Problem Set 1 Solutions

How to Read This Handout

Over the course of the quarter, we'll be releasing solution sets for our problem sets. These solution sets are designed to give you

- a sense of the major insights and techniques behind each problem,
- some examples of what we consider good solutions to each problem, and
- an understanding of why each question was specifically asked of you.

When you're looking over solutions, don't just look at what we have and compare it blindly against what you have. There are many equivalently good ways to write the same proof, and our short-answer questions rarely have a single "best" answer. Instead, try to get an understanding of the structure and formatting of the solution. Did our solution route use the same insights as your solution, or did we solve the problem in a totally different way? If we solved it the same way, compare how we wrote up our solution to how you wrote up yours. Is there anything you can gain from doing so? If we solved it in a different way, take a minute to think about our solution strategy, and compare it against yours. Did you find an alternative, equally valid approach to solving the problem? Or did you make use of a different set of observations?

As you're thinking these details over, feel free to ask questions on Campuswire or to stop by office hours to chat with us. We're happy to help out!

Once you've finished doing that, read over the "Why We Asked This Question" section at the end of each problem. The problem sets in this course are specifically calibrated to get you to think about particular nuances or to get you to engage with certain topics. It's important to make sure you understand what each problem was going for, since commonly the questions we've included are designed to set you up to solve future problems in the same vein.

Problem One: Much Ado About Nothing

i. What is $\emptyset \cup \{\emptyset\}$?

$\emptyset \cup \{\emptyset\} = \{\emptyset\}$. There are no elements in $\emptyset$ and just one element of $\{\emptyset\}$ (namely, $\emptyset$), so gathering all the elements common to the two sets together gives us that $\emptyset \cup \{\emptyset\} = \{\emptyset\}$. This line of reasoning generalizes: $\emptyset \cup S = S$ for *any* set S.

ii. What is $\emptyset \cap \{\emptyset\}$?

$\emptyset \cap \{\emptyset\} = \emptyset$. There are no elements of $\emptyset$, so there cannot be any elements in both $\emptyset$ and $\{\emptyset\}$, meaning that their intersection must be empty. More generally, $\emptyset \cap S = \emptyset$ for *any* set S.

iii. What is $\{\emptyset\} \cup \{\{\emptyset\}\}$?

$\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$. There's one element in $\{\emptyset\}$, namely $\emptyset$, and one element in $\{\{\emptyset\}\}$, namely $\{\emptyset\}$, so if we make a set containing all these elements we get $\{\emptyset, \{\emptyset\}\}$.

iv. What is $\{\emptyset\} \cap \{\{\emptyset\}\}$?

$\{\emptyset\} \cap \{\{\emptyset\}\} = \emptyset$. There are no elements common to two these two sets, since one contains $\emptyset$ and the other contains $\{\emptyset\}$. Remember that $\emptyset \neq \{\emptyset\}$!

v. What is $\wp(\wp(\emptyset))$?

Notice that $\wp(\emptyset) = \{\emptyset\}$. Therefore, we need to find $\wp(\{\emptyset\})$. The set $\{\emptyset\}$ has two subsets: the empty set, which is a subset of all sets, and itself. Therefore, $\wp(\wp(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.

vi. What is $\wp(\wp(\wp(\emptyset)))$?

To determine $\wp(\wp(\wp(\emptyset)))$, let's begin by replacing $\wp(\wp(\emptyset))$ by $\{\emptyset, \{\emptyset\}\}$. That means that we need to determine $\wp(\{\emptyset, \{\emptyset\}\})$. This set has four subsets: one subset of size zero ($\emptyset$), two subsets of size one ($\{\emptyset\}$ and $\{\{\emptyset\}\}$), and one subset of size two ($\{\emptyset, \{\emptyset\}\}$). Gathering all these together gives us this final result:

$$\wp(\wp(\wp(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

Why we asked this question: We introduced a lot of notation in the first lecture (inclusion, subset, power set, cardinality, and set operators) and wanted to make sure everyone was comfortable with them before moving on. Typically, we don't ask questions that just test whether you understand the notation, but in this case we felt it would be good to “stress-test” it in isolation to make sure that everyone had a firmer grasp of set theory before jumping into the next problem.

Problem Three: Describing the World in Set Theory

i. Let's have C be the set of US citizens, S the set of people who live in a US state, M be the set of all people eighteen and older, and V be the set of people who are allowed to vote in US presidential elections. Write an expression that says that every US citizen age eighteen and older who lives in a US state can vote in a US presidential election.

Here's one option:

$$C \cap S \cap M \subseteq V.$$

This says that any US citizen who also lives in a US state and also is 18 or older can vote in a US presidential election. Note that it doesn't say that these are the *only* people who can vote in such an election, just that they are allowed, which is why we used subset instead of equality. There may be people who aren't in *C, S,* and *M* who are allowed to vote; our statement didn't describe them one way or the other.

ii. Suppose you're on an exciting first date. Let Y represent your hobbies and D represent your date's hobbies. Write an expression that says that you have a hobby that your date doesn't have.

There are a few ways to do this. One option is

$$Y \nsubseteq D,$$

which says that there must be some hobby of yours (an element of Y) that isn't also a hobby that your date has (an element of D). You could also write

$$Y - D \neq \emptyset,$$

which says that if you take your hobbies and filter out your date's hobbies, there's still something left.

iii. The song "I am Moana" from the movie *Moana* starts off with the following lyrics:

"I know a girl from an island / She stands apart from the crowd."

Let K be the set of all people I know, G be the set of all girls, L be the set of all people from an island, and C be the set of all people who stand apart from the crowd. Write an expression that expresses the above lyric using the notation of set theory.

There are many different ways to write this expression. The main observation is that to talk about the idea that *there exists* some person with some properties, we can talk about the set of all people with those properties, then argue that it isn't empty.

The set of all girls I know from an island who stand apart from the crowd is $K \cap G \cap L \cap C$. To say that this set isn't empty, we could write something like

$$K \cap G \cap L \cap C \neq \emptyset,$$

which directly says that the set isn't empty, or we could use set cardinalities and write something like

$$|K \cap G \cap L \cap C| \neq 0$$

or

$$|K \cap G \cap L \cap C| > 0.$$

I personally prefer the first one, but these other ones are completely reasonable as well.

iv. Let's say that a ***committee*** is a group of people, which we can think of as being represented by the set of people on that committee. Let's have S represent the set of all students at Stanford and let F represent the set of all faculty at Stanford. Write an expression representing the set of all committees you can make from Stanford students and faculty that contain at least one student and at least one faculty member. You can assume no one is both a student and a faculty member.

Here's one possibility:

$$\wp(S \cup F) - \wp(S) - \wp(F).$$

This says "start with all possible committees, then remove from consideration any committee that has no faculty members on it ($\wp(S)$) and any committee that has no students on it ($\wp(F)$)."

Another option is

$$\wp(S \cup F) - (\wp(S) \cup \wp(F)),$$

which expresses the same idea but groups together the two sets to remove.

Why we asked this question: We included this question partially to give you more practice sorting out the differences between all the different operations and connections we've introduced between sets. We hoped you'd learn to recognize concepts like subsets, power sets, unions, and intersections in cases where the plain English statement says nothing to that effect.

We also included this question to give you more experience expressing your ideas using the notation that we've provided. As we continue on in our exploration of discrete mathematics, we'll often use set theory notation to convey nuanced ideas in a small space. By practicing set theory translations with higher-level ideas, we hoped that you'd build up this skill and be ready to see some more advanced applications later on.

Problem Four: Proof Critiques, Part One

i. Critique this proof about parities (the ***parity*** of a number is whether it's even or odd.)

Theorem: The sum of an even integer and an odd integer is odd.

Proof: This proof will talk about adding together different kinds of numbers. An even integer is an integer that can be written as $2k$ for some integer k. Therefore, $m = 2k$. Similarly, an odd integer is one that can be written as $2k+1$ for some integer k. So $n = 2k+1$. $m + n = 2k + 2k + 1 = 4k + 1$. Therefore $m + n$ is odd. ■

First, let's focus on the proof's style. We'll go one point at a time through the proofwriting checklist.

Clearly articulate your start and end points: While the end of this proof ("Therefore, $m + n$ is odd") is well-articulated, the starting point is not very clear. We only learn implicitly that the proof is doing something with two natural numbers by looking at the context in which the variables m and n appear.

Make each sentence "load-bearing." Does the first sentence of this proof actually clarify anything about the argument being made? I'd say that it's not sufficiently precise to warrant keeping around. The statement of the theorem, and the remainder of the proof, each tell us that that we're going to be adding numbers together, and this isn't something elucidated well by that first sentence.

Scope and properly introduce variables. The variables m and n in this proof appear to come out of nowhere. The first appearance of m ("Therefore, $m = 2k$") only makes sense if you understand that it's supposed to be defining a new variable, as is the case with n. Similarly, the variable k is at times a placeholder ("an integer that can be written as $2k$ for some integer k"), but that's something we'll revisit later.

Make specific claims about specific variables. Take a look at the sentences about even integers and about odd integers. Those claims are very broad and talk about how numbers *in general* work, rather than focusing on specific numbers and their properties.

Don't repeat definitions – use them instead. These same sentences flagged in the preceding point also are, essentially, just restatement of definitions in the abstract. They can be safely removed.

Write in complete sentences and complete paragraphs. Sentences like "$m + n = 2k + 2k + 1 = 4k + 1$" fail the "mugga mugga" test because they have no subject and no verb. They need to be cleaned up a bit.

Next, let's look at correctness. The statement being proved here is actually true, but there's a logic error in the proof. We state that n is odd by writing it as $n = 2k + 1$. However, at this point, the variable k is already in use (we defined it so that $m = 2k$), so this doesn't say "n is twice *something* plus one," but rather that $n = m + 1$. We can easily patch that by using a different name for that integer. Here's a restructured version of the proof:

Theorem: The sum of an even integer and an odd integer is odd.

Proof: Let m be an even integer and n be an odd integer. Since m is even, we see that $m = 2k$ for some integer k. Similarly, since n is odd, we know that $n = 2r + 1$ for some integer r. Combining these expressions, we see that

$$\begin{aligned} m + n &= 2k + 2r + 1 \\ &= 2(k + r) + 1. \end{aligned}$$

Therefore, there is an integer q, namely, $k + r$, such that $m + n = 2q + 1$. Therefore, $m + n$ is odd, as required. ■

ii. Critique this proof about natural numbers.

Theorem: Every natural number is odd.

Proof: Assume for the sake of contradiction that every natural number is even. In particular, that would mean that 137 is even. Since $137 = 2 \cdot 68 + 1$ and 68 is a natural number, we see that 137 is odd. We know that there is no integer n where n is both odd and even. However, $n = 137$ is both even and odd. This is impossible. We've reached a contradiction, so our assumption must have been wrong. Therefore, every natural number is odd. ■

Let's begin by going through the proofwriting checklist. Here's the violations we noted:

Scope and properly introduce your variables. This particular proof doesn't have many variables in it, but there is an issue with the variable n. The first time we see n, it's a placeholder variable – a variable that's a stand-in for "integers in general" rather that something that refers to a single value. This issue arises from an issue that we'll dive into later on.

Make specific claims about specific variables. The sentence we flagged above, while nominally referring to a variable n, isn't really a claim about a specific variable. Instead, it's a discussion about how integers in general behave. Although this isn't a "definition" per se, this does also run afoul of the the spirit behind the "Don't restate definitions; use them instead" rule in the checklist. We really should just come out and say "137 is both odd and even, which is impossible" rather than adding the layer of indirection.

And now for the correctness checks. Unfortunately, this theorem isn't true, since we can quickly check that there are infinitely many natural numbers that aren't odd (say, 0, 2, 4, or 6). The specific logic error being made here is in the setup of the proof by contradiction. The negation of the statement "every natural number is odd" is *not* "every natural number is even," but rather "*some* natural number is even." This means that we aren't assuming the negation of the original statement, and so what we end up with isn't a correct proof by contradiction.

iii. Critique this proof about modular arithmetic.

Theorem: If n is an integer and $n \equiv_2 0$, then $n \equiv_4 0$.

Proof: We will prove the contrapositive of this statement and prove that if n is an integer where $n \equiv_4 0$, then $n \equiv_2 0$. Since $n \equiv_4 0$, we know $n = 0 + 4q$. This in turn tells us that $n = 2(2q)$, so there is an integer m such that $n = 2m$. Consequently, we see that $n = 0 + 2m$, and so $n \equiv_2 0$, as required. ■

Let's begin with a style critique.

Scope and properly introduce variables: There are three spots in this proof where the variable definitions could use some work. First, notice that in the first sentence, introducing what the proof is going to do, it mentions that the proof will demonstrate something of the form "if n has some property, it has some other property." It's fine to do this here, because we're announcing what we're going to do. However, in transitioning from that sentence to the next one, we begin working with the variable n as though it had been given some specific value. We really should do a better job introducing that variable, perhaps with something as simple as "choose an arbitrary n." Second, there's the (lack of) an introduction of the variable q. In expanding out the definition of $n \equiv_4 0$, we really should introduce the variable q in some way (for example, "there is an integer q such that..."), since otherwise it appears to come out of nowhere. Finally, there's the (lack of) an introduction of the variable m. The proof says that there is an integer m where $n = 2m$, but doesn't explain how it arrives at this conclusion. In this case, we should probably point out that we're picking $m = 2q$ so that the reader can see what's going on.

And now for the logic analysis. The logic in this proof is, unfortunately, incorrect from the get-go. The contrapositive of the statement "if n is an integer and $n \equiv_2 0$, then $n \equiv_4 0$" isn't the one stated here. While we have interchanged the antecedent and consequent, we've forgotten to negate them, so what we have isn't the contrapositive. The new implication we get back is indeed true, and so the rest of the proof works out. It's just not equivalent to the original statement.

In fact, the original statement is false. Notice, for example, that $2 \equiv_2 0$, but that $2 \not\equiv_4 0$.

Why we asked this question: Have you ever listened to an interview with a famous musician, or author, or film director, about where they get their inspiration from? While in each of these cases the creator is likely to add their own flavor to the mix, almost universally you'll find them saying things like "I was inspired by the subversiveness of *Get Out*" or "I saw Nick Cage make these crazy suits and thought about what that would be like in music form," etc. In other words, artists are always looking at what other artists are doing, and they use that to inform their craft. They produce better works by seeing other successful works.

Proofwriting works very much the same way. You get better at writing proofs by seeing other proofs and figuring out what about them you like and don't like. This is one of the reasons we allow – and encourage! – you to work in pairs. It gives you a chance to see other approaches and bounce ideas off one another.

At the same time, by having you read proofs that *aren't* great, or that contain subtle issues, we're hoping that you'll get a better appreciation for why we've introduced the specific criteria that we use during grading. These items are there both to force you to write better proofs, but also to make it harder for you to make mistakes. The logic error in part (i) of this problem is closely tied to the poor introduction of the variable n, for example.

Problem Five: Modular Arithmetic, Part Two

Prove or disprove: for any integers x, y, z, and k, if $x \equiv_k y$ then $xz \equiv_k yz$.

Proof: Let x, y, and k be any integers where $x \equiv_k y$. Now, choose an arbitrary integer z. We will prove that $xz \equiv_k yz$.

Since $x \equiv_k y$, we know that there is an integer q where $x = y + kq$. Multiplying both sides of this equality by z tells us that

$$\begin{aligned} xz &= yz + kqz \\ &= yz + k(qz). \end{aligned}$$

Therefore, there is an integer r such that $xz = yz + kr$, and so $xz \equiv_k yz$, as required. ■

Prove or disprove: for any integers x, y, z, and k, if $z \neq 0$ and $xz \equiv_k yz$, then $x \equiv_k y$.

Disproof: Consider the integers $x = 3$, $y = 6$, $z = 8$ and $k = 12$. Choosing $q = -2$, we see that

$$\begin{aligned} xz &= 3 * 8 \\ &= 24 \\ &= 48 - 24 \\ &= (6*8) + (12 * -2) \\ &= yz + kq. \end{aligned}$$

Thus, we can conclude that $xz \equiv_k yz$. However, we note that there is no integer r that satisfies

$$3 = 6 + 12 * r$$

as the only real solution is $r = -1/4$. Thus, x is not equivalent to y modulo k. Therefore, it is not true for any integers x, y, z and k that if $z \neq 0$ and $xz \equiv_k yz$, then $x \equiv_k y$. ■

Why we asked this question: We included this question for a number of reasons. We wanted to give you some practice looking at mathematical expressions and trying to tease out whether they're true or false. That's something that, as we hinted in the problem, isn't something you can necessarily do simply by inspection. You often have to get your hands dirty and just see what ends up happening.

The above justification is a good one for why we'd want you to do a prove-or-disprove type problem at all, but why this particular problem? We chose this problem because we wanted to give you a chance to show us, once more, that you have a good handle on how to write proofs involving modular congruence. The checkpoint proofs are somewhat tricky, and we hoped that, by this point, you'd have internalized your feedback and gotten a solid handle on these styles of arguments.

The pair of these problems also helps think about how small alterations in statements can change a proof into a disproof!

Problem Six: Properties of Sets

i. For all sets A, B, and C, if $A \in B$ and $B \in C$, then $A \in C$.

This statement is false.

Claim: For all sets A, B, and C, if $A \in B$ and $B \in C$, then $A \in C$.

Disproof: We will show that there exist sets A, B, and C where $A \in B$ and $B \in C$, but $A \notin C$. Choose $A = \emptyset$, $B = \{A\}$, and $C = \{B\}$. In this case, we see $A \in B$ and $B \in C$, but that $A \notin C$ because C only contains one element, namely B, and that element isn't A. ■

ii. If A, B, and C are sets where $A - C = B - C$, then $A = B$.

This statement is false.

Claim: If A, B, and C are sets where $A - C = B - C$, then $A = B$.

Disproof: We will show that there are sets A, B, and C where $A - C = B - C$, but $A \neq B$. For example, choose $A = \mathbb{N}$, $B = \mathbb{Z}$, and $C = \mathbb{R}$. Then we see that $\mathbb{N} - \mathbb{R} = \emptyset$ and $\mathbb{Z} - \mathbb{R} = \emptyset$, but also $\mathbb{N} \neq \mathbb{Z}$. ■

iii. Prove or disprove: if A and B are sets where $\wp(A) = \wp(B)$, then $A = B$.

This statement is true.

Theorem: If A and B are sets where $\wp(A) = \wp(B)$, then $A = B$.

Proof: Consider any sets A and B where $\wp(A) = \wp(B)$. We will prove that $A = B$. To do so, we will prove that $A \subseteq B$ and that $B \subseteq A$. Because the roles of A and B in this proof are symmetric, we can just show that $A \subseteq B$.

Consider an arbitrary $x \in A$. We will show that $x \in B$. Since $x \in A$, $\{x\} \subseteq A$, and thus $\{x\} \in \wp(A)$. But this means that $\{x\} \in \wp(B)$. And thus $\{x\} \subseteq B$, so $x \in B$ which is what we wanted to show. Thus, $A \subseteq B$. ■

iv. Prove or disprove: if A, B, and C are sets where $A \Delta C = B \Delta C$, then $A = B$.

This statement is, surprisingly, true! Here's one way to see this.

Theorem: If A, B, and C are sets where $A \Delta C = B \Delta C$, then $A = B$.

Proof: Consider any sets A, B, and C where $A \Delta C = B \Delta C$. We will prove that $A = B$. To do so, we will prove that $A \subseteq B$ and that $B \subseteq A$. Because the roles of A and B in this proof are symmetric, we can just show that $A \subseteq B$.

Consider an arbitrary $a \in A$. We need to show that $a \in B$. To do so, we consider two cases.

Case 1: $a \in C$. In that case, we know that $a \in A$ and $a \in C$, so $a \notin A \Delta C$. Since $A \Delta C = B \Delta C$, this means that $a \notin B \Delta C$. This in turn means that either $a \in B$ and $a \in C$ or that $a \notin B$ and $a \notin C$. But since we know that $a \in C$, this latter case is not possible, and so we learn that both $a \in B$ and $a \in C$. That, in particular, means that $a \in B$, as required.

Case 2: $a \notin C$. This means that $a \in A$ and $a \notin C$, and so $a \in A \Delta C$. Since $A \Delta C = B \Delta C$, we conclude that $a \in B \Delta C$. Therefore, either $a \in B$ and $a \notin C$ or $a \notin B$ and $a \in C$. In this case we're assuming that $a \notin C$, and so the latter option is not possible. This means $a \in B$ and $a \notin C$, so in particular $a \in B$.

In both cases we see that $a \in B$, which is what we needed to show. ■

Why we asked this question: We included this question for a number of reasons.

First, at a high level, we wanted to give you more practice reading statements in set theory and building an understanding of what they mean. This problem is sneaky in that the first two statements both seem fairly plausible – it seems like the ∈ relation ought to be transitive like ⊆ is, and from your experience with regular old numbers it seems like if $A - C = B - C$, then A should equal C. However, in both cases, those claims were actually not true due to nuances of how the formal definitions work. The remaining statements are a bit more nuanced and require some trial and error to sort out, and we hoped this would help you get comfortable taking complex statements and just playing and tinkering around with them to see what you find.

Second, we wanted to give you practice writing proofs on set theory that call back to the appropriate definitions. You've gotten practice writing general proofs that refer back to definitions in the course of the problems on modular arithmetic, but we've found historically that it takes people a while to get used to doing this properly in the land of set theory. We hope that this problem gave you a better sense for how those proofs work. Specifically, we wanted you to get practice working with power sets, which in part (iii) are used to switch between element-of and subset-of, and to prove that sets are equal by showing they're subsets of one another (parts (iv) and (v)).

Third, we thought that the last two parts of this problem (the proofs) were a great testbed for different styles of proofs. The most natural way we know to prove the result from part (iv) involves working with a proof by cases, and the result from part (iii) can be done in several different ways.

Fourth, these problems push hard on the difference between the ∈ and ⊆ relations, something we wanted to give you more practice with. It's certainly interesting to compare and contrast parts (iii) and (iv).

Finally, at least one of the theorems you proved here might show up later on in this course as a building block in a larger result. You might want to keep an eye out for where power sets or symmetric differences come up. 😃

Problem Seven: Tiling with Polyominoes

i. Prove or disprove: it's possible to tile a 9 x 10 chessboard with Tetris pieces.

This statement is false.

As a general rule, it is good to try to do both parts of a "prove or disprove" question. Draw some pictures, make some examples, and see if you can either find a working tiling or if it seems like no tiling is going to work out.

Statement: It's possible to tile a 9 x 10 chessboard with tetris pieces.

Disroof: Suppose that we have a tiling of a 9 x 10 chessboard that uses n pieces for some integer n. Since each tetris piece has 4 squares, this tiling covers $4n$ squares as overlapping is not allowed. However, there is no integer n such that $90 = 4n$, which would be necessary if this tiling did exist. We have reached a contradiction, so it is impossible to tile a 9 x 10 chessboard with tetris pieces. ■

ii. Prove or disprove: it's possible to perfectly tile a 4 x 7 chessboard with seven Tetris pieces.

This statement is false.

The reasoning behind this is subtle. It is a good idea to play around with some examples of this, and try to visualize what putting these pieces onto a chessboard looks like.

Statement: it's possible to perfectly tile a 4 x 7 chessboard with seven Tetris pieces.

Disproof: On a 4 x 7 chessboard, their will be 14 black squares and 14 white squares. When placed on a chess board, each Tetris piece will cover 2 black squares and 2 white squares, except for the "T" piece, which will cover 3 squares of one color and one square of the other color. We can then see that any tiling using all seven Tetris pieces will cover an odd number of black squares and an odd number of white squares. Thus it is impossible to perfectly tile a 4 x 7 chessboard with seven Tetris pieces. ■

Why we asked this question: This question further expands the idea of modular congruence by invoking *parity arguments*. The first part has no solution because 90 has a different "4-parity". It is impossible to express 90 as a multiple of 4. The second follows a similar idea using "2-parity". Being able to classify objects into certain categories and then make claims about those categories is a large part of mathematics. Here, we tested the ability to go from a problem statement to thinking about these categories. Don't worry if you didn't immediately understand this question, even after looking at the solution! The point of these problems is to encourage different avenues of thought and teach the experience of what to do when you're stuck on a problem. Now you have a new set of tools for considering similar questions in the future.

Problem Eight: Yablo's Paradox

i. Prove that if any statement in this list is true, it results in a contradiction.

Theorem: If any statement in this list is true, it results in a contradiction.

Proof: Suppose that some statement (S_n) in the list is true. Then every statement that comes after it must be false. In particular, this means that statement (S_{n+1}) must be false. Since statement (S_{n+1}) states that every statement after it is false, this means that there must actually be some true statement after statement (S_{n+1}); let's call that statement (S_m). But since we assumed statement (S_n) is true, every statement after (S_n) should be false, including (S_m). This results in a contradiction. ■

ii. Prove that every statement in this list is a paradox.

Theorem: Every statement (S_n) in this list is a paradox.

Proof: Consider any statement (S_n) in this list. If statement (S_n) is true, then by our result from part (i) we get a contradiction. So suppose instead that statement (S_n) is false. Because statement (S_n) asserts that every statement after it is false, our assumption that statement (S_n) is false means that there must be some later statement (S_m) that is true. By our previous lemma, since some statement in the list is true, we get a contradiction.

Collectively, we've shown that if any statement in the list is true, it results in a contradiction, and if any statement in the list is false, it results in a contradiction. Therefore, every statement in this list is a paradox. ■

iii. For each statement in the above list, determine whether it's true or false and explain why your choices are consistent with one another.

First, note that statement ($T_{9,999,999,999}$) is vacuously true because there are no statements that come after it. Therefore, every statement (T_n) that comes before it is false, since they incorrectly claim that all future statements, including statement ($T_{9,999,999,999}$), are false. This means that none of the statements are paradoxes – the first 9,999,999,999 of them are false and the last of them is true.

Why we asked this question: We chose to ask parts (i) and (ii) of this question to test your ability to reason about the negations of universal statements. Every statement in the list is a universal statement that makes a claim about every statement that comes after it. In order to see why the statements are all paradoxes, you needed to recognize that the negation of the statement "Every statement after this one is false" is the statement "Some statement after this one is true." Additionally, we chose to ask this question to see how you handled a multipart proof. The logic for why a statement being false causes a contradiction follows from the fact that if any statement is true, it results in a contradiction. We expected that it would take a few tries before you'd find the most elegant way of organizing your thoughts.

We chose to ask part (iii) of this question to explore vacuous truths and some of the weirdness that arises with infinity. Isn't infinity weird?

As a general policy, we don't release solutions to the Optional Fun Problems. After all, they're here to get you thinking about the concepts in new ways! However, we'll try to include some hints about how to solve them in the solution sets so that if you hopped on the Struggle Bus, tried to crack them, and didn't quite see it, you can get a pointer or two about where to look.

Optional Fun Problem One: Hat Colors

What happens if you number the colors 0, 1, 2, …, 9? Now you're dealing with numbers rather than colors, which makes things a bit easier. Then think about the sums of these numbers modulo ten.

Optional Fun Problem Two: Infinite Deviation

Convince yourself of the following: to solve this problem, you need a way of picking one entry out of each *column* in the table such that each row has infinitely many items picked out of it. Then see if you can find a way to accomplish this. A hint: try counting from 0, lots and lots of times, for different lengths of times. Or write the numbers in binary and think about which bit changes as you count.