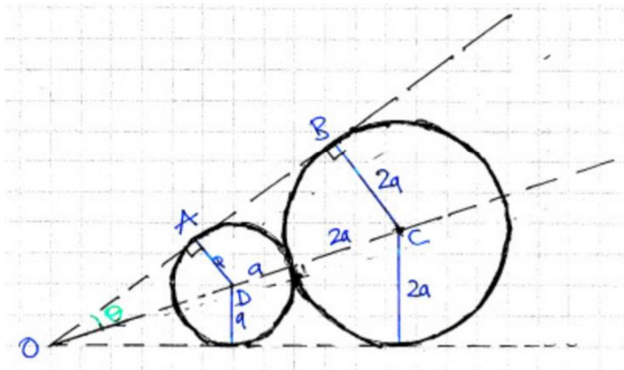
Maths / FM Extended Solutions

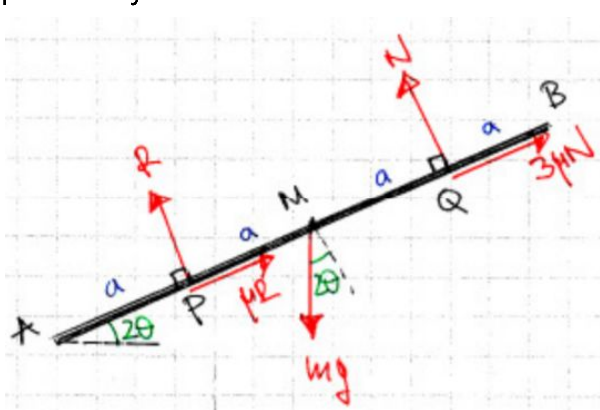
Part A: Maths Content

1. Starting with a diagram:



Let |OC| = *d*. Since triangles OAD and OBC are similar, we have
sin θ = |BC| / |OC| = |AD| / |OD|.
→ 2a / d = a / (d - 3a)
→ 2d - 6a = d
→ d = 6a.
Now, it is clear (by subbing d back into the definition) that sin θ = 1/3. [1 mark]

Drawing a free-body diagram of the rod, in its limiting equilibrium state (friction takes its maximum value of F = μR) where P and Q are the points of contact previously labelled A and B:



Taking moments about P,
→ mg cos 2θ * a = N * 2a [2 marks]
→ N = (1/2) mg cos 2θ [1 mark]

Taking moments about Q,
→ $mg \cos 2\theta * a = R * 2a$
→ $N = (1/2)\, mg \cos 2\theta$ [1 mark]

Next, resolve forces parallel (along) the rod.
→ $\mu R + 3\mu N = mg \sin 2\theta$ [1 mark]
Since $R = N$,
→ $4\mu R = mg \sin 2\theta$
→ $2\mu\, mg \cos 2\theta = mg \sin 2\theta$ [1 mark]
→ $\mu = (1/2) \tan 2\theta$ [1 mark]

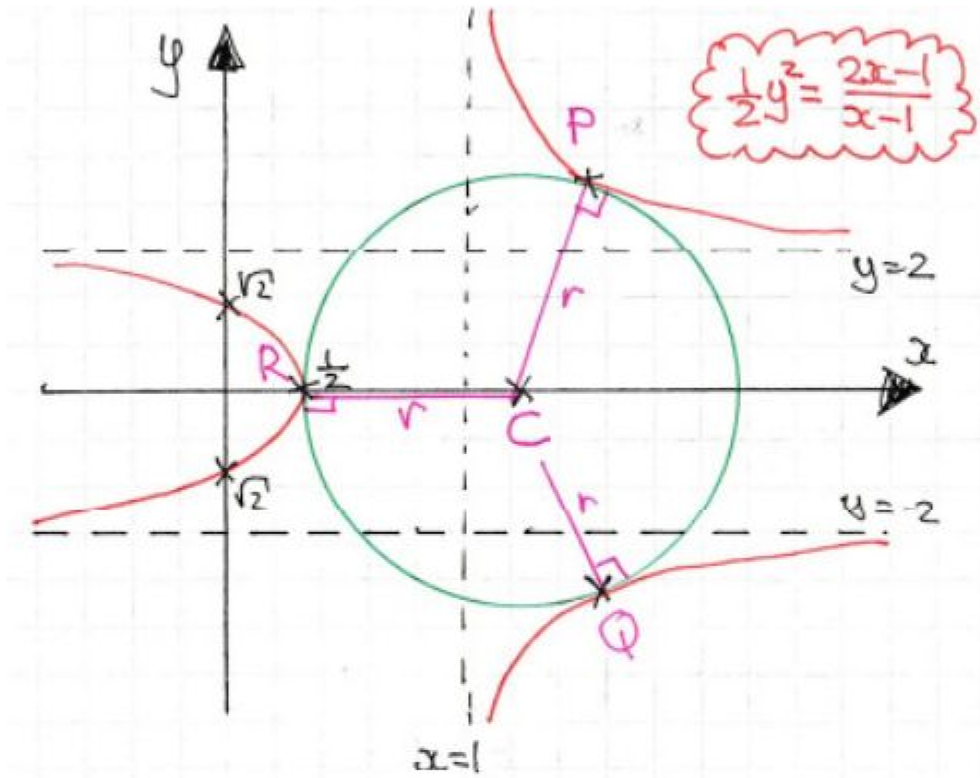
By trigonometry, we have
→ $\sin \theta = 1/3$
→ $\cos \theta = 2\sqrt{2} / 3$
→ $\tan \theta = 1 / (2\sqrt{2})$ [1 mark]
Using the double-angle identity,
→ $\tan 2\theta = 2 \tan \theta / (1 - \tan^2 \theta)$
→ $\tan 2\theta = (2 / 2\sqrt{2}) / (1 - (1 / (2\sqrt{2})^2))$
→ $\tan 2\theta = (1 / \sqrt{2}) / (7 / 8)$
→ $\tan 2\theta = 8\sqrt{2} / 7$

Putting this back in, we get
→ $\mu = 4\sqrt{2} / 7$
Since this is the limiting value, friction must be at least this, so
→ $\mu \geq \frac{4\sqrt{2}}{7}$ i.e. $k = 4/7$. [1 mark]

(Solution by T. Madas at https://madasmaths.com/archive/maths_booklets/mechanics/m2_equilibrium_of_rigid_bodies.pdf, Question 45)

2. **Step 1: Setup**

Let the circle C have centre (a, b) and radius r.
Since the curve function contains a y^2 term, it must be symmetrical in the x-axis and therefore, the circle must also be symmetrical, so its centre lies on the X-axis. Therefore, b = 0 → centre (a, 0), radius r.



The left-most point (which lies on the x-axis) can be found by putting y = 0:
1/2 $(0)^2$ = (2x - 1)/(x - 1) → 0 = 2x - 1 → x = 1/2. This will be a helpful solution later.

Step 2: Finding the centre

Differentiating the curve function implicitly, using quotient rule for RHS,

$$\Rightarrow y\frac{dy}{dx} = \frac{2(x-1)-(2x-1)}{(x-1)^2}$$

$$\Rightarrow y\frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{y(x-1)^2}$$

NORMAL GRADIENT FUNCTION

$$\Rightarrow -\frac{dx}{dy} = y(x-1)^2$$

$$\Rightarrow -\frac{dx}{dy} = \sqrt{2}\,\frac{(2x-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}}(x-1)^2$$

$$\Rightarrow -\frac{dx}{dy} = \sqrt{2}\,(2x-1)^{\frac{1}{2}}(x-1)^{\frac{3}{2}}$$

[4 marks]

Let the x-coordinate of the P be *k*. Then the gradient of the normal at P is

$$\sqrt{2}\,(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}$$

Using point-slope form, the equation of the radius (line CP) is

$$\Rightarrow y-\sqrt{2}\,\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} = \sqrt{2}\,(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}(x-k)$$

[1 mark]

To find the centre of the circle, let y = 0 and solve for x:

$$\Rightarrow -\cancel{\sqrt{2}}\,\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} = \cancel{\sqrt{2}}\,(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}(x-k)$$

$$\Rightarrow -\frac{1}{(k-1)^2} = x-k$$

$$\Rightarrow x = k-\frac{1}{(k-1)^2}$$

[2 marks]

Also, |CP| = |CQ| = |CR| = r. Any of these can be used to form the next equation. For simplicity, use $|CP|^2 = |CR|^2 = r^2$.

$$\Rightarrow \underbrace{\left[k - \left(k - \frac{1}{(k-1)^2}\right)\right]^2 + \left[\sqrt{2}\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} - 0\right]^2}_{|CP|^2} = \underbrace{\left[k - \frac{1}{(k-1)^2} - \frac{1}{2}\right]^2}_{|CR|^2}$$

$$\Rightarrow \frac{1}{(k-1)^4} + \frac{2(2k-1)}{k-1} = \left[\frac{2k-1}{2} - \frac{1}{(k-1)^2}\right]^2$$

$$\Rightarrow \cancel{\frac{1}{(k-1)^4}} + \frac{2(2k-1)}{k-1} = \frac{(2k-1)^2}{4} - \frac{2k-1}{(k-1)^2} + \cancel{\frac{1}{(k-1)^4}}$$

$$\Rightarrow \frac{2}{k-1} = \frac{2k-1}{4} - \frac{1}{(k-1)^2}$$

$$\Rightarrow 2(k-1) = \frac{1}{4}(2k-1)(k-1)^2 - 1$$

$$\Rightarrow 8(k-1) = (2k-1)(k^2-2k+1) - 4$$

$$\Rightarrow 8k - 8 = 2k^3 - 4k^2 + 2k - k^2 + 2k - 1 - 4$$

$$\Rightarrow 0 = 2k^3 - 5k^2 - 4k + 3$$

[7 marks]

We know that k = 1/2 is a solution (since this corresponds to point R) so therefore (2k - 1) will be a factor. By polynomial division, or directly solving, we find the other solutions will be k = 3 and k = -1. [1 mark]

From the graph, P is clearly to the right of the asymptote x = 1 so reject k = -1 → k = 3. [1 mark]

Putting this back in, the centre of the circle is then:
$3 - 1/(3 - 1)^2 = 11/4$. → Centre (11/4, 0). [1 mark]

Step 3: Finding the radius

Since r = |CR|, we have

$$r = k - \frac{1}{(k-1)^2} - \frac{1}{2}$$

$$r = 3 - \frac{1}{(3-1)^2} - \frac{1}{2}$$

$$r = 3 - \frac{1}{4} - \frac{1}{2}$$

$$r = 3 - \frac{3}{4}$$

$$r = \frac{9}{4}$$

[2 marks]

Therefore, the equation of the circle C is:

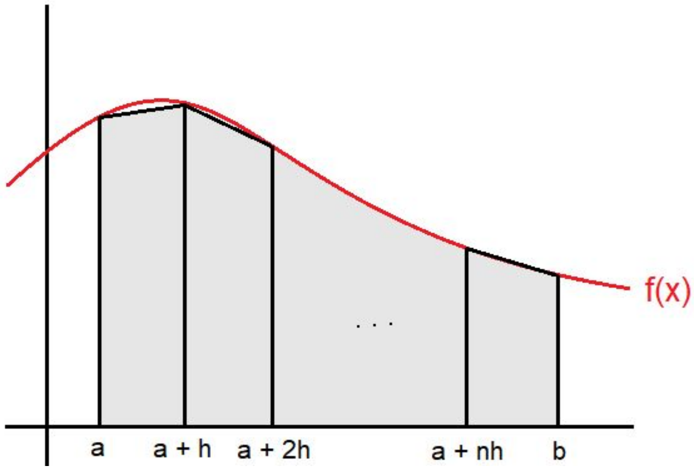
$$\left(x - \frac{11}{4}\right)^2 + y^2 = \frac{81}{16}$$

. [1 mark]

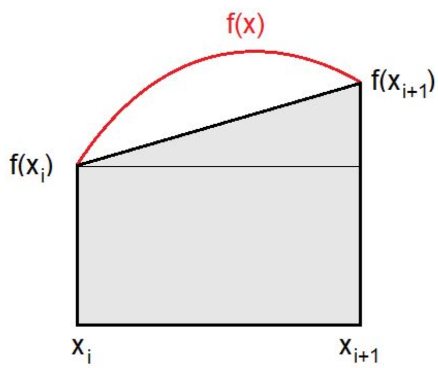
(Solution by T. Madas at https://madasmaths.com/archive/maths_booklets/standard_topics/various/differeNtiation_ii_exam_questions.pdf, Question 261)

3. **Step 1: Setup**

Begin by sketching the trapezium rule in use on a function f(x) between a and b.
Let h = (b - a)/n = step size:



Label each x-coordinate x_i, with x_0 = a and x_n = b. Then,
Now, zooming in on one particular trapezium,



[1 mark]

Let the area of this trapezium be S_i. Then, considering it as the sum of the rectangle and the small triangle as advised,

$S_i = h\, f(x_i) + h(f(x_{i+1}) - f(x_i))/2 = (h/2)(f(x_{i+1}) - f(x_i))$ [1 mark]

This is as far as we can go with the approximation.

Step 2: Finding an expression for I

Let us consider the actual area, I_i:

$$I_i = \int_{x_i}^{x_{i+1}} f(x)\ \mathrm{d}x$$

To simplify the algebra, substitute t = x - x_i → x = t + x_i,
then the bounds become 0 to t:

$$I_i = \int_0^h f(t + x_i)\ \mathrm{d}t$$

Since we cannot integrate the function, but the final result contains derivatives, it makes sense to integrate by parts twice to obtain an expression in terms of second derivatives.

First time:

↳ by parts:

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

↳ let $u = f(t + x_i)$ $\qquad dv = dt$

$du = f'(t + x_i)$ $\qquad v = t + C_1$ ← constant

↳ $dv = dt + 0$

$$\int_0^h f(t + x_i)dt = (t + C_1)f(t + x_i)\Big]_0^h - \int_0^h (t + C_1)f'(t + x_i)dt$$

[3 marks]

Second time (on the RHS integral):

$$u = f'(t+x_i) \qquad dv = (t + C_1)\,dt$$

$$du = f''(t+x_i) \qquad v = \frac{t^2}{2} + \frac{2C_1 t}{2} + C_*$$

complete square: $(t + C_1)^2 = t^2 + 2C_1 t + C_1^2$

$$= \frac{t^2 + 2C_1 t}{2} + C_*$$

$$= \frac{(t + C_1)^2 - C_1^2}{2} + C_*$$

$$= \frac{(t + C_1)^2}{2} \underbrace{- \frac{C_1^2}{2} + C_*}_{= C_2}$$

$$= \frac{(t + C_1)^2}{2} + C_2$$

$$\int_0^h f(t+x_i)\,dt = \overbrace{\Big[(t+C_1) f(t+x_i)\Big]_0^h}^{T_i} - \left(\frac{(t+C_1)^2}{2} + C_2\right) f'(t+x_i)\Big|_0^h$$

$$+ \int_0^h \left(\frac{(t+C_1)^2}{2} + C_2\right) f''(t+x_i)\,dt$$

[4 marks]

This result consists of three terms. Notice that the first one is
$(t + C_1)\, f(t + x_i)\, |_0^h = (h + C_1)\, f(x_{i+1}) - C_1\, f(x_i) = h\, f(x_{i+1}) + C_1\, f(x_{i+1}) - C_1\, f(x_i)$
This contains all terms present in the formula for S_i, and for some constant C_1 it will be equal to it.

It is easy to see that when $C_1 = -h/2$ [1 mark],
this first term is $h\, f(x_{i+1}) - h/2\, f(x_{i+1}) + h/2\, f(x_i)$
$= h/2\, (\, f(x_i) + f(x_{i+1})\,)$ which is exactly S_i. We can choose this constant C_1 freely since it will cancel out when the bounds of integration are applied.

This will mean our result will be $I_i = S_i$ + two other terms.

Since we know therefore, $I_i = S_i + E_i$, the two other terms must add to E_i. Since we are given the formula for E_i as containing $f''(x)$ terms, we must have the term containing $f'(x)$ equal to zero.

Step 3: Finding C_2

Considering the second term (containing first derivative):

we want:

$$\left(\frac{(t+c_1)^2}{2}+c_2\right)f'(t+x_i)\Big]_0^h = 0$$

$$\underbrace{\left(\frac{(h-\frac{h}{2})^2}{2}+c_2\right)}f'(h+x_i)-\underbrace{\left(\frac{(-\frac{h}{2})^2}{2}+c_2\right)}f'(x_i)=0$$

notice: $\longrightarrow = \longleftarrow$

$$\frac{\left(\frac{h}{2}\right)^2}{2}+c_2=0$$

$$c_2=-\frac{h^2}{4\cdot 2}=\boxed{\frac{-h^2}{8}=c_2}$$

[2 marks]

So putting it back together we have $I_i = S_i + E_i$ as required:

$$\int_0^h f(t+x_i)\,dt = \underbrace{\frac{h}{2}\left[f(x_i)+f(x_{i+1})\right]}_{S_i} + \underbrace{\int_0^h\left(\frac{(t-\frac{h}{2})^2}{2}-\frac{h^2}{8}\right)f''(t+x_i)\,dt}_{=\text{Error}=E_i}$$

Step 4: Maximising the error function

The total error E is the sum of each small error: $E = |E_1 + E_2 + \dots + E_{n-1}|$. Writing these as integrals using the form above,

Total Error

$$= \int_0^h \left(\frac{(t-\frac{h}{2})^2}{2} - \frac{h^2}{8}\right) f''(t+\underline{x_0})\,dt$$

$$+ \dots\dots + \dots$$

$$+ \int_0^h \underbrace{\left(\frac{(t-\frac{h}{2})^2}{2} - \frac{h^2}{8}\right)}_{=\text{same}} f''(t+\underline{x_{n-1}})\,dt$$

$$= \int_0^h \left(\frac{(t-\frac{h}{2})^2}{2} - \frac{h^2}{8}\right)\left[f''(t+x_0) + \dots + f''(t+x_{n-1})\right]dt$$

[1 mark]

We need the maximum possible value of this integral to correspond to the result we want to prove. The greatest possible value of f''(x) has been denoted K, so the maximum value of all of them added up will be less than nK. Since these errors must be positive, this matches the definition that $| f''(x) | \leq K$. Applying this,

$$\hookrightarrow \quad |E| = \left|\int_0^h \left[\frac{(t-\frac{h}{2})^2}{2} - \frac{h^2}{8}\right]\cdot\left[f''(t+x_0) + \dots + f''(t+x_{n-1})\right]dt\right|$$

$$\leq \int_0^h \left|\frac{(t-\frac{h}{2})^2}{2} - \frac{h^2}{8}\right| \overbrace{\left[\left|f''(t+x_0)\right| + \dots + \left|f''(t+x_{n-1})\right|\right]}^{\leq nK} dt$$

$$\leq nK \int_0^h \left|\frac{(t-\frac{h}{2})^2}{2} - \frac{h^2}{8}\right| dt$$

[1 mark]

Part 5: Obtaining the formula

All we have left to do is evaluate the remaining integral. To deal with the absolute value bars, we must consider the shape of the graph. Let it equal g(t):

$$\hookrightarrow \quad g(t) = \frac{\left(t - \frac{h}{2}\right)^2}{2} - \frac{h^2}{8} \rightarrow \text{parabola}$$

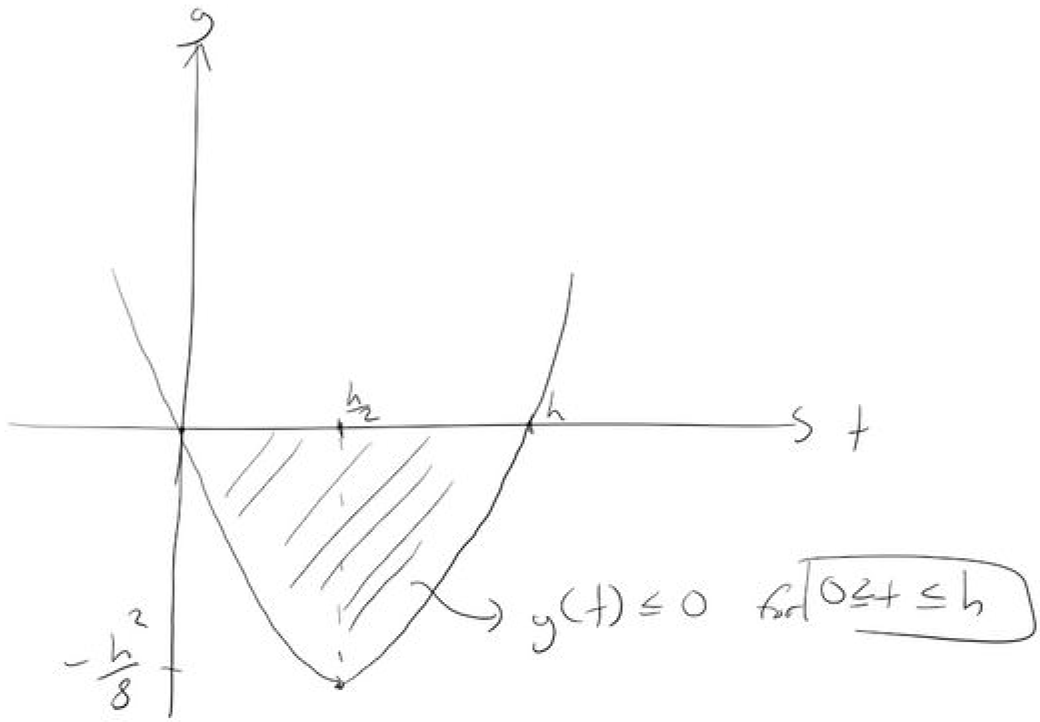
$$@\ t = 0,\ h$$

$$\hookrightarrow \quad g(0) = \frac{\left(-\frac{h}{2}\right)^2}{2} - \frac{h^2}{8}$$

$$= \frac{h^2}{4 \cdot 2} - \frac{h^2}{8} = 0$$

$$g(h) = 0$$

It is clear to see that in the interval, g(t) is negative:



So the absolute value will reflect it in the x-axis, i.e. multiply by -1: [1 mark]

$$\int_0^h |g(t)|\,dt = -\int_0^h g(t)\,dt = \int_0^h \left(\frac{h^2}{8} - \frac{\left(t - \frac{h}{2}\right)^2}{2}\right) dt$$

This is a relatively simple integral - using a substitution,

$$= \left[\frac{h^2}{8}t - \frac{\left(t-\frac{h}{2}\right)^3}{6}\right]_0^h$$

$$= \frac{h^3}{8} - \frac{\left(\frac{h}{2}\right)^3}{6} - \left[-\frac{\left(-\frac{h}{2}\right)^3}{6}\right]$$

$$= \frac{6h^3}{6\cdot 8} - \frac{h^3}{8\cdot 6} - \frac{h^3}{8\cdot 6}$$

$$= \frac{(6-2)h^3}{6\cdot 8} = \frac{4h^3}{6\cdot 2\cdot 4} = \boxed{\frac{h^3}{12}}$$ [2 marks]

So, putting this as the result for the integral we now have:

$$\Rightarrow |E_T| \le nk\frac{h^3}{12} = \frac{nk\left(\frac{b-a}{n}\right)^3}{12}$$

$$\Rightarrow \boxed{|E_T| \le \frac{k(b-a)^3}{12n^2}} \leftarrow$$ [1 mark]

as required.

(Solution by MathEasySolutions at https://hive.blog/mathematics/@mes/approximate-integration-trapezoidal-rule-error-bound-proof)

4. **Step 1: Understanding**

The wording of the statement may be a little strange so it will be useful to get some example values of a and b that work first.

a = 1, b = 1 → $a^2 + b^2$ = 2 and ab + 1 = 2 so 2/2 = 1 = perfect square (1^2).
a = 2, b = 1 → $a^2 + b^2$ = 5 (prime: will not divide to any square other than 1)
a = 3, b = 1 → $a^2 + b^2$ = 10 and ab + 1 = 4 (does not divide)
…
It seems that values which work are quite rare, but eventually we can find one (using table function on calculator) as
a = 8, b = 2 → $a^2 + b^2$ = 68 and ab + 1 = 17 so 68/17 = 4
a = 27, b = 3 → $a^2 + b^2$ = 738 and ab + 1 = 82 so 738/82 = 9
a = 64, b = 4 → $a^2 + b^2$ = 4112 and ab + 1 = 257 so 4112/257 = 16

We see a pattern (a = n^3 and b = n is a solution), but cannot prove anything since we don't know if there are other solutions which don't obey the rule.
What we see is that ($a^2 + b^2$) / (ab + 1) is **always** either a perfect square or not an integer, never just an integer.

Step 2: Starting the proof

Let $k = \frac{a^2+b^2}{ab+1}$. Assume there exists some integers *a* and *b* such that *k* is an integer but not a perfect square. [1 mark]

Let $a = a_1$ and $b = b_1$ be the 'smallest' solution to the equation $\frac{a^2+b^2}{ab+1} = k$, where *k* is an integer and we will define 'smallest' later. WLOG (without loss of generality),
assume further that $a_1 \geq b_1 > 0$. [2 marks]
→ $\frac{a_1{}^2+b_1{}^2}{a_1 b_1+1}$ = k
Manipulating the algebra to form a quadratic in terms of a^2,
→ $a_1^2 - kb_1 a_1 + b_1^2 - k = 0$ [1 mark]
Notice a_1 is a solution of the equation $x^2 - kb_1 x + b_1^2 - k = 0$ (Eqn 1)
Since it is a quadratic, there must be another solution, call it a_2. [1 mark]

Step 3: Defining 'smallest'

Consider these two solution sets, (a_1, b_1) and (a_2, b_2). We need to define what it means for one of them to be 'smaller' than the other since we will need to manipulate to find a contradiction.

We already said that $a_1 \geq b_1$. Consider, for example, two possible pairs: (3, 2) and (4, 1). Since we already have been working with *a*, we can define 'smallest' as the one with the smallest *a*-coordinate. In this example, (3, 2) would be considered smaller than (4, 1), since 3 < 4.

Other choices of 'smallest' can be done. The manipulation required for each would be slightly different, for example we could minimise a + b*.* [1 mark]

We will now need to show that the presence of a_1 implies that the other solution a_2 is strictly smaller than a_1, hence a contradiction to (a_1, b_1) being the 'smallest'.

Step 4: Investigating the other solution

Since a_2 is a solution of Eqn 1, we now have $a_2^2 - kb_1 a_2 + b_1^2 - k = 0$. [1 mark]
Considering the sum of the roots,
$\rightarrow a_1 + a_2 = kb_1$ [1 mark]
$\rightarrow a_2 = kb_1 - a_1$ (Eqn 2) [1 mark]
Considering the product of the roots,
$\rightarrow a_1 a_2 = b_1^2 - k$ [1 mark]
$\rightarrow a_2 = (b_1^2 - k) / a_1$ (Eqn 3) [1 mark]

From Eqn 2, we see a_2 must be an integer since a_1, b_1 and k are assumed/defined to be integers. [1 mark]
From Eqn 3, we see the numerator is never 0 (since k is not a perfect square) and so we see a_2 must be a non-zero integer. [1 mark]
Putting $a_2^2 - kb_1 a_2 + b_1^2 - k = 0$ back into the form of the question, we get $\frac{a_2^2 + b_1^2}{a_2 b_1 + 1}$ = k. Since the numerator must be positive (sum of squares), and k is defined positive, the denominator must also be positive. [2 marks]
$\rightarrow a_2 b_1 + 1 > 0 \rightarrow a_2 b_1 > -1 \rightarrow a_2 > -1/b_1$
Since $b_1 \geq 1$, the RHS is between -1 and 0, so we have

$\rightarrow a_2 > -1$.
Combining with the other results for a_2, we see that a_2 must be a positive integer. [1 mark]

Since we defined $a_1 \geq b_1 > 0$
$\rightarrow a_1^2 \geq b_1^2$ [1 mark]
$\rightarrow a_1^2 > b_1^2 - k$
$\rightarrow a_1 > (b_1^2 - k) / a_1$ [1 mark]
The RHS is defined as a_2:
$\rightarrow a_1 > a_2 \rightarrow a_2 < a_1$. [1 mark]

This is a contradiction - we have shown the existence of a_2 which is smaller than a_1, which we assumed was the smallest solution. [1 mark]

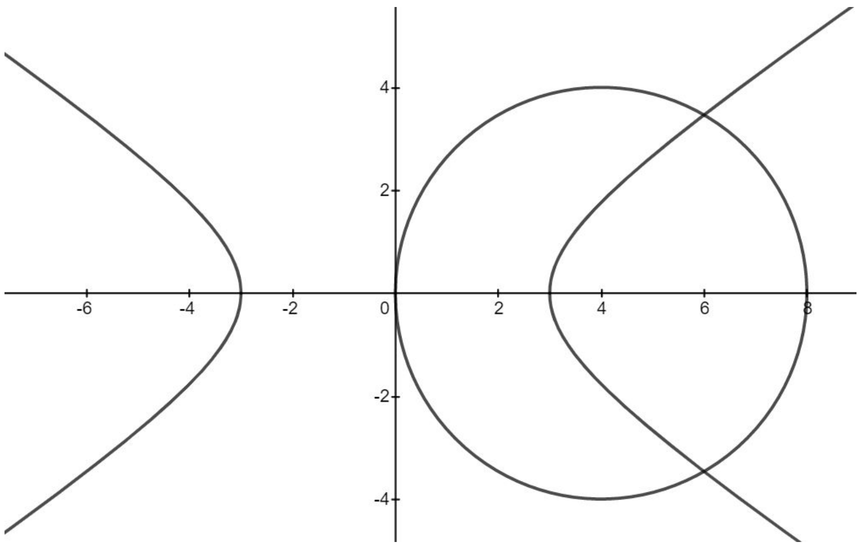
The original assumption of k being a non-square number must be false.
This proves the original statement. [1 mark]

(Solution by blackpenredpen at https://youtu.be/usEQRx4J_ew)

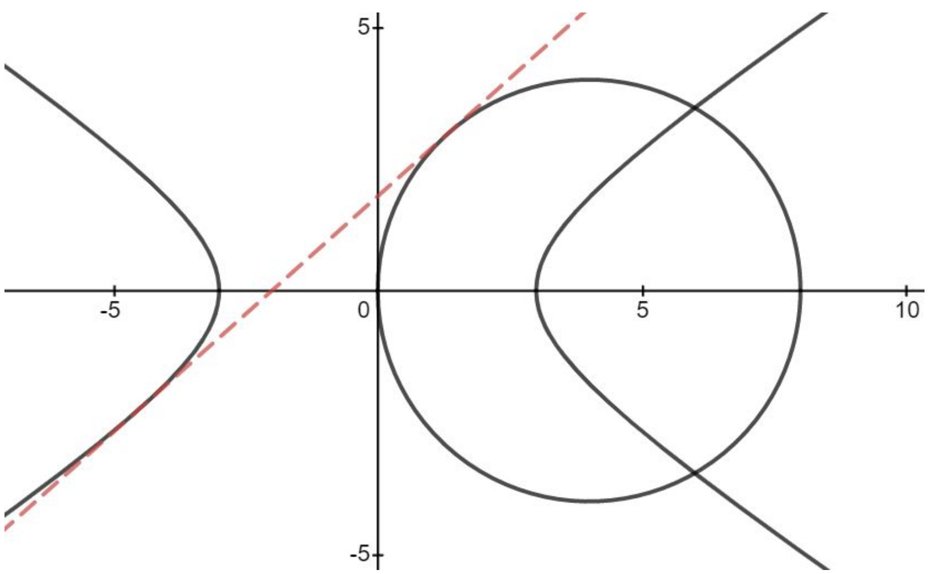
Part B: Further Maths Content

1. **Planning**

Start with a sketch / plot of these two curves:



It becomes clear that the line (of **positive**) gradient can only be tangent if it touches the bottom-left branch of the hyperbola and the top-left quadrant of the circle, i.e.



We will give this line its general form of $y = mx + c$ and continue.

Finding the equation of the line

If L is tangent, it intersects each curve in one place only.
For its intersection with the circle,
$x^2 + y^2 - 8x = 0 \rightarrow x^2 + (mx + c)^2 - 8x = 0$
$\rightarrow x^2 + m^2x^2 + 2cmx + c^2 - 8x = 0$
$\rightarrow (m^2 + 1)x^2 + (2mc - 8)x + c^2 = 0$ [2 marks]

The discriminant of this quadratic must be 0 since there is only one point:
$\rightarrow (2mc - 8)^2 - 4c^2(m^2 + 1) = 0$
$\rightarrow 4m^2c^2 - 32mc + 64 - 4m^2c^2 - 4c^2 = 0$
$\rightarrow c^2 + 8mc - 16 = 0$ [1 mark]

Repeating the process with the hyperbola,
$4x^2 - 9y^2 = 36 \rightarrow 4x^2 - 9(mx + c)^2 = 36$
$\rightarrow 4x^2 - 9m^2x^2 - 18cmx - 9c^2 - 36 = 0$
$\rightarrow (4 - 9m^2)x^2 + (-18cm)x - 9(c^2 + 4) = 0$ [2 marks]
Discriminant = 0:
$\rightarrow 324c^2m^2 + 36(4 - 9m^2)(c^2 + 4) = 0$
$\rightarrow 324c^2m^2 + 144c^2 - 324c^2m^2 + 576 - 1296m^2 = 0$
$\rightarrow c^2 - 9m^2 + 4 = 0$ [1 mark]

We now have two nonlinear equations in c and m which we solve simultaneously by substitution. Rearrange the first formula for m:
$c^2 + 8mc - 16 = 0 \rightarrow m = (16 - c^2) / 8c$ [1 mark]
Substitute into the second equation,
$$\rightarrow c^2 - \frac{9(16 - c^2)^2}{64c^2} + 4 = 0$$
$\rightarrow 64c^4 - 9(16 - c^2)^2 + 256c^2 = 0$
$\rightarrow 64c^4 - 9c^4 + 288c^2 - 2304 + 256c^2 = 0$
$\rightarrow 55c^4 + 544c^2 - 2304 = 0$
$\rightarrow c^2 = 16/5$ or $c^2 = -144/11$ [1 mark]
Since c^2 must be positive, reject $-144/11 \rightarrow c^2 = 16/5 \rightarrow c = \pm 4/\sqrt{5}$
From the diagram in planning, it is clear that $c > 0 \rightarrow c = 4/\sqrt{5}$. [1 mark]
Subbing back in for m,
$m = (16 - 16/5) / (32/\sqrt{5}) \rightarrow m = 2\sqrt{5}/5$ [1 mark]

So the line is $y = \frac{2\sqrt{5}}{5}x + \frac{4}{\sqrt{5}}$, or multiplying by $\sqrt{5}$, we get $2x - \sqrt{5}\,y + 4 = 0$.

Finding the intersection of the curves

We require the solutions to the nonlinear simultaneous equations
$x^2 + y^2 - 8x = 0$ and $4x^2 - 9y^2 = 36$. The approach is similar to before.

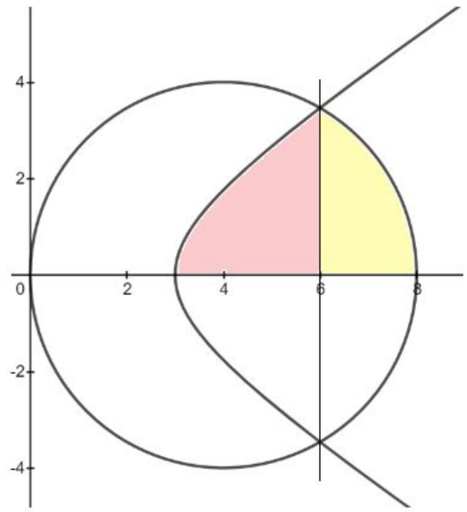
Rearrange first equation for y^2: $y^2 = 8x - x^2$
Substitute into second equation:
$\rightarrow 4x^2 - 9(8x - x^2) = 36$
$\rightarrow 4x^2 - 72x + 9x^2 - 36 = 0$
$\rightarrow 13x^2 - 72x - 36 = 0$
$\rightarrow x = 6$ or $x = -6/13$ [2 marks]

Substituting back in for y,
$\rightarrow y^2 = 8(6) - 6^2$ or $y^2 = 8(-6/13) - (-6/13)^2$
$\rightarrow y^2 = 12$ or $y^2 = -660/169$
$\rightarrow y = \pm 2\sqrt{3}$ or $\pm (2i\sqrt{165}) / 13$ [2 marks]
(reject complex solutions since coordinates are real, or from graph since clearly no intersections with $x < 0$)

So the intersections are $(6, 2\sqrt{3})$ and $(6, -2\sqrt{3})$. [1 mark]

Finding the area of the bounded region

The half-region of interest can be seen as the sum of two areas under the hyperbola (pink) and circle (yellow), then doubled (since symmetrical):



We will set up two integrals. Since the question asks for a decimal answer, we do not need to worry about how messy the integration is as the calculator will integrate numerically.

Rearranging for y in the hyperbola equation,
$4x^2 - 9y^2 = 36 \rightarrow y = \sqrt{[(4x^2 - 36) / 9]} \rightarrow y = (2/3)\sqrt{(x^2 - 9)}$ [1 mark]
Rearranging for y in the circle equation,
$x^2 + y^2 - 8x = 0 \rightarrow y = \sqrt{(8x - x^2)}$ [1 mark]

The bounds can be easily seen from previous work. The total area is

$$= 2\left(\int_3^6 \frac{2}{3}\sqrt{x^2-9}\,\mathrm{d}x + \int_6^8 \sqrt{8x - x^2}\,\mathrm{d}x\right)$$ [1 mark]

which has a numerical value of 22.7098199…

The exact form is

$$= 8\sqrt{3} + \frac{16\pi}{3} + 6\ln(2-\sqrt{3})$$

which can be derived by lengthy substitutions of x = 3 sec u in the first integral and x = 4 sin u in the second integral. [1 mark]

Either way the area is 22.7 square units to 3 significant figures as required.

2. **Part 1: Forming the differential equations**

Let A(t) and B(t) be the mass of contaminant in grams in tanks A and B respectively at time t hours after the initial conditions are recorded.
Concentration (g / L) = mass of contaminant (g) / volume of water (L)
Net flow of water into the system = 7 + 3 = 10 L / hr
Net flow of water out of the system = 10 L / hr
Flow in = flow out → volume of water in each tank remains constant. [1 mark]

Net rate of change = rate of flow in - rate of flow out
In tank 1: change = (added in from tap) + (from P) - (into Q)
→ dA/dt = (7 * 1.5) + (5 * (B/100)) - (12 * (A/50))
→ dA/dt = 21/2 + B/20 - 6A/25
→ 100 dA/dt = 1050 + 5 B - 24 A [2 marks]
In tank 2: change = (added in from tap) + (from R) - (into P)
→ dB/at = 0 + (2 * (A/50)) - (5 * (B/100))
→ dB/dt = A/25 - B/20
→ 100 dB/dt = 4 A - 5 B [2 marks]
Initial conditions: A(0) = 25, B(0) = 75
So the system of coupled differential equations to be solved is:

$$100\ \frac{\mathrm{d}A}{\mathrm{d}t} = 1050 + 5B - 24A \qquad 100\ \frac{\mathrm{d}B}{\mathrm{d}t} = 4A - 5B$$

$$A(0) = 25 \qquad B(0) = 75$$

[1 mark]

The assumptions made here are:

1. The contaminant is spread evenly throughout each tank (perfect mixing)
2. The water flows instantly through the pipes
3. The inflow and outflow from each tank are exactly equal (no net change in volume of water in the system)
4. The contaminant is fully dissolved and is transported without resistance (contaminant flowing through pipes remains proportional to flow rate)
[2 marks for any two of these assumptions]

Part 2: Converting the system to a 2nd-order DE

Differentiate the second equation:
$100\ d^2B/dt^2 = 4\ dA/dt - 5\ dB/dt \rightarrow 100\ dA/dt = 2500\ d^2B/dt^2 + 125\ dB/dt$ [1 mark]
Rearrange for A in second equation:
$4\ A = 100\ dB/dt + 5\ B \rightarrow 24\ A = 600\ dB/dt + 30\ B$ [1 mark]
Sub back into the first equation:
$2500\ d^2B/dt^2 + 125\ dB/dt = 1050 + 5\ B - 600\ dB/dt - 30\ B$ [1 mark]
$\rightarrow 2500\ d^2B/dt^2 + 725\ dB/dt + 25\ B = 1050$
$\rightarrow 100\ d^2B/dt^2 + 29\ dB/dt + B = 42$ [1 mark]
Initial conditions: At $t = 0$, $100\ B'(0) = 4(25) - 5(75) \rightarrow B'(0) = -11/4$. [1 mark]

Part 3: Solving the equations

We have $100\ d^2B/dt^2 + 29\ dB/dt + B = 42$ subject to $B(0) = 75$, $B'(0) = -11/4$.
This is a nonhomogeneous second order differential equation.
The complementary solution is: $100\ B'' + 29\ B' + B = 0 \rightarrow \lambda = -1/4$ and $\lambda = -1/25$.
$\rightarrow B_C(t) = c_1 \exp(-1/4\ t) + c_2 \exp(-1/25\ t)$ [1 mark]
The particular solution is easily deduced to be the constant 42, so
$B(t) = c_1 \exp(-1/4\ t) + c_2 \exp(-1/25\ t) + 42$ [1 mark]
Using initial conditions:
$c_1 + c_2 = 33$, $-1/4\ c_1 + -1/25\ c_2 = -11/4$ [1 mark]
Solution is $c_1 = 143/21$, $c_2 = 550/21$
$\rightarrow B(t) = 143/21 \exp(-1/4\ t) + 550/21 \exp(-1/25\ t) + 42$ [1 mark]
Using the second equation, $100\ B' = 4A - 5B \rightarrow A = (100\ B' + 5B)/4$
$A = (100\ (-143/84 \exp(-1/4\ t) - 550/525 \exp(-1/25\ t))$
$\quad + \ 5(143/21 \exp(-1/4\ t) + 550/21 \exp(-1/25\ t) + 42))/4$
$\rightarrow A = -715/21 \exp(-1/4\ t) + 275/42 \exp(-1/25\ t) + 105/2$ [2 marks]
Factoring out constant multiples, the final solutions are

$$A(t) = \frac{5}{42}\left(55\exp\left(-\frac{t}{25}\right) - 286\exp\left(-\frac{t}{4}\right) + 441\right)$$

$$B(t) = \frac{1}{21}\left(550\exp\left(-\frac{t}{25}\right) + 143\exp\left(-\frac{t}{4}\right) + 2\right)$$

[1 mark]

(Based on a problem by Paul Dawkins at
http://tutorial.math.lamar.edu/Classes/DE/SystemsModeling.aspx)

3. **Part 1: Making the substitution**

Starting with $u = (\tan x)^{2/3} \rightarrow \tan x = u^{3/2}$:
$du/dx = (2/3) (\tan x)^{-1/3} \sec^2 x$ [1 mark] $\rightarrow du = (2/3) (\tan x)^{-1/3} \sec^2 x\, dx$
$\rightarrow dx = (3/2) \tan^{1/3} x \cos^2 x\, du$ [1 mark]
Expressing this in terms of u,
$\rightarrow dx = (3/2) (\tan^{1/3} x) (1/\sec^2 x) = (3/2) (\tan^{1/3} x) (1/(1 + \tan^2 x))$
$= (3/2) (u) (1 + u^3)^{-1}$ [2 marks]
The bounds become $[\tan 0]^{2/3} = 0$ and $[\tan \pi/4]^{2/3} = 1$
So the integral is

$$\frac{3}{2}\int_0^1 \frac{u}{1+u^3}\,\mathrm{d}u$$ [1 mark]

Part 2: Partial fractions

Using partial fractions. Factorise the bottom: $u^3 + 1 = (u + 1)(u^2 - u + 1)$:

$$\frac{3}{2}\int_0^1 \frac{u}{(u+1)(u^2-u+1)}\,\mathrm{d}u = \frac{3}{2}\int_0^1 \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}\,\mathrm{d}u$$ [1 mark]

Let $u = -1$: $A = (-1) / (1 + 1 + 1) = -1/3$.
Let $u = 0$: $0 = -1/3 + C \rightarrow C = 1/3$.
Let $u = 1$: $1/2 = -1/6 + B + 1/3 \rightarrow B = 1/3$. Putting these in and factoring out 1/3,

$$= \frac{1}{2}\int_0^1 \frac{u+1}{u^2-u+1} - \frac{1}{u+1}\,\mathrm{d}u$$ [3 marks]

Part 3: The first integral

Focussing only on the first fraction (since the second is much simpler), complete the square in the denominator:

$$\int_0^1 \frac{u+1}{u^2-u+1}\,\mathrm{d}u = \int_0^1 \frac{u+1}{(u-\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}u$$ [1 mark]

Substitute t = u - 1/2 → du = dt. Bounds are now -1/2 to 1/2:

$$= \int_{-1/2}^{1/2} \frac{t+\frac{3}{2}}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t \;= \int_{-1/2}^{1/2} \frac{t}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t + \frac{3}{2}\int_{-1/2}^{1/2} \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t$$

[2 marks]

In the first fraction, integrate by substitution, giving

$$\int_{-1/2}^{1/2} \frac{t}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t = \frac{1}{2}\left[\ln\left|t^2+\frac{3}{4}\right|\right]_{-1/2}^{1/2} = 0$$

[2 marks]

In the second fraction, integrate by the standard result, using a = √3/2:

$$\frac{3}{2}\int_{-1/2}^{1/2} \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t = \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)\right]_{-1/2}^{1/2}$$
$$= \sqrt{3}\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) = \frac{\pi}{\sqrt{3}}$$

[3 marks]

Part 4: Overall integral

Now, the second integral is

$$\int_0^1 \frac{1}{u+1}\,\mathrm{d}u = [\ln|u+1|]_0^1 = \ln 2$$ [1 mark]

So overall, the integral is then,

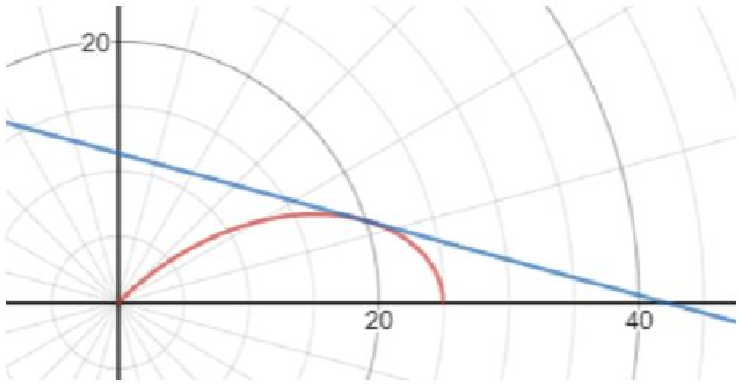
$$\frac{1}{2}\left(\frac{\pi}{\sqrt{3}} - \ln 2\right) = \frac{\pi\sqrt{3}}{6} - \frac{1}{2}\ln 2$$. [2 marks]

(Solution by blackpenredpen at

https://www.youtube.com/watch?v=NO693oP7nHQ)

4. **Part 1: Finding the gradient of the curve**

Start with a sketch:



From Q13 we know that the gradient of a polar curve is given by

$$\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

, where r = 25 cos 2θ → dr/dθ = -50 sin 2θ

[2 marks; or derive again]

So in this case the gradient is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{25\cos 2\theta\cos\theta - 50\sin 2\theta\sin\theta}{-25\cos 2\theta\sin\theta - 50\sin 2\theta\cos\theta} = \frac{2\sin 2\theta\sin\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}$$

[1 mark]

Part 2: Finding the point where the curve meets the tangent

So the point at which the curve meets the tangent is when

$$\frac{2\sin 2\theta\sin\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta} = -\frac{3}{11}$$ [1 mark]

$$22\sin 2\theta\sin\theta - 11\cos 2\theta\cos\theta = -6\sin 2\theta\cos\theta - 3\cos 2\theta\sin\theta$$

Factor out/divide by cos θ cos 2θ. The lost solutions will be θ = π/2 and θ = π/4, but these cannot be correct since the domain of the curve is 0 ≤ θ ≤ π/4 and θ = π/4 is at the pole. [2 marks]

The remaining equation is $22 \tan 2\theta \tan \theta - 11 = -6 \tan 2\theta - 3 \tan \theta$.
→ $(22 \tan \theta + 6) \tan 2\theta = 11 - 3 \tan \theta$ [2 marks]
Using double angle identity,

$$(22\tan\theta + 6)\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = 11 - 3\tan\theta$$

$$44\tan^2\theta + 12\tan\theta = (11 - 3\tan\theta)(1-\tan^2\theta)$$

$$3\tan^3\theta - 55\tan^2\theta - 15\tan\theta + 11 = 0$$ [2 marks]

Solving on a calculator, we find a root at $\tan \theta = 1/3$, and the other two roots can be found by factoring as $\tan \theta = 9 \pm \sqrt{92}$. [2 marks]

Part 3: Justifying which root is correct

The point of tangency is to the right of the turning point of the curve. This fact can be used to identify which root is correct. Turning point → $dy/dx = 0$, so
$2 \sin 2\theta \sin \theta - \cos 2\theta \cos \theta = 0$ [1 mark]
→ $4 \sin^2 \theta \cos \theta - \cos \theta + 2 \sin^2 \theta \cos \theta = 0$
→ $\cos \theta (6 \sin^2 \theta - 1) = 0$ → $\cos \theta = 0$, $\sin \theta = \pm 1/\sqrt{6}$,
→ $\theta = 90^\circ, 24.1^\circ, -24.1^\circ, 204.1^\circ$… Only solution in range is $\theta = 24.1^\circ$, so the correct root must have $\theta < 24.1^\circ$ [1 mark]
(since the curve is traced out right to left).

Taking inverse tangents on all roots gives $\theta = 18.4^\circ$, -30.6° and 86.9°. So the correct root is $\tan \theta = 1/3$ → $\theta = \tan^{-1}(1/3)$. [1 mark]

Part 4: Finding Cartesian coordinates of points of interest

$\tan \theta = 1/3$ → $\sin \theta = 1/\sqrt{10}$ and $\cos \theta = 3/\sqrt{10}$ (by Pythagorean identities)
→ $\cos 2\theta = 2(9/10) - 1 = 4/5$ and $\sin 2\theta = 2(1/\sqrt{10})(3/\sqrt{10}) = 3/5$ [1 mark]
→ $r = 25 \cos 2\theta = 25(4/5) = 20$.
→ $x = r \cos \theta = 20(3/\sqrt{10}) = 6\sqrt{10}$
→ $y = r \sin \theta = 20(1/\sqrt{10}) = 2\sqrt{10}$ [2 marks]
So the point of tangency is $(2\sqrt{10}, 6\sqrt{10})$.
Using the point-slope formula for line L,
$y - 6\sqrt{10} = -3/11(x - 2\sqrt{10})$ → $3x + 11y = 40\sqrt{10}$. [1 mark]
Intercept with initial line: $y = 0$ → $x = (40/3)\sqrt{10}$

So the area of the triangle is 1/2 * x * y = (1/2) (40/3) (2√10) (√10) = 400/3.
[1 mark]

Part 5: Area of the region

The desired area is the area of the triangle minus the area of the polar curve up to the point of tangency. The bounds are θ = 0 and θ = $\tan^{-1}(1/3)$. The area is,

$$= \frac{400}{3} - \frac{625}{2}\int_0^{\tan^{-1}\left(\frac{1}{3}\right)} \cos^2 2\theta \ d\theta$$ [1 mark]

$$= \frac{400}{3} - \frac{625}{4}\int_0^{\tan^{-1}\left(\frac{1}{3}\right)} \cos 4\theta + 1 \ d\theta$$

$$= \frac{400}{3} - \frac{625}{4}\left[\frac{1}{4}\sin 4\theta + \theta\right]_0^{\tan^{-1}\left(\frac{1}{3}\right)}$$ [1 mark]

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{1}{4}\sin\left[4\tan^{-1}\left(\frac{1}{3}\right)\right] + \tan^{-1}\left(\frac{1}{3}\right)\right)$$

Using double-angle identity,

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{1}{2}\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]\cos\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] + \tan^{-1}\left(\frac{1}{3}\right)\right)$$ [1 mark]

Recall from earlier calculations, $\tan^{-1}(1/3)$ = θ and sin 2θ = 3/5 and cos 2θ = 4/5,

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{4}{5} + \tan^{-1}\left(\frac{1}{3}\right)\right)$$ [1 mark]

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{6}{25} + \tan^{-1}\left(\frac{1}{3}\right)\right)$$

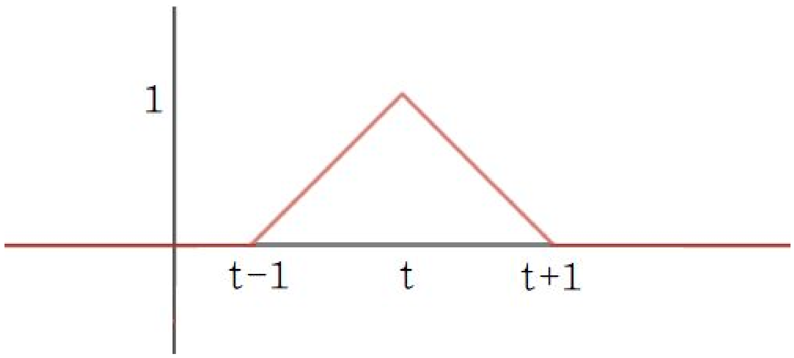
$$= \frac{575}{6} - \frac{625}{4}\tan^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{25}{12}\left[46 - 75\tan^{-1}\left(\frac{1}{3}\right)\right]$$. [1 mark]

(Solution by T. Madas at https://madasmaths.com/archive/maths_booklets/further_topics/various/polar_coOrdinates_exam_questions.pdf, Question 47)

5. **Part 1: Working with f(x)**

Begin by sketching a graph of f(x) to help later understanding.
It has the form of y = -|x| translated by [t; 1]:



[1 mark for sketch]

The lines have gradients 1 and -1 and pass through (t, 1) so using point-slope forms gives (splitting modulus function into two parts):

$$f(x)=\begin{cases} (x-t)+1= \quad x-t+1 & (t-1\le x<t) \\ -(x-t)+1=-x+t+1 & (t\le x\le t+1) \\ 0 & \text{otherwise} \end{cases}$$

[1 mark]

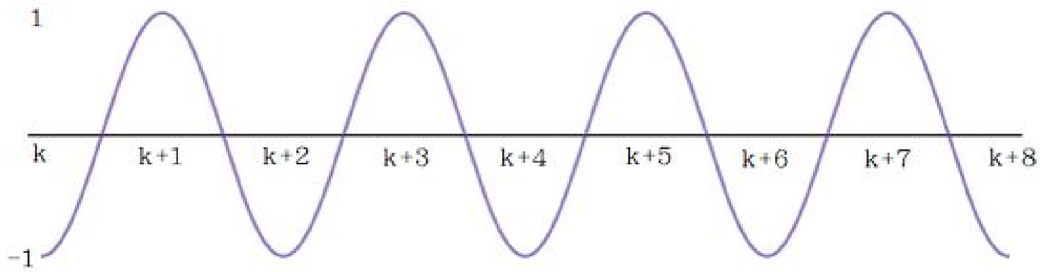
This form is much easier to work with when integrating it.

Part 2: Planning work with g(t)

Next, study g(t). This is a function of t and not x since the x's are all replaced by k when the integration bounds are done. The cosine function in g(t) has a period of 2. Since the integration period is a distance of (k + 8) - k = 8, we consider 8/2 = 4 periods of this function.

Since k is odd, the range of integration starts from the trough (bottom) of the wave and ends at another trough. The product of this wave with f(x) gives the function to integrate. When f(x) is zero, the product is also zero and we can ignore these intervals. There are multiple different possibilities depending on the value of t since this is what moves f(x) left and right through the intervals.

Generalising, the cosine function looks like,



We must consider each possible case of how f(x) lies within this range. For the purpose of speeding calculations up, we should calculate the indefinite integral forms of g(t) first, then put in bounds when needed as these will vary significantly.

The first integral to evaluate is when dealing with the increasing part of f(x):

$$\int (x - t + 1)\cos(\pi x)\ dx$$

$$\int x\cos(\pi x)\ dx - (t-1)\int \cos(\pi x)\ dx$$

Integrating by parts on the first one, (+C omitted since these results will be used with definite integrals later on)

$$\int x\cos(\pi x)\ dx = \frac{x}{\pi}\sin(\pi x) - \frac{1}{\pi^2}\cos(\pi x)$$

The second integral is easily evaluated,

$$(t-1)\int \cos(\pi x)\ dx = \frac{(t-1)}{\pi}\sin(\pi x)$$

Combining and factorising,

$$\int (x - t + 1)\cos(\pi x)\ dx = \frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}$$

. [2 mar ks]

The second integral covers the decreasing part of f(x):

$$\int (-x + t + 1)\cos(\pi x)\ dx$$

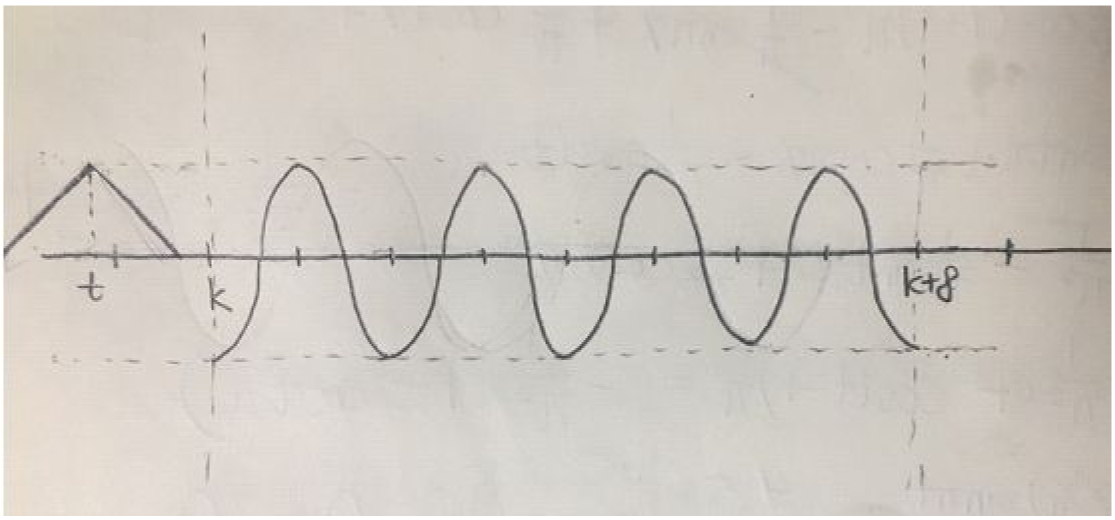
Repeating the same method and factorising again,

$$\int (-x + t + 1)\cos(\pi x)\ dx = \frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}$$. [2 marks]

Part 3: Defining g(t)

Case 1: Fully outside range

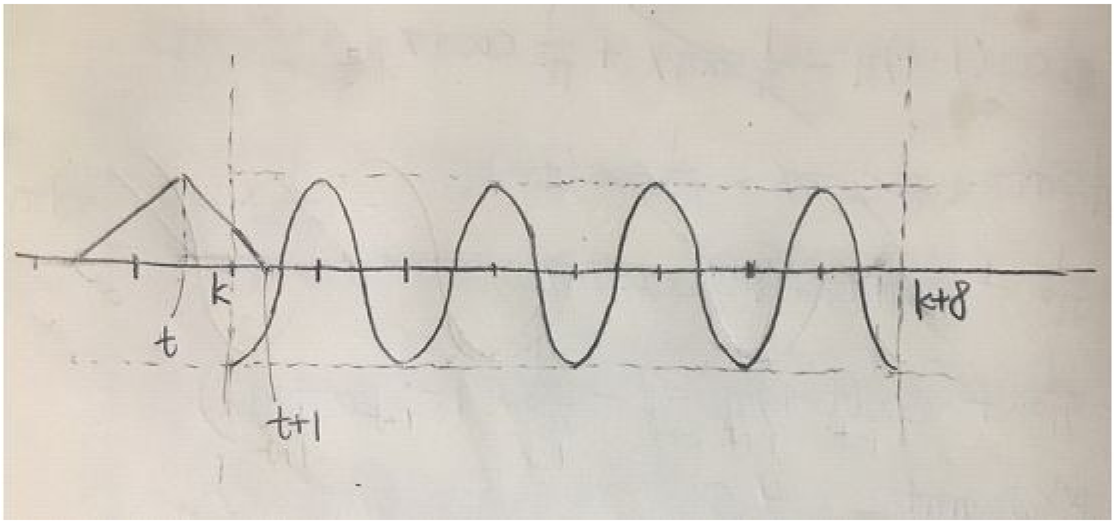
If the triangle part (non-zero) of f(x) is located fully outside this range, then the integral is zero everywhere:



This happens when either t + 1 ≤ k or k + 8 ≤ t - 1. These conditions can be written in terms of t as t < k - 1 or t ≥ k + 9.
In these cases, f(x) cos(πx) = 0 and therefore g(t) = 0. [1 mark]

Case 2: The decreasing part of f(x) is partly in the interval

If f(x) shifts to the right such that part of its decreasing part is in the interval, the integral is evaluated on the overlapping part using the decreasing section of f(x):



This happens when k - 1 ≤ t < k. The function becomes

$$g(t) = \int_{k}^{t+1} (-x + t + 1)\cos(\pi x)\ dx$$

Using the found result from part 2,

$$g(t) = \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=k}^{x=t+1}$$

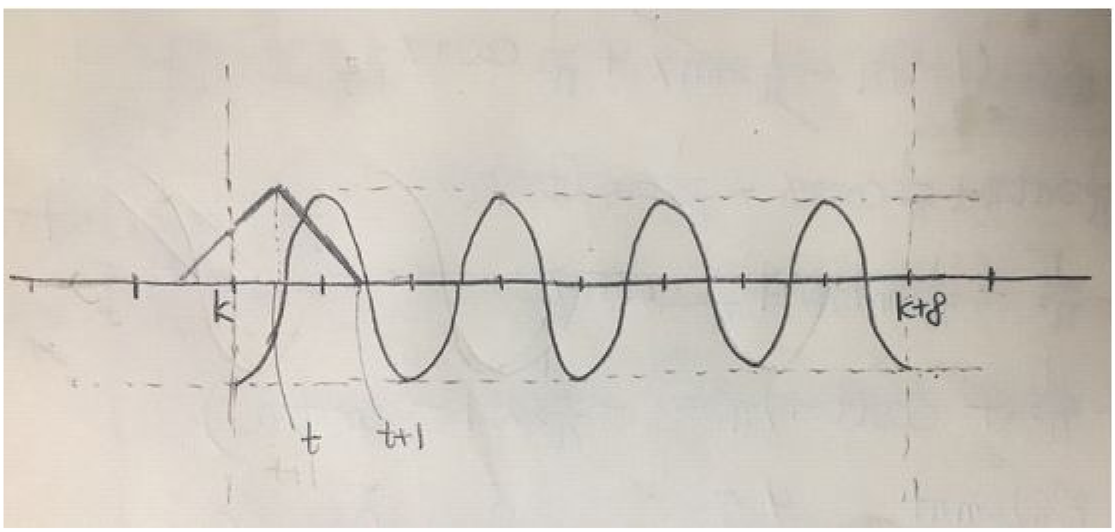
Using translation identities, we can simplify the input of x = t + 1 using
sin(x + π) = -sin(x) and cos(x + π) = -cos(x).
We can also simplify the input of x = k since k is odd, so
sin(kπ) = 0 and cos(kπ) = -1. Therefore

$$g(t) = \frac{1}{\pi^2}(\cos(\pi t) - 1)$$

[3 marks]

Case 3: The increasing part of f(x) is partly in the interval

Shifting f(x) more to the right puts the downward part fully in, but the upward part not fully in:



This happens when $k \le t < k + 1$.
In this case, we have two integrals to consider, from each part of the function:

$$g(t) = \int_k^t (x - t + 1)\cos(\pi x)\ dx + \int_t^{t+1} (-x + t + 1)\cos(\pi x)\ dx$$

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=k}^{x=t} + \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=t}^{x=t+1}$$

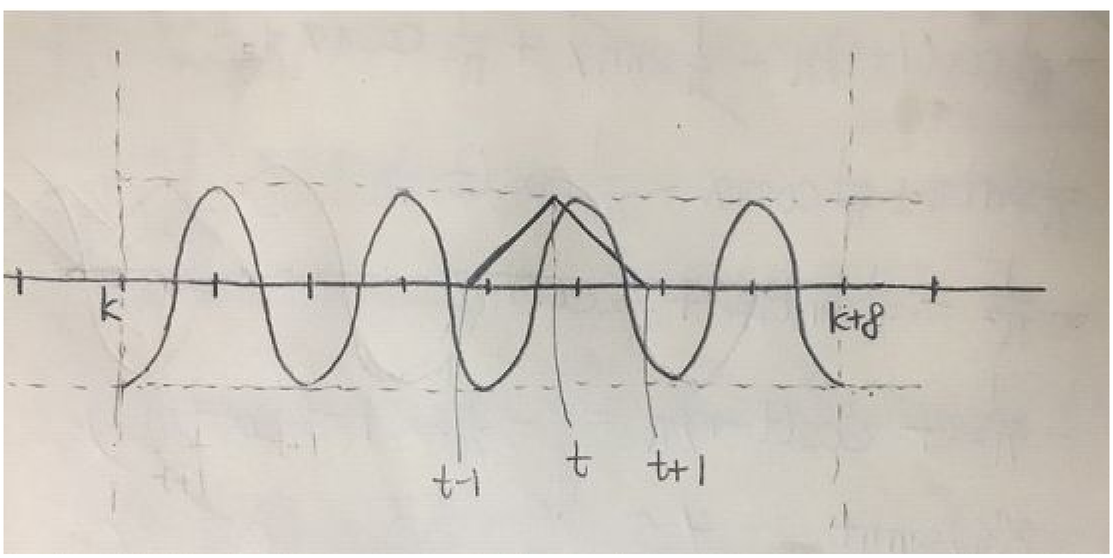
$$g(t) = \left(\frac{(\pi\sin(\pi t) + \cos(\pi t)) + 1}{\pi^2}\right) + \left(\frac{\cos(\pi t) - (\pi\sin(\pi t) - \cos(\pi t))}{\pi^2}\right)$$

$$g(t) = \frac{1}{\pi^2}(1 + 3\cos(\pi t))$$

[3 marks]

Case 4: f(x) is fully within interval

This is the 'standard' case, with no sticking out parts of f(x) at the boundaries.



This happens when k + 1 ≤ t < k + 7. This produces two full integrals, one for each part of f(x):

$$g(t) = \int_{t-1}^{t} (x - t + 1)\cos(\pi x)\ dx + \int_{t}^{t+1} (-x + t + 1)\cos(\pi x)\ dx$$

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=t-1}^{x=t} + \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=t}^{x=t+1}$$

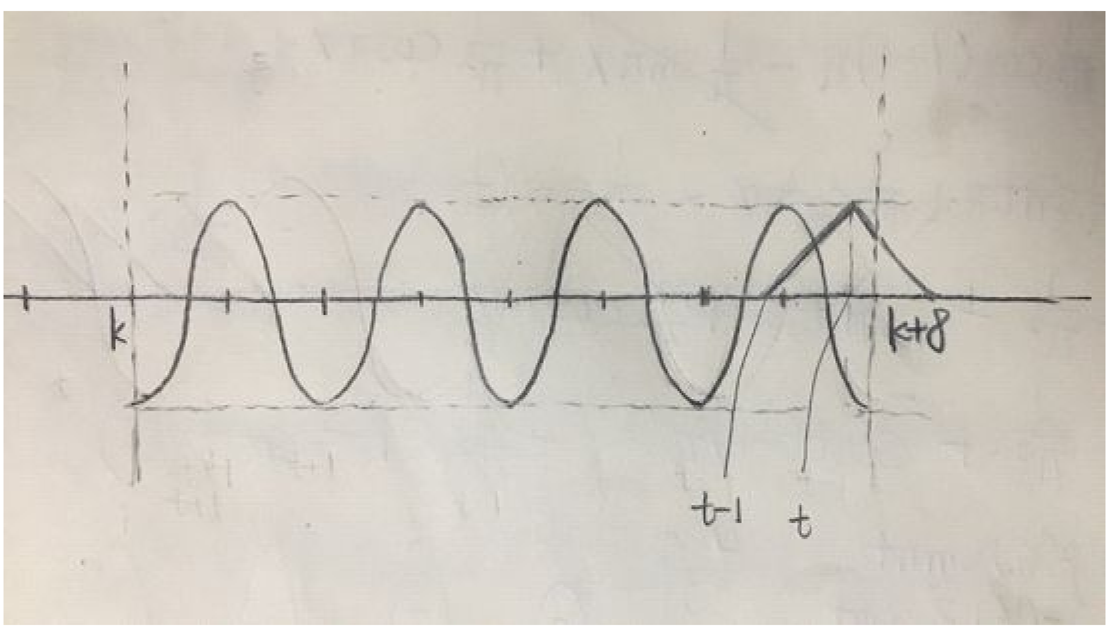
$$g(t) = \left(\frac{(\pi\sin(\pi t) + \cos(\pi t)) + \cos(\pi t)}{\pi^2}\right) + \left(\frac{\cos(\pi t) - (\pi\sin(\pi t) - \cos(\pi t))}{\pi^2}\right)$$

$$g(t) = \frac{4}{\pi^2}\cos(\pi t)$$

[3 marks]

Case 5: The 'other' decreasing part of f(x) is partly in the interval

This is the mirror image of case 3:



This happens when k + 7 ≤ t < k + 8 and the integrals are

$$g(t) = \int_{t-1}^{t} (x - t + 1)\cos(\pi x)\ dx + \int_{t}^{k+8} (-x + t + 1)\cos(\pi x)\ dx$$

Since k is odd, k + 8 is also odd, so we can use the same rules when substituting its limit in:

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=t-1}^{x=t} + \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=t}^{x=k+8}$$

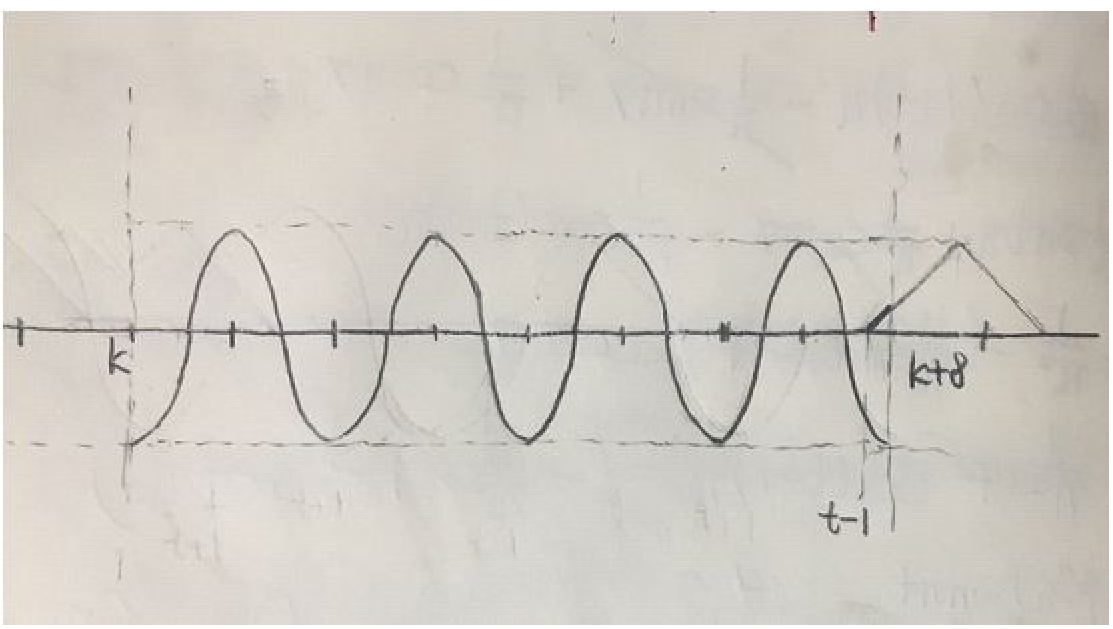
$$g(t) = \left(\frac{\pi\sin(\pi t) + 2\cos(\pi t)}{\pi^2}\right) + \left(\frac{1 - (\pi\sin(\pi t) - \cos(\pi t))}{\pi^2}\right)$$

$$g(t) = \frac{1}{\pi^2}(1 + 3\cos(\pi t))$$

[3 marks]

Case 6: The 'other' increasing part of f(x) is partly in the interval

This is the mirror image of case 2:



This happens when k + 8 ≤ t < k + 9. There is only one part of f(x) in the interval so only one integral:

$$g(t) = \int_{t-1}^{k+8} (x - t + 1)\cos(\pi x)\ dx$$
$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=t-1}^{x=k+8}$$
$$g(t) = \left(\frac{-1 + \cos(\pi t)}{\pi^2}\right)$$
$$g(t) = \frac{1}{\pi^2}(\cos(\pi t) - 1)$$

[2 marks]

Part 4: Putting it all together

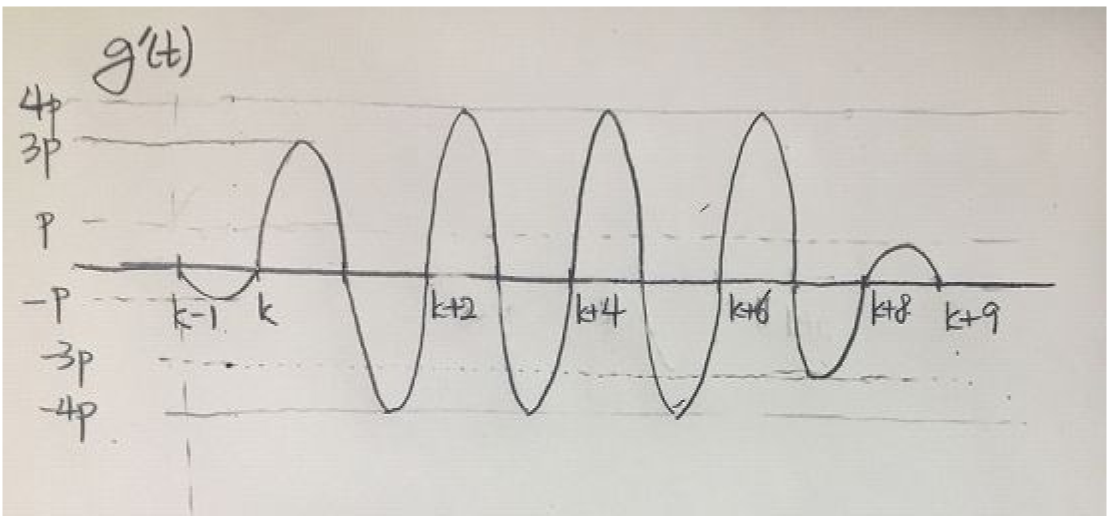
Now, combine all these possible cases into a piecewise function

$$g(t) = \begin{cases} -\frac{1}{\pi^2}(1 - \cos\pi t) & (k-1 \le t < k) \\ \frac{1}{\pi^2}(1 + 3\cos\pi t) & (k \le t < k+1) \\ \frac{4}{\pi^2}\cos\pi t & (k+1 \le t < k+7) \\ \frac{1}{\pi^2}(1 + 3\cos\pi t) & (k+7 \le t < k+8) \\ -\frac{1}{\pi^2}(1 - \cos\pi t) & (k+8 \le t < k+9) \\ 0 & \text{otherwise} \end{cases}$$

We need the minima of this function. Differentiate each function:

$$g'(t)=\begin{cases} -\dfrac{1}{\pi}\sin\pi t & (k-1\le t<k) \\ -\dfrac{3}{\pi}\sin\pi t & (k\le t<k+1) \\ -\dfrac{4}{\pi}\sin\pi t & (k+1\le t<k+7) \\ -\dfrac{3}{\pi}\sin\pi t & (k+7\le t<k+8) \\ -\dfrac{1}{\pi}\sin\pi t & (k+8\le t<k+9) \\ 0 & \text{otherwise} \end{cases}$$

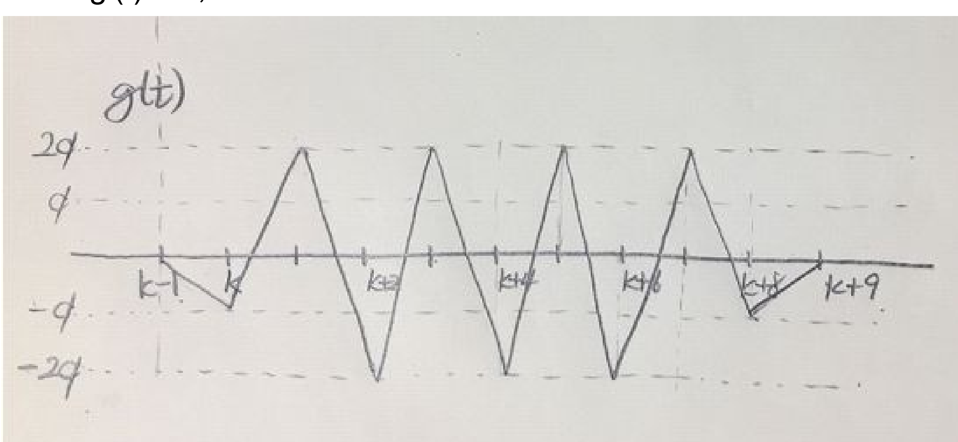
This graph is much easier to sketch:



[1 mark]

We can use this graph to approximately sketch g(t). When g'(t) is negative, g(t) will be decreasing, and when g'(t) is positive, g(t) will be increasing.

When g'(t) = 0, we have a minimum or maximum:



(In the above sketches, p and q are used for simplicity as proportionality constants: p = $1/\pi$ and q = $1/\pi^2$.) [1 mark]

Part 5: Getting the desired value

From the graph, we the minima of g(t) such that g(t) < 0 are
t = k, k + 2, k + 4, k + 6 and k + 8.
The sum of these values is 45:
k + (k + 2) + (k + 4) + (k + 6) + (k + 8) = 45 → 5k + 20 = 45 → k = 5. [1 mark]
The values of α are then α = {5, 7, 9, 11, 13}
Subbing into g(t), we get

$$g(\alpha) = \{-\frac{2}{\pi^2}, -\frac{4}{\pi^2}, -\frac{4}{\pi^2}, -\frac{4}{\pi^2}, -\frac{2}{\pi^2}, \}$$

$$\sum_{i=1}^{5} g(\alpha_i) = -\frac{2}{\pi^2} - \frac{4}{\pi^2} - \frac{4}{\pi^2} - \frac{4}{\pi^2} - \frac{2}{\pi^2}$$

$$\sum_{i=1}^{5} g(\alpha_i) = -\frac{16}{\pi^2}$$

Finally, calculating the required value

$$k - \pi^2 \sum_{i=1}^{5} g(\alpha_i) = 5 - \pi^2 \left(-\frac{16}{\pi^2}\right) = 5 + 16 = 21.$$

[1 mark]

(Solution by 돌핀 at https://blog.naver.com/dak219324/221147969311)