Test title

Definitions and Notation

Let:

- P = **Total population size**
- H = **Number of households**
- M_p = **Person-level mean household size**

  For every person j in a population P living in a household of m_j members, the **person-level** mean household size is a simple average across person-level observations:

  $$M_p = \frac{1}{P}\sum_{j=1}^{H} m_j^2 \tag{1}$$

- M_h = **Household-level mean household size**

  The **household-level** mean household size is a simple average across households as individual observations:

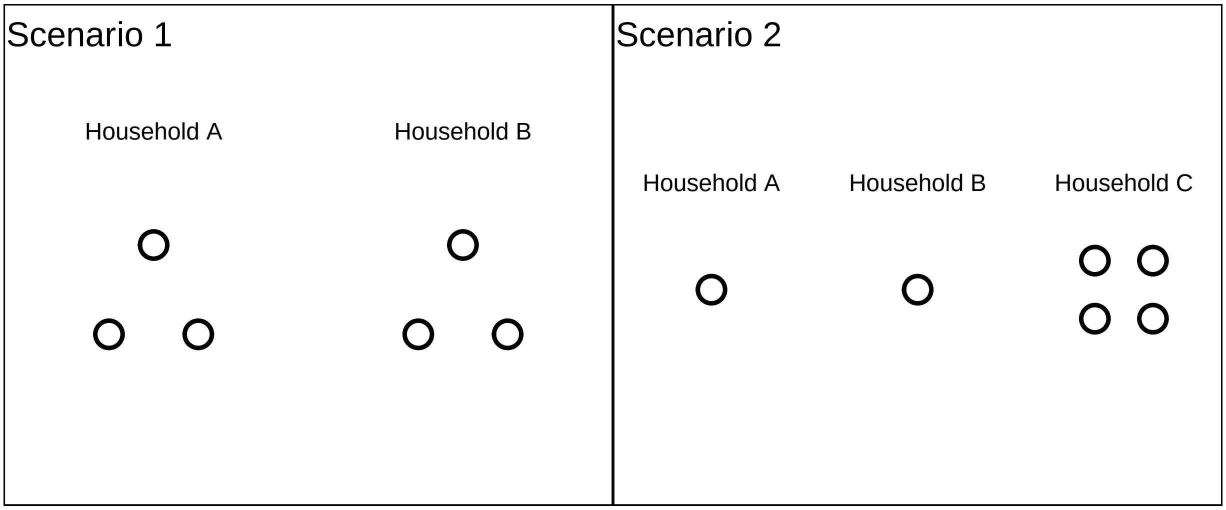
  $$M_h = \frac{P}{H} \tag{2}$$

Context

Equation (2) shows that household-level mean household size and population is adequate to deduce the number of households. However, person-level household size does not lend itself to this equivalency: the same person-level mean and population can be consistent with multiple values for the number of households.

Figure 1.2 illustrates this point in a hypothetical population with $P = 6$ and $M_p = 3$. In Scenario 1, average person-level household size is $\frac{3^2+3^2}{6} = 3$ and there are 2 households. In Scenario 2, average person-level household size is $\frac{1^2+1^2+4^2}{6} = 3$ and there are 3 households. Thus, population and person-level household size alone do not singularly determine the number of households.

Figure 1.2



Circles represent individual people. In both scenarios, P = 6 and M_p = 3, yet the number of households differs.

Assertion

There is a range of possible number of households H consistent with a known population size P and person-level average household size M_p. The range on H is bounded by a minumum of

$$H_{min} = \frac{P}{M_p} \tag{1}$$

and a maximum of

[FILL IN HERE]

Lemma 1: Minimum Number of Households

The minimum number of households consistent with a population size P and person-level average household size M_p is:

$$H_{\text{min}} = \frac{P}{M_p} \tag{2}$$

Proof

Suppose the population P is distributed into H households with the measured household size in the jth household equal to m_j. As defined above, **person-level** mean household size is:

$$M_p = \frac{1}{P} \sum_{j=1}^{H} m_j^2 \tag{3}$$

Any configuration of P individuals across households can be transformed into any other configuration through a finite sequence of single-person transfers. The initial count of households, H, need not match the final count of households, H': we conceptualize a 'reservoir' of zero-person households to account for changes between H and H'. Forming a new household can be conceptualized as moving an individual to a zero-person household, which remains uncounted unless populated. Conversely, a household can be dissolved by moving all of its members to other households, thus turning a household with a positive number of members into a zero-person household. For example, in Figure 1.2, once can incrementally create Scenario 2 after starting from Scenario 1 through the following steps:

1. Move one individual from Household A to a newly formed Household C.
2. Move another individual from Household A to Household C.
3. Move two individuals from Household B to Household C.

Next, we calculate how the person-level mean household size changes after a single-person transfer. Suppose we begin with an arbitrary configuration of P individuals in H households, with each household m_j containing some whole number of members $m_1, m_2, \ldots, m_{H-1}, m_H$. We define a single-person transfer as a move of one individual from household $j = k$ to household $j = k+1$. Mean household size M_p changes from:

$$M_p = \frac{1}{P}\left(\sum_{j=1}^{k-1} m_j^2 + m_k^2 + m_{k+1}^2 + \sum_{j=k+2}^{H} m_j^2\right) \tag{4}$$

to

$$M_p' = \frac{1}{P}\left(\sum_{j=1}^{k-1} m_j'^2 + m_k'^2 + m_{k+1}'^2 + \sum_{j=k+2}^{H} m_j'^2\right) \tag{5}$$

Person-level household size increases if $M_p' - M_p > 0$, remains the same if $M_p' - M_p = 0$, and shrinks if $M_p' - M_p < 0$. Since all household sizes $m_j = m_j'$ are unaffected by the shift, aside from the special case of $j = k$ and $j = k+1$, the remaining terms cancel out due to equivalence.

$$\begin{aligned} M_p' - M_p &= \frac{1}{P}\left(\sum_{j=1}^{k-1} m_j^2 + m_k^2 + m_{k+1}^2 + \sum_{j=k+2}^{H} m_j^2\right) - \frac{1}{P}\left(\sum_{j=1}^{k-1} m_j'^2 + m_k'^2 + m_{k+1}'^2 + \sum_{j=k+2}^{H} m_j'^2\right) && (6) \\ &= \frac{1}{P}\Big((m_k^2 + m_{k+1}^2) - (m_k'^2 + m_{k+1}'^2)\Big) && (3) \\ & && (7) \end{aligned}$$

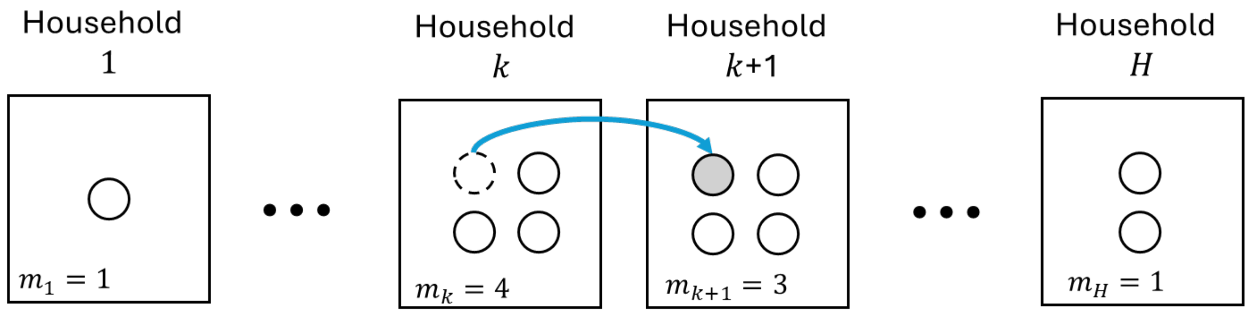
We now transition to exploring how the value $M_p' - M_p$ varies under each of the three circumstances:

1. $m_{k+1} = m_k - 1$
2. $m_{k+1} < m_k - 1$
3. $m_{k+1} > m_k - 1$

Since a single-person transfer involves a move of one person from household k to household $k+1$, the post-transfer number of household members in household k is $m_k' = m_k - 1$ while the post-transfer number of household members in household $k+1$ is $m_{k+1}' = m_{k+1} + 1$.

Case 1: a plain-English explanation of what this type of move is

Figure 1.3: $m_{k+1} = m_k - 1$



An individual is moved from household k, with $m_k = 4$ members, to household $k+1$, with $m_{k+1} = 3$ members. This results in a configuration where $m'_k = 3$ and $m'_{k+1} = 4$.
In this particular case, the initial number of occupants of household k is exactly one larger than the initial number of occupants in household $k+1$. Thus:

$$
\begin{aligned}
M'_p - M_p &= \frac{1}{P}\Bigg((m_k^2 + m_{k+1}^2) - (m_k'^2 + m_{k+1}'^2)\Bigg) && (8)\\
&= \frac{1}{P}\Bigg((m_k^2 + (m_k - 1)^2) - ((m_k - 1)^2 + m_k^2)\Bigg) && (9)\\
&= 0\Bigg) && (10)\\
& && (11)
\end{aligned}
$$

add case 1, 2, 3 here

Lemma 2: Maximum Number of Households

Lemma:
The maximum number of households consistent with a population size P and person-level average household size M_p is:

$$TBD \tag{12}$$

Proof:
In this section, we show that the sparsest configuraiton of households is given in a case where you produce some number of 1-person households and then put the rest of the population into one remaining large household.

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conclusion

Reiterate the minima and maxima, take it back to the example at the begining and show that the two configurations listed in Figure 1.2 are actually the two special cases of minima and maxima.