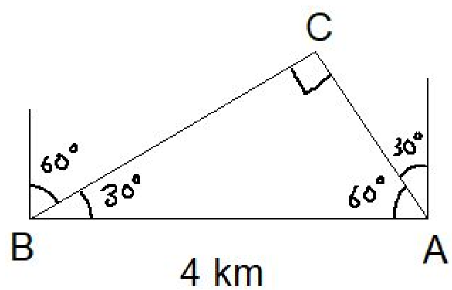
Maths Solutions (Mechanics)

Section A: Multiple Choice

1. **Answer:** $2\sqrt{3}$ km

Working: Start with a diagram:



Distance BC = 4 sin 60 = $2\sqrt{3}$ km

2. **Answer:** 108 N

Working: T - 5g = 5 * 0.8
→ T = 54 N
54 N tension forces pull on **both** sides of the pulley, so the coupling must be able to withstand at least 54 + 54 = 108 N.

3. **Answer:** 9.6 kN

Working: Resolving forces acting on the crate: T - 800g = 800 * 2
→ T = 9600 N = 9.6 kN

4. **Answer:** Distance from pivot = 0.4 m on right of pivot; Force / N = 1100

Working: Resolving forces, reaction force at pivot = (15 + 35 + 60)g = 1100 N
Let x be the distance of centre of gravity to the pivot. Taking moments about pivot:
15g * x + 35g * 1.2 = 60g * 0.8 → x = +0.4 m (to the right)

5. **Answer:** There must be a frictional force acting to the left between the cube and the table.

Working: The other forces acting on the cube are weight and normal reaction force. Both of these act vertically, so there must be another force acting horizontally to counteract force *P*. This must be friction.

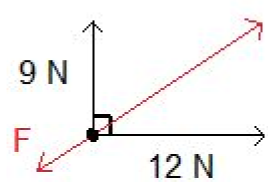
The moment of *P* about the bottom corner is $Pd \times (\sin 30 + \sin 60)$, where $d(\sin 30 + \sin 60)$ is the vertical (perpendicular) distance from *P* to the corner.

6. **Answer:** Height / m = 3.0; time / s = 1.0

Working: s = ?, u = 8, v = -2, a = -10, t = ?
$v^2 - u^2 = 2as \rightarrow s = ((-2)^2 - 8^2) / (2 * -10) = 3.0$ m
$v = u + at \rightarrow t = (-2 - 8) / (-10) = 1.0$ s

7. **Answer:** 5 ms^{-2}

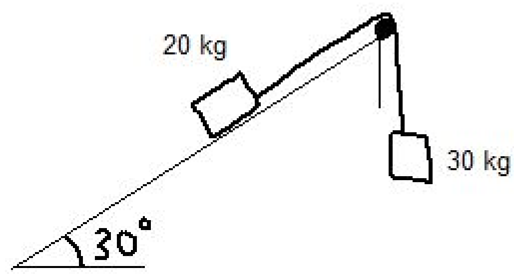
Working: Top-down view:



Resultant force without friction = $\sqrt{(9^2 + 12^2)} = 15$ N
Frictional force = 0.25 * 2g = 5 N
Resultant acceleration = (15 - 5) / 2 = 5 ms^{-2}

8. **Answer:**

Working:



Resolving on 30 kg mass: 30g - T = 30 * 2.5 → T = 225 N
Resolving on 20 kg mass: 225 - 20g sin 30 - F = 20 * 2.5
→ F = 75 N

9. **Answer:** 500 m

Working: Decelerating when gradient of velocity-time graph is negative, from t = 110 to 130 s.
Distance = area under graph:
Distance = 1/2 * 20 * (30 + 20) = 500 m

10. **Answer:** 1.0 m to the right

Working: Taking moments about pivot,
10 * 2000g = 10 * 400g + moment due to crane weight
→ moment due to crane weight = 160000 Nm
Taking moments again in new position,
x * 2000g = 15 * 400g + 160000
→ x = 11 m
Original position was 10 m, so must move 1 m to the right

Alternative method

Additional anti-clockwise moment from load: 5 * 400g = 20000 Nm
Additional clockwise moment needed to balance = 20000 Nm
→ 20000 = x * 2000g → x = 1 m
Direction is to the right to provide a greater clockwise moment.

Section B: Standard Questions

11. Differentiate twice to obtain acceleration:
$v(t) = x'(t) = -t^2 + 6t$ [1 mark]
$a(t) = x''(t) = -2t + 6 = 0$ [1 mark] $\rightarrow t = 3$ [1 mark]
$x(3) = 40 \rightarrow -1/3(3)^3 + 3(3)^2 + k = 40 \rightarrow k = 22$ [1 mark]

12.

a. D = area under graph
= 2 * (1/2 * T/6 * V) + (T - 2T/6) * V [1 mark]
= VT/6 + 2VT/3
= (5/6) VT [1 mark]

m = V / (T/6) = 6V/T [1 mark] → V = (1/6) mT [1 mark]
→ D = 5(mT/6)T/6 = (5/36) mT^2 [1 mark]

b. The areas under both velocity-time graphs are equal since *D* is constant.
Let U be the new top speed. Let kT be the time spent accelerating.
Time spent at speed U = (7/9) T - 2kT. [1 mark] Area under new graph:
D = 2 * (1/2 * kT * U) + [(7/9) T - 2kT] * U [1 mark]
= kTU + (7/9 - 2k) TU
= (7/9 - k) TU [1 mark]

Equate area with previous result:
D = (5/36) mT^2
→ (5/36) mT^2 = (7/9 - k) TU [1 mark]
Eliminating *U*, use m = U / (kT) → U = mkT [1 mark]
→ (5/36) mT^2 = (7/9 - k) mkT^2 [1 mark]
→ 5/36 = (7/9 - k) k
→ k^2 - (7/9) k + (5/36) = 0 [1 mark]
→ k = 1/2 , k = 5/18 [1 mark]
Since k is the fraction of time, we must have k < 1/2 since otherwise the total time would be T which is not within (7/9)T. So take k = 5/18.
→ time spent accelerating = kT = (5/18) T. [1 mark]

c. Old top speed = V = (1/6) mT [1 mark]
New top speed = U = mkT = (5/18) mT [1 mark]
Ratio = (5/18) / (1/6) = 5/3 = 66.6...% = 67% increase [1 mark]

13.

a. Setting up a SUVAT table for a particular particle, using θ as angle of elevation,

	S	U	V	A	T
horizontal	*r*	*v* cos θ	*v* cos θ	0	*t*
vertical	0	*v* sin θ	-*v* sin θ	-*g*	*t*

In the horizontal direction,
r = vt cos θ [1 mark] → v cos θ = r/t [1 mark]
In the vertical direction,
-v sin θ = v sin θ - gt [1 mark] → v sin θ = gt/2 [1 mark]

Squaring both sides of each equation,
$v^2 \sin^2 \theta = g^2t^2/4$ and $v^2 \cos^2 \theta = r^2/t^2$
Adding together,
$v^2 \sin^2 \theta + v^2 \cos^2 \theta = g^2t^2/4 + r^2/t^2$ [1 mark]
$v^2 (\sin^2 \theta + \cos^2 \theta) = g^2t^2/4 + r^2/t^2$
$v^2 = g^2t^2/4 + r^2/t^2$ [1 mark]
$\rightarrow 4v^2t^2 = g^2t^4 + 4r^2$
$\rightarrow g^2 t^4 - 4v^2 t^2 + 4r^2 = 0$ [1 mark]

$$t^2 = \frac{4v^2 \pm \sqrt{16v^4 - 16g^2r^2}}{2g^2}$$ [1 mark]

$$t^2 = \frac{4v^2 \pm 4\sqrt{v^4 - g^2r^2}}{2g^2}$$ [1 mark]

$$\frac{1}{2}t^2 = \frac{v^2 \pm \sqrt{v^4 - g^2r^2}}{g^2}$$

$$\frac{1}{2}g^2t^2 = v^2 \pm \sqrt{v^4 - g^2r^2}$$ [1 mark]

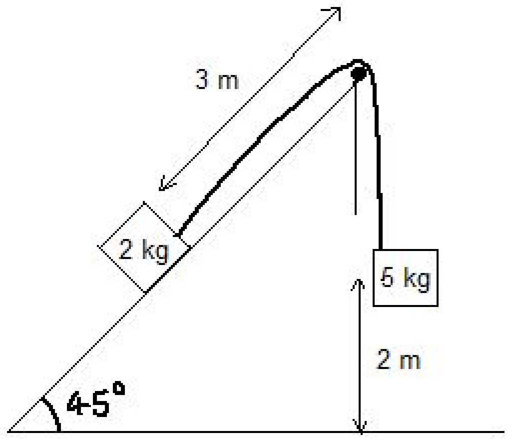
b. Considering vertical motion, V = 0 since instantaneously at rest at max height.

Using $V^2 - U^2 = 2AS$ in the vertical direction,
$-v^2 \sin^2 \theta = -2gh$ [1 mark]
$\rightarrow v^2 \sin^2 \theta = 2gh$ [1 mark]

Using V = U + AT in the vertical direction,
$0 = v \sin \theta - gt/2$ [1 mark]
$\rightarrow v \sin \theta = gt/2$ [1 mark]
$\rightarrow 2v^2 \sin^2 \theta = g^2t^2/2$ [1 mark]

Replacing the RHS with the previous result in part (a),
$2v^2 \sin^2 \theta = v^2 \pm \sqrt{v^4 - g^2r^2}$ [1 mark]
Substituting in the result for $v^2 \sin^2 \theta$,
$4gh = v^2 \pm \sqrt{v^4 - g^2r^2}$ [1 mark]
$h = \dfrac{v^2 \pm \sqrt{v^4 - g^2r^2}}{4g}$ [1 mark]

14. Diagram of the setup:



a. Forces on 5 kg mass: $5g - T = 5a \rightarrow 5a + T = 5g$ [1 mark]
Forces on 2 kg mass: $T - 2g \sin 45 - 0.5 * 2g \cos 45 = 2a$ [1 mark]
$\rightarrow T - \sqrt{2}\, g - \sqrt{2}/2\, g = 2a \rightarrow T - 3\sqrt{2}/2\, g = 2a \rightarrow -2a + T = 3\sqrt{2}/2\, g$ [1 mark]

Solving simultaneously,
$a = 4.0\ ms^{-2}$
$T = 29$ N [1 mark]

b. When Q hits the ground, the string goes slack so the tension force disappears.
Since Q has fallen 2 m, P must move 2 m so it is 1 m away from the pulley.

Speed of P = speed of Q
$s = -2, u = 0, v = ?, a = -9.81, t = ?$
$v^2 = 2as \rightarrow v = \sqrt{(2 * 9.81 * 2)} = 6.26\ ms^{-1}$ [1 mark]

Forces acting on P are: F = weight + friction = $-2g \sin 45 - 0.5 * 2g \cos 45$
$F = -3g\sqrt{3}/2 = 2a$ [1 mark] $\rightarrow a = -3g\sqrt{3}/4$ (deceleration) $= -12.74\ ms^{-2}$ [1 mark]

$s = ?, u = 6.26, v = 0, a = -12.74, t = ?$
$-6.26^2 = 2(-12.74)s$ [1 mark] $\rightarrow s = 1.54$ m [1 mark]
Since 1.54 > 1, the block P will reach the pulley / collide with the pulley. [1 mark]

15.

a. Let tension in string PQ be T_1 and tension in string QR be T_2. Let the coefficient of friction between Q and the table be μ. Since mass of P > mass of R (5 > 3), the system will accelerate anti-clockwise and so friction will act to the right on Q.

Forces on P:	$5g - T_1 = 5a$	(weight - tension)	[1 mark]
Forces on Q:	$T_1 - T_2 - \mu * 8g = 8a$	(tension - friction)	[1 mark]
Forces on R:	$T_2 - 3g = 3a$	(tension - weight)	[1 mark]

Rearranging for tensions in P and R,
$T_1 = 5g - 5a$ [1 mark] and $T_2 = 3g + 3a$ [1 mark]
Subbing into equation Q,
$5g - 5a - 3g - 3a - 8\mu g = 8a$ [1 mark]
$\rightarrow 2g - 8a - 8\mu g = 8a$
$\rightarrow 16a = (2 - 8\mu)g$
$\rightarrow a = (1 - 4\mu)g/8$ (acceleration is positive in downwards direction) [2 marks]

$s = 0.80$, $u = 0$, $v = ?$, $a = (1 - 4\mu)g/8$, $t = 1.32$
Using $s = ut + 1/2\ at^2$,
$0.8 = 1/2 * (1 - 4\mu)g/8 * 1.32^2$
$\rightarrow \mu = 0.063$ [1 mark]

b. Speed of R = speed of P
$v = (1 - 4 * 0.0628)g/8 * 1.32 = 1.21\ ms^{-1}$ [1 mark]
When P hits the ground the string goes slack:
$F = -3g \rightarrow a = -g$ [1 mark]
$s = ?$, $u = 1.21$, $v = 0$, $a = -9.81$, $t = ?$
$-1.21^2 = 2(-9.81)s \rightarrow s = 0.0746\ m = 7.46\ cm$ [1 mark]
Total height gained = 80 + 7.46 = 87.46 = 87 cm [1 mark]

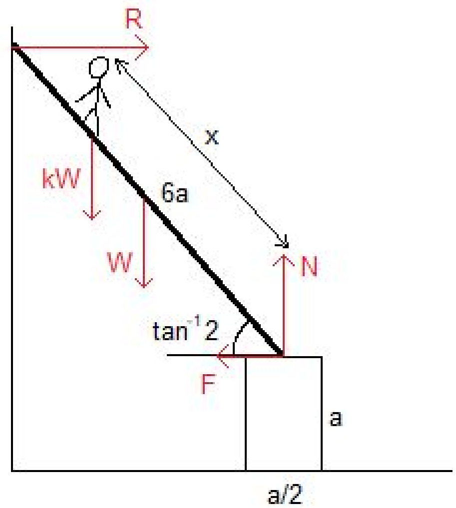
c. i) They have no size (modelled as a particle) / no air resistance / they fall vertically (string is attached above centre of gravity of masses)

ii) It has no size / no air resistance / it does not hit a pulley

(**Not** any assumption about the pulleys or strings)

16.

a. Suppose the painter has climbed a distance *x* up the ladder. The forces acting are the weight of the painter (kW), ladder (W), reaction forces at the table (N) and wall (R), and friction at the table (F):



In order to not slip the ladder, the system must both have no resultant force and no resultant moment / torque (else it would rotate and hence slip).

Resolving forces vertically: $N = kW + W \rightarrow N = (k + 1)W$ [1 mark]
Resolving forces horizontally: $R = F$

Angle of inclination of ladder = $\theta = \tan^{-1} 2$
$\cos\theta = 1/\sqrt{5}\ a$ and $\sin\theta = 2/\sqrt{5}\ a$ [1 mark]

Taking moments about the point of contact with the wall (force x perp. dist.):
$(kW * (6a - x) \cos\theta) + (W * 3a \cos\theta) + (F * 6a \sin\theta) = N * 6a \cos\theta$ [1 mark]
$(kW * (6a - x) \cos\theta) + (W * 3a \cos\theta) + (F * 6a \sin\theta) = (k + 1)W * 6a \cos\theta$
$(kWa\ (6a - x) / \sqrt{5}) + (3a^2W / \sqrt{5}) + (12a^2F / \sqrt{5}) = 6a^2(k + 1)W / \sqrt{5}$
$kW(6a - x) + 3aW + 12aF = 6a(k + 1)W$ [1 mark]
$\rightarrow 12aF = 6aW(k + 1) - kW(6a - x) - 3aW$
$\rightarrow 12aF = 6aWk + 6aW - 6aWk + xkW - 3aW$
$\rightarrow 12aF = xkW + 3aW$
$\rightarrow 12aF = W(kx + 3a)$ [1 mark]

If the ladder slips, then friction takes its maximum value of

$F = \mu N = 1/2 * (k + 1)W$:

$\rightarrow 6aW(k + 1) = W(kx + 3a)$ [1 mark]

$\rightarrow 6ak + 6a = kx + 3a$

$\rightarrow kx = 6ak + 3a$

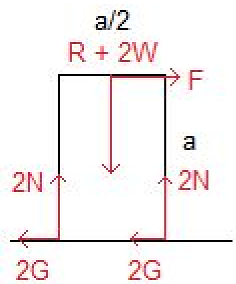
$\rightarrow x = (6ak + 3a) / k$

$\rightarrow x = 6a + 3a/k$ [1 mark]

$\rightarrow x > 6a$

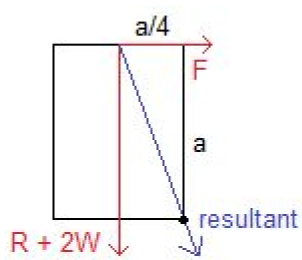
Since the ladder’s length is 6a, the painter’s position (x) cannot be more than 6a along it. So the ladder does not slip at any point. [1 mark]

b. The table is a square so it has 4 points of contact with the ground. Let each reaction force at each leg be N. Let the friction acting at each leg between the leg and the ground be G. Since we are considering the table instead of the ladder, all forces from the ladder are reaction forces in the opposite direction on the table. Then the force diagram for the table is



Reaction due to ladder: k = 9 → R = (9 + 1)W = 10W
Friction due to ladder: 12aF = W(9x + 3a) → F = W(a + 3x)/(4a)
Resolving forces vertically: 4N = R + 2W = 10W + 2W = 12W → N = 3W [1 mark]
Resolving forces horizontally: F = 4G → G = W(a + 3x)/(16a) [1 mark]
If the table does **not** slip, then friction must be sufficient i.e. G < 1/3 * N
→ W(a + 3x)/(16a) < 1/3 * 3W [1 mark]
→ (a + 3x)/(16a) < 1
→ a + 3x < 16a
→ 3x < 15a
→ x < 5a (the table will slip when the painter walks past this point) [1 mark]

If the table does **not** tilt, there must be no moment beyond the bottom-right corner of the table, as this would be the pivot if it tilted.



(R + 2W) / F = a / (a/4) [1 mark]
→ R + 2W = 4F [1 mark]
→ 12W = W(a + 3x) / a
→ 12a = a + 3x
→ 3x = 11a
→ x = 11/3 a (the table will tilt when the painter walks past this point) [1 mark]

Since 11/3 < 5, the table will tilt before it slips. [1 mark]