

# hw01\_KalmanFilter

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## 0.1 Name and ID

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## 0.2 Comp 150 Probabilistic Robotics Homework 1:

### 1 Kalman Filter

#### 1.0.1 Import required libraries.

```
[1]: import numpy as np
from numpy.linalg import inv
import matplotlib.pyplot as plt
import seaborn as sns
from statistics import mean
import copy
```

#### 1.1 1. Kalman Filter: Prediction

##### 1.1.1 Question 1.1

What is the minimal state vector for the Kalman filter so that the resulting system is Markovian?

For the Markovian Property to hold, the state vector for the Kalman filter has to include all the information necessary to predict the future state based only on the current state and control inputs. Here, this means that the state vector should include the position ( $x$ ) and velocity ( $v$ ) of the drone.

Therefore, the minimal state vector for the Kalman filter is  $X = [x, v]^T$ .

##### 1.1.2 Question 1.2

Design the state transition probability function  $p(x_t|u_t, x_{t1})$ . The transition function should contain linear matrices A and B and a noise covariance R.

Assuming  $\Delta x$  is a horizontal displacement, the gravity does not impact it. By applying the second law of Newton we have the following equations of motion:

$$x = x_0 + v * t + 0.5 * a * t^2 \text{ (Eq. 1)}$$

$$v = v_0 + a * t \text{ (Eq. 2)}$$

With  $a$  being the acceleration of the drone.

By rewriting this equation in a state representation using the vector  $X$  as defined previously, we have:

$$X_t = A_t * X_{t-1} + B_t * u_t + \epsilon_t \text{ (Eq. 3)}$$

otherwise:

$$X_t = A_t * X_{t-1} + B_t * u_t + G_t * a_t \text{ (Eq. 4)}$$

We note that from the problem enonciation, the control  $u$  is set to 0 initially, meaning that we can set  $B$  to a 2by2 0 matrice for now. We can also note that epsilon is the gaussian random vector that models the randomness of the wind, affecting the acceleration  $a \sim \mathcal{N}(0, \sigma_{\text{wind}}^2 = 1)$ . The random variable epsilon follows a gaussian distribution  $\epsilon \sim \mathcal{N}(0, R)$ , with  $R = G * (G^T) * (\sigma_{\text{wind}}^2)$ .

By replacing Eq.1 and Eq.2 in Eq.3 we have:

$$A = [[1, d_t], [0, 1]]$$

$$B = [[0], [0]]$$

$$G = [[0.5 * d_t * 2], [d_t]] , \text{ and}$$

$$R = [[0.25 * (d_t^4), 0.5 * (d_t^3)], [0.5 * (d_t^3), d_t^2]] * (\sigma_{\text{wind}}^2)$$

We then have  $p(x_t | u_t, x_{t1})$  given by equation 3.4 in the book Probabilistic Robotics using  $A$ ,  $B$  and  $R$ .

$$\begin{aligned} p(x_t | u_t, x_{t-1}) & \\ = \det(2\pi R_t)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t) \right\} & \end{aligned} \quad (3.4)$$

### 1.1.3 Question 1.3

Implement the state prediction step of the Kalman filter, assuming that at time  $t = 0$ , we start at rest, i.e.,  $x_t = \dot{x}_t = \ddot{x}_t = 0.0$ . Use your code to calculate the state distribution for times  $t = 1, 2, \dots, 5$ .

Here is the Kalman\_filter Algorithm:

```

1: Algorithm Kalman_filter( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
2:    $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$ 
3:    $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$ 
4:    $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$ 
5:    $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$ 
6:    $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$ 
7:   return  $\mu_t, \Sigma_t$ 

```

We will apply lines 1 to 3 to answer this question.

#### 1.1.4 EXTRA CREDITS 1

You can change the value `delta_t` to be any you wish and re-run the notebook. Please Reset the Kernel before hand. **### EXTRA CREDITS 2** You can change the value `sigma_wind` to be any you wish and re-run the notebook. Please Reset the Kernel before hand.

```
[2]: # Define Initial Conditions
a = 0 # Acceleration
v = 0 # Velocity
x = 0 # Position
u = 0
delta_t = 1 # Difference in time (timestep delta t)

sigma_wind = 1

# Define Required Matrices
A = np.array([[1, delta_t],
              [0, 1]])

B = np.array([[0], [0]])

G = np.array([[0.5 * delta_t ** 2], [delta_t]])

R = np.array([[0.25 * (delta_t ** 4), 0.5 * (delta_t ** 3)],
              [0.5 * (delta_t ** 3), delta_t**2]]).dot(sigma_wind)

X = np.array([[0], [0]])

COV = np.array([[0, 0],
                [0, 0]]) # Covariance Matrice, we know the exact position and
# velocity,
# we give it a zero covariance matrix

# Kalman Filter Algorithm lines 1-3
def Kalman_filter(X, COV, u, z=None):
    X = A.dot(X) + B.dot(u)
    COV = A.dot(COV).dot(A.T) + R
    return X, COV

# State Distribution Computation for t = [1:5]
state_distribution = (X, COV)

mean_list = list()
cov_list = list()
```

```

for t in range(1, 6):
    X, COV = state_distribution[0], state_distribution[1]
    state_distribution = Kalman_filter(X, COV, u=0)
    mean_list.append(state_distribution[0])
    cov_list.append(state_distribution[1])
    print("State Distribution for t={}: \nMean = {}, \nCovariance = {}\n".
        ↪format(t, X, COV))

```

State Distribution for t=1:

```

Mean = [[0]
        [0]],
Covariance = [[0 0]
              [0 0]]

```

State Distribution for t=2:

```

Mean = [[0]
        [0]],
Covariance = [[0.25 0.5 ]
              [0.5  1.  ]]

```

State Distribution for t=3:

```

Mean = [[0]
        [0]],
Covariance = [[2.5 2. ]
              [2.  2.  ]]

```

State Distribution for t=4:

```

Mean = [[0]
        [0]],
Covariance = [[8.75 4.5 ]
              [4.5  3.  ]]

```

State Distribution for t=5:

```

Mean = [[0]
        [0]],
Covariance = [[21.  8.]
              [ 8.  4.]]

```

### 1.1.5 Question 1.4

For each posterior, you are asked to plot the uncertainty ellipse which is the ellipse of points that are one standard deviation away from the mean

```

[3]: plt.style.use('seaborn-dark')
plt.rcParams['figure.figsize']=14,6
def plot_ellipse(mean_list, cov_list, t=None, xlim=8, ylim=3, labels=None, ↪
    ↪title=None):

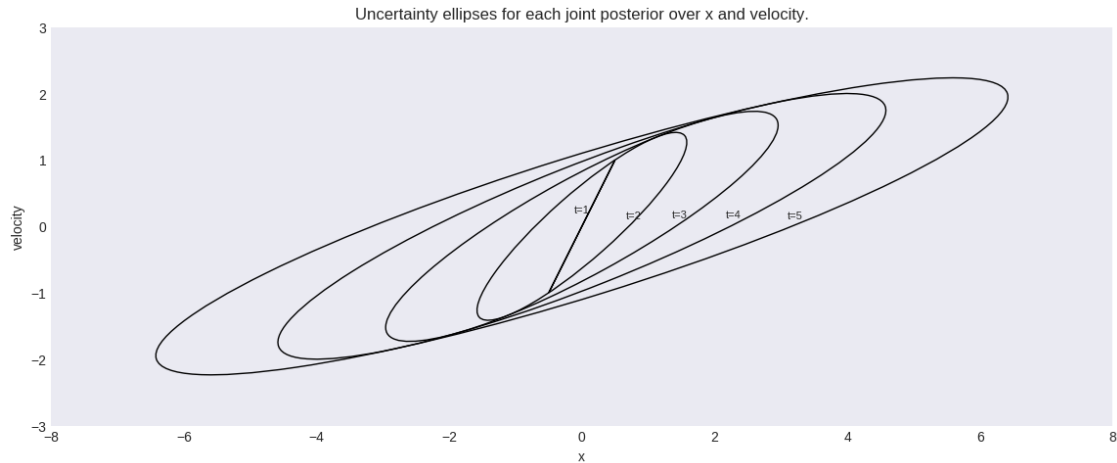
```

```

# Plot the ellipse
fig, ax = plt.subplots(subplot_kw={'aspect': 'equal'})
for i in range(len(mean_list)):
    mean = mean_list[i]
    sigma = cov_list[i]
    # Compute the eigenvalues and eigenvectors of the covariance matrix
    eigenvalues, eigenvectors = np.linalg.eig(sigma)
    # Compute the angle of the ellipse
    angle = np.arctan2(eigenvectors[1, 0], eigenvectors[0, 0])
    # Compute the length of the major and minor axes
    length = np.sqrt(eigenvalues)
    ellipse = plt.matplotlib.patches.Ellipse(mean, 2 * length[0], 2 *
↪length[1], angle=angle * 180 / np.pi, fill=False)
    ax.add_artist(ellipse)
    if t == None and labels == None:
        ax.text(mean[0] + 0.5 * length[0] * np.cos(angle), mean[1] + 0.5 *
↪length[1] * np.sin(angle), f't={i+1}', ha='center', va='center', fontsize=8)
    elif labels != None:
        ax.text(mean[0] + 0.5 * length[0] * np.cos(angle), mean[1] + 0.5 *
↪length[1] * np.sin(angle), f'{labels[i]}', ha='center', va='center',
↪fontsize=8)
    else:
        ax.text(mean[0] + 0.5 * length[0] * np.cos(angle), mean[1] + 0.5 *
↪length[1] * np.sin(angle), f't={t}', ha='center', va='center', fontsize=8)
    if xlim != None:
        plt.xlim(-xlim, xlim)
        plt.ylim(-ylim, ylim)
    plt.xlabel('x')
    plt.ylabel('velocity')
    if title == None:
        plt.title("Uncertainty ellipses for each joint posterior over x and
↪velocity.")
    else:
        plt.title(title)
    plt.show()

plot_ellipse(mean_list, cov_list)

```



## 1.2 2. Kalman Filter: Measurement

### 1.2.1 Question 2.1

Define the measurement model. You will need to define matrices  $C$  and  $Q$ .

We have that:

$$z_t = C_t x_t + \delta_t . \quad (3.5)$$

with  $\delta_t$  following a multivariate Gaussian with zero mean and covariance  $Q$ . Thus:

$C = [1, 0]^T$  as the GPS measures in expectation the true position, and

$$Q = [\sigma_{\text{gps}}^2] = [8.0]$$

### 1.2.2 Question 2.2

Implement the measurement update. Suppose at time  $t = 5$ , the drone's computer has rebooted and we query our sensor for the first time to obtain the measurement  $z = 10$ . State the parameters of the Gaussian estimate before and after incorporating the measurement.

We still have the mean and covariance estimates at time  $t=5$  from Question 1.3. Let's reboot the computer and add the measurement  $z=10$ .

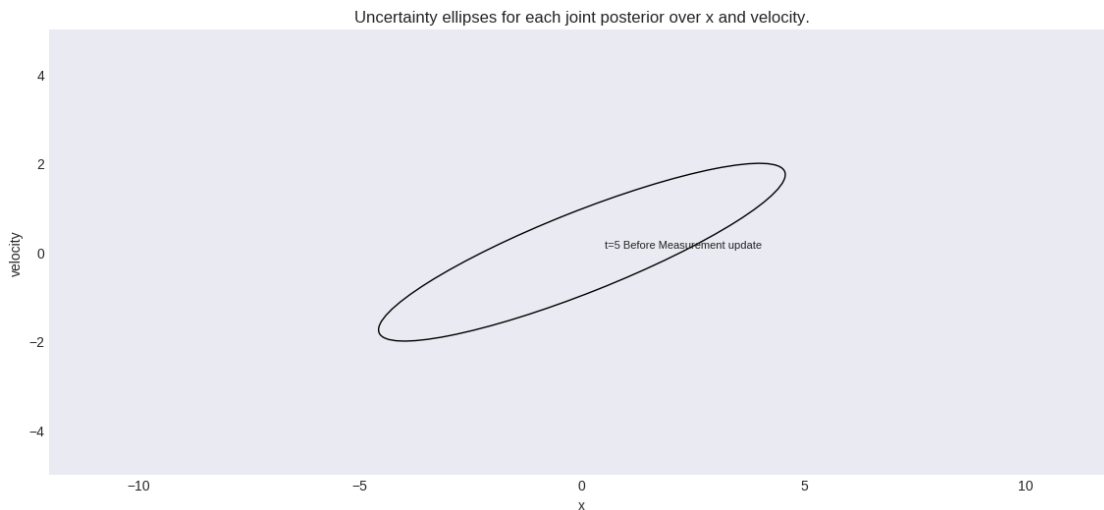
Thus, before the measurement we have:

```
[4]: print("State Distribution for t={}: \nMean = {}, \nCovariance = {}\n".format(5,
    ↪X, COV))
plot_ellipse([np.around(X, decimals=2)], [np.around(COV, decimals=2)], t="5_
    ↪Before Measurement update", xlim=12, ylim=5)
```

State Distribution for t=5:

Mean =  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,

Covariance =  $\begin{bmatrix} 21. & 8. \\ 8. & 4. \end{bmatrix}$



The lines 4-7 of the Kalman Filter are the measurement update. They are coded as follows:

```
[5]: X_5 = copy.deepcopy(X)
      COV_5 = copy.deepcopy(COV)
```

```
[6]: # Define required matrices
      z = 10
      C = np.array([[1], [0]]).T
      print(C)
      Q = np.array([8.0])

      # Calculate the Kalman Gain lines 4-7
      S = C.dot(COV_5).dot(C.T) + Q
      K = COV_5.dot(C.T).dot(inv(S))
      # Update the State Matrix
      X_5_updated = X_5 + K.dot(z-C.dot(X_5))
      # Update the Covariance Matrix
      COV_5_updated = (np.identity(len(K)) - K.dot(C)).dot(COV_5)

      # Round up the values to 1 decimal
      X_5_updated = np.around(X_5_updated, decimals=1)
      COV_5_updated = np.around(COV_5_updated, decimals=1)
```

```

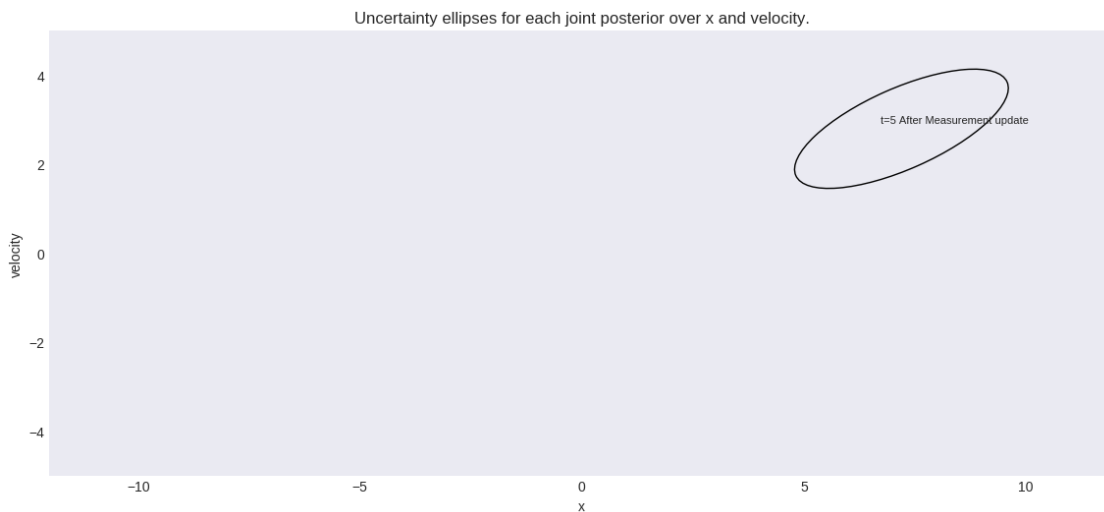
print("State Distribution for t={}: \nMean = {}, \nCovariance = {}\n".format("5_
    ↳After Measurement Update", X_5_updated, COV_5_updated))
plot_ellipse([X_5_updated], [COV_5_updated], t="5 After Measurement update",
    ↳xlim=12, ylim=5)

```

```

[[1 0]]
State Distribution for t=5 After Measurement Update:
Mean = [[7.2]
        [2.8]],
Covariance = [[5.8 2.2]
              [2.2 1.8]]

```



### 1.2.3 Let's write the entire Kalman Filter function

```

[7]: # Kalman Filter Algorithm lines 1-7 (entire algorithm)
def Kalman_filter(X, COV, u, z=None):
    X = A.dot(X) + B.dot(u)
    COV = A.dot(COV).dot(A.T) + R
    if z != None:
        # Calculate the Kalman Gain
        S = C.dot(COV).dot(C.T) + Q
        K = COV.dot(C.T).dot(inv(S))
        # Update the State Matrix
        X = X + K.dot(z-C.dot(X))
        # Update the Covariance Matrix
        COV = (np.identity(len(K)) - K.dot(C)).dot(COV)

    return X, COV

```



Afterwards, implement the sensor modal to randomly sample the true position, corrupted with noise  $\sigma_2$  gps. `###` Let's write the environment class

```
[8]: class Environment:
    def __init__(self, prob_gps_fail=0, sigma_square_noise_gps=8, init_X=np.
    ↪array([[0],[0]])):
        self.prob_gps_fail = prob_gps_fail
        self.std_noise_gps = np.sqrt(sigma_square_noise_gps)
        self.init_X = init_X

    def reset(self):
        self.X = self.init_X
        self.X_measured = self.init_X

    def step(self, control):
        if np.random.random() > self.prob_gps_fail:
            # EXTRA: can change sigma_wind at the begining of the notebook
            accel_wind = np.random.normal(loc=0.0, scale=sigma_wind) # loc is ↪
            ↪the mean, scale is the std
            # Equation of motion. True value of X.
            X = A.dot(self.X) + B.dot(control) + G.dot(accel_wind)
            self.X = X
            # Add noise
            noise = np.random.normal(loc=0.0, scale=self.std_noise_gps)
            z_measured = X[0][0] + noise
            self.X_measured = np.array([[z_measured],[X[1][0]]])
            return z_measured
        else:
            return None

    def get_true_measured_X(self):
        return self.X_measured

    def get_true_X(self):
        return self.X
```

### 1.2.4 Question 2.3

All of a sudden, the sky gets cloudy which may cause the sensor to fail and not produce a measurement with probability `pgpsfail`. For three different values of this probability (e.g., 0.1, 0.5, and 0.9), compute and plot the expected error from the true position at time  $t = 20$ . You may do so by running up to  $N$  simulations and use the observed errors to obtain the expected error empirically.

```
[9]: probas_failure = [0.1, 0.5, 0.9]

    number_simulations = 10000
```

```

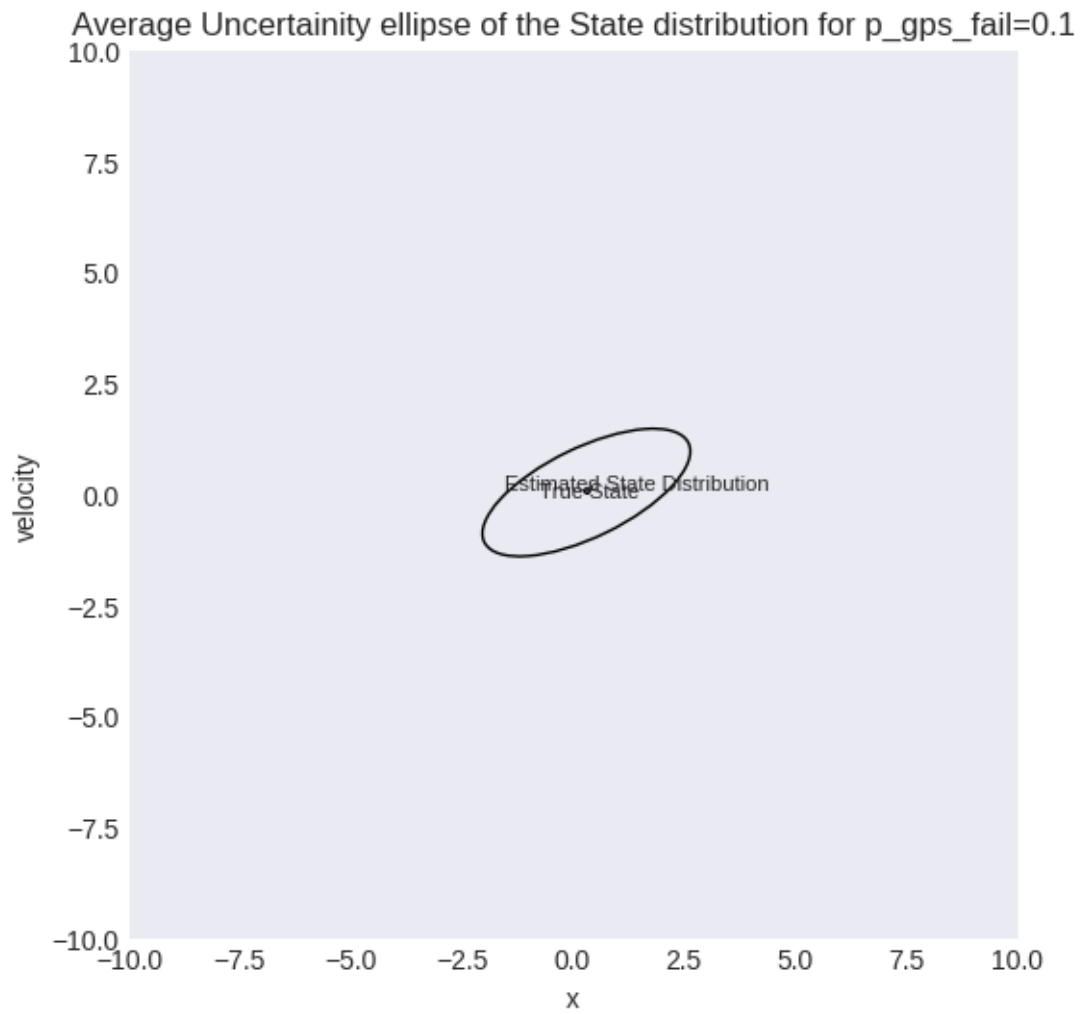
# Experiment loop through gps_failure_proba, number of simulation per
→gps_failure_proba and 20 time steps
for prob in probas_failure:
    obs_errors = list()
    X_list = list()
    COV_list = list()
    X_true_list = list()
    for n in range(0, number_simulations): # N simulations loop
        # Re-Initialize the state space
        X = np.array([[0],[0]])

        COV = np.array([[0, 0],
                        [0, 0]])

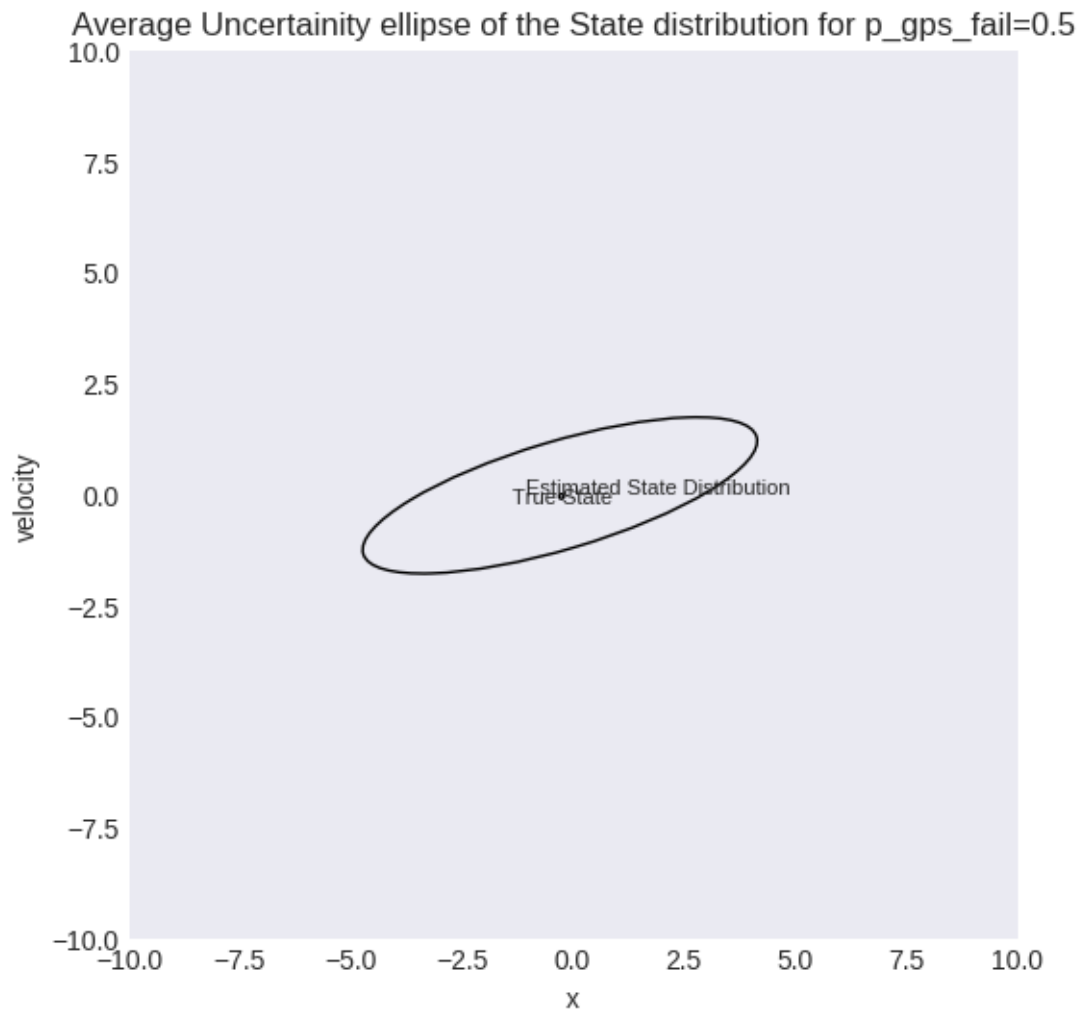
        env = Environment(prob_gps_fail=prob)
        env.reset()
        for t in range(0,20): # timesteps loop
            z_measured = env.step(control=0)
            X, COV = Kalman_filter(X, COV, u=0, z=z_measured)
            true_X = env.get_true_X()
            true_COV = np.array([[0.003, 0],[0, 0.003]]) # very small numbers just
→to hack a mean point in the graph
            obs_errors.append(abs(X[0][0]-true_X[0][0]).astype(np.float64))
            X_list.append(X)
            X_true_list.append(true_X)
            COV_list.append(COV)
        avg_X = np.mean(X_list, axis=0)
        avg_true_X = np.mean(X_true_list, axis=0)
        avg_COV = np.mean(COV_list, axis=0)
        print("Expected Error for p_gps_failure {}: {}".format(prob, np.
→around(mean(obs_errors), decimals=2)))
        plot_ellipse([avg_X, avg_true_X], [avg_COV, true_COV], title="Average
→Uncertainty ellipse of the State distribution for p_gps_fail={}".
→format(prob), labels=["Estimated State Distribution", "True State"], xlim=10,
→ylim=10)

```

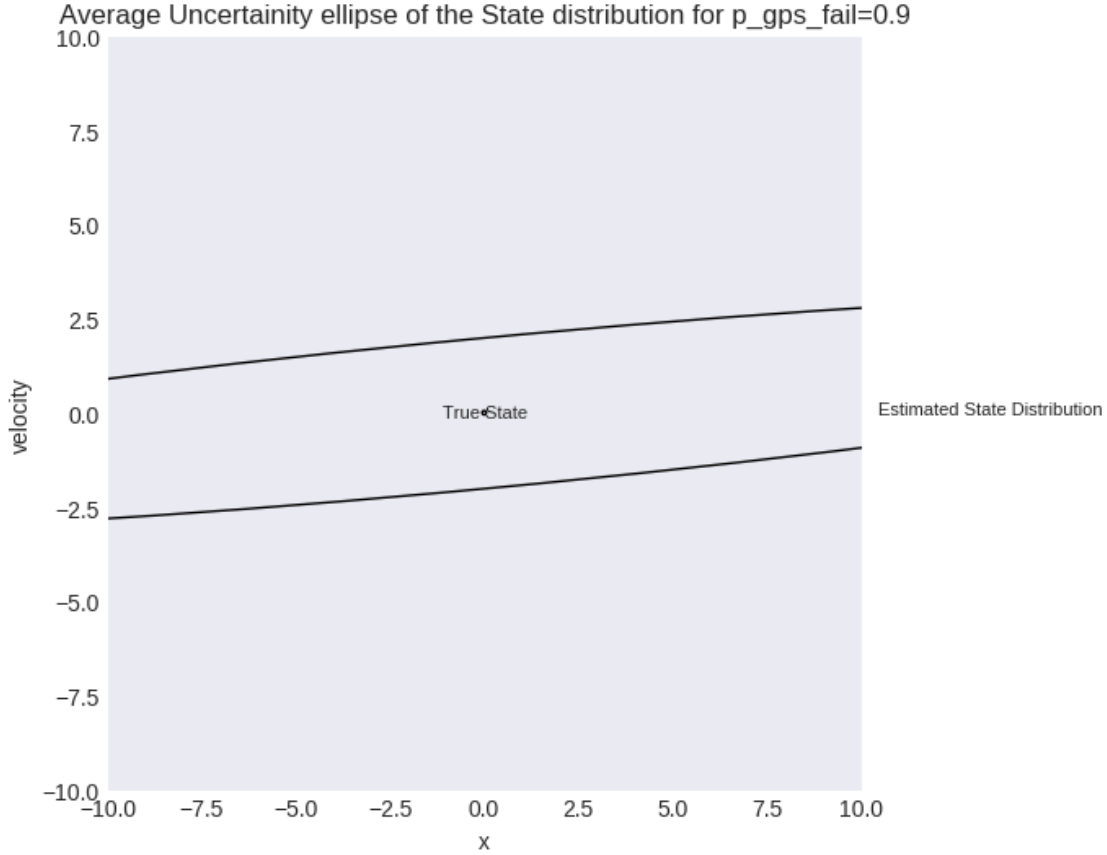
Expected Error for p\_gps\_failure 0.1: 1.97



Expected Error for  $p_{\text{gps\_failure}}$  0.5: 2.8



Expected Error for  $p_{\text{gps\_failure}}$  0.9: 4.35



### 1.3 3. Kalman Filter: Movement

#### 1.3.1 Question 3.1

The drone is now fully operational and can not only take measurements, but also issue motor commands in the form of acceleration commands to its propeller. For example, a command of 1.0 will increase the drone's velocity by 1.0. Revisit Question 1.3 to provide the matrix  $B$ . If at time  $t_1$ , the drone's position and velocity are 5.0 and 1.0, compute the mean estimate for the state at time  $t$  given a motor command of 1.0. Your answer should be based on the constants provided but also include a random variable due to the wind effects. State the distribution of that random variable.

We still have the following equations:

$$x = x_0 + v * t + 0.5 * a * t^2$$

$$v = v_0 + a * t$$

Except now the acceleration of the drone can be decomposed into a random acceleration due to the wind and a command input acceleration  $u$ .

We thus have

$$X_t = A_t * X_{t-1} + B_t * u_t + \epsilon_t \text{ (Eq. 3)}$$

with:

$$B = [[0.5 * d_t * 2], [d_t]]$$

```
[10]: # Overwrite B
      B = np.array([[0.5 * delta_t ** 2], [delta_t]])

      # Initial state according to the question
      X = np.array([[5], [1]])

      COV = np.array([[0, 0],
                      [0, 0]])
```

Kalman filter pass but now with  $u=1$ , the wind effect is included in the function First Without measurement:

```
[11]: X_update, COV_update = Kalman_filter(X, COV, u=1, z=None)
      print("Mean estimate for the state at time t (after one time step without_
      ↳measurement): position={}, velocity={}".format(X_update[0][0], X_update[1][0]))
```

Mean estimate for the state at time t (after one time step without measurement):  
position=6.5, velocity=2.0

Now With environment measurement:

```
[12]: env = Environment(prob_gps_fail=0)
      env.reset()
      z_measured = env.step(control=1)
      X_update, COV_update = Kalman_filter(X, COV, u=1, z=z_measured)
      print("Mean estimate for the state at time t (after one time step without_
      ↳measurement): position={}, velocity={}".format(X_update[0][0], X_update[1][0]))
```

Mean estimate for the state at time t (after one time step without measurement):  
position=6.25198907406916, velocity=1.5039781481383203