A-Level Further Maths - Exam Style Questions

Section A:	Multiple Choice Questions	90 minutes in total
Section B:	Proof, Complex Numbers, Matrices	30 minutes per question
Section C:	Further Algebra and Functions	30 minutes per question
Section D:	Polar Coordinates and Further Vectors	30 minutes per question
Section E:	Further Calculus	30 minutes per question
Section F:	Mechanics	30 minutes per question
Section G:	Statistics	30 minutes per question

A1. The differential equation

$$4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 2^{-x}$$

can be classified as

I	first-order	**II**	second-order
III	linear	**IV**	homogeneous

① **I** only
② **I**, **III** and **IV** only
③ **II** only
④ **II** and **III** only [1 mark]

A2. The correct identity for cosh$(x + y)$, for all real x and y, is

① $\cosh(x + y) \equiv \sinh x \cosh x + \sinh y \cosh y$
② $\cosh(x + y) \equiv \sinh x \sinh y + \cosh x \cosh y$
③ $\cosh(x + y) \equiv \sinh x \cosh x - \sinh y \cosh y$
④ $\cosh(x + y) \equiv \sinh x \sinh y - \cosh x \cosh y$ [1 mark]

A3. The plane Π is such that any point on Π is equidistant from the points (2, -1, 3) and (8, 11, -5). The equation of Π in its simplest Cartesian form is

① $3x + 6y - 4z = 49$
② $3x + 6y - 4z = 98$
③ $x + 2y - z = 0$
④ $x + 2y - z = 98$ [1 mark]

A4. Matrices **A**, **B**, **C** have sizes 2 × 2, 2 × 3 and 3 × 2 respectively.

Which of the following matrix multiplications is defined?

① **CBA**
② **ACB**
③ **ABC**
④ none of the above [1 mark]

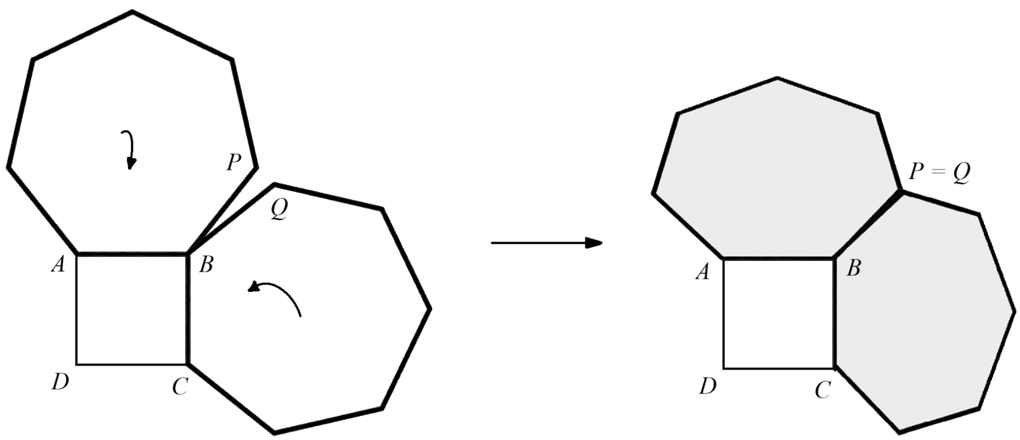
A5. $\mathbf{r}$ is a position vector in 3D space. $\mathbf{n}$ is a constant 3D unit vector, a and α are scalar constants and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are the orthonormal Cartesian basis vectors.

Which row in the table correctly gives a geometric description of the set of vectors $\mathbf{r}$ satisfying each equation?

[2 marks]

	equation	*description of geometric surface*
①	$(\mathbf{r} \bullet \mathbf{n})^2 = \mathbf{r} \bullet \mathbf{r} \cos^2 \alpha$	A pair of cones sharing their vertex at the origin, each opening at half-angle α to a common axis parallel to vector $\mathbf{n}$.
②	$(\mathbf{r} - \mathbf{n}) \bullet (\mathbf{r} - \mathbf{n}) = a$	A sphere with radius a, having centre point $\mathbf{n}$.
③	$(\mathbf{r} \times \mathbf{n}) \bullet (\mathbf{r} \times \mathbf{n}) = a$	A cylinder with radius a with an axis parallel to vector $\mathbf{n}$.
④	$(\mathbf{r} \bullet \mathbf{i})(\mathbf{r} \bullet \mathbf{j})(\mathbf{r} \bullet \mathbf{k}) = 0$	The union of three perpendicular planes, all intersecting at the unique point (1, 1, 1), with normal vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

A6. A fixed square $ABCD$ has side length 1 unit. Two regular heptagons, which each share sides AB and BC with the square, extend outwards from the square in the same flat plane. The heptagonal faces are then both folded up over the square edge by the same angle of rotation θ such that the two adjacent vertices of the heptagons, P and Q, touch each other exactly as shown.



By considering the foldings as rotations represented by suitable transformation matrices, or otherwise, find sec θ.

(The angle measures given are in radians.)

① $\sec\theta = \tan\dfrac{\pi}{7}$

② $\sec\theta = \sec\dfrac{\pi}{7}\csc\dfrac{\pi}{7}$

③ $\sec\theta = \csc\dfrac{\pi}{7}\cot\dfrac{\pi}{7}$

④ $\sec\theta = \tan\dfrac{2\pi}{7}$ [2 marks]

***A7.** Let

$$I(m, n) = \int_0^1 x^n (1-x)^m \, \mathrm{d}x \,, \qquad m,\ n \in \mathbb{N}.$$

By seeking a reduction formula for $I(m, n)$ in terms of $I(m - 1, n + 1)$ and further manipulating suitably, it can be deduced that

① $I(m, n) = \dfrac{m!\, n!}{(m+n)!}$

② $I(m, n) = \dfrac{m!\, n!}{(mn)!}$

③ $I(m, n) = \dfrac{m!\, n!}{(1+m+n)!}$

④ $I(m, n) = \dfrac{(mn)! - 1}{(1+mn)!}$ [4 marks]

***A8.** Define the function $f(x)$, for some real t,

$$f(x) = \begin{cases} 1 - |x - t|, & \text{if } |x - t| \leq 1 \\ 0, & \text{otherwise} \end{cases}, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}.$$

Define also the function $g(t) = \int_k^{k+8} f(x) \cos \pi x \, \mathrm{d}x$, where k is an **odd** integer.

Consider all local minimum points of $g(t)$ for which $g(t) < 0$. The values of $t = \alpha$ corresponding to each of these m distinct points are denoted $\alpha_1, \alpha_2, \ldots, \alpha_m$.

Given that $\sum_{i=1}^{m} \alpha_i = 45$, find the value of $k - \pi^2 \sum_{i=1}^{m} g(\alpha_i)$.

① -11

② 15

③ 25

④ 21 [4 marks]

A9. A rock falling vertically experiences an air resistance force of 12 N at an instant when its acceleration is 2.0 ms^{-2} downwards.

Taking gravitational field strength as 10 N kg^{-1}, find the mass of the rock.

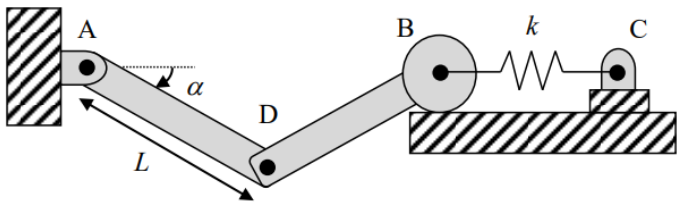
① 1.0 kg
② 1.2 kg
③ 1.5 kg
④ 2.0 kg [1 mark]

A10. A hollow spherical football of total mass 300 g and radius 11.0 cm is cut in half. Before the football is stitched back together, water of density 1.00 g cm^{-3} is poured into the football so that it becomes exactly half full of water.

When the football is at rest, by how much has the position of its centre of mass changed?

① 2.40 cm
② 4.97 cm
③ 5.36 cm
④ 6.19 cm [1 mark]

A11. A mechanism consisting of two uniform bars is held in place by a linear spring of spring constant k. Points A and C are fixed, while B is constrained to move horizontally. Bars AD and BD each have mass m and length L. The spring is unstretched when the angle α is 0°. Gravitational acceleration is g.



If the spring extension is x, then the total potential energy V of the system is

(Take the datum for $V(x)$ to be such that $V = 0$ when $x = 0$.)

① $V(x) = \frac{1}{2}kx^2 + mg\sqrt{Lx + \frac{1}{4}x^2}$

② $V(x) = \frac{1}{2}kx^2 + mg\sqrt{Lx - \frac{1}{4}x^2}$

③ $V(x) = \frac{1}{2}kx^2 - mg\sqrt{Lx + \frac{1}{4}x^2}$

④ $V(x) = \frac{1}{2}kx^2 - mg\sqrt{Lx - \frac{1}{4}x^2}$ [1 mark]

A12. A smooth rod of length L is rotating anti-clockwise in a horizontal plane about a fixed point O at constant angular velocity Ω. A particle P, which is free to slide along the rod, is attached to a light inextensible wire of length $\frac{L}{2}$ whose other end is at O. At time $t = 0$, the wire snaps and particle P is free to move along the rod.

If the distance OP is x at time t while P is on the rod, find an expression for $x(t)$.

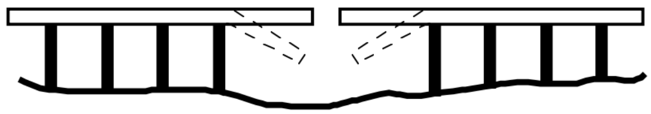
① $x(t) = \frac{L}{2}\cos \Omega t$

② $x(t) = \frac{L}{2}\cosh \Omega t$

③ $x(t) = \frac{L}{2}(1 + \Omega t)$

④ $x(t) = \frac{L}{2}(1 + \Omega^2 t^2)$ [1 mark]

A13. A construction company is required to build a long road bridge across a marsh. There are many short spans but there is to be one long span in the middle. The ground conditions are such that no supporting structure can be built.



The chief engineer is concerned that the individual arms of the incomplete long span may collapse under their own weight before they can be joined together, in the manner indicated by the broken lines above. The bridge will be constructed using girders made from steel, which has a density ρ of 7843 kg m^{-3} and can withstand a maximum stress σ of 400 MN m^{-2}.

A simple model of one half of the long span is constructed out of aluminium. The model half-span is 500 mm. The aluminium has a density ρ of 2720 kg m^{-3} and can withstand a maximum stress σ of 70 MN m^{-2}. Using a centrifuge to vary the effective acceleration due to gravity applied to the model, it is found that the scale model collapses when the acceleration reaches 400 ms^{-2}.

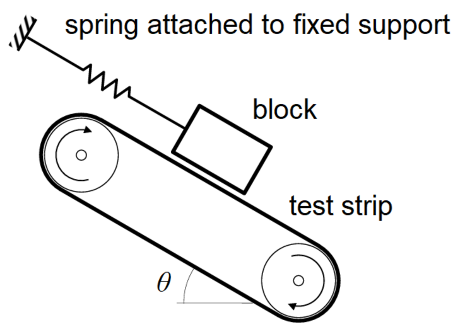
Using dimensional analysis, estimate the half-span of the largest steel bridge that can be constructed using this particular design.

(Take g = 9.81 ms^{-2}, and assume a constant power-law relationship between the dependent and independent variables.)

① 18.6 m
② 21.7 m
③ 40.4 m
④ 49.0 m [2 marks]

A14. An experimental setup used to determine the coefficient of friction μ between a block and a test strip of material is shown below. The test strip moves around the conveyor belt as shown, at an angle of θ to the horizontal. The block, which has mass m, is restrained by a linear elastic spring of stiffness k.

In the absence of friction, the spring would extend by an amount e_0. Therefore, its extension Δe beyond this can be used to determine the coefficient of friction.



The correct expression for the coefficient of friction μ is

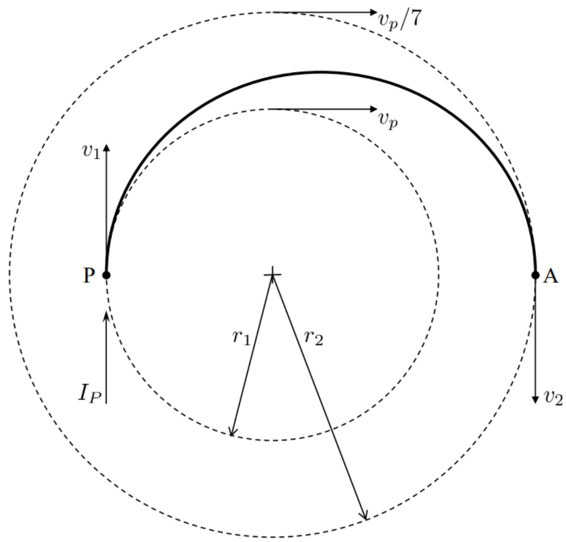
① $\dfrac{\Delta e}{e_0} \tan\theta$

② $\dfrac{\Delta e}{e_0} \cot\theta$

③ $\dfrac{e_0 - \Delta e}{e_0} \tan\theta$

④ $\dfrac{e_0 + \Delta e}{e_0} \cot\theta$ [2 marks]

A15. An artificial satellite of mass m orbiting Earth is transferred from one circular orbit at speed v_p to another at speed $v_p / 7$ by exerting a short-duration thrust of impulse I_P tangentially at P. The centre of the Earth is marked with a cross +.



The impulse I_P at P increases the speed of the satellite from v_p to v_1. The satellite then makes an elliptical path to the point A, arriving with speed v_2 such that $v_1 r_1 = v_2 r_2$. On arriving at point A, a second short-duration thrust of impulse I_A is required to enter a stable circular orbit of radius r_2.

At any time, the gravitational force F acting on the satellite is proportional to its mass, and inversely proportional to the square of the distance between the satellite and the centre of the Earth. The gravitational potential energy V of the satellite can be found by computing

$$V(r) = -\int_r^\infty F(x)\, dx.$$

Which statement is true?

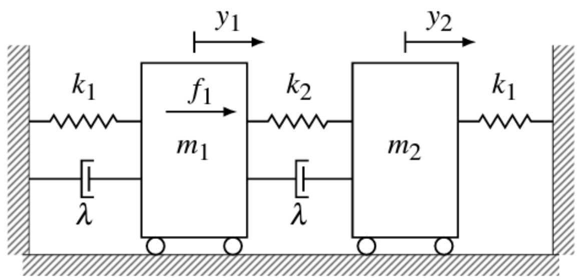
① The orbital radii are in the ratio $r_1 : r_2 = 1 : \sqrt{7}$.

② The satellite arrives at point A with speed $v_2 = \dfrac{v_p}{25}$.

③ The first impulse is $I_P = \dfrac{7}{5} m v_p$, in the direction of v_1.

④ The second impulse is $I_A = \dfrac{4}{35} m v_p$, in the direction of v_2. [4 marks]

***A16.** Consider the mass–spring–dashpot system below, consisting of two trolleys of mass m_1 and m_2, joined to each other and two fixed supports by linear viscoelastic elements as shown. The trolleys roll on a frictionless flat surface.



y_1 and y_2 are the displacements of m_1 and m_2 relative to their equilibrium positions. f_1 is a time-varying force, applied only to m_1, defined as positive in the direction of positive y_1. k_1 and k_2 are force constants for the springs. λ is the damping rate for both dashpots.

The system of differential equations for the motion of the trolleys can be summarised in the condensed form

$$\mathbf{M}\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}+\mathbf{K}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}f_1(t)\\ 0\end{bmatrix}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are matrices of constant coefficients relating to the masses, dashpots and springs respectively. These matrices are given by [4 marks]

	$\mathbf{M}$	$\mathbf{C}$	$\mathbf{K}$
①	$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}$	$\begin{bmatrix}2\lambda & -\lambda\\ -\lambda & \lambda\end{bmatrix}$	$\begin{bmatrix}k_1+k_2 & -k_2\\ -k_2 & k_1+k_2\end{bmatrix}$
②	$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}$	$\begin{bmatrix}2\lambda & -\lambda\\ -\lambda & \lambda\end{bmatrix}$	$\begin{bmatrix}k_1-k_2 & k_2\\ k_2 & k_1-k_2\end{bmatrix}$
③	$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}$	$\begin{bmatrix}-2\lambda & \lambda\\ \lambda & -\lambda\end{bmatrix}$	$\begin{bmatrix}k_1+k_2 & -k_2\\ -k_2 & k_1+k_2\end{bmatrix}$
④	$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}$	$\begin{bmatrix}-2\lambda & \lambda\\ \lambda & -\lambda\end{bmatrix}$	$\begin{bmatrix}k_1-k_2 & k_2\\ k_2 & k_1-k_2\end{bmatrix}$

A17. Which of these statistical distributions is symmetric about its mean?

1 Poisson distribution
2 Rectangular distribution
3 Chi-squared (χ^2) distribution
4 t-distribution

① **1** and **3** only
② **2** and **4** only
③ **1** and **2** only
④ **3** and **4** only [1 mark]

A18. The number of buses arriving at a particular bus stop is modelled by a Poisson distribution with a mean of 8 buses per hour.

The time interval **in minutes** between any two consecutive buses arriving is best modelled by

① an exponential distribution with parameter $\lambda = 7.5$.

② a Poisson distribution with parameter $\lambda = 7.5$.

③ an exponential distribution with parameter $\lambda = \frac{2}{15}$.

④ a Poisson distribution with parameter $\lambda = \frac{2}{15}$. [1 mark]

A19. Applying Yates' continuity correction to a Chi-squared (χ^2) test for association between discrete variables results in a(n)

① decrease to the power of the test.
② increase to the significance level.
③ increase to the value of the test statistic.
④ decrease to the p-value. [1 mark]

A20. When an aerosol canister explodes on level ground, a large number of small pieces of debris are launched as projectiles in random directions.

Analysis of the impacts showed that all debris landed with a speed of approximately $u = 28$ ms^{-1}, while the continuously-varying angles Θ to the horizontal are assumed to be uniformly distributed between 0° and 90°.

The horizontal range of a projectile is given by $X = \dfrac{u^2 \sin 2\Theta}{g}$.

What is the expected range of any given piece of debris? (Use $g = 9.8$ ms^{-1}.)

① 51 m
② 63 m
③ 80 m
④ 100 m [1 mark]

A21. A team of researchers have collected a random sample of data on which they performed a statistical hypothesis test at a 5% significance level. The results of the test provided sufficient evidence to reject the null hypothesis. They publish their findings with a relevant conclusion within the context of their investigation.

One member of the team, Alex, notices that the p-value for the test was much lower than the significance level, at $p = 0.0045$. Alex decides to repeat the test, using the exact same dataset, this time using a significance level of $\alpha = 0.5\%$. As expected, the outcome of the test is again a rejection of the null hypothesis.

Alex asks the team to update the research findings with the new significance level, claiming that "we now have stronger evidence to validate the alternative hypothesis".

Does Alex's test support his conclusion? Why or why not?

① **Alex's claim is valid** because his test was done on the exact same data as the team's, so the team could have originally chosen $\alpha = 0.5\%$ and reached the same conclusion.

② **Alex's claim is valid but** should not be published because it shows that the team was biassed to reject the null hypothesis from the beginning.

③ **Alex's claim may not be valid** because he cannot be sure that if he obtained a new set of data instead of re-using the previous data, the *p*-value would still be less than 0.5%.

④ **Alex's claim is invalid** because he has created his own conclusion to fit the known (no longer random) data, which he could not have done without the first test. [2 marks]

A22. The four entries of a 2 × 2 random matrix $\mathbf{Q}$ are independently and uniformly distributed on the continuous interval between 0 and 1. Let Λ be the sum of the eigenvalues of $\mathbf{Q}$. The mean μ and variance σ^2 of Λ is

① $\mu = 1/2$, $\sigma^2 = 1/6$

② $\mu = 1/2$, $\sigma^2 = 1/12$

③ $\mu = 1$, $\sigma^2 = 1/6$

④ $\mu = 1$, $\sigma^2 = 1/12$ [2 marks]

A23. A box contains n cards. The distinct integers from 1 to n are written on each card.

Two cards are drawn from the box randomly without replacement.

Let X be a discrete random variable representing the product of the values on the two chosen cards.

Find an expression for $\mathrm{E}[X]$.

① $\dfrac{n(n+1)}{3}$

② $\dfrac{(n+1)^2}{4}$

③ $\dfrac{n(n^2-1)(3n+2)}{24}$

④ $\dfrac{(n+1)(3n+2)}{12}$ [4 marks]

***A24.** Parallel light rays illuminate a cube of side length 1 cm from directly above, casting a shadow below the cube on an infinite horizontal plane.

The cube is tossed randomly into the air in different orientations above the plane and the area of the shadow is the continuous random variable A cm^2.

Find the value of $\mathrm{E}[A]$.

① $\dfrac{3}{2}$

② $\dfrac{4}{3}$

③ $\dfrac{\sqrt{6}}{2}$

④ $\dfrac{3\sqrt{3}}{4}$ [4 marks]

B1.

a. Define the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} 2 & a \\ 0 & 1 \end{bmatrix}$$

i) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. [4 marks]

ii) By considering the diagonalised form of $\mathbf{A}$, show that

$$\mathbf{A}^n = \begin{bmatrix} 2^n & (2^n - 1)a \\ 0 & 1 \end{bmatrix}, \quad n \geq 1,\ n \in \mathbb{N}.$$

[4 marks]

iii) Prove the result in part a.ii) by induction. [4 marks]

b. Using induction, prove that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^x \sin \sqrt{3}x\right) = 2^n e^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right), \quad n \geq 1,\ n \in \mathbb{N}.$$

[8 marks]

B2.

a. Define a set S of complex numbers z as

$$S = \left\{ z \in \mathbb{C} : 2 < |z - 2 + i| \leq 3 \right\} \cap \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \arg\left[(z - 2 + i)^2\right] \leq \frac{2\pi}{3} \right\}$$

i) Sketch the locus of $z \in S$ on an Argand diagram. [5 marks]

ii) Find the smallest possible value of $\operatorname{Re} z$ - $\operatorname{Im} z$ for all $z \in S$.

Give your answer in exact form. [2 marks]

b. Prove that for all positive integer n and nonzero real θ,

$$\sum_{r=1}^{n} \cos(r\theta) = \frac{\cos(\frac{1}{2}(n+1)\theta)\sin(\frac{1}{2}n\theta)}{\sin(\frac{1}{2}\theta)}$$

[6 marks]

c. Show that the solutions $z \in \mathbb{C}$ to the equation

$$\sin z = k\,\mathrm{i},$$

where k is real, are of the form

$$z = n\pi + i(-1)^n \sinh^{-1} k, \qquad n \in \mathbb{Z}.$$

[7 marks]

B3. A 3×3 matrix $\mathbf{M}$ represents a transformation in 3D space. $\mathbf{M}$ has an eigenvalue 2 with corresponding eigenvector $\mathbf{i} - 2\mathbf{j}$. $\mathbf{M}$ also has a repeated nonzero real eigenvalue λ, with corresponding independent unit eigenvectors $\mathbf{u}$ and $\mathbf{v}$.

a. $\mathbf{M}$ acts on the following geometric objects, with given vector equations

Plane Π: $\mathbf{r} \bullet (\mathbf{u} \times \mathbf{v}) = 0$;
Line L: $\mathbf{r} \times (\mathbf{i} - 2\mathbf{j}) = \mathbf{0}$.

Explain why:

i) the plane Π is invariant under $\mathbf{M}$ [2 marks]

ii) if $\lambda = 1$, then Π is a plane of invariant points under $\mathbf{M}$ [2 marks]

iii) the line L is **not** a line of invariant points under $\mathbf{M}$ [2 marks]

iv) if $\lambda = 2$, then $\mathbf{M}$ represents an enlargement about the origin [2 marks]

b. Eigenvectors $\mathbf{u}$ and $\mathbf{v}$ of $\mathbf{M}$ have arbitrary real components of the form

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^\mathrm{T} \quad \text{and} \quad \mathbf{v} = [v_1 \quad v_2 \quad v_3]^\mathrm{T}.$$

i) Explain how the diagonalised form $\mathbf{M} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$, where $\mathbf{S}$ is a nonsingular matrix and $\mathbf{\Lambda}$ is a diagonal matrix, implies that the determinant of $\mathbf{M}$ does not depend on the components of its eigenvectors. [3 marks]

ii) Hence show that $|\mathbf{M}| = 2\lambda^2$. [2 marks]

c. It is given that $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{k}$ and $\lambda = -1$.

i) Show that $\mathbf{M}$ satisfies the equation $\mathbf{M}^3 = 3\mathbf{M} + 2\mathbf{I}$. [6 marks]

ii) Find the eigenvalues of $(\mathbf{M}^3 - 2\mathbf{I})^{-1}$. [2 marks]

B4. A complex-valued nonsingular 3 × 3 matrix $\mathbf{Z}$ has the form

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}, \qquad z_{ij} \in \mathbb{C}.$$

Denote the eigenvalues of $\mathbf{Z}$ as λ_1, λ_2 and λ_3, any of which may be complex.

a. By considering the characteristic equation of $\mathbf{Z}$, prove that

$$\mathrm{tr}(\mathbf{Z}) = \lambda_1 + \lambda_2 + \lambda_3.$$ [3 marks]

b. Matrix $\mathbf{\Psi}$ is related to $\mathbf{Z}$ and its eigenvalues by $\lambda_1\lambda_2\lambda_3\,\mathbf{Z}^{-1} = \mathbf{\Psi}^{\mathrm{T}}$.

i) Explain why $\mathbf{\Psi}$ is the matrix of cofactors of $\mathbf{Z}$. [2 marks]

*ii) Prove that

$$\mathrm{tr}(\mathbf{\Psi}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3.$$ [4 marks]

*c. Define three regions D_i for $i \in \{1, 2, 3\}$ in the complex plane which satisfy

$$D_i: \qquad |z - z_{ii}| \le \sum_{j \ne i} |z_{ij}|.$$

By considering the components of the equation $\mathbf{Zu} = \lambda\mathbf{u}$, prove that every eigenvalue $\lambda \in \mathbb{C}$ of $\mathbf{Z}$, when plotted on an Argand diagram, lies within the union of all D_i. [7 marks]

d.

$$\mathbf{Z} = \begin{bmatrix} 400 & \sqrt{2}+i & \sqrt{2}-i \\ \sqrt{2}+i & 360 & 2\sqrt{3}-i \\ \sqrt{2}-i & 2\sqrt{3}-i & 480 \end{bmatrix}.$$

i) Estimate the median eigenvalue of $\mathbf{Z}$. [2 marks]

ii) Comment on the accuracy of estimating the eigenvector corresponding to the eigenvalue found in part d.i).

[2 marks]

C1. The polynomial $P(x)$ is defined for all real x,

$$P(x) = mx^3 - nx^2 + 7x - 1$$

where m and n are real constants with $m \neq 0$. The real non-zero roots of the equation $P(x) = 0$ are α, β and γ.

a. By considering $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$, prove that $n^2 \geq 21m$. [4 marks]

b. If $m = n = 1$, find the cubic polynomial $Q(x)$ where the coefficient of x^3 is 1 and the roots of the equation $Q(x) = 0$ are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

[3 marks]

c. The equation $P(x) = \frac{1}{x}$ has four real non-zero solutions δ, ε, ζ and η.

i) If $m = n = -1$, find the value of $(\delta + \varepsilon + \zeta)(\delta + \varepsilon + \eta)(\delta + \zeta + \eta)(\varepsilon + \zeta + \eta)$.

[5 marks]

ii) Find the values of m and n such that $\delta = \varepsilon = 1$ is a repeated root.

[3 marks]

iii) If $n = 0$, find in terms of m, the value of

$$(\delta - \varepsilon)^2 + (\varepsilon - \zeta)^2 + (\zeta - \eta)^2 + (\eta - \delta)^2 + (\delta - \zeta)^2 + (\varepsilon - \eta)^2$$

[5 marks]

C2.

a. Find the following limits in exact form, where $x \in \mathbb{R},\ n \in \mathbb{N}$.

Give a brief justification for each of your answers. It may be assumed that all given limits exist.

i) $$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$$ [3 marks]

ii) $$\lim_{x\to\infty}\frac{x+1}{x^2+6x}\exp\left[\frac{x^2}{1+x^2}(2+\ln x)\right]$$ [3 marks]

iii) $$\lim_{x\to\infty}\left(\sqrt{x^2+4x+100}-\sqrt{x^2-6x}\right)$$ [3 marks]

iv) $$\lim_{x\to\frac{\pi}{2}}\frac{\ln(x-\frac{\pi}{2})}{\tan x}$$ [3 marks]

v) $$\lim_{x\to 0}\frac{1}{x^4}\int_0^x \sin^3 t\ \mathrm{d}t$$ [3 marks]

*vi) $$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{3n}{3n+k}}$$ [3 marks]

b. Find the exact value of $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n!}$. [2 marks]

C3.

a. i) Solve the equation $\frac{x^2+5x+6}{1-x^2} = 1 - \frac{5(x+1)}{5x+6}$.

Give the value(s) of x in exact form. [3 marks]

ii) Sketch the graphs of $y = \frac{x^2+5x+6}{1-x^2}$ and $y = 1 - \frac{5(x+1)}{5x+6}$ on the same set of axes, identifying all asymptotes and axis intersection points.

[4 marks]

iii) Solve the inequality $\frac{x^2+5x+6}{1-x^2} \geq 1 - \frac{5(x+1)}{5x+6}$.

Give the range(s) of x in exact form. [3 marks]

iv) Without calculus, find the coordinates of the turning points of the graph of $y = \frac{x^2+5x+6}{1-x^2}$, and identify the nature of each turning point.

Give each coordinate in exact form. [4 marks]

b. i) Express $\frac{1}{r(r+2)}$ as the simplest sum of partial fractions. [1 mark]

ii) Hence, find the value of

$$\frac{1}{1\times 3} - \frac{1}{2\times 4} + \frac{1}{3\times 5} - \frac{1}{4\times 6} + \cdots$$

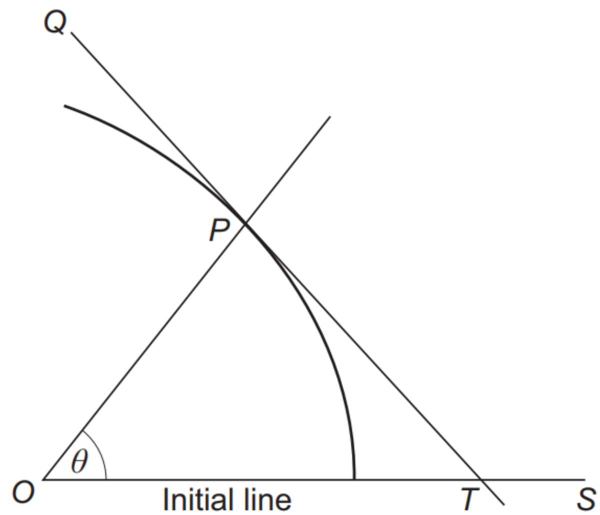
[3 marks]

c. Let $f(x) = \left|\frac{x^5 - x + 1}{(x^2+x+1)(x-1)^2}\right|$ for all valid real x.

Find the equations of the two oblique asymptotes to the graph of $y = f(x)$, indicating whether each corresponds to the case where x approaches either $+\infty$ or $-\infty$.

[2 marks]

D1. The diagram shows part of a spiral curve. The point P has polar coordinates (r, θ) where $0 \leq \theta \leq \frac{\pi}{2}$. The points T and S lie on the initial line and O is the pole. TPQ is the tangent to the curve at P.



a. Show that the gradient of TPQ is equal to

$$\frac{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

[4 marks]

b. The curve has polar equation $r = \mathrm{e}^{(\cot b)\,\theta}$ where b is a constant and $0 < b < \frac{\pi}{2}$.

Use the result of part a) to show that the angle between the line OP and the tangent TPQ does not depend on θ. [7 marks]

c. If instead $\frac{\pi}{2} < b < \pi$, the direction of the spiral is inwards for increasing θ.

i) Derive a formula for the arc length along the spiral between the points corresponding to $\theta = 0$ and $\theta = n\pi$ for any $n \in \mathbb{N}$. [8 marks]

ii) Evaluate the total arc length of the inward spiral as $n \to \infty$ for all $\theta \geq 0$. [1 mark]

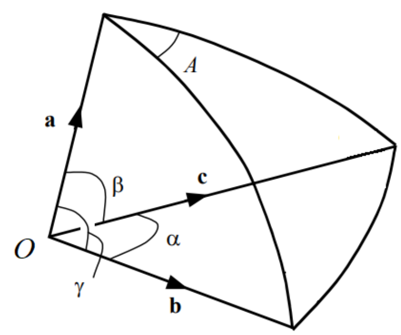
D2.

a. A and B are fixed points in 3D space. A third point P is located such that the position vectors between the points satisfy the following three conditions:

$$|\overrightarrow{AB}| = 2 \qquad \overrightarrow{AP} \cdot \overrightarrow{PB} = 0 \qquad \overrightarrow{AB} \cdot \overrightarrow{AP} \geq 2 + \sqrt{3}$$

By considering a surface of revolution about the axis AB, show that the surface area of the locus of all points P is $(2 - \sqrt{3})\pi$. [7 marks]

b. Consider three **unit** vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, which define three non-collinear points on a sphere of radius 1 unit, centred at the origin O, with angles α, β and γ as shown. The acute angle between the tangent lines at the tip of $\mathbf{a}$ parallel to $\mathbf{b}$ and $\mathbf{c}$ is A.



i) Explain why the vector $\mathbf{b} \times (\mathbf{a} \times \mathbf{c})$ can be written in the form $\lambda\mathbf{a} + \mu\mathbf{c}$, where λ and μ are scalar parameters. [3 marks]

ii) Using both of the following vector product identities:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \bullet \mathbf{z})\mathbf{y} - (\mathbf{x} \bullet \mathbf{y})\mathbf{z} \qquad \text{and} \qquad \mathbf{x} \bullet (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \bullet (\mathbf{z} \times \mathbf{x})$$

which are valid for **any** three vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, prove that the angles α, β, γ and A are related by

$$\cos A = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

[10 marks]

D3. The skew straight lines L_1 and L_2 have vector equations

$$\mathbf{r_1} = (-13\mathbf{j} + \mathbf{k}) + \lambda(-3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k})$$

$$\mathbf{r_2} = (5\mathbf{i} + 25\mathbf{j}) + \mu(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

respectively, where λ and μ are scalar parameters.

a. Find the coordinates of the points on each of the two lines such that the distance between them is the smallest.

[7 marks]

b. The segments of lines L_1 and L_2 for which $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$ represent two edges AB and CD of an irregular tetrahedron $ABCD$, with the points A, B, C and D corresponding to where $\lambda = 0$, $\lambda = 1$, $\mu = 0$ and $\mu = 1$, respectively.

i) Find the Cartesian equation of the plane containing the face ACD.

[5 marks]

ii) Find the acute angle between face ACD and edge AB.

Give your answer in degrees to 2 decimal places. [3 marks]

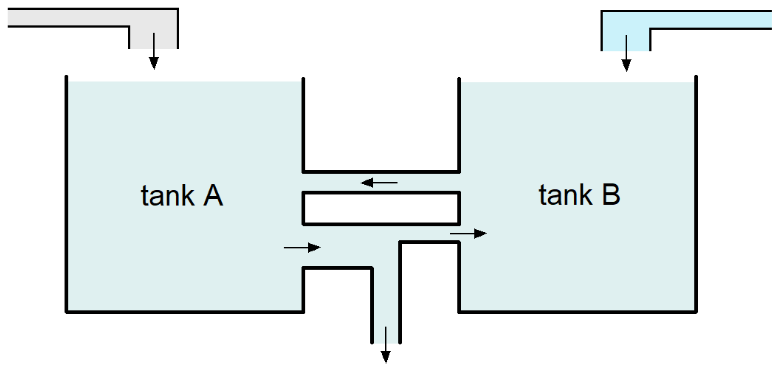
iii) Using a geometric argument, explain why the volume V enclosed by the tetrahedron $ABCD$ is given by

$$V = \frac{1}{6}\left|\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})\right|$$

[3 marks]

iv) Hence find the volume of the tetrahedron. [2 marks]

E1. Two interconnected tanks of water are being mixed.



Initially, tank A contains 50 litres of water and 25 grams of contaminant, while tank B contains 100 litres of water and 75 grams of contaminant. Water with a uniform concentration of 1.5 grams of contaminant per litre enters tank A at a constant rate of 7 litres per hour, and fresh (uncontaminated) water enters tank B at a constant rate of 3 litres per hour.

Water from tank B enters tank A through a pipe at 5 litres per hour. 12 litres per hour is removed from tank A of which 2 litres are pumped back into tank B while the remaining 10 litres is drained from the system via an outlet exit pipe.

a. Write down a system of coupled differential equations describing the changes in masses of contaminant x and y grams in tanks A and B respectively over time t. [4 marks]

b. Solve the system of equations to obtain functions for predicting the masses of contaminant in tanks A and B for any given time $t \geq 0$. [8 marks]

c. Using the "Improved Euler method" given in the Formula Booklet with a time step of 1 hour, approximate the mass of contaminant in tank B after two hours, and give the percentage error based on the model in part b). [5 marks]

d. Find the total mass of contaminant in the system as $t \to \infty$. [2 marks]

e. State one assumption made when forming the model. [1 mark]

E2. Let

$$I_n = \int \operatorname{sech}^n x \ \mathrm{d}x, \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N},\ n \geq 2.$$

a. Derive a reduction formula for I_n in terms of I_{n-2} and n, valid for the given range.

[8 marks]

b. Find the exact value of $\int_{-\infty}^{\infty} \operatorname{sech}^2 x \ \mathrm{d}x$, clearly showing the limiting process.

[3 marks]

c. Using a suitable hyperbolic substitution and the reduction formula derived in part a), or otherwise, show that

$$\int_0^{\infty} \frac{1}{\sqrt{(1+x^2)^7}} \ \mathrm{d}x = \frac{8}{15}$$

[9 marks]

E3.

a. For each of the following differential equations, write down the general form of the complementary function y_{CF} and particular integral y_{PI} components of $y(x)$.

You are not required to find the undetermined coefficients of y_{PI}.

i) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - y = \sinh x$ [2 marks]

*ii) $10\dfrac{\mathrm{d}^2y}{\mathrm{d}x} - 6\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 2\sin^2 x - 4\cos 3x \cos 5x$ [2 marks]

*iii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = e^{-x}(1+\sin x)$ [2 marks]

b. Consider the differential equation

$$y'' + y = \tan x, \qquad y(0) = y'(0) = 0, \qquad 0 \le x < \frac{\pi}{2}.$$

i) It is given that the Maclaurin series expansion up to order x^5 for tan x is

$$\tan x \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5, \qquad |x| < \frac{\pi}{2}.$$

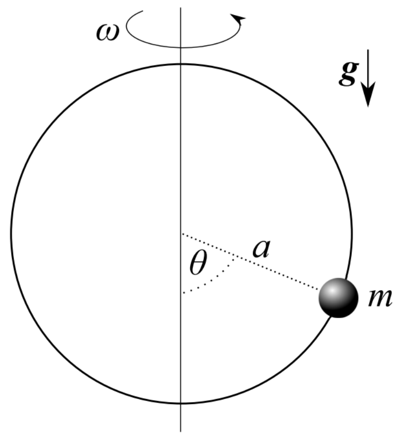
Find an approximate solution to the differential equation. [7 marks]

ii) Estimate the value of $y(\frac{\pi}{6})$ using the solution in part b.i). [1 mark]

iii) Verify that $y_{\text{PI}} = -\cos x \ln(\sec x + \tan x)$ is a valid solution. [4 marks]

iv) Using the solution in part b.iii), find the true value of $y(\frac{\pi}{6})$, and write down the percentage error in the approximation in part b.ii). [2 marks]

F1. A wire ring of radius *a* spins around a vertical diameter at fixed angular velocity ω. A bead, of mass m, is free to slide on the ring without friction under gravity g.



The position of the bead at any time t is defined by the angle θ, which is taken positive measured anticlockwise from the downward vertical.

*a. By applying Newton's Second Law to the bead, derive the equation of motion,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{a}\sin\theta - \omega^2 \sin\theta\cos\theta = 0.$$

[5 marks]

b. If θ is small, and ω is **smaller** than a critical value ($\omega < \omega_c$), find the approximate time period of oscillation for the motion of the bead. [3 marks]

c. The motion of the bead may be different if it starts near the top of the ring.

i) If $\pi - \theta$ is small and $\omega > \omega_c$, describe the motion of the bead. [4 marks]

ii) Describe the motion of the bead if it starts near the bottom of the ring while $\omega > \omega_c$. [3 marks]

d. It is now assumed that an air resistance force $\mathbf{F} = -c\mathbf{v}$ acts on the bead. c is a positive constant and $\mathbf{v}$ is the velocity of the bead **relative** to the ring.

If $\omega < \omega_c$, find ω in terms of c such that the low-amplitude oscillation of the bead around the equilibrium point $\theta = 0$ is critically damped, and describe the motion of the bead in this case. [5 marks]

F2. *In all parts of this question, g is acceleration due to gravity, assumed constant.*

a. A climber of mass m is attached to a rope which is attached firmly to a point B on a vertical rock face. When at a point A, a distance L above B, the climber falls.

The rope has modulus of elasticity λ and natural unstretched length L. The maximum allowable acceleration to ensure the safety of the climber during their fall is $25g$.

i) Show that the rope must be able stretch by at least $\frac{L}{6}$ without breaking. [5 marks]

ii) A different rope is found to break under a force equal to 25 times the weight of the climber with a final extension of 20% of its original length.

Determine whether this rope could be used by the climber. [3 marks]

b. Two balls are released from rest at the same time, with one positioned directly above the other. The upper ball has mass m while the lower ball has mass M, where $M >> m$ and there is a small gap between the balls at the time of release.

The lower ball is released from a height h. The collisions between the ground and the lower ball have coefficient of restitution e, while the collisions between the two balls are perfectly elastic. The balls only move in a vertical line.

Find the approximate maximum height reached by the upper ball in the subsequent motion in terms of e and h. [6 marks]

*c. Coffee beans are dropped from rest onto the scale pan of an electronic balance in a continuous stream from fixed height h and at a constant mass flow rate μ kilograms per second. The beans do not bounce.

The flow of beans is cut off at source when the balance reads the required mass.

Show that the weight of beans in the air at that instant will exactly compensate for the **false** reading of the balance caused by the change in momentum of the falling beans. [6 marks]

F3.

a. The finite region R is bounded by the circle and the line with respective equations $x^2 + y^2 = 16$ and $x = 2$. This region is fully revolved about the x-axis to form a solid spherical cap S.

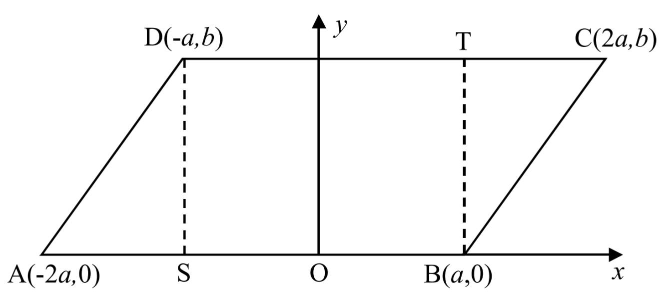
i) Find the x-coordinate of the centre of mass of S. [3 marks]

The solid S is carefully placed with a point on its curved surface in contact with a rough plane inclined at 30° to the horizontal and remains in equilibrium without sliding, toppling or turning.

ii) Find the angle between the plane face of S and the vertical. [4 marks]

iii) Find the range of the coefficient of friction μ between the plane and S in order for S to either slide or topple when placed, and determine which occurs first when μ is within this range. [2 marks]

b. A uniform lamina $ABCD$ is in the shape of a parallelogram and has mass per unit area σ, with vertices at the coordinates shown.



Triangles ASD and BCT are now folded over lines SD and BT respectively so that they lie flat on the rectangular region $SBTD$, and the new body is pivoted freely at the point $(\frac{3}{4}a, \frac{1}{3}b)$ and is in equilibrium with its plane vertical.

i) Find the magnitude of the couple that must be applied to the body for it to rest in equilibrium with the edge DT vertical. [4 marks]

ii) Face BCT is now partially unfolded so that it lies perpendicular to the face $SBTD$, while face ASD remains folded in-plane. The body is dropped from height, with the face BCT parallel to the horizontal. Air resistance causes the body to rotate during its fall. Describe and explain the direction(s) and extent of rotation of the body. [6 marks]

G1.

a. The standard normal distribution $Z \sim \mathrm{N}(0, 1)$ has probability density function (p.d.f.) $\phi(z)$ given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), \qquad z \in \mathbb{R}.$$

i) Write down the value of $\int_{-3}^{3} \phi(z)\, dz$, and state its statistical interpretation.

[1 mark]

ii) If $W = Z^2$, find the p.d.f. $f(w)$ of W. [4 marks]

iii) Show that the value of $P(W \leq x)$ for some value of x is consistent with W having a chi-squared (χ^2) distribution with one degree of freedom.

[3 marks]

*iv) Let S be the sum of two independent observations of W, so $S = W_1 + W_2$.

Explain why the p.d.f. of S is $g(s) = \int_0^s f(s - w)\, f(w)\, dw$ for all real $s \geq 0$.

[2 marks]

b. A projectile is launched at a target at position $(x, y) = (0, 0)$. The actual landing point of the projectile is modelled as a coordinate point (X, Y) metres, where the x and y coordinates are both independent with $X \sim \mathrm{N}(0, 1)$ and $Y \sim \mathrm{N}(0, 1)$.

i) Find the probability that the projectile lands **outside** a square of length 6 m, centred on the target, with the edges aligned with the x-y axes.

[1 mark]

*ii) Using the results derived in part a), show that the **squared** distance D^2 from the target has an Exponential distribution, stating the parameter λ.

[8 marks]

iii) Find the probability that the projectile lands within 3 m of the target.

[1 mark]

G2. The number of disconnections experienced by a company server follows a Poisson distribution with a mean rate of one disconnection every four hours.

a. i) State **one** assumption required for a Poisson model to be valid, and comment on the likelihood of its validity in this scenario. [1 mark]

ii) Find the probability that, for the next five 8-hour working days, the server does not disconnect more than once during the day of any day of the week. [2 marks]

iii) New servers are installed at the company, and it is observed that within the next 24-hour working period, there were 3 disconnections.
Test, at the 10% significance level, the claim that the new servers disconnect less frequently than before. [4 marks]

iv) After a longer period at the company, it is found that the new servers disconnect on average once every five hours.

Comment on the power of the hypothesis test in part a.iii). [3 marks]

b. The connection speed of the servers is now compared. A series of tests is performed and the upload speed is measured in units of Mbps. It is assumed that connection speeds are Normally distributed and rounded to the nearest integer.

Test number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Old server	75	67	69	80	77	71	67	80
New server	76	70	67	79	82	76	73	77

i) Test, at the 5% level of significance, whether there is a **difference** in the mean upload speed between the new and old server. [5 marks]

ii) Tests **5-8** were performed with a different setting enabled on the servers. Test, at the 5% significance level, whether there is an association between the upload speeds of the two servers and the usage of the server setting.
[5 marks]

G3. A shop sells bottled drinks. As part of a planned promotion, the interior of the bottle is to be made of a liquid-sensitive coating so that once the drink is consumed, a picture of a famous singer appears on the bottle. There are n different singers which may appear, each selected randomly, with equal probability and independently at the point of manufacturing. If a person is able to collect all n singers' bottles, they can present their collection at the shop to win a prize.

a. Suppose that a person buys and consumes m bottles of the drink.

i) Show that there are $\frac{(n+m-1)!}{m!\,(n-1)!}$ different possible resulting collections. [3 marks]

ii) Consider the case $m = n$. Show that the probability of having all n different singers in this collection is $\frac{n\,[(n-1)!]^2}{(2n-1)!}$. Does this probability increase or decrease with n? [2 marks]

b. Let T be a discrete random variable representing the number of bottles bought until the full collection of the n different singers is obtained.

Suppose that k - 1 different singers have already been collected. Let X_k be a discrete random variable representing the number of bottles bought in order to obtain the next new singer i.e. to have k different singers for the first time, where $1 \le k \le n$.

i) Explain why $E[T] = \sum_{k=1}^{n} E[X_k]$. [1 mark]

*ii) Find an expression for $E[X_k]$. [3 marks]

*iii) For all integers $n > 1$, prove that $n \ln n + 1 < \mathrm{E}[T] < n \ln n + n$. [5 marks]

*c. It is useful to know that $\mathrm{E}[T] = n \sum_{k=1}^{n} \frac{1}{k}$ and $\mathrm{Var}[T] = n^2 \sum_{k=1}^{n} \frac{1}{k^2}$, which may be used.

In the real scenario, $n = 7$. Each bottle is sold for a profit of £0.50. It is specified that 100,000 bottles will be sold during the promotional period. The prizes for collecting all the bottles are valued at £Q each, to be awarded to every complete collection.

i) It is required that the shop makes a net 25% profit with at least 95% confidence. Stating and justifying your assumptions clearly, estimate the maximum value of Q. [5 marks]

ii) Suggest a reason why the true profit may vary significantly. [1 mark]

Additional Questions

G4. In a game involving a deck of cards, the identity of a card is modelled by a discrete variable N takes integer values in the range $1 \leq N \leq 52$.

The distribution of N within this range is such that $\mathrm{P}(N = n) = k(53 - n)$.

a. Find the values of

i) k. [1 mark]

ii) $\mathrm{E}[N]$. [1 mark]

iii) $\mathrm{Var}[N]$. [2 marks]

iv) the median of N. [3 marks]

b. The score of a card, X, is calculated by taking the **remainder** when $(N - 1)$ is divided by 13, so that X takes integer values between 0 and 12 inclusive.

i) Find a simplified function for $\mathrm{P}(X = x)$, for all valid values of x. [5 marks]

ii) Describe the skewness of the distribution of X. [1 mark]

iii) Find the value of $\mathrm{P}(N = 37 \mid X = 10)$. [3 marks]

c. Describe a practical method with which the distribution of N given could have been determined experimentally. [4 marks]

***A29.** A plane and a cone have Cartesian equations as given below.

Plane Π: $x + 2z - 5 = 0$

Cone Λ: $z = 5 - \sqrt{x^2 + y^2}$

Π and Λ intersect to form an ellipse E with vector equation given by

$$\mathbf{r} = \mathbf{c} + \mathbf{u}\cos\theta + \mathbf{v}\sin\theta \qquad \text{for } 0 \leq \theta < 2\pi,$$

where $\mathbf{c}$ is the centre of E and $\mathbf{u}$ and $\mathbf{v}$ are vectors such that $\mathbf{u} \bullet \mathbf{v} = 0$.

Find the value of $|\, \mathbf{c} \bullet (\mathbf{u} \times \mathbf{v}) \,|$.

① $\dfrac{125\sqrt{3}}{9}$

② $\dfrac{27\sqrt{3}}{2}$

③ $\dfrac{54\sqrt{6}}{5}$

④ $\dfrac{113\sqrt{6}}{9}$ [4 marks]

A30. The function $f(x)$ is defined as

$$f(x) = \begin{cases} 1 + |x|, & \text{if } |x| < \frac{1}{2} \\ 2 - |x|, & \text{if } \frac{1}{2} \leq |x| \leq 2 \\ 0, & \text{if } |x| > 2. \end{cases}$$

How many of the following statements are true?

A $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ for all real a.

B $f(-x) = f(x)$ for all real x.

C $\int_{-\infty}^{\infty} f(x)\, dx = \frac{15}{4}$.

D $f'(-x) + f'(x) = 0$ for all real x.

① 1
② 2
③ 3
④ 4 [4 marks]

A31. Six students - two from China, two from Japan, two from Korea - sit down randomly in six seats. The seats are numbered as shown.

11	12	13

21	22	23

Two students are said to be sitting adjacent to each other if the difference between their seat numbers is either 1 or 10.

Let $X \in \{0, 1, 2, 3\}$ be the number of pairs of students from the same country which are sitting adjacent to each other.

Which row correctly tabulates the probability distribution of X? [4 marks]

		x			
		0	1	2	3
①	$P(X=x)$	$\frac{1}{6}$	$\frac{2}{3}$	0	$\frac{1}{6}$
②		$\frac{2}{15}$	$\frac{2}{3}$	0	$\frac{1}{5}$
③		$\frac{2}{15}$	$\frac{8}{15}$	$\frac{2}{15}$	$\frac{1}{5}$
④		$\frac{1}{6}$	$\frac{2}{5}$	$\frac{4}{15}$	$\frac{1}{6}$

***A32.** A bag contains 50 balls, of which 20 are red and the rest are white.

Balls are taken randomly from the bag **without** replacement until all of the red balls have been found.

Find the probability that, among the balls which have been taken, there are no more white balls than red balls.

① 0.001885
② 0.002301
③ 0.002925
④ 0.003910 [4 marks]

A33. A mass-spring system is oscillating with simple harmonic motion with constant amplitude a and frequency.

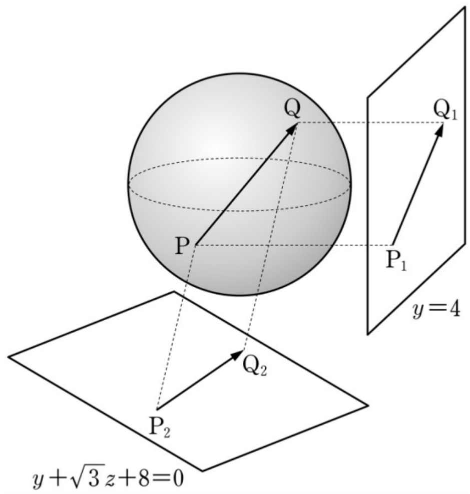
The displacement of the particle from its equilibrium point, $-a \leq x \leq a$, is measured at a random time t, which is uniformly distributed in the interval $[0, T)$, where T is the time period of oscillation.

Find the probability density function $f_X(x)$ for the particle's measured displacement, valid in the interval $-a \leq x \leq a$ and zero elsewhere.

① $f_X(x) = \frac{1}{\pi}\sqrt{a^2 - x^2}$
② $f_X(x) = \frac{1}{(\pi - 2)a} \sin^{-1}\left(\frac{|x|}{a}\right)$
③ $f_X(x) = \frac{\pi}{4a} \cos\left(\frac{\pi x}{2a}\right)$
④ $f_X(x) = \frac{3}{4a^3}\left(a^2 - x^2\right)$ [4 marks]

A34. Two points P and Q are moving on the surface of the sphere with equation $x^2 + y^2 + z^2 = 4$. Two planes, having equations $y = 4$ and $y + \sqrt{3}z + 8 = 0$ lie outside the sphere.

Define points P_1, Q_1, P_2, Q_2 as the projections of P and Q onto each plane.

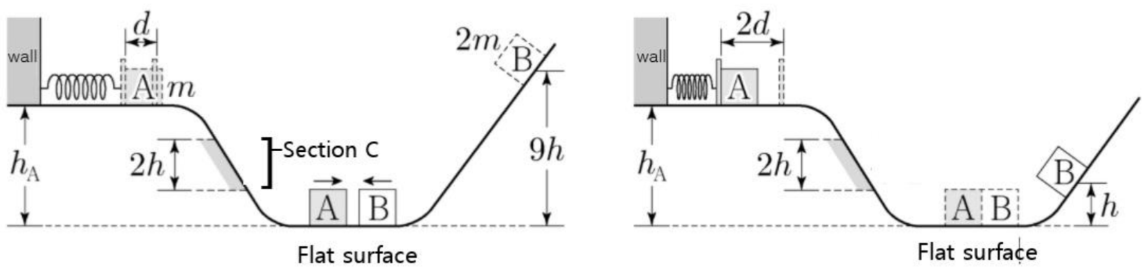


Find the maximum possible value of $2\,|PQ|^2 - \left|P_1Q_1\right|^2 - \left|P_2Q_2\right|^2$.

① 8
② 16
③ 18
④ 24

B5.3. As shown in figure (A), object A is placed in contact with a spring on a platform at height h_A above a flat surface, compressing the spring by a distance d from its natural length. Object B is placed on an incline at an initial height $9h$ from the flat surface, and both objects are released from rest. During the descent of object A, it passes over a rough section (labelled section C) of track, over which it travels at constant speed through a decrease in vertical height of $2h$. Immediately before their collision on the flat surface, objects A and B have the same speed.

After collision, as shown in figure (B), object A recoils up the slope and compresses the spring by a maximum displacement of $2d$, and object B reaches a maximum height of h on the other side.



(A) (B)

The mass of object B is twice that of object A. Both objects remain in full contact with the surface on which they slide. Neglect air resistance and assume the spring obeys Hooke's law.

a. Give and explain **two** conditions under which it is reasonable to assume that the total mechanical energy lost due to sliding of object A in the rough section C, is equal in the cases of its descent and ascent along the slope. [4 marks]

b. Assuming the conditions in part a), find h_A as a multiple of h. [12 marks]

c. Prove that the collision was perfectly elastic. [2 marks]

d. Explain why the coefficient of kinetic friction between object A and the rough section C must be equal to tan θ, where θ is the angle of inclination of the slope. [2 marks]

Question Sources

Sampled from A-Level past papers (AQA/EdExcel/OCR), as well as:

A2, A6, B2c, **D2**b	Cambridge Engineering Part IA Tripos
A4, A8, C2a.iii & vi, **D2**a	수학 짝수형 (Korean Suneung Maths)
A5	NSAA (Cambridge Natural Sciences Entrance Exam)
B1, F2a, **F3**b	CUED (Cambridge Engineering) Preparatory Problems
F1a, **F2**b & c	Cambridge Engineering Part IB Tripos

More Questions

A-Level	www.aqa.org.uk/find-past-papers-and-mark-schemes https://madasmaths.com/
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