Logic and Proof, Chapter Exercises

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1 SECTION 1

1.1 Mark each statement true or false. Justify your answer.

(a) *False.* A sentence need not be true in order to be classified a "statement". We require only that it "can be clearly labeled" true *or* false.
We should specify the conditions that constitute clarity, and who's doing the labeling. We can condition truth and falsity on any number of assumptions—at best, explicit; at worst, implied—and the sentence's author ordinarily assumes a reader with a mental model that mirrors their own, for better or worse.

(b) *False.* In binary logic, a statement cannot be true and false at the same time. Its truth value may be as-yet unknown—it might be a conjecture—but it has to follow the rules of mathematical proof. That is, to be a statement, the rules must be able to label a sentence exclusively "true" or "false" under *some* set of specified conditions. Those conditions may change: $x+6=0$ is a statement, for certain values of x, but not for all.

(c) *True.* The negation of a statement need only have the opposite truth value to the statement itslf. So if p is true, $\neg p$ is false, and $\neg(\neg p)$ is true.

(d) *False.* The negation of a statement is by definition the logical complement of the given proposition. Thus, we can see that either p can be false or $\neg p$ can be false, but they cannot both be false at the same time.

(e) *True.* In logic, "or" is inclusive by default, and we must specify that we mean "exclusive or" ($\oplus$) (or its negation, the biconditional ($\iff$)) to denote $(p \vee q) \wedge \neg(p \wedge q)$.

1.2 Mark each statement True or False. Justify each answer.

(a) *False.* We refer to p as the antecedent (that which comes (*"ced-"*) before (*"ante-"*). In contrast, q is the consequent, or logical "result". In other words, p is called the sufficient condition, and q is called the necessary condition.

(b) *True.* The implication applies only to worlds in which p is true. Since we have only binary truth, we cannot rule out "truth" in worlds not containing p. Hence, these cases are true, and the only false one in that in which p exists, but the necessary condition q does not.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

(c) *False.* This formulation customarily evaluates to the same thing as "q whenever p", because p is a sufficient condition to "trigger" the requirement of q. In other words, whenever we have p, we must also have its necessary condition, q. But as the truth table in (b) illustrates, the statement is true when q is true but p is false, and so the reverse dependency does not hold. Thus, it's incorrect to say that "if p, then q" is equivalent to "p whenever q".

(d) *True.* The negation of a conjunction is logically equivalent to the disjunction of the negations of its individual components.

p	q	$\neg(p \wedge q)$	$\iff$	$[(\neg p) \vee (\neg q)]$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

(e) *False.* The negation of $p \implies q$ is not $q \implies p$.

p	q	$\neg(p \implies q)$	$\not\Longleftrightarrow$	$q \implies p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	F	F	T

1.3 Write the negation of each statement.

(a) The $3x3$ identity matrix is not singular.
(b) The function $f(x) = sin(x)$ is not bounded on $\mathbb{R}$.
(c) The functions f and g are nonlinear.
(d) Six is not prime and seven is even.
(e) x is in D, and $f(x) \geq 5$.
(f) (a_n) is monotone and bounded, and (a_n) is divergent.
(g) f is injective, but S is not finite and S is not denumerable.

1.4 Write the negation of each statement.

(a) $f(x)$ is not continuous at $x = 3$.
(b) R is not reflexive and not symmetric.

(c) 4 and 9 are not relatively prime.
(d) $x \notin A \vee x \in B$.
(e) $x < 7$ and $f(x)$ is in C.
(f) (a_n) is convergent and (a_n) is unbounded or not monotone.
(g) f is continuous and A is open, but $f^{-1}(A)$ is closed.

1.5 Identify the antecedent and the consequent in each statement.

(a)
- *Antecedent:* M is singular.
- *Consequent:* M has a zero eigenvalue.

(b)
- *Antecedent:* Linearity.
- *Consequent:* Continuity.

(c)
- *Antecedent:* A sequence is Cauchy.
- *Consequent:* It is bounded.

(d)
- *Antecedent:* $y > 5$.
- *Consequent:* $x < 3$.

1.6 Identify the antecedent and the consequent in each statement.

(a)
- *Antecedent:* [A sequence] is Cauchy.
- *Consequent:* [It] is convergent.

(b)
- *Antecedent:* Boundedness.
- *Consequent:* Convergence.

(c)
- *Antecedent:* Orthogonality.
- *Consequent:* Invertibility.

(d)
- *Antecedent:* K is closed and bounded.
- *Consequent:* K is compact.

1.7 Construct a truth table for each statement.

(a)

p	q	$p \implies \neg q$
T	T	F
T	F	T
F	T	T
F	F	T

(b)

p	q	$p \wedge (p \implies q)$	$\implies$	q
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	T	F

(c)

p	q	$p \implies (q \wedge \neg q)$	$\iff$	$\neg p$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

1.8 Construct a truth table for each statement.

(a)

p	q	$p \vee \neg q$
T	T	T
T	F	T
F	T	F
F	F	T

(b)

p	$p \wedge \neg p$
T	F
F	F

(c)

p	q	$[(\neg q) \wedge (p \implies q)]$	$\implies$	$\neg p$
T	T	F	T	F
T	F	F	T	F
F	T	F	T	T
F	F	T	T	T

1.9 Indicate whether each statement is True or False.

(a) $\mathrm{T} \wedge \mathrm{T} \implies \mathrm{T}$.
(b) $\mathrm{F} \vee \mathrm{T} \implies \mathrm{T}$.
(c) $\mathrm{F} \vee \mathrm{F} \implies \mathrm{F}$.
(d) $(\mathrm{T} \implies \mathrm{T}) \implies \mathrm{T}$.
(e) $(\mathrm{F} \implies \mathrm{F}) \implies \mathrm{T}$.
(f) $(\mathrm{T} \implies \mathrm{F}) \implies \mathrm{F}$.
(g) $[(\mathrm{T} \wedge \mathrm{F}) \implies \mathrm{T}\,] \implies \mathrm{T}$.
(h) $[(\mathrm{T} \vee \mathrm{F}) \implies \mathrm{F}\,] \implies \mathrm{F}$.
(i) $[(\mathrm{T} \wedge \mathrm{F}) \implies \mathrm{F}\,] \implies \mathrm{T}$.
(j) $\neg(\mathrm{F} \vee \mathrm{T}) \iff \mathrm{T} \wedge \mathrm{F} \implies \mathrm{F}$.

1.10 Indicate whether each statement is True or False.

(a) $\mathrm{T} \wedge \mathrm{F} \implies \mathrm{F}$.
(b) $\mathrm{F} \vee \mathrm{F} \implies \mathrm{F}$.
(c) $\mathrm{F} \vee \mathrm{T} \implies \mathrm{T}$.
(d) $(\mathrm{T} \implies \mathrm{F}) \implies \mathrm{F}$.
(e) $(\mathrm{F} \implies \mathrm{F}) \implies \mathrm{T}$.
(f) $(\mathrm{F} \implies \mathrm{T}) \implies \mathrm{T}$.
(g) $[\,(\mathrm{F} \vee \mathrm{T}) \implies \mathrm{F}\,] \implies \mathrm{F}$.
(h) $[\,(\mathrm{T} \iff \mathrm{F}) \implies \mathrm{T}\,] \implies \mathrm{T}$.
(i) $[\,(\mathrm{T} \wedge \mathrm{T}) \implies \mathrm{F}\,] \implies \mathrm{F}$.
(j) $\neg(\mathrm{F} \wedge \mathrm{T}) \iff \mathrm{T} \vee \mathrm{F} \implies \mathrm{T}$.

1.11 Let p be the statement "The figure is a polygon," and let q be the statement "The figure is a circle". Express each of the following statements in symbols.

(a) $p \wedge \neg q$.
(b) $(p \vee q) \wedge \neg(p \wedge q)$.

(c) $\neg q \implies p$.
(d) $q \implies \neg p$.
(e) $p \iff \neg q$.

1.12 Let m be the statement "x is perpendicular to M", and let n be the statement "x is perpendicular to N". Express each of the following statements in symbols.

(a) $n \wedge \neg m$.
(b) $\neg(m \vee n)$.
(c) $n \implies m$
(d) $m \implies \neg n$
(e) $\neg(m \wedge n) \iff \neg m \vee \neg n$

1.13 Define "nor" (∇) as:

p	q	$p \nabla q$
T	T	F
T	F	F
F	T	F
F	F	T

(a)

p	$\neg p$	$\iff$	$p \nabla p$
T	F	T	F
F	T	T	T

(b)

p	q	$(p \nabla p)$	$\iff$	$(q \nabla q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	F	F
F	F	T	F	T

(c)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1.14 Use truth tables to verify that each of the following is a tautology.

Commutative laws

(a)

p	q	$(p \wedge q)$	$\iff$	$(q \wedge p$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

(b)

p	q	$(p \vee q)$	$\iff$	$(q \vee p$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

Associative laws

(c)

p	q	r	$[p \wedge (q \wedge r)]$	$\iff$	$[(p \wedge q) \wedge r$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

(d)

p	q	r	$[p \vee (q \vee r)]$	$\iff$	$[(p \vee q) \vee r$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	T	F

Distributive laws

(e)

p	q	r	$[p \wedge (q \vee r)]$	$\iff$	$[(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

(f)

p	q	r	$[p \vee (q \wedge r)]$	$\iff$	$[(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

2 Section 2

2.2 Mark each statement true or false. Justify your answer.

(a) *False.* $\exists$ denotes the existence at least one. $\forall$ is the universal quantifier and means "for each, for every".
(b) *True.* Universal quantification is assumed if a variable is used in the antecedent of an implication without an explicit quantifier.
(c) *True.* For instance, $\forall x \exists y$ corresponds to the statement "For every disease, there exists a cure", while $\exists x \forall y$ corresponds to the statement "There is a panacea for all diseases."

2.4 Write the negation of each statement.

(a) Someone does not like Robert.
(b) No students work part-time.
(c) Some square matrix is triangular.
(d) $\forall x \in B, f(x) \leq k$.
(e) $\exists x \ni (x > 5) \wedge [(f(x) \geq 3) \wedge (f(x) \leq 7)]$.
(f) $\exists x \in A \ni \forall y \in B, f(x) \geq f(y)$.

2.6 Determine the truth value of each statement, assuming x is a real number. Justify your answer.

(a) *True.* Let $x = 4$. $3 < 4$ and $4 < 5$, so 4 is included in the interval.
(b) *False.* Let $x = 3$ Then $3 \in [3,5]$ and $3 < 4$.
(c) *True.* Let $x = 3$. Then $3^2 = 9 \neq 3$.
(d) *False.* Let $x = \sqrt{3}$. Then x is a real number and $(\sqrt{3})^2 = 3$.
(e) *False.* Let $x = 0$. Then $0^2 = 0$.
Let $x > 0$. Then for all x, x^2 is positive.
Let $x < 0$. Then for all x, x^2 is also positive.
Therefore, for all real x, $x \neq -5$.
(f) *False.* Let $x = -5$. Then $(-5)^2 = 25 \neq -5$.
(g) *True.* Let $x = x$. $x - x = 0 \iff x = x$. (This is the additive identity of a field.)
(h) *True.* By the same token, it is always true that for all x, $x - x = 0$.

2.8 Which of the following best identifies f as a constant function, where x and y are real numbers?

$\exists y \ni \forall x, f(x) = y$ represents a constant function.

2.10 Determine the truth value of each statement, assuming that x and y are realnumbers. Justify your answer.

(a) *True.* For all $x \in \mathbb{R}$, let $y = 0$. Then $xy = 0$.
(b) *False.* Let $x = 0$. Then for all y, $xy = 0 \neq 1$.
(c) *False.* For all $y \in \mathbb{R}$, there is a real number $x = 0$ such that $xy = 0(y) = 0 \neq 1$.

(d) *True.* Let $y = 1$. Then for all $x \in \mathbb{R}$, $xy = x(1) = x$.

2.12 Determine the truth value of each statement, assuming that x, y, and z are real numbers. Justify your answer.

(a) *True.* Suppose by contradiction that for all $x, y, z \in \mathbb{Z}$, there exist some x and y such that $x + y \neq z$.
Then some integer would have no successor, which violates the properties of $\mathbb{Z}$.
So there must be some x and y such that $x + y = z$.

(b) *False.* Let $x = 0$.
Then, for any y, let $z = y + 0$, or $z = y$.
Let $z = 1$. Since $y \neq 1$ for all $y \in \mathbb{R}$, the statement is false.

(c) *True.* Let x be any real number other than zero. Then, given any y, let $z = \frac{y}{x}$.

(d) *False.* Let $x = 2$. Then $z \leq 2 + y$.
Then for all y, we must show there exists a z such that $y < z \leq x + y$, or $0 < z - y \leq x$.
Assuming $x = 2$, we must show there is some z such that $y < z \leq 2 + y$.
Let $z = y + 1$. Then $y < y + 1 < y + 2$. Therefore, the statement is false, because $z < x + y$.

(e) *False.* Assume by contradiction that $y < z \leq x + y$, or $0 < z - y \leq x$.
Let $x = 2$. Then for all z, $y < z \leq y + 2$.
Let $y = z - 1$, such that $z - 1 < z \leq z + 1$.
Then $z > z - 1$ and $z \leq 2 + (z - 1) = z + 1$ for all z.
Since the negation is true, the statement is false.

2.14 Rewrite the defining conditions in logical symbolism, and write the negation.

(a) $\exists k > 0 \ni \forall x, f(x + k) = f(x) \implies$ f is periodic.

(b) $\neg$(f is periodic) $\implies \forall x > 0, \exists x \ni f(x + k) \neq f(x)$.

2.16 Rewrite the defining conditions in logical symbolism, and write the negation.

(a) $[\forall x, y, x < y \implies f(x) > f(y)] \implies f$ is strictly decreasing.

(b) $\neg(f$ is strictly decreasing) $\implies \exists x, y \ni (x < y) \wedge f(x) \leq f(y)$.

2.18 Rewrite the defining conditions in logical symbolism, and write the negation.

(a) $[\forall y \in B \exists x \in A \ni f(x) = y] \implies f$ is surjective.

(b) $\neg(f$ is surjective) $\implies \exists y \in B \ni \forall x \in A, f(x) \neq y$.

2.20 Rewrite the defining conditions in logical symbolism, and write the negation.

(a) $\forall \epsilon > 0 \exists \delta > 0 \ni \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
(b) $\exists \epsilon > 0 \ni \forall \delta > 0, \exists x, y \in S \ni (|x - y| < \delta) \wedge (|f(x) - f(y)| \geq \epsilon$.

3 Section 3

3.2 Mark each statement True or False. Justify each answer.

(a) *True.* In this context the antecedent is called the conclusion.
(b) *False.* A statement that is always false is called a contradiction.
(c) *True.* The converse of $p \implies q$ is $q \implies p$.
(d) *True.* A universal statement can be proved false using a single example.
(e) *False.* To prove an existential statement is false, we must show $\forall n, \neg p(n)$.

3.4 Write the converse of each implication in exercise 1.3.3.

(a) If some violets are blue, then all roses are red.
(b) If A is not invertible, then there exists a nontrivial solution to $A\mathbf{x} = 0$.
(c) If $f(C)$ is not connected, then f is not continuous or C is not connected.

3.5 Write the inverse of each implication in exercise 1.3.3.

(a) If some roses are not red, then all violets are not blue.
(b) If there exists no nontrivial solution to $A\mathbf{x} = 0$, then A is invertible.
(c) If f is not continuous or C is not connected, then $f(C)$ is not connected.

3.6 Provide a counterexample for each statement.

(a) Let $x = -4$.
(b) Let $n = -2$.
(c) Let $x = \frac{1}{3}$.
(d) An equilateral triangle has all three angles equal to $\frac{\pi}{3}$ rad.
(e) Let $n = 41$.
(f) 2 is prime and not odd.
(g) Let $x = 101$.
(h) Let $n = 5$.
(i) Let $n = 5$.
(j) Let $x = 12$ and $y = 3$.
(k) Let $x = 0$.
(l) Let $x = 1$.
(m) Let $x = 0$.
(n) Let $x = -1$.

3.10 Using tautologies from Example 3.12, establish the desired conclusion from the list of hypotheses.

(a)

$$
\begin{array}{llr}
\neg r & \text{Hypothesis} & (1)\\
\neg r \implies (s \implies r) & \text{Hypothesis and 3.12(n)} & (2)\\
s \implies r & \text{Lines 1,2 and 3.12(h)} & (3)\\
\neg r \implies \neg s & \text{Contrapositive of Line 3, 3.12(c)} & (4)\\
\neg s & \text{Lines 1,4 and 3.12(h)} & (5)
\end{array} \tag{3.1}
$$

(b)

$$
\begin{array}{llr}
\neg t & \text{Hypothesis} & (1)\\
(r \implies t) \wedge (s \implies t) & \text{Hypothesis and 3.12(q)} & (2)\\
(r \implies t) \wedge (\neg t \implies \neg s) & \text{Line 2 and 3.12(c)} & (3)\\
t \vee \neg s & \text{Lines 1, 3 and 3.12(o)} & (4)\\
\neg s & \text{Line 4 and 3.12(j)} & (5)
\end{array} \tag{3.2}
$$

(c)

$$
\begin{array}{llr}
s \vee t & \text{Hypothesis} & (1)\\
s \implies \neg r & \text{Hypothesis and 3.12(c)} & (2)\\
t \implies u & \text{Hypothesis} & (3)\\
\neg r \vee u & \text{Lines 1-3 and 3.12(o)} & (4)
\end{array} \tag{3.3}
$$

4 SECTION 4

4.2 Mark each statement True or False. Justify each answer.

(a) *True.* The contrapositive of the antecedent $\neg c \implies p$ will be true iff p is true, because c is always false. Thus, it shares the same truth table as p.

(b) *False.* $([p \vee \neg q) \implies c$ is logically equivalent to $[(p \implies c) \wedge (\neg q \implies c)]$. This statement has a different truth table from $p \implies q$.

(c) *True.* Without definitions, mathematics is only so much rhetoric.

4.4 For every $\epsilon > 0$ there exists a $\delta > 0$ such that $(1 - \delta < x < 1 + \delta)$ implies $(2 - \epsilon < 7 - 5x < 2 + \epsilon$.

Proof. Let $\delta = \frac{\epsilon}{5}$. Since $\epsilon > 0$, $\delta > 0$ as well.
Then we have $1 - \delta < x < 1 + \delta$ as $1 - \frac{\epsilon}{5} < x < 1 + \frac{\epsilon}{5}$. In other words, $5 - \epsilon < 5x < 5 + \epsilon$. Adding 7, we see that $2 + \epsilon > 7 - 5x > 2 - \epsilon$. This is equivalent to the proposition $2 - \epsilon < 7 - 5x < 2 + \epsilon$. □

4.18 Consider the following theorem: "If $xy = 0$, then $x = 0$ or $y = 0$." Indicate what, if anything, is wrong with each of the following "proofs."

1. This proof takes the valid form $p \implies (q \vee r) \iff (p \wedge \neg q) \implies r)$.
2. This fails to prove what the theorem assumes. "If $x = 0$ or $y = 0$ then $xy = 0$" is the converse of the theorem, so proving it has no bearing on the truth value of the original statement.

4.20 Suppose x and y are real numbers. Prove or give a counterexample.

(a)

Theorem. *If x is irrational and y is irrational, then $x + y$ is irrational.*

Proof. If $x + y$ is irrational, then it can't be expressed as a fraction $\frac{a}{b}$ in lowest terms, where a and b are integers.
Let $x = \sqrt{5}$ and $y = -\sqrt{5}$. Then $x + y = 0$ But 0 can be expressed as a fraction $\frac{0}{n}$, where $n \in \{\mathbb{Z} \neq 0\}$. So 0 is rational, and therefore $x + y$ is rational. So the theorem is false. □

(b)

Theorem. *If $x + y$ is irrational, then x is irrational or y is irrational.*

Proof. We prove the contrapositive. If x is rational and y is rational, then $x + y$ is rational.
Let $x = \frac{a}{b}$ and $y = \frac{m}{n}$ for some $a, b, m, n \in \{\mathbb{Z} \neq\}$. Then

$$x + y = \frac{a}{b} + \frac{m}{n} \tag{4.1}$$
$$= \frac{an}{nb} + \frac{mb}{nb} \tag{4.2}$$
$$= \frac{an + mb}{nb} \tag{4.3}$$

Thus, $x + y$ can be expressed as a ratio of integers and is rational as required. □

(c)

Theorem. *If x is irrational and y is irrational, then xy is irrational.*

Proof. Let $x = \sqrt{2}$ and $y = \frac{1}{\sqrt{2}}$. Then

$$xy = \sqrt{2} * \frac{1}{\sqrt{2}} \tag{4.4}$$
$$= \frac{\sqrt{2}}{\sqrt{2}} \tag{4.5}$$
$$= 1 \tag{4.6}$$

Since 1 is rational, the stated theorem is false. □