

Logic and Proof, Chapter Exercises

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1 SECTION 1

1.1 Mark each statement true or false. Justify your answer.

- (a) *False.* A sentence need not be true in order to be classified a “statement”. We require only that it “can be clearly labeled” true *or* false.
We should specify the conditions that constitute clarity, and who’s doing the labeling. We can condition truth and falsity on any number of assumptions—at best, explicit; at worst, implied—and the sentence’s author ordinarily assumes a reader with a mental model that mirrors their own, for better or worse.
- (b) *False.* In binary logic, a statement cannot be true and false at the same time. Its truth value may be as-yet unknown—it might be a conjecture—but it has to follow the rules of mathematical proof. That is, to be a statement, the rules must be able to label a sentence exclusively “true” or “false” under *some* set of specified conditions. Those conditions may change: $x + 6 = 0$ is a statement, for certain values of x , but not for all.
- (c) *True.* The negation of a statement need only have the opposite truth value to the statement itself. So if p is true, $\neg p$ is false, and $\neg(\neg p)$ is true.
- (d) *False.* The negation of a statement is by definition the logical complement of the given proposition. Thus, we can see that either p can be false or $\neg p$ can be false, but they cannot both be false at the same time.
- (e) *True.* In logic, “or” is inclusive by default, and we must specify that we mean “exclusive or” (\oplus) (or its negation, the biconditional (\iff)) to denote $(p \vee q) \wedge \neg(p \wedge q)$.

1.2 Mark each statement True or False. Justify each answer.

- (a) *False.* We refer to p as the antecedent (that which comes (“ced-”) before (“ante-”). In contrast, q is the consequent, or logical “result”. In other words, p is called the

sufficient condition, and q is called the necessary condition.

- (b) *True*. The implication applies only to worlds in which p is true. Since we have only binary truth, we cannot rule out “truth” in worlds not containing p . Hence, these cases are true, and the only false one in that in which p exists, but the necessary condition q does not.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- (c) *False*. This formulation customarily evaluates to the same thing as “ q whenever p ”, because p is a sufficient condition to “trigger” the requirement of q . In other words, whenever we have p , we must also have its necessary condition, q . But as the truth table in (b) illustrates, the statement is true when q is true but p is false, and so the reverse dependency does not hold. Thus, it’s incorrect to say that “if p , then q ” is equivalent to “ p whenever q ”.
- (d) *True*. The negation of a conjunction is logically equivalent to the disjunction of the negations of its individual components.

p	q	$\neg(p \wedge q)$	\iff	$[(\neg p) \vee (\neg q)]$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

- (e) *False*. The negation of $p \Rightarrow q$ is not $q \Rightarrow p$.

p	q	$\neg(p \Rightarrow q)$	\iff	$q \Rightarrow p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	F	F	T

1.3 Write the negation of each statement.

- The 3×3 identity matrix is not singular.
- The function $f(x) = \sin(x)$ is unbounded on \mathbb{R} .
- The functions f and g are nonlinear.
- Six is not prime and seven is even.
- x is in D , and $f(x) \geq 5$.
- (a_n) is monotone and bounded, and (a_n) is divergent.
- f is injective, but S is not finite and S is not denumerable.

1.4 Write the negation of each statement.

- $f(x)$ is discontinuous at $x = 3$.
- R is not reflexive and not symmetric.

- (c) 4 and 9 are not relatively prime.
 (d) $x \notin A \vee x \in B$.
 (e) $x < 7$ and $f(x)$ is in C .
 (f) (a_n) is convergent and (a_n) is unbounded or not monotone.
 (g) f is continuous and A is open, but $f^{-1}(A)$ is closed.

1.5 Identify the antecedent and the consequent in each statement.

- (a) • *Antecedent:* M is singular.
 • *Consequent:* M has a zero eigenvalue.
 (b) • *Antecedent:* Linearity.
 • *Consequent:* Continuity.
 (c) • *Antecedent:* A sequence is Cauchy.
 • *Consequent:* It is bounded.
 (d) • *Antecedent:* $y > 5$.
 • *Consequent:* $x < 3$.

1.6 Identify the antecedent and the consequent in each statement.

- (a) • *Antecedent:* [A sequence] is Cauchy.
 • *Consequent:* [It] is convergent.
 (b) • *Antecedent:* Boundedness.
 • *Consequent:* Convergence.
 (c) • *Antecedent:* Orthogonality.
 • *Consequent:* Invertibility.
 (d) • *Antecedent:* K is closed and bounded.
 • *Consequent:* K is compact.

1.7 Construct a truth table for each statement.

	p	q	$p \Rightarrow \neg q$		
(a)	T	T	F		
	T	F	T		
	F	T	T		
	F	F	T		
	p	q	$p \wedge (p \Rightarrow q)$	\Rightarrow	q
(b)	T	T	T	T	T
	T	F	F	T	F
	F	T	F	T	T
	F	F	F	T	F
	p	q	$p \Rightarrow (q \wedge \neg q)$	\Longleftrightarrow	$\neg p$
(c)	T	T	F	T	F
	T	F	F	T	F
	F	T	T	T	T
	F	F	T	T	T

1.8 Construct a truth table for each statement.

	p	q	$p \vee \neg q$
	T	T	T
(a)	T	F	T
	F	T	F
	F	F	T

	p	$p \wedge \neg p$
(b)	T	F
	F	F

	p	q	$[(\neg q) \wedge (p \Rightarrow q)]$	\Rightarrow	$\neg p$
	T	T	F	T	F
(c)	T	F	F	T	F
	F	T	F	T	T
	F	F	T	T	T

1.9 Indicate whether each statement is True or False.

- (a) $T \wedge T \Rightarrow T$.
- (b) $F \vee T \Rightarrow T$.
- (c) $F \vee F \Rightarrow F$.
- (d) $(T \Rightarrow T) \Rightarrow T$.
- (e) $(F \Rightarrow F) \Rightarrow T$.
- (f) $(T \Rightarrow F) \Rightarrow F$.
- (g) $[(T \wedge F) \Rightarrow T] \Rightarrow T$.
- (h) $[(T \vee F) \Rightarrow F] \Rightarrow F$.
- (i) $[(T \wedge F) \Rightarrow F] \Rightarrow T$.
- (j) $\neg(F \vee T) \Leftrightarrow T \wedge F \Rightarrow F$.

1.10 Indicate whether each statement is True or False.

- (a) $T \wedge F \Rightarrow F$.
- (b) $F \vee F \Rightarrow F$.
- (c) $F \vee T \Rightarrow T$.
- (d) $(T \Rightarrow F) \Rightarrow F$.
- (e) $(F \Rightarrow F) \Rightarrow T$.
- (f) $(F \Rightarrow T) \Rightarrow T$.
- (g) $[(F \vee T) \Rightarrow F] \Rightarrow F$.
- (h) $[(T \Leftrightarrow F) \Rightarrow T] \Rightarrow T$.
- (i) $[(T \wedge T) \Rightarrow F] \Rightarrow F$.
- (j) $\neg(F \wedge T) \Leftrightarrow T \vee F \Rightarrow T$.

1.11 Let p be the statement “The figure is a polygon,” and let q be the statement “The figure is a circle”. Express each of the following statements in symbols.

- (a) $p \wedge \neg q$.
- (b) $(p \vee q) \wedge \neg(p \wedge q)$.

- (c) $\neg q \implies p$.
- (d) $q \implies \neg p$.
- (e) $p \iff \neg q$.

1.12 Let m be the statement “ x is perpendicular to M ”, and let n be the statement “ x is perpendicular to N ”. Express each of the following statements in symbols.

2 SECTION 2

2.1 *