ALGEBRA

Common Graphs

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Chapter 4 : Common Graphs

We started the process of graphing in the previous chapter. However, since the main focus of that chapter was functions we didn't graph all that many equations or functions. In this chapter we will now look at graphing a wide variety of equations and functions.

Here is a listing of the topics that we'll be looking at in this chapter.

Section 4-1 : Lines, Circles and Piecewise Functions

We're not really going to do any graphing in this section. In fact, this section is here only to acknowledge that we've already looked at these equations and functions in the previous chapter.

Here are the appropriate sections to see for these.

Lines : Graphing and Functions – **Lines**

Circles : Graphing and Functions – **Circles**

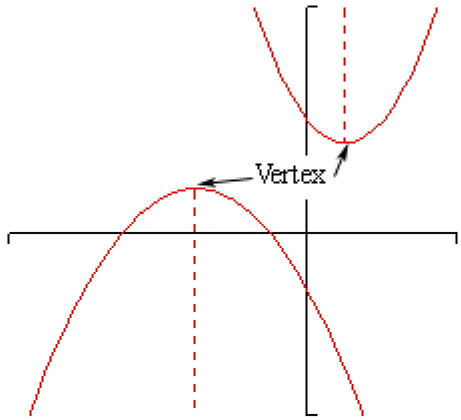
Piecewise Functions : Graphing and Functions – **Graphing Functions**

Section 4-1 : Parabolas

In this section we want to look at the graph of a quadratic function. The most general form of a quadratic function is,

$$f(x) = ax^2 + bx + c$$

The graphs of quadratic functions are called **parabolas**. Here are some examples of parabolas.



All parabolas are vaguely "U" shaped and they will have a highest or lowest point that is called the **vertex**. Parabolas may open up or down and may or may not have *x*-intercepts and they will always have a single *y*-intercept.

Note as well that a parabola that opens down will always open down and a parabola that opens up will always open up. In other words, a parabola will not all of a sudden turn around and start opening up if it has already started opening down. Similarly, if it has already started opening up it will not turn around and start opening down all of a sudden.

The dashed line with each of these parabolas is called the **axis of symmetry**. Every parabola has an axis of symmetry and, as the graph shows, the graph to either side of the axis of symmetry is a mirror image of the other side. This means that if we know a point on one side of the parabola we will also know a point on the other side based on the axis of symmetry. We will see how to find this point once we get into some examples.

We should probably do a quick review of intercepts before going much farther. Intercepts are the points where the graph will cross the *x* or *y*-axis. We also saw a graph in the section where we introduced intercepts where an intercept just touched the axis without actually crossing it.

Finding intercepts is a fairly simple process. To find the *y*-intercept of a function $y = f(x)$ all we need to do is set $x = 0$ and evaluate to find the *y* coordinate. In other words, the *y*-intercept is the point $\left(0, f(0)\right)$. We find *x*-intercepts in pretty much the same way. We set $y = 0$ and solve the resulting equation for the *x* coordinates. So, we will need to solve the equation,

$$f(x) = 0$$

Now, let's get back to parabolas. There is a basic process we can always use to get a pretty good sketch of a parabola. Here it is.

Sketching Parabolas

1. Find the vertex. We'll discuss how to find this shortly. It's fairly simple, but there are several methods for finding it and so will be discussed separately.

2. Find the *y*-intercept, $\left(0, f\left(0\right)\right)$.

3. Solve $f\left(x\right)=0$ to find the *x* coordinates of the *x*-intercepts if they exist. As we will see in our examples we can have 0, 1, or 2 *x*-intercepts.

4. Make sure that you've got at least one point to either side of the vertex. This is to make sure we get a somewhat accurate sketch. If the parabola has two *x*-intercepts then we'll already have these points. If it has 0 or 1 *x*-intercept we can either just plug in another *x* value or use the *y*-intercept and the axis of symmetry to get the second point.

5. Sketch the graph. At this point we've gotten enough points to get a fairly decent idea of what the parabola will look like.

Now, there are two forms of the parabola that we will be looking at. This first form will make graphing parabolas very easy. Unfortunately, most parabolas are not in this form. The second form is the more common form and will require slightly (and only slightly) more work to sketch the graph of the parabola.

Let's take a look at the first form of the parabola.

$$f\left(x\right)=a\left(x-h\right)^2+k$$

There are two pieces of information about the parabola that we can instantly get from this function. First, if *a* is positive then the parabola will open up and if *a* is negative then the parabola will open down. Secondly, the vertex of the parabola is the point $\left(h,k\right)$. Be very careful with signs when getting the vertex here.

So, when we are lucky enough to have this form of the parabola we are given the vertex for free.

Let's see a couple of examples here.

Example 1 Sketch the graph of each of the following parabolas.

(a) $f(x) = 2(x+3)^2 - 8$

(b) $g(x) = -(x-2)^2 - 1$

(c) $h(x) = x^2 + 4$

Solution

Okay, in all of these we will simply go through the process given above to find the needed points and the graph.

(a) $f(x) = 2(x+3)^2 - 8$

First, we need to find the vertex. We will need to be careful with the signs however. Comparing our equation to the form above we see that we must have $h = -3$ and $k = -8$ since that is the only way to get the correct signs in our function. Therefore, the vertex of this parabola is,

$$(-3,-8)$$

Now let's find the *y*-intercept. This is nothing more than a quick function evaluation.

$$f(0) = 2(0+3)^2 - 8 = 2(9) - 8 = 10 \qquad y-\text{intercept}: (0,10)$$

Next, we need to find the *x*-intercepts. This means we'll need to solve an equation. However, before we do that we can actually tell whether or not we'll have any before we even start to solve the equation.

In this case we have $a = 2$ which is positive and so we know that the parabola opens up. Also the vertex is a point below the *x*-axis. So, we know that the parabola will have at least a few points below the *x*-axis and it will open up. Therefore, since once a parabola starts to open up it will continue to open up eventually we will have to cross the *x*-axis. In other words, there are *x*-intercepts for this parabola.

To find them we need to solve the following equation.

$$0 = 2(x+3)^2 - 8$$

We solve equations like this back when we were solving quadratic equations so hopefully you remember how to do them.

$$\begin{aligned} 2(x+3)^2 &= 8 \\ (x+3)^2 &= 4 \\ x+3 &= \pm\sqrt{4} = \pm 2 \\ x &= -3 \pm 2 \qquad \Rightarrow \qquad x = -1,\ x = -5 \end{aligned}$$

The two *x*-intercepts are then,

$$(-5,0) \quad \text{and} \quad (-1,0)$$

Now, at this point we've got points on either side of the vertex so we are officially done with finding the points. However, let's talk a little bit about how to find a second point using the *y*-intercept and the axis of symmetry since we will need to do that eventually.

First, notice that the *y*-intercept has an *x* coordinate of 0 while the vertex has an *x* coordinate of -3. This means that the *y*-intercept is a distance of 3 to the right of the axis of symmetry since that will move straight up from the vertex.

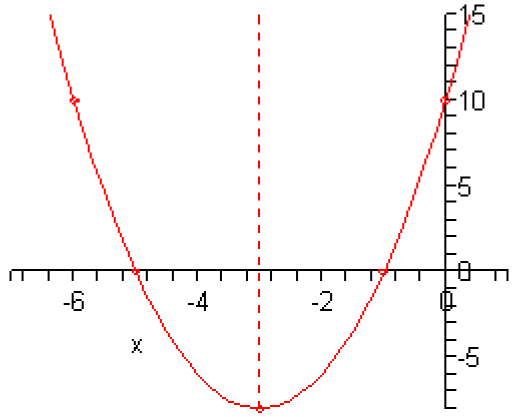
Now, the left part of the graph will be a mirror image of the right part of the graph. So, since there is a point at $y = 10$ that is a distance of 3 to the right of the axis of symmetry there must also be a point at $y = 10$ that is a distance of 3 to the left of the axis of symmetry.

So, since the *x* coordinate of the vertex is -3 and this new point is a distance of 3 to the left its *x* coordinate must be -6. The coordinates of this new point are then $\left(-6,10\right)$. We can verify this by evaluating the function at $x = -6$. If we are correct we should get a value of 10. Let's verify this.

$$f\left(-6\right) = 2\left(-6+3\right)^2 - 8 = 2\left(-3\right)^2 - 8 = 2\left(9\right) - 8 = 10$$

So, we were correct. Note that we usually don't bother with the verification of this point.

Okay, it's time to sketch the graph of the parabola. Here it is.



Note that we included the axis of symmetry in this graph and typically we won't. It was just included here since we were discussing it earlier.

(b) $g\left(x\right) = -\left(x-2\right)^2 - 1$

Okay with this one we won't put in quite a much detail. First let's notice that $a = -1$ which is negative and so we know that this parabola will open downward.

Next, by comparing our function to the general form we see that the vertex of this parabola is $\left(2,-1\right)$. Again, be careful to get the signs correct here!

Now let's get the *y*-intercept.

$$g(0) = -(0-2)^2 - 1 = -(-2)^2 - 1 = -4 - 1 = -5$$

The *y*-intercept is then $(0,-5)$.

Now, we know that the vertex starts out below the *x*-axis and the parabola opens down. This means that there can't possibly be *x*-intercepts since the *x* axis is above the vertex and the parabola will always open down. This means that there is no reason, in general, to go through the solving process to find what won't exist.

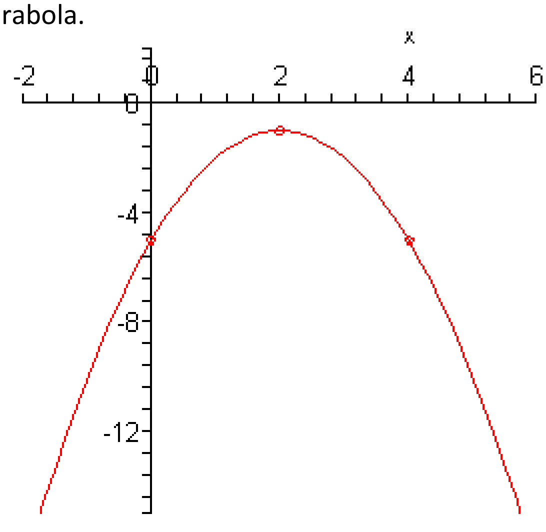
However, let's do it anyway. This will show us what to look for if we don't catch right away that they won't exist from the vertex and direction the parabola opens. We'll need to solve,

$$\begin{aligned} 0 &= -(x-2)^2 - 1 \\ (x-2)^2 &= -1 \\ x - 2 &= \pm i \\ x &= 2 \pm i \end{aligned}$$

So, we got complex solutions. Complex solutions will always indicate no *x*-intercepts.

Now, we do want points on either side of the vertex so we'll use the *y*-intercept and the axis of symmetry to get a second point. The *y*-intercept is a distance of two to the left of the axis of symmetry and is at $y = -5$ and so there must be a second point at the same *y* value only a distance of 2 to the right of the axis of symmetry. The coordinates of this point must then be $(4,-5)$.

Here is the sketch of this parabola.



(c) $h(x) = x^2 + 4$

This one is actually a fairly simple one to graph. We'll first notice that it will open upwards.

Now, the vertex is probably the point where most students run into trouble here. Since we have x^2 by itself this means that we must have $h = 0$ and so the vertex is $(0,4)$.

Note that this means there will not be any *x*-intercepts with this parabola since the vertex is above the *x*-axis and the parabola opens upwards.

Next, the *y*-intercept is,

$$h(0) = (0)^2 + 4 = 4 \qquad\qquad y - \text{intercept : } (0,4)$$

The *y*-intercept is exactly the same as the vertex. This will happen on occasion so we shouldn't get too worried about it when that happens. Although this will mean that we aren't going to be able to use the *y*-intercept to find a second point on the other side of the vertex this time. In fact, we don't even have a point yet that isn't the vertex!

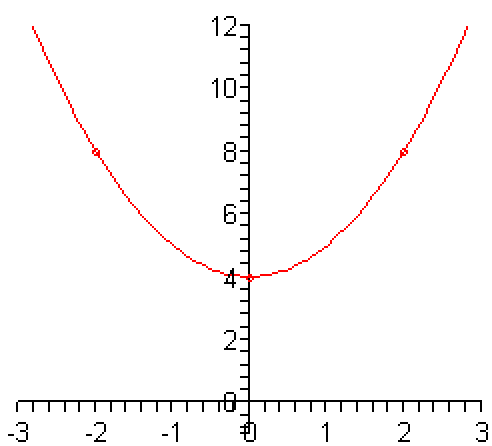
So, we'll need to find a point on either side of the vertex. In this case since the function isn't too bad we'll just plug in a couple of points.

$$h(-2) = (-2)^2 + 4 = 8 \qquad \Rightarrow \qquad (-2,8)$$

$$h(2) = (2)^2 + 4 = 8 \qquad \Rightarrow \qquad (2,8)$$

Note that we could have gotten the second point here using the axis of symmetry if we'd wanted to.

Here is a sketch of the graph.



Okay, we've seen some examples now of this form of the parabola. However, as noted earlier most parabolas are not given in that form. So, we need to take a look at how to graph a parabola that is in the general form.

$$f(x) = ax^2 + bx + c$$

In this form the sign of *a* will determine whether or not the parabola will open upwards or downwards just as it did in the previous set of examples. Unlike the previous form we will not get the vertex for free this time. However, it is will easy to find. Here is the vertex for a parabola in the general form.

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

To get the vertex all we do is compute the *x* coordinate from *a* and *b* and then plug this into the function to get the *y* coordinate. Not quite as simple as the previous form, but still not all that difficult.

Note as well that we will get the *y*-intercept for free from this form. The *y*-intercept is,

$$f(0) = a(0)^2 + b(0) + c = c \qquad \Rightarrow \qquad (0, c)$$

so we won't need to do any computations for this one.

Let's graph some parabolas.

Example 2 Sketch the graph of each of the following parabolas.

(a) $g(x) = 3x^2 - 6x + 5$

(b) $f(x) = -x^2 + 8x$

(c) $f(x) = x^2 + 4x + 4$

Solution

(a) For this parabola we've got $a = 3$, $b = -6$ and $c = 5$. Make sure that you're careful with signs when identifying these values. So we know that this parabola will open up since *a* is positive.

Here are the evaluations for the vertex.

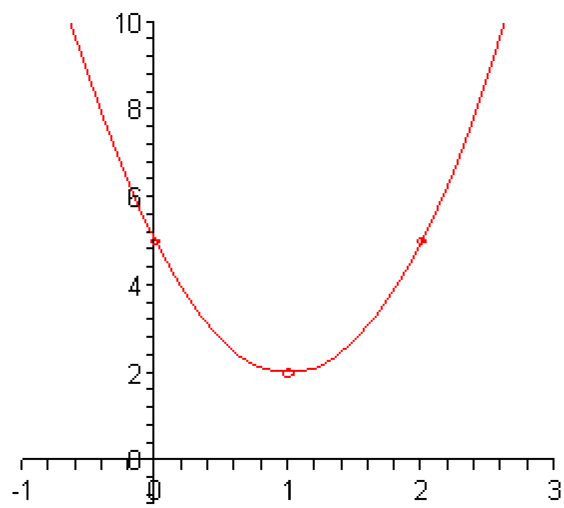
$$x = -\frac{-6}{2(3)} = -\frac{-6}{6} = 1$$

$$y = g(1) = 3(1)^2 - 6(1) + 5 = 3 - 6 + 5 = 2$$

The vertex is then $(1, 2)$. Now at this point we also know that there won't be any *x*-intercepts for this parabola since the vertex is above the *x*-axis and it opens upward.

The *y*-intercept is $(0, 5)$ and using the axis of symmetry we know that $(2, 5)$ must also be on the parabola.

Here is a sketch of the parabola.



(b) In this case $a = -1$, $b = 8$ and $c = 0$. From these we see that the parabola will open downward since *a* is negative. Here are the vertex evaluations.

$$x = -\frac{8}{2(-1)} = -\frac{8}{-2} = 4$$

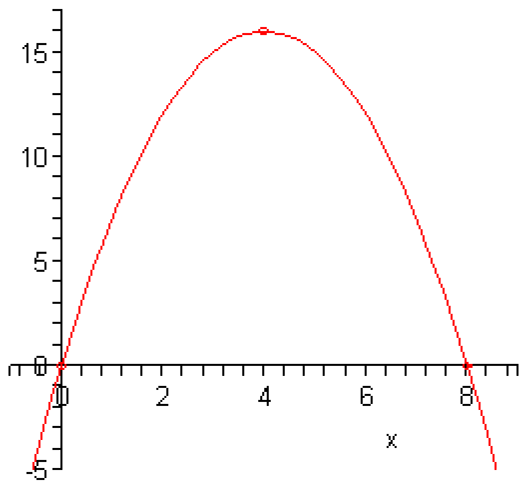
$$y = f(4) = -(4)^2 + 8(4) = 16$$

So, the vertex is $(4,16)$ and we also can see that this time there will be *x*-intercepts. In fact, let's go ahead and find them now.

$$\begin{aligned} 0 &= -x^2 + 8x \\ 0 &= x(-x+8) \qquad \Rightarrow \qquad x = 0,\ x = 8 \end{aligned}$$

So, the *x*-intercepts are $(0,0)$ and $(8,0)$. Notice that $(0,0)$ is also the *y*-intercept. This will happen on occasion so don't get excited about it when it does.

At this point we've got all the information that we need in order to sketch the graph so here it is,



(c) In this final part we have $a = 1$, $b = 4$ and $c = 4$. So, this parabola will open up.

Here are the vertex evaluations.

$$x = -\frac{4}{2(1)} = -\frac{4}{2} = -2$$

$$y = f(-2) = (-2)^2 + 4(-2) + 4 = 0$$

So, the vertex is $(-2,0)$. Note that since the *y* coordinate of this point is zero it is also an *x*-intercept. In fact it will be the only *x*-intercept for this graph. This makes sense if we consider the fact that the vertex, in this case, is the lowest point on the graph and so the graph simply can't touch the *x*-axis anywhere else.

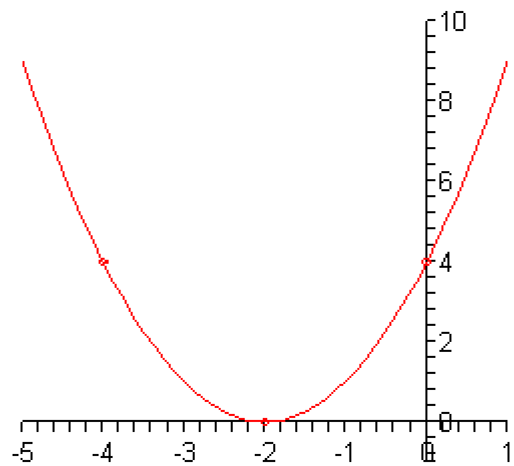
The fact that this parabola has only one *x*-intercept can be verified by solving as we've done in the other examples to this point.

$$0 = x^2 + 4x + 4$$
$$0 = (x+2)^2 \qquad \Rightarrow \qquad x = -2$$

Sure enough there is only one *x*-intercept. Note that this will mean that we're going to have to use the axis of symmetry to get a second point from the *y*-intercept in this case.

Speaking of which, the *y*-intercept in this case is $(0,4)$. This means that the second point is $(-4,4)$.

Here is a sketch of the graph.



As a final topic in this section we need to briefly talk about how to take a parabola in the general form and convert it into the form

$$f(x) = a(x-h)^2 + k$$

This will use a modified completing the square process. It's probably best to do this with an example.

Example 3 Convert each of the following into the form $f(x) = a(x-h)^2 + k$.

(a) $f(x) = 2x^2 - 12x + 3$

(b) $f(x) = -x^2 + 10x - 1$

Solution

Okay, as we pointed out above we are going to complete the square here. However, it is a slightly different process than the other times that we've seen it to this point.

(a) The thing that we've got to remember here is that we must have a coefficient of 1 for the x^2 term in order to complete the square. So, to get that we will first factor the coefficient of the x^2 term out of the whole right side as follows.

$$f(x) = 2\left(x^2 - 6x + \frac{3}{2}\right)$$

Note that this will often put fractions into the problem that is just something that we'll need to be able to deal with. Also note that if we're lucky enough to have a coefficient of 1 on the x^2 term we won't have to do this step.

Now, this is where the process really starts differing from what we've seen to this point. We still take one-half the coefficient of x and square it. However, instead of adding this to both sides we do the following with it.

$$\left(-\frac{6}{2}\right)^2 = (-3)^2 = 9$$

$$f(x) = 2\left(x^2 - 6x + 9 - 9 + \frac{3}{2}\right)$$

We add and subtract this quantity inside the parenthesis as shown. Note that all we are really doing here is adding in zero since 9-9=0! The order listed here is important. We MUST add first and then subtract.

The next step is to factor the first three terms and combine the last two as follows.

$$f(x) = 2\left((x-3)^2 - \frac{15}{2}\right)$$

As a final step we multiply the 2 back through.

$$f(x) = 2(x-3)^2 - 15$$

And there we go.

(b) Be careful here. We don't have a coefficient of 1 on the x^2 term, we've got a coefficient of -1. So, the process is identical outside of that so we won't put in as much detail this time.

$$\begin{aligned} f(x) &= -\left(x^2 - 10x + 1\right) \qquad \left(-\frac{10}{2}\right)^2 = (-5)^2 = 25 \\ &= -\left(x^2 - 10x + 25 - 25 + 1\right) \\ &= -\left((x-5)^2 - 24\right) \\ &= -(x-5)^2 + 24 \end{aligned}$$

Section 4-3 : Ellipses

In a previous section we looked at graphing circles and since circles are really special cases of ellipses we've already got most of the tools under our belts to graph ellipses. All that we really need here to get us started is then **standard form** of the ellipse and a little information on how to interpret it.

Here is the standard form of an ellipse.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Note that the right side MUST be a 1 in order to be in standard form. The point (h,k) is called the **center** of the ellipse.

To graph the ellipse all that we need are the right most, left most, top most and bottom most points. Once we have those we can sketch in the ellipse. Here are formulas for finding these points.

$$\begin{aligned}&\text{right most point}: (h+a,k)\\&\text{left most point}: (h-a,k)\\&\text{top most point}: (h,k+b)\\&\text{bottom most point}: (h,k-b)\end{aligned}$$

Note that *a* is the square root of the number under the *x* term and is the amount that we move right and left from the center. Also, *b* is the square root of the number under the *y* term and is the amount that we move up or down from the center.

Let's sketch some graphs.

Example 1 Sketch the graph of each of the following ellipses.

(a) $\dfrac{(x+2)^2}{9}+\dfrac{(y-4)^2}{25}=1$

(b) $\dfrac{x^2}{49}+\dfrac{(y-3)^2}{4}=1$

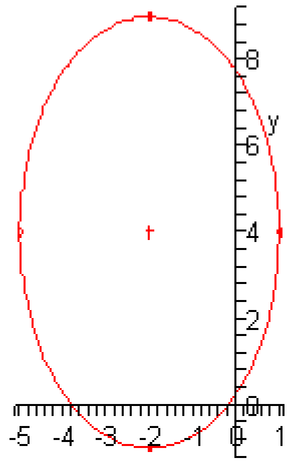
(c) $4(x+1)^2+(y+3)^2=1$

Solution

(a) So, the center of this ellipse is $(-2,4)$ and as usual be careful with signs here! Also, we have $a=3$ and $b=5$. So, the points are,

$$\begin{aligned}&\text{right most point}: (1,4)\\&\text{left most point}: (-5,4)\\&\text{top most point}: (-2,9)\\&\text{bottom most point}: (-2,-1)\end{aligned}$$

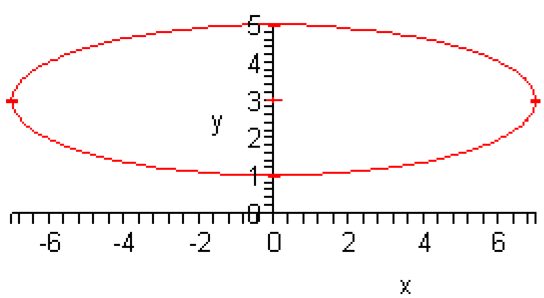
Here is a sketch of this ellipse.



(b) The center for this part is $(0,3)$ and we have $a = 7$ and $b = 2$. The points we need are,

$$\begin{aligned}\text{right most point : } & (7,3)\\ \text{left most point : } & (-7,3)\\ \text{top most point : } & (0,5)\\ \text{bottom most point : } & (0,1)\end{aligned}$$

Here is the sketch of this ellipse.



(c) Now with this ellipse we're going to have to be a little careful as it isn't quite in standard form yet. Here is the standard form for this ellipse.

$$\frac{(x+1)^2}{\frac{1}{4}} + (y+3)^2 = 1$$

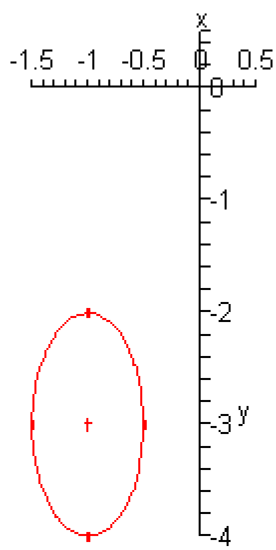
Note that in order to get the coefficient of 4 in the numerator of the first term we will need to have a $\frac{1}{4}$ in the denominator. Also, note that we don't even have a fraction for the *y* term. This implies that there is in fact a 1 in the denominator. We could put this in if it would be helpful to see what is going on here.

$$\frac{(x+1)^2}{\frac{1}{4}} + \frac{(y+3)^2}{1} = 1$$

So, in this form we can see that the center is $(-1,-3)$ and that $a = \frac{1}{2}$ and $b = 1$. The points for this ellipse are,

$$\text{right most point} : \left(-\frac{1}{2},-3\right)$$
$$\text{left most point} : \left(-\frac{3}{2},-3\right)$$
$$\text{top most point} : (-1,-2)$$
$$\text{bottom most point} : (-1,-4)$$

Here is this ellipse.



Finally, let's address a comment made at the start of this section. We said that circles are really nothing more than a special case of an ellipse. To see this let's assume that $a = b$. In this case we have,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1$$

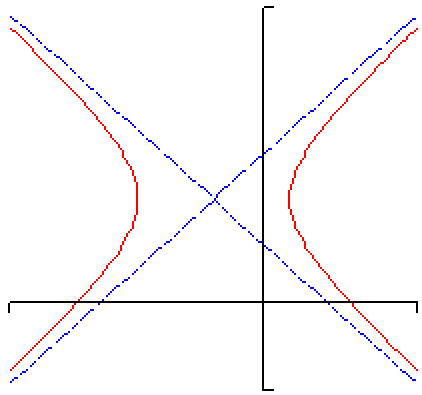
Note that we acknowledged that $a = b$ and used *a* in both cases. Now if we clear denominators we get,

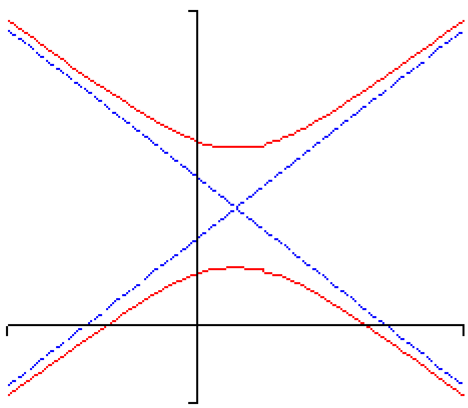
$$(x-h)^2+(y-k)^2=a^2$$

This is the standard form of a circle with center (h,k) and radius *a*. So, circles really are special cases of ellipses.

Section 4-4 : Hyperbolas

The next graph that we need to look at is the hyperbola. There are two basic forms of a hyperbola. Here are examples of each.





Hyperbolas consist of two vaguely parabola shaped pieces that open either up and down or right and left. Also, just like parabolas each of the pieces has a vertex. Note that they aren't really parabolas, they just resemble parabolas.

There are also two lines on each graph. These lines are called asymptotes and as the graphs show as we make x large (in both the positive and negative sense) the graph of the hyperbola gets closer and closer to the asymptotes. The asymptotes are not officially part of the graph of the hyperbola. However, they are usually included so that we can make sure and get the sketch correct. The point where the two asymptotes cross is called the center of the hyperbola.

There are two **standard forms** of the hyperbola, one for each type shown above. Here is a table giving each form as well as the information we can get from each one.

Form	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$
Center	(h,k)	(h,k)
Opens	Opens left and right	Opens up and down
Vertices	$(h+a,k)$ and $(h-a,k)$	$(h,k+b)$ and $(h,k-b)$
Slope of Asymptotes	$\pm\frac{b}{a}$	$\pm\frac{b}{a}$
Equations of Asymptotes	$y=k\pm\frac{b}{a}(x-h)$	$y=k\pm\frac{b}{a}(x-h)$

Note that the difference between the two forms is which term has the minus sign. If the *y* term has the minus sign then the hyperbola will open left and right. If the *x* term has the minus sign then the hyperbola will open up and down.

We got the equations of the asymptotes by using the point-slope form of the line and the fact that we know that the asymptotes will go through the center of the hyperbola.

Let's take a look at a couple of these.

Example 1 Sketch the graph of each of the following hyperbolas.

(a) $\frac{(x-3)^2}{25}-\frac{(y+1)^2}{49}=1$

(b) $\frac{y^2}{9}-(x+2)^2=1$

Solution

(a) Now, notice that the *y* term has the minus sign and so we know that we're in the first column of the table above and that the hyperbola will be opening left and right.

The first thing that we should get is the center since pretty much everything else is built around that. The center in this case is $(3,-1)$ and as always watch the signs! Once we have the center we can get the vertices. These are $(8,-1)$ and $(-2,-1)$.

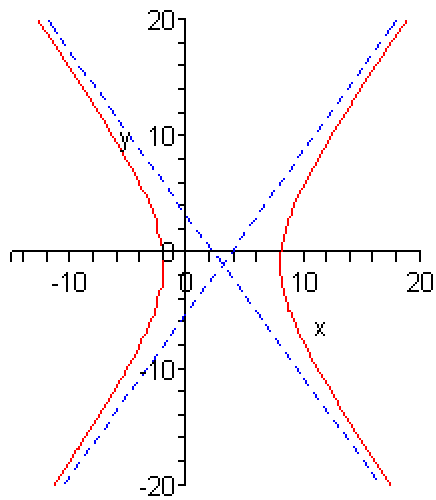
Next, we should get the slopes of the asymptotes. These are always the square root of the number under the *y* term divided by the square root of the number under the *x* term and there will always be a positive and a negative slope. The slopes are then $\pm\frac{7}{5}$.

Now that we've got the center and the slopes of the asymptotes we can get the equations for the asymptotes. They are,

$$y = -1 + \frac{7}{5}(x-3) \qquad \text{and} \qquad y = -1 - \frac{7}{5}(x-3)$$

We can now start the sketching. We start by sketching the asymptotes and the vertices. Once these are done we know what the basic shape should look like so we sketch it in making sure that as *x* gets large we move in closer and closer to the asymptotes.

Here is the sketch for this hyperbola.



(b) In this case the hyperbola will open up and down since the *x* term has the minus sign. Now, the center of this hyperbola is $(-2,0)$. Remember that since there is a y^2 term by itself we had to have $k=0$. At this point we also know that the vertices are $(-2,3)$ and $(-2,-3)$.

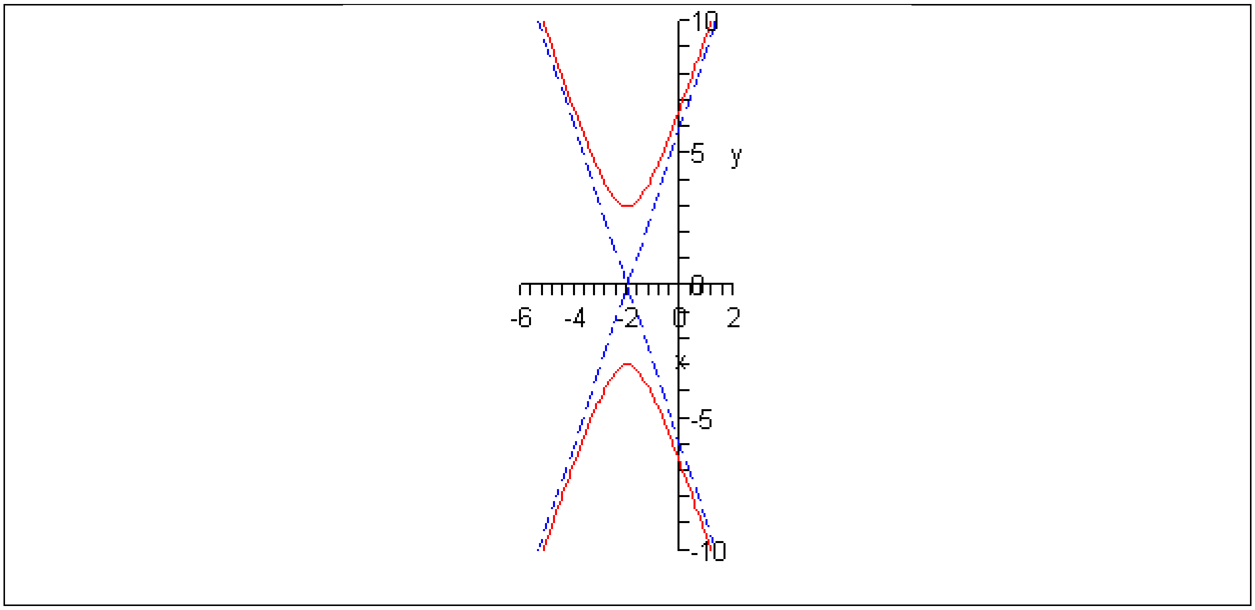
In order to see the slopes of the asymptotes let's rewrite the equation a little.

$$\frac{y^2}{9} - \frac{(x+2)^2}{1} = 1$$

So, the slopes of the asymptotes are $\pm\frac{3}{1} = \pm 3$. The equations of the asymptotes are then,

$$y = 0 + 3(x+2) = 3x+6 \qquad \text{and} \qquad y = 0 - 3(x+2) = -3x-6$$

Here is the sketch of this hyperbola.


10
5
y
0
-6
-4
-2
0
2
x
-5
-10

Section 4-5 : Miscellaneous Functions

The point of this section is to introduce you to some other functions that don't really require the work to graph that the ones that we've looked at to this point in this chapter. For most of these all that we'll need to do is evaluate the function as some *x*'s and then plot the points.

Constant Function
This is probably the easiest function that we'll ever graph and yet it is one of the functions that tend to cause problems for students.

The most general form for the constant function is,

$$f\left(x\right)=c$$

where *c* is some number.

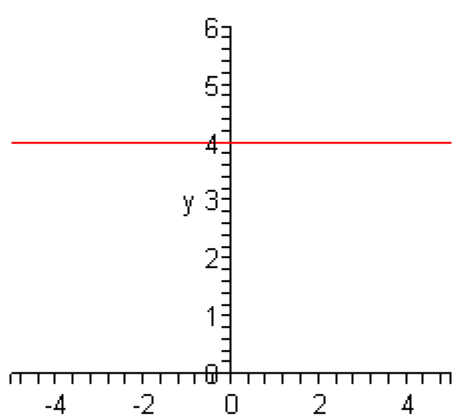
Let's take a look at $f\left(x\right)=4$ so we can see what the graph of constant functions look like. Probably the biggest problem students have with these functions is that there are no *x*'s on the right side to plug into for evaluation. However, all that means is that there is no substitution to do. In other words, no matter what *x* we plug into the function we will always get a value of 4 (or *c* in the general case) out of the function.

So, every point has a *y* coordinate of 4. This is exactly what defines a horizontal line. In fact, if we recall that $f\left(x\right)$ is nothing more than a fancy way of writing *y* we can rewrite the function as,

$$y=4$$

And this is exactly the equation of a horizontal line.

Here is the graph of this function.



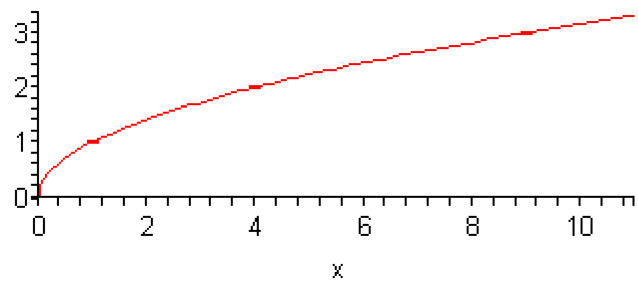
Square Root

Next, we want to take a look at $f(x) = \sqrt{x}$. First, note that since we don't want to get complex numbers out of a function evaluation we have to restrict the values of *x* that we can plug in. We can only plug in value of *x* in the range $x \geq 0$. This means that our graph will only exist in this range as well.

To get the graph we'll just plug in some values of *x* and then plot the points.

x	*f(x)*
0	0
1	1
4	2
9	3

The graph is then,



Absolute Value

We've dealt with this function several times already. It's now time to graph it. First, let's remind ourselves of the definition of the absolute value function.

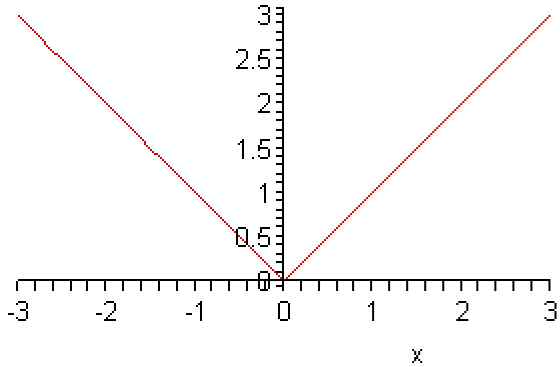
$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is a piecewise function and we've seen how to graph these already. All that we need to do is get some points in both ranges and plot them.

Here are some function evaluations.

x	*f(x)*
0	0
1	1
-1	1
2	2
-2	2

Here is the graph of this function.



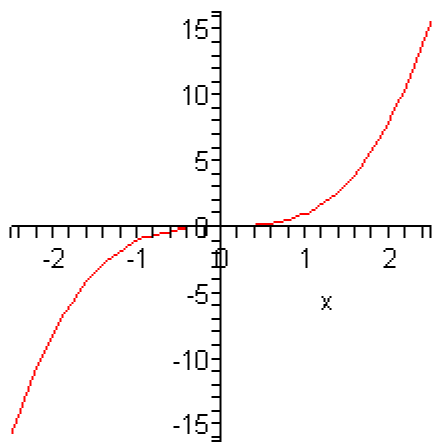
So, this is a "V" shaped graph.

Cubic Function

We're not actually going to look at a general cubic polynomial here. We'll do some of those in the next chapter. Here we are only going to look at $f\left(x\right)=x^3$. There really isn't much to do here other than just plugging in some points and plotting.

x	$f(x)$
0	0
1	1
-1	-1
2	8
-2	-8

Here is the graph of this function.



We will need some of these in the next section so make sure that you can identify these when you see them and can sketch their graphs fairly quickly.

Section 4- 6 : Transformations

In this section we are going to see how knowledge of some fairly simple graphs can help us graph some more complicated graphs. Collectively the methods we're going to be looking at in this section are called **transformations**.

Vertical Shifts

The first transformation we'll look at is a vertical shift.

Given the graph of $f(x)$ the graph of $g(x) = f(x) + c$ will be the graph of $f(x)$ shifted up by *c* units if *c* is positive and or down by *c* units if *c* is negative.

So, if we can graph $f(x)$ getting the graph of $g(x)$ is fairly easy. Let's take a look at a couple of examples.

Example 1 Using transformations sketch the graph of the following functions.

(a) $g(x) = x^2 + 3$

(b) $f(x) = \sqrt{x} - 5$

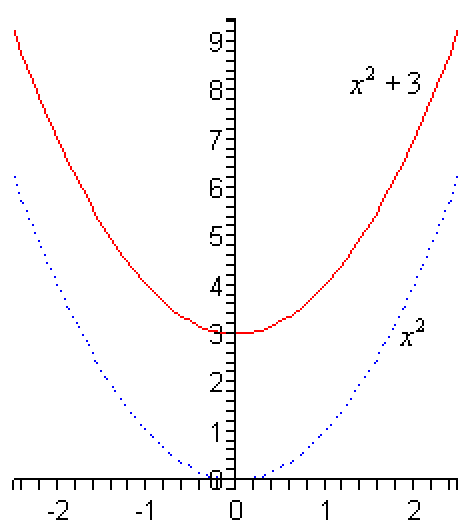
Solution

The first thing to do here is graph the function without the constant which by this point should be fairly simple for you. Then shift accordingly.

(a) $g(x) = x^2 + 3$

In this case we first need to graph x^2 (the dotted line on the graph below) and then pick this up and shift it upwards by 3. Coordinate wise this will mean adding 3 onto all the *y* coordinates of points on x^2.

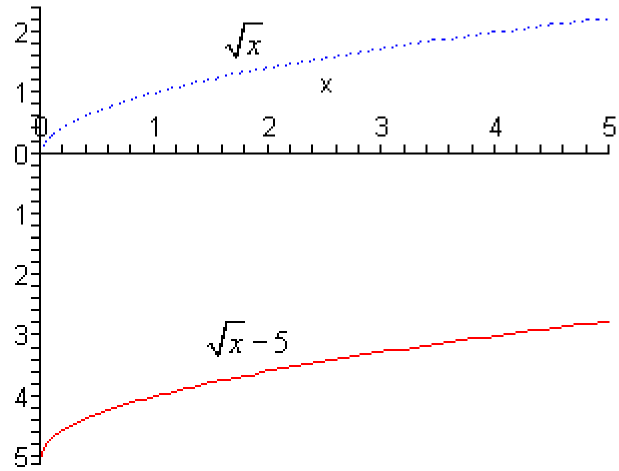
Here is the sketch for this one.



(b) $f(x) = \sqrt{x} - 5$

Okay, in this case we're going to be shifting the graph of $\sqrt{x}$ (the dotted line on the graph below) down by 5. Again, from a coordinate standpoint this means that we subtract 5 from the *y* coordinates of points on $\sqrt{x}$.

Here is this graph.



So, vertical shifts aren't all that bad if we can graph the "base" function first. Note as well that if you're not sure that you believe the graphs in the previous set of examples all you need to do is plug a couple values of *x* into the function and verify that they are in fact the correct graphs.

Horizontal Shifts

These are fairly simple as well although there is one bit where we need to be careful.

Given the graph of $f(x)$ the graph of $g(x) = f(x + c)$ will be the graph of $f(x)$ shifted left by *c* units if *c* is positive and or right by *c* units if *c* is negative.

Now, we need to be careful here. A positive *c* shifts a graph in the negative direction and a negative *c* shifts a graph in the positive direction. They are exactly opposite than vertical shifts and it's easy to flip these around and shift incorrectly if we aren't being careful.

Example 2 Using transformations sketch the graph of the following functions.

(a) $h(x) = (x + 2)^3$

(b) $g(x) = \sqrt{x - 4}$

Solution

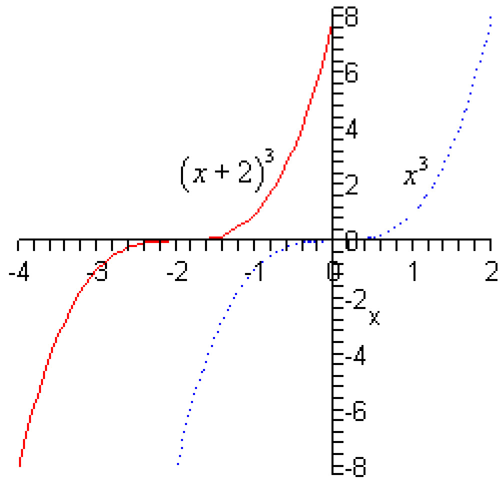
(a) $h(x) = (x + 2)^3$

Okay, with these we need to first identify the "base" function. That is the function that's being shifted. In this case it looks like we are shifting $f(x) = x^3$. We can then see that,

$$h(x) = (x+2)^3 = f(x+2)$$

In this case $c = 2$ and so we're going to shift the graph of $f(x) = x^3$ (the dotted line on the graph below) and move it 2 units to the left. This will mean subtracting 2 from the *x* coordinates of all the points on $f(x) = x^3$.

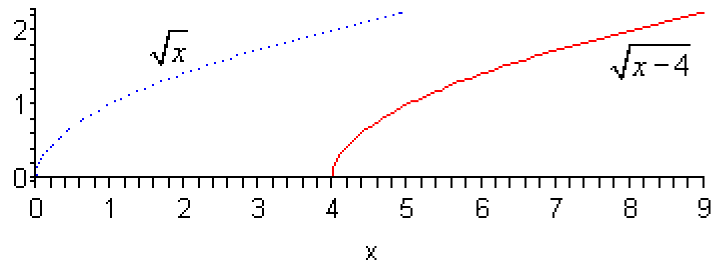
Here is the graph for this problem.



(b) $g(x) = \sqrt{x-4}$

In this case it looks like the base function is $\sqrt{x}$ and it also looks like $c = -4$ and so we will be shifting the graph of $\sqrt{x}$ (the dotted line on the graph below) to the right by 4 units. In terms of coordinates this will mean that we're going to add 4 onto the *x* coordinate of all the points on $\sqrt{x}$.

Here is the sketch for this function.



Vertical and Horizontal Shifts

Now we can also combine the two shifts we just got done looking at into a single problem. If we know the graph of $f(x)$ the graph of $g(x) = f(x+c)+k$ will be the graph of $f(x)$ shifted left or right by *c* units depending on the sign of *c* and up or down by *k* units depending on the sign of *k*.

Let's take a look at a couple of examples.

Example 3 Use transformation to sketch the graph of each of the following.

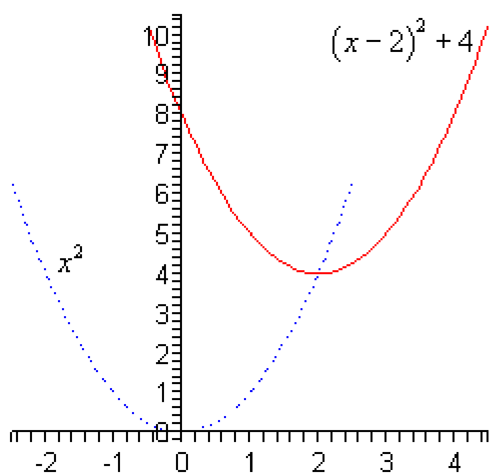
(a) $f(x) = (x-2)^2 + 4$

(b) $g(x) = |x+3| - 5$

Solution

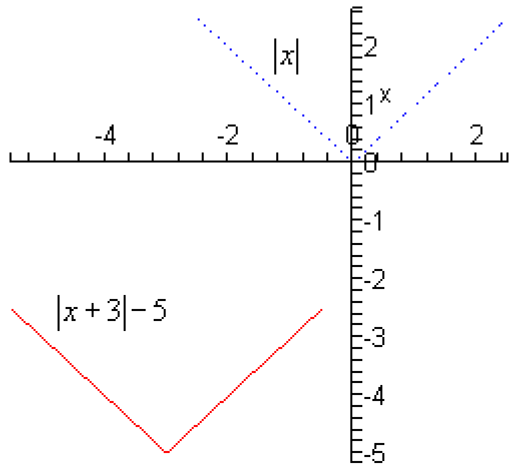
(a) $f(x) = (x-2)^2 + 4$

In this part it looks like the base function is x^2 and it looks like will be shift this to the right by 2 (since $c = -2$) and up by 4 (since $k = 4$). Here is the sketch of this function.



(b) $g(x) = |x+3| - 5$

For this part we will be shifting $|x|$ to the left by 3 (since $c = 3$) and down 5 (since $k = -5$). Here is the sketch of this function.



Reflections

The final set of transformations that we're going to be looking at in this section aren't shifts, but instead they are called reflections and there are two of them.

Reflection about the *x*-axis.

Given the graph of $f(x)$ then the graph of $g(x) = -f(x)$ is the graph of $f(x)$ *reflected* about the *x*-axis. This means that the signs on the all the *y* coordinates are changed to the opposite sign.

Reflection about the *y*-axis.

Given the graph of $f(x)$ then the graph of $g(x) = f(-x)$ is the graph of $f(x)$ *reflected* about the *y*-axis. This means that the signs on the all the *x* coordinates are changed to the opposite sign.

Here is an example of each.

Example 4 Using transformation sketch the graph of each of the following.

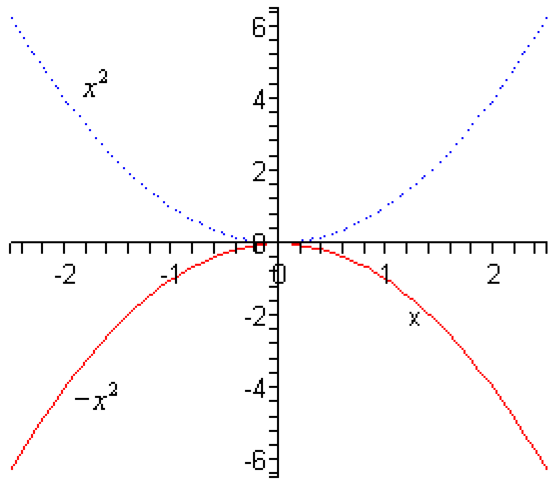
(a) $g(x) = -x^2$

(b) $h(x) = \sqrt{-x}$

Solution

(a) Based on the placement of the minus sign (*i.e.* it's outside the square and NOT inside the square, or $(-x)^2$) it looks like we will be reflecting x^2 about the *x*-axis. So, again, the means that all we do is change the sign on all the *y* coordinates.

Here is the sketch of this graph.



(b) Now with this one let's first address the minus sign under the square root in more general terms. We know that we can't take the square roots of negative numbers, however the presence of that minus sign doesn't necessarily cause problems. We won't be able to plug positive values of *x* into the function since that would give square roots of negative numbers. However, if *x* were negative, then the negative of a negative number is positive and that is okay. For instance,

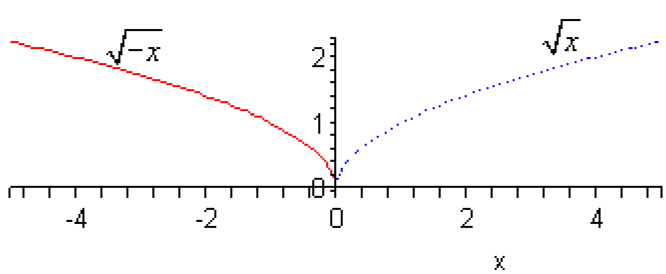
$$h(-4) = \sqrt{-(-4)} = \sqrt{4} = 2$$

So, don't get all worried about that minus sign.

Now, let's address the reflection here. Since the minus sign is under the square root as opposed to in front of it we are doing a reflection about the *y*-axis. This means that we'll need to change all the signs of points on $\sqrt{x}$.

Note as well that this syncs up with our discussion on this minus sign at the start of this part.

Here is the graph for this function.



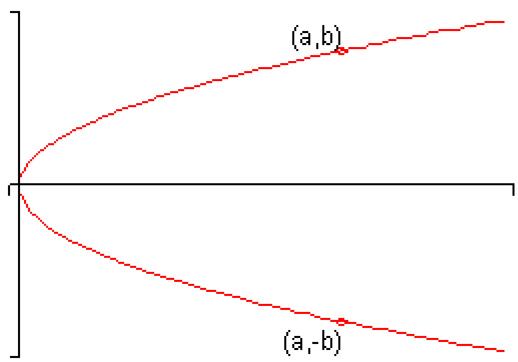
Section 4-7 : Symmetry

In this section we are going to take a look at something that we used back when we were graphing parabolas. However, we're going to take a more general view of it this section. Many graphs have **symmetry** to them.

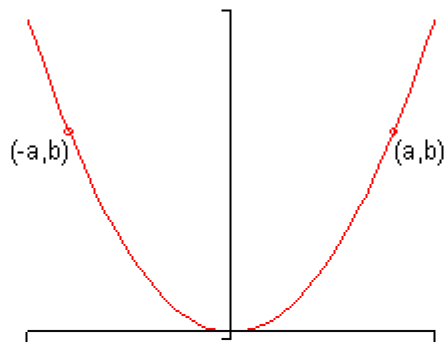
Symmetry can be useful in graphing an equation since it says that if we know one portion of the graph then we will also know the remaining (and symmetric) portion of the graph as well. We used this fact when we were graphing parabolas to get an extra point of some of the graphs.

In this section we want to look at three types of symmetry.

1. A graph is said to be **symmetric about the *x*-axis** if whenever (a,b) is on the graph then so is $(a,-b)$. Here is a sketch of a graph that is symmetric about the *x*-axis.



2. A graph is said to be **symmetric about the *y*-axis** if whenever (a,b) is on the graph then so is $(-a,b)$. Here is a sketch of a graph that is symmetric about the *y*-axis.



3. A graph is said to be **symmetric about the origin** if whenever (a,b) is on the graph then so is $(-a,-b)$. Here is a sketch of a graph that is symmetric about the origin.

Note that most graphs don't have any kind of symmetry. Also, it is possible for a graph to have more than one kind of symmetry. For example, the graph of a circle centered at the origin exhibits all three symmetries.

Tests for Symmetry

We've some fairly simply tests for each of the different types of symmetry.

1. A graph will have symmetry about the x-axis if we get an equivalent equation when all the y's are replaced with $-y$.

2. A graph will have symmetry about the y-axis if we get an equivalent equation when all the x's are replaced with $-x$.

3. A graph will have symmetry about the origin if we get an equivalent equation when all the y's are replaced with $-y$ and all the x's are replaced with $-x$.

We will define just what we mean by an "equivalent equation" when we reach an example of that. For the majority of the examples that we're liable to run across this will mean that it is exactly the same equation.

Let's test a few equations for symmetry. Note that we aren't going to graph these since most of them would actually be fairly difficult to graph. The point of this example is only to use the tests to determine the symmetry of each equation.

Example 1 Determine the symmetry of each of the following equations.

(a) $y = x^2 - 6x^4 + 2$

(b) $y = 2x^3 - x^5$

(c) $y^4 + x^3 - 5x = 0$

(d) $y = x^3 + x^2 + x + 1$

(e) $x^2 + y^2 = 1$

Solution

(a) $y = x^2 - 6x^4 + 2$

We'll first check for symmetry about the *x*-axis. This means that we need to replace all the *y*'s with –*y*. That's easy enough to do in this case since there is only one *y*.

$$-y = x^2 - 6x^4 + 2$$

Now, this is not an equivalent equation since the terms on the right are identical to the original equation and the term on the left is the opposite sign. So, this equation doesn't have symmetry about the *x*-axis.

Next, let's check symmetry about the *y*-axis. Here we'll replace all *x*'s with –*x*.

$$\begin{aligned} y &= (-x)^2 - 6(-x)^4 + 2 \\ y &= x^2 - 6x^4 + 2 \end{aligned}$$

After simplifying we got exactly the same equation back out which means that the two are equivalent. Therefore, this equation does have symmetry about the *y*-axis.

Finally, we need to check for symmetry about the origin. Here we replace both variables.

$$\begin{aligned} -y &= (-x)^2 - 6(-x)^4 + 2 \\ -y &= x^2 - 6x^4 + 2 \end{aligned}$$

So, as with the first test, the left side is different from the original equation and the right side is identical to the original equation. Therefore, this isn't equivalent to the original equation and we don't have symmetry about the origin.

(b) $y = 2x^3 - x^5$

We'll not put in quite as much detail here. First, we'll check for symmetry about the *x*-axis.

$$-y = 2x^3 - x^5$$

We don't have symmetry here since the one side is identical to the original equation and the other isn't. So, we don't have symmetry about the *x*-axis.

Next, check for symmetry about the *y*-axis.

$$y = 2(-x)^3 - (-x)^5$$
$$y = -2x^3 + x^5$$

Remember that if we take a negative to an odd power the minus sign can come out in front. So, upon simplifying we get the left side to be identical to the original equation, but the right side is now the opposite sign from the original equation and so this isn't equivalent to the original equation and so we don't have symmetry about the *y*-axis.

Finally, let's check symmetry about the origin.

$$-y = 2(-x)^3 - (-x)^5$$
$$-y = -2x^3 + x^5$$

Now, this time notice that all the signs in this equation are exactly the opposite form the original equation. This means that it IS equivalent to the original equation since all we would need to do is multiply the whole thing by "-1" to get back to the original equation.

Therefore, in this case we have symmetry about the origin.

(c) $y^4 + x^3 - 5x = 0$

First, check for symmetry about the *x*-axis.

$$(-y)^4 + x^3 - 5x = 0$$
$$y^4 + x^3 - 5x = 0$$

This is identical to the original equation and so we have symmetry about the *x*-axis.

Now, check for symmetry about the *y*-axis.

$$y^4 + (-x)^3 - 5(-x) = 0$$
$$y^4 - x^3 + 5x = 0$$

So, some terms have the same sign as the original equation and other don't so there isn't symmetry about the *y*-axis.

Finally, check for symmetry about the origin.

$$(-y)^4 + (-x)^3 - 5(-x) = 0$$
$$y^4 - x^3 + 5x = 0$$

Again, this is not the same as the original equation and isn't exactly the opposite sign from the original equation and so isn't symmetric about the origin.

(d) $y = x^3 + x^2 + x + 1$

First, symmetry about the *x*-axis.

$$-y = x^3 + x^2 + x + 1$$

It looks like no symmetry about the *x*-axis

Next, symmetry about the *y*-axis.

$$y = (-x)^3 + (-x)^2 + (-x) + 1$$
$$y = -x^3 + x^2 - x + 1$$

So, no symmetry here either.

Finally, symmetry about the origin.

$$-y = (-x)^3 + (-x)^2 + (-x) + 1$$
$$-y = -x^3 + x^2 - x + 1$$

And again, no symmetry here either.

This function has no symmetry of any kind. That's not unusual as most functions don't have any of these symmetries.

(e) $x^2 + y^2 = 1$

Check *x*-axis symmetry first.

$$x^2 + (-y)^2 = 1$$
$$x^2 + y^2 = 1$$

So, it's got symmetry about the *x*-axis symmetry.

Next, check for *y*-axis symmetry.

$$(-x)^2 + y^2 = 1$$
$$x^2 + y^2 = 1$$

Looks like it's also got *y*-axis symmetry.

Finally, symmetry about the origin.

$$(-x)^2 + (-y)^2 = 1$$
$$x^2 + y^2 = 1$$

So, it's also got symmetry about the origin.

Note that this is a circle centered at the origin and as noted when we first started talking about symmetry it does have all three symmetries.

Section 4-8 : Rational Functions

In this final section we need to discuss graphing rational functions. It's is probably best to start off with a fairly simple one that we can do without all that much knowledge on how these work.

Let's sketch the graph of $f\left(x\right)=\dfrac{1}{x}$. First, since this is a rational function we are going to have to be careful with division by zero issues. So, we can see from this equation that we'll have to avoid $x=0$ since that will give division by zero.

Now, let's just plug in some values of *x* and see what we get.

x	*f(x)*
-4	-0.25
-2	-0.5
-1	-1
-0.1	-10
-0.01	-100
0.01	100
0.1	10
1	1
2	0.5
4	0.25

So, as *x* get large (positively and negatively) the function keeps the sign of *x* and gets smaller and smaller. Likewise, as we approach $x=0$ the function again keeps the same sign as *x* but starts getting quite large. Here is a sketch of this graph.

First, notice that the graph is in two pieces. Almost all rational functions will have graphs in multiple pieces like this.

Next, notice that this graph does not have any intercepts of any kind. That's easy enough to check for ourselves.

Recall that a graph will have a *y*-intercept at the point $\left(0, f\left(0\right)\right)$. However, in this case we have to avoid $x = 0$ and so this graph will never cross the *y*-axis. It does get very close to the *y*-axis, but it will never cross or touch it and so no *y*-intercept.

Next, recall that we can determine where a graph will have *x*-intercepts by solving $f\left(x\right) = 0$. For rational functions this may seem like a mess to deal with. However, there is a nice fact about rational functions that we can use here. A rational function will be zero at a particular value of *x* only if the numerator is zero at that *x* and the denominator isn't zero at that *x*. In other words, to determine if a rational function is ever zero all that we need to do is set the numerator equal to zero and solve. Once we have these solutions we just need to check that none of them make the denominator zero as well.

In our case the numerator is one and will never be zero and so this function will have no *x*-intercepts. Again, the graph will get very close to the *x*-axis but it will never touch or cross it.

Finally, we need to address the fact that graph gets very close to the *x* and *y*-axis but never crosses. Since there isn't anything special about the axis themselves we'll use the fact that the *x*-axis is really the line given by $y = 0$ and the *y*-axis is really the line given by $x = 0$.

In our graph as the value of *x* approaches $x = 0$ the graph starts gets very large on both sides of the line given by $x = 0$. This line is called a **vertical asymptote.**

Also, as *x* get very large, both positive and negative, the graph approaches the line given by $y = 0$. This line is called a **horizontal asymptote**.

Here are the general definitions of the two asymptotes.

1. The line $x = a$ is a **vertical asymptote** if the graph increases or decreases without bound on one or both sides of the line as *x* moves in closer and closer to $x = a$.

2. The line $y = b$ is a **horizontal asymptote** if the graph approaches $y = b$ as *x* increases or decreases without bound. Note that it doesn't have to approach $y = b$ as *x* BOTH increases and decreases. It only needs to approach it on one side in order for it to be a horizontal asymptote.

Determining asymptotes is actually a fairly simple process. First, let's start with the rational function,

$$f\left(x\right) = \frac{ax^n + \cdots}{bx^m + \cdots}$$

where *n* is the largest exponent in the numerator and *m* is the largest exponent in the denominator.

We then have the following facts about asymptotes.

1. The graph will have a vertical asymptote at $x = a$ if the denominator is zero at $x = a$ and the numerator isn't zero at $x = a$.

2. If $n < m$ then the *x*-axis is the horizontal asymptote.

3. If $n = m$ then the line $y = \dfrac{a}{b}$ is the horizontal asymptote.

4. If $n > m$ there will be no horizontal asymptotes.

The process for graphing a rational function is fairly simple. Here it is.

Process for Graphing a Rational Function

1. Find the intercepts, if there are any. Remember that the *y*-intercept is given by $\left(0, f\left(0\right)\right)$ and we find the *x*-intercepts by setting the numerator equal to zero and solving.

2. Find the vertical asymptotes by setting the denominator equal to zero and solving.

3. Find the horizontal asymptote, if it exists, using the fact above.

4. The vertical asymptotes will divide the number line into regions. In each region graph at least one point in each region. This point will tell us whether the graph will be above or below the horizontal asymptote and if we need to we should get several points to determine the general shape of the graph.

5. Sketch the graph.

Note that the sketch that we'll get from the process is going to be a fairly rough sketch but that is okay. That's all that we're really after is a basic idea of what the graph will look at.

Let's take a look at a couple of examples.

Example 1 Sketch the graph of the following function.

$$f\left(x\right) = \frac{3x+6}{x-1}$$

Solution

So, we'll start off with the intercepts. The *y*-intercept is,

$$f\left(0\right) = \frac{6}{-1} = -6 \qquad \Rightarrow \qquad \left(0,-6\right)$$

The *x*-intercepts will be,

$$\begin{aligned} 3x+6 &= 0 \\ x &= -2 \qquad \Rightarrow \qquad \left(-2,0\right) \end{aligned}$$

Now, we need to determine the asymptotes. Let's first find the vertical asymptotes.

$$x - 1 = 0 \qquad \Rightarrow \qquad x = 1$$

So, we've got one vertical asymptote. This means that there are now two regions of *x*'s. They are $x < 1$ and $x > 1$.

Now, the largest exponent in the numerator and denominator is 1 and so by the fact there will be a horizontal asymptote at the line.

$$y = \frac{3}{1} = 3$$

Now, we just need points in each region of *x*'s. Since the *y*-intercept and *x*-intercept are already in the left region we won't need to get any points there. That means that we'll just need to get a point in the right region. It doesn't really matter what value of *x* we pick here we just need to keep it fairly small so it will fit onto our graph.

$$f(2) = \frac{3(2)+6}{2-1} = \frac{12}{1} = 12 \qquad \Rightarrow \qquad (2,12)$$

Okay, putting all this together gives the following graph.

Note that the asymptotes are shown as dotted lines.

Example 2 Sketch the graph of the following function.

$$f(x) = \frac{9}{x^2 - 9}$$

Solution

Okay, we'll start with the intercepts. The *y*-intercept is,

$$f(0) = \frac{9}{-9} = -1 \qquad \Rightarrow \qquad (0,-1)$$

The numerator is a constant and so there won't be any *x*-intercepts since the function can never be zero.

Next, we'll have vertical asymptotes at,

$$x^2 - 9 = 0 \qquad \Rightarrow \qquad x = \pm 3$$

So, in this case we'll have three regions to our graph : $x < -3$, $-3 < x < 3$, $x > 3$.

Also, the largest exponent in the denominator is 2 and since there are no *x*'s in the numerator the largest exponent is 0, so by the fact the *x*-axis will be the horizontal asymptote.

Finally, we need some points. We'll use the following points here.

$$f(-4) = \frac{9}{7} \qquad \left(-4, \frac{9}{7}\right)$$

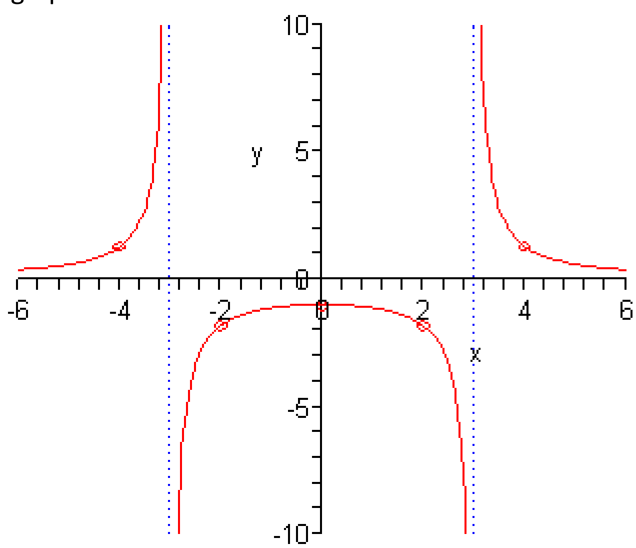
$$f(-2) = -\frac{9}{5} \qquad \left(-2, -\frac{9}{5}\right)$$

$$f(2) = -\frac{9}{5} \qquad \left(2, -\frac{9}{5}\right)$$

$$f(4) = \frac{9}{7} \qquad \left(4, \frac{9}{7}\right)$$

Notice that along with the *y*-intercept we actually have three points in the middle region. This is because there are a couple of possible behaviors in this region and we'll need to determine the actual behavior. We'll see the other main behaviors in the next examples and so this will make more sense at that point.

Here is the sketch of the graph.



Example 3 Sketch the graph of the following function.

$$f(x) = \frac{x^2 - 4}{x^2 - 4x}$$

Solution

This time notice that if we were to plug in $x = 0$ into the denominator we would get division by zero. This means there will not be a *y*-intercept for this graph. We have however, managed to find a vertical asymptote already.

Now, let's see if we've got *x*-intercepts.

$$x^2 - 4 = 0 \qquad \Rightarrow \qquad x = \pm 2$$

So, we've got two of them.

We've got one vertical asymptote, but there may be more so let's go through the process and see.

$$x^2 - 4x = x\left(x-4\right) = 0 \qquad \Rightarrow \qquad x = 0,\ x = 4$$

So, we've got two again and the three regions that we've got are $x < 0$, $0 < x < 4$ and $x > 4$.

Next, the largest exponent in both the numerator and denominator is 2 so by the fact there will be a horizontal asymptote at the line,

$$y = \frac{1}{1} = 1$$

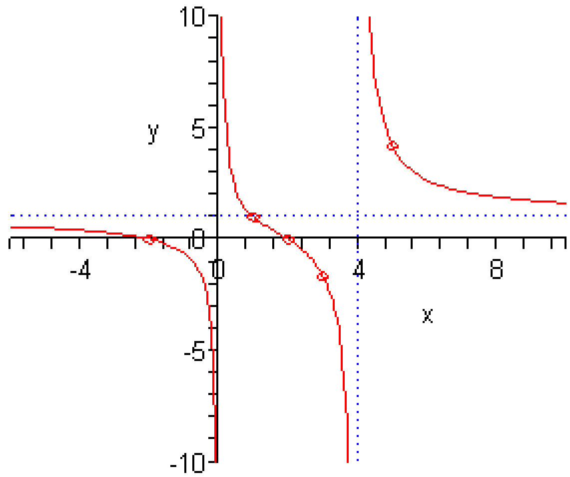
Now, one of the *x*-intercepts is in the far left region so we don't need any points there. The other *x*-intercept is in the middle region. So, we'll need a point in the far right region and as noted in the previous example we will want to get a couple more points in the middle region to completely determine its behavior.

$$f\left(1\right) = 1 \qquad \left(1,1\right)$$

$$f\left(3\right) = -\frac{5}{3} \qquad \left(3,-\frac{5}{3}\right)$$

$$f\left(5\right) = \frac{21}{5} \qquad \left(5,\frac{21}{5}\right)$$

Here is the sketch for this function.



Notice that this time the middle region doesn't have the same behavior at the asymptotes as we saw in the previous example. This can and will happen fairly often. Sometimes the behavior at the two asymptotes will be the same as in the previous example and sometimes it will have the opposite behavior at each asymptote as we see in this example. Because of this we will always need to get a couple of points in these types of regions to determine just what the behavior will be.