Physics 230 Notes

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Contents

Chapter 1

Electric Charge and Electric Field

1.1 Electric Charge

Electromagnetism (EM) affects only charged particles, mainly electrons and protons. All particles have charges that are integer multiples of the elementary charge e such that the charge is given by

$$q = ne, \tag{1.1}$$

where q represents charge (C), n represents an integer, and $e = 1.6022 \times 10^{-19}$ C represents the elementary charge.

Electric charge is conserved. This means that the total charge of any isolated system with no charge moving in or out stays the same – charge is never created or destroyed.

Coulomb's Law states that charges of the same sign repel and charges of the opposite sign attract. Furthermore, the force F produced by charges can be calculated via

$$F = k\frac{q_1 q_2}{r^2}, \tag{1.2}$$

where $k = 9.0 \times 10^9 \,\frac{\mathrm{Nm}^2}{\mathrm{C}^2}$, q_1 represents the first charge, q_2 represents the second charge, and r represents the separation distance. Note that k can also be represtented as

$$k = \frac{1}{4\pi\epsilon_0}. \tag{1.3}$$

The **principle of superposition of forces** is the vector sum of the individual forces:

$$F = k\frac{q_1 q_2}{(r_{12})^2} r_{12} + k\frac{q_1 q_2}{(r_{13})^2} r_{13} = q_1 \vec{E}. \tag{1.4}$$

1.2 Electric Field and Electric Forces

The **electric field of a point charge** at $\vec{r}$ is given by

$$\vec{E}(\vec{r}) = \frac{kq}{r^3}\vec{r} = \frac{\vec{F}}{q}, \tag{1.5}$$

where q denotes the charge of the point source (C), r denotes the radial distance of the point charge from the origin, and $\vec{r}$ represents the displacement vector of the point from the origin. We can also calculate the magnitude of the eletric field via

$$E = k\frac{|q|}{r^2}. \tag{1.6}$$

The **electric field of a group of charges** is the superposition of all the electric forces from all the charges. This can be approximated with volume charge density (ρ) via

$$\vec{E}(\vec{r_o}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r})(\vec{r_0}-\vec{r})}{|\vec{r_0}-\vec{r}|^3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{1.7}$$

where $\epsilon_0 = 8.854\times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}$, $\rho(\vec{r})$ is the surface charge density, and $\vec{r}$ is the radius.

For **a conductor without current flow**, the charge all resides on the surface. Using the surface charge density $\sigma(\vec{r})$, we can approximate the electric field via

$$\vec{E}(\vec{r_o}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r})(\vec{r_0}-\vec{r})}{|\vec{r_0}-\vec{r}|^3}\,\mathrm{d}A. \tag{1.8}$$

If one has a **thin wire** where all the charge resides, with linear charge density $\lambda(\vec{r}) = \frac{\mathrm{d}q}{\mathrm{d}l}$, where $\mathrm{d}l$ is the element of length along the wire, then

$$\vec{E}(\vec{r_o}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{r})(\vec{r_0}-\vec{r})}{|\vec{r_0}-\vec{r}|^3}\,\mathrm{d}l. \tag{1.9}$$

Example. For a ring of total charge Q on a circle of radius a, the circumference is $2\pi a$ and therefore the linear charge density is $\lambda = \frac{Q}{2\pi a}$. The electric field is therefore

$$\begin{aligned}
\vec{E}(\vec{r_0}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\vec{r_0}-\vec{r}}{|\vec{r_0}-\vec{r}|^3}\lambda(\vec{r})\,\mathrm{d}l = E_x\hat{i} &(1.10)\\
&= \frac{1}{4\pi\epsilon}\frac{Qx}{(x^2+a^2)^{3/2}}\hat{i} &(1.11)\\
&= \frac{Q\hat{i}}{4\pi\epsilon_0 x^2} \quad (\text{For } x >> a). &(1.12)
\end{aligned}$$

1.3 Electric Field Lines

Electric field lines have tangent vectors parallel to the electric field and begin only on positive charges and end only on negative charges, though they can also go to infinity in either direction.

1.4 Electric Dipoles

An **electric dipole** is a pair of point charges of equal magnitude and opposite sign (a positive charge q and a negative charge $-q$ separated by a distance d.

An **electric dipole moment** is the product of the positive charge q and the displacement d it is separated from the negative charge $-q$, given by

$$\vec{p} = q\vec{d}. \tag{1.13}$$

Using the volume charge density formula $\rho(\vec{r}) = \rho(x\vec{i} + y\vec{j} + z\vec{k}) = \rho(x, y, z)$, we can approximate the electric dipole of a huge number of elementary charges (total charge is neutral) via

$$\vec{P} = \int \rho(\vec{r})\vec{r}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1.14}$$

The total force on an electric dipole is just the net force from an external electric field (the electric field from all the other charges that are not part of the electric dipole). This is true for the total torque.

In a uniform electric field $\vec{E}$, the net force on a electric dipole is 0. However, if the electric dipole monent vector $\vec{p}$ is not parallel to $\vec{E}$, then a torque is exerted on the dipole that changes its angular momentum $\vec{L}$ by $\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\tau}$. This vector torque is calculated via

$$\vec{\tau} = \vec{P} \times \vec{E}, \tag{1.15}$$

where $\vec{P}$ is the electric dipole moment, $\vec{E}$ is the electric field, and the direction of τ is perpendicular to both $\vec{P}$ and $\vec{E}$. The magnitude of torque can also be found via

$$\tau = pE\sin\phi, \tag{1.16}$$

where p is the magnitude of the electric dipole moment $\vec{p}$, E is the magnitude of the electric field $\vec{E}$, and ϕ is the angle between $\vec{p}$ and $\vec{E}$.

To calculate the potential energy of a dipole, we use

$$U = -\vec{p} \cdot \vec{E}. \tag{1.17}$$

where $\vec{p}$ is the electric dipole moment and $\vec{E}$ is the electric field.

We can approximate the field of an electric dipole at $r >> d$ with binomial expansion using

$$\vec{E}(\vec{r}) \approx \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{4\pi\epsilon_0 r^3}, \tag{1.18}$$

where $\vec{p}$ is the electric dipole moment, $\hat{r}$ is the unit vector in the direction or $\vec{r}$, and $\epsilon_0 = 8.854 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N{\cdot}m^2}}$ (we probably don't need to know this formula for the exam tbh).

1.5 Gauss's Law

Chapter 2

Electric Potential

Chapter 3

Electric Potential

Chapter 4

Capacitance and Dielectrics

Chapter 5

Current, Resistance, and Electromotive Force

Chapter 6

Direct-Current Circuits

Chapter 7

Direct-Current Circuits

Now we shall look in more detail at electrical circuits, which have a network of conductors and other devices connected together. Generally, the conductors can be idealized as thin wires, each of which carries a constant current in a stationary situation, and junctions where all the incoming currents get divided up into all the outgoing currents, with no charge of either sign building up at the junction. This was given by **Kirchhoff's junction rule** or **current law** or **first law: Sum of currents into a junction is zero**.

Since in a stationary situation the electric force is a **conservative force**, with a **potential** (potential energy per charge, measured in the SI unit of $1\,\mathrm{V} = 1\,\mathrm{J/C}$) that is defined up to one overall additive constant (so that the potential differences or voltages between two points are uniquely defined), we get **Kirchhoff's loop rule** or **voltage law** or **second law: the algebraic sum of potential differences around any closed loop is zero** (the algebraic sum means keeping track of signs. If we have a potential increase from a to b, so $V_b - V_a = V_{ba} = -V_{ab} > 0$, the potential difference is positive; if $V_b < V_a$, $V_b - V_a = V_{ba} = -V_{ab} < 0$, this is negative).

Now let us use Kirchhoff's rules to get the effective resistance R_{eff} of N resistors, of resistances R_i with i running from 1 to N, that is, $R_1, R_2, \ldots, R_N$, when the resistors are connected in **series** and in **parallel**. In series, we have

$$R_{\text{eff}} = R_{\text{eq}} = \sum R_i = R_1 + R_2 + \cdots + R_N. \tag{7.1}$$

In parallel, we have

$$\frac{1}{R_{\text{eff}}} = \frac{1}{R_{\text{eq}}} = \sum \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}. \tag{7.2}$$

A wire of cross section A can be considered as a parallel arrangement of wires of cross sections adding up to A, which fits $\frac{1}{R} = \frac{A}{\rho L} = \frac{\sigma}{L} A$, adding up the A's.

7.1 Electrical Measuring Instruments

Electrical measuring instruments for current (**ammeters**), potential difference or voltage (**voltemeters**), and resistance (**ohmmeters**) generally have a basic unit for measuring current, such as a **d'Arsonval galvanometer**. This meter has a coil of wire on a pivot that is in a magnetic field of a permanent magnet that is also part of the meter. The coil has a spring attached to it that supplies a restoring force when the coil is twisted away from its equilibrium position (since this force tries to restore the **orientation** of the coil, with its center at a fixed **position,** what is more relevant is the restoring **torque** of the spring). When a current passes through the coil, the magnetic field exerts a torque proportional to the current. Therefore, after the coil settles down to a fixed orientation (due to frictional torque, so that the system is like a damped harmonic oscillator for orientation) the restoring torque (generally approximately proportional to the angle of the coil orientation relative to the equilibrium orientation) balances the torque produced by the magnetic field on the current in the coil (which is proportional to the current). Thus the angle that the coil has twisted from its equilibrium orientation is approximately proportional to the current in the coil and thus gives a measure of that current. A pointer needle attached to the coil then gives a reading on a calibrated scale (even if the angle of twist is not very precisely proportional to the current, during manufacture one can use currents otherwise calibrated to calibrate the scale of the meter).

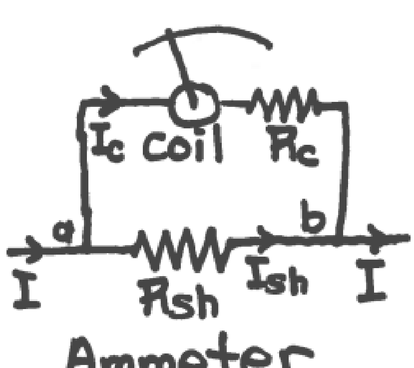
An **ammeter** uses such a d'Arsonval galvanometer directly to measure the current through it. If one has a circuit element with current running through it that one wishes to measure, one needs to break the circuit and attach the two separated ends to the two terminals of the ammeter, so that the current that was just running through the circuit now runs through the ammeter and is measured by it.

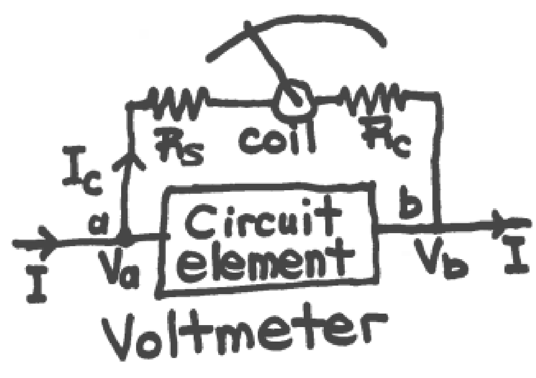
Of course, inserting an ammeter into a circuit adds the additional resistance of the ammeter to the circuit and thus reduces the current reading from what it actually was in the circuit without the ammeter inserted into it. To minimize this error of the ammeter reading, we want the ammeter resistance as **low** as possible, but unless we use superconducting coils at very low temperatures, there is a practical limitation of how low the resistance can be.

For an ammeter to be able to measure a larger range of currents, it may have a **shunt resistor** that can be put in **parallel** to the coil, so that a larger total current can pass through the complete ammeter (coil plus resistor) than what would give a full-scale deflection of the coil if all of the current passed through the coil. For example, suppose $I_{\rm fs}$ is the current passing through the coil that would give full-scale deflection, but one wants to be able to measure currents up to I_a passing through the full ammeter, so that $I_{\rm fs}$ passes through the coil with resistance R_c, and $I_a - I_{\rm fs}$ passes through the shunt with resistance $R_{\rm sh}$:

Since the voltage drop $V_{ab} = I_{\rm fs}R_c = (I_a - I_{\rm fs})R_{\rm sh}$, so with R_c fixed, we need $R_{\rm sh} = \frac{I_{\rm fs}R_c}{I_a - I_{\rm fs}}$. Shunts of **decreasing** resistances $R_{\rm sh}$ then allow an ammeter to measure increasing currents I_a.

A **voltmeter** is designed to measure the potential difference or voltage $V_{ab} = V_a - V_b$ between two points in a circuit that one connects the two terminals of the voltmeter to without breaking the circuit as one does with an ammeter. In other words, a voltmeter leaves the original circuit intact and provides a parallel path for a tiny amount of current to pass through it to deflect the coil and give a meter reading. To avoid having a large amount of current passing through it (which would increase the voltage drop in other parts of the circuit not parallel to the voltmeter and hence make the voltage measured across the voltmeter lower than what it would be if the voltmeter were not attached), one wants the voltmeter resistance to be as **high** as possible, though it must be low enough to give enough current through the coil that one can get a meter reading.





To be able to measure higher voltages tha nwhat would give full-scale deflection with just the coil resistance R_c (which with the full-scale current $I_{\rm fs}$ would be a voltage of $V_0 = I_{\rm fs}R_c$), one can include in the voltmeter an additional resistance R_s in **series** with the coil, so that then a full-scale deflection at current I_{fs} would occur for a voltage of $V_v = I_{\mathrm{fs}}(R_c + R_s) = V_0\left(1 + \frac{R_s}{R_c}\right)$. Resistors in a series of **increasing** resistances R_s then allow a voltmeter to measure increasing voltages V_v.

Thus with a resistor R_s in series with that of the coil (R_c), a voltmeter has a total resistance $R_v = R_c + R_s$, whereas with a shunt resistor R_{sh} in parallel with the coil R_c, an ammeter has resistance $\frac{1}{R_A} = \frac{1}{R_c} + \frac{1}{R_{\mathrm{sh}}}$.

Suppose we originally have the circuit at the bottom, with a source of emf ε and internal resistance r in series with a resistance R. The total resistance is $R+r$, so in this situation the current is $I_0 = \frac{\varepsilon}{R+r}$. The total power produce by the emf (not counting the negative power produce by the internal resistance r) is $I_0\varepsilon = \frac{\varepsilon^2}{R+r}$. The power dissipated in the internal resistance is the current through it, I_0 (the same everywhere around this simple circuit of one loop), multiplied by the voltage drop through r, which is $I_0 r$, so the product is $I_0^2 r = \frac{\varepsilon^2 r}{(R+r)^2}$.

The power delivered to the external resistance R is $I_0^2 R = \frac{\varepsilon^2 R}{(R+r)^2}$. We can then check that the total power dissipated both internally and externally is $I_0^2 r + I_0^2 R = I_0^2(r + R) = \frac{\varepsilon^2(r+R)}{(R+r)^2} = \frac{\varepsilon^2}{R+r}$, which matches the power generated, $I_o\varepsilon = \frac{\varepsilon^2}{R+r}$. Assuming r is fixed, if we change the external resistance R, the total power dissipated, $\frac{\varepsilon^2}{R+r}$, is a monotonically decreasing function of R and so is maximized when R is minimized, say to $R = 0$ when the two terminals of the source of emf are connected together with no resistor. This "shorts out" the source of emf, and if r is low enough, this can give a high enough current $I_{0_{\max}} = \frac{\varepsilon}{r}$ and power $P_{\max} = \dfrac{\varepsilon^2}{r}$ to wreck a battery or perhaps even to start a fire.

Suppose we do not want to maximize the total power dissipated by both resistors, but just the power delivered to the external resistor R to be dissipated there (turned into heat), $P_{\mathrm{ext}} = \frac{\varepsilon^2 R}{(R+r)^2}$, with r kept fixed. This maximum occurs where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}P_{\mathrm{ext}} &= \varepsilon^2\left[\frac{\mathrm{d}}{\mathrm{d}R}\frac{R}{(R+r)^2}\right]\\ &= \varepsilon^2\frac{r-R}{(R+r)^3} = 0,\end{aligned}$$

hence the maximum power that source of emf ε with internal resistance r can deliver to an external resistance R whose magnitude can be varied is $\frac{\varepsilon^2}{4r}$.

Now let us add an ammeter and a voltmeter to the circuit to try to measure the current I_0, resistance R, voltage $V_{ac} = I_0 R$ across the resistor, and power $I_0 V_{ac} = I_0^2 R$ dissipated by the resistor. If the ammeter had zero resistance, $R_A = 0$, it would not add to the resistance of the rest of the circuit that it is in series with, and if the voltmeter and infinite resistance, $R_V = \infty$, it would not decrease the resistance of the part of the circuit where it is in parallel with the original circuit, so in this case the measured current would indeed be $I = I_0$, and the measured voltage across R would indeed by $V = I_0 R$, so one could get $I_0 = I$, $V_{ab} = V$, $R = \frac{V}{I}$, and $P_{\text{ext}} = IV$. But with an actual ammeter with $R_A > 0$, and an actual voltmeter with $1/R_v > 0$, things are a bit different.

With nonzero resistance of the ammeter and the nonzero current through the voltmeter (due to the fact that it does not have infinite resistance), they do not read the same current and voltage as would be present in their absence.

7.2 Ohmmeters

An **ohmmeter** is used to measure resistance directly of an electrical device that is passive (no source of emf within it) and is not connected to as circuit with current flowing (not counting the circuit formed by connecting the device to the ohmmeter. The ohmmeter has its own source of emf within it (ex. a battery) unlike an ammeter or voltmeter that uses the current or voltage produced by an external source of emf.

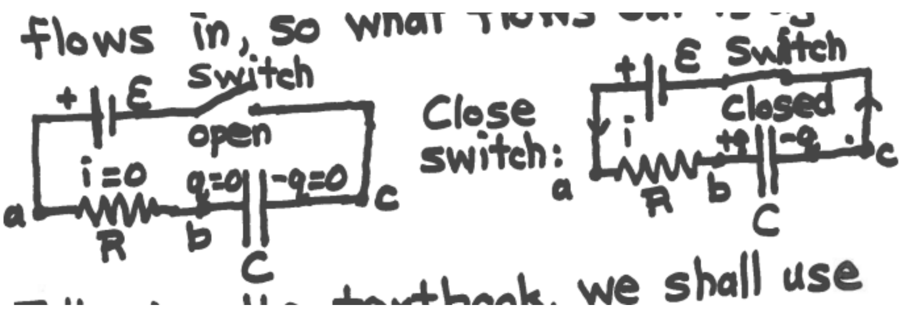
An ohmmeter has a source of emf ε, a d'Arsonaval galvanometer to measure the current (and so acting like an ammeter though without any shunt resistance), and a variable resistance R_s. First, the two terminals are connected together and R_s set to give full-scale deflection. Then, the two terminals are placed at opposite ends of the resistance R to be measured, so the total resistance becomes $R_s + R$. Therefore, when the two terminals of the ohmmeter are connected together with only the resistance R_s, $I_0 = I_{\text{fs}} = \frac{\varepsilon}{R_s}$. Then when the resistance R is added to the circuit, the meter reads $\frac{\varepsilon}{R_s+R} = \frac{R_s I_{\text{fs}}}{R_s+R}$, giving $R = R_s\left(\frac{I_{\text{fs}}}{I} - 1\right)$. The scale on the meter gives $\frac{I_{\text{fs}}}{I}$, which with known R_s (from the way the meter is constructed) one can measure R.

Because there is no other source of emf giving a current through the resistor R other than the emf in the ohmmeter, its measurement of R directly gives the correct value, unlike the current measured by an ammeter of positive resistance and by a voltmeter of non-infinite resistance.

Unlike a voltmeter, a **potentiometer** can measure the emf of a source without drawing any current that would change the voltage across it and its internal resistance. The potentiometer has a source of emf ε_1 that produces a nonzero current I_1 through the bottom loop of total resistance $R_{ab} + r_1$, so $I_1 = \frac{\varepsilon_1}{R_{ab}+r_1}$. The potentiometer also includes the d'Arsonval galvanometer between terminal d and a point c that can be moved along the resistor R_{ab} to give a resistance $R_{cb} = \frac{cb}{ab} R_{ab}$ proportional to the length cb. This length, and hence R_{cb}, is adjusted until the galvanometer reads zero current $I = 0$ in the top loop, where the source of emf ε and its internal resistance have terminals b and d attached to the potentiometer. With $I = 0$, $V_{cb} = \varepsilon$. But $V_{cb} = I_1 R_{cb} = \frac{\varepsilon_1}{R_{ab}+r} \frac{cb}{ab} R_{ab}$, giving ε.

7.3 R-C Circuits

We now consider a nonstationary (time-dependent) situation in which we have two capacitor plates on which opposite charges can be a function of time, $+q(t)$ on one side and $-q(t)$ on the other. We also have a time-dependent current $i(t) = \frac{dq}{dt}$ flows into the capacitor on one side. On the opposite side, $i'(t) = -\frac{dq}{dt}$ flows in, so what flows out is again $i(t) = \frac{dq}{dt}$.



Following the textbook, we shall use lowercase q, i, and v for time-dependent charge ($+q$ on the left side of the capacitor of capacitance C, $-q$ on the right side), current $i(t) = \dfrac{dq(t)}{dt}$, and voltage. When the time dependence is slow compared with the speed of light, the electric force is very nearly conservative, $\vec{E} \approx -\vec{\nabla} V$, and Kirchhoff's loop law applies.

After the switch is closed, $q(t)$ increases, so $i(t) > 0$. $V_{ab} = V_a - V_b = iR$, $V_{bc} = \frac{q}{C}$, and $V_{ca} = -\varepsilon$ through the source of emf, so Kirchhoff's loop law says $0 = V_{ab} + V_{bc} + V_{ca} = Ri + \frac{q}{C} - \varepsilon = R\frac{dq}{dt} + \frac{1}{C} q - \varepsilon$. This is a linear 1st-order differential equation, which we can solve by separation of variables. Thus, we have

$$\boxed{q = C\varepsilon \left(1 - e^{-t/RC}\right).} \tag{7.3}$$

Then the current is

$$\boxed{i = \frac{\varepsilon}{R} e^{-t/RC} = I_0 e^{-t/RC},} \tag{7.4}$$

where $I_0 = \frac{\varepsilon}{R}$ is the current right after the switch is closed, at $t = 0$, when $e^{-t/RC} = e^0 = 1$. Let's check that this satisfies Kirchhoff's loop law:

$$0 \stackrel{?}{=} Ri + \frac{q}{C} - \varepsilon \tag{7.5}$$

$$= R\frac{\varepsilon}{R}e^{-t/RC} + \varepsilon\left(1 - e^{-t/RC}\right) - \varepsilon \tag{7.6}$$

$$= \varepsilon\left(e^{-t/RC} + 1 - e^{-t/RC} - 1\right) \tag{7.7}$$

$$= 0. \tag{7.8}$$

When t gets much larger than the **time constant** $\tau = RC$ (crudely, $\frac{t}{\tau}$ "goes to ∞"), $e^{-t/RC} = e^{-t/\tau} \to 0$, so $q \to Q_f = C\varepsilon$, $i \to 0$. Then $V_{ab} = iR = 0$, $V_{bc} = \dfrac{Q_f}{C} = \dfrac{C\varepsilon}{C} = \varepsilon$, and $V_{ca} = -\varepsilon$. In contrast, at the initial time, $t = 0$, $q = 0$ but $i = I_0 = \frac{\varepsilon}{R}$, so $V_{ab} = iR = \varepsilon$, $V_{bc} = \frac{q}{C} = 0$, and $V_{ca} = -\varepsilon$. The time constant $\tau = RC$ is also called the **relaxation constant**.

After reaching the steady-state situation (after a very long time $t >> \tau = RC$ of charging) with $I = 0$ and $q = Q_f = \varepsilon$ on the left plate, suppose we through the switch to get a new circuit bypassing the emf. Then still $V_{ab} = iR$ and $V_{bc} = \frac{q}{C}$, but $V_{ca} = 0$, so $Ri + \frac{q}{C} = R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0$, showing that $i = -\frac{q}{RC}$ is now negative (positive current now flowing clockwise around the circuit). Then $\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{q}{RC}$, $\mathrm{d}q = -\frac{q}{RC}\,\mathrm{d}t$, $\frac{\mathrm{d}q}{q} = -\frac{\mathrm{d}t}{RC}$, $\int \frac{\mathrm{d}q}{q} = \ln(q) + \text{const} = -\frac{t}{RC}$. If we reset the time to $t = 0$ when we throw this switch, then at $t = 0$, $\ln(q) + \text{const} = \ln(q) - \ln(Q_f) = \ln\left(\frac{q}{Q_f}\right)$. Since now Q_f is the initial charge, call it Q_0.

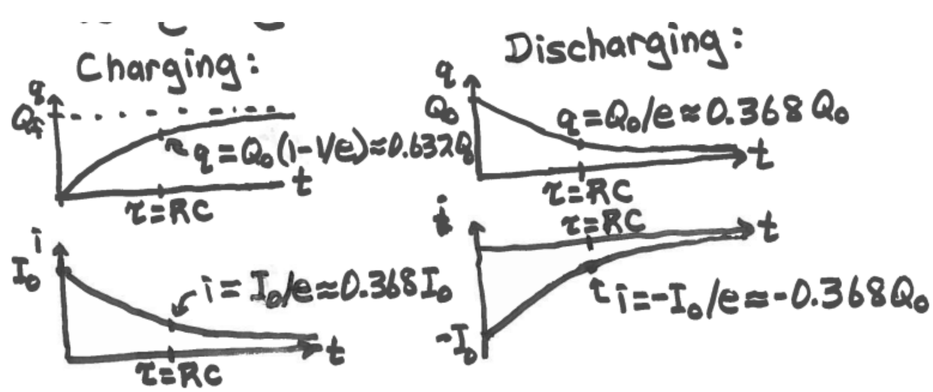
Thus, while discharging through the resistor (but with no emf in the circuit) with $q = Q_0$ at the reset $t = 0$, we get $\ln\left(\frac{q}{Q_0}\right) = -\frac{t}{RC}$. Exponentiating both sides, we get

$$\boxed{q = Q_0 e^{-t/RC},} \tag{7.9}$$

and

$$\boxed{i = \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{Q_0}{RC}e^{-t/RC}.} \tag{7.10}$$

In this case, both the charge $q(t)$ and the current $i(t)$ exponentially decrease toward zero with the same time constant $\tau = RC$, but their coefficients are different, so $\frac{i}{q} = -\frac{1}{RC} = -\frac{1}{\tau}$. $V_{ab} = iR = -\frac{Q_0}{C}e^{-t/RC}$, $V_{bc} = \frac{q}{C} = \frac{Q_0}{C}e^{-t/RC}$, $V_{ab} + V_{bc} = 0$.


Charging:
Discharging:
q=Q0(1-1/e)≈0.632Q0
q=Q0/e≈0.368Q0
τ=RC
i=I0/e≈0.368I0
i=-I0/e≈-0.368Q0
t

Chapter 8

Magnetic Field and Magnetic Forces

So far we have discussed electric fields $\vec{E}$, which give force $\vec{F}_E = q\vec{E}$ on a charge q, no matter what its velocity. But the electromagnetic field also includes another vector field, the magnetic field $\vec{B}$, which gives a force $\vec{F}_B = q\vec{v} \times \vec{B}$ on a charge q that depends on its velocity $\vec{v}$ and is zero if the velocity is zero.

The magnitude of the magnetic force is

$$F_B = \left|\vec{F}_B\right| = \left|q\vec{v} \times \vec{B}\right| = |q|\, v_\perp B = |q|\, vB\sin\theta, \tag{8.1}$$

where $v_\perp = v\sin\theta$ is the component of the velocity perpendicular to $\vec{B}$. $\vec{F}_B$ is perpendicular to both $\vec{v}$ and $\vec{B}$ and with the sense that one's right thumb would point if one's fingers coiled from first being in the direction of $\vec{v}$ and then towards the tips in the $\vec{B}$ direction.

$\vec{v} \times \vec{B}$ has magnitude $\vec{v}$ and $\vec{B}$. It is the vector 'area' $\vec{A} = \vec{v} \times \vec{B}$ associated with this parallelogram, perpendicular to the area itself. In this case the vector $\vec{v} \times \vec{B}$ points into the page (or screen when the image is projected), as indicated by $\otimes$, which symbolizes the tail feathers of an arrow pointing away. The opposite arrow perpendicular to the paper, say $-\vec{v} \times \vec{B} = \vec{B} \times \vec{v}$, would be indicated by $\odot$, which symbolizes the tip of an arrow pointing toward the viewer.

The sense can also be given by the direction the tip of a screw would move if it's axis wer perpendicular to $\vec{v}$ and $\vec{B}$ and its head (parallel to the plane of $\vec{v}$ and $\vec{B}$) were turned by $< 180\,\deg$ from $\vec{v}$ to $\vec{B}$.

When both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is present, the total electromagnetic field force on a particle of charge q and velocity $\vec{v}$ is called the **Lorentz force**,

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}. \tag{8.2}$$

Electric fields are produced by charges whether or not they are moving (though the electric fields deviate from Coulomb's law if the charges are accelerating or have velocities comparable to the speed of light, $299\,792\,456\,\frac{\mathrm{m}}{\mathrm{s}}$), but magnetic fields are produced only by charges that are moving and/or accelerating. Later we shall discuss the analogues of Coulomb's law for how magnetic fields are produced. Charges moving within atoms and molecules are one way. Another is that even a charge not moving can have a spin, which produce a dipole magnetic field.

Just as the electric field $\vec{E}$ has electric field lines that are everywhere parallel to $\vec{E}$ and that give a flux $\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A}$ through a surface, so a magnetic field $\vec{B}$ has **magnetic field lines** everywhere parallel to $\vec{B}$ and that give a flux $\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$ through a surface. For the electric field, Gauss's law gives the electric flux outward through a closed surface as

$$\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}, \tag{8.3}$$

where Q_{encl} is the total electric charge inside the volume enclosed by the surface. The analogue for the magnetic field would be $\Phi_B = \oint \vec{B} \cdot \mathrm{d}\vec{A} \propto$ (magnetic charge Q_B enclosed). However, no magnetic charge Q_B has ever been observed, so we shall say Gauss's law for magnetism is $\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0}$ through any closed surface.

Analogous to the way that Maxwell in 1865 completed the development of **electromagnetism** started by Coulomb, Gauss, Ampere, Faraday, and others, the 1960s saw the unification of electromagnetism and the **weak interaction** (radioacivity) by Glashow, Weinberg, and Salam. When combined with the theory of **quantum chromodynamics** for the strong nuclear reactions, including the Higgs mechanism for generating the masses of quarks (the building blocks for hadrons such as protons and neutrons) and of electrons and related other leptons, this gives what is known as the **Standard Model** of particle physics. However, this treats rather separately, and does not fully unify, the **electroweak** and **strong** interactions. Such a unification would be a **Grand Unified Theory**.

Such a **Grand Unified Theory** (GUT) would explain why the electron charge seems to be exactly the same magnitutde (but opposite sign) as the proton charge, and, related to this, it predicts the existence of **magnetic monopoles**, which would have nonzero magnetic charge. However, these magnetic monopoles are predicted to have energies about a trillion times larger than the energies that can be produced in the largest accerlerator today, the **Large Hadronic Collider (LHC)** at CERN, where several UofA professors work. So unless these estimates are wrong, it might be too difficult for humans ever to produce magnetic monopoles. Therefore, in this course we shall assume $\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0$ or no magnetic monopoles.

Gauss's Law for magnetism with no magnetic monopoles, or $\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0}$, implies that magnetic field lines have no beginning or end. Generally, they come in from infinity and go back out to infinity, but they can also wrap around and around in a finite volume without ever reconnecting, unlike what the textbook says near the bottom of p.889: "Like all other magnetic field lines, they form closed loops." Only in special situations, such as symmetry of rotation about an axis, or other situations in which a set of field lines stay in one plane without going to infinity, do the field lines form closed loops.

Asymmetric field lines do not stay on any single non-branching surface, and they need not form closed loops. A permanent magnet longer along the direction of the magnetic field inside has a north pole end (N) and a south pole end (S). Magnetic field lines run through the magnet from the S end to the N end and then emerge to loop back outside (though) they need not stay in a plane or distorted plane to reconnect precisely). However, because magnetic field lines nowhere begin or end, if one cuts the magnet in two, one will not get an isolated north pole or south pole, but rather the two new magnets will each have a north pole end and a south pole end, with the magnetic field passing through each magnet from south to north and then looping back outside, without ever ending.

A compass needle is a magnet free to rotate in the horizontal plane (when the compass is level), and then the north pole end of the compass points in the direction of the horizonal component of the magnetic field. From a Natural Resources Canada magnetic declination calulator online, when I entered the date 2019 March 12 and latitude $\frac{\pi}{2} - \frac{2}{\pi}$ radians $= 53.5244°$N and longitude $\frac{5\pi}{6} - \frac{2}{\pi}$ radians $= 113.5244°$W for the University of Alberta ("colatitude equator's reciprocal", and longitude $\pi/3 = 60°$ greater than latitude), 1 got a magnetic declination of 14.094°E, meaning that a compass would point 14.094° east of true north. It also said that this declination is decreasing about 1.04° every 6 years. Where I lived in Manokotak, Alaska, the declination decreased from 20°E when I first went there in late summer 1959 to 12.2°E today, a decreased of 7.8° in 60 years, an average decrease of 0.13°/year or 1°/(8 years).

The earth itself acts like a relaively permanent magnet, though its magnetic field is gradually changing with time (for example, with the North Magnetic Pole, where $\vec{B}$ is vertically downward, moving across northern Canadian islands during my lifetime, now being north of the Canadian Arctic territorial claim and within 4° of the Geographic North Pole and moving toward Russia at 55-60 km/year) and also reverses at random time intervals ranging between 100000 years and 50000000 years, the most recent being about 780000 years ago.

The earth's magnetic field is crudely a dipole field, with the dipole part having an axis that passes through the North Geomagnetic Pole that in 2015 was located at 80.37°N, on Ellesmere Island, Nunavut, Canada. But because the field is not purely a dipole, the North Magnetic Pole and the North Geomagnetic Pole do not coincide, and a compass needle does not point to either.

When two magnets are placed close together end to end, the magnetic field energy is lower if the magnetic field on the axis is aligned: $\boxed{\mathrm{S} \rightarrow \mathrm{N}}\boxed{\mathrm{S} \rightarrow \mathrm{N}}$ Thus the north pole of one magnet attracts the south pole of another, so opposite poles attract, rather similar to the way opposite charges attract and charges of the same sign repel, except that for magnets, one cannot have isolated poles. But since the north pole of a compass is attracted towards the north magnetic pole of the earth (where the magnetic field enters the earth vertically downwards), if one views the earth as a magnet, the North Magnetic Pole, or perhaps slightly better, the North Geomagnetic Pole (intersection of the dipole axis with the earth in the north), is a magnetic south pole, where magnetic field lines enter the earth, as they do in the south pole of a magnet.

Since the magnetic force magnitude is

$$F_B = \left|\vec{F}_B\right| = \left|q\vec{v} \times \vec{B}\right| = |q|\, v_\perp B = |q|\, v B_\perp, \tag{8.4}$$

the units of magnetic field are those of $\frac{F}{qv} = \frac{\mathrm{kg \cdot m/s^2}}{\mathrm{C \cdot m/s}} = \frac{\mathrm{kg}}{\mathrm{C \cdot s}} = \frac{\mathrm{kg}}{\mathrm{A \cdot s^2}} = \mathrm{T}$, the **tesla**, (in honor of Nikola Tesla, 1856-1943, the prominent Serbian-American scientist and inventor). $1\,\mathrm{T} = 1\,\frac{\mathrm{kg}}{\mathrm{A \cdot s^2}} = 1\,\frac{\mathrm{N}}{\mathrm{A \cdot m}}$.

Another common unit is the **gauss (G)**, $1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$, a more convenient unit for the earth's magnetic field, since tover the earth's surface, B ranges from 0.25 G to 0.65 G. For Edmonton 2019 March 12, $B = 0.567\,97\,\mathrm{G}$, with an inclination angle of 75.096°, a vertical component of 0.548 86 G downward, a horizontal component of 0.548 86 G 14.094°E of north, given a northward component of 0.141 69 G and an eastward component of 0.035 57 G.

I tried to get a crude estimate for the magnetic flux outward through the part of the earth's surface where the outward radial (locally vertical) component B_r of the magnetic field is positive (mainly in the southern hemisphere, south of what might call the 'magnetic equator'), where the magnetic field has $B_r = 0$ and is parallel to the earth's surface, by which 1 mean the surface if the **geoid** which is the gravitational equipotential surface at mean sea level in the rotating frame of the earth, the rotation causing the geoid to be approximately ellipsoidal, and in fact given by a nominl ellipsoid to first approximation that has been chosen to have a semi-major axis $a = 6\,378\,137\,\mathrm{m}$ and a semi-minor axis $b = a(1 - 1/298257223563)$.

A dipole approximation for the field gives

$$B_r = -2B_o \left(\frac{R_\oplus}{r}\right)^3 \cos\theta, \tag{8.5}$$

where θ is the angle from the north end of the dipole axis and $B_o = 3.12 \times 10^{-5}\,\mathrm{T}$ is the value of the dipole field at $\theta = \frac{\pi}{2}$, where it is purely tangential (parallel to the sphere at that value of r). Then since a narrow band on a sphere of radius $r = R_\oplus =$ earth's radius of angular width $\mathrm{d}\theta$ has width $r\,\mathrm{d}\theta$ and length $2\pi r \sin\theta$, its area is $\mathrm{d}A = 2\pi r^2 \sin\theta\,\mathrm{d}\theta$. This gives the total area of the sphere as

Chapter 9

Sources of Magnetic Field

We have seen that the electromagnetic force of sturctureless point particle of charge q is found by using the Lorentz force equation

$$\text{ffl}\boxed{\vec{F} = q(\vec{E} + \vec{V} \times \vec{B})}. \tag{9.1}$$

We have also learned Gauss's law for $\vec{E}$,

$$\boxed{\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}}, \tag{9.2}$$

and Gauss's law for $\vec{B}$

$$\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0}, \tag{9.3}$$

assuming no magnetic monopoles (none seen in nature and probably requiring too much energy to be produced by humans). These three laws are exact and have exactly the same form in special relativity (though they can be written slightly more elegantly in 4-dimensional realtivistic notation, such as $m\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}$). We also learned an approcimate relation when we ignored relativistic effects such as electromagnetic waves:

$$\boxed{\oint \vec{E} \cdot \mathrm{d}l = 0} \leftrightarrow \exists \text{ potential } V \text{ such that } \boxed{\vec{E} = -\Delta V}. \tag{9.4}$$

The exact law $\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon + 0}$ and the nonrelativistic law $\oint \vec{E} \cdot \mathrm{d}l = 0$ allows one to deduce the nonrelativistic Coulomb's law (the approximation for v ¡¡ c) given first for the potential V and then for the electric field $\vec{E}$ ato location $\vec{r}_0$:

$$V(\vec{r}_0) = \frac{U(\vec{r}_0)}{q_0} = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r})(\vec{r}_0 - \vec{r})}{|\vec{r}_0 - \vec{r}|^3} \,\mathrm{d}l, \tag{9.5}$$

where q_i is an individual charge at location $\vec{r}_i$, $\mathrm{d}q(\vec{r})$ is an infinitesimal elemnt of charge $\rho(\vec{r})$ is a volume charge density, $\sigma(\vec{r})$ a surface charge density, $sigma(\vec{r})$ a surface charge density, $\lambda(\vec{r})$ linear charge density.

Just as we need two laws for the electric field, $\oint \vec{E}\cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$ and $\vec{E}\cdot \mathrm{d}\vec{l} = 0$ (this being a nonrelativistic approximation) to get Coulomb's law for how nonrelaticistic charges produce an electric field, we also need a second law beside $\oint \vec{B}\cdot \mathrm{d}\vec{A} = 0$ for how to get the magnetic field in a nonrelativistic situation. The form most nearly similar to $\oint \vec{E}\cdot \mathrm{d}\vec{l} = 0$ (conservative nature of the electric forve in the nonrelativistic approximation of no electromagnetic waves carrying momentum) is **Ampere's Law**,

$$\boxed{\oint \vec{B}\cdot \mathrm{d}\vec{l} = \mu_0 I = \mu_0 \int \vec{J}\cdot \mathrm{d}\vec{A}} \tag{9.6}$$

, where $I = \int \vec{J}\mathrm{d}\vec{A}$ is the current through any open surface with edge the loop around which $\oint \vec{B}\cdot \mathrm{d}\vec{l}$ is calculated (and with $\mathrm{d}\vec{A}$ up as seen from above when $\mathrm{d}\vec{l}$ goes counterclockwise around the loop).

Ampere's Law actually works only in a stationary situation in which charge is not building up anywhere, so $\oint \vec{J}\cdot \mathrm{d}\vec{A} = -\frac{\mathrm{d}Q_{encl}}{\mathrm{d}t} = 0$ and so that then $\int \vec{J}\mathrm{d}\vec{A}$ is independent of the open surface bounded by the loop around which $\oint \vec{B}\cdot \mathrm{d}\vec{l}$ is taken. When charges do build up, what is 0 by charge conservation is $0 = \oint \vec{J}\cdot \mathrm{d}\vec{A} + \frac{\mathrm{d}Q_{encl}}{\mathrm{d}t} = \oint \vec{J}\cdot \mathrm{d}\vec{A} + \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_0 \oint \vec{E}\cdot \mathrm{d}\vec{A} = \oint \left(\vec{J} + \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\right)\cdot \mathrm{d}\vec{A}$ so James Clerk Maxwell in his 1865 completion of the equations of electromagnetism brilliantly modified Ampere's Law to become

$$\boxed{\oint \vec{B}\cdot \mathrm{d}\vec{l} = \mu_0 \int \left(\vec{J} + \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\right)\cdot \mathrm{d}\vec{A}} \tag{9.7}$$

Later we shall learn Faraday's modification of $\oint \vec{E}\cdot \mathrm{d}\vec{l} = 0$

$$\boxed{\oint \vec{E}\cdot \mathrm{d}\cdot \mathrm{d}\vec{l} = -\int \frac{\mathrm{d}B}{\mathrm{d}t}\cdot \mathrm{d}\vec{A}} \tag{9.8}$$

These plus the following two equations

$$\boxed{\oint \vec{E}\cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}} \text{ and } \boxed{\oint \vec{B}\cdot \mathrm{d}\vec{A} = 0} \tag{9.9}$$

are **Maxwell's equations for EM**

From Maxwell's equatios and appropriate boundary conditions (such as no electromagnetic waves coming from the distant past, which are really approximately valid for waves of frequency high compared with the microwave frequencies of the cosmic microwave background of CMB radiation that is highly thermal, with a present temperature of 2.7255(6)K [k - kelvin, degrees above absolute zero] that is a hundred times colder than the freezing point of water at $0°C = 273.15\,\mathrm{K}$ and which gives an energy density $100^4 = 100000000 = 10^8$ times less than thermal radiation at $0°C$), one can get the electromagnetic fields produced by charges either at rest or in motion. For charges at rest, one only gets electric fields, by Coulomb's law.

To get magnetic fields from structureless point charges, one needs charges in motion. (One can also get magnetic fields from particles with both charge and spin angular momentum even if they are believed to be point particles such as electrons and quarks). Analogous to Coulomb's law $\vec{E}(\vec{r}_0) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i(\vec{r}_0-\vec{r}_i)}{|\vec{r}_0-\vec{r}_i)|^3}$,

$$\boxed{\vec{B}(\vec{r}_0) = \frac{\mu_0}{4\pi}\sum_{i=1}^{N}\frac{q_i\vec{v}_1 \times (\vec{r}_0 - \vec{r}_i)}{|\vec{r}_0 - \vec{r}_i|^3}} \tag{9.10}$$

is the nonrelativistic approximation ($|\vec{v}_i| << c$) for the magnetic field for a collection of particles of charge q_i and velocity $\vec{v}_i$ at locations $\vec{r}_i$ (where $\vec{r}_0$ is the location of the field point). For a single moving charge, if we define the unit vector $\vec{r} = \frac{(\vec{r}_0-\vec{r})}{|\vec{r}_0-\vec{r}|}$ that points from the source point $\vec{r}$to the field point $\vec{r}_0$ at distance $r = |\vec{r}_0 - \vec{r}|$, $\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$, an inverse-square law like Coulomb's.

just as ε_0 is the **electric constant** (also called the vacuum permittivity, the permittivity of free space, the distributed capacitance of the vacuum, or simply epsilon nought or epsilon zero), so μ_0 is the **magnetic constant** (also called the vacuum permeability, the permeability of free space, the permeability of the vacuum. or simply mu nought or mu zero).

What are the units? The SI unit for B is the tesla (T), with $\vec{F} = q\vec{v} \times \vec{B}$ giving $1\,\mathrm{N} = 1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}^2} = (1\,\mathrm{C})(1\,\frac{\mathrm{m}}{\mathrm{s}})(1\,\mathrm{T}) = 1\,\mathrm{T}\frac{\mathrm{C\cdot m}}{\mathrm{s}}$ or $1\,\mathrm{T} = 1\,\frac{\mathrm{kg}}{\mathrm{C\cdot s}}$. Then

Chapter 10

Electromagnetic Induction

Chapter 11

Inductance

Chapter 12

Electromagnetic Waves