DIFFERENTIAL EQUATIONS

Basic Concepts

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Preface

Here are my notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self-contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general, I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

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Chapter 1 : Basic Concepts

There isn't really a whole lot to this chapter it is mainly here so we can get some basic definitions and concepts out of the way. Most of the definitions and concepts introduced here can be introduced without any real knowledge of how to solve differential equations. Most of them are terms that we'll use throughout a class so getting them out of the way right at the beginning is a good idea.

During an actual class I tend to hold off on a many of the definitions and introduce them at a later point when we actually start solving differential equations. The reason for this is mostly a time issue. In this class time is usually at a premium and some of the definitions/concepts require a differential equation and/or its solution so we use the first couple differential equations that we will solve to introduce the definition or concept.

Here is a quick list of the topics in this Chapter.

Definitions – In this section some of the common definitions and concepts in a differential equations course are introduced including order, linear vs. nonlinear, initial conditions, initial value problem and interval of validity.

Direction Fields – In this section we discuss direction fields and how to sketch them. We also investigate how direction fields can be used to determine some information about the solution to a differential equation without actually having the solution.

Final Thoughts – In this section we give a couple of final thoughts on what we will be looking at throughout this course including brief discussions of the existence and uniqueness questions.

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Section 1-1 : Definitions

Differential Equation

The first definition that we should cover should be that of **differential equation**. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass m is moving with acceleration a and being acted on with force F then Newton's Second Law tells us.

$$F = ma \tag{1}$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, a, in one of two ways.

$$a = \frac{dv}{dt} \qquad \text{OR} \qquad a = \frac{d^2u}{dt^2} \tag{2}$$

Where *v* is the velocity of the object and *u* is the position function of the object at any time *t*. We should also remember at this point that the force, *F* may also be a function of time, velocity, and/or position.

So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, *v*, or the position, *u*, of the object as follows.

$$m\frac{dv}{dt} = F\left(t, v\right) \tag{3}$$

$$m\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right) \tag{4}$$

So, here is our first differential equation. We will see both forms of this in later chapters.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g\left(t\right) \tag{5}$$

$$\sin\left(y\right)\frac{d^2y}{dx^2} = \left(1 - y\right)\frac{dy}{dx} + y^2\mathbf{e}^{-5y} \tag{6}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos\left(t\right) \tag{7}$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{8}$$

$$a^2 u_{xx} = u_{tt} \tag{9}$$

$$\frac{\partial^3 u}{\partial^2 x \partial t} = 1 + \frac{\partial u}{\partial y} \tag{10}$$

Order

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The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above (3) is a first order differential equation, (4), (5), (6), (8), and (9) are second order differential equations, (10) is a third order differential equation and (7) is a fourth order differential equation.

Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

We will be looking almost exclusively at first and second order differential equations in these notes. As you will see most of the solution techniques for second order differential equations can be easily (and naturally) extended to higher order differential equations and we'll discuss that idea later on.

Ordinary and Partial Differential Equations

A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has partial derivatives in it. In the differential equations above (3) - (7) are ode's and (8) - (10) are pde's.

The vast majority of these notes will deal with ode's. The only exception to this will be the last chapter in which we'll take a brief look at a common and basic solution technique for solving pde's.

Linear Differential Equations

A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n\left(t\right)y^{(n)}\left(t\right)+a_{n-1}\left(t\right)y^{(n-1)}\left(t\right)+\cdots+a_1\left(t\right)y'\left(t\right)+a_0\left(t\right)y\left(t\right)=g\left(t\right) \qquad \textbf{(11)}$$

The important thing to note about linear differential equations is that there are no products of the function, $y\left(t\right)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are "inside" another function, for example, $\sqrt{y'}$ or $\mathbf{e}^y$.

The coefficients $a_0\left(t\right), \ldots, a_n\left(t\right)$ and $g\left(t\right)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y\left(t\right)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

In (5) - (7) above only (6) is non-linear, the other two are linear differential equations. We can't classify (3) and (4) since we do not know what form the function *F* has. These could be either linear or non-linear depending on *F*.

Solution

A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function $y\left(t\right)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$. It is important to note that solutions are

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often accompanied by intervals and these intervals can impart some important information about the solution. Consider the following example.

Example 1 Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2 y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution
We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad\qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$\begin{aligned} 4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) &= 0 \\ 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} &= 0 \\ 0 &= 0 \end{aligned}$$

So, $y(x) = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution. Why then did we include the condition that $x > 0$? We did not use this condition anywhere in the work showing that the function would satisfy the differential equation.

To see why recall that

$$y(x) = x^{-\frac{3}{2}} = \frac{1}{\sqrt{x^3}}$$

In this form it is clear that we'll need to avoid $x = 0$ at the least as this would give division by zero.

Also, there is a general rule of thumb that we're going to run with in this class. This rule of thumb is : Start with real numbers, end with real numbers. In other words, if our differential equation only contains real numbers then we don't want solutions that give complex numbers. So, in order to avoid complex numbers we will also need to avoid negative values of *x*.

So, we saw in the last example that even though a function may symbolically satisfy a differential equation, because of certain restrictions brought about by the solution we cannot use all values of the independent variable and hence, must make a restriction on the independent variable. This will be the case with many solutions to differential equations.

In the last example, note that there are in fact many more possible solutions to the differential equation given. For instance, all of the following are also solutions

$$y(x) = x^{-\frac{1}{2}}$$
$$y(x) = -9x^{-\frac{3}{2}}$$
$$y(x) = 7x^{-\frac{1}{2}}$$
$$y(x) = -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}}$$

We'll leave the details to you to check that these are in fact solutions. Given these examples can you come up with any other solutions to the differential equation? There are in fact an infinite number of solutions to this differential equation.

So, given that there are an infinite number of solutions to the differential equation in the last example (provided you believe us when we say that anyway….) we can ask a natural question. Which is the solution that we want or does it matter which solution we use? This question leads us to the next definition in this section.

Initial Condition(s)
Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s when we're feeling lazy…) are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given initial conditions.

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Example 2 $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \dfrac{1}{8}$, and $y'(4) = -\dfrac{3}{64}$.

Solution
As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$
$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4) = \frac{1}{8}$ and $y'(4) = -\frac{3}{64}$. In fact, $y(x) = x^{-\frac{3}{2}}$ is the only solution to this differential equation that satisfies these two initial conditions.

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Initial Value Problem

An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example 3 The following is an IVP.

$$4x^2 y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8}, \quad y'(4) = -\frac{3}{64}$$

Example 4 Here's another IVP.

$$2t\, y' + 4y = 3 \qquad y(1) = -4$$

As we noted earlier the number of initial conditions required will depend on the order of the differential equation.

Interval of Validity

The **interval of validity** for an IVP with initial condition(s)

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

is the largest possible interval on which the solution is valid and contains t_0. These are easy to define, but can be difficult to find, so we're going to put off saying anything more about these until we get into actually solving differential equations and need the interval of validity.

General Solution

The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Example 5 $y(t) = \frac{3}{4} + \frac{c}{t^2}$ is the general solution to

$$2t\, y' + 4y = 3$$

We'll leave it to you to check that this function is in fact a solution to the given differential equation. In fact, all solutions to this differential equation will be in this form. This is one of the first differential equations that you will learn how to solve and you will be able to verify this shortly for yourself.

Actual Solution

The **actual solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the actual solution to the following IVP?

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

Solution

This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe our solution to this example…) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of *c* that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

From this last example we can see that once we have the general solution to a differential equation finding the actual solution is nothing more than applying the initial condition(s) and solving for the constant(s) that are in the general solution.

Implicit/Explicit Solution

In this case it's easier to define an explicit solution, then tell you what an implicit solution isn't, and then give you an example to show you the difference. So, that's what we'll do.

An **explicit solution** is any solution that is given in the form $y = y(t)$. In other words, the only place that *y* actually shows up is once on the left side and only raised to the first power. An **implicit solution** is any solution that isn't in explicit form. Note that it is possible to have either general implicit/explicit solutions and actual implicit/explicit solutions.

Example 7 $y^2 = t^2 - 3$ is the actual implicit solution to $y' = \frac{t}{y}, \quad y(2) = -1$

At this point we will ask that you trust us that this is in fact a solution to the differential equation. You will learn how to get this solution in a later section. The point of this example is that since there is a y^2 on the left side instead of a single $y(t)$ this is not an explicit solution!

Example 8 Find an actual explicit solution to $y' = \frac{t}{y}, \quad y(2) = -1$.

Solution

We already know from the previous example that an implicit solution to this IVP is $y^2 = t^2 - 3$. To find the explicit solution all we need to do is solve for $y(t)$.

$$y(t) = \pm\sqrt{t^2 - 3}$$

Now, we've got a problem here. There are two functions here and we only want one and in fact only one will be correct! We can determine the correct function by reapplying the initial condition. Only one of them will satisfy the initial condition.

In this case we can see that the "-" solution will be the correct one. The actual explicit solution is then

$$y(t) = -\sqrt{t^2 - 3}$$

In this case we were able to find an explicit solution to the differential equation. It should be noted however that it will not always be possible to find an explicit solution.

Also, note that in this case we were only able to get the explicit actual solution because we had the initial condition to help us determine which of the two functions would be the correct solution.

We've now gotten most of the basic definitions out of the way and so we can move onto other topics.

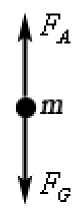
Section 1-2 : Direction Fields

This topic is given its own section for a couple of reasons. First, understanding direction fields and what they tell us about a differential equation and its solution is important and can be introduced without any knowledge of how to solve a differential equation and so can be done here before we get into solving them. So, having some information about the solution to a differential equation without actually having the solution is a nice idea that needs some investigation.

Next, since we need a differential equation to work with, this is a good section to show you that differential equations occur naturally in many cases and how we get them. Almost every physical situation that occurs in nature can be described with an appropriate differential equation. The differential equation may be easy or difficult to arrive at depending on the situation and the assumptions that are made about the situation and we may not ever be able to solve it, however it will exist.

The process of describing a physical situation with a differential equation is called modeling. We will be looking at modeling several times throughout this class.

One of the simplest physical situations to think of is a falling object. So, let's consider a falling object with mass *m* and derive a differential equation that, when solved, will give us the velocity of the object at any time, *t*. We will assume that only gravity and air resistance will act upon the object as it falls. Below is a figure showing the forces that will act upon the object.



Before defining all the terms in this problem we need to set some conventions. We will assume that forces acting in the downward direction are positive forces while forces that act in the upward direction are negative. Likewise, we will assume that an object moving downward (*i.e.* a falling object) will have a positive velocity.

Now, let's take a look at the forces shown in the diagram above. F_G is the force due to gravity and is given by $F_G = mg$ where *g* is the acceleration due to gravity. In this class we use *g* = 9.8 m/s^2 or *g* = 32 ft/s^2 depending on whether we will use the metric or Imperial system. F_A is the force due to air resistance and for this example we will assume that it is proportional to the velocity, *v*, of the mass. Therefore, the force due to air resistance is then given by $F_A = -\gamma v$, where $\gamma > 0$. Note that the "–" is required to get the correct sign on the force. Both γ and *v* are positive and the force is acting upward and hence must be negative. The "–" will give us the correct sign and hence direction for this force.

Recall from the previous section that Newton's Second Law of motion can be written as

$$m\frac{dv}{dt} = F\left(t, v\right)$$

where $F\left(t, v\right)$ is the sum of forces that act on the object and may be a function of the time *t* and the velocity of the object, *v*. For our situation we will have two forces acting on the object gravity, $F_G = mg$. acting in the downward direction and hence will be positive, and air resistance, $F_A = -\gamma v$, acting in the upward direction and hence will be negative. Putting all of this together into Newton's Second Law gives the following.

$$m\frac{dv}{dt} = mg - \gamma v$$

To simplify the differential equation let's divide out the mass, *m*.

$$\frac{dv}{dt} = g - \frac{\gamma v}{m} \qquad \textbf{(1)}$$

This then is a first order linear differential equation that, when solved, will give the velocity, *v* (in m/s), of a falling object of mass *m* that has both gravity and air resistance acting upon it.

In order to look at direction fields (that is after all the topic of this section....) it would be helpful to have some numbers for the various quantities in the differential equation. So, let's assume that we have a mass of 2 kg and that γ = 0.392. Plugging this into (1) gives the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad \textbf{(2)}$$

Let's take a geometric view of this differential equation. Let's suppose that for some time, *t*, the velocity just happens to be *v* = 30 m/s. Note that we're not saying that the velocity ever will be 30 m/s. All that we're saying is that let's suppose that by some chance the velocity does happen to be 30 m/s at some time *t*. So, if the velocity does happen to be 30 m/s at some time *t* we can plug *v* = 30 into (2) to get.

$$\frac{dv}{dt} = 3.92$$

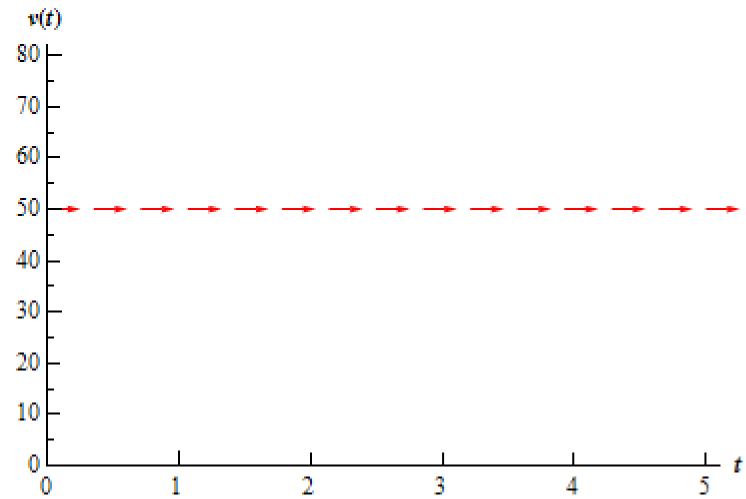
Recall from your Calculus I course that a positive derivative means that the function in question, the velocity in this case, is increasing, so if the velocity of this object is ever 30m/s for any time *t* the velocity must be increasing at that time.

Also, recall that the value of the derivative at a particular value of *t* gives the slope of the tangent line to the graph of the function at that time, *t*. So, if for some time *t* the velocity happens to be 30 m/s the slope of the tangent line to the graph of the velocity is 3.92.

We could continue in this fashion and pick different values of *v* and compute the slope of the tangent line for those values of the velocity. However, let's take a slightly more organized approach to this. Let's first identify the values of the velocity that will have zero slope or horizontal tangent lines. These are easy enough to find. All we need to do is set the derivative equal to zero and solve for *v*.

In the case of our example we will have only one value of the velocity which will have horizontal tangent lines, $v = 50$ m/s. What this means is that IF (again, there's that word if), for some time *t*, the velocity happens to be 50 m/s then the tangent line at that point will be horizontal. What the slope of the tangent line is at times before and after this point is not known yet and has no bearing on the slope at this particular time, t.

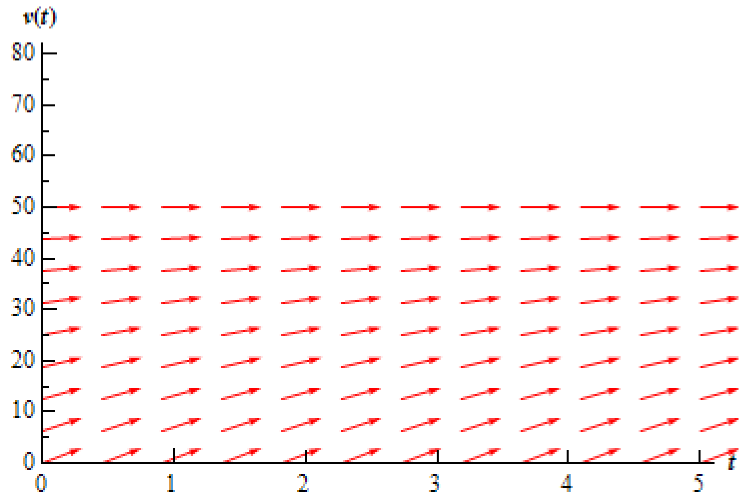
So, if we have $v = 50$, we know that the tangent lines will be horizontal. We denote this on an axis system with horizontal arrows pointing in the direction of increasing t at the level of $v = 50$ as shown in the following figure.



Now, let's get some tangent lines and hence arrows for our graph for some other values of v. At this point the only exact slope that is useful to us is where the slope horizontal. So instead of going after exact slopes for the rest of the graph we are only going to go after general trends in the slope. Is the slope increasing or decreasing? How fast is the slope increasing or decreasing? For this example those types of trends are very easy to get.

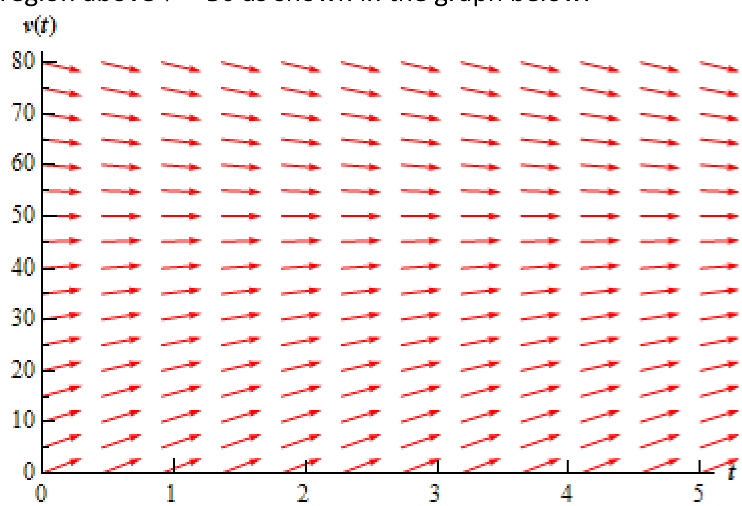
First, notice that the right hand side of (2) is a polynomial and hence continuous. This means that it can only change sign if it first goes through zero. So, if the derivative will change signs (no guarantees that it will) it will do so at $v = 50$ and the only place that it may change sign is $v = 50$. This means that for $v > 50$ the slope of the tangent lines to the velocity will have the same sign. Likewise, for $v < 50$ the slopes will also have the same sign. The slopes in these ranges may have (and probably will) have different values, but we do know what their signs must be.

Let's start by looking at $v < 50$. We saw earlier that if $v = 30$ the slope of the tangent line will be 3.92, or positive. Therefore, for all values of $v < 50$ we will have positive slopes for the tangent lines. Also, by looking at (2) we can see that as v approaches 50, always staying less than 50, the slopes of the tangent lines will approach zero and hence flatten out. If we move v away from 50, staying less than 50, the slopes of the tangent lines will become steeper. If you want to get an idea of just how steep the tangent lines become you can always pick specific values of v and compute values of the derivative. For instance, we know that at $v = 30$ the derivative is 3.92 and so arrows at this point should have a slope of around 4. Using this information, we can now add in some arrows for the region below $v = 50$ as shown in the graph below.



Now, let's look at $v > 50$. The first thing to do is to find out if the slopes are positive or negative. We will do this the same way that we did in the last bit, *i.e.* pick a value of v, plug this into (2) and see if the derivative is positive or negative. Note, that you should NEVER assume that the derivative will change signs where the derivative is zero. It is easy enough to check so you should always do so.

We need to check the derivative so let's use $v = 60$. Plugging this into (2) gives the slope of the tangent line as -1.96, or negative. Therefore, for all values of $v > 50$ we will have negative slopes for the tangent lines. As with v < 50, by looking at (2) we can see that as v approaches 50, always staying greater than 50, the slopes of the tangent lines will approach zero and flatten out. While moving v away from 50 again, staying greater than 50, the slopes of the tangent lines will become steeper. We can now add in some arrows for the region above $v = 50$ as shown in the graph below.



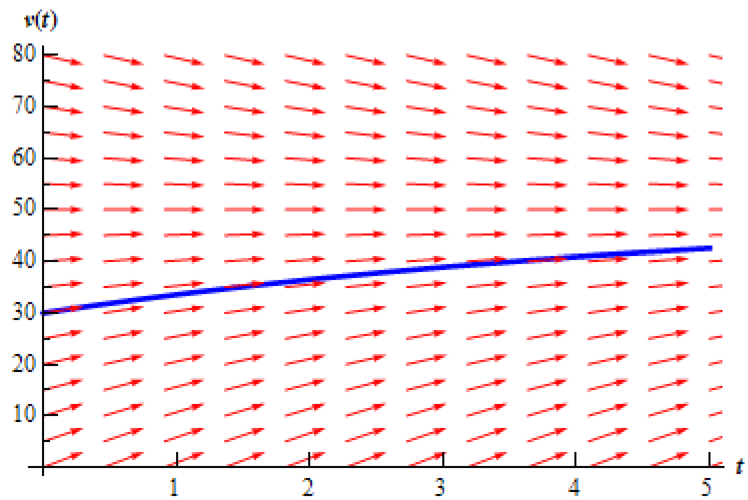
This graph above is called the **direction field** for the differential equation.

So, just why do we care about direction fields? There are two nice pieces of information that can be readily found from the direction field for a differential equation.

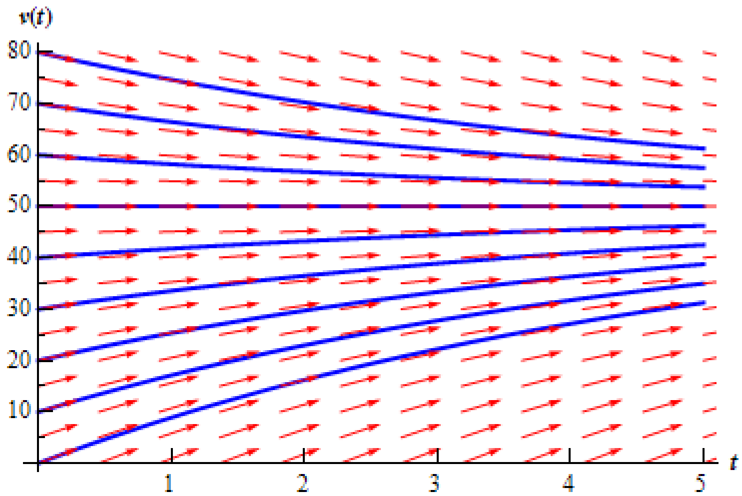
1. **Sketch of solutions**. Since the arrows in the direction fields are in fact tangents to the actual solutions to the differential equations we can use these as guides to sketch the graphs of solutions to the differential equation.

2. **Long Term Behavior**. In many cases we are less interested in the actual solutions to the differential equations as we are in how the solutions behave as t increases. Direction fields, if we can get our hands on them, can be used to find information about this long term behavior of the solution.

So, back to the direction field for our differential equation. Suppose that we want to know what the solution that has the value $v(0) = 30$ looks like. We can go to our direction field and start at 30 on the vertical axis. At this point we know that the solution is increasing and that as it increases the solution should flatten out because the velocity will be approaching the value of $v = 50$. So we start drawing an increasing solution and when we hit an arrow we just make sure that we stay parallel to that arrow. This gives us the figure below.



To get a better idea of how all the solutions are behaving, let's put a few more solutions in. Adding some more solutions gives the figure below. The set of solutions that we've graphed below is often called the **family of solution curves** or the set of **integral curves**. The number of solutions that is plotted when plotting the integral curves varies. You should graph enough solution curves to illustrate how solutions in all portions of the direction field are behaving.



Now, from either the direction field, or the direction field with the solution curves sketched in we can see the behavior of the solution as *t* increases. For our falling object, it looks like all of the solutions will approach *v* = 50 as *t* increases.

We will often want to know if the behavior of the solution will depend on the value of *v*(0). In this case the behavior of the solution will not depend on the value of *v*(0), but that is probably more of the exception than the rule so don't expect that.

Let's take a look at a more complicated example.

Example 1 Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation. Determine how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency.

$$y' = \left(y^2 - y - 2\right)\left(1 - y\right)^2$$

Solution

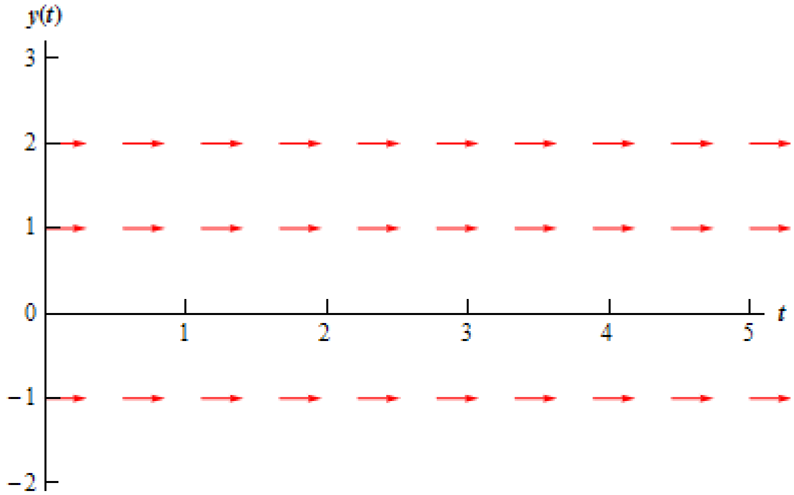
First, do not worry about where this differential equation came from. To be honest, we just made it up. It may, or may not describe an actual physical situation.

This differential equation looks somewhat more complicated than the falling object example from above. However, with the exception of a little more work, it is not much more complicated. The first step is to determine where the derivative is zero.

$$0 = \left(y^2 - y - 2\right)\left(1 - y\right)^2$$

$$0 = \left(y - 2\right)\left(y + 1\right)\left(1 - y\right)^2$$

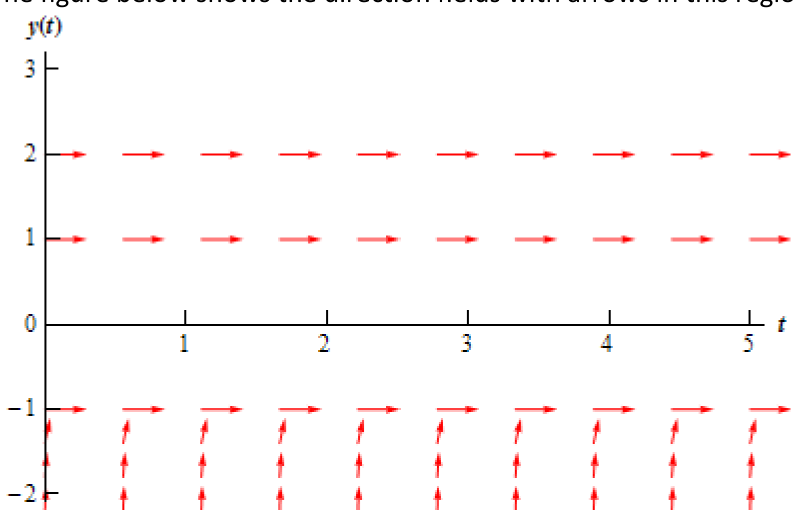
We can now see that we have three values of y in which the derivative, and hence the slope of tangent lines, will be zero. The derivative will be zero at $y = -1$, 1, and 2. So, let's start our direction field with drawing horizontal tangents for these values. This is shown in the figure below.



Now, we need to add arrows to the four regions that the graph is now divided into. For each of these regions I will pick a value of y in that region and plug it into the right hand side of the differential equation to see if the derivative is positive or negative in that region. Again, to get an accurate direction fields you should pick a few more values over the whole range to see how the arrows are behaving over the whole range.

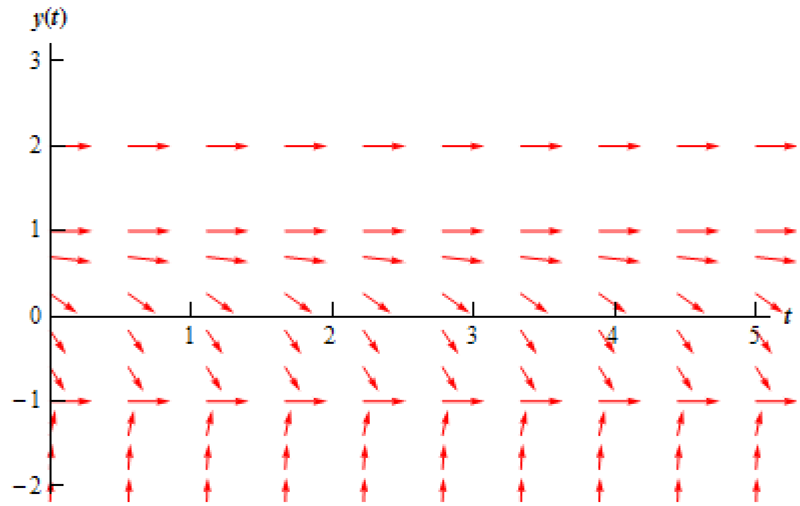
$y < -1$

In this region we can use y = -2 as the test point. At this point we have $y' = 36$. So, tangent lines in this region will have very steep and positive slopes. Also as $y \to -1$ the slopes will flatten out while staying positive. The figure below shows the direction fields with arrows in this region.



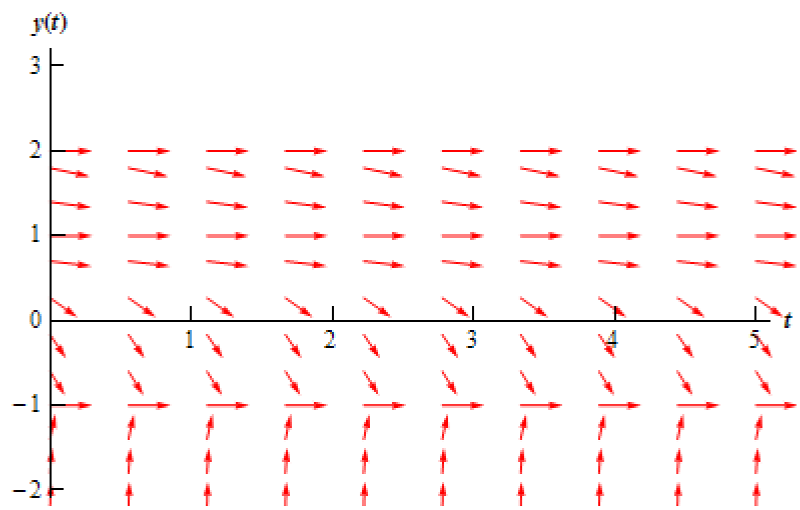
$-1 < y < 1$

In this region we can use y = 0 as the test point. At this point we have $y' = -2$. Therefore, tangent lines in this region will have negative slopes and apparently not be very steep. So what do the arrows look like in this region? As $y \to 1$ staying less than 1 of course, the slopes should be negative and approach zero. As we move away from 1 and towards -1 the slopes will start to get steeper (and stay negative), but eventually flatten back out, again staying negative, as $y \to -1$ since the derivative must approach zero at that point. The figure below shows the direction fields with arrows added to this region.



$1 < y < 2$

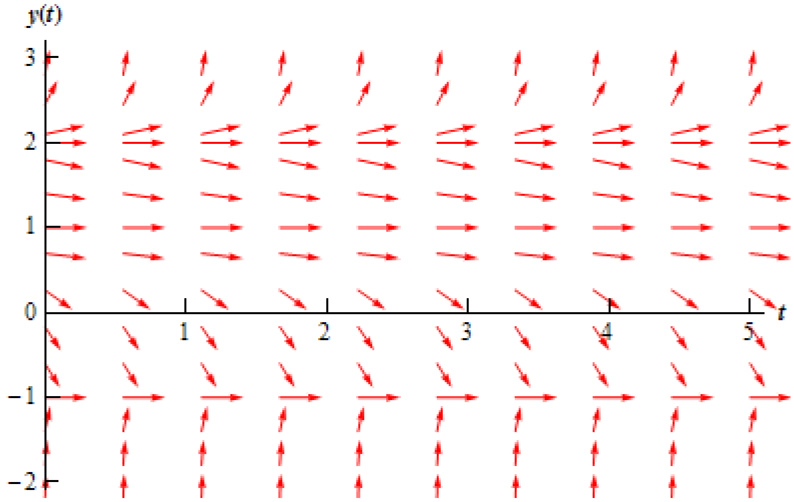
In this region we will use $y = 1.5$ as the test point. At this point we have $y' = -0.3125$. Tangent lines in this region will also have negative slopes and apparently not be as steep as the previous region. Arrows in this region will behave essentially the same as those in the previous region. Near $y = 1$ and $y = 2$ the slopes will flatten out and as we move from one to the other the slopes will get somewhat steeper before flattening back out. The figure below shows the direction fields with arrows added to this region.



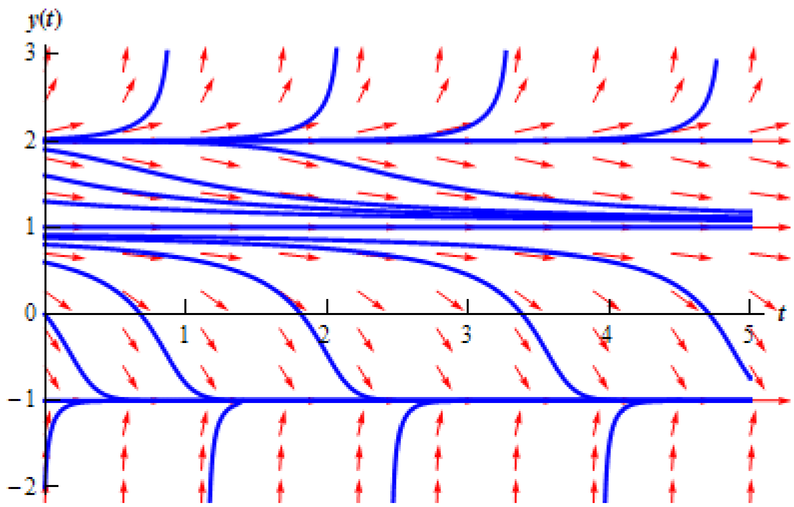
$y > 2$

In this last region we will use $y = 3$ as the test point. At this point we have $y' = 16$. So, as we saw in the first region tangent lines will start out fairly flat near $y = 2$ and then as we move way from $y = 2$ they will get fairly steep.

The complete direction field for this differential equation is shown below.



Here is the set of integral curves for this differential equation.



Finally, let's take a look at long term behavior of all solutions. Unlike the first example, the long term behavior in this case will depend on the value of y at $t = 0$. By examining either of the previous two figures we can arrive at the following behavior of solutions as $t \to \infty$.

Value of $y(0)$	Behavior as $t \to \infty$
$y(0) < 1$	$y \to -1$
$1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$

Do not forget to acknowledge what the horizontal solutions are doing. This is often the most missed portion of this kind of problem.

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In both of the examples that we've worked to this point the right hand side of the derivative has only contained the function and NOT the independent variable. When the right hand side of the differential equation contains both the function and the independent variable the behavior can be much more complicated and sketching the direction fields by hand can be very difficult. Computer software is very handy in these cases.

In some cases they aren't too difficult to do by hand however. Let's take a look at the following example.

Example 2 Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation.

$$y' = y - x$$

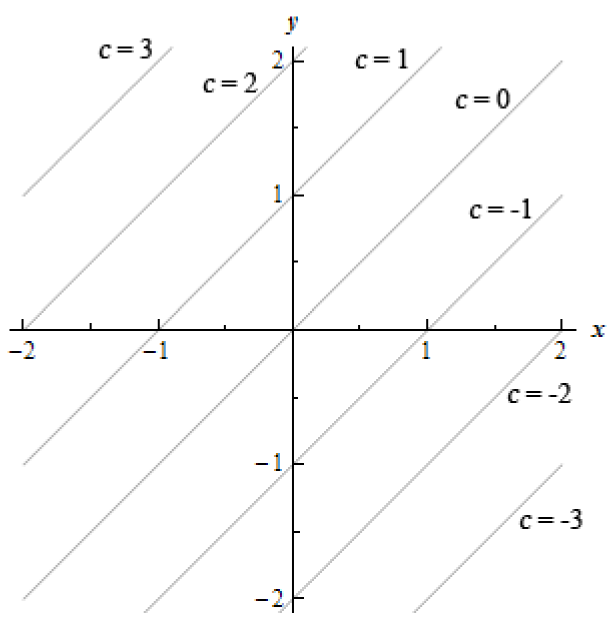
Solution

To sketch direction fields for this kind of differential equation we first identify places where the derivative will be constant. To do this we set the derivative in the differential equation equal to a constant, say c. This gives us a family of equations, called **isoclines**, that we can plot and on each of these curves the derivative will be a constant value of c.

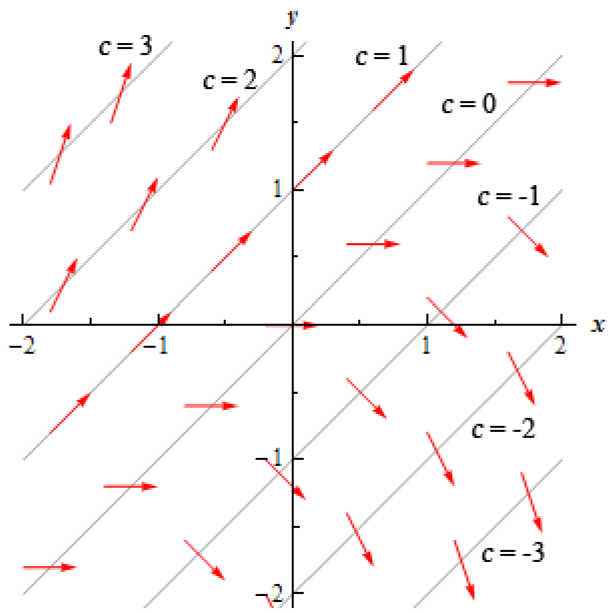
Notice that in the previous examples we looked at the isocline for $c = 0$ to get the direction field started. For our case the family of isoclines is.

$$c = y - x$$

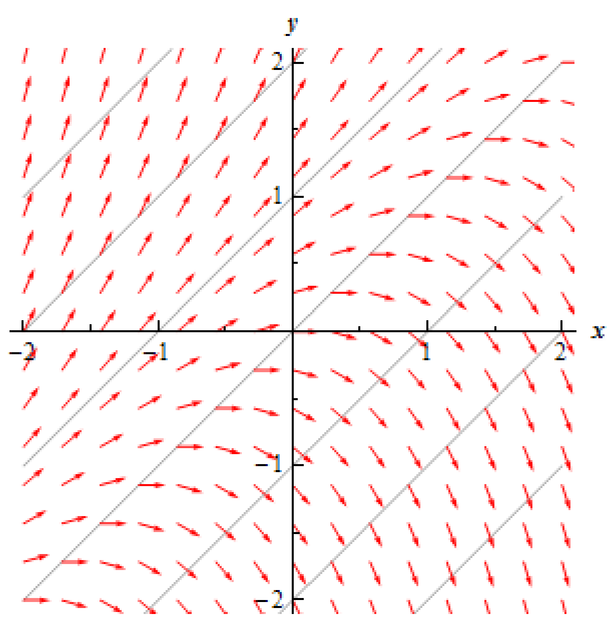
The graph of these curves for several values of c is shown below.



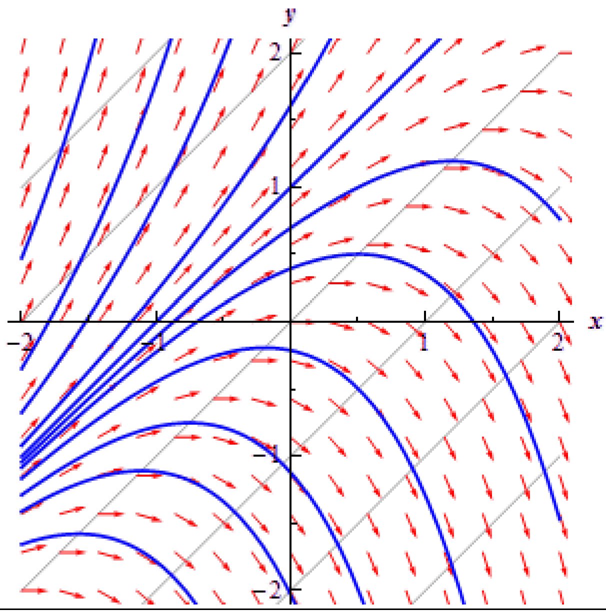
Now, on each of these lines, or isoclines, the derivative will be constant and will have a value of c. On the $c = 0$ isocline the derivative will always have a value of zero and hence the tangents will all be horizontal. On the $c = 1$ isocline the tangents will always have a slope of 1, on the $c = -2$ isocline the tangents will always have a slope of -2, *etc.* Below are a few tangents put in for each of these isoclines.



To add more arrows for those areas between the isoclines start at say, $c = 0$ and move up to $c = 1$ and as we do that we increase the slope of the arrows (tangents) from 0 to 1. This is shown in the figure below.



We can then add in integral curves as we did in the previous examples. This is shown in the figure below.


y
x
2
1
−1
−2
−2
−1
1
2

Section 1-3 : Final Thoughts

Before moving on to learning how to solve differential equations we want to give a few final thoughts. Any differential equations course will concern itself with answering one or more of the following questions.

1. **Given a differential equation will a solution exist?**
   Not all differential equations will have solutions so it's useful to know ahead of time if there is a solution or not. If there isn't a solution why waste our time trying to find something that doesn't exist?

   This question is usually called the **existence question** in a differential equations course.

2. **If a differential equation does have a solution how many solutions are there?**
   As we will see eventually, it is possible for a differential equation to have more than one solution. We would like to know how many solutions there will be for a given differential equation.

   There is a sub question here as well. What condition(s) on a differential equation are required to obtain a single unique solution to the differential equation?

   Both this question and the sub question are more important than you might realize. Suppose that we derive a differential equation that will give the temperature distribution in a bar of iron at any time t. If we solve the differential equation and end up with two (or more) completely separate solutions we will have problems. Consider the following situation to see this.

   If we subject 10 identical iron bars to identical conditions they should all exhibit the same temperature distribution. So only one of our solutions will be accurate, but we will have no way of knowing which one is the correct solution.

   It would be nice if, during the derivation of our differential equation, we could make sure that our assumptions would give us a differential equation that upon solving will yield a single unique solution.

   This question is usually called the **uniqueness question** in a differential equations course.

3. **If a differential equation does have a solution can we find it?**
   This may seem like an odd question to ask and yet the answer is not always yes. Just because we know that a solution to a differential equations exists does not mean that we will be able to find it.

In a first course in differential equations (such as this one) the third question is the question that we will concentrate on. We will answer the first two questions for special, and fairly simple, cases, but most of our efforts will be concentrated on answering the third question for as wide a variety of differential equations as possible.

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