1 EQUATIONS TO OBTAIN SATELLITE COORDINATES FROM THE KEPLERIAN BROADCAST EPHEMERIS

The algorithm to calculate the satellite coordinates starts with the computation of the mean anomaly (SEEBER, 2003):

$$M = M_0 + n(t - t_e), \tag{01}$$

at the epoch of interest from the semimajor axis

$$a \begin{cases} (\sqrt{a})^2 & (LNAV, INAV\ and\ FNAV) \\ a_{ref} + \Delta a & (CNAV)' \end{cases} \tag{02}$$

the perturbed mean motion (n) will be (equation 09):

$$n = \sqrt{\frac{GM}{a^3}} + \begin{cases} \Delta n & (LNAV, INAV\ and\ FNAV) \\ \Delta n + \Delta n(t - t_e) & (CNAV)' \end{cases} \tag{03}$$

the solution of Kepler's equation can be obtained from equation (10), after it will be computed the true anomaly (υ):

$$\upsilon = 2tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{(E)}{2}\right), \tag{04}$$

and the unperturbed argument of latitude $\bar{u} = \omega + \upsilon$ are obtained. It is used to evaluate the periodic corrections presented on equations 5, 6, and 7:

$$\delta_r = C_{rs}\sin(2\bar{u}) + C_{us}\sin(2\bar{u}), \tag{05}$$
$$\delta_u = C_{us}\sin(2\bar{u}) + C_{uc}\sin(2\bar{u}), \tag{06}$$
$$\delta_i = C_{is}\sin(2\bar{u}) + C_{ic}\sin(2\bar{u}), \tag{07}$$

δ_r is the radius correction, δ_u is the correction of the argument of latitude and δ_i is the inclination correction. The parameters corrected by the perturbations are given by:

$$r = a(1 - e\cos E) + \delta_r, \tag{08}$$
$$u = \bar{u} + \delta_u, \tag{09}$$
$$i = i_0 + \frac{di}{dt}(t - t_e) + \delta_i, \tag{10}$$

r is the perturbed radius, u is the perturbed argument of latitude and i is the perturbed inclination.

The Greenwich longitude of the ascending node (λ_Ω) is given by equation (11) and the nodal rate ($\dot{\Omega}$) is either given directly or obtained from the reference for the CNAV, xp and yp are the orbit-plane position (equations 13 and 14) coordinates before transformation to the ECEF (Earth Centered Earth Fixed) coordinate system

$$\lambda_\Omega = \Omega_0 + (\dot{\Omega} - \omega_0)(t - t_a) - \omega_0 t, \tag{11}$$
$$\dot{\Omega} = \dot{\Omega}_{ref} + \Delta\dot{\Omega}, \tag{12}$$

$$xp = r\cos u, \tag{13}$$
$$yp = r\sin u, \tag{14}$$

the earth-fixed position is finally given by equation 15:

$$r_{ECEF} = R_3(-\lambda_\Omega)R_1(-i)\begin{bmatrix} xp \\ yp \\ 0 \end{bmatrix}, \tag{15}$$

complementary to the GNSS satellite position, its velocity is required as part of the navigation solution process. The velocity can be found by differentiation of the above expressions for the position with respect to time t. The resulting ECEF velocity equations are given by 16, 17, and 18 (THOMPSON et al., 2019):

$$Vx = -xp\dot{\Omega}_k \sin\Omega - yp\left(\dot{\Omega}_k \cos\Omega\cos i - \frac{di_k}{dt}\sin\Omega\sin i\right) + Vx'\cos\Omega - Vy'\sin\Omega\cos i, \tag{16}$$
$$Vy = xp\,\Omega_k \cos\Omega - yp\left(\dot{\Omega}_k \sin\Omega\sin i - \frac{di_k}{dt}\cos\Omega\sin i\right) + Vx'\sin\Omega - Vy'\cos\Omega\cos i, \tag{17}$$
$$Vz = yp\left(\frac{di_k}{dt}\right)\cos i + Vy'\sin i, \tag{18}$$

to fully implement these equations, several additional derivatives are required. These derivatives (equations 19 to 26) are functions of the parameters of the broadcast navigation message and parameters computed by the SV position equations

$$\dot{E} = \frac{n}{1 - e\cos E}, \tag{19}$$
$$\dot{v} = \frac{\dot{E}\sqrt{1-e^2}}{1 - e\cos E}, \tag{20}$$
$$\frac{di_k}{dt} = \frac{di}{dt} + 2\dot{v}(C_{is}\cos 2u - C_{ic}\sin 2u), \tag{21}$$
$$\dot{u} = \dot{v} + 2\dot{v}(C_{us}\cos 2u - C_{uc}\sin 2u), \tag{22}$$
$$\dot{r} = eA\dot{E}\sin E + 2\dot{v}(C_{rs}\cos 2u - C_{rc}\sin 2u), \tag{23}$$
$$\dot{\Omega}_k = \dot{\Omega} + \dot{\Omega}_k, \tag{24}$$
$$Vx' = \dot{r}\cos\dot{u} - \dot{r}\sin\dot{u}, \tag{25}$$
$$Vy' = \dot{r}\sin\dot{u} + \dot{r}\cos\dot{u}. \tag{26}$$

2 EQUATIONS TO OBTAIN SATELLITE COORDINATES FROM THE CATERIAN BROADCAST EPHEMERIS

Following 6 first order differential equations are needed to compute the satellite positions in any time:

$$\frac{dx}{dt} = v_x(t) \tag{27}$$
$$\frac{dy}{dt} = v_y(t) \tag{28}$$

$$\frac{dz}{dt} = v_z(t) \tag{29}$$

$$\frac{dv_x}{dt} = -\frac{\mu x}{r^3} + \frac{3}{2} C_{20} \frac{\mu a_e^2 x}{r^5}\left(1 - \frac{5z^2}{r^2}\right) + w_e^2 x + 2w_e \frac{dy}{dt} + \ddot{x} \tag{30}$$

$$\frac{dv_y}{dt} = -\frac{\mu y}{r^3} + \frac{3}{2} C_{20} \frac{\mu a_e^2 y}{r^5}\left(1 - \frac{5z^2}{r^2}\right) + w_e^2 y + 2w_e \frac{dx}{dt} + \ddot{y} \tag{31}$$

$$\frac{dv_z}{dt} = -\frac{\mu z}{r^3} + \frac{3}{2} C_{20} \frac{\mu a_e^2 z}{r^5}\left(3 - \frac{5z^2}{r^2}\right) + \ddot{z} \tag{32}$$

Following the Runge Kutta-4 formula presented in section XX. Detailed algorithm implementation steps in Python 3 are given below. The first values (w10, w20, w30, w40, w50 and w60) in the algorithm are obtained from ephemeris considering Earth rotation during the time of signal travel from satellite to the user. For each time interval, that will be determined (i.e. 1ms, 10ms, 1 sec, etc.), between desired time and tb, integration is applied:

$$t_0 = t_b;\ w_{10} = x;\ w_{20} = y; w_{30} = z, w_{40} = \dot{x}, w_{50} = \dot{y}, w_{60} = \dot{z} \tag{33}$$

$$r = \sqrt{w_{10}^2 + w_{20}^2 + w_{30}^2} \tag{34}$$

$$k_{11} = hw_{40} \tag{35}$$

$$k_{12} = hw_{50} \tag{36}$$

$$k_{13} = hw_{60} \tag{37}$$

$$k_{14} = h(-\frac{\mu w_{10}}{r^3} + \frac{3}{2} C_{20} \frac{\mu a_e^2 w_{10}}{r^5}\left(1 - \frac{5w_{30}^2}{r^2}\right) + w_e^2 w_{10} + 2w_e w_{50} + \ddot{x} \tag{38}$$

$$k_{15} = h(-\frac{\mu w_{20}}{r^3} + \frac{3}{2} C_{20} \frac{\mu a_e^2 w_{20}}{r^5}\left(1 - \frac{5w_{30}^2}{r^2}\right) + w_e^2 w_{20} + 2w_e w_{40} + \ddot{y} \tag{39}$$

$$k_{16} = h(-\frac{\mu w_{30}}{r^3} + \frac{3}{2} C_{20} \frac{\mu a_e^2 w_{30}}{r^5}\left(1 - \frac{5w_{30}^2}{r^2}\right) + \ddot{z} \tag{40}$$

$$k_{21} = h\left(w_{40} + \frac{1}{2}k_{14}\right) \tag{41}$$

$$k_{22} = h\left(w_{50} + \frac{1}{2}k_{15}\right) \tag{42}$$

$$k_{23} = h\left(w_{60} + \frac{1}{2}k_{16}\right) \tag{43}$$

$$\begin{aligned} k_{24} = h(&-\frac{\mu\left(w_{10} + \frac{1}{2}k_{11}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{10} + \frac{1}{2}k_{11}\right)}{r^5}\left(1 - \frac{5\left(w_{30} + \frac{1}{2}k_{13}\right)^2}{r^2}\right) \\ &+ w_e^2\left(w_{10} + \frac{1}{2}k_{11}\right) + 2w_e\left(w_{50} + \frac{1}{2}k_{15}\right) + \ddot{x} \end{aligned} \tag{44}$$

$$\begin{aligned} k_{25} = h(&-\frac{\mu\left(w_{20} + \frac{1}{2}k_{12}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{20} + \frac{1}{2}k_{12}\right)}{r^5}\left(1 - \frac{5\left(w_{30} + \frac{1}{2}k_{13}\right)^2}{r^2}\right) \\ &+ w_e^2\left(w_{20} + \frac{1}{2}k_{12}\right) + 2w_e\left(w_{40} + \frac{1}{2}k_{14}\right) + \ddot{y} \end{aligned} \tag{45}$$

$$k_{26} = h(-\frac{\mu\left(w_{30}+\frac{1}{2}k_{13}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{30}+\frac{1}{2}k_{13}\right)}{r^5}\left(3-\frac{5\left(w_{30}+\frac{1}{2}k_{13}\right)^2}{r^2}\right) + \ddot{z} \tag{46}$$

$$k_{31} = h\left(w_{40}+\frac{1}{2}k_{24}\right) \tag{47}$$

$$k_{32} = h\left(w_{50}+\frac{1}{2}k_{25}\right) \tag{48}$$

$$k_{33} = h\left(w_{60}+\frac{1}{2}k_{26}\right) \tag{49}$$

$$\begin{aligned} k_{34} = h(&-\frac{\mu\left(w_{10}+\frac{1}{2}k_{21}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{10}+\frac{1}{2}k_{21}\right)}{r^5}\left(1-\frac{5\left(w_{30}+\frac{1}{2}k_{23}\right)^2}{r^2}\right) \\ &+ w_e^2\left(w_{10}+\frac{1}{2}k_{21}\right) + 2w_e\left(w_{50}+\frac{1}{2}k_{25}\right) + \ddot{x} \end{aligned} \tag{50}$$

$$\begin{aligned} k_{35} = h(&-\frac{\mu\left(w_{20}+\frac{1}{2}k_{22}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{20}+\frac{1}{2}k_{22}\right)}{r^5}\left(1-\frac{5\left(w_{30}+\frac{1}{2}k_{23}\right)^2}{r^2}\right) \\ &+ w_e^2\left(w_{20}+\frac{1}{2}k_{22}\right) + 2w_e\left(w_{40}+\frac{1}{2}k_{24}\right) + \ddot{y} \end{aligned} \tag{51}$$

$$k_{36} = h(-\frac{\mu\left(w_{30}+\frac{1}{2}k_{23}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{30}+\frac{1}{2}k_{23}\right)}{r^5}\left(3-\frac{5\left(w_{30}+\frac{1}{2}k_{23}\right)^2}{r^2}\right) + \ddot{z} \tag{52}$$

$$k_{41} = h(w_{40}+k_{34}) \tag{53}$$

$$k_{32} = h(w_{50}+k_{35}) \tag{54}$$

$$k_{33} = h(w_{60}+k_{36}) \tag{55}$$

$$\begin{aligned} k_{44} = h(&-\frac{\mu\left(w_{10}+\frac{1}{2}k_{31}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{10}+\frac{1}{2}k_{31}\right)}{r^5}\left(1-\frac{5\left(w_{30}+\frac{1}{2}k_{33}\right)^2}{r^2}\right) \\ &+ w_e^2\left(w_{10}+\frac{1}{2}k_{31}\right) + 2w_e\left(w_{50}+\frac{1}{2}k_{35}\right) + \ddot{x} \end{aligned} \tag{56}$$

$$\begin{aligned} k_{45} = h(&-\frac{\mu\left(w_{20}+\frac{1}{2}k_{32}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{20}+\frac{1}{2}k_{32}\right)}{r^5}\left(1-\frac{5\left(w_{30}+\frac{1}{2}k_{33}\right)^2}{r^2}\right) \\ &+ w_e^2\left(w_{20}+\frac{1}{2}k_{32}\right) + 2w_e\left(w_{40}+\frac{1}{2}k_{34}\right) + \ddot{y} \end{aligned} \tag{57}$$

$$k_{46} = h(-\frac{\mu\left(w_{30}+\frac{1}{2}k_{33}\right)}{r^3} + \frac{\frac{3}{2}C_{20}\mu a_e^2\left(w_{30}+\frac{1}{2}k_{33}\right)}{r^5}\left(3-\frac{5\left(w_{30}+\frac{1}{2}k_{33}\right)^2}{r^2}\right) + \ddot{z} \tag{58}$$

$$w_{11} = w_{10} + \frac{1}{6}(k_{11}+2k_{21}+2k_{31}+k_{41}) \tag{59}$$

$$w_{21} = w_{20} + \frac{1}{6}(k_{12}+2k_{22}+2k_{32}+k_{42}) \tag{60}$$

$$w_{31} = w_{30} + \frac{1}{6}(k_{13} + 2k_{23} + 2k_{33} + k_{43}) \tag{61}$$

$$w_{41} = w_{40} + \frac{1}{6}(k_{14} + 2k_{24} + 2k_{34} + k_{44}) \tag{62}$$

$$w_{51} = w_{50} + \frac{1}{6}(k_{15} + 2k_{25} + 2k_{35} + k_{45}) \tag{63}$$

$$w_{61} = w_{60} + \frac{1}{6}(k_{16} + 2k_{26} + 2k_{36} + k_{46}) \tag{64}$$

After all the steps are done until the t of interest, the final coordinates for that epoch is:

$$x_s = w_{11} \tag{65}$$

$$y_s = w_{21} \tag{66}$$

$$z_s = w_{31} \tag{67}$$

$$\dot{x}_s = w_{41} \tag{68}$$

$$\dot{y}_s = w_{51} \tag{69}$$

$$\dot{z}_s = w_{61} \tag{70}$$

3 SATELLITE REFERENCE FRAMES: BODY-FIXED FRAME AND LOCAL ORBITAL FRAME

The local reference frame ($\mathcal{R}_{RTN}$) is defined by the cross-track (N), along-track (T) and radial (R) components. For a satellite in space with a vector $\boldsymbol{r}$ from the center of mass of the Earth to the satellite and an inertial velocity $\boldsymbol{v}$, the corresponding unit vectors in $\mathcal{R}_{RTN}$ (P_{RNT}) are described on equations 71, 72 and 73 (MONTENBRUCK et al., 2015):

$$\mathbf{e_R} = \frac{\mathbf{r}}{|\mathbf{r}|}, \tag{71}$$

$$\mathbf{e_N} = \frac{\mathbf{r} \times \mathbf{v}}{|\mathbf{r} \times \mathbf{v}|}, \tag{72}$$

$$\mathbf{e_T} = \mathbf{e_N} \times \mathbf{e_R}, \tag{73}$$

from the local frame unit vector (P_{RNT}), the body-fixed frame will be given as:

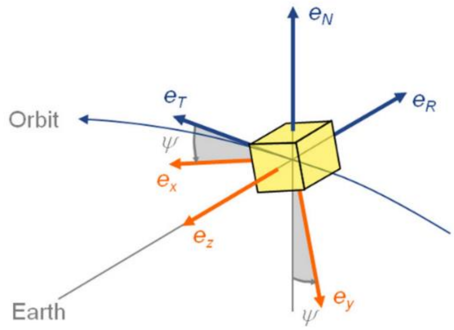
$$\mathbf{P_{BF}} = \mathbf{A_{BF}^{RTN}} \cdot \mathbf{P_{RNT}}, \tag{74}$$

the transformation between the orbital frame and the body-fixed frame is commonly described by a series of three elementary rotations through the angles ϕ (roll), ϑ (pitch) and ψ (yaw):

$$\mathbf{A_{BF}^{RTN}} = \mathbf{R_1}(\phi)\mathbf{R_2}(\vartheta)\mathbf{R_3}(\psi). \tag{75}$$

Considering the GNSS characteristic of nadir-pointing spacecraft, the pitch and roll angles vanish, and the spacecraft attitude can be fully described by the yaw angle. Figure 8 shows ψ as being the angle between the $\mathbf{e_T}$ and $\mathbf{e_{X,BF}}$, which is the body-fixed frame unit vector for the x-axis (MONTENBRUCK et al., 2015).

Figure 1 - Definition of the yaw-angle



Source: Montenbruck et al. (2015)

For a given Sun elevation β above the orbital plane and an orbital angle μ (measured from the midnight point), the yaw angle can so be calculated by equation 76:

$$\psi = \tan^{-1}(-\tan\beta/\ \sin\mu). \tag{76}$$

The geometrical representations of the satellite attitude and the angles β and μ are presented in figure 2. β is the complement of the angle between the sun unit vector and the orbit normal vector (e_N) . μ will be the angle in the orbit plane between the projection of the sun vector onto the orbit plane and the satellite position vector. Crossing the orbital normal with the sun unit vector results in the vector $\vec{n}$, perpendicular to both vectors (PENINA; LISA, 2003):

$$\mathbf{n} = \mathbf{e_N} \times \mathbf{r_{sun}}, \tag{77}$$

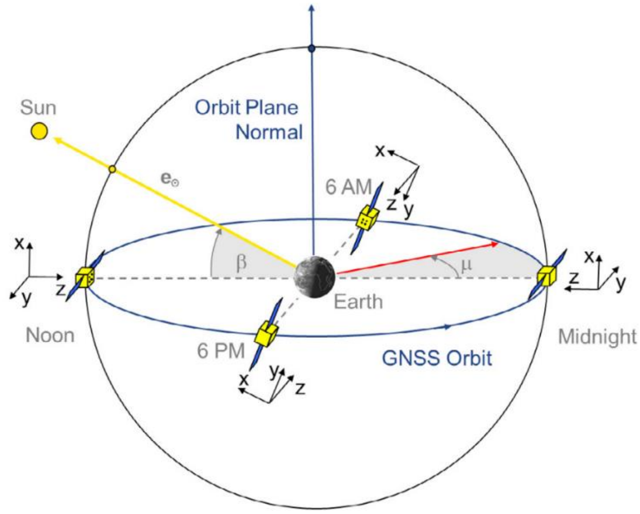
Montenbruck; Gill (2000) described the algorithm to determine the sun vector. Crossing $\vec{n}$ with e_N it will yield the projection of the sun unit vector in the orbit plane ($\boldsymbol{r_{sun}^{proj}}$):

$$\mathbf{r_{sun}^{proj}} = \mathbf{n} \times \mathbf{e_N}. \tag{78}$$

Figure 2 describes the geometry of β and μ which can then be determined using:

$$\alpha = \tan^{-1}\left(\frac{\mathbf{r_{sat}}.\,\mathbf{n}}{\mathbf{r_{sat}}.\,\mathbf{r_{sun}^{proj}}}\right). \tag{79}$$

Figure 1 - GNSS satellite orientation in nominal yaw-steering mode



Source: Montenbruck et al. (2015)