CALCULUS III
Applications of Partial Derivatives

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Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 3 : Applications of Partial Derivatives

In this chapter we will take a look at a several applications of partial derivatives. Most of the applications will be extensions to applications to ordinary derivatives that we saw back in Calculus I. For instance, we will be looking at finding the absolute and relative extrema of a function and we will also be looking at optimization. Both (all three?) of these subjects were major applications back in Calculus I. They will, however, be a little more work here because we now have more than one variable.

Here is a list of the topics in this chapter.

Tangent Planes and Linear Approximations – In this section formally define just what a tangent plane to a surface is and how we use partial derivatives to find the equations of tangent planes to surfaces that can be written as $z = f(x, y)$. We will also see how tangent planes can be thought of as a linear approximation to the surface at a given point.

Gradient Vector, Tangent Planes and Normal Lines – In this section discuss how the gradient vector can be used to find tangent planes to a much more general function than in the previous section. We will also define the normal line and discuss how the gradient vector can be used to find the equation of the normal line.

Relative Minimums and Maximums – In this section we will define critical points for functions of two variables and discuss a method for determining if they are relative minimums, relative maximums or saddle points (*i.e.* neither a relative minimum or relative maximum).

Absolute Minimums and Maximums – In this section we will how to find the absolute extrema of a function of two variables when the independent variables are only allowed to come from a region that is bounded (*i.e.* no part of the region goes out to infinity) and closed (*i.e.* all of the points on the boundary are valid points that can be used in the process).

Lagrange Multipliers – In this section we'll see discuss how to use the method of Lagrange Multipliers to find the absolute minimums and maximums of functions of two or three variables in which the independent variables are subject to one or more constraints. We also give a brief justification for how/why the method works.

Section 3-1 : Tangent Planes and Linear Approximations

Earlier we saw how the two partial derivatives f_x and f_y can be thought of as the slopes of traces. We want to extend this idea out a little in this section. The graph of a function $z = f(x, y)$ is a surface in $\mathbb{R}^3$ (three dimensional space) and so we can now start thinking of the plane that is "tangent" to the surface as a point.

Let's start out with a point (x_0, y_0) and let's let C_1 represent the trace to $f(x, y)$ for the plane $y = y_0$ (i.e. allowing x to vary with y held fixed) and we'll let C_2 represent the trace to $f(x, y)$ for the plane $x = x_0$ (i.e. allowing y to vary with x held fixed). Now, we know that $f_x(x_0, y_0)$ is the slope of the tangent line to the trace C_1 and $f_y(x_0, y_0)$ is the slope of the tangent line to the trace C_2. So, let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The tangent plane will then be the plane that contains the two lines L_1 and L_2. Geometrically this plane will serve the same purpose that a tangent line did in Calculus I. A tangent line to a curve was a line that just touched the curve at that point and was "parallel" to the curve at the point in question. Well tangent planes to a surface are planes that just touch the surface at the point and are "parallel" to the surface at the point. Note that this gives us a point that is on the plane. Since the tangent plane and the surface touch at (x_0, y_0) the following point will be on both the surface and the plane.

$$(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$$

What we need to do now is determine the equation of the tangent plane. We know that the general equation of a plane is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point that is on the plane, which we have. Let's rewrite this a little. We'll move the x terms and y terms to the other side and divide both sides by c. Doing this gives,

$$z - z_0 = -\frac{a}{c}(x - x_0) - \frac{b}{c}(y - y_0)$$

Now, let's rename the constants to simplify up the notation a little. Let's rename them as follows,

$$A = -\frac{a}{c} \qquad B = -\frac{b}{c}$$

With this renaming the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

and we need to determine values for A and B.

Let's first think about what happens if we hold y fixed, i.e. if we assume that $y = y_0$. In this case the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0)$$

This is the equation of a line and this line must be tangent to the surface at (x_0, y_0) (since it's part of the tangent plane). In addition, this line assumes that $y = y_0$ (*i.e.* fixed) and *A* is the slope of this line. But if we think about it this is exactly what the tangent to C_1 is, a line tangent to the surface at (x_0, y_0) assuming that $y = y_0$. In other words,

$$z - z_0 = A(x - x_0)$$

is the equation for L_1 and we know that the slope of L_1 is given by $f_x(x_0, y_0)$. Therefore, we have the following,

$$A = f_x(x_0, y_0)$$

If we hold *x* fixed at $x = x_0$ the equation of the tangent plane becomes,

$$z - z_0 = B(y - y_0)$$

However, by a similar argument to the one above we can see that this is nothing more than the equation for L_2 and that it's slope is *B* or $f_y(x_0, y_0)$. So,

$$B = f_y(x_0, y_0)$$

The equation of the tangent plane to the surface given by $z = f(x, y)$ at (x_0, y_0) is then,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Also, if we use the fact that $z_0 = f(x_0, y_0)$ we can rewrite the equation of the tangent plane as,

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We will see an easier derivation of this formula (actually a more general formula) in the next section so if you didn't quite follow this argument hold off until then to see a better derivation.

Example 1 Find the equation of the tangent plane to $z = \ln(2x + y)$ at $(-1, 3)$.

Solution
There really isn't too much to do here other than taking a couple of derivatives and doing some quick evaluations.

$$f(x, y) = \ln(2x + y) \qquad\qquad z_0 = f(-1, 3) = \ln(1) = 0$$

$$f_x(x, y) = \frac{2}{2x + y} \qquad\qquad f_x(-1, 3) = 2$$

$$f_y(x, y) = \frac{1}{2x + y} \qquad\qquad f_y(-1, 3) = 1$$

The equation of the plane is then,
$$z - 0 = 2(x+1) + (1)(y-3)$$
$$z = 2x + y - 1$$

One nice use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point. Because of this we define the **linear approximation** to be,

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and as long as we are "near" (x_0, y_0) then we should have that,

$$f(x,y) \approx L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

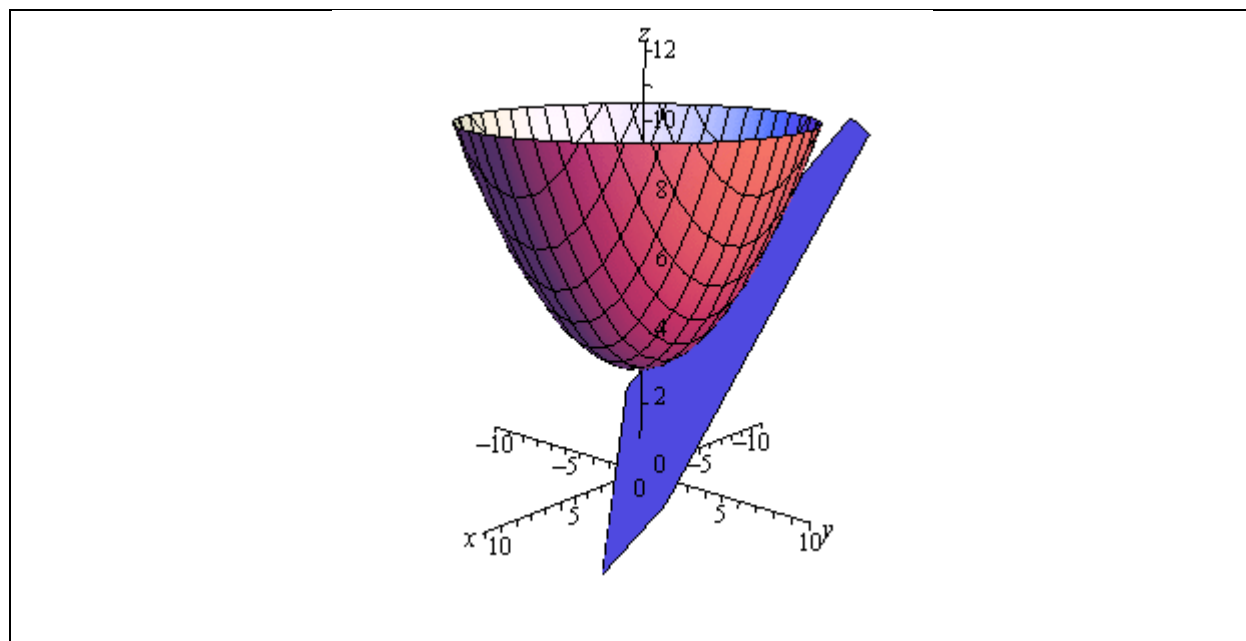***Example 2*** Find the linear approximation to $z = 3 + \dfrac{x^2}{16} + \dfrac{y^2}{9}$ at $(-4, 3)$.

Solution
So, we're really asking for the tangent plane so let's find that.

$$f(x,y) = 3 + \frac{x^2}{16} + \frac{y^2}{9} \qquad\qquad f(-4,3) = 3 + 1 + 1 = 5$$

$$f_x(x,y) = \frac{x}{8} \qquad\qquad f_x(-4,3) = -\frac{1}{2}$$

$$f_y(x,y) = \frac{2y}{9} \qquad\qquad f_y(-4,3) = \frac{2}{3}$$

The tangent plane, or linear approximation, is then,

$$L(x,y) = 5 - \frac{1}{2}(x+4) + \frac{2}{3}(y-3)$$

For reference purposes here is a sketch of the surface and the tangent plane/linear approximation.

Section 3-2 : Gradient Vector, Tangent Planes and Normal Lines

In this section we want to revisit tangent planes only this time we'll look at them in light of the gradient vector. In the process we will also take a look at a normal line to a surface.

Let's first recall the equation of a plane that contains the point (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a, b, c \rangle$ is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

When we introduced the gradient vector in the section on directional derivatives we gave the following fact.

Fact

> The gradient vector $\nabla f(x_0, y_0)$ is orthogonal (or perpendicular) to the level curve $f(x, y) = k$ at the point (x_0, y_0). Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0).

Actually, all we need here is the last part of this fact. This says that the gradient vector is always orthogonal, or *normal*, to the surface at a point.

Also recall that the gradient vector is,

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

So, the tangent plane to the surface given by $f(x, y, z) = k$ at (x_0, y_0, z_0) has the equation,

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

This is a much more general form of the equation of a tangent plane than the one that we derived in the previous section.

Note however, that we can also get the equation from the previous section using this more general formula. To see this let's start with the equation $z = f(x, y)$ and we want to find the tangent plane to the surface given by $z = f(x, y)$ at the point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$. In order to use the formula above we need to have all the variables on one side. This is easy enough to do. All we need to do is subtract a *z* from both sides to get,

$$f(x, y) - z = 0$$

Now, if we define a new function

$$F(x, y, z) = f(x, y) - z$$

we can see that the surface given by $z = f(x, y)$ is identical to the surface given by $F(x, y, z) = 0$ and this new equivalent equation is in the correct form for the equation of the tangent plane that we derived in this section.

So, the first thing that we need to do is find the gradient vector for *F*.

$$\nabla F = \langle F_x, F_y, F_z \rangle = \langle f_x, f_y, -1 \rangle$$

Notice that

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x \qquad\qquad F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = -1$$

The equation of the tangent plane is then,

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Or, upon solving for *z*, we get,

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

which is identical to the equation that we derived in the previous section.

We can get another nice piece of information out of the gradient vector as well. We might on occasion want a line that is orthogonal to a surface at a point, sometimes called the **normal line**. This is easy enough to get if we recall that the equation of a line only requires that we have a point and a parallel vector. Since we want a line that is at the point (x_0, y_0, z_0) we know that this point must also be on the line and we know that $\nabla f(x_0, y_0, z_0)$ is a vector that is normal to the surface and hence will be parallel to the line. Therefore, the equation of the normal line is,

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla f(x_0, y_0, z_0)$$

Example 1 Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 30$ at the point $(1, -2, 5)$.

Solution
For this case the function that we're going to be working with is,

$$F(x, y, z) = x^2 + y^2 + z^2$$

and note that we don't have to have a zero on one side of the equal sign. All that we need is a constant. To finish this problem out we simply need the gradient evaluated at the point.

$$\nabla F = \langle 2x, 2y, 2z \rangle$$
$$\nabla F(1, -2, 5) = \langle 2, -4, 10 \rangle$$

The tangent plane is then,

$$2(x-1)-4(y+2)+10(z-5)=0$$

The normal line is,

$$\vec{r}(t)=\langle 1,-2,5\rangle+t\langle 2,-4,10\rangle=\langle 1+2t,-2-4t,5+10t\rangle$$

Section 3-3 : Relative Minimums and Maximums

In this section we are going to extend one of the more important ideas from Calculus I into functions of two variables. We are going to start looking at trying to find minimums and maximums of functions. This in fact will be the topic of the following two sections as well.

In this section we are going to be looking at identifying relative minimums and relative maximums. Recall as well that we will often use the word extrema to refer to both minimums and maximums.

The definition of relative extrema for functions of two variables is identical to that for functions of one variable we just need to remember now that we are working with functions of two variables. So, for the sake of completeness here is the definition of relative minimums and relative maximums for functions of two variables.

Definition

1. A function $f(x,y)$ has a **relative minimum** at the point (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in some region around (a,b).

2. A function $f(x,y)$ has a **relative maximum** at the point (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some region around (a,b).

Note that this definition does not say that a relative minimum is the smallest value that the function will ever take. It only says that in some region around the point (a,b) the function will always be larger than $f(a,b)$. Outside of that region it is completely possible for the function to be smaller. Likewise, a relative maximum only says that around (a,b) the function will always be smaller than $f(a,b)$. Again, outside of the region it is completely possible that the function will be larger.

Next, we need to extend the idea of **critical points** up to functions of two variables. Recall that a critical point of the function $f(x)$ was a number $x = c$ so that either $f'(c) = 0$ or $f'(c)$ doesn't exist. We have a similar definition for critical points of functions of two variables.

Definition

The point (a,b) is a **critical point** (or a **stationary point**) of $f(x,y)$ provided one of the following is true,
   1. $\nabla f(a,b) = \vec{0}$ (this is equivalent to saying that $f_x(a,b) = 0$ and $f_y(a,b) = 0$),
   2. $f_x(a,b)$ and/or $f_y(a,b)$ doesn't exist.

To see the equivalence in the first part let's start off with $\nabla f = \vec{0}$ and put in the definition of each part.

$$\nabla f(a,b) = \vec{0}$$

$$\langle f_x(a,b), f_y(a,b) \rangle = \langle 0,0 \rangle$$

The only way that these two vectors can be equal is to have $f_x(a,b) = 0$ and $f_y(a,b) = 0$. In fact, we will use this definition of the critical point more than the gradient definition since it will be easier to find the critical points if we start with the partial derivative definition.

Note as well that BOTH of the first order partial derivatives must be zero at (a,b). If only one of the first order partial derivatives are zero at the point then the point will NOT be a critical point.

We now have the following fact that, at least partially, relates critical points to relative extrema.

Fact

> If the point (a,b) is a relative extrema of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at (a,b) then (a,b) is also a critical point of $f(x,y)$ and in fact we'll have $\nabla f(a,b) = \vec{0}$.

Proof

> This is a really simple proof that relies on the single variable version that we saw in Calculus I version, often called Fermat's Theorem.
>
> Let's start off by defining $g(x) = f(x,b)$ and suppose that $f(x,y)$ has a relative extrema at (a,b). However, this also means that $g(x)$ also has a relative extrema (of the same kind as $f(x,y)$) at $x = a$. By Fermat's Theorem we then know that $g'(a) = 0$. But we also know that $g'(a) = f_x(a,b)$ and so we have that $f_x(a,b) = 0$.
>
> If we now define $h(y) = f(a,y)$ and going through exactly the same process as above we will see that $f_y(a,b) = 0$.
>
> So, putting all this together means that $\nabla f(a,b) = \vec{0}$ and so $f(x,y)$ has a critical point at (a,b).

Note that this does NOT say that all critical points are relative extrema. It only says that relative extrema will be critical points of the function. To see this let's consider the function

$$f(x,y) = xy$$

The two first order partial derivatives are,

$$f_x(x,y) = y \qquad\qquad f_y(x,y) = x$$

The only point that will make both of these derivatives zero at the same time is $(0,0)$ and so $(0,0)$ is a critical point for the function. Here is a graph of the function.
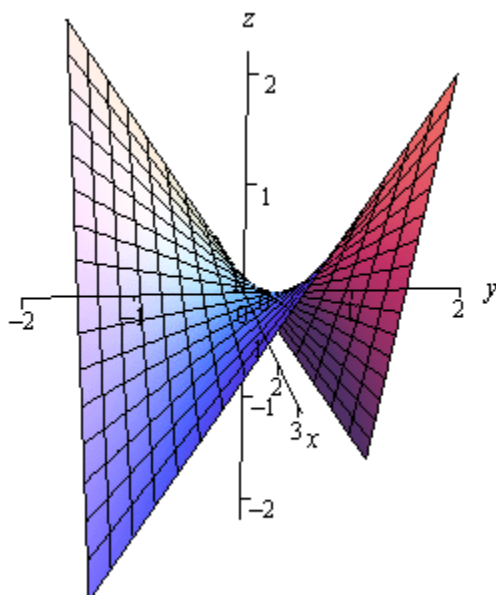
Note that the axes are not in the standard orientation here so that we can see more clearly what is happening at the origin, *i.e.* at $(0,0)$. If we start at the origin and move into either of the quadrants where both *x* and *y* are the same sign the function increases. However, if we start at the origin and move into either of the quadrants where *x* and *y* have the opposite sign then the function decreases. In other words, no matter what region you take about the origin there will be points larger than $f(0,0)=0$ and points smaller than $f(0,0)=0$. Therefore, there is no way that $(0,0)$ can be a relative extrema.

Critical points that exhibit this kind of behavior are called **saddle points**.

While we have to be careful to not misinterpret the results of this fact it is very useful in helping us to identify relative extrema. Because of this fact we know that if we have all the critical points of a function then we also have every possible relative extrema for the function. The fact tells us that all relative extrema must be critical points so we know that if the function does have relative extrema then they must be in the collection of all the critical points. Remember however, that it will be completely possible that at least one of the critical points won't be a relative extrema.

So, once we have all the critical points in hand all we will need to do is test these points to see if they are relative extrema or not. To determine if a critical point is a relative extrema (and in fact to determine if it is a minimum or a maximum) we can use the following fact.

Fact

Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b). Next define,

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

We then have the following classifications of the critical point.
1. If $D > 0$ and $f_{xx}(a,b) > 0$ then there is a relative minimum at (a,b).
2. If $D > 0$ and $f_{xx}(a,b) < 0$ then there is a relative maximum at (a,b).
3. If $D < 0$ then the point (a,b) is a saddle point.
4. If $D = 0$ then the point (a,b) may be a relative minimum, relative maximum or a saddle point. Other techniques would need to be used to classify the critical point.

Note that if $D > 0$ then both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ will have the same sign and so in the first two cases above we could just as easily replace $f_{xx}(a,b)$ with $f_{yy}(a,b)$. Also note that we aren't going to be seeing any cases in this class where $D = 0$ as these can often be quite difficult to classify. We will be able to classify all the critical points that we find.

Let's see a couple of examples.

Example 1 Find and classify all the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.

Solution
We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so let's get those.

$$f_x = 3x^2 - 3y \qquad\qquad f_y = 3y^2 - 3x$$
$$f_{xx} = 6x \qquad\qquad f_{yy} = 6y \qquad\qquad f_{xy} = -3$$

Let's first find the critical points. Critical points will be solutions to the system of equations,
$$f_x = 3x^2 - 3y = 0$$
$$f_y = 3y^2 - 3x = 0$$

This is a non-linear system of equations and these can, on occasion, be difficult to solve. However, in this case it's not too bad. We can solve the first equation for *y* as follows,

$$3x^2 - 3y = 0 \qquad \Rightarrow \qquad y = x^2$$

Plugging this into the second equation gives,

$$3\left(x^2\right)^2 - 3x = 3x\left(x^3 - 1\right) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points.

$$x = 0: \quad y = 0^2 = 0 \qquad \Rightarrow \qquad (0,0)$$
$$x = 1: \quad y = 1^2 = 1 \qquad \Rightarrow \qquad (1,1)$$

So, we get two critical points. All we need to do now is classify them. To do this we will need *D*. Here is the general formula for *D*.

$$\begin{aligned} D(x,y) &= f_{xx}(x,y) f_{yy}(x,y) - \left[f_{xy}(x,y) \right]^2 \\ &= (6x)(6y) - (-3)^2 \\ &= 36xy - 9 \end{aligned}$$

To classify the critical points all that we need to do is plug in the critical points and use the fact above to classify them.
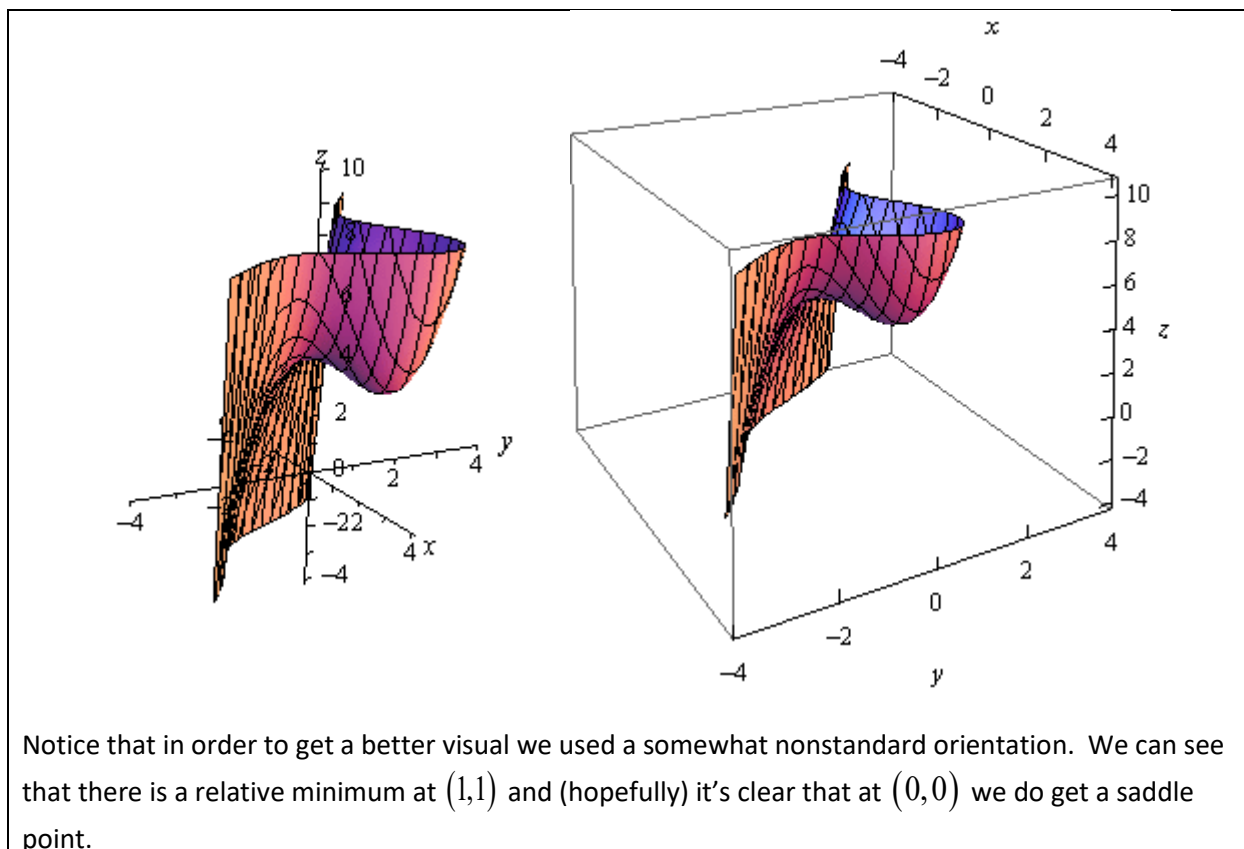
$(0,0)$:

$$D = D(0,0) = -9 < 0$$

So, for $(0,0)$ *D* is negative and so this must be a saddle point.

$(1,1)$:

$$D = D(1,1) = 36 - 9 = 27 > 0 \qquad\qquad f_{xx}(1,1) = 6 > 0$$

For $(1,1)$ *D* is positive and f_{xx} is positive and so we must have a relative minimum.

For the sake of completeness here is a graph of this function.

Notice that in order to get a better visual we used a somewhat nonstandard orientation. We can see that there is a relative minimum at $(1,1)$ and (hopefully) it's clear that at $(0,0)$ we do get a saddle point.

Example 2 Find and classify all the critical points for $f(x,y) = 3x^2 y + y^3 - 3x^2 - 3y^2 + 2$

Solution
As with the first example we will first need to get all the first and second order derivatives.

$$f_x = 6xy - 6x \qquad\qquad f_y = 3x^2 + 3y^2 - 6y$$
$$f_{xx} = 6y - 6 \qquad\qquad f_{yy} = 6y - 6 \qquad\qquad f_{xy} = 6x$$

We'll first need the critical points. The equations that we'll need to solve this time are,

$$6xy - 6x = 0$$
$$3x^2 + 3y^2 - 6y = 0$$

These equations are a little trickier to solve than the first set, but once you see what to do they really aren't terribly bad.

First, let's notice that we can factor out a 6*x* from the first equation to get,

$$6x(y-1) = 0$$

So, we can see that the first equation will be zero if $x = 0$ or $y = 1$. Be careful to not just cancel the *x* from both sides. If we had done that we would have missed $x = 0$.

To find the critical points we can plug these (individually) into the second equation and solve for the remaining variable.

$x = 0$:
$$3y^2 - 6y = 3y(y-2) = 0 \qquad \Rightarrow \qquad y = 0, \; y = 2$$

$y = 1$:
$$3x^2 - 3 = 3(x^2 - 1) = 0 \qquad \Rightarrow \qquad x = -1, \; x = 1$$

So, if $x = 0$ we have the following critical points,
$$(0,0) \quad (0,2)$$

and if $y = 1$ the critical points are,
$$(1,1) \quad (-1,1)$$

Now all we need to do is classify the critical points. To do this we'll need the general formula for *D*.
$$D(x,y) = (6y-6)(6y-6) - (6x)^2 = (6y-6)^2 - 36x^2$$

$(0,0)$:
$$D = D(0,0) = 36 > 0 \qquad f_{xx}(0,0) = -6 < 0$$

$(0,2)$:
$$D = D(0,2) = 36 > 0 \qquad f_{xx}(0,2) = 6 > 0$$

$(1,1)$:
$$D = D(1,1) = -36 < 0$$

$(-1,1)$:
$$D = D(-1,1) = -36 < 0$$

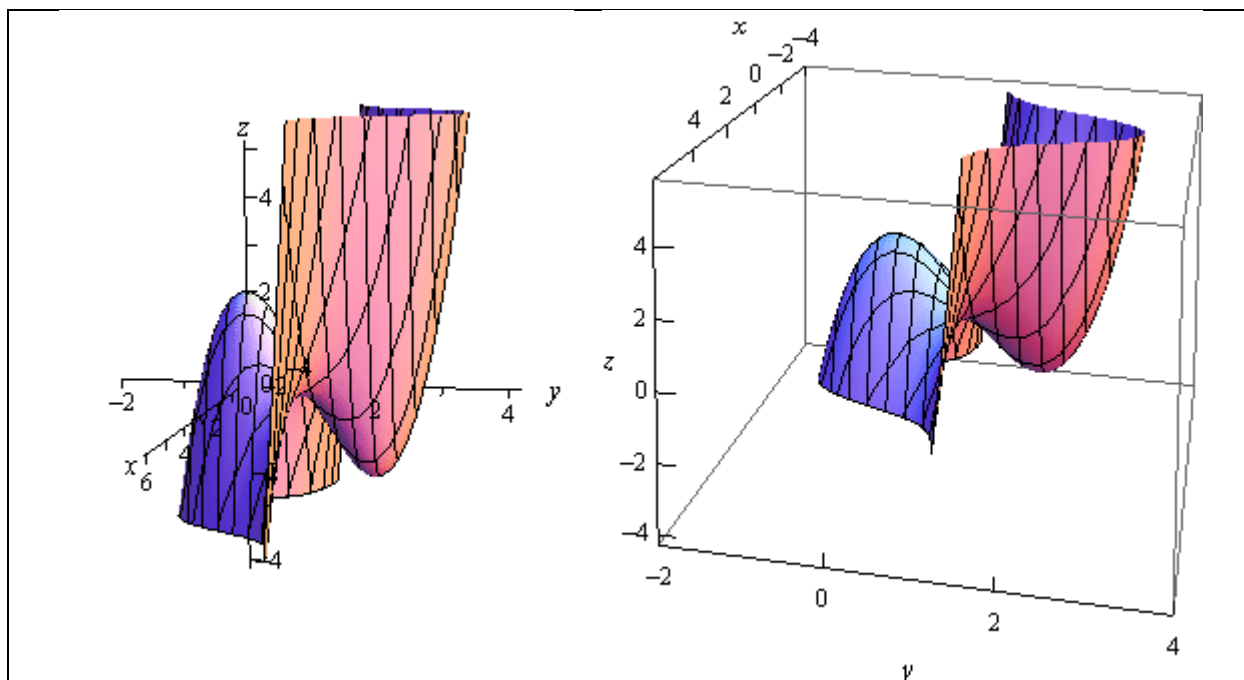So, it looks like we have the following classification of each of these critical points.
$$(0,0) \quad : \quad \text{Relative Maximum}$$
$$(0,2) \quad : \quad \text{Relative Minimum}$$
$$(1,1) \quad : \quad \text{Saddle Point}$$
$$(-1,1) \quad : \quad \text{Saddle Point}$$

Here is a graph of the surface for the sake of completeness.

Let's do one more example that is a little different from the first two.

Example 3 Determine the point on the plane $4x - 2y + z = 1$ that is closest to the point $(-2, -1, 5)$.

Solution
Note that we are NOT asking for the critical points of the plane. In order to do this example we are going to need to first come up with the equation that we are going to have to work with.

First, let's suppose that (x, y, z) is any point on the plane. The distance between this point and the point in question, $(-2, -1, 5)$, is given by the formula,

$$d = \sqrt{(x+2)^2 + (y+1)^2 + (z-5)^2}$$

What we are then asked to find is the minimum value of this equation. The point (x, y, z) that gives the minimum value of this equation will be the point on the plane that is closest to $(-2, -1, 5)$.

There are a couple of issues with this equation. First, it is a function of *x*, *y* and *z* and we can only deal with functions of *x* and *y* at this point. However, this is easy to fix. We can solve the equation of the plane to see that,

$$z = 1 - 4x + 2y$$

Plugging this into the distance formula gives,

$$d = \sqrt{(x+2)^2 + (y+1)^2 + (1-4x+2y-5)^2}$$
$$= \sqrt{(x+2)^2 + (y+1)^2 + (-4-4x+2y)^2}$$

Now, the next issue is that there is a square root in this formula and we know that we're going to be differentiating this eventually. So, in order to make our life a little easier let's notice that finding the minimum value of d will be equivalent to finding the minimum value of d^2.

So, let's instead find the minimum value of
$$f(x,y) = d^2 = (x+2)^2 + (y+1)^2 + (-4-4x+2y)^2$$

Now, we need to be a little careful here. We are being asked to find the closest point on the plane to $(-2,-1,5)$ and that is not really the same thing as what we've been doing in this section. In this section we've been finding and classifying critical points as relative minimums or maximums and what we are really asking is to find the smallest value the function will take, or the absolute minimum. Hopefully, it does make sense from a physical standpoint that there will be a closest point on the plane to $(-2,-1,5)$. This point should also be a relative minimum in addition to being an absolute minimum.

So, let's go through the process from the first and second example and see what we get as far as relative minimums go. If we only get a single relative minimum then we will be done since that point will also need to be the absolute minimum of the function and hence the point on the plane that is closest to $(-2,-1,5)$.

We'll need the derivatives first.
$$f_x = 2(x+2) + 2(-4)(-4-4x+2y) = 36 + 34x - 16y$$
$$f_y = 2(y+1) + 2(2)(-4-4x+2y) = -14 - 16x + 10y$$
$$f_{xx} = 34$$
$$f_{yy} = 10$$
$$f_{xy} = -16$$

Now, before we get into finding the critical point let's compute D quickly.
$$D = 34(10) - (-16)^2 = 84 > 0$$

So, in this case D will always be positive and also notice that $f_{xx} = 34 > 0$ is always positive and so any critical points that we get will be guaranteed to be relative minimums.

Now let's find the critical point(s). This will mean solving the system.
$$36 + 34x - 16y = 0$$
$$-14 - 16x + 10y = 0$$

To do this we can solve the first equation for *x*.

$$x = \frac{1}{34}(16y - 36) = \frac{1}{17}(8y - 18)$$

Now, plug this into the second equation and solve for *y*.

$$-14 - \frac{16}{17}(8y - 18) + 10y = 0 \qquad \Rightarrow \qquad y = -\frac{25}{21}$$

Back substituting this into the equation for *x* gives $x = -\frac{34}{21}$.

So, it looks like we get a single critical point : $\left(-\frac{34}{21}, -\frac{25}{21}\right)$. Also, since we know this will be a relative minimum and it is the only critical point we know that this is also the *x* and *y* coordinates of the point on the plane that we're after. We can find the *z* coordinate by plugging into the equation of the plane as follows,

$$z = 1 - 4\left(-\frac{34}{21}\right) + 2\left(-\frac{25}{21}\right) = \frac{107}{21}$$

So, the point on the plane that is closest to $\left(-2, -1, 5\right)$ is $\left(-\frac{34}{21}, -\frac{25}{21}, \frac{107}{21}\right)$.

Section 3-4 : Absolute Minimums and Maximums

In this section we are going to extend the work from the previous section. In the previous section we were asked to find and classify all critical points as relative minimums, relative maximums and/or saddle points. In this section we want to optimize a function, that is identify the absolute minimum and/or the absolute maximum of the function, on a given region in $\mathbb{R}^2$. Note that when we say we are going to be working on a region in $\mathbb{R}^2$ we mean that we're going to be looking at some region in the *xy*-plane.

In order to optimize a function in a region we are going to need to get a couple of definitions out of the way and a fact. Let's first get the definitions out of the way.

Definitions

1. A region in $\mathbb{R}^2$ is called **closed** if it includes its boundary. A region is called **open** if it doesn't include any of its boundary points.

2. A region in $\mathbb{R}^2$ is called **bounded** if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

Let's think a little more about the definition of closed. We said a region is closed if it includes its boundary. Just what does this mean? Let's think of a rectangle. Below are two definitions of a rectangle, one is closed and the other is open.

$$\begin{array}{cc} \text{Open} & \text{Closed} \\ -5 < x < 3 & -5 \le x \le 3 \\ 1 < y < 6 & 1 \le y \le 6 \end{array}$$

In this first case we don't allow the ranges to include the endpoints (*i.e.* we aren't including the edges of the rectangle) and so we aren't allowing the region to include any points on the edge of the rectangle. In other words, we aren't allowing the region to include its boundary and so it's open.

In the second case we are allowing the region to contain points on the edges and so will contain its entire boundary and hence will be closed.

This is an important idea because of the following fact.

Extreme Value Theorem

If $f(x,y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum of the function in D.

Note that this theorem does NOT tell us where the absolute minimum or absolute maximum will occur. It only tells us that they will exist. Note as well that the absolute minimum and/or absolute maximum may occur in the interior of the region or it may occur on the boundary of the region.

The basic process for finding absolute maximums is pretty much identical to the process that we used in Calculus I when we looked at finding absolute extrema of functions of single variables. There will however, be some procedural changes to account for the fact that we now are dealing with functions of two variables. Here is the process.

Finding Absolute Extrema

1. Find all the critical points of the function that lie in the region *D* and determine the function value at each of these points.

2. Find all extrema of the function on the boundary. This usually involves the Calculus I approach for this work.

3. The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

The main difference between this process and the process that we used in Calculus I is that the "boundary" in Calculus I was just two points and so there really wasn't a lot to do in the second step. For these problems the majority of the work is often in the second step as we will often end up doing a Calculus I absolute extrema problem one or more times.

Let's take a look at an example or two.

Example 1 Find the absolute minimum and absolute maximum of $f(x, y) = x^2 + 4y^2 - 2x^2 y + 4$ on the rectangle given by $-1 \le x \le 1$ and $-1 \le y \le 1$.

Solution
Let's first get a quick picture of the rectangle for reference purposes.
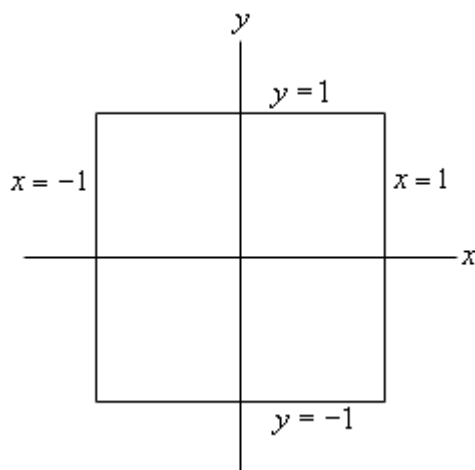
The boundary of this rectangle is given by the following conditions.

$$\text{right side :} \quad x = 1, \ -1 \le y \le 1$$
$$\text{left side :} \quad x = -1, \ -1 \le y \le 1$$
$$\text{upper side :} \quad y = 1, \ -1 \le x \le 1$$
$$\text{lower side :} \quad y = -1, \ -1 \le x \le 1$$

These will be important in the second step of our process.

We'll start this off by finding all the critical points that lie inside the given rectangle. To do this we'll need the two first order derivatives.

$$f_x = 2x - 4xy \qquad f_y = 8y - 2x^2$$

Note that since we aren't going to be classifying the critical points we don't need the second order derivatives. To find the critical points we will need to solve the system,

$$2x - 4xy = 0$$
$$8y - 2x^2 = 0$$

We can solve the second equation for *y* to get,

$$y = \frac{x^2}{4}$$

Plugging this into the first equation gives us,

$$2x - 4x\left(\frac{x^2}{4}\right) = 2x - x^3 = x\left(2 - x^2\right) = 0$$

This tells us that we must have $x = 0$ or $x = \pm\sqrt{2} = \pm 1.414\ldots$. Now, recall that we only want critical points in the region that we're given. That means that we only want critical points for which $-1 \le x \le 1$. The only value of *x* that will satisfy this is the first one so we can ignore the last two for this problem. Note however that a simple change to the boundary would include these two so don't forget to always check if the critical points are in the region (or on the boundary since that can also happen).

Plugging $x = 0$ into the equation for *y* gives us,

$$y = \frac{0^2}{4} = 0$$

The single critical point, in the region (and again, that's important), is $(0,0)$. We now need to get the value of the function at the critical point.

$$f(0,0) = 4$$

Eventually we will compare this to values of the function found in the next step and take the largest and smallest as the absolute extrema of the function in the rectangle.

Now we have reached the long part of this problem. We need to find the absolute extrema of the function along the boundary of the rectangle. What this means is that we're going to need to look at what the function is doing along each of the sides of the rectangle listed above.

Let's first take a look at the right side. As noted above the right side is defined by
$$x = 1, \ -1 \le y \le 1$$

Notice that along the right side we know that $x = 1$. Let's take advantage of this by defining a new function as follows,
$$g(y) = f(1, y) = 1^2 + 4y^2 - 2(1^2)y + 4 = 5 + 4y^2 - 2y$$

Now, finding the absolute extrema of $f(x, y)$ along the right side will be equivalent to finding the absolute extrema of $g(y)$ in the range $-1 \le y \le 1$. Hopefully you recall how to do this from Calculus I. We find the critical points of $g(y)$ in the range $-1 \le y \le 1$ and then evaluate $g(y)$ at the critical points and the end points of the range of y's.

Let's do that for this problem.
$$g'(y) = 8y - 2 \qquad \Rightarrow \qquad y = \frac{1}{4}$$

This is in the range and so we will need the following function evaluations.
$$g(-1) = 11 \qquad\qquad g(1) = 7 \qquad\qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

Notice that, using the definition of $g(y)$ these are also function values for $f(x, y)$.
$$g(-1) = f(1, -1) = 11$$
$$g(1) = f(1, 1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(1, \frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now do the left side of the rectangle which is defined by,
$$x = -1, \ -1 \le y \le 1$$

Again, we'll define a new function as follows,
$$g(y) = f(-1, y) = (-1)^2 + 4y^2 - 2(-1)^2 y + 4 = 5 + 4y^2 - 2y$$

Notice however that, for this boundary, this is the same function as we looked at for the right side. This will not always happen, but since it has let's take advantage of the fact that we've already done the work for this function. We know that the critical point is $y = \frac{1}{4}$ and we know that the function value at the critical point and the end points are,

$$g(-1) = 11 \qquad\qquad g(1) = 7 \qquad\qquad g\left(\frac{1}{4}\right) = \frac{19}{4} = 4.75$$

The only real difference here is that these will correspond to values of $f(x, y)$ at different points than for the right side. In this case these will correspond to the following function values for $f(x, y)$.

$$g(-1) = f(-1, -1) = 11$$
$$g(1) = f(-1, 1) = 7$$
$$g\left(\frac{1}{4}\right) = f\left(-1, \frac{1}{4}\right) = \frac{19}{4} = 4.75$$

We can now look at the upper side defined by,
$$y = 1, \ -1 \le x \le 1$$

We'll again define a new function except this time it will be a function of *x*.
$$h(x) = f(x, 1) = x^2 + 4(1^2) - 2x^2(1) + 4 = 8 - x^2$$

We need to find the absolute extrema of $h(x)$ on the range $-1 \le x \le 1$. First find the critical point(s).
$$h'(x) = -2x \qquad\qquad \Rightarrow \qquad\qquad x = 0$$

The value of this function at the critical point and the end points is,
$$h(-1) = 7 \qquad\qquad h(1) = 7 \qquad\qquad h(0) = 8$$
and these in turn correspond to the following function values for $f(x, y)$
$$h(-1) = f(-1, 1) = 7$$
$$h(1) = f(1, 1) = 7$$
$$h(0) = f(0, 1) = 8$$

Note that there are several "repeats" here. The first two function values have already been computed when we looked at the right and left side. This will often happen.

Finally, we need to take care of the lower side. This side is defined by,
$$y = -1, \ -1 \le x \le 1$$

The new function we'll define in this case is,
$$h(x) = f(x, -1) = x^2 + 4(-1)^2 - 2x^2(-1) + 4 = 8 + 3x^2$$

The critical point for this function is,
$$h'(x) = 6x \qquad\qquad \Rightarrow \qquad\qquad x = 0$$

The function values at the critical point and the endpoint are,
$$h(-1) = 11 \qquad\qquad h(1) = 11 \qquad\qquad h(0) = 8$$
and the corresponding values for $f(x, y)$ are,
$$h(-1) = f(-1, -1) = 11$$
$$h(1) = f(1, -1) = 11$$
$$h(0) = f(0, -1) = 8$$

The final step to this (long...) process is to collect up all the function values for $f(x, y)$ that we've computed in this problem. Here they are,
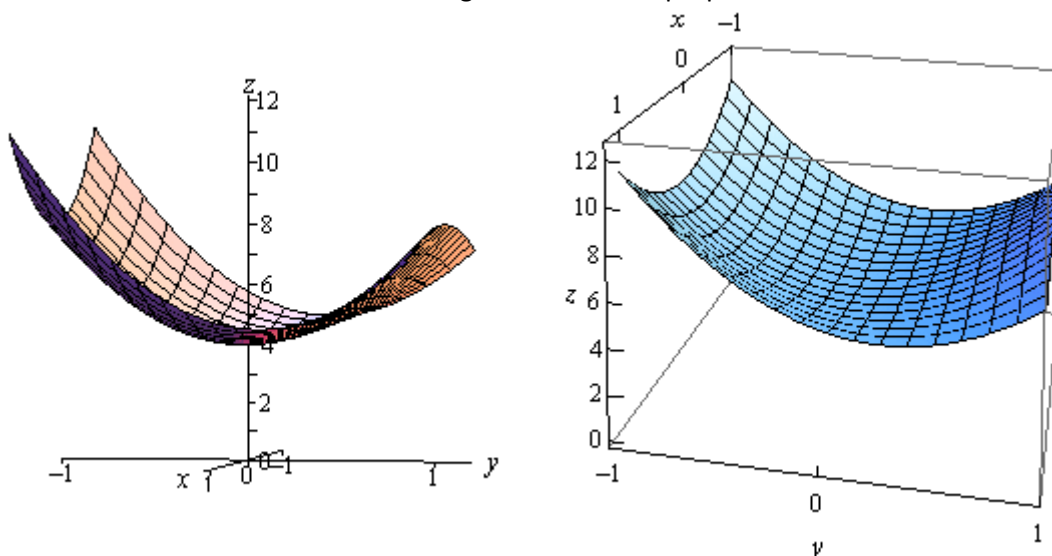
$$f(0, 0) = 4 \qquad\qquad f(1, -1) = 11 \qquad\qquad f(1, 1) = 7$$

$$f\left(1, \frac{1}{4}\right) = 4.75 \qquad\qquad f(-1, 1) = 7 \qquad\qquad f(-1, -1) = 11$$

$$f\left(-1, \frac{1}{4}\right) = 4.75 \qquad\qquad f(0, 1) = 8 \qquad\qquad f(0, -1) = 8$$

The absolute minimum is at $(0, 0)$ since gives the smallest function value and the absolute maximum occurs at $(1, -1)$ and $(-1, -1)$ since these two points give the largest function value.

Here is a sketch of the function on the rectangle for reference purposes.

As this example has shown these can be very long problems on occasion. Let's take a look at an easier, well shorter anyway, problem with a different kind of boundary.

Example 2 Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \leq 16$

Solution
First note that a disk of radius 4 is given by the inequality in the problem statement. The "less than" inequality is included to get the interior of the disk and the equal sign is included to get the boundary. Of course, this also means that the boundary of the disk is a circle of radius 4.

Let's first find the critical points of the function that lies inside the disk. This will require the following two first order partial derivatives.
$$f_x = 4x \qquad f_y = -2y + 6$$

To find the critical points we'll need to solve the following system.
$$4x = 0$$
$$-2y + 6 = 0$$

This is actually a fairly simple system to solve however. The first equation tells us that $x = 0$ and the second tells us that $y = 3$. So, the only critical point for this function is $(0,3)$ and this is inside the disk of radius 4. The function value at this critical point is,
$$f(0,3) = 9$$

Now we need to look at the boundary. This one will be somewhat different from the previous example. In this case we don't have fixed values of x and y on the boundary. Instead we have,
$$x^2 + y^2 = 16$$

We can solve this for x^2 and plug this into the x^2 in $f(x,y)$ to get a function of y as follows.
$$x^2 = 16 - y^2$$
$$g(y) = 2(16 - y^2) - y^2 + 6y = 32 - 3y^2 + 6y$$

We will need to find the absolute extrema of this function on the range $-4 \leq y \leq 4$ (this is the range of y's for the disk....). We'll first need the critical points of this function.
$$g'(y) = -6y + 6 \qquad \Rightarrow \qquad y = 1$$

The value of this function at the critical point and the endpoints are,
$$g(-4) = -40 \qquad\qquad g(4) = 8 \qquad\qquad g(1) = 35$$

Unlike the first example we will still need to find the values of x that correspond to these. We can do this by plugging the value of y into our equation for the circle and solving for x.

$$y = -4 : \qquad x^2 = 16 - 16 = 0 \qquad \Rightarrow \qquad x = 0$$

$$y = 4 \quad : \qquad x^2 = 16 - 16 = 0 \qquad \Rightarrow \qquad x = 0$$

$$y = 1 \quad : \qquad x^2 = 16 - 1 = 15 \Rightarrow \qquad x = \pm\sqrt{15} = \pm 3.87$$

The function values for $g(y)$ then correspond to the following function values for $f(x,y)$.

$$g(-4) = -40 \quad \Rightarrow \qquad f(0,-4) = -40$$

$$g(4) = 8 \qquad \Rightarrow \qquad f(0,4) = 8$$

$$g(1) = 35 \qquad \Rightarrow \qquad f\left(-\sqrt{15},1\right) = 35 \ \text{ and } \ f\left(\sqrt{15},1\right) = 35$$

Note that the third one actually corresponded to two different values for $f(x,y)$ since that y also produced two different values of x.

So, comparing these values to the value of the function at the critical point of $f(x,y)$ that we found earlier we can see that the absolute minimum occurs at $(0,-4)$ while the absolute maximum occurs twice at $\left(-\sqrt{15},1\right)$ and $\left(\sqrt{15},1\right)$.

Here is a sketch of the region for reference purposes.
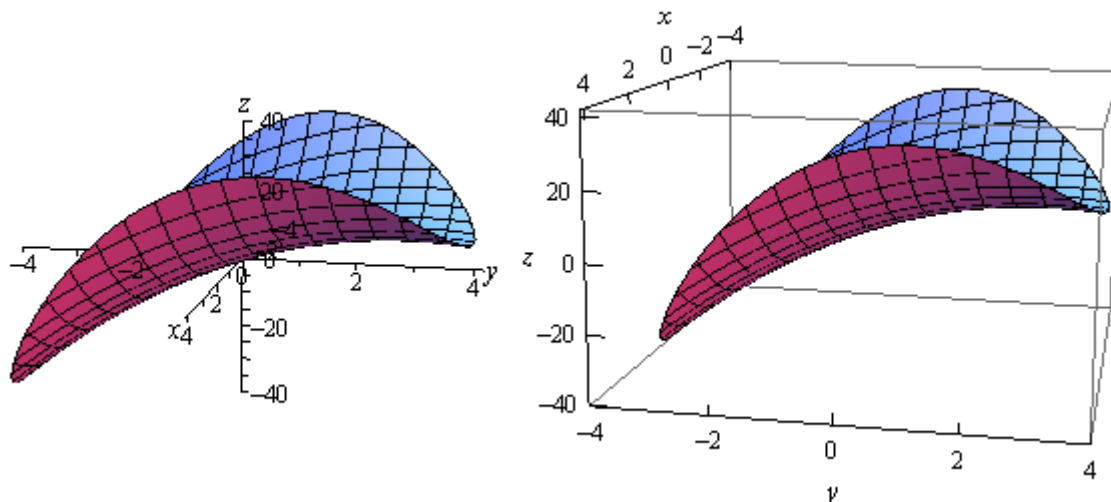
In both of these examples one of the absolute extrema actually occurred at more than one place. Sometimes this will happen and sometimes it won't so don't read too much into the fact that it happened in both examples given here.

Also note that, as we've seen, absolute extrema will often occur on the boundaries of these regions, although they don't have to occur at the boundaries. Had we given much more complicated examples with multiple critical points we would have seen examples where the absolute extrema occurred interior to the region and not on the boundary.

Section 3-5 : Lagrange Multipliers

In the previous section we optimized (*i.e.* found the absolute extrema) a function on a region that contained its boundary. Finding potential optimal points in the interior of the region isn't too bad in general, all that we needed to do was find the critical points and plug them into the function. However, as we saw in the examples finding potential optimal points on the boundary was often a fairly long and messy process.

In this section we are going to take a look at another way of optimizing a function subject to given constraint(s). The constraint(s) may be the equation(s) that describe the boundary of a region although in this section we won't concentrate on those types of problems since this method just requires a general constraint and doesn't really care where the constraint came from.

So, let's get things set up. We want to optimize (*i.e.* find the minimum and maximum value of) a function, $f(x, y, z)$, subject to the constraint $g(x, y, z) = k$. Again, the constraint may be the equation that describes the boundary of a region or it may not be. The process is actually fairly simple, although the work can still be a little overwhelming at times.

Method of Lagrange Multipliers

1. Solve the following system of equations.
$$\nabla f(x, y, z) = \lambda \, \nabla g(x, y, z)$$
$$g(x, y, z) = k$$

2. Plug in all solutions, (x, y, z), from the first step into $f(x, y, z)$ and identify the minimum and maximum values, provided they exist and $\nabla g \neq \vec{0}$ at the point.

The constant, λ, is called the **Lagrange Multiplier**.

Notice that the system of equations from the method actually has four equations, we just wrote the system in a simpler form. To see this let's take the first equation and put in the definition of the gradient vector to see what we get.
$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

In order for these two vectors to be equal the individual components must also be equal. So, we actually have three equations here.
$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

These three equations along with the constraint, $g(x, y, z) = c$, give four equations with four unknowns *x*, *y*, *z*, and λ.

Note as well that if we only have functions of two variables then we won't have the third component of the gradient and so will only have three equations in three unknowns *x*, *y*, and λ.

As a final note we also need to be careful with the fact that in some cases minimums and maximums won't exist even though the method will seem to imply that they do. In every problem we'll need to make sure that minimums and maximums will exist before we start the problem.

To see a physical justification for the formulas above. Let's consider the minimum and maximum value of $f(x,y) = 8x^2 - 2y$ subject to the constraint $x^2 + y^2 = 1$. In the practice problems for this section (problem #2 to be exact) we will show that minimum value of $f(x, y)$ is -2 which occurs at $(0,1)$ and the maximum value of $f(x, y)$ is 8.125 which occurs at $\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$ and $\left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$.

Here is a sketch of the constraint as well as $f(x.y) = k$ for various values of k.
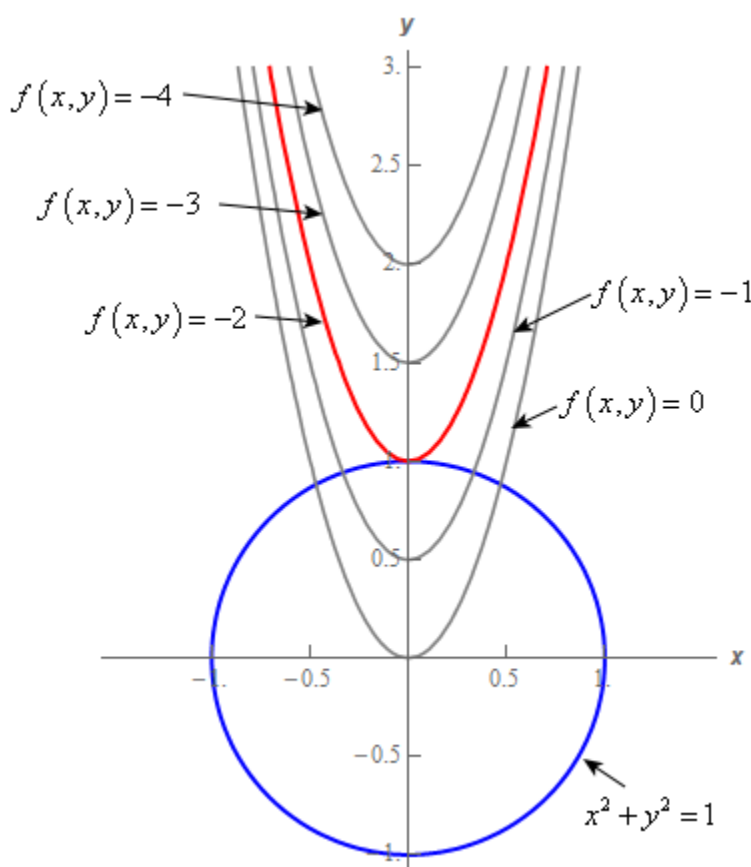
First remember that solutions to the system must be somewhere on the graph of the constraint, $x^2 + y^2 = 1$ in this case. Because we are looking for the minimum/maximum value of $f(x, y)$ this, in turn, means that the location of the minimum/maximum value of $f(x, y)$, i.e. the point (x, y), must occur where the graph of $f(x, y) = k$ intersects the graph of the constraint when k is either the minimum or maximum value of $f(x, y)$.

Now, we can see that the graph of $f(x,y)=-2$, *i.e.* the graph of the minimum value of $f(x,y)$, just touches the graph of the constraint at $(0,1)$. In fact, the two graphs at that point are tangent.

If the two graphs are tangent at that point then their normal vectors must be parallel, *i.e.* the two normal vectors must be scalar multiples of each other. Mathematically, this means,

$$\nabla f(x,y,z) = \lambda \ \nabla g(x,y,z)$$

for some scalar λ and this is exactly the first equation in the system we need to solve in the method.

Note as well that if *k* is smaller than the minimum value of $f(x,y)$ the graph of $f(x,y)=k$ doesn't intersect the graph of the constraint and so it is not possible for the function to take that value of *k* at a point that will satisfy the constraint.

Likewise, if *k* is larger than the minimum value of $f(x,y)$ the graph of $f(x,y)=k$ will intersect the graph of the constraint but the two graphs are not tangent at the intersection point(s). This means that the method will not find those intersection points as we solve the system of equations.

Next, the graph below shows a different set of values of *k*. In this case, the values of *k* include the maximum value of $f(x,y)$ as well as a few values on either side of the maximum value.

Again, we can see that the graph of $f(x,y)=8.125$ will just touch the graph of the constraint at two points. This is a good thing as we know the solution does say that it should occur at two points. Also note that at those points again the graph of $f(x,y)=8.125$ and the constraint are tangent and so, just as with the minimum values, the normal vectors must be parallel at these points.

Likewise, for value of *k* greater than 8.125 the graph of $f(x,y)=k$ does not intersect the graph of the constraint and so it will not be possible for $f(x,y)$ to take on those larger values at points that are on the constraint.

Also, for values of *k* less than 8.125 the graph of $f(x,y)=k$ does intersect the graph of the constraint but will not be tangent at the intersection points and so again the method will not produce these intersection points as we solve the system of equations.
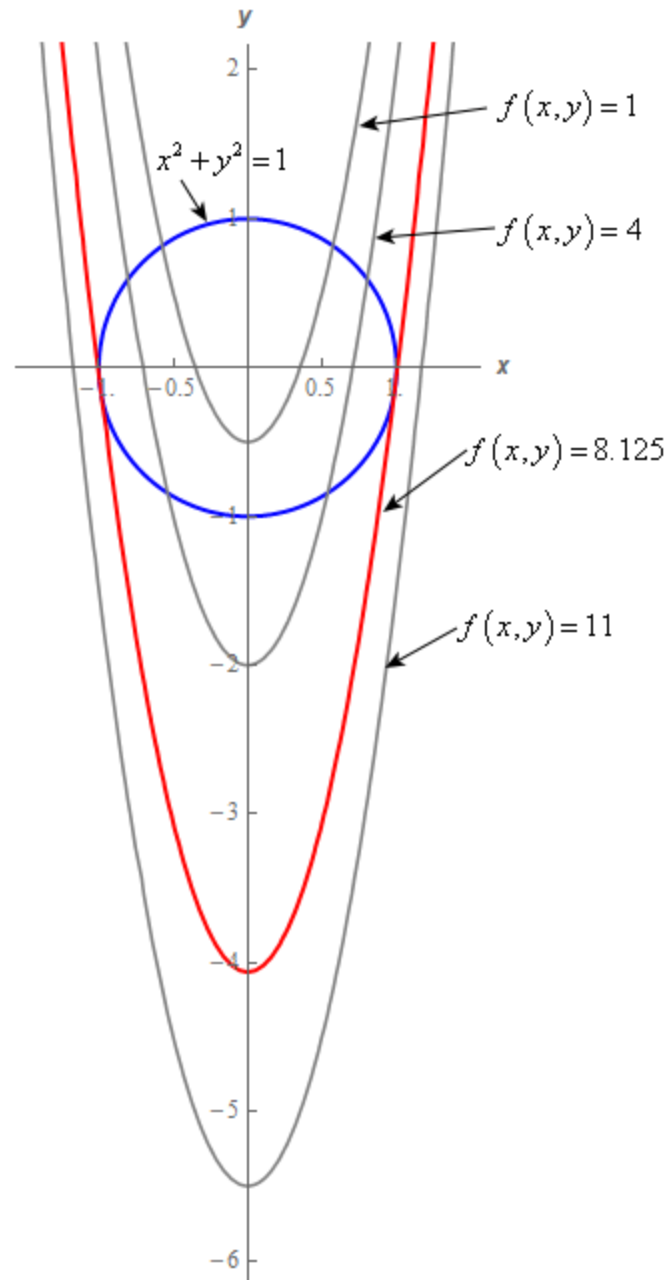
So, with these graphs we've seen that the minimum/maximum values of $f(x, y)$ will come where the graph of $f(x, y) = k$ and the graph of the constraint are tangent and so their normal vectors are parallel. Also, because the point must occur on the constraint itself. In other words, the system of equations we need to solve to determine the minimum/maximum value of $f(x, y)$ are exactly those given in the above when we introduced the method.

Note that the physical justification above was done for a two dimensional system but the same justification can be done in higher dimensions. The difference is that in higher dimensions we won't be

working with curves. For example, in three dimensions we would be working with surfaces. However, the same ideas will still hold. At the points that give minimum and maximum value(s) of $f(x,y,z)$ the surfaces would be parallel and so the normal vectors would also be parallel.

Let's work a couple of examples.

Example 1 Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution
Before we start the process here note that we also saw a way to solve this kind of problem in Calculus I, except in those problems we required a condition that related one of the sides of the box to the other sides so that we could get down to a volume and surface area function that only involved two variables. We no longer need this condition for these problems.

Now, let's get on to solving the problem. We first need to identify the function that we're going to optimize as well as the constraint. Let's set the length of the box to be *x*, the width of the box to be *y* and the height of the box to be *z*. Let's also note that because we're dealing with the dimensions of a box it is safe to assume that *x*, *y*, and *z* are all positive quantities.

We want to find the largest volume and so the function that we want to optimize is given by,
$$f(x,y,z) = xyz$$

Next, we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides so the constraint is given by,

$$2xy + 2xz + 2yz = 64 \qquad \Rightarrow \qquad xy + xz + yz = 32$$

Note that we divided the constraint by 2 to simplify the equation a little. Also, we get the function $g(x,y,z)$ from this.

$$g(x,y,z) = xy + xz + yz$$

The function itself, $f(x,y,z) = xyz$ will clearly have neither minimums or maximums unless we put some restrictions on the variables. The only real restriction that we've got is that all the variables must be positive. This, of course, instantly means that the function does have a minimum, zero, even though this is a silly value as it also means we pretty much don't have a box. It does however mean that we know the minimum of $f(x,y,z)$ does exist.

So, let's now see if $f(x,y,z)$ will have a maximum. Clearly, hopefully, $f(x,y,z)$ will not have a maximum if all the variables are allowed to increase without bound. That however, can't happen because of the constraint,

$$xy + xz + yz = 32$$

Here we've got the sum of three positive numbers (remember that we x, y, and z are positive because we are working with a box) and the sum must equal 32. So, if one of the variables gets very large, say x, then because each of the products must be less than 32 both y and z must be very small to make sure the first two terms are less than 32. So, there is no way for all the variables to increase without bound and so it should make some sense that the function, $f(x,y,z) = xyz$, will have a maximum.

This is not an exact proof that $f(x,y,z)$ will have a maximum but it should help to visualize that $f(x,y,z)$ should have a maximum value as long as it is subject to the constraint.

Here are the four equations that we need to solve.

$$yz = \lambda(y+z) \qquad\qquad (f_x = \lambda g_x) \qquad\qquad (1)$$
$$xz = \lambda(x+z) \qquad\qquad (f_y = \lambda g_y) \qquad\qquad (2)$$
$$xy = \lambda(x+y) \qquad\qquad (f_z = \lambda g_z) \qquad\qquad (3)$$
$$xy + xz + yz = 32 \qquad\qquad (g(x,y,z) = 32) \qquad\qquad (4)$$

There are many ways to solve this system. We'll solve it in the following way. Let's multiply equation (1) by x, equation (2) by y and equation (3) by z. This gives,

$$xyz = \lambda x(y+z) \qquad\qquad (5)$$
$$xyz = \lambda y(x+z) \qquad\qquad (6)$$
$$xyz = \lambda z(x+y) \qquad\qquad (7)$$

Now notice that we can set equations (5) and (6) equal. Doing this gives,

$$\lambda x(y+z) = \lambda y(x+z)$$
$$\lambda(xy+xz) - \lambda(yx+yz) = 0$$
$$\lambda(xz - yz) = 0 \qquad \Rightarrow \qquad \lambda = 0 \quad \text{or} \quad xz = yz$$

This gave two possibilities. The first, $\lambda = 0$ is not possible since if this was the case equation (1) would reduce to

$$yz = 0 \qquad \Rightarrow \qquad y = 0 \text{ or } z = 0$$

Since we are talking about the dimensions of a box neither of these are possible so we can discount $\lambda = 0$. This leaves the second possibility.

$$xz = yz$$

Since we know that $z \neq 0$ (again since we are talking about the dimensions of a box) we can cancel the z from both sides. This gives,

$$x = y \qquad\qquad (8)$$

Next, let's set equations (6) and (7) equal. Doing this gives,

$$\lambda y (x+z) = \lambda z (x+y)$$

$$\lambda (yx + yz - zx - zy) = 0$$

$$\lambda (yx - zx) = 0 \qquad \Rightarrow \qquad \lambda = 0 \text{ or } yx = zx$$

As already discussed we know that $\lambda = 0$ won't work and so this leaves,

$$yx = zx$$

We can also say that $x \neq 0$ since we are dealing with the dimensions of a box so we must have,

$$z = y \tag{9}$$

Plugging equations (8) and (9) into equation (4) we get,

$$y^2 + y^2 + y^2 = 3y^2 = 32 \qquad\qquad y = \pm\sqrt{\frac{32}{3}} = \pm 3.266$$

However, we know that *y* must be positive since we are talking about the dimensions of a box. Therefore, the only solution that makes physical sense here is

$$x = y = z = 3.266$$

So, it looks like we've got a cube.

We should be a little careful here. Since we've only got one solution we might be tempted to assume that these are the dimensions that will give the largest volume. Anytime we get a single solution we really need to verify that it is a maximum (or minimum if that is what we are looking for).

This is actually pretty simple to do. First, let's note that the volume at our solution above is,

$$V = f\left(\sqrt{\frac{32}{3}}, \sqrt{\frac{32}{3}}, \sqrt{\frac{32}{3}} \right) = \left(\sqrt{\frac{32}{3}} \right)^3 = 34.8376$$

Now, we know that a maximum of $f(x,y,z)$ will exist ("proved" that earlier in the solution) and so to verify that that this really is a maximum all we need to do if find another set of dimensions that satisfy our constraint and check the volume. If the volume of this new set of dimensions is smaller that the volume above then we know that our solution does give a maximum.

If, on the other hand, the new set of dimensions give a larger volume we have a problem. We only have a single solution and we know that a maximum exists and the method should generate that maximum. So, in this case, the likely issue is that we will have made a mistake somewhere and we'll need to go back and find it.

So, let's find a new set of dimensions for the box. The only thing we need to worry about is that they will satisfy the constraint. Outside of that there aren't other constraints on the size of the dimensions. So, we can freely pick two values and then use the constraint to determine the third value.

Let's choose $x = y = 1$. No reason for these values other than they are "easy" to work with. Plugging these into the constraint gives,

$$1 + z + z = 32 \quad \rightarrow \quad 2z = 31 \quad \rightarrow \quad z = \frac{31}{2}$$

So, this is a set of dimensions that satisfy the constraint and the volume for this set of dimensions is,

$$V = f\left(1, 1, \frac{31}{2}\right) = \frac{31}{2} = 15.5 < 34.8376$$

So, the new dimensions give a smaller volume and so our solution above is, in fact, the dimensions that will give a maximum volume of the box are $x = y = z = 3.266$

Notice that we never actually found values for λ in the above example. This is fairly standard for these kinds of problems. The value of λ isn't really important to determining if the point is a maximum or a minimum so often we will not bother with finding a value for it. On occasion we will need its value to help solve the system, but even in those cases we won't use it past finding the point.

Example 2 Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Solution
This one is going to be a little easier than the previous one since it only has two variables. Also, note that it's clear from the constraint that region of possible solutions lies on a disk of radius $\sqrt{136}$ which is a closed and bounded region, $-\sqrt{136} \le x, y \le \sqrt{136}$, and hence by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system that we need to solve.
$$5 = 2\lambda x$$
$$-3 = 2\lambda y$$
$$x^2 + y^2 = 136$$

Notice that, as with the last example, we can't have $\lambda = 0$ since that would not satisfy the first two equations. So, since we know that $\lambda \ne 0$ we can solve the first two equations for x and y respectively. This gives,

$$x = \frac{5}{2\lambda} \qquad\qquad y = -\frac{3}{2\lambda}$$

Plugging these into the constraint gives,

$$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = \frac{17}{2\lambda^2} = 136$$

We can solve this for λ.

$$\lambda^2 = \frac{1}{16} \qquad \Rightarrow \qquad \lambda = \pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums.

If $\lambda = -\frac{1}{4}$ we get,

$$x = -10 \qquad\qquad\qquad y = 6$$

and if $\lambda = \frac{1}{4}$ we get,

$$x = 10 \qquad\qquad\qquad y = -6$$

To determine if we have maximums or minimums we just need to plug these into the function. Also recall from the discussion at the start of this solution that we know these will be the minimum and maximums because the Extreme Value Theorem tells us that minimums and maximums will exist for this problem.

Here are the minimum and maximum values of the function.

$$f(-10,6) = -68 \qquad \text{Minimum at } (-10,6)$$
$$f(10,-6) = 68 \qquad \text{Maximum at } (10,-6)$$

In the first two examples we've excluded $\lambda = 0$ either for physical reasons or because it wouldn't solve one or more of the equations. Do not always expect this to happen. Sometimes we will be able to automatically exclude a value of λ and sometimes we won't.

Let's take a look at another example.

Example 3 Find the maximum and minimum values of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

Solution
First note that our constraint is a sum of three positive or zero number and it must be 1. Therefore, it is clear that our solution will fall in the range $0 \leq x, y, z \leq 1$ and so the solution must lie in a closed and bounded region and so by the Extreme Value Theorem we know that a minimum and maximum value must exist.

Here is the system of equation that we need to solve.

$$yz = \lambda \qquad\qquad\qquad\qquad (10)$$
$$xz = \lambda \qquad\qquad\qquad\qquad (11)$$
$$xy = \lambda \qquad\qquad\qquad\qquad (12)$$

$$x + y + z = 1 \tag{13}$$

Let's start this solution process off by noticing that since the first three equations all have λ they are all equal. So, let's start off by setting equations (10) and (11) equal.

$$yz = xz \qquad \Rightarrow \qquad z(y - x) = 0 \qquad \Rightarrow \qquad z = 0 \ \text{ or } \ y = x$$

So, we've got two possibilities here. Let's start off with by assuming that $z = 0$. In this case we can see from either equation (10) or (11) that we must then have $\lambda = 0$. From equation (12) we see that this means that $xy = 0$. This in turn means that either $x = 0$ or $y = 0$.

So, we've got two possible cases to deal with there. In each case two of the variables must be zero. Once we know this we can plug into the constraint, equation (13), to find the remaining value.

$$z = 0, \ x = 0 : \qquad \Rightarrow \qquad y = 1$$
$$z = 0, \ y = 0 : \qquad \Rightarrow \qquad x = 1$$

So, we've got two possible solutions $(0, 1, 0)$ and $(1, 0, 0)$.

Now let's go back and take a look at the other possibility, $y = x$. We also have two possible cases to look at here as well.

This first case is $x = y = 0$. In this case we can see from the constraint that we must have $z = 1$ and so we now have a third solution $(0, 0, 1)$.

The second case is $x = y \neq 0$. Let's set equations (11) and (12) equal.

$$xz = xy \qquad \Rightarrow \qquad x(z - y) = 0 \qquad \Rightarrow \qquad x = 0 \ \text{ or } \ z = y$$

Now, we've already assumed that $x \neq 0$ and so the only possibility is that $z = y$. However, this also means that,

$$x = y = z$$

Using this in the constraint gives,

$$3x = 1 \qquad \Rightarrow \qquad x = \frac{1}{3}$$

So, the next solution is $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3} \right)$.

We got four solutions by setting the first two equations equal.

To completely finish this problem out we should probably set equations (10) and (12) equal as well as setting equations (11) and (12) equal to see what we get. Doing this gives,

$$yz = xy \qquad \Rightarrow \qquad y(z-x) = 0 \qquad \Rightarrow \qquad y = 0 \text{ or } z = x$$

$$xz = xy \qquad \Rightarrow \qquad x(z-y) = 0 \qquad \Rightarrow \qquad x = 0 \text{ or } z = y$$

Both of these are very similar to the first situation that we looked at and we'll leave it up to you to show that in each of these cases we arrive back at the four solutions that we already found.

So, we have four solutions that we need to check in the function to see whether we have minimums or maximums.

$$f(0,0,1) = 0 \quad f(0,1,0) = 0 \quad f(1,0,0) = 0 \quad \text{All Minimums}$$

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27} \qquad\qquad\qquad \text{Maximum}$$

So, in this case the maximum occurs only once while the minimum occurs three times.

Note as well that we never really used the assumption that $x, y, z \geq 0$ in the actual solution to the problem. We used it to make sure that we had a closed and bounded region to guarantee we would have absolute extrema. To see why this is important let's take a look at what might happen without this assumption. Without this assumption it wouldn't be too difficult to find points that give both larger and smaller values of the functions. For example.

$$x = -100, \ y = 100, \ z = 1 : \ -100 + 100 + 1 = 1 \quad f(-100, 100, 1) = -10000$$

$$x = -50, \ y = -50, \ z = 101 : \ -50 - 50 + 101 = 1 \quad f(-50, -50, 101) = 252500$$

With these examples you can clearly see that it's not too hard to find points that will give larger and smaller function values. However, all of these examples required negative values of *x*, *y* and/or *z* to make sure we satisfy the constraint. By eliminating these we will know that we've got minimum and maximum values by the Extreme Value Theorem.

Before we proceed we need to address a quick issue that the last example illustrates about the method of Lagrange Multipliers. We found the absolute minimum and maximum to the function. However, what we did not find is all the locations for the absolute minimum. For example, assuming $x, y, z \geq 0$, consider the following sets of points.

$$(0, y, z) \qquad \text{where } y + z = 1$$

$$(x, 0, z) \qquad \text{where } x + z = 1$$

$$(x, y, 0) \qquad \text{where } x + y = 1$$

Every point in this set of points will satisfy the constraint from the problem and in every case the function will evaluate to zero and so also give the absolute minimum.

So, what is going on? Recall from the previous section that we had to check both the critical points and the boundaries to make sure we had the absolute extrema. The same was true in Calculus I. We had to check both critical points and end points of the interval to make sure we had the absolute extrema.

It turns out that we really need to do the same thing here if we want to know that we've found all the locations of the absolute extrema. The method of Lagrange multipliers will find the absolute extrema, it just might not find all the locations of them as the method does not take the end points of variables ranges into account (note that we might luck into some of these points but we can't guarantee that).

So, after going through the Lagrange Multiplier method we should then ask what happens at the end points of our variable ranges. For the example that means looking at what happens if $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, and $z = 1$. In the first three cases we get the points listed above that do happen to also give the absolute minimum. For the later three cases we can see that if one of the variables are 1 the other two must be zero (to meet the constraint) and those were actually found in the example. Sometimes that will happen and sometimes it won't.

In the case of this example the end points of each of the variable ranges gave absolute extrema but there is no reason to expect that to happen every time. In Example 2 above, for example, the end points of the ranges for the variables do not give absolute extrema (we'll let you verify this).

The moral of this is that if we want to know that we have every location of the absolute extrema for a particular problem we should also check the end points of any variable ranges that we might have. If all we are interested in is the value of the absolute extrema then there is no reason to do this.

Okay, it's time to move on to a slightly different topic. To this point we've only looked at constraints that were equations. We can also have constraints that are inequalities. The process for these types of problems is nearly identical to what we've been doing in this section to this point. The main difference between the two types of problems is that we will also need to find all the critical points that satisfy the inequality in the constraint and check these in the function when we check the values we found using Lagrange Multipliers.

Let's work an example to see how these kinds of problems work.

Example 4 Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \le 4$.

Solution
Note that the constraint here is the inequality for the disk. Because this is a closed and bounded region the Extreme Value Theorem tells us that a minimum and maximum value must exist.

The first step is to find all the critical points that are in the disk (*i.e.* satisfy the constraint). This is easy enough to do for this problem. Here are the two first order partial derivatives.

$$f_x = 8x \qquad \Rightarrow \qquad 8x = 0 \qquad \Rightarrow \qquad x = 0$$
$$f_y = 20y \qquad \Rightarrow \qquad 20y = 0 \qquad \Rightarrow \qquad y = 0$$

So, the only critical point is $(0,0)$ and it does satisfy the inequality.

At this point we proceed with Lagrange Multipliers and we treat the constraint as an equality instead of the inequality. We only need to deal with the inequality when finding the critical points.

So, here is the system of equations that we need to solve.

$$8x = 2\lambda x$$
$$20y = 2\lambda y$$
$$x^2 + y^2 = 4$$

From the first equation we get,

$$2x(4 - \lambda) = 0 \qquad \Rightarrow \qquad x = 0 \quad \text{or} \quad \lambda = 4$$

If we have $x = 0$ then the constraint gives us $y = \pm 2$.

If we have $\lambda = 4$ the second equation gives us,

$$20y = 8y \qquad \Rightarrow \qquad y = 0$$

The constraint then tells us that $x = \pm 2$.

If we'd performed a similar analysis on the second equation we would arrive at the same points.

So, Lagrange Multipliers gives us four points to check : $(0, 2)$, $(0, -2)$, $(2, 0)$, and $(-2, 0)$.

To find the maximum and minimum we need to simply plug these four points along with the critical point in the function.

$$f(0, 0) = 0 \qquad\qquad\qquad \text{Minimum}$$
$$f(2, 0) = f(-2, 0) = 16$$
$$f(0, 2) = f(0, -2) = 40 \qquad \text{Maximum}$$

In this case, the minimum was interior to the disk and the maximum was on the boundary of the disk.

The final topic that we need to discuss in this section is what to do if we have more than one constraint. We will look only at two constraints, but we can naturally extend the work here to more than two constraints.

We want to optimize $f(x, y, z)$ subject to the constraints $g(x, y, z) = c$ and $h(x, y, z) = k$. The system that we need to solve in this case is,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$
$$g(x, y, z) = c$$
$$h(x, y, z) = k$$

So, in this case we get two Lagrange Multipliers. Also, note that the first equation really is three equations as we saw in the previous examples. Let's see an example of this kind of optimization problem.

Example 5 Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Solution
Verifying that we will have a minimum and maximum value here is a little trickier. Clearly, because of the second constraint we've got to have $-1 \le x, y \le 1$. With this in mind there must also be a set of limits on *z* in order to make sure that the first constraint is met. If one really wanted to determine that range you could find the minimum and maximum values of $2x - y$ subject to $x^2 + y^2 = 1$ and you could then use this to determine the minimum and maximum values of *z*. We won't do that here. The point is only to acknowledge that once again the possible solutions must lie in a closed and bounded region and so minimum and maximum values must exist by the <u>Extreme Value Theorem</u>.

Here is the system of equations that we need to solve.

$$0 = 2\lambda + 2\mu x \qquad \left(f_x = \lambda g_x + \mu h_x\right) \qquad\qquad (14)$$

$$4 = -\lambda + 2\mu y \qquad \left(f_y = \lambda g_y + \mu h_y\right) \qquad\qquad (15)$$

$$-2 = -\lambda \qquad \left(f_z = \lambda g_z + \mu h_z\right) \qquad\qquad (16)$$

$$2x - y - z = 2 \qquad\qquad (17)$$

$$x^2 + y^2 = 1 \qquad\qquad (18)$$

First, let's notice that from equation (16) we get $\lambda = 2$. Plugging this into equation (14) and equation (15) and solving for *x* and *y* respectively gives,

$$0 = 4 + 2\mu x \qquad \Rightarrow \qquad x = -\frac{2}{\mu}$$

$$4 = -2 + 2\mu y \qquad \Rightarrow \qquad y = \frac{3}{\mu}$$

Now, plug these into equation (18).

$$\frac{4}{\mu^2} + \frac{9}{\mu^2} = \frac{13}{\mu^2} = 1 \qquad \Rightarrow \qquad \mu = \pm\sqrt{13}$$

So, we have two cases to look at here. First, let's see what we get when $\mu = \sqrt{13}$. In this case we know that,

$$x = -\frac{2}{\sqrt{13}} \qquad\qquad y = \frac{3}{\sqrt{13}}$$

Plugging these into equation (17) gives,

$$-\frac{4}{\sqrt{13}} - \frac{3}{\sqrt{13}} - z = 2 \qquad \Rightarrow \qquad z = -2 - \frac{7}{\sqrt{13}}$$

So, we've got one solution.

Let's now see what we get if we take $\mu = -\sqrt{13}$. Here we have,

$$x = \frac{2}{\sqrt{13}} \qquad\qquad y = -\frac{3}{\sqrt{13}}$$

Plugging these into equation (17) gives,

$$\frac{4}{\sqrt{13}} + \frac{3}{\sqrt{13}} - z = 2 \qquad \Rightarrow \qquad z = -2 + \frac{7}{\sqrt{13}}$$

and there's a second solution.

Now all that we need to is check the two solutions in the function to see which is the maximum and which is the minimum.

$$f\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right) = 4 + \frac{26}{\sqrt{13}} = 11.2111$$

$$f\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right) = 4 - \frac{26}{\sqrt{13}} = -3.2111$$

So, we have a maximum at $\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right)$ and a minimum at $\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right)$.