Section A - Quick Reference Answer Key

Q	A	Marks	Q	A	Marks	Q	A	Marks
1	②	1	9	③	1	17	②	1
2	②	1	10	②	1	18	③	1
3	①	1	11	④	1	19	①	1
4	③	1	12	②	1	20	①	1
5	①	2	13	③	2	21	④	2
6	④	2	14	①	2	22	③	2
7	③	4	15	④	4	23	④	4
8	④	4	16	①	4	24	①	4

A1. Answer: ①

First order because the highest derivative is a first derivative (dy/dx)
Nonlinear due to the $(dy/dx)^2$ term
Nonhomogeneous due to the 2^{-x} term
→ **I** only (①)

A2. Answer: ②

Using Osborne's rule, compare to the equivalent trig identity:
$\cos(x + y) \equiv \cos x \cos y - \sin x \sin y$
The product of sines changes its sign:
$\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$ (②)

A3. Answer: ①

A point $\mathbf{a}$ on the plane is at the midpoint, which is (5, 5, -1).
The normal vector $\mathbf{n}$ is given by the direction vector between, which is (6, 12, -8).
Use $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \rightarrow \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \rightarrow 6x + 12y - 8z = 98 \rightarrow 3x + 6y - 4z = 49$. (①)

A4. Answer: ③

The matrices formed by the multiplications with these dimensions are
$\mathbf{BC}$: $(2 \times 3) \times (3 \times 2) = 2 \times 2$.
$\mathbf{CB}$: $(3 \times 2) \times (2 \times 3) = 3 \times 3$.
$\mathbf{CBA}$ is undefined $(3 \times 3) \times (2 \times 2)$, $\mathbf{ACB}$ is undefined $(2 \times 2) \times (3 \times 3)$,
$\mathbf{ABC}$ is defined $(2 \times 2) \times (2 \times 2)$. (③)

A5. Answer: ①

$(\mathbf{r} \bullet \mathbf{n})^2 = \mathbf{r} \bullet \mathbf{r} \cos^2 \alpha \;\rightarrow (|\mathbf{r}|\,|\mathbf{n}| \cos \theta)^2 = |\mathbf{r}|^2 \cos^2 \alpha \;\rightarrow\; |\mathbf{r}| \cos \theta = |\mathbf{r}| \cos \alpha$
$\rightarrow \theta = \alpha \rightarrow$ angle between $\mathbf{r}$ and $\mathbf{n}$ is always α
$\rightarrow$ cone along $\mathbf{n}$ axis, opening at half-angle α (①)

$(\mathbf{r} - \mathbf{n}) \bullet (\mathbf{r} - \mathbf{n}) = a \;\rightarrow\; |\mathbf{r} - \mathbf{n}|^2 = a \;\rightarrow\; |\mathbf{r} - \mathbf{n}| = \sqrt{a}$
$\rightarrow$ distance between $\mathbf{r}$ and $\mathbf{n}$ is always $\sqrt{a}$
$\rightarrow$ sphere centred at $\mathbf{n}$, radius $\sqrt{a}$

$(\mathbf{r} \times \mathbf{n}) \bullet (\mathbf{r} \times \mathbf{n}) = a \;\rightarrow\; |\mathbf{r} \times \mathbf{n}|^2 = a \;\rightarrow\; |\mathbf{r} \times \mathbf{n}| = \sqrt{a}$
$\rightarrow |\mathbf{r}| \sin \theta$ is always $\sqrt{a}$
$\rightarrow$ cylinder along $\mathbf{n}$ axis, radius $\sqrt{a}$

$(\mathbf{r} \bullet \mathbf{i})(\mathbf{r} \bullet \mathbf{j})(\mathbf{r} \bullet \mathbf{k}) = 0 \;\rightarrow\; \mathbf{r} \bullet \mathbf{i} = 0$ or $\mathbf{r} \bullet \mathbf{j} = 0$ or $\mathbf{r} \bullet \mathbf{k} = 0$
$\rightarrow$ three planes, each passing through (0, 0, 0) and with normal vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

A6. Answer: ④

Use a Cartesian coordinate system with B as the origin, AB as the x-axis, BC as the y-axis, and the z-axis coming out of the page.

External angle of heptagon = $\pi - \angle ABP = \pi - \angle CBQ = \frac{2\pi}{7}$.

Coordinates of P before folding = $\mathbf{p} = \left(cos\frac{2\pi}{7},\ sin\frac{2\pi}{7},\ 0\right)^{\mathsf{T}}$

Coordinates of Q before folding = $\mathbf{q} = \left(sin\frac{2\pi}{7},\ cos\frac{2\pi}{7},\ 0\right)^{\mathsf{T}}$

Foldings can be represented as rotations about the x and y axes for the heptagons containing points P and Q respectively. Note that the senses of rotation are reversed - take $\mathbf{p}$ as rotating anticlockwise and $\mathbf{q}$ as rotating clockwise (looking inwards).

The folded points are then,

$$\mathbf{p}' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation about } AB} \underbrace{\begin{bmatrix} \cos\frac{2\pi}{7} \\ \sin\frac{2\pi}{7} \\ 0 \end{bmatrix}}_{P}, \quad \mathbf{q}' = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{\text{rotation about } BC} \underbrace{\begin{bmatrix} \sin\frac{2\pi}{7} \\ \cos\frac{2\pi}{7} \\ 0 \end{bmatrix}}_{Q}.$$

$\mathbf{p}' = \left(cos\frac{2\pi}{7},\ cos\,\theta\ sin\frac{2\pi}{7},\ sin\,\theta\ sin\frac{2\pi}{7}\right)^{\mathsf{T}}$

$\mathbf{q}' = \left(cos\,\theta\ sin\frac{2\pi}{7},\ cos\frac{2\pi}{7},\ sin\,\theta\ sin\frac{2\pi}{7}\right)^{\mathsf{T}}$

Equating coordinates,

$$cos\,\theta\ sin\,\frac{2\pi}{7} = cos\frac{2\pi}{7} \ \Rightarrow\ cos\,\theta = cot\,\frac{2\pi}{7} \ \Rightarrow\ sec\,\theta = tan\,\frac{2\pi}{7}. \quad (④)$$

A7. Answer: ③

Use integration by parts with $u = (1 - x)^m$ and $du = -m(1 - x)^{m-1}$:

$$I(m,\ n) = \int_0^1 x^n (1 - x)^m\, dx = \frac{1}{n+1}\left[x^{n+1}(1 - x)^m\right]_0^1 + \frac{m}{n+1}\int_0^1 x^{n+1}(1 - x)^{m-1}\, dx$$

The first term is zero, so we get the reduction formula

$I(m,\ n) = \frac{m}{n+1} I(m - 1,\ n + 1).$

Applying this formula repeatedly to itself, we get

$I(m,\ n) = \frac{m}{n+1} \times \frac{m-1}{n+2} I(m - 2,\ n + 2)$

$I(m,\ n) = \frac{m}{n+1} \times \frac{m-1}{n+2} \times \frac{m-2}{n+3} I(m - 3,\ n + 3) \ldots$

$I(m,\ n) = \frac{m}{n+1} \times \frac{m-1}{n+2} \times \ \ldots\ \times \frac{1}{n+m}\ I(0,\ m + n)$

$I(m,\ n) = \frac{m(m-1)(m-2)\ldots 1}{(n+m)(n+m-1)\ldots(n+1)} I(0,\ m + n) = \frac{m!\,n!}{(m+n)!}\ I(0,\ m + n)$

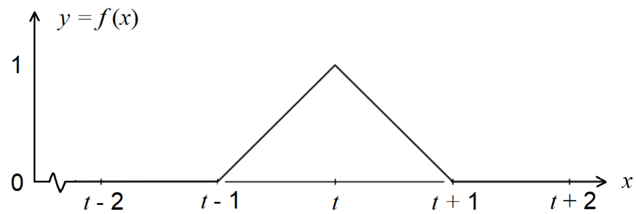
Calculate the value of $I(0, m + n)$:

$I(0,\ m + n) = \int_0^1 x^{m+n}\, dx = \frac{1}{m+n+1}$, therefore the final answer is

$I(m,\ n) = \frac{m!\,n!}{(m+n+1)(m+n)!} = \frac{m!\,n!}{(m+n+1)!}$. (③)

A8. **Answer:** ④

The graph of $f(x)$ is a triangle of width 2 centred at t:

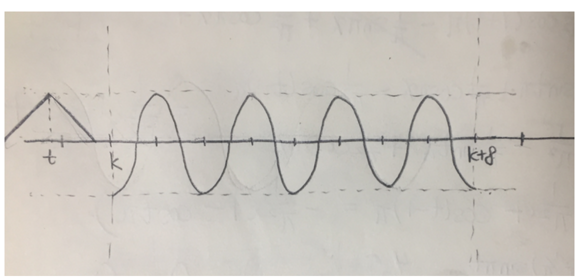


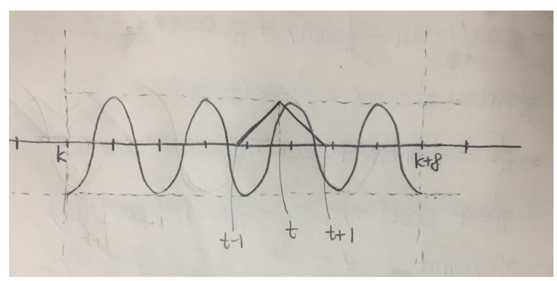
The graph of $y = cos\, \pi x$ is a sinusoid with period 2, alternating between -1 and 1 at odd and even integers respectively.

Multiplying $f(x)$ by this function has the effect of retaining a triangular ‘pulse’ wave around $x = t$, decaying to zero when $x < t - 1$ and $x > t + 1$.

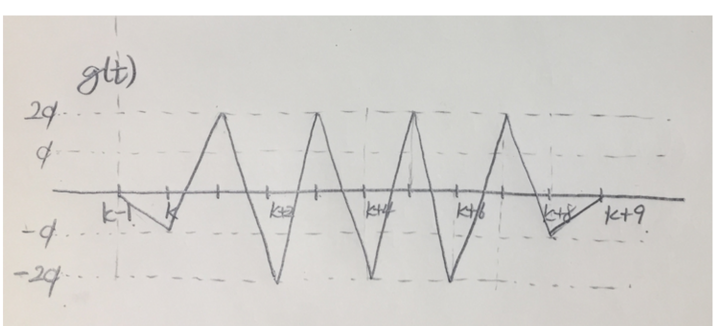
Since the bounds of integration are $k < x < k + 8$, the graph of $g(t)$, which integrates this triangular pulse, must be zero when $t < k - 1$ and $t > k + 9$.

Consider the product $f(x) \cos \pi x$ for various ranges of t, looking at the signs in the leading edge of $f(x)$ as it moves from left to right (as t increases):





The product of the two functions will have the **minimum** (maximum negative) area when the minima of the cosine wave line up with the peak (maximum) of the triangular pulse i.e. when t is odd.



By inspection, the minimum points for which $g(t) < 0$ are at positions $t = k$, $k + 2$, $k + 4$, $k + 6$ and $k + 8$.

It is given that the sum of these is 45, so

$k + (k + 2) + (k + 4) + (k + 6) + (k + 8) = 45;$
$5k + 20 = 45 \Rightarrow k = 5.$

Now by considering a simple transformation to functions with equivalent areas,
$$g(k) = -\int_0^1 (1 - x)\cos \pi x\, dx = \frac{1}{\pi}[(x - 1)\sin \pi x]_0^1 - \frac{1}{\pi}\int_0^1 \sin \pi x\, dx = \frac{-2}{\pi^2}.$$
By symmetry, the central minima are twice those at ends, so
$g(k + 2) = g(k + 4) = g(k + 6) = 2g(k) = \frac{-4}{\pi^2}$ and $g(k + 8) = g(k) = \frac{-2}{\pi^2}$.

Therefore $k - \pi^2 \sum_{i=1}^{m} g(\alpha_i) = 5 - \pi^2\left(-\frac{2}{\pi^2} - \frac{4}{\pi^2} - \frac{4}{\pi^2} - \frac{4}{\pi^2} - \frac{2}{\pi^2}\right) = 21.$ (④)

Graphs drawn by 돌핀 (dak219324) via Naver:
[4점공략] 2018 수능 수학 가형 30번 : 네이버 블로그

A detailed video solution (English) for this question:
LEC 004 2018 Korean SAT (#CSAT) No. 30 Solution

A9. Answer: ③

Resolving forces on the rock, $mg - F_R = ma \rightarrow 10m - 12 = 2m$
$\rightarrow m = 1.5$ kg. (③)

A10. Answer: ②

Since the centre of mass (COM) of the spherical shell remains unchanged, taking mass-moments about the centre gives the relation
$(total\ mass) \times (new\ COM) = (water\ COM) \times (water\ mass)$

The water forms a 'solid' hemisphere, which has a COM a distance $\frac{r}{2}$ below the surface (from the formula book) = 5.5 cm.

Volume of water = $\frac{2\pi r^3}{3} = 2787.64$ cm^3 → mass = 2787.64 g.

→ new COM = $\frac{2787.64}{2787.64 + 300} \times 5.5$ = 4.9656 cm = 4.97 cm. (②)

A11. Answer: ④

When the spring extends by x, the length AB decreases from $2L$ to $2L$ - x.

The vertical distance fallen by the COM of each bar is $\frac{1}{2}\sqrt{L^2 - \left(\frac{2L-x}{2}\right)^2}$.

Therefore, GPE = $-2mg \times \frac{1}{2}\sqrt{L^2 - \left(\frac{2L-x}{2}\right)^2} = -mg\sqrt{Lx - \frac{1}{4}x^2}$.

The strain energy (EPE) in the spring is $\frac{1}{2}kx^2$.

Therefore the total potential energy is GPE + EPE = $\frac{1}{2}kx^2 - mg\sqrt{Lx - \frac{1}{4}x^2}$ (④)

(Alternatively, could check that the GPE term is -mgL when $x = 2L$).

A12. Answer: ②

While held by the wire, the particle experiences a centripetal force $F = \frac{mL\Omega^2}{2}$ towards the centre. This produces a centripetal acceleration $\frac{L\Omega^2}{2}$ towards the centre relative to the inertial frame, or zero acceleration relative to the rod.

When the wire snaps, the force is removed, effectively subtracting $x\Omega^2$ from the relative acceleration, giving an **outward** acceleration of $x\Omega^2$.

Therefore the differential equation of motion is $x'' - \Omega^2 x = 0$.

Solution: $x = A\,e^{\Omega t} + B\,e^{-\Omega t} = C\cosh\Omega t + D\sinh\Omega t$.

From initial conditions, $x(0) = \frac{L}{2}$ and $x'(0) = 0$ so D = 0 and C = $\frac{L}{2}$.

→ $x(t) = \frac{L}{2}\cosh\Omega t$ (②)

A13. Answer: ③

Let x be the maximum-half span. Assume that $x = k\,\rho^a\,\sigma^b\,g^c$, where k is a dimensionless proportionality constant, and (a, b, c) are constant powers.

Evaluating the dimensions, $[x] = \mathbf{L}$, $[\rho] = \mathbf{ML^{-3}}$, $[\sigma] = \mathbf{FL^{-2}} = \mathbf{ML^{-1}T^{-2}}$, $[g] = \mathbf{LT^{-2}}$.

$\rightarrow \mathbf{M^0L^1T^0} = \mathbf{M^aL^{-3a}}\ \mathbf{M^bL^{-b}T^{-2b}}\ \mathbf{L^cT^{-2c}} \ \rightarrow\ \{a + b = 0,\ -3a - b + c = 1,\ -2b - 2c = 0\}$.

$\rightarrow a = -1, b = 1, c = -1 \rightarrow x = \dfrac{k\sigma}{\rho g} \rightarrow \dfrac{\rho g x}{\sigma}$ is a constant dimensionless group.

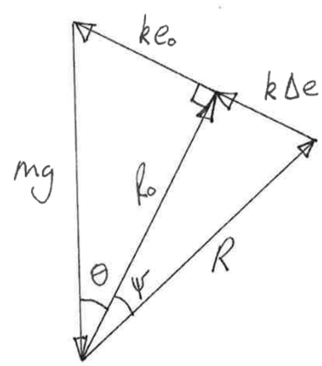
Using the model test results and applying proportionality,

$$\left(\frac{\rho g x}{\sigma}\right)_{model} = \left(\frac{\rho g x}{\sigma}\right)_{real} \Rightarrow \frac{2720 \times 400 \times 0.5}{70 \times 10^6} = \frac{7843 \times 9.81 \times x_{real}}{400 \times 10^6} \Rightarrow x_{real} = 40.4\text{ m (③)}$$

(**Note:** In reality, this analysis is wrong - there is a dependence on the thickness of the girders t, and the limit is given by $x = \sqrt{\frac{\sigma t}{3\rho g}}$, (assuming $x >> t$) but this could not have been found by this method of dimensional analysis alone as there are now 4 independent variables in 3 dimensions.)

A14. Answer: ①

Consider a limiting angle of friction $\psi = \tan^{-1}\mu$. If R_0 is the normal contact force without friction and R is the net limiting reaction force (friction + normal contact), then the force polygon is



By trigonometry, $\mu = \tan\psi = \frac{k\,\Delta e}{R_0}$ and $R_0 = \frac{ke_0}{\tan\theta}$, so $\mu = \frac{\Delta e}{e_0}\tan\theta$. (①)

A15. Answer: ④

Centripetal forces before and after: $\frac{km}{r_1^{\ 2}} = \frac{mv_P^{\ 2}}{r_1}$ and $\frac{km}{r_2^{\ 2}} = \frac{m(v_P/7)^2}{r_2}$

$\rightarrow\ k = v_P^{\ 2} r_1 = \frac{1}{49} v_P^{\ 2} r_2\ \rightarrow\ r_2 = 49 r_1\ \rightarrow\ r_1 : r_2 = 1 : 49.$

Potential energy = $-\ km \int_r^{\infty} \frac{1}{x^2}\,dx = km\left[\frac{1}{x}\right]_r^{\infty} = -\ \frac{km}{r}$.

Conservation of energy between orbits: $\frac{1}{2}mv_1^{\ 2} - \frac{km}{r_1} = \frac{1}{2}mv_2^{\ 2} - \frac{km}{r_2}$

$\rightarrow \frac{1}{2}v_1^{\ 2}\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) = \frac{k}{r_1}\left(1 - \frac{r_1}{r_2}\right).$

Now using $v_1 r_1 = v_2 r_2$ (derived using conservation of angular momentum), we get $\frac{v_2}{v_1} = \frac{r_1}{r_2} = \frac{1}{49}$, so substituting this back in, $\frac{1}{2}v_1^{\ 2}\left(1 - \frac{1}{49^2}\right) = \frac{k}{r_1}\left(1 - \frac{1}{49}\right).$

$\rightarrow v_1 = \sqrt{\frac{2k}{r_1} \times \frac{49}{50}} = \sqrt{\frac{2v_P^{\ 2} r_1}{r_1} \times \frac{49}{50}} = \frac{7}{5}v_P\ \rightarrow\ v_2 = \frac{7}{5}v_P \times \frac{1}{49} = \frac{1}{35}v_P.$

First impulse: $I_P = m(v_1 - v_P) = m\left(\frac{7}{5} - 1\right)v_P = \frac{2}{5}mv_P.$

Second impulse: $I_A = m(v_P/7 - v_2) = m\left(\frac{1}{7} - \frac{1}{35}\right)v_P = \frac{4}{35}mv_P.$ (④)

A16. Answer: ①

Look at the forces acting on each object. Take positive quantities to the right.

The forces acting on m_1 are:

Spring k_2:	exerts force $k_2(y_2 - y_1)$	Spring k_1:	exerts force $-k_1y_1$
Damper (L):	exerts force $-\lambda y_1'$	Damper (R):	exerts force $\lambda(y_2' - y_1')$
External force:	$f_1(t)$	Resultant:	my_1''

The forces acting on m_2 are:

Spring k_2:	exerts force $k_2(y_1 - y_2)$	Spring k_1:	exerts force $-k_1y_2$
Damper (R):	exerts force $\lambda(y_1' - y_2')$	Resultant:	my_2''

The force equations are then

m_1: $\quad k_2(y_2 - y_1) - k_1y_1 - \lambda y_1' + \lambda(y_2' - y_1') + f_1(t) = my_1''$

m_2: $\quad k_2(y_1 - y_2) - k_1y_2 + \lambda(y_1' - y_2') = my_2''$

Rearranging,

m_1: $\quad my_1'' + 0y_2'' + 2\lambda y_1' - \lambda y_2' + (k_1 + k_2)y_1 - k_2y_2 = f_1(t)$

m_2: $\quad 0y_1'' + my_2'' - \lambda y_1' + \lambda y_2' - k_2y_1 + (k_1 + k_2)y_2 = 0$

This can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 2\lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}. \quad (①)$$

A17. Answer: ②

Poisson distribution and Chi-Squared distribution are both positively skewed. Rectangular and t (like Normal) are both symmetric. (②)

A18. Answer: ③

The time delay between Poisson events has an Exponential distribution, with the same λ such that the mean is 1 / λ.

Mean time delay (minutes) = $\frac{60}{8}$ = 7.5 $\rightarrow$ $\lambda = \frac{1}{7.5} = \frac{2}{15}$. (③)

A19. Answer: ①

Yates' correction subtracts 0.5 from each $O - E$ entry. This results in a decrease to the χ^2 test statistic value obtained, making it more likely to exceed the critical value, making a rejection of H_0 more likely.

This represents an increase to the p-value, a decrease to P(Type I error), an increase to P(Type II error) and a decrease to the power (①)

The significance level does not change.

A20. Answer: ①

The distribution of Θ, in radians, will be $f(\theta) = \frac{1}{\pi/2} = \frac{2}{\pi}$ if $0 \leq \theta \leq \frac{\pi}{2}$ else 0.

Using LOTUS with $g(\Theta) = \frac{u^2 \sin 2\Theta}{g}$, we have

$E[X] = E[g(\Theta)] = \int_0^{\pi/2} \frac{u^2 \sin 2\theta}{g} \times \frac{2}{\pi}\, d\theta$ = 51 m. (①)

A21. Answer: ④

This is a form of the Texas Sharpshooter Fallacy. If the team just reports a *p*-value and let the reader decide if it is significant or not (or even if they want to declare significance or not) and don't make any objective decisions/declarations based on the *p*-value, then that is reasonable.

But when you indicate that the test was done at a specific significance level, 5%, then you are saying that before collecting the data (before randomisation) there was a 5% chance of declaring significance (rejecting the null hypothesis) if in fact the null hypothesis is true.

In Alex's scenario, with a true null hypothesis a p-value of 0.049 would have lead to a declaration of significance, so the probability of rejecting when the null is true is 5%, to say that it was 0.5% is dishonest and objectively false. (④)

A22. Answer: ③

From the characteristic equation, we have $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. Considering the sum of the roots, the sum of the eigenvalues is $\Lambda = a + d$, which is a sum of two independent rectangular distributions on [0, 1].

For one such rectangularly distributed CRV: $\mathrm{E}[a] = \frac{1}{2}$, $\mathrm{Var}[a] = \frac{1}{12}$.

Using the formulas $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$ and $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ for independent variables, we find that $\mathrm{E}[\Lambda] = 1$ and $\mathrm{Var}[\Lambda] = \frac{1}{6}$. ③

A23. Answer: ④

The number of ways of choosing 2 cards from n cards is ${}^{n}C_2 = \frac{n!}{2(n-2)!} = \frac{n(n-1)}{2}$.

From the definition of expectation, $E[X] = \frac{\sum_i X_i}{{}^{n}C_2} = \frac{2}{n(n-1)} \sum_i X_i$ where X_i are the possible values of the product. $\sum_i X_i$ represents all possible products, i.e.

$$\sum_i X_i = \sum_{r=2}^{n} [r \times (1 + 2 + 3 + \dots + (r-1)] = \sum_{r=2}^{n} \left[r \times \frac{(r-1)r}{2} \right] = \frac{1}{2} \sum_{r=2}^{n} r^2 (r-1)$$

$$= \frac{1}{2} \sum_{r=1}^{n} r^3 - r^2 = \frac{1}{2} \left(\frac{1}{4} n^2 (n+1)^2 - \frac{n}{6} (n+1)(2n+1) \right) \quad \text{(standard results)}$$

$$= \frac{1}{24} n(n+1)(3n(n+1) - 2(2n+1)) = \frac{1}{24} n(n+1)(3n^2 - n - 2)$$

$$= \frac{1}{24} n(n+1)(3n+2)(n-1)$$

Therefore $E[X] = \frac{2}{n(n-1)} \times \frac{n(n+1)(3n+2)(n-1)}{24} = \frac{(n+1)(3n+2)}{12}$. (④)

(Evaluating $E[AB] = E[A]\, E[B]$ where *A* and *B* are each card is not the correct approach because *A* and *B* are not independent.)

A24. Answer: ①

The average total area of the shadow will be 3 times the average area of the shadow cast by one of its top faces, by symmetry. Let Θ be the c.r.v. for the angle between the normal vector of a top face and the vertical line. Then Θ is uniformly distributed between 0 and π.

Area of shadow of one face with its normal at angle θ to vertical = $|\cos\theta|$.
Therefore $A = 3\,|\cos\Theta|$.

Suppose that the vector traces out a sphere when rotated through all possible angles. The probability of the face's vector being at θ to the vertical is

$\frac{\text{area of ring on sphere at } \theta}{\text{total area of sphere}} = \frac{2\pi r \sin\theta \times r\, d\theta}{4\pi r^2} = \frac{1}{2} \sin\theta\, d\theta$ i.e. the p.d.f. of Θ is ½ $\sin\theta$.

Using LOTUS,

$$E[A] = \int_0^{\pi} 3\,|\cos\theta|\, \frac{1}{2} \sin\theta\, d\theta = 3 \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \frac{3}{2}. \ (①)$$

Question B1

a. i) Eigenvalues:

$$\begin{vmatrix} 2-\lambda & a \\ 0 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)(1-\lambda) = 0 \quad \Rightarrow \quad \lambda = 1, 2$$

Eigenvectors:

$\lambda_1 = 1$: $x + ay = 0$. Let $x = a$: $\mathbf{v}_1 = \begin{bmatrix} a \\ -1 \end{bmatrix}$

$\lambda_2 = 2$: $y = 0$. Let $x = 1$: $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

ii) Using the diagonalised form, $\mathbf{A} = \mathbf{UDU}^{-1} \rightarrow \mathbf{A}^n = \mathbf{UD}^n\mathbf{U}^{-1}$:

$$\mathbf{A}^n = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

Evaluate $\mathbf{U}^{-1}$ and $\mathbf{D}^n$:

$$\mathbf{U}^{-1} = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix} \qquad \mathbf{D}^n = \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}$$

Multiply out matrices:

$$\mathbf{A}^n = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix} = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2^n & a \cdot 2^n \end{bmatrix} = \begin{bmatrix} 2^n & (2^n - 1)a \\ 0 & 1 \end{bmatrix}$$

iii) Base case: let $n = 1$.

$$\text{LHS} = \begin{bmatrix} 2 & a \\ 0 & 1 \end{bmatrix} \quad \text{RHS} = \begin{bmatrix} 2^1 & (2^1 - 1)a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & a \\ 0 & 1 \end{bmatrix}$$

LHS = RHS → true for $n = 1$.

Inductive hypothesis: assume true for some $n = k, k \in \mathbb{N}$. For $n = k + 1$:

$$\mathbf{A}^{k+1} = \mathbf{A}\mathbf{A}^k = \begin{bmatrix} 2 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^k & (2^k - 1)a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^{k+1} & (2^{k+1} - 2)a + a \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{k+1} & (2^{k+1} - 1)a \\ 0 & 1 \end{bmatrix} = \text{RHS for } n = k + 1.$$

The statement is true for $n = 1$, and if true for $n = k$ then true for $n = k + 1$
⇒ the statement is true for all natural n by induction.

b. Base case: let $n = 1$:

$$\text{LHS} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x \sin\sqrt{3}x\right) = e^x\left(\sqrt{3}\cos\sqrt{3}x + \sin\sqrt{3}x\right)$$

$$= 2e^x \sin\left(\sqrt{3}x + \frac{\pi}{3}\right)$$

(conversion to $R\sin(\theta + \alpha)$ form)

LHS = RHS → true for $n = 1$.

Inductive hypothesis: assume true for some $n = k, k \in \mathbb{N}$. For $n = k + 1$:

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(e^x \sin\sqrt{3}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(e^x \sin\sqrt{3}x\right)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(2^k e^x \sin\left(\sqrt{3}x + \frac{k\pi}{3}\right)\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(2^k e^x \sin\left(\sqrt{3}x + \frac{k\pi}{3}\right)\right) = 2^k e^x\left(\sin\left(\sqrt{3}x + \frac{k\pi}{3}\right) + \sqrt{3}\cos\left(\sqrt{3}x + \frac{k\pi}{3}\right)\right)$$

$$= 2^{k+1} e^x \sin\left(\left(\sqrt{3}x + \frac{k\pi}{3}\right) + \frac{\pi}{3}\right) = 2^{k+1} e^x \sin\left(\sqrt{3}x + \frac{(k+1)\pi}{3}\right)$$

$$= \text{RHS for } n = k + 1.$$

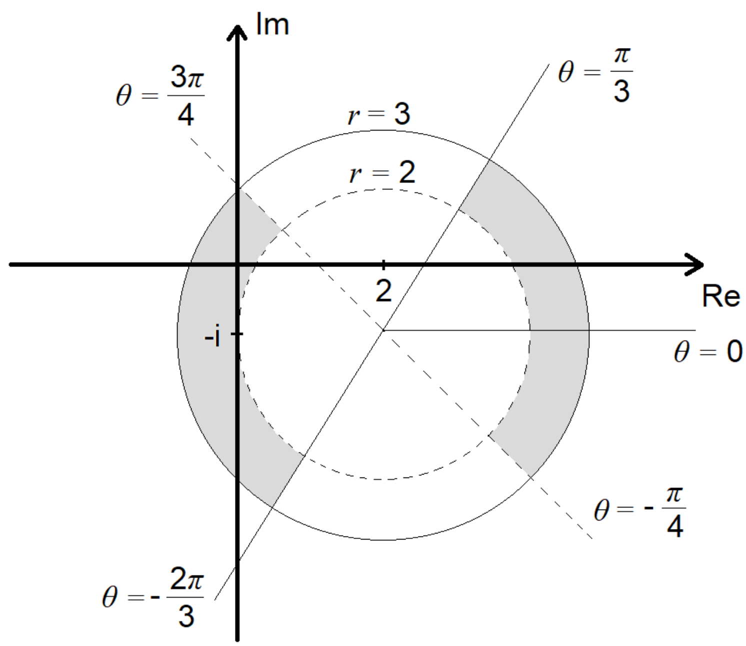
The statement is true for $n = 1$, and if true for $n = k$ then true for $n = k + 1$
⇒ the statement is true for all natural n by induction.

Question B2

a. i) Circles: centre 2 - i, radius $2 < r \leq 3$

Angles: $\arg[(z - (2 - \mathrm{i}))^2] = 2\arg(z - (2 - \mathrm{i}))$ (using $(e^{i\theta})^2 = e^{i2\theta}$)

This is the set of all numbers such that when their argument relative to the point 2 - i is doubled, they are mapped to the region with $-\frac{\pi}{2} < \theta \leq \frac{2\pi}{3}$:



ii) Let $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z \rightarrow$ smallest $k = x - y \rightarrow y = x - k$.

The line $y = x + (-k)$ is tangent to the uppermost point top-left half of the locus, at which point the value of the intercept $-k$ is largest.

This point is $z = 2 - i + 3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$.

Therefore the value of $x - y$ is

$$\Rightarrow Re(z) - Im(z) = (2 + 3\cos\frac{3\pi}{4}) - (-1 + 3\sin\frac{3\pi}{4})$$

$$\Rightarrow Re(z) - Im(z) = 3 - 3\sqrt{2}.$$

b. $$\sum_{r=1}^{n} \cos r\theta = Re\left(\sum_{r=1}^{n} e^{ir\theta}\right)$$ (De Moivre's theorem)

$$= Re\left(e^{i\theta} \cdot \frac{e^{in\theta}-1}{e^{i\theta}-1}\right)$$ (sum of finite geometric series: $a = r = e^{i\theta}$)

$$= Re\left(\frac{e^{i\theta}e^{\frac{in\theta}{2}}}{e^{\frac{i\theta}{2}}} \cdot \frac{e^{\frac{in\theta}{2}}-e^{-\frac{in\theta}{2}}}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}\right)$$ (take out factors of $e^{in\theta/2}$ from top and $e^{i\theta/2}$ from bottom)

$$= Re\left(e^{\frac{1}{2}i(n+1)\theta} \cdot \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right)$$ ($z - z^{-1} = 2i \sin \theta$)

$$= \frac{\cos\frac{(n+1)\theta}{2} \cdot \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$ ($Re\ e^{i\theta} = \cos \theta$)

Alternative method: proof by induction. Requires product-to-sum formula, derived by the converse of the sum-to-product formulas in the Formula Booklet.

Base case: LHS = $\sum_{r=1}^{1} \cos r\theta = \cos\theta$, RHS = $\frac{\cos\theta \sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \cos\theta$ = LHS.
Inductive hypothesis: assume true for some integer $n = k$. Then for $n = k + 1$,

$$\sum_{r=1}^{k+1} \cos\theta = \sum_{r=1}^{k} \cos\theta + \cos(k+1)\theta = \frac{\cos\frac{1}{2}(k+1)\theta \sin\frac{1}{2}k\theta}{\sin\frac{1}{2}\theta} + \cos(k+1)\theta$$

$$= \frac{\cos\frac{1}{2}(k+1)\theta \sin\frac{1}{2}k\theta}{\sin\frac{1}{2}\theta} + \frac{\cos(k+1)\theta \sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$$ (get common denominators)

$$= \frac{\sin\frac{1}{2}(2k+1)\theta - \sin\frac{1}{2}\theta + \sin\frac{1}{2}(2k+3)\theta - \sin\frac{1}{2}(2k+1)\theta}{2\sin\frac{1}{2}\theta}$$ (product-to-sum identity)

$$= \frac{\sin\frac{1}{2}(2k+3)\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \frac{\cos\frac{1}{2}(k+2)\theta \sin\frac{1}{2}(k+1)\theta}{\sin\frac{1}{2}\theta}$$

(sum-to-product identity)

= RHS for $n = k + 1$
→ proven for all integers $n \geq 1$ by induction.

c. Let $z = a + bi$. Then $sin\ z = sin(a + bi) = sin\ a\ cos\ bi + sin\ bi\ cos\ a$.

From De Moivre's theorem,

$cos\ bi = \frac{1}{2}\left(e^{(bi)i} + e^{-(bi)i}\right) = \frac{1}{2}\left(e^{-b} + e^{b}\right)$ = cosh *b*, and

$sin\ bi = \frac{1}{2i}\left(e^{(bi)i} - e^{-(bi)i}\right) = \ - i\ \frac{1}{2}\left(e^{-b} - e^{b}\right)$ = i sinh *b*, so

sin *z* = sin *a* cosh *b* + i sinh *b* cos *a*

Substituting into the equation, sin *a* cosh *b* + i sinh *b* cos *a* = *k* i.

Equating real parts: sin a cosh $b = 0 \rightarrow$ sin $a = 0 \rightarrow a = n\pi$. (cosh $x \neq 0$)

Equating imaginary parts: sinh b cos $a = k \rightarrow$ cos $n\pi$ sinh $b = k$

$\rightarrow (-1)^n$ sinh $b = k$ (cos $n\pi = (-1)^n$ for integer n)

$\rightarrow$ sinh $b = (-1)^n k$

$\rightarrow b = \sinh^{-1} [(-1)^n k]$

$\rightarrow b = \sinh^{-1} k$ {if n is even}; - $\sinh^{-1} k$ {if n is odd} (sinh x and $\sinh^{-1} x$ are odd functions so $\sinh^{-1}(-x) = -\sinh^{-1} x$)

$\rightarrow b = (-1)^n \sinh^{-1} k$.

Therefore the solutions are $z = n\pi + (-1)^n \sinh^{-1} k$ for any integer n.

Question B3

a. i) The plane Π has its normal vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, so both eigenvectors must lie in the plane. Since $\mathbf{u}$ and $\mathbf{v}$ have the same eigenvalue λ, all vectors in this plane are scaled by this factor so the plane is invariant under the transformation.

ii) If $\lambda = 1$ then the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ do not scale in length, so any vector in the plane also does not scale, so $\mathbf{u}$ and $\mathbf{v}$ span a plane of invariant points.

iii) The direction vector of line L has eigenvalue 2, so all vectors on this line are scaled by a factor of 2. Therefore L is an invariant line but **not** a line of invariant points.

iv) If $\lambda = 2$ then all three eigenvalues are equal, so the eigenvectors can be freely chosen to be an orthogonal set with $\mathbf{M} = 2\mathbf{I}$, which is the matrix for an enlargement of linear scale factor 2 about the origin.

b. i) $\mathbf{M} = \mathbf{S\Lambda S}^{-1} \Rightarrow |\mathbf{M}| = |\mathbf{S}||\mathbf{\Lambda}||\mathbf{S}^{-1}| = |\mathbf{S}||\mathbf{S}|^{-1}|\mathbf{\Lambda}| = |\mathbf{\Lambda}|$ = product of eigenvalues, since $\mathbf{\Lambda}$ is a diagonal matrix, which only depends on the eigenvalues of $\mathbf{M}$, not the eigenvectors.

ii)

$$|\mathbf{M}| = |\mathbf{\Lambda}| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 2\lambda^2.$$

c. i) Expanding from the diagonalised form,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

$$\mathbf{M} = \begin{bmatrix} -1 & -\frac{3}{2} & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{M}^3 = \begin{bmatrix} -1 & -\frac{9}{2} & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$3\mathbf{M} + 2\mathbf{I} = \begin{bmatrix} -3 & -\frac{9}{2} & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{9}{2} & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore $\mathbf{M}^3 = 3\mathbf{M} + 2\mathbf{I}$.

ii) $\mathbf{M}^3 - 2\mathbf{I} = 3\mathbf{M} \Rightarrow (\mathbf{M}^3 - 2\mathbf{I})^{-1} = \frac{1}{3}\mathbf{M}^{-1}$

In diagonalised form: $(\mathbf{M}^3 - 2\mathbf{I})^{-1} = \frac{1}{3}(\mathbf{S\Lambda S}^{-1})^{-1} = \frac{1}{3}\mathbf{S\Lambda}^{-1}\mathbf{S}^{-1}$.

$\rightarrow$ eigenvalues are $\frac{1}{3}\lambda^{-1} = \frac{1}{3\lambda}$

$\rightarrow -\frac{1}{3}$ (repeated), $\frac{1}{6}$.

Question B4

a. Characteristic equation: $det(Z - \lambda I) = 0$.

$$(z_{11} - \lambda)[(z_{22} - \lambda)(z_{33} - \lambda) - z_{23}z_{32}] - z_{12}[z_{21}(z_{33} - \lambda) - z_{23}z_{31}] + z_{13}[z_{21}z_{32} - (z_{22} - \lambda)z_{31}] = 0$$

$$-\lambda^3 + (z_{11} + z_{22} + z_{33})\lambda^2 + \ldots\lambda + \ldots = 0.$$

The sum of the roots is the sum of the eigenvalues, so

$$\Sigma\,\lambda = \lambda_1 + \lambda_2 + \lambda_3 = -\frac{b}{a} = z_{11} + z_{22} + z_{33} = sum\,of\,diagonals = tr\,Z.$$

b. i) Product of eigenvalues: $\lambda_1\lambda_2\lambda_3 = -\frac{d}{a} = d = constant\ term$.
The constant term results when $\lambda = 0$, which is given by the determinant,
So $\lambda_1\lambda_2\lambda_3 = det\,Z$. Therefore the given equation is $\mathbf{Z}^{-1} = \frac{1}{det\,Z}\mathbf{\Psi}^{\mathrm{T}}$, which is the definition of an inverse matrix when $\mathbf{\Psi}$ is the cofactor matrix for $\mathbf{Z}$.

ii) Coefficient of λ in characteristic equation:
$c = -z_{11}(z_{22} + z_{33}) + z_{23}z_{32} - z_{22}z_{33} + z_{12}z_{21} + z_{13}z_{31}$
Sum of product pairs of roots $\Sigma\,\lambda_i\lambda_j = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \frac{c}{a} = -c$:

$$\begin{aligned}\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 &= (z_{22}z_{33} - z_{23}z_{32}) + (z_{11}z_{33} - z_{13}z_{31}) + (z_{11}z_{22} - z_{12}z_{21})\\ &= \psi_{11} + \psi_{22} + \psi_{33}\\ &= tr\,\Psi.\end{aligned}$$

where ψ_{ij} are the cofactors of $\mathbf{Z}$.

c. Let λ be an eigenvalue of $\mathbf{Z}$ with corresponding eigenvector $\mathbf{u} = [u_1 \ u_2 \ u_3]^T$.

Then $\mathbf{Zu} = \lambda\mathbf{u} \ \rightarrow \ z_{i1}u_1 + z_{i2}u_2 + z_{i3}u_3 = \lambda u_i$ for $i \in \{1, 2, 3\}$.

$\rightarrow \sum_{j=1}^{3} z_{ij}u_j = \lambda u_i$ (write as summation over indices)

$\rightarrow \sum_{j \neq i} z_{ij}u_j = (\lambda - z_{ii})u_i$ (subtract $z_{ii}u_i$ from both sides)

$\rightarrow |\lambda - z_{ii}| = \left|\frac{\sum_{j \neq i} z_{ij}u_j}{u_i}\right| = \left|\sum_{j \neq i} \frac{z_{ij}u_j}{u_i}\right|$ (take absolute value on both sides)

$\rightarrow |\lambda - z_{ii}| \leq \sum_{j \neq i} \left|\frac{z_{ij}u_j}{u_i}\right|$ (triangle inequality: $|a + b| \leq |a| + |b|$)

Choose the value of i for which $|u_i|$ is the largest of $\{u_1, u_2, u_3\}$ i.e. the row which has the largest component of the eigenvector. Then $\frac{u_j}{u_i} \leq 1$ for all j and so

$\rightarrow |\lambda - z_{ii}| \leq \sum_{j \neq i} \left|\frac{z_{ij}u_j}{u_i}\right| \leq \sum_{j \neq i} |z_{ij}|$ which matches the locus given with $z = \lambda$,

and so all eigenvalues always obey the locus bounds.

(This is the 'Gershgorin Circle Theorem', commonly written as $\lambda \in \bigcup_{i=1}^{3} D_i$.)

d. i) Using the theorem derived in part c), we see that the eigenvalues must lie in circles centred at the diagonal entries, with radii equal to the sum of the other entries on that row. In this matrix $\mathbf{Z}$, the off-diagonals are very small compared to the diagonals, so the eigenvalues will lie very close to these diagonals (very small radii of circles). So the eigenvalues are approximately {400, 360, 480}, of which the median is approximately 400.

ii) Suppose that the error in this eigenvalue is ε. From the theorem, we must have that ε is bounded with $\varepsilon \leq 2\sqrt{2} \approx 2.8$, which means an eigenvector equation will be $\varepsilon x + (\sqrt{2} + \mathrm{i})y + (\sqrt{2} - \mathrm{i})z = 0$. Since ε is close in magnitude to the other coefficients, approximating it as zero is **very inaccurate** and will not likely produce anywhere near the correct eigenvector.

Question C1

a. Since all roots are real, the square of the differences between any two roots cannot be negative.

$$(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$$
$$\alpha^2 - 2\alpha\beta + \beta^2 + \beta^2 - 2\beta\gamma + \gamma^2 + \gamma^2 - 2\gamma\alpha + \alpha^2 \geq 0$$
$$2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \geq 0$$
$$2[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \geq 0$$
$$2(\alpha + \beta + \gamma)^2 - 6(\alpha\beta + \beta\gamma + \gamma\alpha) \geq 0$$

Using the Vieta formulas $\alpha + \beta + \gamma = \frac{n}{m}$ and $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{7}{m}$:

$\frac{2n^2}{m^2} \geq \frac{42}{m} \Rightarrow n^2 \geq 21m.$ ($m^2 > 0 \rightarrow$ can multiply through)

b. Vieta formula for product of roots: $\alpha\beta\gamma = \frac{1}{m} = 1$.

Therefore the roots of $Q(x)$ are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Let $z = \frac{1}{x} \rightarrow$ a function with the given roots is $P(z) = P(\frac{1}{x})$:

$$\frac{1}{x^3} - \frac{1}{x^2} + \frac{7}{x} - 1 = 0$$

$Q(x) = x^3 - 7x^2 + x - 1$ (multiply through by x^3 and negate)

c. i) $P(x) = \frac{1}{x} \Rightarrow xP(x) - 1 = 0 \Rightarrow -x^4 + x^3 + 7x^2 - x - 1 = 0$

Since $\delta + \varepsilon + \zeta + \eta = 1$:

$(\delta + \varepsilon + \zeta)(\delta + \varepsilon + \eta)(\delta + \zeta + \eta)(\varepsilon + \zeta + \eta) = (1 - \delta)(1 - \varepsilon)(1 - \zeta)(1 - \eta)$

An equation with roots equal to these factors is given by substituting $z = 1 - x$ into the quartic. The product of these factors will then be the product of the roots, i.e. the constant term divided by the x^4 term.

$-(1 - x)^4 + (1 - x)^3 + 7(1 - x)^2 - (1 - x) - 1 = 0$

x^4 term: -1

constant term: $-1 + 1 + 7 - 1 - 1 = 5$

Therefore the value is -5.

ii) The new quartic is $mx^4 - nx^3 + 7x^2 - x - 1 = 0$.

Substitute $x = 1$: $n - m = 5$

If the root is repeated then the root must be a stationary point on the graph of the polynomial:

Derivative: $4mx^3 - 3nx^2 + 14x - 1 = 0$

Substitute $x = 1$: $3n - 4m = 13$

Solving simultaneously, $m = 2$ and $n = 7$.

iii) $(\delta - \varepsilon)^2 + (\varepsilon - \zeta)^2 + (\zeta - \eta)^2 + (\eta - \delta)^2 + (\delta - \zeta)^2 + (\varepsilon - \eta)^2$

$= \delta^2 - 2\delta\varepsilon + \varepsilon^2 + \varepsilon^2 - 2\varepsilon\zeta + \zeta^2 + \zeta^2 - 2\zeta\eta + \eta^2 + \eta^2 - 2\eta\delta + \delta^2 + \delta^2 - 2\delta\zeta + \zeta^2 + \varepsilon^2 - 2\varepsilon\eta + \eta^2.$

$= 3(\delta^2 + \varepsilon^2 + \zeta^2 + \eta^2) - 2(\delta\varepsilon + \varepsilon\zeta + \zeta\eta + \eta\delta + \delta\zeta + \varepsilon\eta)$

$= 3[(\delta + \varepsilon + \zeta + \eta)^2 - 2(\delta\varepsilon + \varepsilon\zeta + \zeta\eta + \eta\delta + \delta\zeta + \varepsilon\eta)] - 2(\delta\varepsilon + \varepsilon\zeta + \zeta\eta + \eta\delta + \delta\zeta + \varepsilon\eta)$

$= -8(\delta\varepsilon + \varepsilon\zeta + \zeta\eta + \eta\delta + \delta\zeta + \varepsilon\eta)$ $(\sum \alpha = \frac{n}{m} = 0)$

$= -8(\frac{7}{m})$ $(\sum \alpha\beta = \frac{7}{m})$

$= -\frac{56}{m}.$

Question C2

a.

i) $$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\left(\frac{x\ln x-x+1}{(x-1)\ln x}\right)$$

Both the numerator and denominator evaluate to zero, so apply L'Hopital's rule:

$$=\lim_{x\to 1}\frac{\ln x}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{\ln x}{\ln x+1-\frac{1}{x}}$$

The numerator and denominator are still zero - use L'Hopital's rule again:

$$=\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\lim_{x\to 1}\frac{x}{x+1}=\frac{1}{1+1}=\frac{1}{2}.$$

Alternative method: substitute $u = x - 1$ and use the Maclaurin series expansion for $\ln(1 + u)$ up to at least second-order.

ii) Consider asymptotic values:

$$\lim_{x\to\infty}\frac{x+1}{x^2+6x}\exp\left[\frac{x^2}{1+x^2}(2+\ln x)\right]$$

When x is very large:

- $\frac{x^2}{1+x^2} \to 1$
- $exp[\frac{x^2}{1+x^2}(2 + ln\,x)] \to exp(2 + ln\,x) = e^2 e^{ln\,x} = xe^2$
- $\frac{x+1}{x^2+6x}exp[...] \to \frac{x(x+1)}{x(x+6)}e^2 \to e^2.$

Therefore the limiting value is e^2.

Alternative method: substitute $u = \frac{1}{x}$, write $2 = ln(e^2)$ and simplify

iii) Complete the squares:

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x+100}-\sqrt{x^2-6x}\right)=\lim_{x\to\infty}\left(\sqrt{(x+2)^2+96}-\sqrt{(x-3)^2-9}\right)$$

When x is very large, $(x\ +\ 2)^2 \gg 96$ and $(x\ -\ 3)^2 \gg\ \ -9$:

$$=\lim_{x\to\infty}\left(\sqrt{(x+2)^2}-\sqrt{(x-3)^2}\right)=\lim_{x\to\infty}((x+2)-(x-3))=2-(-3)=5$$

Alternative method: multiply the expression by $\frac{\sqrt{x^2+4x+100}+\sqrt{x^2-6x}}{\sqrt{x^2+4x+100}+\sqrt{x^2-6x}}$ and simplify the resulting numerator with difference of two squares. This is more algebraically intensive than using the asymptotic arguments given here, however.

iv) Let $u\ =\ x\ -\ \frac{\pi}{2}$:

$$\lim_{x\to\frac{\pi}{2}}\frac{\ln(x-\frac{\pi}{2})}{\tan x}=\lim_{u\to 0}\frac{\ln u}{\tan(u+\frac{\pi}{2})}$$

Using addition and translation identities,

$$tan(u\ +\frac{\pi}{2})\ =\frac{sin(u+\frac{\pi}{2})}{cos(u+\frac{\pi}{2})}=\frac{cos\,u}{-sin\,u}=\ \ -\frac{1}{tan\,u}.$$

(Alternative: informally using

$$tan(u\ +\frac{\pi}{2})\ =\frac{tan\,u+tan\frac{\pi}{2}}{1-tan\,u\,tan\frac{\pi}{2}}\sim\frac{\infty}{-\infty\ tan\,u}=\ \ -\frac{1}{tan\,u})$$

Therefore the limit is $-\lim_{u\to 0}\ ln\,u\ \ tan\,u$.

When u is small, $tan\,u\ \approx\ u\ \to\ \ -\lim_{u\to 0}\ u\,tan\,u\ =\ 0.$ (standard limit)

v) Both the integral and x^4 approach zero, so apply L'Hopital's rule.

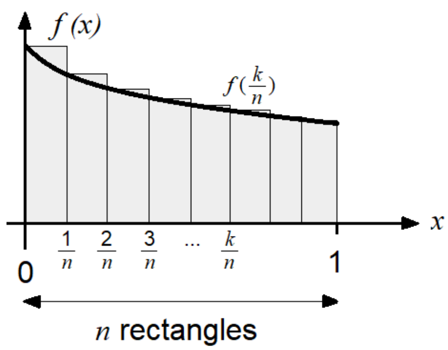
(Fundamental Theorem of Calculus: $\frac{d}{dx}\int_a^x f(t)\ dt=f(x)$; standard: $\lim_{x\to 0}\frac{sin\,x}{x}=1$)

$$\lim_{x\to 0}\frac{1}{x^4}\int_0^x sin^3\ t\ dt\ =\ \lim_{x\to 0}\frac{sin^3\ x}{4x^3}\approx\frac{x^3}{4x^3}=\frac{1}{4}.$$

vi) Divide top and bottom of the fraction by n:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{3n}{3n+k}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{3}{3+\frac{k}{n}}}$$

Observe that this summation matches the form of a Riemann sum, with $\Delta x = \frac{1}{n}$ and $x_k = \frac{k}{n}$:



(rectangular rule: $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n})$)

$$= \int_0^1 \sqrt{\frac{3}{3+x}}\,\mathrm{d}x$$
$$= \sqrt{3}\int_0^1 (x+3)^{-1/2}\,\mathrm{d}x = 2\sqrt{3}\left[\sqrt{x+3}\right]_0^1 = 4\sqrt{3} - 6.$$

b. $$\sum_{n=1}^{\infty} \frac{n+1}{n!} = \sum_{n=1}^{\infty} \left(\frac{n}{n!} + \frac{1}{n!}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)!} + \frac{1}{n!}\right) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{n!} = 2\left(\sum_{n=0}^{\infty} \frac{1}{n!}\right) - 1$$

Using the Maclaurin series for e^x at $x = 1$, $e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Therefore the value of the series is $2e - 1$.

Question C3

a. i) Simplifying,

$$\frac{x^2+5x+6}{1-x^2} = \frac{1}{5x+6}$$

$$(x^2 + 5x + 6)(5x + 6) = 1 - x^2$$

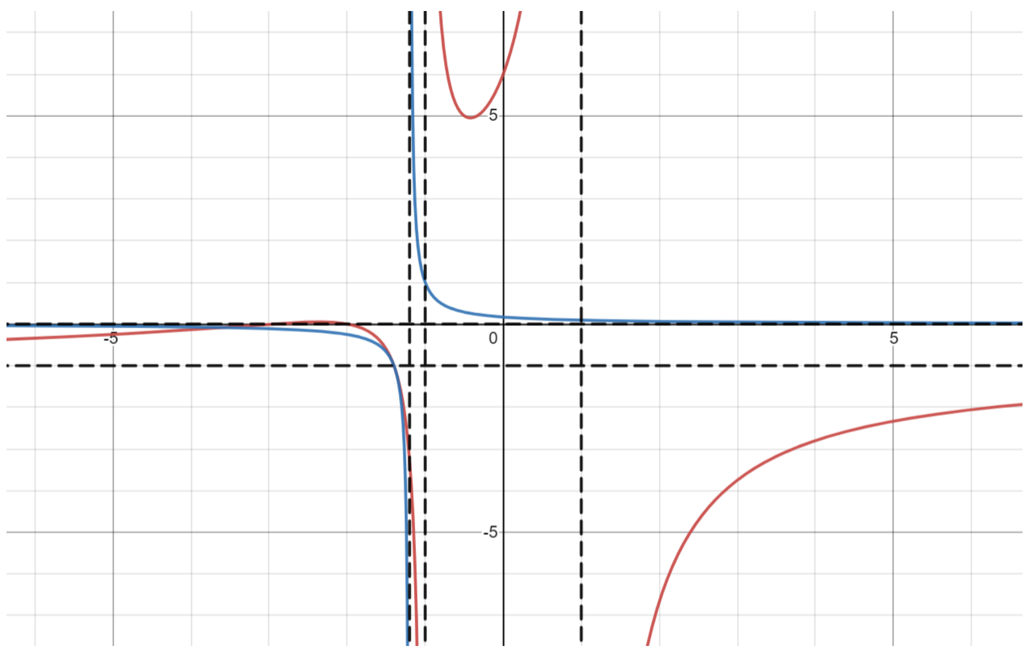
$$5x^3 + 31x^2 + 60x + 36 = 1 - x^2$$

$$5x^3 + 32x^2 + 60x + 35 = 0$$ (calculator cubic solver to find $x = -\frac{7}{5}$)

$$(5x + 7)(x^2 + 5x + 5) = 0$$ (synthetic division / equate coefficients)

$$\Rightarrow x = -\frac{5+\sqrt{5}}{2},\ x = -\frac{7}{5},\ x = \frac{\sqrt{5}-5}{2}$$ (in ascending order)

ii)



Curve	*Asymptotes*	*Intersections*
Red curve: $y = \frac{x^2+5x+6}{1-x^2}$	Vertical: x = -1, x = 1 Horizontal: y = -1	x-axis: x = -3, x = -2 y-axis: y = 6
Blue curve: $y = 1 - \frac{5(x+1)}{5x+6}$	Vertical: $x = -\frac{6}{5}$ Horizontal: y = 0	x-axis: none y-axis: $y = \frac{1}{6}$

iii) Writing the intersections in decimal: x = -3.618, x = -1.4, x = -1.382.

From the graph, the intervals where red ≥ blue are:

- first intersection $\leq x \leq$ second intersection
- third intersection $\leq x <$ first asymptote
- second asymptote $< x <$ third asymptote

Using the found results, the complete solution is

$$\left\{-\frac{5+\sqrt{5}}{2} \leq x \leq -\frac{7}{5}\right\} \cup \left\{\frac{\sqrt{5}-5}{2} \leq x < -\frac{6}{5}\right\} \cup \{-1 < x < 1\}.$$

iv) Let $\frac{x^2+5x+6}{1-x^2} = y$.

$x^2 + 5x + 6 = y - yx^2$

$(1 + y)x^2 + 5x + (6 - y) = 0$

At a turning point, there is only one intersection with a horizontal line at this y-coordinate, so the discriminant must be zero:

$\Rightarrow \ 25 - 4(1 + y)(6 - y) = 0$

$\Rightarrow \ 4y^2 - 20y + 1 = 0$

$\Rightarrow \ y_1 = \frac{5}{2} - \sqrt{6}$ and $y_2 = \frac{5}{2} + \sqrt{6}$

Substituting back in to the quadratic to solve for the x-coordinates,

$\Rightarrow \ x_1 = -\frac{7+2\sqrt{6}}{5}$ and $x_2 = \frac{2\sqrt{6}-7}{5}$

Since there are no other intersections, the higher point must be a minimum and the lower point must be a maximum:

local maximum: $\left(-\frac{7+2\sqrt{6}}{5}, \frac{5-2\sqrt{6}}{2}\right)$, and

local minimum: $\left(\frac{2\sqrt{6}-7}{5}, \frac{5+2\sqrt{6}}{2}\right)$

b. i) $$\frac{1}{r(r+2)} = \frac{1/2}{r} + \frac{-1/2}{r+2} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right).$$

ii) The positive terms in the sequence are $\frac{1}{1(1+2)} + \frac{1}{3(3+2)} + \frac{1}{5(5+2)} + \ldots$

The negative terms in the sequence are $-\left(\frac{1}{2(2+2)} + \frac{1}{4(4+2)} + \frac{1}{6(6+2)} + \ldots\right)$

Applying the partial fractions to each,

Positive terms: $\frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots$

Negative terms: $-\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) - \frac{1}{2}\left(\frac{1}{6} - \frac{1}{8}\right) - \ldots$

Both are telescoping series so use method of differences.

Taking out factors of ½, the positive series approaches $\frac{1}{2}$ and the negative series approaches $-\frac{1}{4}$, so the overall sum approaches $\frac{1}{4}$.

c. Expand denominator:

$$\left|\frac{x^5 - x + 1}{(x^2 + x + 1)(x - 1)^2}\right| = \left|\frac{x^5 - x + 1}{(x^2 + x + 1)(x^2 - 2x + 1)}\right| = \left|\frac{x^5 - x + 1}{x^4 - x^3 - x + 1}\right|$$

$$= \left|x + 1 + \frac{x^3 - x^2 - x}{x^4 - x^3 - x + 1}\right| \qquad \text{(polynomial long division)}$$

As $x \to \pm\infty$, the remainder part approaches zero in both cases, so the asymptote is $y = |x + 1|$.

When $x \to \infty$, $\quad y = x + 1.$

When $x \to -\infty$, $\quad y = -x - 1.$

Question D1

a. Using Cartesian coordinates, let $x = r\,cos\,\theta$ and $y = r\,sin\,\theta$.

Differentiate each using product rule as r depends on θ:

$\Rightarrow$ $\frac{dx}{d\theta} = \frac{dr}{d\theta} cos\,\theta - r\,sin\,\theta$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta} sin\,\theta + r\,cos\,\theta$

Using chain rule, the gradient of the tangent is $\frac{dy}{dx} = \frac{(\frac{dy}{d\theta})}{(\frac{dx}{d\theta})} = \frac{\frac{dr}{d\theta} sin\,\theta + r\,cos\,\theta}{\frac{dr}{d\theta} cos\,\theta - r\,sin\,\theta}$.

b. $\angle OPT = \pi - \theta - \angle OTP$

By considering supplementary angles, $\angle OTP = \pi - \arctan \frac{dy}{dx}$

Evaluating the gradient,

$$\frac{dy}{dx} = \frac{(cot\,b)e^{(cot\,b)\theta}\,sin\,\theta + e^{(cot\,b)\theta}cos\,\theta}{(cot\,b)e^{(cot\,b)\theta}\,cos\,\theta - e^{(cot\,b)\theta}\,sin\,\theta} = \frac{cot\,b\,sin\,\theta + cos\,\theta}{cot\,b\,cos\,\theta - sin\,\theta}$$

$$= \frac{cot\,b\,tan\,\theta + 1}{cot\,b - tan\,\theta} = \frac{tan\,\theta + tan\,b}{1 - tan\,\theta\,tan\,b} = \tan(\theta + b).$$

Therefore

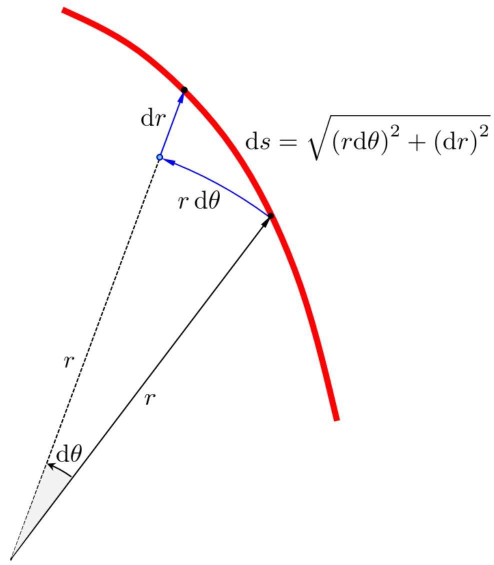
$\angle OPT = \pi - \theta - \angle OTP$

$= \pi - \theta - (\pi - tan^{-1}(tan(\theta + b)))$

$= b$ (or $b - \pi$ if $\theta + b > \frac{\pi}{2}$)

which does not depend on θ.

c. i) Consider an infinitesimal section of a polar curve between $(r,\ \theta)$ and $(r\ +\ dr,\ \theta\ +\ d\theta)$ and find the arc length in terms of the differential elements using circular arc sections and Pythagoras' theorem.



Alternative method: start from the formula for the arc length of a parametric curve and substitute the derived expressions from part a), $\frac{dx}{d\theta} = \frac{dr}{d\theta}cos\,\theta\ -\ r\ sin\,\theta$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta}sin\,\theta\ +\ r\ cos\,\theta$.

Writing ds in terms of dθ,

$$ds = \sqrt{(r\ d\theta)^2\ +\ (dr)^2} = \sqrt{r^2\ +\ \left(\frac{dr}{d\theta}\right)^2}\ d\theta \qquad \text{(factor out } (d\theta)^2\text{)}$$

Therefore the total arc length is

$$\int_0^{n\pi} ds = \int_0^{n\pi} \sqrt{r^2\ +\ \left(\frac{dr}{d\theta}\right)^2}$$

Substituting in the given expression for r,

$$= \int_0^{n\pi} \sqrt{e^{2(cot\,b)\theta}(1\ +\ cot^2 b)}\ d\theta\ =\ csc^2 b\ \int_0^{n\pi} e^{(cot\,b)\theta}\ d\theta$$

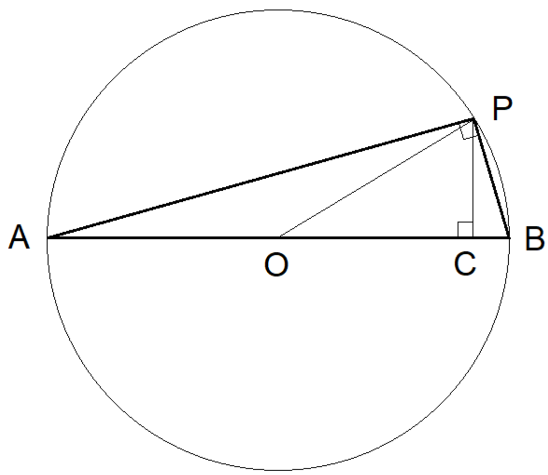
$$= sec\,b\ csc\,b\ \left(e^{(cot\,b)n\pi}\ -\ 1\right)$$

ii) Since $\frac{\pi}{2} < b < \pi$, the value of cot b is negative, so the exponential term decays as $n \to \infty$, leaving

Total arc length = sec b csc b = 2 csc 2b.

Question D2

a. If $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$, then $AP \perp BP$ and so by the converse of 'the angle at the circumference of a semicircle is 90°' circle theorem, AB is a diameter of a circle passing through A, B and P. The radius is $AO = OB = 1$.



From the condition $\overrightarrow{AB} \cdot \overrightarrow{AP} \geq 2 + \sqrt{3}$, it follows that

$$2\,|AP|\,\cos\angle BAP \geq 2 + \sqrt{3} \Rightarrow |AC| \geq 1 + \frac{\sqrt{3}}{2} \Rightarrow |OC| \geq \frac{\sqrt{3}}{2}$$
$$\Rightarrow |\angle COP| \leq \cos^{-1}\frac{|OC|}{|OP|} = \cos^{-1}\frac{\sqrt{3}}{2} \Rightarrow |\angle COP| \leq \frac{\pi}{6}.$$

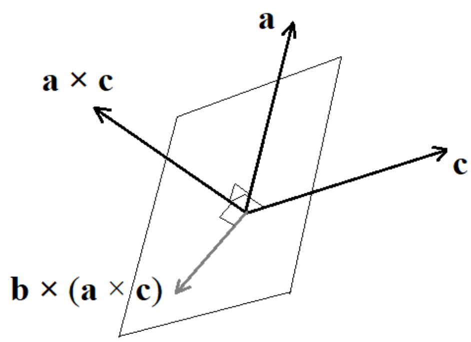
The locus of P is anywhere along the arc extending up to $\pm\frac{\pi}{6} = \pm 30°$ around the circle from B, revolved in 3D about the axis AB.

Equation of locus: $y = \pm\sqrt{1 - x^2}$ for $\frac{\sqrt{3}}{2} \leq x \leq 1$

Surface area:

$$= 2\pi \int_{\frac{\sqrt{3}}{2}}^{1} y\ ds = 2\pi \int_{\frac{\sqrt{3}}{2}}^{1} \sqrt{1 - x^2}\sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$
$$= 2\pi \int_{\frac{\sqrt{3}}{2}}^{1} 1\ \mathrm{d}x = 2\pi\left(1 - \frac{\sqrt{3}}{2}\right) = \left(2 - \sqrt{3}\right)\pi.$$

b. i) The vector **a** × **c** is perpendicular to both **a** and **c**:



The vector **b** × (**a** × **c**) must be perpendicular to **a** × **c**, so the vector **b** × (**a** × **c**) must lie in the plane with normal vector **a** × **c**, which is the same plane containing **a** and **c**. This plane contains the origin.

Using the scalar equation of a plane spanned by two vectors, **b** × (**a** × **c**) = λ**a** + μ**c.**

ii) Angle A is the angle between the plane containing **a** and **b**, and the plane containing **a** and **c**. Therefore A is also the angle between the normal vectors of these planes:

$$\cos A = \frac{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})}{|\mathbf{a} \times \mathbf{b}||\mathbf{a} \times \mathbf{c}|}$$

Manipulate numerator (all magnitudes are 1)

$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{c} \times (\mathbf{a} \times \mathbf{b}))$ (using $\mathbf{x} \bullet (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \bullet (\mathbf{z} \times \mathbf{x})$ with $\mathbf{x} = \mathbf{a} \times \mathbf{b}$, $\mathbf{y} = \mathbf{a}$, $\mathbf{z} = \mathbf{c}$)

$= \mathbf{a} \cdot [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}]$ (using $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \bullet \mathbf{z})\mathbf{y}$ - $(\mathbf{x} \bullet \mathbf{y})\mathbf{z}$ with $\mathbf{x} = \mathbf{c}$, $\mathbf{y} = \mathbf{a}$, $\mathbf{z} = \mathbf{b}$)

$= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b})$ (the scalar product is distributive)

$= \cos\alpha - \cos\beta\cos\gamma.$ (all magnitudes are 1)

The denominator is $|\mathbf{a} \times \mathbf{b}||\mathbf{a} \times \mathbf{c}| = \sin\beta\sin\gamma$, so

$$\cos A = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

Question D3

a. A generalised vector from one point on L_1 to a point on L_2 is $\mathbf{r}_2$ - $\mathbf{r}_1$:

$$\mathbf{r}_2 - \mathbf{r}_1 = \begin{bmatrix} 5+3\lambda+2\mu \\ 38-4\lambda-2\mu \\ -1+7\lambda+3\mu \end{bmatrix}$$

At the point of closest approach, the relative position vector must be perpendicular to both lines, so

$$\begin{bmatrix} 5+3\lambda+2\mu \\ 38-4\lambda-2\mu \\ -1+7\lambda+3\mu \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 4 \\ -7 \end{bmatrix} = 0 \qquad \text{and} \qquad \begin{bmatrix} 5+3\lambda+2\mu \\ 38-4\lambda-2\mu \\ -1+7\lambda+3\mu \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = 0$$

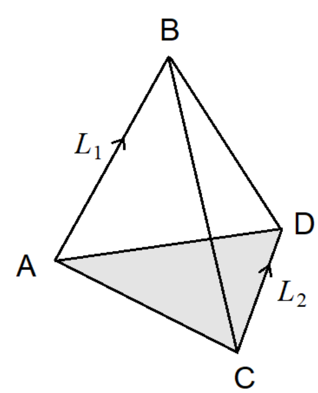
$$-15 - 9\lambda - 6\mu + 152 - 16\lambda - 8\mu + 7 - 49\lambda - 21\mu = 0$$
$$10 + 6\lambda + 4\mu - 76 + 8\lambda + 4\mu - 3 + 21\lambda + 9\mu = 0$$

$74\lambda + 35\mu = 144$ and $35\lambda + 17\mu = 69$
$\Rightarrow \lambda = 1$ and $\mu = 2$.

Therefore the points of closest approach are L_1: (-3, -9, -6) and L_2: (9, 21, 6).

b. i) Vector $\overrightarrow{AC}$ = [5 38 -1]^T

Normal vector of plane *ACD* = $\overrightarrow{AC} \times \overrightarrow{CD}$
= [5 38 -1]^T × [2 -2 3]^T
= [112 -17 -86]^T



A point on the plane is at *A*: $\mathbf{a}$ = [0 -13 1]^T

Therefore an equation for plane *ACD* is

$$\left(\mathbf{r} - \begin{bmatrix} 0 \\ -13 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 112 \\ -17 \\ -86 \end{bmatrix} = 0 \qquad \Rightarrow \qquad 112x - 17y - 86z = 135.$$

ii) The angle θ between the normal vector of *ACD* and L_1 is

$$\sin\theta = \frac{\begin{bmatrix}112\\-17\\-86\end{bmatrix}\cdot\begin{bmatrix}-3\\4\\-7\end{bmatrix}}{\left|\begin{bmatrix}112\\-17\\-86\end{bmatrix}\right|\left|\begin{bmatrix}-3\\4\\-7\end{bmatrix}\right|} = \frac{198}{\sqrt{1496946}} = 0.16183...$$

$\Rightarrow \theta = 9.31°$ (2 d.p.)

iii) The tetrahedron can be considered a triangular-based pyramid with base *ACD* and a perpendicular height normal to this face.

Area of base *ACD* = $\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{AD}\right|$

The unit normal vector in direction of perpendicular height is

$$\hat{\mathbf{n}} = \frac{\overrightarrow{AC}\times\overrightarrow{AD}}{\left|\overrightarrow{AC}\times\overrightarrow{AD}\right|}$$

The perpendicular height is the magnitude of the component of *AB* parallel to $\hat{\mathbf{n}}$, given by

$$h = \left|\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right| = \frac{\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|}{\left|\overrightarrow{AC}\times\overrightarrow{AD}\right|}$$

The volume of the pyramid, given by $V = \frac{1}{3}Ah$, is then

$$V = \frac{1}{6}\left|\overrightarrow{AC}\times\overrightarrow{AD}\right|\frac{\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|}{\left|\overrightarrow{AC}\times\overrightarrow{AD}\right|} = \frac{1}{6}\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|.$$

iv) *AB* = $[-3 \quad 4 \quad -7]^T$, *AC* × *AD* = $[112 \quad -17 \quad -86]^T$
AB • (*AC* × *AD*) = 198
V = 33 (length units cubed).

Question E1

a. For each tank, apply conservation of mass for the contaminant:

Rate of change of mass of contaminant
= (flow rate in × concentration in) - (flow rate out × concentration out).

Net flow rate in = net flow rate out for both tanks and whole system, so the volumes of water in each tank do not change from their initial values.

Concentration of contaminant in tank A = $\frac{x}{50}$
Concentration of contaminant in tank B = $\frac{y}{100}$

The differential equations modelling the mass transfer are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \underbrace{1.5 \times 7}_{\text{mass from tap}} + \underbrace{\frac{y}{100} \times 5}_{\text{mass from B}} - \underbrace{\frac{x}{50} \times 12}_{\text{mass leaving}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \underbrace{0 \times 3}_{\text{mass from tap}} - \underbrace{\frac{y}{100} \times 5}_{\text{mass leaving}} + \underbrace{\frac{x}{50} \times 2}_{\text{mass returning}}$$

Simplifying

$\frac{dx}{dt} = -\frac{6}{25}x + \frac{1}{20}y + \frac{21}{2}$ and $\frac{dy}{dt} = \frac{1}{25}x - \frac{1}{20}y.$

$100x' = -24x + 5y + 1050$ and $100y' = 4x - 5y.$

b. Rearrange for 5*y* in first equation: $5y = 100x' + 24x - 1050$

Differentiate: $5y' = 100x'' + 24x'$

Substitute into second equation:

$2000x'' + 480x' = 4x - (100x' + 24x - 1050)$

$200x'' + 58x' + 2x = 105$

$x(t) = Ae^{-\frac{1}{25}t} + Be^{-\frac{1}{4}t} + \frac{105}{2}$. (real roots, constant particular integral)

Initial conditions: $x(0) = 25$ and $x'(0) = \frac{33}{4}$

$\Rightarrow A + B = -\frac{55}{2}$ and $-\frac{1}{25}A - \frac{1}{4}B = \frac{33}{4}$

$\Rightarrow A = \frac{275}{42}$ and $B = -\frac{715}{21}$

$\Rightarrow x(t) = \frac{275}{42}e^{-\frac{1}{25}t} - \frac{715}{21}e^{-\frac{1}{4}t} + \frac{105}{2}$.

Differentiate and substitute back into the second differential equation:

$\Rightarrow x'(t) = -\frac{11}{42}e^{-\frac{1}{25}t} + \frac{715}{84}e^{-\frac{1}{4}t}$

$\Rightarrow y(t) = 20\left(-\frac{11}{42}e^{-\frac{1}{25}t} + \frac{715}{84}e^{-\frac{1}{4}t}\right) + \frac{24}{5}\left(\frac{275}{42}e^{-\frac{1}{25}t} - \frac{715}{21}e^{-\frac{1}{4}t} + \frac{105}{2}\right) - 210$

$\Rightarrow y(t) = \frac{550}{21}e^{-\frac{1}{25}t} + \frac{143}{21}e^{-\frac{1}{4}t} + 42$.

Alternative method: write the complementary functions as $A \cosh \lambda t + B \sinh \lambda t$.

c. First step: use Euler's method in each differential equation:

$x_1 = x_0 + 1 \cdot x_0'$ $\quad y_1 = y_0 + 1 \cdot y_0'$

Second step: use improved Euler method:

$x_2 = x_0 + 2 \cdot x_1'$ $\quad y_2 = y_0 + 2 \cdot y_1'$

n	t	x_n	y_n	$\frac{dx}{dt} = x_n'$	$\frac{dy}{dt} = y_n'$
0	0	25	75	$\frac{33}{4}$	$-\frac{11}{4}$
1	1	$\frac{133}{4}$	$\frac{289}{4}$	$\frac{2453}{400}$	$-\frac{913}{400}$
2	2	$\frac{7453}{200}$	$\frac{14087}{200}$		

Approximation for the mass in tank B ≈ y_2 = 70.435 grams

Actual value, from solution to differential equation: $y(2)$ = 70.30704166… grams
⇒ Percentage error = 0.1819993238…%
= 0.182% (3 s.f.)

d. Total mass = $x(t) + y(t)$, limiting value is $\lim_{t \to \infty} (x(t) + y(t))$.

All exponential terms in x and y will decay to zero, leaving only the constants:

$x + y \to \frac{105}{2} + 42 = 94.5$ grams

e. Any one of the following:

- Assumes perfect mixing / contaminant is uniformly distributed throughout the tanks / contaminant dissolves fully within the water
- Assumes the pipes are short / no contaminant in the pipes / water takes no time to travel from one tank to another
- Assumes contaminant does not interact with the water / no chemical reactions / volume of water in each tank remains constant
- Assumes full, unimpeded flow in pipes / no osmotic pressure across pipes / concentration in pipes matches concentration in source tank / contaminant does not get stuck in pipes / no backflow in pipes

Question E2

a. Integrate by parts, with u = sech^{n-2} x and dv = sech^2 x dx.

Relevant derivatives: $\frac{d}{dx}\ tanh\ x = sech^2 x$ and $\frac{d}{dx}\ sech\ x = \ -\ sech\ x\ tanh\ x$.

Relevant hyperbolic identity: $tanh^2\ x = 1 - sech^2\ x$.

$$I_n = \int \operatorname{sech}^n x\ dx = \int \operatorname{sech}^{n-2} x\ .\operatorname{sech}^2 x\ dx = \int \operatorname{sech}^{n-2} x\ .\frac{d}{dx} tanh\, x\ dx$$

$$= \operatorname{sech}^{n-2} x\ tanh\ x\ \ dx - (n-2) \int \operatorname{sech}^{n-3} x\ [-\operatorname{sech} x \tanh x]\ .tanh\, x\ dx$$

$$= \operatorname{sech}^{n-2} x\ tanh\ x\ \ dx + (n-2) \int \operatorname{sech}^{n-2} x\quad tanh^2\, x\ dx$$

$$= \operatorname{sech}^{n-2} x\ tanh\ x\ \ dx + (n-2) \int \operatorname{sech}^{n-2} x\quad [1 - sech^2\, x]\ dx$$

$$= \operatorname{sech}^{n-2} x\ tanh\ x\ \ dx + (n-2) \int \operatorname{sech}^{n-2} x\ dx - (n-2) \int \operatorname{sech}^{n} x\ dx$$

$$= \operatorname{sech}^{n-2} x\ tanh\ x\ \ dx + (n-2)\, I_{n-2}\ - (n-2)\, I_n$$

Solve for I_n: $(n - 1)I_n = sech^{n-2} x\ tanh\ x + (n - 2)I_{n-2}$

Therefore

$$I_n = \frac{1}{n-1} \operatorname{sech}^{n-2} x \tanh x + \frac{n-2}{n-1} I_{n-2}.$$

b. $\int_{-\infty}^{\infty} sech^2 x\ dx = \lim_{a\to\infty} \int_{-a}^{a} sech^2 x\ dx = \lim_{a\to\infty} [tanh\ x]^{a}_{-a}$

$= \lim_{a\to\infty} (tanh\ a - tanh(-\ a))$

$= 1 - (-\ 1)$

$= 2.$

c. Let $x = \sinh t \;\rightarrow\; dx = \cosh t \, dt$.

The new bounds are $\sinh^{-1} 0 = 0$ and $\sinh^{-1} a$ as $a \rightarrow \infty$, which approaches ∞.

The integral is then

$$\int_0^\infty \frac{1}{\sqrt{(1+x^2)^7}}\,\mathrm{d}x = \int_0^\infty \frac{\cosh t}{\sqrt{(1+\sinh^2 t)^7}}\,\mathrm{d}t$$

$$= \int_0^\infty \frac{\cosh t}{\cosh^7 t}\,\mathrm{d}t = \int_0^\infty \operatorname{sech}^6\, t\;\mathrm{d}t$$

($\sinh^2 t + \cosh^2 t = 1$)

Using the reduction formula in part a) with $n = 6$:

$$\int_0^\infty \operatorname{sech}^6\, t\;\mathrm{d}t = \frac{1}{5}\lim_{a\to\infty}\left[\operatorname{sech}^4\, x\tanh x\right]_0^a + \frac{4}{5}\int_0^\infty \operatorname{sech}^4\, t\;\mathrm{d}t$$

The first term is $\frac{1}{5}\lim\limits_{a\rightarrow\infty}\,(sech^4 a\; tanh\; a\; -\; sech^4\; 0\; tanh\; 0)\; = \frac{1}{5}(0\; -\; 0)\; =\; 0$.

Using the reduction formula again with $n = 4$, the first term is of similar form with the same limit, so

$$\int_0^\infty \operatorname{sech}^6\, t\;\mathrm{d}t = \frac{4}{5}\int_0^\infty \operatorname{sech}^4\, t\;\mathrm{d}t = \frac{4}{5}\times\frac{2}{3}\int_0^\infty \operatorname{sech}^2\, t\;\mathrm{d}t$$

From part b): $\int\limits_{-\infty}^{\infty} sech^2 x\; dx\; =\; 2\;\; \Rightarrow\;\; \int\limits_{0}^{\infty} sech^2 x\; dx\; =\; 1$ (sech x = sech -x)

Therefore the value of the integral is $\frac{4}{5}\times\frac{2}{3}\times\, 1\; = \frac{8}{15}$.

Question E3

a.

i) $\lambda^2 - 1 = 0 \Rightarrow \lambda = 1, \ -1 \Rightarrow y_{CF} = Ae^x + Be^{-x}$ or $y_{CF} = A\cosh x + B\sinh x$

Nonhomogeneous part = sinh x = $\frac{1}{2}(e^x - e^{-x})$

Neither of these are linearly independent with the CF, so multiply by x:

$\Rightarrow \ y_{PI} = Cxe^x + Dxe^{-x}$ or $y_{PI} = Cx\cosh x + Dx\sinh x$

ii) $10\lambda^2 - 6\lambda + 1 = 0 \Rightarrow \lambda = \frac{3 \pm i}{10} \Rightarrow \alpha = \frac{3}{10}, \ \beta = \frac{1}{10}$

$\Rightarrow \ y_{CF} = e^{\frac{3}{10}x}(A\cos\frac{1}{10}x + B\sin\frac{1}{10}x)$ or $y_{CF} = Ae^{\frac{3}{10}x}\cos(\frac{1}{10}x + B)$

Nonhomogeneous part = $2\sin^2 x - 4\cos 3x\cos 5x$

$= 2\left(\frac{1-\cos 2x}{2}\right) - 4\left(\frac{1}{2}(\cos 8x + \cos 2x\right)$ (double angle + product to sum formula)

$= 1 - 3\cos 2x - 2\cos 8x$

$\Rightarrow \ y_{PI} = C + D\cos 2x + E\sin 2x + F\cos 8x + G\sin 8x$

iii) $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = \ -1$ (repeated) $\Rightarrow \ y_{CF} = Ae^{-x} + Bxe^{-x}$

Nonhomogeneous part = $e^{-x} + e^{-x}\sin x$

The e^{-x} term is not linearly independent, and neither is $x\,e^{-x}$, so use $x^2\,e^{-x}$.

The other term has derivatives of the form $e^{-x}(A\cos x + B\sin x)$.

$\Rightarrow \ y_{PI} = Cx^2e^{-x} + e^{-x}(D\cos x + E\sin x)$.

(Any solution of the form $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ can also be represented using $A\cosh\lambda_1 x + B\sinh\lambda_2 x$, and any solution of the form $A\cos\beta x + B\sin\beta x$ can also be represented using $R\cos(\beta x + \phi)$ or $R\sin(\beta x + \phi)$.)

b. i) Use a PI of the form $y_{PI} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$.

$\Rightarrow\ y_{PI}' = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4$

$\Rightarrow\ y_{PI}'' = 2C + 6Dx + 12Ex^2 + 20Fx^3$

Substituting into the DE,

$(A + 2C) + (B + 6D)x + (C + 12Ex^2) + (D + 20Fx^3) + Ex^4 + Fx^5 =$
$x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.

Equating coefficients, $E = 0$ and $F = \frac{2}{15}$, so

$D = -\frac{7}{3}$, $C = 0$, $B = 15$, $A = 0$.

CF: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_{CF} = G\cos x + H\sin x$

General solution: $y = G\cos x + H\sin x + 15x - \frac{7}{3}x^3 + \frac{2}{15}x^5$.

Initial conditions: $y(0) = 0 \Rightarrow G = 0$ and $y'(0) = 0 \Rightarrow H = -15$

Particular solution: $y = -15\sin x + 15x - \frac{7}{3}x^3 + \frac{2}{15}x^5$.

ii) Let $x = \frac{\pi}{6}$ in the particular solution:

$y(\frac{\pi}{6}) = 0.02428453823...$

iii) $y_{PI} = -\cos x\ \ln(\sec x + \tan x)$

$y_{PI}' = \frac{-\cos x\,(\sec x \tan x + \sec^2 x)}{\sec x + \tan x} + \sin x\ \ln(\sec x + \tan x)$

$= -1 + \sin x\ \ln(\sec x + \tan x)$ (factorise sec x from top)

$y_{PI}'' = \frac{\sin x(\sec x \tan x + \sec^2 x)}{\sec x + \tan x} + \cos x\ \ln(\sec x + \tan x)$

$= \sin x \sec x + \cos x\ \ln(\sec x + \tan x)$

$\Rightarrow\ y_{PI}'' + y_{PI} = \sin x \sec x = \tan x$

which matches the nonhomogeneous part (RHS) of the DE, so the given PI is a valid solution to the DE.

iv) General solution: $y = G\cos x + H\sin x - \cos x\ \ln(\sec x + \tan x)$

Initial conditions: $y(0) = 0 \Rightarrow G = 0$ and $y'(0) = 0 \Rightarrow H = 1$.

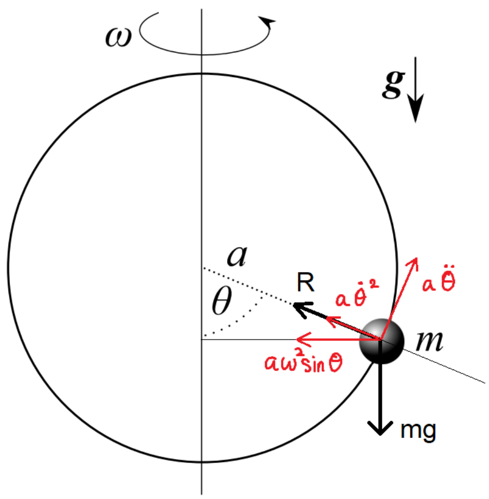
Particular solution: $y = \sin x - \cos x\ \ln(\sec x + \tan x)$

$\Rightarrow\ y(\frac{\pi}{6}) = 0.02428692455...$

$\Rightarrow$ percentage error = $0.00982554143...$ % (about 1 in 10^4)

Question F1

a.



The forces acting on the bead are gravity and the reaction force.

The **accelerations** of the bead are $-a\dot{\theta}^2$ (centripetal in the vertical plane), $a\ddot{\theta}$ (tangentially) and $-a\omega^2\sin\theta$ (centripetal in the horizontal plane, due to the ring rotation).

Resolving with *F* = *ma* in the tangential direction, being careful with signs and using the correct factors for the components,

$$\underbrace{-mg\sin\theta}_{\text{weight component}} = m\times(\underbrace{a\ddot{\theta}}_{\text{tangential acceleration}} - \underbrace{a\omega^2\sin\theta}_{\text{centripetal acceleration due to }\omega}\cos\theta)$$

Dividing both sides by *ma* and rearranging,

$-\frac{g}{a}\sin\theta = \ddot{\theta} - \omega^2\sin\theta\cos\theta \Rightarrow \ddot{\theta} + \frac{g}{a}\sin\theta - \omega^2\sin\theta\cos\theta = 0.$

(Resolving in the radial direction produces another equation: $R - mg\cos\theta = ma\dot{\theta}^2 + ma\omega^2\sin^2\theta$, but it involves *R* and is therefore not useful in the derivation.)

Alternative method: find expressions for the kinetic energy E_k and potential energy E_p. Since total energy is conserved (constant), the equation of motion is found by working out $\frac{d}{dt}(E_k + E_p) = 0$ and rearranging. Be careful when working out E_k to use both components of the velocity, so that $E_k = \frac{1}{2}m|\mathbf{v}|^2$.

b. θ is small $\rightarrow \sin\theta \approx \theta$ and $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \approx \theta$ (neglect θ^2 terms or higher)

Replacing these in the differential equation,

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{a} - \omega^2\right)\theta = 0$$

If $\omega^2 < \frac{g}{a}$, then the solution to this differential equation will have oscillations, so $\omega_c = \sqrt{\frac{g}{a}}$.

This is a S.H.M. equation with effective angular velocity $\sqrt{\frac{g}{a} - \omega^2}$ so the time period of these oscillations is $T = \frac{2\pi}{\sqrt{\frac{g}{a} - \omega^2}} = 2\pi\sqrt{\frac{a}{g - a\omega^2}}$.

c. i) When θ is measured from the top instead, $\sin\theta$ has the same value, while $\cos\theta$ and $\frac{d^2\theta}{dt^2}$ have the negative of their values since it is now $\pi - \theta$.

The new differential equation becomes

$$-\frac{d^2\theta}{dt^2} + \frac{g}{a}\sin\theta + \omega^2 \sin\theta\cos\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \left(\omega^2 - \frac{g}{a}\right)\theta = 0$$

This time, if $\omega > \omega_c$, the bead will oscillate about the **top** of the ring (new $\theta = 0$, old $\theta = \pi$), with angular frequency $\sqrt{\omega^2 - \frac{g}{a}}$.

ii) Returning to the original differential equation, the solution will contain real positive exponentials if $\omega > \omega_c$, so the bead will be repelled away from the bottom of the ring, up towards the top where it oscillates about the top position as seen in part b.i).

d. The speed of the bead (relative to the ring, so neglect the component perpendicular to the plane of the page) is $v = a\dot{\theta}$. The force to be added is $F = -cv = -ac\dot{\theta}$.

The new (approximated) differential equation becomes (either re-derive or 'guess' by matching the units of each quantity):

$$\frac{d^2\theta}{dt^2} + \left(\frac{c}{m}\right)\frac{d\theta}{dt} + \left(\frac{g}{a} - \omega^2\right)\theta = 0$$

For critical damping, the discriminant of the characteristic equation must be zero:

$$\frac{c^2}{m^2} - 4\left(\frac{g}{a} - \omega^2\right) = 0$$

$$\Rightarrow \frac{g}{a} - \omega^2 = \frac{c^2}{4m^2}$$

$$\Rightarrow \omega = \sqrt{\frac{g}{a} - \frac{c^2}{4m^2}} \text{ or } \sqrt{\frac{4gm^2 - ac^2}{4am^2}}$$

In this case, the bead decays to its equilibrium position at the bottom of the ring in the shortest time possible, and without oscillation.

Question F2

a. i) Apply conservation of energy between the start and the lowest point.

At point *A* (the top), we have $E_k = 0$, $E_p = 0$ (define the datum to be at A), $E_e = 0$ (just unstretched). At the lowest point, suppose the extension of the rope is x, so the distance below *A* is $x + 2L$. We have $E_k = 0$ (instantaneously at rest), $E_p = -mg(x + 2L)$, $E_e = \frac{\lambda x^2}{2L}$.

Therefore $\frac{\lambda x^2}{2L} = mg(x + 2L)$.

The maximum acceleration occurs at the bottom, as this is when the rope is exerting the largest force. At this point, the force balance gives

$\frac{\lambda x}{L} - mg = ma.$ (Hooke's law with $\lambda = kL$)

Since $a < 25g$, this becomes $\frac{\lambda x}{L} < 26mg \rightarrow \lambda < \frac{26mgL}{x}$. Substituting this into the energy equation to eliminate λ, we get

$13mgx > mg(x + 2L) \Rightarrow 12mgx > 2mgL \Rightarrow x > \frac{1}{6}L.$

ii) The rope cannot sustain more than a $25mg$ force so can never produce an acceleration more than $24g \rightarrow$ safe considering the acceleration.

At failure, $F = \frac{\lambda x}{L}$ so $25mg = 0.2\lambda$ (since $\frac{x}{L} = 20\%$)

$\rightarrow \lambda = 125mg.$

Substituting into the energy equation, we get $\frac{125mgx^2}{2L} = mgx + 2mgL$

$\Rightarrow \frac{125}{2}\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) - 2 = 0 \Rightarrow \frac{x}{L} = 0.1871$ (positive solution only)

Since 0.1871 < 0.2, the rope does not break, so is suitable.

b. Speed just before hitting the ground = $v_0 = \sqrt{2gh}$. (KE gained = GPE lost)

The lower ball collides with the ground first, rebounding with speed $ev_0 = e\sqrt{2gh}$.

The balls are now instantaneously moving in opposite directions.

Conservation of momentum: $Me\sqrt{2gh} - m\sqrt{2gh} = Mv_{M1} + mv_{m1}$

Coefficient of restitution: $e_{FM} = 1 = \frac{v_{m1} - v_{M1}}{e\sqrt{2gh} + \sqrt{2gh}}$ (floor is elastic)

$\Rightarrow\ v_{m1} - v_{M1} = (1 + e)\sqrt{2gh}$

Eliminating v_{M1}, we get $v_{M1} = v_{m1} - (1 + e)\sqrt{2gh}$.

Substituting into the momentum equation gives

$\sqrt{2gh}(Me - m) = M(v_{m1} - (1 + e)\sqrt{2gh}) + mv_{m1}$

$v_{m1}(M + m) = \sqrt{2gh}(M(2e + 1) - m)$

$v_{m1} = \frac{\sqrt{2gh}\,(M(2e+1) - m)}{M + m}$

Since $M >> m$, the denominator is approximately M and the numerator is approximately $(2e + 1)M\sqrt{2gh} \rightarrow v_{m1} \approx (2e + 1)\sqrt{2gh}$

Height gained $H = \frac{v_{m1}^2}{2g} = \frac{(2e+1)^2\, 2gh}{2g} = (2e + 1)^2 h$.

(This will be the maximum overall height because from the second bounce onwards, the lower ball will lose speed due to having $e < 1$. Since $m << M$, the momentum gained by the upper ball falling from higher has a negligible effect on its subsequent heights.)

c. Speed of beans as they hit the pan = $v_0 = \sqrt{2gh}$. (KE gained = GPE lost)

Rate of momentum loss of beans = $\mu v_0 = \mu\sqrt{2gh}$. (bean speed $\Delta v = -v_0$)

Additional downward force on balance = additional reaction force exerted on beans = rate of change of momentum of beans, so the erroneous additional reading on the balance is $\mu\sqrt{2gh}$. $(F = \frac{dp}{dt} = \frac{d}{dt}(mv) = mg + \mu v)$

Time taken for beans to fall = $t = \frac{v_0}{g} = \sqrt{\frac{2h}{g}}$ ($v = u + at$)

Mass of beans dropped in this time = $\mu t = \mu\sqrt{\frac{2h}{g}}$ (definition of flow rate)

Weight of beans not yet on the pan (in the air falling) = $\mu t g = \mu\sqrt{2gh}$. ($W = mg$)

These two quantities are equivalent, so the rate of change of momentum of the decelerating beans exactly compensates for the weight of the falling beans.

Question F3

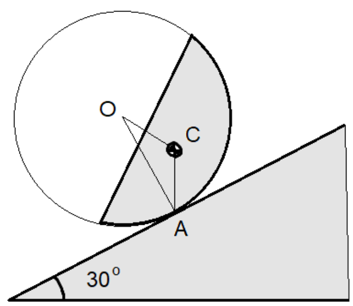
a. i) The centre of mass is given by $M\bar{x} = \sum x\, dm$. The region *R* is a segment of a circle, so has equation $y = \pm\sqrt{16 - x^2}$ ($-4 \le x \le 4$).

The centre of mass is then (evaluating integrals numerically)

$$\left[\int_2^4 \pi\left(\sqrt{16 - x^2}\right)^2 dx\right]\bar{x} = \int_2^4 x \times \pi\left(\sqrt{16 - x^2}\right)^2 dx \;\Rightarrow\; \bar{x} = \frac{27}{10} = 2.7.$$

ii) By symmetry, $\bar{y} = \bar{z} = 0$ so the spherical cap will lie aligned with the plane.

The object will balance when its COM is directly above the point of contact, so the weight passes through it (no moment about the contact).



Since *OA* is perpendicular to the tangent plane, angle *OAC* = 30°.

From the found measurements, |*OC*| = 2.7 and |*OA*| = 4. Therefore,

$$\frac{\sin ACO}{4} = \frac{\sin 30^o}{2.7} \;\Rightarrow\; ACO = 44.7945^o$$

(sine rule)

$$\Rightarrow\; AOC = 90 - ACO = angle\ of\ chord = 42.2^o.$$

iii) Sliding occurs if $\mu < \tan\theta$ $\Rightarrow$ need $\mu < \frac{1}{\sqrt{3}} \approx 0.577$ (angle of friction)

The body can never topple, as it will always be able to roll into a position so that the COM is above the contact point → sliding occurs first.

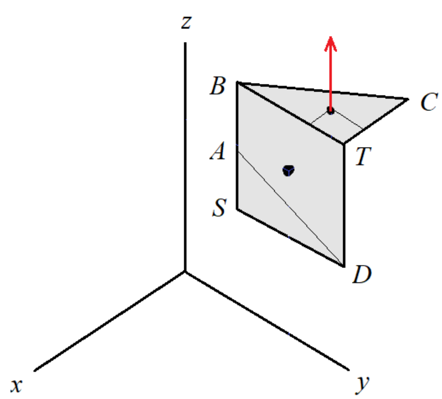
(As the body slides, its contact point may change, possibly causing it to later topple onto the plane face if the movement is large enough, but in theory, the body could slide down indefinitely.)

b. i) By symmetry, the centre of mass is at $\left(0,\ \frac{1}{2}b\right)$. The distances to the pivot are then $\left(\frac{3}{4}a,\ \frac{1}{6}b\right)$. Weight = $mg = \sigma A g = 3ab\sigma g$.

If *DT* is vertical, then the weight must act parallel to it, so the moment of this force is $W \times \frac{1}{6}b$.

The magnitude of the couple (torque) is therefore $\frac{1}{2}ab^2\sigma g$.

ii) The body is no longer symmetric, and so the air resistance force exerts a moment about the COM, causing it to rotate. In the new chosen coordinates (below), the *xy* plane is horizontal and the positive *z*-axis is vertical. The unfolded triangle *BCT* is horizontal and the rest of the body (*BASDT*) is vertical.



The net air resistance force (**red**) acts at the COM of triangle *BCT*. This is at a position $\frac{1}{3}b$ along *BT* (parallel to *y*-axis) from *T* and a position $\frac{1}{3}a$ along *CT* (parallel to *x*-axis) from *T*.

Due to the presence of the square and other triangle, the COM of the whole object must be (referring to the axes):

- at a larger *x* coordinate (closer to edge *BT*, due to combined body *BASDT*)
- at a smaller *z* coordinate (further down from edge *BT*, again due to *BASDT*)
- at the same *y* coordinate (due to the COM of *ASD* being the same distance from *S* as the COM of *BCT* is to *T*)

Therefore, with orientations defined as looking inwards towards the origin along the coordinate axes, the body will initially rotate anticlockwise about the *y*-axis due to the larger *x* coordinate. (The smaller *z* coordinate does **not** induce a rotation, because the line of action of the force is along this direction.)

⇒ the body rotates about lines parallel to *DS* (and *BT*) passing through the COM until the COM is vertically in line with the net line of action of the air resistance force(s) on each face.

(As rotation continues, the square will also have a force acting on it, which partially covers the triangular face, so modelling the precise rotation is tricky.)

Question G1

a. i) $\int_{-3}^{3} \phi(z)\, dz = P(-3 \le Z \le 3) = 0.9973$ (4 s.f.)

This represents the probability of a Normally distributed variable taking a value within 3 standard deviations of the mean.

ii) c.d.f. of *W*: $F(w) = P(W \le w) = P(Z^2 \le w) = P(|Z| \le \sqrt{w})$

$= \Phi(\sqrt{w}) - \Phi(-\sqrt{w})$

p.d.f. of *W*: $f(w) = \frac{d\Phi}{dw} = \phi(\sqrt{2}) \cdot \frac{1}{\sqrt{w}}$

Using the given form for ϕ,

$$f(w) = \frac{1}{\sqrt{2\pi w}} \exp\left(-\frac{1}{2}w\right), \qquad w \ge 0.$$

iii) If *W* has a chi-squared distribution, then the value of $P(W \le x)$ represents the *p*-value of a χ^2 test statistic with critical value *x*.

Choose a suitable *x* from the χ^2 table in the Formula Booklet:
e.g. $x = 2.706$ when $v = 1$ and $p = 0.9$ (i.e. 10% significance level).

Calculate *p* using the obtained p.d.f.:

$$P(W \le a) = \int_{0}^{2.706} \frac{1}{\sqrt{2\pi w}} exp\left(-\frac{1}{2}w\right) dw$$

This is an improper integral at the lower bound, so write as

$$= 1 - \int_{2.706}^{\infty} \frac{1}{\sqrt{2\pi w}} exp\left(-\frac{1}{2}w\right) dw$$

and evaluate using the calculator with a large upper bound, i.e.

$$\approx 1 - \int_{2.706}^{1000} \frac{1}{\sqrt{2\pi w}} \exp\left(-\frac{1}{2}w\right) dw = 0.9000286219\ldots$$

This matches the above *p*-value to 4 significant figures, so this agrees with *W* having a one degree-of-freedom chi-squared distribution.

iv) For a particular value of *w* in the first observation, it is required for the second observation to be *s* - *w* to ensure their sum equals *s*. This represents the conditional probability

$$P(S = s \mid W = w) = f(s - w)\, dw$$

The total probability is found by integrating over all possible values of *w* (from 0 to *s* since *W* ≥ 0) and multiplying by the probability of each event:

$$P(S = s) = \left(\int_0^s P(S = s \mid W = w)\, P(W = w)\, dw \right) ds$$

$$= \left(\int_0^s f(s - w)\, f(w)\, dw \right) ds$$

The p.d.f. of *S* is therefore $g(s) = \int_0^s f(s - w)\, f(w)\, dw$.

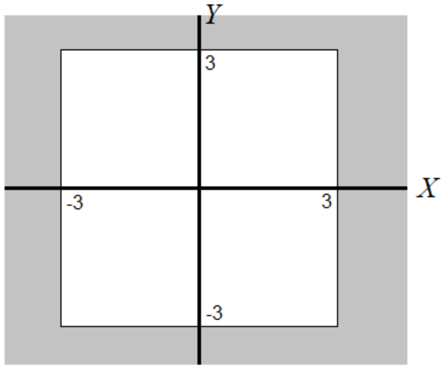
Alternative method: a less rigorous interpretation:

$$P(S = s) \simeq \int_0^s P(W_1 = w \ \cap \ W_2 = s - w)\, dw \simeq \int_0^s f(w)\, f(s - w)\, dw.$$

(≃ represents an informal equivalence, representing the same idea)

(This can be made more rigorous by considering the derivative of the c.d.f. i.e. $G(s) = \int_0^s P(W \le s - w \mid W = w)\, dw$ but this requires more complicated analysis.)

b. i)



$P(outside\ square) = 1 - P(inside\ square)$
$= 1 - P(-3 \le X \le 3 \ \cap \ -3 \le Y \le 3)$
$= 1 - [\,P(-3 \le X \le 3)\,]^2$ (X and Y are i.i.d. with N(0, 1)).
$= 1 - (0.9973...)^2$
$= 0.005393$

(Working out $P(|X| > 3) + P(|Y| > 3)$ is incorrect as the corner regions overlap.)

ii) $D^2 = (\sqrt{X^2 + Y^2})^2 = X^2 + Y^2,\ D^2 \ge 0$

The distributions of X^2 and Y^2 are identical, with p.d.f.

$X^2 \sim Y^2 \sim f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}x}$ (from part a.ii)

The distribution of the sum $D^2 = X^2 + Y^2$ has p.d.f. (from part a.iv)

$$g(s) = \int_0^s f_X(s - x)\, f_Y(x)\, dx = \int_0^s \frac{1}{\sqrt{4\pi^2 x(s - x)}} e^{-\frac{1}{2}(s-x)} e^{-\frac{1}{2}x}\, dx$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}s} \int_0^s \frac{1}{\sqrt{x(s-x)}}\, dx = \frac{1}{2\pi} e^{-\frac{1}{2}s} \int_0^s \frac{1}{\sqrt{sx - x^2}}\, dx = \frac{1}{2\pi} e^{-\frac{1}{2}s} \int_0^s \frac{1}{\sqrt{\left(\frac{s}{2}\right)^2 - (x - \frac{s}{2})^2}}\, dx$$

Integrate with a trig-substitution $x - \frac{s}{2} = \frac{s}{2} \sin\theta \rightarrow dx = \frac{s}{2} \cos\theta\, d\theta$:

(or substitute $u = x - \frac{s}{2}$ and use $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a} + C$)

$$g(s) = \frac{1}{2\pi} e^{-\frac{1}{2}s} \int_{-\pi/2}^{\pi/2} 1\, d\theta = \frac{1}{2} e^{-\frac{1}{2}s}$$ for all $s \ge 0$, otherwise 0.

This matches the form of an Exponential distribution $\lambda e^{-\lambda x}$ with $\lambda = \frac{1}{2}$.

iii) $P(D \le 3) = P(D^2 \le 9)$.

Using the c.d.f. of the Exponential distribution $1 - e^{-\lambda x}$, or by integration,

$= 1 - e^{-\frac{9}{2}} = 0.9890$ (4 s.f.)

(This is slightly less than P(inside square) in part a.i), since ◯ ⊂ □.)

Question G2

a. i) Any **one** assumption and comment from:

- Disconnections are independent: this may not be true as disconnections may have a common root cause which leads to more frequent disconnections if the issue is left unresolved.
- Disconnections have a constant average rate: the rate of disconnections could vary over time due to different operating conditions, such as higher server usage at peak times of the day.
- Disconnections are point events in time: this is not true, it will take a little time for the server to either recover itself or be fixed, which effectively delays the next disconnection event.

ii) Let X be the number of disconnections in one 8-hour workday.
Then $X \sim \text{Po}(2)$ since we expect 2 disconnections per 8 hours.
$P(X \leq 1) = 0.406005879...$

For this to occur for five days consecutively,
P(5 days with no more than 1 disconnection) = $0.40600...^5$
$= 0.01103$ (4 s.f.).

iii) Define X as the number of disconnections in a 24-hour period,
so that $X \sim \text{Po}(6)$ on the old server.

H_0: mean rate of disconnections is once per four hours ($\lambda = 6$)
H_1: mean rate of disconnections is less than once per four hours ($\lambda < 6$)

Under H_0, $P(X \leq 2) = 0.1512...$

Since this is more than 0.10, we accept H_0 as there is insufficient evidence to support a decrease in the mean rate of disconnections at the 10% significance level.

iv) True distribution of X: $X \sim \text{Po}(4.8)$.
Acceptance region of test (region where $P(X \geq x \mid X \sim \text{Po}(6)) > 0.10$) is $X \geq 3$
Power = 1 - P(Type II error) = 1 - $P(X \geq 3 \mid X \sim \text{Po}(4.8))$
$= P(X \leq 2 \mid \lambda = 4.8) = 0.1425\ldots$

This is a very low-powered test because of the closeness of the true rate to the null hypothesis rate.

(This is **not** due to the high significance level - this has the opposite effect, lowering P(Type II error) at the expense of a high P(Type I error).)

b. i) Let X be the difference in speeds. Under H_0, X is Normally distributed with zero mean (if the mean speeds are equal), where $X \sim N(\mu, \sigma^2)$.

H_0: no difference in mean speed ($\mu = 0$)
H_1: there is a difference in mean speed ($\mu \neq 0$)

Old server	75	67	69	80	77	71	67	80
New server	76	70	67	79	82	76	73	77
Difference, *X*	-1	-3	2	1	-5	-5	-6	3

Sample mean = $\overline{X} = -1.75$
Sample variance = $s_x^2 = \frac{171}{14} = 12.2143\ldots$
The sample is small and drawn from a distribution with unknown variance so using the t-distribution with 7 degrees of freedom,
$$t = \frac{\overline{X} - \mu}{s_x/\sqrt{n}} = \frac{-1.75 - 0}{\sqrt{12.2143 / 8}} = -1.41628\ldots$$
Critical value at $p = 0.025$ (two tail) = -(Critical value at $p = 0.975$) = -2.365.
Since $|-1.416| < |-2.365|$, accept H_0.
There is insufficient evidence to suggest a difference in mean upload speed between the two servers.

Alternative method: test for the mean of the new sample against the null hypothesis being that it is equal to the mean of the old sample. Results may differ slightly but the conclusion should be the same.

ii) H_0: no association between server and setting

H_1: there is an association between server and setting

Observed contingency table (no pooling possible or required)

Observed	Without setting	With setting	Total
Old server	291	295	586
New server	292	308	600
Total	583	603	1186

Expected values assuming no association:

Expected	Without setting	With setting	Total
Old server	288.059	297.941	586
New server	294.941	305.059	600
Total	583	603	1186

Chi-Square table: $\chi_i^2 = \frac{(O_i - E_i)^2}{E_i}$ or $\chi_i^2 = \frac{(|O_i - E_i| - 0.5)^2}{E_i}$

(Yates' continuity correction is optional as entries are large despite being a 2×2 contingency table)

χ^2 (without Yates)	Without setting	With setting
Old server	0.030027	0.029031
New server	0.029326	0.028353

$\chi^2 = 0.116737$ (or less if using Yates' correction)

Critical value (1 degree of freedom, $p = 0.95$) = 3.841

Since $0.116737 < 3.841$, there is insufficient evidence to suggest any association between the server and the setting used for upload speed.

Question G3

a. i) Let the different types of bottles (singers) be numbered 1 to n.
Consider arranging the bottles in order, and drawing a line between each group with the same number (singer). For example, a selection could be:

1 1 1 | 2 2 | | 4 | 5 (here, m = 7 and n = 5; singer #3 has not yet been found)

In any given collection, we can place n - 1 lines (|) to separate the groups.

Considering a different edge case such as 1 1 1 1 1 1 1 | | | |, we see that there are $m + n$ - 1 different places where we could place a line.

So there are ${}^{m+n-1}C_{n-1}$ different ways to divide the list, and hence the same number of collections. Evaluating this in terms of factorials,

$$ {}^{m+n-1}C_{n-1} = {}^{m+n-1}C_{n-1} = \frac{(m+n-1)!}{(n-1)!\,(m+n-1-(n-1))!} = \frac{(m+n-1)!}{(n-1)!\,m!}. $$

ii) Let $m = n$ in the formula found.

Number of combinations: $\frac{(m+n-1)!}{(n-1)!\,m!} = \frac{(2n-1)!}{(n-1)!\,n!} = \frac{(2n-1)!}{(n-1)!\,n(n-1)!} = \frac{(2n-1)!}{n\,(n-1)!^2}$

With $m = n$, there is only one winning combination (one of each kind), so the probability is $1 \div \frac{(2n-1)!}{n\,(n-1)!^2} = \frac{n\,(n-1)!^2}{(2n-1)!}$.

Consider $\lim_{n\to\infty} \frac{n\,(n-1)!^2}{(2n-1)!}$

$= \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)...1 \times (n-1)(n-2)(n-3)...1}{(2n-1)(2n-2)(2n-3)...n(n-1)(n-2)(n-3)...1}$

$= \lim_{n\to\infty} \frac{n!}{(2n-1)(2n-2)(2n-3)...(2n-n)}$, denominator is larger than $(n)(n)(n)...(n) = n^n$

$\leq \lim_{n\to\infty} \frac{n!}{n^n} = 0$

so the sequence must be decreasing as the limit is zero.

b. i) Since the events are disjoint and mutually exhaustive, we have

$$T = X_1 + X_2 + \dots + X_n = \sum_{k=1}^{n} X_k$$

By linearity of the expectation operator, $E[T] = E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} E[X_k]$.

ii) Suppose that k - 1 singers out of n have already been found, so that there are n - k + 1 singers left to be found. Then the probability of finding one of these new singers in any given bottle at this point is $p = \frac{n-k+1}{n}$.

The complement probability (probability of finding a singer we already have) is then $1 - p = 1 - \frac{n-k+1}{n} = 1 - (1 - \frac{k-1}{n}) = \frac{k-1}{n}$.

Therefore, for a particular value of X_k, we have $P(X_k = x) = (1-p)^{x-1}p$.

The expected value is then $E[X_k] = \sum_{x=1}^{\infty} x\, P(X_k = x) = \sum_{x=1}^{\infty} x\,(1-p)^{x-1}p$.

$$= p\left(\sum_{x=1}^{\infty}(1-p)^{x-1}p + \sum_{x=2}^{\infty}(1-p)^{x-1}p + \sum_{x=3}^{\infty}(1-p)^{x-1}p + \dots\right)$$

$$= p\left(\frac{1}{1-(1-p)} + \frac{1-p}{1-(1-p)} + \frac{(1-p)^2}{1-(1-p)} + \dots\right)$$

$$= p\left(\frac{1}{p} + \frac{1-p}{p} + \frac{(1-p)^2}{p} + \dots\right)$$

$$= (1-p)^0 + (1-p)^1 + (1-p)^2 + (1-p)^3 + \dots = \frac{1}{1-(1-p)} = \frac{1}{p}$$

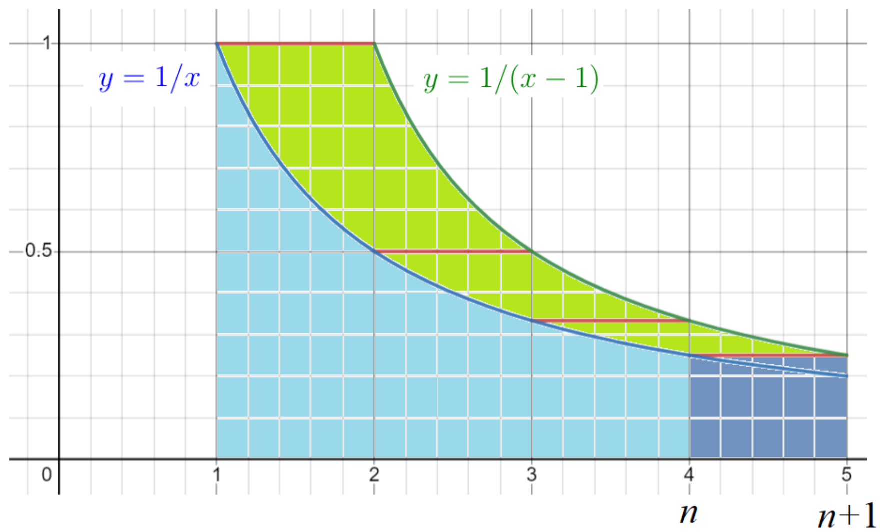
$$= \frac{n}{n-k+1}.$$

iii) $E[T] = \sum_{k=1}^{n} E[X_k] = \sum_{k=1}^{n} \frac{n}{n-k+1} = n \sum_{k=1}^{n} \frac{1}{n-k+1} = n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + ... \frac{1}{1}\right)$

$= n \sum_{k=1}^{n} \frac{1}{k}$

We therefore need to prove that $ln(n) + \frac{1}{n} < \sum_{k=1}^{n} \frac{1}{k} < ln(n) + 1.$

Notice that the value of $\sum_{k=1}^{n} \frac{1}{k}$ can be represented as the rectangular rule approximation to its continuous function in two ways, one of which is always smaller and one of which is always larger:



Since the overestimate of $\int_1^x \frac{1}{x}\, dx$ is always decreasing (it is convex / concave up), we can say that the overestimate in the range $n < x < n+1$ is smaller than the sum of the overestimates in the range $1 < x < n$, and therefore we can use the blue (light + dark) areas as the lower bound of the sum. The green area is a clear upper bound, improved by taking the first rectangle as its true value (1).

Expressing these areas, we have

$$\underbrace{\int_1^n \frac{1}{x}\, dx}_{\text{light blue}} + \underbrace{\frac{1}{n}}_{\text{dark blue}} < \underbrace{\sum_{k=1}^{n} \frac{1}{n}}_{\text{rectangle sum}} < \underbrace{1}_{\text{first green rectangle}} + \underbrace{\int_2^{n+1} \frac{1}{x-1}\, dx}_{\text{green}}$$

Evaluating these integrals gives $ln(n) + \frac{1}{n} < \sum_{k=1}^{n} \frac{1}{k} < 1 + ln(n).$

Multiplying through by n gives $n\, ln(n) + 1 < E[T] < n\, ln(n) + n.$

c. i) 100,000 bottles sold at £0.50 profit per bottle → shop income = £50,000.

Require 25% profit, so we need the payout to be less than £40,000.

The actual payout is given by $Q \times \frac{100{,}000}{\overline{T}}$, where $\overline{T}$ is the mean value of T (bottles bought per prize). From the central limit theorem, we know that the $\overline{T}$ should have a Normal distribution since the number of bottles sold (100,000) is very large. We therefore require $Q < 0.4\overline{T}$ with 95% probability. This is a left-tail probability with a z-score of $z = \Phi^{-1}(0.05) = -1.644853667\ldots$ so if $\overline{T} \sim N(\mu, \sigma^2)$ then $Q < 0.4(\mu + z\sigma)$.

We know that $\mu = \mathrm{E}[T] = n \sum_{k=1}^{n} \frac{1}{k} = 7 \sum_{k=1}^{7} \frac{1}{k} = \frac{363}{20} = 18.15$.

We know that $Var[T] = n^2 \sum_{k=1}^{n} \frac{1}{k^2} = 7^2 \sum_{k=1}^{7} \frac{1}{k^2} = \frac{266381}{3600} = 74.07805556$.

We now make a non-standard assumption: the number of prizes awarded will be very close to its mean value i.e. the standard deviation is much smaller than the mean. In this case, the number of prizes awarded will be $\frac{100{,}000}{E[T]} = 5509.64$.

We can **approximate** the variance of $\overline{T}$ as

$$Var[\overline{T}] \approx \frac{Var[T]}{N_T} = \frac{74.07805556}{5509.64} = 0.01343.$$

This is only an approximation because we do not have certainty in the value of N_T. The standard deviation of $\overline{T}$ is then $\sqrt{Var[\overline{T}]} = 0.115888$. This verifies our assumption that the standard deviation is very small, and therefore it is very unlikely that the variation in N_T would affect our estimate of the sample mean's variance by very much.

Our limiting prize value is then

$Q < 0.4(\mu + z\sigma) = 0.4(18.15 - 1.64485 \times 0.11588)$

which gives $Q_{max} = 7.183709505... = £7.18$.

ii) Possible reasons for differing outcomes are:

- Not all buyers may be trying to complete their collections. Casual buyers who buy less than 7 bottles will not have a chance and will never win a prize. This would increase the expected profits and result in a required increase to Q.
- Buyers may be cooperating or trading in order to help complete each others' collections. This would decrease the expected profits and result in a required decrease to Q.
- If bottles are being sold at different shops, there may be unequal distributions of the different singers at each shop, which could affect the probability of completing a collection if a buyer only shops at one place.

G4

a. i) All probabilities must add to 1, so

$$(53 - 1)k + (53 - 2)k + (53 - 3)k + \dots (53 - 52)k = 1$$

$$k + 2k + 3k + \dots 52k = 1$$

$$1378k = 1 \qquad \left(\sum_{k=1}^{n} k = \tfrac{1}{2}n(n + 1)\right)$$

$$k = \tfrac{1}{1378}.$$

ii) $$E[N] = \frac{1}{1378}\sum_{n=1}^{52} n(53 - n) = \frac{24804}{1378} = 18.$$

iii) $$Var[N] = E[N^2] - E[N]^2 = \frac{1}{1378}\sum_{n=1}^{52}\left(n^2(53 - n)\right) - 18^2$$

$$= 477 - 324 = 153.$$

iv) $$\frac{1}{1378}\sum_{n=1}^{m}(53 - n) = \frac{1}{2} \quad \Rightarrow \quad \sum_{n=1}^{m}(53 - n) = 689$$

$$\Rightarrow \quad 53m - \tfrac{1}{2}m(m + 1) = 689 \quad \Rightarrow \quad \tfrac{1}{2}m^2 - \tfrac{105}{2}m + 689 = 0$$

$\Rightarrow \quad m = 89.625$ (positive solution)

The median is defined to be halfway between the lower and upper integer values when the probabilities cannot be exactly 0.5, so the median is 89.5.

b. i) If $X = x$, then the values of N which could map to this value are

$N = \{x + 1,\ x + 1 + 13,\ x + 1 + 26,\ x + 1 + 39\}$.

$N = \{x + 1,\ x + 14,\ x + 27,\ x + 40\}$.

(For example, if $X = 3$, then N could have been 4, 17, 30 or 43 because subtracting 1 from these numbers and dividing by 13 gives a remainder 3.)

Therefore, $P(X = x) = P(N = x + 1) + P(N = x + 14)$
$+ P(N = x + 27) + P(N = x + 40)$.

Using the given probability mass function for N,

$P(X = x) = \frac{1}{1378}(53 - (x + 1) + 53 - (x + 14)$
$+ 53 - (x + 27) + 53 - (x + 40))$.

$P(X = x) = \frac{5}{53} - \frac{2}{689}x$ for all integers $0 \le x \le 12$ (otherwise 0).

ii) Positively skewed (probability decreases with larger x).

iii) The possible values and probabilities of N for $X = 10$ are

$P(N = 11) = \frac{21}{689}$, $P(N = 24) = \frac{29}{1378}$, $P(N = 37) = \frac{8}{689}$, $P(N = 50) = \frac{3}{1378}$.

Therefore $P(N = 37 \mid X = 10) = \frac{8/689}{21/689 + 29/1378 + 8/689 + 3/1378} = \frac{8}{45}$.

c. Perform a large number of trials, recording the value of N at each trial.

Plot the number of occurrences of each value of N against N on a scatter plot.

Find the equation of the line of best fit through the points.

Convert this equation to a probability mass function as given.

A29.

$$\mathbf{r} = \begin{bmatrix} 5/3 \\ 0 \\ 5/3 \end{bmatrix} + \cos\theta \begin{bmatrix} -10/3 \\ 0 \\ 5/3 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 \\ 5/\sqrt{3} \\ 0 \end{bmatrix}, \qquad 0 \le \theta < 2\pi$$

A31. Video solution: 미통기 확률 26번

A32. Since the endpoint of each trial depends on its own outcome, this is a geometric-distribution-like problem and should be analysed by separating the last draw from all previous draws.

Let $n = 50$ and $r = 20$. Let X be a random variable for the total number of balls chosen, so that r of them are red and $X - r$ are white. There will be no more white balls than red balls if $X \le 2r$, so we need to find out $\mathrm{P}(X \le 2r)$. Since we must have $r \le X \le n$, this probability is also $\mathrm{P}(r \le X \le 2r)$ (since in our specific case, $n > 2r$).

$\mathrm{P}(X = x) = \mathrm{P}$(draw r - 1 red in x - 1 draws) × P(draw red on the last draw)

$$= \frac{\#(choose\ x{-}r\ from\ n{-}r\ [whites]) \times \#(choose\ r{-}1\ from\ r\ [reds])}{\#(choose\ x{-}1\ from\ n\ [total])} \times \frac{1\ remaining\ red}{n{-}(x{-}1)\ remaining\ balls}$$

$$= \frac{{}^{n-r}C_{x-r} \times {}^{r}C_{r-1}}{{}^{n}C_{x-1}} \times \frac{1}{n-x+1}$$

$$= \frac{{}^{n-r}C_{x-r}\, r}{{}^{n}C_{x-1}(n-x+1)}$$

So our required probability is (use calculator summation operator)

$$\mathrm{P}(r \le X \le 2r) = \sum_{x=r}^{2r} \frac{{}^{n-r}C_{x-r}\, r}{{}^{n}C_{x-1}(n-x+1)} = r \sum_{x=r}^{2r} \frac{{}^{n-r}C_{x-r}}{{}^{n}C_{x-1}(n-x+1)}$$

$$= 20 \sum_{x=20}^{40} \frac{{}^{30}C_{x-20}}{{}^{50}C_{x-1}\,(51-x)} = 2.924864\ldots \times 10^{-3} = 0.002925. \quad (③)$$

(The distribution of X is known as the negative hypergeometric distribution.)

A34. Refer to the three diagrams to help with the solution:

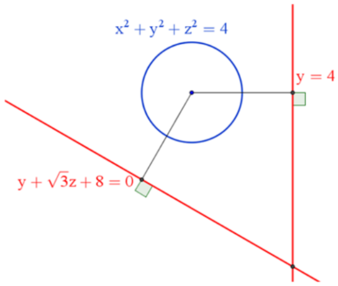


Figure 1

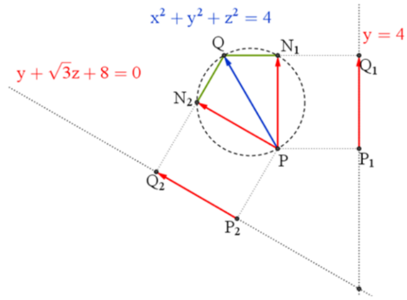


Figure 2

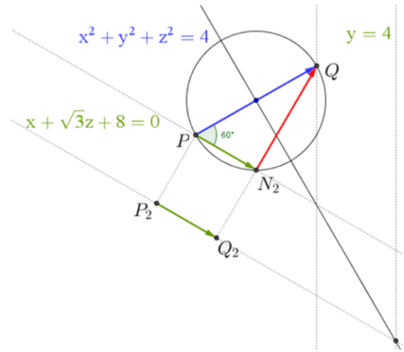


Figure 3

The unit normal vectors for each plane are $(y = 4)$: $\mathbf{n}_1 = \mathbf{j}$; and

$(y + \sqrt{3}z + 8 = 0)$: $\frac{1}{\sqrt{1^2 + (\sqrt{3})^2}}(\mathbf{j} + \sqrt{3}\mathbf{k}) = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$.

The distances between the planes and the centre of the sphere are then

$(y = 4)$: $d_1 = \frac{|4|}{\sqrt{1^2}} = 4$ (obvious); and $(y + \sqrt{3}z + 8 = 0)$: $d_2 = \frac{|-8|}{\sqrt{1^2 + (\sqrt{3})^2}} = 4$.

So the two planes are equidistant to the sphere, so there is symmetry.

The two planes intersect at an angle $\cos\theta = (\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}) \cdot \mathbf{j} = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3} = 60^\circ$.

Consider viewing the problem in the plane of the two plane normals as in **Figure 1.**

We require the maximum value of $2\,|PQ|^2 - |P_1Q_1|^2 - |P_2Q_2|^2$. Intuitively, this will be largest when the distance PQ is largest (i.e. P and Q are diametrically opposite) while the projection lengths P_1Q_1 and P_2Q_2 are smallest. If these two occur simultaneously, then the optimal orientation is found. We will prove that this is in fact the case:

Assume initially that P and Q are diametrically opposite (**Figure 2**). Then:

$= 2\,|PQ|^2 - |P_1Q_1|^2 - |P_2Q_2|^2 = (|PQ|^2 - |P_1Q_1|^2) + (|PQ|^2 - |P_2Q_2|^2)$

$= |N_1Q|^2 + |N_2Q|^2$ (i.e. the sum of the green lengths).

By the semicircle theorem (Thales' theorem), PQN_1 and PQN_2 are congruent right-triangles. Therefore, the lengths are constrained via Pythagoras' theorem. Since $(\text{green lengths})^2 + (\text{projection lengths})^2 = \text{constant} = \text{diameter}^2$, we can see that maximising PQ implies minimising P_1Q_1 and P_2Q_2, so this is indeed optimal.

By trigonometry (**Figure 3**), the required value is then $2 \times \left(4\,sin\,\frac{\pi}{3}\right)^2 = 24.$ (④)

Images and solution by 김우섭: https://m.blog.naver.com/sodong212/130179803559