

CONTINUITY

1. DEFINITION

A function $f(x)$ is said to be continuous at $x = a$; where $a \in \text{domain of } f(x)$, if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

i.e., LHL = RHL = value of a function at $x = a$

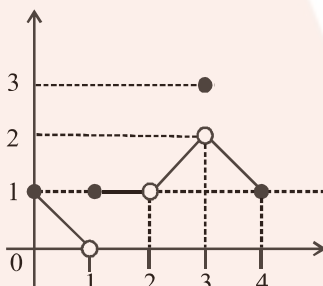
or $\lim_{x \rightarrow a} f(x) = f(a)$

1.1 Reasons of discontinuity

If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

There are following possibilities of discontinuity :

1. $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist but they are not equal.
2. $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exists and are equal but not equal to $f(a)$.
3. $f(a)$ is not defined.
4. At least one of the limits does not exist. Geometrically, the graph of the function will exhibit a break at the point of discontinuity.



The graph as shown is discontinuous at $x = 1, 2$ and 3 .

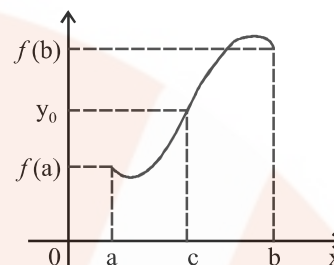
2. PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be continuous functions at $x = a$. Then,

1. $cf(x)$ is continuous at $x = a$, where c is any constant.
2. $f(x) \pm g(x)$ is continuous at $x = a$.
3. $f(x) \cdot g(x)$ is continuous at $x = a$.
4. $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.
5. If $f(x)$ is continuous on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of equation $f(x) = 0$ in the open interval (a, b) .

3. THE INTERMEDIATE VALUE THEOREM

Suppose $f(x)$ is continuous on an interval I , and a and b are any two points of I . Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.



The Function f , being continuous on (a, b) takes on every value between $f(a)$ and $f(b)$

Note..

That a function f which is continuous in $[a, b]$ possesses the following properties :

- (i) If $f(a)$ and $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b) .
- (ii) If K is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b) .

4. CONTINUITY IN AN INTERVAL

- (a) A function f is said to be continuous in (a, b) if f is continuous at each and every point $\in (a, b)$.
- (b) A function f is said to be continuous in a closed interval $[a, b]$ if:
 - (1) f is continuous in the open interval (a, b) and
 - (2) f is right continuous at 'a' i.e. $\lim_{x \rightarrow a^+} f(x) = f(a) = \text{a finite quantity}$.
 - (3) f is left continuous at 'b'; i.e. $\lim_{x \rightarrow b^-} f(x) = f(b) = \text{a finite quantity}$.

5. A LIST OF CONTINUOUS FUNCTIONS

Function $f(x)$	Interval in which $f(x)$ is continuous
1. constant c	$(-\infty, \infty)$
2. x^n , n is an integer ≥ 0	$(-\infty, \infty)$
3. x^{-n} , n is a positive integer	$(-\infty, \infty) - \{0\}$
4. $ x-a $	$(-\infty, \infty)$
5. $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$	$(-\infty, \infty)$
6. $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty) - \{x; q(x)=0\}$
7. $\sin x$	$(-\infty, \infty)$
8. $\cos x$	$(-\infty, \infty)$
9. $\tan x$	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10. $\cot x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
11. $\sec x$	$(-\infty, \infty) - \{(2n+1)\frac{\pi}{2} : n \in I\}$
12. $\operatorname{cosec} x$	$(-\infty, \infty) - \{n\pi : n \in I\}$
13. e^x	$(-\infty, \infty)$
14. $\log_e x$	$(0, \infty)$

6. TYPES OF DISCONTINUITIES

Type-1 : (Removable type of discontinuities)

In case, $\lim_{x \rightarrow c} f(x)$ exists but is not equal to $f(c)$ then the function is said to have a **removable discontinuity or discontinuity of the first kind**. In this case, we can redefine the function such that $\lim_{x \rightarrow c} f(x) = f(c)$ and make it continuous at $x = c$. Removable type of discontinuity can be further classified as :

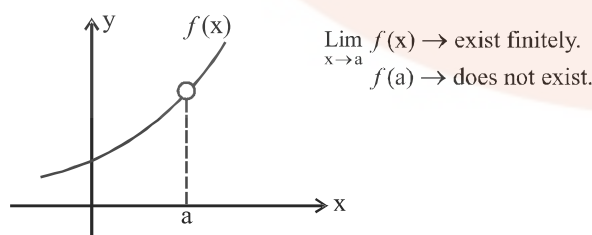
(a) Missing Point Discontinuity :

Where $\lim_{x \rightarrow a} f(x)$ exists finitely but $f(a)$ is not defined.

E.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity

at $x = 1$, and

$f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at $x = 0$.



missing point discontinuity at $x = a$

(b) Isolated Point Discontinuity :

Where $\lim_{x \rightarrow a} f(x)$ exists & $f(a)$ also exists but;

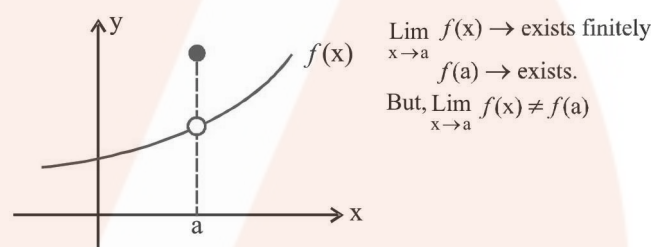
$\lim_{x \rightarrow a} f(x) \neq f(a)$.

E.g. $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$ and $f(4) = 9$ has an isolated point

discontinuity at $x = 4$.

Similarly $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$ has an isolated

point discontinuity at all $x \in I$.

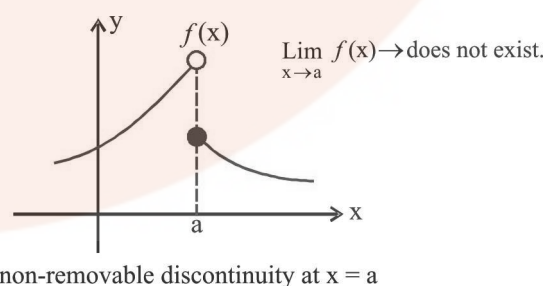


Isolated point discontinuity at $x = a$

Type-2 : (Non-Removable type of discontinuities)

In case, $\lim_{x \rightarrow a} f(x)$ does not exist, then it is not possible to

make the function continuous by redefining it. Such discontinuities are known as **non-removable discontinuity or discontinuity of the 2nd kind**. Non-removable type of discontinuity can be further classified as :



non-removable discontinuity at $x = a$

(a) Finite Discontinuity :

E.g., $f(x) = x - [x]$ at all integral x ; $f(x) = \tan^{-1} \frac{1}{x}$ at $x = 0$ and

$$f(x) = \frac{1}{1 + 2^x} \text{ at } x = 0 \text{ (note that } f(0^+) = 0 ; f(0^-) = 1)$$

(b) Infinite Discontinuity :

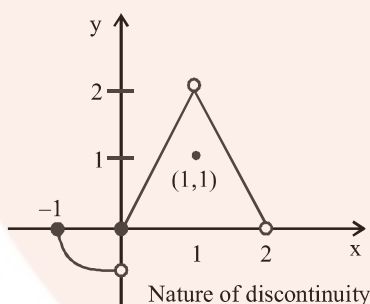
E.g., $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$

at $x = \frac{\pi}{2}$ and $f(x) = \frac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory Discontinuity :

E.g., $f(x) = \sin \frac{1}{x}$ at $x = 0$.

In all these cases the value of $f(a)$ of the function at $x = a$ (point of discontinuity) may or may not exist but $\lim_{x \rightarrow a}$ does not exist.



From the adjacent graph note that

- f is continuous at $x = -1$
- f has isolated discontinuity at $x = 1$
- f has missing point discontinuity at $x = 2$
- f has non-removable (finite type) discontinuity at the origin.

Note...

- (a) In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at $x = a$ and LHL at $x = a$ is called the **jump of discontinuity**. A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.
- (b) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.
- (c) If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$$f(x) = x \text{ and } g(x) = \begin{cases} \sin \frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- (d) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

- (e) Point functions are to be treated as discontinuous eg.
 $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at $x = 1$.
- (f) A continuous function whose domain is closed must have a range also in closed interval.
- (g) If f is continuous at $x = a$ and g is continuous at $x = f(a)$ then the composite $g[f(x)]$ is continuous at

$x = a$ E.g $f(x) = \frac{x \sin x}{x^2 + 2}$ and $g(x) = |x|$ are continuous at x

$= 0$, hence the composite $(g \circ f)(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ will also be continuous at $x = 0$.

DIFFERENTIABILITY

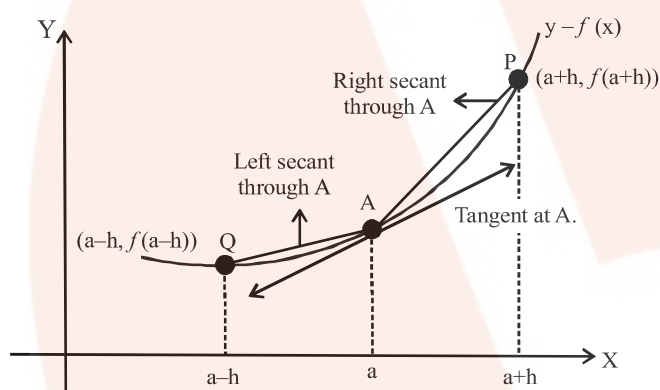
1. DEFINITION

Let $f(x)$ be a real valued function defined on an open interval (a, b) where $c \in (a, b)$. Then $f(x)$ is said to be differentiable or derivable at $x = c$,

iff, $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{(x - c)}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at $x = c$, and is denoted by

$$f'(c) \text{ or } \frac{d}{dx}(f(x))_{x=c}.$$



- Slope of Right hand secant = $\frac{f(a+h) - f(a)}{h}$ as $h \rightarrow 0$, $P \rightarrow A$ and secant (AP) \rightarrow tangent at A
- \Rightarrow Right hand derivative = $\lim_{h \rightarrow 0} \left(\frac{f(a+h) - f(a)}{h} \right)$
- = Slope of tangent at A (when approached from right) $f'(a^+)$.
- Slope of Left hand secant = $\frac{f(a-h) - f(a)}{-h}$ as $h \rightarrow 0$, $Q \rightarrow A$ and secant AQ \rightarrow tangent at A

$$\Rightarrow \text{Left hand derivative} = \lim_{h \rightarrow 0} \left(\frac{f(a-h) - f(a)}{-h} \right)$$

= Slope of tangent at A (when approached from left) $f'(a^-)$.

Thus, $f(x)$ is differentiable at $x = c$.

$$\Leftrightarrow \lim_{x \rightarrow c} \frac{f(x) - f(c)}{(x - c)} \text{ exists finitely}$$

$$\Leftrightarrow \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{(x - c)} = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{(x - c)}$$

$$\Leftrightarrow \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

$$\text{Hence, } \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{(x - c)} = \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} \text{ is}$$

called the **left hand derivative** of $f(x)$ at $x = c$ and is denoted by $f'(c^-)$ or $Lf'(c)$.

$$\text{While, } \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{(x - c)} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \text{ is}$$

called the **right hand derivative** of $f(x)$ at $x = c$ and is denoted by $f'(c^+)$ or $Rf'(c)$.

If $f'(c^-) \neq f'(c^+)$, we say that $f(x)$ is not differentiable at $x = c$.

2. DIFFERENTIABILITY IN A SET

- A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) , if it is differentiable at each point of (a, b) .
- A function $f(x)$ defined on closed interval $[a, b]$ is said to be differentiable or derivable. "If f is derivable in the open interval (a, b) and also the end points a and b , then f is said to be derivable in the closed interval $[a, b]$ ".

$$\text{i.e., } \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{(x - a)} \text{ and } \lim_{x \rightarrow b^-} \frac{f(x) - f(b)}{(x - b)}, \text{ both exist.}$$

A function f is said to be a differentiable function if it is differentiable at every point of its domain.



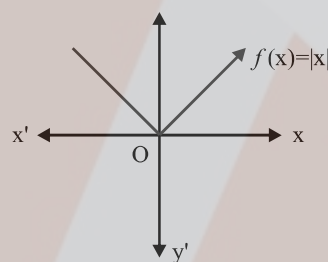
1. If $f(x)$ and $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$ will also be derivable at $x = a$ and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x = a$.
2. If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$. E.g. $f(x) = x$ and $g(x) = |x|$.
3. If $f(x)$ and $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$. E.g., $f(x) = |x|$ and $g(x) = |x|$.
4. If $f(x)$ and $g(x)$ both are not differentiable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function. E.g., $f(x) = |x|$ and $g(x) = -|x|$.
5. If $f(x)$ is derivable at $x = a$
 $\Rightarrow f'(x)$ is continuous at $x = a$.

$$\text{e.g. } f(x) = \begin{cases} 2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Converse : The converse of the above theorem is not necessarily true i.e., a function may be continuous at a point but may not be differentiable at that point.

E.g., The function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$, as shown in the figure.



The figure shows that sharp edge at $x = 0$ hence, function is not differentiable but continuous at $x = 0$.

3. RELATION B/W CONTINUITY & DIFFERENTIABILITY

In the previous section we have discussed that if a function is differentiable at a point, then it should be continuous at that point and a discontinuous function cannot be differentiable. This fact is proved in the following theorem.

Theorem : If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true,

or $f(x)$ is differentiable at $x = c$

$\Rightarrow f(x)$ is continuous at $x = c$.



(a) Let $f^{r+}(a) = p$ & $f^{r-}(a) = q$ where p & q are finite then

:

(i) $p = q \Rightarrow f$ is derivable at $x = a$

$\Rightarrow f$ is continuous at $x = a$.

(ii) $p \neq q \Rightarrow f$ is not derivable at $x = a$.

It is very important to note that f may be still continuous at $x = a$.

In short, for a function f :

Differentiable \Rightarrow Continuous;

Not Differentiable \nRightarrow Not Continuous

(i.e., function may be continuous)

But,

Not Continuous \Rightarrow Not Differentiable.

(b) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner at $x = a$.

Theorem 2 : Let f and g be real functions such that fg is defined if g is continuous at $x = a$ and f is continuous at $g(a)$, show that fg is continuous at $x = a$.

DIFFERENTIATION

1. DEFINITION

- (a) Let us consider a function $y=f(x)$ defined in a certain interval. It has a definite value for each value of the independent variable x in this interval.

Now, the ratio of the increment of the function to the increment in the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Now, as $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$ and $\frac{\Delta y}{\Delta x} \rightarrow$ finite quantity, then

derivative $f'(x)$ exists and is denoted by y' or $f'(x)$ or $\frac{dy}{dx}$

$$\text{Thus, } f'(x) = \lim_{x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(if it exists)

for the limit to exist,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h}$$

(Right Hand derivative)

(Left Hand derivative)

- (b) The derivative of a given function f at a point $x = a$ of its domain is defined as :

$$\text{Limit}_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}, \text{ provided the limit exists \& is}$$

denoted by $f'(a)$.

Note that alternatively, we can define

$$f'(a) = \text{Limit}_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

This method is called first principle of finding the derivative of $f(x)$.

2. DERIVATIVE OF STANDARD FUNCTION

$$(i) \quad \frac{d}{dx}(x^n) = n \cdot x^{n-1}; x \in \mathbb{R}, n \in \mathbb{R}, x > 0$$

$$(ii) \quad \frac{d}{dx}(e^x) = e^x$$

$$(iii) \quad \frac{d}{dx}(a^x) = a^x \cdot \ln a \quad (a > 0)$$

$$(iv) \quad \frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

$$(v) \quad \frac{d}{dx}(\log_a|x|) = \frac{1}{x} \log_a e$$

$$(vi) \quad \frac{d}{dx}(\sin x) = \cos x$$

$$(vii) \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$(viii) \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$(ix) \quad \frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$(x) \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

$$(xi) \quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$(xii) \quad \frac{d}{dx}(\text{constant}) = 0$$

$$(xiii) \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$(xiv) \quad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$(xv) \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$(xvi) \quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, \quad x \in \mathbb{R}$$

$$(xvii) \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

$$(xviii) \quad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

(xix) **Results :**

If the inverse functions f & g are defined by

$y = f(x)$ & $x = g(y)$. Then $g(f(x)) = x$.

$$\Rightarrow g'(f(x)) \cdot f'(x) = 1.$$

This result can also be written as, if $\frac{dy}{dx}$ exists & $\frac{dy}{dx} \neq 0$, then

$$\frac{dx}{dy} = 1 / \left(\frac{dy}{dx} \right) \text{ or } \frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \text{ or } \frac{dy}{dx} = 1 / \left(\frac{dx}{dy} \right) \left[\frac{dx}{dy} \neq 0 \right]$$

3. THEOREMS ON DERIVATIVES

If u and v are derivable functions of x , then,

$$(i) \quad \text{Term by term differentiation : } \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$(ii) \quad \text{Multiplication by a constant } \frac{d}{dx}(K u) = K \frac{du}{dx}, \text{ where } K \text{ is any constant}$$

$$(iii) \quad \text{“Product Rule” } \frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx} \text{ known as}$$

In general,

$$(a) \quad \text{If } u_1, u_2, u_3, u_4, \dots, u_n \text{ are the functions of } x, \text{ then}$$

$$\begin{aligned} & \frac{d}{dx}(u_1 \cdot u_2 \cdot u_3 \cdot u_4 \dots u_n) \\ &= \left(\frac{du_1}{dx} \right) (u_2 u_3 u_4 \dots u_n) + \left(\frac{du_2}{dx} \right) (u_1 u_3 u_4 \dots u_n) \end{aligned}$$

$$+ \left(\frac{du_3}{dx} \right) (u_1 u_2 u_4 \dots u_n) + \left(\frac{du_4}{dx} \right) (u_1 u_2 u_3 u_5 \dots u_n)$$

$$+ \dots + \left(\frac{du_n}{dx} \right) (u_1 u_2 u_3 \dots u_{n-1})$$

$$(iv) \quad \text{“Quotient Rule” } \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \left(\frac{du}{dx} \right) - u \left(\frac{dv}{dx} \right)}{v^2} \text{ where } v \neq 0$$

known as

$$(b) \quad \text{Chain Rule : If } y = f(u), u = g(w), w = h(x)$$

$$\text{then } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$$

$$\text{or } \frac{dy}{dx} = f'(u) \cdot g'(w) \cdot h'(x)$$



$$\text{In general if } y = f(u) \text{ then } \frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}.$$

4. METHODS OF DIFFERENTIATION

4.1 Derivative by using Trigonometrical Substitution

Using trigonometrical transformations before differentiation shorten the work considerably. Some important results are given below :

$$(i) \quad \sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$$

$$(ii) \quad \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

$$(iii) \quad \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

$$(iv) \quad \sin 3x = 3 \sin x - 4 \sin^3 x$$

$$(v) \quad \cos 3x = 4 \cos^3 x - 3 \cos x$$

$$(vi) \quad \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$(vii) \quad \tan \left(\frac{\pi}{4} + x \right) = \frac{1 + \tan x}{1 - \tan x}$$

DIFFERENTIATION

$$(viii) \tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$$

$$(ix) \sqrt{1 \pm \sin x} = \left| \cos \frac{x}{2} \pm \sin \frac{x}{2} \right|$$

$$(x) \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left(\frac{x \pm y}{1 \mp xy} \right)$$

$$(xi) \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \left\{ x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right\}$$

$$(xii) \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} \left\{ xy \mp \sqrt{1-x^2} \sqrt{1-y^2} \right\}$$

$$(xiii) \sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$$

$$(xiv) \sin^{-1} x = \operatorname{cosec}^{-1}(1/x); \cos^{-1} x = \sec^{-1}(1/x); \tan^{-1} x = \cot^{-1}(1/x)$$



Some standard substitutions :

Expressions Substitutions

$$\sqrt{a^2 - x^2} \quad x = a \sin \theta \text{ or } a \cos \theta$$

$$\sqrt{a^2 + x^2} \quad x = a \tan \theta \text{ or } a \cot \theta$$

$$\sqrt{x^2 - a^2} \quad x = a \sec \theta \text{ or } a \operatorname{cosec} \theta$$

$$\sqrt{\frac{a+x}{a-x}} \text{ or } \sqrt{\frac{a-x}{a+x}} \quad x = a \cos \theta \text{ or } a \cos 2\theta$$

$$\sqrt{(a-x)(x-b)} \text{ or } \quad x = a \cos^2 \theta + b \sin^2 \theta$$

$$\sqrt{\frac{a-x}{x-b}} \text{ or } \sqrt{\frac{x-a}{a-x}}$$

$$\sqrt{(x-a)(x-b)} \text{ or } \quad x = a \sec^2 \theta - b \tan^2 \theta$$

$$\sqrt{\frac{x-a}{x-b}} \text{ or } \sqrt{\frac{x-a}{x-b}}$$

$$\sqrt{2ax - x^2} \quad x = a(1 - \cos \theta)$$

4.2 Logarithmic Differentiation

To find the derivative of :

$$\text{If } y = \{f_1(x)\}^{f_2(x)} \text{ or } y = f_1(x) \cdot f_2(x) \cdot f_3(x) \dots$$

$$\text{or } y = \frac{f_1(x) \cdot f_2(x) \cdot f_3(x) \dots}{g_1(x) \cdot g_2(x) \cdot g_3(x) \dots}$$

then it is convenient to take the logarithm of the function first and then differentiate. This is called derivative of the logarithmic function.

Important Notes (Alternate methods)

$$1. \text{ If } y = \{f(x)\}^{g(x)} = e^{g(x) \ln f(x)} \text{ ((variable)}^{(\text{variable})}) \{ \because x = e^{\ln x} \}$$

$$\therefore \frac{dy}{dx} = e^{g(x) \ln f(x)} \cdot \left\{ g(x) \cdot \frac{d}{dx} \ln f(x) + \ln f(x) \cdot \frac{d}{dx} g(x) \right\}$$

$$= \{f(x)\}^{g(x)} \cdot \left\{ g(x) \cdot \frac{f'(x)}{f(x)} + \ln f(x) \cdot g'(x) \right\}$$

$$2. \text{ If } y = \{f(x)\}^{g(x)}$$

$$\therefore \frac{dy}{dx} = \text{Derivative of } y \text{ treating } f(x) \text{ as constant} + \text{Derivative of } y \text{ treating } g(x) \text{ as constant}$$

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \{f(x)\}^{g(x)-1} \cdot \frac{d}{dx} f(x)$$

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot g'(x) + g(x) \cdot \{f(x)\}^{g(x)-1} \cdot f'(x)$$

4.3 Implicit Differentiation : $\phi(x, y) = 0$

(i) In order to find dy/dx in the case of implicit function, we differentiate each term w.r.t. x , regarding y as a function of x & then collect terms in dy/dx together on one side to finally find dy/dx .

(ii) In answers of dy/dx in the case of implicit function, both x & y are present.

Alternate Method : If $f(x, y) = 0$

$$\text{then } \frac{dy}{dx} = - \frac{\left(\frac{\partial f}{\partial x} \right)}{\left(\frac{\partial f}{\partial y} \right)} = - \frac{\text{diff. of } f \text{ w.r.t. } x \text{ treating } y \text{ as constant}}{\text{diff. of } f \text{ w.r.t. } y \text{ treating } x \text{ as constant}}$$

4.4 Parametric Differentiation

If $y = f(t)$ & $x = g(t)$ where t is a Parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \dots(1)$$

Note...

$$\begin{aligned} 1. \quad \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ 2. \quad \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} \quad \left(\because \frac{dy}{dx} \text{ in terms of } t \right) \\ &= \frac{d}{dt} \left(\frac{f'(t)}{g'(t)} \right) \cdot \frac{1}{f'(t)} \quad \{ \text{From (1)} \} \\ &= \frac{f''(t)g'(t) - g''(t)f'(t)}{\{f'(t)\}^3} \end{aligned}$$

4.5 Derivative of a Function w.r.t. another Function

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

4.6 Derivative of Infinite Series

If taking out one or more than one terms from an infinite series, it remains unchanged. Such that

(A) If $y = \sqrt{f(x)} + \sqrt{f(x)} + \sqrt{f(x)} + \dots \infty$

then $y = \sqrt{f(x) + y} \Rightarrow (y^2 - y) = f(x)$

Differentiating both sides w.r.t. x , we get $(2y - 1) \frac{dy}{dx} = f'(x)$

(B) If $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{\dots \infty}}}$ then $y = \{f(x)\}^y \Rightarrow y = e^{y \ln f(x)}$

Differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{y \{f(x)\}^{y-1} \cdot f'(x)}{1 - \{f(x)\}^y \cdot \ln f(x)} = \frac{y^2 f'(x)}{f(x) \{1 - y \ln f(x)\}}$$

5. DERIVATIVE OF ORDER TWO & THREE

Let a function $y = f(x)$ be defined on an open interval (a, b) . It's derivative, if it exists on (a, b) , is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w.r.t. x . If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y'' .

Similarly, the 3rd order derivative of y w.r.t. x , if it exists, is

defined by $\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right)$ it is also denoted by $f'''(x)$ or y''' .

Some Standard Results :

(i) $\frac{d^m}{dx^m} (ax + b)^m = \frac{m!}{(m-n)!} \cdot a^n \cdot (ax + b)^{m-n}, m \geq n.$

(ii) $\frac{d^n}{dx^n} x^n = n!$

(iii) $\frac{d^n}{dx^n} (e^{mx}) = m^n \cdot e^{mx}, m \in \mathbb{R}$

(iv) $\frac{d^n}{dx^n} (\sin(ax + b)) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right), n \in \mathbb{N}$

(v) $\frac{d^n}{dx^n} (\cos(ax + b)) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right), n \in \mathbb{N}$

(vi) $\frac{d^n}{dx^n} \{e^{ax} \sin(bx + c)\} = r^n \cdot e^{ax} \cdot \sin(bx + c + n\phi), n \in \mathbb{N}$

where $r = \sqrt{a^2 + b^2}, \phi = \tan^{-1}(b/a).$

(vii) $\frac{d^n}{dx^n} \{e^{ax} \cdot \cos(bx + c)\} = r^n \cdot e^{ax} \cdot \cos(bx + c + n\phi), n \in \mathbb{N}$

where $r = \sqrt{a^2 + b^2}, \phi = \tan^{-1}(b/a).$

DIFFERENTIATION

6. DIFFERENTIATION OF DETERMINANTS

$$\text{If } F(X) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix},$$

where $f, g, h, \ell, m, n, u, v, w$ are differentiable function of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

7. L' HOSPITAL'S RULE

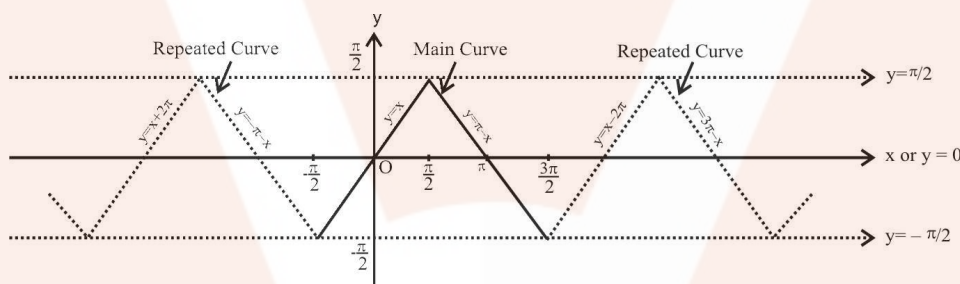
If $f(x)$ & $g(x)$ are functions of x such that :

- $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$ or $\lim_{x \rightarrow a} f(x) = \infty = \lim_{x \rightarrow a} g(x)$ and
- Both $f(x)$ & $g(x)$ are continuous at $x = a$ and
- Both $f(x)$ & $g(x)$ are differentiable at $x = a$ and
- Both $f'(x)$ & $g'(x)$ are continuous at $x = a$, Then

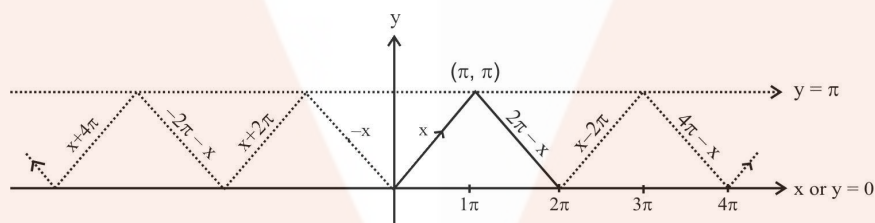
Limit $\frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$ & so on till indeterminate form vanishes..

8. ANALYSIS & GRAPHS OF SOME USEFUL FUNCTION

$$(i) \quad y = \sin^{-1}(\sin x) \quad x \in \mathbb{R}; y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$(ii) \quad y = \cos^{-1}(\cos x) \quad x \in \mathbb{R}; y \in [0, \pi]$$



$$(iii) \quad y = \tan^{-1}(\tan x) \quad x \in \mathbb{R} - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}; y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

