ALGEBRA

Graphing and Functions

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Preface

Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it's been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Algebra or needing a refresher for Algebra. I've tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you've had some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 3 : Graphing and Functions

In this chapter we will be introducing two topics that are very important in an algebra class. We will start off the chapter with a brief discussion of graphing. This is not really the main topic of this chapter, but we need the basics down before moving into the second topic of this chapter. The next chapter will contain the remainder of the graphing discussion.

The second topic that we'll be looking at is that of functions. This is probably one of the more important ideas that will come out of an Algebra class. When first studying the concept of functions many students don't really understand the importance or usefulness of functions and function notation. The importance and/or usefulness of functions and function notation will only become apparent in later chapters and later classes. In fact, there are some topics that can only be done easily with function and function notation.

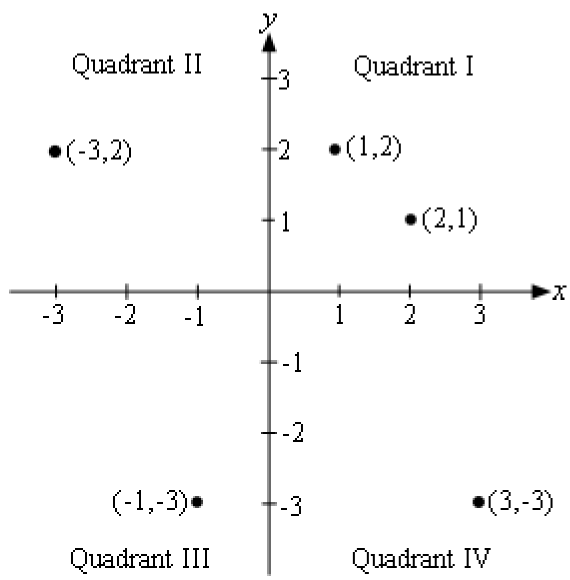
Here is a brief listing of the topics in this chapter.

Graphing – In this section we will introduce the Cartesian (or Rectangular) coordinate system. We will define/introduce ordered pairs, coordinates, quadrants, and *x* and *y*-intercepts. We will illustrate these concepts with a couple of quick examples
Lines – In this section we will discuss graphing lines. We will introduce the concept of slope and discuss how to find it from two points on the line. In addition, we will introduce the standard form of the line as well as the point-slope form and slope-intercept form of the line. We will finish off the section with a discussion on parallel and perpendicular lines.
Circles – In this section we discuss graphing circles. We introduce the standard form of the circle and show how to use completing the square to put an equation of a circle into standard form.
The Definition of a Function – In this section we will formally define relations and functions. We also give a "working definition" of a function to help understand just what a function is. We introduce function notation and work several examples illustrating how it works. We also define the domain and range of a function. In addition, we introduce piecewise functions in this section.
Graphing Functions – In this section we discuss graphing functions including several examples of graphing piecewise functions.
Combining functions – In this section we will discuss how to add, subtract, multiply and divide functions. In addition, we introduce the concept of function composition.
Inverse Functions – In this section we define one-to-one and inverse functions. We also discuss a process we can use to find an inverse function and verify that the function we get from this process is, in fact, an inverse function.

Section 3-1 : Graphing

In this section we need to review some of the basic ideas in graphing. It is assumed that you've seen some graphing to this point and so we aren't going to go into great depth here. We will only be reviewing some of the basic ideas.

We will start off with the Rectangular or Cartesian coordinate system. This is just the standard axis system that we use when sketching our graphs. Here is the Cartesian coordinate system with a few points plotted.



The horizontal and vertical axes, typically called the ***x*-axis** and the ***y*-axis** respectively, divide the coordinate system up into quadrants as shown above. In each quadrant we have the following signs for *x* and *y*.

Quadrant I	$x > 0$, or *x* positive	$y > 0$, or *y* positive
Quadrant II	$x < 0$, or *x* negative	$y > 0$, or *y* positive
Quadrant III	$x < 0$, or *x* negative	$y < 0$, or *y* negative
Quadrant IV	$x > 0$, or *x* positive	$y < 0$, or *y* negative

Each point in the coordinate system is defined by an **ordered pair** of the form (x, y). The first number listed is the ***x*-coordinate** of the point and the second number listed is the ***y*-coordinate** of the point. The ordered pair for any given point, (x, y), is called the **coordinates** for the point.

The point where the two axes cross is called the **origin** and has the coordinates $(0,0)$.

Note as well that the order of the coordinates is important. For example, the point $\left(2,1\right)$ is the point that is two units to the right of the origin and then 1 unit up, while the point $\left(1,2\right)$ is the point that is 1 unit to the right of the origin and then 2 units up.

We now need to discuss graphing an equation. The first question that we should ask is what exactly is a graph of an equation? A graph is the set of all the ordered pairs whose coordinates satisfy the equation.

For instance, the point $\left(2,-3\right)$ is a point on the graph of $y=\left(x-1\right)^2-4$ while $\left(1,5\right)$ isn't on the graph. How do we tell this? All we need to do is take the coordinates of the point and plug them into the equation to see if they satisfy the equation. Let's do that for both to verify the claims made above.

$\left(2,-3\right)$:

In this case we have $x=2$ and $y=-3$ so plugging in gives,

$$\begin{aligned} -3 &\stackrel{?}{=} \left(2-1\right)^2-4 \\ -3 &\stackrel{?}{=} \left(1\right)^2-4 \\ -3 &= -3 \qquad\qquad \text{OK} \end{aligned}$$

So, the coordinates of this point satisfies the equation and so it is a point on the graph.

$\left(1,5\right)$:

Here we have $x=1$ and $y=5$. Plugging these in gives,

$$\begin{aligned} 5 &\stackrel{?}{=} \left(1-1\right)^2-4 \\ 5 &\stackrel{?}{=} \left(0\right)^2-4 \\ 5 &\neq -4 \qquad\qquad \text{NOT OK} \end{aligned}$$

The coordinates of this point do NOT satisfy the equation and so this point isn't on the graph.

Now, how do we sketch the graph of an equation? Of course, the answer to this depends on just how much you know about the equation to start off with. For instance, if you know that the equation is a line or a circle we've got simple ways to determine the graph in these cases. There are also many other kinds of equations that we can usually get the graph from the equation without a lot of work. We will see many of these in the next chapter.

However, let's suppose that we don't know ahead of time just what the equation is or any of the ways to quickly sketch the graph. In these cases we will need to recall that the graph is simply all the points that satisfy the equation. So, all we can do is plot points. We will pick values of *x*, compute *y* from the equation and then plot the ordered pair given by these two values.

How, do we determine which values of *x* to choose? Unfortunately, the answer there is we guess. We pick some values and see what we get for a graph. If it looks like we've got a pretty good sketch we stop. If not we pick some more. Knowing the values of *x* to choose is really something that we can only get with experience and some knowledge of what the graph of the equation will *probably* look like. Hopefully, by the end of this course you will have gained some of this knowledge.

Let's take a quick look at a graph.

Example 1 Sketch the graph of $y = \left(x-1\right)^2 - 4$.

Solution
Now, this is a parabola and after the next chapter you will be able to quickly graph this without much effort. However, we haven't gotten that far yet and so we will need to choose some values of *x*, plug them in and compute the *y* values.

As mentioned earlier, it helps to have an idea of what this graph is liable to look like when picking values of *x*. So, don't worry at this point why we chose the values that we did. After the next chapter you would also be able to choose these values of *x*.

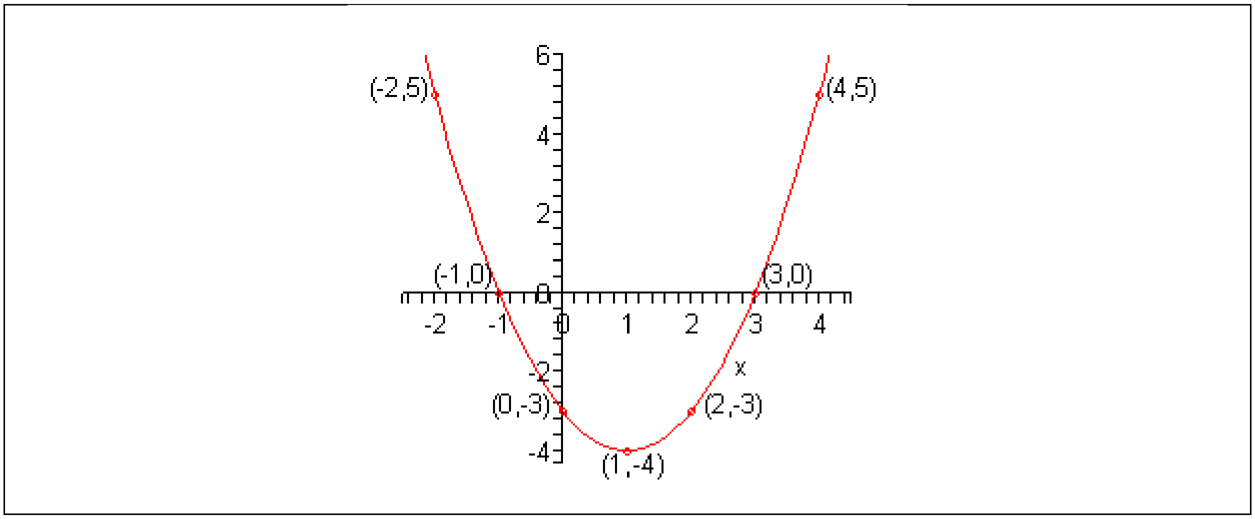
Here is a table of values for this equation.

x	y	(x,y)
-2	5	$(-2,5)$
-1	0	$(-1,0)$
0	-3	$(0,-3)$
1	-4	$(1,-4)$
2	-3	$(2,-3)$
3	0	$(3,0)$
4	5	$(4,5)$

Let's verify the first one and we'll leave the rest to you to verify. For the first one we simply plug $x = -2$ into the equation and compute *y*.

$$\begin{aligned} y &= \left(-2-1\right)^2 - 4 \\ &= \left(-3\right)^2 - 4 \\ &= 9 - 4 \\ &= 5 \end{aligned}$$

Here is the graph of this equation.



Notice that when we set up the axis system in this example, we only set up as much as we needed. For example, since we didn't go past -2 with our computations we didn't go much past that with our axis system.

Also, notice that we used a different scale on each of the axes. With the horizontal axis we incremented by 1's while on the vertical axis we incremented by 2. This will often be done in order to make the sketching easier.

The final topic that we want to discuss in this section is that of **intercepts**. Notice that the graph in the above example crosses the *x*-axis in two places and the *y*-axis in one place. All three of these points are called intercepts. We can, and often will be, more specific however.

We often will want to know if an intercept crosses the *x* or *y*-axis specifically. So, if an intercept crosses the *x*-axis we will call it an ***x*-intercept**. Likewise, if an intercept crosses the *y*-axis we will call it a ***y*-intercept**.

Now, since the *x*-intercept crosses *x*-axis then the *y* coordinates of the *x*-intercept(s) will be zero. Also, the *x* coordinate of the *y*-intercept will be zero since these points cross the *y*-axis. These facts give us a way to determine the intercepts for an equation. To find the *x*-intercepts for an equation all that we need to do is set $y = 0$ and solve for *x*. Likewise, to find the *y*-intercepts for an equation we simply need to set $x = 0$ and solve for *y*.

Let's take a quick look at an example.

Example 2 Determine the *x*-intercepts and *y*-intercepts for each of the following equations.

(a) $y = x^2 + x - 6$

(b) $y = x^2 + 2$

(c) $y = (x+1)^2$

Solution

As verification for each of these we will also sketch the graph of each function. We will leave the details of the sketching to you to verify. Also, these are all parabolas and as mentioned earlier we will be looking at these in detail in the next chapter.

(a) $y = x^2 + x - 6$

Let's first find the *y*-intercept(s). Again, we do this by setting $x = 0$ and solving for *y*. This is usually the easier of the two. So, let's find the *y*-intercept(s).

$$y = (0)^2 + 0 - 6 = -6$$

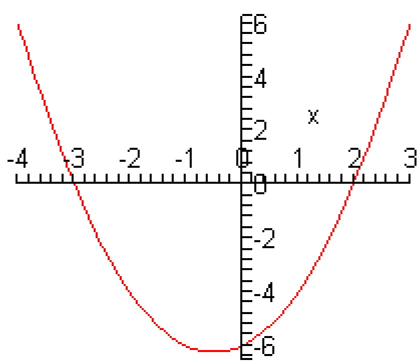
So, there is a single *y*-intercept : $(0,-6)$.

The work for the *x*-intercept(s) is almost identical except in this case we set $y = 0$ and solve for *x*. Here is that work.

$$\begin{aligned} 0 &= x^2 + x - 6 \\ 0 &= (x+3)(x-2) \qquad \Rightarrow \qquad x = -3,\ x = 2 \end{aligned}$$

For this equation there are two *x*-intercepts : $(-3,0)$ and $(2,0)$. Oh, and you do remember how to solve quadratic equations right?

For verification purposes here is sketch of the graph for this equation.



(b) $y = x^2 + 2$

First, the *y*-intercepts.

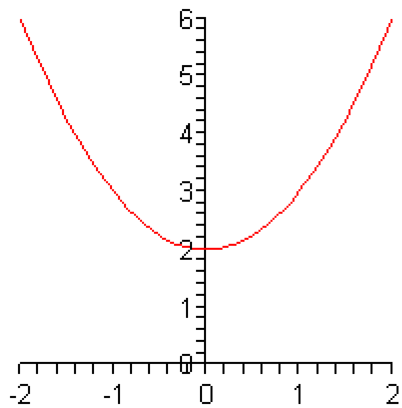
$$y = (0)^2 + 2 = 2 \qquad \Rightarrow \qquad (0,2)$$

So, we've got a single *y*-intercepts. Now, the *x*-intercept(s).

$$0 = x^2 + 2$$
$$-2 = x^2 \qquad \Rightarrow \qquad x = \pm\sqrt{2}\,i$$

Okay, we got complex solutions from this equation. What this means is that we will not have any *x*-intercepts. Note that it is perfectly acceptable for this to happen so don't worry about it when it does happen.

Here is the graph for this equation.



Sure enough, it doesn't cross the *x*-axis.

(c) $y = (x+1)^2$

Here is the *y*-intercept work for this equation.

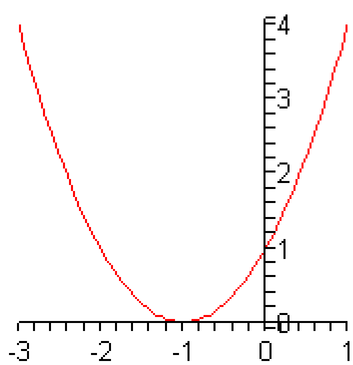
$$y = (0+1)^2 = 1 \qquad \Rightarrow \qquad (0,1)$$

Now the *x*-intercept work.

$$0 = (x+1)^2 \qquad \Rightarrow \qquad x = -1 \qquad \Rightarrow \qquad (-1,0)$$

In this case we have a single *x*-intercept.

Here is a sketch of the graph for this equation.



Now, notice that in this case the graph doesn't actually cross the *x*-axis at $x = -1$. This point is still called an *x*-intercept however.

We should make one final comment before leaving this section. In the previous set of examples all the equations were quadratic equations. This was done only because they exhibited the range of behaviors that we were looking for and we would be able to do the work as well. You should not walk away from this discussion of intercepts with the idea that they will only occur for quadratic equations. They can, and do, occur for many different equations.

Section 3-2 : Lines

Let's start this section off with a quick mathematical definition of a line. Any equation that can be written in the form,

$$Ax + By = C$$

where we can't have both *A* and *B* be zero simultaneously is a line. It is okay if one of them is zero, we just can't have both be zero. Note that this is sometimes called the **standard form** of the line.

Before we get too far into this section it would probably be helpful to recall that a line is defined by any two points that are on the line. Given two points that are on the line we can graph the line and/or write down the equation of the line. This fact will be used several times throughout this section.

One of the more important ideas that we'll be discussing in this section is that of **slope**. The slope of a line is a measure of the *steepness* of a line and it can also be used to measure whether a line is increasing or decreasing as we move from left to right. Here is the precise definition of the slope of a line.

Given any two points on the line say, $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$, the slope of the line is given by,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

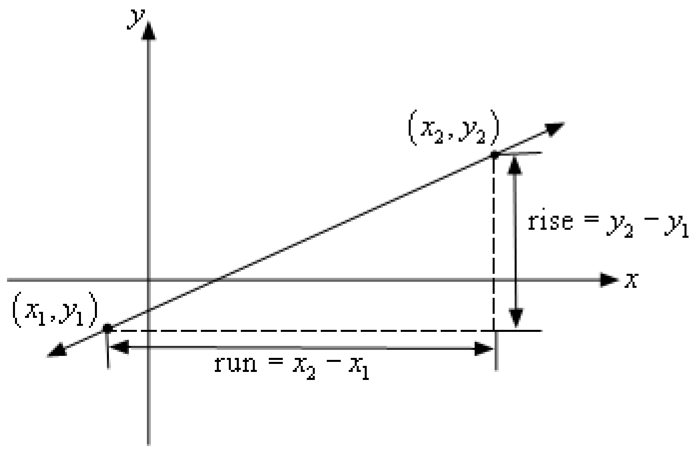
In other words, the slope is the difference in the *y* values divided by the difference in the *x* values. Also, do not get worried about the subscripts on the variables. These are used fairly regularly from this point on and are simply used to denote the fact that the variables are both *x* or *y* values but are, in all likelihood, different.

When using this definition do not worry about which point should be the first point and which point should be the second point. You can choose either to be the first and/or second and we'll get exactly the same value for the slope.

There is also a geometric "definition" of the slope of the line as well. You will often hear the slope as being defined as follows,

$$m = \frac{\text{rise}}{\text{run}}$$

The two definitions are identical as the following diagram illustrates. The numerators and denominators of both definitions are the same.



Note as well that if we have the slope (written as a fraction) and a point on the line, say (x_1, y_1), then we can easily find a second point that is also on the line. Before seeing how this can be done let's take the convention that if the slope is negative we will put the minus sign on the numerator of the slope. In other words, we will assume that the *rise* is negative if the slope is negative. Note as well that a negative *rise* is really a *fall*.

So, we have the slope, written as a fraction, and a point on the line, (x_1, y_1). To get the coordinates of the second point, (x_2, y_2) all that we need to do is start at (x_1, y_1) then move to the right by the *run* (or denominator of the slope) and then up/down by *rise* (or the numerator of the slope) depending on the sign of the *rise*. We can also write down some equations for the coordinates of the second point as follows,

$$\begin{aligned} x_2 &= x_1 + \text{run} \\ y_2 &= y_1 + \text{rise} \end{aligned}$$

Note that if the slope is negative then the *rise* will be a negative number.

Let's compute a couple of slopes.

Example 1 Determine the slope of each of the following lines. Sketch the graph of each line.

(a) The line that contains the two points $(-2,-3)$ and $(3,1)$.

(b) The line that contains the two points $(-1,5)$ and $(0,-2)$.

(c) The line that contains the two points $(-3,2)$ and $(5,2)$.

(d) The line that contains the two points $(4,3)$ and $(4,-2)$.

Solution

Okay, for each of these all that we'll need to do is use the slope formula to find the slope and then plot the two points and connect them with a line to get the graph.

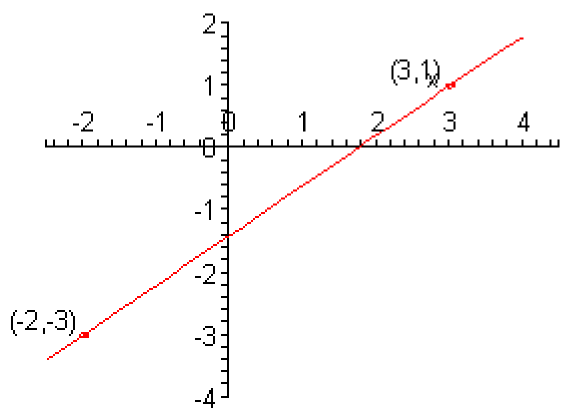
(a) The line that contains the two points $(-2,-3)$ and $(3,1)$.

Do not worry which point gets the subscript of 1 and which gets the subscript of 2. Either way will get the same answer. Typically, we'll just take them in the order listed. So, here is the slope for this part.

$$m = \frac{1-(-3)}{3-(-2)} = \frac{1+3}{3+2} = \frac{4}{5}$$

Be careful with minus signs in these computations. It is easy to lose track of them. Also, when the slope is a fraction, as it is here, leave it as a fraction. Do not convert to a decimal unless you absolutely have to.

Here is a sketch of the line.



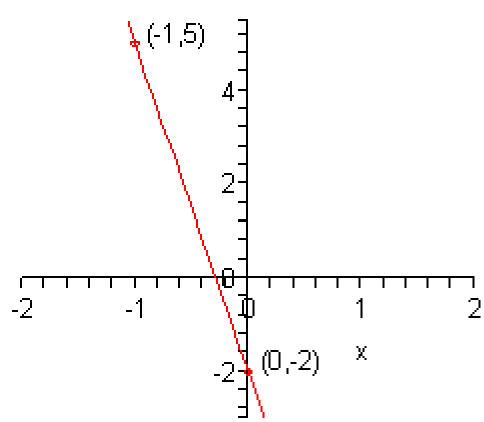
Notice that this line increases as we move from left to right.

(b) The line that contains the two points $(-1,5)$ and $(0,-2)$.

Here is the slope for this part.

$$m = \frac{-2-5}{0-(-1)} = \frac{-7}{1} = -7$$

Again, watch out for minus signs. Here is a sketch of the graph.



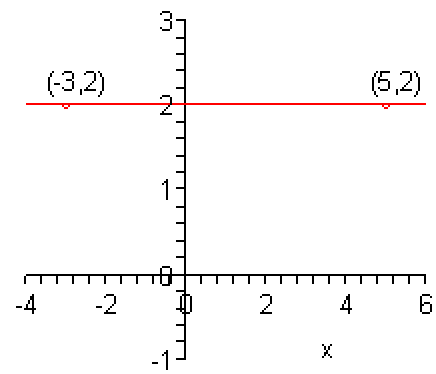
This line decreases as we move from left to right.

(c) The line that contains the two points $\left(-3,2\right)$ and $\left(5,2\right)$.

Here is the slope for this line.

$$m = \frac{2-2}{5-\left(-3\right)} = \frac{0}{8} = 0$$

We got a slope of zero here. That is okay, it will happen sometimes. Here is the sketch of the line.



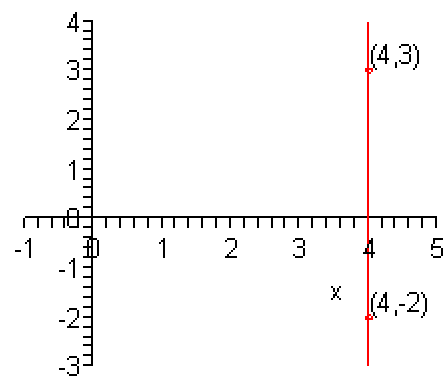
In this case we've got a horizontal line.

(d) The line that contains the two points $\left(4,3\right)$ and $\left(4,-2\right)$.

The final part. Here is the slope computation.

$$m = \frac{-2-3}{4-4} = \frac{-5}{0} = \text{undefined}$$

In this case we get division by zero which is undefined. Again, don't worry too much about this it will happen on occasion. Here is a sketch of this line.



This final line is a vertical line.

We can use this set of examples to see some general facts about lines.

First, we can see from the first two parts that the larger the number (ignoring any minus signs) the steeper the line. So, we can use the slope to tell us something about just how steep a line is.

Next, we can see that if the slope is a positive number then the line will be increasing as we move from left to right. Likewise, if the slope is a negative number then the line will decrease as we move from left to right.

We can use the final two parts to see what the slopes of horizontal and vertical lines will be. A horizontal line will always have a slope of zero and a vertical line will always have an undefined slope.

We now need to take a look at some special forms of the equation of the line.

We will start off with horizontal and vertical lines. A horizontal line with a *y*-intercept of *b* will have the equation,

$$\boxed{y = b}$$

Likewise, a vertical line with an *x*-intercept of *a* will have the equation,

$$\boxed{x = a}$$

So, if we go back and look that the last two parts of the previous example we can see that the equation of the line for the horizontal line in the third part is

$$y = 2$$

while the equation for the vertical line in the fourth part is

$$x = 4$$

The next special form of the line that we need to look at is the **point-slope form** of the line. This form is very useful for writing down the equation of a line. If we know that a line passes through the point $\left(x_1, y_1\right)$ and has a slope of *m* then the point-slope form of the equation of the line is,

$$\boxed{y - y_1 = m\left(x - x_1\right)}$$

Sometimes this is written as,

$$\boxed{y = y_1 + m\left(x - x_1\right)}$$

The form it's written in usually depends on the instructor that is teaching the class.

As stated earlier this form is particularly useful for writing down the equation of a line so let's take a look at an example of this.

Example 2 Write down the equation of the line that passes through the two points $\left(-2, 4\right)$ and $\left(3, -5\right)$.

Solution

At first glance it may not appear that we'll be able to use the point-slope form of the line since this requires a single point (we've got two) and the slope (which we don't have). However, that fact that we've got two points isn't really a problem; in fact, we can use these two points to determine the missing slope of the line since we do know that we can always find that from any two points on the line.

So, let's start off my finding the slope of the line.

$$m = \frac{-5-4}{3-(-2)} = -\frac{9}{5}$$

Now, which point should we use to write down the equation of the line? We can actually use either point. To show this we will use both.

First, we'll use $(-2,4)$. Now that we've gotten the point all that we need to do is plug into the formula. We will use the second form.

$$y = 4 - \frac{9}{5}\left(x-(-2)\right) = 4 - \frac{9}{5}(x+2)$$

Now, let's use $(3,-5)$.

$$y = -5 - \frac{9}{5}(x-3)$$

Okay, we claimed that it wouldn't matter which point we used in the formula, but these sure look like different equations. It turns out however, that these really are the same equation. To see this all that we need to do is distribute the slope through the parenthesis and then simplify.

Here is the first equation.

$$\begin{aligned} y &= 4 - \frac{9}{5}(x+2) \\ &= 4 - \frac{9}{5}x - \frac{18}{5} \\ &= -\frac{9}{5}x + \frac{2}{5} \end{aligned}$$

Here is the second equation.

$$\begin{aligned} y &= -5 - \frac{9}{5}(x-3) \\ &= -5 - \frac{9}{5}x + \frac{27}{5} \\ &= -\frac{9}{5}x + \frac{2}{5} \end{aligned}$$

So, sure enough they are the same equation.

The final special form of the equation of the line is probably the one that most people are familiar with. It is the **slope-intercept form**. In this case if we know that a line has slope *m* and has a *y*-intercept of $(0,b)$ then the slope-intercept form of the equation of the line is,

$$y = mx + b$$

This form is particularly useful for graphing lines. Let's take a look at a couple of examples.

Example 3 Determine the slope of each of the following equations and sketch the graph of the line.

(a) $2y - 6x = -2$

(b) $3y + 4x = 6$

Solution

Okay, to get the slope we'll first put each of these in slope-intercept form and then the slope will simply be the coefficient of the *x* (including sign). To graph the line we know the *y*-intercept of the line, that's the number without an *x* (including sign) and as discussed above we can use the slope to find a second point on the line. At that point there isn't anything to do other than sketch the line.

(a) $2y - 6x = -2$

First solve the equation for *y*. Remember that we solved equations like this back in the previous chapter.

$$\begin{aligned} 2y &= 6x - 2 \\ y &= 3x - 1 \end{aligned}$$

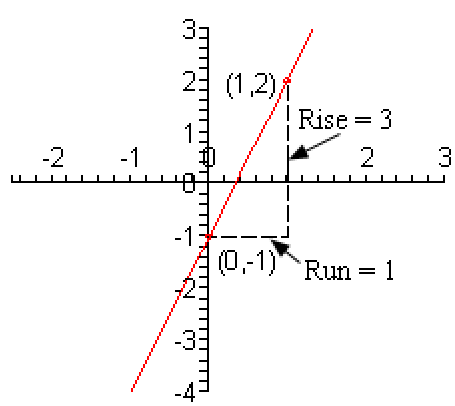
So, the slope for this line is 3 and the *y*-intercept is the point $(0, -1)$. Don't forget to take the sign when determining the *y*-intercept. Now, to find the second point we usually like the slope written as a fraction to make it clear what the *rise* and *run* are. So,

$$m = 3 = \frac{3}{1} = \frac{\text{rise}}{\text{run}} \qquad \Rightarrow \qquad \text{rise} = 3, \quad \text{run} = 1$$

The second point is then,

$$x_2 = 0 + 1 = 1 \qquad\qquad y_2 = -1 + 3 = 2 \qquad\qquad \Rightarrow \qquad\qquad (1, 2)$$

Here is a sketch of the graph of the line.



(b) $3y + 4x = 6$

Again, solve for *y*.

$$3y = -4x + 6$$

$$y = -\frac{4}{3}x + 2$$

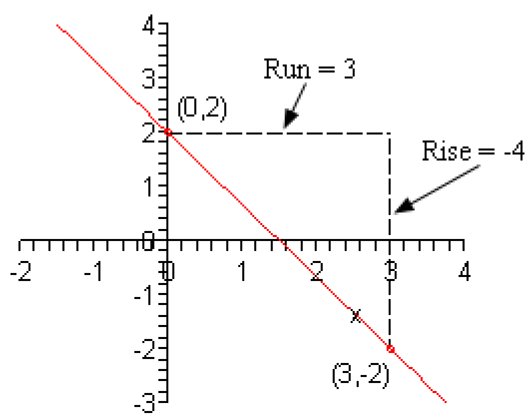
In this case the slope is $-\frac{4}{3}$ and the *y*-intercept is $(0,2)$. As with the previous part let's first determine the *rise* and the *run*.

$$m = -\frac{4}{3} = \frac{-4}{3} = \frac{\text{rise}}{\text{run}} \qquad \Rightarrow \qquad \text{rise} = -4, \ \ \text{run} = 3$$

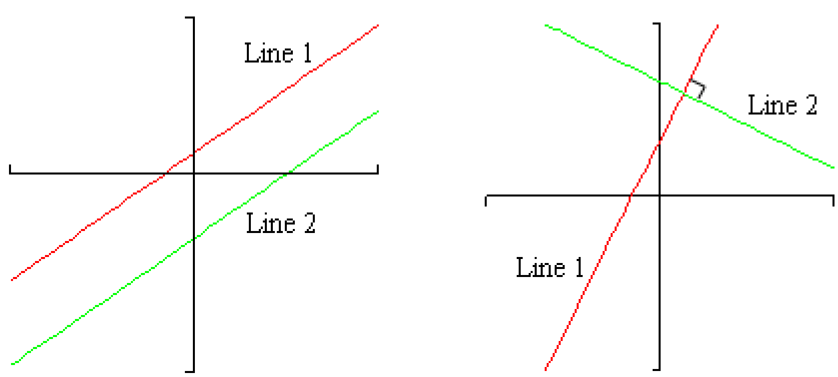
Again, remember that if the slope is negative make sure that the minus sign goes with the numerator. The second point is then,

$$x_2 = 0 + 3 = 3 \qquad y_2 = 2 + (-4) = -2 \qquad \Rightarrow \qquad (3,-2)$$

Here is the sketch of the graph for this line.



The final topic that we need to discuss in this section is that of parallel and perpendicular lines. Here is a sketch of parallel and perpendicular lines.



Parallel Lines

Perpendicular Lines

Suppose that the slope of Line 1 is m_1 and the slope of Line 2 is m_2. We can relate the slopes of parallel lines and we can relate slopes of perpendicular lines as follows.

$$\text{parallel} : m_1 = m_2$$
$$\text{perpendicular} : m_1 m_2 = -1 \quad \text{or} \quad m_2 = -\frac{1}{m_1}$$

Note that there are two forms of the equation for perpendicular lines. The second is the more common and in this case we usually say that m_2 is the negative reciprocal of m_1.

Example 4 Determine if the line that passes through the points $\left(-2,-10\right)$ and $\left(6,-1\right)$ is parallel, perpendicular or neither to the line given by $7y - 9x = 15$.

Solution
Okay, in order to do answer this we'll need the slopes of the two lines. Since we have two points for the first line we can use the formula for the slope,

$$m_1 = \frac{-1-\left(-10\right)}{6-\left(-2\right)} = \frac{9}{8}$$

We don't actually need the equation of this line and so we won't bother with it.

Now, to get the slope of the second line all we need to do is put it into slope-intercept form.

$$7y = 9x + 15$$
$$y = \frac{9}{7}x + \frac{15}{7} \qquad \Rightarrow \qquad m_2 = \frac{9}{7}$$

Okay, since the two slopes aren't the same (they're close, but still not the same) the two lines are not parallel. Also,

$$\left(\frac{9}{8}\right)\left(\frac{9}{7}\right) = \frac{81}{56} \neq -1$$

so the two lines aren't perpendicular either.

Therefore, the two lines are neither parallel nor perpendicular.

Example 5 Determine the equation of the line that passes through the point $\left(8,2\right)$ and is,

(a) parallel to the line given by $10y + 3x = -2$

(b) perpendicular to the line given by $10y + 3x = -2$.

Solution
In both parts we are going to need the slope of the line given by $10y + 3x = -2$ so let's actually find that before we get into the individual parts.

$$10y = -3x - 2$$
$$y = -\frac{3}{10}x - \frac{1}{5} \qquad \Rightarrow \qquad m_1 = -\frac{3}{10}$$

Now, let's work the example.

(a) parallel to the line given by $10y + 3x = -2$

In this case the new line is to be parallel to the line given by $10y + 3x = -2$ and so it must have the same slope as this line. Therefore, we know that,

$$m_2 = -\frac{3}{10}$$

Now, we've got a point on the new line, $\left(8,2\right)$, and we know the slope of the new line, $-\frac{3}{10}$, so we can use the point-slope form of the line to write down the equation of the new line. Here is the equation,

$$\begin{aligned} y &= 2 - \frac{3}{10}\left(x-8\right) \\ &= 2 - \frac{3}{10}x + \frac{24}{10} \\ &= -\frac{3}{10}x + \frac{44}{10} \\ y &= -\frac{3}{10}x + \frac{22}{5} \end{aligned}$$

(b) perpendicular to the line given by $10y + 3x = -2$

For this part we want the line to be perpendicular to $10y + 3x = -2$ and so we know we can find the new slope as follows,

$$m_2 = -\frac{1}{-\frac{3}{10}} = \frac{10}{3}$$

Then, just as we did in the previous part we can use the point-slope form of the line to get the equation of the new line. Here it is,

$$\begin{aligned} y &= 2 + \frac{10}{3}\left(x-8\right) \\ &= 2 + \frac{10}{3}x - \frac{80}{3} \\ y &= \frac{10}{3}x - \frac{74}{3} \end{aligned}$$

Section 3-3 : Circles

In this section we are going to take a quick look at circles. However, before we do that we need to give a quick formula that hopefully you'll recall seeing at some point in the past.

Given two points (x_1, y_1) and (x_2, y_2) the distance between them is given by,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

So, why did we remind you of this formula? Well, let's recall just what a circle is. A circle is all the points that are the same distance, *r* – called the radius, from a point, (h,k) - called the center. In other words, if (x, y) is any point that is on the circle then it has a distance of *r* from the center, (h,k).

If we use the distance formula on these two points we would get,

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

Or, if we square both sides we get,

$$(x-h)^2 + (y-k)^2 = r^2$$

This is the **standard form** of the equation of a circle with radius *r* and center (h,k).

Example 1 Write down the equation of a circle with radius 8 and center $(-4,7)$.

Solution

Okay, in this case we have $r = 8$, $h = -4$ and $k = 7$ so all we need to do is plug them into the standard form of the equation of the circle.

$$\begin{aligned} (x-(-4))^2 + (y-7)^2 &= 8^2 \\ (x+4)^2 + (y-7)^2 &= 64 \end{aligned}$$

Do not square out the two terms on the left. Leaving these terms as they are will allow us to quickly identify the equation as that of a circle and to quickly identify the radius and center of the circle.

Graphing circles is a fairly simple process once we know the radius and center. In order to graph a circle all we really need is the right most, left most, top most and bottom most points on the circle. Once we know these it's easy to sketch in the circle.

Nicely enough for us these points are easy to find. Since these are points on the circle we know that they must be a distance of *r* from the center. Therefore, the points will have the following coordinates.

$$\begin{aligned}&\text{right most point : } (h+r,k)\\&\text{left most point : } (h-r,k)\\&\text{top most point : } (h,k+r)\\&\text{bottom most point : } (h,k-r)\end{aligned}$$

In other words all we need to do is add *r* on to the *x* coordinate or *y* coordinate of the point to get the right most or top most point respectively and subtract *r* from the *x* coordinate or *y* coordinate to get the left most or bottom most points.

Let's graph some circles.

Example 2 Determine the center and radius of each of the following circles and sketch the graph of the circle.

(a) $x^2 + y^2 = 1$

(b) $x^2 + (y-3)^2 = 4$

(c) $(x-1)^2 + (y+4)^2 = 16$

Solution

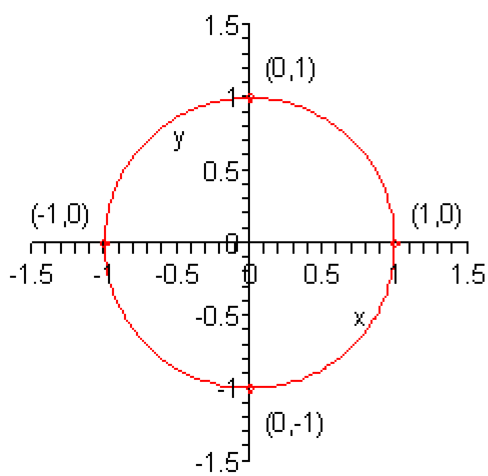
In all of these all that we really need to do is compare the equation to the standard form and identify the radius and center. Once that is done find the four points talked about above and sketch in the circle.

(a) $x^2 + y^2 = 1$

In this case it's just *x* and *y* squared by themselves. The only way that we could have this is to have both *h* and *k* be zero. So, the center and radius is,

$$\text{center} = (0,0) \qquad \text{radius} = \sqrt{1} = 1$$

Don't forget that the radius is the square root of the number on the other side of the equal sign. Here is a sketch of this circle.



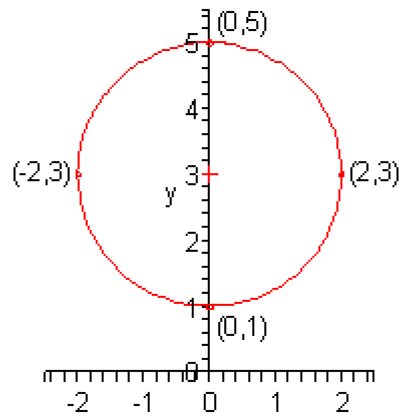
A circle centered at the origin with radius 1 (*i.e.* this circle) is called the **unit circle**. The unit circle is very useful in a Trigonometry class.

(b) $x^2 + (y-3)^2 = 4$

In this part, it looks like the *x* coordinate of the center is zero as with the previous part. However, this time there is something more with the *y* term and so comparing this term to the standard form of the circle we can see that the *y* coordinate of the center must be 3. The center and radius of this circle is then,

$$\text{center} = (0,3) \qquad\qquad \text{radius} = \sqrt{4} = 2$$

Here is a sketch of the circle. The center is marked with a red cross in this graph.



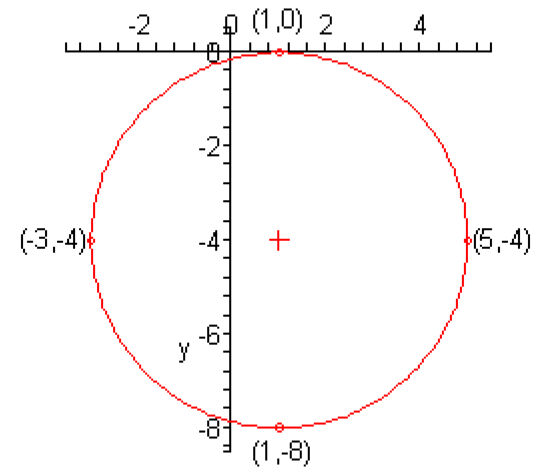
(c) $(x-1)^2 + (y+4)^2 = 16$

For this part neither of the coordinates of the center are zero. By comparing our equation with the standard form it's fairly easy to see (hopefully…) that the *x* coordinate of the center is 1. The *y* coordinate isn't too bad either, but we do need to be a little careful. In this case the term is $(y+4)^2$ and in the standard form the term is $(y-k)^2$. Note that the signs are different. The only way that this can happen is if *k* is negative. So, the *y* coordinate of the center must be -4.

The center and radius for this circle are,

$$\text{center} = (1,-4) \qquad\qquad \text{radius} = \sqrt{16} = 4$$

Here is a sketch of this circle with the center marked with a red cross.



So, we've seen how to deal with circles that are already in the standard form. However, not all circles will start out in the standard form. So, let's take a look at how to put a circle in the standard form.

Example 3 Determine the center and radius of each of the following.

(a) $x^2 + y^2 + 8x + 7 = 0$

(b) $x^2 + y^2 - 3x + 10y - 1 = 0$

Solution

Neither of these equations are in standard form and so to determine the center and radius we'll need to put it into standard form. We actually already know how to do this. Back when we were solving quadratic equations we saw a way to turn a quadratic polynomial into a perfect square. The process was called completing the square.

This is exactly what we want to do here, although in this case we aren't solving anything and we're going to have to deal with the fact that we've got both *x* and *y* in the equation. Let's step through the process with the first part.

(a) $x^2 + y^2 + 8x + 7 = 0$

We'll go through the process in a step by step fashion with this one.

Step 1 : First get the constant on one side by itself and at the same time group the *x* terms together and the *y* terms together.

$$x^2 + 8x + y^2 = -7$$

In this case there was only one term with a *y* in it and two with *x*'s in them.

Step 2 : For each variable with two terms complete the square on those terms.

So, in this case that means that we only need to complete the square on the *x* terms. Recall how this is done. We first take half the coefficient of the *x* and square it.

$$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$$

We then add this to both sides of the equation.

$$x^2 + 8x + 16 + y^2 = -7 + 16 = 9$$

Now, the first three terms will factor as a perfect square.

$$\left(x+4\right)^2 + y^2 = 9$$

Step 3 : This is now the standard form of the equation of a circle and so we can pick the center and radius right off this. They are,

$$\text{center} = \left(-4, 0\right) \qquad\qquad \text{radius} = \sqrt{9} = 3$$

(b) $x^2 + y^2 - 3x + 10y - 1 = 0$

In this part we'll go through the process a little quicker. First get terms properly grouped and placed.

$$\underbrace{x^2 - 3x}_{\text{complete the square}} + \underbrace{y^2 + 10y}_{\text{complete the square}} = 1$$

Now, as noted above we'll need to complete the square twice here, once for the *x* terms and once for the *y* terms. Let's first get the numbers that we'll need to add to both sides.

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4} \qquad\qquad \left(\frac{10}{2}\right)^2 = \left(5\right)^2 = 25$$

Now, add these to both sides of the equation.

$$\underbrace{x^2 - 3x + \frac{9}{4}}_{\text{factor this}} + \underbrace{y^2 + 10y + 25}_{\text{factor this}} = 1 + \frac{9}{4} + 25 = \frac{113}{4}$$

When adding the numbers to both sides make sure and place them properly. This means that we need to put the number from the coefficient of the *x* with the *x* terms and the number from the coefficient of the *y* with the *y* terms. This placement is important since this will be the only way that the quadratics will factor as we need them to factor.

Now, factor the quadratics as show above. This will give the standard form of the equation of the circle.

$$\left(x - \frac{3}{2}\right)^2 + \left(y+5\right)^2 = \frac{113}{4}$$

This looks a little messier than the equations that we've seen to this point. However, this is something that will happen on occasion so don't get excited about it. Here is the center and radius for this circle.

$$\text{center} = \left(\frac{3}{2}, -5\right) \qquad\qquad \text{radius} = \sqrt{\frac{113}{4}} = \frac{\sqrt{113}}{2}$$

Do not get excited about the messy radius or fractions in the center coordinates.

Section 3-4 : The Definition of a Function

We now need to move into the second topic of this chapter. Before we do that however we need a quick definition taken care of.

Definition of Relation

A **relation** is a set of ordered pairs.

This seems like an odd definition but we'll need it for the definition of a function (which is the main topic of this section). However, before we actually give the definition of a function let's see if we can get a handle on just what a relation is.

Think back to Example 1 in the Graphing section of this chapter. In that example we constructed a set of ordered pairs we used to sketch the graph of $y = (x-1)^2 - 4$. Here are the ordered pairs that we used.

$$(-2,5) \quad (-1,0) \quad (0,-3) \quad (1,-4) \quad (2,-3) \quad (3,0) \quad (4,5)$$

Any of the following are then relations because they consist of a set of ordered pairs.

$$\begin{array}{l} \{(-2,5) \quad (-1,0) \quad (2,-3)\} \\ \{(-1,0) \quad (0,-3) \quad (2,-3) \quad (3,0) \quad (4,5)\} \\ \{(3,0) \quad (4,5)\} \\ \{(-2,5) \quad (-1,0) \quad (0,-3) \quad (1,-4) \quad (2,-3) \quad (3,0) \quad (4,5)\} \end{array}$$

There are of course many more relations that we could form from the list of ordered pairs above, but we just wanted to list a few possible relations to give some examples. Note as well that we could also get other ordered pairs from the equation and add those into any of the relations above if we wanted to.

Now, at this point you are probably asking just why we care about relations and that is a good question. Some relations are very special and are used at almost all levels of mathematics. The following definition tells us just which relations are these special relations.

Definition of a Function

A **function** is a relation for which each value from the set the first components of the ordered pairs is associated with exactly one value from the set of second components of the ordered pair.

Okay, that is a mouth full. Let's see if we can figure out just what it means. Let's take a look at the following example that will hopefully help us figure all this out.

Example 1 The following relation is a function.

$$\{(-1,0)\ \ (0,-3)\ \ (2,-3)\ \ (3,0)\ \ (4,5)\}$$

Solution

From these ordered pairs we have the following sets of first components (*i.e.* the first number from each ordered pair) and second components (*i.e.* the second number from each ordered pair).

$$1^{\text{st}} \text{ components : } \{-1,0,2,3,4\} \qquad\qquad 2^{\text{nd}} \text{ components : } \{0,-3,0,5\}$$

For the set of second components notice that the "-3" occurred in two ordered pairs but we only listed it once.

To see why this relation is a function simply pick any value from the set of first components. Now, go back up to the relation and find every ordered pair in which this number is the first component and list all the second components from those ordered pairs. The list of second components will consist of exactly one value.

For example, let's choose 2 from the set of first components. From the relation we see that there is exactly one ordered pair with 2 as a first component, $(2,-3)$. Therefore, the list of second components (*i.e.* the list of values from the set of second components) associated with 2 is exactly one number, -3.

Note that we don't care that -3 is the second component of a second ordered par in the relation. That is perfectly acceptable. We just don't want there to be any more than one ordered pair with 2 as a first component.

We looked at a single value from the set of first components for our quick example here but the result will be the same for all the other choices. Regardless of the choice of first components there will be exactly one second component associated with it.

Therefore, this relation is a function.

In order to really get a feel for what the definition of a function is telling us we should probably also check out an example of a relation that is not a function.

Example 2 The following relation is not a function.

$$\{(6,10)\ \ (-7,3)\ \ (0,4)\ \ (6,-4)\}$$

Solution

Don't worry about where this relation came from. It is just one that we made up for this example.

Here is the list of first and second components

$$1^{\text{st}} \text{ components : } \{6,-7,0\} \qquad\qquad 2^{\text{nd}} \text{ components : } \{10,3,4,-4\}$$

From the set of first components let's choose 6. Now, if we go up to the relation we see that there are two ordered pairs with 6 as a first component : $(6,10)$ and $(6,-4)$. The list of second components associated with 6 is then : 10, -4.

The list of second components associated with 6 has two values and so this relation is not a function.

Note that the fact that if we'd chosen -7 or 0 from the set of first components there is only one number in the list of second components associated with each. This doesn't matter. The fact that we found even a single value in the set of first components with more than one second component associated with it is enough to say that this relation is not a function.

As a final comment about this example let's note that if we removed the first and/or the fourth ordered pair from the relation we would have a function!

So, hopefully you have at least a feeling for what the definition of a function is telling us.

Now that we've forced you to go through the actual definition of a function let's give another "working" definition of a function that will be much more useful to what we are doing here.

The actual definition works on a relation. However, as we saw with the four relations we gave prior to the definition of a function and the relation we used in Example 1 we often get the relations from some equation.

It is important to note that not all relations come from equations! The relation from the second example for instance was just a set of ordered pairs we wrote down for the example and didn't come from any equation. This can also be true with relations that are functions. They do not have to come from equations.

However, having said that, the functions that we are going to be using in this course do all come from equations. Therefore, let's write down a definition of a function that acknowledges this fact.

Before we give the "working" definition of a function we need to point out that this is NOT the actual definition of a function, that is given above. This is simply a good "working definition" of a function that ties things to the kinds of functions that we will be working with in this course.

"Working Definition" of Function

A **function** is an equation for which any *x* that can be plugged into the equation will yield exactly one *y* out of the equation.

There it is. That is the definition of functions that we're going to use and will probably be easier to decipher just what it means.

Before we examine this a little more note that we used the phrase "*x* that can be plugged into" in the definition. This tends to imply that not all *x*'s can be plugged into an equation and this is in fact correct. We will come back and discuss this in more detail towards the end of this section, however at this point just remember that we can't divide by zero and if we want real numbers out of the equation we can't take the square root of a negative number. So, with these two examples it is clear that we will not always be able to plug in every *x* into any equation.

Further, when dealing with functions we are always going to assume that both *x* and *y* will be real numbers. In other words, we are going to forget that we know anything about complex numbers for a little bit while we deal with this section.

Okay, with that out of the way let's get back to the definition of a function and let's look at some examples of equations that are functions and equations that aren't functions.

Example 3 Determine which of the following equations are functions and which are not functions.

(a) $y = 5x + 1$

(b) $y = x^2 + 1$

(c) $y^2 = x + 1$

(d) $x^2 + y^2 = 4$

Solution

The "working" definition of function is saying is that if we take all possible values of *x* and plug them into the equation and solve for *y* we will get exactly one value for each value of *x*. At this stage of the game it can be pretty difficult to actually show that an equation is a function so we'll mostly talk our way through it. On the other hand, it's often quite easy to show that an equation isn't a function.

(a) $y = 5x + 1$

So, we need to show that no matter what *x* we plug into the equation and solve for *y* we will only get a single value of *y*. Note as well that the value of *y* will probably be different for each value of *x*, although it doesn't have to be.

Let's start this off by plugging in some values of *x* and see what happens.

$$\begin{aligned} x = -4: & \qquad y = 5(-4) + 1 = -20 + 1 = -19 \\ x = 0: & \qquad y = 5(0) + 1 = 0 + 1 = 1 \\ x = 10: & \qquad y = 5(10) + 1 = 50 + 1 = 51 \end{aligned}$$

So, for each of these values of *x* we got a single value of *y* out of the equation. Now, this isn't sufficient to claim that this is a function. In order to officially prove that this is a function we need to show that this will work no matter which value of *x* we plug into the equation.

Of course, we can't plug all possible value of *x* into the equation. That just isn't physically possible. However, let's go back and look at the ones that we did plug in. For each *x*, upon plugging in, we first multiplied the *x* by 5 and then added 1 onto it. Now, if we multiply a number by 5 we will get a single value from the multiplication. Likewise, we will only get a single value if we add 1 onto a number. Therefore, it seems plausible that based on the operations involved with plugging *x* into the equation that we will only get a single value of *y* out of the equation.

So, this equation is a function.

(b) $y = x^2 + 1$
Again, let's plug in a couple of values of *x* and solve for *y* to see what happens.

$$x = -1: \qquad y = (-1)^2 + 1 = 1 + 1 = 2$$
$$x = 3: \qquad y = (3)^2 + 1 = 9 + 1 = 10$$

Now, let's think a little bit about what we were doing with the evaluations. First, we squared the value of *x* that we plugged in. When we square a number there will only be one possible value. We then add 1 onto this, but again, this will yield a single value.

So, it seems like this equation is also a function.

Note that it is okay to get the same *y* value for different *x*'s. For example,
$$x = -3: \qquad y = (-3)^2 + 1 = 9 + 1 = 10$$
We just can't get more than one *y* out of the equation after we plug in the *x*.

(c) $y^2 = x + 1$
As we've done with the previous two equations let's plug in a couple of value of *x*, solve for *y* and see what we get.
$$x = 3: \qquad y^2 = 3 + 1 = 4 \qquad \Rightarrow \qquad y = \pm 2$$
$$x = -1: \qquad y^2 = -1 + 1 = 0 \qquad \Rightarrow \qquad y = 0$$
$$x = 10: \qquad y^2 = 10 + 1 = 11 \qquad \Rightarrow \qquad y = \pm\sqrt{11}$$

Now, remember that we're solving for *y* and so that means that in the first and last case above we will actually get two different *y* values out of the *x* and so this equation is NOT a function.

Note that we can have values of *x* that will yield a single *y* as we've seen above, but that doesn't matter. If even one value of *x* yields more than one value of *y* upon solving the equation will not be a function.

What this really means is that we didn't need to go any farther than the first evaluation, since that gave multiple values of *y*.

(d) $x^2 + y^2 = 4$
With this case we'll use the lesson learned in the previous part and see if we can find a value of *x* that will give more than one value of *y* upon solving. Because we've got a y^2 in the problem this shouldn't be too hard to do since solving will eventually mean using the square root property which will give more than one value of *y*.
$$x = 0: \qquad 0^2 + y^2 = 4 \qquad \Rightarrow \qquad y^2 = 4 \qquad \Rightarrow \qquad y = \pm 2$$

So, this equation is not a function. Recall, that from the previous section this is the equation of a circle. Circles are never functions.

Hopefully these examples have given you a better feel for what a function actually is.

We now need to move onto something called **function notation**. Function notation will be used heavily throughout most of the remaining chapters in this course and so it is important to understand it.

Let's start off with the following quadratic equation.

$$y = x^2 - 5x + 3$$

We can use a process similar to what we used in the previous set of examples to convince ourselves that this is a function. Since this is a function we will denote it as follows,

$$f\left(x\right) = x^2 - 5x + 3$$

So, we replaced the *y* with the notation $f\left(x\right)$. This is read as "f of *x*". Note that there is nothing special about the *f* we used here. We could just have easily used any of the following,

$$g\left(x\right) = x^2 - 5x + 3 \qquad h\left(x\right) = x^2 - 5x + 3 \qquad R\left(x\right) = x^2 - 5x + 3$$

The letter we use does not matter. What is important is the "$\left(x\right)$" part. The letter in the parenthesis must match the variable used on the right side of the equal sign.

It is very important to note that $f\left(x\right)$ is really nothing more than a really fancy way of writing *y*. If you keep that in mind you may find that dealing with function notation becomes a little easier.

Also, this is **NOT** a multiplication of *f* by *x*! This is one of the more common mistakes people make when they first deal with functions. This is just a notation used to denote functions.

Next we need to talk about **evaluating functions**. Evaluating a function is really nothing more than asking what its value is for specific values of *x*. Another way of looking at it is that we are asking what the *y* value for a given *x* is.

Evaluation is really quite simple. Let's take the function we were looking at above

$$f\left(x\right) = x^2 - 5x + 3$$

and ask what its value is for $x = 4$. In terms of function notation we will "ask" this using the notation $f\left(4\right)$. So, when there is something other than the variable inside the parenthesis we are really asking what the value of the function is for that particular quantity.

Now, when we say the value of the function we are really asking what the value of the equation is for that particular value of *x*. Here is $f\left(4\right)$.

$$f\left(4\right) = \left(4\right)^2 - 5\left(4\right) + 3 = 16 - 20 + 3 = -1$$

Notice that evaluating a function is done in exactly the same way in which we evaluate equations. All we do is plug in for *x* whatever is on the inside of the parenthesis on the left. Here's another evaluation for this function.

$$f\left(-6\right)=\left(-6\right)^2-5\left(-6\right)+3=36+30+3=69$$

So, again, whatever is on the inside of the parenthesis on the left is plugged in for *x* in the equation on the right. Let's take a look at some more examples.

Example 4 Given $f\left(x\right)=x^2-2x+8$ and $g\left(x\right)=\sqrt{x+6}$ evaluate each of the following.

(a) $f\left(3\right)$ and $g\left(3\right)$
(b) $f\left(-10\right)$ and $g\left(-10\right)$
(c) $f\left(0\right)$
(d) $f\left(t\right)$
(e) $f\left(t+1\right)$ and $f\left(x+1\right)$
(f) $f\left(x^3\right)$
(g) $g\left(x^2-5\right)$

Solution

(a) $f\left(3\right)$ **and** $g\left(3\right)$

Okay we've got two function evaluations to do here and we've also got two functions so we're going to need to decide which function to use for the evaluations. The key here is to notice the letter that is in front of the parenthesis. For $f\left(3\right)$ we will use the function $f\left(x\right)$ and for $g\left(3\right)$ we will use $g\left(x\right)$. In other words, we just need to make sure that the variables match up.

Here are the evaluations for this part.

$$f\left(3\right)=\left(3\right)^2-2\left(3\right)+8=9-6+8=11$$
$$g\left(3\right)=\sqrt{3+6}=\sqrt{9}=3$$

(b) $f\left(-10\right)$ **and** $g\left(-10\right)$

This one is pretty much the same as the previous part with one exception that we'll touch on when we reach that point. Here are the evaluations.

$$f\left(-10\right)=\left(-10\right)^2-2\left(-10\right)+8=100+20+8=128$$

Make sure that you deal with the negative signs properly here. Now the second one.

$$g\left(-10\right)=\sqrt{-10+6}=\sqrt{-4}$$

We've now reached the difference. Recall that when we first started talking about the definition of functions we stated that we were only going to deal with real numbers. In other words, we only plug in real numbers and we only want real numbers back out as answers. So, since we would get a complex number out of this we can't plug -10 into this function.

(c) $f\left(0\right)$

Not much to this one.

$$f\left(0\right)=\left(0\right)^2-2\left(0\right)+8=8$$

Again, don't forget that this isn't multiplication! For some reason students like to think of this one as multiplication and get an answer of zero. Be careful.

(d) $f\left(t\right)$

The rest of these evaluations are now going to be a little different. As this one shows we don't need to just have numbers in the parenthesis. However, evaluation works in exactly the same way. We plug into the *x*'s on the right side of the equal sign whatever is in the parenthesis. In this case that means that we plug in *t* for all the *x*'s.

Here is this evaluation.

$$f\left(t\right)=t^2-2t+8$$

Note that in this case this is pretty much the same thing as our original function, except this time we're using *t* as a variable.

(e) $f\left(t+1\right)$ **and** $f\left(x+1\right)$

Now, let's get a little more complicated, or at least they appear to be more complicated. Things aren't as bad as they may appear however. We'll evaluate $f\left(t+1\right)$ first. This one works exactly the same as the previous part did. All the *x*'s on the left will get replaced with $t+1$. We will have some simplification to do as well after the substitution.

$$\begin{aligned} f\left(t+1\right) &= \left(t+1\right)^2-2\left(t+1\right)+8 \\ &= t^2+2t+1-2t-2+8 \\ &= t^2+7 \end{aligned}$$

Be careful with parenthesis in these kinds of evaluations. It is easy to mess up with them.

Now, let's take a look at $f\left(x+1\right)$. With the exception of the *x* this is identical to $f\left(t+1\right)$ and so it works exactly the same way.

$$\begin{aligned} f\left(x+1\right) &= \left(x+1\right)^2-2\left(x+1\right)+8 \\ &= x^2+2x+1-2x-2+8 \\ &= x^2+7 \end{aligned}$$

Do not get excited about the fact that we reused *x*'s in the evaluation here. In many places where we will be doing this in later sections there will be *x*'s here and so you will need to get used to seeing that.

(f) $f\left(x^3\right)$

Again, don't get excited about the *x*'s in the parenthesis here. Just evaluate it as if it were a number.

$$\begin{aligned} f\left(x^3\right) &= \left(x^3\right)^2 - 2\left(x^3\right) + 8 \\ &= x^6 - 2x^3 + 8 \end{aligned}$$

(g) $g\left(x^2 - 5\right)$

One more evaluation and this time we'll use the other function.

$$\begin{aligned} g\left(x^2 - 5\right) &= \sqrt{x^2 - 5 + 6} \\ &= \sqrt{x^2 + 1} \end{aligned}$$

Function evaluation is something that we'll be doing a lot of in later sections and chapters so make sure that you can do it. You will find several later sections very difficult to understand and/or do the work in if you do not have a good grasp on how function evaluation works.

While we are on the subject of function evaluation we should now talk about **piecewise functions**. We've actually already seen an example of a piecewise function even if we didn't call it a function (or a piecewise function) at the time. Recall the mathematical definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is a function and if we use function notation we can write it as follows,

$$f\left(x\right) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is also an example of a piecewise function. A piecewise function is nothing more than a function that is broken into pieces and which piece you use depends upon value of *x*. So, in the absolute value example we will use the top piece if *x* is positive or zero and we will use the bottom piece if *x* is negative.

Let's take a look at evaluating a more complicated piecewise function.

Example 5 Given,

$$g(t) = \begin{cases} 3t^2 + 4 & \text{if } t \le -4 \\ 10 & \text{if } -4 < t \le 15 \\ 1 - 6t & \text{if } t > 15 \end{cases}$$

evaluate each of the following.

(a) $g(-6)$

(b) $g(-4)$

(c) $g(1)$

(d) $g(15)$

(e) $g(21)$

Solution

Before starting the evaluations here let's notice that we're using different letters for the function and variable than the ones that we've used to this point. That won't change how the evaluation works. Do not get so locked into seeing *f* for the function and *x* for the variable that you can't do any problem that doesn't have those letters.

Now, to do each of these evaluations the first thing that we need to do is determine which inequality the number satisfies, and it will only satisfy a single inequality. When we determine which inequality the number satisfies we use the equation associated with that inequality.

So, let's do some evaluations.

(a) $g(-6)$

In this case -6 satisfies the top inequality and so we'll use the top equation for this evaluation.

$$g(-6) = 3(-6)^2 + 4 = 112$$

(b) $g(-4)$

Now we'll need to be a little careful with this one since -4 shows up in two of the inequalities. However, it only satisfies the top inequality and so we will once again use the top function for the evaluation.

$$g(-4) = 3(-4)^2 + 4 = 52$$

(c) $g(1)$

In this case the number, 1, satisfies the middle inequality and so we'll use the middle equation for the evaluation. This evaluation often causes problems for students despite the fact that it's actually one of the easiest evaluations we'll ever do. We know that we evaluate functions/equations by plugging in the number for the variable. In this case there are no variables. That isn't a problem. Since there aren't any variables it just means that we don't actually plug in anything and we get the following,

$$g(1) = 10$$

(d) $g(15)$

Again, like with the second part we need to be a little careful with this one. In this case the number satisfies the middle inequality since that is the one with the equal sign in it. Then like the previous part we just get,

$$g(15)=10$$

Don't get excited about the fact that the previous two evaluations were the same value. This will happen on occasion.

(e) $g(21)$

For the final evaluation in this example the number satisfies the bottom inequality and so we'll use the bottom equation for the evaluation.

$$g(21)=1-6(21)=-125$$

Piecewise functions do not arise all that often in an Algebra class however, they do arise in several places in later classes and so it is important for you to understand them if you are going to be moving on to more math classes.

As a final topic we need to come back and touch on the fact that we can't always plug every *x* into every function. We talked briefly about this when we gave the definition of the function and we saw an example of this when we were evaluating functions. We now need to look at this in a little more detail.

First, we need to get a couple of definitions out of the way.

Domain and Range

The **domain** of an equation is the set of all *x*'s that we can plug into the equation and get back a real number for *y*. The **range** of an equation is the set of all *y*'s that we can ever get out of the equation.

Note that we did mean to use equation in the definitions above instead of functions. These are really definitions for equations. However, since functions are also equations we can use the definitions for functions as well.

Determining the range of an equation/function can be pretty difficult to do for many functions and so we aren't going to really get into that. We are much more interested here in determining the domains of functions. From the definition the domain is the set of all *x*'s that we can plug into a function and get back a real number. At this point, that means that we need to avoid division by zero and taking square roots of negative numbers.

Let's do a couple of quick examples of finding domains.

Example 6 Determine the domain of each of the following functions.

(a) $g\left(x\right)=\dfrac{x+3}{x^2+3x-10}$

(b) $f\left(x\right)=\sqrt{5-3x}$

(c) $h\left(x\right)=\dfrac{\sqrt{7x+8}}{x^2+4}$

(d) $R\left(x\right)=\dfrac{\sqrt{10x-5}}{x^2-16}$

Solution

The domains for these functions are all the values of *x* for which we don't have division by zero or the square root of a negative number. If we remember these two ideas finding the domains will be pretty easy.

(a) $g\left(x\right)=\dfrac{x+3}{x^2+3x-10}$

So, in this case there are no square roots so we don't need to worry about the square root of a negative number. There is however a possibility that we'll have a division by zero error. To determine if we will we'll need to set the denominator equal to zero and solve.

$$x^2+3x-10=\left(x+5\right)\left(x-2\right)=0 \qquad x=-5,\ x=2$$

So, we will get division by zero if we plug in $x=-5$ or $x=2$. That means that we'll need to avoid those two numbers. However, all the other values of *x* will work since they don't give division by zero. The domain is then,

$$\text{Domain : All real numbers except } x=-5 \text{ and } x=2$$

(b) $f\left(x\right)=\sqrt{5-3x}$

In this case we won't have division by zero problems since we don't have any fractions. We do have a square root in the problem and so we'll need to worry about taking the square root of a negative numbers.

This one is going to work a little differently from the previous part. In that part we determined the value(s) of *x* to avoid. In this case it will be just as easy to directly get the domain. To avoid square roots of negative numbers all that we need to do is require that

$$5-3x\geq 0$$

This is a fairly simple linear inequality that we should be able to solve at this point.

$$5\geq 3x \qquad \Rightarrow \qquad x\leq\frac{5}{3}$$

The domain of this function is then,

$$\text{Domain : } x\leq\frac{5}{3}$$

(c) $h\left(x\right)=\dfrac{\sqrt{7x+8}}{x^2+4}$

In this case we've got a fraction, but notice that the denominator will never be zero for any real number since x^2 is guaranteed to be positive or zero and adding 4 onto this will mean that the denominator is always at least 4. In other words, the denominator won't ever be zero. So, all we need to do then is worry about the square root in the numerator.

To do this we'll require,

$$\begin{aligned} 7x+8 &\geq 0 \\ 7x &\geq -8 \\ x &\geq -\frac{8}{7} \end{aligned}$$

Now, we can actually plug in any value of *x* into the denominator, however, since we've got the square root in the numerator we'll have to make sure that all *x*'s satisfy the inequality above to avoid problems. Therefore, the domain of this function is

$$\text{Domain}: x \geq -\frac{8}{7}$$

(d) $R\left(x\right)=\dfrac{\sqrt{10x-5}}{x^2-16}$

In this final part we've got both a square root and division by zero to worry about. Let's take care of the square root first since this will probably put the largest restriction on the values of *x*. So, to keep the square root happy (*i.e.* no square root of negative numbers) we'll need to require that,

$$\begin{aligned} 10x-5 &\geq 0 \\ 10x &\geq 5 \\ x &\geq \frac{1}{2} \end{aligned}$$

So, at the least we'll need to require that $x \geq \dfrac{1}{2}$ in order to avoid problems with the square root.

Now, let's see if we have any division by zero problems. Again, to do this simply set the denominator equal to zero and solve.

$$x^2-16=\left(x-4\right)\left(x+4\right)=0 \qquad \Rightarrow \qquad x=-4,\ x=4$$

Now, notice that $x=-4$ doesn't satisfy the inequality we need for the square root and so that value of *x* has already been excluded by the square root. On the other hand, $x=4$ does satisfy the inequality. This means that it is okay to plug $x=4$ into the square root, however, since it would give division by zero we will need to avoid it.

The domain for this function is then,

$$\text{Domain}: x \geq \frac{1}{2} \quad \text{except} \quad x=4$$

Section 3-5 : Graphing Functions

Now we need to discuss graphing functions. If we recall from the previous section we said that $f(x)$ is nothing more than a fancy way of writing *y*. This means that we already know how to graph functions. We graph functions in exactly the same way that we graph equations. If we know ahead of time what the function is a graph of we can use that information to help us with the graph and if we don't know what the function is ahead of time then all we need to do is plug in some *x*'s compute the value of the function (which is really a *y* value) and then plot the points.

Example 1 Sketch the graph of $f(x) = (x-1)^3 + 1$.

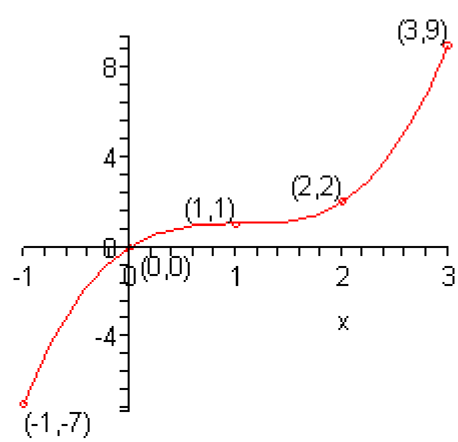
Solution

Now, as we talked about when we first looked at graphing earlier in this chapter we'll need to pick values of *x* to plug in and knowing the values to pick really only comes with experience. Therefore, don't worry so much about the values of *x* that we're using here. By the end of this chapter you'll also be able to correctly pick these values.

Here are the function evaluations.

x	*f(x)*	(x,y)
-1	-7	$(-1,-7)$
0	0	$(0,0)$
1	1	$(1,1)$
2	2	$(2,2)$
3	9	$(3,9)$

Here is the sketch of the graph.



So, graphing functions is pretty much the same as graphing equations.

There is one function that we've seen to this point that we didn't really see anything like when we were graphing equations in the first part of this chapter. That is piecewise functions. So, we should graph a couple of these to make sure that we can graph them as well.

Example 2 Sketch the graph of the following piecewise function.

$$g(x) = \begin{cases} -x^2 + 4 & \text{if } x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$$

Solution

Okay, now when we are graphing piecewise functions we are really graphing several functions at once, except we are only going to graph them on very specific intervals. In this case we will be graphing the following two functions,

$$\begin{array}{lll} -x^2 + 4 & \text{on} & x < 1 \\ 2x - 1 & \text{on} & x \geq 1 \end{array}$$

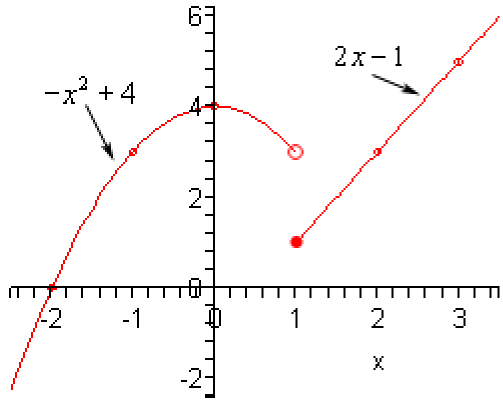
We'll need to be a little careful with what is going on right at $x = 1$ since technically that will only be valid for the bottom function. However, we'll deal with that at the very end when we actually do the graph. For now, we will use $x = 1$ in both functions.

The first thing to do here is to get a table of values for each function on the specified range and again we will use $x = 1$ in both even though technically it only should be used with the bottom function.

x	$-x^2+4$	(x,y)
-2	0	$(-2,0)$
-1	3	$(-1,3)$
0	4	$(0,4)$
1	3	$(1,3)$

x	$2x-1$	(x,y)
1	1	$(1,1)$
2	3	$(2,3)$
3	5	$(3,5)$

Here is a sketch of the graph and notice how we denoted the points at $x = 1$. For the top function we used an open dot for the point at $x = 1$ and for the bottom function we used a closed dot at $x = 1$. In this way we make it clear on the graph that only the bottom function really has a point at $x = 1$.



Notice that since the two graphs didn't meet at $x=1$ we left a blank space in the graph. Do NOT connect these two points with a line. There really does need to be a break there to signify that the two portions do not meet at $x=1$.

Sometimes the two portions will meet at these points and at other times they won't. We shouldn't ever expect them to meet or not to meet until we've actually sketched the graph.

Let's take a look at another example of a piecewise function.

Example 3 Sketch the graph of the following piecewise function.

$$h(x)=\begin{cases} x+3 & \text{if } x<-2 \\ x^2 & \text{if } -2\le x<1 \\ -x+2 & \text{if } x\ge 1 \end{cases}$$

Solution

In this case we will be graphing three functions on the ranges given above. So, as with the previous example we will get function values for each function in its specified range and we will include the endpoints of each range in each computation. When we graph we will acknowledge which function the endpoint actually belongs with by using a closed dot as we did previously. Also, the top and bottom functions are lines and so we don't really need more than two points for these two. We'll get a couple more points for the middle function.

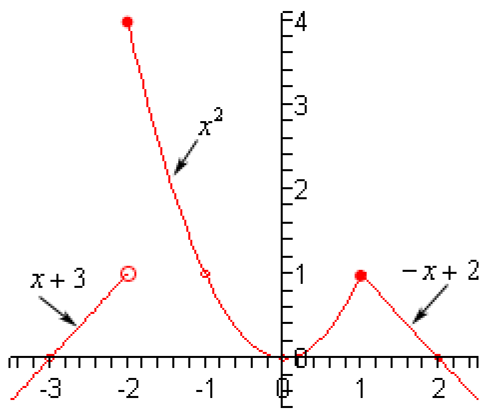
x	$x+3$	(x,y)
-3	0	$(-3,0)$
-2	1	$(-2,1)$

x	x^2	(x,y)
-2	4	$(-2,4)$

-1	1	$(-1,1)$
0	0	$(0,0)$
1	1	$(1,1)$

x	$-x+2$	(x,y)
1	1	$(1,1)$
2	0	$(2,0)$

Here is the sketch of the graph.



Note that in this case two of the portions met at the breaking point $x=1$ and at the other breaking point, $x=-2$, they didn't meet up. As noted in the previous example sometimes they meet up and sometimes they won't.

Section 3-6 : Combining Functions

The topic with functions that we need to deal with is combining functions. For the most part this means performing basic arithmetic (addition, subtraction, multiplication, and division) with functions. There is one new way of combing functions that we'll need to look at as well.

Let's start with basic arithmetic of functions. Given two functions $f\left(x\right)$ and $g\left(x\right)$ we have the following notation and operations.

$$\left(f+g\right)\left(x\right)=f\left(x\right)+g\left(x\right) \qquad \left(f-g\right)\left(x\right)=f\left(x\right)-g\left(x\right)$$

$$\left(fg\right)\left(x\right)=f\left(x\right)g\left(x\right) \qquad \left(\frac{f}{g}\right)\left(x\right)=\frac{f\left(x\right)}{g\left(x\right)}$$

Sometimes we will drop the $\left(x\right)$ part and just write the following,

$$f+g=f\left(x\right)+g\left(x\right) \qquad f-g=f\left(x\right)-g\left(x\right)$$

$$fg=f\left(x\right)g\left(x\right) \qquad \frac{f}{g}=\frac{f\left(x\right)}{g\left(x\right)}$$

Note as well that we put *x*'s in the parenthesis, but we will often put in numbers as well. Let's take a quick look at an example.

Example 1 Given $f\left(x\right)=2+3x-x^2$ and $g\left(x\right)=2x-1$ evaluate each of the following.

(a) $\left(f+g\right)\left(4\right)$

(b) $g-f$

(c) $\left(fg\right)\left(x\right)$

(d) $\left(\dfrac{f}{g}\right)\left(0\right)$

Solution

By evaluate we mean one of two things depending on what is in the parenthesis. If there is a number in the parenthesis then we want a number. If there is an *x* (or no parenthesis, since that implies an *x*) then we will perform the operation and simplify as much as possible.

(a) $\left(f+g\right)\left(4\right)$

In this case we've got a number so we need to do some function evaluation.

$$\begin{aligned}\left(f+g\right)\left(4\right)&=f\left(4\right)+g\left(4\right)\\&=\left(2+3\left(4\right)-4^2\right)+\left(2\left(4\right)-1\right)\\&=-2+7\\&=5\end{aligned}$$

(b) $g - f$

Here we don't have an *x* or a number so this implies the same thing as if there were an *x* in parenthesis. Therefore, we'll subtract the two functions and simplify. Note as well that this is written in the opposite order from the definitions above, but it works the same way.

$$\begin{aligned} g - f &= g(x) - f(x) \\ &= 2x - 1 - \left(2 + 3x - x^2\right) \\ &= 2x - 1 - 2 - 3x + x^2 \\ &= x^2 - x - 3 \end{aligned}$$

(c) $(fg)(x)$

As with the last part this has an *x* in the parenthesis so we'll multiply and then simplify.

$$\begin{aligned} (fg)(x) &= f(x)g(x) \\ &= \left(2 + 3x - x^2\right)(2x - 1) \\ &= 4x + 6x^2 - 2x^3 - 2 - 3x + x^2 \\ &= -2x^3 + 7x^2 + x - 2 \end{aligned}$$

(d) $\left(\dfrac{f}{g}\right)(0)$

In this final part we've got a number so we'll once again be doing function evaluation.

$$\begin{aligned} \left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} \\ &= \frac{2 + 3(0) - (0)^2}{2(0) - 1} \\ &= \frac{2}{-1} \\ &= -2 \end{aligned}$$

Now we need to discuss the new method of combining functions. The new method of combining functions is called **function composition**. Here is the definition.

Given two functions $f(x)$ and $g(x)$ we have the following two definitions.

1. The **composition** of $f(x)$ and $g(x)$ (note the order here) is,

$$(f \circ g)(x) = f\left[g(x)\right]$$

2. The **composition of** $g(x)$ and $f(x)$ (again, note the order) is,

$$(g \circ f)(x) = g\left[f(x)\right]$$

We need to note a couple of things here about function composition. First this is **NOT** multiplication. Regardless of what the notation may suggest to you this is simply not multiplication.

Second, the order we've listed the two functions is very important since more often than not we will get different answers depending on the order we've listed them.

Finally, function composition is really nothing more than function evaluation. All we're really doing is plugging the second function listed into the first function listed. In the definitions we used $[\ ]$ for the function evaluation instead of the standard $(\ )$ to avoid confusion with too many sets of parenthesis, but the evaluation will work the same.

Let's take a look at a couple of examples.

Example 2 Given $f(x) = 2 + 3x - x^2$ and $g(x) = 2x - 1$ evaluate each of the following.

(a) $(fg)(x)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

Solution

(a) These are the same functions that we used in the first set of examples and we've already done this part there so we won't redo all the work here. It is here only here to prove the point that function composition is NOT function multiplication.

Here is the multiplication of these two functions.

$$(fg)(x) = -2x^3 + 7x^2 + x - 2$$

(b) Now, for function composition all you need to remember is that we are going to plug the second function listed into the first function listed. If you can remember that you should always be able to write down the basic formula for the composition.

Here is this function composition.

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f[2x-1]\end{aligned}$$

Now, notice that since we've got a formula for $g(x)$ we went ahead and plugged that in first. Also, we did this kind of function evaluation in the first section we looked at for functions. At the time it probably didn't seem all that useful to be looking at that kind of function evaluation, yet here it is.

Let's finish this problem out.

$$\begin{aligned}(f\circ g)(x) &= f\left[g(x)\right]\\ &= f\left[2x-1\right]\\ &= 2+3(2x-1)-(2x-1)^2\\ &= 2+6x-3-\left(4x^2-4x+1\right)\\ &= -1+6x-4x^2+4x-1\\ &= -4x^2+10x-2\end{aligned}$$

Notice that this is very different from the multiplication! Remember that function composition is NOT function multiplication.

(c) We'll not put in the detail in this part as it works essentially the same as the previous part.

$$\begin{aligned}(g\circ f)(x) &= g\left[f(x)\right]\\ &= g\left[2+3x-x^2\right]\\ &= 2\left(2+3x-x^2\right)-1\\ &= 4+6x-2x^2-1\\ &= -2x^2+6x+3\end{aligned}$$

Notice that this is NOT the same answer as that from the second part. In most cases the order in which we do the function composition will give different answers.

The ideas from the previous example are important enough to make again. First, function composition is NOT function multiplication. Second, the order in which we do function composition is important. In most case we will get different answers with a different order. Note however, that there are times when we will get the same answer regardless of the order.

Let's work a couple more examples.

Example 3 Given $f(x)=x^2-3$ and $h(x)=\sqrt{x+1}$ evaluate each of the following.

(a) $(f\circ h)(x)$

(b) $(h\circ f)(x)$

(c) $(f\circ f)(x)$

(d) $(h\circ h)(8)$

(e) $(f\circ h)(4)$

Solution

(a) $(f\circ h)(x)$

Not much to do here other than run through the formula.

$$\begin{aligned}(f\circ h)(x) &= f\left[h(x)\right]\\ &= f\left[\sqrt{x+1}\right]\\ &= \left(\sqrt{x+1}\right)^2-3\\ &= x+1-3\\ &= x-2\end{aligned}$$

(b) $(h\circ f)(x)$

Again, not much to do here.

$$\begin{aligned}(h\circ f)(x) &= h\left[f(x)\right]\\ &= h\left[x^2-3\right]\\ &= \sqrt{x^2-3+1}\\ &= \sqrt{x^2-2}\end{aligned}$$

(c) $(f\circ f)(x)$

Now in this case do not get excited about the fact that the two functions here are the same. Composition works the same way.

$$\begin{aligned}(f\circ f)(x) &= f\left[f(x)\right]\\ &= f\left[x^2-3\right]\\ &= \left(x^2-3\right)^2-3\\ &= x^4-6x^2+9-3\\ &= x^4-6x^2+6\end{aligned}$$

(d) $(h\circ h)(8)$

In this case, unlike all the previous examples, we've got a number in the parenthesis instead of an *x*, but it works in exactly the same manner.

$$\begin{aligned}(h\circ h)(8) &= h\left[h(8)\right]\\ &= h\left[\sqrt{8+1}\right]\\ &= h\left[\sqrt{9}\right]\\ &= h[3]\\ &= \sqrt{3+1}\\ &= 2\end{aligned}$$

(e) $\left(f \circ h\right)\left(4\right)$

Again, we've got a number here. This time there are actually two ways to do this evaluation. The first is to simply use the results from the first part since that is a formula for the general function composition.

If we do the problem that way we get,

$$\left(f \circ h\right)\left(4\right) = 4 - 2 = 2$$

We could also do the evaluation directly as we did in the previous part. The answers should be the same regardless of how we get them. So, to get another example down of this kind of evaluation let's also do the evaluation directly.

$$\begin{aligned}\left(f \circ h\right)\left(4\right) &= f\left[h\left(4\right)\right] \\ &= f\left[\sqrt{4+1}\right] \\ &= f\left[\sqrt{5}\right] \\ &= \left(\sqrt{5}\right)^2 - 3 \\ &= 5 - 3 \\ &= 2\end{aligned}$$

So, sure enough we got the same answer, although it did take more work to get it.

Example 4 Given $f\left(x\right) = 3x - 2$ and $g\left(x\right) = \frac{x}{3} + \frac{2}{3}$ evaluate each of the following.

(a) $\left(f \circ g\right)\left(x\right)$

(b) $\left(g \circ f\right)\left(x\right)$

Solution

(a) Hopefully, by this point these aren't too bad.

$$\begin{aligned}\left(f \circ g\right)\left(x\right) &= f\left[g\left(x\right)\right] \\ &= f\left[\frac{x}{3} + \frac{2}{3}\right] \\ &= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

Looks like things simplified down considerable here.

(b) All we need to do here is use the formula so let's do that.

$$\begin{aligned}\left(g\circ f\right)\left(x\right)&=g\left[f\left(x\right)\right]\\&=g\left[3x-2\right]\\&=\frac{1}{3}\left(3x-2\right)+\frac{2}{3}\\&=x-\frac{2}{3}+\frac{2}{3}\\&=x\end{aligned}$$

So, in this case we get the same answer regardless of the order we did the composition in.

So, as we've seen from this last example it is possible to get the same answer from both compositions on occasion. In fact when the answer from both composition is *x*, as it is in this case, we know that these two functions are very special functions. In fact, they are so special that we're going to devote the whole next section to these kinds of functions. So, let's move onto the next section.

Section 3-7 : Inverse Functions

In the last example from the previous section we looked at the two functions $f\left(x\right)=3x-2$ and $g\left(x\right)=\frac{x}{3}+\frac{2}{3}$ and saw that

$$\left(f\circ g\right)\left(x\right)=\left(g\circ f\right)\left(x\right)=x$$

and as noted in that section this means that these are very special functions. Let's see just what makes them so special. Consider the following evaluations.

$$f\left(-1\right)=3\left(-1\right)-2=-5 \qquad \Rightarrow \qquad g\left(-5\right)=\frac{-5}{3}+\frac{2}{3}=\frac{-3}{3}=-1$$

$$g\left(2\right)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right)=3\left(\frac{4}{3}\right)-2=4-2=2$$

In the first case we plugged $x=-1$ into $f\left(x\right)$ and got a value of -5. We then turned around and plugged $x=-5$ into $g\left(x\right)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x=2$ into $g\left(x\right)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f\left(x\right)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$\left(g\circ f\right)\left(-1\right)=g\left[f\left(-1\right)\right]=g\left[-5\right]=-1$$

and the second case is really,

$$\left(f\circ g\right)\left(2\right)=f\left[g\left(2\right)\right]=f\left[\frac{4}{3}\right]=2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x=-1$ into $f\left(x\right)$ and then plugged the result from this function evaluation back into $g\left(x\right)$ and in some way $g\left(x\right)$ undid what $f\left(x\right)$ had done to $x=-1$ and gave us back the original *x* that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of *x* produce the same *y*. This is a fairly simple definition of one-to-one but it takes an example of a function that isn't one-to-one to show just what it means. Before doing that however we should note that this definition of one-to-one is not really the mathematically correct definition of one-to-one. It is identical to the mathematically correct definition it just doesn't use all the notation from the formal definition.

Now, let's see an example of a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words, there are two different values of *x* that produce the same value of *y*. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \le x < \infty$. This can sometimes be done with functions.

Showing that a function is one-to-one is often a tedious and difficult process. For the most part we are going to assume that the functions that we're going to be dealing with in this section are one-to-one. We did need to talk about one-to-one functions however since only one-to-one functions can be inverse functions.

Now, let's formally define just what inverse functions are.

Inverse Functions

Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = x \qquad \text{AND} \qquad (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are **inverses** of each other. More specifically we will say that $g(x)$ is the **inverse** of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise, we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$$f(x) = 3x - 2 \qquad f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

$$g(x) = \frac{x}{3} + \frac{2}{3} \qquad g^{-1}(x) = 3x - 2$$

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that is sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there is a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this section if one is true then the other will also be true. However, there are functions (they are far beyond the scope of this course however) for which it is possible for only of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f(x)$ with *y*.

$$y = 3x - 2$$

Next, replace all *x*'s with *y* and all *y's* with *x*.

$$x = 3y - 2$$

Now, solve for *y*.

$$\begin{aligned} x+2 &= 3y \\ \frac{1}{3}(x+2) &= y \\ \frac{x}{3}+\frac{2}{3} &= y \end{aligned}$$

Finally replace *y* with $f^{-1}(x)$.

$$f^{-1}(x)=\frac{x}{3}+\frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $(f \circ f^{-1})(x)=x$ is true.

$$\begin{aligned} (f \circ f^{-1})(x) &= f\left[f^{-1}(x)\right] \\ &= f\left[\frac{x}{3}+\frac{2}{3}\right] \\ &= 3\left(\frac{x}{3}+\frac{2}{3}\right)-2 \\ &= x+2-2 \\ &= x \end{aligned}$$

Example 2 Given $g(x)=\sqrt{x-3}$ find $g^{-1}(x)$, $x \ge 0$.

Solution

Now the fact that we're now using $g(x)$ instead of $f(x)$ doesn't change how the process works. Here are the first few steps.

$$\begin{aligned} y &= \sqrt{x-3} \\ x &= \sqrt{y-3} \end{aligned}$$

Now, to solve for *y* we will need to first square both sides and then proceed as normal.

$$\begin{aligned} x &= \sqrt{y-3} \\ x^2 &= y-3 \\ x^2+3 &= y \end{aligned}$$

This inverse is then,

$$g^{-1}(x)=x^2+3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned}\left(g^{-1}\circ g\right)(x) &= g^{-1}\left[g(x)\right]\\ &= g^{-1}\left(\sqrt{x-3}\right)\\ &= \left(\sqrt{x-3}\right)^2+3\\ &= x-3+3\\ &= x\end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

Before we move on we should also acknowledge the restrictions of $x \ge 0$ that we gave in the problem statement but never apparently did anything with. Note that this restriction is required to make sure that the inverse, $g^{-1}(x)$ given above is in fact one-to-one.

Without this restriction the inverse would not be one-to-one as is easily seen by a couple of quick evaluations.

$$g^{-1}(1) = (1)^2+3 = 4 \qquad\qquad g^{-1}(-1) = (-1)^2+3 = 4$$

Therefore, the restriction is required in order to make sure the inverse is one-to-one.

The next example can be a little messy so be careful with the work here.

Example 3 Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$\begin{aligned}y &= \frac{x+4}{2x-5}\\ x &= \frac{y+4}{2y-5}\end{aligned}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$\begin{aligned}x(2y-5) &= y+4\\ 2xy-5x &= y+4\\ 2xy-y &= 4+5x\\ (2x-1)y &= 4+5x\\ y &= \frac{4+5x}{2x-1}\end{aligned}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}\left(x\right)=\frac{4+5x}{2x-1}$$

Finally, we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\begin{aligned}\left(h\circ h^{-1}\right)\left(x\right)&=h\left[h^{-1}\left(x\right)\right]\\&=h\left[\frac{4+5x}{2x-1}\right]\\&=\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}\end{aligned}$$

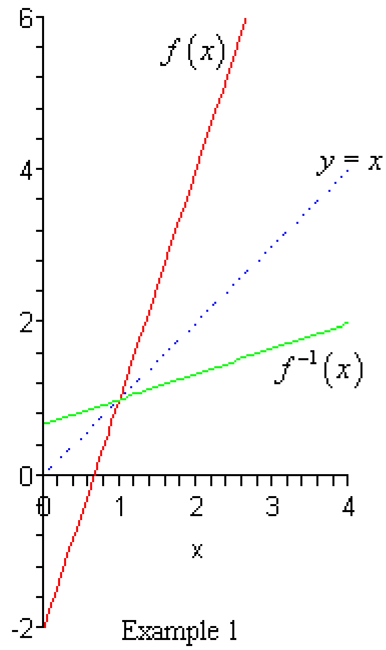
Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

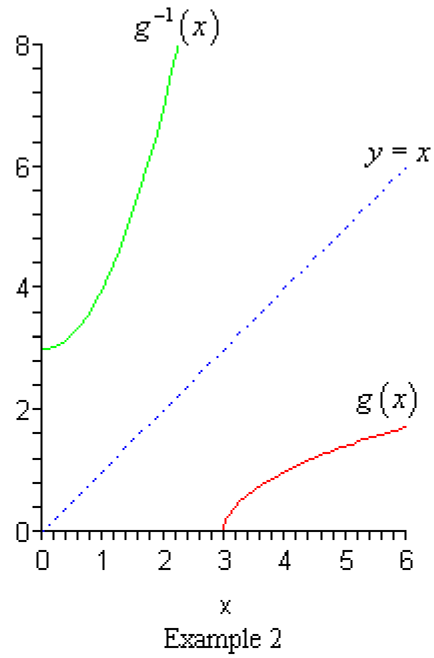
$$\begin{aligned}\left(h\circ h^{-1}\right)\left(x\right)&=\frac{2x-1}{2x-1}\,\frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}\\&=\frac{\left(2x-1\right)\left(\frac{4+5x}{2x-1}+4\right)}{\left(2x-1\right)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)}\\&=\frac{4+5x+4\left(2x-1\right)}{2\left(4+5x\right)-5\left(2x-1\right)}\\&=\frac{4+5x+8x-4}{8+10x-10x+5}\\&=\frac{13x}{13}\\&=x\end{aligned}$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and its inverse.

Here is the graph of the function and inverse from the first two examples. We'll not deal with the final example since that is a function that we haven't really talked about graphing yet.





In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.