CALCULUS II

Applications of Integrals

Paul Dawkins

Table of Contents

Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don't have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can't just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you've got it memorized doesn't mean that you can use it. You'll need to be able to look at an integral and realize that integration by parts can be used (which isn't always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you'll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than

are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 2 : Applications of Integrals

In this section we're going to take a look at some of the applications of integration. It should be noted as well that these applications are presented here, as opposed to Calculus I, simply because many of the integrals that arise from these applications tend to require techniques that we discussed in the previous chapter.

Here is a list of applications that we'll be taking a look at in this chapter.

Arc Length – In this section we'll determine the length of a curve over a given interval.

Surface Area – In this section we'll determine the surface area of a solid of revolution, *i.e.* a solid obtained by rotating a region bounded by two curves about a vertical or horizontal axis.

Center of Mass – In this section we will determine the center of mass or centroid of a thin plate where the plate can be described as a region bounded by two curves (one of which may the *x* or *y*-axis).

Hydrostatic Pressure and Force – In this section we'll determine the hydrostatic pressure and force on a vertical plate submerged in water. The plates used in the examples can all be described as regions bounded by one or more curves/lines.

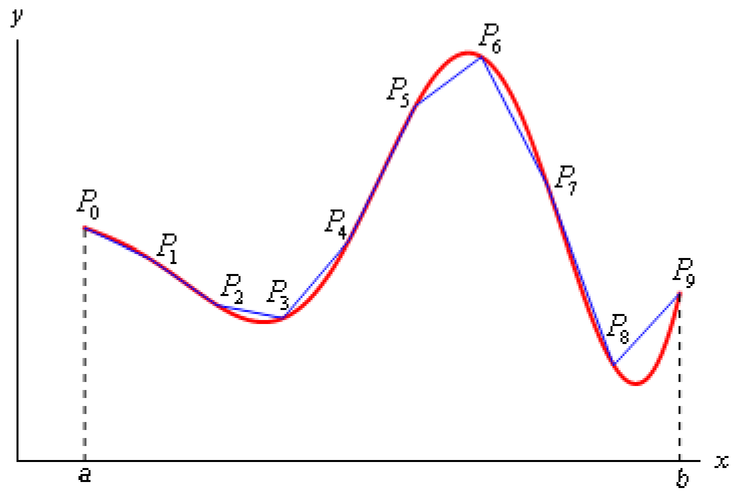
Probability – Many quantities can be described with probability density functions. For example, the length of time a person waits in line at a checkout counter or the life span of a light bulb. None of these quantities are fixed values and will depend on a variety of factors. In this section we will look at probability density functions and computing the mean (think average wait in line or average life span of a light blub) of a probability density function.

Section 2-1 : Arc Length

In this section we are going to look at computing the arc length of a function. Because it's easy enough to derive the formulas that we'll use in this section we will derive one of them and leave the other to you to derive.

We want to determine the length of the continuous function $y = f(x)$ on the interval $[a,b]$. We'll also need to assume that the derivative is continuous on $[a,b]$.

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into *n* equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

and we can get the exact length by taking *n* larger and larger. In other words, the exact length will be,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. We can then compute directly the length of the line segments as follows.

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the Mean Value Theorem we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty}\sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x \end{aligned}$$

However, using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

A slightly more convenient notation (in our opinion anyway) is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In a similar fashion we can also derive a formula for $x = h(y)$ on $[c,d]$. This formula is,

$$L = \int_c^d \sqrt{1 + \left[h'(y)\right]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Again, the second form is probably a little more convenient.

Note the difference in the derivative under the square root! Don't get too confused. With one we differentiate with respect to *x* and with the other we differentiate with respect to *y*. One way to keep the two straight is to notice that the differential in the "denominator" of the derivative will match up with the differential in the integral. This is one of the reasons why the second form is a little more convenient.

Before we work any examples we need to make a small change in notation. Instead of having two formulas for the arc length of a function we are going to reduce it, in part, to a single formula.

From this point on we are going to use the following formula for the length of the curve.

Arc Length Formula(s)

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f\left(x\right),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h\left(y\right),\ c \le y \le d$$

Note that no limits were put on the integral as the limits will depend upon the *ds* that we're using. Using the first *ds* will require *x* limits of integration and using the second *ds* will require *y* limits of integration.

Thinking of the arc length formula as a single integral with different ways to define *ds* will be convenient when we run across arc lengths in future sections. Also, this *ds* notation will be a nice notation for the next section as well.

Now that we've derived the arc length formula let's work some examples.

Example 1 Determine the length of $y = \ln\left(\sec x\right)$ between $0 \le x \le \frac{\pi}{4}$.

Solution

In this case we'll need to use the first *ds* since the function is in the form $y = f\left(x\right)$. So, let's get the derivative out of the way.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \qquad\qquad \left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

Let's also get the root out of the way since there is often simplification that can be done and there's no reason to do that inside the integral.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = \left|\sec x\right| = \sec x$$

Note that we could drop the absolute value bars here since secant is positive in the range given.

The arc length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec x\,dx \\ &= \ln\left|\sec x + \tan x\right|\Big|_0^{\frac{\pi}{4}} \\ &= \ln\left(\sqrt{2} + 1\right) \end{aligned}$$

Example 2 Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \le y \le 4$.

Solution

There is a very common mistake that students make in problems of this type. Many students see that the function is in the form $x = h(y)$ and they immediately decide that it will be too difficult to work with it in that form so they solve for *y* to get the function into the form $y = f(x)$. While that can be done here it will lead to a messier integral for us to deal with.

Sometimes it's just easier to work with functions in the form $x = h(y)$. In fact, if you can work with functions in the form $y = f(x)$ then you can work with functions in the form $x = h(y)$. There really isn't a difference between the two so don't get excited about functions in the form $x = h(y)$.

Let's compute the derivative and the root.

$$\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+y-1} = \sqrt{y}$$

As you can see keeping the function in the form $x = h(y)$ is going to lead to a very easy integral. To see what would happen if we tried to work with the function in the form $y = f(x)$ see the next example.

Let's get the length.

$$\begin{aligned} L &= \int_1^4 \sqrt{y}\,dy \\ &= \left.\frac{2}{3}y^{\frac{3}{2}}\right|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

As noted in the last example we really do have a choice as to which *ds* we use. Provided we can get the function in the form required for a particular *ds* we can use it. However, as also noted above, there will often be a significant difference in difficulty in the resulting integrals. Let's take a quick look at what would happen in the previous example if we did put the function into the form $y = f(x)$.

Example 3 Redo the previous example using the function in the form $y = f(x)$ instead.

Solution

In this case the function and its derivative would be,

$$y = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad\qquad \frac{dy}{dx} = \left(\frac{3x}{2}\right)^{-\frac{1}{3}}$$

The root in the arc length formula would then be.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \sqrt{\frac{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}$$

All the simplification work above was just to put the root into a form that will allow us to do the integral.

Now, before we write down the integral we'll also need to determine the limits. This particular *ds* requires *x* limits of integration and we've got *y* limits. They are easy enough to get however. Since we know *x* as a function of *y* all we need to do is plug in the original *y* limits of integration and get the *x* limits of integration. Doing this gives,

$$0 \le x \le \frac{2}{3}(3)^{\frac{3}{2}}$$

Not easy limits to deal with, but there they are.

Let's now write down the integral that will give the length.

$$L = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}\,dx$$

That's a really unpleasant looking integral. It can be evaluated however using the following substitution.

$$u = \left(\frac{3x}{2}\right)^{\frac{2}{3}}+1 \qquad\qquad du = \left(\frac{3x}{2}\right)^{-\frac{1}{3}}dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1$$

$$x = \frac{2}{3}(3)^{\frac{3}{2}} \qquad \Rightarrow \qquad u = 4$$

Using this substitution the integral becomes,

$$\begin{aligned} L &= \int_1^4 \sqrt{u}\,du \\ &= \left.\frac{2}{3}u^{\frac{3}{2}}\right|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

So, we got the same answer as in the previous example. Although that shouldn't really be all that surprising since we were dealing with the same curve.

From a technical standpoint the integral in the previous example was not that difficult. It was just a Calculus I substitution. However, from a practical standpoint the integral was significantly more difficult than the integral we evaluated in Example 2. So, the moral of the story here is that we can use either formula (provided we can get the function in the correct form of course) however one will often be significantly easier to actually evaluate.

Okay, let's work one more example.

Example 4 Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$. Assume that *y* is positive.

Solution

We'll use the second *ds* for this one as the function is already in the correct form for that one. Also, the other *ds* would again lead to a particularly difficult integral. The derivative and root will then be,

$$\frac{dx}{dy} = y \qquad \Rightarrow \qquad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$$

Before writing down the length notice that we were given *x* limits and we will need *y* limits for this *ds*. With the assumption that *y* is positive these are easy enough to get. All we need to do is plug *x* into our equation and solve for *y*. Doing this gives,

$$0 \le y \le 1$$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

This integral will require the following trig substitution.

$$\begin{array}{ccccc} y = \tan\theta & & & dy = \sec^2\theta\, d\theta & \\ y = 0 & \Rightarrow & 0 = \tan\theta & \Rightarrow & \theta = 0 \\ y = 1 & \Rightarrow & 1 = \tan\theta & \Rightarrow & \theta = \frac{\pi}{4} \end{array}$$

$$\sqrt{1 + y^2} = \sqrt{1 + \tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right| = \sec\theta$$

The length is then,

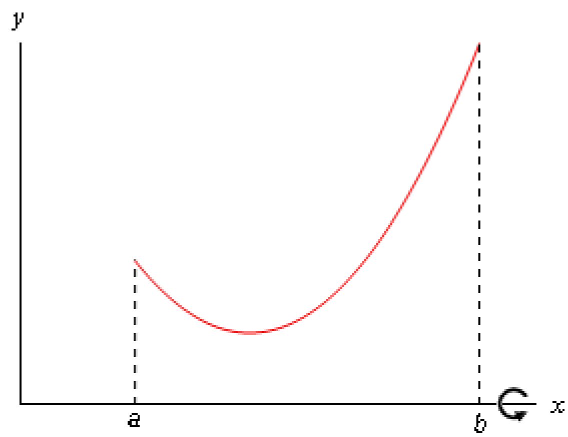
$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta \\ &= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

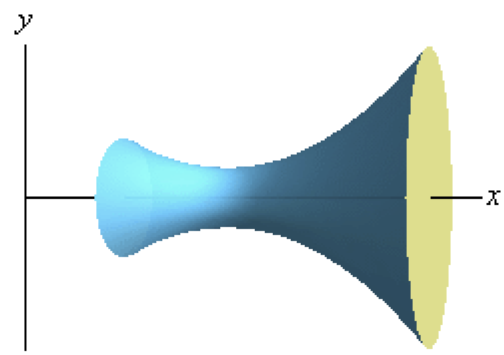
The first couple of examples ended up being fairly simple Calculus I substitutions. However, as this last example had shown we can end up with trig substitutions as well for these integrals.

Section 2-2 : Surface Area

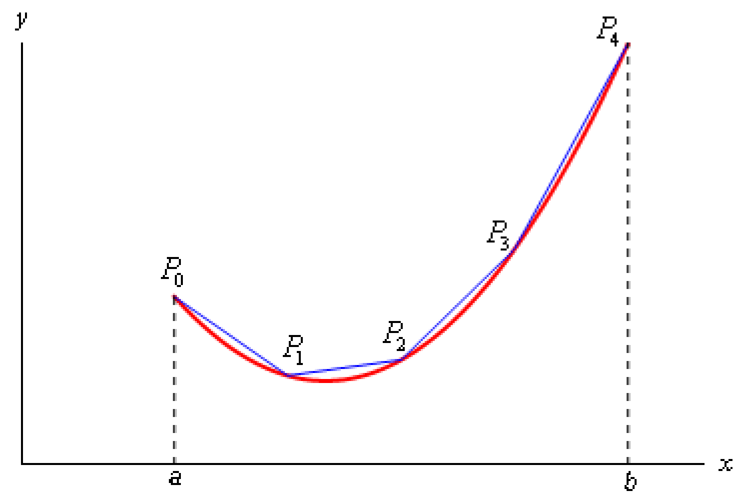
In this section we are going to look once again at solids of revolution. We first looked at them back in Calculus I when we found the volume of the solid of revolution. In this section we want to find the surface area of this region.

So, for the purposes of the derivation of the formula, let's look at rotating the continuous function $y = f\left(x\right)$ in the interval $\left[a,b\right]$ about the *x*-axis. We'll also need to assume that the derivative is continuous on $\left[a,b\right]$. Below is a sketch of a function and the solid of revolution we get by rotating the function about the *x*-axis.

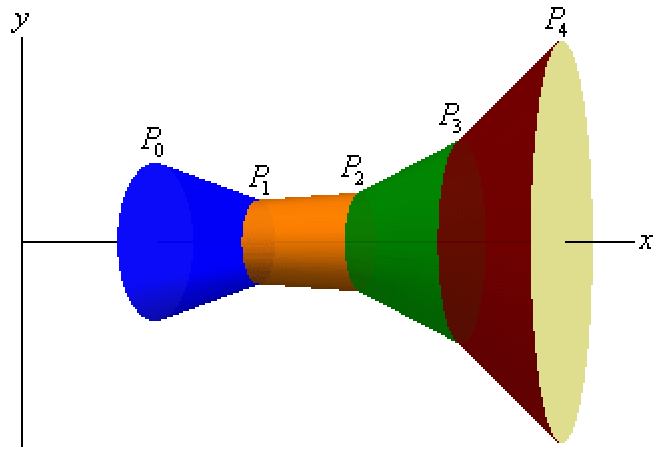




We can derive a formula for the surface area much as we derived the formula for arc length. We'll start by dividing the interval into *n* equal subintervals of width Δx. On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of each interval. Here is a sketch of that for our representative function using $n = 4$.



Now, rotate the approximations about the *x*-axis and we get the following solid.



The approximation on each interval gives a distinct portion of the solid and to make this clear each portion is colored differently. Each of these portions are called frustums and we know how to find the surface area of frustums.

The surface area of a frustum is given by,

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad r_1 = \text{radius of right end}$$
$$r_2 = \text{radius of left end}$$

and *l* is the length of the slant of the frustum.

For the frustum on the interval $[x_{i-1}, x_i]$ we have,

$$r_1 = f(x_i)$$
$$r_2 = f(x_{i-1})$$
$$l = |P_{i-1}\ P_i| \quad (\text{length of the line segment connecting } P_i \text{ and } P_{i-1})$$

and we know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

So, the surface area of the frustum on the interval $[x_{i-1}, x_i]$ is approximately,

$$
\begin{aligned}
A_i &= 2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)|P_{i-1}\ P_i| \\
&\approx 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \ \Delta x
\end{aligned}
$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \ \Delta x$$

and we can get the exact surface area by taking the limit as *n* goes to infinity.

$$
\begin{aligned}
S &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \ \Delta x \\
&= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\, dx
\end{aligned}
$$

If we wanted to we could also derive a similar formula for rotating $x = h(y)$ on $[c,d]$ about the *y*-axis. This would give the following formula.

$$S = \int_c^d 2\pi\, h(y)\sqrt{1+\left[h'(y)\right]^2}\, dy$$

These are not the "standard" formulas however. Notice that the roots in both of these formulas are nothing more than the two *ds*'s we used in the previous section. Also, we will replace $f(x)$ with *y* and $h(y)$ with *x*. Doing this gives the following two formulas for the surface area.

Surface Area Formulas

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

There are a couple of things to note about these formulas. First, notice that the variable in the integral itself is always the opposite variable from the one we're rotating about. Second, we are allowed to use either *ds* in either formula. This means that there are, in some way, four formulas here. We will choose the *ds* based upon which is the most convenient for a given function and problem.

Now let's work a couple of examples.

Example 1 Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}$, $-2 \le x \le 2$ about the *x*-axis.

Solution
The formula that we'll be using here is,

$$S = \int 2\pi y \, ds$$

since we are rotating about the *x*-axis and we'll use the first *ds* in this case because our function is in the correct form for that *ds* and we won't gain anything by solving it for *x*.

Let's first get the derivative and the root taken care of.

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}\left(-2x\right) = -\frac{x}{\left(9 - x^2\right)^{\frac{1}{2}}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

Here's the integral for the surface area,

$$S = \int_{-2}^{2} 2\pi y \frac{3}{\sqrt{9 - x^2}} \, dx$$

There is a problem however. The *dx* means that we shouldn't have any *y*'s in the integral. So, before evaluating the integral we'll need to substitute in for *y* as well.

The surface area is then,

$$\begin{aligned} S &= \int_{-2}^{2} 2\pi \sqrt{9 - x^2} \, \frac{3}{\sqrt{9 - x^2}} \, dx \\ &= \int_{-2}^{2} 6\pi \, dx \\ &= 24\pi \end{aligned}$$

Previously we made the comment that we could use either *ds* in the surface area formulas. Let's work an example in which using either *ds* won't create integrals that are too difficult to evaluate and so we can check both *ds*'s.

Example 2 Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the *y*-axis. Use both *ds*'s to compute the surface area.

Solution
Note that we've been given the function set up for the first *ds* and limits that work for the second *ds*.

Solution 1
This solution will use the first *ds* listed above. We'll start with the derivative and root.

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9x^{\frac{4}{3}}}} = \sqrt{\frac{9x^{\frac{4}{3}}+1}{9x^{\frac{4}{3}}}} = \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}$$

We'll also need to get new limits. That isn't too bad however. All we need to do is plug in the given *y*'s into our equation and solve to get that the range of *x*'s is $1 \le x \le 8$. The integral for the surface area is then,

$$\begin{aligned} S &= \int_1^8 2\pi x \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}\,dx \\ &= \frac{2\pi}{3}\int_1^8 x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}}+1}\,dx \end{aligned}$$

Note that this time we didn't need to substitute in for the *x* as we did in the previous example. In this case we picked up a *dx* from the *ds* and so we don't need to do a substitution for the *x*. In fact, if we had substituted for *x* we would have put *y*'s into the integral which would have caused problems.

Using the substitution

$$u = 9x^{\frac{4}{3}}+1 \qquad\qquad du = 12x^{\frac{1}{3}}\,dx$$

the integral becomes,

$$\begin{aligned} S &= \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du \\ &= \left.\frac{\pi}{27}u^{\frac{3}{2}}\right|_{10}^{145} \\ &= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48 \end{aligned}$$

Solution 2

This time we'll use the second *ds*. So, we'll first need to solve the equation for *x*. We'll also go ahead and get the derivative and root while we're at it.

$$x = y^3 \qquad\qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9y^4}$$

The surface area is then,

$$S = \int_1^2 2\pi x \sqrt{1+9y^4}\, dy$$

We used the original *y* limits this time because we picked up a *dy* from the *ds*. Also note that the presence of the *dy* means that this time, unlike the first solution, we'll need to substitute in for the *x*. Doing that gives,

$$\begin{aligned} S &= \int_1^2 2\pi y^3 \sqrt{1+9y^4}\, dy \qquad\qquad u = 1+9y^4 \\ &= \frac{\pi}{18}\int_{10}^{145} \sqrt{u}\, du \\ &= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48 \end{aligned}$$

Note that after the substitution the integral was identical to the first solution and so the work was skipped.

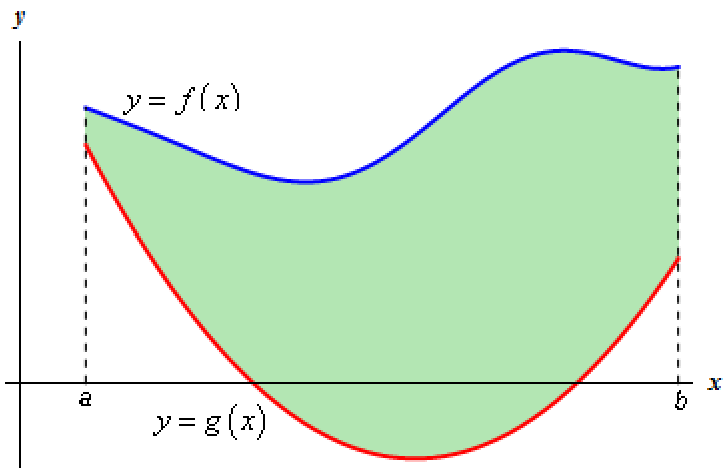
As this example has shown we can use either *ds* to get the surface area. It is important to point out as well that with one *ds* we had to do a substitution for the *x* and with the other we didn't. This will always work out that way.

Note as well that in the case of the last example it was just as easy to use either *ds*. That often won't be the case. In many examples only one of the *ds* will be convenient to work with so we'll always need to determine which *ds* is liable to be the easiest to work with before starting the problem.

Section 2-3 : Center of Mass

In this section we are going to find the **center of mass** or **centroid** of a thin plate with uniform density ρ. The center of mass or centroid of a region is the point in which the region will be perfectly balanced horizontally if suspended from that point.

So, let's suppose that the plate is the region bounded by the two curves $f(x)$ and $g(x)$ on the interval $[a,b]$. So, we want to find the center of mass of the region below.



We'll first need the mass of this plate. The mass is,

$$\begin{aligned} M &= \rho\left(\text{Area of plate}\right) \\ &= \rho\int_a^b f(x) - g(x)\,dx \end{aligned}$$

Next, we'll need the **moments** of the region. There are two moments, denoted by M_x and M_y. The moments measure the tendency of the region to rotate about the *x* and *y*-axis respectively. The moments are given by,

Equations of Moments

$$M_x = \rho\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

$$M_y = \rho\int_a^b x\left(f(x) - g(x)\right)dx$$

The coordinates of the center of mass, $(\bar{x}, \bar{y})$, are then,

Center of Mass Coordinates

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x\left(f(x) - g(x)\right)dx}{\int_a^b f(x) - g(x)\,dx} = \frac{1}{A}\int_a^b x\left(f(x) - g(x)\right)dx$$

$$\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx}{\int_a^b f(x) - g(x)\,dx} = \frac{1}{A}\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

where,

$$A = \int_a^b f(x) - g(x)\,dx$$

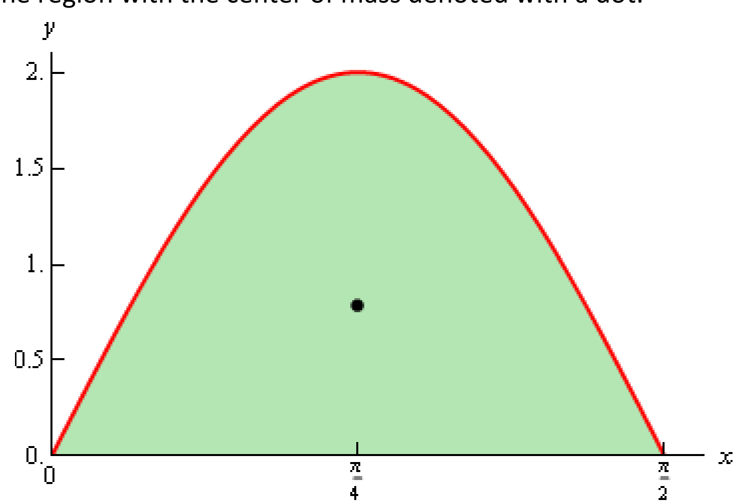
Note that the density, ρ, of the plate cancels out and so isn't really needed.

Let's work a couple of examples.

Example 1 Determine the center of mass for the region bounded by $y = 2\sin(2x)$, $y = 0$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution

Here is a sketch of the region with the center of mass denoted with a dot.



Let's first get the area of the region.

$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{2}} 2\sin\left(2x\right)dx \\
&= -\cos\left(2x\right)\Big|_0^{\frac{\pi}{2}} \\
&= 2
\end{aligned}
$$

Now, the moments (without density since it will just drop out) are,

$$
\begin{aligned}
M_x &= \int_0^{\frac{\pi}{2}} 2\sin^2\left(2x\right)dx \\
&= \int_0^{\frac{\pi}{2}} 1-\cos\left(4x\right)dx \\
&= \left(x-\frac{1}{4}\sin\left(4x\right)\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \int_0^{\frac{\pi}{2}} 2x\sin\left(2x\right)dx \qquad \text{integrating by parts...} \\
&= -x\cos\left(2x\right)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos\left(2x\right)dx \\
&= -x\cos\left(2x\right)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\sin\left(2x\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}
\end{aligned}
$$

The coordinates of the center of mass are then,

$$
\begin{aligned}
\overline{x} &= \frac{\pi/2}{2} = \frac{\pi}{4} \\
\overline{y} &= \frac{\pi/2}{2} = \frac{\pi}{4}
\end{aligned}
$$

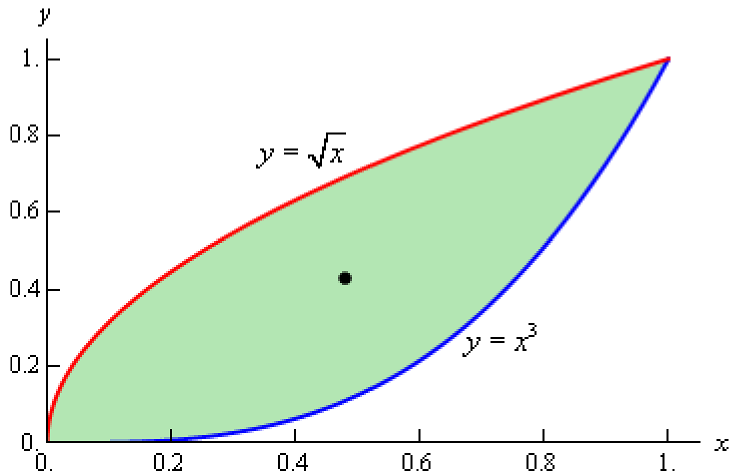
Again, note that we didn't put in the density since it will cancel out.

So, the center of mass for this region is $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$.

Example 2 Determine the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$.

Solution

The two curves intersect at $x = 0$ and $x = 1$ and here is a sketch of the region with the center of mass marked with a box.



We'll first get the area of the region.

$$A = \int_0^1 \sqrt{x} - x^3\,dx = \left. \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^4 \right) \right|_0^1 = \frac{5}{12}$$

Now the moments, again without density, are

$$M_x = \int_0^1 \frac{1}{2}\left(x - x^6 \right) dx = \left. \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{7}x^7 \right) \right|_0^1 = \frac{5}{28}$$

$$M_y = \int_0^1 x\left(\sqrt{x} - x^3 \right) dx = \int_0^1 x^{\frac{3}{2}} - x^4\,dx = \left. \left(\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{5}x^5 \right) \right|_0^1 = \frac{1}{5}$$

The coordinates of the center of mass is then,

$$\bar{x} = \frac{1/5}{5/12} = \frac{12}{25}$$

$$\bar{y} = \frac{5/28}{5/12} = \frac{3}{7}$$

The coordinates of the center of mass are then, $\left(\frac{12}{25}, \frac{3}{7} \right)$.

Section 2-4 : Hydrostatic Pressure and Force

In this section we are going to submerge a vertical plate in water and we want to know the force that is exerted on the plate due to the pressure of the water. This force is often called the hydrostatic force.

There are two basic formulas that we'll be using here. First, if we are *d* meters below the surface then the hydrostatic pressure is given by,

$$P = \rho g d$$

where, ρ is the density of the fluid and *g* is the gravitational acceleration. We are going to assume that the fluid in question is water and since we are going to be using the metric system these quantities become,

$$\rho = 1000 \text{ kg/m}^3 \qquad\qquad g = 9.81 \text{ m/s}^2$$

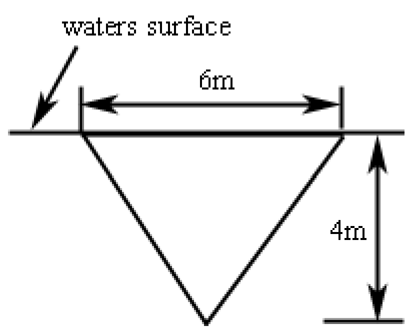
The second formula that we need is the following. Assume that a constant pressure *P* is acting on a surface with area *A*. Then the hydrostatic force that acts on the area is,

$$F = PA$$

Note that we won't be able to find the hydrostatic force on a vertical plate using this formula since the pressure will vary with depth and hence will not be constant as required by this formula. We will however need this for our work.

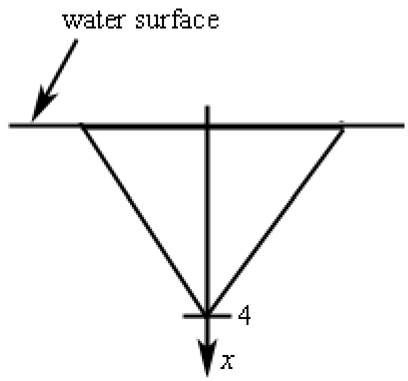
The best way to see how these problems work is to do an example or two.

Example 1 Determine the hydrostatic force on the following triangular plate that is submerged in water as shown.



Solution

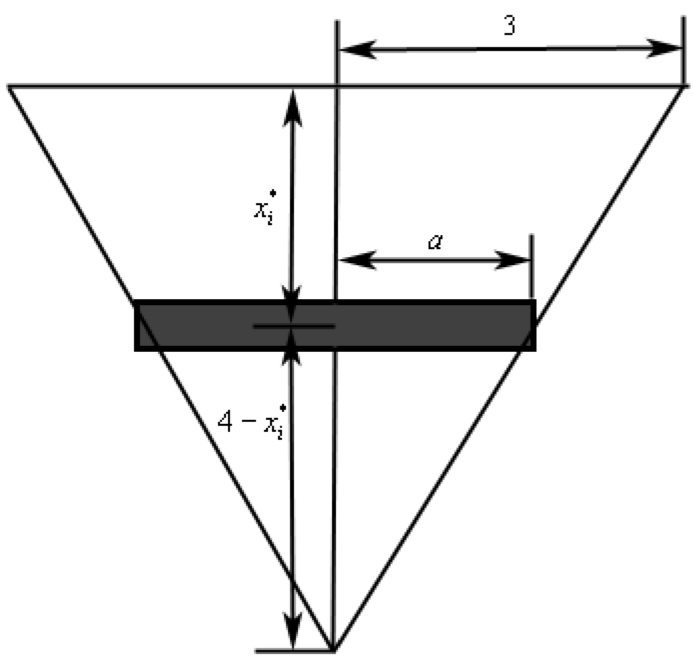
The first thing to do here is set up an axis system. So, let's redo the sketch above with the following axis system added in.



So, we are going to orient the *x*-axis so that positive *x* is downward, $x = 0$ corresponds to the water surface and $x = 4$ corresponds to the depth of the tip of the triangle.

Next we break up the triangle into *n* horizontal strips each of equal width Δx and in each interval $\left[x_{i-1}, x_i\right]$ choose any point x_i^*. In order to make the computations easier we are going to make two assumptions about these strips. First, we will ignore the fact that the ends are actually going to be slanted and assume the strips are rectangular. If Δx is sufficiently small this will not affect our computations much. Second, we will assume that Δx is small enough that the hydrostatic pressure on each strip is essentially constant.

Below is a representative strip.



The height of this strip is Δx and the width is 2*a*. We can use similar triangles to determine *a* as follows,

$$\frac{3}{4} = \frac{a}{4 - x_i^*} \qquad \Rightarrow \qquad a = 3 - \frac{3}{4}x_i^*$$

Now, since we are assuming the pressure on this strip is constant, the pressure is given by,

$$P_i = \rho g d = 1000(9.81)x_i^* = 9810x_i^*$$

and the hydrostatic force on each strip is,

$$F_i = P_i A = P_i(2a\Delta x) = 9810x_i^*(2)\left(3 - \frac{3}{4}x_i^*\right)\Delta x = 19620x_i^*\left(3 - \frac{3}{4}x_i^*\right)\Delta x$$

The approximate hydrostatic force on the plate is then the sum of the forces on all the strips or,

$$F \approx \sum_{i=1}^{n} 19620x_i^*\left(3 - \frac{3}{4}x_i^*\right)\Delta x$$

Taking the limit will get the exact hydrostatic force,

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} 19620x_i^*\left(3 - \frac{3}{4}x_i^*\right)\Delta x$$

Using the definition of the definite integral this is nothing more than,

$$F = \int_0^4 19620\left(3x - \frac{3}{4}x^2\right)dx$$

The hydrostatic force is then,

$$\begin{aligned} F &= \int_0^4 19620\left(3x - \frac{3}{4}x^2\right)dx \\ &= 19620\left(\frac{3}{2}x^2 - \frac{1}{4}x^3\right)\bigg|_0^4 \\ &= 156960\,N \end{aligned}$$

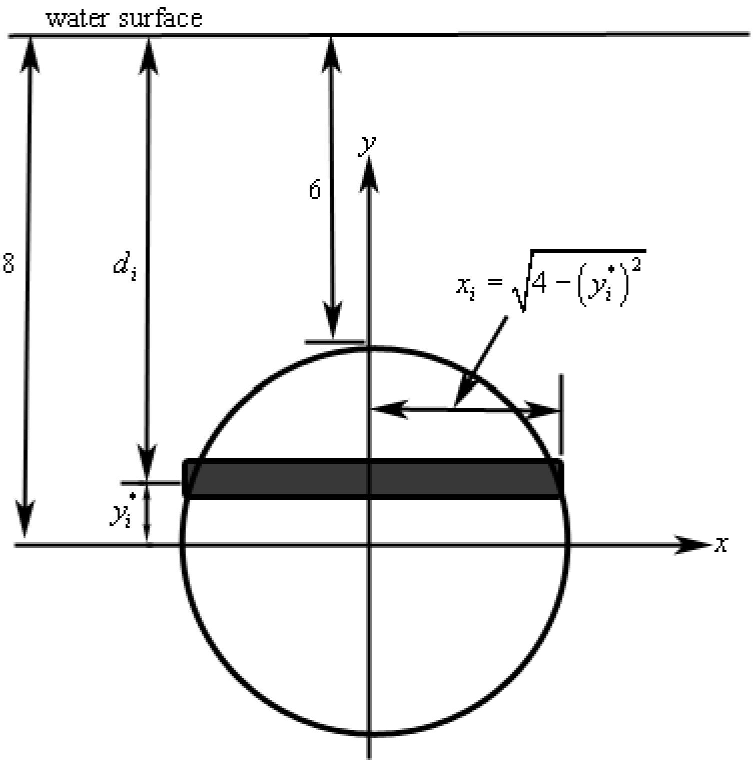
Let’s take a look at another example.

Example 2 Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.

Solution

First, we’re going to assume that the top of the circular plate is 6 meters under the water. Next, we will set up the axis system so that the origin of the axis system is at the center of the plate. Setting the axis system up in this way will greatly simplify our work.

Finally, we will again split up the plate into *n* horizontal strips each of width Δy and we’ll choose a point y_i^* from each strip. We’ll also assume that the strips are rectangular again to help with the computations. Here is a sketch of the setup.



The depth below the water surface of each strip is,

$$d_i = 8 - y_i^*$$

and that in turn gives us the pressure on the strip,

$$P_i = \rho g d_i = 9810\left(8 - y_i^*\right)$$

The area of each strip is,

$$A_i = 2\sqrt{4-\left(y_i^*\right)^2}\;\Delta y$$

The hydrostatic force on each strip is,

$$F_i = P_i A_i = 9810\left(8 - y_i^*\right)(2)\sqrt{4-\left(y_i^*\right)^2}\;\Delta y$$

The total force on the plate is,

$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n} 19620\left(8 - y_i^*\right)\sqrt{4-\left(y_i^*\right)^2}\;\Delta y \\ &= 19620\int_{-2}^{2}\left(8-y\right)\sqrt{4-y^2}\,dy \end{aligned}$$

To do this integral we'll need to split it up into two integrals.

$$F = 19620\int_{-2}^{2} 8\sqrt{4-y^2}\,dy - 19620\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

The first integral requires the trig substitution $y = 2\sin\theta$ and the second integral needs the substitution $v = 4 - y^2$. After using these substitutions we get,

$$\begin{aligned} F &= 627840\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta + 9810\int_0^0 \sqrt{v}\, dv \\ &= 313920\int_{-\pi/2}^{\pi/2} 1 + \cos\left(2\theta\right) d\theta + 0 \\ &= 313920\left(\theta + \frac{1}{2}\sin\left(2\theta\right)\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 313920\pi \end{aligned}$$

Note that after the substitution we know the second integral will be zero because the upper and lower limit is the same.

Section 2-5 : Probability

In this last application of integrals that we'll be looking at we're going to look at probability. Before actually getting into the applications we need to get a couple of definitions out of the way.

Suppose that we wanted to look at the age of a person, the height of a person, the amount of time spent waiting in line, or maybe the lifetime of a battery. Each of these quantities have values that will range over an interval of integers. Because of this these are called **continuous random variables**. Continuous random variables are often represented by *X*.

Every continuous random variable, *X*, has a **probability density function**, $f(x)$. Probability density functions satisfy the following conditions.

1. $f(x) \geq 0$ for all *x*.
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Probability density functions can be used to determine the probability that a continuous random variable lies between two values, say *a* and *b*. This probability is denoted by $P(a \leq X \leq b)$ and is given by,

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Let's take a look at an example of this.

Example 1 Let $f(x) = \dfrac{x^3}{5000}(10-x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of *x*.
Answer each of the following questions about this function.

(a) Show that $f(x)$ is a probability density function.

(b) Find $P(1 \leq X \leq 4)$

(c) Find $P(x \geq 6)$

Solution

(a) Show that $f(x)$ is a probability density function.

First note that in the range $0 \leq x \leq 10$ is clearly positive and outside of this range we've defined it to be zero.

So, to show this is a probability density function we'll need to show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \frac{x^3}{5000}(10-x)\,dx$$
$$= \left(\frac{x^4}{2000} - \frac{x^5}{25000}\right)\Bigg|_0^{10}$$
$$= 1$$

Note the change in limits on the integral. The function is only non-zero in these ranges and so the integral can be reduced down to only the interval where the function is not zero.

(b) Find $P(1 \le X \le 4)$

In this case we need to evaluate the following integral.

$$P(1 \le X \le 4) = \int_1^4 \frac{x^3}{5000}(10-x)\,dx$$
$$= \left(\frac{x^4}{2000} - \frac{x^5}{25000}\right)\Bigg|_1^4$$
$$= 0.08658$$

So the probability of *X* being between 1 and 4 is 8.658%.

(c) Find $P(x \ge 6)$

Note that in this case $P(x \ge 6)$ is equivalent to $P(6 \le X \le 10)$ since 10 is the largest value that *X* can be. So the probability that *X* is greater than or equal to 6 is,

$$P(X \ge 6) = \int_6^{10} \frac{x^3}{5000}(10-x)\,dx$$
$$= \left(\frac{x^4}{2000} - \frac{x^5}{25000}\right)\Bigg|_6^{10}$$
$$= 0.66304$$

This probability is then 66.304%.

Probability density functions can also be used to determine the mean of a continuous random variable. The mean is given by,

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

Let's work one more example.

Example 2 It has been determined that the probability density function for the wait in line at a counter is given by,

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.1\mathbf{e}^{-\frac{t}{10}} & \text{if } t \ge 0 \end{cases}$$

where *t* is the number of minutes spent waiting in line. Answer each of the following questions about this probability density function.

(a) Verify that this is in fact a probability density function.
(b) Determine the probability that a person will wait in line for at least 6 minutes.
(c) Determine the mean wait in line.

Solution
(a) Verify that this is in fact a probability density function.

This function is clearly positive or zero and so there's not much to do here other than compute the integral.

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\,dt &= \int_0^{\infty} 0.1\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \int_0^{u} 0.1\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \left.\left(-\mathbf{e}^{-\frac{t}{10}}\right)\right|_0^u \\ &= \lim_{u\to\infty} \left(1-\mathbf{e}^{-\frac{u}{10}}\right) = 1 \end{aligned}$$

So it is a probability density function.

(b) Determine the probability that a person will wait in line for at least 6 minutes.
The probability that we're looking for here is $P(X \ge 6)$.

$$\begin{aligned} P(X \ge 6) &= \int_6^{\infty} 0.1\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \int_6^{u} 0.1\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty} \left.\left(-\mathbf{e}^{-\frac{t}{10}}\right)\right|_6^u \\ &= \lim_{u\to\infty} \left(\mathbf{e}^{-\frac{6}{10}} - \mathbf{e}^{-\frac{u}{10}}\right) = \mathbf{e}^{-\frac{3}{5}} = 0.548812 \end{aligned}$$

So the probability that a person will wait in line for more than 6 minutes is 54.8811%.

(c) Determine the mean wait in line.

Here's the mean wait time.

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} t\, f\left(t\right) dt \\ &= \int_{0}^{\infty} 0.1t\,\mathbf{e}^{-\frac{t}{10}}\,dt \\ &= \lim_{u\to\infty}\int_{0}^{u} 0.1t\,\mathbf{e}^{-\frac{t}{10}}\,dt \qquad\qquad \text{integrating by parts....} \\ &= \lim_{u\to\infty}\left(-\left(t+10\right)\mathbf{e}^{-\frac{t}{10}}\right)\Bigg|_{0}^{u} \\ &= \lim_{u\to\infty}\left(10-\left(u+10\right)\mathbf{e}^{-\frac{u}{10}}\right) = 10\end{aligned}$$

So, it looks like the average wait time is 10 minutes.