial_l X_i{}^{jl} \tag{9}$$

$$\hat{S}_{kl}{}^j = S_{kl}{}^j - 2X_{[kl]}{}^j + \partial_i Y_{kl}{}^{ji} \tag{10}$$

where the tensors $X_i{}^{jl}$ and $Y_{kl}{}^{ji}$ only have the constraints of X being skew-symmetric in the upper two indices, and Y being skew-symmetric in the lower two indices. Exploiting this freedom, we choose X and Y such that $\hat{S}_{kl}{}^j = 0$. The stress energy tensor under this particular transformation is known as the BR stress energy tensor:

$$\sigma_i{}^j := \Sigma_i{}^j + \frac{1}{2}\partial_k(S^{jk}_{\ \ i} + S_i{}^{kj} - S_i{}^{jk}) \tag{11}$$

By definition, the BR tensor and the canonical stress energy tensor satisfy the same balance equations. Further, by construction:

$$2\sigma_{[ij]} = 2\Sigma_{[ij]} - \partial_k S_{ij}{}^k \tag{12}$$

Under this particular relocalization, the corresponding total angular momentum density is purely orbital:

$$l_{kl}^{\ j} := x_k \sigma_l{}^j - x_l \sigma_k{}^j \tag{13}$$

Note this is not the generator of orbital angular momentum, which is a second-rank tensor. Under this, the balance equations simplify quite a bit.

We further define open and closed systems: A closed system does not have any external fields in the lagrangian: all the fields are internal (dynamical). Being internal to the system, these fields must satisfy the Euler-Lagrange equation:

$$\frac{\delta \mathcal{L}}{\delta \phi^A} = 0 \tag{14}$$

Considering this partition of our fields into dynamical and external, our final balance equations for the outlined relocalization simplify to:

$$\partial_j {\sigma_i}^j = -\frac{\delta \mathcal{L}}{\delta \phi^A_{\text{ext}}} \partial_i \phi^A_{\text{ext}} \tag{15}$$

$$\partial_j l_{kl}^{\ \ j} = -\frac{\delta \mathcal{L}}{\delta \phi^A_{\text{ext}}} (j_{kl})^A{}_B \phi^B_{\text{ext}} \tag{16}$$

$$2\sigma_{[kl]} = \frac{\delta \mathcal{L}}{\delta \phi^A_{\text{ext}}} (s_{kl})^A{}_B \phi^B_{\text{ext}} \tag{17}$$

Having constructed the BR stress energy tensor, we now analyse the balance equations in closed and open systems.

2.2 Closed system

For a closed system, the RHS of all three balance equations reduce to 0. Thus, for a closed system, the BR stress energy tensor and the corresponding angular momentum densities are conserved. Further, the BR stress energy tensor is always symmetric! This is not true for the canonical stress energy tensor, which is symmetric in general only if the fields have no microstructure:

$$2\Sigma_{kl} = \partial_j {S_{kl}}^j \neq 0 \tag{18}$$

2.3 Open system

If we have external (non-dynamical/background) fields as well in our lagrangian, the RHS of our balance equation need not be 0. Thus the BR tensor is not symmetric, and along with the angular momentum density is not conserved in general. However, under specific conditions on the external fields, continuity equations can still be obtained.

We decompose the BR stress energy tensor as:

$${\sigma_i}^j := \begin{pmatrix} \mathcal{U} & S^a \\ -\pi_a & -{p_a}^b \end{pmatrix} \tag{19}$$

where $\mathcal{U}$ is the energy density, S^a the energy flux density, π_a the momentum density and ${p_a}^b$ the momentum flux density of open system.

Now, we consider the case in which the external fields are time-independent. The $i = 0$ balance equation for the BR tensor gives us an energy continuity equation:

$$\frac{\partial \mathcal{U}}{\partial t} + \partial_a S^a = 0 \tag{20}$$

We similarly obtain a momentum continuity equation for invariance under spatial translations along the x^a direction:

$$\frac{\partial \pi_{\dashv}}{\partial t} + \partial_b {p^b}_a = 0 \tag{21}$$

Similarly, for invariance under spatial rotations in the $a - b$ plane, we obtain an angular momentum continuity equation:

$$\frac{\partial {l_{ab}}^0}{\partial t} + \partial_c {l_{ab}}^c \tag{22}$$

and for invariance under boosts in x_a direction:

$$\frac{\partial {l_{0a}}^0}{\partial t} + \partial_b {l_{0a}}^b = 0 \tag{23}$$

However, we will see that in a medium, the symmetry under boosts is broken, hence the last equation never holds.

3 Electromagnetic field in matter

3.1 Macroscopic Maxwell Equations

We are familiar with the $\mathbf{H}, \mathbf{D}$ fields used to simplify electromagnetism problems in the presence of a medium. They are incorporated into what are known as the Macroscopic Maxwell Equations, where only the free charges and currents contribute to the source terms:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{24}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{25}$$
$$\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \tag{26}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{27}$$

The name arises from the fact that we are bundling up all microscopic details of the material response into the $\mathbf{H}, \mathbf{D}$ fields and only looking at the effective behaviour.

In this section, we also assume that the material medium's dynamics are either only negligibly affected by the fields, or an external agent keeps the medium in a predetermined state of motion. This allows us to treat only the electromagnetic 4-potential's components as the dynamical fields of the system, and treat the medium as an external component in an open system.

The $\mathbf{H}, \mathbf{D}$ fields are defined using **constitutive relations** which completely characterise the response of the material:

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B}) \tag{28}$$
$$\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B}) \tag{29}$$

Assuming these laws are 1. local in spacetime and 2. linear in the fields, the general form becomes:

$$D^a = \varepsilon_0 \varepsilon^{ab} E_b + \beta^b_{\ a} B^b \tag{30}$$
$$H_a = \alpha_a^{\ b} E_b + \mu_0^{-1} \left(\mu^{-1}\right)_{ab} B^b \tag{31}$$

While slightly restrictive, this still allows for discussion of anisotropy, magneto-electric effects, and the effects of a moving medium.

ε^{ab} is the relative permittivity three-tensor, $\left(\mu^{-1}\right)_{ab}$ is the inverse relative permeability tensor and $\alpha_a^{\ b}, \beta^b_{\ a}$ are the linear magneto-electric coupling coefficients. In general, all four are spacetime-dependent fields.

3.2 tensorial form of constitutive relations

We can repackage these two equations, and the 36 independent component fields of the coefficients of $\mathbf{E}, \mathbf{B}$ in the same, into a single neat equation using the field strength tensor and the excitation tensor.

The field strength tensor is well known to be (with both indices lowered):

$$F^{ij} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{32}$$

$$\implies F_{ij} = \begin{pmatrix} 0 & E_x c & E_y c & E_z c \\ -E_x c & 0 & -B_z & B_y \\ -E_y c & B_z & 0 & -B_x \\ -E_z c & -B_y & B_x & 0 \end{pmatrix} \tag{33}$$

Similarly, define the electromagnetic excitation four-tensor in terms of D, H:

$$H^{ij} = \begin{pmatrix} 0 & -D_x/c & -D_y/c & -D_z/c \\ D_x/c & 0 & H_z & -H_y \\ D_y/c & -H_z & 0 & H_x \\ D_z/c & H_y & -H_x & 0 \end{pmatrix} \tag{34}$$

Both of these are antisymmetric, and hence the constitutive relations can be packaged together by defining the constitutive tensor χ^{ijkl} (antisymmetric in its first two and last two indices separately):

$$H^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl} \tag{35}$$

$$\chi^{ijkl} = -\chi^{jikl} = \chi^{jilk} \tag{36}$$

In the case of a non-dissipative medium, it becomes real and gains the symmetry $\chi^{ijkl} = \chi^{klij}$.

It is worth noting the constitutive tensors for the two simplest cases: For a vacuum,

$$\chi^{ijkl}_{\text{vac}} := \frac{1}{\mu_0}(g^{ik}g^{jl} - g^{il}g^{jk}) \tag{37}$$

Defined using the inverse Minkowski metric, $g^{ij} = \text{diag}\left(\frac{1}{c^2}, -1, -1, -1\right)$. It is simple to then obtain the relations $\mathbf{D} = \varepsilon_0\mathbf{E}, \mathbf{H} = \mathbf{B}/\mu_0$.

The other simple case is that of a linear, homogeneous and isotropic medium at rest:

$$\chi^{ijkl} = \frac{1}{\mu\mu_0}(\gamma^{ik}\gamma^{jl} - \gamma^{il}\gamma^{jk}) \tag{38}$$

$$\gamma^{ij} = \text{diag}\left(\frac{n^2}{c^2}, -1, -1, -1\right) \tag{39}$$

$$\text{Where } n = \sqrt{\varepsilon\mu} \tag{40}$$

3.3 Lagrangian Formalism of Electromagnetism

The lagrangian density for the electromagnetic fields in matter is:

$$\mathcal{L}'^e = \mathcal{L}^e + \mathcal{L}^J \tag{41}$$

$$\mathcal{L}^e = -\frac{1}{4}F_{ij}H^{ij} = -\frac{1}{8}\chi^{ijkl}F_{ij}F_{kl} \tag{42}$$

$$\mathcal{L}^J = J^iA_i \tag{43}$$

Where $F_{ij} = \partial_iA_j - \partial_jA_i$ is the expression of the field strength tensor in terms of the electromagnetic 4-potential $A^i = (\phi/c, \mathbf{A})$, and $J^i = (\rho/c, \mathbf{J})$ is the free 4-current density (not considering bound charges of the medium, which are encoded into χ).

The dynamical fields are A^i only - χ^{ijkl}, J^i will be treated as external fields here.

3.3.1 Gauge Invariance

A transformation of the type $A^i \to A^i + \partial^i \lambda$ leaves F^{ij}, and hence the source-free lagrangian $\mathcal{L}^e$, unchanged. This is the electromagnetic gauge invariance.

3.4 Obtaining Maxwell's equations

The homogeneous Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, can be derived using the Bianchi identity of the stress-energy tensor arising from the theorem of mixed partials:

$$\partial_{(i} F_{jk)} = 0 \tag{44}$$

$$\implies \varepsilon^{lijk} \partial_i F_{jk} = 0 \tag{45}$$

$$\implies \partial_i \underbrace{\left(\frac{1}{2} \varepsilon^{lijk} F_{jk} \right)}_{\tilde{F}^{li}} = 0 \tag{46}$$

Where $\tilde{F}^{ij}$ is the dual field strength tensor, and $\partial_j \tilde{F}^{ij} = 0$ encode the homogeneous Maxwell equations.

To obtain the inhomogeneous Maxwell equations, we use the Euler-Lagrange equations of motion:

$$\frac{\delta \mathcal{L}}{\delta A_i} = 0 \tag{47}$$

Where we use the variational derivative, which can be written in the form

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \right) \tag{48}$$

Applied to the electromagnetic lagrangian (with source terms), we obtain:

$$\partial_j H^{ij} = J^i \tag{49}$$

3.5 Stress-Energy tensor

We can compute the canonical stress-energy tensor for $\mathcal{L}^e$ (without source terms) using (1):

$$\overset{\mathrm{e}}{\Sigma}{}_i^{\ j} = \Theta_i^{\ j} + H^{kj}(\partial_k A_i) \tag{50}$$

Which includes $\Theta_i^{\ j}$, the Minkowski energy-momentum tensor, defined in any medium as

$$\Theta_i^{\ j} := F_{ik}H^{kj} + \frac{1}{4}\delta_i^{\ j} F_{kl}H^{kl} \tag{51}$$

Notice that while $\Theta_i^{\ j}$ is gauge invariant, $\overset{\mathrm{e}}{\Sigma}{}_i^{\ j}$ is not. This suggests an importance of $\Theta_i^{\ j}$, confirmed by calculating the BR tensor for the electromagnetic source-free lagrangian density. Calculate the spin density using (2):

$$\overset{\mathrm{e}}{S}{}_{kl}^{\ \ j} = H^j_{\ l}A_k - H^j_{\ k}A_l \tag{52}$$

And then, attempting to derive the BR tensor from the canonical stress tensor $\overset{\mathrm{e}}{\Sigma}{}_i^{\ j}$ and the corresponding spin density, we obtain that the BR tensor equals the Minkowski tensor.

$$\overset{\mathrm{e}}{\sigma}{}_i^{\ j} = \Theta_i^{\ j} \tag{53}$$

This is gauge-invariant, but if we include the source terms in the lagrangian and then calculate the BR tensor (the Minkowski tensor remaining unchanged as its definition does not include source terms), it is gauge non-invariant.

4 Electromagnetic balance equations and conserved Minkowski quantities

4.1 Energy and momentum of the electromagnetic field in a background medium

Analyzing the BR tensor in the case of an electromagnetic field in background matter using (53) and (**??**) in (15).

Again considering an electromagnetic field in background matter i.e. open system. Since the external fields don't satisfy the Euler Lagrange equation, $\partial\mathcal{L}/\partial\Phi^{\alpha}_{ext} \neq 0$, they'll contribute to RHS

One way to obtain the BR tensor balance equations is to substitute theta in the above BR tensor expression and find out the divergence, i.e.

$$\sigma_i{}^j = F_{ik}H^{kj} + \frac{1}{4}\delta_i{}^j F_{kl}H^{kl} - \delta^i{}_j J^k A_k + A_i J^i \tag{54}$$

and

$$H^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl} \tag{55}$$

Implies, χ^{ijkl} itself acts like an external field

$$\sigma_i{}^j = \Theta_i^j - \delta_i^j J^k A_k + A_i J^i \tag{56}$$

and the Minkowski (and BR) energy-momentum tensor is given as

$$\Theta_i{}^j := F_{ik}H^{kj} + \frac{1}{4}\delta_i{}^j F_{kl}H^{kl} \tag{57}$$

(we had earlier shown that the Miknowski tensor coincides with the BR tensor for electromagnetic field in matter, i.e. $\sigma_j{}^j = \Theta_i{}^j$)

$$\partial_j \overset{\mathrm{e}}{\sigma}_i{}^{j\prime} = \frac{1}{8}F_{mj}F_{kl}\partial_i\chi^{mjkl} - A_j\partial_i J^j \tag{58}$$

$$\partial_j\Theta_i^j + \mathcal{F}_i^J + \mathcal{F}_i^m = 0 \tag{59}$$

where the source terms are four-force density corresponding to

$$\mathcal{F}^{\mathcal{J}} := -F_{ij}J^j \tag{60}$$

[Lorenz force density] is the rate of energy and momentum transfer from the electromagnetic field to the free external charges and currents inside an infinitesimal volume element.

components of $\mathcal{F}_i^J$ are $\mathcal{F}_i^J := (w^J, -\mathbf{f}^J)$; $w^J := j.E$ is the electromagnetic work done on the external currents and $\mathbf{f}^J := \rho E + j \times B$ is the Lorenz force on external charges and currents (since $J^i = (\rho, \mathbf{j})$ is the external current and charge density).

In accordance with equation (59), conversion from electromagnetic into mechanical energy and momentum is balanced by a corresponding rate of decrease of energy and momentum of the electromagnetic field in matter, which is quantified by the four-divergence of the Minkowski energy–momentum tensor ${\Theta_i}^j$.

$$\mathcal{F}_i^m := -\frac{1}{8} F_{mj} F_{kl} \partial_i \chi^{mjkl} \tag{61}$$

similarly to interpret the term $\mathcal{F}_i^m$ [effective material four-force density] $\mathcal{F}_i^m := (w^m, -\mathbf{f}^m)$ describing the macroscopic transfer of energy and momentum from em field to bound charges and currents of the background material medium.

Special case:

- Material is homogeneous in space, $\partial_a \chi^{ijkl} = 0$
- and time-independent $\partial_t \chi^{ijkl} = 0$, if both considered, $\mathcal{F}_i^m$ vanishes
- if external charges and currents are not present, i.e. no free charges and currents i.e. $J^i = 0 \implies \mathcal{F}_i^J$ also vanishes, $\implies \partial_j {\Theta_i}^j = 0$, i.e. the Minkowski tensor is conserved

Hence for vanishing free charges and currents, the conservation of Minkowski tensor of light in matter are directly determined by the space and time translation symmetry of the background material medium

Light propagating in an homogeneous region of the medium would not exert any macroscopic 'effective' force on it

In a linear, non-dissipative, non-dispersive, homogeneous and time-independent medium at rest, with no external charges and currents, a light pulse propagates with constant amplitude and velocity $\nu = c\hat{k}/n$

Note: There do exist microscopic interactions between the electromagnetic field and atoms/molecules of the medium, but macroscopically, the light in this kind of medium moves as if there was no effective force. - However if light propagates through an inhomogeneous media for ex it encounters the interface between two homogeneous mediums, then the *material would exert an effective macroscopic force on the em wave* and change its velocity of propagation. The action of this effective force only at the boundary is consistent with the change of Minkowski momentum there and also with its conservation when there is no force in a locally homogeneous medium.

4.1.1 Vacuum

For he trivial case of vacuum, the energy-momentum balance equation reduces to the standard one (considering there might be free charges and currents)

$$\chi^{ijkl}_{\mathrm{vac}} := (g^{ik}g^{jl} - g^{il}g^{jk})/\mu_0 \tag{62}$$

$$\partial_j \overset{\mathrm{vac}}{\Theta}{}_i{}^j = F_{ij}J^j \tag{63}$$

$$\overset{\mathrm{vac}}{\Theta}{}_i{}^j := (F_{ik}F^{kj} + (1/4)\partial_i^{\,j} F_{kl}F^{kl})/\mu_0 \tag{64}$$

Equation (64) is the well-known symmetric Minkowski tenso r in vacuum. As a consequence of equation (20), when $J^i = 0$ the Minkowski tensor in vacuum will be always conserved

for a bit more general case, for an isotropic, homogeneous medium at rest, the constitutive tensor reads,

$$\chi^{ijkl}_{\mathrm{iso,hom}} := (\gamma^{ik}\gamma^{jl} - \gamma^{il}\gamma^{jk})/\mu_0\mu \tag{65}$$

where the optical matrix is given as $\gamma^{ij} := \mathrm{diag}(n^2/c^2, -1, -1, -1)$

4.1.2 Explicit Minkowski energy balance equation

Decomposing the energy-momentum equation in space and time components and separating the balance equations, (valid for any linear, non-dissipative and non-dispersive background medium, not necessarily static)

$$\frac{\partial \mathcal{U}^M}{\partial t} + \nabla . S^M + w^J + w^m = 0 \tag{66}$$

where Minkowski energy density $\mathcal{U}^M$

Energy flux density S^M

Power density transferred to the external currents w^J

Effective material power density transferred to bound charges and currents w^m

$$\mathcal{U}^M = \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B}) \tag{67}$$

$$\mathbf{S}^M = \mathbf{E} \times \mathbf{H} \tag{68}$$

$$w^J = j.E \tag{69}$$

$$w^m = \frac{1}{2}\epsilon_0 E_a E_b \frac{\partial}{\partial t}\epsilon^{ab} - \frac{1}{2\mu_0} B^a B^b \frac{\partial}{\partial t}(\mu^{-1})_{ab} + E_a B^b \frac{\partial}{\partial t}\beta^a{}_b \tag{70}$$

Considering a spatial volume V bounded by a closed surface ∂V inside the medium with time-independent properties, and free of external currents,, w^J, w^m vanish and

$$\frac{d}{dt}\int_V d^3x\, \mathcal{U}^M = -\oint_{\partial V} dS(\hat{n}.S^M) \tag{71}$$

When the electromagnetic energy flux $\mathbf{S}^M$ through the boundary of the volume V vanishes, the **Minkowski energy** inside this region is a time-independent quantity:

$$E^M := \frac{1}{2}\int_V d^3x\ (\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B}) = \text{constant} \tag{72}$$

4.1.3 Explicit Minkowski momentum balance equation

for the spatial components, the momentum Balance equation is given as

$$\frac{\partial \pi_a^M}{\partial t} + \partial_b \overset{M}{p}{}_a{}^b + f_a^J + f_a^m = 0 \tag{73}$$

where Minkowski momentum density of the em field π^M

Minkowski electromagnetic stress tensor $p_a{}^b$

Lorenz force density exerted on the external charges and currents $\mathbf{f}^J$

Effective material force density exerted by field on bound charges and currents of the medium $\mathbf{f}^M$

$$\pi^M = \mathbf{D} \times \mathbf{B} \tag{74}$$

$$\overset{M}{p}{}_a{}^b = -E_a D^b - H_a B^b + \frac{1}{2}\delta_a^b (E_c D^c + H_c B^c) \tag{75}$$

$$\mathbf{f}^J = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \tag{76}$$

$$f_a^m = -\frac{1}{2}\epsilon_0 E_b E_c \partial_a \epsilon^{bc} + \frac{1}{2\mu_0} B^b B^c \partial_a (\mu^{-1})_{bc} - E_b B^c \partial_a \beta^b{}_c \tag{77}$$

Specializing again to a spatial region V within an homogeneous medium without external charges and currents ($\rho = 0, \mathbf{j} = 0$) with all derivatives in last equation vanishing (similar to the energy conservation equation)

$$\frac{d}{dt}\int_V d^3x\ \pi_a^M = -\oint_{\partial V} dS\ \overset{M}{p}{}_a{}^b\ \hat{n}_b \tag{78}$$

In the absence of electromagnetic momentum flux through the boundary, the integrated Minkowski momentum is a time-independent quantity

$$p^M := \int_M d^3x\ \mathbf{D} \times \mathbf{B} = \text{constant} \tag{79}$$

Hence, the two constants of motion of this open system (under translational symmetry conditions of the medium in space and time) turn out to be the **Minkowski energy** and the **Minkowski momentum**. They are both effective quantities which depend both on fields **E, B** and medium properties ϵ^{ab}, $(\mu^{-1})_{ab}$and β_b^a through the electric and magnetic excitations **D** and **H** encoded in the consitutive tensor.

Assumptions until now:

- homogeneous medium without external charges and currents $| = 0$ and
- medium inside the closed surface with time-independent properties and no local variations of χ^{ijkl}, leading to $\partial_i \chi^{ijkl} = 0$ for i = 0, 1, 2, 3 and
- in absence of electromagnetic energy and momentum flux through the boundary ∂V

With the above assumptions, we end up with two constants of motion of this open system. Lets observe the angular momentum now

4.2 Angular momentum and asymmetry of the Minkowski tensor

For the electromagnetic Lagrangian, Minkowski angular momentum tensor density can be defined as :

$$m_{kl}^{\ \ j} := x_k \Theta_l^{\ j} - x_l \Theta_k^{\ j} \tag{80}$$

$$\mathbf{l}^M := \mathbf{x} \times \pi^{\mathbf{M}} \tag{81}$$

$m_{kl}^{\ \ j}$ being the Minkowski angular momentum tensor density, whose spatial components are directly identified with vectorial Minkowski angular momentum density l^M and the angular momentum flux $\overset{M}{K}{}_a^{\ b}$

$$m_{ab}^{\ \ 0} = -\epsilon_{abc}\delta^{cd}l_d^M \tag{82}$$

$$m_{ab}^{\ \ c} = -\epsilon_{abd}\delta^{ce}K_e^{\ d} \tag{83}$$

The angular momentum Balance equation is then

$$\partial_j m_{kl}^{\ \ j} + \overset{J}{\mathcal{T}}_{kl} + \overset{m}{\mathcal{T}}_{kl} = 0 \tag{84}$$

Here $\overset{J}{\mathcal{T}}_{kl}$ is the antisymmetric Lorentz four-torque density tensor, directly constructed from the Lorentz four-force $\mathcal{F}_i^J$:

$$\overset{J}{\mathcal{T}}_{kl} := x_k\mathcal{F}_l^J - x_l\mathcal{F}_k^J \tag{85}$$

$$\tau^J := x \times f^J \tag{86}$$

$$\overset{J}{\mathcal{T}}_{kl} := -\epsilon_{abc}\delta^{cd}\tau_d^J \tag{87}$$

τ^J being the Lorentz torque density on the external charges and currents. $\overset{m}{\mathcal{T}}_{kl}$ can be interpreted as the effective material four-torque density exerted by electromagnetic field on bound charges and currents of the material medium. But NOTE: its not just constructed as the four-torque of $\mathcal{F}_i^m$, but also depends on the anti-symmetric part of the Minkowski tensor, this implies that in order to vanish this term, only ensuring that spatial components of effective material four-force tensor to zero won't be enough,

$$\overset{m}{\mathcal{T}}_{kl} := x_k \mathcal{F}^m_l - x_l \mathcal{F}^m_k + 2\Theta_{[kl]} \tag{88}$$

In terms of the general constitutive tensor, it can be written as:

$$\begin{aligned}\overset{m}{\mathcal{T}}_{kl} = -\frac{1}{8} F^{ij} F^{mn} [& x_k(\partial_l \chi_{ijmn}) - x_l(\partial_k \chi_{ijmn}) \\ & + g_{ik}\chi_{ljmn} - g_{il}\chi_{kjmn} \\ & + g_{jk}\chi_{ilmn} - g_{jl}\chi_{ikmn} \\ & + g_{mk}\chi_{ijln} - g_{ml}\chi_{ijkn} \\ & + g_{nk}\chi_{ijml} - g_{nl}\chi_{ijmk}\end{aligned} \tag{89}$$

Comparing it with equation (16) and putting $J^i = 0$, the total angular momentum operator acting on the fourth rank constitutive tensor of the background medium is

$$\overset{m}{\mathcal{T}}_{kl} = \frac{1}{8} \frac{\partial \mathcal{L}^e}{\partial \chi^{ijmn}} (j_{kl})^{ijmn}{}_{opqr} \chi^{opqr} \tag{90}$$

Hear ye, hear ye, we got a 10 rank tensor to deal with

Hence the interpretation of $\overset{m}{\mathcal{T}}_{kl}$ is completely analogous to $\mathcal{F}^m_i$, but additionally related to the symmetry of the medium under spatial rotations and boosts (Lorentz transformations) since its just

When the medium is isotropic, the corresponding χ^{ijkl} is invariant under 3-D rotations i.e. $(j_{ab})^{ijmn}{}_{opqr}\chi^{opqr} = 0$ thus the spatial components $\overset{m}{\mathcal{T}}_{ab}$ in equation (84) in the angular momentum balance equation vanishes.

As for $\mathcal{F}^m_i$, in an isotropic media, the light and matter do interact microscopically but they do it in such a way that there is no net macroscopic angular momentum transfer, i.e. $\overset{m}{\mathcal{T}}_{kl} = 0$. This implies that the Minkowski angular momentum of light

$\updownarrow^M$, related to $m^{\,0}_{ab}$ through $m^{\,0}_{ab} = -\epsilon_{abc}\delta^{cd} l^M_d$ is a conserved quantity, provided $J^i = 0$.

But if the medium does possess local anisotropy, there will be a net material macroscopic torque and angular momentum of light will change. But note that just ensuring that the rotations are symmetric is enough to make spatial components of effective material four-torque density = 0

Hence the conservation of the Minkowski angular momentum of light in matter is not violated by the lack of symmetry with respect to all components of the Minkowski four-tensor. Only the symmetry of the 3*3 block of spatial components Θ_{ab} is important to make $\overset{m}{\mathcal{T}}_{ab}= 0$. - The vanishing of the space-time components of the material $\overset{m}{\mathcal{T}}_{0a}$ is independently determined by the invariance of χ^{ijkl} under boosts (only true for when the material is vacuum)

More generally, if there is a background medium present, the constitutive tensor will *not be invariant under boosts.* (corresponding to the remaining 7 components of Θ). This breaks the invariance of the open system under boosts, since an external agent is always needed to keep the medium fixed in its state of motion (which is related to variation of the constitutive tensor of the medium χ^{ijkl}) in spite of the interaction between electromagnetic field and the medium.

Under the action of spin operator:

$$2\Theta_{[0a]} = -\frac{1}{8}\frac{\partial\mathcal{L}^e}{\partial\chi^{ijmn}}(s_{0a})^{ijmn}{}_{opqr}\chi^{opqr} \neq 0 \tag{91}$$

If χ^{ijkl} is invariant under rotations and boosts then there is no net torque on the medium. But this is the case only for vacuum.

4.2.1 Explicit Minkowski angular momentum balance equation

For completeness, writing down the Minkowski angular momentum balance equation,

$$\frac{\partial l_a^M}{\partial t} + \partial_b \overset{M}{K_a^{\ b}} + \tau_a^m + \tau_a^J = 0 \tag{92}$$

Here, the Minkowski angular momentum density l^M, the Minkowski angular momentum flux $\overset{M}{K_a^{\ b}}$, the Lorentz torque on the external charges and currents τ^J and the effective material torque τ^m are explicitly given as

$$l^M = x \times \pi^M = (x.B)D - (x.D)B \tag{93}$$

$$\begin{aligned} \overset{M}{K_a^{\ b}} &= \epsilon_{acd}\delta^{de}x^c E_e D^b + \epsilon_{acd}\delta^{de}x^c H_e B^b \\ &\quad - \frac{1}{2}\epsilon_{acd}\delta^{db}x^c(E_e D^e + H_e B^e) \end{aligned} \tag{94}$$

$$\tau^J = x \times f^J \tag{95}$$

$$\begin{aligned} \tau_a^m = -\frac{1}{2}\epsilon_{abc}x^b\delta^{cd}[D^e(\partial_d E_e) - E_e(\partial_d D^e) + B^e(\partial_d H_e) \\ -H_e(\partial_d B^e) + \epsilon_{abc}\delta^{bd}(E_d D^c + H_d B^c)] \end{aligned} \tag{96}$$

For the case of a spatial volume V without external charges and currents ($\tau_a^J = 0$) and an isotropic medium ($\tau_a^m = 0$), the integrated Minkowski angular momentum is a time independent quantity provided the net angular momentum flux though the boundary ∂V vanishes:

$$\mathbf{L}^M = \int_V d^3x[(x.B)D - (x.D)B] = \text{constant} \tag{97}$$

Hence we have got three constants of motion i.e. **Minkowski energy**, **Minkowski momentum** and **Minkowski angular momentum** for the system having translational symmetry conditions and the same assumptions as in section 4.1 carried through.

4.3 Abraham tensor of light in matter

The Abraham stress-energy tensor for electromagnetic fields in matter is defined in terms of the Minkowski tensor as:

$$\Omega_i^{\ j} := \Theta_i^{\ j} - P_i^{\ k}\Theta_{[k}^{\ \ j]} - u_k u^j \Theta_{[i}^{\ \ k]}/c^2 \tag{98}$$

Here, $u^i := (\gamma, \gamma\nu)$ is the four-velocity field of the medium with $\gamma := (1-\nu^2/c^2)^{-1/2}$ being the Lorentz factor. $P_i^{\ k}$ is the projector on the local rest frame of the medium:

$$P_i^{\ k} := \delta_i^k - u_i u^k/c^2 \tag{99}$$

In essence, the Abraham tensor is merely a symmetrisation of the Minkowski tensor, at the cost of requiring u^i explicitly in its definition.

Inserting the general definition of the Minkowski tensor (51) into equation (98), we find that

$$\begin{aligned}\Omega_i^{\ j} = \frac{1}{2}(F_{ik}H^{kj} + H_{ik}F^{kj}) + \frac{1}{4}\delta_i^{\ j} H^{kl}F_{kl} \\ +\frac{1}{2c^2}u_i u^l (F^{kj}H_{kl} - H^{kj}F_{kl}) \\ +\frac{1}{2c^2}u^j u_l (F_{ki}H^{kl} - H_{kl}F^{kl})\end{aligned} \tag{100}$$

Four-divergences of the Minkowski and Abraham tensors have the following relation, where the difference terms are absorbed into the Abraham four-force density:

$$\partial_j \Theta_i^{\ j} = \partial_j \Omega_i^{\ j} + \mathcal{F}_i^{\ A} \tag{101}$$

$$\begin{aligned}\mathcal{F}_i^{\ A} := \frac{1}{2}\partial_j [F_{ik}H^{kj} + (F_{ik}H^{kl} - H_{ik}F^{kl})\frac{u^j u_l}{c^2} \\ - H_{ik}F^{kj} - (F_{lk}H^{kj} - H_{lk}F^{kj})\frac{u_i u^l}{c^2}]\end{aligned} \tag{102}$$

Thus the Minkowski energy-momentum balance equation can be written in terms of the Abraham tensor and the Abraham four-force (102) using (101),

$$\partial_j \Omega_i^{\ j} + \mathcal{F}_i^J + \mathcal{F}_i^m + \mathcal{F}_i^A = 0 \tag{103}$$

While this is merely a reformulation of the Minkowski balance equation, it is worth noting the implication that even when the background medium has symmetries such as spatial homogeneity, time independence and spatial isotropy and there are no free charges, the Abraham tensor is still not conserved. Even for the simple case of a linear isotropic medium with constant and uniform ε, μ, the Abraham force is $f^A \propto \frac{n^2-1}{c^2}$, and hence non-zero unless in a vacuum.

5 Electromagnetic field and dynamical medium as a closed system

Till now, we considered the material medium as a background in which the EM wave propagates. Now, we look at the theory including the dynamics of the material medium, such that the total system is closed.

We are generalizing our theory to include any non-dissipative fluid medium with linear and non-dispersive optical properties.

5.1 Lagrangian Model for a dynamical medium interacting with light

The fields which satisfy the Euler Lagrange equations $\frac{\delta \mathcal{L}}{\delta \Phi^A_{dyn}} = 0$ are called dynamical fields. In a closed system, all fields are dynamical.

We express our constitutive tensor χ^{ijkl} in terms of dynamical matter fields.

$$\chi^{ijkl} = \chi^{ijkl}(\nu, u^i, \Psi^A)$$

Where u^i is the four velocity field of the medium, ν is the particle number density and Ψ^A are **dynamical fields**, which represent material variables.

The material variables can be any quantity used to describe the microstructure of the medium, and we also assume that χ does not depend on its derivatives.

5.1.1 Example of Dynamical Fields

An example of the dynamical matter field Ψ^A is the four director field of a nematic liquid crystal.
A liquid crustal has its properties between that of a conventional liquid and solid crystal. It may "flow" as a liquid but may have its molecules oriented in a crystal like way. Nematic just means that all the molecules of the crystal are aligned such that their long axes are roughly parallel but not in well defined planes - the more important point here is that they have microstructure.
The liquid crystal is a medium of "stretched" particles whose orientational motion is described by an additional hydrodynamic variable. Mathematically, this additional degree of freedom is a unit vector field $n = n(x, t), n^2 = 1$, which is called director. The director is a microscopic variable that is assigned to every material point of the medium. [5]

For a system without any anisotropy or a microstructure, all the dynamical fields become scalars, since there is no notion of direction anymore. From 18, we know that asymmetry of the canonical stress energy tensor is because of the tensorial nature of fields. So if one assumes that all dynamical fields are scalar fields, RHS goes to 0.

We use the laws of relative hydrodynamics to model our material medium. The material variables describing the translational dynamics obey the following equations.[3]

$$\partial_i(\nu u^i) = 0 \tag{104}$$

$$u^i \partial_i s = 0 \tag{105}$$

$$u^i \partial_i X = 0 \tag{106}$$

$$u_i u^i = c^2 \tag{107}$$

Where s is the **entropy density** of the medium and X is called the **Identity (Lin) Coordinate**. Equation 104 characterizes the particle number conservation, equation 105 implies entropy conservation along the streamlines of the fluid and since this entropy need not be the same along different streamlines, we introduce the Lin coordinate X to identify the particles in different streamlines. Equation 107 is the relativistic normalization of the four velocity field.

The total Lagrangian for the system (Electromagnetic wave + Material) is given by

$$\mathcal{L} = \mathcal{L}^e + \mathcal{L}^m \tag{108}$$

where $\mathcal{L}^e$ is the electromagnetic Lagrangian defined earlier, and $\mathcal{L}^m$ is the dynamical material Lagrangian given by:

$$\mathcal{L}^m = -\rho(\nu, s) + \Lambda_0(u^i u_i - c^2) - \nu^i \partial_i \Lambda_1 + \Lambda_2 u^i \partial_i s + \Lambda_3 u^i \partial_i X + \mathcal{L}^a \tag{109}$$

Here, $\rho(\nu, s)$ is the **thermodynamic internal energy density** of the fluid and Λ_i are the Lagrange multipliers that impose the conditions mentioned earlier, equations 104 to 107.

The matter Lagrangian $\mathcal{L}^a$ is the anisotropic contribution describing the nature of material elements in the medium and has the form $\mathcal{L}^a = \mathcal{L}^a(\nu, \partial_i \nu, u^i, \partial_j u^i, , \Psi^A, \partial_i \Psi^A)$

5.2 Equations of Motion

In this section, we derive the equations of motion corresponding to the total Lagrangian for the closed system. From $\mathcal{L}^e$, we just get back the Maxwell's equations as the Euler-Lagrange equations, and the ones corresponding to the Lagrange multipliers give us our conservation equations. In addition to this, we have the variation of $\mathcal{L}$ with respect to Ψ^a, ν and u^i which give us the following:

$$\frac{\delta \mathcal{L}^{\mathrm{a}}}{\delta \Psi^A} + \frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial \Psi^A} = 0 \tag{110}$$

$$-\nu u^i \partial_i \Lambda_1 = \nu \frac{\partial \rho}{\partial \nu} - \nu \frac{\delta \mathcal{L}^{\mathrm{a}}}{\delta \nu} - \nu \frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial \nu} \tag{111}$$

$$\partial_i \Lambda_1 - \Lambda_2 \partial_i s - \Lambda_3 \partial_i X = 2\Lambda_0 u_i + \frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial u^i} + \frac{\delta \mathcal{L}^{\mathrm{a}}}{\delta u^i} \tag{112}$$

Using the Gibbs law of thermodynamics, we introduce the temperature T and pressure p of the material

$$\mathrm{d}\left(\frac{\rho}{\nu}\right) = T\ \mathrm{d}s - p\ \mathrm{d}\left(\frac{1}{\nu}\right) \tag{113}$$

From this, we get $\nu\left(\frac{\partial \rho}{\partial \nu}\right) = \rho + p$.

Substituting this into 111 and using equations 104 to 107, we can simplify the $\mathcal{L}^m$ tensor to the following form

$$\mathcal{L}^m = \tilde{p} + \mathcal{L}^a \tag{114}$$

$$\tilde{p} = p - \nu\frac{d\mathcal{L}^e}{d\nu} - \nu\frac{\delta\mathcal{L}^a}{\delta\nu} \tag{115}$$

Where we defined $\tilde{p}$ as the effective pressure.

5.3 Canonical and BR tensors for the matter subsystem

We use the same formalism as before and substitute the matter Lagrangian $\mathcal{L}^m$ into the general definition of the canonical momentum tensor. We obtain the canonical energy–momentum tensor for matter $\overset{m}{\Sigma}{}_i^j$.

$$\overset{m}{\Sigma}{}_i^j = -2\Lambda_0 u_i u^j - u^j\left(\frac{\delta\mathcal{L}^{\mathrm{e}}}{\delta u^i} + \frac{\delta\mathcal{L}^{\mathrm{a}}}{\delta u^i}\right) - \delta_i^j\tilde{p} + \overset{a}{\Sigma}{}_i^j \tag{116}$$

The anisotropic energy-momentum tensor $\overset{a}{\Sigma}{}_i^j$ is defined as

$$\overset{a}{\Sigma}{}_i^j = \frac{\partial\mathcal{L}^{\mathrm{a}}}{\partial\left(\partial_j\Psi^A\right)}\partial_i\Psi^A + \frac{\partial\mathcal{L}^{\mathrm{a}}}{\partial\left(\partial_j u^k\right)}\partial_i u^k + \frac{\partial\mathcal{L}^{\mathrm{a}}}{\partial\left(\partial_j\nu\right)}\partial_i\nu - \delta_i^j\mathcal{L}^{\mathrm{a}} \tag{117}$$

Using 112 and the equations of motion obtained, we get

$$-2\Lambda_0 = (\rho + \tilde{p})/c^2 + \left(\frac{u^i}{c^2}\right)\left(\frac{\partial\mathcal{L}^e}{\partial u^i} + \frac{\delta\mathcal{L}^{\mathrm{a}}}{\delta u^i}\right) \tag{118}$$

Substituting this into the equation we obtained earlier for $\overset{mj}{\Sigma}_i$ and eliminating the Lagrange multipliers, we get

$$\overset{mj}{\Sigma}_i = \rho\frac{u_i u^j}{c^2} - P_i^j\tilde{p} + \overset{a\,j}{\Sigma}_i + \overset{\mathrm{a}j}{i} - u^j P_i^k\left(\frac{\partial\mathcal{L}^{\mathrm{e}}}{\partial u^k} + \frac{\delta\mathcal{L}^{\mathrm{a}}}{\delta u^k}\right) \tag{119}$$

Here, P_i^j is the same projector defined in section 4.

Our next goal is to construct the BR tensor $\overset{mj}{\sigma}_i$ for the material. To do this (from equation 11), we require the forms of the corresponding spin density and its derivatives. We derive them using equations 2 and 6 and write the obtained equations below:

$$\overset{mk}{S}_{ij} = \frac{\partial\mathcal{L}^{\mathrm{a}}}{\partial\left(\partial_k\Psi^A\right)}\left(s_{ij}\right)^A{}_B\Psi^B + 2\frac{\partial\mathcal{L}^{\mathrm{a}}}{\partial\left(\partial_k u^{[i}\right)}u_{j]} \tag{120}$$

From the Euler-Lagrange equations for a closed system and the angular momentum identity, we obtain

$$\partial_k\overset{mk}{S}_{ij} = 2\overset{m}{\Sigma}_{[ij]} + 2\frac{\partial\mathcal{L}^{\mathrm{a}}}{\partial u^{[i}}u_{j]} - \frac{\delta\mathcal{L}^a}{\delta\Psi^A}(s_{ij})^A_B\Psi^B \tag{121}$$

Calculating the BR tensor from these equations, we get

$$\overset{mj}{\Sigma}_i = \rho\frac{u_i u^j}{c^2} - P_i^j\tilde{p} - u^j P_i^k\frac{\partial\mathcal{L}^e}{\partial u^k} - u^{(j}P_{i)}^k\frac{\delta\mathcal{L}^{\mathrm{a}}}{\delta u^k} + \overset{a\,j)}{\Sigma}_{(i} + \frac{1}{2}\frac{\delta\mathcal{L}^a}{\delta\Psi^A}(s_{ij})^A_B\Psi^B + \frac{1}{2}\partial_k\left(\overset{mjk}{S}_i + \overset{mkj}{S}_i\right) \tag{122}$$

The important thing to note here is that the BR tensor for matter depends on both the material fields Ψ^A **as well as the electromagnetic field** F_{ij} (through terms containing $\mathcal{L}^e$)

5.4 Total BR tensor of the closed system

Here, we find the total BR tensor of the closed system containing the electromagnetic field subsystem and the material medium. In section 3, we found that the

Minkowski tensor is the BR tensor for the electromagnetic field subsystem when there are no currents.

The total BR tensor $\overset{t}{\sigma}{}_i^{\,j}$ of the closed system is obtained by the sum of the BR tensor of the material medium and the Minkowski tensor for the electromagnetic field.

$$\overset{t}{\sigma}{}_i^{\,j} = \Theta_i^j + \overset{m}{\sigma}{}_i^{\,j} \tag{123}$$

We can see that this formulation is valid because we can consider each subset to be an open system of a total closed system. Hence, we can use the Minkowski tensor developed earlier for the electromagnetic wave as an open system.

The total BR tensor is the physically relevant quantity to describe the energy, momentum and angular momentum content of the closed system, since it always satisfies a conservation equation $\partial_j \overset{t}{\sigma}{}_i^{\,j} = 0$. For the associated angular momentum $\overset{t}{l}{}_{kl}^{\,j} = x_k \overset{t}{\sigma}{}_l^{\,j} = x_l \overset{t}{\sigma}{}_k^{\,j}$, we have $\partial_j \overset{t}{l}{}_{kl}^{\,j} = 0$

The BR tensor is symmetric (by construction) and depends on all the dynamical fields of the closed system.

6 General Abraham Decomposition of the total BR tensor

In the previous section, we obtained the total BR tensor $\overset{t}{\sigma}{}_i^{\,j}$ for the electromagnetic field and dynamical medium as a closed system, as the sum of the Minkowski tensor for the electromagnetic fields and the BR tensor for the material fields. We will now rewrite this sum using the Abraham tensor.

We can apply the angular momentum identity (17) obtained from the balance equations and apply it on only the Electromagnetic Lagrangian $\mathcal{L}^e$. Using the equations of motion to cancel some terms in the expression, we obtain the asymmetric part of the Electromagnetic BR tensor as:

$$\Theta_{[ij]} = \frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial u^{[i}} u_{j]} - \frac{1}{2} \frac{\delta \mathcal{L}^{\mathrm{a}}}{\delta \Psi^A} \left(s_{ij}\right)^A{}_B \Psi^B \tag{124}$$

We substitute this into the relation between the Minkowski and Abraham tensors, to obtain

$$\Omega_i{}^j = \Theta_i{}^j - u^j P_i{}^k \frac{\partial \mathcal{L}^{\mathrm{e}}}{\partial u^k} + \frac{1}{2}\frac{\delta \mathcal{L}^{\mathrm{a}}}{\delta \Psi^A}\left(s_i^j\right)^A{}_B \Psi^B \tag{125}$$

$$+ \frac{1}{c^2} u^k u^{(j} \frac{\delta \mathcal{L}^{\mathrm{a}}}{\delta \Psi^A}\left(s_{i)k}\right)^A{}_B \Psi^B \tag{126}$$

Thus, we can conveniently decompose the total BR tensor as

$$\overset{t}{\sigma}{}_i^{\,j} = \Omega_i^{\,j} + \overset{m}{\kappa}{}_i^{\,j} \tag{127}$$

where $\overset{m}{\kappa}{}_i^{\,j} = \overset{t}{\sigma}{}_i^{\,j} - \Omega_i^{\,j}$ is the *kinetic energy-momentum tensor of matter*, except for the effective pressure $\tilde{p}$ term, contains only matter field dependent terms. However, the magneto- and electrostriction (expansion/contraction of materials with zero polarization) effects can be ignored in most practical cases, making $\tilde{p} = 0$. Thus $\overset{m}{\kappa}{}_i^{\,j}$ will contain only a material field dependence.

Despite being physically equivalent (since the physically measurable quantity is the total BR tensor), the Abraham Decomposition has more benefits compared to the Minkowski Decomposition, in terms of interpretability and the properties of their individual tensors.

First, both the terms in the Abraham Decomposition are individually symmetric tensors (the total BR tensor is clearly symmetric as we are considering a closed system). Secondly, due to the fact that the two terms contain only Electromagnetic field-dependent and material field-dependent terms respectively (after taking $\tilde{p} = 0$), the continuity equation for the system can be conveniently written as

$$\partial_j \Omega_i^{\,j} = -\partial_j \overset{m}{\kappa}{}_i^{\,j} \tag{128}$$

Which can be interpreted as energy/momentum transfers between two "almost decoupled" (fully decoupled if $\tilde{p} = 0$) systems. An analogous interpretation in the Minkowski decomposition is not possible, as the BR tensor of matter always contains the field strength tensor F_{ij}.

7 Conclusion

After developing the general Lagrangian formalism for the electromagnetic in matter using classical field theory, we examined two different systems: the first being an *open* system with the material fields taken to be external fields in the background, and the second being a *closed* system taking into account both the electromagnetic and material fields in the total Lagrangian and the dynamic coupling between the two types of fields.

In the open system with non-dynamic, external material fields, the conservation of the Minkowski tensor was determined by the symmetries of the background medium. We showed that the macroscopic Maxwell's equations implied balance equations for the Minkowski energy-momentum and Minkowski angular momentum, and demonstrated that the asymmetry in the Minkowski tensor is necessary in order to correctly describe the interaction of light with a general medium. In contrast, the Abraham tensor for this system was *not* conserved when the medium has symmetries. To resolve this problem, an ad-hoc 'Abraham force' can be introduced, but there are no observable consequences with the theoretical introduction of such a force; experiments aiming to measure the Abraham force can only verify the form of the balance equation, not distinguish between the Abraham and Minkowski tensors, since we are simply re-balancing the contributions of the terms within the total balance equation when we decide to measure a particular term, which will clearly favor the *a priori* set distribution of terms - which we intended to determine *a posteriori* by experiment.

When we allow the material to move in response to the dynamic interaction with the electromagnetic field and consider the entire set of electromagnetic and material fields to be a closed system, the Abraham tensor has a more convenient interpretation, even though we show that the total BR tensor for the combined system has equivalent decompositions in which either the Minkowski tensor or the Abraham tensor appears as a term. The Abraham Decomposition is a more favourable choice here since the two terms obtained are only dependent on the Electromagnetic and the material fields, respectively (if we also have the effective pressure to be zero, which is true for most situations). However, despite the convenience of the Abraham decomposition, it is important to note that the experimentally measurable quantity is the total BR tensor, and any decomposition of the same is arbitrary and one cannot measure these component terms via experiment.

Thus, we can conclude that instead of there being a 'correct' energy-momentum tensor for electromagnetic fields in a medium, the division of momentum between the material fields and electromagnetic fields is arbitrary and cannot be measured in an exclusive fashion without selecting a division *a priori*. However, some decompositions might be more convenient than others, depending upon the choice and formulation of the system.

8 Tutorial Question

Question

Starting from the lagrangian density $\mathcal{L} = -\frac{1}{4}F_{ij}H^{ij} = -\frac{1}{8}\chi^{ijkl}F_{ij}F_{kl}$, derive the macroscopic Maxwell equations in the absence of free charges and in a non-dissipative medium (i.e. $\chi^{ijkl} = \chi^{klij}$).

Useful formulae:

1. $F_{ij} = \partial_i A_j - \partial_j A_i$, $A_i = (c\phi, \mathbf{A})$ (4-potential formulation of electromagnetism)
2. $H^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}$ is the definition of the excitation 4-tensor H^{ij}.
3. Euler-Lagrange equations in field theory: $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \right) = 0$ for all internal fields ϕ of an open system.
4. We are using the metric $g_{ij} = \mathrm{diag}(c^2, -1, -1, -1)$, and hence our strength and excitation tensors are:

$$F_{ij} = \begin{pmatrix} 0 & E_x c & E_y c & E_z c \\ -E_x c & 0 & -B_z & B_y \\ -E_y c & B_z & 0 & -B_x \\ -E_z c & -B_y & B_x & 0 \end{pmatrix} \tag{129}$$

$$H^{ij} = \begin{pmatrix} 0 & -D_x/c & -D_y/c & -D_z/c \\ D_x/c & 0 & H_z & -H_y \\ D_y/c & -H_z & 0 & H_x \\ D_z/c & H_y & -H_x & 0 \end{pmatrix} \tag{130}$$

Answer

The derivation must reach either the Bianchi identity or dual field strength form of the homogeneous Maxwell equations, and the divergence of the excitation tensor being 0 ($\partial_j H^{ij} = 0$) for the inhomogeneous Maxwell equations.

Solution

For the homogeneous Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, we do not need the lagrangian density at all. They only involve spacetime derivatives of the EM field, and can be derived essentially from the fact that the 4-potential formulation is valid for electromagnetism.

$$\partial_i F_{jk} = \partial_i \partial_j A_k - \partial_i \partial_k A_j$$

We need to equate these derivatives to something to get an identity, and it is highly suggestive from their form that we use the symmetry of mixed partials, so we add similar terms with indices cyclically permuted to obtain an identity.

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = \tag{131}$$

$$\partial_i \partial_j A_k - \partial_i \partial_k A_j + \partial_j \partial_k A_i - \partial_j \partial_i A_k + \partial_k \partial_i A_j - \partial_k \partial_j A_i = 0 \tag{132}$$

$$\implies \partial_{(i} F_{jk)} = 0 \tag{133}$$

Note that in general $X_{(ij)} = \frac{1}{2}(X_{ij} + X_{ji})$ represents the symmetrisation of a tensor over the indices in the round brackets. Similarly, $X_{[ij]} = \frac{1}{2}(X_{ij} - X_{ji})$ is the antisymmetrisation.

Thus we have the Bianchi identity for the stress-energy tensor, and this is equivalent to the homogenous Maxwell equations, as is clear by writing this in terms of the dual field strength tensor.

$$\partial_{(i}F_{jk)} = 0 \tag{134}$$
$$\implies \varepsilon^{lijk}\partial_i F_{jk} = 0 \tag{135}$$
$$\implies \partial_i \underbrace{\left(\frac{1}{2}\varepsilon^{lijk}F_{jk}\right)}_{\tilde{F}^{li}} = 0 \tag{136}$$

The dual field strength tensor $\tilde{F}^{ij}$ has the form:

$$\tilde{F}^{ij} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \tag{137}$$

And thus $\partial_i \tilde{F}^{ij} = 0$ are clearly the homogeneous Maxwell equations.

The inhomogeneous Maxwell equations without free charges become

$$\nabla \cdot \mathbf{D} = 0 \tag{138}$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{139}$$

To obtain the inhomogeneous Maxwell equations, we use the Euler-Lagrange equations of motion for classical field theory. In field theory, the lagrangian density is a function of fields over spacetime and their spacetime derivatives, so

$\mathcal{L} \equiv \mathcal{L}(\phi^A, \partial_i \phi^A)$. Hence to minimise the action and obtain the Euler-Lagrange equations, we take derivatives with respect to ϕ^As and $\partial_i \phi^A$s - the fields replace the generalised coordinates from classical mechanics. Note that $\partial_i \phi \equiv \frac{\partial \phi}{\partial x^i}$, where $x^i = (ct, \mathbf{x})$.

Thus the Euler-Lagrange equations for our problem:

$$\frac{\delta \mathcal{L}}{\delta A_i} = 0$$

Where we use the variational derivative, which can be written in the form:

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \right)$$

Note that this only becomes 0 when ϕ is an internal - dynamical - field of the system. Our lagrangian includes the fields A_i but also χ^{ijkl}. However, χ^{ijkl} is not part of our system, it is an external field - the medium acts as a background to the system of EM fields.

Using the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = 0 \tag{140}$$

$$\implies \frac{1}{8} \partial_\nu \left(\frac{\partial(\chi^{ijkl} F_{ij} F_{kl})}{\partial(\partial_\nu A_\mu)} \right) = 0 \tag{141}$$

$$\implies \partial_\nu \left[\chi^{ijkl} \frac{\partial}{\partial(\partial_\nu A_\mu)} (F_{ij} F_{kl}) \right] = 0 \tag{142}$$

To use the product rule, first we'll evaluate:

$$\frac{\partial F_{ij}}{\partial(\partial_\nu A_\mu)} = \frac{\partial}{\partial(\partial_\nu A_\mu)} (\partial_i A_j - \partial_j A_i) \tag{143}$$

$$= \delta_i^{\ \nu} \delta_j^{\ \mu} - \delta_j^{\ \nu} \delta_i^{\ \mu} \tag{144}$$

Thus the E-L equation simplifies to

$$\partial_\nu \left[\chi^{ijkl}(\delta_i^{\ \nu}\delta_j^{\ \mu} - \delta_j^{\ \nu}\delta_i^{\ \mu})F_{kl} + \chi^{ijkl}(\delta_k^{\ \nu}\delta_l^{\ \mu} - \delta_l^{\ \nu}\delta_k^{\ \mu})F_{ij}\right] = 0 \tag{145}$$

$$\implies \partial_\nu \left[\chi^{\nu\mu kl}F_{kl} - \chi^{\mu\nu kl}F_{kl} + \chi^{ij\nu\mu}F_{ij} - \chi^{ij\mu\nu}F_{ij}\right] = 0 \tag{146}$$

$$\implies \partial_\nu \left[2\chi^{\nu\mu kl}F_{kl} + 2\chi^{\nu\mu ij}F_{ij}\right] = 0 \tag{147}$$

$$\implies \partial_\nu (8H^{\nu\mu}) = 0 \tag{148}$$

$$\implies \partial_j H^{ij} = 0 \tag{149}$$

And hence we have obtained the Lorentz-covariant form of the inhomogeneous Maxwell equations, which can be easily reduced to the 3-vector form. Note that important here were the following properties of the constitutive tensor χ:

1. Antisymmetry in the first two indices, $\chi^{ijkl} = -\chi^{jikl} = \chi^{jilk}$

   This can be seen from the fact that both F_{ij} and H^{ij} are antisymmetric, and that the equation $H^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl}$ holds.

2. Symmetry in the exchange of the first pair of indices with the second pair, $\chi^{ijkl} = \chi^{klij}$.

   This is true only for non-dissipative media, and the general proof is out of the scope of this presentation or this question, hence this was given in the question itself.

References

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Contributions

Introduction and Interpretation : Bhavana

Section 2 : Samyak and Namita

Section 3 : Rehmat

Section 4 : Rehmat and Namita

Section 5 : Bhavana

Section 6 : Kaustav

Conclusion and Summary : Kaustav

Tutorial Question : Rehmat and Samyak