DIFFERENTIAL EQUATIONS

Higher Order Differential Equations

Paul Dawkins

Table of Contents

Chapter 7 : Higher Order Differential Equations

In this chapter we're going to take a look at higher order differential equations. This chapter will actually contain more than most text books tend to have when they discuss higher order differential equations.

We will definitely cover the same material that most text books do here. However, in all the previous chapters all of our examples were 2nd order differential equations or 2 x 2 systems of differential equations. So, in this chapter we're also going to do a couple of examples here dealing with 3rd order or higher differential equations with Laplace transforms and series as well as a discussion of some larger systems of differential equations.

Here is a brief listing of the topics in this chapter.

Basic Concepts for *n*th Order Linear Equations – In this section we'll start the chapter off with a quick look at some of the basic ideas behind solving higher order linear differential equations. Included will be updated definitions/facts for the Principle of Superposition, linearly independent functions and the Wronskian.

Linear Homogeneous Differential Equations – In this section we will extend the ideas behind solving 2nd order, linear, homogeneous differential equations to higher order. As we'll most of the process is identical with a few natural extensions to repeated real roots that occur more than twice. We will also need to discuss how to deal with repeated complex roots, which are now a possibility. In addition, we will see that the main difficulty in the higher order cases is simply finding all the roots of the characteristic polynomial.

Undetermined Coefficients – In this section we work a quick example to illustrate that using undetermined coefficients on higher order differential equations is no different that when we used it on 2nd order differential equations with only one small natural extension.

Variation of Parameters – In this section we will give a detailed discussion of the process for using variation of parameters for higher order differential equations. We will also develop a formula that can be used in these cases. We will also see that the work involved in using variation of parameters on higher order differential equations can be quite involved on occasion.

Laplace Transforms – In this section we will work a quick example using Laplace transforms to solve a differential equation on a 3rd order differential equation just to say that we looked at one with order higher than 2nd. As we'll see, outside of needing a formula for the Laplace transform of y''', which we can get from the general formula, there is no real difference in how Laplace transforms are used for higher order differential equations.

Systems of Differential Equations – In this section we'll take a quick look at extending the ideas we discussed for solving 2 x 2 systems of differential equations to systems of size 3 x 3. As we will see they are mostly just natural extensions of what we already know who to do. We will also make a couple of quick comments about 4 x 4 systems.

Series Solutions –In this section we are going to work a quick example illustrating that the process of finding series solutions for higher order differential equations is pretty much the same as that used on 2^{nd} order differential equations.

Section 7-1 : Basic Concepts for n^{th} Order Linear Equations

We'll start this chapter off with the material that most text books will cover in this chapter. We will take the material from the Second Order chapter and expand it out to n^{th} order linear differential equations. As we'll see almost all of the 2nd order material will very naturally extend out to n^{th} order with only a little bit of new material.

So, let's start things off here with some basic concepts for n^{th} order linear differential equations. The most general n^{th} order linear differential equation is,

$$P_n(t)\,y^{(n)} + P_{n-1}(t)\,y^{(n-1)} + \cdots + P_1(t)\,y' + P_0(t)\,y = G(t) \qquad \textbf{(1)}$$

where you'll hopefully recall that,

$$y^{(m)} = \frac{d^m y}{dx^m}$$

Many of the theorems and ideas for this material require that $y^{(n)}$ has a coefficient of 1 and so if we divide out by $P_n(t)$ we get,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = g(t) \qquad \textbf{(2)}$$

As we might suspect an IVP for an n^{th} order differential equation will require the following n initial conditions.

$$y(t_0) = \overline{y}_0, \qquad y'(t_0) = \overline{y}_1, \qquad \cdots, \qquad y^{(n-1)}(t_0) = \overline{y}_{n-1} \qquad \textbf{(3)}$$

The following theorem tells us when we can expect there to be a unique solution to the IVP given by (2) and (3).

Theorem 1

Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ and $g(t)$ are all continuous in some open interval *I* containing t_0 then there is a unique solution to the IVP given by (2) and (3) and the solution will exist for all *t* in *I*.

This theorem is a very natural extension of a similar theorem we saw in the 1st order material.

Next we need to move into a discussion of the n^{th} order linear homogeneous differential equation,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = 0 \qquad \textbf{(4)}$$

Let's suppose that we know $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions to (4) then by the an extension of the Principle of Superposition we know that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

will also be a solution to (4). The real question here is whether or not this will form a general solution to (4).

In order for this to be a general solution then we will have to be able to find constants $c_1, c_2, \ldots, c_n$ for any choice of t_0 (in the interval *I* from Theorem 1) and any choice of $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$. Or, in other words we need to be able to find $c_1, c_2, \ldots, c_n$ that will solve,

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) &= \bar{y}_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) &= \bar{y}_1 \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= \bar{y}_{n-1} \end{aligned}$$

Just as we did for 2nd order differential equations, we can use Cramer's Rule to solve this and the denominator of each the answers will be the following determinant of an *n* x *n* matrix.

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

As we did back with the 2nd order material we'll define this to be the **Wronskian** and denote it by,

$$W(y_1, y_2, \ldots y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Now that we have the definition of the Wronskian out of the way we need to get back to the question at hand. Because the Wronskian is the denominator in the solution to each of the c_i we can see that we'll have a solution provided it is not zero for any value of $t = t_0$ that we chose to evaluate the Wronskian at. The following theorem summarizes all this up.

Theorem 2

Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ are all continuous on the open interval *I* and further suppose that $y_1(t), y_2(t), \ldots y_n(t)$ are all solutions to (4). If $W(y_1, y_2, \ldots y_n)(t) \neq 0$ for every *t* in *I* then $y_1(t), y_2(t), \ldots y_n(t)$ form a **Fundamental Set of Solutions** and the general solution to (4) is,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Recall as well that if a set of solutions form a fundamental set of solutions then they will also be a set of linearly independent functions.

We'll close this section off with a quick reminder of how we find solutions to the nonhomogeneous differential equation, (2). We first need the *n*th order version of a theorem we saw back in the 2nd order material.

Theorem 3

Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to (2) and that $y_1(t), y_2(t), \ldots y_n(t)$ are a fundamental set of solutions to the homogeneous differential equation (4) then,

$$Y_1(t) - Y_2(t)$$

is a solution to (4) and it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Now, just as we did with the 2nd order material if we let $Y(t)$ be the general solution to (2) and if we let $Y_P(t)$ be any solution to (2) then using the result of this theorem we see that we must have,

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y_P(t) = y_c(t) + Y_P(t)$$

where, $y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ is called the **complementary solution** and $Y_P(t)$ is called a **particular solution**.

Over the course of the next couple of sections we'll discuss the differences in finding the complementary and particular solutions for n^{th} order differential equations in relation to what we know about 2nd order differential equations. We'll see that, for the most part, the methods are the same. The amount of work involved however will often be significantly more.

Section 7-2 : Linear Homogeneous Differential Equations

As with 2nd order differential equations we can't solve a nonhomogeneous differential equation unless we can first solve the homogeneous differential equation. We'll also need to restrict ourselves down to constant coefficient differential equations as solving non-constant coefficient differential equations is quite difficult and so we won't be discussing them here. Likewise, we'll only be looking at linear differential equations.

So, let's start off with the following differential equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Now, assume that solutions to this differential equation will be in the form $y(t) = \mathbf{e}^{rt}$ and plug this into the differential equation and with a little simplification we get,

$$\mathbf{e}^{rt}\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right) = 0$$

and so in order for this to be zero we'll need to require that

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

This is called the **characteristic polynomial/equation** and its roots/solutions will give us the solutions to the differential equation. We know that, including repeated roots, an n^{th} degree polynomial (which we have here) will have *n* roots. So, we need to go through all the possibilities that we've got for roots here.

This is where we start to see differences in how we deal with n^{th} order differential equations versus 2nd order differential equations. There are still the three main cases: real distinct roots, repeated roots and complex roots (although these can now also be repeated as we'll see). In 2nd order differential equations each differential equation could only involve one of these cases. Now, however, that will not necessarily be the case. We could very easily have differential equations that contain each of these cases.

For instance, suppose that we have an 9th order differential equation. The complete list of roots could have 3 roots which only occur once in the list (*i.e.* real distinct roots), a root with multiplicity 4 (*i.e.* occurs 4 times in the list) and a set of complex conjugate roots (recall that because the coefficients are all real complex roots will always occur in conjugate pairs).

So, for each n^{th} order differential equation we'll need to form a set of *n* linearly independent functions (*i.e.* a fundamental set of solutions) in order to get a general solution. In the work that follows we'll discuss the solutions that we get from each case but we will leave it to you to verify that when we put everything together to form a general solution that we do indeed get a fundamental set of solutions. Recall that in order to this we need to verify that the Wronskian is not zero.

So, let's get started with the work here. Let's start off by assuming that in the list of roots of the characteristic equation we have $r_1, r_2, \ldots, r_k$ and they only occur once in the list. The solution from each of these will then be,

$$\mathbf{e}^{r_1 t}, \quad \mathbf{e}^{r_2 t}, \quad \cdots, \quad \mathbf{e}^{r_k t}$$

There's nothing really new here for real distinct roots.

Now let's take a look at repeated roots. The result here is a natural extension of the work we saw in the 2nd order case. Let's suppose that r is a root of multiplicity k (*i.e.* r occurs k times in the list of roots). We will then get the following k solutions to the differential equation,

$$\mathbf{e}^{rt}, \qquad t\,\mathbf{e}^{rt}, \qquad \cdots, \qquad t^{k-1}\mathbf{e}^{rt}$$

So, for repeated roots we just add in a t for each of the solutions past the first one until we have a total of k solutions. Again, we will leave it to you to compute the Wronskian to verify that these are in fact a set of linearly independent solutions.

Finally, we need to deal with complex roots. The biggest issue here is that we can now have repeated complex roots for 4th order or higher differential equations. We'll start off by assuming that $r = \lambda \pm \mu\, i$ occurs only once in the list of roots. In this case we'll get the standard two solutions,

$$\mathbf{e}^{\lambda t}\cos\left(\mu\, t\right) \qquad \mathbf{e}^{\lambda t}\sin\left(\mu\, t\right)$$

Now let's suppose that $r = \lambda \pm \mu\, i$ has a multiplicity of k (*i.e.* they occur k times in the list of roots). In this case we can use the work from the repeated roots above to get the following set of $2k$ complex-valued solutions,

$$\begin{array}{llll} \mathbf{e}^{(\lambda+\mu i)t}, t\,\mathbf{e}^{(\lambda+\mu i)t}, & \cdots, & t^{k-1}\mathbf{e}^{(\lambda+\mu i)t} \\ \mathbf{e}^{(\lambda-\mu i)t}, t\,\mathbf{e}^{(\lambda-\mu i)t}, & \cdots, & t^{k-1}\mathbf{e}^{(\lambda-\mu i)t} \end{array}$$

The problem here of course is that we really want real-valued solutions. So, recall that in the case where they occurred once all we had to do was use Euler's formula on the first one and then take the real and imaginary part to get two real valued solutions. We'll do the same thing here and use Euler's formula on the first set of complex-valued solutions above, split each one into its real and imaginary parts to arrive at the following set of $2k$ real-valued solutions.

$$\begin{array}{l} \mathbf{e}^{\lambda t}\cos\left(\mu\, t\right), \quad \mathbf{e}^{\lambda t}\sin\left(\mu\, t\right), \quad t\mathbf{e}^{\lambda t}\cos\left(\mu\, t\right), \quad t\mathbf{e}^{\lambda t}\sin\left(\mu\, t\right), \quad \cdots, \\ \qquad\qquad\qquad\qquad\qquad\qquad t^{k-1}\mathbf{e}^{\lambda t}\cos\left(\mu\, t\right), \quad t^{k-1}\mathbf{e}^{\lambda t}\sin\left(\mu\, t\right) \end{array}$$

Once again, we'll leave it to you to verify that these do in fact form a fundamental set of solutions.

Before we work a couple of quick examples here we should point out that the characteristic polynomial is now going to be at least a 3rd degree polynomial and finding the roots of these by hand is often a very difficult and time consuming process and in many cases if the roots are not rational (*i.e.* in the form $\frac{p}{q}$) it can be almost impossible to find them all by hand. To see a process for determining all the rational roots of a polynomial check out the Finding Zeroes of Polynomials page in the Algebra notes. In practice however, we usually use some form of computation aid such as Maple or Mathematica to find all the roots.

So, let's work a couple of example here to illustrate at least some of the ideas discussed here.

Example 1 Solve the following IVP.

$$y^{(3)} - 5y'' - 22y' + 56y = 0 \qquad y(0) = 1 \quad y'(0) = -2 \quad y''(0) = -4$$

Solution

The characteristic equation is,

$$r^3 - 5r^2 - 22r + 56 = (r+4)(r-2)(r-7) = 0 \qquad \Rightarrow \qquad r_1 = -4,\ r_2 = 2,\ r_3 = 7$$

So, we have three real distinct roots here and so the general solution is,

$$y(t) = c_1\mathbf{e}^{-4t} + c_2\mathbf{e}^{2t} + c_3\mathbf{e}^{7t}$$

Differentiating a couple of times and applying the initial conditions gives the following system of equations that we'll need to solve in order to find the coefficients.

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 \\ -2 &= y'(0) = -4c_1 + 2c_2 + 7c_3 \\ -4 &= y''(0) = 16c_1 + 4c_2 + 49c_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= \tfrac{14}{33} \\ c_2 &= \tfrac{13}{15} \\ c_3 &= -\tfrac{16}{55} \end{aligned}$$

The actual solution is then,

$$y(t) = \tfrac{14}{33}\mathbf{e}^{-4t} + \tfrac{13}{15}\mathbf{e}^{2t} - \tfrac{16}{55}\mathbf{e}^{7t}$$

So, outside of needing to solve a cubic polynomial (which we left the details to you to verify) and needing to solve a system of 3 equations to find the coefficients (which we also left to you to fill in the details) the work here is pretty much identical to the work we did in solving a 2nd order IVP.

Because the initial condition work is identical to work that we should be very familiar with to this point with the exception that it involved solving larger systems we're going to not bother with solving IVP's for the rest of the examples. The main point of this section is the new ideas involved in finding the general solution to the differential equation anyway and so we'll concentrate on that for the remaining examples.

Also note that we'll not be showing very much work in solving the characteristic polynomial. We are using computational aids here and would encourage you to do the same here. Solving these higher degree polynomials is just too much work and would obscure the point of these examples.

So, let's move into a couple of examples where we have more than one case involved in the solution.

Example 2 Solve the following differential equation.

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

Solution

The characteristic equation is,

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r-1)(r+2)^3 = 0$$

So, we have two roots here, $r_1 = \frac{1}{2}$ and $r_2 = -2$ which is multiplicity of 3. Remember that we'll get three solutions for the second root and after the first one we add *t*'s only the solution until we reach three solutions.

The general solution is then,

$$y(t) = c_1\mathbf{e}^{\frac{1}{2}t} + c_2\mathbf{e}^{-2t} + c_3t\mathbf{e}^{-2t} + c_4t^2\mathbf{e}^{-2t}$$

Example 3 Solve the following differential equation.

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

Solution

The characteristic equation is,

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = r\left(r^2 + 6r + 34\right)^2 = 0$$

So, we have one real root $r = 0$ and a pair of complex roots $r = -3 \pm 5i$ each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions, 2 will be the *normal* solutions and two will be the normal solution each multiplied by t.

The general solution is,

$$y(t) = c_1 + c_2\mathbf{e}^{-3t}\cos(5t) + c_3\mathbf{e}^{-3t}\sin(5t) + c_4\,t\,\mathbf{e}^{-3t}\cos(5t) + c_5\,t\,\mathbf{e}^{-3t}\sin(5t)$$

Let's now work an example that contains all three of the basic cases just to say that we that we've got one work here.

Example 4 Solve the following differential equation.

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

Solution

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2\left(r^2 - 4r + 5\right) = 0$$

In this case we've got one real distinct root, $r = 1$, and double root, $r = 5$, and a pair of complex roots, $r = 2 \pm i$ that only occur once.

The general solution is then,

$$y(t) = c_1\mathbf{e}^{t} + c_2\mathbf{e}^{5t} + c_3\,t\,\mathbf{e}^{5t} + c_4\mathbf{e}^{2t}\cos(t) + c_5\mathbf{e}^{2t}\sin(t)$$

We've got one final example to work here that on the surface at least seems almost too easy. The problem here will be finding the roots as well see.

Example 5 Solve the following differential equation.

$$y^{(4)} + 16y = 0$$

Solution

The characteristic equation is

$$r^4 + 16 = 0$$

So, a really simple characteristic equation. However, in order to find the roots we need to compute the fourth root of -16 and that is something that most people haven't done at this point in their mathematical career. We'll just give the formula here for finding them, but if you're interested in seeing a little more about this you might want to check out the Powers and Roots section of the Complex Numbers Primer.

The 4 (and yes there are 4!) 4th roots of -16 can be found by evaluating the following,

$$\sqrt[4]{-16} = \left(-16\right)^{\frac{1}{4}} = \sqrt[4]{16}\mathbf{e}^{\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)i} = 2\left(\cos\left(\tfrac{\pi}{4}+\tfrac{\pi k}{2}\right)+i\sin\left(\tfrac{\pi}{4}+\tfrac{\pi k}{2}\right)\right) \qquad k = 0,1,2,3$$

Note that each value of *k* will give a distinct 4th root of -16. Also, note that for the 4th root (and ONLY the 4th root) of any negative number all we need to do is replace the 16 in the above formula with the absolute value of the number in question and this formula will work for those as well.

Here are the 4th roots of -16.

$$k=0:\quad 2\left(\cos\left(\tfrac{\pi}{4}\right)+i\sin\left(\tfrac{\pi}{4}\right)\right) = 2\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right) = \sqrt{2}+i\sqrt{2}$$

$$k=1:\quad 2\left(\cos\left(\tfrac{3\pi}{4}\right)+i\sin\left(\tfrac{3\pi}{4}+\tfrac{\pi k}{2}\right)\right) = 2\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right) = -\sqrt{2}+i\sqrt{2}$$

$$k=2:\quad 2\left(\cos\left(\tfrac{5\pi}{4}\right)+i\sin\left(\tfrac{5\pi}{4}\right)\right) = 2\left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = -\sqrt{2}-i\sqrt{2}$$

$$k=3:\quad 2\left(\cos\left(\tfrac{7\pi}{4}\right)+i\sin\left(\tfrac{7\pi}{4}\right)\right) = 2\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = \sqrt{2}-i\sqrt{2}$$

So, we have two sets of complex roots : $r = \sqrt{2} \pm i\sqrt{2}$ and $r = -\sqrt{2} \pm i\sqrt{2}$. The general solution is,

$$y\left(t\right) = c_1\mathbf{e}^{\sqrt{2}t}\cos\left(\sqrt{2}\,t\right) + c_2\mathbf{e}^{\sqrt{2}t}\sin\left(\sqrt{2}\,t\right) + c_3\mathbf{e}^{-\sqrt{2}t}\cos\left(\sqrt{2}\,t\right) + c_4\mathbf{e}^{-\sqrt{2}t}\sin\left(\sqrt{2}\,t\right)$$

So, we've worked a handful of examples here of higher order differential equations that should give you a feel for how these work in most cases.

There are of course a great many different kinds of combinations of the basic cases than what we did here and of course we didn't work any case involving 6th order or higher, but once you've got an idea on how these work it's pretty easy to see that they all work pretty in pretty much the same manner. The biggest problem with the higher order differential equations is that the work in solving the characteristic polynomial and the system for the coefficients on the solution can be quite involved.

Section 7-3 : Undetermined Coefficients

We now need to start looking into determining a particular solution for n^{th} order differential equations. The two methods that we'll be looking at are the same as those that we looked at in the 2^{nd} order chapter.

In this section we'll look at the method of Undetermined Coefficients and this will be a fairly short section. With one small extension, which we'll see in the lone example in this section, the method is identical to what we saw back when we were looking at undetermined coefficients in the 2^{nd} order differential equations chapter.

Given the differential equation,

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = g(t)$$

if $g(t)$ is an exponential function, polynomial, sine, cosine, sum/difference of one of these and/or a product of one of these then we guess the form of a particular solution using the same guidelines that we used in the 2^{nd} order material. We then plug the guess into the differential equation, simplify and set the coefficients equal to solve for the constants.

The one thing that we need to recall is that we first need the complementary solution prior to making our guess for a particular solution. If any term in our guess is in the complementary solution then we need to multiply the portion of our guess that contains that term by a *t*. This is where the one extension to the method comes into play. With a 2^{nd} order differential equation the most we'd ever need to multiply by is t^2. With higher order differential equations this may need to be more than t^2.

The work involved here is almost identical to the work we've already done and in fact it isn't even that much more difficult unless the guess is particularly messy and that makes for more mess when we take the derivatives and solve for the coefficients. Because there isn't much difference in the work here we're only going to do a single example in this section illustrating the extension. So, let's take a look at the lone example we're going to do here.

Example 1 Solve the following differential equation.

$$y^{(3)} - 12y'' + 48y' - 64y = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$$

Solution

We first need the complementary solution so the characteristic equation is,

$$r^3 - 12r^2 + 48r - 64 = (r-4)^3 = 0 \qquad \Rightarrow \qquad r = 4 \ (\text{multiplicity } 3)$$

We've got a single root of multiplicity 3 so the complementary solution is,

$$y_c(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t}$$

Now, our first guess for a particular solution is,

$$Y_P = A + B\mathbf{e}^{-8t} + C\mathbf{e}^{4t}$$

Notice that the last term in our guess is in the complementary solution so we'll need to add one at least one t to the third term in our guess. Also notice that multiplying the third term by either t or t^2 will result in a new term that is still in the complementary solution and so we'll need to multiply the third term by t^3 in order to get a term that is not contained in the complementary solution.

Our final guess is then,

$$Y_P = A + B\mathbf{e}^{-8t} + Ct^3\mathbf{e}^{4t}$$

Now all we need to do is take three derivatives of this, plug this into the differential equation and simplify to get (we'll leave it to you to verify the work here),

$$-64A - 1728B\mathbf{e}^{-8t} + 6C\mathbf{e}^{4t} = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$$

Setting coefficients equal and solving gives,

$$\begin{array}{rrcr} t^0: & -64A = 12 & & A = -\tfrac{3}{16} \\ \mathbf{e}^{-8t}: & -1728B = -32 & \Rightarrow & B = \tfrac{1}{54} \\ \mathbf{e}^{4t}: & 6C = 2 & & C = \tfrac{1}{3} \end{array}$$

A particular solution is then,

$$Y_P = -\tfrac{3}{16} + \tfrac{1}{54}\mathbf{e}^{-8t} + \tfrac{1}{3}t^3\mathbf{e}^{4t}$$

The general solution to this differential equation is then,

$$y(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t} - \tfrac{3}{16} + \tfrac{1}{54}\mathbf{e}^{-8t} + \tfrac{1}{3}t^3\mathbf{e}^{4t}$$

Okay, we've only worked one example here, but remember that we mentioned earlier that with the exception of the extension to the method that we used in this example the work here is identical to work we did the 2nd order material.

Section 7-4 : Variation of Parameters

We now need to take a look at the second method of determining a particular solution to a differential equation. As we did when we first saw Variation of Parameters we'll go through the whole process and derive up a set of formulas that can be used to generate a particular solution.

However, as we saw previously when looking at 2nd order differential equations this method can lead to integrals that are not easy to evaluate. So, while this method can always be used, unlike Undetermined Coefficients, to at least write down a formula for a particular solution it is not always going to be possible to actually get a solution.

So let's get started on the process. We'll start with the differential equation,

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = g(t) \qquad \textbf{(1)}$$

and assume that we've found a fundamental set of solutions, $y_1(t), y_2(t), \ldots, y_n(t)$, for the associated homogeneous differential equation.

Because we have a fundamental set of solutions to the homogeneous differential equation we now know that the complementary solution is,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

The method of variation of parameters involves trying to find a set of new functions, $u_1(t), u_2(t), \ldots, u_n(t)$ so that,

$$Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t) + \cdots + u_n(t)\, y_n(t) \qquad \textbf{(2)}$$

will be a solution to the nonhomogeneous differential equation. In order to determine if this is possible, and to find the $u_i(t)$ if it is possible, we'll need a total of *n* equations involving the unknown functions that we can (hopefully) solve.

One of the equations is easy. The guess, (2), will need to satisfy the original differential equation, (1). So, let's start taking some derivatives and as we did when we first looked at variation of parameters we'll make some assumptions along the way that will simplify our work and in the process generate the remaining equations we'll need.

The first derivative of (2) is,

$$Y'(t) = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n' + u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n$$

Note that we rearranged the results of the differentiation process a little here and we dropped the (t) part on the *u* and *y* to make this a little easier to read. Now, if we keep differentiating this it will quickly become unwieldy and so let's make as assumption to simplify things here. Because we are after the $u_i(t)$ we should probably try to avoid letting the derivatives on these become too large. So, let's make the assumption that,

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0$$

The natural question at this point is does this even make sense to do? The answer is, if we end up with a system of *n* equations that we can solve for the $u_i\left(t\right)$ then yes it does make sense to do. Of course, the other answer is, we wouldn't be making this assumption if we didn't know that it was going to work. However, to accept this answer requires that you trust us to make the correct assumptions so maybe the first answer is the best at this point.

At this point the first derivative of (2) is,

$$Y'\left(t\right) = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'$$

and so we can now take the second derivative to get,

$$Y''\left(t\right) = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n'' + u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'$$

This looks an awful lot like the original first derivative prior to us simplifying it so let's again make a simplification. We'll again want to keep the derivatives on the $u_i\left(t\right)$ to a minimum so this time let's assume that,

$$u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0$$

and with this assumption the second derivative becomes,

$$Y''\left(t\right) = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n''$$

Hopefully you're starting to see a pattern develop here. If we continue this process for the first $n-1$ derivatives we will arrive at the following formula for these derivatives.

$$Y^{(k)}\left(t\right) = u_1 y_1^{(k)} + u_2 y_2^{(k)} + \cdots + u_n y_n^{(k)} \qquad k = 1, 2, \ldots, n-1 \qquad \textbf{(3)}$$

To get to each of these formulas we also had to assume that,

$$u_1' y_1^{(k)} + u_2' y_2^{(k)} + \cdots + u_n' y_n^{(k)} = 0 \qquad k = 0, 1, \ldots n-2 \qquad \textbf{(4)}$$

and recall that the 0th derivative of a function is just the function itself. So, for example, $y_2^{(0)}\left(t\right) = y_2\left(t\right)$
.

Notice as well that the set of assumptions in (4) actually give us $n-1$ equations in terms of the derivatives of the unknown functions : $u_1\left(t\right), u_2\left(t\right), \ldots, u_n\left(t\right)$.

All we need to do then is finish generating the first equation we started this process to find (*i.e.* plugging (2) into (1)). To do this we'll need one more derivative of the guess. Differentiating the $\left(n-1\right)^{\text{st}}$ derivative, which we can get from (3), to get the *n*th derivative gives,

$$Y^{(n)}\left(t\right) = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}$$

This time we'll also not be making any assumptions to simplify this but instead just plug this along with the derivatives given in (3) into the differential equation, (1)

$$\begin{aligned}u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} + \\ p_{n-1}(t)\left[u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right] + \\ \vdots \qquad \\ p_1(t)\left[u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'\right] + \\ p_0(t)\left[u_1 y_1 + u_2 y_2 + \cdots + u_n y_n\right] = g(t)\end{aligned}$$

Next, rearrange this a little to get,

$$\begin{aligned}&u_1\left[y_1^{(n)} + p_{n-1}(t) y_1^{(n-1)} + \cdots + p_1(t) y_1' + p_0(t) y_1\right] + \\ &u_2\left[y_2^{(n)} + p_{n-1}(t) y_2^{(n-1)} + \cdots + p_1(t) y_2' + p_0(t) y_2\right] + \\ &\qquad \vdots \\ &u_n\left[y_n^{(n)} + p_{n-1}(t) y_n^{(n-1)} + \cdots + p_1(t) y_n' + p_0(t) y_n\right] + \\ &\qquad u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g(t)\end{aligned}$$

Recall that $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions to the homogeneous differential equation and so all the quantities in the $[\;]$ are zero and this reduces down to,

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g(t)$$

So, this equation, along with those given in (4), give us the *n* equations that we needed. Let's list them all out here for the sake of completeness.

$$\begin{aligned}u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n &= 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' &= 0 \\ u_1' y_1'' + u_2' y_2'' + \cdots + u_n' y_n'' &= 0 \\ &\vdots \\ u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} &= 0 \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= g(t)\end{aligned}$$

So, we've got *n* equations, but notice that just like we got when we did this for 2nd order differential equations the unknowns in the system are not $u_1(t), u_2(t), \ldots, u_n(t)$ but instead they are the derivatives, $u_1'(t), u_2'(t), \ldots, u_n'(t)$. This isn't a major problem however. Provided we can solve this system we can then just integrate the solutions to get the functions that we're after.

Also, recall that the $y_1(t), y_2(t), \ldots, y_n(t)$ are assumed to be known functions and so they along with their derivatives (which appear in the system) are all known quantities in the system.

Now, we need to think about how to solve this system. If there aren't too many equations we can just solve it directly if we want to. However, for large *n* (and it won't take much to get large here) that could be quite tedious and prone to error and it won't work at all for general *n* as we have here.

The best solution method to use at this point is then Cramer's Rule. We've used Cramer's Rule several times in this course, but the best reference for our purposes here is when we used it when we first defined [Fundamental Sets of Solutions] back in the 2nd order material.

Upon using Cramer's Rule to solve the system the resulting solution for each u_i' will be a quotient of two determinants of *n* x *n* matrices. The denominator of each solution will be the determinant of the matrix of the known coefficients,

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W\left(y_1, y_2, \ldots y_n\right)\left(t\right)$$

This however, is just the Wronskian of $y_1\left(t\right), y_2\left(t\right), \ldots, y_n\left(t\right)$ as noted above and because we have assumed that these form a fundamental set of solutions we also know that the Wronskian will not be zero. This in turn tells us that the system above is in fact solvable and all of the assumptions we apparently made out of the blue above did in fact work.

The numerators of the solution for u_i' will be the determinant of the matrix of coefficients with the *i*th column replaced with the column $\left(0, 0, 0, \ldots, 0, g\left(t\right)\right)$. For example, the numerator for the first one, u_1' is,

$$\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ g\left(t\right) & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Now, by a nice property of determinants if we factor something out of one of the columns of a matrix then the determinant of the resulting matrix will be the factor times the determinant of new matrix. In other words, if we factor $g\left(t\right)$ out of this matrix we arrive at,

$$\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ g\left(t\right) & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = g\left(t\right)\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

We did this only for the first one, but we could just as easily done this with any of the *n* solutions. So, let W_i represent the determinant we get by replacing the *i*th column of the Wronskian with the column $\left(0,0,0,\ldots,0,1\right)$ and the solution to the system can then be written as,

$$u_1' = \frac{g\left(t\right)W_1\left(t\right)}{W\left(t\right)}, \quad u_2' = \frac{g\left(t\right)W_2\left(t\right)}{W\left(t\right)}, \quad \cdots, \quad u_n' = \frac{g\left(t\right)W_n\left(t\right)}{W\left(t\right)}$$

Wow! That was a lot of effort to generate and solve the system but we're almost there. With the solution to the system in hand we can now integrate each of these terms to determine just what the unknown functions, $u_1\left(t\right),u_2\left(t\right),\ldots,u_n\left(t\right)$ we've after all along are.

$$u_1 = \int \frac{g\left(t\right)W_1\left(t\right)}{W\left(t\right)}\,dt, \quad u_2 = \int \frac{g\left(t\right)W_2\left(t\right)}{W\left(t\right)}\,dt, \quad \cdots, \quad u_n = \int \frac{g\left(t\right)W_n\left(t\right)}{W\left(t\right)}\,dt$$

Finally, a particular solution to (1) is then given by,

$$Y\left(t\right) = y_1\left(t\right)\int \frac{g\left(t\right)W_1\left(t\right)}{W\left(t\right)}\,dt + y_2\left(t\right)\int \frac{g\left(t\right)W_2\left(t\right)}{W\left(t\right)}\,dt + \cdots + y_n\left(t\right)\int \frac{g\left(t\right)W_n\left(t\right)}{W\left(t\right)}\,dt$$

We should also note that in the derivation process here we assumed that the coefficient of the $y^{\left(n\right)}$ term was a one and that has been factored into the formula above. If the coefficient of this term is not one then we'll need to make sure and divide it out before trying to use this formula.

Before we work an example here we really should note that while we can write this formula down actually computing these integrals may be all but impossible to do.

Okay let's take a look at a quick example.

Example 1 Solve the following differential equation.

$$y^{(3)} - 2y'' - 21y' - 18y = 3 + 4\mathbf{e}^{-t}$$

Solution

The characteristic equation is,

$$r^3 - 2r^2 - 21r - 18 = \left(r+3\right)\left(r+1\right)\left(r-6\right) = 0 \qquad \Rightarrow \qquad r_1 = -3,\ r_2 = -1,\ r_3 = 6$$

So, we have three real distinct roots here and so the complimentary solution is,

$$y_c\left(t\right) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t}$$

Okay, we've now got several determinants to compute. We'll leave it to you to verify the following determinant computations.

$$W = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 126\mathbf{e}^{2t} \qquad W_1 = \begin{vmatrix} 0 & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ 0 & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 1 & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 7\mathbf{e}^{5t}$$

$$W_2 = \begin{vmatrix} \mathbf{e}^{-3t} & 0 & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & 0 & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & 1 & 36\mathbf{e}^{6t} \end{vmatrix} = -9\mathbf{e}^{3t} \qquad W_3 = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & 0 \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 0 \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 1 \end{vmatrix} = 2\mathbf{e}^{-4t}$$

Now, given that $g(t) = 3 + 4\mathbf{e}^{-t}$ we can compute each of the u_i. Here are those integrals.

$$u_1 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(7\mathbf{e}^{5t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{18}\int 3\mathbf{e}^{3t} + 4\mathbf{e}^{2t}\,dt = \frac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)$$

$$u_2 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(-9\mathbf{e}^{3t}\right)}{126\mathbf{e}^{2t}}\,dt = -\frac{1}{14}\int 3\mathbf{e}^{t} + 4\,dt = -\frac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)$$

$$u_3 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(2\mathbf{e}^{-4t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{63}\int 3\mathbf{e}^{-6t} + 4\mathbf{e}^{-7t}\,dt = \frac{1}{63}\left(-\frac{1}{2}\mathbf{e}^{-6t} - \frac{4}{7}\mathbf{e}^{-7t}\right)$$

Note that we didn't include the constants of integration in each of these because including them would just have introduced a term that would get absorbed into the complementary solution just as we saw when we were dealing with 2nd order differential equations.

Finally, a particular solution for this differential equation is then,

$$\begin{aligned} Y_P &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= \tfrac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)\mathbf{e}^{-3t} - \tfrac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)\mathbf{e}^{-t} + \tfrac{1}{63}\left(-\tfrac{1}{2}\mathbf{e}^{-6t} - \tfrac{4}{7}\mathbf{e}^{-7t}\right)\mathbf{e}^{6t} \\ &= -\tfrac{1}{6} + \tfrac{5}{49}\mathbf{e}^{-t} - \tfrac{2}{7}t\mathbf{e}^{-t} \end{aligned}$$

The general solution is then,

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t} - \tfrac{1}{6} + \tfrac{5}{49}\mathbf{e}^{-t} - \tfrac{2}{7}t\mathbf{e}^{-t}$$

We're only going to do a single example in this section to illustrate the process more than anything so with that we'll close out this section.

Section 7-5 : Laplace Transforms

There really isn't all that much to this section. All we're going to do here is work a quick example using Laplace transforms for a 3rd order differential equation so we can say that we worked at least one problem for a differential equation whose order was larger than 2.

Everything that we know from the Laplace Transforms chapter is still valid. The only new bit that we'll need here is the Laplace transform of the third derivative. We can get this from the general formula that we gave when we first started looking at solving IVP's with Laplace transforms. Here is that formula,

$$\mathcal{L}\{y'''\} = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

Here's the example for this section.

Example 1 Solve the following IVP.

$$y''' - 4y'' = 4t + 3u_6(t)\mathbf{e}^{30-5t}, \quad y(0) = -3 \;\; y'(0) = 1 \quad y''(0) = 4$$

Solution

As always, we first need to make sure the function multiplied by the Heaviside function has been properly shifted.

$$y''' - 4y'' = 4t + 3u_6(t)\mathbf{e}^{-5(t-6)}$$

It has been properly shifted and we can see that we're shifting $\mathbf{e}^{-5t}$. All we need to do now is take the Laplace transform of everything, plug in the initial conditions and solve for $Y(s)$. Doing all of this gives,

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 4\left(s^2Y(s) - sy(0) - y'(0)\right) = \frac{4}{s^2} + \frac{3\mathbf{e}^{-6s}}{s+5}$$

$$(s^3 - 4s^2)Y(s) + 3s^2 - 13s = \frac{4}{s^2} + \frac{3\mathbf{e}^{-6s}}{s+5}$$

$$(s^3 - 4s^2)Y(s) = \frac{4}{s^2} - 3s^2 + 13s + \frac{3\mathbf{e}^{-6s}}{s+5}$$

$$(s^3 - 4s^2)Y(s) = \frac{4 - 3s^4 + 13s^3}{s^2} + \frac{3\mathbf{e}^{-6s}}{s+5}$$

$$Y(s) = \frac{4 - 3s^4 + 13s^3}{s^4(s-4)} + \frac{3\mathbf{e}^{-6s}}{s^2(s-4)(s+5)}$$

$$Y(s) = F(s) + 3\mathbf{e}^{-6s}G(s)$$

Now we need to partial fraction and inverse transform *F(s)* and *G(s)*. We'll leave it to you to verify the details.

$$F\left(s\right)=\frac{4-3s^4+13s^3}{s^4\left(s-4\right)}=\frac{\frac{17}{64}}{s-4}-\frac{\frac{209}{64}}{s}-\frac{\frac{1}{16}}{s^2}-\frac{\frac{1}{4}\left(\frac{2!}{2!}\right)}{s^3}-\frac{1\left(\frac{3!}{3!}\right)}{s^4}$$

$$f\left(t\right)=\tfrac{17}{64}\mathbf{e}^{4t}-\tfrac{209}{64}-\tfrac{1}{16}t-\tfrac{1}{8}t^2-\tfrac{1}{6}t^3$$

$$G\left(s\right)=\frac{1}{s^2\left(s-4\right)\left(s+5\right)}=\frac{\frac{1}{144}}{s-4}-\frac{\frac{1}{225}}{s+5}-\frac{\frac{1}{400}}{s}-\frac{\frac{1}{20}}{s^2}$$

$$g\left(t\right)=\tfrac{1}{144}\mathbf{e}^{4t}-\tfrac{1}{225}\mathbf{e}^{-5t}-\tfrac{1}{400}-\tfrac{1}{20}t$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y\left(s\right)=F\left(s\right)+3\mathbf{e}^{-6s}G\left(s\right)\qquad\Rightarrow\qquad y\left(t\right)=f\left(t\right)+3u_6\left(t\right)g\left(t-6\right)$$

where *f(t)* and *g(t)* are the functions shown above.

Okay, there is the one Laplace transform example with a differential equation with order greater than 2. As you can see the work in identical except for the fact that the partial fraction work (which we didn't show here) is liable to be messier and more complicated.

Section 7-6 : Systems of Differential Equations

In this section we want to take a brief look at systems of differential equations that are larger than 2 x 2. The problem here is that unlike the first few sections where we looked at n^{th} order differential equations we can't really come up with a set of formulas that will always work for every system. So, with that in mind we're going to look at all possible cases for a 3 x 3 system (leaving some details for you to verify at times) and then a couple of quick comments about 4 x 4 systems to illustrate how to extend things out to even larger systems and then we'll leave it to you to actually extend things out if you'd like to.

We will also not be doing any actual examples in this section. The point of this section is just to show how to extend out what we know about 2 x 2 systems to larger systems.

Initially the process is identical regardless of the size of the system. So, for a system of 3 differential equations with 3 unknown functions we first put the system into matrix form,

$$\vec{x}\,' = A\vec{x}$$

where the coefficient matrix, *A*, is a 3 x 3 matrix. We next need to determine the eigenvalues and eigenvectors for *A* and because *A* is a 3 x 3 matrix we know that there will be 3 eigenvalues (including repeated eigenvalues if there are any).

This is where the process from the 2 x 2 systems starts to vary. We will need a total of 3 linearly independent solutions to form the general solution. Some of what we know from the 2 x 2 systems can be brought forward to this point. For instance, we know that solutions corresponding to simple eigenvalues (*i.e.* they only occur once in the list of eigenvalues) will be linearly independent. We know that solutions from a set of complex conjugate eigenvalues will be linearly independent. We also know how to get a set of linearly independent solutions from a double eigenvalue with a single eigenvector.

There are also a couple of facts about eigenvalues/eigenvectors that we need to review here as well. First, provided *A* has only real entries (which it always will here) all complex eigenvalues will occur in conjugate pairs (*i.e.* $\lambda = \alpha \pm \beta\, i$) and their associated eigenvectors will also be complex conjugates of each other. Next, if an eigenvalue has multiplicity $k \ge 2$ (*i.e.* occurs at least twice in the list of eigenvalues) then there will be anywhere from 1 to *k* linearly independent eigenvectors for the eigenvalue.

With all these ideas in mind let's start going through all the possible combinations of eigenvalues that we can possibly have for a 3 x 3 case. Let's also note that for a 3 x 3 system it is impossible to have only 2 real distinct eigenvalues. The only possibilities are to have 1 or 3 real distinct eigenvalues.

Here are all the possible cases.

3 Real Distinct Eigenvalues

In this case we'll have the real, distinct eigenvalues $\lambda_1 \ne \lambda_2 \ne \lambda_3$ and their associated eigenvectors, $\vec{\eta}_1$, $\vec{\eta}_2$ and $\vec{\eta}_3$ are guaranteed to be linearly independent and so the three linearly independent solutions we get from this case are,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1 \qquad\qquad \mathbf{e}^{\lambda_2 t}\vec{\eta}_2 \qquad\qquad \mathbf{e}^{\lambda_3 t}\vec{\eta}_3$$

1 Real and 2 Complex Eigenvalues
From the real eigenvalue/eigenvector pair, λ_1 and $\vec{\eta}_1$, we get one solution,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1$$

We get the other two solutions in the same manner that we did with the 2 x 2 case. If the eigenvalues are $\lambda_{2,3} = \alpha \pm \beta\, i$ with eigenvectors $\vec{\eta}_2$ and $\vec{\eta}_3 = \overline{(\vec{\eta}_2)}$ we can get two real-valued solution by using Euler's formula to expand,

$$\mathbf{e}^{\lambda_2 t}\vec{\eta}_2 = \mathbf{e}^{(\alpha+\beta i)t}\vec{\eta}_2 = \mathbf{e}^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)\vec{\eta}_2$$

into its real and imaginary parts, $\vec{u} + i\,\vec{v}$. The final two real valued solutions we need are then,

$$\vec{u} \qquad\qquad \vec{v}$$

1 Real Distinct and 1 Double Eigenvalue with 1 Eigenvector
From the real eigenvalue/eigenvector pair, λ_1 and $\vec{\eta}_1$, we get one solution,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1$$

From our work in the 2 x 2 systems we know that from the double eigenvalue λ_2 with single eigenvector, $\vec{\eta}_2$, we get the following two solutions,

$$\mathbf{e}^{\lambda_2 t}\vec{\eta}_2 \qquad\qquad t\,\mathbf{e}^{\lambda_2 t}\vec{\xi} + \mathbf{e}^{\lambda_2 t}\vec{\rho}$$

where $\vec{\xi}$ and $\vec{\rho}$ must satisfy the following equations,

$$\left(A - \lambda_2 I\right)\vec{\xi} = \vec{0} \qquad\qquad \left(A - \lambda_2 I\right)\vec{\rho} = \vec{\xi}$$

Note that the first equation simply tells us that $\vec{\xi}$ must be the single eigenvector for this eigenvalue, $\vec{\eta}_2$, and we usually just say that the second solution we get from the double root case is,

$$t\,\mathbf{e}^{\lambda_2 t}\vec{\eta}_2 + \mathbf{e}^{\lambda_2 t}\vec{\rho} \qquad\qquad \text{where } \vec{\rho} \text{ satisfies } \left(A - \lambda_2 I\right)\vec{\rho} = \vec{\eta}_2$$

1 Real Distinct and 1 Double Eigenvalue with 2 Linearly Independent Eigenvectors
We didn't look at this case back when we were examining the 2 x 2 systems but it is easy enough to deal with. In this case we'll have a single real distinct eigenvalue/eigenvector pair, λ_1 and $\vec{\eta}_1$, as well as a double eigenvalue λ_2 and the double eigenvalue has two linearly independent eigenvectors, $\vec{\eta}_2$ and $\vec{\eta}_3$.

In this case all three eigenvectors are linearly independent and so we get the following three linearly independent solutions,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1 \qquad \mathbf{e}^{\lambda_2 t}\vec{\eta}_2 \qquad \mathbf{e}^{\lambda_2 t}\vec{\eta}_3$$

We are now out of the cases that compare to those that we did with 2 x 2 systems and we now need to move into the brand new case that we pick up for 3 x 3 systems. This new case involves eigenvalues with multiplicity of 3. As we noted above we can have 1, 2, or 3 linearly independent eigenvectors and so we actually have 3 sub cases to deal with here. So, let's go through these final 3 cases for a 3 x 3 system.

1 Triple Eigenvalue with 1 Eigenvector

The eigenvalue/eigenvector pair in this case are λ and $\vec{\eta}$. Because the eigenvalue is real we know that the first solution we need is,

$$\mathbf{e}^{\lambda t}\vec{\eta}$$

We can use the work from the double eigenvalue with one eigenvector to get that a second solution is,

$$t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho} \qquad \text{where } \vec{\rho} \text{ satisfies } \left(A - \lambda I\right)\vec{\rho} = \vec{\eta}$$

For a third solution we can take a clue from how we dealt with n^{th} order differential equations with roots multiplicity 3. In those cases, we multiplied the original solution by a t^2. However, just as with the double eigenvalue case that won't be enough to get us a solution. In this case the third solution will be,

$$\tfrac{1}{2}t^2\mathbf{e}^{\lambda t}\vec{\xi} + t\mathbf{e}^{\lambda t}\vec{\rho} + \mathbf{e}^{\lambda t}\vec{\mu}$$

where $\vec{\xi}$, $\vec{\rho}$, and $\vec{\mu}$ must satisfy,

$$\left(A - \lambda I\right)\vec{\xi} = \vec{0} \qquad \left(A - \lambda I\right)\vec{\rho} = \vec{\xi} \qquad \left(A - \lambda I\right)\vec{\mu} = \vec{\rho}$$

You can verify that this is a solution and the conditions by taking a derivative and plugging into the system.

Now, the first condition simply tells us that $\vec{\xi} = \vec{\eta}$ because we only have a single eigenvector here and so we can reduce this third solution to,

$$\tfrac{1}{2}t^2\mathbf{e}^{\lambda t}\vec{\eta} + t\mathbf{e}^{\lambda t}\vec{\rho} + \mathbf{e}^{\lambda t}\vec{\mu}$$

where $\vec{\rho}$, and $\vec{\mu}$ must satisfy,

$$\left(A - \lambda I\right)\vec{\rho} = \vec{\eta} \qquad \left(A - \lambda I\right)\vec{\mu} = \vec{\rho}$$

and finally notice that we would have solved the new first condition in determining the second solution above and so all we really need to do here is solve the final condition.

As a final note in this case, the $\frac{1}{2}$ is in the solution solely to keep any extra constants from appearing in the conditions which in turn allows us to reuse previous results.

1 Triple Eigenvalue with 2 Linearly Independent Eigenvectors

In this case we'll have the eigenvalue λ with the two linearly independent eigenvectors $\vec{\eta}_1$ and $\vec{\eta}_2$ so we get the following two linearly independent solutions,

$$\mathbf{e}^{\lambda t}\vec{\eta}_1 \qquad\qquad \mathbf{e}^{\lambda t}\vec{\eta}_2$$

We now need a third solution. The third solution will be in the form,

$$t\,\mathbf{e}^{\lambda t}\vec{\xi} + \mathbf{e}^{\lambda t}\vec{\rho}$$

where $\vec{\xi}$ and $\vec{\rho}$ must satisfy the following equations,

$$\left(A - \lambda I\right)\vec{\xi} = \vec{0} \qquad\qquad \left(A - \lambda I\right)\vec{\rho} = \vec{\xi}$$

We've already verified that this will be a solution with these conditions in the double eigenvalue case (that work only required a repeated eigenvalue, not necessarily a double one).

However, unlike the previous times we've seen this we can't just say that $\vec{\xi}$ is an eigenvector. In all the previous cases in which we've seen this condition we had a single eigenvector and this time we have two linearly independent eigenvectors. This means that the most general possible solution to the first condition is,

$$\vec{\xi} = c_1\vec{\eta}_1 + c_2\vec{\eta}_2$$

This creates problems in solving the second condition. The second condition will not have solutions for every choice of c_1 and c_2 and the choice that we use will be dependent upon the eigenvectors. So upon solving the first condition we would need to plug the general solution into the second condition and then proceed to determine conditions on c_1 and c_2 that would allow us to solve the second condition.

1 Triple Eigenvalue with 3 Linearly Independent Eigenvectors

In this case we'll have the eigenvalue λ with the three linearly independent eigenvectors $\vec{\eta}_1$, $\vec{\eta}_2$, and $\vec{\eta}_3$ so we get the following three linearly independent solutions,

$$\mathbf{e}^{\lambda t}\vec{\eta}_1 \qquad\qquad \mathbf{e}^{\lambda t}\vec{\eta}_2 \qquad\qquad \mathbf{e}^{\lambda t}\vec{\eta}_3$$

4 x 4 Systems

We'll close this section out with a couple of comments about 4 x 4 systems. In these cases we will have 4 eigenvalues and will need 4 linearly independent solutions in order to get a general solution. The vast majority of the cases here are natural extensions of what 3 x 3 systems cases and in fact will use a vast majority of that work.

Here are a couple of new cases that we should comment briefly on however. With 4 x 4 systems it will now be possible to have two different sets of double eigenvalues and two different sets of complex conjugate eigenvalues. In either of these cases we can treat each one as a separate case and use our previous knowledge about double eigenvalues and complex eigenvalues to get the solutions we need.

It is also now possible to have a "double" complex eigenvalue. In other words, we can have $\lambda = \alpha \pm \beta\, i$ each occur twice in the list of eigenvalues. The solutions for this case aren't too bad. We get two solutions in the normal way of dealing with complex eigenvalues. The remaining two solutions will come from the work we did for a double eigenvalue. The work we did in that case did not require that the eigenvalue/eigenvector pair to be real. Therefore, if the eigenvector associated with $\lambda = \alpha + \beta\, i$ is $\vec{\eta}$ then the second solution will be,

$$t\,\mathbf{e}^{(\alpha+\beta i)t}\vec{\eta} + \mathbf{e}^{(\alpha+\beta i)t}\vec{\rho} \qquad\qquad \text{where } \vec{\rho} \text{ satisfies } \left(A - \lambda I\right)\vec{\rho} = \vec{\eta}$$

and once we've determined $\vec{\rho}$ we can again split this up into its real and imaginary parts using Euler's formula to get two new real valued solutions.

Finally, with 4 x 4 systems we can now have eigenvalues with multiplicity of 4. In these cases, we can have 1, 2, 3, or 4 linearly independent eigenvectors and we can use our work with 3 x 3 systems to see how to generate solutions for these cases. The one issue that you'll need to pay attention to is the conditions for the 2 and 3 eigenvector cases will have the same complications that the 2 eigenvector case has in the 3 x 3 systems.

So, we've discussed some of the issues involved in systems larger than 2 x 2 and it is hopefully clear that when we move into larger systems the work can be become vastly more complicated.

Section 7-7 : Series Solutions

The purpose of this section is not to do anything new with a series solution problem. Instead it is here to illustrate that moving into a higher order differential equation does not really change the process outside of making it a little longer.

Back in the Series Solution chapter we only looked at 2nd order differential equations so we're going to do a quick example here involving a 3rd order differential equation so we can make sure and say that we've done at least one example with an order larger than 2.

Example 1 Find the series solution around $x_0 = 0$ for the following differential equation.

$$y''' + x^2 y' + xy = 0$$

Solution

Recall that we can only find a series solution about $x_0 = 0$ if this point is an ordinary point, or in other words, if the coefficient of the highest derivative term is not zero at $x_0 = 0$. That is clearly the case here so let's start with the form of the solutions as well as the derivatives that we'll need for this solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

Plugging into the differential equation gives,

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

Now, move all the coefficients into the series and do appropriate shifts so that all the series are in terms of x^n.

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Next, let's notice that we can start the second series at $n = 1$ since that term will be zero. So let's do that and then we can combine the second and third terms to get,

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=1}^{\infty} \left[(n-1)+1\right] a_{n-1} x^n = 0$$

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=1}^{\infty} n a_{n-1} x^n = 0$$

So, we got a nice simplification in the new series that will help with some further simplification. The new second series can now be started at $n = 0$ and then combined with the first series to get,

$$\sum_{n=0}^{\infty}\left[(n+3)(n+2)(n+1)a_{n+3}+na_{n-1}\right]x^n=0$$

We can now set the coefficients equal to get a fairly simply recurrence relation.

$$(n+3)(n+2)(n+1)a_{n+3}+na_{n-1}=0 \qquad n=0,1,2,\ldots$$

Solving the recurrence relation gives,

$$a_{n+3}=\frac{-na_{n-1}}{(n+1)(n+2)(n+3)} \qquad n=0,1,2,\ldots$$

Now we need to start plugging in values of *n* and this will be one of the main areas where we can see a somewhat significant increase in the amount of work required when moving into a higher order differential equation.

$$n=0: \quad a_3=0$$

$$n=1: \quad a_4=\frac{-a_0}{(2)(3)(4)}$$

$$n=2: \quad a_5=\frac{-2a_1}{(3)(4)(5)}$$

$$n=3: \quad a_6=\frac{-3a_2}{(4)(5)(6)}$$

$$n=4: \quad a_7=\frac{-4a_3}{(5)(6)(7)}=0$$

$$n=5: \quad a_8=\frac{-5a_4}{(6)(7)(8)}=\frac{5a_0}{(2)(3)(4)(6)(7)(8)}$$

$$n=6: \quad a_9=\frac{-6a_5}{(7)(8)(9)}=\frac{(2)(6)a_1}{(3)(4)(5)(7)(8)(9)}$$

$$n=7: \quad a_{10}=\frac{-7a_6}{(8)(9)(10)}=\frac{(3)(7)a_2}{(4)(5)(6)(8)(9)(10)}$$

$$n=8: \quad a_{11}=\frac{-8a_7}{(9)(10)(11)}=0$$

$$n=9: \quad a_{12}=\frac{-9a_8}{(10)(11)(12)}=\frac{-(5)(9)a_0}{(2)(3)(4)(6)(7)(8)(10)(11)(12)}$$

$$n=10: a_{13}=\frac{-10a_9}{(11)(12)(13)}=\frac{-(2)(6)(10)a_1}{(3)(4)(5)(7)(8)(9)(11)(12)(13)}$$

$$n=11: a_{14}=\frac{-11a_{10}}{(12)(13)(14)}=\frac{-(3)(7)(11)a_2}{(4)(5)(6)(8)(9)(10)(12)(13)(14)}$$

Okay, we can now break the coefficients down into 4 sub cases given by a_{4k}, a_{4k+1}, a_{4k+2} and a_{4k+3} for $k = 0,1,2,3,\ldots$ We'll give a semi-detailed derivation for a_{4k} and then leave the rest to you with only couple of comments as they are nearly identical derivations.

First notice that all the a_{4k} terms have a_0 in them and they will alternate in sign. Next notice that we can turn the denominator into a factorial, $(4k)!$ to be exact, if we multiply top and bottom by the numbers that are already in the numerator and so this will turn these numbers into squares. Next notice that the product in the top will start at 1 and increase by 4 until we reach $4k-3$. So, taking all of this into account we get,

$$a_{4k} = \frac{(-1)^k (1)^2 (5)^2 \cdots (4k-3)^2 a_0}{(4k)!} \qquad k = 1,2,3,\ldots$$

and notice that this will only work starting with $k=1$ as we won't get a_0 out of this formula as we should by plugging in $k=0$.

Now, for a_{4k+1} the derivation is almost identical and so the formula is,

$$a_{4k+1} = \frac{(-1)^k (2)^2 (6)^2 \cdots (4k-2)^2 a_1}{(4k+1)!} \qquad k = 1,2,3,\ldots$$

and again notice that this won't work for $k=0$

The formula for a_{4k+2} is again nearly identical except for this one note that we also need to multiply top and bottom by a 2 in order to get the factorial to appear in the denominator and so the formula here is,

$$a_{4k+2} = \frac{2(-1)^k (3)^2 (7)^2 \cdots (4k-1)^2 a_2}{(4k+2)!} \qquad k = 1,2,3,\ldots$$

noticing yet one more time that this won't work for $k=0$.

Finally, we have $a_{4k+3} = 0$ for $k = 0,1,2,3,\ldots$

Now that we have all the coefficients let's get the solution,

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{4k} x^{4k} + a_{4k+1} x^{4k+1} + a_{4k+3} x^{4k+3} + a_{4k+3} x^{4k+3} + \cdots \\ &= a_0 + a_1 x + a_2 x^2 \cdots + \frac{(-1)^k (1)^2 (5)^2 \cdots (4k-3)^2 a_0}{(4k)!} x^{4k} + \\ &\qquad \frac{(-1)^k (2)^2 (6)^2 \cdots (4k-2)^2 a_1}{(4k+1)!} x^{4k+1} + \\ &\qquad \frac{2(-1)^k (3)^2 (7)^2 \cdots (4k-1)^2 a_2}{(4k+2)!} x^{4k+2} + \cdots \end{aligned}$$

Collecting up the terms that contain the same coefficient (except for the first one in each case since the formula won't work for those) and writing everything as a set of series gives us our solution,

$$\begin{aligned} y(x) = a_0 &\left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1)^2 (5)^2 \cdots (4k-3)^2 x^{4k}}{(4k)!} \right\} + \\ &a_1 \left\{ x + \sum_{k=1}^{\infty} \frac{(-1)^k (2)^2 (6)^2 \cdots (4k-2)^2 x^{4k+1}}{(4k+1)!} \right\} + \\ &a_2 \left\{ x^2 + \sum_{k=1}^{\infty} \frac{2(-1)^k (3)^2 (7)^2 \cdots (4k-1)^2 x^{4k+2}}{(4k+2)!} \right\} \end{aligned}$$

So, there we have it. As we can see the work in getting formulas for the coefficients was a little messy because we had three formulas to get, but individually they were not as bad as even some of them could be when dealing with 2nd order differential equations. Also note that while we got lucky with this problem and we were able to get general formulas for the terms the higher the order the less likely this will become.