Paper 2, Section II, K

(a) Let $(X_i, \mathcal{A}_i)$ for $i = 1, 2$ be two measurable spaces. Define the product σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ on the Cartesian product $X_1 \times X_2$. Given a probability measure μ_i on $(X_i, \mathcal{A}_i)$ for each $i = 1, 2$, define the product measure $\mu_1 \otimes \mu_2$. Assuming the existence of a product measure, explain why it is unique. [You may use standard results from the course if clearly stated.]

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which the real random variables U and V are defined. Explain what is meant when one says that U has law μ. On what measurable space is the measure μ defined? Explain what it means for U and V to be independent random variables.

(c) Now let $X = \left[-\frac{1}{2}, \frac{1}{2}\right]$, let $\mathcal{A}$ be its Borel σ-algebra and let μ be Lebesgue measure. Give an example of a measure η on the product $(X \times X, \mathcal{A} \otimes \mathcal{A})$ such that $\eta(X \times A) = \mu(A) = \eta(A \times X)$ for every Borel set A, but such that η is not Lebesgue measure on $X \times X$.

(d) Let η be as in part (c) and let $I, J \subset X$ be intervals of length x and y respectively. Show that

$$x + y - 1 \leqslant \eta(I \times J) \leqslant \min\{x, y\}$$

(e) Let X be as in part (c). Fix $d \geqslant 2$ and let Π_i denote the projection $\Pi_i\left(x_1, \ldots, x_d\right) = \left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d\right)$ from X^d to X^{d-1}. Construct a probability measure η on X^d, such that the image under each Π_i coincides with the $(d-1)$-dimensional Lebesgue measure, while η itself is not the d-dimensional Lebesgue measure. [Hint: Consider the following collection of $2d-1$ independent random variables: $U_1, \ldots, U_d$ uniformly distributed on $\left[0, \frac{1}{2}\right]$, and $\varepsilon_1, \ldots, \varepsilon_{d-1}$ such that $\mathbb{P}\left(\varepsilon_i = 1\right) = \mathbb{P}\left(\varepsilon_i = -1\right) = \frac{1}{2}$ for each i.]