ALGEBRA

Polynomial Functions

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Chapter 5 : Polynomial Functions

In this chapter we are going to take a more in depth look at polynomials. We've already solved and graphed second degree polynomials (*i.e.* quadratic equations/functions) and we now want to extend things out to more general polynomials. We will take a look at finding solutions to higher degree polynomials and how to get a rough sketch for a higher degree polynomial.

We will also be looking at Partial Fractions in this chapter. It doesn't really have anything to do with graphing polynomials but needed to be put somewhere and this chapter seemed like as good a place as any.

Here is a brief listing of the material in this chapter.

Dividing Polynomials – In this section we'll review some of the basics of dividing polynomials. We will define the remainder and divisor used in the division process and introduce the idea of synthetic division. We will also give the Division Algorithm.
Zeroes/Roots of Polynomials – In this section we'll define the zero or root of a polynomial and whether or not it is a simple root or has multiplicity k. We will also give the Fundamental Theorem of Algebra and The Factor Theorem as well as a couple of other useful Facts.
Graphing Polynomials – In this section we will give a process that will allow us to get a rough sketch of the graph of some polynomials. We discuss how to determine the behavior of the graph at x-intercepts and the leading coefficient test to determine the behavior of the graph as we allow x to increase and decrease without bound.
Finding Zeroes of Polynomials – As we saw in the previous section in order to sketch the graph of a polynomial we need to know what it's zeroes are. However, if we are not able to factor the polynomial we are unable to do that process. So, in this section we'll look at a process using the Rational Root Theorem that will allow us to find some of the zeroes of a polynomial and in special cases all of the zeroes.
Partial Fractions – In this section we will take a look at the process of partial fractions and finding the partial fraction decomposition of a rational expression. What we will be asking here is what "smaller" rational expressions did we add and/or subtract to get the given rational expression. This is a process that has a lot of uses in some later math classes. It can show up in Calculus and Differential Equations for example.

Section 5-1 : Dividing Polynomials

In this section we're going to take a brief look at dividing polynomials. This is something that we'll be doing off and on throughout the rest of this chapter and so we'll need to be able to do this.

Let's do a quick example to remind us how long division of polynomials works.

Example 1 Divide $5x^3 - x^2 + 6$ by $x - 4$.

Solution

Let's first get the problem set up.

$$x-4\overline{\left)5x^3 - x^2 + 0x + 6\right.}$$

Recall that we need to have the terms written down with the exponents in decreasing order and to make sure we don't make any mistakes we add in any missing terms with a zero coefficient.

Now we ask ourselves what we need to multiply $x - 4$ to get the first term in first polynomial. In this case that is $5x^2$. So multiply $x - 4$ by $5x^2$ and subtract the results from the first polynomial.

$$\begin{array}{r} 5x^2 \phantom{{}- x^2 + 0x + 6} \\ x-4\overline{\left)5x^3 - x^2 + 0x + 6\right.} \\ \underline{-\left(5x^3 - 20x^2\right)} \phantom{{}+ 0x + 6} \\ 19x^2 + 0x + 6 \end{array}$$

The new polynomial is called the **remainder**. We continue the process until the degree of the remainder is less than the degree of the **divisor**, which is $x - 4$ in this case. So, we need to continue until the degree of the remainder is less than 1.

Recall that the **degree** of a polynomial is the highest exponent in the polynomial. Also, recall that a constant is thought of as a polynomial of degree zero. Therefore, we'll need to continue until we get a constant in this case.

Here is the rest of the work for this example.

$$
\begin{array}{r}
5x^2+19x+76 \\
x-4\overline{\big)5x^3-x^2+0x+6} \\
\underline{-\left(5x^3-20x^2\right)} \\
19x^2+0x+6 \\
\underline{-\left(19x^2-76x\right)} \\
76x+6 \\
\underline{-\left(76x-304\right)} \\
310
\end{array}
$$

Okay, now that we've gotten this done, let's remember how we write the actual answer down. The answer is,

$$\frac{5x^3-x^2+6}{x-4}=5x^2+19x+76+\frac{310}{x-4}$$

There is actually another way to write the answer from the previous example that we're going to find much more useful, if for no other reason that it's easier to write down. If we multiply both sides of the answer by $x-4$ we get,

$$5x^3-x^2+6=(x-4)\left(5x^2+19x+76\right)+310$$

In this example we divided the polynomial by a linear polynomial in the form of $x-r$ and we will be restricting ourselves to only these kinds of problems. Long division works for much more general division, but these are the kinds of problems we are going to seeing the later sections.

In fact, we will be seeing these kinds of divisions so often that we'd like a quicker and more efficient way of doing them. Luckily there is something out there called **synthetic division** that works wonderfully for these kinds of problems. In order to use synthetic division we must be dividing a polynomial by a linear term in the form $x-r$. If we aren't then it won't work.

Let's redo the previous problem with synthetic division to see how it works.

Example 2 Use synthetic division to divide $5x^3-x^2+6$ by $x-4$.

Solution

Okay with synthetic division we pretty much ignore all the *x*'s and just work with the numbers in the polynomials.

First, let's notice that in this case *r=4*.

Now we need to set up the process. There are many different notations for doing this. We'll be using the following notation.

$$\underline{4}\rfloor \;\; 5 \quad -1 \quad 0 \quad 6$$

The numbers to the right of the vertical bar are the coefficients of the terms in the polynomial written in order of decreasing exponent. Also notice that any missing terms are acknowledged with a coefficient of zero.

Now, it will probably be easier to write down the process and then explain it so here it is.

$$\begin{array}{r|rrrr} 4 & 5 & -1 & 0 & 6 \\ & & 20 & 76 & 304 \\ \hline & 5 & 19 & 76 & 310 \end{array}$$

The first thing we do is drop the first number in the top line straight down as shown. Then along each diagonal we multiply the starting number by *r* (which is 4 in this case) and put this number in the second row. Finally, add the numbers in the first and second row putting the results in the third row. We continue this until we get reach the final number in the first row.

Now, notice that the numbers in the bottom row are the coefficients of the quadratic polynomial from our answer written in order of decreasing exponent and the final number in the third row is the remainder.

The answer is then the same as the first example.

$$5x^3 - x^2 + 6 = (x-4)(5x^2 + 19x + 76) + 310$$

We'll do some more examples of synthetic division in a bit. However, we really should generalize things out a little first with the following fact.

Division Algorithm

Given a polynomial *P(x)* with degree at least 1 and any number *r* there is another polynomial *Q(x)*, called the **quotient**, with degree one less than the degree of *P(x)* and a number *R*, called the **remainder**, such that,

$$P(x) = (x-r)Q(x) + R$$

Note as well that *Q(x)* and *R* are unique, or in other words, there is only one *Q(x)* and *R* that will work for a given *P(x)* and *r*.

So, with the one example we've done to this point we can see that,

$$Q(x) = 5x^2 + 19x + 76 \qquad \text{and} \qquad R = 310$$

Now, let's work a couple more synthetic division problems.

Example 3 Use synthetic division to do each of the following divisions.

(a) Divide $2x^3 - 3x - 5$ by $x + 2$

(b) Divide $4x^4 - 10x^2 + 1$ by $x - 6$

Solution

(a) Divide $2x^3 - 3x - 5$ by $x + 2$

Okay in this case we need to be a little careful here. We MUST divide by a term in the form $x - r$ in order for this to work and that minus sign is absolutely required. So, we're first going to need to write $x + 2$ as,

$$x + 2 = x - (-2)$$

and in doing so we can see that $r = -2$.

We can now do synthetic division and this time we'll just put up the results and leave it to you to check all the actual numbers.

$$\begin{array}{r|rrrr} \underline{-2} & 2 & 0 & -3 & -5 \\ & & -4 & 8 & -10 \\ \hline & 2 & -4 & 5 & -15 \end{array}$$

So, in this case we have,

$$2x^3 - 3x - 5 = (x + 2)(2x^2 - 4x + 5) - 15$$

(b) Divide $4x^4 - 10x^2 + 1$ by $x - 6$

In this case we've got r=6. Here is the work.

$$\begin{array}{r|rrrrr} \underline{6} & 4 & 0 & -10 & 0 & 1 \\ & 0 & 24 & 144 & 804 & 4824 \\ \hline & 4 & 24 & 134 & 804 & 4825 \end{array}$$

In this case we then have.

$$4x^4 - 10x^2 + 1 = (x - 6)(4x^3 + 24x^2 + 134x + 804) + 4825$$

So, just why are we doing this? That's a natural question at this point. One answer is that, down the road in a later section, we are going to want to get our hands on the *Q(x)*. Just why we might want to do that will have to wait for an explanation until we get to that point.

There is also another reason for this that we are going to make heavy usage of later on. Let's first start out with the division algorithm.

$$P(x) = (x - r)Q(x) + R$$

Now, let's evaluate the polynomial *P(x)* at *r*. If we had an actual polynomial here we could evaluate *P(x)* directly of course, but let's use the division algorithm and see what we get,

$$\begin{aligned} P(r) &= (r-r)Q(r)+R \\ &= (0)Q(r)+R \\ &= R \end{aligned}$$

Now, that's convenient. The remainder of the division algorithm is also the value of the polynomial evaluated at *r*. So, from our previous examples we now know the following function evaluations.

$$\begin{aligned} &\text{If } P(x)=5x^3-x^2+6 \text{ then } P(4)=310 \\ &\text{If } P(x)=2x^3-3x-5 \text{ then } P(-2)=-15 \\ &\text{If } P(x)=4x^4-10x^2+1 \text{ then } P(6)=4825 \end{aligned}$$

This is a very quick method for evaluating polynomials. For polynomials with only a few terms and/or polynomials with "small" degree this may not be much quicker that evaluating them directly. However, if there are many terms in the polynomial and they have large degrees this can be much quicker and much less prone to mistakes than computing them directly.

As noted, we will be using this fact in a later section to greatly reduce the amount of work we'll need to do in those problems.

Section 5-2 : Zeroes/Roots of Polynomials

We'll start off this section by defining just what a **root** or **zero** of a polynomial is. We say that $x = r$ is a root or zero of a polynomial, $P(x)$, if $P(r) = 0$. In other words, $x = r$ is a root or zero of a polynomial if it is a solution to the equation $P(x) = 0$.

In the next couple of sections we will need to find all the zeroes for a given polynomial. So, before we get into that we need to get some ideas out of the way regarding zeroes of polynomials that will help us in that process.

The process of finding the zeroes of $P(x)$ really amount to nothing more than solving the equation $P(x) = 0$ and we already know how to do that for second degree (quadratic) polynomials. So, to help illustrate some of the ideas were going to be looking at let's get the zeroes of a couple of second degree polynomials.

Let's first find the zeroes for $P(x) = x^2 + 2x - 15$. To do this we simply solve the following equation.

$$x^2 + 2x - 15 = (x+5)(x-3) = 0 \qquad \Rightarrow \qquad x = -5,\ x = 3$$

So, this second degree polynomial has two zeroes or roots.

Now, let's find the zeroes for $P(x) = x^2 - 14x + 49$. That will mean solving,

$$x^2 - 14x + 49 = (x-7)^2 = 0 \qquad \Rightarrow \qquad x = 7$$

So, this second degree polynomial has a single zero or root. Also, recall that when we first looked at these we called a root like this a **double root**.

We solved each of these by first factoring the polynomial and then using the zero factor property on the factored form. When we first looked at the zero factor property we saw that it said that if the product of two terms was zero then one of the terms had to be zero to start off with.

The zero factor property can be extended out to as many terms as we need. In other words, if we've got a product of *n* terms that is equal to zero, then at least one of them had to be zero to start off with. So, if we could factor higher degree polynomials we could then solve these as well.

Let's take a look at a couple of these.

Example 1 Find the zeroes of each of the following polynomials.

(a) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

(b) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

(c) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x+4)$

Solution

In each of these the factoring has been done for us. Do not worry about factoring anything like this. You won't be asked to do any factoring of this kind anywhere in this material. There are only here to make the point that the zero factor property works here as well. We will also use these in a later example.

(a) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2 (x-2)^3$

Okay, in this case we do have a product of 3 terms however the first is a constant and will not make the polynomial zero. So, from the final two terms it looks like the polynomial will be zero for $x = -1$ and $x = 2$. Therefore, the zeroes of this polynomial are,

$$x = -1 \text{ and } x = 2$$

(b) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4 (x-3)^3 (x+5)$

We've also got a product of three terms in this polynomial. However, since the first is now an *x* this will introduce a third zero. The zeroes for this polynomial are,

$$x = -5, \; x = 0, \text{ and } \; x = 3$$

because each of these will make one of the terms, and hence the whole polynomial, zero.

(c) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3 (x-1)^2 (x+5)(x+4)$

With this polynomial we have four terms and the zeroes here are,

$$x = -5, \; x = -1, \; x = 1, \text{ and } \; x = -4$$

Now, we've got some terminology to get out of the way. If *r* is a zero of a polynomial and the exponent on the term that produced the root is *k* then we say that *r* has **multiplicity *k***. Zeroes with a multiplicity of 1 are often called **simple** zeroes.

For example, the polynomial $P(x) = x^2 - 10x + 25 = (x-5)^2$ will have one zero, $x = 5$, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as,

$$P(x) = x^2 - 10x + 25 = (x-5)(x-5)$$

Written this way the term $x - 5$ shows up twice and each term gives the same zero, $x = 5$. Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

Example 2 List the multiplicities of the zeroes of each of the following polynomials.

(a) $P(x) = x^2 + 2x - 15$

(b) $P(x) = x^2 - 14x + 49$

(c) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

(d) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

(e) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x+4)$

Solution

We've already determined the zeroes of each of these in previous work or examples in this section so we won't redo that work. In each case we will simply write down the previously found zeroes and then go back to the factored form of the polynomial, look at the exponent on each term and give the multiplicity.

(a) In this case we've got two simple zeroes : $x = -5,\ x = 3$.

(b) Here $x = 7$ is a zero of multiplicity 2.

(c) There are two zeroes for this polynomial : $x = -1$ with multiplicity 2 and $x = 2$ with multiplicity 3.

(d) We have three zeroes in this case. : $x = -5$ which is simple, $x = 0$ with multiplicity of 4 and $x = 3$ with multiplicity 3.

(e) In the final case we've got four zeroes. $x = -5$ which is simple, $x = -1$ with multiplicity of 3, $x = 1$ with multiplicity 2 and $x = -4$ which is simple.

This example leads us to several nice facts about polynomials. Here is the first and probably the most important.

Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree *n* then $P(x)$ will have exactly *n* zeroes, some of which may repeat.

This fact says that if you list out all the zeroes and listing each one *k* times where *k* is its multiplicity you will have exactly *n* numbers in the list. Another way to say this fact is that the multiplicity of all the zeroes must add to the degree of the polynomial.

We can go back to the previous example and verify that this fact is true for the polynomials listed there.

This will be a nice fact in a couple of sections when we go into detail about finding all the zeroes of a polynomial. If we know an upper bound for the number of zeroes for a polynomial then we will know when we've found all of them and so we can stop looking.

Note as well that some of the zeroes may be complex. In this section we have worked with polynomials that only have real zeroes but do not let that lead you to the idea that this theorem will only apply to real zeroes. It is completely possible that complex zeroes will show up in the list of zeroes.

The next fact is also very useful at times.

The Factor Theorem

For the polynomial $P(x)$,

1. If *r* is a zero of $P(x)$ then $x-r$ will be a factor of $P(x)$.
2. If $x-r$ is a factor of $P(x)$ then *r* will be a zero of $P(x)$.

Again, if we go back to the previous example we can see that this is verified with the polynomials listed there.

The factor theorem leads to the following fact.

Fact 1

If $P(x)$ is a polynomial of degree *n* and *r* is a zero of $P(x)$ then $P(x)$ can be written in the following form.

$$P(x)=(x-r)Q(x)$$

where $Q(x)$ is a polynomial with degree $n-1$. $Q(x)$ can be found by dividing $P(x)$ by $x-r$.

There is one more fact that we need to get out of the way.

Fact 2

If $P(x)=(x-r)Q(x)$ and $x=t$ is a zero of $Q(x)$ then $x=t$ will also be a zero of $P(x)$.

This fact is easy enough to verify directly. First, if $x=t$ is a zero of $Q(x)$ then we know that,

$$Q(t)=0$$

since that is what it means to be a zero. So, if $x=t$ is to be a zero of $P(x)$ then all we need to do is show that $P(t)=0$ and that's actually quite simple. Here it is,

$$P(t)=(t-r)Q(t)=(t-r)(0)=0$$

and so $x=t$ is a zero of $P(x)$.

Let's work an example to see how these last few facts can be of use to us.

Example 3 Given that $x=2$ is a zero of $P(x)=x^3+2x^2-5x-6$ find the other two zeroes.

Solution

First, notice that we really can say the other two since we know that this is a third degree polynomial and so by The Fundamental Theorem of Algebra we will have exactly 3 zeroes, with some repeats possible.

So, since we know that $x = 2$ is a zero of $P(x) = x^3 + 2x^2 - 5x - 6$ the Fact 1 tells us that we can write $P(x)$ as,

$$P(x) = (x-2)Q(x)$$

and $Q(x)$ will be a quadratic polynomial. Then we can find the zeroes of $Q(x)$ by any of the methods that we've looked at to this point and by Fact 2 we know that the two zeroes we get from $Q(x)$ will also by zeroes of $P(x)$. At this point we'll have 3 zeroes and so we will be done.

So, let's find $Q(x)$. To do this all we need to do is a quick synthetic division as follows.

$$\begin{array}{r|rrrr} \underline{2} & 1 & 2 & -5 & -6 \\ & & 2 & 8 & 6 \\ \hline & 1 & 4 & 3 & 0 \end{array}$$

Before writing down $Q(x)$ recall that the final number in the third row is the remainder and that we know that $P(2)$ must be equal to this number. So, in this case we have that $P(2) = 0$. If you think about it, we should already know this to be true. We were given in the problem statement the fact that $x = 2$ is a zero of $P(x)$ and that means that we must have $P(2) = 0$.

So, why go on about this? This is a great check of our synthetic division. Since we know that $x = 2$ is a zero of $P(x)$ and we get any other number than zero in that last entry we will know that we've done something wrong and we can go back and find the mistake.

Now, let's get back to the problem. From the synthetic division we have,

$$P(x) = (x-2)(x^2 + 4x + 3)$$

So, this means that,

$$Q(x) = x^2 + 4x + 3$$

and we can find the zeroes of this. Here they are,

$$Q(x) = x^2 + 4x + 3 = (x+3)(x+1) \qquad \Rightarrow \qquad x = -3,\ x = -1$$

So, the three zeroes of $P(x)$ are $x = -3$, $x = -1$ and $x = 2$.

As an aside to the previous example notice that we can also now completely factor the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$. Substituting the factored form of $Q(x)$ into $P(x)$ we get,

$$P(x) = (x-2)(x+3)(x+1)$$

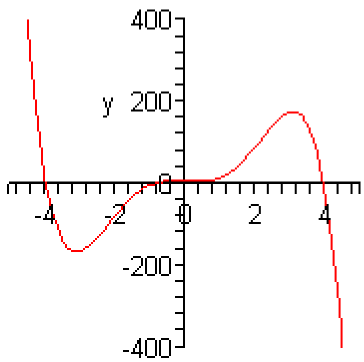
This is how the polynomials in the first set of examples were factored by the way. Those require a little more work than this, but they can be done in the same manner.

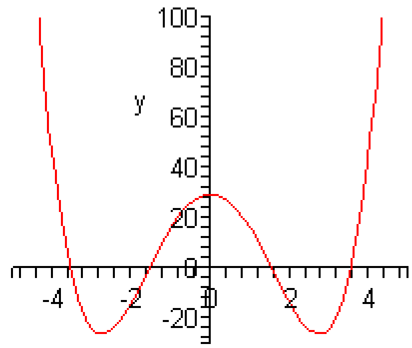
Section 5-3 : Graphing Polynomials

In this section we are going to look at a method for getting a rough sketch of a general polynomial. The only real information that we're going to need is a complete list of all the zeroes (including multiplicity) for the polynomial.

In this section we are going to either be given the list of zeroes or they will be easy to find. In the next section we will go into a method for determining a large portion of the list for most polynomials. We are graphing first since the method for finding all the zeroes of a polynomial can be a little long and we don't want to obscure the details of this section in the mess of finding the zeroes of the polynomial.

Let's start off with the graph of couple of polynomials.





Do not worry about the equations for these polynomials. We are giving these only so we can use them to illustrate some ideas about polynomials.

First, notice that the graphs are nice and smooth. There are no holes or breaks in the graph and there are no sharp corners in the graph. The graphs of polynomials will always be nice smooth curves.

Secondly, the "humps" where the graph changes direction from increasing to decreasing or decreasing to increasing are often called **turning points**. If we know that the polynomial has degree *n* then we will know that there will be at most $n-1$ turning points in the graph.

While this won't help much with the actual graphing process it will be a nice check. If we have a fourth degree polynomial with 5 turning point then we will know that we've done something wrong since a fourth degree polynomial will have no more than 3 turning points.

Next, we need to explore the relationship between the *x*-intercepts of a graph of a polynomial and the zeroes of the polynomial. Recall that to find the [*x*-intercepts] of a function we need to solve the equation

$$P(x)=0$$

Also, recall that $x=r$ is a zero of the polynomial, $P(x)$, provided $P(r)=0$. But this means that $x=r$ is also a solution to $P(x)=0$.

In other words, the zeroes of a polynomial are also the *x*-intercepts of the graph. Also, recall that *x*-intercepts can either cross the *x*-axis or they can just touch the *x*-axis without actually crossing the axis.

Notice as well from the graphs above that the *x*-intercepts can either flatten out as they cross the *x*-axis or they can go through the *x*-axis at an angle.

The following fact will relate all of these ideas to the multiplicity of the zero.

Fact

If $x = r$ is a zero of the polynomial $P(x)$ with multiplicity *k* then,

1. If *k* is odd then the *x*-intercept corresponding to $x = r$ will cross the *x*-axis.
2. If *k* is even then the *x*-intercept corresponding to $x = r$ will only touch the *x*-axis and not actually cross it.

Furthermore, if $k > 1$ then the graph will flatten out at $x = r$.

Finally, notice that as we let *x* get large in both the positive or negative sense (*i.e.* at either end of the graph) then the graph will either increase without bound or decrease without bound. This will always happen with every polynomial and we can use the following test to determine just what will happen at the endpoints of the graph.

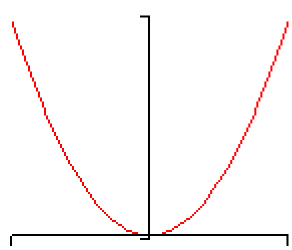
Leading Coefficient Test

Suppose that $P(x)$ is a polynomial with degree *n*. So we know that the polynomial must look like,

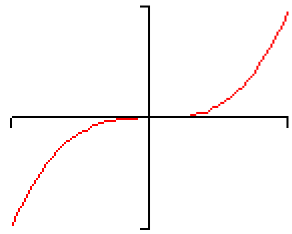
$$P(x) = ax^n + \cdots$$

We don't know if there are any other terms in the polynomial, but we do know that the first term will have to be the one listed since it has degree *n*. We now have the following facts about the graph of $P(x)$ at the ends of the graph.

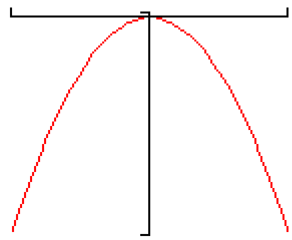
1. If $a > 0$ and *n* is even then the graph of $P(x)$ will increase without bound at both endpoints. A good example of this is the graph of x^2.



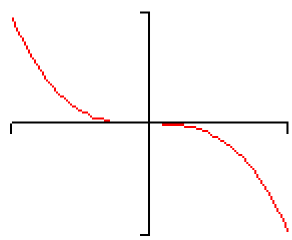
2. If $a > 0$ and *n* is odd then the graph of $P(x)$ will increase without bound at the right end and decrease without bound at the left end. A good example of this is the graph of x^3.



3. If $a < 0$ and *n* is even then the graph of $P(x)$ will decrease without bound at both endpoints. A good example of this is the graph of $-x^2$.



4. If $a < 0$ and *n* is odd then the graph of $P(x)$ will decrease without bound at the right end and increase without bound at the left end. A good example of this is the graph of $-x^3$.



Okay, now that we've got all that out of the way we can finally give a process for getting a rough sketch of the graph of a polynomial.

Process for Graphing a Polynomial

1. Determine all the zeroes of the polynomial and their multiplicity. Use the fact above to determine the *x*-intercept that corresponds to each zero will cross the *x*-axis or just touch it and if the *x*-intercept will flatten out or not.

2. Determine the *y*-intercept, $\left(0, P(0)\right)$.

3. Use the leading coefficient test to determine the behavior of the polynomial at the end of the graph.

4. Plot a few more points. This is left intentionally vague. The more points that you plot the better the sketch. At the least you should plot at least one at either end of the graph and at least one point between each pair of zeroes.

We should give a quick warning about this process before we actually try to use it. This process assumes that all the zeroes are real numbers. If there are any complex zeroes then this process may miss some pretty important features of the graph.

Let's sketch a couple of polynomials.

Example 1 Sketch the graph of $P\left(x\right)=5x^5-20x^4+5x^3+50x^2-20x-40$.

Solution

We found the zeroes and multiplicities of this polynomial in the previous section so we'll just write them back down here for reference purposes.

$$\begin{aligned} x &= -1 \quad \left(\text{multiplicity } 2\right) \\ x &= 2 \quad\;\; \left(\text{multiplicity } 3\right) \end{aligned}$$

So, from the fact we know that $x=-1$ will just touch the *x*-axis and not actually cross it and that $x=2$ will cross the *x*-axis. Also, both will be flat as they cross the *x*-axis since the multiplicity for both is greater than 1.

Next, the *y*-intercept is $\left(0,-40\right)$.

The coefficient of the 5th degree term is positive and since the degree is odd we know that this polynomial will increase without bound at the right end and decrease without bound at the left end.

Finally, we just need to evaluate the polynomial at a couple of points. The points that we pick aren't really all that important. We just want to pick points according to the guidelines in the process outlined above and points that will be fairly easy to evaluate. Here are some points. We will leave it to you to verify the evaluations.

$$P\left(-2\right)=-320 \qquad\qquad P\left(1\right)=-20 \qquad\qquad P\left(3\right)=80$$

Now, to actually sketch the graph we'll start on the left end and work our way across to the right end. First, we know that on the left end the graph decreases without bound as we make *x* more and more negative and this agrees with the point that we evaluated at $x=-2$.

So, as we move to the right the function will actually be increasing at $x=-2$ and we will continue to increase until we hit the first *x*-intercept at $x=-1$. At this point we know that the graph just touches the *x*-axis without actually crossing it and will be flat as it does this. This means that at $x=-1$ the graph must be a turning point.

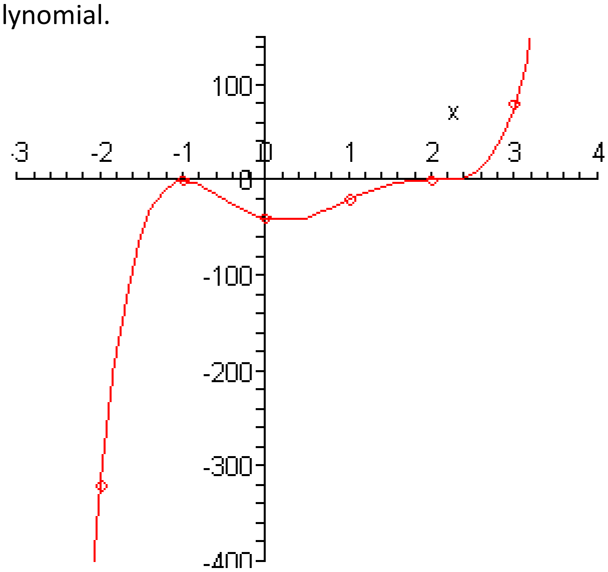
The graph is now decreasing as we move to the right. Again, this agrees with the next point that we'll run across, the *y*-intercept.

Now, according to the next point that we've got, $x = 1$, the graph must have another turning point somewhere between $x = 0$ and $x = 1$ since the graph is higher at $x = 1$ than at $x = 0$. Just where this turning point will occur is very difficult to determine at this level so we won't worry about trying to find it. In fact, determining this point usually requires some Calculus.

So, we are moving to the right and the function is increasing. The next point that we hit is the *x*-intercept at $x = 2$ and this one crosses the *x*-axis so we know that there won't be a turning point here as there was at the first *x*-intercept. Also, the graph will be flat as it touches the *x*-axis because the multiplicity is greater than one. So, the graph will continue to increase through this point, briefly flattening out as it touches the *x*-axis, until we hit the final point that we evaluated the function at $x = 3$.

At this point we've hit all the *x*-intercepts and we know that the graph will increase without bound at the right end and so it looks like all we need to do is sketch in an increasing curve.

Here is a sketch of the polynomial.



Note that one of the reasons for plotting points at the ends is to see just how fast the graph is increasing or decreasing. We can see from the evaluations that the graph is decreasing on the left end much faster than it's increasing on the right end.

Okay, let's take a look at another polynomial. This time we'll go all the way through the process of finding the zeroes.

Example 2 Sketch the graph of $P(x) = x^4 - x^3 - 6x^2$.

Solution

First, we'll need to factor this polynomial as much as possible so we can identify the zeroes and get their multiplicities.

$$P(x) = x^4 - x^3 - 6x^2 = x^2\left(x^2 - x - 6\right) = x^2(x-3)(x+2)$$

Here is a list of the zeroes and their multiplicities.

$$\begin{aligned} x &= -2 \quad (\text{multiplicity } 1) \\ x &= 0 \quad\;\; (\text{multiplicity } 2) \\ x &= 3 \quad\;\; (\text{multiplicity } 1) \end{aligned}$$

So, the zeroes at $x = -2$ and $x = 3$ will correspond to *x*-intercepts that cross the *x*-axis since their multiplicity is odd and will do so at an angle since their multiplicity is NOT at least 2. The zero at $x = 0$ will not cross the *x*-axis since its multiplicity is even but will be flat as it touches the *x*-axis since the multiplicity is greater than one.

The *y*-intercept is $(0,0)$ and notice that this is also an *x*-intercept.

The coefficient of the 4th degree term is positive and so since the degree is even we know that the polynomial will increase without bound at both ends of the graph.

Finally, here are some function evaluations.

$$P(-3) = 54 \qquad P(-1) = -4 \qquad P(1) = -6 \qquad P(4) = 96$$

Now, starting at the left end we know that as we make *x* more and more negative the function must increase without bound. That means that as we move to the right the graph will actually be decreasing.

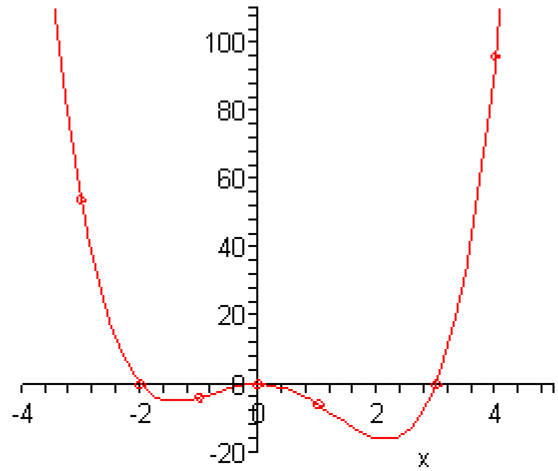
At $x = -3$ the graph will be decreasing and will continue to decrease when we hit the first *x*-intercept at $x = -2$ since we know that this *x*-intercept will cross the *x*-axis.

Next, since the next *x*-intercept is at $x = 0$ we will have to have a turning point somewhere so that the graph can increase back up to this *x*-intercept. Again, we won't worry about where this turning point actually is.

Once we hit the *x*-intercept at $x = 0$ we know that we've got to have a turning point since this *x*-intercept doesn't cross the *x*-axis. Therefore, to the right of $x = 0$ the graph will now be decreasing. Recall however, that because the multiplicity is greater than one it will be flat as it touches the *x*-axis.

It will continue to decrease until it hits another turning point (at some unknown point) so that the graph can get back up to the *x*-axis for the next *x*-intercept at $x = 3$. This is the final *x*-intercept and since the graph is increasing at this point and must increase without bound at this end we are done.

Here is a sketch of the graph.



Example 3 Sketch the graph of $P(x) = -x^5 + 4x^3$.

Solution

As with the previous example we'll first need to factor this as much as possible.

$$P(x) = -x^5 + 4x^3 = -\left(x^5 - 4x^3\right) = -x^3\left(x^2 - 4\right) = -x^3\left(x-2\right)\left(x+2\right)$$

Notice that we first factored out a minus sign to make the rest of the factoring a little easier. Here is a list of all the zeroes and their multiplicities.

$$\begin{aligned} x &= -2 \quad (\text{multiplicity } 1) \\ x &= 0 \quad\;\;\, (\text{multiplicity } 3) \\ x &= 2 \quad\;\;\, (\text{multiplicity } 1) \end{aligned}$$

So, all three zeroes correspond to *x*-intercepts that actually cross the *x*-axis since all their multiplicities are odd, however, only the *x*-intercept at $x = 0$ will cross the *x*-axis flattened out.

The *y*-intercept is $(0,0)$ and as with the previous example this is also an *x*-intercept.

In this case the coefficient of the 5th degree term is negative and so since the degree is odd the graph will increase without bound on the left side and decrease without bound on the right side.

Here are some function evaluations.

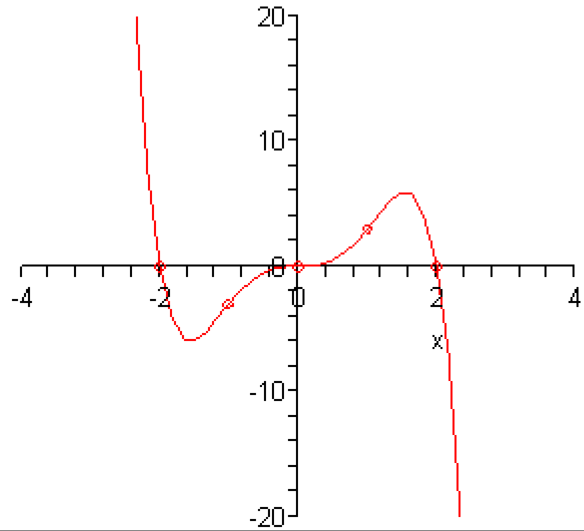
$$P(-3) = 135 \qquad P(-1) = -3 \qquad P(1) = 3 \qquad P(3) = -135$$

Alright, this graph will start out much as the previous graph did. At the left end the graph will be decreasing as we move to the right and will decrease through the first *x*-intercept at $x = -2$ since know that this *x*-intercept crosses the *x*-axis.

Now at some point we'll get a turning point so the graph can get back up to the next *x*-intercept at $x = 0$ and the graph will continue to increase through this point since it also crosses the *x*-axis. Note as well that the graph should be flat at this point as well since the multiplicity is greater than one.

Finally, the graph will reach another turning point and start decreasing so it can get back down to the final *x*-intercept at $x = 2$. Since we know that the graph will decrease without bound at this end we are done.

Here is the sketch of this polynomial.



The process that we've used in these examples can be a difficult process to learn. It takes time to learn how to correctly interpret the results.

Also, as pointed out at various spots there are several situations that we won't be able to deal with here. To find the majority of the turning points we would need some Calculus, which we clearly don't have. Also, the process does require that we have all the zeroes and that they all be real numbers.

Even with these drawbacks however, the process can at least give us an idea of what the graph of a polynomial will look like.

Section 5-4 : Finding Zeroes of Polynomials

We've been talking about zeroes of polynomial and why we need them for a couple of sections now. We haven't, however, really talked about how to actually find them for polynomials of degree greater than two. That is the topic of this section. Well, that's kind of the topic of this section. In general, finding all the zeroes of any polynomial is a fairly difficult process. In this section we will give a process that will find all rational (*i.e.* integer or fractional) zeroes of a polynomial. We will be able to use the process for finding all the zeroes of a polynomial provided all but at most two of the zeroes are rational. If more than two of the zeroes are not rational then this process will not find all of the zeroes.

We will need the following theorem to get us started on this process.

Rational Root Theorem

If the rational number $x = \frac{b}{c}$ is a zero of the n$^{\text{th}}$ degree polynomial,

$$P\left(x\right) = sx^n + \cdots + t$$

where all the coefficients are integers then *b* will be a factor of *t* and *c* will be a factor of *s*.

Note that in order for this theorem to work then the zero must be reduced to lowest terms. In other words, it will work for $\frac{4}{3}$ but not necessarily for $\frac{20}{15}$.

Let's verify the results of this theorem with an example.

Example 1 Verify that the roots of the following polynomial satisfy the rational root theorem.

$$P\left(x\right) = 12x^3 - 41x^2 - 38x + 40 = \left(x-4\right)\left(3x-2\right)\left(4x+5\right)$$

Solution

From the factored form we can see that the zeroes are,

$$x = 4 = \frac{4}{1} \qquad x = \frac{2}{3} \qquad x = -\frac{5}{4}$$

Notice that we wrote the integer as a fraction to fit it into the theorem. Also, with the negative zero we can put the negative onto the numerator or denominator. It won't matter.

So, according to the rational root theorem the numerators of these fractions (with or without the minus sign on the third zero) must all be factors of 40 and the denominators must all be factors of 12.

Here are several ways to factor 40 and 12.

$$\begin{array}{llll} 40 = \left(4\right)\left(10\right) & 40 = \left(2\right)\left(20\right) & 40 = \left(5\right)\left(8\right) & 40 = \left(-5\right)\left(-8\right) \\ 12 = \left(1\right)\left(12\right) & 12 = \left(3\right)\left(4\right) & 12 = \left(-3\right)\left(-4\right) & \end{array}$$

From these we can see that in fact the numerators are all factors of 40 and the denominators are all factors of 12. Also note that, as shown, we can put the minus sign on the third zero on either the numerator or the denominator and it will still be a factor of the appropriate number.

So, why is this theorem so useful? Well, for starters it will allow us to write down a list of *possible* rational zeroes for a polynomial and more importantly, any rational zeroes of a polynomial WILL be in this list.

In other words, we can quickly determine all the rational zeroes of a polynomial simply by checking all the numbers in our list.

Before getting into the process of finding the zeroes of a polynomial let's see how to come up with a list of possible rational zeroes for a polynomial.

Example 2 Find a list of all possible rational zeroes for each of the following polynomials.

(a) $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$

(b) $P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9$

Solution

(a) $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$

Now, just what does the rational root theorem say? It says that if $x = \frac{b}{c}$ is to be a zero of $P(x)$ then *b* must be a factor of 6 and *c* must be a factor of 1. Also, as we saw in the previous example we can't forget negative factors.

So, the first thing to do is actually to list all possible factors of 1 and 6. Here they are.

$$\begin{array}{ll} 6: & \pm 1, \pm 2, \pm 3, \pm 6 \\ 1: & \pm 1 \end{array}$$

Now, to get a list of possible rational zeroes of the polynomial all we need to do is write down all possible fractions that we can form from these numbers where the numerators must be factors of 6 and the denominators must be factors of 1. This is actually easier than it might at first appear to be.

There is a very simple shorthanded way of doing this. Let's go through the first one in detail then we'll do the rest quicker. First, take the first factor from the numerator list, including the $\pm$, and divide this by the first factor (okay, only factor in this case) from the denominator list, again including the $\pm$. Doing this gives,

$$\frac{\pm 1}{\pm 1}$$

This looks like a mess, but it isn't too bad. There are four fractions here. They are,

$$\frac{+1}{+1} = 1 \qquad \frac{+1}{-1} = -1 \qquad \frac{-1}{+1} = -1 \qquad \frac{-1}{-1} = 1$$

Notice however, that the four fractions all reduce down to two possible numbers. This will always happen with these kinds of fractions. What we'll do from now on is form the fraction, do any simplification of the numbers, ignoring the $\pm$, and then drop one of the $\pm$.

So, the list possible rational zeroes for this polynomial is,

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 6}{\pm 1} = \pm 6$$

So, it looks there are only 8 possible rational zeroes and in this case they are all integers. Note as well that any rational zeroes of this polynomial WILL be somewhere in this list, although we haven't found them yet.

(b) $P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9$

We'll not put quite as much detail into this one. First get a list of all factors of -9 and 2. Note that the minus sign on the 9 isn't really all that important since we will still get a $\pm$ on each of the factors.

$$\begin{aligned} -9: &\quad \pm 1, \pm 3, \pm 9 \\ 2: &\quad \pm 1, \pm 2 \end{aligned}$$

Now, the factors of -9 are all the possible numerators and the factors of 2 are all the possible denominators.

Here then is a list of all possible rational zeroes of this polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 9}{\pm 1} = \pm 9$$

$$\frac{\pm 1}{\pm 2} = \pm \frac{1}{2} \qquad \frac{\pm 3}{\pm 2} = \pm \frac{3}{2} \qquad \frac{\pm 9}{\pm 2} = \pm \frac{9}{2}$$

So, we've got a total of 12 possible rational zeroes, half are integers and half are fractions.

The following fact will also be useful on occasion in finding the zeroes of a polynomial.

Fact

If $P(x)$ is a polynomial and we know that $P(a) > 0$ and $P(b) < 0$ then somewhere between *a* and *b* is a zero of $P(x)$.

What this fact is telling us is that if we evaluate the polynomial at two points and one of the evaluations gives a positive value (*i.e.* the point is above the *x*-axis) and the other evaluation gives a negative value (*i.e.* the point is below the *x*-axis), then the only way to get from one point to the other is to go through the *x*-axis. Or, in other words, the polynomial must have a zero, since we know that zeroes are where a graph touches or crosses the *x*-axis.

Note that this fact doesn't tell us what the zero is, it only tells us that one will exist. Also, note that if both evaluations are positive or both evaluations are negative there may or may not be a zero between them.

Here is the process for determining all the rational zeroes of a polynomial.

Process for Finding Rational Zeroes

1. Use the rational root theorem to list all possible rational zeroes of the polynomial $P(x)$.

2. Evaluate the polynomial at the numbers from the first step until we find a zero. Let's suppose the zero is $x = r$, then we will know that it's a zero because $P(r) = 0$. Once this has been determined that it is in fact a zero write the original polynomial as
$$P(x) = (x - r)Q(x)$$

3. Repeat the process using $Q(x)$ this time instead of $P(x)$. This repeating will continue until we reach a second degree polynomial. At this point we can solve this directly for the remaining zeroes.

To simplify the second step we will use synthetic division. This will greatly simplify our life in several ways. First, recall that the last number in the final row is the polynomial evaluated at *r* and if we do get a zero the remaining numbers in the final row are the coefficients for $Q(x)$ and so we won't have to go back and find that.

Also, in the evaluation step it is usually easiest to evaluate at the possible integer zeroes first and then go back and deal with any fractions if we have to.

Let's see how this works.

Example 3 Determine all the zeroes of $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$.

Solution

We found the list of all possible rational zeroes in the previous example. Here they are.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

We now need to start evaluating the polynomial at these numbers. We can start anywhere in the list and will continue until we find zero.

To do the evaluations we will build a **synthetic division table**. In a synthetic division table do the multiplications in our head and drop the middle row just writing down the third row and since we will be going through the process multiple times we put all the rows into a table.

Here is the first synthetic division table for this problem.

$$\begin{array}{r|rrrrrl} & 1 & -7 & 17 & -17 & 6 & \\ \hline -1 & 1 & -8 & 25 & -42 & 48 & = P(-1) \neq 0 \\ 1 & 1 & -6 & 11 & -6 & 0 & = P(1) = 0\,!! \end{array}$$

So, we found a zero. Before getting into that let's recap the computations here to make sure you can do them.

The top row is the coefficients from the polynomial and the first column is the numbers that we're evaluating the polynomial at.

Each row (after the first) is the third row from the synthetic division process. Let's quickly look at the first couple of numbers in the second row. The number in the second column is the first coefficient dropped down. The number in the third column is then found by multiplying the -1 by 1 and adding to the -7. This gives the -8. For the fourth number is then -1 times -8 added onto 17. This is 25, *etc.*

You can do regular synthetic division if you need to, but it's a good idea to be able to do these tables as it can help with the process.

Okay, back to the problem. We now know that $x = 1$ is a zero and so we can write the polynomial as,

$$P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 = (x-1)\left(x^3 - 6x^2 + 11x - 6\right)$$

Now we need to repeat this process with the polynomial $Q(x) = x^3 - 6x^2 + 11x - 6$. So, the first thing to do is to write down all possible rational roots of this polynomial and in this case we're lucky enough to have the first and last numbers in this polynomial be the same as the original polynomial, that usually won't happen so don't always expect it. Here is the list of all possible rational zeroes of this polynomial.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

Now, before doing a new synthetic division table let's recall that we are looking for zeroes to $P(x)$ and from our first division table we determined that $x = -1$ is NOT a zero of $P(x)$ and so there is no reason to bother with that number again.

This is something that we should always do at this step. Take a look at the list of new possible rational zeros and ask are there any that can't be rational zeroes of the original polynomial. If there are some, throw them out as we will already know that they won't work. So, a reduced list of numbers to try here is,

$$1, \pm 2, \pm 3, \pm 6$$

Note that we do need to include $x = 1$ in the list since it is possible for a zero to occur more that once (*i.e.* multiplicity greater than one).

Here is the synthetic division table for this polynomial.

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ \hline 1 & 1 & -5 & 6 & 0 \end{array} = P(1) = 0!!$$

So, $x=1$ is also a zero of $Q(x)$ and we can now write $Q(x)$ as,

$$Q(x)=x^3-6x^2+11x-6=(x-1)(x^2-5x+6)$$

Now, technically we could continue the process with x^2-5x+6, but this is a quadratic equation and we know how to find zeroes of these without a complicated process like this so let's just solve this like we normally would.

$$x^2-5x+6=(x-2)(x-3)=0 \qquad \Rightarrow \qquad x=2,\ x=3$$

Note that these two numbers are in the list of possible rational zeroes.

Finishing up this problem then gives the following list of zeroes for $P(x)$.

$$\begin{aligned} x&=1 \quad (\text{multiplicity } 2)\\ x&=2 \quad (\text{multiplicity } 1)\\ x&=3 \quad (\text{multiplicity } 1)\end{aligned}$$

Note that $x=1$ has a multiplicity of 2 since it showed up twice in our work above.

Before moving onto the next example let's also note that we can now completely factor the polynomial $P(x)=x^4-7x^3+17x^2-17x+6$. We know that each zero will give a factor in the factored form and that the exponent on the factor will be the multiplicity of that zero. So, the factored form is,

$$P(x)=x^4-7x^3+17x^2-17x+6=(x-1)^2(x-2)(x-3)$$

Let's take a look at another example.

Example 4 Find all the zeroes of $P(x)=2x^4+x^3+3x^2+3x-9$.

Solution

From the second example we know that the list of all possible rational zeroes is,

$$\frac{\pm1}{\pm1}=\pm1 \qquad \frac{\pm3}{\pm1}=\pm3 \qquad \frac{\pm9}{\pm1}=\pm9$$

$$\frac{\pm1}{\pm2}=\pm\frac{1}{2} \qquad \frac{\pm3}{\pm2}=\pm\frac{3}{2} \qquad \frac{\pm9}{\pm2}=\pm\frac{9}{2}$$

The next step is to build up the synthetic division table. When we've got fractions it's usually best to start with the integers and do those first. Also, this time we'll start with doing all the negative integers first. We are doing this to make a point on how we can use the fact given above to help us identify zeroes.

$$\begin{array}{r|rrrrl} & 2 & 1 & 3 & 3 & -9 \\ \hline -9 & 2 & -17 & 156 & -1401 & 12600 = P(-9) \neq 0 \\ -3 & 2 & -5 & 18 & -51 & 144 = P(-3) \neq 0 \\ -1 & 2 & -1 & 4 & -1 & -8 = P(-1) \neq 0 \end{array}$$

Now, we haven't found a zero yet, however let's notice that $P(-3) = 144 > 0$ and $P(-1) = -8 < 0$ and so by the fact above we know that there must be a zero somewhere between $x = -3$ and $x = -1$. Now, we can also notice that $x = -\frac{3}{2} = -1.5$ is in this range and is the only number in our list that is in this range and so there is a chance that this is a zero. Let's run through synthetic division real quick to check and see if it's a zero and to get the coefficients for $Q(x)$ if it is a zero.

$$\begin{array}{r|rrrrr} & 2 & 1 & 3 & 3 & -9 \\ \hline -\frac{3}{2} & 2 & -2 & 6 & -6 & 0 \end{array}$$

So, we got a zero in the final spot which tells us that this was a zero and $Q(x)$ is,

$$Q(x) = 2x^3 - 2x^2 + 6x - 6$$

We now need to repeat the whole process with this polynomial. Also, unlike the previous example we can't just reuse the original list since the last number is different this time. So, here are the factors of -6 and 2.

$$\begin{array}{rl} -6: & \pm 1, \pm 2, \pm 3, \pm 6 \\ 2: & \pm 1, \pm 2 \end{array}$$

Here is a list of all possible rational zeroes for $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 6}{\pm 1} = \pm 6$$

$$\frac{\pm 1}{\pm 2} = \pm \frac{1}{2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \pm \frac{3}{2} \qquad \frac{\pm 6}{\pm 2} = \pm 3$$

Notice that some of the numbers appear in both rows and so we can shorten the list by only writing them down once. Also, remember that we are looking for zeroes of $P(x)$ and so we can exclude any number in this list that isn't also in the original list we gave for $P(x)$. So, excluding previously checked numbers that were not zeros of $P(x)$ as well as those that aren't in the original list gives the following list of possible number that we'll need to check.

$$1,\ 3,\ \pm\frac{1}{2},\ \pm\frac{3}{2}$$

Again, we've already checked $x=-3$ and $x=-1$ and know that they aren't zeroes so there is no reason to recheck them. Let's again start with the integers and see what we get.

$$\begin{array}{r|rrrr} & 2 & -2 & 6 & -6 \\ \hline 1 & 2 & 0 & 6 & 0 \end{array} = P(1) = 0\ !\,!$$

So, $x=1$ is a zero of $Q(x)$ and we can now write $Q(x)$ as,

$$Q(x) = 2x^3 - 2x^2 + 6x - 6 = (x-1)(2x^2+6)$$

and as with the previous example we can solve the quadratic by other means.

$$\begin{aligned} 2x^2 + 6 &= 0 \\ x^2 &= -3 \\ x &= \pm\sqrt{3}\,i \end{aligned}$$

So, in this case we get a couple of complex zeroes. That can happen.

Here is a complete list of all the zeroes for $P(x)$ and note that they all have multiplicity of one.

$$x = -\frac{3}{2},\ \ x = 1,\ \ x = -\sqrt{3}\ i,\ \ x = \sqrt{3}\ i$$

So, as you can see this is a fairly lengthy process and we only did the work for two 4th degree polynomials. The larger the degree the longer and more complicated the process. With that being said, however, it is sometimes a process that we've got to go through to get zeroes of a polynomial.

Section 5-5 : Partial Fractions

This section doesn't really have a lot to do with the rest of this chapter, but since the subject needs to be covered and this was a fairly short chapter it seemed like as good a place as any to put it.

So, let's start with the following. Let's suppose that we want to add the following two rational expressions.

$$\begin{aligned}\frac{8}{x+1}-\frac{5}{x-4}&=\frac{8\left(x-4\right)}{\left(x+1\right)\left(x-4\right)}-\frac{5\left(x+1\right)}{\left(x+1\right)\left(x-4\right)}\\&=\frac{8x-32-\left(5x+5\right)}{\left(x+1\right)\left(x-4\right)}\\&=\frac{3x-37}{\left(x+1\right)\left(x-4\right)}\end{aligned}$$

What we want to do in this section is to start with rational expressions and ask what simpler rational expressions did we add and/or subtract to get the original expression. The process of doing this is called **partial fractions** and the result is often called the **partial fraction decomposition**.

The process can be a little long and on occasion messy, but it is actually fairly simple. We will start by trying to determine the partial fraction decomposition of,

$$\frac{P\left(x\right)}{Q\left(x\right)}$$

where both *P(x)* and *Q(x)* are polynomials and the degree of *P(x)* is smaller than the degree of *Q(x)*. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$\left(ax+b\right)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{\left(ax+b\right)^2}+\cdots+\dfrac{A_k}{\left(ax+b\right)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$\left(ax^2+bx+c\right)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\dfrac{A_kx+B_k}{\left(ax^2+bx+c\right)^k}$

Notice that the first and third cases are really special cases of the second and fourth cases respectively if we let $k=1$. Also, it will always be possible to factor any polynomial down into a product of linear factors ($ax+b$) and quadratic factors (ax^2+bx+c) some of which may be raised to a power.

There are several methods for determining the coefficients for each term and we will go over each of those as we work the examples. Speaking of which, let's get started on some examples.

Example 1 Determine the partial fraction decomposition of each of the following.

(a) $\dfrac{8x-42}{x^2+3x-18}$

(b) $\dfrac{9-9x}{2x^2+7x-4}$

(c) $\dfrac{4x^2}{(x-1)(x-2)^2}$

(d) $\dfrac{9x+25}{(x+3)^2}$

Solution

We'll go through the first one in great detail to show the complete partial fraction process and then we'll leave most of the explanation out of the remaining parts.

(a) $\dfrac{8x-42}{x^2+3x-18}$

The first thing to do is factor the denominator as much as we can.

$$\frac{8x-42}{x^2+3x-18}=\frac{8x-42}{(x+6)(x-3)}$$

So, by comparing to the table above it looks like the partial fraction decomposition must look like,

$$\frac{8x-42}{x^2+3x-18}=\frac{A}{x+6}+\frac{B}{x-3}$$

Note that we've got different coefficients for each term since there is no reason to think that they will be the same.

Now, we need to determine the values of *A* and *B*. The first step is to actually add the two terms back up. This is usually simpler than it might appear to be. Recall that we first need the least common denominator, but we've already got that from the original rational expression. In this case it is,

$$LCD=(x+6)(x-3)$$

Now, just look at each term and compare the denominator to the LCD. Multiply the numerator and denominator by whatever is missing then add. In this case this gives,

$$\frac{8x-42}{x^2+3x-18}=\frac{A(x-3)}{(x+6)(x-3)}+\frac{B(x+6)}{(x+6)(x-3)}=\frac{A(x-3)+B(x+6)}{(x+6)(x-3)}$$

We need values of *A* and *B* so that the numerator of the expression on the left is the same as the numerator of the term on the right. Or,

$$8x - 42 = A(x-3) + B(x+6)$$

This needs to be true regardless of the *x* that we plug into this equation. As noted above there are several ways to do this. One way will always work but can be messy and will often require knowledge that we don't have yet. The other way will not always work, but when it does it will greatly reduce the amount of work required.

In this set of examples, the second (and easier) method will always work so we'll be using that here. Here we are going to make use of the fact that this equation must be true regardless of the *x* that we plug in.

So, let's pick an *x*, plug it in and see what happens. For no apparent reason let's try plugging in $x = 3$. Doing this gives,

$$\begin{aligned} 8(3) - 42 &= A(3-3) + B(3+6) \\ -18 &= 9B \\ -2 &= B \end{aligned}$$

Can you see why we choose this number? By choosing $x = 3$ we got the term involving *A* to drop out and we were left with a simple equation that we can solve for *B*.

Now, we could also choose $x = -6$ for exactly the same reason. Here is what happens if we use this value of *x*.

$$\begin{aligned} 8(-6) - 42 &= A(-6-3) + B(-6+6) \\ -90 &= -9A \\ 10 &= A \end{aligned}$$

So, by correctly picking *x* we were able to quickly and easily get the values of *A* and *B*. So, all that we need to do at this point is plug them in to finish the problem. Here is the partial fraction decomposition for this part.

$$\frac{8x-42}{x^2+3x-18} = \frac{10}{x+6} + \frac{-2}{x-3} = \frac{10}{x+6} - \frac{2}{x-3}$$

Notice that we moved the minus sign on the second term down to make the addition a subtraction. We will always do that.

(b) $\dfrac{9-9x}{2x^2+7x-4}$

Okay, in this case we won't put quite as much detail into the problem. We'll first factor the denominator and then get the form of the partial fraction decomposition.

$$\frac{9-9x}{2x^2+7x-4} = \frac{9-9x}{(2x-1)(x+4)} = \frac{A}{2x-1} + \frac{B}{x+4}$$

In this case the LCD is $\left(2x-1\right)\left(x+4\right)$ and so adding the two terms back up give,

$$\frac{9-9x}{2x^2+7x-4}=\frac{A\left(x+4\right)+B\left(2x-1\right)}{\left(2x-1\right)\left(x+4\right)}$$

Next, we need to set the two numerators equal.

$$9-9x=A\left(x+4\right)+B\left(2x-1\right)$$

Now all that we need to do is correctly pick values of *x* that will make one of the terms zero and solve for the constants. Note that in this case we will need to make one of them a fraction. This is fairly common so don't get excited about it. Here is this work.

$$\begin{array}{llll} x=-4: & 45=-9B & \Rightarrow & B=-5 \\ x=\dfrac{1}{2}: & \dfrac{9}{2}=A\left(\dfrac{9}{2}\right) & \Rightarrow & A=1 \end{array}$$

The partial fraction decomposition for this expression is,

$$\frac{9-9x}{2x^2+7x-4}=\frac{1}{2x-1}-\frac{5}{x+4}$$

(c) $\dfrac{4x^2}{\left(x-1\right)\left(x-2\right)^2}$

In this case the denominator has already been factored for us. Notice as well that we've now got a linear factor to a power. So, recall from our table that this means we will get 2 terms in the partial fraction decomposition from this factor. Here is the form of the partial fraction decomposition for this expression.

$$\frac{4x^2}{\left(x-1\right)\left(x-2\right)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{\left(x-2\right)^2}$$

Now, remember that the LCD is just the denominator of the original expression so in this case we've got $\left(x-1\right)\left(x-2\right)^2$. Adding the three terms back up gives us,

$$\frac{4x^2}{\left(x-1\right)\left(x-2\right)^2}=\frac{A\left(x-2\right)^2+B\left(x-1\right)\left(x-2\right)+C\left(x-1\right)}{\left(x-1\right)\left(x-2\right)^2}$$

Remember that we just need to add in the factors that are missing to each term.

Now set the numerators equal.

$$4x^2=A\left(x-2\right)^2+B\left(x-1\right)\left(x-2\right)+C\left(x-1\right)$$

In this case we've got a slightly different situation from the previous two parts. Let's start by picking a couple of values of *x* and seeing what we get since there are two that should jump right out at us as being particularly useful.

$$x = 1: \qquad 4 = A\left(-1\right)^2 \qquad \Rightarrow \qquad A = 4$$
$$x = 2: \qquad 16 = C\left(1\right) \qquad \Rightarrow \qquad C = 16$$

So, we can get *A* and *C* in the same manner that we've been using to this point. However, there is no value of *x* that will allow us to eliminate the first and third term leaving only the middle term that we can use to solve for *B*. While this may appear to be a problem it actually isn't. At this point we know two of the three constants. So all we need to do is chose any other value of *x* that would be easy to work with ($x = 0$ seems particularly useful here), plug that in along with the values of *A* and *C* and we'll get a simple equation that we can solve for *B*.

Here is that work.

$$\begin{aligned} 4\left(0\right)^2 &= \left(4\right)\left(-2\right)^2 + B\left(-1\right)\left(-2\right) + 16\left(-1\right) \\ 0 &= 16 + 2B - 16 \\ 0 &= 2B \\ 0 &= B \end{aligned}$$

In this case we got $B = 0$ this will happen on occasion, but do not expect it to happen in all cases. Here is the partial fraction decomposition for this part.

$$\frac{4x^2}{\left(x-1\right)\left(x-2\right)^2} = \frac{4}{x-1} + \frac{16}{\left(x-2\right)^2}$$

(d) $\dfrac{9x+25}{\left(x+3\right)^2}$

Again, the denominator has already been factored for us. In this case the form of the partial fraction decomposition is,

$$\frac{9x+25}{\left(x+3\right)^2} = \frac{A}{x+3} + \frac{B}{\left(x+3\right)^2}$$

Adding the two terms together gives,

$$\frac{9x+25}{\left(x+3\right)^2} = \frac{A\left(x+3\right)+B}{\left(x+3\right)^2}$$

Notice that in this case the second term already had the LCD under it and so we didn't need to add anything in that time.

Setting the numerators equal gives,

$$9x + 25 = A\left(x+3\right) + B$$

Now, again, we can get *B* for free by picking $x=-3$.

$$9(-3)+25=A(-3+3)+B$$
$$-2=B$$

To find *A* we will do the same thing that we did in the previous part. We'll use $x=0$ and the fact that we know what *B* is.

$$\begin{aligned}25&=A(3)-2\\27&=3A\\9&=A\end{aligned}$$

In this case, notice that the constant in the numerator of the first isn't zero as it was in the previous part. Here is the partial fraction decomposition for this part.

$$\frac{9x+25}{(x+3)^2}=\frac{9}{x+3}-\frac{2}{(x+3)^2}$$

Now, we need to do a set of examples with quadratic factors. Note however, that this is where the work often gets fairly messy and in fact we haven't covered the material yet that will allow us to work many of these problems. We can work some simple examples however, so let's do that.

Example 2 Determine the partial fraction decomposition of each of the following.

(a) $\dfrac{8x^2-12}{x(x^2+2x-6)}$

(b) $\dfrac{3x^3+7x-4}{(x^2+2)^2}$

Solution

(a) $\dfrac{8x^2-12}{x(x^2+2x-6)}$

In this case the *x* that sits in the front is a linear term since we can write it as,

$$x=x+0$$

and so the form of the partial fraction decomposition is,

$$\frac{8x^2-12}{x(x^2+2x-6)}=\frac{A}{x}+\frac{Bx+C}{x^2+2x-6}$$

Now we'll use the fact that the LCD is $x(x^2+2x-6)$ and add the two terms together,

$$\frac{8x^2-12}{x(x^2+2x-6)}=\frac{A(x^2+2x-6)+x(Bx+C)}{x(x^2+2x-6)}$$

Next, set the numerators equal.

$$8x^2 - 12 = A\left(x^2 + 2x - 6\right) + x\left(Bx + C\right)$$

This is where the process changes from the previous set of examples. We could choose $x = 0$ to get the value of *A*, but that's the only constant that we could get using this method and so it just won't work all that well here.

What we need to do here is multiply the right side out and then collect all the like terms as follows,

$$8x^2 - 12 = Ax^2 + 2Ax - 6A + Bx^2 + Cx$$
$$8x^2 - 12 = \left(A + B\right)x^2 + \left(2A + C\right)x - 6A$$

Now, we need to choose *A*, *B*, and *C* so that these two are equal. That means that the coefficient of the x^2 term on the right side will have to be 8 since that is the coefficient of the x^2 term on the left side. Likewise, the coefficient of the *x* term on the right side must be zero since there isn't an *x* term on the left side. Finally the constant term on the right side must be -12 since that is the constant on the left side.

We generally call this **setting coefficients equal** and we'll write down the following equations.

$$\begin{aligned} A + B &= 8 \\ 2A + C &= 0 \\ -6A &= -12 \end{aligned}$$

Now, we haven't talked about how to solve systems of equations yet, but this is one that we can do without that knowledge. We can solve the third equation directly for *A* to get that $A = 2$. We can then plug this into the first two equations to get,

$$\begin{aligned} 2 + B &= 8 & \Rightarrow \quad B &= 6 \\ 2\left(2\right) + C &= 0 & \Rightarrow \quad C &= -4 \end{aligned}$$

So, the partial fraction decomposition for this expression is,

$$\frac{8x^2 - 12}{x\left(x^2 + 2x - 6\right)} = \frac{2}{x} + \frac{6x - 4}{x^2 + 2x - 6}$$

(b) $\dfrac{3x^3 + 7x - 4}{\left(x^2 + 2\right)^2}$

Here is the form of the partial fraction decomposition for this part.

$$\frac{3x^3 + 7x - 4}{\left(x^2 + 2\right)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{\left(x^2 + 2\right)^2}$$

Adding the two terms up gives,

$$\frac{3x^3+7x-4}{\left(x^2+2\right)^2}=\frac{\left(Ax+B\right)\left(x^2+2\right)+Cx+D}{\left(x^2+2\right)^2}$$

Now, set the numerators equal and we might as well go ahead and multiply the right side out and collect up like terms while we're at it.

$$3x^3+7x-4=\left(Ax+B\right)\left(x^2+2\right)+Cx+D$$
$$3x^3+7x-4=Ax^3+2Ax+Bx^2+2B+Cx+D$$
$$3x^3+7x-4=Ax^3+Bx^2+\left(2A+C\right)x+2B+D$$

Setting coefficients equal gives,

$$\begin{aligned} A&=3\\ B&=0\\ 2A+C&=7\\ 2B+D&=-4 \end{aligned}$$

In this case we got *A* and *B* for free and don't get excited about the fact that $B=0$. This is not a problem and in fact when this happens the remaining work is often a little easier. So, plugging the known values of *A* and *B* into the remaining two equations gives,

$$\begin{aligned} 2\left(3\right)+C&=7 &&\Rightarrow && C=1\\ 2\left(0\right)+D&=-4 &&\Rightarrow && D=-4 \end{aligned}$$

The partial fraction decomposition is then,

$$\frac{3x^3+7x-4}{\left(x^2+2\right)^2}=\frac{3x}{x^2+2}+\frac{x-4}{\left(x^2+2\right)^2}$$