# CALCULUS I

## Review

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# Preface

Here are the notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self-contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

# Chapter 1 : Review

Technically a student coming into a Calculus class is supposed to know both Algebra and Trigonometry. Unfortunately, the reality is often much different. Most students enter a Calculus class woefully unprepared for both the algebra and the trig that is in a Calculus class. This is very unfortunate since good algebra skills are absolutely vital to successfully completing any Calculus course and if your Calculus course includes trig (as this one does) good trig skills are also important in many sections.

The intent of this chapter is to do a very cursory review of some algebra and trig skills that are vital to a calculus course. This chapter does not include all the algebra and trig skills that are needed to be successful in a Calculus course. It only includes those topics that most students are particularly deficient in. For instance, factoring is also vital to completing a standard calculus class but is not included here as it is assumed that if you are taking a Calculus course then you do know how to factor. For a more in depth review you should check out the full set of Algebra notes at **http://tutorial.math.lamar.edu**.

Note that even though these topics are very important to a Calculus class I rarely cover all of them in the actual class itself. We simply don't have the time to do that. I do cover certain portions of this chapter in class, but for the most part I leave it to the students to read this chapter on their own.

Here is a list of topics that are in this chapter.

**Functions** – In this section we will cover function notation/evaluation, determining the domain and range of a function and function composition.

**Inverse Functions** – In this section we will define an inverse function and the notation used for inverse functions. We will also discuss the process for finding an inverse function.

**Trig Functions** – In this section we will give a quick review of trig functions. We will cover the basic notation, relationship between the trig functions, the right triangle definition of the trig functions. We will also cover evaluation of trig functions as well as the unit circle (one of the most important ideas from a trig class!) and how it can be used to evaluate trig functions.

**Solving Trig Equations** – In this section we will discuss how to solve trig equations. The answers to the equations in this section will all be one of the "standard" angles that most students have memorized after a trig class. However, the process used here can be used for any answer regardless of it being one of the standard angles or not.

**Solving Trig Equations with Calculators, Part I** – In this section we will discuss solving trig equations when the answer will (generally) require the use of a calculator (i.e. they aren't one of the standard angles). Note however, the process used here is identical to that for when the answer is one of the standard angles. The only difference is that the answers in here can be a little messy due to the need of a calculator. Included is a brief discussion of inverse trig functions.

**Solving Trig Equations with Calculators, Part II** – In this section we will continue our discussion of solving trig equations when a calculator is needed to get the answer. The equations in this section tend to be a little trickier than the "normal" trig equation and are not always covered in a trig class.

**Exponential Functions** – In this section we will discuss exponential functions. We will cover the basic definition of an exponential function, the natural exponential function, i.e. $\mathbf{e}^x$, as well as the properties and graphs of exponential functions.

**Logarithm Functions** – In this section we will discuss logarithm functions, evaluation of logarithms and their properties. We will discuss many of the basic manipulations of logarithms that commonly occur in Calculus (and higher) classes. Included is a discussion of the natural (\(\ln(x)\)) and common logarithm (\(\log(x)\)) as well as the change of base formula.

**Exponential and Logarithm Equations** – In this section we will discuss various methods for solving equations that involve exponential functions or logarithm functions.

**Common Graphs** – In this section we will do a very quick review of many of the most common functions and their graphs that typically show up in a Calculus class.

# Section 1-1 : Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class so you will need to be able to deal with them.

First, what exactly is a function? The simplest definition is an equation will be a function if, for any *x* in the domain of the equation (the domain is all the *x*'s that can be plugged into the equation), the equation will yield exactly one value of *y* when we evaluate the equation at a specific *x*.

This is usually easier to understand with an example.

***Example 1*** Determine if each of the following are functions.

**(a)** $y = x^2 + 1$

**(b)** $y^2 = x + 1$

***Solution***

**(a)** This first one is a function. Given an *x*, there is only one way to square it and then add 1 to the result. So, no matter what value of *x* you put into the equation, there is only one possible value of *y* when we evaluate the equation at that value of *x*.

**(b)** The only difference between this equation and the first is that we moved the exponent off the *x* and onto the *y*. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of *x*, say $x = 3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of *y* that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of *y* that we get from a single *x* this equation isn't a function.

Note that this only needs to be the case for a single value of *x* to make an equation not be a function. For instance, we could have used $x = -1$ and in this case, we would get a single *y* ( $y = 0$ ). However, because of what happens at $x = 3$ this equation will not be a function.

Next, we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the *y* in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation, we can write this as any of the following.

$$\begin{array}{ll} f\left(x\right) = 2x^2 - 5x + 3 & g\left(x\right) = 2x^2 - 5x + 3 \\ h\left(x\right) = 2x^2 - 5x + 3 & R\left(x\right) = 2x^2 - 5x + 3 \\ w\left(x\right) = 2x^2 - 5x + 3 & y\left(x\right) = 2x^2 - 5x + 3 \\ & \vdots \end{array}$$

Recall that this is NOT a letter times *x*, this is just a fancy way of writing *y*.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x=-3$. Using function notation we represent the value of the function at $x=-3$ as $f(-3)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an *x* on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$\begin{aligned} f(-3) &= 2(-3)^2 - 5(-3) + 3 \\ &= 2(9) + 15 + 3 \\ &= 36 \end{aligned}$$

Let's take a look at some more function evaluation.

***Example 2*** Given $f(x) = -x^2 + 6x - 11$ find each of the following.

**(a)** $f(2)$
**(b)** $f(-10)$
**(c)** $f(t)$
**(d)** $f(t-3)$
**(e)** $f(x-3)$
**(f)** $f(4x-1)$

***Solution***

**(a)** $f(2) = -(2)^2 + 6(2) - 11 = -3$

**(b)** $f(-10) = -(-10)^2 + 6(-10) - 11 = -100 - 60 - 11 = -171$

Be careful when squaring negative numbers!

**(c)** $f(t) = -t^2 + 6t - 11$

Remember that we substitute for the *x*'s WHATEVER is in the parenthesis on the left. Often this will be something other than a number. So, in this case we put *t*'s in for all the *x*'s on the left.

**(d)** $f(t-3) = -(t-3)^2 + 6(t-3) - 11 = -t^2 + 12t - 38$

Often instead of evaluating functions at numbers or single letters we will have some fairly complex evaluations so make sure that you can do these kinds of evaluations.

**(e)** $f(x-3) = -(x-3)^2 + 6(x-3) - 11 = -x^2 + 12x - 38$

The only difference between this one and the previous one is that we changed the *t* to an *x*. Other than that, there is absolutely no difference between the two! Don't get excited if an *x* appears inside the parenthesis on the left.

**(f)** $f\left(4x-1\right)=-\left(4x-1\right)^2+6\left(4x-1\right)-11=-16x^2+32x-18$
This one is not much different from the previous part. All we did was change the equation that we were plugging into the function.

All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, $g\left(x\right)$, is equivalent to solving

$$g\left(x\right)=0$$

***Example 3*** Determine all the roots of $f\left(t\right)=9t^3-18t^2+6t$

***Solution***
So, we will need to solve,

$$9t^3-18t^2+6t=0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t\left(3t^2-6t+2\right)=0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t=0 \qquad \text{OR,}$$
$$3t^2-6t+2=0$$

From the first it's clear that one of the roots must then be $t=0$. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$
\begin{aligned}
t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\
&= \frac{6 \pm \sqrt{12}}{6} \\
&= \frac{6 \pm \sqrt{(4)(3)}}{6} \\
&= \frac{6 \pm 2\sqrt{3}}{6} \\
&= \frac{3 \pm \sqrt{3}}{3} \\
&= 1 \pm \frac{1}{3}\sqrt{3} \\
&= 1 \pm \frac{1}{\sqrt{3}}
\end{aligned}
$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0,\ t = \frac{3+\sqrt{3}}{3},\ \ t = \frac{3-\sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the last time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above example are not really all that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

One of the more important ideas about functions is that of the **domain** and **range** of a function. In simplest terms the domain of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and logarithms of negative numbers (if not familiar with logarithms we'll take a look at them a little **later**), *etc*. The range of a function is simply the set of all possible values that a function can take.

Let's find the domain and range of a few functions.

***Example 4*** Find the domain and range of each of the following functions.

**(a)** $f\left(x\right)=5x-3$

**(b)** $g\left(t\right)=\sqrt{4-7t}$

**(c)** $h\left(x\right)=-2x^2+12x+5$

**(d)** $f\left(z\right)=\left|z-6\right|-3$

**(e)** $g\left(x\right)=8$

***Solution***

**(a)** $f\left(x\right)=5x-3$

We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero…). This means that this function can take on any value and so the range is all real numbers. Using "mathematical" notation this is,

$$\text{Range}: \left(-\infty,\infty\right)$$

This is more generally a polynomial and we know that we can plug any value into a polynomial and so the domain in this case is also all real numbers or,

$$\text{Domain}: -\infty < x < \infty \qquad \text{or} \qquad \left(-\infty,\infty\right)$$

**(b)** $g\left(t\right)=\sqrt{4-7t}$

This is a square root and we know that square roots are always positive or zero. We know then that the range will be,

$$\text{Range}: \left[0,\infty\right)$$

For the domain we have a little bit of work to do, but not much. We need to make sure that we don't take square roots of any negative numbers, so we need to require that,

$$\begin{aligned} 4-7t &\geq 0 \\ 4 &\geq 7t \\ \tfrac{4}{7} &\geq t \qquad \Rightarrow \qquad t \leq \tfrac{4}{7} \end{aligned}$$

The domain is then,

$$\text{Domain}: t \leq \tfrac{4}{7} \qquad \text{or} \qquad \left(-\infty,\tfrac{4}{7}\right]$$

**(c)** $h(x) = -2x^2 + 12x + 5$

Here we have a quadratic, which is a polynomial, so we again know that the domain is all real numbers or,

$$\text{Domain}: \ -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

In this case the range requires a little bit of work. From an Algebra class we know that the graph of this will be a **parabola** that opens down (because the coefficient of the $x^2$ is negative) and so the vertex will be the highest point on the graph. If we know the vertex we can then get the range. The vertex is then,

$$x = -\frac{12}{2(-2)} = 3 \qquad y = h(3) = -2(3)^2 + 12(3) + 5 = 23 \qquad \Rightarrow \qquad (3, 23)$$

So, as discussed, we know that this will be the highest point on the graph or the largest value of the function and the parabola will take all values less than this, so the range is then,

$$\text{Range}: \ (-\infty, 23]$$

**(d)** $f(z) = |z - 6| - 3$

This function contains an absolute value and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if $z = 6$ and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting 3 from the absolute value portion and so we then know that the range will be,

$$\text{Range}: \ [-3, \infty)$$

We can plug any value into an absolute value and so the domain is once again all real numbers or,

$$\text{Domain}: \ -\infty < z < \infty \quad \text{or} \quad (-\infty, \infty)$$

**(e)** $g(x) = 8$

This function may seem a little tricky at first but is actually the easiest one in this set of examples. This is a constant function and so any value of *x* that we plug into the function will yield a value of 8. This means that the range is a single value or,

$$\text{Range}: \ 8$$

The domain is all real numbers,

$$\text{Domain}: \ -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

In general, determining the range of a function can be somewhat difficult. As long as we restrict ourselves down to “simple” functions, some of which we looked at in the previous example, finding the range is not too bad, but for most functions it can be a difficult process.

Because of the difficulty in finding the range for a lot of functions we had to keep those in the previous set somewhat simple, which also meant that we couldn’t really look at some of the more complicated domain examples that are liable to be important in a Calculus course. So, let's take a look at another set of functions only this time we'll just look for the domain.

***Example 5*** Find the domain of each of the following functions.

**(a)** $f(x) = \dfrac{x-4}{x^2-2x-15}$

**(b)** $g(t) = \sqrt{6+t-t^2}$

**(c)** $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

***Solution***

**(a)** $f(x) = \dfrac{x-4}{x^2-2x-15}$

Okay, with this problem we need to avoid division by zero, so we need to determine where the denominator is zero which means solving,

$$x^2-2x-15 = (x-5)(x+3) = 0 \qquad \Rightarrow \qquad x=-3,\ x=5$$

So, these are the only values of *x* that we need to avoid and so the domain is,

$$\text{Domain : All real numbers except } x=-3 \ \& \ x=5$$

**(b)** $g(t) = \sqrt{6+t-t^2}$

In this case we need to avoid square roots of negative numbers and so need to require that,

$$6+t-t^2 \ge 0 \qquad \Rightarrow \qquad t^2-t-6 \le 0$$

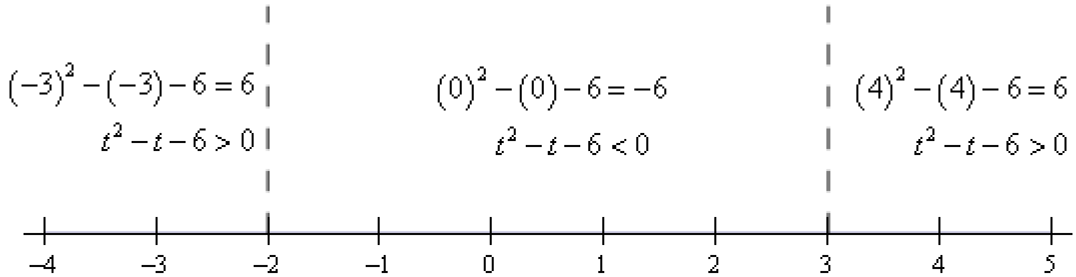
Note that we multiplied the whole inequality by -1 (and remembered to switch the direction of the inequality) to make this easier to deal with. You'll need to be able to solve inequalities like this more than a few times in a Calculus course so let's make sure you can solve these.

The first thing that we need to do is determine where the function is zero and that's not too difficult in this case.

$$t^2-t-6 = (t-3)(t+2) = 0$$

So, the function will be zero at $t=-2$ and $t=3$. Recall that these points will be the only place where the function *may* change sign. It's not required to change sign at these points, but these will be the only points where the function can change sign. This means that all we need to do is break up a number line into the three regions that avoid these two points and test the sign of the function at a single point in each of the regions. If the function is positive at a single point in the region it will be positive at all points in that region because it doesn't contain the any of the points where the function may change sign. We'll have a similar situation if the function is negative for the test point.

So, here is a number line showing these computations.




From this we can see that the only region in which the quadratic (in its modified form) will be negative is in the middle region. Recalling that we got to the modified region by multiplying the quadratic by a -1 this means that the quadratic under the root will only be positive in the middle region and so the domain for this function is then,

$$\text{Domain}: \ -2 \le t \le 3 \qquad \text{or} \qquad [-2,3]$$

**(c)** $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

In this case we have a mixture of the two previous parts. We have to worry about division by zero and square roots of negative numbers. We can cover both issues by requiring that,

$$x^2-9>0$$

Note that we need the inequality here to be strictly greater than zero to avoid the division by zero issues. We can either solve this by the method from the previous example or, in this case, it is easy enough to solve by inspection. The domain is this case is,

$$\text{Domain}: \ x<-3 \ \& \ x>3 \qquad \text{or} \qquad (-\infty,-3) \ \& \ (3,\infty)$$

The next topic that we need to discuss here is that of **function composition**. The composition of $f(x)$ and $g(x)$ is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will more often than not result in a different answer.

***Example 6*** Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

**(a)** $(f \circ g)(5)$

**(b)** $(f \circ g)(x)$

**(c)** $(g \circ f)(x)$

**(d)** $(g \circ g)(x)$

***Solution***

**(a)** $(f \circ g)(5)$

In this case we've got a number instead of an *x* but it works in exactly the same way.

$$\begin{aligned}(f \circ g)(5) &= f(g(5)) \\ &= f(-99) = 29512\end{aligned}$$

**(b)** $(f \circ g)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1-20x) \\ &= 3(1-20x)^2 - (1-20x) + 10 \\ &= 3(1 - 40x + 400x^2) - 1 + 20x + 10 \\ &= 1200x^2 - 100x + 12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

**(c)** $(g \circ f)(x)$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x^2 - x + 10) \\ &= 1 - 20(3x^2 - x + 10) \\ &= -60x^2 + 20x - 199\end{aligned}$$

And just to make the point one more time. This answer is different from the previous part. Order is important in composition.

**(d)** $(g \circ g)(x)$

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$
\begin{aligned}
(g \circ g)(x) &= g(g(x)) \\
&= g(1-20x) \\
&= 1-20(1-20x) \\
&= 400x-19
\end{aligned}
$$

Let's work one more example that will lead us into the next section.

***Example 7*** Given $f(x)=3x-2$ and $g(x)=\frac{1}{3}x+\frac{2}{3}$ find each of the following.

**(a)** $(f \circ g)(x)$

**(b)** $(g \circ f)(x)$

***Solution***

**(a)**

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{1}{3}x+\frac{2}{3}\right) \\
&= 3\left(\frac{1}{3}x+\frac{2}{3}\right)-2 \\
&= x+2-2 \\
&= x
\end{aligned}
$$

**(b)**

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(3x-2) \\
&= \frac{1}{3}(3x-2)+\frac{2}{3} \\
&= x-\frac{2}{3}+\frac{2}{3} \\
&= x
\end{aligned}
$$

In this case the two compositions were the same and in fact the answer was very simple.

$$(f \circ g)(x) = (g \circ f)(x) = x$$

This will usually not happen. However, when the two compositions are both *x* there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.

# Section 1-2 : Inverse Functions

In the last **example** from the previous section we looked at the two functions $f\left(x\right)=3x-2$ and $g\left(x\right)=\frac{x}{3}+\frac{2}{3}$ and saw that

$$\left(f\circ g\right)\left(x\right)=\left(g\circ f\right)\left(x\right)=x$$

and as noted in that section this means that there is a nice relationship between these two functions. Let's see just what that relationship is. Consider the following evaluations.

$$f\left(-1\right)=3\left(-1\right)-2=-5 \qquad \Rightarrow \qquad g\left(-5\right)=\frac{-5}{3}+\frac{2}{3}=\frac{-3}{3}=-1$$

$$g\left(2\right)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right)=3\left(\frac{4}{3}\right)-2=4-2=2$$

In the first case we plugged $x=-1$ into $f\left(x\right)$ and got a value of -5. We then turned around and plugged $x=-5$ into $g\left(x\right)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x=2$ into $g\left(x\right)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f\left(x\right)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$\left(g\circ f\right)\left(-1\right)=g\left[f\left(-1\right)\right]=g\left[-5\right]=-1$$

and the second case is really,

$$\left(f\circ g\right)\left(2\right)=f\left[g\left(2\right)\right]=f\left[\frac{4}{3}\right]=2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x=-1$ into $f\left(x\right)$ and then plugged the result from this function evaluation back into $g\left(x\right)$ and in some way $g\left(x\right)$ undid what $f\left(x\right)$ had done to $x=-1$ and gave us back the original *x* that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of *x* produce the same *y*. Mathematically this is the same as saying,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Sometimes it is easier to understand this definition if we see a function that isn't one-to-one. Let's take a look at a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words, there are two different values of *x* that produce the same value of *y*. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leq x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often tedious and/or difficult. For the most part we are going to assume that the functions that we're going to be dealing with in this course are either one-to-one or we have restricted the domain of the function to get it to be a one-to-one function.

Now, let's formally define just what inverse functions are. Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = x \qquad \text{AND} \qquad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise, we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f(x) = 3x - 2 \qquad f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

$$g(x) = \frac{x}{3} + \frac{2}{3} \qquad g^{-1}(x) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that it sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process

**Finding the Inverse of a Function**

Given the function $f\left(x\right)$ we want to find the inverse function, $f^{-1}\left(x\right)$.

1. First, replace $f\left(x\right)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}\left(x\right)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $\left(f\circ f^{-1}\right)\left(x\right)=x$ and $\left(f^{-1}\circ f\right)\left(x\right)=x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with since it is easy to make mistakes in those steps.

In the verification step we technically really do need to check that both $\left(f\circ f^{-1}\right)\left(x\right)=x$ and $\left(f^{-1}\circ f\right)\left(x\right)=x$ are true. For all the functions that we are going to be looking at in this course if one is true then the other will also be true. However, there are functions (they are beyond the scope of this course however) for which it is possible for only one of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

***Example 1*** Given $f\left(x\right)=3x-2$ find $f^{-1}\left(x\right)$.

***Solution***

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f\left(x\right)$ with *y*.

$$y=3x-2$$

Next, replace all *x*'s with *y* and all *y's* with *x*.

$$x=3y-2$$

Now, solve for *y*.

$$
\begin{aligned}
x+2 &= 3y \\
\frac{1}{3}(x+2) &= y \\
\frac{x}{3}+\frac{2}{3} &= y
\end{aligned}
$$

Finally replace *y* with $f^{-1}(x)$.

$$f^{-1}(x)=\frac{x}{3}+\frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $\left(f \circ f^{-1}\right)(x)=x$ is true.

$$
\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left[f^{-1}(x)\right] \\
&= f\left[\frac{x}{3}+\frac{2}{3}\right] \\
&= 3\left(\frac{x}{3}+\frac{2}{3}\right)-2 \\
&= x+2-2 \\
&= x
\end{aligned}
$$

***Example 2*** Given $g(x)=\sqrt{x-3}$ find $g^{-1}(x)$.

***Solution***

The fact that we're using $g(x)$ instead of $f(x)$ doesn't change how the process works. Here are the first few steps.

$$y=\sqrt{x-3} \qquad \Rightarrow \qquad x=\sqrt{y-3}$$

Now, to solve for *y* we will need to first square both sides and then proceed as normal.

$$
\begin{aligned}
x &= \sqrt{y-3} \\
x^2 &= y-3 \\
x^2+3 &= y
\end{aligned}
$$

This inverse is then,

$$g^{-1}(x)=x^2+3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned}\left(g^{-1}\circ g\right)(x) &= g^{-1}\left[g(x)\right]\\ &= g^{-1}\left(\sqrt{x-3}\right)\\ &= \left(\sqrt{x-3}\right)^2+3\\ &= x-3+3\\ &= x\end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

The next example can be a little messy so be careful with the work here.

***Example 3*** Given $h(x)=\dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

***Solution***

The first couple of steps are pretty much the same as the previous examples so here they are,

$$y=\frac{x+4}{2x-5} \qquad \Rightarrow \qquad x=\frac{y+4}{2y-5}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$\begin{aligned}x(2y-5) &= y+4\\ 2xy-5x &= y+4\\ 2xy-y &= 4+5x\\ (2x-1)y &= 4+5x\\ y &= \frac{4+5x}{2x-1}\end{aligned}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x)=\frac{4+5x}{2x-1}$$

Finally, we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\left(h \circ h^{-1}\right)(x) = h\left[h^{-1}(x)\right]$$
$$= h\left[\frac{4+5x}{2x-1}\right]$$
$$= \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}$$

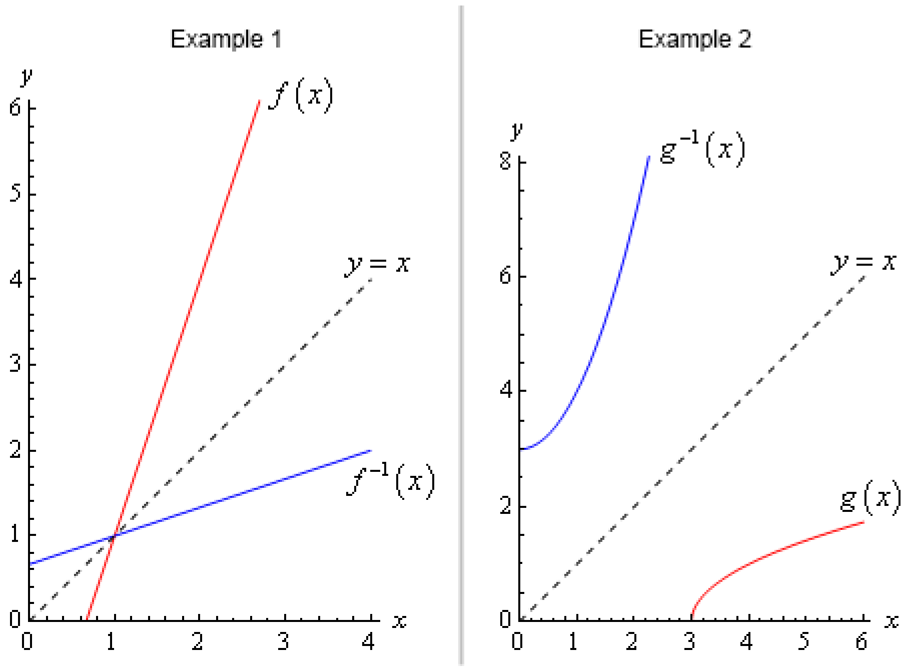
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$\left(h \circ h^{-1}\right)(x) = \frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}$$
$$= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)}$$
$$= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)}$$
$$= \frac{4+5x+8x-4}{8+10x-10x+5}$$
$$= \frac{13x}{13} = x$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.




In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

# Section 1-3 : Trig Functions

The intent of this section is to remind you of some of the more important (from a Calculus standpoint…) topics from a trig class. One of the most important (but not the first) of these topics will be how to use the unit circle. We will leave the most important topic to the next section.

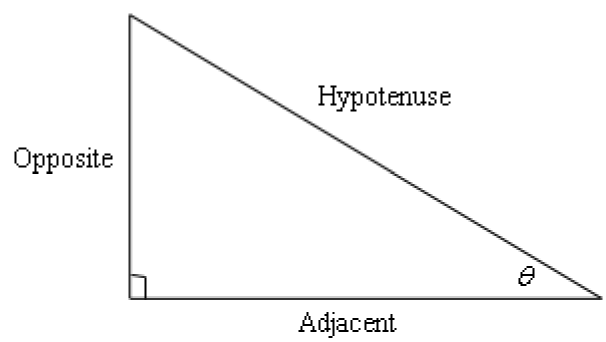
First let's start with the six trig functions and how they relate to each other.

$$\cos(x) \qquad\qquad \sin(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad\qquad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

$$\sec(x) = \frac{1}{\cos(x)} \qquad\qquad \csc(x) = \frac{1}{\sin(x)}$$

Recall as well that all the trig functions can be defined in terms of a right triangle.




From this right triangle we get the following definitions of the six trig functions.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad\qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad\qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad\qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

Remembering both the relationship between all six of the trig functions and their right triangle definitions will be useful in this course on occasion.

Next, we need to touch on radians. In most trig classes instructors tend to concentrate on doing everything in terms of degrees (probably because it's easier to visualize degrees). The same is true in many science classes. However, in a calculus course almost everything is done in radians. The following table gives some of the basic angles in both degrees and radians.

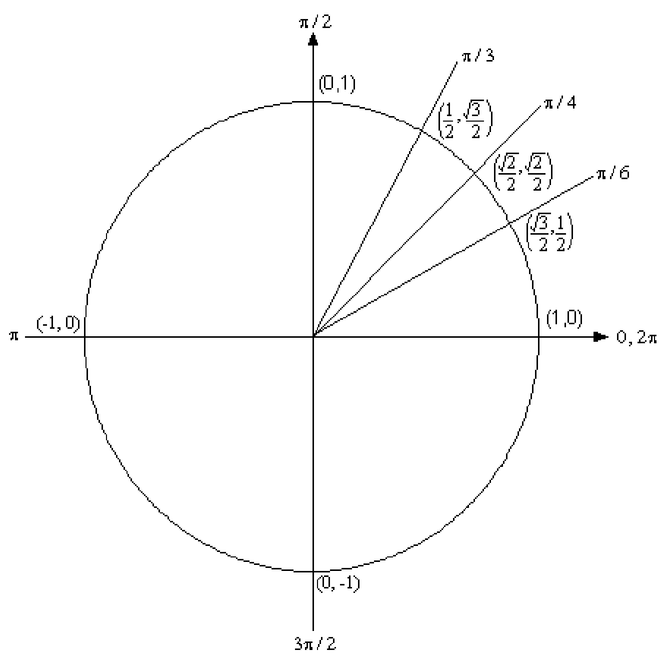
| Degree | 0 | 30 | 45 | 60 | 90 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|
| Radians | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

Know this table! We may not see these specific angles all that much when we get into the Calculus portion of these notes, but knowing these can help us to visualize each angle. Now, one more time just make sure this is clear.

**Be forewarned, everything in most calculus classes will be done in radians!**

Let's next take a look at one of the most overlooked ideas from a trig class. The unit circle is one of the more useful tools to come out of a trig class. Unfortunately, most people don't learn it as well as they should in their trig class.

Below is unit circle with just the first quadrant filled in with the "standard" angles. The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate, *i.e.* the *x*-coordinate, is the cosine of that angle and the second coordinate, *i.e.* the *y*-coordinate, is the sine of that angle. We've put some of the *basic* angles along with the coordinates of their intersections on the unit circle.




So, from the unit circle above we can see that $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$.

Also, remember how the signs of angles work. If you rotate in a counter clockwise direction the angle is positive and if you rotate in a clockwise direction the angle is negative.

Recall as well that one complete revolution is $2\pi$, so the positive *x*-axis can correspond to either an angle of 0 or $2\pi$ (or $4\pi$, or $6\pi$, or $-2\pi$, or $-4\pi$, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$\frac{\pi}{6}+2\pi=\frac{13\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around counter clockwise)

$\frac{\pi}{6}+4\pi=\frac{25\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice counter clockwise)

$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around clockwise)

$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice clockwise)

*etc.*

In fact, $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi n,\ n=0,\pm 1,\pm 2,\pm 3,\ldots$ In this case *n* is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of *n* correspond to counter clockwise rotations and negative values of *n* correspond to clockwise rotations.

So, why did we only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first with a small application of geometry. You'll see how this is done in the following set of examples.

***Example 1*** Evaluate each of the following.

**(a)** $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$

**(b)** $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

**(c)** $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

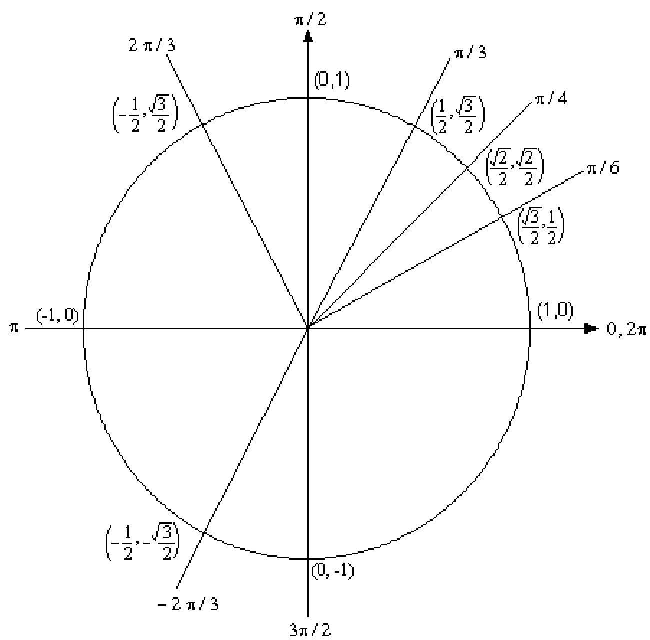
**(d)** $\sec\left(\frac{25\pi}{6}\right)$

***Solution***

**(a)** The first evaluation in this part uses the angle $\frac{2\pi}{3}$. That's not on our unit circle above, however notice that $\frac{2\pi}{3}=\pi-\frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the *x* coordinate will be negative.

Likewise, for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3} = -\pi + \frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles, along with the coordinates of the intersection points, are shown on the following unit circle.



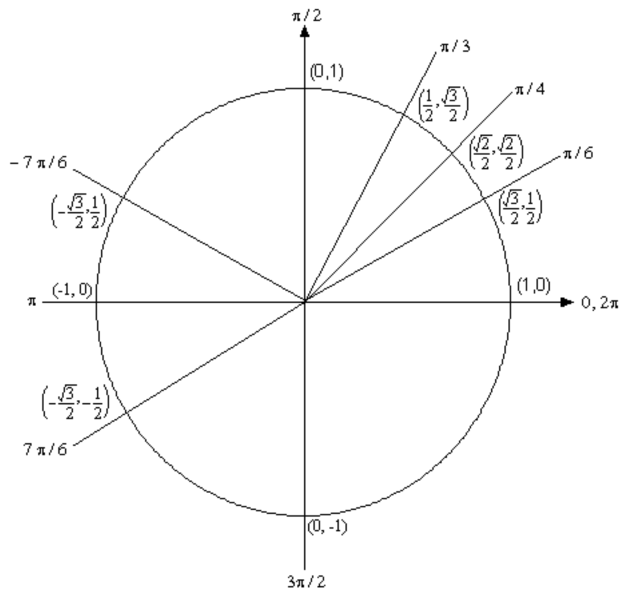

From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

**(b)** For this example, notice that $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. Also $-\frac{7\pi}{6} = -\pi - \frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. So, as with the last part, both of these angles will be mirror images of $\frac{\pi}{6}$ in the third and second quadrants respectively and we can use this to determine the coordinates for both of these new angles.

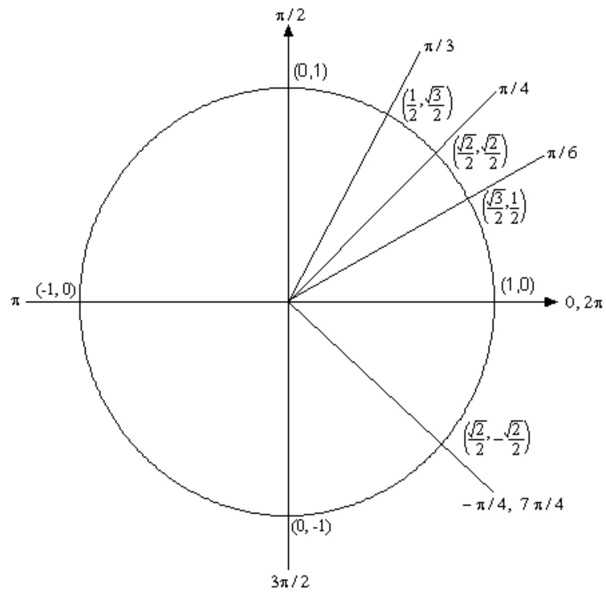
Both of these angles are shown on the following unit circle along with the coordinates for the intersection points.




From this unit circle we can see that $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta).$$

**(c)** Here we should note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle! Also note that this angle will be the mirror image of $\frac{\pi}{4}$ in the fourth quadrant. The unit circle for this angle is




Now, if we remember that $\tan\left(x\right)=\dfrac{\sin\left(x\right)}{\cos\left(x\right)}$ we can use the unit circle to find the values of the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right)=\tan\left(-\frac{\pi}{4}\right)=\frac{\sin\left(-\pi/4\right)}{\cos\left(-\pi/4\right)}=\frac{-\sqrt{2}/2}{\sqrt{2}/2}=-1\,.$$

On a side note, notice that $\tan\left(\dfrac{\pi}{4}\right)=1$ and we can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan\left(-\theta\right)=-\tan\left(\theta\right).$$

**(d)** Here we need to notice that $\dfrac{25\pi}{6}=4\pi+\dfrac{\pi}{6}$. In other words, we've started at $\dfrac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(4\pi+\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec\left(x\right)=\frac{1}{\cos\left(x\right)}$$

and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}$$

So, in the last example we saw how the unit circle can be used to determine the value of the trig functions at any of the "common" angles. It's important to notice that all of these examples used the fact that if you know the first quadrant of the unit circle and can relate all the other angles to "mirror images" of one of the first quadrant angles you don't really need to know whole unit circle. If you'd like to see a complete unit circle I've got one on my **Trig Cheat Sheet** that is available at **http://tutorial.math.lamar.edu**.

Another important idea from the last example is that when it comes to evaluating trig functions all that you really need to know is how to evaluate sine and cosine. The other four trig functions are defined in terms of these two so if you know how to evaluate sine and cosine you can also evaluate the remaining four trig functions.

We've not covered many of the topics from a trig class in this section, but we did cover some of the more important ones from a calculus standpoint. There are many important trig formulas that you will use occasionally in a calculus class. Most notably are the half-angle and double-angle formulas. If you need reminded of what these are, you might want to download my **Trig Cheat Sheet** as most of the important facts and formulas from a trig class are listed there.

# Section 1-4 : Solving Trig Equations

In this section we will take a look at solving trig equations. This is something that you will be asked to do on a fairly regular basis in many classes.

Let's just jump into the examples and see how to solve trig equations.

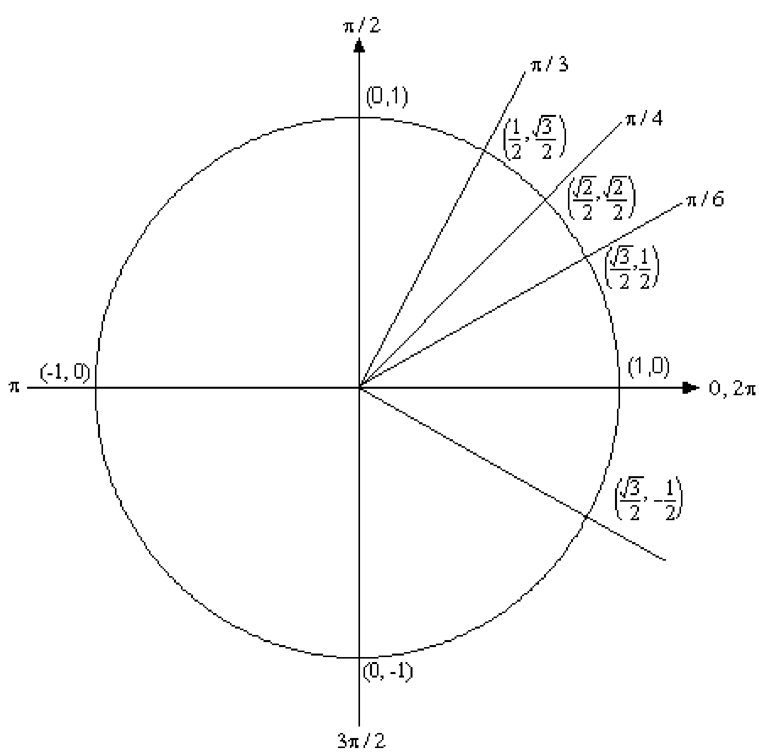
***Example 1*** Solve $2\cos(t) = \sqrt{3}$.

***Solution***

There's really not a whole lot to do in solving this kind of trig equation. We first need to get the trig function on one side by itself. To do this all we need to do is divide both sides by 2.

$$2\cos(t) = \sqrt{3}$$
$$\cos(t) = \frac{\sqrt{3}}{2}$$

We are looking for all the values of *t* for which cosine will have the value of $\frac{\sqrt{3}}{2}$. So, let's take a look at the following unit circle.




From quick inspection we can see that $t = \frac{\pi}{6}$ is a solution. However, as we have shown on the unit circle there is another angle which will also be a solution. We need to determine what this angle is.

When we look for these angles we typically want *positive* angles that lie between 0 and $2\pi$. This angle will not be the only possibility of course, but we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive *x*-axis, then so must the angle in the fourth quadrant. So, we have two options. We could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles. To get a positive angle all we need to do is use the fact that the angle is $\frac{\pi}{6}$ with the positive *x*-axis (as noted above) and a positive angle will be $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

One way to remember how to get the positive form of the second angle is to think of making one full revolution from the positive *x*-axis (*i.e.* $2\pi$) and then backing off (*i.e.* subtracting) $\frac{\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Recall from the previous **section** and you'll see there that we used

$$\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

to represent all the possible angles that can end at the same location on the unit circle, *i.e.* angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction (*n* is positive) or clockwise direction (*n* is negative) for *n* complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

$$\begin{aligned} &\frac{\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \\ &\frac{11\pi}{6} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

Now, in a calculus class this is not a typical trig equation that we'll be asked to solve. A more typical example is the next one.

***Example 2*** Solve $2\cos\left(t\right)=\sqrt{3}$ on $[-2\pi,2\pi]$.

***Solution***

In a calculus class we are often more interested in only the solutions to a trig equation that fall in a certain interval. The first step in this kind of problem is to find all possible solutions. We did this in the previous example.

$$\frac{\pi}{6}+2\pi n,\ \ n=0,\pm 1,\pm 2,\pm 3,\ldots$$
$$\frac{11\pi}{6}+2\pi n,\ \ n=0,\pm 1,\pm 2,\pm 3,\ldots$$

Now, to find the solutions in the interval all we need to do is start picking values of *n*, plugging them in and getting the solutions that will fall into the interval that we've been given.

*n=0*.

$$\frac{\pi}{6}+2\pi\left(0\right)=\frac{\pi}{6}<2\pi$$
$$\frac{11\pi}{6}+2\pi\left(0\right)=\frac{11\pi}{6}<2\pi$$

Now, notice that if we take any positive value of *n* we will be adding on positive multiples of $2\pi$ onto a positive quantity and this will take us past the upper bound of our interval so we don't need to take any positive value of *n*.

However, just because we aren't going to take any positive value of *n* doesn't mean that we shouldn't also look at negative values of *n*.

*n=-1*.

$$\frac{\pi}{6}+2\pi\left(-1\right)=-\frac{11\pi}{6}>-2\pi$$
$$\frac{11\pi}{6}+2\pi\left(-1\right)=-\frac{\pi}{6}>-2\pi$$

These are both greater than $-2\pi$ and so are solutions, but if we subtract another $2\pi$ off (*i.e* use $n=-2$) we will once again be outside of the interval so we've found all the possible solutions that lie inside the interval $[-2\pi,2\pi]$.

So, the solutions are : $\frac{\pi}{6},\frac{11\pi}{6},-\frac{\pi}{6},-\frac{11\pi}{6}$.

So, let's see if you've got all this down.

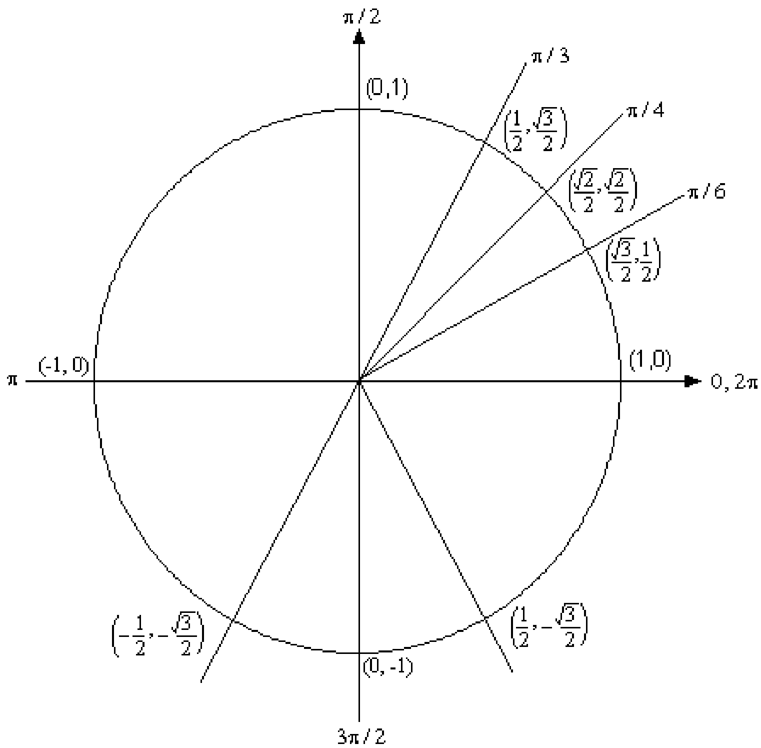
***Example 3*** Solve $2\sin\left(5x\right) = -\sqrt{3}$ on $[-\pi, 2\pi]$

***Solution***

This problem is very similar to the other problems in this section with a very important difference. We'll start this problem in exactly the same way as we did in the first example. So, first get the sine on one side by itself.

$$2\sin(5x) = -\sqrt{3}$$
$$\sin(5x) = \frac{-\sqrt{3}}{2}$$

We are looking for angles that will give $-\frac{\sqrt{3}}{2}$ out of the sine function. Let's again go to our trusty unit circle.




Now, there are no angles in the first quadrant for which sine has a value of $-\frac{\sqrt{3}}{2}$. However, there are two angles in the lower half of the unit circle for which sine will have a value of $-\frac{\sqrt{3}}{2}$. So, what are these angles?

Notice that $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$. Given this we now know that the angle in the third quadrant will be $\frac{\pi}{3}$ below the ***negative*** *x*-axis or $\pi+\frac{\pi}{3}=\frac{4\pi}{3}$. An easy way to remember this is to notice that we'll rotate half a revolution from the positive *x*-axis to get to the negative *x*-axis then add on $\frac{\pi}{3}$ to reach the angle we are looking for.

Likewise, the angle in the fourth quadrant will $\frac{\pi}{3}$ below the ***positive*** *x*-axis. So, we could use $-\frac{\pi}{3}$ or $2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$. Remember that we're typically looking for positive angles between 0 and $2\pi$ so we'll use the positive angle. An easy way to remember how to the positive angle here is to rotate one full revolution from the positive *x*-axis (*i.e.* $2\pi$ ) and then backing off (*i.e.* subtracting) $\frac{\pi}{3}$.

Now we come to the very important difference between this problem and the previous problems in this section. The solution is **NOT**

$$x=\frac{4\pi}{3}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$
$$x=\frac{5\pi}{3}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

This is not the set of solutions because we are NOT looking for values of *x* for which $\sin(x)=-\frac{\sqrt{3}}{2}$, but instead we are looking for values of *x* for which $\sin(5x)=-\frac{\sqrt{3}}{2}$. Note the difference in the arguments of the sine function! One is *x* and the other is $5x$. This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

$$5x=\frac{4\pi}{3}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$
$$5x=\frac{5\pi}{3}+2\pi n,\quad n=0,\pm1,\pm2,\ldots$$

Well, actually, that's not quite the solution. We are looking for values of *x* so divide everything by 5 to get.

$$x=\frac{4\pi}{15}+\frac{2\pi n}{5},\quad n=0,\pm1,\pm2,\ldots$$
$$x=\frac{\pi}{3}+\frac{2\pi n}{5},\quad n=0,\pm1,\pm2,\ldots$$

Notice that we also divided the $2\pi n$ by 5 as well! This is important! If we don't do that you **WILL** miss solutions. For instance, take $n=1$.

$$x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{2\pi}{3}\right)\right) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \qquad \Rightarrow \qquad \sin\left(5\left(\frac{11\pi}{15}\right)\right) = \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

We'll leave it to you to verify our work showing they are solutions. However, it makes the point. If you didn't divide the $2\pi n$ by 5 you would have missed these solutions!

Okay, now that we've gotten all possible solutions it's time to find the solutions on the given interval. We'll do this as we did in the previous problem. Pick values of *n* and get the solutions.

*n* = 0.

$$x = \frac{4\pi}{15} + \frac{2\pi(0)}{5} = \frac{4\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(0)}{5} = \frac{\pi}{3} < 2\pi$$

*n* = 1.

$$x = \frac{4\pi}{15} + \frac{2\pi(1)}{5} = \frac{2\pi}{3} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(1)}{5} = \frac{11\pi}{15} < 2\pi$$

*n* = 2.

$$x = \frac{4\pi}{15} + \frac{2\pi(2)}{5} = \frac{16\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(2)}{5} = \frac{17\pi}{15} < 2\pi$$

*n* = 3.

$$x = \frac{4\pi}{15} + \frac{2\pi(3)}{5} = \frac{22\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(3)}{5} = \frac{23\pi}{15} < 2\pi$$

*n* = 4.

$$x = \frac{4\pi}{15} + \frac{2\pi(4)}{5} = \frac{28\pi}{15} < 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi(4)}{5} = \frac{29\pi}{15} < 2\pi$$

*n* = 5.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(5\right)}{5} = \frac{34\pi}{15} > 2\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi\left(5\right)}{5} = \frac{35\pi}{15} > 2\pi$$

Okay, so we finally got past the right endpoint of our interval so we don't need any more positive *n*. Now let's take a look at the negative *n* and see what we've got.

*n* = –1 .

$$x = \frac{4\pi}{15} + \frac{2\pi\left(-1\right)}{5} = -\frac{2\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi\left(-1\right)}{5} = -\frac{\pi}{15} > -\pi$$

*n* = –2.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(-2\right)}{5} = -\frac{8\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi\left(-2\right)}{5} = -\frac{7\pi}{15} > -\pi$$

*n* = –3.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(-3\right)}{5} = -\frac{14\pi}{15} > -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi\left(-3\right)}{5} = -\frac{13\pi}{15} > -\pi$$

*n* = –4.

$$x = \frac{4\pi}{15} + \frac{2\pi\left(-4\right)}{5} = -\frac{4\pi}{3} < -\pi$$

$$x = \frac{\pi}{3} + \frac{2\pi\left(-4\right)}{5} = -\frac{19\pi}{15} < -\pi$$

And we're now past the left endpoint of the interval. Sometimes, there will be many solutions as there were in this example. Putting all of this together gives the following set of solutions that lie in the given interval.

$$\frac{4\pi}{15}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{11\pi}{15}, \frac{16\pi}{15}, \frac{17\pi}{15}, \frac{22\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{29\pi}{15}$$

$$-\frac{\pi}{15}, -\frac{2\pi}{15}, -\frac{7\pi}{15}, -\frac{8\pi}{15}, -\frac{13\pi}{15}, -\frac{14\pi}{15}$$

Let's work another example.

***Example 4*** Solve $\sin(2x) = -\cos(2x)$ on $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

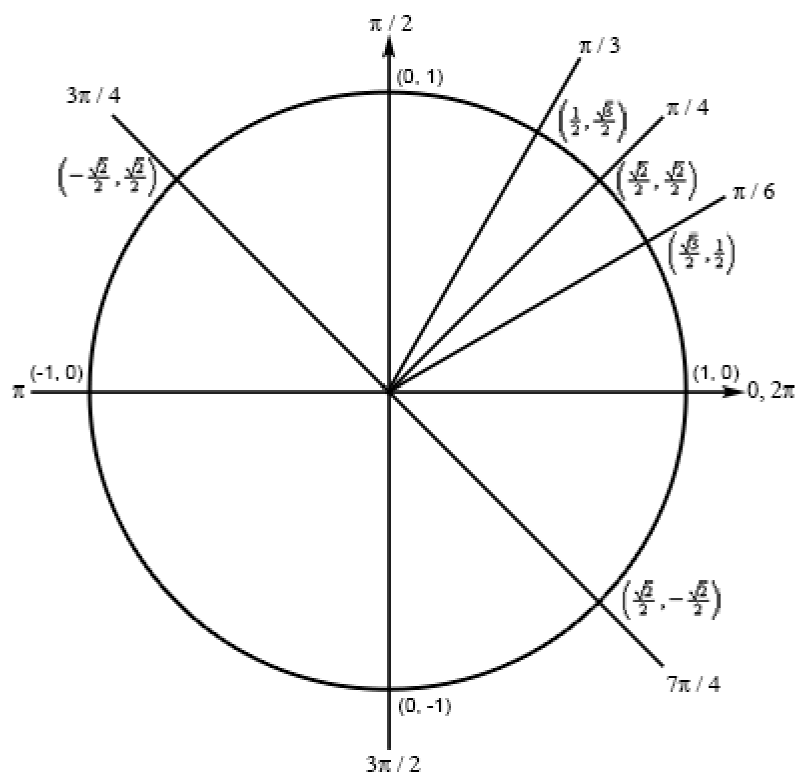
***Solution***

This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

$$\begin{aligned} \sin(2x) &= -\cos(2x) \\ \frac{\sin(2x)}{\cos(2x)} &= -1 \\ \tan(2x) &= -1 \end{aligned}$$

So, solving $\sin(2x) = -\cos(2x)$ is the same as solving $\tan(2x) = -1$. Hopefully, you'll recall that the smallest positive angle where tangent it -1 is $\frac{3\pi}{4}$ and this angle is in the 2nd quadrant.

There is also a second angle for which tangent will be -1 and we can use the unit circle to illustrate this second angle. Let's take a look at the following unit circle.




As shown in this unit circle if we add $\pi$ to our first angle we get $\frac{3\pi}{4} + \pi = \frac{7\pi}{4}$ and we get an angle that is in the fourth quadrant and has the same coordinates except for opposite signs. This means that tangent will also have a value of -1 here and so is a second angle.

This will always be true when solving tangent equations. Once we have one angle that will solve the equation a second angle will always be $\pi$ plus the first angle.

All possible angles are then,

$$2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Or, upon dividing by the 2 we get all possible solutions.

$$x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now, let's determine the solutions that lie in the given interval.

*n* = 0.

$$x = \frac{3\pi}{8} + \pi(0) = \frac{3\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(0) = \frac{7\pi}{8} < \frac{3\pi}{2}$$

*n* = 1.

$$x = \frac{3\pi}{8} + \pi(1) = \frac{11\pi}{8} < \frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(1) = \frac{15\pi}{8} > \frac{3\pi}{2}$$

Unlike the previous example only one of these will be in the interval. This will happen occasionally so don't always expect both answers from a particular *n* to work. Also, we should now check *n=2* for the first to see if it will be in or out of the interval. I'll leave it to you to check that it's out of the interval.

Now, let's check the negative *n*.

*n* = –1.

$$x = \frac{3\pi}{8} + \pi(-1) = -\frac{5\pi}{8} > -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(-1) = -\frac{\pi}{8} > -\frac{3\pi}{2}$$

*n* = –2.

$$x = \frac{3\pi}{8} + \pi(-2) = -\frac{13\pi}{8} < -\frac{3\pi}{2}$$
$$x = \frac{7\pi}{8} + \pi(-2) = -\frac{9\pi}{8} > -\frac{3\pi}{2}$$

Again, only one will work here. I'll leave it to you to verify that $n = -3$ will give two answers that are both out of the interval.

The complete list of solutions is then,

$$-\frac{9\pi}{8}, -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$$

Before moving on we need to address one issue about the previous example. The solution method used there is not the "standard" solution method. Because the second angle is just $\pi$ plus the first and if we added $\pi$ onto the second angle we'd be back at the line representing the first angle the more standard solution method is to just add $\pi n$ onto the first angle to get,

$$2x = \frac{3\pi}{4} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Then dividing by 2 to get the full set of solutions,

$$x = \frac{3\pi}{8} + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

This set of solutions is identical to the set of solutions we got in the example (we'll leave it to you to plug in some $n$'s and verify that). So, why did we not use the method in the previous example? Simple. The method in the previous example more closely mirrors the solution method for cosine and sine (*i.e.* they both, generally, give two sets of angles) and so for students that aren't comfortable with solving trig equations this gives a "consistent" solution method.

Let's work one more example so that we can make a point that needs to be understood when solving some trig equations.

***Example 5*** Solve $\cos\left(3x\right) = 2$.

***Solution***
This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \cos\left(\theta\right) \le 1$ and $-1 \le \sin\left(\theta\right) \le 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

In this section we solved some simple trig equations. There are more complicated trig equations that we can solve so don't leave this section with the feeling that there is nothing harder out there in the world to solve. In fact, we'll see at least one of the more complicated problems in the next section. Also, every one of these problems came down to solutions involving one of the "common" or "standard" angles. Most trig equations won't come down to one of those and will in fact need a calculator to solve. The next section is devoted to this kind of problem.

# Section 1-5 : Solving Trig Equations with Calculators, Part I

In the previous section we started solving trig equations. The only problem with the equations we solved in there is that they pretty much all had solutions that came from a handful of "standard" angles and of course there are many equations out there that simply don't. So, in this section we are going to take a look at some more trig equations, the majority of which will require the use of a calculator to solve (a couple won't need a calculator).

The fact that we are using calculators in this section does not however mean that the problems in the previous section aren't important. It is going to be assumed in this section that the basic ideas of solving trig equations are known and that we don't need to go back over them here. In particular, it is assumed that you can use a unit circle to help you find all answers to the equation (although the process here is a little different as we'll see) and it is assumed that you can find answers in a given interval. If you are unfamiliar with these ideas you should first go to the previous section and go over those problems.

Before proceeding with the problems we need to go over how our calculators work so that we can get the correct answers. Calculators are great tools but if you don't know how they work and how to interpret their answers you can get in serious trouble.

First, as already pointed out in previous sections, everything we are going to be doing here will be in radians so make sure that your calculator is set to radians before attempting the problems in this section. Also, we are going to use 4 decimal places of accuracy in the work here. You can use more if you want, but in this class, we'll always use at least 4 decimal places of accuracy.

Next, and somewhat more importantly, we need to understand how calculators give answers to inverse trig functions. We didn't cover inverse trig functions in this review, but they are just inverse functions and we have talked a little bit about inverse functions in a review **section**. The only real difference is that we are now using trig functions. We'll only be looking at three of them and they are:

$$\begin{aligned}
&\text{Inverse Cosine} &&: \cos^{-1}(x) = \arccos(x) \\
&\text{Inverse Sine} &&: \sin^{-1}(x) = \arcsin(x) \\
&\text{Inverse Tangent} &&: \tan^{-1}(x) = \arctan(x)
\end{aligned}$$

As shown there are two different notations that are commonly used. In these notes we'll be using the first form since it is a little more compact. Most calculators these days will have buttons on them for these three so make sure that yours does as well.

We now need to deal with how calculators give answers to these. Let's suppose, for example, that we wanted our calculator to compute $\cos^{-1}\left(\frac{3}{4}\right)$. First, remember that what the calculator is computing is the angle, let's say *x*, that we would plug into cosine to get a value of $\frac{3}{4}$, or

$$x = \cos^{-1}\left(\frac{3}{4}\right) \qquad \Rightarrow \qquad \cos(x) = \frac{3}{4}$$

So, in other words, when we are using our calculator to compute an inverse trig function we are really solving a simple trig equation.

Having our calculator compute $\cos^{-1}\left(\frac{3}{4}\right)$ and hence solve $\cos\left(x\right)=\frac{3}{4}$ gives,

$$x=\cos^{-1}\left(\frac{3}{4}\right)=0.7227$$

From the previous section we know that there should in fact be an infinite number of answers to this including a second angle that is in the interval $\left[0,2\pi\right]$. However, our calculator only gave us a single answer. How to determine what the other angles are will be covered in the following examples so we won't go into detail here about that. We did need to point out however, that the calculators will only give a single answer and that we're going to have more work to do than just plugging a number into a calculator.

Since we know that there are supposed to be an infinite number of solutions to $\cos\left(x\right)=\frac{3}{4}$ the next question we should ask then is just how did the calculator decide to return the answer that it did? Why this one and not one of the others? Will it give the same answer every time?

There are rules that determine just what answer the calculator gives when computing inverse trig functions. All calculators will give answers in the following ranges.

$$0\le\cos^{-1}\left(x\right)\le\pi \qquad -\frac{\pi}{2}\le\sin^{-1}\left(x\right)\le\frac{\pi}{2} \qquad -\frac{\pi}{2}<\tan^{-1}\left(x\right)<\frac{\pi}{2}$$

If you think back to the unit circle and recall that we think of cosine as the horizontal axis then we can see that we'll cover all possible values of cosine in the upper half of the circle and this is exactly the range given above for the inverse cosine. Likewise, since we think of sine as the vertical axis in the unit circle we can see that we'll cover all possible values of sine in the right half of the unit circle and that is the range given above.

For the tangent range look back to the graph of the tangent function itself and we'll see that one branch of the tangent is covered in the range given above and so that is the range we'll use for inverse tangent. Note as well that we don't include the endpoints in the range for inverse tangent since tangent does not exist there.

So, if we can remember these rules we will be able to determine the remaining angle in $\left[0,2\pi\right]$ that also works for each solution.

As a final quick topic let's note that it will, on occasion, be useful to remember the decimal representations of some basic angles. So here they are,

$$\frac{\pi}{2}=1.5708 \qquad \pi=3.1416 \qquad \frac{3\pi}{2}=4.7124 \qquad 2\pi=6.2832$$

Using these we can quickly see that $\cos^{-1}\left(\frac{3}{4}\right)$ must be in the first quadrant since 0.7227 is between 0 and 1.5708. This will be of great help when we go to determine the remaining angles

So, once again, we can't stress enough that calculators are great tools that can be of tremendous help to us, but if you don't understand how they work you will often get the answers to problems wrong.

So, with all that out of the way let's take a look at our first problem.

***Example 1*** Solve $4\cos(t)=3$ on [-8,10].

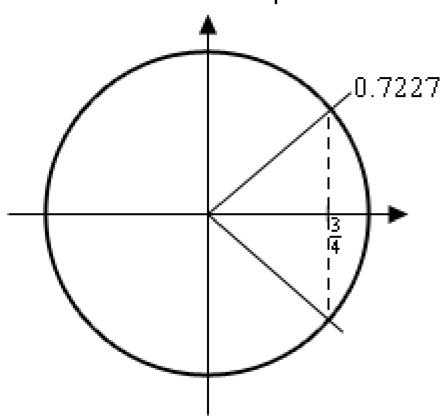
***Solution***

Okay, the first step here is identical to the problems in the previous section. We first need to isolate the cosine on one side by itself and then use our calculator to get the first answer.

$$\cos(t)=\frac{3}{4} \qquad \Rightarrow \qquad t=\cos^{-1}\left(\frac{3}{4}\right)=0.7227$$

So, this is the one we were using above in the opening discussion of this section. At the time we mentioned that there were infinite number of answers and that we'd be seeing how to find them later. Well that time is now.

First, let's take a quick look at a unit circle for this example.




The angle that we've found is shown on the circle as well as the other angle that we know should also be an answer. Finding this angle here is just as easy as in the previous section. Since the line segment in the first quadrant forms an angle of 0.7227 radians with the positive *x*-axis then so does the line segment in the fourth quadrant. This means that we can use either -0.7227 as the second angle or $2\pi-0.7227=5.5605$ . Which you use depends on which you prefer. We'll pretty much always use the positive angle to avoid the possibility that we'll lose the minus sign.

So, all possible solutions, ignoring the interval for a second, are then,

$$\begin{aligned} t&=0.7227+2\pi n \\ t&=5.5605+2\pi n \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Now, all we need to do is plug in values of *n* to determine the angle that are actually in the interval. Here's the work for that.

$$\begin{array}{llll} n=-2 : & t=\cancel{-11.8437} & \text{and} & -7.0059 \\ n=-1 : & t=-5.5605 & \text{and} & -0.7227 \\ n=0 : & t=0.7227 & \text{and} & 5.5605 \\ n=1 : & t=7.0059 & \text{and} & \cancel{11.8437} \end{array}$$

So, the solutions to this equation, in the given interval, are,

$$t=-7.0059,\ -5.5605,\ -0.7227,\ 0.7227,\ 5.5605,\ 7.0059$$

Note that we had a choice of angles to use for the second angle in the previous example. The choice of angles there will also affect the value(s) of *n* that we'll need to use to get all the solutions. In the end, regardless of the angle chosen, we'll get the same list of solutions, but the value(s) of *n* that give the solutions will be different depending on our choice.

Also, in the above example we put in a little more explanation than we'll show in the remaining examples in this section to remind you how these work.

***Example 2*** Solve $-10\cos\left(3t\right)=7$ on [-2,5].

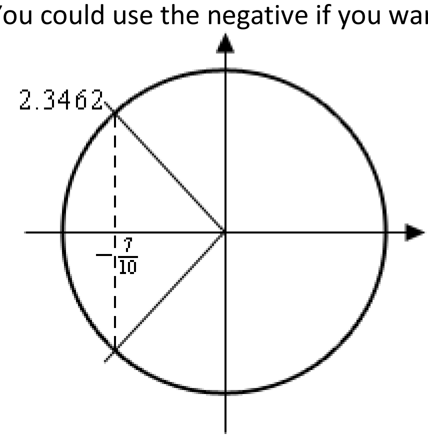
***Solution***

Okay, let's first get the inverse cosine portion of this problem taken care of.

$$\cos\left(3t\right)=-\frac{7}{10} \qquad \Rightarrow \qquad 3t=\cos^{-1}\left(-\frac{7}{10}\right)=2.3462$$

Don't forget that we still need the "3"!

Now, let's look at a quick unit circle for this problem. As we can see the angle 2.3462 radians is in the second quadrant and the other angle that we need is in the third quadrant. We can find this second angle in exactly the same way we did in the previous example. We can use either -2.3462 or we can use $2\pi-2.3462=3.9370$. As with the previous example we'll use the positive choice, but that is purely a matter of preference. You could use the negative if you wanted to.



So, let's now finish out the problem. First, let's acknowledge that the values of 3*t* that we need are,

$$\begin{aligned} 3t &= 2.3462 + 2\pi n \\ 3t &= 3.9370 + 2\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Now, we need to properly deal with the 3, so divide that out to get all the solutions to the trig equation.

$$\begin{aligned} t &= 0.7821 + \frac{2\pi n}{3} \\ t &= 1.3123 + \frac{2\pi n}{3} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Finally, we need to get the values in the given interval.

$$\begin{array}{llll} n = -2 : & t = \cancel{-3.4067} & \text{and} & \cancel{-2.8765} \\ n = -1 : & t = -1.3123 & \text{and} & -0.7821 \\ n = 0 : & t = 0.7821 & \text{and} & 1.3123 \\ n = 1 : & t = 2.8765 & \text{and} & 3.4067 \\ n = 2 : & t = 4.9709 & \text{and} & \cancel{5.5011} \end{array}$$

The solutions to this equation, in the given interval are then,

$$t = -1.3123,\ -0.7821,\ 0.7821,\ 1.3123,\ 2.8765,\ 3.4067,\ 4.9709$$

We've done a couple of basic problems with cosines, now let's take a look at how solving equations with sines work.

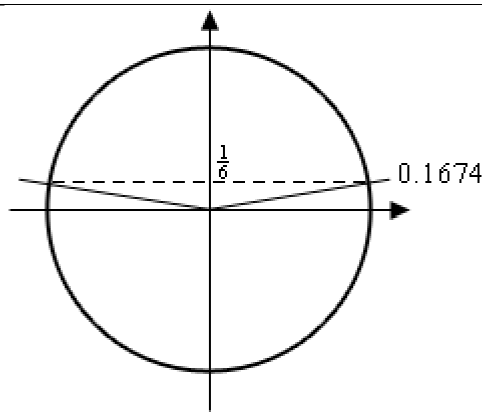
***Example 3*** Solve $6\sin\left(\frac{x}{2}\right) = 1$ on [-20,30]

***Solution***

Let's first get the calculator work out of the way since that isn't where the difference comes into play.

$$\sin\left(\frac{x}{2}\right) = \frac{1}{6} \qquad \Rightarrow \qquad \frac{x}{2} = \sin^{-1}\left(\frac{1}{6}\right) = 0.1674$$

Here's a unit circle for this example.




To find the second angle in this case we can notice that the line in the first quadrant makes an angle of 0.1674 with the positive *x*-axis and so the angle in the second quadrant will then make an angle of 0.1674 with the negative *x*-axis. So, if we start at the positive *x*-axis we rotate a half revolution and then back off 0.1674. Therefore, the angle that we're after is then, $\pi - 0.1674 = 2.9742$ .

Here's the rest of the solution for this example. We're going to assume from this point on that you can do this work without much explanation.

$$\begin{aligned} \frac{x}{2} &= 0.1674 + 2\pi n \\ \frac{x}{2} &= 2.9742 + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= 0.3348 + 4\pi n \\ x &= 5.9484 + 4\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\begin{array}{llll} n = -2 : & x = \cancel{-24.7980} & \text{and} & -19.1844 \\ n = -1 : & x = -12.2316 & \text{and} & -6.6180 \\ n = 0 : & x = 0.3348 & \text{and} & 5.9484 \\ n = 1 : & x = 12.9012 & \text{and} & 18.5148 \\ n = 2 : & x = 25.4676 & \text{and} & \cancel{31.0812} \end{array}$$

The solutions to this equation are then,

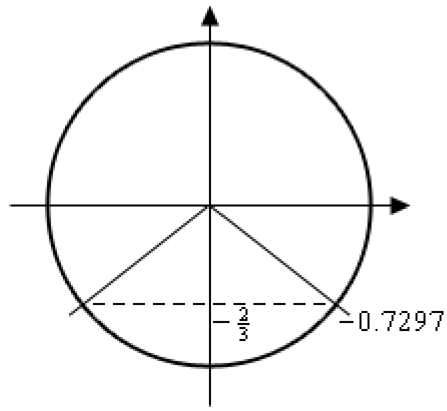
$$x = -19.1844,\ -12.2316,\ -6.6180,\ 0.3348,\ 5.9484,\ 12.9012,\ 18.5128,\ 25.4676$$

***Example 4*** Solve $3\sin\left(5z\right) = -2$ on [0,1].

***Solution***

You should be getting pretty good at these by now, so we won't be putting much explanation in for this one. Here we go.

$$\sin\left(5z\right) = -\frac{2}{3} \qquad \Rightarrow \qquad 5z = \sin^{-1}\left(-\frac{2}{3}\right) = -0.7297$$




Okay, with this one we're going to do a little more work than with the others. For the first angle we could use the answer our calculator gave us. However, it's easy to lose minus signs so we'll instead use $2\pi - 0.7297 = 5.5535$. Again, there is no reason to this other than a worry about losing the minus sign in the calculator answer. If you'd like to use the calculator answer you are more than welcome to. For the second angle we'll note that the lines in the third and fourth quadrant make an angle of 0.7297 with the *x*-axis. So, if we start at the positive *x*-axis we rotate a half revolution and then add on 0.7297 for the second angle. Therefore, the second angle is $\pi + 0.7297 = 3.8713$.

Here's the rest of the work for this example.

$$\begin{aligned} 5z &= 5.5535 + 2\pi n \\ 5z &= 3.8713 + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} z &= 1.1107 + \frac{2\pi n}{5} \\ z &= 0.7743 + \frac{2\pi n}{5} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\begin{array}{lll} n = -1 \; : & \cancel{x = -0.1460} \quad \text{and} & \cancel{-0.4823} \\ n = 0 \quad : & \cancel{x = 1.1107} \quad \text{and} & 0.7743 \end{array}$$

So, in this case we get a single solution of 0.7743.

Note that in the previous example we only got a single solution. This happens on occasion so don't get worried about it. Also, note that it was the second angle that gave this solution and so if we'd just relied on our calculator without worrying about other angles we would not have gotten this solution. Again, it can't be stressed enough that while calculators are a great tool if we don't understand how to correctly interpret/use the result we can (and often will) get the solution wrong.

To this point we've only worked examples involving sine and cosine. Let's now work a couple of examples that involve other trig functions to see how they work.

***Example 5*** Solve $9\sin(2x) = -5\cos(2x)$ on [-10,0].

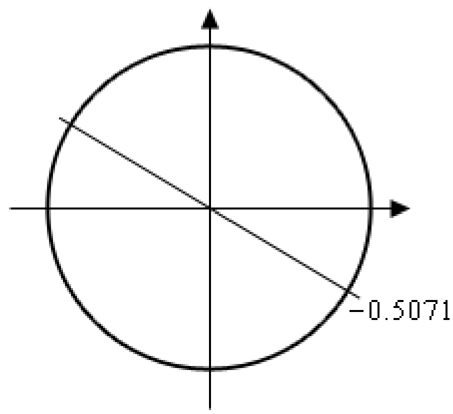
***Solution***
At first glance this problem seems to be at odds with the sentence preceding the example. However, it really isn't.

First, when we have more than one trig function in an equation we need a way to get equations that only involve one trig function. There are many ways of doing this that depend on the type of equation we're starting with. In this case we can simply divide both sides by a cosine and we'll get a single tangent in the equation. We can now see that this really is an equation that doesn't involve a sine or a cosine.

So, let's get started on this example.

$$\frac{\sin(2x)}{\cos(2x)} = \tan(2x) = -\frac{5}{9} \qquad \Rightarrow \qquad 2x = \tan^{-1}\left(-\frac{5}{9}\right) = -0.5071$$

Now, the unit circle doesn't involve tangents, however we can use it to illustrate the second angle in the range $[0, 2\pi]$.




The angles that we're looking for here are those whose quotient of $\frac{\text{sine}}{\text{cosine}}$ is the same. The second angle where we will get the same value of tangent will be exactly opposite of the given point. For this angle the values of sine and cosine are the same except they will have opposite signs. In the quotient however, the difference in signs will cancel out and we'll get the same value of tangent. So, the second angle will always be the first angle plus $\pi$.

Before getting the second angle let's also note that, like the previous example, we'll use the $2\pi - 0.5071 = 5.7761$ for the first angle. Again, this is only because of a concern about losing track of the minus sign in our calculator answer. We could just as easily do the work with the original angle our calculator gave us.

Now, this is where is seems like we're just randomly making changes and doing things for no reason. The second angle that we're going to use is,

$$\pi + \left(-0.5071\right) = \pi - 0.5071 = 2.6345$$

The fact that we used the calculator answer here seems to contradict the fact that we used a different angle for the first above. The reason for doing this here is to give a second angle that is in the range $\left[0, 2\pi\right]$. Had we used 5.7761 to find the second angle we'd get $\pi + 5.7761 = 8.9177$. This is a perfectly acceptable answer; however, it is larger than $2\pi$ (6.2832) and the general rule of thumb is to keep the initial angles as small as possible.

Here are all the solutions to the equation.

$$\begin{aligned} 2x &= 5.7761 + 2\pi n \\ 2x &= 2.6345 + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= 2.8881 + \pi n \\ x &= 1.3173 + \pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\begin{array}{lll} n = -4 : & x = -9.6783 & \text{and} \quad \cancel{-11.2491} \\ n = -3 : & x = -6.5367 & \text{and} \quad -8.1075 \\ n = -2 : & x = -3.3951 & \text{and} \quad -4.9659 \\ n = -1 : & x = -0.2535 & \text{and} \quad -1.8243 \\ n = 0 \ \ : & \cancel{x = 2.8881} & \text{and} \quad \cancel{1.3173} \end{array}$$

The seven solutions to this equation are then,

$$-0.2535, \quad -1.8243, \quad -3.3951, \quad -4.9659, \quad -6.5367, \quad -8.1075, \quad -9.6783$$

Note as well that we didn't need to do the $n = 0$ computation since we could see from the given interval that we only wanted negative answers and these would clearly give positive answers.

Before moving on we need to address one issue about the previous example. The solution method used there is not the "standard" solution method. Because the second angle is just $\pi$ plus the first and if we added $\pi$ onto the second angle we'd be back at the line representing the first angle the more standard solution method is to just add $\pi n$ onto the first angle.

If using the calculator answer this would give,

$$2x = -0.5071 + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

If using the positive angle corresponding to the calculator answer this would give,

$$2x = 5.7761 + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

Then dividing by 2 either of the following sets of solutions,

$$x = -0.2415 + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x = 2.8881 + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Either of these sets of solutions is identical to the set of solutions we got in the example (we'll leave it to you to plug in some *n*'s and verify that). So, why did we not use the method in the previous example? Simple. The method in the previous example more closely mirrors the solution method for cosine and sine (*i.e.* they both, generally, give two sets of angles) and so for students that aren't comfortable with solving trig equations this gives a "consistent" solution method.

Many calculators today can only do inverse sine, inverse cosine, and inverse tangent. So, let's see an example that uses one of the other trig functions.

***Example 6*** Solve $7\sec\left(3t\right) = -10$.

***Solution***
We'll start this one in exactly the same way we've done all the others.

$$\sec\left(3t\right) = -\frac{10}{7} \qquad \Rightarrow \qquad 3t = \sec^{-1}\left(-\frac{10}{7}\right)$$

Now we reach the problem. As noted above, many calculators can't handle inverse secant so we're going to need a different solution method for this one. To finish the solution here we'll simply recall the definition of secant in terms of cosine and convert this into an equation involving cosine instead and we already know how to solve those kinds of trig equations.

$$\frac{1}{\cos\left(3t\right)} = \sec\left(3t\right) = -\frac{10}{7} \qquad \Rightarrow \qquad \cos\left(3t\right) = -\frac{7}{10}$$

Now, we solved this equation in the second example above so we won't redo our work here. The solution is,

$$\begin{aligned} t &= 0.7821 + \frac{2\pi n}{3} \\ t &= 1.3123 + \frac{2\pi n}{3} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

We weren't given an interval in this problem so here is nothing else to do here.

For the remainder of the examples in this section we're not going to be finding solutions in an interval to save some space. If you followed the work from the first few examples in which we were given intervals you should be able to do any of the remaining examples if given an interval.

Also, we will no longer be including sketches of unit circles in the remaining solutions. We are going to assume that you can use the above sketches as guides for sketching unit circles to verify the claims in the following examples.

The next three examples don't require a calculator but are important enough or cause enough problems for students to include in this section in case you run across them and haven't seen them anywhere else.

***Example 7*** Solve $\cos\left(4\theta\right) = -1$.

***Solution***
There really isn't too much to do with this problem. It is, however, different from all the others done to this point. All the others done to this point have had two angles in the interval $\left[0, 2\pi\right]$ that were solutions to the equation. This only has one. If you aren't sure you believe this sketch a quick unit circle and you'll see that in fact there is only one angle for which cosine is -1.

Here is the solution to this equation.

$$4\theta = \pi + 2\pi n \qquad \Rightarrow \qquad \theta = \frac{\pi}{4} + \frac{\pi n}{2} \qquad n = 0, \pm 1, \pm 2, \ldots$$

***Example 8*** Solve $\sin\left(\frac{\alpha}{7}\right) = 0$.

***Solution***
Again, not much to this problem. Using a unit circle it isn't too hard to see that the solutions to this equation are,

$$\begin{aligned} \frac{\alpha}{7} &= 0 + 2\pi n \\ \frac{\alpha}{7} &= \pi + 2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \alpha &= 14\pi n \\ \alpha &= 7\pi + 14\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

This next example has an important point that needs to be understood when solving some trig equations.

***Example 9*** Solve $\sin\left(3t\right) = 2$.

***Solution***
This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \le \sin\left(\theta\right) \le 1$ and $-1 \le \cos\left(\theta\right) \le 1$. Therefore, since sine will never be greater that 1 it definitely can't be 2. So, **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

Because this document is also being prepared for viewing on the web we're going to split this section in two in order to keep the page size (and hence load time in a browser) to a minimum. In the next section we're going to take a look at some slightly more "complicated" equations. Although, as you'll see, they aren't as complicated as they may at first seem.

# Section 1-6 : Solving Trig Equations with Calculators, Part II

Because this document is also being prepared for viewing on the web we split this section into two parts to keep the size of the pages to a minimum.

Also, as with the last few examples in the previous part of this section we are not going to be looking for solutions in an interval in order to save space. The important part of these examples is to find the solutions to the equation. If we'd been given an interval it would be easy enough to find the solutions that actually fall in the interval.

In all the examples in the previous section all the arguments, the $3t$, $\frac{\alpha}{7}$, *etc*., were fairly simple. Let's take a look at an example that has a slightly more complicated looking argument.

***Example 1*** Solve $5\cos\left(2x-1\right)=-3$.

***Solution***
Note that the argument here is not really all that complicated but the addition of the "-1" often seems to confuse people so we need to a quick example with this kind of argument. The solution process is identical to all the problems we've done to this point so we won't be putting in much explanation. Here is the solution.

$$\cos\left(2x-1\right)=-\frac{3}{5} \qquad \Rightarrow \qquad 2x-1=\cos^{-1}\left(-\frac{3}{5}\right)=2.2143$$

This angle is in the second quadrant and so we can use either -2.2143 or $2\pi-2.2143=4.0689$ for the second angle. As usual for these notes we'll use the positive one. Therefore the two angles are,

$$\begin{aligned}2x-1&=2.2143+2\pi n\\2x-1&=4.0689+2\pi n\end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Now, we still need to find the actual values of *x* that are the solutions. These are found in the same manner as all the problems above. We'll first add 1 to both sides and then divide by 2. Doing this gives,

$$\begin{aligned}x&=1.6072+\pi n\\x&=2.5345+\pi n\end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

So, in this example we saw an argument that was a little different from those seen previously, but not all that different when it comes to working the problems so don't get too excited about it.
We now need to move into a different type of trig equation. All of the trig equations solved to this point (the previous example as well as the previous section) were, in some way, more or less the "standard" trig equation that is usually solved in a trig class. There are other types of equations involving trig

functions however that we need to take a quick look at. The remaining examples show some of these different kinds of trig equations.

***Example 2*** Solve $2\cos\left(6y\right)+11\cos\left(6y\right)\sin\left(3y\right)=0$.

***Solution***

So, this definitely doesn't look like any of the equations we've solved to this point and initially the process is different as well. First, notice that there is a $\cos\left(6y\right)$ in each term, so let's factor that out and see what we have.

$$\cos\left(6y\right)\left(2+11\sin\left(3y\right)\right)=0$$

We now have a product of two terms that is zero and so we know that we must have,

$$\cos\left(6y\right)=0 \qquad \text{OR} \qquad 2+11\sin\left(3y\right)=0$$

Now, at this point we have two trig equations to solve and each is identical to the type of equation we were solving earlier. Because of this we won't put in much detail about solving these two equations.

First, solving $\cos\left(6y\right)=0$ gives,

$$\begin{aligned} 6y &= \frac{\pi}{2}+2\pi n \\ 6y &= \frac{3\pi}{2}+2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} y &= \frac{\pi}{12}+\frac{\pi n}{3} \\ y &= \frac{\pi}{4}+\frac{\pi n}{3} \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Next, solving $2+11\sin\left(3y\right)=0$ gives,

$$\begin{aligned} 3y &= 6.1004+2\pi n \\ 3y &= 3.3244+2\pi n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} y &= 2.0335+\frac{2\pi n}{3} \\ y &= 1.1081+\frac{2\pi n}{3} \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Remember that in these notes we tend to take positive angles and so the first solution here is in fact $2\pi-0.1828$ where our calculator gave us -0.1828 as the answer when using the inverse sine function.

The solutions to this equation are then,

$$y = \frac{\pi}{12} + \frac{\pi n}{3}$$
$$y = \frac{\pi}{4} + \frac{\pi n}{3} \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$y = 2.0335 + \frac{2\pi n}{3}$$
$$y = 1.1081 + \frac{2\pi n}{3}$$

This next example also involves "factoring" trig equations but in a slightly different manner than the previous example.

***Example 3*** Solve $4\sin^2\left(\frac{t}{3}\right) - 3\sin\left(\frac{t}{3}\right) = 1$.

***Solution***
Before solving this equation let's solve an apparently unrelated equation.

$$4x^2 - 3x = 1 \qquad \Rightarrow \qquad 4x^2 - 3x - 1 = (4x+1)(x-1) = 0 \qquad \Rightarrow \qquad x = -\frac{1}{4}, 1$$

This is an easy (or at least we hope it's easy at this point) equation to solve. The obvious question then is, why did we do this? We'll, if you compare the two equations you'll see that the only real difference is that the one we just solved has an *x* everywhere and the equation we want to solve has a sine. What this tells us is that we can work the two equations in exactly the same way.

We, will first "factor" the equation as follows,

$$4\sin^2\left(\frac{t}{3}\right) - 3\sin\left(\frac{t}{3}\right) - 1 = \left(4\sin\left(\frac{t}{3}\right) + 1\right)\left(\sin\left(\frac{t}{3}\right) - 1\right) = 0$$

Now, set each of the two factors equal to zero and solve for the sine,

$$\sin\left(\frac{t}{3}\right) = -\frac{1}{4} \qquad \sin\left(\frac{t}{3}\right) = 1$$

We now have two trig equations that we can easily (hopefully…) solve at this point. We'll leave the details to you to verify that the solutions to each of these and hence the solutions to the original equation are,

$$t = 18.0915 + 6\pi n$$
$$t = 10.1829 + 6\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$t = \frac{3\pi}{2} + 6\pi n$$

The first two solutions are from the first equation and the third solution is from the second equation.

Let's work one more trig equation that involves solving a quadratic equation. However, this time, unlike the previous example this one won't factor and so we'll need to use the quadratic formula.

***Example 4*** Solve $8\cos^2(1-x)+13\cos(1-x)-5=0$.

***Solution***
Now, as mentioned prior to starting the example this quadratic does not factor. However, that doesn't mean all is lost. We can solve the following equation with the quadratic formula (you do **remember** this and how to use it right?),

$$8t^2+13t-5=0 \qquad \Rightarrow \qquad t=\frac{-13\pm\sqrt{329}}{16}=0.3211,\ -1.9461$$

So, if we can use the quadratic formula on this then we can also use it on the equation we're asked to solve. Doing this gives us,

$$\cos(1-x)=0.3211 \qquad \text{OR} \qquad \cos(1-x)=-1.9461$$

Now, recall **Example 9** from the previous section. In that example we noted that $-1\le\cos(\theta)\le 1$ and so the second equation will have no solutions. Therefore, the solutions to the first equation will yield the only solutions to our original equation. Solving this gives the following set of solutions,

$$\begin{aligned} x&=-0.2439-2\pi n \\ x&=-4.0393-2\pi n \end{aligned} \qquad n=0,\pm1,\pm2,\ldots$$

Note that we did get some negative numbers here and that does seem to violate the general form that we've been using in most of these examples. However, in this case the "-" are coming about when we solved for *x* after computing the inverse cosine in our calculator.

There is one more example in this section that we need to work that illustrates another way in which factoring can arise in solving trig equations. This equation is also the only one where the variable appears both inside and outside of the trig equation. Not all equations in this form can be easily solved, however some can so we want to do a quick example of one.

***Example 5*** Solve $5x\tan(8x)=3x$.

***Solution***
First, before we even start solving we need to make one thing clear. **DO NOT CANCEL AN *x* FROM BOTH SIDES!!!** While this may seem like a natural thing to do it **WILL** cause us to lose a solution here.

So, to solve this equation we'll first get all the terms on one side of the equation and then factor an *x* out of the equation. If we can cancel an *x* from all terms then it can be factored out. Doing this gives,

$$5x\tan(8x)-3x=x\left(5\tan(8x)-3\right)=0$$

Upon factoring we can see that we must have either,

$$x = 0 \qquad \text{OR} \qquad \tan\left(8x\right) = \frac{3}{5}$$

Note that if we'd canceled the *x* we would have missed the first solution. Now, we solved an equation with a tangent in it in **Example 5** of the previous section so we'll not go into the details of this solution here. Here is the solution to the trig equation.

$$\begin{aligned} x &= 0.0676 + \frac{\pi n}{4} \\ x &= 0.4603 + \frac{\pi n}{4} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

The complete set of solutions then to the original equation are,

$$\begin{aligned} x &= 0 \\ x &= 0.0676 + \frac{\pi n}{4} \\ x &= 0.4603 + \frac{\pi n}{4} \end{aligned} \qquad n = 0, \pm 1, \pm 2, \ldots$$

# Section 1-7 : Exponential Functions

In this section we're going to review one of the more common functions in both calculus and the sciences. However, before getting to this function let's take a much more general approach to things.

Let's start with $b > 0$, $b \neq 1$. An exponential function is then a function in the form,

$$f\left(x\right) = b^x$$

Note that we avoid $b = 1$ because that would give the constant function, $f\left(x\right) = 1$. We avoid $b = 0$ since this would also give a constant function and we avoid negative values of *b* for the following reason.

Let's, for a second, suppose that we did allow *b* to be negative and look at the following function.

$$g\left(x\right) = \left(-4\right)^x$$

Let's do some evaluation.

$$g\left(2\right) = \left(-4\right)^2 = 16 \qquad\qquad g\left(\frac{1}{2}\right) = \left(-4\right)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of *x* we will get real numbers and for other values of *x* we will get complex numbers. We want to avoid this so if we require $b > 0$ this will not be a problem.

Let's take a look at a couple of exponential functions.

***Example 1*** Sketch the graph of $f\left(x\right) = 2^x$ and $g\left(x\right) = \left(\frac{1}{2}\right)^x$

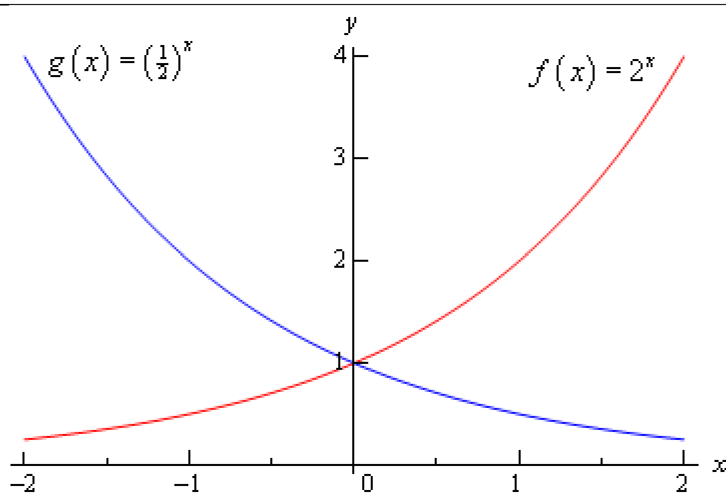
***Solution***

Let's first get a table of values for these two functions.

| *x* | *f(x)* | *g(x)* |
|---|---|---|
| -2 | $f\left(-2\right) = 2^{-2} = \frac{1}{4}$ | $g\left(-2\right) = \left(\frac{1}{2}\right)^{-2} = 4$ |
| -1 | $f\left(-1\right) = 2^{-1} = \frac{1}{2}$ | $g\left(-1\right) = \left(\frac{1}{2}\right)^{-1} = 2$ |
| 0 | $f\left(0\right) = 2^0 = 1$ | $g\left(0\right) = \left(\frac{1}{2}\right)^0 = 1$ |
| 1 | $f\left(1\right) = 2$ | $g\left(1\right) = \frac{1}{2}$ |
| 2 | $f\left(2\right) = 4$ | $g\left(2\right) = \frac{1}{4}$ |

Here's the sketch of both of these functions.




This graph illustrates some very nice properties about exponential functions in general.

**Properties of** $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.
2. $f(x) \neq 0$. An exponential function will never be zero.
3. $f(x) > 0$. An exponential function is always positive.
4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.
5. The domain of an exponential function is $(-\infty, \infty)$. In other words, you can plug every *x* into an exponential function.
6. If $0 < b < 1$ then,
   a. $f(x) \to 0 \quad \text{as} \quad x \to \infty$
   b. $f(x) \to \infty \quad \text{as} \quad x \to -\infty$
7. If $b > 1$ then,
   a. $f(x) \to \infty \quad \text{as} \quad x \to \infty$
   b. $f(x) \to 0 \quad \text{as} \quad x \to -\infty$

These will all be very useful properties to recall at times as we move throughout this course (and later Calculus courses for that matter…).

There is a very important exponential function that arises naturally in many places. This function is called the **natural exponential function**. However, for most people, this is simply the exponential function.

**Definition** : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where, $\mathbf{e} = 2.71828182845905\ldots$.

So, since $\mathbf{e} > 1$ we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

Let's take a quick look at an example.

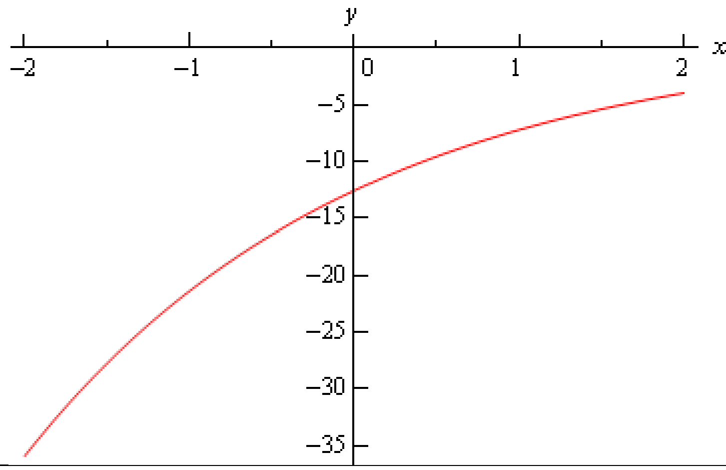
***Example 2*** Sketch the graph of $h(t) = 1 - 5\mathbf{e}^{1-\frac{t}{2}}$

***Solution***

Let's first get a table of values for this function.

| $t$ | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $h(t)$ | -35.9453 | -21.4084 | -12.5914 | -7.2436 | -4 | -2.0327 |

Here is the sketch.




The main point behind this problem is to make sure you can do this type of evaluation so make sure that you can get the values that we graphed in this example. You will be asked to do this kind of evaluation on occasion in this class.

You will be seeing exponential functions in pretty much every chapter in this class so make sure that you are comfortable with them.

## Section 1-8 : Logarithm Functions

In this section we'll take a look at a function that is related to the exponential functions we looked at in the last section. We will look at logarithms in this section. Logarithms are one of the functions that students fear the most. The main reason for this seems to be that they simply have never really had to work with them. Once they start working with them, students come to realize that they aren't as bad as they first thought.

We'll start with $b > 0$, $b \neq 1$ just as we did in the last section. Then we have

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The first is called logarithmic form and the second is called the exponential form. Remembering this equivalence is the key to evaluating logarithms. The number, *b*, is called the base.

Let's do some quick evaluations.

***Example 1*** Without a calculator give the exact value of each of the following logarithms.

**(a)** $\log_2 16$

**(b)** $\log_4 16$

**(c)** $\log_5 625$

**(d)** $\log_9 \dfrac{1}{531441}$

**(e)** $\log_{\frac{1}{6}} 36$

**(f)** $\log_{\frac{3}{2}} \dfrac{27}{8}$

***Solution***

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

**(a)** $\log_2 16$

First, let's convert to exponential form.

$$\log_2 16 = ? \qquad \text{is equivalent to} \qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad \text{because} \qquad 2^4 = 16$$

We'll not do the remaining parts in quite this detail, but they were all work in this way.

**(b)** $\log_4 16$

$$\log_4 16 = 2 \qquad \text{because} \qquad 4^2 = 16$$

Note the difference between the first and second logarithm! The base is important! It can completely change the answer.

**(c)** $\log_5 625 = 4 \qquad \text{because} \qquad 5^4 = 625$

**(d)** $\log_9 \frac{1}{531441} = -6$ because $9^{-6} = \frac{1}{9^6} = \frac{1}{531441}$

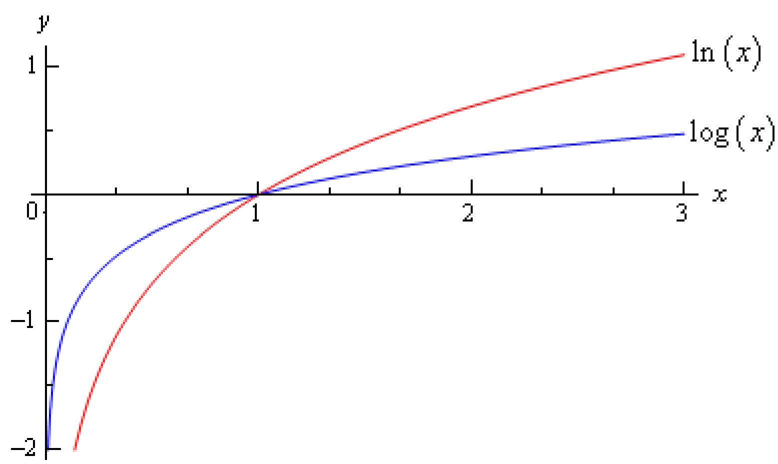
**(e)** $\log_{\frac{1}{6}} 36 = -2$ because $\left(\frac{1}{6}\right)^{-2} = 6^2 = 36$

**(f)** $\log_{\frac{3}{2}} \frac{27}{8} = 3$ because $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$

There are a couple of special logarithms that arise in many places. These are,

$$\ln x = \log_{\mathbf{e}} x \qquad \text{This log is called the natural logarithm}$$
$$\log x = \log_{10} x \qquad \text{This log is called the common logarithm}$$

In the natural logarithm the base **e** is the same number as in the natural exponential logarithm that we saw in the last **section**. Here is a sketch of both of these logarithms.




From this graph we can get a couple of very nice properties about the natural logarithm that we will use many times in this and later Calculus courses.

$$\ln x \to \infty \quad \text{as} \quad x \to \infty$$
$$\ln x \to -\infty \quad \text{as} \quad x \to 0,\ x > 0$$

Let's take a look at a couple of more logarithm evaluations. Some of which deal with the natural or common logarithm and some of which don't.

***Example 2*** Without a calculator give the exact value of each of the following logarithms.

**(a)** $\ln \sqrt[3]{\mathbf{e}}$

**(b)** $\log 1000$

**(c)** $\log_{16} 16$

**(d)** $\log_{23} 1$

**(e)** $\log_2 \sqrt[7]{32}$

***Solution***

These work exactly the same as previous example so we won't put in too many details.

**(a)** $\ln \sqrt[3]{\mathbf{e}} = \frac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

**(b)** $\log 1000 = 3$ because $10^3 = 1000$

**(c)** $\log_{16} 16 = 1$ because $16^1 = 16$

**(d)** $\log_{23} 1 = 0$ because $23^0 = 1$

**(e)** $\log_2 \sqrt[7]{32} = \frac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^5\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

This last set of examples leads us to some of the basic properties of logarithms.

**Properties**

1. The domain of the logarithm function is $\left(0,\infty\right)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. The range of the logarithm function is $\left(-\infty,\infty\right)$.
3. $\log_b b = 1$
4. $\log_b 1 = 0$
5. $\log_b b^x = x$
6. $b^{\log_b x} = x$

The last two properties will be especially useful in the next **section**. Notice as well that these last two properties tell us that,

$$f\left(x\right) = b^x \qquad \text{and} \qquad g\left(x\right) = \log_b x$$

are **inverses** of each other.

Here are some more properties that are useful in the manipulation of logarithms.

**More Properties**

7. $\log_b xy = \log_b x + \log_b y$
8. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
9. $\log_b \left(x^r\right) = r \log_b x$

Note that there is no equivalent property to the first two for sums and differences. In other words,

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

***Example 3*** Write each of the following in terms of simpler logarithms.

**(a)** $\ln x^3 y^4 z^5$

**(b)** $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

**(c)** $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

***Solution***

What the instructions really mean here is to use as many of the properties of logarithms as we can to simplify things down as much as we can.

**(a)** $\ln x^3 y^4 z^5$

Property 7 above can be extended to products of more than two functions. Once we've used Property 7 we can then use Property 9.

$$\begin{aligned}\ln x^3 y^4 z^5 &= \ln x^3 + \ln y^4 + \ln z^5 \\ &= 3\ln x + 4\ln y + 5\ln z\end{aligned}$$

**(b)** $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

When using property 8 above make sure that the logarithm that you subtract is the one that contains the denominator as its argument. Also, note that that we'll be converting the root to fractional exponents in the first step.

$$\begin{aligned}\log_3\left(\frac{9x^4}{\sqrt{y}}\right) &= \log_3 9x^4 - \log_3 y^{\frac{1}{2}} \\ &= \log_3 9 + \log_3 x^4 - \log_3 y^{\frac{1}{2}} \\ &= 2 + 4\log_3 x - \frac{1}{2}\log_3 y\end{aligned}$$

**(c)** $\log\left(\dfrac{x^2+y^2}{(x-y)^3}\right)$

The point to this problem is mostly the correct use of property 9 above.

$$\begin{aligned}\log\left(\frac{x^2+y^2}{(x-y)^3}\right) &= \log\left(x^2+y^2\right) - \log(x-y)^3 \\ &= \log\left(x^2+y^2\right) - 3\log(x-y)\end{aligned}$$

You can use Property 9 on the second term because the WHOLE term was raised to the 3, but in the first logarithm, only the individual terms were squared and not the term as a whole so the 2's must stay where they are!

The last topic that we need to look at in this section is the **change of base** formula for logarithms. The change of base formula is,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This is the most general change of base formula and will convert from base *b* to base *a*. However, the usual reason for using the change of base formula is to compute the value of a logarithm that is in a base that you can't easily deal with. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can deal with. The two most common change of base formulas are

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{and} \qquad \log_b x = \frac{\log x}{\log b}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the first part of this section we computed the value of a few logarithms, but we could do these without the change of base formula because all the arguments could be written in terms of the base to a power. For instance,

$$\log_7 49 = 2 \qquad \text{because} \qquad 7^2 = 49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7 50 = \frac{\ln 50}{\ln 7} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7 50 = \frac{\log 50}{\log 7} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

So, it doesn't matter which we use, we will get the same answer regardless of the logarithm that we use in the change of base formula.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7 49 = \frac{\ln 49}{\ln 7} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

So, in this section we saw how logarithms work and took a look at some of the properties of logarithms. We will run into logarithms on occasion so make sure that you can deal with them when we do run into them.

## Section 1-9 : Exponential and Logarithm Equations

In this section we'll take a look at solving equations with exponential functions or logarithms in them.

We'll start with equations that involve exponential functions. The main property that we'll need for these equations is,

$$\log_b b^x = x$$

***Example 1*** Solve $7 + 15\mathbf{e}^{1-3z} = 10$.

***Solution***

The first step is to get the exponential all by itself on one side of the equation with a coefficient of one.

$$\begin{aligned} 7 + 15\mathbf{e}^{1-3z} &= 10 \\ 15\mathbf{e}^{1-3z} &= 3 \\ \mathbf{e}^{1-3z} &= \frac{1}{5} \end{aligned}$$

Now, we need to get the *z* out of the exponent so we can solve for it. To do this we will use the property above. Since we have an **e** in the equation we'll use the natural logarithm. First, we take the logarithm of both sides and then use the property to simplify the equation.

$$\begin{aligned} \ln\left(\mathbf{e}^{1-3z}\right) &= \ln\left(\frac{1}{5}\right) \\ 1 - 3z &= \ln\left(\frac{1}{5}\right) \end{aligned}$$

All we need to do now is solve this equation for *z*.

$$\begin{aligned} 1 - 3z &= \ln\left(\frac{1}{5}\right) \\ -3z &= -1 + \ln\left(\frac{1}{5}\right) \\ z &= -\frac{1}{3}\left(-1 + \ln\left(\frac{1}{5}\right)\right) = 0.8698126372 \end{aligned}$$

***Example 2*** Solve $10^{t^2 - t} = 100$.

***Solution***

Now, in this case it looks like the best logarithm to use is the common logarithm since left hand side has a base of 10. There's no initial simplification to do, so just take the log of both sides and simplify.

$$\begin{aligned} \log 10^{t^2 - t} &= \log 100 = \log 10^2 = 2 \\ t^2 - t &= 2 \end{aligned}$$

At this point, we've just got a quadratic that can be solved

$$t^2 - t - 2 = 0$$
$$(t-2)(t+1) = 0$$

So, it looks like the solutions in this case are $t = 2$ and $t = -1$.

Now that we've seen a couple of equations where the variable only appears in the exponent we need to see an example with variables both in the exponent and out of it.

***Example 3*** Solve $x - x\mathbf{e}^{5x+2} = 0$.

***Solution***

The first step is to factor an *x* out of both terms.

**DO NOT DIVIDE AN *x* FROM BOTH TERMS!!!!**

Note that it is very tempting to "simplify" the equation by dividing an *x* out of both terms. However, if you do that you'll miss a solution as we'll see.

$$\begin{aligned} x - x\mathbf{e}^{5x+2} &= 0 \\ x\left(1 - \mathbf{e}^{5x+2}\right) &= 0 \end{aligned}$$

So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.

$$x = 0 \qquad \text{OR} \qquad 1 - \mathbf{e}^{5x+2} = 0$$

The first possibility has nothing more to do, except notice that if we had divided both sides by an *x* we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first two that we did in this section.

$$\begin{aligned} \mathbf{e}^{5x+2} &= 1 \\ 5x + 2 &= \ln 1 \\ 5x + 2 &= 0 \\ x &= -\frac{2}{5} \end{aligned}$$

Don't forget that $\ln 1 = 0$.

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

The next equation is a more complicated (looking at least…) example similar to the previous one.

***Example 4*** Solve $5\left(x^2-4\right)=\left(x^2-4\right)\mathbf{e}^{7-x}$.

***Solution***

As with the previous problem do NOT divide an $x^2-4$ out of both sides. Doing this will lose solutions even though it "simplifies" the equation. Note however, that if you can divide a term out then you can also factor it out if the equation is written properly.

So, the first step here is to move everything to one side of the equation and then to factor out the $x^2-4$.

$$\begin{aligned} 5\left(x^2-4\right)-\left(x^2-4\right)\mathbf{e}^{7-x} &= 0 \\ \left(x^2-4\right)\left(5-\mathbf{e}^{7-x}\right) &= 0 \end{aligned}$$

At this point all we need to do is set each factor equal to zero and solve each.

$$\begin{aligned} x^2-4 &= 0 & \qquad 5-\mathbf{e}^{7-x} &= 0 \\ x &= \pm 2 & \mathbf{e}^{7-x} &= 5 \\ & & 7-x &= \ln(5) \\ & & x &= 7-\ln(5)=5.390562088 \end{aligned}$$

The three solutions are then $x=\pm 2$ and $x=5.3906$.

As a final example let's take a look at an equation that contains two different exponentials.

***Example 5*** Solve $4\mathbf{e}^{1+3x}-9\mathbf{e}^{5-2x}=0$.

***Solution***

The first step here is to get one exponential on each side and then we'll divide both sides by one of them (which doesn't matter for the most part) so we'll have a quotient of two exponentials. The quotient can then be simplified and we'll finally get both coefficients on the other side. Doing all of this gives,

$$\begin{aligned} 4\mathbf{e}^{1+3x} &= 9\mathbf{e}^{5-2x} \\ \frac{\mathbf{e}^{1+3x}}{\mathbf{e}^{5-2x}} &= \frac{9}{4} \\ \mathbf{e}^{1+3x-(5-2x)} &= \tfrac{9}{4} \\ \mathbf{e}^{5x-4} &= \tfrac{9}{4} \end{aligned}$$

Note that while we said that it doesn't really matter which exponential we divide out by doing it the way we did here we'll avoid a negative coefficient on the *x*. Not a major issue, but those minus signs on coefficients are really easy to lose on occasion.

This is now in a form that we can deal with so here's the rest of the solution.

$$\begin{aligned}\mathbf{e}^{5x-4} &= \tfrac{9}{4}\\ 5x-4 &= \ln\left(\tfrac{9}{4}\right)\\ 5x &= 4+\ln\left(\tfrac{9}{4}\right)\\ x &= \tfrac{1}{5}\left(4+\ln\left(\tfrac{9}{4}\right)\right) = 0.9621860432\end{aligned}$$

This equation has a single solution of $x = 0.9622$.

Now let's take a look at some equations that involve logarithms. The main property that we'll be using to solve these kinds of equations is,

$$b^{\log_b x} = x$$

***Example 6*** Solve $3+2\ln\left(\frac{x}{7}+3\right) = -4$.

***Solution***

This first step in this problem is to get the logarithm by itself on one side of the equation with a coefficient of 1.

$$2\ln\left(\frac{x}{7}+3\right) = -7$$

$$\ln\left(\frac{x}{7}+3\right) = -\frac{7}{2}$$

Now, we need to get the *x* out of the logarithm and the best way to do that is to "exponentiate" both sides using **e**. In other words,

$$\mathbf{e}^{\ln\left(\frac{x}{7}+3\right)} = \mathbf{e}^{-\frac{7}{2}}$$

So, using the property above with **e**, since there is a natural logarithm in the equation, we get,

$$\frac{x}{7}+3 = \mathbf{e}^{-\frac{7}{2}}$$

Now all that we need to do is solve this for *x*.

$$\begin{aligned}\frac{x}{7}+3 &= \mathbf{e}^{-\frac{7}{2}}\\ \frac{x}{7} &= -3+\mathbf{e}^{-\frac{7}{2}}\\ x &= 7\left(-3+\mathbf{e}^{-\frac{7}{2}}\right) = -20.78861832\end{aligned}$$

At this point we might be tempted to say that we're done and move on. However, we do need to be careful. Recall from the previous **section** that we can't plug a negative number into a logarithm. This, by itself, doesn't mean that our answer won't work since its negative. What we need to do is plug it into the logarithm and make sure that $\frac{x}{7}+3$ will not be negative. I'll leave it to you to verify that this is in fact positive upon plugging our solution into the logarithm and so $x = -20.78861832$ is a solution to the equation.

Let’s now take a look at a more complicated equation. Often there will be more than one logarithm in the equation. When this happens we will need to use one or more of the following properties to combine all the logarithms into a single logarithm. Once this has been done we can proceed as we did in the previous example.

$$\log_b xy = \log_b x + \log_b y \qquad \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \qquad \log_b\left(x^r\right) = r\log_b x$$

***Example 7*** Solve $2\ln\left(\sqrt{x}\right) - \ln\left(1-x\right) = 2$.

***Solution***

First get the two logarithms combined into a single logarithm.

$$\begin{aligned} 2\ln\left(\sqrt{x}\right) - \ln\left(1-x\right) &= 2 \\ \ln\left(\left(\sqrt{x}\right)^2\right) - \ln\left(1-x\right) &= 2 \\ \ln\left(x\right) - \ln\left(1-x\right) &= 2 \\ \ln\left(\frac{x}{1-x}\right) &= 2 \end{aligned}$$

Now, exponentiate both sides and solve for *x*.

$$\begin{aligned} \frac{x}{1-x} &= \mathbf{e}^2 \\ x &= \mathbf{e}^2\left(1-x\right) \\ x &= \mathbf{e}^2 - \mathbf{e}^2 x \\ x\left(1+\mathbf{e}^2\right) &= \mathbf{e}^2 \\ x &= \frac{\mathbf{e}^2}{1+\mathbf{e}^2} = 0.8807970780 \end{aligned}$$

The solution work here was a little messy but this is work that you will need to be able to do on occasion so make sure you can do it!

Finally, we just need to make sure that the solution, $x = 0.8807970780$, doesn’t produce negative numbers in both of the original logarithms. It doesn’t, so this is in fact our solution to this problem.

Let’s take a look at another example.

***Example 8*** Solve $\log x + \log(x-3) = 1$.

***Solution***
As with the last example, first combine the logarithms into a single logarithm.

$$\begin{aligned} \log x + \log(x-3) &= 1 \\ \log\left(x(x-3)\right) &= 1 \end{aligned}$$

Now exponentiate, using 10 this time instead of **e** because we've got common logs in the equation, both sides.

$$\begin{aligned} 10^{\log\left(x^2-3x\right)} &= 10^1 \\ x^2-3x &= 10 \\ x^2-3x-10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

So, potential solutions are $x=5$ and $x=-2$. Note, however that if we plug $x=-2$ into either of the two original logarithms we would get negative numbers so this can't be a solution. We can however, use $x=5$.

Therefore, the solution to this equation is $x=5$.

When solving equations with logarithms it is important to check your potential solutions to make sure that they don't generate logarithms of negative numbers or zero. It is also important to make sure that you do the checks in the **original** equation. If you check them in the second logarithm above (after we've combined the two logs) both solutions will appear to work! This is because in combining the two logarithms we've actually changed the problem. In fact, it is this change that introduces the extra solution that we couldn't use!

Also, be careful in solving equations containing logarithms to not get locked into the idea that you will get two potential solutions and only one of these will work. It is possible to have problems where both are solutions and where neither are solutions.

There is one more problem that we should work.

***Example 9*** Solve $\ln(x-2) + \ln(x+1) = 2$.

***Solution***
The first step of this problem is the same as we've been doing up to this point. So, let's combine the logarithms.

$$\begin{aligned} \ln\left((x-2)(x+1)\right) &= 2 \\ \ln\left(x^2-x-2\right) &= 2 \end{aligned}$$

Now we can exponentiate both sides with respect to **e** to eliminate the logarithm. Doing this along with a little simplification gives,

$$x^2 - x - 2 = \mathbf{e}^2$$
$$x^2 - x - 2 - \mathbf{e}^2 = 0$$

We've reached the point of this problem. We need to solve this quadratic and without the $\mathbf{e}^2$ everyone would be able to do that. However, with the $\mathbf{e}^2$ people tend to decide that they can't do it.

This is just a quadratic equation and everyone in this class should be able to solve that. The only difference between this quadratic equation and those you are probably used to seeing is that there are numbers in it that are not integers, or at worst, fractions. In this case the constant in the quadratic is just $-2-\mathbf{e}^2$ and so all we need to do is use the quadratic formula to get the solutions.

The solutions to this quadratic equation are,

$$x = \frac{1 \pm \sqrt{1 - 4(1)\left(-2-\mathbf{e}^2\right)}}{2} = \frac{1 \pm \sqrt{9 + 4\mathbf{e}^2}}{2} = -2.6047,\ 3.6047$$

Do not get excited about the "messy" solutions to this quadratic. We will get these kinds of solutions on occasion.

The last step to this problem is to check the two solutions to the quadratic equation in the original equation. Doing that we can see that the first solution, -2.6047, will give negative numbers in the logarithms and so can't be a solution. On the other hand, the second solution, 3.6047, does not give negative numbers in the logarithms and so is okay.

The solution to the original equation is $x = 3.6047$ .

# Section 1-10 : Common Graphs

The purpose of this section is to make sure that you're familiar with the graphs of many of the basic functions that you're liable to run across in a calculus class.

***Example 1*** Graph $y = -\frac{2}{5}x + 3$.

***Solution***

This is a line in the slope intercept form

$$y = mx + b$$

In this case the line has a *y* intercept of (0,*b*) and a slope of *m*. Recall that slope can be thought of as

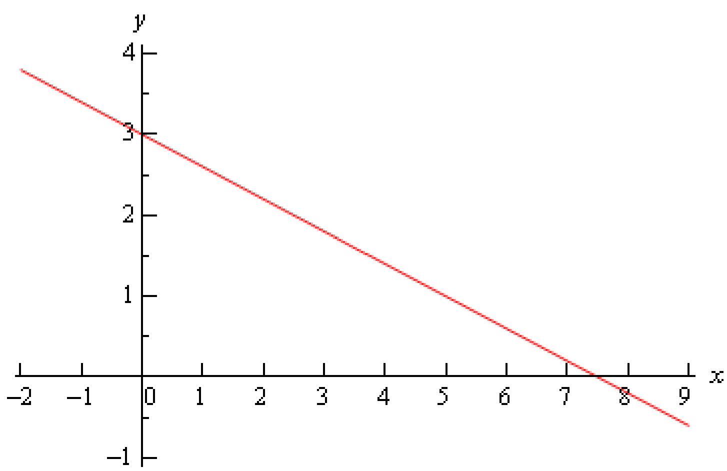
$$m = \frac{\text{rise}}{\text{run}}$$

Note that if the slope is negative we tend to think of the rise as a fall.

The slope allows us to get a second point on the line. Once we have any point on the line and the slope we move right by *run* and up/down by *rise* depending on the sign. This will be a second point on the line.

In this case we know (0,3) is a point on the line and the slope is $-\frac{2}{5}$. So starting at (0,3) we'll move 5 to the right (*i.e.* $0 \to 5$) and down 2 (*i.e.* $3 \to 1$) to get (5,1) as a second point on the line. Once we've got two points on a line all we need to do is plot the two points and connect them with a line.

Here's the sketch for this line.



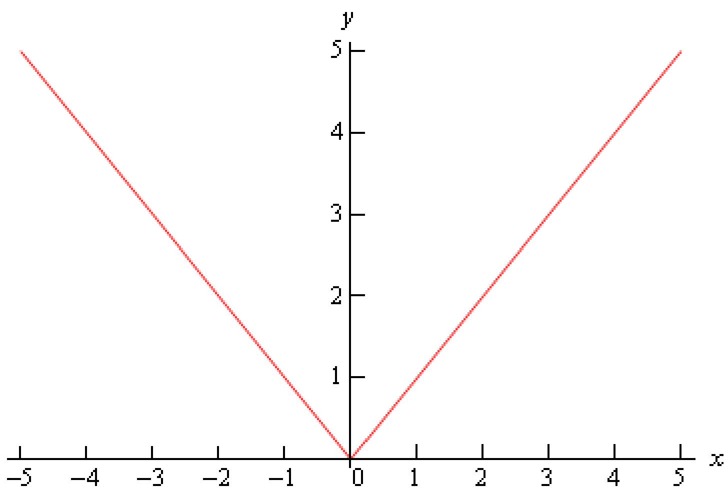
***Example 2*** Graph $f\left(x\right)=\left|x\right|$

***Solution***

There really isn't much to this problem outside of reminding ourselves of what absolute value is. Recall that the absolute value function is defined as,

$$\left|x\right|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

The graph is then,




***Example 3*** Graph $f\left(x\right)=-x^2+2x+3$.

***Solution***

This is a parabola in the general form.

$$f\left(x\right)=ax^2+bx+c$$

In this form, the *x*-coordinate of the vertex (the highest or lowest point on the parabola) is $x=-\frac{b}{2a}$ and the *y*-coordinate is $y=f\left(-\frac{b}{2a}\right)$. So, for our parabola the coordinates of the vertex will be.

$$x=-\frac{2}{2\left(-1\right)}=1$$

$$y=f\left(1\right)=-\left(1\right)^2+2\left(1\right)+3=4$$

So, the vertex for this parabola is (1,4).

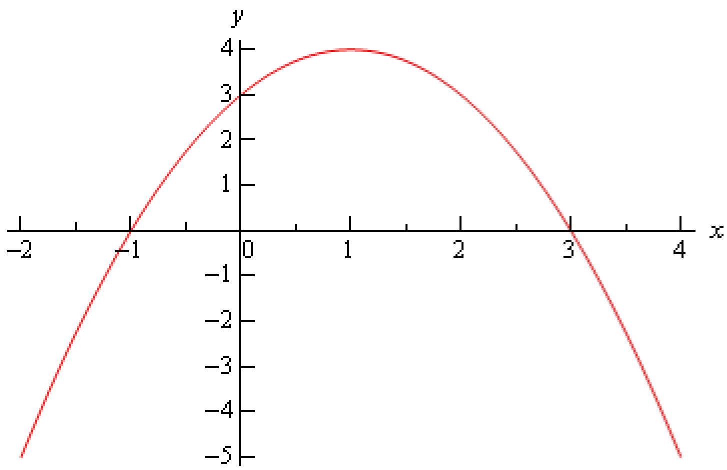
We can also determine which direction the parabola opens from the sign of *a*. If *a* is positive the parabola opens up and if *a* is negative the parabola opens down. In our case the parabola opens down.

Now, because the vertex is above the *x*-axis and the parabola opens down we know that we'll have *x*-intercepts (*i.e.* values of *x* for which we'll have $f\left(x\right)=0$) on this graph. So, we'll solve the following.

$$-x^2 + 2x + 3 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$

So, we will have *x*-intercepts at $x = -1$ and $x = 3$. Notice that to make our life easier in the solution process we multiplied everything by -1 to get the coefficient of the $x^2$ positive. This made the factoring easier.

Here's a sketch of this parabola.




***Example 4*** Graph $f(y) = y^2 - 6y + 5$

***Solution***

Most people come out of an Algebra class capable of dealing with functions in the form $y = f(x)$. However, many functions that you will have to deal with in a Calculus class are in the form $x = f(y)$ and can only be easily worked with in that form. So, you need to get used to working with functions in this form.

The nice thing about these kinds of function is that if you can deal with functions in the form $y = f(x)$ then you can deal with functions in the form $x = f(y)$ even if you aren't that familiar with them.

Let's first consider the equation.

$$y = x^2 - 6x + 5$$

This is a parabola that opens up and has a vertex of (3,-4), as we know from our work in the previous example.

For our function we have essentially the same equation except the *x* and *y*'s are switched around. In other words, we have a parabola in the form,

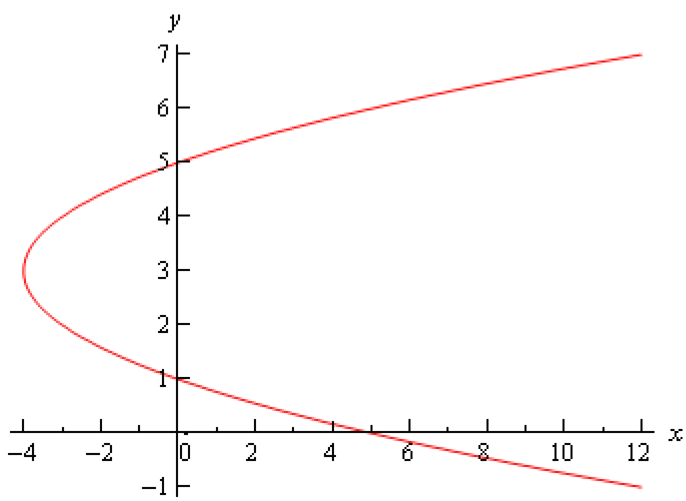
$$x = ay^2 + by + c$$

This is the general form of this kind of parabola and this will be a parabola that opens left or right depending on the sign of *a*. The *y*-coordinate of the vertex is given by $y = -\dfrac{b}{2a}$ and we find the *x*-coordinate by plugging this into the equation. So, you can see that this is very similar to the type of parabola that you're already used to dealing with.

Now, let's get back to the example. Our function is a parabola that opens to the right (*a* is positive) and has a vertex at (-4,3). The vertex is to the left of the *y*-axis and opens to the right so we'll need the *y*-intercepts (*i.e.* values of *y* for which we'll have $f\left(y\right) = 0$)). We find these just like we found *x*-intercepts in the previous problem.

$$\begin{gathered} y^2 - 6y + 5 = 0 \\ \left(y-5\right)\left(y-1\right) = 0 \end{gathered}$$

So, our parabola will have *y*-intercepts at $y = 1$ and $y = 5$. Here's a sketch of the graph.




***Example 5*** Graph $x^2 + 2x + y^2 - 8y + 8 = 0$.

***Solution***

To determine just what kind of graph we've got here we need to complete the square on both the *x* and the *y*.

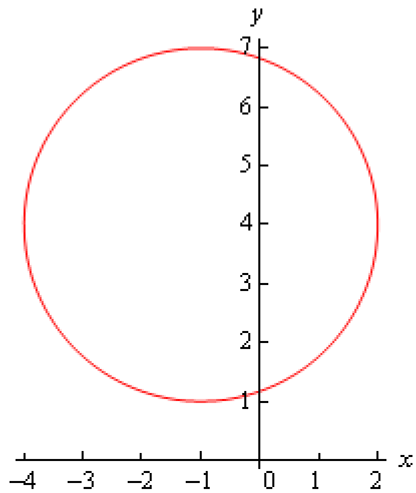
$$\begin{aligned} x^2 + 2x + y^2 - 8y + 8 &= 0 \\ x^2 + 2x + 1 - 1 + y^2 - 8y + 16 - 16 + 8 &= 0 \\ \left(x+1\right)^2 + \left(y-4\right)^2 &= 9 \end{aligned}$$

Recall that to complete the square we take the half of the coefficient of the *x* (or the *y*), square this and then add and subtract it to the equation.

Upon doing this we see that we have a circle and it's now written in standard form.

$$\left(x-h\right)^2 + \left(y-k\right)^2 = r^2$$

When circles are in this form we can easily identify the center (*h, k*) and radius *r*. Once we have these we can graph the circle simply by starting at the center and moving right, left, up and down by *r* to get the rightmost, leftmost, top most and bottom most points respectively.
Our circle has a center at (-1, 4) and a radius of 3. Here's a sketch of this circle.




***Example 6*** Graph $\dfrac{(x-2)^2}{9}+4(y+2)^2=1$

***Solution***
This is an ellipse. The standard form of the ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

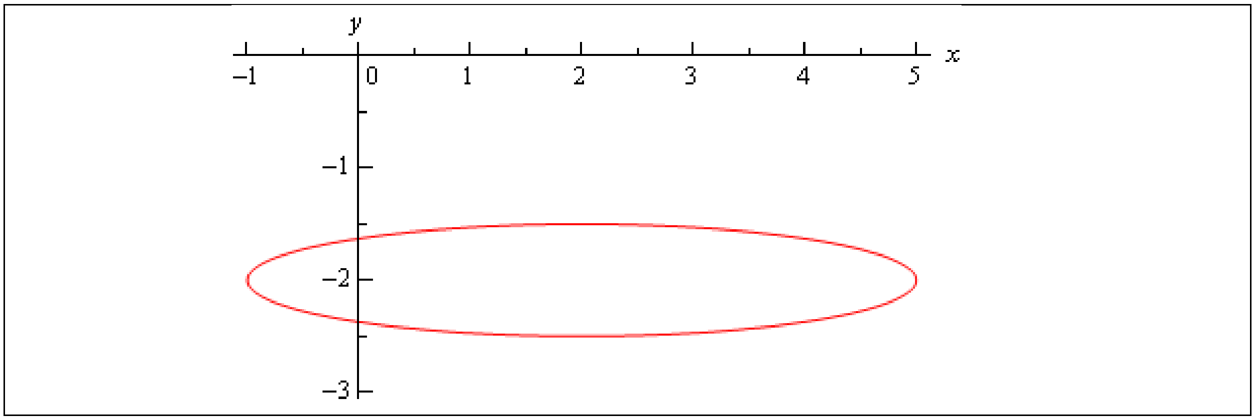
This is an ellipse with center (*h, k*) and the right most and left most points are a distance of *a* away from the center and the top most and bottom most points are a distance of *b* away from the center.

The ellipse for this problem has center (2, -2) and has $a=3$ and $b=\frac{1}{2}$. Note that to get the *b* we're really rewriting the equation as,

$$\frac{(x-2)^2}{9}+\frac{(y+2)^2}{1/4}=1$$

to get it into standard from.

Here's a sketch of the ellipse.




***Example 7*** Graph $\dfrac{(x+1)^2}{9}-\dfrac{(y-2)^2}{4}=1$

***Solution***
This is a hyperbola. There are actually two standard forms for a hyperbola. Here are the basics for each form.

| Form | $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ | $\dfrac{(y-k)^2}{b^2}-\dfrac{(x-h)^2}{a^2}=1$ |
|---|---|---|
| Center | (*h, k*) | (*h, k*) |
| Opens | Opens right and left | Opens up and down |
| Vertices | *a* units right and left of center. | *b* units up and down from center. |
| Slope of Asymptotes | $\pm\dfrac{b}{a}$ | $\pm\dfrac{b}{a}$ |

So, what does all this mean? First, notice that one of the terms is positive and the other is negative. This will determine which direction the two parts of the hyperbola open. If the *x* term is positive the hyperbola opens left and right. Likewise, if the *y* term is positive the parabola opens up and down.

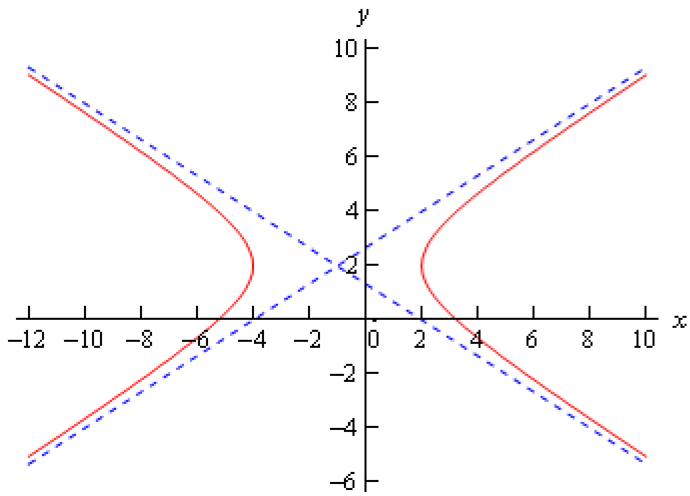
Both have the same "center". Note that hyperbolas don't really have a center in the sense that circles and ellipses have centers. The center is the starting point in graphing a hyperbola. It tells us how to get to the vertices and how to get the asymptotes set up.

The asymptotes of a hyperbola are two lines that intersect at the center and have the slopes listed above. As you move farther out from the center the graph will get closer and closer to the asymptotes.

For the equation listed here the hyperbola will open left and right. Its center is

(-1, 2). The two vertices are (-4, 2) and (2, 2). The asymptotes will have slopes $\pm\frac{2}{3}$.

Here is a sketch of this hyperbola. Note that the asymptotes are denoted by the two dashed lines.

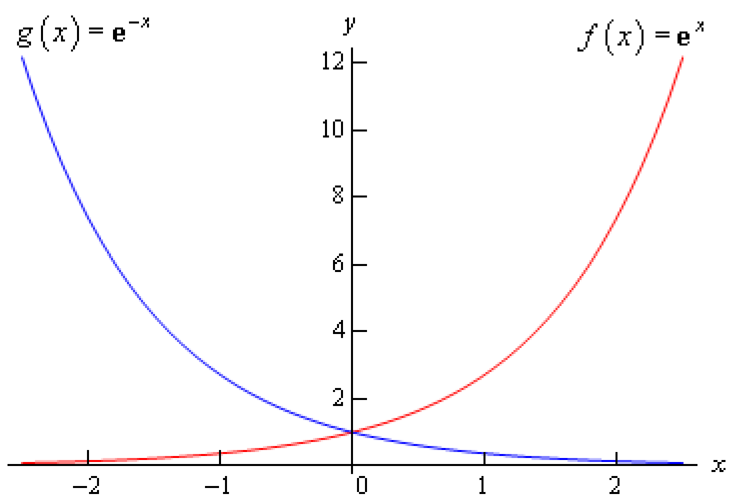


***Example 8*** Graph $f\left(x\right)=\mathbf{e}^{x}$ and $g\left(x\right)=\mathbf{e}^{-x}$.

***Solution***

There really isn't a lot to this problem other than making sure that both of these exponentials are graphed somewhere.

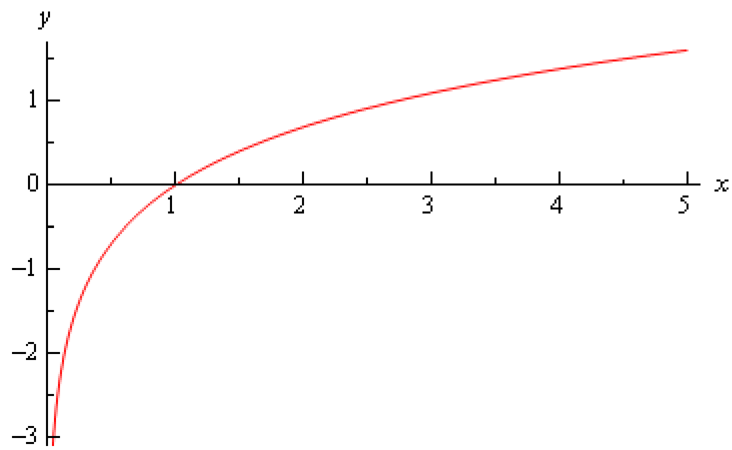
These will both show up with some regularity in later sections and their behavior as *x* goes to both plus and minus infinity will be needed and from this graph we can clearly see this behavior.



***Example 9*** Graph $f\left(x\right) = \ln\left(x\right)$.

***Solution***

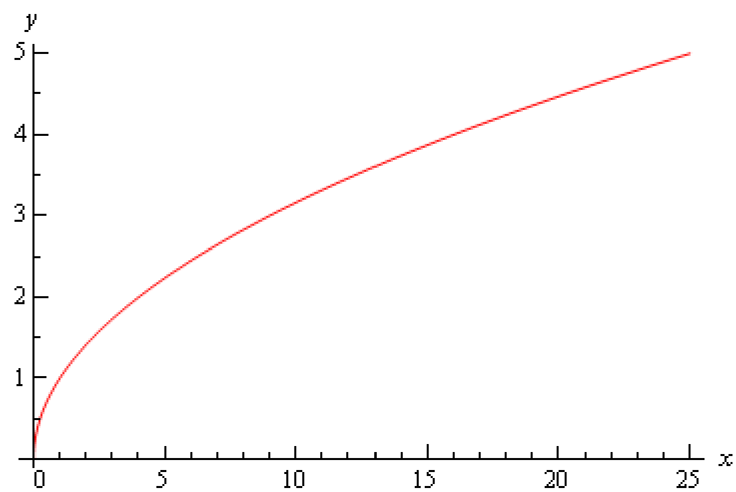
This has already been graphed once in this review, but this puts it here with all the other "important" graphs.




***Example 10*** Graph $y = \sqrt{x}$.

***Solution***

This one is fairly simple, we just need to make sure that we can graph it when need be.

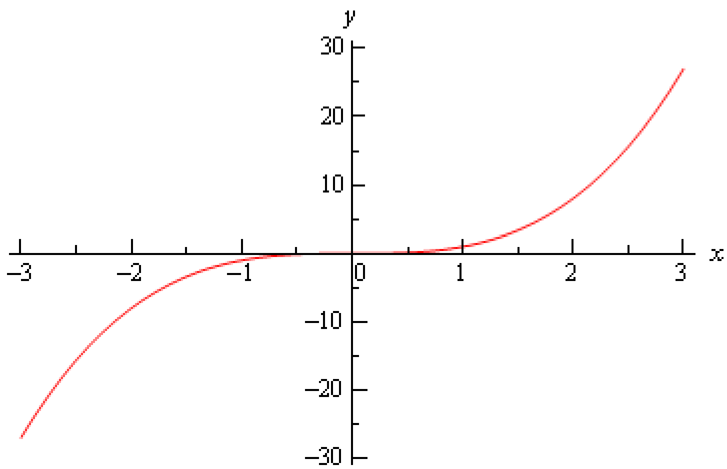



Remember that the domain of the square root function is $x \ge 0$.

***Example 11*** Graph $y = x^3$

***Solution***

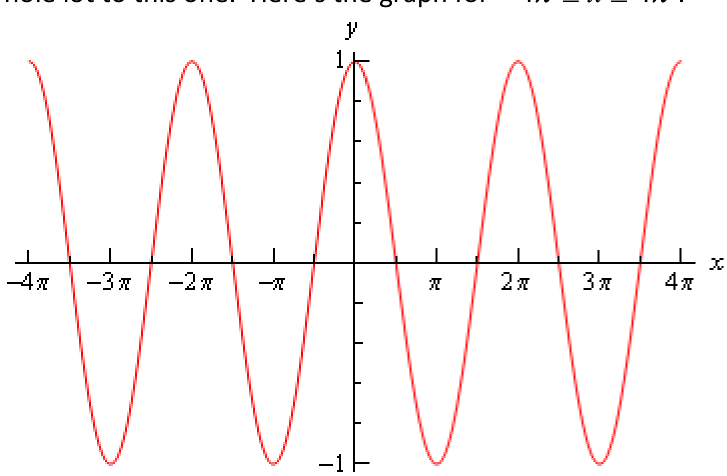
Again, there really isn't much to this other than to make sure it's been graphed somewhere so we can say we've done it.




***Example 12*** Graph $y = \cos(x)$

***Solution***

There really isn't a whole lot to this one. Here's the graph for $-4\pi \le x \le 4\pi$ .




Let's also note here that we can put all values of *x* into cosine (which won't be the case for most of the trig functions) and so the domain is all real numbers. Also note that

$$-1 \le \cos(x) \le 1$$

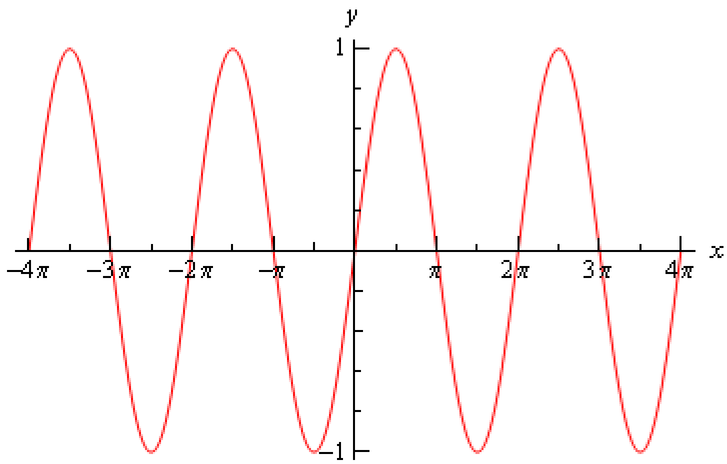
It is important to notice that cosine will never be larger than 1 or smaller than -1. This will be useful on occasion in a calculus class. In general we can say that

$$-R \le R\cos(\omega x) \le R$$

***Example 13*** Graph $y = \sin\left(x\right)$

***Solution***

As with the previous problem there really isn't a lot to do other than graph it. Here is the graph for $-4\pi \le x \le 4\pi$.




From this graph we can see that sine has the same range that cosine does. In general

$$-R \le R\sin\left(\omega x\right) \le R$$

As with cosine, sine itself will never be larger than 1 and never smaller than -1. Also the domain of sine is all real numbers.

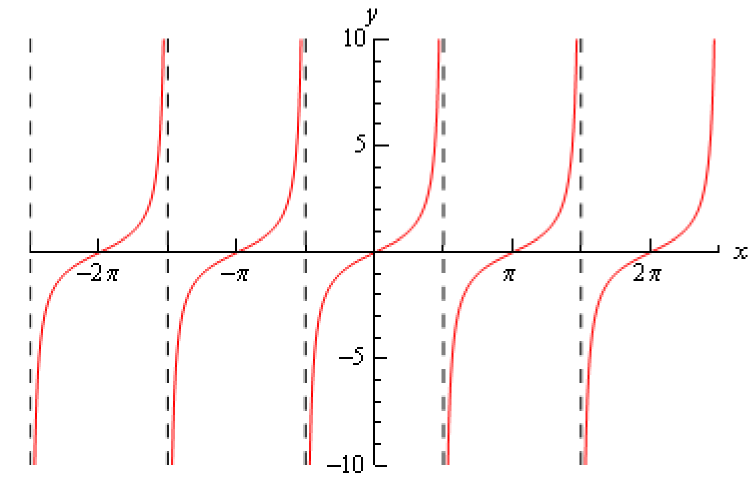
***Example 14*** Graph $y = \tan\left(x\right)$.

***Solution***

In the case of tangent we have to be careful when plugging *x*'s in since tangent doesn't exist wherever cosine is zero (remember that $\tan x = \dfrac{\sin x}{\cos x}$). Tangent will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of tangent on the range $-\dfrac{5\pi}{2} < x < \dfrac{5\pi}{2}$.




*Example 15* Graph $y = \sec(x)$

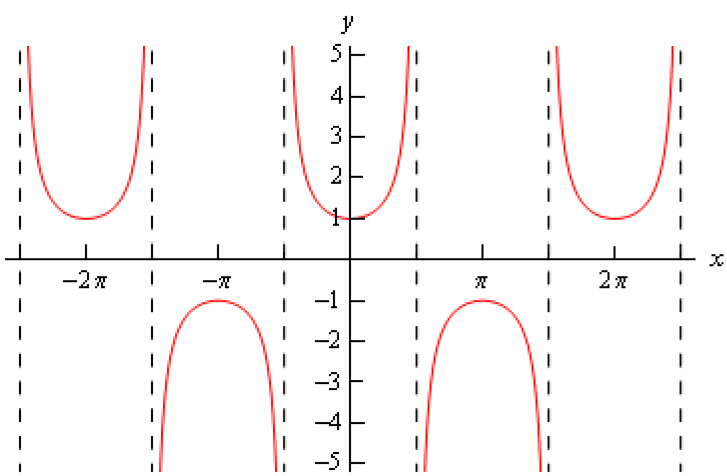
***Solution***

As with tangent we will have to avoid *x*'s for which cosine is zero (remember that $\sec x = \dfrac{1}{\cos x}$).

Secant will not exist at

$$x = \cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and the graph will have asymptotes at these points. Here is the graph of secant on the range $-\dfrac{5\pi}{2} < x < \dfrac{5\pi}{2}$.




Notice that the graph is always greater than 1 or less than -1. This should not be terribly surprising. Recall that $-1 \le \cos(x) \le 1$. So, one divided by something less than one will be greater than 1. Also, $1/\pm 1 = \pm 1$ and so we get the following ranges for secant.

$$\sec(\omega x) \ge 1 \qquad \text{and} \qquad \sec(\omega x) \le -1$$

Note that we did not graph cotangent or cosecant here. However, they are similar to the graphs of tangent and secant and you should be able to do quick sketches of them given the work above if needed.

Finally, note that we did not cover any of the basic transformations that are often used in graphing functions here. The practice problems for this section have quite a few problems designed to help you remember them. If you know the basic transformations it often makes graphing a much simpler process so if you are not comfortable with them you should work through the practice problems for this section.