Sets and Functions, Chapter Exercises

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1 SECTION 1

1.2 Mark each statement True or False. Justify each answer.

(a) *False.* Let $A = \{x \in \mathbb{R} : x \text{ is prime}\}$ and let $B = \{x \in \mathbb{R} : x \text{ is divisible by } 6\}$. Then $A \cap B = \emptyset$, but $A \neq \emptyset$ and $B \neq \emptyset$.

(b) *True.* Let $A = \{1,3,5\}$ and let $B = \{2,4,6\}$. Then $A \cup B = \{1,2,3,4,5,6\}$. If $x \in A \cup B$, then x is equal to one of those values. If $x = 4$, then it is also true that $x \in B$. By the same logic more generally, if $x \in A \cup B$, then $x \in A$ or $x \in B$.

(c) *False.* If $x \in A \setminus B$, then $x \in A$ *and* $x \notin B$. Let $C = \{x : (x \in A) \wedge (x \in B)\}$. If $x \in C$, then if $x \in A \setminus B \overset{?}{\Longrightarrow} (x \in A) \vee (x \notin B)$, it is also true $x \in A \setminus B$. But this is at odds with the definition of the relative complement of B with respect to A.

(d) *False.* It's fine to begin this way; the only nontrivial proof is one in which S is nonempty.

1.4 Let $A = \{2,4,6,8\}$, $B = \{6,8,10\}$, and $C = \{5,6,7,8\}$. Find the following sets.

(a) $\{6,8\}$
(b) $\{2,4,6,8,10\}$
(c) $\{2,4\}$
(d) $\{6,8\}$
(e) $\{10\}$
(f) $\{5,7,10\}$
(g) $\emptyset$
(h) $\{5,7\}$

1.6 Let A and B be subsets of a universal set U. Simplify each of the following expressions.

(a) U
(b) $\emptyset$
(c) $A \cap B$
(d) $A \cup B$
(e) A
(f) A

1.8 Let $S = \{\emptyset, \{\emptyset\}\}$. Determine whether each of the following is True or False. Explain your answers.

(a) *True.* $\forall x \in \emptyset$, $x \in S$. Therefore, $\emptyset \subseteq S$.
(b) *True.* S is strictly enumerated, and $\emptyset$ is one of its elements.
(c) *True.* $\{\emptyset\} \subseteq S$ because for all $x \in \{\emptyset\}$, $x \in S$. Unlike in (a), though, the proof is not vacuous; $\emptyset$ is an element of both sets.
(d) *True.* $\{\emptyset\}$ is an element in S, as defined by enumeration. (To contrast (c) and (d), $\{\{\emptyset\}\}$ would also be a valid subset of S.)

1.10 Fill in the blanks in the proof of the following theorem.

Theorem. $A \subseteq B$ *iff* $A \cup B = B$.

Proof. Suppose that $A \subseteq B = B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. Since $A \subseteq B$, in either case, we have $x \in B$. Thus $A \subseteq B$. On the other hand, if $x \in B$, then $x \in A \cup B$, so $A \subseteq B$. Hence $A \cup B$ B.
Conversely, suppose that $A \cup B = B$. If $x \in A$, then $x \in A \cup B$. But $A \cup B = B$, so $x \in B$. Thus $A \subseteq B$. □

1.14 Which statements below would enable one to conclude that $x \in A \cup B$?

(a) If $x \in A$ and $x \in B$, then $x \in A \cup B$.
(b) If $x \in A$ or $x \in B$, then $x \in A \cup B$.
(c) This statement tells us $A \subseteq B$, but does not let us conclude anything about x.
(d) If $x \notin A$, then $x \in B$ implies that if $x \notin B$, then $x \in A$. Since x must always be either in A or B, we can conclude that $x \in A \cup B$.

1.16 Which statements below would enable one to conclude that $x \in A \setminus B$?

(a) If $x \in A$ and $x \notin B \setminus A$, then x could be in $A \cap B$, so we cannot conclude that $x \in A \setminus B$.
(b) If $x \in A \cup B$ and $x \notin B$, then $(A \cup B) \cap U \setminus B$. Equivalently, $A \cap U \setminus B \cup B \cap U \setminus B$, and $A \setminus B \cup \emptyset$, or $A \setminus B$. Therefore we can conclude that $x \in A \setminus B$.
(c) If $x \in A \cup B$ and $x \notin A \cap B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \setminus B$. But if $x \in B$, then $x \notin A \setminus B$. So we cannot conclude that $x \in A \setminus B$.

(d) If $x \in A$ and $x \notin A \cap B$, then $x \notin B$. Thus $x \in A \setminus B$

1.18 Prove that the empty set is unique. That is, suppose that A and B are empty and prove that $A = B$.

Proof. Suppose that A and B are empty sets. If $A = B$, then $A \subseteq B$ and $B \subseteq A$.
If $A \subseteq B$, then if $x \in A$, $x \in B$. By the definition of an empty set, for all x, $x \notin \emptyset$. Therefore, for all x, $x \notin A$ and $x \notin B$. By logical implication, then, if $x \in A$, $x \in B$. This implies that $A \subseteq B$.
On the other hand. if $x \in B$, then $x \in A$. By the same argument, we conclude $B \subseteq A$. So, we have $A \subseteq B$ and $B \subseteq A$. Thus $A = B$. □

1.20 Prove: $A \cap B$ and $A \setminus B$ are disjoint and $A = (A \cap B) \cup (A \setminus B)$.

Proof. Assume the conclusion that $(A \cap B) \cap (A \setminus B) = \emptyset$.
If $x \in (A \cap B)$, then $x \in A$ and $x \in B$. But then $x \notin (A \setminus B)$, because if $x \in A \setminus B$, then $x \in A$ and $x \notin B$. In other words, if $x \in (A \cap B)$, $x \in (A \setminus (A \setminus B))$. So $(A \cap B) \nsubseteq (A \setminus B)$.
Conversely, if $x \in (A \setminus B)$, then $x \in A$ and $x \in B$. But then $x \notin (A \cap B)$. So $(A \setminus B) \nsubseteq (A \cap B)$.
Thus, $(A \cap B) \cap (A \setminus B) = \emptyset$. Therefore $(A \cap B)$ and $(A \setminus B)$ are disjoint.
In other words:

$$
\begin{aligned}
&(A \cap B) \cap (A \setminus B) && (1.1)\\
&= (A \cap B) \cap (A \cap U \setminus B) && (1.2)\\
&= (U \setminus A \cup U \setminus B) \cup (U \setminus A \cup B) && (1.3)\\
&= (U \setminus A) \cup (U \setminus B \cup B) && (1.4)\\
&= A \cap (B \cap U \setminus B) && (1.5)\\
&= A \cap \emptyset && (1.6)\\
&= \emptyset && (1.7)
\end{aligned}
$$

□

Proof. To prove $A = (A \cap B) \cup (A \setminus B)$, first show that if $x \in (A \cap B) \cup (A \setminus B)$, then $x \in A$.
If $x \in (A \cap B) \cup (A \setminus B)$, then $x \in (A \cup A \setminus B)$ and $x \in (B \cup A \setminus B)$. Suppose $x \in (A \cap B) \cup (A \setminus B)$.
If $x \in A$, then $x \in (A \cup A \setminus B)$ and $x \in (B \cup A \setminus B)$, so $x \in (A \cap B) \cup (A \setminus B)$.
If $x \in B$, then $x \in (B \cup A \setminus B)$ (equivalently, $(A \cup B)$), but $x \notin (A \cup A \setminus B)$ (equivalently, A).
So by necessity, we have $x \in A$, and $(A \cap B) \cup (A \setminus B) = A$.

On the other hand, suppose $x \in A$. Then $x \in (A \cap B)$ or $x \in (A \setminus B)$:

$$
\begin{aligned}
&A && (1.8)\\
&= A \cap U && (1.9)\\
&= A \cap (B \cup U \setminus B) && (1.10)\\
&= (A \cap B) \cup (A \cap U \setminus B) && (1.11)\\
&= (A \cap B) \cup (A \setminus B) && (1.12)
\end{aligned}
$$

Either way, $x \in (A \cap B) \cup (A \setminus B)$. So we have $A = (A \cap B) \cup (A \setminus B)$.

□