

Logic and Proof, Chapter Exercises

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March 28, 2020

1 SECTION 1

1.1 Mark each statement true or false. Justify your answer.

- (a) *False.* A sentence need not be true in order to be classified a “statement”. We require only that it “can be clearly labeled” true *or* false.
We should specify the conditions that constitute clarity, and who’s doing the labeling. We can condition truth and falsity on any number of assumptions—at best, explicit; at worst, implied—and the sentence’s author ordinarily assumes a reader with a mental model that mirrors their own, for better or worse.
- (b) *False.* In binary logic, a statement cannot be true and false at the same time. Its truth value may be as-yet unknown—it might be a conjecture—but it has to follow the rules of mathematical proof. That is, to be a statement, the rules must be able to label a sentence exclusively “true” or “false” under *some* set of specified conditions. Those conditions may change: $x + 6 = 0$ is a statement, for certain values of x , but not for all.
- (c) *True.* The negation of a statement need only have the opposite truth value to the statement itself. So if p is true, $\neg p$ is false, and $\neg(\neg p)$ is true.
- (d) *False.* The negation of a statement is by definition the logical complement of the given proposition. Thus, we can see that either p can be false or $\neg p$ can be false, but they cannot both be false at the same time.
- (e) *True.* In logic, “or” is inclusive by default, and we must specify that we mean “exclusive or” (\oplus) (or its negation, the biconditional (\iff)) to denote $(p \vee q) \wedge \neg(p \wedge q)$.

1.2 Mark each statement True or False. Justify each answer.

- (a) *False.* We refer to p as the antecedent (that which comes (“ced-”) before (“ante-”). In contrast, q is the consequent, or logical “result”. In other words, p is called the

sufficient condition, and q is called the necessary condition.

- (b) *True*. The implication applies only to worlds in which p is true. Since we have only binary truth, we cannot rule out “truth” in worlds not containing p . Hence, these cases are true, and the only false one in that in which p exists, but the necessary condition q does not.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- (c) *False*. This formulation customarily evaluates to the same thing as “ q whenever p ”, because p is a sufficient condition to “trigger” the requirement of q . In other words, whenever we have p , we must also have its necessary condition, q . But as the truth table in (b) illustrates, the statement is true when q is true but p is false, and so the reverse dependency does not hold. Thus, it’s incorrect to say that “if p , then q ” is equivalent to “ p whenever q ”.
- (d) *True*. The negation of a conjunction is logically equivalent to the disjunction of the negations of its individual components.

p	q	$\neg(p \wedge q)$	\iff	$[(\neg p) \vee (\neg q)]$
T	T	F	T	F
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

- (e) *False*. The negation of $p \Rightarrow q$ is not $q \Rightarrow p$.

p	q	$\neg(p \Rightarrow q)$	\iff	$q \Rightarrow p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	F	F	T

1.3 Write the negation of each statement.

- The 3×3 identity matrix is not singular.
- The function $f(x) = \sin(x)$ is not bounded on \mathbb{R} .
- The functions f and g are nonlinear.
- Six is not prime and seven is even.
- x is in D , and $f(x) \geq 5$.
- (a_n) is monotone and bounded, and (a_n) is divergent.
- f is injective, but S is not finite and S is not denumerable.

1.4 Write the negation of each statement.

- $f(x)$ is not continuous at $x = 3$.
- R is not reflexive and not symmetric.

- (c) 4 and 9 are not relatively prime.
 (d) $x \notin A \vee x \in B$.
 (e) $x < 7$ and $f(x)$ is in C .
 (f) (a_n) is convergent and (a_n) is unbounded or not monotone.
 (g) f is continuous and A is open, but $f^{-1}(A)$ is closed.

1.5 Identify the antecedent and the consequent in each statement.

- (a) • *Antecedent:* M is singular.
 • *Consequent:* M has a zero eigenvalue.
 (b) • *Antecedent:* Linearity.
 • *Consequent:* Continuity.
 (c) • *Antecedent:* A sequence is Cauchy.
 • *Consequent:* It is bounded.
 (d) • *Antecedent:* $y > 5$.
 • *Consequent:* $x < 3$.

1.6 Identify the antecedent and the consequent in each statement.

- (a) • *Antecedent:* [A sequence] is Cauchy.
 • *Consequent:* [It] is convergent.
 (b) • *Antecedent:* Boundedness.
 • *Consequent:* Convergence.
 (c) • *Antecedent:* Orthogonality.
 • *Consequent:* Invertibility.
 (d) • *Antecedent:* K is closed and bounded.
 • *Consequent:* K is compact.

1.7 Construct a truth table for each statement.

	p	q	$p \Rightarrow \neg q$		
(a)	T	T	F		
	T	F	T		
	F	T	T		
	F	F	T		
	p	q	$p \wedge (p \Rightarrow q)$	\Rightarrow	q
(b)	T	T	T	T	T
	T	F	F	T	F
	F	T	F	T	T
	F	F	F	T	F
	p	q	$p \Rightarrow (q \wedge \neg q)$	\Longleftrightarrow	$\neg p$
(c)	T	T	F	T	F
	T	F	F	T	F
	F	T	T	T	T
	F	F	T	T	T

1.8 Construct a truth table for each statement.

	p	q	$p \vee \neg q$
	T	T	T
(a)	T	F	T
	F	T	F
	F	F	T

	p	$p \wedge \neg p$
(b)	T	F
	F	F

	p	q	$[(\neg q) \wedge (p \Rightarrow q)]$	\Rightarrow	$\neg p$
	T	T	F	T	F
(c)	T	F	F	T	F
	F	T	F	T	T
	F	F	T	T	T

1.9 Indicate whether each statement is True or False.

- (a) $T \wedge T \Rightarrow T$.
- (b) $F \vee T \Rightarrow T$.
- (c) $F \vee F \Rightarrow F$.
- (d) $(T \Rightarrow T) \Rightarrow T$.
- (e) $(F \Rightarrow F) \Rightarrow T$.
- (f) $(T \Rightarrow F) \Rightarrow F$.
- (g) $[(T \wedge F) \Rightarrow T] \Rightarrow T$.
- (h) $[(T \vee F) \Rightarrow F] \Rightarrow F$.
- (i) $[(T \wedge F) \Rightarrow F] \Rightarrow T$.
- (j) $\neg(F \vee T) \Leftrightarrow T \wedge F \Rightarrow F$.

1.10 Indicate whether each statement is True or False.

- (a) $T \wedge F \Rightarrow F$.
- (b) $F \vee F \Rightarrow F$.
- (c) $F \vee T \Rightarrow T$.
- (d) $(T \Rightarrow F) \Rightarrow F$.
- (e) $(F \Rightarrow F) \Rightarrow T$.
- (f) $(F \Rightarrow T) \Rightarrow T$.
- (g) $[(F \vee T) \Rightarrow F] \Rightarrow F$.
- (h) $[(T \Leftrightarrow F) \Rightarrow T] \Rightarrow T$.
- (i) $[(T \wedge T) \Rightarrow F] \Rightarrow F$.
- (j) $\neg(F \wedge T) \Leftrightarrow T \vee F \Rightarrow T$.

1.11 Let p be the statement “The figure is a polygon,” and let q be the statement “The figure is a circle”. Express each of the following statements in symbols.

- (a) $p \wedge \neg q$.
- (b) $(p \vee q) \wedge \neg(p \wedge q)$.

- (c) $\neg q \implies p$.
 (d) $q \implies \neg p$.
 (e) $p \iff \neg q$.

1.12 Let m be the statement “ x is perpendicular to M ”, and let n be the statement “ x is perpendicular to N ”. Express each of the following statements in symbols.

- (a) $n \wedge \neg m$.
 (b) $\neg(m \vee n)$.
 (c) $n \implies m$
 (d) $m \implies \neg n$
 (e) $\neg(m \wedge n) \iff \neg m \vee \neg n$

1.13 Define “nor” (∇) as:

p	q	$p \nabla q$
T	T	F
T	F	F
F	T	F
F	F	T

	p	$\neg p$	\iff	$p \nabla p$
(a)	T	F	T	F
	F	T	T	T

	p	q	$(p \nabla p)$	\iff	$(q \nabla q)$
(b)	T	T	F	T	F
	T	F	F	F	T
	F	T	T	F	F
	F	F	T	F	T

	p	q	$p \wedge q$
(c)	T	T	T
	T	F	F
	F	T	F
	F	F	F

1.14 Use truth tables to verify that each of the following is a tautology.

Commutative laws

(a)	p	q	$(p \wedge q)$	\iff	$(q \wedge p)$
	T	T	T	T	T
	T	F	F	T	F
	F	T	F	T	F
	F	F	F	T	F

p	q	$(p \vee q)$	\iff	$(q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

(b) *Associative laws*

p	q	r	$[p \wedge (q \wedge r)]$	\iff	$[(p \wedge q) \wedge r]$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

p	q	r	$[p \vee (q \vee r)]$	\iff	$[(p \vee q) \vee r]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	T	F

(d) *Distributive laws*

p	q	r	$[p \wedge (q \vee r)]$	\iff	$[(p \wedge q) \vee (p \wedge r)]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

p	q	r	$[p \vee (q \wedge r)]$	\iff	$[(p \vee q) \wedge (p \vee r)]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

(f)					
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2 SECTION 2

2.2 Mark each statement true or false. Justify your answer.

- (a) *False.* \exists denotes the existence at least one. \forall is the universal quantifier and means “for each, for every”.
- (b) *True.* Universal quantification is assumed if a variable is used in the antecedent of an implication without an explicit quantifier.
- (c) *True.* For instance, $\forall x \exists y$ corresponds to the statement “For every disease, there exists a cure”, while $\exists x \forall y$ corresponds to the statement “There is a panacea for all diseases.”

2.4 Write the negation of each statement.

- (a) Someone does not like Robert.
- (b) No students work part-time.
- (c) Some square matrix is triangular.
- (d) $\forall x \in B, f(x) \leq k$.
- (e) $\exists x \exists (x > 5) \wedge [(f(x) \geq 3) \wedge (f(x) \leq 7)]$.
- (f) $\exists x \in A \exists \forall y \in B, f(x) \geq f(y)$.

2.6 Determine the truth value of each statement, assuming x is a real number. Justify your answer.

- (a) *True.* Let $x = 4$. $3 < 4$ and $4 < 5$, so 4 is included in the interval.
- (b) *False.* Let $x = 3$. Then $3 \in [3, 5]$ and $3 < 4$.
- (c) *True.* Let $x = 3$. Then $3^2 = 9 \neq 3$.
- (d) *False.* Let $x = \sqrt{3}$. Then x is a real number and $(\sqrt{3})^2 = 3$.
- (e) *False.* Let $x = 0$. Then $0^2 = 0$.
Let $x > 0$. Then for all x , x^2 is positive.
Let $x < 0$. Then for all x , x^2 is also positive.
Therefore, for all real x , $x \neq -5$.
- (f) *False.* Let $x = -5$. Then $(-5)^2 = 25 \neq -5$.
- (g) *True.* Let $x = x$. $x - x = 0 \iff x = x$. (This is the additive identity of a field.)
- (h) *True.* By the same token, it is always true that for all x , $x - x = 0$.

2.8 Which of the following best identifies f as a constant function, where x and y are real numbers?

$\exists y \exists \forall x, f(x) = y$ represents a constant function.

2.10 Determine the truth value of each statement, assuming that x and y are real numbers. Justify your answer.

- (a) *True.* For all $x \in \mathbb{R}$, let $y = 0$. Then $xy = 0$.
- (b) *False.* Let $x = 0$. Then for all y , $xy = 0 \neq 1$.
- (c) *False.* For all $y \in \mathbb{R}$, there is a real number $x = 0$ such that $xy = 0(y) = 0 \neq 1$.

- (d) *True*. Let $y = 1$. Then for all $x \in \mathbb{R}$, $xy = x(1) = x$.

2.12 Determine the truth value of each statement, assuming that x , y , and z are real numbers. Justify your answer.

- (a) *True*. Suppose by contradiction that for all $x, y, z \in \mathbb{Z}$, there exist some x and y such that $x + y \neq z$.
Then some integer would have no successor, which violates the properties of \mathbb{Z} .
So there must be some x and y such that $x + y = z$.
- (b) *False*. Let $x = 0$.
Then, for any y , let $z = y + 0$, or $z = y$.
Let $z = 1$. Since $y \neq 1$ for all $y \in \mathbb{R}$, the statement is false.
- (c) *True*. Let x be any real number other than zero. Then, given any y , let $z = \frac{y}{x}$.
- (d) *False*. Let $x = 2$. Then $z \leq 2 + y$.
Then for all y , we must show there exists a z such that $y < z \leq x + y$, or $0 < z - y \leq x$.
Assuming $x = 2$, we must show there is some z such that $y < z \leq 2 + y$.
Let $z = y + 1$. Then $y < y + 1 < y + 2$. Therefore, the statement is false, because $z < x + y$.
- (e) *False*. Assume by contradiction that $y < z \leq x + y$, or $0 < z - y \leq x$.
Let $x = 2$. Then for all z , $y < z \leq y + 2$.
Let $y = z - 1$, such that $z - 1 < z \leq z + 1$.
Then $z > z - 1$ and $z \leq 2 + (z - 1) = z + 1$ for all z .
Since the negation is true, the statement is false.

2.14 Rewrite the defining conditions in logical symbolism, and write the negation.

- (a) $\exists k > 0 \exists \forall x, f(x + k) = f(x) \implies f \text{ is periodic.}$
(b) $\neg(f \text{ is periodic}) \implies \forall x > 0, \exists x \exists f(x + k) \neq f(x).$

2.16 Rewrite the defining conditions in logical symbolism, and write the negation.

- (a) $[\forall x, y, x < y \implies f(x) > f(y)] \implies f \text{ is strictly decreasing.}$
(b) $\neg(f \text{ is strictly decreasing}) \implies \exists x, y \ni (x < y) \wedge f(x) \leq f(y).$

2.18 Rewrite the defining conditions in logical symbolism, and write the negation.

- (a) $[\forall y \in B \exists x \in A \ni f(x) = y] \implies f \text{ is surjective.}$
(b) $\neg(f \text{ is surjective}) \implies \exists y \in B \ni \forall x \in A, f(x) \neq y.$

2.20 Rewrite the defining conditions in logical symbolism, and write the negation.

- (a) $\forall \epsilon > 0 \exists \delta > 0 \ni \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$
- (b) $\exists \epsilon > 0 \ni \forall \delta > 0, \exists x, y \in S \ni (|x - y| < \delta) \wedge (|f(x) - f(y)| \geq \epsilon).$

3 SECTION 3

3.2 Mark each statement True or False. Justify each answer.

- (a) *True.* In this context the antecedent is called the conclusion.
- (b) *False.* A statement that is always false is called a contradiction.
- (c) *True.* The converse of $p \implies q$ is $q \implies p$.
- (d) *True.* A universal statement can be proved false using a single example.
- (e) *False.* To prove an existential statement is false, we must show $\forall n, \neg p(n)$.

3.4 Write the converse of each implication in exercise 1.3.3.

- (a) If some violets are blue, then all roses are red.
- (b) If A is not invertible, then there exists a nontrivial solution to $A\mathbf{x} = 0$.
- (c) If $f(C)$ is not connected, then f is not continuous or C is not connected.

3.5 Write the inverse of each implication in exercise 1.3.3.

- (a) If some roses are not red, then all violets are not blue.
- (b) If there exists no nontrivial solution to $A\mathbf{x} = 0$, then A is invertible.
- (c) If f is not continuous or C is not connected, then $f(C)$ is not connected.

3.6 Provide a counterexample for each statement.

- (a) Let $x = -4$.
- (b) Let $n = -2$.
- (c) Let $x = \frac{1}{3}$.
- (d) An equilateral triangle has all three angles equal to $\frac{\pi}{3}$ rad.
- (e) Let $n = 41$.
- (f) 2 is prime and not odd.
- (g) Let $x = 101$.
- (h) Let $n = 5$.
- (i) Let $n = 5$.
- (j) Let $x = 12$ and $y = 3$.
- (k) Let $x = 0$.
- (l) Let $x = 1$.
- (m) Let $x = 0$.
- (n) Let $x = -1$.

3.10 Using tautologies from Example 3.12, establish the desired conclusion from the list of hypotheses.

(a)

$\neg r$	Hypothesis	(1)
$\neg r \Rightarrow (s \Rightarrow r)$	Hypothesis and 3.12(n)	(2)
$s \Rightarrow r$	Lines 1,2 and 3.12(h)	(3)
$\neg r \Rightarrow \neg s$	Contrapositive of Line 3, 3.12(c)	(4)
$\neg s$	Lines 1,4 and 3.12(h)	(5)
(3.1)		

(b)

$\neg t$	Hypothesis	(1)
$(r \Rightarrow t) \wedge (s \Rightarrow t)$	Hypothesis and 3.12(q)	(2)
$(r \Rightarrow t) \wedge (\neg t \Rightarrow \neg s)$	Line 2 and 3.12(c)	(3)
$t \vee \neg s$	Lines 1, 3 and 3.12(o)	(4)
$\neg s$	Line 4 and 3.12(j)	(5)
(3.2)		

(c)

$s \vee t$	Hypothesis	(1)
$s \Rightarrow \neg r$	Hypothesis and 3.12(c)	(2)
$t \Rightarrow u$	Hypothesis	(3)
$\neg r \vee u$	Lines 1-3 and 3.12(o)	(4)
(3.3)		