DIFFERENTIAL EQUATIONS

Series Solutions to Differential Equations

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Chapter : 6 : Series Solutions to Differential Equations

In this chapter we will finally be looking at nonconstant coefficient differential equations. While we won't cover all possibilities in this chapter we will be looking at two of the more common methods for dealing with this kind of differential equation.

The first method that we'll be taking a look at, series solutions, will actually find a series representation for the solution instead of the solution itself. You first saw something like this when you looked at Taylor series in your Calculus class. As we will see however, these won't work for every differential equation.

The second method that we'll look at will only work for a special class of differential equations. This special case will cover some of the cases in which series solutions can't be used.

Here is a brief listing of the topics in this chapter.

Review : Power Series – In this section we give a brief review of some of the basics of power series. Included are discussions of using the Ratio Test to determine if a power series will converge, adding/subtracting power series, differentiating power series and index shifts for power series.

Review : Taylor Series – In this section we give a quick reminder on how to construct the Taylor series for a function. Included are derivations for the Taylor series of $\mathbf{e}^x$ and $\cos\left(x\right)$ about $x = 0$ as well as showing how to write down the Taylor series for a polynomial.

Series Solutions – In this section we define ordinary and singular points for a differential equation. We also show who to construct a series solution for a differential equation about an ordinary point. The method illustrated in this section is useful in solving, or at least getting an approximation of the solution, differential equations with coefficients that are not constant.

Euler Equations – In this section we will discuss how to solve Euler's differential equation, $ax^2 y'' + bxy' + cy = 0$. Note that while this does not involve a series solution it is included in the series solution chapter because it illustrates how to get a solution to at least one type of differential equation at a singular point.

Section 6-1 : Review : Power Series

Before looking at series solutions to a differential equation we will first need to do a cursory review of power series. A power series is a series in the form,

$$f\left(x\right)=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \tag{1}$$

where, x_0 and a_n are numbers. We can see from this that a power series is a function of x. The function notation is not always included, but sometimes it is so we put it into the definition above.

Before proceeding with our review we should probably first recall just what series really are. Recall that series are really just summations. One way to write our power series is then,

$$\begin{aligned} f\left(x\right)&=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \\ &=a_0+a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+a_3\left(x-x_0\right)^3+\cdots \end{aligned} \tag{2}$$

Notice as well that if we needed to for some reason we could always write the power series as,

$$\begin{aligned} f\left(x\right)&=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \\ &=a_0+a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+a_3\left(x-x_0\right)^3+\cdots \\ &=a_0+\sum_{n=1}^{\infty}a_n\left(x-x_0\right)^n \end{aligned}$$

All that we're doing here is noticing that if we ignore the first term (corresponding to $n=0$) the remainder is just a series that starts at $n = 1$. When we do this we say that we've stripped out the $n=0$, or first, term. We don't need to stop at the first term either. If we strip out the first three terms we'll get,

$$\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n=a_0+a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+\sum_{n=3}^{\infty}a_n\left(x-x_0\right)^n$$

There are times when we'll want to do this so make sure that you can do it.

Now, since power series are functions of x and we know that not every series will in fact exist, it then makes sense to ask if a power series will exist for all x. This question is answered by looking at the convergence of the power series. We say that a power series **converges** for $x = c$ if the series,

$$\sum_{n=0}^{\infty}a_n\left(c-x_0\right)^n$$

converges. Recall that this series will converge if the limit of partial sums,

$$\lim_{N\to\infty}\sum_{n=0}^{N}a_n\left(c-x_0\right)^n$$

exists and is finite. In other words, a power series will converge for $x = c$ if

$$\sum_{n=0}^{\infty} a_n \left(c - x_0\right)^n$$

is a finite number.

Note that a power series will always converge if $x = x_0$. In this case the power series will become

$$\sum_{n=0}^{\infty} a_n \left(x_0 - x_0\right)^n = a_0$$

With this we now know that power series are guaranteed to exist for at least one value of x. We have the following fact about the convergence of a power series.

Fact

Given a power series, (1), there will exist a number $0 \le \rho \le \infty$ so that the power series will converge for $\left|x - x_0\right| < \rho$ and diverge for $\left|x - x_0\right| > \rho$. This number is called the **radius of convergence**.

Determining the radius of convergence for most power series is usually quite simple if we use the ratio test.

Ratio Test

Given a power series compute,

$$L = \left|x - x_0\right| \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$$

then,

$$\begin{aligned} L &< 1 && \Rightarrow && \text{the series converges} \\ L &> 1 && \Rightarrow && \text{the series diverges} \\ L &= 1 && \Rightarrow && \text{the series may converge or diverge} \end{aligned}$$

Let's take a quick look at how this can be used to determine the radius of convergence.

Example 1 Determine the radius of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{\left(-3\right)^n}{n\,7^{n+1}} \left(x - 5\right)^n$$

Solution

So, in this case we have,

$$a_n = \frac{\left(-3\right)^n}{n\,7^{n+1}} \qquad\qquad a_{n+1} = \frac{\left(-3\right)^{n+1}}{\left(n+1\right)7^{n+2}}$$

Remember that to compute a_{n+1} all we do is replace all the n's in a_n with n+1. Using the ratio test then gives,

$$
\begin{aligned}
L &= |x-5| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= |x-5| \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(n+1)\,7^{n+2}} \, \frac{n\,7^{n+1}}{(-3)^n} \right| \\
&= |x-5| \lim_{n\to\infty} \left| \frac{-3}{(n+1)\,7} \, \frac{n}{1} \right| \\
&= \frac{3}{7} |x-5|
\end{aligned}
$$

Now we know that the series will converge if,

$$\frac{3}{7}|x-5| < 1 \qquad \Rightarrow \qquad |x-5| < \frac{7}{3}$$

and the series will diverge if,

$$\frac{3}{7}|x-5| > 1 \qquad \Rightarrow \qquad |x-5| > \frac{7}{3}$$

In other words, the radius of the convergence for this series is,

$$\rho = \frac{7}{3}$$

As this last example has shown, the radius of convergence is found almost immediately upon using the ratio test.

So, why are we worried about the convergence of power series? Well in order for a series solution to a differential equation to exist at a particular *x* it will need to be convergent at that *x*. If it's not convergent at a given *x* then the series solution won't exist at that *x*. So, the convergence of power series is fairly important.

Next, we need to do a quick review of some of the basics of manipulating series. We'll start with addition and subtraction.

There really isn't a whole lot to addition and subtraction. All that we need to worry about is that the two series start at the same place and both have the same exponent of the x-x_0. If they do then we can perform addition and/or subtraction as follows,

$$\sum_{n=n_0}^{\infty} a_n (x-x_0)^n \pm \sum_{n=n_0}^{\infty} b_n (x-x_0)^n = \sum_{n=n_0}^{\infty} (a_n \pm b_n)(x-x_0)^n$$

In other words, all we do is add or subtract the coefficients and we get the new series.

One of the rules that we're going to have when we get around to finding series solutions to differential equations is that the only *x* that we want in a series is the *x* that sits in $(x-x_0)^n$. This means that we will need to be able to deal with series of the form,

$$\left(x-x_0\right)^c \sum_{n=0}^{\infty} a_n \left(x-x_0\right)^n$$

where *c* is some constant. These are actually quite easy to deal with.

$$\begin{aligned}\left(x-x_0\right)^c \sum_{n=0}^{\infty} a_n \left(x-x_0\right)^n &= \left(x-x_0\right)^c \left(a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + \cdots\right) \\ &= a_0\left(x-x_0\right)^c + a_1\left(x-x_0\right)^{1+c} + a_2\left(x-x_0\right)^{2+c} + \cdots \\ &\sum_{n=0}^{\infty} a_n \left(x-x_0\right)^{n+c}\end{aligned}$$

So, all we need to do is to multiply the term in front into the series and add exponents. Also note that in order to do this both the coefficient in front of the series and the term inside the series must be in the form $x\text{-}x_0$. If they are not the same we can't do this, we will eventually see how to deal with terms that aren't in this form.

Next, we need to talk about differentiation of a power series. By looking at (2) it should be fairly easy to see how we will differentiate a power series. Since a series is just a giant summation all we need to do is differentiate the individual terms. The derivative of a power series will be,

$$\begin{aligned}f'\left(x\right) &= a_1 + 2a_2\left(x-x_0\right) + 3a_3\left(x-x_0\right)^2 + \cdots \\ &= \sum_{n=1}^{\infty} n a_n \left(x-x_0\right)^{n-1} \\ &= \sum_{n=0}^{\infty} n a_n \left(x-x_0\right)^{n-1}\end{aligned}$$

So, all we need to do is just differentiate the term inside the series and we're done. Notice as well that there are in fact two forms of the derivative. Since the *n=0* term of the derivative is zero it won't change the value of the series and so we can include it or not as we need to. In our work we will usually want the derivative to start at *n=1*, however there will be the occasional problem were it would be more convenient to start it at *n=0*.

Following how we found the first derivative it should make sense that the second derivative is,

$$\begin{aligned}f''\left(x\right) &= \sum_{n=2}^{\infty} n\left(n-1\right) a_n \left(x-x_0\right)^{n-2} \\ &= \sum_{n=1}^{\infty} n\left(n-1\right) a_n \left(x-x_0\right)^{n-2} \\ &= \sum_{n=0}^{\infty} n\left(n-1\right) a_n \left(x-x_0\right)^{n-2}\end{aligned}$$

In this case since the *n=0* and *n=1* terms are both zero we can start at any of three possible starting points as determined by the problem that we're working.

Next, we need to talk about **index shifts**. As we will see eventually we are going to want our power series written in terms of $\left(x-x_0\right)^n$ and they often won't, initially at least, be in that form. To get them into the form we need we will need to perform an index shift.

Index shifts themselves really aren't concerned with the exponent on the *x* term, they instead are concerned with where the series starts as the following example shows.

Example 2 Write the following as a series that starts at *n=0* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2}$$

Solution

An index shift is a fairly simple manipulation to perform. First, we will notice that if we define *i=n-3* then when *n=3* we will have *i=0*. So, what we'll do is rewrite the series in terms of *i* instead of *n*. We can do this by noting that *n=i+3*. So, everywhere we see an *n* in the actual series term we will replace it with an *i+3*. Doing this gives,

$$\begin{aligned}\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} &= \sum_{i=0}^{\infty}\left(i+3\right)^2 a_{i+3-1}\left(x+4\right)^{i+3+2}\\ &= \sum_{i=0}^{\infty}\left(i+3\right)^2 a_{i+2}\left(x+4\right)^{i+5}\end{aligned}$$

The upper limit won't change in this process since infinity minus three is still infinity.

The final step is to realize that the letter we use for the index doesn't matter and so we can just switch back to *n*'s.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} = \sum_{n=0}^{\infty}\left(n+3\right)^2 a_{n+2}\left(x+4\right)^{n+5}$$

Now, we usually don't go through this process to do an index shift. All we do is notice that we dropped the starting point in the series by 3 and everywhere else we saw an *n* in the series we increased it by 3. In other words, all the *n*'s in the series move in the opposite direction that we moved the starting point.

Example 3 Write the following as a series that starts at *n=5* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2}$$

Solution

To start the series to start at *n=5* all we need to do is notice that this means we will increase the starting point by 2 and so all the other *n*'s will need to decrease by 2. Doing this for the series in the previous example would give,

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} = \sum_{n=5}^{\infty}\left(n-2\right)^2 a_{n-3}\left(x+4\right)^{n}$$

Now, as we noted when we started this discussion about index shift the whole point is to get our series into terms of $\left(x-x_0\right)^n$. We can see in the previous example that we did exactly that with an index shift. The original exponent on the *(x+4)* was *n+2*. To get this down to an *n* we needed to decrease the exponent by 2. This can be done with an index that increases the starting point by 2.

Let's take a look at a couple of more examples of this.

Example 4 Write each of the following as a single series in terms of $\left(x-x_0\right)^n$.

(a) $\left(x+2\right)^2 \sum_{n=3}^{\infty} n a_n \left(x+2\right)^{n-4} - \sum_{n=1}^{\infty} n a_n \left(x+2\right)^{n+1}$

(b) $x \sum_{n=0}^{\infty} \left(n-5\right)^2 b_{n+1} \left(x-3\right)^{n+3}$

Solution

(a) $\left(x+2\right)^2 \sum_{n=3}^{\infty} n a_n \left(x+2\right)^{n-4} - \sum_{n=1}^{\infty} n a_n \left(x+2\right)^{n+1}$

First, notice that there are two series here and the instructions clearly ask for only a single series. So, we will need to subtract the two series at some point in time. The vast majority of our work will be to get the two series prepared for the subtraction. This means that the two series can't have any coefficients in front of them (other than one of course…), they will need to start at the same value of *n* and they will need the same exponent on the $x-x_0$.

We'll almost always want to take care of any coefficients first. So, we have one in front of the first series so let's multiply that into the first series. Doing this gives,

$$\sum_{n=3}^{\infty} n a_n \left(x+2\right)^{n-2} - \sum_{n=1}^{\infty} n a_n \left(x+2\right)^{n+1}$$

Now, the instructions specify that the new series must be in terms of $\left(x-x_0\right)^n$, so that's the next thing that we've got to take care of. We will do this by an index shift on each of the series. The exponent on the first series needs to go up by two so we'll shift the first series down by 2. On the second series will need to shift up by 1 to get the exponent to move down by 1. Performing the index shifts gives us the following,

$$\sum_{n=1}^{\infty} \left(n+2\right) a_{n+2} \left(x+2\right)^{n} - \sum_{n=2}^{\infty} \left(n-1\right) a_{n-1} \left(x+2\right)^{n}$$

Finally, in order to subtract the two series we'll need to get them to start at the same value of *n*. Depending on the series in the problem we can do this in a variety of ways. In this case let's notice that since there is an *n-1* in the second series we can in fact start the second series at *n=1* without changing its value. Also note that in doing so we will get both of the series to start at *n=1* and so we can do the subtraction. Our final answer is then,

$$\sum_{n=1}^{\infty} \left(n+2\right) a_{n+2} \left(x+2\right)^{n} - \sum_{n=1}^{\infty} \left(n-1\right) a_{n-1} \left(x+2\right)^{n} = \sum_{n=1}^{\infty} \left[\left(n+2\right) a_{n+2} - \left(n-1\right) a_{n-1}\right] \left(x+2\right)^{n}$$

(b) $x\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3}$

In this part the main issue is the fact that we can't just multiply the coefficient into the series this time since the coefficient doesn't have the same form as the term inside the series. Therefore, the first thing that we'll need to do is correct the coefficient so that we can bring it into the series. We do this as follows,

$$\begin{aligned} x\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} &= (x-3+3)\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} \\ &= (x-3)\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} + 3\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} \end{aligned}$$

We can now move the coefficient into the series, but in the process of we managed to pick up a second series. This will happen so get used to it. Moving the coefficients of both series in gives,

$$\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+4} + \sum_{n=0}^{\infty}3(n-5)^2 b_{n+1}(x-3)^{n+3}$$

We now need to get the exponent in both series to be an *n*. This will mean shifting the first series up by 4 and the second series up by 3. Doing this gives,

$$\sum_{n=4}^{\infty}(n-9)^2 b_{n-3}(x-3)^{n} + \sum_{n=3}^{\infty}3(n-8)^2 b_{n-2}(x-3)^{n}$$

In this case we can't just start the first series at $n=3$ because there is not an $n-3$ sitting in that series to make the $n=3$ term zero. So, we won't be able to do this part as we did in the first part of this example.

What we'll need to do in this part is strip out the $n=3$ from the second series so they will both start at $n=4$. We will then be able to add the two series together. Stripping out the $n=3$ term from the second series gives,

$$\sum_{n=4}^{\infty}(n-9)^2 b_{n-3}(x-3)^{n} + 3(-5)^2 b_1(x-3)^3 + \sum_{n=4}^{\infty}3(n-8)^2 b_{n-2}(x-3)^{n}$$

We can now add the two series together.

$$75b_1(x-3)^3 + \sum_{n=4}^{\infty}\left[(n-9)^2 b_{n-3} + 3(n-8)^2 b_{n-2}\right](x-3)^{n}$$

This is what we're looking for. We won't worry about the extra term sitting in front of the series. When we finally get around to finding series solutions to differential equations we will see how to deal with that term there.

There is one final fact that we need take care of before moving on. Before giving this fact for power series let's notice that the only way for

$$a + bx + cx^2 = 0$$

to be zero for all x is to have $a=b=c=0$.

We've got a similar fact for power series.

Fact

If,

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = 0$$

for all x then,

$$a_n = 0, \;\; n = 0, 1, 2, \ldots$$

This fact will be key to our work with differential equations so don't forget it.

Section 6-2 : Review : Taylor Series

We are not going to be doing a whole lot with Taylor series once we get out of the review, but they are a nice way to get us back into the swing of dealing with power series. By time most students reach this stage in their mathematical career they've not had to deal with power series for at least a semester or two. Remembering how Taylor series work will be a very convenient way to get comfortable with power series before we start looking at differential equations.

Taylor Series

If *f(x)* is an infinitely differentiable function then the Taylor Series of *f(x)* about $x = x_0$ is,

$$f\left(x\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(x_0\right)}{n!}\left(x - x_0\right)^n$$

Recall that

$$f^{(0)}\left(x\right) = f\left(x\right) \qquad\qquad f^{(n)}\left(x\right) = \text{ n}^{\text{th}} \text{ derivative of } f\left(x\right)$$

Let's take a look at an example.

Example 1 Determine the Taylor series for $f\left(x\right) = \mathbf{e}^x$ about *x=0*.

Solution

This is probably one of the easiest functions to find the Taylor series for. We just need to recall that,

$$f^{(n)}\left(x\right) = \mathbf{e}^x \qquad n = 0,1,2,\ldots$$

and so we get,

$$f^{(n)}\left(0\right) = 1 \qquad n = 0,1,2,\ldots$$

The Taylor series for this example is then,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Of course, it's often easier to find the Taylor series about *x=0* but we don't always do that.

Example 2 Determine the Taylor series for $f\left(x\right) = \mathbf{e}^x$ about $x = -4$.

Solution

This problem is virtually identical to the previous problem. In this case we just need to notice that,

$$f^{(n)}\left(-4\right) = \mathbf{e}^{-4} \quad n = 0,1,2,\ldots$$

The Taylor series for this example is then,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{\mathbf{e}^{-4}}{n!}\left(x+4\right)^n$$

Let's now do a Taylor series that requires a little more work.

Example 3 Determine the Taylor series for $f\left(x\right)=\cos\left(x\right)$ about $x=0$.

Solution

This time there is no formula that will give us the derivative for each *n* so let's start taking derivatives and plugging in $x=0$.

$$\begin{aligned} f^{(0)}(x) &= \cos(x) & \qquad f^{(0)}(0) &= 1 \\ f^{(1)}(x) &= -\sin(x) & f^{(1)}(0) &= 0 \\ f^{(2)}(x) &= -\cos(x) & f^{(2)}(0) &= -1 \\ f^{(3)}(x) &= \sin(x) & f^{(3)}(0) &= 0 \\ f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\ &\ \vdots & &\ \vdots \end{aligned}$$

Once we reach this point it's fairly clear that there is a pattern emerging here. Just what this pattern is has yet to be determined, but it does seem fairly clear that a pattern does exist.

Let's plug what we've got into the formula for the Taylor series and see what we get.

$$\begin{aligned} \cos(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots \\ &= \frac{1}{0!} + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + 0 + \frac{x^8}{8!} + \cdots \end{aligned}$$

So, every other term is zero.

We would like to write this in terms of a series, however finding a formula that is zero every other term and gives the correct answer for those that aren't zero would be unnecessarily complicated. So, let's rewrite what we've got above and while were at it renumber the terms as follows,

$$\cos(x) = \underbrace{\frac{1}{0!}}_{n=0} - \underbrace{\frac{x^2}{2!}}_{n=1} + \underbrace{\frac{x^4}{4!}}_{n=2} - \underbrace{\frac{x^6}{6!}}_{n=3} + \underbrace{\frac{x^8}{8!}}_{n=4} + \cdots$$

With this "renumbering" we can fairly easily get a formula for the Taylor series of the cosine function about $x=0$.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

For practice you might want to see if you can verify that the Taylor series for the sine function about $x=0$ is,

$$\sin\left(x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}x^{2n+1}}{\left(2n+1\right)!}$$

We need to look at one more example of a Taylor series. This example is both tricky and very easy.

Example 4 Determine the Taylor series for $f\left(x\right)=3x^2-8x+2$ about $x=2$.

Solution

There's not much to do here except to take some derivatives and evaluate at the point.

$$\begin{aligned} f\left(x\right)&=3x^2-8x+2 & f\left(2\right)&=-2\\ f'\left(x\right)&=6x-8 & f'\left(2\right)&=4\\ f''\left(x\right)&=6 & f''\left(2\right)&=6\\ f^{(n)}\left(x\right)&=0,\ n\geq 3 & f^{(n)}\left(2\right)&=0,\ n\geq 3\end{aligned}$$

So, in this case the derivatives will all be zero after a certain order. That happens occasionally and will make our work easier. Setting up the Taylor series then gives,

$$\begin{aligned} 3x^2-8x+2&=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(2\right)}{n!}\left(x-2\right)^n\\ &=\frac{f^{(0)}\left(2\right)}{0!}+\frac{f^{(1)}\left(2\right)}{1!}\left(x-2\right)+\frac{f^{(2)}\left(2\right)}{2!}\left(x-2\right)^2+\frac{f^{(3)}\left(2\right)}{3!}\left(x-2\right)^3+\cdots\\ &=-2+4\left(x-2\right)+\frac{6}{2}\left(x-2\right)^2+0\\ &=-2+4\left(x-2\right)+3\left(x-2\right)^2\end{aligned}$$

In this case the Taylor series terminates and only had three terms. Note that since we are after the Taylor series we do not multiply the 4 through on the second term or square out the third term. All the terms with the exception of the constant should contain an $x-2$.

Note in this last example that if we were to multiply the Taylor series we would get our original polynomial. This should not be too surprising as both are polynomials and they should be equal.

We now need a quick definition that will make more sense to give here rather than in the next section where we actually need it since it deals with Taylor series.

Definition

A function, *f(x)*, is called **analytic** at $x=a$ if the Taylor series for *f(x)* about *x=a* has a positive radius of convergence and converges to *f(x)*.

We need to give one final note before proceeding into the next section. We started this section out by saying that we weren't going to be doing much with Taylor series after this section. While that is correct it is only correct because we are going to be keeping the problems fairly simple. For more complicated problems we would also be using quite a few Taylor series.

Section 6-3 : Series Solutions

Before we get into finding series solutions to differential equations we need to determine when we can find series solutions to differential equations. So, let's start with the differential equation,

$$p(x)y'' + q(x)y' + r(x)y = 0 \qquad \textbf{(1)}$$

This time we really do mean nonconstant coefficients. To this point we've only dealt with constant coefficients. However, with series solutions we can now have nonconstant coefficient differential equations. Also, in order to make the problems a little nicer we will be dealing only with polynomial coefficients.

Now, we say that $x=x_0$ is an **ordinary point** if provided both

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)}$$

are analytic at $x=x_0$. That is to say that these two quantities have Taylor series around $x=x_0$. We are going to be only dealing with coefficients that are polynomials so this will be equivalent to saying that

$$p(x_0) \neq 0$$

for most of the problems.

If a point is not an ordinary point we call it a **singular point**.

The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \textbf{(2)}$$

and then try to determine what the a_n's need to be. We will only be able to do this if the point $x=x_0$, is an ordinary point. We will usually say that (2) is a series solution around $x=x_0$.

Let's start with a very basic example of this. In fact, it will be so basic that we will have constant coefficients. This will allow us to check that we get the correct solution.

Example 1 Determine a series solution for the following differential equation about $x_0 = 0$.

$$y'' + y = 0$$

Solution

Notice that in this case $p(x)=1$ and so every point is an ordinary point. We will be looking for a solution in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We will need to plug this into our differential equation so we'll need to find a couple of derivatives.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad\qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Recall from the power series review section on power series that we can start these at *n=0* if we need to, however it's almost always best to start them where we have here. If it turns out that it would have been easier to start them at *n=0* we can easily fix that up when the time comes around.

So, plug these into our differential equation. Doing this gives,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

The next step is to combine everything into a single series. To do this requires that we get both series starting at the same point and that the exponent on the x be the same in both series.

We will always start this by getting the exponent on the x to be the same. It is usually best to get the exponent to be an n. The second series already has the proper exponent and the first series will need to be shifted down by 2 in order to get the exponent up to an n. If you don't recall how to do this take a quick look at the first review section where we did several of these types of problems.

Shifting the first power series gives us,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Notice that in the process of the shift we also got both series starting at the same place. This won't always happen, but when it does we'll take it. We can now add up the two series. This gives,

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

Now recalling the fact from the power series review section we know that if we have a power series that is zero for all x (as this is) then all the coefficients must have been zero to start with. This gives us the following,

$$(n+2)(n+1)a_{n+2} + a_n = 0, \qquad n = 0,1,2,\ldots$$

This is called the **recurrence relation** and notice that we included the values of n for which it must be true. We will always want to include the values of n for which the recurrence relation is true since they won't always start at $n = 0$ as it did in this case.

Now let's recall what we were after in the first place. We wanted to find a series solution to the differential equation. In order to do this, we needed to determine the values of the a_n's. We are almost to the point where we can do that. The recurrence relation has two different a_n's in it so we can't just solve this for a_n and get a formula that will work for all n. We can however, use this to determine what all but two of the a_n's are.

To do this we first solve the recurrence relation for the a_n that has the largest subscript. Doing this gives,

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad n = 0,1,2,\ldots$$

Now, at this point we just need to start plugging in some value of *n* and see what happens,

n=0 $a_2 = \dfrac{-a_0}{(2)(1)}$ *n=1* $a_3 = \dfrac{-a_1}{(3)(2)}$

n=2
$$\begin{aligned} a_4 &= -\frac{a_2}{(4)(3)} \\ &= \frac{a_0}{(4)(3)(2)(1)} \end{aligned}$$

n=3
$$\begin{aligned} a_5 &= -\frac{a_3}{(5)(4)} \\ &= \frac{a_1}{(5)(4)(3)(2)} \end{aligned}$$

n=4
$$\begin{aligned} a_6 &= -\frac{a_4}{(6)(5)} \\ &= \frac{-a_0}{(6)(5)(4)(3)(2)(1)} \\ &\vdots \\ a_{2k} &= \frac{(-1)^k a_0}{(2k)!}, \ k = 1,2,\ldots \end{aligned}$$

n=5
$$\begin{aligned} a_7 &= -\frac{a_5}{(7)(6)} \\ &= \frac{-a_1}{(7)(6)(5)(4)(3)(2)} \\ &\vdots \\ a_{2k+1} &= \frac{(-1)^k a_1}{(2k+1)!}, \ k = 1,2,\ldots \end{aligned}$$

Notice that at each step we always plugged back in the previous answer so that when the subscript was even we could always write the a_n in terms of a_0 and when the coefficient was odd we could always write the a_n in terms of a_1. Also notice that, in this case, we were able to find a general formula for a_n's with even coefficients and a_n's with odd coefficients. This won't always be possible to do.

There's one more thing to notice here. The formulas that we developed were only for *k=1,2,…* however, in this case again, they will also work for *k=0*. Again, this is something that won't always work, but does here.

Do not get excited about the fact that we don't know what a_0 and a_1 are. As you will see, we actually need these to be in the problem to get the correct solution.

Now that we've got formulas for the a_n's let's get a solution. The first thing that we'll do is write out the solution with a couple of the a_n's plugged in.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \cdots \\ &= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \cdots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^k a_1}{(2k+1)!} x^{2k+1} + \cdots \end{aligned}$$

The next step is to collect all the terms with the same coefficient in them and then factor out that coefficient.

$$y(x) = a_0 \left\{ 1 - \frac{x^2}{2!} \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \right\} + a_1 \left\{ x - \frac{x^3}{3!} + \cdots + \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \cdots \right\}$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

In the last step we also used the fact that we knew what the general formula was to write both portions as a power series. This is also our solution. We are done.

Before working another problem let's take a look at the solution to the previous example. First, we started out by saying that we wanted a series solution of the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and we didn't get that. We got a solution that contained two different power series. Also, each of the solutions had an unknown constant in them. This is not a problem. In fact, it's what we want to have happen. From our work with second order constant coefficient differential equations we know that the solution to the differential equation in the last example is,

$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

Solutions to second order differential equations consist of two separate functions each with an unknown constant in front of them that are found by applying any initial conditions. So, the form of our solution in the last example is exactly what we want to get. Also recall that the following Taylor series,

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad\qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Recalling these we very quickly see that what we got from the series solution method was exactly the solution we got from first principles, with the exception that the functions were the Taylor series for the actual functions instead of the actual functions themselves.

Now let's work an example with nonconstant coefficients since that is where series solutions are most useful.

Example 2 Find a series solution around $x_0 = 0$ for the following differential equation.

$$y'' - xy = 0$$

Solution

As with the first example *p(x)=1* and so again for this differential equation every point is an ordinary point. Now we'll start this one out just as we did the first example. Let's write down the form of the solution and get its derivatives.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Plugging into the differential equation gives,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

Unlike the first example we first need to get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we will need to shift the first series down by 2 and the second series up by 1 to get both of the series in terms of x^n.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Next, we need to get the two series starting at the same value of n. The only way to do that for this problem is to strip out the $n=0$ term.

$$(2)(1)a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right] x^n = 0$$

We now need to set all the coefficients equal to zero. We will need to be careful with this however. The $n=0$ coefficient is in front of the series and the $n=1,2,3\ldots$ are all in the series. So, setting coefficient equal to zero gives,

$$\begin{aligned} n &= 0: & 2a_2 &= 0 \\ n &= 1,2,3,\ldots & (n+2)(n+1)a_{n+2} - a_{n-1} &= 0 \end{aligned}$$

Solving the first as well as the recurrence relation gives,

$$\begin{aligned} n &= 0: & a_2 &= 0 \\ n &= 1,2,3,\ldots & a_{n+2} &= \frac{a_{n-1}}{(n+2)(n+1)} \end{aligned}$$

Now we need to start plugging in values of n.

$$a_3 = \frac{a_0}{(3)(2)} \qquad a_4 = \frac{a_1}{(4)(3)} \qquad a_5 = \frac{a_2}{(5)(4)} = 0$$

$$a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)} \qquad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)} \qquad a_8 = \frac{a_5}{(8)(7)} = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_{3k} = \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \qquad a_{3k+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \qquad a_{3k+2} = 0$$

$$k = 1,2,3,\cdots \qquad k = 1,2,3,\cdots \qquad k = 0,1,2,\cdots$$

There are a couple of things to note about these coefficients. First, every third coefficient is zero. Next, the formulas here are somewhat unpleasant and not all that easy to see the first time around. Finally, these formulas will not work for *k=0* unlike the first example.

Now, get the solution,

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{3k} x^{3k} + a_{3k+1} x^{3k+1} + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 \cdots + \frac{a_0 x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} + \\ &\qquad \frac{a_1 x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} + \cdots \end{aligned}$$

Again, collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series,

$$y(x) = a_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right\} + a_1 \left\{ x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right\}$$

We couldn't start our series at *k=0* this time since the general term doesn't hold for *k=0*.

Now, we need to work an example in which we use a point other that *x=0*. In fact, let's just take the previous example and rework it for a different value of x_0. We're also going to need to change up the instructions a little for this example.

Example 3 Find the first four terms in each portion of the series solution around $x_0 = -2$ for the following differential equation.

$$y'' - xy = 0$$

Solution

Unfortunately for us there is nothing from the first example that can be reused here. Changing to $x_0 = -2$ completely changes the problem. In this case our solution will be,

$$y(x) = \sum_{n=0}^{\infty} a_n (x+2)^n$$

The derivatives of the solution are,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x+2)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2}$$

Plug these into the differential equation.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - x\sum_{n=0}^{\infty} a_n(x+2)^n = 0$$

We now run into our first real difference between this example and the previous example. In this case we can't just multiply the *x* into the second series since in order to combine with the series it must be *x+2*. Therefore, we will first need to modify the coefficient of the second series before multiplying it into the series.

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - (x+2-2)\sum_{n=0}^{\infty} a_n(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - (x+2)\sum_{n=0}^{\infty} a_n(x+2)^n + 2\sum_{n=0}^{\infty} a_n(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x+2)^{n-2} - \sum_{n=0}^{\infty} a_n(x+2)^{n+1} + \sum_{n=0}^{\infty} 2a_n(x+2)^n = 0$$

We now have three series to work with. This will often occur in these kinds of problems. Now we will need to shift the first series down by 2 and the second series up by 1 the get common exponents in all the series.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}(x+2)^n + \sum_{n=0}^{\infty} 2a_n(x+2)^n = 0$$

In order to combine the series we will need to strip out the *n=0* terms from both the first and third series.

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}(x+2)^n + \sum_{n=1}^{\infty} 2a_n(x+2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n\right](x+2)^n = 0$$

Setting coefficients equal to zero gives,

$$\begin{aligned} n &= 0 & \quad 2a_2 + 2a_0 &= 0 \\ n &= 1,2,3,\ldots & \quad (n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n &= 0 \end{aligned}$$

We now need to solve both of these. In the first case there are two options, we can solve for a_2 or we can solve for a_0. Out of habit I'll solve for a_0. In the recurrence relation we'll solve for the term with the largest subscript as in previous examples.

$$\begin{aligned} n &= 0 & \quad a_2 &= -a_0 \\ n &= 1,2,3,\ldots & \quad a_{n+2} &= \frac{a_{n-1} - 2a_n}{(n+2)(n+1)} \end{aligned}$$

Notice that in this example we won't be having every third term drop out as we did in the previous example.

At this point we'll also acknowledge that the instructions for this problem are different as well. We aren't going to get a general formula for the a_n's this time so we'll have to be satisfied with just getting the first couple of terms for each portion of the solution. This is often the case for series solutions. Getting general formulas for the a_n's is the exception rather than the rule in these kinds of problems.

To get the first four terms we'll just start plugging in terms until we've got the required number of terms. Note that we will already be starting with an a_0 and an a_1 from the first two terms of the solution so all we will need are three more terms with an a_0 in them and three more terms with an a_1 in them.

$$n = 0 \qquad\qquad a_2 = -a_0$$

We've got two a_0's and one a_1.

$$n = 1 \qquad\qquad a_3 = \frac{a_0 - 2a_1}{(3)(2)} = \frac{a_0}{6} - \frac{a_1}{3}$$

We've got three a_0's and two a_1's.

$$n = 2 \qquad\qquad a_4 = \frac{a_1 - 2a_2}{(4)(3)} = \frac{a_1 - 2(-a_0)}{(4)(3)} = \frac{a_0}{6} + \frac{a_1}{12}$$

We've got four a_0's and three a_1's. We've got all the a_0's that we need, but we still need one more a_1. So, we'll need to do one more term it looks like.

$$n = 3 \qquad\qquad a_5 = \frac{a_2 - 2a_3}{(5)(4)} = -\frac{a_0}{20} - \frac{1}{10}\left(\frac{a_0}{6} - \frac{a_1}{3}\right) = -\frac{a_0}{15} + \frac{a_1}{30}$$

We've got five a_0's and four a_1's. We've got all the terms that we need.

Now, all that we need to do is plug into our solution.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x+2)^n \\ &= a_0 + a_1(x+2) + a_2(x+2)^2 + a_3(x+2)^3 + a_4(x+2)^4 + a_5(x+2)^5 + \cdots \\ &= a_0 + a_1(x+2) - a_0(x+2)^2 + \left(\frac{a_0}{6} - \frac{a_1}{3}\right)(x+2)^3 + \\ &\qquad\qquad \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x+2)^4 + \left(-\frac{a_0}{15} + \frac{a_1}{30}\right)(x+2)^5 + \cdots \end{aligned}$$

Finally collect all the terms up with the same coefficient and factor out the coefficient to get,

$$\begin{aligned} y(x) = a_0 &\left\{1 - (x+2)^2 + \frac{1}{6}(x+2)^3 + \frac{1}{6}(x+2)^4 - \frac{1}{15}(x+2)^5 + \cdots\right\} + \\ a_1 &\left\{(x+2) - \frac{1}{3}(x+2)^3 + \frac{1}{12}(x+2)^4 + \frac{1}{30}(x+2)^5 + \cdots\right\} \end{aligned}$$

That's the solution for this problem as far as we're concerned. Notice that this solution looks nothing like the solution to the previous example. It's the same differential equation but changing x_0 completely changed the solution.

Let's work one final problem.

Example 4 Find the first four terms in each portion of the series solution around $x_0 = 0$ for the following differential equation.

$$\left(x^2+1\right)y'' - 4xy' + 6y = 0$$

Solution

We finally have a differential equation that doesn't have a constant coefficient for the second derivative.

$$p\left(x\right) = x^2 + 1 \qquad\qquad p\left(0\right) = 1 \neq 0$$

So $x_0 = 0$ is an ordinary point for this differential equation. We first need the solution and its derivatives,

$$y\left(x\right) = \sum_{n=0}^{\infty} a_n x^n \qquad y'\left(x\right) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''\left(x\right) = \sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2}$$

Plug these into the differential equation.

$$\left(x^2+1\right)\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Now, break up the first term into two so we can multiply the coefficient into the series and multiply the coefficients of the second and third series in as well.

$$\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^n + \sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2} - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

We will only need to shift the second series down by two to get all the exponents the same in all the series.

$$\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^n + \sum_{n=0}^{\infty} \left(n+2\right)\left(n+1\right) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

At this point we could strip out some terms to get all the series starting at *n=2*, but that's actually more work than is needed. Let's instead note that we could start the third series at *n=0* if we wanted to because that term is just zero. Likewise, the terms in the first series are zero for both *n=1* and *n=0* and so we could start that series at *n=0*. If we do this all the series will now start at *n=0* and we can add them up without stripping terms out of any series.

$$\sum_{n=0}^{\infty} \left[n\left(n-1\right) a_n + \left(n+2\right)\left(n+1\right) a_{n+2} - 4n a_n + 6 a_n \right] x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2 - 5n + 6\right) a_n + \left(n+2\right)\left(n+1\right) a_{n+2} \right] x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n-2\right)\left(n-3\right) a_n + \left(n+2\right)\left(n+1\right) a_{n+2} \right] x^n = 0$$

Now set coefficients equal to zero.

$$\left(n-2\right)\left(n-3\right)a_n+\left(n+2\right)\left(n+1\right)a_{n+2}=0, \qquad n=0,1,2,\ldots$$

Solving this gives,

$$a_{n+2}=-\frac{\left(n-2\right)\left(n-3\right)a_n}{\left(n+2\right)\left(n+1\right)}, \qquad n=0,1,2,\ldots$$

Now, we plug in values of *n*.

$$\begin{aligned} n=0: &\qquad a_2=-3a_0 \\ n=1: &\qquad a_3=-\frac{1}{3}a_1 \\ n=2: &\qquad a_4=-\frac{0}{12}a_2=0 \\ n=3: &\qquad a_5=-\frac{0}{20}a_3=0 \end{aligned}$$

Now, from this point on all the coefficients are zero. In this case both of the series in the solution will terminate. This won't always happen, and often only one of them will terminate.

The solution in this case is,

$$y\left(x\right)=a_0\left\{1-3x^2\right\}+a_1\left\{x-\frac{1}{3}x^3\right\}$$

Section 6-4 : Euler Equations

In this section we want to look for solutions to

$$ax^2 y'' + bxy' + cy = 0 \qquad \textbf{(1)}$$

around $x_0 = 0$. These types of differential equations are called **Euler Equations**.

Recall from the previous section that a point is an ordinary point if the quotients,

$$\frac{bx}{ax^2} = \frac{b}{ax} \qquad \text{and} \qquad \frac{c}{ax^2}$$

have Taylor series around $x_0 = 0$. However, because of the *x* in the denominator neither of these will have a Taylor series around $x_0 = 0$ and so $x_0 = 0$ is a singular point. So, the method from the previous section won't work since it required an ordinary point.

However, it is possible to get solutions to this differential equation that aren't series solutions. Let's start off by assuming that *x>0* (the reason for this will be apparent after we work the first example) and that all solutions are of the form,

$$y(x) = x^r \qquad \textbf{(2)}$$

Now plug this into the differential equation to get,

$$\begin{aligned} ax^2(r)(r-1)x^{r-2} + bx(r)x^{r-1} + cx^r &= 0 \\ ar(r-1)x^r + b(r)x^r + cx^r &= 0 \\ \left(ar(r-1) + b(r) + c\right)x^r &= 0 \end{aligned}$$

Now, we assumed that *x>0* and so this will only be zero if,

$$ar(r-1) + b(r) + c = 0 \qquad \textbf{(3)}$$

So solutions will be of the form (2) provided *r* is a solution to (3). This equation is a quadratic in *r* and so we will have three cases to look at : Real, Distinct Roots, Double Roots, and Complex Roots.

Real, Distinct Roots

There really isn't a whole lot to do in this case. We'll get two solutions that will form a fundamental set of solutions (we'll leave it to you to check this) and so our general solution will be,

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

Example 1 Solve the following IVP

$$2x^2 y'' + 3xy' - 15y = 0, \qquad y(1) = 0 \quad y'(1) = 1$$

Solution

We first need to find the roots to (3).

$$2r(r-1) + 3r - 15 = 0$$

$$2r^2 + r - 15 = (2r-5)(r+3) = 0 \qquad \Rightarrow \qquad r_1 = \frac{5}{2},\ r_2 = -3$$

The general solution is then,

$$y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}$$

To find the constants we differentiate and plug in the initial conditions as we did back in the second order differential equations chapter.

$$y'(x) = \frac{5}{2} c_1 x^{\frac{3}{2}} - 3c_2 x^{-4}$$

$$\left. \begin{aligned} 0 &= y(1) = c_1 + c_2 \\ 1 &= y'(1) = \frac{5}{2} c_1 - 3c_2 \end{aligned} \right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{11},\ c_2 = -\frac{2}{11}$$

The actual solution is then,

$$y(x) = \frac{2}{11} x^{\frac{5}{2}} - \frac{2}{11} x^{-3}$$

With the solution to this example we can now see why we required *x>0*. The second term would have division by zero if we allowed *x=0* and the first term would give us square roots of negative numbers if we allowed *x<0*.

Double Roots

This case will lead to the same problem that we've had every other time we've run into double roots (or double eigenvalues). We only get a single solution and will need a second solution. In this case it can be shown that the second solution will be,

$$y_2(x) = x^r \ln x$$

and so the general solution in this case is,

$$y(x) = c_1 x^r + c_2 x^r \ln x = x^r \left(c_1 + c_2 \ln x\right)$$

We can again see a reason for requiring *x>0*. If we didn't we'd have all sorts of problems with that logarithm.

Example 2 Find the general solution to the following differential equation.

$$x^2 y'' - 7xy' + 16y = 0$$

Solution

First the roots of (3).

$$\begin{aligned} r(r-1) - 7r + 16 &= 0 \\ r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \qquad \Rightarrow \qquad r = 4 \end{aligned}$$

So, the general solution is then,

$$y(x) = c_1 x^4 + c_2 x^4 \ln x$$

Complex Roots

In this case we'll be assuming that our roots are of the form,

$$r_{1,2} = \lambda \pm \mu i$$

If we take the first root we'll get the following solution.

$$x^{\lambda + \mu i}$$

This is a problem since we don't want complex solutions, we only want real solutions. We can eliminate this by recalling that,

$$x^r = \mathbf{e}^{\ln x^r} = \mathbf{e}^{r \ln x}$$

Plugging the root into this gives,

$$\begin{aligned} x^{\lambda + \mu i} &= \mathbf{e}^{(\lambda + \mu i)\ln x} \\ &= \mathbf{e}^{\lambda \ln x} \mathbf{e}^{\mu i \ln x} \\ &= \mathbf{e}^{\ln x^{\lambda}} \left(\cos(\mu \ln x) + i \sin(\mu \ln x) \right) \\ &= x^{\lambda} \cos(\mu \ln x) + i x^{\lambda} \sin(\mu \ln x) \end{aligned}$$

Note that we had to use Euler formula as well to get to the final step. Now, as we've done every other time we've seen solutions like this we can take the real part and the imaginary part and use those for our two solutions.

So, in the case of complex roots the general solution will be,

$$y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x) = x^{\lambda} \left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x) \right)$$

Once again, we can see why we needed to require $x > 0$.

Example 3 Find the solution to the following differential equation.

$$x^2 y'' + 3xy' + 4y = 0$$

Solution

Get the roots to (3) first as always.

$$r(r-1) + 3r + 4 = 0$$

$$r^2 + 2r + 4 = 0 \qquad \Rightarrow \qquad r_{1,2} = -1 \pm \sqrt{3}\,i$$

The general solution is then,

$$y(x) = c_1 x^{-1} \cos\left(\sqrt{3}\ln x\right) + c_2 x^{-1} \sin\left(\sqrt{3}\ln x\right)$$

We should now talk about how to deal with $x < 0$ since that is a possibility on occasion. To deal with this we need to use the variable transformation,

$$\eta = -x$$

In this case since $x < 0$ we will get $\eta > 0$. Now, define,

$$u(\eta) = y(x) = y(-\eta)$$

Then using the chain rule we can see that,

$$u'(\eta) = -y'(x) \qquad \text{and} \qquad u''(\eta) = y''(x)$$

With this transformation the differential equation becomes,

$$a(-\eta)^2 u'' + b(-\eta)(-u') + cu = 0$$

$$a\eta^2 u'' + b\eta u' + cu = 0$$

In other words, since *η>0* we can use the work above to get solutions to this differential equation. We'll also go back to *x*'s by using the variable transformation in reverse.

$$\eta = -x$$

Let's just take the real, distinct case first to see what happens.

$$u(\eta) = c_1 \eta^{r_1} + c_2 \eta^{r_2}$$

$$y(x) = c_1 (-x)^{r_1} + c_2 (-x)^{r_2}$$

Now, we could do this for the rest of the cases if we wanted to, but before doing that let's notice that if we recall the definition of absolute value,

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

we can combine both of our solutions to this case into one and write the solution as,

$$y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}, \qquad x \ne 0$$

Note that we still need to avoid $x = 0$ since we could still get division by zero. However, this is now a solution for any interval that doesn't contain $x = 0$.

We can do likewise for the other two cases and the following solutions for any interval not containing $x = 0$.

$$y(x) = c_1 |x|^r + c_2 |x|^r \ln|x|$$
$$y(x) = c_1 |x|^\lambda \cos(\mu \ln|x|) + c_2 |x|^\lambda \sin(\mu \ln|x|)$$

We can make one more generalization before working one more example. A more general form of an Euler Equation is,

$$a(x - x_0)^2 y'' + b(x - x_0) y' + cy = 0$$

and we can ask for solutions in any interval not containing $x = x_0$. The work for generating the solutions in this case is identical to all the above work and so isn't shown here.

The solutions in this general case for any interval not containing $x = a$ are,

$$y(x) = c_1 |x - a|^{r_1} + c_2 |x - a|^{r_2}$$
$$y(x) = |x - a|^r \left(c_1 + c_2 \ln|x - a|\right)$$
$$y(x) = |x - a|^\lambda \left(c_1 \cos(\mu \ln|x - a|) + c_2 \sin(\mu \ln|x - a|)\right)$$

Where the roots are solutions to

$$ar(r - 1) + b(r) + c = 0$$

Example 4 Find the solution to the following differential equation on any interval not containing $x = -6$.

$$3(x + 6)^2 y'' + 25(x + 6) y' - 16y = 0$$

Solution

So, we get the roots from the identical quadratic in this case.

$$3r(r - 1) + 25r - 16 = 0$$
$$3r^2 + 22r - 16 = 0$$
$$(3r - 2)(r + 8) = 0 \qquad \Rightarrow \qquad r_1 = \frac{2}{3},\ r_2 = -8$$

The general solution is then,

$$y(x) = c_1 |x + 6|^{\frac{2}{3}} + c_2 |x + 6|^{-8}$$