Sets and Functions, Chapter Exercises

L. J.

May 26, 2020

1 SECTION 1

1.2 Mark each statement True or False. Justify each answer.

(a) *False*. Let $A = \{x \in \mathbb{R} : x \text{ is prime}\}$ and let $B = \{x \in \mathbb{R} : x \text{ is divisible by } 6\}$. Then $A \cap B = \emptyset$, but $A \neq \emptyset$ and $B \neq \emptyset$.

(b) *True*. Let $A = \{1, 3, 5\}$ and let $B = \{2, 4, 6\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$. If $x \in A \cup B$, then x is equal to one of those values. If $x = 4$, then it is also true that $x \in B$. By the same logic more generally, if $x \in A \cup B$, then $x \in A$ or $x \in B$.

(c) *False*. If $x \in A \setminus B$, then $x \in A$ *and* $x \notin B$. Let $C = \{x : (x \in A) \wedge (x \in B)\}$. If $x \in C$, then if $x \in A \setminus B \overset{?}{\Longrightarrow} (x \in A) \vee (x \notin B)$, it is also true $x \in A \setminus B$. But this is at odds with the definition of the relative complement of B with respect to A.

(d) *False*. It's fine to begin this way; the only nontrivial proof is one in which S is nonempty.

1.4 Let $A = \{2, 4, 6, 8\}$, $B = \{6, 8, 10\}$, and $C = \{5, 6, 7, 8\}$. Find the following sets.

(a) $\{6, 8\}$
(b) $\{2, 4, 6, 8, 10\}$
(c) $\{2, 4\}$
(d) $\{6, 8\}$
(e) $\{10\}$
(f) $\{5, 7, 10\}$
(g) $\emptyset$
(h) $\{5, 7\}$

1.6 Let A and B be subsets of a universal set U. Simplify each of the following expressions.

(a) U
(b) $\emptyset$
(c) $A \cap B$
(d) $A \cup B$
(e) A
(f) A

1.8 Let $S = \{\emptyset, \{\emptyset\}\}$. Determine whether each of the following is True or False. Explain your answers.

(a) *True.* $\forall x \in \emptyset$, $x \in S$. Therefore, $\emptyset \subseteq S$.
(b) *True.* S is strictly enumerated, and $\emptyset$ is one of its elements.
(c) *True.* $\{\emptyset\} \subseteq S$ because for all $x \in \{\emptyset\}$, $x \in S$. Unlike in (a), though, the proof is not vacuous; $\emptyset$ is an element of both sets.
(d) *True.* $\{\emptyset\}$ is an element in S, as defined by enumeration. (To contrast (c) and (d), $\{\{\emptyset\}\}$ would also be a valid subset of S.)

1.10 Fill in the blanks in the proof of the following theorem.

Theorem. $A \subseteq B$ *iff* $A \cup B = B$.

Proof. Suppose that $A \subseteq B = B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. Since $A \subseteq B$, in either case, we have $x \in B$. Thus $A \subseteq B$. On the other hand, if $x \in B$, then $x \in A \cup B$, so $A \subseteq B$. Hence $A \cup B$ B.
Conversely, suppose that $A \cup B = B$. If $x \in A$, then $x \in A \cup B$. But $A \cup B = B$, so $x \in B$. Thus $A \subseteq B$. □

1.14 Which statements below would enable one to conclude that $x \in A \cup B$?

(a) If $x \in A$ and $x \in B$, then $x \in A \cup B$.
(b) If $x \in A$ or $x \in B$, then $x \in A \cup B$.
(c) This statement tells us $A \subseteq B$, but does not let us conclude anything about x.
(d) If $x \notin A$, then $x \in B$ implies that if $x \notin B$, then $x \in A$. Since x must always be either in A or B, we can conclude that $x \in A \cup B$.

1.16 Which statements below would enable one to conclude that $x \in A \setminus B$?

(a) If $x \in A$ and $x \notin B \setminus A$, then x could be in $A \cap B$, so we cannot conclude that $x \in A \setminus B$.
(b) If $x \in A \cup B$ and $x \notin B$, then $(A \cup B) \cap U \setminus B$. Equivalently, $A \cap U \setminus B \cup B \cap U \setminus B$, and $A \setminus B \cup \emptyset$, or $A \setminus B$. Therefore we can conclude that $x \in A \setminus B$.
(c) If $x \in A \cup B$ and $x \notin A \cap B$, then $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \setminus B$. But if $x \in B$, then $x \notin A \setminus B$. So we cannot conclude that $x \in A \setminus B$.

(d) If $x \in A$ and $x \notin A \cap B$, then $x \notin B$. Thus $x \in A \setminus B$

1.18 Prove that the empty set is unique. That is, suppose that A and B are empty and prove that $A = B$.

Proof. Suppose that A and B are empty sets. If $A = B$, then $A \subseteq B$ and $B \subseteq A$.
If $A \subseteq B$, then if $x \in A$, $x \in B$. By the definition of an empty set, for all x, $x \notin \emptyset$. Therefore, for all x, $x \notin A$ and $x \notin B$. By logical implication, then, if $x \in A$, $x \in B$. This implies that $A \subseteq B$.
On the other hand. if $x \in B$, then $x \in A$. By the same argument, we conclude $B \subseteq A$. So, we have $A \subseteq B$ and $B \subseteq A$. Thus $A = B$. □

1.20 Prove: $A \cap B$ and $A \setminus B$ are disjoint and $A = (A \cap B) \cup (A \setminus B)$.

Proof. Assume the conclusion that $(A \cap B) \cap (A \setminus B) = \emptyset$.
If $x \in (A \cap B)$, then $x \in A$ and $x \in B$. But then $x \notin (A \setminus B)$, because if $x \in A \setminus B$, then $x \in A$ and $x \notin B$. In other words, if $x \in (A \cap B)$, $x \in (A \setminus (A \setminus B))$. So $(A \cap B) \nsubseteq (A \setminus B)$.
Conversely, if $x \in (A \setminus B)$, then $x \in A$ and $x \in B$. But then $x \notin (A \cap B)$. So $(A \setminus B) \nsubseteq (A \cap B)$.
Thus, $(A \cap B) \cap (A \setminus B) = \emptyset$. Therefore $(A \cap B)$ and $(A \setminus B)$ are disjoint.
In other words:

$$(A \cap B) \cap (A \setminus B) \tag{1.1}$$
$$= (A \cap B) \cap (A \cap U \setminus B) \tag{1.2}$$
$$= (U \setminus A \cup U \setminus B) \cup (U \setminus A \cup B) \tag{1.3}$$
$$= (U \setminus A) \cup (U \setminus B \cup B) \tag{1.4}$$
$$= A \cap (B \cap U \setminus B) \tag{1.5}$$
$$= A \cap \emptyset \tag{1.6}$$
$$= \emptyset \tag{1.7}$$

□

Proof. To prove $A = (A \cap B) \cup (A \setminus B)$, first show that if $x \in (A \cap B) \cup (A \setminus B)$, then $x \in A$.
If $x \in (A \cap B) \cup (A \setminus B)$, then $x \in (A \cup A \setminus B)$ and $x \in (B \cup A \setminus B)$. Suppose $x \in (A \cap B) \cup (A \setminus B)$.
If $x \in A$, then $x \in (A \cup A \setminus B)$ and $x \in (B \cup A \setminus B)$, so $x \in (A \cap B) \cup (A \setminus B)$.
If $x \in B$, then $x \in (B \cup A \setminus B)$ (equivalently, $(A \cup B)$), but $x \notin (A \cup A \setminus B)$ (equivalently, A).
So by necessity, we have $x \in A$, and $(A \cap B) \cup (A \setminus B) = A$.

On the other hand, suppose $x \in A$. Then $x \in (A \cap B)$ or $x \in (A \setminus B)$:

$$A \tag{1.8}$$
$$= A \cap U \tag{1.9}$$
$$= A \cap (B \cup U \setminus B) \tag{1.10}$$
$$= (A \cap B) \cup (A \cap U \setminus B) \tag{1.11}$$
$$= (A \cap B) \cup (A \setminus B) \tag{1.12}$$

Either way, $x \in (A \cap B) \cup (A \setminus B)$. So we have $A = (A \cap B) \cup (A \setminus B)$.

□

2 Section 2

2.2 Mark each statement True or False. Justify each answer.

(a) *True.* If $\mathscr{P}$ is a partition of S, then if x and y are in the same part of the partition, we can define xRy as an equivalence relation such that the equivalence classes are the same as $\mathscr{P}$.
(b) *False.* If xRx for all $x \in S$, then R is reflexive. But not all relations are reflexive.
(c) *False.* A partition of S can be defined by $\{E_x : x \in S\}$, where E_x represents an equivalence class. An equivalence class is defined by $E_x = \{y \in S : yRx\}$.
(d) *True.* By definition, a partition $\mathscr{P}$ consists of nonempty, disjoint sets such that for all $x \in S$, x belongs to some subset of $\mathscr{P}$.

2.4 Show that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

(a, b) is the set whose members are $\{a\}$ and $\{a, b\}$.

$$(a, b) = \{\{a\}, \{a, b\}\}$$

And the product of two sets A and B is

$$A \times B\ \{(a, b) : a \in A \text{and} b \in B\}$$

Then,

$$\begin{aligned}\{a\} \times \{a\} &= \{(a, a)\} \\ &= \{\{\{a\}, \{a, a\}\}\}.\end{aligned}$$

Since $a = a$, $\{a, b\} = \{a\}$, and the above simplifies to

$$\begin{aligned}&= \{\{\{a\}\{a\}\}\} \\ &= \{\{\{a\}\}\}.\end{aligned}$$

2.6 Let A be be any set and let $B = \emptyset$. What can you conclude about $A \times B$?

Let A be any set, and let B be $\emptyset$. By definition,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

and

$$\emptyset = \{x : \forall x, x \notin \emptyset\}.$$

Then if $B = \emptyset$, the product $A \times B$ requires that all the ordered pairs in the product have a first element in A and a second element in $\emptyset$. But now, $\forall b, b \notin B$, so there are no ordered pairs that satisfy the definition of the product of $A \times B$.
Therefore, $A \times B = \emptyset$.

2.8 Let $A = \{1,2\}$.

(a) **How many elements are in the set $A \times A$?** $|A \times A| = 4$.

(b) **How many possible relations are there on the set A?** A relation on A is any subset R of $A \times A$ ($R \subset A \times A$); any element can be in or out of the relation. So, what is the power set of A?

$$2^{|A\times A|} = 2^4 = 16$$

(c) **How many possible relations are there on the set $\{1,2,3\}$?** For A, we had $|A| = 2$. Let S be a generic set, $n = |S|$ be the number of elements in the set, and $|R|$ be the total number of possible relations on S. Then for S_n, we have $|R| = 2^{n^2}$. If $S = \{1,2,3\}$, then $n = 3$ and we have

$$|R| = 2^{(3^2)} = 2^9 = 512$$

possible relations.

2.10 Prove or give a counterexample.

(a) Let $A = \{2,3\}$ and $B = \{0,-5\}$. Then

$$A \times B = \{(2,0),(2,-5),(3,0),(3,5)\}$$

and

$$B \times A = \{(0,2),(0,3),(-5,2),(-5,3)\}.$$

Therefore

$$A \times B \neq B \times A.$$

(b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
Let

$$A = \{x : x \in [0,3)\}$$

$$B = \{x : x \in (2,4]\}$$

$$C = \{x : x \in [-2,2]\}.$$

Then

$$(A \cup B) \times C = \{(x,y) : x \in [0,4] \text{ and } y \in [-2,2]$$

We have:

$$A \times C = \{(x,y) : x \in [0,3) \text{ and } y \in [-2,2]\}$$

$$B \times C\ \{(x,y) : x \in (2,4] \text{ and } y \in [-2,2]\}$$

and

$$(A \times C) \cup (B \times C) = \{(x,y) : x \in [0,4] \text{ and } y \in [-2,2]\}.$$

(c) TK

(d) TK

2.12 Find examples of relations with the following properties.

Let $S = \{p, q, r\}$. Then define the following relations R on S:

(a) **R is reflexive, but not symmetric or transitive.**
$R = \{(p, p), (q, q), (r, r,)\}$

(b) **R is symmetric, but not reflexive or transitive.**
$R = \{(p, q), (q, p), (r, q), (q, r)\}$

(c) **R is transitive, but not symmetric or reflexive.**
$R = \{(p, q), (q, r), (p, r)\}$

(d) **R is reflexive and symmetric, but not transitive.**
$R = \{(p, p), (q, q), (r, r), (p, q), (q, p), (q, r), (r, q)\}$

(e) **R is reflexive and transitive, but not symmetric.**
$R = \{(p, p), (q, q), (r, r), (p, q), (q, r), (p, r)\}$

(f) **R is symmetric and transitive, but not reflexive.**
$R = \{(p, q), (q, p), (q, r), (r, q), (p, r), (r, p)\}$

2.14 Let $S = \mathbb{R} \times \mathbb{R}$. Verify that the relation $(a, b)R(c, d)$ iff $a + d = b + c$ is an equivalence relation. Describe the equivalence class $E_{(7,3)}$.

Rearrange the relation as:

$$\begin{aligned} a + d &= b + c \\ a + d - c &= b \\ a - c &= b - d \end{aligned}$$

The relation is reflexive: $(a, b)R(a, b)$ gives

$$a - a = b - b$$

$$0 = 0$$

The relation is symmetric:

$$\begin{gathered} (c, d)R(a, b) \\ c - a = d - b \\ -c + a = -d + b \\ a - c = b - d \end{gathered}$$

The relation is transitive:

$$(a, b)R(c, d) \wedge (c, d)R(m, n) \implies (a, b)R(m, n)$$

$$TKTKTKTK$$

The equivalence class $E_{(7,3)} = \{d \in S : d = c - 4\}$:

$$\begin{aligned} 7 + d &= 3 + c \\ d &= c - 4 \end{aligned}$$

It describes a linear relationship.

2.16 Let $S = \mathbb{R} \times \mathbb{R}$. Define the equivalence relation R on S by $(a,b)R(c,d)$ iff $b - 3a = d - 3c$.

(a) We can rearrange the relation as:

$$
\begin{aligned}
b - d &= 3a - 3c \\
b - d &= 3(a - c)
\end{aligned}
$$

So, R partitions S into parallel lines of the form $y = 3x\ (+k)$.

(b) $E_{(2,5)}$ would be the part of the partition containing the point $(2,5)$, which we obtain by substituting it for one-half of the ordered pairs:

$$
\begin{aligned}
b - 3a &= 5 - 3(2) \\
b - 3a &= -1x && = 3a - 1
\end{aligned}
$$

So the equivalence class is described by the linear relationship $y = 3x - 1$.

2.18 Let R be the relation $\{(1,1),(2,2),(3,1),(3,3)\}$ on the set $\{1,2,3\}$. If R is an equivalence relation, list the pieces of the partition determined by R, and if not, state why.

R is not an equivalence relation because it is not symmetric. $(3,1) \in R$ but $(1,3) \notin R$.

2.20 Let $S = \{a,b,c,d\}$ and let $P = \{\{a\},\{b,c,d\}\}$. Describe the equivalence relation R on S determined by P by listing the ordered pairs in the relation.

We see that a is in a partition alone, but b, c and d are in a relation with each other, so the relation consists of the pairs

$$
\begin{aligned}
&\{(a,a), \\
&\ (b,b),(c,c),(d,d), \\
&\ (b,c),(b,d), \\
&\ (c,b),(c,d), \\
&\ (d,c),(d,b)\}.
\end{aligned}
$$

3 Section 3

3.2 Mark each statement True or False. Justify your answer.

(a) *True.* By the existence requirement of the definition of a function, for all a in A, there must exist some b in B such that $f(a) = b$. Since $C \subset A$, it follows that every

c in C is also in A, and so for all c in C, there must be some b in B such that $f(c) = b$. Thus $f(C)$ is a nonempty subset of B.

(b) *False.* If f is surjective, it is a mapping form all elements in the domain A to all elements in B, that is to say the range of f is equal to its codomain. But though $f^{-1}(y)$ is guaranteed to exist for $y \in B$, it is not guaranteed to be one-to-one unless f is also injective. Therefore $f^{-1}(y)$ is a subset ($\subset$) of A, rather than an element in ($\in$) A.

(c) *False.* If B is the codomain of f, and D is a subset of B, then unless f is surjective, there is no guarantee that D intersects the image of A in B.

(d) *True.* Let $f : A \to B$ and $g : B \to C$ be surjective functions. Since g is surjective, the range of g is equal to C, and for all $b \in B$, there exists $c \in C$. Since f is surjective, the range of f is equal to B, and for all $a \in A$, there exists $b \in B$. So $g \circ f(a) = g(f(a)) = g(b) = c$. Therefore $g \circ f$ is surjective.

(e) *True.* If f is bijective, then its domain is A and its range and codomain are B. Since f is a function, each x in f corresponds to only one y in B. Therefore, if $(x, y) \in f$, it must follow that the inverse of f is bijective as well, and $(y, x) \in f^{-1}$.

(f) *False.* The identity function maps $\mathbb{R}$ onto $\mathbb{R}$ such that for all $x \in \mathbb{R}$, $(i)x = x$.

3.4 Find all possible functions $f : A \to B$ in each case below. Describe the functions by listing their ordered pairs.

(a) $f = \{(1,5),(2,5),(3,5)\}$.

(b) $f_1 = \{(4,5)\}, f_2 = \{(4,6)\}$.

(c) $f_1 = \{(1,5),(2,6)\}, f_2 = \{(1,5),(2,5)\}, f_3 = \{(2,5),(1,6)\}, f_4 = \{(2,5),(1,6)\}$

3.6 Let $A \subset \mathbb{R}$ and define $f : A \to B$ as given below. In each case describe an A that is as large as possible while making f injective.

(a) $A = [5,\infty)$ or $A = (-\infty,5]$.

(b) $A = [-0.5,\infty)$ or $A = (-\infty,-0.5]$.

(c) $A = [-\frac{\pi}{2},\frac{\pi}{2}]$ or $A = [\frac{\pi}{2},\frac{3\pi}{2}]$ (for example).

3.8 Circles in the plane

(a) Let S be the set of all circles in the plane. Define $f : S \to [0,\infty)$ by $f(C)$ = the area of C for all $C \in S$.
Assuming a circle can have a radius $r = 0$, a function that maps all circles to the value of their area is surjective but not injective, because two circles at different coordinate positions can nonetheless have equal area.

(b) Let T be the set of all circles in the plane that are centered at the origin. Define $g : T \to [0,\infty)$ by $g(C)$ = the area of C for all $C \in T$.
Now, g is both injective and surjective; for every circle defined by radius length r in the interval $[0,\infty)$ there is an area $g(C)$ in $[0,\infty)$, and since all circles are centered at the origin, any circle with the same area has the same radius length and is the same circle.

3.10 In each part, find a function $\mathbb{N} \to \mathbb{N}$ that has the desired properties.

(a) Surjective, but not injective:

$$f(x) = \begin{cases} x-1, & \text{for } x > 1, \\ 1 & \text{for } x = 1. \end{cases} \tag{3.1}$$

(b) Injective, but not surjective:

$$f(x) = 2x. \tag{3.2}$$

(c) Neither surjective nor injective:

$$f(x) = 1 \tag{3.3}$$

(d) Bijective:

$$f(x) = x. \tag{3.4}$$

3.14 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$, Find $f^{-1}(T)$ for each of the following.

(a) $\{-3, 3\}$
(b) $(-3, -2] \cup [2, 3)$
(c) $[-3, 3]$

3.16 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Find examples to show that equality does not hold in parts (a), (b), and (c) of Theorem 2.3.16.

(a) Show that $C \neq f^{-1}[f(C)]$.
Let C and C' be nonempty subsets of R such that $C \cap C' = \emptyset$ and $f(C) = f(C')$. In that case, $f^{-1}[f(C)] = C \cup C'$, and $C \subset C \cup C'$, so $C \neq f^{-1}[f(C)]$.

3.20 Suppose that $f : A \to B$ and suppose that $C \subset A$ and $D \subset B$.

(a) Prove or give a counterexample: $f(C) \subset D \iff C \subset f^{-1}(D)$.
Suppose $f(C) \subset D$ and suppose $x \in C$. Since $C \subset A$, $x \in A$. Since $x \in C$, $f(x) \in f(C)$ and therefore $f(x) \in D$, by supposition.
So $x \in A$, and $f(x) \in D$, meaning that $x \in f^{-1}(D)$, where $f^{-1}(D) = \{x \in C : f(x) \in D\}$.