Logic and Proof, Chapter Exercises

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1 SECTION 1

1.1 Mark each statement true or false. Justify your answer.

(a) *False.* A sentence need not be true in order to be classified a "statement". We require only that it "can be clearly labeled" true *or* false.
We should specify the conditions that constitute clarity, and who's doing the labeling. We can condition truth and falsity on any number of assumptions—at best, explicit; at worst, implied—and the sentence's author ordinarily assumes a reader with a mental model that mirrors their own, for better or worse.

(b) *False.* In binary logic, a statement cannot be true and false at the same time. Its truth value may be as-yet unknown—it might be a conjecture—but it has to follow the rules of mathematical proof. That is, to be a statement, the rules must be able to label a sentence exclusively "true" or "false" under *some* set of specified conditions. Those conditions may change: $x + 6 = 0$ is a statement, for certain values of x, but not for all.

(c) *True.* The negation of a statement need only have the opposite truth value to the statement itslf. So if p is true, $\neg p$ is false, and $\neg(\neg p)$ is true.

(d) *False.* The negation of a statement is by definition the logical complement of the given proposition. Thus, we can see that either p can be false or $\neg p$ can be false, but they cannot both be false at the same time.

(e) *True.* In logic, "or" is inclusive by default, and we must specify that we mean "exclusive or" ($\oplus$) (or its negation, the biconditional ($\iff$)) to denote $(p \vee q) \wedge \neg(p \wedge q)$.

1.2 Mark each statement True or False. Justify each answer.

(a) *False.* We refer to p as the antecedent (that which comes (*"ced-"*) before (*"ante-"*). In contrast, q is the consequent, or logical "result". In other words, p is called the sufficient condition, and q is called the necessary condition.

(b) *True.* The implication applies only to worlds in which p is true. Since we have only binary truth, we cannot rule out "truth" in worlds not containing p. Hence, these cases are true, and the only false one in that in which p exists, but the necessary condition q does not.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

(c) *False.* This formulation customarily evaluates to the same thing as "q whenever p", because p is a sufficient condition to "trigger" the requirement of q. In other words, whenever we have p, we must also have its necessary condition, q. But as the truth table in (*b*) illustrates, the statement is true when q is true but p is false, and so the reverse dependency does not hold. Thus, it's incorrect to say that "if p, then q" is equivalent to "p whenever q".

(d) *True.* The negation of a conjunction is logically equivalent to the disjunction of the negations of its individual components.

p	q	$\neg(p \wedge q)$	$\iff$	$[(\neg p) \vee (\neg q)]$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

(e) *False.* The negation of $p \implies q$ is not $q \implies p$.

p	q	$\neg(p \implies q)$	$\not\Longleftrightarrow$	$q \implies p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	F	F	T

1.3 Write the negation of each statement.

(a) The $3x3$ identity matrix is not singular.
(b) The function $f(x) = sin(x)$ is unbounded on $\mathbb{R}$.
(c) The functions f and g are nonlinear.
(d) Six is not prime and seven is even.
(e) x is in D, and $f(x) \geq 5$.
(f) (a_n) is monotone and bounded, and (a_n) is divergent.
(g) f is injective, but S is not finite and S is not denumerable.

1.4 Write the negation of each statement.

(a) $f(x)$ is discontinuous at $x = 3$.
(b) R is not reflexive and not symmetric.

(c) 4 and 9 are not relatively prime.
(d) $x \notin A \vee x \in B$.
(e) $x < 7$ and $f(x)$ is in C.
(f) (a_n) is convergent and (a_n) is unbounded or not monotone.
(g) f is continuous and A is open, but $f^{-1}(A)$ is closed.

1.5 Identify the antecedent and the consequent in each statement.

(a)
- *Antecedent:* M is singular.
- *Consequent:* M has a zero eigenvalue.

(b)
- *Antecedent:* Linearity.
- *Consequent:* Continuity.

(c)
- *Antecedent:* A sequence is Cauchy.
- *Consequent:* It is bounded.

(d)
- *Antecedent:* $y > 5$.
- *Consequent:* $x < 3$.

1.6 Construct a truth table for each statement.

(a)

p	q	$p \implies \neg q$
T	T	F
T	F	T
F	T	T
F	F	T

(b)

1.7 Identify the antecedent and the consequent in each statement.

2 Section 2

2.1 *