A-Level Further Maths: Practice Paper 5

Focus: Multiple Choice

Difficulty: Hard

Marks:

Section A (fast): 40 questions (40 marks)

Section B (standard): 40 questions (80 marks)

Section C (challenging): 16 questions (64 marks)

The questions are in order of increasing difficulty. There is a mixture of Pure, Mechanics and Statistics questions. Each question has **one** correct answer, marked with a circle. Don't spend too long on one question in sections A and B. Read the questions carefully - look out for tricks. Section C is harder than the A-level standard. Take a break in between each section. Check the fully worked solutions for any questions you missed.

Section A: Fast. Aim to answer 1 question per minute in this section.

1. The sum of all positive integers *x* such that

$$\left(\frac{1}{2}\right)^{|x-5|} \geq \log_3 81$$

is

O 1
O 3
O 6
O undefined; there are no values of *x* [1 mark]

2. Four matrices **A**, **B**, **C** and **D** are defined as

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 0 & 7 \\ 4 & 4 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ -9 & 8 \\ 1 & 8 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 5 \\ 2 \\ -10 \end{bmatrix}$$

Which of the following products is **not** defined?

O **ABC**
O **CB**
O **CD**
O **BD** [1 mark]

3. The vectors **a** and **b** are such that |**a**| = $\sqrt{10}$, |**b**| = 10 and **a** • **b** = 30.

The value of |**a** × **b**| is

O 10
O $\sqrt{10}$
O 30
O $\sqrt{30}$ [1 mark]

4. The implicit Cartesian form of the polar equation $r^2 = \cot\theta$ for $0 < \theta < \pi$ is

O $x^2y + y^3 - x = 0$, for $x > 0$
O $x^2y - y^3 + x = 0$, for $x < 0$
O $x^2y + y^3 - x = 0$, for all real x
O $x^2y - y^3 + x = 0$, for $|x| < 1$ [1 mark]

5. Consider the differential equation, where $y(x)$ is defined for all real $x > 0$:

$$x^2y' + (1-x)^2y = 1 + x^2$$

If $A(x)$ is a correct integrating factor to solve this differential equation, then

O $\ln A = \dfrac{x^3}{3} + x^2 - x$
O $\ln A = \dfrac{x^3}{3} - x^2 + x$
O $\ln A = x + \dfrac{1}{x} - 2\ln|x|$
O $\ln A = x - \dfrac{1}{x} - 2\ln|x|$ [1 mark]

6. **M** is the matrix transformation in 3D space representing a clockwise rotation by 90° about the z-axis. Which of the following is true?

1 **M** has no real eigenvectors
2 The z-axis is the only invariant line under **M**
3 $\mathbf{M}^2 = \mathbf{I}$

O **1** and **2** only
O **2** only
O **1** and **3** only
O **2** and **3** only [1 mark]

7. The polynomial $P(x)$ is defined as $P(x) = x^3 - 8x^2 + 7x - 1$. The three distinct roots of $P(x)$ are α, β and γ.

Which of these functions has roots $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$?

- O $P(7/x)$
- O $P(-7/x)$
- O $P(1/x)$
- O $P(-1/x)$ [1 mark]

8. Part of the Maclaurin series expansion for a function $f(x)$ is shown below.

$$f(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + ... \quad \text{for all } x \in \mathbb{R}$$

Which of these could be $f(x)$?

- O $f(x) = \sinh x$
- O $f(x) = e^x \sin x$
- O $f(x) = \cos 2x$
- O $f(x) = \cos^{-1} x$ [1 mark]

9. The completed-square form of the function $f(z) = z^2 - (4 - 2i)z + (3 - 4i)$, where z is complex, is

- O $f(z) = (z + 2 + i)^2$
- O $f(z) = (z + 2 - i)^2$
- O $f(z) = (z - 2 + i)^2$
- O $f(z) = (z - 2 - i)^2$ [1 mark]

10. Consider the function $f(x) = 2 \sinh x$ defined for all real x.
If $f(x) \geq ax$ for all $x \geq 0$, then the possible values of a are

O $a \geq 2$
O $a \leq e$
O $a \leq 2$
O $|a| \leq 2$ [1 mark]

11. The correct identity for $\cosh(x + y)$, for all real x and y, is

O $\cosh(x + y) \equiv \sinh x \cosh x + \sinh y \cosh y$
O $\cosh(x + y) \equiv \sinh x \sinh y + \cosh x \cosh y$
O $\cosh(x + y) \equiv \sinh x \cosh x - \sinh y \cosh y$
O $\cosh(x + y) \equiv \sinh x \sinh y - \cosh x \cosh y$ [1 mark]

12. The plane Π is such that any point on Π is equidistant from the points (2, -1, 3) and (8, 11, -5). The equation of Π in its simplest Cartesian form is

O $3x + 6y - 4z = 49$
O $3x + 6y - 4z = 98$
O $x + 2y - z = 0$
O $x + 2y - z = 98$ [1 mark]

13. Matrix **M** is a square, nonsingular 3 x 3 matrix representing a sequence of two transformations in 3D space.

Given that det **M** = -1, and the second transformation is a reflection in the plane $y = 0$, which of these could describe the first transformation?

O Enlargement with respect to origin, scale factor -1
O Rotation about the y-axis by 180°
O Reflection in the plane $x = 0$
O Projection onto the plane $x = y$ by the mapping $(x, y, z) \Rightarrow (x + y, x + y, z)$

[1 mark]

14. A student attempts to use proof by induction to show that $n^2 - n$ is odd for all $n \in \mathbb{R}$. They argue as follows:

> Assume true for $n = k$, where k is a positive integer.
>
> For $k + 1$, we have $(k + 1)^2 - (k + 1) = k^2 + 2k + 1 - k - 1$
> $= k^2 + k$
> $= k^2 - k + 2k$
> which must be odd, since $k^2 - k$ is assumed to be odd and $2k$ is even.
>
> Therefore, true for $n = k \Rightarrow$ true for $n = k + 1$.
> Hence by induction, $n^2 - n$ is odd for all positive integers n.

What mistake has the student made in this argument?

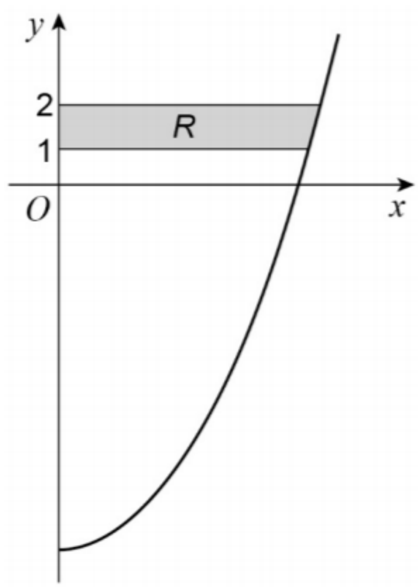
O It was assumed that $k^2 - k$ is odd before it was proven
O The general form for an odd integer, i.e. $n = 2k + 1$, was not used
O The argument is incomplete since there is no established base case
O There is an algebraic error in the inductive step

[1 mark]

15. Which of these is **not** an asymptote to the curve $y = \dfrac{1}{x^2 - 4}$?

O $x = 0$
O $y = 0$
O $x = 2$
O $x = -2$ [1 mark]

16. The diagram shows the curve $y = 18x^2 - 9$ for $x \geq 0$.



A solid is formed when the region R is rotated through 360° about the y-axis.

The volume of this solid is $k\pi$, where the value of k is

O $\dfrac{7}{6}$

O $\dfrac{7}{12}$

O 33

O $\dfrac{33}{4}$ [1 mark]

17. If $z = \cos\theta + i\sin\theta$, then which of the following is true for all valid θ ?

O $z^2 = 2z$

O $z + \dfrac{1}{z} = 2i\cos\theta$

O $z - \dfrac{1}{z} = 2i\sin\theta$

O $\dfrac{z^2-1}{z^2+1} = -i\tan\theta$ [1 mark]

18. A student is trying to prove the following result for all $\{\theta \in \mathbb{R} : \theta \neq 0\}$:

$$\sum_{r=1}^{n} \cos(r\theta) = \frac{\cos(\frac{1}{2}(n+1)\theta)\sin(\frac{1}{2}n\theta)}{\sin(\frac{1}{2}\theta)}$$

What technique should the student use to start their proof?

(Assume the student is starting from the left-hand side.)

O Use the method of differences

O Use a Maclaurin series

O Use De Moivre's theorem

O Use a trigonometric identity [1 mark]

19. An eagle has caught a salmon of mass *m* kg to take to its nest. When the eagle is flying with speed *v* ms^{-1}, it drops the salmon. The salmon falls a vertical distance of *h* metres back into the sea. The salmon's weight *mg* is the only force that acts on it as it falls.

The potential energy lost by the salmon as it falls into the sea is given by

O mgh

O $\frac{1}{2}mv^2$

O $-mgh$

O $-\frac{1}{2}mv^2$ [1 mark]

20. The earth takes approximately 365 days to orbit the sun.

What is the angular velocity of the earth in radians per second?

O 365

O 2×10^{-7}

O 2π

O 1×10^{-5} [1 mark]

21. A train which is moving at a constant speed of 15 ms^{-1} on a horizontal track. A passenger has placed their mobile phone on a rough horizontal table fixed to the floor in the train. The coefficient of friction between the phone and the table is 0.2. The train moves round a bend of constant radius of curvature *r* m.

The mobile phone does not slide on the table as the train moves around the bend. The smallest possible value of *r* to a suitable degree of accuracy is

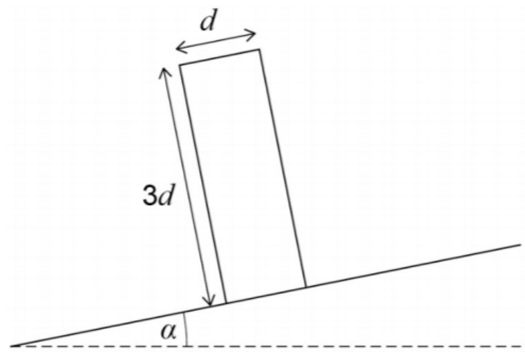
O 100

O 110

O 120

O 130 [1 mark]

22. A solid uniform cylinder of height $3d$ and diameter d is placed on a rough plane (coefficient of friction μ) inclined at angle α to the horizontal as shown.



Which of these is true?

- O If $\mu > \tan\alpha$ then the cylinder will slide as soon as it is released
- O If $\mu < \tan\alpha$ then the cylinder will topple as soon as it is released
- O Both of the above
- O None of the above [1 mark]

23. The continuous random variable T is modelled by an exponential distribution with parameter λ. The probability density function of T is $f(t)$ and the cumulative distribution function is $F(t)$.

Which of these correctly defines $f(t)$ or $F(t)$ for all $t \geq 0$?

- O $f(t) = \lambda e^{\lambda t}$
- O $f(t) = e^{-\lambda t}$
- O $F(t) = 1 - \lambda e^{-\lambda t}$
- O $F(t) = 1 - e^{-\lambda t}$ [1 mark]

24. Which of these statistical distributions is symmetric about its mean?

1 Poisson distribution
2 Rectangular distribution
3 Chi-squared distribution
4 *t*-distribution

O **1** and **3** only
O **2** and **4** only
O **1** and **2** only
O **3** and **4** only [1 mark]

25. The number of buses arriving at a particular bus stop is modelled by a Poisson distribution with a mean of 8 buses per hour.

The time interval **in minutes** between any two consecutive busses arriving is modelled by an exponential distribution with parameter λ, where the value of λ is

O $\frac{15}{2}$
O 8
O $\frac{2}{15}$
O $\frac{1}{8}$ [1 mark]

26. A line of invariant points of the transformation represented by the matrix

$$\begin{bmatrix} -3 & 2 \\ -10 & 6 \end{bmatrix}$$

is

O $y = \frac{1}{2}x$

O $y = \frac{2}{5}x$

O $y = 2x$

O $y = \frac{5}{2}x$ [1 mark]

27. The smallest value of *N* for which $3^n \leq n!$ for all $n \in \mathbb{Z}, n \geq N$ is

O 5

O 6

O 7

O 8 [1 mark]

28. The differential equation

$$4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}y}{\mathrm{d}x} + 8y = \tan x$$

can be classified as

1 first-order

2 second-order

3 linear

4 homogeneous

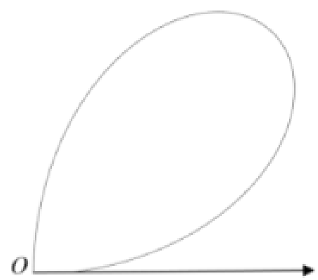
O **1** only

O **1**, **3** and **4** only

O **2** only

O **2** and **3** only [1 mark]

29. Which of these gives the area of the region bounded by the curve with polar equation $r = \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$?



O $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 2\theta \; \mathrm{d}\theta$

O $\displaystyle\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta \; \mathrm{d}\theta$

O $\displaystyle\frac{1}{4}\int_0^{\frac{\pi}{2}} 1 - \cos 4\theta \; \mathrm{d}\theta$

O $\displaystyle\int_0^{\frac{\pi}{2}} \cos 2\theta \; \mathrm{d}\theta$ [1 mark]

30. The sum of the two distinct eigenvalues of a 2 × 2 matrix **M** equals

O the sum of the top-left and bottom-right entries of **M**
O the sum of the top-right and bottom-left entries of **M**
O the sum of the four entries of **M**
O the product of the four entries of **M** [1 mark]

31. **M** is a square $n \times n$ diagonalisable matrix with n distinct eigenvalues.

By considering the determinant of **M** in its diagonalised form, it follows that

O |**M**| must be non-zero
O |**M**| equals the product of the n eigenvalues
O |**M**| equals the product of the magnitudes of the n eigenvectors
O |**M**| equals the coefficient of λ^n in the characteristic equation of **M**

[1 mark]

32. Find the value of

$$\lim_{y \to 0} \left(\frac{1}{y^4} \int_0^y \sin^3 x \, \mathrm{d}x \right).$$

(You may assume the limit exists.)

O 0

O $\frac{1}{24}$

O $\frac{1}{4}$

O $\frac{1}{2}$

[1 mark]

33. Consider a function *f*(*x*) whose first and second derivatives are both positive and continuous over the interval [0, 1].

When the trapezium rule and Simpson's rule are used to approximate the value of $\int_0^1 f(x)\, dx$ using a single interval, how do the results compare to the true value?

	Trapezium rule	Simpson's rule
O	underestimate	underestimate
O	overestimate	overestimate
O	overestimate	underestimate
O	overestimate	not enough information

[1 mark]

34. Which of the following is **not** a necessary condition for a discrete random variable to be modelled by a Poisson distribution?

O Outcomes of the event occur at a constant average rate

O Outcomes of the event occur independently and at random

O Once an event has occurred, it is possible for the next event to happen at any time from that instant

O The number of events occuring in a given interval must be small

35. For which of these is the Poisson distribution likely to be a valid model?

1 The number of cars passing a fixed point on a free flowing motorway during a given 5-minute interval.

2 The number of telephone calls arriving at the switchboard of a football club during the first hour of telephone sales of tickets ‘going live’ for the next round of the cup competition they are in, after they have just won the previous match.

3 The number of cars passing a point on a single carriage road, where there are road works with traffic light control, on the way in to a town between 8:00 am and 8:30 am on a Tuesday in April.

O **1** only
O **1** and **2** only
O **2** only
O **2** and **3** only [1 mark]

36. The power of a statistical hypothesis test is

O P(Type I error)
O 1 – P(Type I error)
O P(Type II error)
O 1 – P(Type II error) [1 mark]

37. A continuous random variable *X* has the following probability density function:

$$f(x)=\begin{cases}\frac{4}{5}x^2 & 0\le x\le 1\\ \frac{1}{20}(x-3)(3x-11) & 1\le x\le a\\ 0 & \text{otherwise}\end{cases}$$

In the interval $1 \le x \le a$, the cumulative distribution function of *X* is

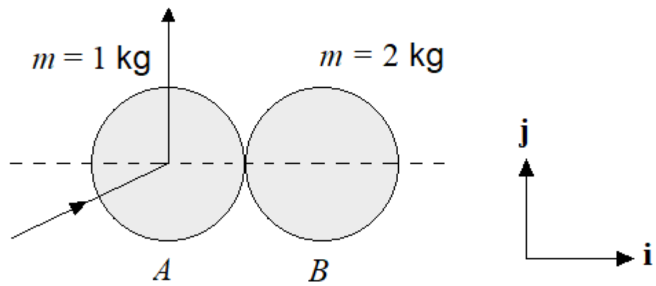
O $F(x) = \frac{1}{20}(x^3 - 10x^2 + 33x - 16)$

O $F(x) = \frac{1}{60}(3x^3 - 30x^2 + 99x - 41)$

O $F(x) = \frac{1}{20}(x^3 - 10x^2 + 33x - 14)$

O $F(x) = \frac{1}{60}(3x^3 - 30x^2 + 99x - 56)$ [1 mark]

38. Two smooth spheres, *A* and *B*, of equal radius and masses 1 kg and 2 kg respectively are placed on a smooth horizontal table. Sphere *A* is set in motion towards sphere *B* and the spheres collide.



Just before the collision, the velocity of sphere *A* is 4**i** + 3**j** and just after, it is moving in the direction perpendicular to the line through the spheres' centres.

What is the coefficient of restitution between the spheres?

O 0.30
O 0.50
O 0.70
O 0.90 [1 mark]

39. A book of mass 1 kg is placed on a rough plane inclined at 30° to the horizontal. The book slides down with an acceleration of 1 ms^{-2}.

The angle of friction for this book is (Take g = 10 ms^{-2}; round to nearest degree)

O 20°
O 25°
O 30°
O 45° [1 mark]

40. An elastic cord has modulus of elasticity λ. The dimensions of λ are

O $\mathbf{MT^{-2}}$
O $\mathbf{MLT^{-1}}$
O $\mathbf{MLT^{-2}}$
O $\mathbf{ML^{-1}T^{-2}}$ [1 mark]

Section B: Standard. Aim to answer 1 question per 2 minutes in this section.

41. The domain of $f(x)$ is all real numbers and satisfies $f(x+1) = 2f(x)$ and also,

$$f(x) = x(x-1) \quad \text{for } x \in (0, 1]$$

The greatest value of m such that $f(x) \geq \frac{8}{9}$ for all $x \leq m$ is

O 9/4
O 8/3
O 7/3
O 5/2 [2 marks]

42. A random variable X is normally distributed and is related to the standard normal variable Z by $X = a + 2Z$ for some constant a.

Given further that $\mathrm{P}(X \leq 1) = \mathrm{P}(X \geq 5)$, find the value of $\mathrm{P}(a - 2 \leq X \leq 2a + 1)$.

O 0.5328
O 0.6687
O 0.7745
O 0.8185 [2 marks]

43. The function $f(x)$ satisfies

$$f(x) = \int_a^x \frac{1}{[f(t)]^2} \, \mathrm{d}t$$

for some real constant a and defined for all $x > a$. If $f(1) = 0$, then

O $f(x) = (3x - 3)^{1/3}$
O $f(x) = (3x + 3)^{1/3}$
O $f(x) = (3x - 3)^{-2/3}$
O $f(x) = (3x + 3)^{-2/3}$ [2 marks]

44. The four roots of a quartic polynomial $P(x)$ form a square when plotted in an Argand diagram. Which of these could **not** be $P(x)$?

O $P(x) = x^4 - 1$
O $P(x) = x^4 - 2x^2 + 2$
O $P(x) = x^4 + 4x^3 + 6x^2 + 4x + 2$
O $P(x) = x^4 - 4\mathrm{i}x^3 - 6x^2 + 4\mathrm{i}x + 2$ [2 marks]

45. The circle with centre (0, 5) and radius 1 unit is revolved by 2π radians about the x-axis. The area, S, of the resulting closed surface is given by

O $$S = 2\pi \int_{-1}^{1} \left(5 + \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$
O $$S = 4\pi \int_{0}^{1} \left(5 + \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$
O $$S = 20\pi \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$
O $$S = 40\pi \int_{0}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$
[2 marks]

46. Consider the function $f(x)$, defined as

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\ \mathrm{d}t$$

for all $x > 0$. Which of these functions has a discontinuity in the interval $(0, 2\pi]$?

O $f^{-1}(x)$
O $f'(x)$
O $f''(x)$
O $\ln f(x+1)$ [2 marks]

47. The circle C is defined as the plot of all z satisfying $\left|z - \sqrt{3} - i\right| = 1$ on an Argand diagram.

Which of the following is true?

O $|z|$ takes its minimum value on C when $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

O arg z takes its maximum value on C when $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$.

O For all z on C, $\frac{3}{2} \leq |z| \leq \frac{7}{2}$.

O For all z on C, $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$. [2 marks]

48. The roots of the equation $x^4 - px^3 + qx^2 - pqx + 1 = 0$ are α, β, γ and δ.

What is the value of $(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta)$?

O -1

O 1

O pq

O p^2q^2 [2 marks]

49. A rod, of length x m and moment of inertia I kg m^2, is free to rotate in a vertical plane about a fixed smooth horizontal axis through one end. When the rod is hanging at rest, its lower end receives an impulse of magnitude J Ns, which is just sufficient for the rod to complete full revolutions.

It is thought that there is a relationship between J, x, I, the acceleration due to gravity, g ms^{-2} such that $J \propto x^{\alpha} I^{\beta} g^{\gamma}$, for some α, β and γ and the proportionality constant is dimensionless.

When α, β and γ are fixed such that this equation is dimensionally consistent,

O $\alpha = 1$
O $\beta = -1/2$
O $\gamma = 1/2$
O None of the above [2 marks]

50. The force acting on a basketball when it collides directly with the backboard is given by the equation $4 \times 10^4\, t^2(1 - 2t)$ where t is the time in seconds since the basketball first made contact with the backboard.

Assuming the equation models the situation accurately, which of these is true?

O The time in contact with the backboard is 0.5 seconds
O The maximum magnitude of the force exerted is 40 kN
O The total impulse exerted on the basketball is about 40 kNs
O The total work done on the ball is about 40 J [2 marks]

51. A bungee jumper, of mass 80 kg, is attached to one end of a light elastic cord, of natural length 16 metres and modulus of elasticity 784 N. The other end of the cord is attached to a horizontal platform, which is at a height of 65 metres above the ground.

The bungee jumper steps off the platform at the point where the cord is attached, falling vertically. The maximum extension of the bungee cord during the fall is x.

Taking $g = 9.8$ N kg^{-1}, which equation is satisfied by x?

O $x^2 - 16x - 128 = 0$
O $x^2 - 32x - 256 = 0$
O $x^2 - 32x - 512 = 0$
O $x^2 - 64x - 256 = 0$ [2 marks]

52. Matrix $\mathbf{M}$ is invertible and satisfies $\mathbf{M} = \mathbf{UDU}^{-1}$ for some diagonal matrix $\mathbf{D}$.

Which of these must be true?

O $|\mathbf{M}| = |\mathbf{U}|$
O $|\mathbf{M}| = |\mathbf{D}|$
O $\mathbf{M}^{-1} = \mathbf{M}$
O $\mathbf{M}^{-1} = \mathbf{U}^{-1}\mathbf{D}^{-1}\mathbf{U}$ [2 marks]

53. One of the defining properties of the exponential distribution is that it is memoryless.

If $T \sim \text{Exp}(\lambda)$ then the memorylessness of T is most explicitly demonstrated by

O $P(T > s + t) \equiv P(T > t)$ for all $s > 0$ and $t > 0$
O $P(T > s + t \mid T > s) \equiv P(T > t)$ for all $s > 0$ and $t > 0$
O $P(T < t - s) \equiv P(T > t + s)$ for all $0 < s < t$
O $P(T < t \mid T > s) \equiv P(T < t - s)$ for all $0 < s < t$ [2 marks]

54. The correct way to start a method to find $\sum_{r=1}^{n} \frac{r}{(r+1)!}$ is

- O Substitute the Maclaurin series for $(1 + x)e^x$
- O Replace $\frac{r}{(r+1)!} \equiv \frac{1}{r!} - \frac{1}{(r+1)!}$ and substitute the Maclaurin series for e^x
- O Replace $\frac{r}{(r+1)!} \equiv \frac{r!}{(r-1)!(r+1)!}$ and rewrite as a binomial series
- O Replace $\frac{r}{(r+1)!} \equiv \frac{1}{r!} - \frac{1}{(r+1)!}$ and use the method of differences

[2 marks]

55. A student is trying to find the value of $\int_0^1 x^2\,dx$ from 'first principles', by considering the integral as the sum of n thin vertical rectangles representing the area under the curve $y = x^2$ between $x = 0$ and $x = 1$, then taking the limit as $n \to \infty$.

Their argument is outlined below, with some algebraic manipulation steps omitted for clarity.

$$\int_0^1 x^2\,\mathrm{d}x$$

Step **1**	$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} \right]$
Step **2**	$= \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{k=1}^{n} k^2 \right)$
Step **3**	$= \lim_{n\to\infty} \left(\frac{n(n+1)(2n+1)}{6n^3} \right)$
Step **4**	$= \lim_{n\to\infty} \left(\frac{2n^3+3n^2+n}{6n^3} \right)$
Step **5**	$= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)$
Step **6**	$= \frac{1}{3}.$

Which of these is true?

O In Step **1**, the kth rectangle from the left has height $\frac{k}{n}$ and width $\Delta x = \frac{1}{n}$.

O In Step **2**, factoring out $\frac{1}{n^3}$ from the summation requires a more rigorous justification because the limit of the summation depends on n.

O Both of the above

O None of the above [2 marks]

56. Given that the general solution to the non-linear differential equation

$$2y\frac{d^2y}{dx^2} - 8y\frac{dy}{dx} + 16y^2 = \left(\frac{dy}{dx}\right)^2, \; y \neq 0,$$

is

$$y = (Ae^{2x} + Bxe^{2x})^2,$$

which of the following substitutions would transform the above into a linear homogeneous second-order differential equation with constant coefficients?

O $z = \sqrt{y}$

O $z = y^2$

O $z = \frac{1}{y}$

O $z = y \cdot \frac{dy}{dx}$ [2 marks]

57. The roots of the polynomial $x^2 + px + q$ are α and β.

A quadratic with roots $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ in terms of *p* and *q* is

O $x^2 - p(1 + q)x + p^2 + q^2 - 2q + 1$

O $x^2 + p(1 + q)x + p^2 + q^2 - 2q + 1$

O $qx^2 + p(1 + q)x + p^2 + q^2 - 2q + 1$

O $qx^2 + p(1 + q)x + p^2 + q^2 - 2q + 1$ [2 marks]

58. T is a continuous random variable with an exponential distribution.

If the mean of T is μ, then

O the variance of T is $\frac{1}{\mu^2}$

O the median of T is $\mu \ln 2$

O $\mathrm{E}[T^n] = \frac{n!}{\mu^n}$ for all natural n

O $\mathrm{P}(T > a) = \mathrm{P}(T < b)$ has no solutions if $a \neq b$ [2 marks]

59. Let

$$I_n = \int \frac{x^n}{\sqrt{ax+b}} \, \mathrm{d}x \ , \quad n \in \mathbb{N} .$$

Using the reduction formula

$$I_n = \frac{2x^n\sqrt{ax+b}}{a(2n+1)} - \frac{2bn}{a(2n+1)} \times I_{n-1} ,$$

(valid for all integer $n \geq 0$, all real a, b, x : $ax + b \geq 0$) for any suitable a and b, or otherwise, find the value of

$$\int_0^1 \frac{x^2}{\sqrt{1-x}} \, \mathrm{d}x .$$

(You may assume the integral converges. Evaluating this integral on a calculator will **not** give a valid answer.)

O $\frac{2}{15}(7\sqrt{2} - 8)$

O $\frac{2}{3}(2 - \sqrt{2})$

O $\frac{16}{15}$

O 2 [2 marks]

60. The cubic equation $2x^3 + px^2 + qx - 2 = 0$ has roots $\alpha = 1 + i$, β and γ, where *p* and *q* are real coefficients.

Which of these is an expression for $\alpha^n + \beta^n + \gamma^n$?

O $2^{\frac{n}{2}} \cos \frac{n\pi}{4} + 2^{-n}$

O $2^{\frac{n}{2}+1} \cos \frac{n\pi}{4} + 2^{-n}$

O $2^{\frac{n}{2}} i \sin \frac{n\pi}{4} + 2^{-n}$

O $2^{\frac{n}{2}+1} \sin \frac{n\pi}{4} + 2^{-n}$ [2 marks]

61. A coin is tossed 18 times and the outcome is heads 11 times. The coin is suspected to be biased towards showing heads and a one-tail hypothesis test is conducted to investigate if this is the case, at a 5% significance level.

The random variable *X* is the number of times heads appears in a sample of 18 coin tosses. If the test defines the null hypothesis as the coin being unbiased, which of these is true?

O Under H_0, X ~ B(18, 0.6111)
O The critical region for this test is $X \leq 5$
O P(Type I error) = 0.0500
O If in fact P(heads) = 0.8, then P(Type II error) = 0.1329

62. The admissions statistics for a college in 2019 were as shown in the contingency table below.

	Admitted	Refused
Male applicants	435	140
Female applicants	302	123

A chi-square test for association was carried out on this data at the 5% significance level.

Which of these is true about the test?

O A valid null hypothesis is “there is no correlation between the gender of applicants and whether they were admitted to this college in 2019”.

O It would be appropriate to apply Yates’ correction to the χ^2 statistic.

O It would be appropriate to pool rows/columns of the contingency table.

O Given that $\chi^2 = 2.6601$ (**without** Yates’ correction), it would be appropriate to reject the null hypothesis.

[2 marks]

63. A researcher wants to test the null hypothesis that a new type of weedkiller is no more effective than an old type of weedkiller.

Considering two of the possible outcomes of this test, what are the consequences of a Type I and Type II error in this context?

A concluding the new weedkiller **is** more effective than the old weedkiller when in fact it is **not** more effective

B **not** concluding the new weedkiller is more effective than the old weedkiller when in fact it **is** more effective.

O A Type I error would be **A**, a Type II error would be **B**

O A Type I error would be **B**, a Type II error would be **A**

O Neither **A** nor **B** is a Type I error.

O Neither **A** nor **B** is a Type II error. [2 marks]

64. The number of busses arriving at a bus stop in one hour, X, is modelled by a Poisson distribution. A hypothesis test is conducted at the 5% significance level with the hypothesis H_0: $\lambda = 3$ and H_1: $\lambda \neq 3$.

The power of this test is given as 0.516. Which of these **must** true?

O The true value of λ is about 5
O If the significance level is decreased, P(Type I error) increases.
O If the significance level is decreased, P(Type II error) decreases.
O An unbiased estimate for the variance of X is 9. [2 marks]

65. The lengths of a type of nail are normally distributed with standard deviation 3 mm and unknown mean μ mm.

A random sample of 10 nails is collected and the lengths, L mm, are measured. The mean length of this sample is 28 mm.

What is the distribution of the random variable L?

O L has a normal distribution with mean 28 mm and variance 9 mm^2.
O L has a normal distribution with mean 28 mm and variance 0.9 mm^2.
O L has a t-distribution with 9 degrees of freedom.
O $\frac{L-28}{3}$ has a t-distribution with 9 degrees of freedom. [2 marks]

66. A fair 6-sided dice numbered with distinct integers 1 through 6 is rolled 3 times. Let the discrete random variable *X* be the **largest** number shown of these rolls.

Which of these is true about the distribution of *X*? (*a*, *b*, *k* ∈ {1, 2, 3, 4, 5, 6}).

O $\mathrm{P}(X = k) = \binom{6}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{6-k}$

O $\mathrm{P}(X = k) = \frac{k^3 - (k-1)^3}{216}$

O $P(X \leq k) = 1 - \left(\frac{k}{6}\right)^3$

O $P(a \leq X \leq b) = \frac{b^3 - a^3}{216}$ [2 marks]

67. The transformation matrix **M** is given by

$$\mathbf{M} = \frac{1}{15}\begin{bmatrix} a & 2 & -10 \\ 2 & b & 5 \\ -10 & 5 & c \end{bmatrix}$$

which, when acting on 3D space, has the geometric interpretation of a reflection in the plane 2*x* - *y* + 5*z* = *d* for some constant *d*.

(You may assume *a*, *b* and *c* are non-zero real constants.)

What are the values of *a*, *b*, *c* and *d* ?

O *a* = $\sqrt{30}$

O *b* = 14

O *c* = $\sqrt{3}$

O *d* = 30 [2 marks]

68. Matrices **A** and **B** are non-singular and represent transformations in 2D space.

If **i** + **j** and **i** - **j** are eigenvectors of **A** with nonzero eigenvalues, and **B** has a repeated eigenvalue of -1, then which of these **must** be true?

O $y = x$ and $y = -x$ are lines of invariant points of **A**.
O $y = x$ and $y = -x$ are invariant lines of **A**.
O **B** represents a reflection in the line $y = x$.
O $|\mathbf{A}| + |\mathbf{AB}| = 0$. [2 marks]

69. The parallelogram *OABC* lies in the *xy* plane with $A = (1, 2)$, $B = (2, 1)$, $C = (3, 3)$ and *O* as the origin.

When transformation matrix **T** acts on the plane, the image of *A* is *A'*, the image of *B* is *B'*, and *C* and *O* are invariant.

Given that $\overrightarrow{OA'} = \overrightarrow{OB}$ and $\overrightarrow{OB'} = \overrightarrow{OA}$, what is the determinant of **T**?

O 0
O 1
O -1
O there is not enough information [2 marks]

70. For suitable *x*, let

$$f(x) = \int_0^x \tanh^{-1} t \ \mathrm{d}t = x \tanh^{-1} x + \frac{1}{2} \ln \left|1 - x^2\right|,$$

Which of these is **false**?

- O *f*(*x*) has a vertical asymptote at *x* = 1
- O *f*(*x*) has no inflection points
- O The integral is improper when *x* = 1
- O The value of $\lim_{x \to 1} f(x)$ is $\ln 2$

[2 marks]

71. Investigate the convergence or divergence of

$$f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... + \frac{1}{n}\ , \ n \in \mathbb{N}$$

as $n \to \infty$.

- O The sum converges because $f(n) - f(n-1) \leq f(n-1) - f(n-2)$.
- O The sum converges because $f(n) \leq \int_1^n \frac{1}{x}\, dx$, which converges as $n \to \infty$.
- O The sum diverges because $f(n) = ln(n)$, which diverges as $n \to \infty$.
- O The sum diverges because $f(n) \geq \int_1^n \frac{1}{x}\, dx$, which diverges as $n \to \infty$.

[2 marks]

72. The continuous random variable X is uniformly distributed on the interval [1, 16].

If $Y = \sqrt{X}$, then the probability density function of Y is

O $f(y) = \frac{2}{15}y, \text{ if } 1 \leq y \leq 4; \ 0, \text{ otherwise.}$

O $f(y) = \frac{1}{21}y^2, \text{ if } 1 \leq y \leq 4; \ 0, \text{ otherwise.}$

O $f(y) = \frac{3}{14}\sqrt{y}, \text{ if } 1 \leq y \leq 4; \ 0, \text{ otherwise.}$

O $f(y) = \frac{2}{255}y, \text{ if } 1 \leq y \leq 16; \ 0, \text{ otherwise.}$

[2 marks]

73. Define the sets

$$P = \{z \in \mathbb{C} : 3 \leq |z - 1 - i| \leq 4\}$$

$$Q = \left\{z \in \mathbb{C} : 0 \leq \arg(z - 1 - i) \leq \frac{\pi}{3}\right\}.$$

What is the maximum value of Re(z) + Im(z) for all $z \in P \cap Q$?

O 8

O $1 + 2\sqrt{3}$

O $12 - 4\sqrt{2}$

O $2 + 4\sqrt{2}$

[2 marks]

74. Find the value of

$$\lim_{x\to\infty} \frac{x+1}{x^2+6x} \exp\left[\frac{x^2}{1+x^2}(\ln x + 2)\right]$$

(*exp*(*x*) denotes the exponential function, e^x. You may assume the limit exists.)

O $4\ln 6$

O 8

O e^2

O $3e$ [2 marks]

75. A simple harmonic oscillator of angular frequency ω is subject to an external force given by $F = \cos\omega t$. The oscillator is free from any effects of damping.

When the displacement of this simple harmonic oscillator is modelled by a second-order linear nonhomogeneous differential equation, the particular integral is denoted $I(t)$.

What is the form of $I(t)$?

O $\frac{1}{2\omega}\sin\omega t$

O $\frac{1}{\sqrt{2\omega}}\cos\omega t$

O $\frac{1}{2\omega}t\sin\omega t$

O $\frac{1}{\sqrt{2\omega}}(\cos\omega t + \sin\omega t)$ [2 marks]

76. A small block of mass 2 kg is attached to one end of a spring of force constant 24 Nm^{-1} which lies above a smooth horizontal table. Attached to the other end of the mass is a viscous dashpot system which provides a linear damping force.

The force exerted by the dashpot is of the form $|F| = cv$, where *v* is the speed of the mass and *c* is a positive constant of suitable dimensions.

Which row in the table below shows the range(s)/value(s) of *c* for which the system exhibits critical damping, underdamping, or overdamping?

(Assume no external forces act other than that of the dashpot.)

	Critical damping	Underdamping	Overdamping
O	$c = 4\sqrt{6}$	$c < 4\sqrt{6}$	$c > 4\sqrt{6}$
O	$c = 4\sqrt{6}$	$c > 4\sqrt{6}$	$c < 4\sqrt{6}$
O	$c = 8\sqrt{3}$	$c < 8\sqrt{3}$	$c > 8\sqrt{3}$
O	$c = 8\sqrt{3}$	$c > 8\sqrt{3}$	$c < 8\sqrt{3}$

77. *x* and *y* are functions of time *t*. They vary according to the coupled system

$$\dot{x} = 2x + 4y \qquad \dot{y} = x - y + t$$

and at $t = 0, x = 12$ and $y = 8$.

Which of these is a second-order differential equation satisfied by *y*, and what are the initial conditions?

O $\ddot{y} - \dot{y} - 4y = 1$, $y(0) = 8,\ \dot{y}(0) = 12$

O $\ddot{y} - \dot{y} - 6y = 1 - 2t$, $y(0) = 8,\ \dot{y}(0) = 12$

O $\ddot{y} - \dot{y} - 4y = 1$, $y(0) = 8,\ \dot{y}(0) = 4$

O $\ddot{y} - \dot{y} - 6y = 1 - 2t$, $y(0) = 8,\ \dot{y}(0) = 4$

[2 marks]

78. $f(x)$ is a continuous function with continuous derivatives on the interval $[a, b]$.

When Simpson's rule is used to approximate the value of $\int_a^b f(x)\,dx$, the result is the value S, while the true value is T.

The formula for the *theoretical maximum error* for S_n, is bounded by

$$|E| \leq \left|\frac{K(b-a)^5}{180n^4}\right|$$

where K is the smallest number such that $K \geq |f^{(4)}(x)|$ for all $x \in [a, b]$.

What is the theoretical maximum error and the actual error when $\int_0^1 x^5$ is approximated using Simpson's rule with a single parabola? (Ignore signs.)

	Theoretical maximum error	Actual error
O	0.0416666...	0.0208333...
O	0.6666666...	0.01041666...
O	0.0416666...	0.0208333...
O	0.6666666...	0.01041666...

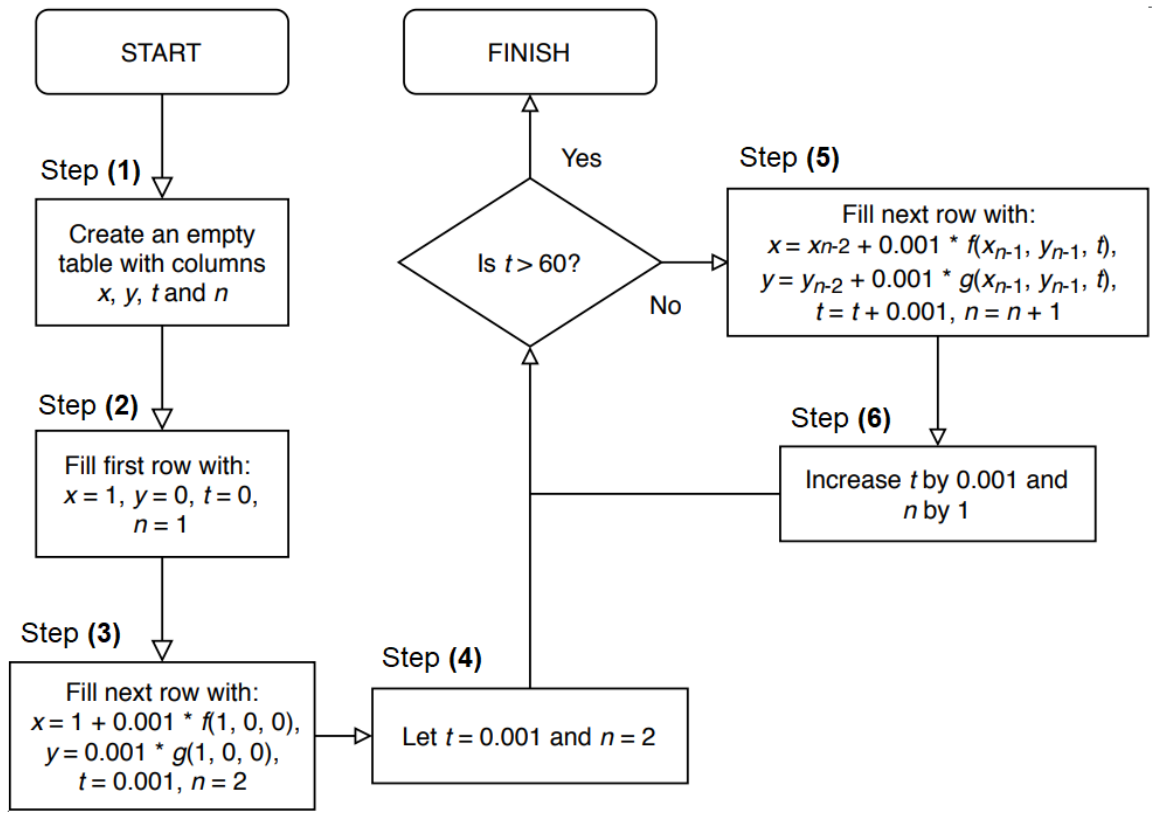
[2 marks]

79. A numerical method for solving differential equations is to be programmed into a computer. The programmer wants to solve a system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, t) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, t) \qquad x = 1 \text{ and } y = 0 \text{ when } t = 0$$

and obtain a table of values for *x* and *y* for *t* in steps of 0.001 up to 60.

The programmer has created a flowchart showing the main process which the computer will follow when executing this method.



Which of these is true?

O In Step **(2)**, the step size for the numerical method is chosen.

O In Steps **(3)** and **(5)**, Euler's method is used on both *x* and *y*.

O In Step **(5)**, Euler's improved method is used on both *x* and *y*.

O When the process finishes, the table will have 60,000 rows.

[2 marks]

80. Which of the following sets of parametric equations, when graphed, forms the parabola $y = 1 - x^2$ for all x?

(The domain of t is all $t \in \mathbb{R}$.)

O $x = \sinh t$ $y = \cosh^2 t$
O $x = \tanh t$ $y = \operatorname{sech}^2 t$
O $x = \operatorname{csch} t$ $y = \coth^2 t$
O None of these [2 marks]

Section C: Hard. Aim to answer 1 question per 4 minutes in this section.

81. Two fixed points A and B lie in 3D Cartesian space such that $|AB| = 2$.

A point P satisfies the following conditions:

$$\overrightarrow{AP} \cdot \overrightarrow{PB} = 0 \qquad \text{and} \qquad \overrightarrow{AB} \cdot \overrightarrow{AP} \geq 2 + \sqrt{3}.$$

Consider the separate cases **1** and **2** detailed below.

Case **1**: P is further restricted to lie in a single 2D plane containing A and B. Let the length of the locus of P in this plane be L.

Case **2**: P is free to move in 3D space (subject to the given conditions). Let the surface area of the locus of P be S.

What are the exact values of L and S?

O $L = \frac{\pi}{3}$, $S = (2 - \sqrt{3})\pi$

O $L = \frac{\pi}{2}$, $S = (2 - \sqrt{3})\pi$

O $L = \frac{\pi}{3}$, $S = (1 + 2\sqrt{3})\pi$

O $L = \frac{\pi}{2}$, $S = (1 + 2\sqrt{3})\pi$ [4 marks]

82. A box contains n cards, where $\{n \in \mathbb{N} : n \geq 2\}$. A different number from 1 to n is written on each card. Two cards are drawn from the box randomly. Let the random variable X be the product of the two numbers written on these cards.

Below shows a process for finding E[X], but some expressions in terms of n are missing and have been replaced by **(a)**, **(b)** and **(c)**.

When choosing 2 cards randomly from n cards, the probability of choosing 2 particular cards is $\dfrac{2}{\textbf{(a)}}$.

Therefore, if we let S be the sum of all products of the chosen cards, $E(X) = \dfrac{2}{\textbf{(a)}} \times S$.

Meanwhile, if we let r $(2 \leq r \leq n)$ be the bigger number between the two chosen numbers,

$$S = \sum_{r=2}^{n} [r \times \{1 + 2 + 3 + \cdots + (r-2) + (r-1)\}]$$

$$= \sum_{r=2}^{n} \left\{ r \times \frac{r(r-1)}{2} \right\} = \frac{1}{2} \sum_{r=2}^{n} (r^3 - r^2) = \frac{1}{2} \sum_{r=1}^{n} (r^3 - r^2)$$

$$= \frac{n(n+1)(n-1)(\ \textbf{(b)}\ )}{24}$$

Therefore, $E(X) = \dfrac{\textbf{(c)}}{12}$.

Which of the following correctly gives the expressions **(a)**, **(b)** and **(c)**?

1 **(a)** is $n(n - 1)$
2 **(b)** is $2n + 3$
3 **(c)** is $(n + 1)(3n + 2)$

O **1** and **2** only
O **2** and **3** only
O **1** and **3** only
O **1**, **2** and **3** [4 marks]

83. **M** is a 3 × 3 matrix representing a transformation in 3D space. **M** has an eigenvalue of 2 with corresponding eigenvector **i** - 2**j**. **M** also has a repeated eigenvalue of λ, with corresponding eigenvectors **u** and **v** (where **u** ≠ **v** ≠ **i** - 2**j**).

Define the plane Π as having equation **r** • (**u** × **v**) = 0 and the line *L* as having equation **r** × (**i** - 2**j**) = 0.

Which of these is **false**?

- O For all nonzero λ, the plane Π is invariant under **M**.
- O If λ = 1, Π is a plane of invariant points under **M**.
- O If λ = 2, **M** represents an enlargement with volume scale factor 8.
- O The line *L* is a line of invariant points under **M**. [4 marks]

84. Consider the general cubic equation $ax^3 + bx^2 + cx + d = 0$, which has three distinct roots α, β and γ.

By considering $(\alpha+\beta)(\beta+\gamma)(\alpha+\gamma)$ and the identities for $\sum\alpha$, $\sum\alpha\beta$ and $\sum\alpha\beta\gamma$, which of these relations between coefficients and roots is/are always true?

(You may assume that the statements below are logically equivalent to their converse, i.e. of the form *A* ⇔ *B*.)

1 If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$ then two of the roots are equal in magnitude but opposite in sign

2 If $\begin{vmatrix} a & b^3 \\ d & c^3 \end{vmatrix} = 0$ then α, β and γ form a geometric sequence

3 If one of the roots is zero then $d \neq 0$

- O **1** only
- O **2** only
- O **1** and **2** only
- O **2** and **3** only [4 marks]

85. Let

$$I(m,n) = \int_0^1 x^n (1-x)^m \, \mathrm{d}x \qquad m,n \in \mathbb{N}.$$

By seeking a reduction formula for $I(m, n)$ in terms of $I(m - 1, n + 1)$ and further manipulating suitably, it can be deduced that

O $I(m,n) = \dfrac{m!n!}{(m+n)!}$

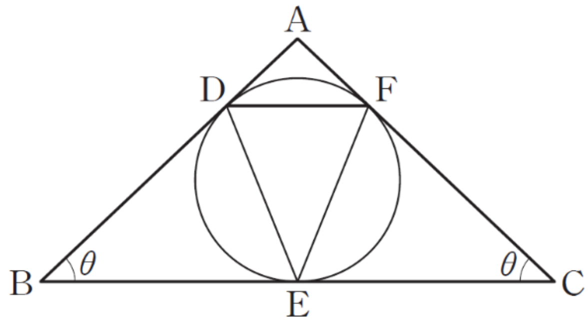
O $I(m,n) = \dfrac{m!n!}{(1+m+n)!}$

O $I(m,n) = \dfrac{m!n!}{(mn-1)!}$

O $I(m,n) = \dfrac{1-m!n!}{(mn+1)!}$

[4 marks]

86. Triangle *ABC* is isosceles with $\angle ABC = \angle BCA = \theta$ radians. A circle is inscribed in $\triangle ABC$ which touches the edges of the triangle at points *D*, *E* and *F* as shown.



The length of side *BC* is 2 units. Let the area of triangle *DEF* be $S(\theta)$.

What is the value of $\lim_{\theta \to 0} \frac{S(\theta)}{\theta^3}$? (Assume $0 < \theta < \frac{\pi}{2}$.)

O $\frac{1}{4}$

O $\frac{1}{2}$

O 2

O 4

[4 marks]

87. A bottle of water is modelled as a hollow cylinder of mass 12 g and height H. It is initially filled with 495 cm^3 of water. The density of water is 1 g cm^{-3}. A small hole is pierced in the bottom of the bottle and the water level gradually falls to zero.

The distance between the **centre of mass** of the water bottle system and the bottom of the bottle is *z* at any time. Due the symmetry of the system, when the bottle is both completely full and completely empty, *z* =$\frac{1}{2}H$in each case, but in between these cases, *z* varies.

What is the **minimum value** of *z* as the bottle empties, and how much water is left in the bottle at this point?

(Keep all values in exact form where possible.)

O minimum *z* =$\frac{2}{15}H$, when there is 66 cm^3 of water left.

O minimum *z* =$\frac{1}{9}H$, when there is 66 cm^3 of water left.

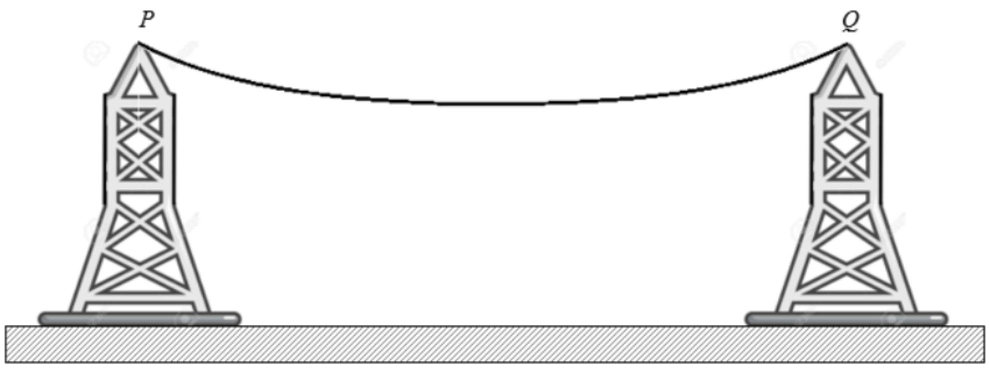
O minimum *z* =$\frac{2}{15}H$, when there is 55 cm^3 of water left.

O minimum *z* =$\frac{1}{9}H$, when there is 55 cm^3 of water left. [4 marks]

88. Two electricity pylons, P and Q, are situated at a distance of 350 metres apart on horizontal ground. An electricity cable running between the two pylons hangs in a catenary curve, whose height h above the ground at a horizontal distance x from P can be described by the equation

$$h = a + \cosh\left(\frac{x}{b} - c\right)$$

for suitable positive constants a, b and c as shown.



The heights of P and Q above the ground is 55 metres. For safety reasons, the minimum distance between the cable and the ground is exactly 45 metres.

Which of these is true? (Round any decimal values to 2 decimal places.)

O The value of a is exactly 45
O The value of b is 111.97 to 2 decimal places
O The value of c is 2.91 to 3 significant figures
O The length of the cable between P and Q rounds to 351 metres

[4 marks]

89. (Notation: $n(X)$ is the number of elements in a finite set X.)

The distinct roots of the equations $P(x) = 0$ and $Q(x) = 0$ are each contained within sets S_P and S_Q respectively, which have $n(S_P) = 7$ and $n(S_Q) = 9$.

Further define sets

$$A = \{(x, y) : P(x)Q(y) = 0 \text{ and } Q(x)P(y) = 0;\ x, y \in \mathbb{R}\}$$
$$B = \{(x, y) : (x, y) \in A \text{ and } x = y\}.$$

where (x, y) denotes an ordered pair.

Set A contains an **infinite** number of elements. $n(B)$ depends on $P(x)$ and $Q(x)$.

What is the largest possible value of $n(B)$?

O 15
O 16
O 36
O 63 [4 marks]

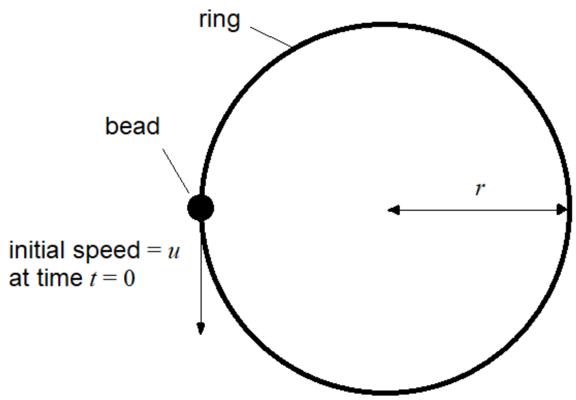
90. The number of real solutions to the equation

$$\sin^{-1}\left(\sum_{r=1}^{\infty} x^{r+1} - x\sum_{r=1}^{\infty}\left(\frac{x}{2}\right)^r\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{r=1}^{\infty}\left(-\frac{x}{2}\right)^r - x\sum_{r=1}^{\infty}(-x)^r\right)$$

in the interval $x \in (-\frac{1}{2}, \frac{1}{2})$ is

O 2
O 3
O 4
O 5 [4 marks]

91. A small bead of mass *m* is threaded onto a thin circular wire of radius *r* which is fixed in a vertical plane. At time $t = 0$, the bead is positioned as shown and has been projected downwards with an initial speed *u*.



Due to friction with the ring, the bead travels around the wire several times and then first comes instantaneously to rest at a time $t = T$ before travelling again in the opposite direction, oscillating to the equilibrium position after a longer time.

The coefficient of friction between the bead and the wire is μ, and at any time the **angular** displacement of the bead from its initial position is *θ*.

Which of these differential equations models the motion of the bead correctly for all *t* in the interval $0 \le t \le T$? (Gravitational acceleration = g.)

O $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + \frac{g}{r}(\mu\sin\theta - \cos\theta) = 0$$

O $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \mu\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + \frac{g}{r}(\mu\sin\theta - \cos\theta) = 0$$

O $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - \mu\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{r}(\mu\sin\theta - \cos\theta) = 0$$

O $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{r}(\mu\sin\theta - \cos\theta) = 0$$

[4 marks]

92. A team of researchers have collected a random sample of data on which they performed a statistical hypothesis test at a 5% significance level. The results of the test provided sufficient evidence to reject the null hypothesis. They publish their findings with a relevant conclusion within the context of their investigation.

One member of the team, Alex, notices that the *p*-value for the test was much lower than the significance level, at $p = 0.0045$. Alex decides to repeat the test, using the exact same dataset, this time using a significance level of $\alpha = 0.5\%$. As expected, the outcome of the test is again a rejection of the null hypothesis.

Alex asks the team to update the research findings with the new significance level, claiming that "we now have stronger evidence to validate the alternative hypothesis".

Does Alex's test support his conclusion? Why or why not?

O **Alex's claim is valid** because his test was done on the exact same data as the team's, so the team could have originally chosen $\alpha = 0.5\%$ and reached the same conclusion.

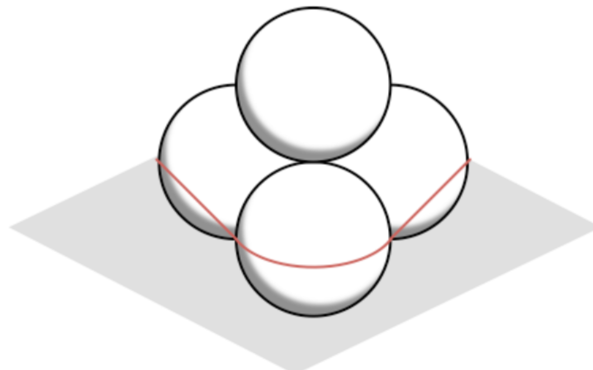
O **Alex's claim is valid but** should not be published because it shows that the team was biased to rejecting the null hypothesis from the beginning.

O **Alex's claim may not be valid** because he cannot be sure that if he obtained a new set of data instead of re-using the previous data, the *p*-value would still be less than 0.5%.

O **Alex's claim is invalid** because he has created his own conclusion to fit the known (no longer random) data, which he could not have done without the first test.

[4 marks]

93. We have four identical frictionless solid spheres of uniform density ρ and radius r. Three of them are on a smooth table adjacent to each other and the remaining sphere is placed on top of the others as shown.



A light, inextensible closed loop of string is wrapped around each of the lower spheres so that it goes round the arrangement at the height of greatest width. The string fits perfectly with no tension when the top sphere is missing.

What is the tension in the string when the top sphere is added?

O $\dfrac{\sqrt{3}}{9}\pi r^3 \rho g$

O $\dfrac{8\sqrt{5}}{45}\pi r^3 \rho g$

O $\dfrac{2\sqrt{6}}{27}\pi r^3 \rho g$

O $\dfrac{2\sqrt{2}}{9}\pi r^3 \rho g$ [4 marks]

94. Let $P_n(x)$ be a sequence of functions. The functions form a pattern of iterated radicals, increasing in degree up to n, as shown:

$$P_n(x) = x \cdot \sqrt{x \cdot \sqrt[3]{x \cdot \sqrt[4]{x \cdot \sqrt[5]{x \cdot \ldots \sqrt[n]{x}}}}}$$

These functions are defined for all positive real x and natural n.

Which of these is true?

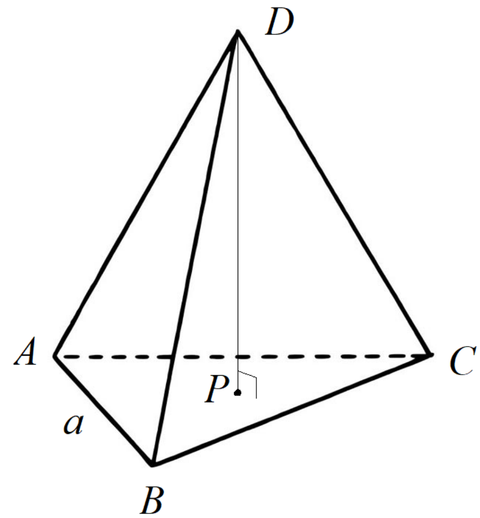
1 $\quad \dfrac{P_n}{P_{n-1}} = \dfrac{1}{x^{n!}}$

2 $\quad \displaystyle\sum_{n=1}^{\infty} P_n\left(\frac{1}{2}\right) = e^{\frac{2}{e}}$

3 $\quad \displaystyle\lim_{n\to\infty} \int_0^1 P_n(x)\ \mathrm{d}x = \frac{1}{1+e}$

O **1** and **2** only
O **2** and **3** only
O **1** and **3** only
O None of them [4 marks]

95. *ABCD* is a regular tetrahedron, constructed from four equilateral triangles with side length *a*. The tetrahedron satisfies the property that line *PD* is perpendicular to plane *ABC*, where *P* is the centre of mass of *ABC*.



The volume of the tetrahedron is proportional to a^3.

The proportionality constant in this relation is

O $\frac{\sqrt{3}}{4}$

O $\frac{\sqrt{2}}{12}$

O $\frac{\sqrt{3}}{6}$

O $\frac{\sqrt{2}}{9}$ [4 marks]

96. $f(x)$ is a quartic polynomial with leading coefficient 1. Let

$$g(x) = \ln \frac{f(x)}{x}, \qquad x > 0$$

$g(x)$ has minimum points at $x = 1$ and $x = 2$, and the value of $g(x)$ at both these points is 0.

What is the value of $f(3)$?

O 4
O 5
O 7
O 10 [4 marks]

End of Questions

Question Sources

Questions sourced from (generally in order of increasing difficulty)

- AQA / Edexcel / OCR A-level past papers
- MCAT past exams (American medical college entrance exam)
- NSAA past papers (Cambridge entrance exam - Natural Sciences)
- Brilliant.org, i-want-to-study-engineering.org, madasmaths.com
- Various international university admissions tests
(UEE - NUS, Singapore; Gaokao - China; CSAT - South Korea; JEE - IIT, India)