# Sets and Functions, Chapter Exercises

L. J.

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## 1 SECTION 1

### 1.2 Mark each statement True or False. Justify each answer.

(a) *False*. Let $A = \{x \in \mathbb{R} : x \text{ is prime}\}$ and let $B = \{x \in \mathbb{R} : x \text{ is divisible by } 6\}$. Then $A \cap B = \emptyset$, but $A \neq \emptyset$ and $B \neq \emptyset$.

(b) *True*. Let $A = \{1,3,5\}$ and let $B = \{2,4,6\}$. Then $A \cup B = \{1,2,3,4,5,6\}$. If $x \in A \cup B$, then $x$ is equal to one of those values. If $x = 4$, then it is also true that $x \in B$. By the same logic more generally, if $x \in A \cup B$, then $x \in A$ or $x \in B$.

(c) *False*. If $x \in A \setminus B$, then $x \in A$ *and* $x \notin B$. Let $C = \{x : (x \in A) \wedge (x \in B)\}$. If $x \in C$, then if $x \in A \setminus B \overset{?}{\Longrightarrow} (x \in A) \vee (x \notin B)$, it is also true $x \in A \setminus B$. But this is at odds with the definition of the relative complement of $B$ with respect to $A$.

(d) *False*. It's fine to begin this way; the only nontrivial proof is one in which $S$ is nonempty.

### 1.4 Let $A = \{2,4,6,8\}$, $B = \{6,8,10\}$, and $C = \{5,6,7,8\}$. Find the following sets.

(a) $\{6,8\}$
(b) $\{2,4,6,8,10\}$
(c) $\{2,4\}$
(d) $\{6,8\}$
(e) $\{10\}$
(f) $\{5,7,10\}$
(g) $\emptyset$
(h) $\{5,7\}$

## 1.6 Let A and B be subsets of a universal set U. Simplify each of the following expressions.

(a) $U$
(b) $\emptyset$
(c) $A \cap B$
(d) $A \cup B$
(e) $A$
(f) $A$

## 1.8 Let $S = \{\emptyset, \{\emptyset\}\}$. Determine whether each of the following is True or False. Explain your answers.

(a) *True.* $\forall x \in \emptyset$, $x \in S$. Therefore, $\emptyset \subseteq S$.
(b) *True.* $S$ is strictly enumerated, and $\emptyset$ is one of its elements.
(c) *True.* $\{\emptyset\} \subseteq S$ because for all $x \in \{\emptyset\}$, $x \in S$. Unlike in (a), though, the proof is not vacuous; $\emptyset$ is an element of both sets.
(d) *True.* $\{\emptyset\}$ is an element in $S$, as defined by enumeration. (To contrast (c) and (d), $\{\{\emptyset\}\}$ would also be a valid subset of $S$.)

## 1.10 Fill in the blanks in the proof of the following theorem.

**Theorem.** $A \subseteq B$ *iff* $A \cup B = B$.

*Proof.* Suppose that $A \subseteq B = B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. Since $A \subseteq B$, in either case, we have $x \in B$. Thus $A \subseteq B$. On the other hand, if $x \in B$, then $x \in A \cup B$, so $A \subseteq B$. Hence $A \cup B$ $B$.
Conversely, suppose that $A \cup B = B$. If $x \in A$, then $x \in A \cup B$. But $A \cup B = B$, so $x \in B$. Thus $A \subseteq B$. □

## 1.14 Which statements below would enable one to conclude that $x \in A \cup B$?

(a) If $x \in A$ and $x \in B$, then $x \in A \cup B$.
(b) If $x \in A$ or $x \in B$, then $x \in A \cup B$.
(c) This statement tells us $A \subseteq B$, but does not let us conclude anything about $x$.
(d) If $x \notin A$, then $x \in B$ implies that if $x \notin B$, then $x \in A$. Since $x$ must always be either in $A$ or $B$, we can conclude that $x \in A \cup B$.

## 1.16 Which statements below would enable one to conclude that $x \in A \setminus B$?

(a) If $x \in A$ and $x \notin B \setminus A$, then $x$ could be in $A \cap B$, so we cannot conclude that $x \in A \setminus B$.
(b) If $x \in A \cup B$ and $x \notin B$, then $(A \cup B) \cap U \setminus B$. Equivalently, $A \cap U \setminus B \cup B \cap U \setminus B$, and $A \setminus B \cup \emptyset$, or $A \setminus B$. Therefore we can conclude that $x \in A \setminus B$.
(c) If $x \in A \cup B$ and $x \notin A \cap B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \setminus B$. But if $x \in B$, then $x \notin A \setminus B$. So we cannot conclude that $x \in A \setminus B$.

(d) If $x \in A$ and $x \notin A \cap B$, then $x \notin B$. Thus $x \in A \setminus B$

## 1.18 Prove that the empty set is unique. That is, suppose that $A$ and $B$ are empty and prove that $A = B$.

*Proof.* Suppose that $A$ and $B$ are empty sets. If $A = B$, then $A \subseteq B$ and $B \subseteq A$.
If $A \subseteq B$, then if $x \in A$, $x \in B$. By the definition of an empty set, for all $x$, $x \notin \emptyset$. Therefore, for all $x$, $x \notin A$ and $x \notin B$. By logical implication, then, if $x \in A$, $x \in B$. This implies that $A \subseteq B$.
On the other hand. if $x \in B$, then $x \in A$. By the same argument, we conclude $B \subseteq A$. So, we have $A \subseteq B$ and $B \subseteq A$. Thus $A = B$. □

## 1.20 Prove: $A \cap B$ and $A \setminus B$ are disjoint and $A = (A \cap B) \cup (A \setminus B)$.

*Proof.* Assume the conclusion that $(A \cap B) \cap (A \setminus B) = \emptyset$.
If $x \in (A \cap B)$, then $x \in A$ and $x \in B$. But then $x \notin (A \setminus B)$, because if $x \in A \setminus B$, then $x \in A$ and $x \notin B$. In other words, if $x \in (A \cap B)$, $x \in (A \setminus (A \setminus B))$. So $(A \cap B) \nsubseteq (A \setminus B)$.
Conversely, if $x \in (A \setminus B)$, then $x \in A$ and $x \in B$. But then $x \notin (A \cap B)$. So $(A \setminus B) \nsubseteq (A \cap B)$.
Thus, $(A \cap B) \cap (A \setminus B) = \emptyset$. Therefore $(A \cap B)$ and $(A \setminus B)$ are disjoint.
In other words:

$$(A \cap B) \cap (A \setminus B) \tag{1.1}$$
$$= (A \cap B) \cap (A \cap U \setminus B) \tag{1.2}$$
$$= (U \setminus A \cup U \setminus B) \cup (U \setminus A \cup B) \tag{1.3}$$
$$= (U \setminus A) \cup (U \setminus B \cup B) \tag{1.4}$$
$$= A \cap (B \cap U \setminus B) \tag{1.5}$$
$$= A \cap \emptyset \tag{1.6}$$
$$= \emptyset \tag{1.7}$$

□

*Proof.* To prove $A = (A \cap B) \cup (A \setminus B)$, first show that if $x \in (A \cap B) \cup (A \setminus B)$, then $x \in A$.
If $x \in (A \cap B) \cup (A \setminus B)$, then $x \in (A \cup A \setminus B)$ and $x \in (B \cup A \setminus B)$. Suppose $x \in (A \cap B) \cup (A \setminus B)$.
If $x \in A$, then $x \in (A \cup A \setminus B)$ and $x \in (B \cup A \setminus B)$, so $x \in (A \cap B) \cup (A \setminus B)$.
If $x \in B$, then $x \in (B \cup A \setminus B)$ (equivalently, $(A \cup B)$), but $x \notin (A \cup A \setminus B)$ (equivalently, $A$).
So by necessity, we have $x \in A$, and $(A \cap B) \cup (A \setminus B) = A$.

On the other hand, suppose $x \in A$. Then $x \in (A \cap B)$ or $x \in (A \setminus B)$:

$$A \tag{1.8}$$
$$= A \cap U \tag{1.9}$$
$$= A \cap (B \cup U \setminus B) \tag{1.10}$$
$$= (A \cap B) \cup (A \cap U \setminus B) \tag{1.11}$$
$$= (A \cap B) \cup (A \setminus B) \tag{1.12}$$

Either way, $x \in (A \cap B) \cup (A \setminus B)$. So we have $A = (A \cap B) \cup (A \setminus B)$.

□

# 2 Section 2

## 2.2 Mark each statement True or False. Justify each answer.

(a) *True*. If $\mathscr{P}$ is a partition of $S$, then if $x$ and $y$ are in the same part of the partition, we can define $xRy$ as an equivalence relation such that the equivalence classes are the same as $\mathscr{P}$.

(b) *False*. If $xRx$ for all $x \in S$, then $R$ is reflexive. But not all relations are reflexive.

(c) *False*. A partition of $S$ can be defined by $\{E_x : x \in S\}$, where $E_x$ represents an equivalence class. An equivalence class is defined by $E_x = \{y \in S : yRx\}$.

(d) *True*. By definition, a partition $\mathscr{P}$ consists of nonempty, disjoint sets such that for all $x \in S$, $x$ belongs to some subset of $\mathscr{P}$.

## 2.4 Show that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

$(a, b)$ is the set whose members are $\{a\}$ and $\{a, b\}$.

$$(a, b) = \{\{a\}, \{a, b\}\}$$

And the product of two sets $A$ and $B$ is

$$A \times B \ \{(a, b) : a \in A \text{and} b \in B\}$$

Then,

$$\begin{aligned} \{a\} \times \{a\} &= \{(a, a)\} \\ &= \{\{\{a\}, \{a, a\}\}\}. \end{aligned}$$

Since $a = a$, $\{a, b\} = \{a\}$, and the above simplifies to

$$\begin{aligned} &= \{\{\{a\}\{a\}\}\} \\ &= \{\{\{a\}\}\}. \end{aligned}$$

## 2.6 Let $A$ be be any set and let $B = \emptyset$. What can you conclude about $A \times B$?

Let $A$ be any set, and let $B$ be $\emptyset$. By definition,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

and

$$\emptyset = \{x : \forall x, x \notin \emptyset\}.$$

Then if $B = \emptyset$, the product $A \times B$ requires that all the ordered pairs in the product have a first element in $A$ and a second element in $\emptyset$. But now, $\forall b, b \notin B$, so there are no ordered pairs that satisfy the definition of the product of $A \times B$.
Therefore, $A \times B = \emptyset$.

## 2.8 Let $A = \{1, 2\}$.

(a) **How many elements are in the set $A \times A$?** $|A \times A| = 4$.

(b) **How many possible relations are there on the set $A$?** A relation on $A$ is any subset $R$ of $A \times A$ ($R \subset A \times A$); any element can be in or out of the relation. So, what is the power set of $A$?

$$2^{|A\times A|} = 2^4 = 16$$

(c) **How many possible relations are there on the set $\{1, 2, 3\}$?** For $A$, we had $|A| = 2$. Let $S$ be a generic set, $n = |S|$ be the number of elements in the set, and $|R|$ be the total number of possible relations on $S$. Then for $S_n$, we have $|R| = 2^{n^2}$. If $S = \{1, 2, 3\}$, then $n = 3$ and we have

$$|R| = 2^{(3^2)} = 2^9 = 512$$

possible relations.

## 2.10 Prove or give a counterexample.

(a) Let $A = \{2, 3\}$ and $B = \{0, -5\}$. Then

$$A \times B = \{(2, 0), (2, -5), (3, 0), (3, 5)\}$$

and

$$B \times A = \{(0, 2), (0, 3), (-5, 2), (-5, 3)\}.$$

Therefore

$$A \times B \neq B \times A.$$

(b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
Let

$$A = \{x : x \in [0, 3)\}$$
$$B = \{x : x \in (2, 4]\}$$
$$C = \{x : x \in [-2, 2]\}.$$

Then

$$(A \cup B) \times C = \{(x, y) : x \in [0, 4] \text{ and } y \in [-2, 2]$$

We have:

$$A \times C = \{(x, y) : x \in [0, 3) \text{ and } y \in [-2, 2]\}$$
$$B \times C\ \{(x, y) : x \in (2, 4] \text{ and } y \in [-2, 2]\}$$

and

$$(A \times C) \cup (B \times C) = \{(x, y) : x \in [0, 4] \text{ and } y \in [-2, 2]\}.$$

(c) TK

(d) TK

## 2.12 Find examples of relations with the following properties.

Let $S = \{p, q, r\}$. Then define the following relations $R$ on $S$:

(a) **$R$ is reflexive, but not symmetric or transitive.**
$R = \{(p, p), (q, q), (r, r,)\}$

(b) **$R$ is symmetric, but not reflexive or transitive.**
$R = \{(p, q), (q, p), (r, q), (q, r)\}$

(c) **$R$ is transitive, but not symmetric or reflexive.**
$R = \{(p, q), (q, r), (p, r)\}$

(d) **$R$ is reflexive and symmetric, but not transitive.**
$R = \{(p, p), (q, q), (r, r), (p, q), (q, p), (q, r), (r, q)\}$

(e) **$R$ is reflexive and transitive, but not symmetric.**
$R = \{(p, p), (q, q), (r, r), (p, q), (q, r), (p, r)\}$

(f) **$R$ is symmetric and transitive, but not reflexive.**
$R = \{(p, q), (q, p), (q, r), (r, q), (p, r), (r, p)\}$

## 2.14 Let $S = \mathbb{R} \times \mathbb{R}$. Verify that the relation $(a, b)R(c, d)$ iff $a + d = b + c$ is an equivalence relation. Describe the equivalence class $E_{(7,3)}$.

Rearrange the relation as:

$$a + d = b + c$$
$$a + d - c = b$$
$$a - c = b - d$$

The relation is reflexive: $(a, b)R(a, b)$ gives

$$a - a = b - b$$

$$0 = 0$$

The relation is symmetric:

$$(c, d)R(a, b)$$
$$c - a = d - b$$
$$-c + a = -d + b$$
$$a - c = b - d$$

The relation is transitive:

$$(a, b)R(c, d) \land (c, d)R(m, n) \implies (a, b)R(m, n)$$

$$TKTKTKTK$$

The equivalence class $E_{(7,3)} = \{d \in S : d = c - 4\}$:

$$7 + d = 3 + c$$
$$d = c - 4$$

It describes a linear relationship.

### 2.16 Let $S = \mathbb{R} \times \mathbb{R}$. Define the equivalence relation $R$ on $S$ by $(a,b)R(c,d)$ iff $b - 3a = d - 3c$.

(a) We can rearrange the relation as:

$$
\begin{aligned}
b - d &= 3a - 3c \\
b - d &= 3(a - c)
\end{aligned}
$$

So, $R$ partitions $S$ into parallel lines of the form $y = 3x\ (+k)$.

(b) $E_{(2,5)}$ would be the part of the partition containing the point $(2,5)$, which we obtain by substituting it for one-half of the ordered pairs:

$$
\begin{aligned}
b - 3a &= 5 - 3(2) \\
b - 3a &= -1x & &= 3a - 1
\end{aligned}
$$

So the equivalence class is described by the linear relationship $y = 3x - 1$.

### 2.18 Let $R$ be the relation $\{(1,1),(2,2),(3,1),(3,3)\}$ on the set $\{1,2,3\}$. If $R$ is an equivalence relation, list the pieces of the partition determined by $R$, and if not, state why.

$R$ is not an equivalence relation because it is not symmetric. $(3,1) \in R$ but $(1,3) \notin R$.

### 2.20 Let $S = \{a,b,c,d\}$ and let $P = \{\{a\},\{b,c,d\}\}$. Describe the equivalence relation $R$ on $S$ determined by $P$ by listing the ordered pairs in the relation.

We see that $a$ is in a partition alone, but $b, c$ and $d$ are in a relation with each other, so the relation consists of the pairs

$$
\begin{aligned}
&\{(a,a), \\
&\ (b,b),(c,c),(d,d), \\
&\ (b,c),(b,d), \\
&\ (c,b),(c,d), \\
&\ (d,c),(d,b)\}.
\end{aligned}
$$

## 3 Section 3

### 3.2 Mark each statement True or False. Justify your answer.

(a) *True.* By the existence requirement of the definition of a function, for all $a$ in $A$, there must exist some $b$ in $B$ such that $f(a) = b$. Since $C \subset A$, it follows that every

$c$ in $C$ is also in $A$, and so for all $c$ in $C$, there must be some $b$ in $B$ such that $f(c) = b$. Thus $f(C)$ is a nonempty subset of $B$.

(b) *False*. If $f$ is surjective, it is a mapping form all elements in the domain $A$ to all elements in $B$, that is to say the range of $f$ is equal to its codomain. But though $f^{-1}(y)$ is guaranteed to exist for $y \in B$, it is not guaranteed to be one-to-one unless $f$ is also injective. Therefore $f^{-1}(y)$ is a subset ($\subset$) of $A$, rather than an element in ($\in$) $A$.

(c) *False*. If $B$ is the codomain of $f$, and $D$ is a subset of $B$, then unless $f$ is surjective, there is no guarantee that $D$ intersects the image of $A$ in $B$.

(d) *True*. Let $f : A \to B$ and $g : B \to C$ be surjective functions. Since $g$ is surjective, the range of $g$ is equal to $C$, and for all $b \in B$, there exists $c \in C$. Since $f$ is surjective, the range of $f$ is equal to $B$, and for all $a \in A$, there exists $b \in B$. So $g \circ f(a) = g(f(a)) = g(b) = c$. Therefore $g \circ f$ is surjective.

(e) *True*. If $f$ is bijective, then its domain is $A$ and its range and codomain are $B$. Since $f$ is a function, each $x$ in $f$ corresponds to only one $y$ in $B$. Therefore, if $(x, y) \in f$, it must follow that the inverse of $f$ is bijective as well, and $(y, x) \in f^{-1}$.

(f) *False*. The identity function maps $\mathbb{R}$ onto $\mathbb{R}$ such that for all $x \in \mathbb{R}$, $(i)x = x$.

## 3.4 Find all possible functions $f : A \to B$ in each case below. Describe the functions by listing their ordered pairs.

(a) $f = \{(1,5),(2,5),(3,5)\}$.

(b) $f_1 = \{(4,5)\}, f_2 = \{(4,6)\}$.

(c) $f_1 = \{(1,5),(2,6)\}, f_2 = \{(1,5),(2,5)\}, f_3 = \{(2,5),(1,6)\}, f_4 = \{(2,5),(1,6)\}$

## 3.6 Let $A \subset \mathbb{R}$ and define $f : A \to B$ as given below. In each case describe an $A$ that is as large as possible while making $f$ injective.

(a) $A = [5,\infty)$ or $A = (-\infty,5]$.

(b) $A = [-0.5,\infty)$ or $A = (-\infty,-0.5]$.

(c) $A = [-\frac{\pi}{2}, \frac{\pi}{2}]$ or $A = [\frac{\pi}{2}, \frac{3\pi}{2}]$ (for example).

## 3.8 Circles in the plane

(a) Let $S$ be the set of all circles in the plane. Define $f : S \to [0,\infty)$ by $f(C)$ = the area of $C$ for all $C \in S$.
Assuming a circle can have a radius $r = 0$, a function that maps all circles to the value of their area is surjective but not injective, because two circles at different coordinate positions can nonetheless have equal area.

(b) Let $T$ be the set of all circles in the plane that are centered at the origin. Define $g : T \to [0,\infty)$ by $g(C)$ = the area of $C$ for all $C \in T$.
Now, $g$ is both injective and surjective; for every circle defined by radius length $r$ in the interval $[0,\infty)$ there is an area $g(C)$ in $[0,\infty)$, and since all circles are centered at the origin, any circle with the same area has the same radius length and is the same circle.

## 3.10 In each part, find a function $\mathbb{N} \to \mathbb{N}$ that has the desired properties.

(a) Surjective, but not injective:

$$f(x) = \begin{cases} x-1, & \text{for } x > 1, \\ 1 & \text{for } x = 1. \end{cases} \tag{3.1}$$

(b) Injective, but not surjective:

$$f(x) = 2x. \tag{3.2}$$

(c) Neither surjective nor injective:

$$f(x) = 1 \tag{3.3}$$

(d) Bijective:

$$f(x) = x. \tag{3.4}$$