

# Logic and Proof, Chapter Exercises

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L. J.

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## 1 SECTION 1

### 1.1 Mark each statement true or false. Justify your answer.

- (a) *False.* A sentence need not be true in order to be classified a “statement”. We require only that it “can be clearly labeled” true *or* false.  
We should specify the conditions that constitute clarity, and who’s doing the labeling. We can condition truth and falsity on any number of assumptions—at best, explicit; at worst, implied—and the sentence’s author ordinarily assumes a reader with a mental model that mirrors their own, for better or worse.
- (b) *False.* In binary logic, a statement cannot be true and false at the same time. Its truth value may be as-yet unknown—it might be a conjecture—but it has to follow the rules of mathematical proof. That is, to be a statement, the rules must be able to label a sentence exclusively “true” or “false” under *some* set of specified conditions. Those conditions may change:  $x + 6 = 0$  is a statement, for certain values of  $x$ , but not for all.
- (c) *True.* The negation of a statement need only have the opposite truth value to the statement itself. So if  $p$  is true,  $\neg p$  is false, and  $\neg(\neg p)$  is true.
- (d) *False.* The negation of a statement is by definition the logical complement of the given proposition. Thus, we can see that either  $p$  can be false or  $\neg p$  can be false, but they cannot both be false at the same time.
- (e) *True.* In logic, “or” is inclusive by default, and we must specify that we mean “exclusive or” ( $\oplus$ ) (or its negation, the biconditional ( $\iff$ )) to denote  $(p \vee q) \wedge \neg(p \wedge q)$ .

### 1.2 Mark each statement True or False. Justify each answer.

- (a) *False.* We refer to  $p$  as the antecedent (that which comes (“ced-”) before (“ante-”). In contrast,  $q$  is the consequent, or logical “result”. In other words,  $p$  is called the

sufficient condition, and  $q$  is called the necessary condition.

- (b) *True*. The implication applies only to worlds in which  $p$  is true. Since we have only binary truth, we cannot rule out “truth” in worlds not containing  $p$ . Hence, these cases are true, and the only false one in that in which  $p$  exists, but the necessary condition  $q$  does not.

$p$	$q$	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- (c) *False*. This formulation customarily evaluates to the same thing as “ $q$  whenever  $p$ ”, because  $p$  is a sufficient condition to “trigger” the requirement of  $q$ . In other words, whenever we have  $p$ , we must also have its necessary condition,  $q$ . But as the truth table in (b) illustrates, the statement is true when  $q$  is true but  $p$  is false, and so the reverse dependency does not hold. Thus, it’s incorrect to say that “if  $p$ , then  $q$ ” is equivalent to “ $p$  whenever  $q$ ”.
- (d) *True*. The negation of a conjunction is logically equivalent to the disjunction of the negations of its individual components.

$p$	$q$	$\neg(p \wedge q)$	$\iff$	$[(\neg p) \vee (\neg q)]$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

- (e) *False*. The negation of  $p \Rightarrow q$  is not  $q \Rightarrow p$ .

$p$	$q$	$\neg(p \Rightarrow q)$	<del><math>\iff</math></del>	$q \Rightarrow p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	F	F	T

### 1.3 Write the negation of each statement.

- The  $3 \times 3$  identity matrix is not singular.
- The function  $f(x) = \sin(x)$  is not bounded on  $\mathbb{R}$ .
- The functions  $f$  and  $g$  are nonlinear.
- Six is not prime and seven is even.
- $x$  is in  $D$ , and  $f(x) \geq 5$ .
- $(a_n)$  is monotone and bounded, and  $(a_n)$  is divergent.
- $f$  is injective, but  $S$  is not finite and  $S$  is not denumerable.

### 1.4 Write the negation of each statement.

- $f(x)$  is not continuous at  $x = 3$ .
- $R$  is not reflexive and not symmetric.

- (c) 4 and 9 are not relatively prime.  
 (d)  $x \notin A \vee x \in B$ .  
 (e)  $x < 7$  and  $f(x)$  is in  $C$ .  
 (f)  $(a_n)$  is convergent and  $(a_n)$  is unbounded or not monotone.  
 (g)  $f$  is continuous and  $A$  is open, but  $f^{-1}(A)$  is closed.

### 1.5 Identify the antecedent and the consequent in each statement.

- (a) • *Antecedent:*  $M$  is singular.  
 • *Consequent:*  $M$  has a zero eigenvalue.  
 (b) • *Antecedent:* Linearity.  
 • *Consequent:* Continuity.  
 (c) • *Antecedent:* A sequence is Cauchy.  
 • *Consequent:* It is bounded.  
 (d) • *Antecedent:*  $y > 5$ .  
 • *Consequent:*  $x < 3$ .

### 1.6 Identify the antecedent and the consequent in each statement.

- (a) • *Antecedent:* [A sequence] is Cauchy.  
 • *Consequent:* [It] is convergent.  
 (b) • *Antecedent:* Boundedness.  
 • *Consequent:* Convergence.  
 (c) • *Antecedent:* Orthogonality.  
 • *Consequent:* Invertibility.  
 (d) • *Antecedent:*  $K$  is closed and bounded.  
 • *Consequent:*  $K$  is compact.

### 1.7 Construct a truth table for each statement.

	$p$	$q$	$p \Rightarrow \neg q$		
(a)	T	T	F		
	T	F	T		
	F	T	T		
	F	F	T		
	$p$	$q$	$p \wedge (p \Rightarrow q)$	$\Rightarrow$	$q$
(b)	T	T	T	T	T
	T	F	F	T	F
	F	T	F	T	T
	F	F	F	T	F
	$p$	$q$	$p \Rightarrow (q \wedge \neg q)$	$\Longleftrightarrow$	$\neg p$
(c)	T	T	F	T	F
	T	F	F	T	F
	F	T	T	T	T
	F	F	T	T	T

**1.8 Construct a truth table for each statement.**

	$p$	$q$	$p \vee \neg q$
	T	T	T
(a)	T	F	T
	F	T	F
	F	F	T

  

	$p$	$p \wedge \neg p$
(b)	T	F
	F	F

  

	$p$	$q$	$[(\neg q) \wedge (p \Rightarrow q)]$	$\Rightarrow$	$\neg p$
	T	T	F	T	F
(c)	T	F	F	T	F
	F	T	F	T	T
	F	F	T	T	T

**1.9 Indicate whether each statement is True or False.**

- (a)  $T \wedge T \Rightarrow T$ .
- (b)  $F \vee T \Rightarrow T$ .
- (c)  $F \vee F \Rightarrow F$ .
- (d)  $(T \Rightarrow T) \Rightarrow T$ .
- (e)  $(F \Rightarrow F) \Rightarrow T$ .
- (f)  $(T \Rightarrow F) \Rightarrow F$ .
- (g)  $[(T \wedge F) \Rightarrow T] \Rightarrow T$ .
- (h)  $[(T \vee F) \Rightarrow F] \Rightarrow F$ .
- (i)  $[(T \wedge F) \Rightarrow F] \Rightarrow T$ .
- (j)  $\neg(F \vee T) \Leftrightarrow T \wedge F \Rightarrow F$ .

**1.10 Indicate whether each statement is True or False.**

- (a)  $T \wedge F \Rightarrow F$ .
- (b)  $F \vee F \Rightarrow F$ .
- (c)  $F \vee T \Rightarrow T$ .
- (d)  $(T \Rightarrow F) \Rightarrow F$ .
- (e)  $(F \Rightarrow F) \Rightarrow T$ .
- (f)  $(F \Rightarrow T) \Rightarrow T$ .
- (g)  $[(F \vee T) \Rightarrow F] \Rightarrow F$ .
- (h)  $[(T \Leftrightarrow F) \Rightarrow T] \Rightarrow T$ .
- (i)  $[(T \wedge T) \Rightarrow F] \Rightarrow F$ .
- (j)  $\neg(F \wedge T) \Leftrightarrow T \vee F \Rightarrow T$ .

**1.11 Let  $p$  be the statement “The figure is a polygon,” and let  $q$  be the statement “The figure is a circle”. Express each of the following statements in symbols.**

- (a)  $p \wedge \neg q$ .
- (b)  $(p \vee q) \wedge \neg(p \wedge q)$ .

- (c)  $\neg q \implies p$ .  
 (d)  $q \implies \neg p$ .  
 (e)  $p \iff \neg q$ .

**1.12 Let  $m$  be the statement “ $x$  is perpendicular to  $M$ ”, and let  $n$  be the statement “ $x$  is perpendicular to  $N$ ”. Express each of the following statements in symbols.**

- (a)  $n \wedge \neg m$ .  
 (b)  $\neg(m \vee n)$ .  
 (c)  $n \implies m$   
 (d)  $m \implies \neg n$   
 (e)  $\neg(m \wedge n) \iff \neg m \vee \neg n$

**1.13 Define “nor” ( $\nabla$ ) as:**

$p$	$q$	$p \nabla q$
T	T	F
T	F	F
F	T	F
F	F	T

(a) 

$p$	$\neg p$	$\iff$	$p \nabla p$
T	F	T	F
F	T	T	T

(b) 

$p$	$q$	$(p \nabla p)$	$\iff$	$(q \nabla q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	F	F
F	F	T	F	T

(c) 

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

**1.14 Use truth tables to verify that each of the following is a tautology.**

*Commutative laws*

(a) 

$p$	$q$	$(p \wedge q)$	$\iff$	$(q \wedge p)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

$p$	$q$	$(p \vee q)$	$\iff$	$(q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

(b) *Associative laws*

$p$	$q$	$r$	$[p \wedge (q \wedge r)]$	$\iff$	$[(p \wedge q) \wedge r]$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

$p$	$q$	$r$	$[p \vee (q \vee r)]$	$\iff$	$[(p \vee q) \vee r]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	T	F

(d) *Distributive laws*

$p$	$q$	$r$	$[p \wedge (q \vee r)]$	$\iff$	$[(p \wedge q) \vee (p \wedge r)]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

$p$	$q$	$r$	$[p \vee (q \wedge r)]$	$\iff$	$[(p \vee q) \wedge (p \vee r)]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

## 2 SECTION 2

### 2.2 Mark each statement true or false. Justify your answer.

- (a) *False.*  $\exists$  denotes the existence at least one.  $\forall$  is the universal quantifier and means “for each, for every”.
- (b) *True.* Universal quantification is assumed if a variable is used in the antecedent of an implication without an explicit quantifier.
- (c) *True.* For instance,  $\forall x \exists y$  corresponds to the statement “For every disease, there exists a cure”, while  $\exists x \forall y$  corresponds to the statement “There is a panacea for all diseases.”

### 2.4 Write the negation of each statement.

- (a) Someone does not like Robert.
- (b) No students work part-time.
- (c) Some square matrix is triangular.
- (d)  $\forall x \in B, f(x) \leq k$ .
- (e)  $\exists x \exists (x > 5) \wedge [(f(x) \geq 3) \wedge (f(x) \leq 7)]$ .
- (f)  $\exists x \in A \exists \forall y \in B, f(x) \geq f(y)$ .

### 2.6 Determine the truth value of each statement, assuming $x$ is a real number. Justify your answer.

- (a) *True.* Let  $x = 4$ .  $3 < 4$  and  $4 < 5$ , so 4 is included in the interval.
- (b) *False.* Let  $x = 3$ . Then  $3 \in [3, 5]$  and  $3 < 4$ .
- (c) *True.* Let  $x = 3$ . Then  $3^2 = 9 \neq 3$ .
- (d) *False.* Let  $x = \sqrt{3}$ . Then  $x$  is a real number and  $(\sqrt{3})^2 = 3$ .
- (e) *False.* Let  $x = 0$ . Then  $0^2 = 0$ .  
Let  $x > 0$ . Then for all  $x$ ,  $x^2$  is positive.  
Let  $x < 0$ . Then for all  $x$ ,  $x^2$  is also positive.  
Therefore, for all real  $x$ ,  $x \neq -5$ .
- (f) *False.* Let  $x = -5$ . Then  $(-5)^2 = 25 \neq -5$ .
- (g) *True.* Let  $x = x$ .  $x - x = 0 \iff x = x$ . (This is the additive identity of a field.)
- (h) *True.* By the same token, it is always true that for all  $x$ ,  $x - x = 0$ .

### 2.8 Which of the following best identifies $f$ as a constant function, where $x$ and $y$ are real numbers?

$\exists y \ni \forall x, f(x) = y$  represents a constant function.

### 2.10 Determine the truth value of each statement, assuming that $x$ and $y$ are real numbers. Justify your answer.

- (a) *True.* For all  $x \in \mathbb{R}$ , let  $y = 0$ . Then  $xy = 0$ .
- (b) *False.* Let  $x = 0$ . Then for all  $y$ ,  $xy = 0 \neq 1$ .
- (c) *False.* For all  $y \in \mathbb{R}$ , there is a real number  $x = 0$  such that  $xy = 0(y) = 0 \neq 1$ .

- (d) *True*. Let  $y = 1$ . Then for all  $x \in \mathbb{R}$ ,  $xy = x(1) = x$ .

**2.12 Determine the truth value of each statement, assuming that  $x$ ,  $y$ , and  $z$  are real numbers. Justify your answer.**

- (a) *True*. Suppose by contradiction that for all  $x, y, z \in \mathbb{Z}$ , there exist some  $x$  and  $y$  such that  $x + y \neq z$ .  
Then some integer would have no successor, which violates the properties of  $\mathbb{Z}$ .  
So there must be some  $x$  and  $y$  such that  $x + y = z$ .
- (b) *False*. Let  $x = 0$ .  
Then, for any  $y$ , let  $z = y + 0$ , or  $z = y$ .  
Let  $z = 1$ . Since  $y \neq 1$  for all  $y \in \mathbb{R}$ , the statement is false.
- (c) *True*. Let  $x$  be any real number other than zero. Then, given any  $y$ , let  $z = \frac{y}{x}$ .
- (d) *False*. Let  $x = 2$ . Then  $z \leq 2 + y$ .  
Then for all  $y$ , we must show there exists a  $z$  such that  $y < z \leq x + y$ , or  $0 < z - y \leq x$ .  
Assuming  $x = 2$ , we must show there is some  $z$  such that  $y < z \leq 2 + y$ .  
Let  $z = y + 1$ . Then  $y < y + 1 < y + 2$ . Therefore, the statement is false, because  $z < x + y$ .
- (e) *False*. Assume by contradiction that  $y < z \leq x + y$ , or  $0 < z - y \leq x$ .  
Let  $x = 2$ . Then for all  $z$ ,  $y < z \leq y + 2$ .  
Let  $y = z - 1$ , such that  $z - 1 < z \leq z + 1$ .  
Then  $z > z - 1$  and  $z \leq 2 + (z - 1) = z + 1$  for all  $z$ .  
Since the negation is true, the statement is false.

**2.14 Rewrite the defining conditions in logical symbolism, and write the negation.**

- (a)  $\exists k > 0 \exists \forall x, f(x + k) = f(x) \implies f \text{ is periodic.}$   
(b)  $\neg(f \text{ is periodic}) \implies \forall x > 0, \exists x \exists f(x + k) \neq f(x).$

**2.16 Rewrite the defining conditions in logical symbolism, and write the negation.**

- (a)  $[\forall x, y, x < y \implies f(x) > f(y)] \implies f \text{ is strictly decreasing.}$   
(b)  $\neg(f \text{ is strictly decreasing}) \implies \exists x, y \ni (x < y) \wedge f(x) \leq f(y).$

**2.18 Rewrite the defining conditions in logical symbolism, and write the negation.**

- (a)  $[\forall y \in B \exists x \in A \ni f(x) = y] \implies f \text{ is surjective.}$   
(b)  $\neg(f \text{ is surjective}) \implies \exists y \in B \ni \forall x \in A, f(x) \neq y.$



**2.20 Rewrite the defining conditions in logical symbolism, and write the negation.**

- (a)  $\forall \epsilon > 0 \exists \delta > 0 \ni \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$
- (b)  $\exists \epsilon > 0 \ni \forall \delta > 0, \exists x, y \in S \ni (|x - y| < \delta) \wedge (|f(x) - f(y)| \geq \epsilon).$

### 3 SECTION 3

**3.2 Mark each statement True or False. Justify each answer.**

- (a) *True.* In this context the antecedent is called the conclusion.
- (b) *False.* A statement that is always false is called a contradiction.
- (c) *True.* The converse of  $p \implies q$  is  $q \implies p$ .
- (d) *True.* A universal statement can be proved false using a single example.
- (e) *False.* To prove an existential statement is false, we must show  $\forall n, \neg p(n)$ .

**3.4 Write the converse of each implication in exercise 1.3.3.**

- (a) If some violets are blue, then all roses are red.
- (b) If  $A$  is not invertible, then there exists a nontrivial solution to  $A\mathbf{x} = 0$ .
- (c) If  $f(C)$  is not connected, then  $f$  is not continuous or  $C$  is not connected.

**3.5 Write the inverse of each implication in exercise 1.3.3.**

- (a) If some roses are not red, then all violets are not blue.
- (b) If there exists no nontrivial solution to  $A\mathbf{x} = 0$ , then  $A$  is invertible.
- (c) If  $f$  is not continuous or  $C$  is not connected, then  $f(C)$  is not connected.

**3.6 Provide a counterexample for each statement.**

- (a) Let  $x = -4$ .
- (b) Let  $n = -2$ .
- (c) Let  $x = \frac{1}{3}$ .
- (d) An equilateral triangle has all three angles equal to  $\frac{\pi}{3}$  rad.
- (e) Let  $n = 41$ .
- (f) 2 is prime and not odd.
- (g) Let  $x = 101$ .
- (h) Let  $n = 5$ .
- (i) Let  $n = 5$ .
- (j) Let  $x = 12$  and  $y = 3$ .
- (k) Let  $x = 0$ .
- (l) Let  $x = 1$ .
- (m) Let  $x = 0$ .
- (n) Let  $x = -1$ .

**3.10 Using tautologies from Example 3.12, establish the desired conclusion from the list of hypotheses.**

(a)

$\neg r$	Hypothesis	(1)
$\neg r \implies (s \implies r)$	Hypothesis and 3.12(n)	(2)
$s \implies r$	Lines 1,2 and 3.12(h)	(3)
$\neg r \implies \neg s$	Contrapositive of Line 3, 3.12(c)	(4)
$\neg s$	Lines 1,4 and 3.12(h)	(5)
(3.1)		

(b)

$\neg t$	Hypothesis	(1)
$(r \implies t) \wedge (s \implies t)$	Hypothesis and 3.12(q)	(2)
$(r \implies t) \wedge (\neg t \implies \neg s)$	Line 2 and 3.12(c)	(3)
$t \vee \neg s$	Lines 1, 3 and 3.12(o)	(4)
$\neg s$	Line 4 and 3.12(j)	(5)
(3.2)		

(c)

$s \vee t$	Hypothesis	(1)
$s \implies \neg r$	Hypothesis and 3.12(c)	(2)
$t \implies u$	Hypothesis	(3)
$\neg r \vee u$	Lines 1-3 and 3.12(o)	(4)
(3.3)		

## 4 SECTION 4

### 4.2 Mark each statement True or False. Justify each answer.

- (a) *True.* The contrapositive of the antecedent  $\neg c \implies p$  will be true iff  $p$  is true, because  $c$  is always false. Thus, it shares the same truth table as  $p$ .
- (b) *False.*  $[(p \vee \neg q) \implies c]$  is logically equivalent to  $[(p \implies c) \wedge (\neg q \implies c)]$ . This statement has a different truth table from  $p \implies q$ .
- (c) *True.* Without definitions, mathematics is only so much rhetoric.

### 4.4 For every $\epsilon > 0$ there exists a $\delta > 0$ such that $(1 - \delta < x < 1 + \delta)$ implies $(2 - \epsilon < 7 - 5x < 2 + \epsilon)$ .

*Proof.* Let  $\delta = \frac{\epsilon}{5}$ . Since  $\epsilon > 0$ ,  $\delta > 0$  as well. Then we have  $1 - \delta < x < 1 + \delta$  as  $1 - \frac{\epsilon}{5} < x < 1 + \frac{\epsilon}{5}$ . In other words,  $5 - \epsilon < 5x < 5 + \epsilon$ . Adding 7, we see that  $2 + \epsilon > 7 - 5x > 2 - \epsilon$ . This is equivalent to the proposition  $2 - \epsilon < 7 - 5x < 2 + \epsilon$ .  $\square$

**4.18 Consider the following theorem: “If  $xy = 0$ , then  $x = 0$  or  $y = 0$ .”**  
**Indicate what, if anything, is wrong with each of the following**  
**“proofs.”**

1. This proof takes the valid form  $p \implies (q \vee r) \iff (p \wedge \neg q) \implies r$ .
2. This fails to prove what the theorem assumes. “If  $x = 0$  or  $y = 0$  then  $xy = 0$ ” is the converse of the theorem, so proving it has no bearing on the truth value of the original statement.

**4.20 Suppose  $x$  and  $y$  are real numbers. Prove or give a counterexample.**

- 1.