Further Maths Solutions (Pure 1)

Section A: Multiple Choice

1. Answer: BD

Working: To multiply matrices, the length (number of columns) of the first matrix must equal the height (number of rows) of the second matrix.
ABC = (AB)C: 3 = 3 (result is 2x3) so then 2 = 2.
CB: 3 = 3.
CD: 3 = 3.
BD: 2 is not equal to 3.

2. **Answer:** $f(x) = (3x - 3)^{1/3}$

Working: Differentiating both sides of the equation, (let g(x) be the integrand)
$f'(x) = d/dx\ (\ G(x) - G(a)\ )$
G(a) is a constant so its derivative is zero, and $G'(x) = g(x)$:
$f'(x) = g(x) \rightarrow f'(x) = (1/f(x))^2 \rightarrow dy/dx = y^{-2}$
This DE is separable:
$y^2\, dy = dx \rightarrow ⅓\, y^3 = x + C$
Using initial condition, $y(1) = 0 \rightarrow 0 = 1 + C \rightarrow C = -1$
$⅓\, y^3 = x - 1 \rightarrow y = (3x - 3)^{1/3}$.

3. **Answer:** 10

Working: |a x b| is the area of the parallelogram formed by a and b.
Angle: $\cos\theta = a\ .\ b\ /\ |a||b| = 30/(10\sqrt{10}) = 3/\sqrt{10} = 3\sqrt{10}/10$
$\rightarrow \sin\theta = \sqrt{1 - (3/\sqrt{10})^2} = \sqrt{1 - 9/10} = 1/\sqrt{10}$.
$\rightarrow |a \times b| = |a||b| \sin\theta = 10\sqrt{10}\ /\ \sqrt{10} = 10$.

4. **Answer:** $x^2y + y^3 - x = 0$, for all $x \in \mathbb{R}$

Working: $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$:
$x^2 + y^2 = \cot(\tan^{-1}(y/x)) \rightarrow x^2 + y^2 = x/y \rightarrow x^2y + y^3 - x = 0$
Bounds:
$\theta = 0 \rightarrow r^2 =$ infinity $\rightarrow r =$ plus or minus infinity
$\rightarrow$ ranges across x-axis $\rightarrow$ all real x.

5. **Answer:** $\ln A = x - x^{-1} - 2 \ln x$

Working: $P(x) = (1 - x)^2 / x^2 = (1 - 2x + x^2)/x^2 = x^{-2} - 2x^{-1} + 1$
Integrating, $\ln A = -x^{-1} - 2 \ln x + x = x - x^{-1} - 2 \ln x$

6. **Answer:** **1** and **2** only

Working: 1: Eigenvector: does not get rotated. The matrix is purely a rotation so all vectors get rotated → no (real) eigenvectors.
2: Invariant line → all points remain on that line. All points move except z-axis so yes z-axis is only invariant line.
3: M^2 = do transformation twice → rotate 180°. This will not return to the original position (requires 360°).

7. **Answer:** $P(1/x)$

Working: By Vieta's formula, product of roots $\alpha\beta\gamma = -d/a = -(-1)/1 = 1$.
The new roots are of the form
$\alpha\beta\gamma/\alpha, \alpha\beta\gamma/\beta, \alpha\beta\gamma/\gamma = 1/\alpha, 1/\beta, 1/\gamma \rightarrow$ new roots are $x' = 1/x$
$\rightarrow x = 1/x'$

8. **Answer:** $f(x) = e^x \sin x$

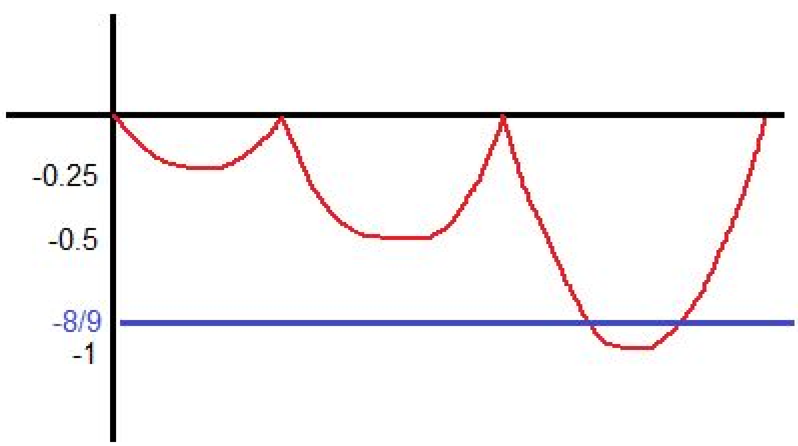
Working: Expansion has no constant term → f(0) = 0
→ must be either sinh(x) or e^x sin(x)
Check sinh(x) first. x term: f'(0) * x / 1! = f'(0) x = cosh(0) x = x.
x^2 term: f''(0) * x^2 / 2 = sinh(0) * x^2 / 2 = 0.
This does not match x^2, so by process of elimination it is e^x sin x.

9. **Answer:** $f(z) = (z - 2 + i)^2$

Working: $z^2 - (4 - 2i)z + (3 - 4i) = (z - (2 - i))^2 - (2 - i)^2 + (3 - 4i)$
$= (z - 2 + i)^2 - (3 - 4i) + (3 - 4i)$
$= (z - 2 + i)^2$

10. **Answer:** 7/3

Working: Sketch the function on 0 < x < 1, it is a parabola with vertex at -0.25. In the interval 1 < x < 2, the function is scaled up by a factor of 2, so its vertex is -0.5. In 2 < x < 3, the vertex is at -1:



From observation, the wanted value of m is the lowest intersection. This is in the third part, so in this interval, y = 4(x - 2)(x - 3) (using transformations of the original) → y = $4x^2 - 20x + 24$
→ $4x^2 - 20x + 24 = -8/9$ → $4x^2 - 20x + 224/9 = 0$
→ x = 7/3 or x = 8/3
The smallest value is then m = 7/3.

Section B: Standard Questions

11.

a. $z = a(1 + 2i)(1 + 3i) + b(1 + i)(1 + 3i) + c(1 + i)(1 + 2i)$ [1 mark]
$= a(5i - 5) + b(4i - 2) + c(3i - 1)$
$= (-5a - 2b - c) + (5a + 4b + 3c)i$ [1 mark]
Real → imaginary component = 0
→ $5a + 4b + 3c = 0$ [1 mark]

b. $\text{Arg } z = \pi/4$ → $\text{Re}(z) = \text{Im}(z)$ → $5a + 4b + 3c = -5a - 2b - c$ [1 mark]
→ $10a + 6b + 4c = 0$ → $\mathbf{5a + 3b + 2c = 0}$ [1 mark]

12.

a. Deduces that squaring a complex number doubles its argument and squares its modulus: $4 < |(z - (2 - i))^2| \leq 9 \rightarrow 4 < |z - (2 - i)|^2 \leq 9 \rightarrow$ **$2 < |z - (2 - i)| \leq 3$**

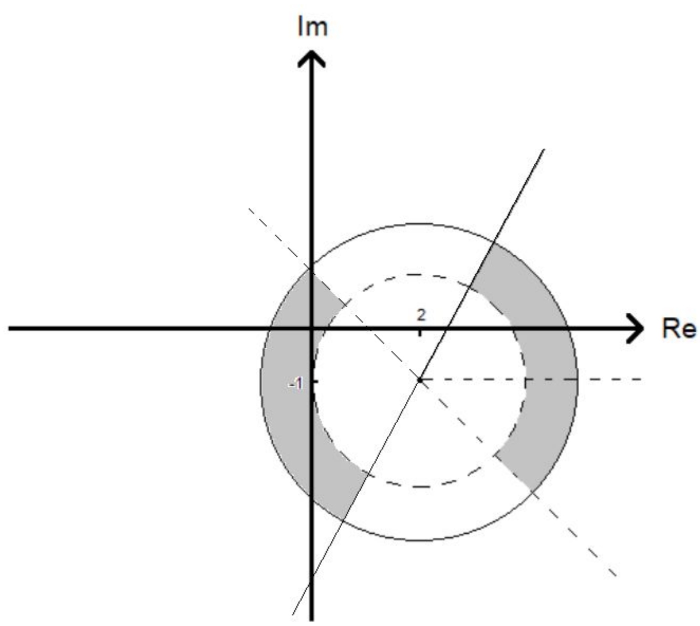
For argument, must include integer multiples of 2π for complete region:

$-\pi/2 + 2k\pi < \arg(z - (2 - i))^2 \leq 2\pi/3 + 2k\pi$, where k is an integer

$\rightarrow -\pi/2 + 2k\pi < 2 \arg(z - (2 - i)) \leq 2\pi/3 + 2k\pi$

$\rightarrow -\pi/4 + k\pi < \arg(z - (2 - i)) \leq \pi/3 + k\pi$. Let $k = 0$ and $k = 1$ (for example),

$\rightarrow -\pi/4 < \arg(z - (2 - i)) \leq \pi/3$ **and** $3\pi/4 \leq \arg(z - (2 - i)) \leq 4\pi/3$ (ignoring convention that arg(z) is less than π, this is equivalent to 4π/3 anticlockwise from initial line as normal.)



[1 mark: centre point labelled or marked on axes]

[1 mark: correct circles and lines, dotted/dashed/full where appropriate]

[1 mark: correct half-region on the right shaded]

[1 mark: correct half-region on the left also shaded]

b. Re(z) - Im(z) = k for the smallest k → Im(z) = Re(z) - k for smallest k

→ Im(z) = Re(z) + k for largest k

→ y = x + C for largest C (in Cartesian coordinates)

This line intersects the top-most point in the left-half of the locus diagram.

This value is $z = (2 - i) + 3[\cos(3\pi/4) + i \sin(3\pi/4)]$ [1 mark]

So we have $\text{Re}(z) - \text{Im}(z) = (2 + 3 \cos 3\pi/4) - (-1 + 3 \sin 3\pi/4)$

= **$3 - 3\sqrt{2}$**. [1 mark]

13.

a. $\ln(1/(2 - x^2)) = -\ln(2 - x^2) = -\ln(1 + (1 - x^2))$

Uses the standard result that

$$\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot u^n$$

Then

$$-\ln(1+(1-x^2)) = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (1-x^2)^n$$

Multiply $(-1) * (-1)^{n+1} = (-1)^{n+2} = (-1)^n$:

$$\ln\left(\frac{1}{2-x^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n(1-x^2)^n}{n}$$

$$\ln\left(\frac{1}{2-x^2}\right) = \sum_{n=1}^{\infty} \frac{(x^2-1)^n}{n}.$$

b. Must have $|x^2 - 1| < 1 \rightarrow -1 < x^2 - 1 < 1 \rightarrow 0 < x^2 < 2 \rightarrow$ $\mathbf{|x| < \sqrt{2}}$ [1 mark]

14. **Base case:**

$n = 1$:

LHS: $d/dx\ (e^x \sin(\sqrt{3}\, x)) = e^x (\sin(\sqrt{3}\, x) + \sqrt{3} \cos(\sqrt{3}\, x))$ [1 mark]

RHS: $2 e^x (\sin(\sqrt{3}\, x + \pi/3)) = 2 e^x (\sin(\sqrt{3}\, x) \cos(\pi/3) + \cos(\sqrt{3}\, x) \sin(\pi/3))$

$= 2 e^x (1/2 \sin(\sqrt{3}\, x) + \sqrt{3}/2 \cos(\sqrt{3}\, x)) = e^x (\sin(\sqrt{3}\, x) + \sqrt{3} \cos(\sqrt{3}\, x))$ [1 mark]

LHS = RHS → true for $n = 1$. [1 mark]

Inductive step:

Assume true for $n = k$, for any natural k. Then,

RHS: $d^k/dx^k (e^x \sin(\sqrt{3}\, x)) = 2^k e^x \sin(\sqrt{3}\, x + k\pi/3)$ [1 mark]

Then, for $n = k + 1$,

LHS: $d^{k+1}/dx^{k+1} (e^x \sin(\sqrt{3}\, x)) = d/dx\ (d^k/dx^k (e^x \sin(\sqrt{3}\, x)))$ [1 mark]

$= d/dx\ (2^k e^x \sin(\sqrt{3}\, x + k\pi/3))$ [1 mark]

$= 2^k e^x (\sin(\sqrt{3}\, x + k\pi/3) + \sqrt{3} \cos(\sqrt{3}\, x + k\pi/3))$ [1 mark]

$= 2^k e^x\, 2 \sin(\sqrt{3}\, x + k\pi/3 + \pi/3)$ [1 mark]

$= 2^{k+1} e^x \sin(\sqrt{3}\, x + (k+1)\pi/3)$

= RHS for $n = k + 1$. [1 mark]

Conclusion:

Since true for $n = 1$, and assuming true for $n = k$ implies true for $n = k + 1$, it is true for all natural numbers n. [1 mark]

15. Integrating by parts,

u = √ ln x dv = dx

du = $\frac{1}{2x\sqrt{ln\ x}}$ v = x [1 mark]

→ $= x\sqrt{\ln x} - \int \frac{1}{2\sqrt{\ln x}}\, \mathrm{d}x$ [1 mark]

Make substitution u = √ln x → $x = e^{u^2}$ [1 mark], $dx = 2u\ e^{u^2}$ [1 mark]

→ $\int \frac{1}{\sqrt{\ln x}}\, \mathrm{d}x = \int \frac{1}{u} \cdot 2ue^{u^2}\ \mathrm{d}u = 2\int e^{u^2}\ \mathrm{d}u$ [1 mark]

Using the definition given,

$\int \frac{1}{2\sqrt{\ln x}}\, \mathrm{d}x = F(u) + C$ [1 mark]

Putting this back in, we get

→ $\int \sqrt{\ln x}\ \mathrm{d}x = x\sqrt{\ln x} - F(\sqrt{\ln x}) + C$ [1 mark]

16.

a. $d\mathbf{p}/dt = \mathbf{Mp}$ [1 mark]

b. Given: $d/dt\ (\ \mathbf{p}_1(t)\ ) = \mathbf{M}\ \mathbf{p}_1(t)$ and $d/dt\ (\ \mathbf{p}_2(t)\ ) = \mathbf{M}\ \mathbf{p}_2(t)$

$d\mathbf{p}/dt = d/dt\ (\ a\mathbf{p}_1(t) + b\mathbf{p}_2(t)\ ) = a\ d/dt\ (\ \mathbf{p}_1(t)\ ) + b\ d/dt\ (\ \mathbf{p}_2(t)\ )$ [1 mark]

$= a\ \mathbf{M}\ \mathbf{p}_1(t) + b\ \mathbf{M}\ \mathbf{p}_2(t)$ [1 mark]
$= \mathbf{M}\ (\ a\ \mathbf{p}_1(t) + b\ \mathbf{p}_2(t)\ )$ [1 mark]
$= \mathbf{M}\ \mathbf{p}$. [1 mark] → solution.

c. $\mathbf{p} = e^{\lambda t}\mathbf{k} \rightarrow d\mathbf{p}/dt = \lambda\ \mathbf{k}\ e^{\lambda t}$ [1 mark] $= \lambda\ \mathbf{p}$
$d\mathbf{p}/dt = \mathbf{M}\ \mathbf{p} \rightarrow \lambda\ \mathbf{p} = \mathbf{M}\ \mathbf{p} \rightarrow \lambda\ e^{\lambda t}\mathbf{k} = \mathbf{M}\ e^{\lambda t}\mathbf{k}$ [1 mark] $\rightarrow \mathbf{M}\ \mathbf{k} = \lambda\ \mathbf{k}$ [1 mark]

d. Eigenvalues:
$(5 - \lambda)(1 - \lambda) - 1(-3) = 0 \rightarrow \lambda^2 - 6\lambda + 8 = 0$ [1 mark] $\rightarrow \lambda = 4$ and $\lambda = 2$. [1 mark]
Eigenvectors:
$\lambda = 4$: $x - 3y = 0 \rightarrow x = 3y \rightarrow$ eigenvector $[3; 1]^T$
$\lambda = 2$: $3x - 3y = 0 \rightarrow x = y \rightarrow$ eigenvector $[1; 1]^T$

Assuming $\mathbf{p} = e^{\lambda t}\mathbf{k}$ is a solution,
$\mathbf{Mk} = \lambda\mathbf{k} \rightarrow \mathbf{k}$ eigenvectors with corresponding eigenvalues λ
$\rightarrow \mathbf{p} = e^{4t}[3; 1]^T$ and $\mathbf{p} = e^{2t}[1; 1]^T$ are solutions [1 mark]
$\rightarrow \mathbf{p} = a\ e^{4t}[3; 1]^T + b\ e^{2t}[1; 1]^T$ is the general solution [1 mark]
1000 rabbits at t = 0: $1000 = 3a + b$ [1 mark]
50 wolves at t = 0: $50 = a + b$ [1 mark]
Solving simultaneously, $a = 475$, $b = -425$ [1 mark]
→ Solution to system is $\mathbf{p} = 475\ e^{4t}[3; 1]^T - 425\ e^{2t}[1; 1]^T$ [1 mark]
→ $\mathbf{r(t) = 1425\ e^{4t} - 425\ e^{2t}}$ [1 mark] and $\mathbf{w(t) = 475\ e^{4t} - 425\ e^{2t}}$ [1 mark]

17.

a. Let coordinates of P and Q be when $\lambda = p$ and $\lambda = q$ respectively.
Coordinates are
$P = (3 + p, 2p, 3 - 2p)$ and $Q = (3 + q, 2q, 3 - 2q)$
$AP = [p, -3 + 2p, 6 - 2p]^T$ and $AQ = [q, -3 + 2q, 6 - 2q]^T$ [1 mark]

$AP. AQ = 0 \rightarrow pq + (2p - 3)(2q - 3) + (6 - 2p)(6 - 2q) = 0$ [1 mark]
$pq + 4pq - 6q - 6p + 9 + 36 - 12p - 12q + 4pq = 0$
$9pq - 18p - 18q + 45 = 0 \rightarrow pq - 2(p + q) + 5 = 0$ [1 mark]

$|AP| = |AQ|$
$|AP| = \sqrt{(p^2 + (2p - 3)^2 + (6 - 2p)^2)} = \sqrt{(p^2 + 4p^2 - 12p + 9 + 4p^2 - 24p + 36)}$
$= 3\sqrt{(p^2 - 4p + 5)}$ [1 mark]
$|AQ| = 3\sqrt{(q^2 - 4q + 5)}$
$\rightarrow 3\sqrt{(p^2 - 4p + 5)} = 3\sqrt{(q^2 - 4q + 5)}$ [1 mark]
$\rightarrow p^2 - 4p = q^2 - 4q \rightarrow p^2 - q^2 = 4(p - q) \rightarrow (p + q)(p - q) = 4(p - q)$
$\rightarrow p + q = 4$ ($p - q$ cannot be 0 since p and q are distinct) [1 mark]

Subbing back in to above, $pq - 2(4) + 5 = 0 \rightarrow pq = 3 \rightarrow p = 3/q$ [1 mark]
$\rightarrow 3/q + q = 4 \rightarrow q^2 - 4q + 3 = 0 \rightarrow q = q = 3$ or $q = 1 \rightarrow p = 1$ or $p = 3$ [1 mark]
$P = (p + 3, 2p, 3 - 2p) = (4, 2, 1)$ or $(6, 6, -3)$
$Q = (q + 3, 2q, 3 - 2q) = (6, 6, -3)$ or $(4, 2, 1)$ [1 mark]
$\sqrt{(4^2 + 2^2 + 1^2)} = \sqrt{21}$ and $\sqrt{(6^2 + 6^2 + (-3)^2)} = \sqrt{79}$.
Since $|OQ| > |OP|$,
P is (4, 2, 1) and Q is (6, 6, -3). [1 mark]

b. Area $= \frac{1}{2} \, | (\mathbf{p} - \mathbf{a}) \times (\mathbf{q} - \mathbf{a}) |$
$= \frac{1}{2} \, | [\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}] \times [3\,\mathbf{i} + 3\,\mathbf{j}] \,|$ [1 mark]
$= \mathbf{9}$. [1 mark]

18. Let tangent be $y = mx + c$.

Then $mx + c = x^3 - x^4$.

$x^4 - x^3 + mx + c = 0$ [1 mark]

Since tangent at two places, need two repeated roots (at $x = A$ and B):

$(x - A)^2(x - B)^2 = 0$

$x^4 - x^3 + mx + c = (x - A)^2(x - B)^2$ [1 mark]

$= (x^2 - 2Ax + A^2)(x^2 - 2Bx + B^2)$

$= x^4 - 2Bx^3 + Bx^2 - 2Ax^3 + 4ABx^2 - 2AB^2x + Ax^2 - 2A^2Bx + A^2B^2$ [1 mark]

$= x^4 + (-2A - 2B)x^3 + (A + B + 4AB)x^2 + (-2AB^2 - 2A^2B)x + A^2B^2$

Equating coefficients:

x^3: $-2A - 2B = -1 \rightarrow 2A + 2B = 1 \rightarrow A + B = \frac{1}{2}$ [1 mark]

x^2: $A + B + 4AB = 0 \rightarrow \frac{1}{2} + 4AB = 0 \rightarrow AB = -\frac{1}{8}$ [1 mark]

x: $-2AB^2 - 2A^2B = m \rightarrow -2AB(A + B) = -2(-\frac{1}{8})(\frac{1}{2}) = \frac{1}{8}$

constant: $A^2B^2 = c \rightarrow (AB)^2 = 1/64$ [1 mark]

$\rightarrow y = 1/8\ x + 1/64$

$\rightarrow$ **$64y = 8x + 1$**. [1 mark]

Section C: Extended Questions

19. **Part 1: Forming the differential equations**

Let A(t) and B(t) be the mass of contaminant in grams in tanks A and B respectively at time t hours after the initial conditions are recorded.
Concentration (g / L) = mass of contaminant (g) / volume of water (L)
Net flow of water into the system = 7 + 3 = 10 L / hr
Net flow of water out of the system = 10 L / hr
Flow in = flow out → volume of water in each tank remains constant. [1 mark]

Net rate of change = rate of flow in - rate of flow out
In tank 1: change = (added in from tap) + (from P) - (into Q)
→ dA/dt = (7 * 1.5) + (5 * (B/100)) - (12 * (A/50))
→ dA/dt = 21/2 + B/20 - 6A/25
→ 100 dA/dt = 1050 + 5 B - 24 A [2 marks]
In tank 2: change = (added in from tap) + (from R) - (into P)
→ dB/at = 0 + (2 * (A/50)) - (5 * (B/100))
→ dB/dt = A/25 - B/20
→ 100 dB/dt = 4 A - 5 B [2 marks]
Initial conditions: A(0) = 25, B(0) = 75
So the system of coupled differential equations to be solved is:

$$100\,\frac{\mathrm{d}A}{\mathrm{d}t} = 1050 + 5B - 24A \qquad 100\,\frac{\mathrm{d}B}{\mathrm{d}t} = 4A - 5B$$

$$A(0) = 25 \qquad B(0) = 75$$

[1 mark]

The assumptions made here are:

1. The contaminant is spread evenly throughout each tank (perfect mixing)
2. The water flows instantly through the pipes
3. The inflow and outflow from each tank are exactly equal (no net change in volume of water in the system)
4. The contaminant is fully dissolved and is transported without resistance (contaminant flowing through pipes remains proportional to flow rate)

[2 marks for any two of these assumptions]

Part 2: Converting the system to a 2nd-order DE

Differentiate the second equation:
$100\ d^2B/dt^2 = 4\ dA/dt - 5\ dB/dt \rightarrow 100\ dA/dt = 2500\ d^2B/dt^2 + 125\ dB/dt$ [1 mark]
Rearrange for A in second equation:
$4\ A = 100\ dB/dt + 5\ B \rightarrow 24\ A = 600\ dB/dt + 30\ B$ [1 mark]
Sub back into the first equation:
$2500\ d^2B/dt^2 + 125\ dB/dt = 1050 + 5\ B - 600\ dB/dt - 30\ B$ [1 mark]
$\rightarrow 2500\ d^2B/dt^2 + 725\ dB/dt + 25\ B = 1050$
$\rightarrow 100\ d^2B/dt^2 + 29\ dB/dt + B = 42$ [1 mark]
Initial conditions: At $t = 0$, $100\ B'(0) = 4(25) - 5(75) \rightarrow B'(0) = -11/4$. [1 mark]

Part 3: Solving the equations

We have $100\ d^2B/dt^2 + 29\ dB/dt + B = 42$ subject to $B(0) = 75$, $B'(0) = -11/4$.
This is a nonhomogeneous second order differential equation.
The complementary solution is: $100\ B'' + 29\ B' + B = 0 \rightarrow \lambda = -1/4$ and $\lambda = -1/25$.
$\rightarrow B_C(t) = c_1 \exp(-1/4\ t) + c_2 \exp(-1/25\ t)$ [1 mark]
The particular solution is easily deduced to be the constant 42, so
$B(t) = c_1 \exp(-1/4\ t) + c_2 \exp(-1/25\ t) + 42$ [1 mark]
Using initial conditions:
$c_1 + c_2 = 33$, $-1/4\ c_1 + -1/25\ c_2 = -11/4$ [1 mark]
Solution is $c_1 = 143/21$, $c_2 = 550/21$
$\rightarrow B(t) = 143/21 \exp(-1/4\ t) + 550/21 \exp(-1/25\ t) + 42$ [1 mark]
Using the second equation, $100\ B' = 4A - 5B \rightarrow A = (100\ B' + 5B)/4$
$A = (100\ (-143/84 \exp(-1/4\ t) - 550/525 \exp(-1/25\ t))$
$\quad + \quad 5(143/21 \exp(-1/4\ t) + 550/21 \exp(-1/25\ t) + 42))/4$
$\rightarrow A = -715/21 \exp(-1/4\ t) + 275/42 \exp(-1/25\ t) + 105/2$ [2 marks]
Factoring out constant multiples, the final solutions are

$$A(t) = \frac{5}{42}\left(55\exp\left(-\frac{t}{25}\right) - 286\exp\left(-\frac{t}{4}\right) + 441\right)$$

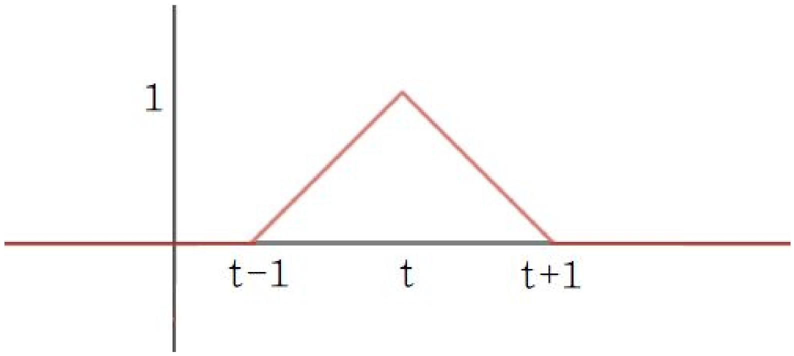
$$B(t) = \frac{1}{21}\left(550\exp\left(-\frac{t}{25}\right) + 143\exp\left(-\frac{t}{4}\right) + 2\right)$$

[1 mark]

(Solution by Paul Dawkins at
http://tutorial.math.lamar.edu/Classes/DE/SystemsModeling.aspx)

20. **Part 1: Working with f(x)**

Begin by sketching a graph of f(x) to help later understanding.
It has the form of y = -|x| translated by [t; 1]:



[1 mark for sketch]

The lines have gradients 1 and -1 and pass through (t, 1) so using point-slope forms gives (splitting modulus function into two parts):

$$f(x)=\begin{cases} (x-t)+1= \quad x-t+1 & (t-1\leq x<t) \\ -(x-t)+1=-x+t+1 & (t\leq x\leq t+1) \\ 0 & \text{otherwise} \end{cases}$$

[1 mark]

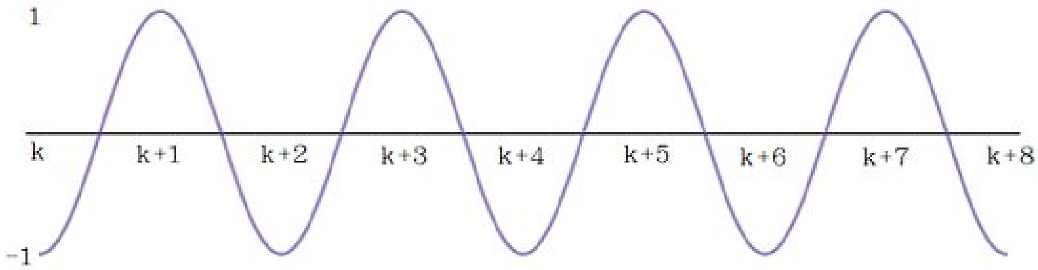
This form is much easier to work with when integrating it.

Part 2: Planning work with g(t)

Next, study g(t). This is a function of t and not x since the x's are all replaced by k when the integration bounds are done. The cosine function in g(t) has a period of 2. Since the integration period is a distance of (k + 8) - k = 8, we consider 8/2 = 4 periods of this function.

Since k is odd, the range of integration starts from the trough (bottom) of the wave and ends at another trough. The product of this wave with f(x) gives the function to integrate. When f(x) is zero, the product is also zero and we can ignore these intervals. There are multiple different possibilities depending on the value of t since this is what moves f(x) left and right through the intervals.

Generalising, the cosine function looks like,



We must consider each possible case of how f(x) lies within this range. For the purpose of speeding calculations up, we should calculate the indefinite integral forms of g(t) first, then put in bounds when needed as these will vary significantly.

The first integral to evaluate is when dealing with the increasing part of f(x):

$$\int (x - t + 1)\cos(\pi x)\ dx$$

$$\int x\cos(\pi x)\ dx - (t-1)\int \cos(\pi x)\ dx$$

Integrating by parts on the first one, (+C omitted since these results will be used with definite integrals later on)

$$\int x\cos(\pi x)\ dx = \frac{x}{\pi}\sin(\pi x) - \frac{1}{\pi^2}\cos(\pi x)$$

The second integral is easily evaluated,

$$(t-1)\int \cos(\pi x)\ dx = \frac{(t-1)}{\pi}\sin(\pi x)$$

Combining and factorising,

$$\int (x - t + 1)\cos(\pi x)\ dx = \frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}$$

. [2 mar ks]

The second integral covers the decreasing part of f(x):

$$\int (-x + t + 1)\cos(\pi x)\ dx$$

Repeating the same method and factorising again,

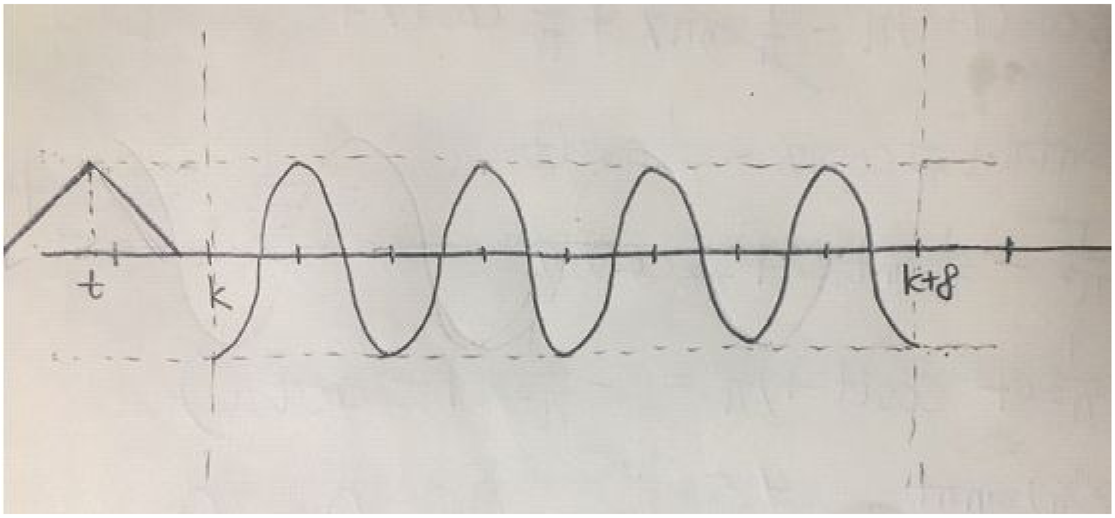
$$\int (-x + t + 1)\cos(\pi x)\ dx = \frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}$$

. [2 marks]

Part 3: Defining g(t)

Case 1: Fully outside range

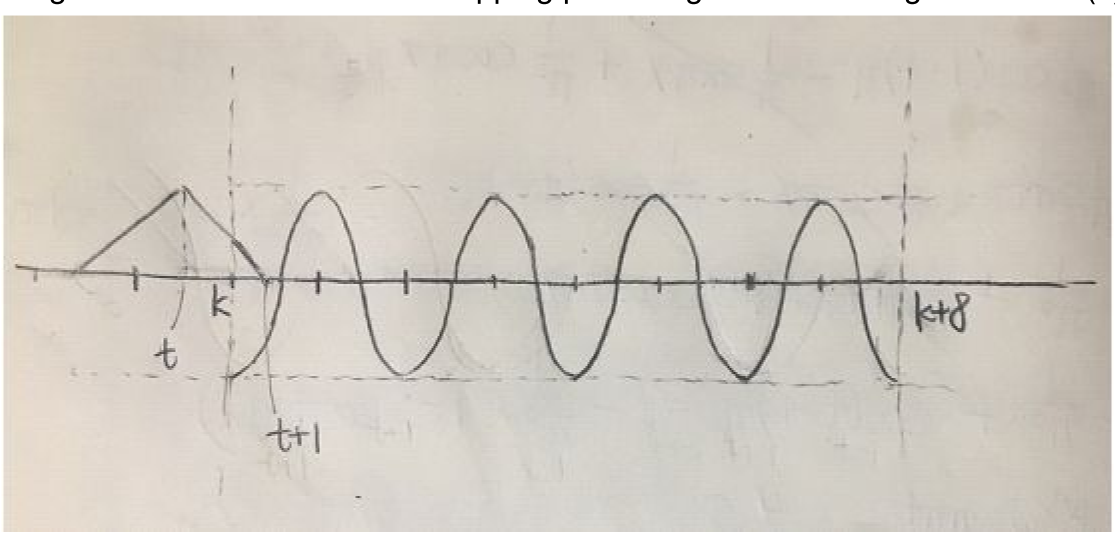
If the triangle part (non-zero) of f(x) is located fully outside this range, then the integral is zero everywhere:



This happens when either $t + 1 \leq k$ or $k + 8 \leq t - 1$. These conditions can be written in terms of t as $t < k - 1$ or $t \geq k + 9$.
In these cases, $f(x) \cos(\pi x) = 0$ and therefore $g(t) = 0$. [1 mark]

Case 2: The decreasing part of f(x) is partly in the interval

If f(x) shifts to the right such that part of its decreasing part is in the interval, the integral is evaluated on the overlapping part using the decreasing section of f(x):



This happens when k - 1 ≤ t < k. The function becomes

$$g(t) = \int_k^{t+1} (-x + t + 1)\cos(\pi x)\ dx$$

Using the found result from part 2,

$$g(t) = \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=k}^{x=t+1}$$

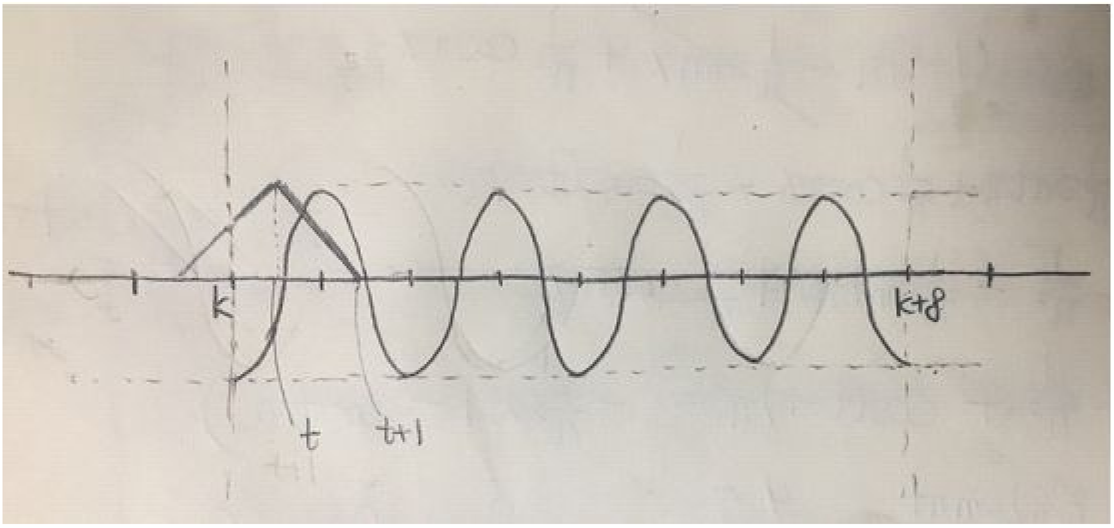
Using translation identities, we can simplify the input of x = t + 1 using sin(x + π) = -sin(x) and cos(x + π) = -cos(x).
We can also simplify the input of x = k since k is odd, so sin(kπ) = 0 and cos(kπ) = -1. Therefore

$$g(t) = \frac{1}{\pi^2}(\cos(\pi t) - 1)$$

[3 marks]

Case 3: The increasing part of f(x) is partly in the interval

Shifting f(x) more to the right puts the downward part fully in, but the upward part not fully in:



This happens when k ≤ t < k + 1.
In this case, we have two integrals to consider, from each part of the function:

$$g(t) = \int_k^t (x - t + 1)\cos(\pi x)\ dx + \int_t^{t+1} (-x + t + 1)\cos(\pi x)\ dx$$

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=k}^{x=t} + \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=t}^{x=t+1}$$

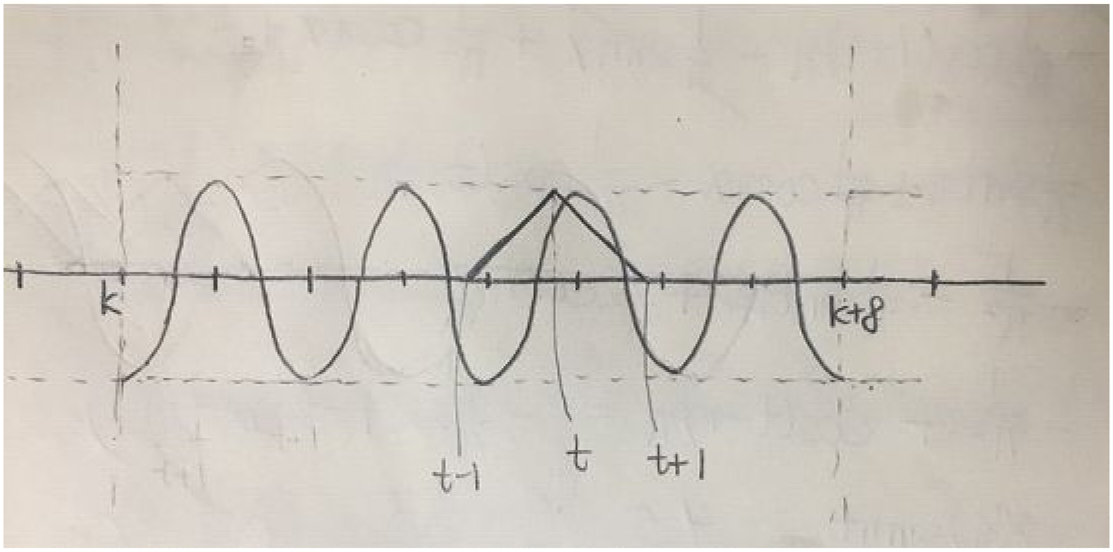
$$g(t) = \left(\frac{(\pi\sin(\pi t) + \cos(\pi t)) + 1}{\pi^2}\right) + \left(\frac{\cos(\pi t) - (\pi\sin(\pi t) - \cos(\pi t))}{\pi^2}\right)$$

$$g(t) = \frac{1}{\pi^2}(1 + 3\cos(\pi t))$$

[3 marks]

Case 4: f(x) is fully within interval

This is the 'standard' case, with no sticking out parts of f(x) at the boundaries.



This happens when k + 1 ≤ t < k + 7. This produces two full integrals, one for each part of f(x):

$$g(t) = \int_{t-1}^{t} (x - t + 1)\cos(\pi x)\ dx + \int_{t}^{t+1} (-x + t + 1)\cos(\pi x)\ dx$$

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=t-1}^{x=t} + \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=t}^{x=t+1}$$

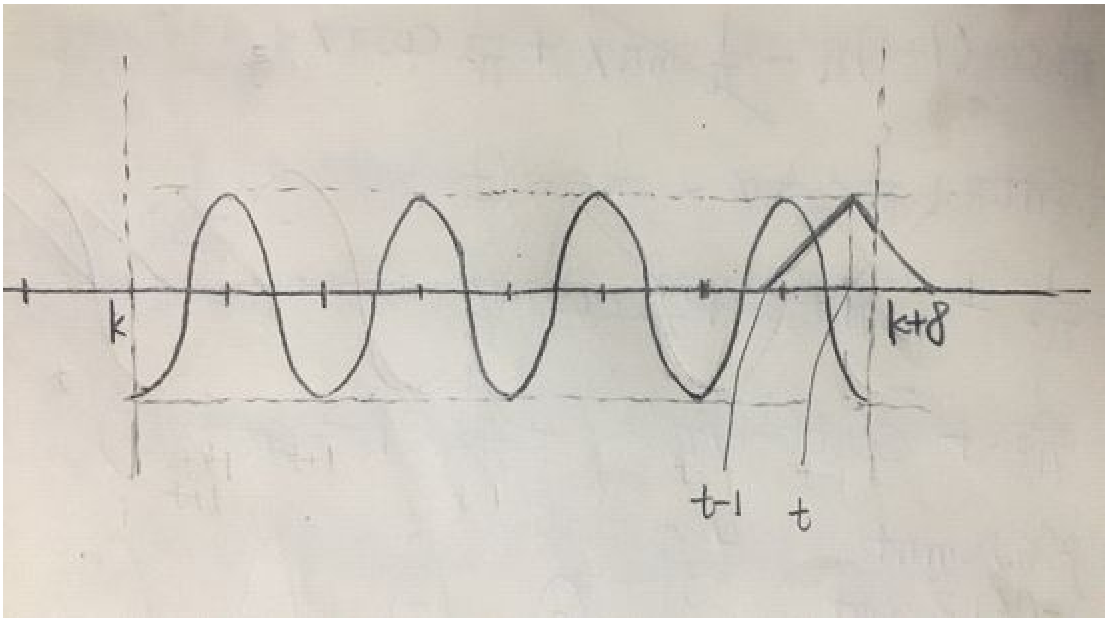
$$g(t) = \left(\frac{(\pi\sin(\pi t) + \cos(\pi t)) + \cos(\pi t)}{\pi^2}\right) + \left(\frac{\cos(\pi t) - (\pi\sin(\pi t) - \cos(\pi t))}{\pi^2}\right)$$

$$g(t) = \frac{4}{\pi^2}\cos(\pi t)$$

[3 marks]

Case 5: The 'other' decreasing part of f(x) is partly in the interval

This is the mirror image of case 3:



This happens when k + 7 ≤ t < k + 8 and the integrals are

$$g(t) = \int_{t-1}^{t} (x - t + 1)\cos(\pi x)\ dx + \int_{t}^{k+8} (-x + t + 1)\cos(\pi x)\ dx$$

Since k is odd, k + 8 is also odd, so we can use the same rules when substituting its limit in:

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=t-1}^{x=t} + \left[\frac{\pi(t - x + 1)\sin(\pi x) - \cos(\pi x)}{\pi^2}\right]_{x=t}^{x=k+8}$$

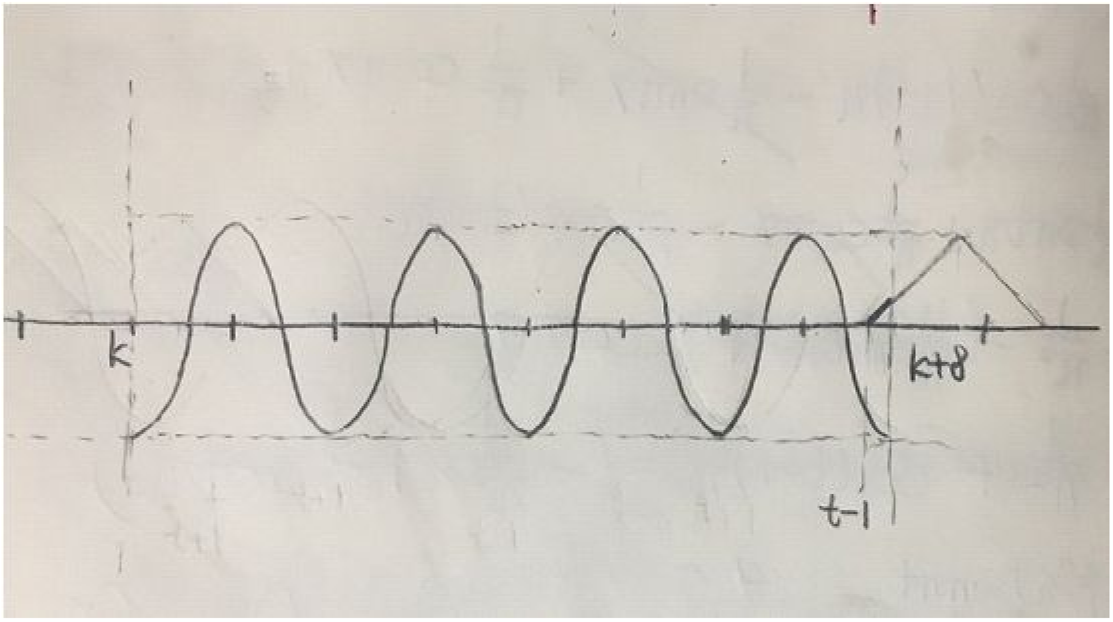
$$g(t) = \left(\frac{\pi\sin(\pi t) + 2\cos(\pi t)}{\pi^2}\right) + \left(\frac{1 - (\pi\sin(\pi t) - \cos(\pi t))}{\pi^2}\right)$$

$$g(t) = \frac{1}{\pi^2}(1 + 3\cos(\pi t))$$

[3 marks]

Case 6: The 'other' increasing part of f(x) is partly in the interval

This is the mirror image of case 2:



This happens when k + 8 ≤ t < k + 9. There is only one part of f(x) in the interval so only one integral:

$$g(t) = \int_{t-1}^{k+8} (x - t + 1)\cos(\pi x)\ dx$$

$$g(t) = \left[\frac{\pi(-t + x + 1)\sin(\pi x) + \cos(\pi x)}{\pi^2}\right]_{x=t-1}^{x=k+8}$$

$$g(t) = \left(\frac{-1 + \cos(\pi t)}{\pi^2}\right)$$

$$g(t) = \frac{1}{\pi^2}(\cos(\pi t) - 1)$$

[2 marks]

Part 4: Putting it all together

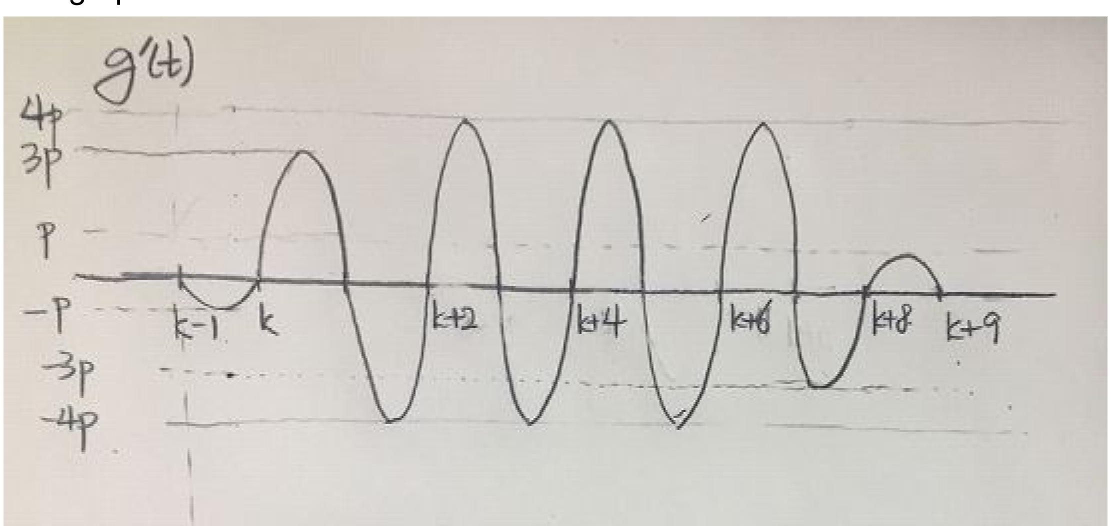
Now, combine all these possible cases into a piecewise function

$$g(t)=\begin{cases} -\frac{1}{\pi^2}(1-\cos\pi t) & (k-1\le t<k) \\ \frac{1}{\pi^2}(1+3\cos\pi t) & (k\le t<k+1) \\ \frac{4}{\pi^2}\cos\pi t & (k+1\le t<k+7) \\ \frac{1}{\pi^2}(1+3\cos\pi t) & (k+7\le t<k+8) \\ -\frac{1}{\pi^2}(1-\cos\pi t) & (k+8\le t<k+9) \\ 0 & \text{otherwise} \end{cases}$$

We need the minima of this function. Differentiate each function:

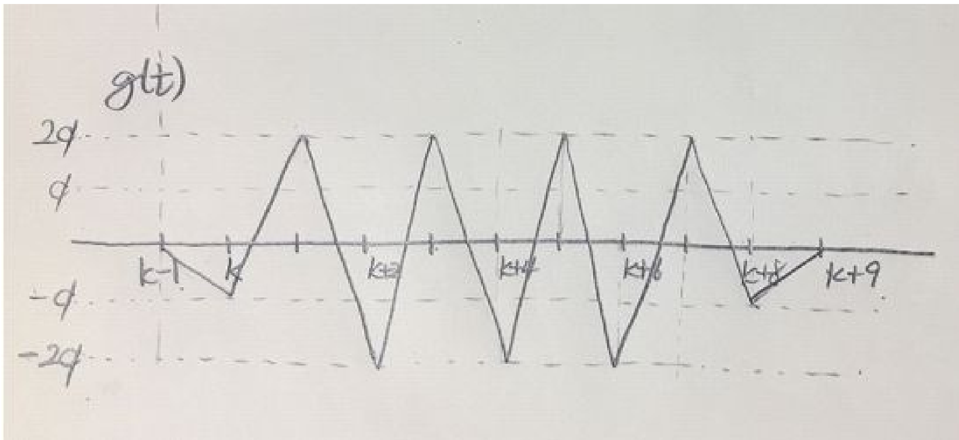
$$g'(t)=\begin{cases} -\frac{1}{\pi}\sin\pi t & (k-1\le t<k) \\ -\frac{3}{\pi}\sin\pi t & (k\le t<k+1) \\ -\frac{4}{\pi}\sin\pi t & (k+1\le t<k+7) \\ -\frac{3}{\pi}\sin\pi t & (k+7\le t<k+8) \\ -\frac{1}{\pi}\sin\pi t & (k+8\le t<k+9) \\ 0 & \text{otherwise} \end{cases}$$

This graph is much easier to sketch:



[1 mark]

We can use this graph to approximately sketch g(t). When g'(t) is negative, g(t) will be decreasing, and when g'(t) is positive, g(t) will be increasing.
When g'(t) = 0, we have a minimum or maximum:



(In the above sketches, p and q are used for simplicity as proportionality constants: $p = 1/\pi$ and $q = 1/\pi^2$.) [1 mark]

Part 5: Getting the desired value

From the graph, we the minima of g(t) such that g(t) < 0 are
t = k, k + 2, k + 4, k + 6 and k + 8.
The sum of these values is 45:
k + (k + 2) + (k + 4) + (k + 6) + (k + 8) = 45 → 5k + 20 = 45 → k = 5. [1 mark]
The values of α are then α = {5, 7, 9, 11, 13}
Subbing into g(t), we get

$$g(\alpha) = \{-\frac{2}{\pi^2}, -\frac{4}{\pi^2}, -\frac{4}{\pi^2}, -\frac{4}{\pi^2}, -\frac{2}{\pi^2}, \}$$

$$\sum_{i=1}^{5} g(\alpha_i) = -\frac{2}{\pi^2} - \frac{4}{\pi^2} - \frac{4}{\pi^2} - \frac{4}{\pi^2} - \frac{2}{\pi^2}$$

$$\sum_{i=1}^{5} g(\alpha_i) = -\frac{16}{\pi^2}$$

Finally, calculating the required value

$$k - \pi^2 \sum_{i=1}^{5} g(\alpha_i) = 5 - \pi^2 \left(-\frac{16}{\pi^2}\right) = 5 + 16 = 21.$$

[1 mark]

(Solution by 돌핀 at https://blog.naver.com/dak219324/221147969311)