Physics 230 Notes

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Chapter 1

Direct-Current Circuits

Now we shall look in more detail at electrical circuits, which have a network of conductors and other devices connected together. Generally, the conductors can be idealized as thin wires, each of which carries a constant current in a stationary situation, and junctions where all the incoming currents get divided up into all the outgoing currents, with no charge of either sign building up at the junction. This was given by **Kirchhoff's junction rule** or **current law** or **first law: Sum of currents into a junction is zero**.

Since in a stationary situation the electric force is a **conservative force**, with a **potential** (potential energy per charge, measured in the SI unit of $1\,\mathrm{V} = 1\,\mathrm{J/C}$) that is defined up to one overall additive constant (so that the potential differences or voltages between two points are uniquely defined), we get **Kirchhoff's loop rule** or **voltage law** or **second law: the algebraic sum of potential differences around any closed loop is zero** (the algebraic sum means keeping track of signs. If we have a potential increase from a to b, so $V_b - V_a = V_{ba} = -V_{ab} > 0$, the potential difference is positive; if $V_b < V_a$, $V_b - V_a = V_{ba} = -V_{ab} < 0$, this is negative).

Now let us use Kirchhoff's rules to get the effective resistance R_{eff} of N resistors, of resistances R_i with i running from 1 to N, that is, $R_1, R_2, \ldots, R_N$, when the resistors are connected in **series** and in **parallel**. In series, we have

$$R_{\mathrm{eff}} = R_{\mathrm{eq}} = \sum R_i = R_1 + R_2 + \cdots + R_N. \tag{1.1}$$

In parallel, we have

$$\frac{1}{R_{\mathrm{eff}}} = \frac{1}{R_{\mathrm{eq}}} = \sum \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}. \tag{1.2}$$

A wire of cross section A can be considered as a parallel arrangement of wires of cross sections adding up to A, which fits $\frac{1}{R} = \frac{A}{\rho L} = \frac{\sigma}{L} A$, adding up the A's.

1.1 Electrical Measuring Instruments

Electrical measuring instruments for current (**ammeters**), potential difference or voltage (**voltemeters**), and resistance (**ohmmeters**) generally have a basic unit for measuring current, such as a **d'Arsonval galvanometer**. This meter has a coil of wire on a pivot that is in a magnetic field of a permanent magnet that is also part of the meter. The coil has a spring attached to it that supplies a restoring force when the coil is twisted away from its equilibrium position (since this force tries to restore the **orientation** of the coil, with its center at a fixed **position,** what is more relevant is the restoring **torque** of the spring). When a current passes through the coil, the magnetic field exerts a torque proportional to the current. Therefore, after the coil settles down to a fixed orientation (due to frictional torque, so that the system is like a damped harmonic oscillator for orientation) the restoring torque (generally approximately proportional to the angle of the coil orientation relative to the equilibrium orientation) balances the torque produced by the magnetic field on the current in the coil (which is proportional to the current). Thus the angle that the coil has twisted from its equilibrium orientation is approximately proportional to the current in the coil and thus gives a measure of that current. A pointer needle attached to the coil then gives a reading on a calibrated scale (even if the angle of twist is not very precisely proportional to the current, during manufacture one can use currents otherwise calibrated to calibrate the scale of the meter).

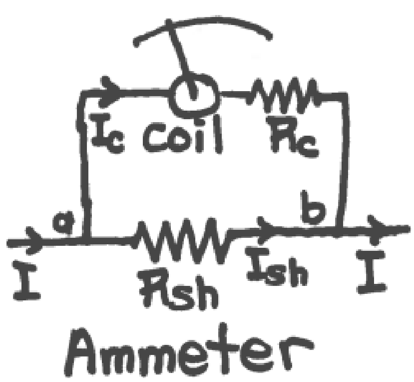
An **ammeter** uses such a d'Arsonval galvanometer directly to measure the current through it. If one has a circuit element with current running through it that one wishes to measure, one needs to break the circuit and attach the two separated ends to the two terminals of the ammeter, so that the current that was just running through the circuit now runs through the ammeter and is measured by it.

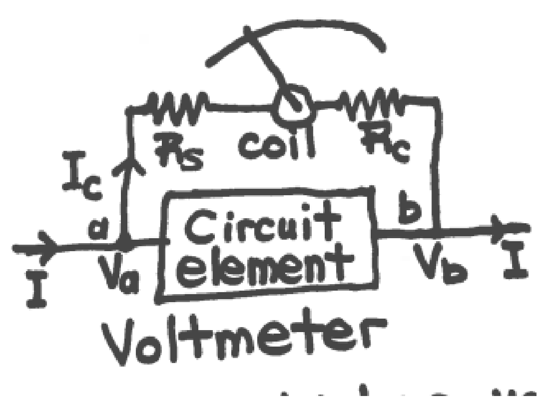
Of course, inserting an ammeter into a circuit adds the additional resistance of the ammeter to the circuit and thus reduces the current reading from what it actually was in the circuit without the ammeter inserted into it. To minimize this error of the ammeter reading, we want the ammeter resistance as **low** as possible, but unless we use superconducting coils at very low temperatures, there is a practical limitation of how low the resistance can be.

For an ammeter to be able to measure a larger range of currents, it may have a **shunt resistor** that can be put in **parallel** to the coil, so that a larger total current can pass through the complete ammeter (coil plus resistor) than what would give a full-scale deflection of the coil if all of the current passed through the coil. For example, suppose $I_{\rm fs}$ is the current passing through the coil that would give full-scale deflection, but one wants to be able to measure currents up to I_a passing through the full ammeter, so that $I_{\rm fs}$ passes through the coil with resistance R_c, and $I_a - I_{\rm fs}$ passes through the shunt with resistance $R_{\rm sh}$:

Since the voltage drop $V_{ab} = I_{\rm fs} R_c = (I_a - I_{\rm fs}) R_{\rm sh}$, so with R_c fixed, we need $R_{\rm sh} = \frac{I_{\rm fs} R_c}{I_a - I_{\rm fs}}$. Shunts of **decreasing** resistances $R_{\rm sh}$ then allow an ammeter to measure increasing currents I_a.

A **voltmeter** is designed to measure the potential difference or voltage $V_{ab} = V_a - V_b$ between two points in a circuit that one connects the two terminals of the voltmeter to without breaking the circuit as one does with an ammeter. In other words, a voltmeter leaves the original circuit intact and provides a parallel path for a tiny amount of current to pass through it to deflect the coil and give a meter reading. To avoid having a large amount of current passing through it (which would increase the voltage drop in other parts of the circuit not parallel to the voltmeter and hence make the voltage measured across the voltmeter lower than what it would be if the voltmeter were not attached), one wants the voltmeter resistance to be as **high** as possible, though it must be low enough to give enough current through the coil that one can get a meter reading.





To be able to measure higher voltages tha nwhat would give full-scale deflection with just the coil resistance R_c (which with the full-scale current $I_{\rm fs}$ would be a voltage of $V_0 = I_{\rm fs} R_c$), one can include in the voltmeter an additional resistance R_s in **series** with the coil, so that then a full-scale deflection at current I_{fs} would occur for a voltage of $V_v = I_{\text{fs}}(R_c + R_s) = V_0\left(1 + \frac{R_s}{R_c}\right)$. Resistors in a series of **increasing** resistances R_s then allow a voltmeter to measure increasing voltages V_v.

Thus with a resistor R_s in series with that of the coil (R_c), a voltmeter has a total resistance $R_v = R_c + R_s$, whereas with a shunt resistor R_{sh} in parallel with the coil R_c, an ammeter has resistance $\frac{1}{R_A} = \frac{1}{R_c} + \frac{1}{R_{\text{sh}}}$.

Suppose we originally have the circuit at the bottom, with a source of emf ε and internal resistance r in series with a resistance R. The total resistance is $R+r$, so in this situation the current is $I_0 = \frac{\varepsilon}{R+r}$. The total power produce by the emf (not counting the negative power produce by the internal resistance r) is $I_0\varepsilon = \frac{\varepsilon^2}{R+r}$. The power dissipated in the internal resistance is the current through it, I_0 (the same everywhere around this simple circuit of one loop), multiplied by the voltage drop through r, which is $I_0 r$, so the product is $I_0^2 r = \frac{\varepsilon^2 r}{(R+r)^2}$.

The power delivered to the external resistance R is $I_0^2 R = \frac{\varepsilon^2 R}{(R+r)^2}$. We can then check that the total power dissipated both internally and externally is $I_0^2 r + I_0^2 R = I_0^2(r+R) = \frac{\varepsilon^2(r+R)}{(R+r)^2} = \frac{\varepsilon^2}{R+r}$, which matches the power generated, $I_o\varepsilon = \frac{\varepsilon^2}{R+r}$. Assuming r is fixed, if we change the external resistance R, the total power dissipated, $\frac{\varepsilon^2}{R+r}$, is a monotonically decreasing function of R and so is maximized when R is minimized, say to $R = 0$ when the two terminals of the source of emf are connected together with no resistor. This "shorts out" the source of emf, and if r is low enough, this can give a high enough current $I_{0_{\max}} = \frac{\varepsilon}{r}$ and power $P_{\max} = \dfrac{\varepsilon^2}{r}$ to wreck a battery or perhaps even to start a fire.

Suppose we do not want to maximize the total power dissipated by both resistors, but just the power delivered to the external resistor R to be dissipated there (turned into heat), $P_{\text{ext}} = \frac{\varepsilon^2 R}{(R+r)^2}$, with r kept fixed. This maximum occurs where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}R}P_{\text{ext}} &= \varepsilon^2\left[\frac{\mathrm{d}}{\mathrm{d}R}\frac{R}{(R+r)^2}\right]\\ &= \varepsilon^2\frac{r-R}{(R+r)^3} = 0,\end{aligned}$$

hence the maximum power that source of emf ε with internal resistance r can deliver to an external resistance R whose magnitude can be varied is $\frac{\varepsilon^2}{4r}$.

Now let us add an ammeter and a voltmeter to the circuit to try to measure the current I_0, resistance R, voltage $V_{ac} = I_0 R$ across the resistor, and power $I_0 V_{ac} = I_0^2 R$ dissipated by the resistor. If the ammeter had zero resistance, $R_A = 0$, it would not add to the resistance of the rest of the circuit that it is in series with, and if the voltmeter and infinite resistance, $R_V = \infty$, it would not decrease the resistance of the part of the circuit where it is in parallel with the original circuit, so in this case the measured current would indeed be $I = I_0$, and the measured voltage across R would indeed by $V = I_0 R$, so one could get $I_0 = I$, $V_{ab} = V$, $R = \frac{V}{I}$, and $P_{\text{ext}} = IV$. But with an actual ammeter with $R_A > 0$, and an actual voltmeter with $1/R_v > 0$, things are a bit different.

With nonzero resistance of the ammeter and the nonzero current through the voltmeter (due to the fact that it does not have infinite resistance), they do not read the same current and voltage as would be present in their absence.

1.2 Ohmmeters

An **ohmmeter** is used to measure resistance directly of an electrical device that is passive (no source of emf within it) and is not connected to as circuit with current flowing (not counting the circuit formed by connecting the device to the ohmmeter. The ohmmeter has its own source of emf within it (ex. a battery) unlike an ammeter or voltmeter that uses the current or voltage produced by an external source of emf.

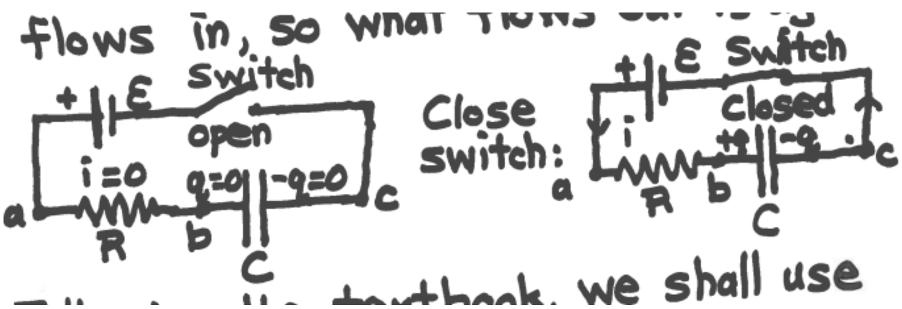
An ohmmeter has a source of emf ε, a d'Arsonaval galvanometer to measure the current (and so acting like an ammeter though without any shunt resistance), and a variable resistance R_s. First, the two terminals are connected together and R_s set to give full-scale deflection. Then, the two terminals are placed at opposite ends of the resistance R to be measured, so the total resistance becomes $R_s + R$. Therefore, when the two terminals of the ohmmeter are connected together with only the resistance R_s, $I_0 = I_{\text{fs}} = \frac{\varepsilon}{R_s}$. Then when the resistance R is added to the circuit, the meter reads $\frac{\varepsilon}{R_s+R} = \frac{R_s I_{\text{fs}}}{R_s+R}$, giving $R = R_s \left(\frac{I_{\text{fs}}}{I} - 1 \right)$. The scale on the meter gives $\frac{I_{\text{fs}}}{I}$, which with known R_s (from the way the meter is constructed) one can measure R.

Because there is no other source of emf giving a current through the resistor R other than the emf in the ohmmeter, its measurement of R directly gives the correct value, unlike the current measured by an ammeter of positive resistance and by a voltmeter of non-infinite resistance.

Unlike a voltmeter, a **potentiometer** can measure the emf of a source without drawing any current that would change the voltage across it and its internal resistance. The potentiometer has a source of emf ε_1 that produces a nonzero current I_1 through the bottom loop of total resistance $R_{ab} + r_1$, so $I_1 = \frac{\varepsilon_1}{R_{ab}+r_1}$. The potentiometer also includes the d'Arsonval galvanometer between terminal d and a point c that can be moved along the resistor R_{ab} to give a resistance $R_{cb} = \frac{cb}{ab} R_{ab}$ proportional to the length cb. This length, and hence R_{cb}, is adjusted until the galvanometer reads zero current $I = 0$ in the top loop, where the source of emf ε and its internal resistance have terminals b and d attached to the potentiometer. With $I = 0$, $V_{cb} = \varepsilon$. But $V_{cb} = I_1 R_{cb} = \frac{\varepsilon_1}{R_{ab}+r} \frac{cb}{ab} R_{ab}$, giving ε.

1.3 R-C Circuits

We now consider a nonstationary (time-dependent) situation in which we have two capacitor plates on which opposite charges can be a function of time, $+q(t)$ on one side and $-q(t)$ on the other. We also have a time-dependent current $i(t) = \frac{\mathrm{d}q}{\mathrm{d}t}$ flows into the capacitor on one side. On the opposite side, $i'(t) = -\frac{\mathrm{d}q}{\mathrm{d}t}$ flows in, so what flows out is again $i(t) = \frac{\mathrm{d}q}{\mathrm{d}t}$.



Following the textbook, we shall use lowercase q, i, and v for time-dependent charge ($+q$ on the left side of the capacitor of capacitance C, $-q$ on the right side), current $i(t) = \dfrac{\mathrm{d}q(t)}{\mathrm{d}t}$, and voltage. When the time dependence is slow compared with the speed of light, the electric force is very nearly conservative, $\vec{E} \approx -\vec{\nabla} V$, and Kirchhoff's loop law applies.

After the switch is closed, $q(t)$ increases, so $i(t) > 0$. $V_{ab} = V_a - V_b = iR$, $V_{bc} = \frac{q}{C}$, and $V_{ca} = -\varepsilon$ through the source of emf, so Kirchhoff's loop law says $0 = V_{ab} + V_{bc} + V_{ca} = Ri + \frac{q}{C} - \varepsilon = R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q - \varepsilon$. This is a linear 1st-order differential equation, which we can solve by separation of variables. Thus, we have

$$\boxed{q = C\varepsilon\left(1 - e^{-t/RC}\right).} \tag{1.3}$$

Then the current is

$$\boxed{i = \frac{\varepsilon}{R} e^{-t/RC} = I_0 e^{-t/RC},} \tag{1.4}$$

where $I_0 = \frac{\varepsilon}{R}$ is the current right after the switch is closed, at $t = 0$, when $e^{-t/RC} = e^0 = 1$. Let's check that this satisfies Kirchhoff's loop law:

$$0 \stackrel{?}{=} Ri + \frac{q}{C} - \varepsilon \tag{1.5}$$
$$= R\frac{\varepsilon}{R}e^{-t/RC} + \varepsilon\left(1 - e^{-t/RC}\right) - \varepsilon \tag{1.6}$$
$$= \varepsilon\left(e^{-t/RC} + 1 - e^{-t/RC} - 1\right) \tag{1.7}$$
$$= 0. \tag{1.8}$$

When t gets much larger than the **time constant** $\tau = RC$ (crudely, $\frac{t}{\tau}$ "goes to ∞"), $e^{-t/RC} = e^{-t/\tau} \to 0$, so $q \to Q_f = C\varepsilon$, $i \to 0$. Then $V_{ab} = iR = 0$, $V_{bc} = \dfrac{Q_f}{C} = \dfrac{C\varepsilon}{C} = \varepsilon$, and $V_{ca} = -\varepsilon$. In contrast, at the initial time, $t = 0$, $q = 0$ but $i = I_0 = \frac{\varepsilon}{R}$, so $V_{ab} = iR = \varepsilon$, $V_{bc} = \frac{q}{C} = 0$, and $V_{ca} = -\varepsilon$. The time constant $\tau = RC$ is also called the **relaxation constant**.

After reaching the steady-state situation (after a very long time $t >> \tau = RC$ of charging) with $I = 0$ and $q = Q_f = \varepsilon$ on the left plate, suppose we through the switch to get a new circuit bypassing the emf. Then still $V_{ab} = iR$ and $V_{bc} = \frac{q}{C}$, but $V_{ca} = 0$, so $Ri + \frac{q}{C} = R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0$, showing that $i = -\frac{q}{RC}$ is now negative (positive current now flowing clockwise around the circuit). Then $\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{q}{RC}$, $\mathrm{d}q = -\frac{q}{RC}\,\mathrm{d}t$, $\frac{\mathrm{d}q}{q} = -\frac{\mathrm{d}t}{RC}$, $\int \frac{\mathrm{d}q}{q} = \ln(q) + \text{const} = -\frac{t}{RC}$. If we reset the time to $t = 0$ when we throw this switch, then at $t = 0$, $\ln(q) + \text{const} = \ln(q) - \ln(Q_f) = \ln\left(\frac{q}{Q_f}\right)$. Since now Q_f is the initial charge, call it Q_0.

Thus, while discharging through the resistor (but with no emf in the circuit) with $q = Q_0$ at the reset $t = 0$, we get $\ln\left(\frac{q}{Q_0}\right) = -\frac{t}{RC}$. Exponentiating both sides, we get

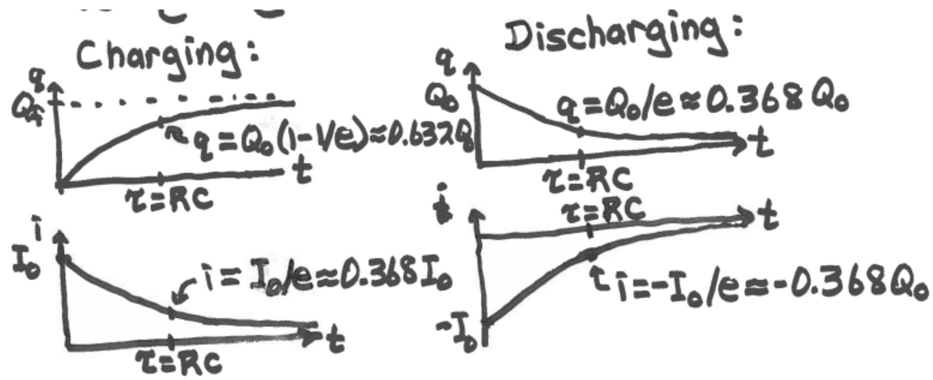
$$\boxed{q = Q_0 e^{-t/RC},} \tag{1.9}$$

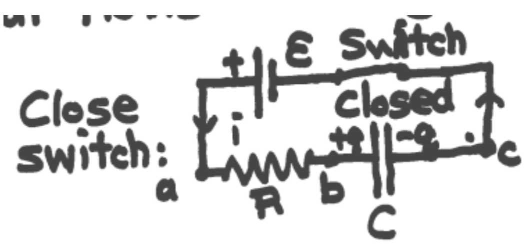
and

$$\boxed{i = \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{Q_0}{RC}e^{-t/RC}.} \tag{1.10}$$

In this case, both the charge $q(t)$ and the current $i(t)$ exponentially decrease toward zero with the same time constant $\tau = RC$, but their coefficients are different, so $\frac{i}{q} = -\frac{1}{RC} = -\frac{1}{\tau}$. $V_{ab} = iR = -\frac{Q_0}{C}e^{-t/RC}$, $V_{bc} = \frac{q}{C} = \frac{Q_0}{C}e^{-t/RC}$, $V_{ab} + V_{bc} = 0$.

In the figure above, when the capacitor is charging, the source of emf is providing power $P = \varepsilon i = \frac{\varepsilon^2}{R}e^{-t/RC}$. The rate at which electrical power is dissipated in the resistor is $V_{ab}i = i^2R = \left(\frac{\varepsilon}{R}e^{-t/RC}\right)^2 R = \frac{\varepsilon^2}{R}e^{-2t/RC}$, initially the same as P, but exponentially decaying at twice the rate. The rate at which energy is being stored in the capacitor is $iv_{bc} = i\frac{q}{C} = \frac{\varepsilon}{R}e^{-t/RC}\frac{C\varepsilon}{C}\left(1 - e^{-t/RC}\right) = \frac{\varepsilon^2}{R}\left(e^{-t/RC} - e^{-2t/RC}\right)$. Thus





the power provided, P, does equal the power dissipated in the resistor, i^2R, plus the power being stored in the capacitor, $i\frac{q}{C}$: $\varepsilon i = i^2R + i\frac{q}{C}$, which is i times $\varepsilon = iR + \frac{q}{C} = R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C}$. The total energy supplied by the emf is εQ_f, of which $\frac{1}{2}\varepsilon Q_f$ goes into the capacitor and an equal amount $\frac{1}{2}\varepsilon Q_f$ gets dissipated in the resistor.

1.4 Power Distribution System

The total power delivered $P = IV$. However, the losses in the transmission lines are $I^2R = \left(\frac{P}{V}\right)^2 R = \frac{P^2R}{V^2}$, so to reduce that, electricity is sent long distances at high voltage. But then it needs to be reduced to safe voltages in houses (usually around 120 V rms in North America, around 240 V in Europe). It is usually simpler to transform voltages by transformers if one uses AC (alternating current).

Inside a house, most outlets have two terminals, one at 120 V rms oscillating, and one grounded or connected to the earth. Often now there is a second ground terminal for extra safety. One connects devices across the terminals so that they are in parallel, all with the same voltage, but with parallel currents going through the different devices. (Christmas tree lights in a string used to be in series, with the nuisance that if one burned out, the whole string of lights went out.)

1.5 Summary of Direct-Current Circuits

1. Kirchhoff's junction rule of current law (conservation of charge in a stationary situation): the algebraic sum of currents into a junction is 0.

2. Kirchhoff's loop rule or voltage law or second law (conservative electric forces, $\vec{E} = -\vec{\nabla}V$): the algebraic sum of potential differences around any closed loop is zero.

3. Resistances in series add: $R_{\text{eq}} = \sum_{i=1}^{N} R_i$.

4. Reciprocals of resistances in parallel add: $\frac{1}{R_{\text{eq}}} = \sum_{i=0}^{N} \frac{1}{R_i}$.

5. In a d'Arsonval galvanometer, deflection is proportional to current (I).

6. With a low resistance (perhaps with a shunt resistor), this gives an ammeter to measure I.

7. With a high resistance (perhaps with a choice of resistances in series), this gives a voltmeter.

8. A device containing a galvanometer and a source of emf can be an ohmmeter, measuring resistance.

9. A potentiometer has a variable resistor also and can measure an emf without drawing current.

10. An R-C circuit has a resistor and capacitor in series, and also an emf when charging, so $q = C\varepsilon\left(1 - e^{-t/RC}\right)$, $i = \frac{\varepsilon}{R}e^{-t/RC}$, time constant $\tau = RC$.

11. After being charged, when the emf is removed, $q = Q_0 e^{-t/RC}$, $i = -\frac{Q_0}{RC}e^{-t/RC}$ when discharging.

12. Power is sent large distances at high voltage V, low I^2R.

Chapter 2

Magnetic Field and Magnetic Forces

So far we have discussed electric fields $\vec{E}$, which give force $\vec{F}_E = q\vec{E}$ on a charge q, no matter what its velocity. But the electromagnetic field also includes another vector field, the magnetic field $\vec{B}$, which gives a force $\vec{F}_B = q\vec{v} \times \vec{B}$ on a charge q that depends on its velocity $\vec{v}$ and is zero if the velocity is zero.

The magnitude of the magnetic force is

$$F_B = \left|\vec{F}_B\right| = \left|q\vec{v} \times \vec{B}\right| = |q|\, v_\perp B = |q|\, vB \sin\theta, \tag{2.1}$$

where $v_\perp = v \sin\theta$ is the component of the velocity perpendicular to $\vec{B}$. $\vec{F}_B$ is perpendicular to both $\vec{v}$ and $\vec{B}$ and with the sense that one's right thumb would point if one's fingers coiled from first being in the direction of $\vec{v}$ and then towards the tips in the $\vec{B}$ direction.

$\vec{v} \times \vec{B}$ has magnitude $\vec{v}$ and $\vec{B}$. It is the vector 'area' $\vec{A} = \vec{v} \times \vec{B}$ associated with this parallelogram, perpendicular to the area itself. In this case the vector $\vec{v} \times \vec{B}$ points into the page (or screen when the image is projected), as indicated by $\otimes$, which symbolizes the tail feathers of an arrow pointing away. The opposite arrow perpendicular to the paper, say $-\vec{v} \times \vec{B} = \vec{B} \times \vec{v}$, would be indicated by $\odot$, which symbolizes the tip of an arrow pointing toward the viewer.

The sense can also be given by the direction the tip of a screw would move if it's axis were perpendicular to $\vec{v}$ and $\vec{B}$ and its head (parallel to the plane of $\vec{v}$ and $\vec{B}$) were turned by $< 180\,\deg$ from $\vec{v}$ to $\vec{B}$.

When both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is present, the total electromagnetic field force on a particle of charge q and velocity $\vec{v}$ is called the **Lorentz force**,

$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}. \tag{2.2}$$

Electric fields are produced by charges whether or not they are moving (though the electric fields deviate from Coulomb's law if the charges are accelerating or have velocities comparable to the speed of light, 299 792 456 $\frac{\mathrm{m}}{\mathrm{s}}$), but magnetic fields are produced only by charges that are moving and/or accelerating. Later we shall discuss the analogues of Coulomb's law for how magnetic fields are produced. Charges moving within atoms and molecules are one way. Another is that even a charge not moving can have a spin, which produce a dipole magnetic field.

Just as the electric field $\vec{E}$ has electric field lines that are everywhere parallel to $\vec{E}$ and that give a flux $\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A}$ through a surface, so a magnetic field $\vec{B}$ has **magnetic field lines** everywhere parallel to $\vec{B}$ and that give a flux $\Phi_B = \int \vec{B} \cdot \mathrm{d}\vec{A}$ through a surface. For the electric field, Gauss's law gives the electric flux outward through a closed surface as

$$\Phi_E = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}, \tag{2.3}$$

where Q_{encl} is the total electric charge inside the volume enclosed by the surface. The analogue for the magnetic field would be $\Phi_B = \oint \vec{B} \cdot \mathrm{d}\vec{A} \propto$ (magnetic charge Q_B enclosed). However, no magnetic charge Q_B has ever been observed, so we shall say Gauss's law for magnetism is $\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0}$ through any closed surface.

Analogous to the way that Maxwell in 1865 completed the development of **electromagnetism** started by Coulomb, Gauss, Ampere, Faraday, and others, the 1960s saw the unification of electromagnetism and the **weak interaction** (radioactivity) by Glashow, Weinberg, and Salam. When combined with the theory of **quantum chromodynamics** for the strong nuclear reactions, including the Higgs mechanism for generating the masses of quarks (the building blocks for hadrons such as protons and neutrons) and of electrons and related other leptons, this gives what is known as the **Standard Model** of particle physics. However, this treats rather separately, and does not fully unify, the **electroweak** and **strong** interactions. Such a unification would be a **Grand Unified Theory**.

Such a **Grand Unified Theory** (GUT) would explain why the electron charge seems to be exactly the same magnitude (but opposite sign) as the proton charge, and, related to this, it predicts the existence of **magnetic monopoles**, which would have nonzero magnetic charge. However, these magnetic monopoles are predicted to have energies about a trillion times larger than the energies that can be produced in the largest accelerator today, the **Large Hadronic Collider (LHC)** at CERN, where several UofA professors work. So unless these estimates are wrong, it might be too difficult for humans ever to produce magnetic monopoles. Therefore, in this course we shall assume $\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0$ or no magnetic monopoles.

Gauss's Law for magnetism with no magnetic monopoles, or $\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0}$, implies that magnetic field lines have no beginning or end. Generally, they come in from infinity and go back out to infinity, but they can also wrap around and around in a finite volume without ever reconnecting, unlike what the textbook says near the bottom of p.889: "Like all other magnetic field lines, they form closed loops." Only in special situations, such as symmetry of rotation about an axis, or other situations in which a set of field lines stay in one plane without going to infinity, do the field lines form closed loops.

Asymmetric field lines do not stay on any single non-branching surface, and they need not form closed loops. A permanent magnet longer along the direction of the magnetic field inside has a north pole end (N) and a south pole end (S). Magnetic field lines run through the magnet from the S end to the N end and then emerge to loop back outside (though) they need not stay in a plane or distorted plane to reconnect precisely). However, because magnetic field lines nowhere begin or end, if one cuts the magnet in two, one will not get an isolated north pole or south pole, but rather the two new magnets will each have a north pole end and a south pole end, with the magnetic field passing through each magnet from south to north and then looping back outside, without ever ending.

A compass needle is a magnet free to rotate in the horizontal plane (when the compass is level), and then the north pole end of the compass points in the direction of the horizontal component of the magnetic field. From a Natural Resources Canada magnetic declination calculator online, when I entered the date 2019 March 12 and latitude $\frac{\pi}{2} - \frac{2}{\pi}$ radians $= 53.5244°$N and longitude $\frac{5\pi}{6} - \frac{2}{\pi}$ radians $= 113.5244°$W for the University of Alberta ("colatitude equator's reciprocal", and longitude $\pi/3 = 60°$ greater than latitude), 1 got a magnetic declination of 14.094°E, meaning that a compass would point 14.094° east of true north. It also said that this declination is decreasing about 1.04° every 6 years. Where I lived in Manokotak, Alaska, the declination decreased from 20°E when I first went there in late summer 1959 to 12.2°E today, a decreased of 7.8° in 60 years, an average decrease of 0.13°/year or 1°/(8 years).

The earth itself acts like a relatively permanent magnet, though its magnetic field is gradually changing with time (for example, with the North Magnetic Pole, where $\vec{B}$ is vertically downward, moving across northern Canadian islands during my lifetime, now being north of the Canadian Arctic territorial claim and within 4° of the Geographic North Pole and moving toward Russia at 55-60 km/year) and also reverses at random time intervals ranging between 100000 years and 50000000 years, the most recent being about 780000 years ago.

The earth's magnetic field is crudely a dipole field, with the dipole part having an axis that passes through the North Geomagnetic Pole that in 2015 was located at 80.37°N, on Ellesmere Island, Nunavut, Canada. But because the field is not purely a dipole, the North Magnetic Pole and the North Geomagnetic Pole do not coincide, and a compass needle does not point to either.

When two magnets are placed close together end to end, the magnetic field energy is lower if the magnetic field on the axis is aligned: $\boxed{\text{S} \rightarrow \text{N}}\,\boxed{\text{S} \rightarrow \text{N}}$ Thus the north pole of one magnet attracts the south pole of another, so opposite poles attract, rather similar to the way opposite charges attract and charges of the same sign repel, except that for magnets, one cannot have isolated poles. But since the north pole of a compass is attracted towards the north magnetic pole of the earth (where the magnetic field enters the earth vertically downwards), if one views the earth as a magnet, the North Magnetic Pole, or perhaps slightly better, the North Geomagnetic Pole (intersection of the dipole axis with the earth in the north), is a magnetic south pole, where magnetic field lines enter the earth, as they do in the south pole of a magnet.

Since the magnetic force magnitude is

$$F_B = \left|\vec{F}_B\right| = \left|q\vec{v} \times \vec{B}\right| = |q|\, v_\perp B = |q|\, v B_\perp, \tag{2.4}$$

the units of magnetic field are those of $\frac{F}{qv} = \frac{\text{kg}\cdot\text{m/s}^2}{\text{C}\cdot\text{m/s}} = \frac{\text{kg}}{\text{C}\cdot\text{s}} = \frac{\text{kg}}{\text{A}\cdot\text{s}^2} = \text{T}$, the **tesla**, (in honor of Nikola Tesla, 1856-1943, the prominent Serbian-American scientist and inventor). $1\,\text{T} = 1\,\frac{\text{kg}}{\text{A}\cdot\text{s}^2} = 1\,\frac{\text{N}}{\text{A}\cdot\text{m}}$.

Another common unit is the **gauss (G)**, $1\,\text{G} = 10^{-4}\,\text{T}$, a more convenient unit for the earth's magnetic field, since over the earth's surface, B ranges from 0.25 G to 0.65 G. For Edmonton 2019 March 12, $B = 0.567\,97\,\text{G}$, with an inclination angle of 75.096°, a vertical component of 0.548 86 G downward, a horizontal component of 0.548 86 G 14.094° E of north, given a northward component of 0.141 69 G and an eastward component of 0.035 57 G.

I tried to get a crude estimate for the magnetic flux outward through the part of the earth's surface where the outward radial (locally vertical) component B_r of the magnetic field is positive (mainly in the southern hemisphere, south of what might call the 'magnetic equator'), where the magnetic field has $B_r = 0$ and is parallel to the earth's surface, by which 1 mean the surface if the **geoid** which is the gravitational equipotential surface at mean sea level in the rotating frame of the earth, the rotation causing the geoid to be approximately ellipsoidal, and in fact given by a nominal ellipsoid to first approximation that has been chosen to have a semi-major axis $a = 6\,378\,137\,\mathrm{m}$ and a semi-minor axis $b = a(1 - 1/298257223563)$.

A dipole approximation for the field gives

$$B_r = -2B_0 \left(\frac{R_\oplus}{r}\right)^3 \cos\theta, \tag{2.5}$$

where θ is the angle from the north end of the dipole axis and $B_0 = 3.12 \times 10^{-5}\,\mathrm{T}$ is the value of the dipole field at $\theta = \frac{\pi}{2}$, where it is purely tangential (parallel to the sphere at that value of r). Then since a narrow band on a sphere of radius $r = R_\oplus =$ earth's radius of angular width $\mathrm{d}\theta$ has width $r\,\mathrm{d}\theta$ and length $2\pi r \sin\theta$, its area is $\mathrm{d}A = 2\pi r^2 \sin\theta\,\mathrm{d}\theta$. This gives the total area of the sphere as

$$\begin{aligned} A &= \int_0^\pi 2\pi r^2 \sin\theta\,\mathrm{d}\theta \\ &= 2\pi r^2[-\cos\theta]_0^\pi \\ &= 4\pi r^2. \end{aligned}$$

The outward magnetic flux through the 'southern magnetic hemisphere,' $\frac{\pi}{2} < \theta < \pi$, is

$$\begin{aligned} \int B_r\,\mathrm{d}A &= \int_{\frac{\pi}{2}}^{\pi} (-2B_0\cos\theta) 2\pi r^2 \sin\theta\,\mathrm{d}\theta \\ &= -4\pi r^2 B_0 \int_{\frac{\pi}{2}}^{\pi} \cos\theta \sin\theta\,\mathrm{d}\theta \\ &= -4\pi r^2 B_0 \left[-\frac{1}{2}\cos^2\theta\right]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{1}{2} A B_0 \\ &= \frac{1}{2}(5.100\,72 \times 10^{14}\,\mathrm{m}^2)(3.12 \times 10^{-5}\,\mathrm{T}) \\ &= 7.96 \times 10^9\,\mathrm{Tm}^2 \\ &= 7.96 \times 10^9\,\mathrm{Wb}. \end{aligned}$$

This is an estimate of the total flux passing through the earth, equal to what exits in the 'southern magnetic hemisphere,' and also equal to what enters the earth in the 'northern magnetic hemisphere,' since the total outward flux through the entire closed surface of the earth is $\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0$.

The largest fluxes passing through the surfaces of nations are almost certainly those through Russia and Canada, because not only do they have the largest areas, but also they are near a magnetic pole, where the radial component B_r of the magnetic field generally is larger. From graphs of the intensity and inclination (angle from vertical) of the earth's magnetic field at the surface taken from the World Magnetic Model of 2015 (168 spherical harmonic coefficients) and given in the Wikipedia article on "Earth's magnetic field," it appeared that the inclination over Russia varied from $61°$ to $83°$, say with a mean about $66°$, giving $\sin\theta \approx 0.91$, and with a field intensity over Russia varying from $51\,000\,\mathrm{nT}$ to $61\,000\,\mathrm{nT}$, say with a mean of about $56\,000\,\mathrm{nT} = 5.6 \times 10^{-5}\,\mathrm{T}$. Multiplying this by $\sin\theta \approx 0.91$ gives a mean vertically downward field component about $5.1 \times 10^{-5}\,\mathrm{T}$. Multiplying that normal component by the area Wikipedia listed for Russia, $17\,125\,192\,\mathrm{km}^2 \approx 1.7125 \times 10^{13}\,\mathrm{m}^2$ (1.7151 times that of Canada) gives a total magnetic flux through Russia about $8.8 \times 10^8\,\mathrm{Wb}$, about 11% of the total for a surface bounded by the 'magnetic equator,' e.g., outward through the 'southern magnetic hemisphere' or inward through the 'northern magnetic hemisphere' that contains both Russia and Canada.

The magnetic inclination over Canada appear to vary from about $70°$ to about $85°$, say with a mean of roughly $78°$ with $\sin 78° \approx 0.98$. The magnetic field intensity varied from roughly $54\,000\,\mathrm{nT}$ in a relatively small region to a peak of just over $59\,000\,\mathrm{nT}$ a bit west of Hudson's Bay, with nearly half of Canada's area (mainly in the north and west of Hudson's Bay) having over $58\,000\,\mathrm{nT}$, so I took an average to be about $58\,000\,\mathrm{nT} = 5.8 \times 10^{-5}\,\mathrm{T}$ and multiplied this by $\sin 78° \approx 0.98$ to get a mean normal (vertically downward) $-B_r \approx 5.7 \times 10^{-5}\,\mathrm{T}$, about 10% more than Russia, but with an area $9\,984\,670\,\mathrm{km}^2$, a total flux of about $5.7 \times 10^8\,\mathrm{Wb}$, about 65% of Russia's flux and about 7% of the total for the earth.

The total Lorentz force on a charged particle q with velocity $\vec{v}$ is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Since the infinitesimal work done by the field on the particle is $\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{r}$, where $\mathrm{d}\vec{r}$ is the infinitesimal vector displacement, the power done by the field is $P = \frac{\mathrm{d}W}{\mathrm{d}t} = \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{F} \cdot \vec{v} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} = q\vec{E} \cdot \vec{v}$, with no contribution from the magnetic field $\vec{B}$, because $(\vec{v} \times \vec{B})$ is perpendicular to $\vec{v}$. (In fact, $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}$ for any vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, this being the volume of the parallelopiped [a hexahedron or polyhedron with six faces, each of which is a parallelogram] with edges at one vertex $\vec{A}$, $\vec{B}$, and $\vec{C}$, up to sign. Therefore with $\vec{A} = \vec{v}$, $\vec{B} = \vec{B}$, and $|vecC = \vec{v}$, $(\vec{v} \times \vec{B}) \cdot \vec{v} = (\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B} = (\vec{v} \times \vec{v}) \cdot \vec{B} = 0$, since $\vec{v} \times \vec{v} = 0$.)

Therefore a magnetic field does no work on structureless point charges, though it can on particles with magnetic moments that change orientation. In particular, if a magnetic force is the only force on a point particle (which I shall use in the context for a structureless point particle, though the electron and quarks are particles with charge that are believed to have no intrinsic size, so in that way they are point particles, but they do have the structure of having magnetic dipole moments, so magnetic fields can do work on them if their orientation changes), then the change in kinetic energy is the work done, and since that is zero with a purely magnetic force, the kinetic energy does not change. Then the speed doesn't change either (even in relativity where $K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$).

For a uniform magnetic field B (say in the z-direction, $\vec{B} = B\hat{k}$, $B_z = B$) and particle velocity $\vec{v}$ in the x-y plane perpendicular to $\vec{B}$, the magnetic force is perpendicular to both $\vec{v}$ and $\vec{B}$ and hence is also in the x-y plane. The particle goes around a circle with constant space, and the magnetic force $F = |q|vB$ provides the centripetal acceleration $m\frac{v^2}{R}$, so $R = \frac{mv^2}{|q|vB} = \frac{mv}{|q|B} = \frac{p}{|q|B}$, where $p = mv$ is the momentum. Actually, $R = \frac{p}{|q|B}$ is true even in special relativity, where $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q(\vec{E} + \vec{v} \times \vec{B})$ remains exactly the same formula as it had nonrelativistically but now $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$, analogous to total energy $\varepsilon = mc^2 + K = \frac{mc^2}{\sqrt{1-v^2/c^2}}$.

For example, circular motion in the x-y plane, with for convenience the center of the circle at $x = y = 0$ and with radius R, can be described by $\vec{r} = \hat{\imath}R\cos(\omega t) + \hat{\jmath}R\sin(\omega t)$, $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = -\hat{\imath}R\omega\sin(\omega t) + \hat{\jmath}R\omega\cos(\omega t)$, $\vec{p} = -\hat{\imath}p\sin(\omega t) + \hat{\jmath}p\cos(\omega t)$ ($\vec{p}$ in the same direction as $\vec{v}$, but without here saying yet how p depends on R and ω, though onec an see that $v^2 = \vec{v}\cdot\vec{v} = v_x^2 + v_y^2 = R^2\omega^2\sin^2(\omega t) + R^2\omega^2\cos^2(\omega t)$, so $v = \sqrt{\vec{v}\cdot\vec{v}} = R|\omega|$. Then in a uniform magnetic field perpendicular to the plane of the circle, $\vec{B} = B\hat{k}$, $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -\hat{\imath}p\omega\cos(\omega t) - \hat{\jmath}p\omega\sin(\omega t) = q\vec{v}\times\vec{B} = q(-\hat{\imath}R\omega\sin(\omega t) + \hat{\jmath}R\omega\cos(\omega t)) \times (B\hat{k}) = \hat{\jmath}qBR\omega\sin(\omega t) + \hat{\imath}BR\omega\cos(\omega t)$, so $p = -qBR$. Because $\vec{p} \propto \vec{v}$ with a positive coefficient (nonrelativisitically m, relativistically $\frac{m}{1-v^2/c^2}$, $p_x = -p\sin(\omega t) \propto v_x = -R\omega\sin(\omega t)$, so p has the same sign as ω. Therefore, if $qBR > 0$, $p = -qBR < 0 \Longrightarrow \omega < 0$, so the motion is clockwise as seen from above (looking down the z-axis) when $qB > 0$ ($\vec{B}$ upward with positive q, or $\vec{B}$ downward with negative q), since we are always taking $R > 0$. On the other hand, if $qBR < 0$ ($\Longleftrightarrow$ $qB < 0$), $p = -qBR > 0 \Longrightarrow \omega > 0$, so the motion is counterclockwise when $qB < 0$ ($\vec{B}$ upward with negative q, or $\vec{B}$ downward with positive q). In any case, $R = \frac{|p|}{|q||B|} = \frac{mv}{|qB|\sqrt{1-v^2/c^2}}$. Thus in relativity, even though $v < c$, there is no limit to the size of the circle. $v = R|\omega|$ is always true, whether or not the motion is nonrelativisitic, so $|\omega| = \frac{v}{R} = \frac{v|qB|}{|p|} = \frac{|qB|}{m}\sqrt{1 - v^2/c^2} \longrightarrow_{v/c\to 0} \frac{|qB}{m}$ nonrelativisitically, the **cyclotron frequency** of a particle of mass m and charge B in a uniform magnetic field. This was the basis of a **cyclotron**, which is a particle accelerator that applies a boost to each particle twice per period $\frac{2\pi}{\omega} = \frac{2\pi m}{|qB|}$, causing the particles to move to larger and larger orbital radii $R = \frac{mv}{|qB|}$ nonrelativisitically as the angular frequency $|\omega| = \frac{|qB|}{m}$ remained the same. but when one tries to accelerate particles to relativistic speeds with v/c not negligible, the angular frequency $|\omega| = \frac{v}{R} = \frac{|qB|}{m}\sqrt{1 - v^2/c^2}$ decreases toward 0 as $\frac{v}{R}$ (with $v < c$) decreases and $\sqrt{1 - v^2/c^2}$ decreases, so the simplest principle of a cyclotron does not work to accelerate particles to relativistic speeds by a fixed frequency of impulses (by electric fields).

If the velocity of a charged particles is not perpendicular to the uniform magnetic field $\vec{B} = B\hat{k}$, since there is no force in the z-direction parallel to the field, the velocity and momentum components v_z and p_z in the z-direction stay constant.

Let us work out in the relativistic case how the angular frequency ω depends on the charge q, the mass m, the magnetic field B, the angle θ of the motion from the x-y plane, and the radius R of the helix in the x-y plane. Here I shall use the absolute values for all of these quantities, avoiding minus signs for ω, p, and r, but just remembering how the motion is clockwise when $qB > 0$ (with B positive when $\vec{B}$ is upward) and counterclockwise when $qB < 0$.

If v is the speed, $v_{\|} = v_z = v\sin\theta$ is the velocity component parallel to the magnetic field, and $v_{\perp} = v\cos\theta$ is the velocity component in the x-y plane perpendicular to the magnetic field. $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$, so $p_{\|} = \frac{mv_{\|}}{\sqrt{1-v^2/c^2}}$, $p_{\perp} - \frac{mv_{\perp}}{\sqrt{1-v^2/c^2}}$. $v_{\perp} = R\omega$, so $F_{\perp} = \frac{\mathrm{d}p_{\perp}}{dt} = p_{\perp}\omega = qv_{\perp}B = qR\omega B$ (since $v^2 = v_{\|}^2 + v_{\perp}^2$ does not change: $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$). Thus $p_{\perp} = qBR$, exactly the same for relativistic spiral motion as for nonrelativistic circular motion. However, $v = \frac{v_{\perp}}{\cos\theta}$, so $\frac{mv_{\perp}}{p_{\perp}} = \sqrt{1-v^2/c^2} = \sqrt{1-\frac{v_{\perp}^2}{c^2\cos^2\theta}}$. Then we get $R = \frac{p_{\perp}}{qB}$, $\omega = \frac{v_{\perp}}{R} = \frac{qBv_{\perp}}{p_{\perp}} = \frac{qB}{m}\sqrt{1-v^2/c^2} = \frac{qB}{m}\sqrt{1-\frac{R^2\omega^2}{c^2\cos^2\theta}}$, $\omega^2 = \frac{q^2B^2}{m^2} - \frac{q^2B^2}{m^2}\frac{R^2\omega^2}{c^2\cos^2\theta}$. Solving for ω^2, $\left(1+\frac{q^2B^2R^2}{m^2c^2\cos^2\theta}\right)\omega^2 = \frac{q^2B^2}{m^2}$, so

$$\boxed{\omega = \frac{qB}{m\sqrt{1+\left(\frac{qBR}{mc\cos\theta}\right)^2}}.} \tag{2.6}$$

If $R << \frac{mc\cos\theta}{qB}$, this reduces to

$$\omega \approx \frac{qB}{m}, \tag{2.7}$$

the cyclotron frequency that to a very good approximation is independent of R for R not too big for given q, B, m, and θ.

Take an electron of $m = 9.109\,383\,56 \times 10^{-31}\,\mathrm{kg}$, charge magnitude (minus the charge that is -e) that is now defined as $e = 1.602\,176\,634 \times 10^{-19}\,\mathrm{C}$, $c = 299\,792\,458\,\frac{\mathrm{m}}{\mathrm{s}}$, $B = 1200\,\mathrm{T}$ record produced by the University of Tokyo in 2018. $\frac{R}{\cos\theta} = \frac{mc}{qB} = \frac{(9.109\,383\,56 \times 10^{-31}\,\mathrm{kg})(299\,792\,458\,\frac{\mathrm{m}}{\mathrm{s}})}{(1.602\,176\,634 \times 10^{-19}\,\mathrm{C})(1200\,\mathrm{T})} = 1.420\,424 \times 10^{-6}\,\mathrm{m}$ (if exactly $SI1200T$) = 1.4 microns, so with that field even very small R gives relativistic effects, but $B\downarrow$ makes $R\uparrow$.

2.1 Velocity Selector

If one has a particle beam (say from a heated cathode or radioactive source) that has a range of speeds (and hence of energies), one often desires to get the beam to have a narrow range of speeds and energies by a **velocity selector** that deflects particles outside that range away from the beam allowed path (that is, so the deflected particles hit some boundary and are removed). This has a magnetic field $\vec{B}$ roughly perpendicular to the allowed path, and an electric field chosen to be $\vec{E} = -\vec{v_0} \times \vec{B}$, where $\vec{v_0}$ is the desired velocity of the particles that get through the allowed beam path, so $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = q(-\vec{v_0} \times \vec{B} + \vec{v} \times \vec{B}) = q(\vec{v} - \vec{v_0}) \times \vec{B}$, nonzero unless $\vec{v} - \vec{v_0}$ is parallel to $\vec{B}$, but then if $\vec{v} - \vec{v_0} \neq 0$, the particle would leave the path.

In 1897 Joseph John (J.J.) Thomson (1856-1940, Nobel Prize 1906, knighted 1908 to become "Sir," Master of Trinity College 1918-1940) accelerated cathode rays by a measured electric potential difference (voltage) V and then used perpendicular $\vec{E}$ and $\vec{B}$ fields, also perpendicular to the path, to measure the velocity. If the rays consisted of particles of charge magnitude e and mass m that were nearly at rest before being accelerated by the voltage V, after the acceleration they would have kinetic energy $\frac{1}{2}mv^2 = eV$, so $v = \sqrt{\frac{2eV}{m}}$. Then with $\vec{E} = -\vec{v_0} \times \vec{B}$ or $E = v_0 B$ when all are perpendicular, and when the rays passed undeflected so $v = v_0 = \frac{E}{B}$, one gets $v = \sqrt{\frac{2eV}{m}} = \frac{E}{B}$, or $\frac{2eV}{m} = \frac{E^2}{B^2}$. Therefore, by measuring V, B, and E on the right hand side, one has a measurement of the charge-to-mass ratio $\frac{e}{m}$ of the particles, which became known as **electrons** after an earlier 1891 suggestion by George Johnstone Stoney. The main significance was that Thomson got a single value for $\frac{e}{m}$, not depending on details of the experimental conditions, so it showed that cathode rays are composed of particles with a unique charge-to-mass ration e/m. (Of course, the negative sign of the charge was also known from the sign of V, so the charge-to-mass ratio is really $-e/m$.)

Note that the only $-e/m$ was measured by J.J. Thomson's experiment, but not either $-e$ or m separately. One can see that this is true without any detailed math, since if each particle has a measured velocity v that is the same, a pair of particles would have the same velocity, and in Thomson's experiment a pair traveling at v would be indistinguishable from a single particle of twice the charge and twice the mass, which would have the same charge-to-mass ratio. The charge itself was measured 12 years later by the 1909 **oil drop experiment** by Robert A. Millikan and Harvey Fletcher. The experiment measured the differences in electric forces on an oil drop that differed from neutrality by one or more electrons.

The argument I presented for why J.J. Thomson's experiment, which did not detect individual electrons, measured only the charge-to-mass ratio. reminds of Galileo's argument against ARistotle's calim that objects would fall at speeds proportional to their weights, in *Dialogues Concerning Two New Sciences:*

"But, even without further experiment, it is possible to prove clearly, by means of a short and conclusive argument, that a heavier body does not move more rapidly than a lighter one, provided both bodies are of the same material.... If then we take two bodies whose natural speeds are different, it is clear that on uniting the two, the more rapid one will be partly retarded by the slower, and the slower will be somewhat hastened by the swifter...

"But, if this is true, and if a large stone moves with a speed of, say, eight, while a smaller stone moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see how, from your assumption that the heavier body moves more rapidly than the light one, I infer that the heavier one moves more slowly.... We infer therefore that large and small bodies move with the same speed, provided they are of the same specific gravity." — Galileo Galilei

Once the electron charge $-e$ was measured, one could measure the masses of singly ionized atoms (one electron lost from the neutral atom, so ion charge $+e$) by their motion in measured electric and magnetic fields. First, the velocity $\vec{v} = \vec{v_0}$ (within some limits of the size of the aperture where the beam emerges) is measured by having the particles traverse a region with $\vec{B}$ and with $\vec{E} = -\vec{v_0} \times \vec{B}$. Next, the (nonrelativistic) particles pass through a region with a different electromagnetic field $\vec{E}' = 0$, $\vec{B}' \neq 0$, where they are bent into circular arcs of radius $R = \frac{p}{qB'} = \frac{mv}{qB'}$, so for $q = +e$, $m = \frac{qB'R}{V} = \frac{qB'R}{E/B}$. In 1913, J.J. Thomson found by this method that neon atoms had two different masses, neon-20 and neon-22, both now known to have 10 protons, but different numbers of neutrons (10 or 12). Thomson's student Francis William Aston made the first fully functional mass spectrometer in 1919.

2.2 Magnetic Force on a Current

If we have a current Density $\vec{J} = qn\vec{v_d}$ or $\sum_{i=1}^{N} q_i n_i \vec{v}_{di}$ with N species or $\left(\sum_{j=1}^{M} q_i \vec{v}_{di}\right)/V$ with M particles in volume V, the total magnetic force is $\vec{F}_M = \sum_{j=1}^{M} q_i \vec{v}_{di} \times \vec{B} = V\vec{J} \times \vec{B}$, or **force density** $\boxed{\vec{f} = \vec{J} \times \vec{B}}$ (magnetic force per unit volume).

Suppose the current is flowing through a fairly thin wire segment of vector length $\mathrm{d}\vec{l}$. The current is $I = \int \vec{J} \cdot \mathrm{d}\vec{A}$, and the current density will be in the direction of $\mathrm{d}\vec{l}$, $\vec{J} = J\frac{\mathrm{d}\vec{l}}{\mathrm{d}l}$, with $\mathrm{d}\vec{l} \cdot \mathrm{d}\vec{A} = \mathrm{d}V$, the volume element of the wire segment, so $I = \int J\frac{\mathrm{d}\vec{l}}{\mathrm{d}l} \cdot \mathrm{d}\vec{A} = \frac{1}{\mathrm{d}l} J\,\mathrm{d}V$, or $J = \frac{I\,\mathrm{d}l}{\mathrm{d}V}$, $\vec{J} = \frac{I\,\mathrm{d}\vec{l}}{\mathrm{d}V}$. Then the force density is $\vec{f} = \vec{J} \times \vec{B} = \frac{I\,\mathrm{d}\vec{l}}{\mathrm{d}V} \times \vec{B}$, and the element of force is $\mathrm{d}\vec{F} = \vec{f}\,\mathrm{d}V$, or $\boxed{\mathrm{d}\vec{F} = I\,\mathrm{d}\vec{l} \times \vec{B}}$, the current I multiplied by the cross product of the length $\mathrm{d}\vec{l}$ and of $\vec{B}$.

2.3 Force and Torque on a Current Loop

The net force in a uniform magnetic field is 0.

Torque is a twisting about some pivot point, the time rate of the transfer of angular momentum in the same way that force is the time rate of the transfer of linear momentum.

$$\boxed{\vec{\tau} = I \oint \vec{r} \times (\mathrm{d}\vec{r} \times \vec{B}).} \tag{2.8}$$

$$\boxed{\vec{\tau} = I \oint \left[\mathrm{d}\vec{r}(\vec{r} \cdot \vec{B}) - \vec{B}(\vec{r} \cdot \mathrm{d}\vec{r})\right].} \tag{2.9}$$

$$\boxed{\vec{\tau} = I\vec{A} \times \vec{B},} \tag{2.10}$$

where I is the current in the loop and $\vec{A}$ is the vector area (the vector sum of the vector areas of all the infinitesimal areas filling in the loop on any smooth surface filling in the loop; if the loop is planar, lying in one plane, the magnitude $A = |\vec{A}|$ of the vector area $\vec{A}$ is the normal to the plane, with the sense that $\vec{A}$ is up when the current goes around counterclockwise as seen from above).

The **magnetic dipole moment** vector of the loop with current I and vector area $\vec{A} = \frac{1}{2}\oint \vec{r} \times \mathrm{d}\vec{r}$ is $\boxed{\vec{\mu} = I\vec{A}}$.

The vector magnetic torque of the magnetic dipole $\vec{\mu}$ by a uniform magnetic field is $\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$.

The magnetic torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ on a magnetic dipole moment $\vec{\mu}$ in a magnetic field leads to a potential energy $\boxed{U = -\vec{\mu} \cdot \vec{B}} = -\mu B \cos\theta$, where θ is the angle between the magnetic dipole moment $\vec{\mu}$ and the magnetic field $\vec{B}$.

2.4 The Hall Effect

When one has a current moving through a wide strip and a magnetic field perpendicular to the strip, the $\vec{f} = \vec{J} \times \vec{B}$ force density pushes charges in that direction (which is independent of the sign of the charge carriers) until the excess charge on that edge (of the same sign as the charge carriers) produces an electric field in the strip that balances the magnetic force, so that in the steady-state situation there is no force on the charge carriers with drift velocity $\vec{v_d}$ when $\vec{E} = -\vec{v_d} \times \vec{B}$. Then from this $\vec{E}$ and $\vec{B}$, one can get the sign of $\vec{v_d}$ (whether going one way or the other along the length of the strip). Knowing the current gives $\vec{J} = qn\vec{v_d}$, so then knowing $\vec{v_d}$ gives qn and hence the sign of q.

Chapter 3

Sources of Magnetic Field

We have seen that the electromagnetic force of sturctureless point particle of charge q is found by using the Lorentz force equation

$$\boxed{\vec{F} = q(\vec{E} + \vec{V} \times \vec{B})}. \tag{3.1}$$

We have also learned Gauss's law for $\vec{E}$,

$$\boxed{\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}}, \tag{3.2}$$

and Gauss's law for $\vec{B}$

$$\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0}, \tag{3.3}$$

assuming no magnetic monopoles (none seen in nature and probably requiring too much energy to be produced by humans). These three laws are exact and have exactly the same form in special relativity (though they can be written slightly more elegantly in 4-dimensional realtivistic notation, such as $m\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}$). We also learned an approcimate relation when we ignored relativistic effects such as electromagnetic waves:

$$\boxed{\oint \vec{E} \cdot \mathrm{d}l = 0} \leftrightarrow \exists \text{ potential } V \text{ such that } \boxed{\vec{E} = -\Delta V}. \tag{3.4}$$

The exact law $\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon + 0}$ and the nonrelativistic law $\oint \vec{E} \cdot \mathrm{d}l = 0$ allows one to deduce the nonrelativistic Coulomb's law (the approximation for v ¡¡ c) given first for the potential V and then for the electric field $\vec{E}$ ato location $\vec{r}_0$:

$$V(\vec{r}_0) = \frac{U(\vec{r}_0)}{q_0} = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r})(\vec{r}_0 - \vec{r})}{|\vec{r}_0 - \vec{r}|^3} \,\mathrm{d}l, \tag{3.5}$$

where q_i is an individual charge at location $\vec{r}_i$, $\mathrm{d}q(\vec{r})$ is an infinitesimal elemnt of charge $\rho(\vec{r})$ is a volume charge density, $\sigma(\vec{r})$ a surface charge density, $sigma(\vec{r})$ a surface charge density, $\lambda(\vec{r})$ linear charge density.

Just as we need two laws for the electric field, $\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$ and $\vec{E} \cdot \mathrm{d}\vec{l} = 0$ (this being a nonrelativistic approximation) to get Coulomb's law for how nonrelaticistic charges produce an electric field, we also need a second law beside $\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0$ for how to get the magnetic field in a nonrelativistic situation. The form most nearly similar to $\oint \vec{E} \cdot \mathrm{d}\vec{l} = 0$ (conservative nature of the electric forve in the nonrelativistic approximation of no electromagnetic waves carrying momentum) is **Ampere's Law**,

$$\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I = \mu_0 \int \vec{J} \cdot \mathrm{d}\vec{A}} \tag{3.6}$$

, where $I = \int \vec{J} \mathrm{d}\vec{A}$ is the current through any open surface with edge the loop around which $\oint \vec{B} \cdot \mathrm{d}\vec{l}$ is calculated (and with $\mathrm{d}\vec{A}$ up as seen from above when $\mathrm{d}\vec{l}$ goes counterclockwise around the loop).

Ampere's Law actually works only in a stationary situation in which charge is not building up anywhere, so $\oint \vec{J} \cdot \mathrm{d}\vec{A} = -\frac{\mathrm{d}Q_{encl}}{\mathrm{d}t} = 0$ and so that then $\int \vec{J} \mathrm{d}\vec{A}$ is independent of the open surface bounded by the loop around which $\oint \vec{B} \cdot \mathrm{d}\vec{l}$ is taken. When charges do build up, what is 0 by charge conservation is $0 = \oint \vec{J} \cdot \mathrm{d}\vec{A} + \frac{\mathrm{d}Q_{encl}}{\mathrm{d}t} = \oint \vec{J} \cdot \mathrm{d}\vec{A} + \frac{\mathrm{d}}{\mathrm{d}t} \varepsilon_0 \oint \vec{E} \cdot \mathrm{d}\vec{A} = \oint \left(\vec{J} + \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\right) \cdot \mathrm{d}\vec{A}$ so James Clerk Maxwell in his 1865 completion of the equations of electromagnetism brilliantly modified Ampere's Law to become

$$\boxed{\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \int \left(\vec{J} + \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\right) \cdot \mathrm{d}\vec{A}} \tag{3.7}$$

Later we shall learn Faraday's modification of $\oint \vec{E} \cdot \mathrm{d}\vec{l} = 0$

$$\boxed{\oint \vec{E} \cdot \mathrm{d} \cdot \mathrm{d}\vec{l} = -\int \frac{\mathrm{d}B}{\mathrm{d}t} \cdot \mathrm{d}\vec{A}} \tag{3.8}$$

These plus the following two equations

$$\boxed{\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\varepsilon_0}} \text{ and } \boxed{\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0} \tag{3.9}$$

are **Maxwell's equations for EM**

From Maxwell's equatios and appropriate boundary conditions (such as no electromagnetic waves coming from the distant past, which are really approximately valid for waves of frequency high compared with the microwave frequencies of the cosmic microwave background of CMB radiation that is highly thermal, with a present temperature of 2.7255(6)K [k - kelvin, degrees above absolute zero] that is a hundred times colder than the freezing point of water at $0°C = 273.15\,\mathrm{K}$ and which gives an energy density $100^4 = 100000000 = 10^8$ times less than thermal radiation at $0°C$), one can get the electromagnetic fields produced by charges either at rest or in motion. For charges at rest, one only gets electric fields, by Coulomb's law.

To get magnetic fields from structureless point charges, one needs charges in motion. (One can also get magnetic fields from particles with both charge and spin angular momentum even if they are believed to be point particles such as electrons and quarks). Analogous to Coulomb's law $\vec{E}(\vec{r}_0) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i(\vec{r}_0-\vec{r}_i)}{|\vec{r}_0-\vec{r}_i|^3}$,

$$\boxed{\vec{B}(\vec{r}_0) = \frac{\mu_0}{4\pi}\sum_{i=1}^{N}\frac{q_i\vec{v}_1\times(\vec{r}_0-\vec{r}_i)}{|\vec{r}_0-\vec{r}_i|^3}} \tag{3.10}$$

is the nonrelativistic approximation ($|\vec{v}_i| << c$) for the magnetic field for a collection of particles of charge q_i and velocity $\vec{v}_i$ at locations $\vec{r}_i$ (where $\vec{r}_0$ is the location of the field point). For a single moving charge, if we define the unit vector $\vec{r} = \frac{(\vec{r}_0-\vec{r})}{|\vec{r}_0-\vec{r}|}$ that points from the source point $\vec{r}$to the field point $\vec{r}_0$ at distance $r = |\vec{r}_0 - \vec{r}|$, $\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$, an inverse-square law like Coulomb's.

just as ε_0 is the **electric constant** (also called the vacuum permittivity, the permittivity of free space, the distributed capacitance of the vacuum, or simply epsilon nought or epsilon zero), so μ_0 is the **magnetic constant** (also called the vacuum permeability, the permeability of free space, the permeability of the vacuum. or simply mu nought or mu zero).

What are the units? The SI unit for B is the tesla (T), with $\vec{F} = q\vec{v}\times\vec{B}$ giving $1\,\mathrm{N} = 1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s^2}} = (1\,\mathrm{C})(1\,\frac{\mathrm{m}}{\mathrm{s}})(1\,\mathrm{T}) = 1\,\mathrm{T}\frac{\mathrm{C\cdot m}}{\mathrm{s}}$ or $1\,\mathrm{T} = 1\,\frac{\mathrm{kg}}{\mathrm{C\cdot s}}$. Then $\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$ implies that μ_0 has units of $\frac{\mathrm{r^2}}{\mathrm{qv}}\mathrm{B}$ or $\frac{\mathrm{m^2}}{\mathrm{Cms^{-1}}}\mathrm{T} = \frac{\mathrm{ms}}{\mathrm{C}}\frac{\mathrm{kg}}{\mathrm{C\cdot s}} = \frac{\mathrm{kg\cdot m}}{\mathrm{C^2}}$

In November 2018, the 26th General Conference on Weights and Measures (CGPM) approved a redefinition of SI base units to come into force 2019 May 20: 1 second = 9192631770 period of the unperturbed ground-state hyperfine transition frequency of the caesium 133 atom (old). 1 metre = distance light travels in $\frac{1\mathrm{s}}{299792458}$ 1 kilogram is defined so $h = 6.626\,070\,15\times10^{-34}\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}}$ an ampere = $1\,\frac{\mathrm{C}}{\mathrm{s}}$ with $e = 1.602\,176\,634\times10^{-19}\,\mathrm{C}$. 1 kelvin = $1\,\frac{\mathrm{J}}{\mathrm{k}}$, $k = 1.380\,649\times10^{-23}\,\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2K}}$(Boltemann). 1 mole = $6.022\,140\,76\times10^{23}$ elementary entities. 1 candela (cd) is defined so that the luminous efficacy of monochromatic radiation of frequency $540\times10^{12}\,\mathrm{Hz}$ is $683\,\frac{\mathrm{cd\cdot sr\cdot s^3}}{\mathrm{kgm^3}}$. Therefore, sa of 2019 May 20, $\frac{\mu_0}{4\pi}$ is not defined to be exactly $10\times10^{-7}\,\frac{\mathrm{kg\cdot m}}{\mathrm{C^2}} = 10\times10^{-7}\,\frac{\mathrm{N}}{\mathrm{A^2}}$ ut will be determined by experiment, using the new definition of the coulomb in terms of the elementary charge e.

Chapter 4

Electromagnetic Induction

4.1 Summary of Electromagnetic Induction

- Faraday's law:
$$\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \tag{4.1}$$
for the induced emf in a closed loop from the time rate of charge of magnetic flux through the loop.

- Lenz's law: The sign in Faraday's law is such that the induced current tends to oppose the change in flux that produced it (stability).

- If the conductor loop moves in a static magnetic field,
$$\varepsilon = \oint (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \oint \vec{B} \cdot \mathrm{d}\vec{A} = -\oint \vec{B} \cdot \frac{\mathrm{d}\vec{A}}{\mathrm{d}t} \tag{4.2}$$
is a motional emf.

- If $\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \neq 0$, $\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\oint \frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \cdot \mathrm{d}\vec{A} \neq 0$ makes the electric field nonconservative, so $\vec{E} \neq -\vec{\nabla} V$.

- Ampere's law as corrected by Maxwell includes a displacement current $i_D = \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$: $\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \int \left(\vec{J} + \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t} \right) \cdot \mathrm{d}\vec{A} = \mu_0 (i_c + i_D)_{\mathrm{encl}}$.

- The other 3 Maxwell equations are

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\mathrm{encl}}}{\varepsilon_0}, \tag{4.3}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0, \tag{4.4}$$

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}. \tag{4.5}$$

- When $Q = 0$, $J = 0$, symmetric under $\vec{E} \to c\vec{B}$, $\vec{B} \to -\frac{1}{c}\vec{E}$.
- In 4D relativistic notation, the 4 Maxwell equations can be reduced to $\mathrm{d}F = 0$ and $\delta F = J$, or, using $F = \mathrm{d}A$ to solve $\mathrm{d}F = 0$, $\delta\, \mathrm{d}A = J$.

Chapter 5

Inductance

5.1 Summary of Inductance

- If we have two nearby coils with currents i_1 and i_2 then current i_1 induces $\varepsilon_2 = -M_{21}\frac{\mathrm{d}i_1}{\mathrm{d}t}$ in the 2nd, and current i_2 induces $\varepsilon_1 = -M_{12}\frac{\mathrm{d}i_2}{\mathrm{d}t}$ in the 1st. $M_{12} = M_{21}$, so we can just call the mutual inductance M.

- In a single circuit with self-inductance L, $\varepsilon = -L\frac{\mathrm{d}i}{\mathrm{d}t}$. M and L have the SI unit 1 henry = $1\,\mathrm{H} = 1\,\frac{\mathrm{Wb}}{\mathrm{A}} = 1\,\frac{\mathrm{V}}{\mathrm{A/s}} = 1\,\frac{\mathrm{Vs}}{\mathrm{A}} = 1\,\Omega\mathrm{s} = 1\,\frac{\mathrm{J}}{\mathrm{A}^2} = 1\,\frac{\mathrm{kgm}^2}{\mathrm{s}^2\mathrm{A}^2} = 1\,\frac{\mathrm{kgm}^2}{\mathrm{C}^2}$.

- $\oint \vec{E} \cdot \mathrm{d}\vec{l} = -L\frac{\mathrm{d}i}{\mathrm{d}t} \neq 0$, so the electric field is **nonconservative**.

- The energy in an inductor with current I: $U = \frac{1}{2}LI^2$.

- This energy is in the magnetic field, with energy density $u = \frac{B^2}{2\mu_0}$, or $u = \frac{B^2}{2\mu}$ if we include magnetic material.

- Electric and magnetic energy densities are equal for plane electromagnetic waves, which have $E = cB$, $\vec{E}$ and $\vec{B}$ perpendicular, and wave direction parallel to $\vec{E} \times \vec{B}$.

- An R-L circuit has $i = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$ with time constant $\tau = \frac{L}{R}$ when charging, or $i = I_0 e^{-t/\tau}$ when discharging.

- An L-C circuit has $q = Q\cos(\omega t + \phi)$, $i = \frac{\mathrm{d}q}{\mathrm{d}t} = -Q\omega\sin(\omega t + \phi)$ with $\omega = \sqrt{\frac{1}{LC}}$, conserved energy, and no damping.

- An L-R-C circuit has $q =$ sum of damped exponentials if overdamped ($R^2 > 4L/C$), $q = e^{-(R/2L)t}(At + B)$ if critically damped ($R^2 = 4L/C$), and $q = Ae^{-(R/2L)t}\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}t + \phi\right)$ if underdamped.

Chapter 6

Electromagnetic Waves

6.1 Summary of Electromagnetic Waves

- We had learned Maxwell's equations as integrals:

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon} = \frac{1}{\varepsilon_0} \int \rho \, \mathrm{d}V, \tag{6.1}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0, \tag{6.2}$$

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \vec{B} \cdot \mathrm{d}\vec{A}, \tag{6.3}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \int \left(\vec{J} + \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t} \right) \cdot \mathrm{d}\vec{A}. \tag{6.4}$$

- Divergence theorem $\oint \vec{V} \cdot \mathrm{d}\vec{A} = \int \left(\vec{\nabla} \cdot \vec{V} \right) \mathrm{d}V$ and Kelvin-Stokes theorem $\oint \vec{V} \cdot \mathrm{d}\vec{l} = \int \left(\vec{\nabla} \times \vec{V} \right) \cdot \mathrm{d}\vec{A}$ give Maxwell's equations as partial differential equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \tag{6.5}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{6.6}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}, \tag{6.7}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right). \tag{6.8}$$

- A general plane wave solution in empty space, $\vec{v} = cR$: $\vec{E} = \hat{\imath} E_x(u) + \hat{\jmath} E_y(u)$, $\vec{B} = -\frac{1}{c} \hat{\imath} E_y(u) + \frac{1}{c} \hat{\jmath} E_x(u)$ in terms of two arbitrary functions of $u = ct - z$, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$.

- $\vec{E} = \vec{a}E_a(u) + \vec{b}E_b(u)$, $\vec{B} = -\frac{1}{c}\vec{a}E_b(u) + \frac{1}{c}\vec{b}E_a(u)$ for $\vec{v} = cR$, $u = ct - \vec{n} \cdot \vec{r}$, $\vec{a} \times \vec{b} = \vec{n}$, $\vec{b} \times \vec{n} = \vec{a}$, $\vec{n} \times \vec{a} = \vec{b}$ (orthonormal).

- A plane wave is transverse ($\vec{E} \perp \vec{n}$, $\vec{B} \perp \vec{n}$), and $\vec{E} \perp \vec{B}$.

- For fixed frequency ω, $\vec{E} = \vec{a}E_1 \cos(\omega t - \frac{\omega}{c}\vec{n} \cdot \vec{r} + \phi_1) + \vec{b}E_2 \cos(\omega t - \frac{\omega}{c}\vec{n} \cdot \vec{r} + \phi_2)$ generally sweeps out an ellipse (elliptical polarization) in the plane $\perp \vec{n}$ (or a line for plane polarization, or a circle for circular polarization), as does $\vec{B} = \frac{1}{c} n \times \vec{E}$.

- Most light is a mixture of different frequencies and polarizations, unpolarized if not favoring any.

- Light scattered or reflected is often partially polarized.

- The energy density in an electromagnetic field is $u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$. For a plane wave, each of the two terms are equal, so $u = \varepsilon_0 E^2$.

- The energy flux in general electromagnetic field is the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$. For a plane wave, $\vec{S} = u\vec{v} = uc\vec{n}$.

- For a plane wave of fixed frequency, time average $< \vec{S} >= \vec{S_{av}} = \frac{1}{2\mu_0 c}\left(E_{\text{max}}^2 + E_{\text{min}}^2\right)\vec{n} = \frac{1}{2}\varepsilon_0 c \left(E_{\text{max}}^2 + E_{\text{min}}^2\right)\vec{n}$.

- A particle of energy ε and velocity $\vec{v}$ has momentum $\vec{p} = \frac{1}{c^2}\varepsilon\vec{v}$. Photon: $\varepsilon = \hbar\omega$, $\vec{p} = \frac{\hbar\omega}{c}\vec{n}$.

- Momentum density of a plane wave: $\frac{\mathrm{d}\vec{p}}{\mathrm{d}V} = \frac{1}{c^2}\vec{S}$.

- Absorption radiation pressure for a plane wave: $p_{\text{rad}} = u = \frac{S}{c}$, $< p_{\text{rad}} >= \frac{1}{\mu_0 c} < E^2 >$.

- For an isotropic distribution, $p_{\text{rad}} = \frac{1}{3}i$.

- Sunlight: $S_O = 1361\,\text{W/m}^2$, $E_{\text{rms}} =< E^2 >^{0.5}= 716\,\text{V/m}$, $B_{\text{rms}} = 2.388 \times 10^{-6}\,\text{T}$, $u = 4.540 \times 10^{-6}\,\text{J/m}^3$, $p_{\text{rad}} = u = 4.540 \times 10^{-6}\,\text{N/m}^2 = 4.540 \times 10^{-6}\,\text{Pa}$, solar luminosity $L_O = 4\pi r^2 S_O = 3.828 \times 10^{26}\,\text{W}$.

- Electromagnetic waves in matter: $E = vB$, $v = \frac{1}{\sqrt{\varepsilon\mu}}$.

- Standing waves of reflected plane waves have nodal plants $z = \pi n/k$ for $\vec{E}$, $z = (\pi/k)(n+0.5) for \vec{B}$. $\vec{E}$ and $\vec{B}$ each vanish everywhere at different times.

- Two infinite parallel perfectly conducting plates with separation L: $k = k_n = \pi n/L$, $\omega = \omega_A = ck_n = \pi cn/L$, $\lambda = \lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$, $f_n = \frac{\omega_n}{2\pi} = \frac{cn}{2L} = \frac{c}{\lambda_n}$, period $T_n = \frac{1}{f_n} = \frac{2L}{cn}$.