CALCULUS I

Extras

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Chapter 7 : Extras

In this chapter is material that didn't fit into other sections for a variety of reasons. Also, in order to not obscure the mechanics of actually working problems, most of the proofs of various facts and formulas are in this chapter as opposed to being in the section with the fact/formula.

This chapter contains those topics.

Proof of Various Limit Properties – In this section we prove several of the limit properties and facts that were given in various sections of the Limits chapter.

Proof of Various Derivative Facts/Formulas/Properties – In this section we prove several of the rules/formulas/properties of derivatives that we saw in Derivatives Chapter.

Proof of Trig Limits – In this section we give proofs for the two limits that are needed to find the derivative of the sine and cosine functions using the definition of the derivative.

Proofs of Derivative Applications Facts/Formulas – In this section we prove many of the facts that we saw in the Applications of Derivatives chapter.

Proof of Various Integral Facts/Formulas/Properties – In this section we prove some of the facts and formulas from the Integral Chapter as well as a couple from the Applications of Integrals chapter.

Area and Volume Formulas – In this section we derive the formulas for finding area between two curves and finding the volume of a solid of revolution.

Types of Infinity – In this section we have a discussion on the types of infinity and how these affect certain limits. Note that there is a lot of theory going on 'behind the scenes' so to speak that we are not going to cover in this section. This section is intended only to give you a feel for what is going on here. To get a fuller understanding of some of the ideas in this section you will need to take some upper level mathematics courses.

Summation Notation – In this section we give a quick review of summation notation. Summation notation is heavily used when defining the definite integral and when we first talk about determining the area between a curve and the x-axis.

Constant of Integration – In this section we have a discussion on a couple of subtleties involving constants of integration that many students don't think about when doing indefinite integrals. Not understanding these subtleties can lead to confusion on occasion when students get different answers to the same integral. We include two examples of this kind of situation.

Section 7-1 : Proof of Various Limit Properties

In this section we are going to prove some of the basic properties and facts about limits that we saw in the **Limits** chapter. Before proceeding with any of the proofs we should note that many of the proofs use the **precise definition** of the limit and it is assumed that not only have you read that section but that you have a fairly good feel for doing that kind of proof. If you're not very comfortable using the definition of the limit to prove limits you'll find many of the proofs in this section difficult to follow.

The proofs that we'll be doing here will not be quite as detailed as those in the precise definition of the limit section. The "proofs" that we did in that section first did some work to get a guess for the δ and then we verified the guess. The reality is that often the work to get the guess is not shown and the guess for δ is just written down and then verified. For the proofs in this section where a δ is actually chosen we'll do it that way. To make matters worse, in some of the proofs in this section work very differently from those that were in the limit definition section.

So, with that out of the way, let's get to the proofs.

Limit Properties

In the **Limit Properties** section we gave several properties of limits. We'll prove most of them here. First, let's recall the properties here so we have them in front of us. We'll also be making a small change to the notation to make the proofs go a little easier. Here are the properties for reference purposes.

Assume that $\lim\limits_{x\to a} f(x) = K$ and $\lim\limits_{x\to a} g(x) = L$ exist and that *c* is any constant. Then,

1. $\lim\limits_{x\to a}\left[cf(x)\right] = c\lim\limits_{x\to a} f(x) = cK$

2. $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = K \pm L$

3. $\lim\limits_{x\to a}\left[f(x)g(x)\right] = \lim\limits_{x\to a} f(x)\;\lim\limits_{x\to a} g(x) = KL$

4. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{K}{L}$, provided $L = \lim\limits_{x\to a} g(x) \neq 0$

5. $\lim\limits_{x\to a}\left[f(x)\right]^n = \left[\lim\limits_{x\to a} f(x)\right]^n = K^n$, where n is any real number

6. $\lim\limits_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

7. $\lim\limits_{x\to a} c = c$

8. $\lim\limits_{x \to a} x = a$

9. $\lim\limits_{x \to a} x^n = a^n$

Note that we added values (*K*, *L*, *etc.*) to each of the limits to make the proofs much easier. In these proofs we'll be using the fact that we know $\lim\limits_{x \to a} f(x) = K$ and $\lim\limits_{x \to a} g(x) = L$ we'll use the definition of the limit to make a statement about $\left|f(x) - K\right|$ and $\left|g(x) - L\right|$ which will then be used to prove what we actually want to prove. When you see these statements do not worry too much about why we chose them as we did. The reason will become apparent once the proof is done.

Also, we're not going to be doing the proofs in the order they are written above. Some of the proofs will be easier if we've got some of the others proved first.

Proof of 7

This is a very simple proof. To make the notation a little clearer let's define the function $f(x) = c$ then what we're being asked to prove is that $\lim\limits_{x \to a} f(x) = c$. So let's do that.

Let $\varepsilon > 0$ and we need to show that we can find a $\delta > 0$ so that

$$\left|f(x) - c\right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left|x - a\right| < \delta$$

The left inequality is trivially satisfied for any *x* however because we defined $f(x) = c$**. S**o simply choose $\delta > 0$ to be any number you want (you generally can't do this with these proofs). Then,

$$\left|f(x) - c\right| = \left|c - c\right| = 0 < \varepsilon$$

Proof of 1

There are several ways to prove this part. If you accept **3** And **7** then all you need to do is let $g(x) = c$ and then this is a direct result of **3** and **7.** However, we'd like to do a more rigorous mathematical proof. So here is that proof.

First, note that if $c = 0$ then $cf(x) = 0$ and so,

$$\lim_{x \to a}\left[0 f(x)\right] = \lim_{x \to a} 0 = 0 = 0 f(x)$$

The limit evaluation is a special case of **7** (with $c = 0$) which we just proved Therefore we know **1** is true for $c = 0$ and so we can assume that $c \ne 0$ for the remainder of this proof.

Let $\varepsilon > 0$ then because $\lim\limits_{x \to a} f(x) = K$ by the definition of the limit there is a $\delta_1 > 0$ such that,

$$\left|f(x) - K\right| < \frac{\varepsilon}{\left|c\right|} \qquad \text{whenever} \qquad 0 < \left|x - a\right| < \delta_1$$

Now choose $\delta = \delta_1$ and we need to show that

$$\left| cf\left(x\right) - cK \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

and we'll be done. So, assume that $0 < \left| x - a \right| < \delta$ and then,

$$\left| cf\left(x\right) - cK \right| = \left| c \right| \left| f\left(x\right) - K \right| < \left| c \right| \frac{\varepsilon}{\left| c \right|} = \varepsilon$$

Proof of 2

Note that we'll need something called the **triangle inequality** in this proof. The triangle inequality states that,

$$\left| a + b \right| \le \left| a \right| + \left| b \right|$$

Here's the actual proof.

We'll be doing this proof in two parts. First let's prove $\lim\limits_{x \to a} \left[f\left(x\right) + g\left(x\right) \right] = K + L$.

Let $\varepsilon > 0$ then because $\lim\limits_{x \to a} f\left(x\right) = K$ and $\lim\limits_{x \to a} g\left(x\right) = L$ there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that,

$$\left| f\left(x\right) - K \right| < \frac{\varepsilon}{2} \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta_1$$

$$\left| g\left(x\right) - L \right| < \frac{\varepsilon}{2} \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta_2$$

Now choose $\delta = \min\left\{ \delta_1, \delta_2 \right\}$. Then we need to show that

$$\left| f\left(x\right) + g\left(x\right) - \left(K + L\right) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < \left| x - a \right| < \delta$$

Assume that $0 < \left| x - a \right| < \delta$. We then have,

$$\begin{aligned} \left| f\left(x\right) + g\left(x\right) - \left(K + L\right) \right| &= \left| \left(f\left(x\right) - K \right) + \left(g\left(x\right) - L \right) \right| \\ &\le \left| f\left(x\right) - K \right| + \left| g\left(x\right) - L \right| \qquad \text{by the triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

In the third step we used the fact that, by our choice of δ, we also have $0 < \left| x - a \right| < \delta_1$ and $0 < \left| x - a \right| < \delta_2$ and so we can use the initial statements in our proof.

Next, we need to prove $\lim\limits_{x \to a} \left[f\left(x\right) - g\left(x\right) \right] = K - L$. We could do a similar proof as we did above for the sum of two functions. However, we might as well take advantage of the fact that we've proven this for a sum and that we've also proven **1.**

$$\begin{aligned}\lim_{x\to a}\left[f(x)-g(x)\right] &= \lim_{x\to a}\left[f(x)+(-1)g(x)\right] \\ &= \lim_{x\to a} f(x)+\lim_{x\to a}(-1)g(x) && \text{by first part of } \mathbf{2}. \\ &= \lim_{x\to a} f(x)+(-1)\lim_{x\to a} g(x) && \text{by } \mathbf{1}. \\ &= K+(-1)L \\ &= K-L\end{aligned}$$

Proof of 3

This one is a little tricky. First, let's note that because $\lim_{x\to a} f(x)=K$ and $\lim_{x\to a} g(x)=L$ we can use **2** and **7** to prove the following two limits.

$$\lim_{x\to a}\left[f(x)-K\right]=\lim_{x\to a} f(x)-\lim_{x\to a} K=K-K=0$$
$$\lim_{x\to a}\left[g(x)-L\right]=\lim_{x\to a} g(x)-\lim_{x\to a} L=L-L=0$$

Now, let $\varepsilon>0$. Then there is a $\delta_1>0$ and a $\delta_2>0$ such that,

$$\begin{aligned}\left|\left(f(x)-K\right)-0\right|&<\sqrt{\varepsilon} && \text{whenever} && 0<|x-a|<\delta_1 \\ \left|\left(g(x)-L\right)-0\right|&<\sqrt{\varepsilon} && \text{whenever} && 0<|x-a|<\delta_2\end{aligned}$$

Choose $\delta=\min\left\{\delta_1,\delta_2\right\}$. If $0<|x-a|<\delta$ we then get,

$$\begin{aligned}\left|\left[f(x)-K\right]\left[g(x)-L\right]-0\right| &= \left|f(x)-K\right|\left|g(x)-L\right| \\ &< \sqrt{\varepsilon}\sqrt{\varepsilon} \\ &= \varepsilon\end{aligned}$$

So, we've managed to prove that,

$$\lim_{x\to a}\left[f(x)-K\right]\left[g(x)-L\right]=0$$

This apparently has nothing to do with what we actually want to prove, but as you'll see in a bit it is needed.

Before launching into the actual proof of **3** let's do a little Algebra. First, expand the following product.

$$\left[f(x)-K\right]\left[g(x)-L\right]=f(x)g(x)-Lf(x)-Kg(x)+KL$$

Rearranging this gives the following way to write the product of the two functions.

$$f(x)g(x)=\left[f(x)-K\right]\left[g(x)-L\right]+Lf(x)+Kg(x)-KL$$

With this we can now proceed with the proof of **3**.

$$\begin{aligned}\lim_{x\to a} f(x)g(x) &= \lim_{x\to a}\Big[\big[f(x)-K\big]\big[g(x)-L\big]+Lf(x)+Kg(x)-KL\Big]\\ &= \lim_{x\to a}\big[f(x)-K\big]\big[g(x)-L\big]+\lim_{x\to a} Lf(x)+\lim_{x\to a} Kg(x)-\lim_{x\to a} KL\\ &= 0+\lim_{x\to a} Lf(x)+\lim_{x\to a} Kg(x)-\lim_{x\to a} KL\\ &= LK+KL-KL\\ &= LK\end{aligned}$$

Fairly simple proof really, once you see all the steps that you have to take before you even start. The second step made multiple uses of property **2.** In the third step we used the limit we initially proved. In the fourth step we used properties **1** and **7.** Finally, we just did some simplification.

Proof of 4

This one is also a little tricky. First, we'll start of by proving,

$$\lim_{x\to a}\frac{1}{g(x)}=\frac{1}{L}$$

Let $\varepsilon>0$. We'll not need this right away, but these proofs always start off with this statement. Now, because $\lim_{x\to a} g(x)=L$ there is a $\delta_1>0$ such that,

$$\left|g(x)-L\right|<\frac{|L|}{2} \qquad \text{whenever} \qquad 0<|x-a|<\delta_1$$

Now, assuming that $0<|x-a|<\delta_1$ we have,

$$\begin{aligned}|L| &= \left|L-g(x)+g(x)\right| && \text{just adding zero to } L\\ &\le \left|L-g(x)\right|+\left|g(x)\right| && \text{using the triangle inequality}\\ &= \left|g(x)-L\right|+\left|g(x)\right| && \left|L-g(x)\right|=\left|g(x)-L\right|\\ &< \frac{|L|}{2}+\left|g(x)\right| && \text{assuming that } 0<|x-a|<\delta_1\end{aligned}$$

Rearranging this gives,

$$|L|<\frac{|L|}{2}+\left|g(x)\right| \qquad \Rightarrow \qquad \frac{|L|}{2}<\left|g(x)\right| \qquad \Rightarrow \qquad \frac{1}{\left|g(x)\right|}<\frac{2}{|L|}$$

Now, there is also a $\delta_2>0$ such that,

$$\left|g(x)-L\right|<\frac{|L|^2}{2}\varepsilon \qquad \text{whenever} \qquad 0<|x-a|<\delta_2$$

Choose $\delta=\min\left\{\delta_1,\delta_2\right\}$. If $0<|x-a|<\delta$ we have,

$$\begin{aligned}
\left|\frac{1}{g(x)}-\frac{1}{L}\right| &= \left|\frac{L-g(x)}{Lg(x)}\right| && \text{common denominators}\\
&= \frac{1}{|Lg(x)|}\left|L-g(x)\right| && \text{doing a little rewriting}\\
&= \frac{1}{|L|}\frac{1}{|g(x)|}\left|g(x)-L\right| && \text{doing a little more rewriting}\\
&< \frac{1}{|L|}\frac{2}{|L|}\left|g(x)-L\right| && \text{assuming that } 0<|x-a|<\delta\le\delta_1\\
&< \frac{2}{|L|^2}\frac{|L|^2}{2}\varepsilon && \text{assuming that } 0<|x-a|<\delta\le\delta_2\\
&= \varepsilon
\end{aligned}$$

Now that we've proven $\lim\limits_{x\to a}\dfrac{1}{g(x)}=\dfrac{1}{L}$ the more general fact is easy.

$$\begin{aligned}
\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] &= \lim_{x\to a}\left[f(x)\frac{1}{g(x)}\right]\\
&= \lim_{x\to a} f(x)\lim_{x\to a}\frac{1}{g(x)} && \text{using property } \mathbf{3}.\\
&= K\frac{1}{L}=\frac{K}{L}
\end{aligned}$$

Proof of 5. for *n* an integer

As noted we're only going to prove **5** for integer exponents. This will also involve proof by induction so if you aren't familiar with induction proofs you can skip this proof.

So, we're going to prove,

$$\lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a}f(x)\right]^n=K^n, \qquad n\ge 2,\ n \text{ is an integer.}$$

For $n=2$ we have nothing more than a special case of property **3.**

$$\lim_{x\to a}\left[f(x)\right]^2=\lim_{x\to a}f(x)f(x)=\lim_{x\to a}f(x)\lim_{x\to a}f(x)=KK=K^2$$

So, **5** is proven for $n=2$. Now assume that **5** is true for $n-1$, or $\lim\limits_{x\to a}\left[f(x)\right]^{n-1}=K^{n-1}$. Then, again using property **3** we have,

$$\begin{aligned}\lim_{x\to a}\left[f(x)\right]^n &= \lim_{x\to a}\left(\left[f(x)\right]^{n-1} f(x)\right)\\ &= \lim_{x\to a}\left[f(x)\right]^{n-1}\lim_{x\to a} f(x)\\ &= K^{n-1}K\\ &= K^n\end{aligned}$$

Proof of 6

As pointed out in the **Limit Properties** section this is nothing more than a special case of the full version of **5** and the proof is given there and so is the proof is not give here.

Proof of 8

This is a simple proof. If we define $f(x) = x$ to make the notation a little easier, we're being asked to prove that $\lim\limits_{x\to a} f(x) = a$.

Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then, if $0 < |x-a| < \delta = \varepsilon$ we have,

$$|f(x) - a| = |x-a| < \delta = \varepsilon$$

So, we've proved that $\lim\limits_{x\to a} x = a$.

Proof of 9

This is just a special case of property **5** with $f(x) = x$ and so we won't prove it here.

Facts, Infinite Limits

Given the functions $f(x)$ and $g(x)$ suppose we have,

$$\lim_{x\to c} f(x) = \infty \qquad\qquad \lim_{x\to c} g(x) = L$$

for some real numbers *c* and *L*. Then,

1. $\lim\limits_{x\to c}\left[f(x) \pm g(x)\right] = \infty$

2. If $L > 0$ then $\lim\limits_{x\to c}\left[f(x)g(x)\right] = \infty$

3. If $L < 0$ then $\lim\limits_{x\to c}\left[f(x)g(x)\right] = -\infty$

4. $\lim\limits_{x \to c} \dfrac{g(x)}{f(x)} = 0$

Partial Proof of 1

We will prove $\lim\limits_{x \to c} \left[f(x) + g(x) \right] = \infty$ here. The proof of $\lim\limits_{x \to c} \left[f(x) - g(x) \right] = \infty$ is nearly identical and is left to you.

Let $M > 0$ then because we know $\lim\limits_{x \to c} f(x) = \infty$ there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ we have,

$$f(x) > M - L + 1$$

Also, because we know $\lim\limits_{x \to c} g(x) = L$ there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ we have,

$$0 < |g(x) - L| < 1 \quad \rightarrow \quad -1 < g(x) - L < 1 \quad \rightarrow \quad L - 1 < g(x) < L + 1$$

Now, let $\delta = \min\{\delta_1, \delta_2\}$ and so if $0 < |x - c| < \delta$ we know from the above statements that we will have both,

$$f(x) > M - L + 1 \qquad\qquad g(x) > L - 1$$

This gives us,

$$\begin{aligned} f(x) + g(x) &> M - L + 1 + L - 1 \\ &= M \qquad\qquad \Rightarrow \qquad\qquad f(x) + g(x) > M \end{aligned}$$

So, we've proved that $\lim\limits_{x \to c} \left[f(x) + g(x) \right] = \infty$.

Proof of 2

Let $M > 0$ then because we know $\lim\limits_{x \to c} f(x) = \infty$ there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ we have,

$$f(x) > \tfrac{2M}{L}$$

Also, because we know $\lim\limits_{x \to c} g(x) = L$ there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ we have,

$$0 < |g(x) - L| < \tfrac{L}{2} \quad \rightarrow \quad -\tfrac{L}{2} < g(x) - L < \tfrac{L}{2} \quad \rightarrow \quad \tfrac{L}{2} < g(x) < \tfrac{3L}{2}$$

Note that because we know that $L > 0$ choosing $\frac{L}{2}$ in the first inequality above is a valid choice because it will also be positive as required by the definition of the limit.

Now, let $\delta = \min\left\{\delta_1, \delta_2\right\}$ and so if $0 < \left|x - c\right| < \delta$ we know from the above statements that we will have both,

$$f\left(x\right) > \tfrac{2M}{L} \qquad\qquad g\left(x\right) > \tfrac{L}{2}$$

This gives us,

$$\begin{aligned} f\left(x\right)g\left(x\right) &> \left(\tfrac{2M}{L}\right)\left(\tfrac{L}{2}\right) \\ &= M \qquad\qquad \Rightarrow \qquad\qquad f\left(x\right)g\left(x\right) > M \end{aligned}$$

So, we've proved that $\lim\limits_{x \to c} f\left(x\right)g\left(x\right) = \infty$.

Proof of 3

Let $M > 0$ then because we know $\lim\limits_{x \to c} f\left(x\right) = \infty$ there exists a $\delta_1 > 0$ such that if $0 < \left|x - c\right| < \delta_1$ we have,

$$f\left(x\right) > \tfrac{-2M}{L}$$

Note that because $L < 0$ in the case we will have $\tfrac{-2M}{L} > 0$ here.

Next, because we know $\lim\limits_{x \to c} g\left(x\right) = L$ there exists a $\delta_2 > 0$ such that if $0 < \left|x - c\right| < \delta_2$ we have,

$$0 < \left|g\left(x\right) - L\right| < -\tfrac{L}{2} \qquad \to \qquad \tfrac{L}{2} < g\left(x\right) - L < -\tfrac{L}{2} \qquad \to \qquad \tfrac{3L}{2} < g\left(x\right) < \tfrac{L}{2}$$

Again, because we know that $L < 0$ we will have $-\tfrac{L}{2} > 0$. Also, for reasons that will shortly be apparent, multiply the final inequality by a minus sign to get,

$$-\tfrac{L}{2} < -g\left(x\right) < -\tfrac{3L}{2}$$

Now, let $\delta = \min\left\{\delta_1, \delta_2\right\}$ and so if $0 < \left|x - c\right| < \delta$ we know from the above statements that we will have both,

$$f\left(x\right) > \tfrac{-2M}{L} \qquad\qquad -g\left(x\right) > -\tfrac{L}{2}$$

This gives us,

$$\begin{aligned} -f\left(x\right)g\left(x\right) &= f\left(x\right)\left[-g\left(x\right)\right] \\ &> \left(-\tfrac{2M}{L}\right)\left(-\tfrac{L}{2}\right) \\ &= M \qquad\qquad \Rightarrow \qquad\qquad -f\left(x\right)g\left(x\right) > M \end{aligned}$$

This may seem to not be what we needed however multiplying this by a minus sign gives,

$$f\left(x\right)g\left(x\right) < -M$$

and because we originally chose $M > 0$ we have now proven that $\lim\limits_{x \to c} f\left(x\right)g\left(x\right) = -\infty$.

Proof of 4

We'll need to do this in three cases. Let's start with the easiest case.

Case 1 : $L = 0$

Let $\varepsilon > 0$ then because we know $\lim\limits_{x \to c} f(x) = \infty$ there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ we have,

$$f(x) > \tfrac{1}{\sqrt{\varepsilon}} > 0$$

Next, because we know $\lim\limits_{x \to c} g(x) = 0$ there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ we have,

$$0 < |g(x)| < \sqrt{\varepsilon}$$

Now, let $\delta = \min\{\delta_1, \delta_2\}$ and so if $0 < |x - c| < \delta$ we know from the above statements that we will have both,

$$f(x) > \tfrac{1}{\sqrt{\varepsilon}} \qquad\qquad |g(x)| < \sqrt{\varepsilon}$$

This gives us,

$$\left| \frac{g(x)}{f(x)} \right| = \frac{|g(x)|}{f(x)} < \frac{\sqrt{\varepsilon}}{f(x)} < \frac{\sqrt{\varepsilon}}{1/\sqrt{\varepsilon}} = \varepsilon$$

In the second step we could remove the absolute value bars from $f(x)$ because we know it is positive.

So, we proved that $\lim\limits_{x \to c} \dfrac{g(x)}{f(x)} = 0$ if $L = 0$.

Case 2 : $L > 0$

Let $\varepsilon > 0$ then because we know $\lim\limits_{x \to c} f(x) = \infty$ there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ we have,

$$f(x) > \tfrac{3L}{2\varepsilon} > 0$$

Next, because we know $\lim\limits_{x \to c} g(x) = L$ there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ we have,

$$0 < |g(x) - L| < \tfrac{L}{2} \quad \to \quad -\tfrac{L}{2} < g(x) - L < \tfrac{L}{2} \quad \to \quad \tfrac{L}{2} < g(x) < \tfrac{3L}{2}$$

Also, because we are assuming that $L > 0$ it is safe to assume that for $0 < |x - c| < \delta_2$ we have $g(x) > 0$.

Now, let $\delta = \min\left\{\delta_1, \delta_2\right\}$ and so if $0 < |x - c| < \delta$ we know from the above statements that we will have both,

$$f\left(x\right) > \tfrac{3L}{2\varepsilon} \qquad\qquad g\left(x\right) < \tfrac{3L}{2}$$

This gives us,

$$\left|\frac{g\left(x\right)}{f\left(x\right)}\right| = \frac{g\left(x\right)}{f\left(x\right)} < \frac{3L/2}{\left|f\left(x\right)\right|} < \frac{3L/2}{3L/2\varepsilon} = \varepsilon$$

In the second step we could remove the absolute value bars because we know or can safely assume (as noted above) that both functions were positive.

So, we proved that $\displaystyle\lim_{x \to c} \frac{g\left(x\right)}{f\left(x\right)} = 0$ if $L > 0$.

Case 3 : $L < 0$.
Let $\varepsilon > 0$ then because we know $\displaystyle\lim_{x \to c} f\left(x\right) = \infty$ there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$ we have,

$$f\left(x\right) > \tfrac{-3L}{2\varepsilon} > 0$$

Next, because we know $\displaystyle\lim_{x \to c} g\left(x\right) = L$ there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$ we have,

$$0 < \left|g\left(x\right) - L\right| < -\tfrac{L}{2} \qquad \to \qquad \tfrac{L}{2} < g\left(x\right) - L < -\tfrac{L}{2} \qquad \to \qquad \tfrac{3L}{2} < g\left(x\right) < \tfrac{L}{2}$$

Next, multiply this be a negative sign to get,

$$-\tfrac{L}{2} < -g\left(x\right) < -\tfrac{3L}{2}$$

Also, because we are assuming that $L < 0$ it is safe to assume that for $0 < |x - c| < \delta_2$ we have $g\left(x\right) < 0$.

Now, let $\delta = \min\left\{\delta_1, \delta_2\right\}$ and so if $0 < |x - c| < \delta$ we know from the above statements that we will have both,

$$f\left(x\right) > \tfrac{-3L}{2\varepsilon} \qquad\qquad -g\left(x\right) < \tfrac{-3L}{2}$$

This gives us,

$$\left|\frac{g\left(x\right)}{f\left(x\right)}\right| = \frac{-g\left(x\right)}{f\left(x\right)} < \frac{-3L/2}{\left|f\left(x\right)\right|} < \frac{-3L/2}{-3L/2\varepsilon} = \varepsilon$$

In the second step we could remove the absolute value bars by adding in the negative because we know that $f\left(x\right) > 0$ and can safely assume that $g\left(x\right) < 0$ (as noted above).

So, we proved that $\displaystyle \lim_{x \to c} \frac{g\left(x\right)}{f\left(x\right)} = 0$ if $L < 0$.

Fact 1, Limits At Infinity, Part 1

1. If r is a positive rational number and c is any real number then,

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

2. If r is a positive rational number, c is any real number and x^r is defined for $x < 0$ then,

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0$$

Proof of 1

This is actually a fairly simple proof but we'll need to do three separate cases.

Case 1 : Assume that $c > 0$. Next, let $\varepsilon > 0$ and define

$$M = \sqrt[r]{\frac{c}{\varepsilon}}$$

Note that because c and ε are both positive we know that this root will exist. Now, assume that we have

$$x > M = \sqrt[r]{\frac{c}{\varepsilon}}$$

Given this assumption we have,

$$\begin{aligned} x &> \sqrt[r]{\frac{c}{\varepsilon}} & & \\ x^r &> \frac{c}{\varepsilon} & & \text{get rid of the root} \\ \frac{c}{x^r} &< \varepsilon & & \text{rearrange things a little} \\ \left| \frac{c}{x^r} - 0 \right| &< \varepsilon & & \text{everything is positive so we can add absolute value bars} \end{aligned}$$

So, provided $c > 0$ we've proven that $\lim\limits_{x\to\infty} \dfrac{c}{x^r} = 0$.

Case 2 : Assume that $c = 0$. Here all we need to do is the following,

$$\lim_{x\to\infty} \frac{c}{x^r} = \lim_{x\to\infty} \frac{0}{x^r} = \lim_{x\to\infty} 0 = 0$$

Case 3 : Finally, assume that $c < 0$. In this case we can then write $c = -k$ where $k > 0$. Then using Case 1 and the fact that we can factor constants out of a limit we get,

$$\lim_{x\to\infty} \frac{c}{x^r} = \lim_{x\to\infty} \frac{-k}{x^r} = -\lim_{x\to\infty} \frac{k}{x^r} = -0 = 0$$

Proof of 2

This is very similar to the proof of **1** so we'll just do the first case (as it's the hardest) and leave the other two cases up to you to prove.

Case 1 : Assume that $c > 0$. Next, let $\varepsilon > 0$ and define

$$N = -\sqrt[r]{\frac{c}{\varepsilon}}$$

Note that because *c* and ε are both positive we know that this root will exist. Now, assume that we have

$$x < N = -\sqrt[r]{\frac{c}{\varepsilon}}$$

Note that this assumption also tells us that *x* will be negative. Give this assumption we have,

$$\begin{aligned}
x &< -\sqrt[r]{\frac{c}{\varepsilon}} \\
|x| &> \left|\sqrt[r]{\frac{c}{\varepsilon}}\right| && \text{take absolute value of both sides} \\
|x^r| &> \left|\frac{c}{\varepsilon}\right| && \text{get rid of the root} \\
\left|\frac{c}{x^r}\right| &< |\varepsilon| = \varepsilon && \text{rearrange things a little and use the fact that } \varepsilon > 0 \\
\left|\frac{c}{x^r} - 0\right| &< \varepsilon && \text{rewrite things a little}
\end{aligned}$$

So, provided $c > 0$ we've proven that $\lim\limits_{x \to \infty} \dfrac{c}{x^r} = 0$. Note that the main difference here is that we need to take the absolute value first to deal with the minus sign. Because both sides are negative we know that when we take the absolute value of both sides the direction of the inequality will have to switch as well.

Case 2, Case 3 : As noted above these are identical to the proof of the corresponding cases in the first proof and so are omitted here.

Fact 2, Limits At Infinity, Part I

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree *n* (*i.e.* $a_n \neq 0$) then,

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n \qquad\qquad \lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} a_n x^n$$

Proof of $\lim\limits_{x \to \infty} p(x) = \lim\limits_{x \to \infty} a_n x^n$

We're going to prove this in an identical fashion to the problems that we worked in this section involving polynomials. We'll first factor out $a_n x^n$ from the polynomial and then make a giant use of **Fact 1** (which we just proved above) and the basic properties of limits.

$$\begin{aligned} \lim_{x \to \infty} p(x) &= \lim_{x \to \infty} \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \right) \\ &= \lim_{x \to \infty} \left[a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) \right] \end{aligned}$$

Now, clearly the limit of the second term is one and the limit of the first term will be either ∞ or $-\infty$ depending upon the sign of a_n. Therefore by the **Facts** from the Infinite Limits section we can see that the limit of the whole polynomial will be the same as the limit of the first term or,

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \right) = \lim_{x \to \infty} a_n x^n$$

Proof of $\lim\limits_{x \to -\infty} p(x) = \lim\limits_{x \to -\infty} a_n x^n$

The proof of this part is literally identical to the proof of the first part, with the exception that all ∞'s are changed to $-\infty$, and so is omitted here.

Fact 2, Continuity

If $f(x)$ is continuous at $x = b$ and $\lim_{x \to a} g(x) = b$ then,

$$\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

Proof

Let $\varepsilon > 0$ then we need to show that there is a $\delta > 0$ such that,

$$\left| f\left(g(x)\right) - f(b) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

Let's start with the fact that $f(x)$ is continuous at $x = b$. Recall that this means that $\lim_{x \to b} f(x) = f(b)$ and so there must be a $\delta_1 > 0$ so that,

$$\left| f(x) - f(b) \right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - b| < \delta_1$$

Now, let's recall that $\lim_{x \to a} g(x) = b$. This means that there must be a $\delta > 0$ so that,

$$\left| g(x) - b \right| < \delta_1 \qquad \text{whenever} \qquad 0 < |x - a| < \delta$$

But all this means that we're done.

Let's summarize up. First assume that $0 < |x - a| < \delta$. This then tells us that,

$$\left| g(x) - b \right| < \delta_1$$

But, we also know that if $0 < |x - b| < \delta_1$ then we must also have $\left| f(x) - f(b) \right| < \varepsilon$. What this is telling us is that if a number is within a distance of δ_1 of *b* then we can plug that number into $f(x)$ and we'll be within a distance of ε of $f(b)$.

So, $\left| g(x) - b \right| < \delta_1$ is telling us that $g(x)$ is within a distance of δ_1 of *b* and so if we plug it into $f(x)$ we'll get,

$$\left| f\left(g(x)\right) - f(b) \right| < \varepsilon$$

and this is exactly what we wanted to show.

Section 7-2 : Proof of Various Derivative Properties

In this section we're going to prove many of the various derivative facts, formulas and/or properties that we encountered in the early part of the **Derivatives** chapter. Not all of them will be proved here and some will only be proved for special cases, but at least you'll see that some of them aren't just pulled out of the air.

Theorem, from Definition of Derivative

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

Proof

Because $f(x)$ is differentiable at $x = a$ we know that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. We'll need this in a bit.

If we next assume that $x \neq a$ we can write the following,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

Then basic properties of limits tells us that we have,

$$\begin{aligned} \lim_{x \to a}\left(f(x) - f(a)\right) &= \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a}(x - a)\right] \\ &= \lim_{x \to a}\frac{f(x) - f(a)}{x - a} \;\; \lim_{x \to a}(x - a) \end{aligned}$$

The first limit on the right is just $f'(a)$ as we noted above and the second limit is clearly zero and so,

$$\lim_{x \to a}\left(f(x) - f(a)\right) = f'(a) \cdot 0 = 0$$

Okay, we've managed to prove that $\lim_{x \to a}\left(f(x) - f(a)\right) = 0$. But just how does this help us to prove that $f(x)$ is continuous at $x = a$?

Let's start with the following.

$$\lim_{x \to a} f(x) = \lim_{x \to a}\left[f(x) + f(a) - f(a)\right]$$

Note that we've just added in zero on the right side. A little rewriting and the use of limit properties gives,

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[f(a) + f(x) - f(a)\right]$$
$$= \lim_{x\to a} f(a) + \lim_{x\to a}\left[f(x) - f(a)\right]$$

Now, we just proved above that $\lim_{x\to a}\left(f(x) - f(a)\right) = 0$ and because $f(a)$ is a constant we also know that $\lim_{x\to a} f(a) = f(a)$ and so this becomes,

$$\lim_{x\to a} f(x) = \lim_{x\to a} f(a) + 0 = f(a)$$

Or, in other words, $\lim_{x\to a} f(x) = f(a)$ but this is exactly what it means for $f(x)$ is continuous at $x = a$ and so we're done.

Proof of Sum/Difference of Two Functions : $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

This is easy enough to prove using the definition of the derivative. We'll start with the sum of two functions. First plug the sum into the definition of the derivative and rewrite the numerator a little.

$$\left(f(x) + g(x)\right)' = \lim_{h\to 0}\frac{f(x+h) + g(x+h) - \left(f(x) + g(x)\right)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

Now, break up the fraction into two pieces and recall that the limit of a sum is the sum of the limits. Using this fact we see that we end up with the definition of the derivative for each of the two functions.

$$\left(f(x) + g(x)\right)' = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

The proof of the difference of two functions in nearly identical so we'll give it here without any explanation.

$$\left(f(x) - g(x)\right)' = \lim_{h\to 0}\frac{f(x+h) - g(x+h) - \left(f(x) - g(x)\right)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h) - f(x) - \left(g(x+h) - g(x)\right)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) - g'(x)$$

Proof of Constant Times a Function : $\left(cf\left(x\right)\right)' = cf'\left(x\right)$

This is property is very easy to prove using the definition provided you recall that we can factor a constant out of a limit. Here's the work for this property.

$$\left(cf\left(x\right)\right)' = \lim_{h\to 0}\frac{cf\left(x+h\right)-cf\left(x\right)}{h} = c\lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h} = cf'\left(x\right)$$

Proof of the Derivative of a Constant : $\dfrac{d}{dx}\left(c\right) = 0$

This is very easy to prove using the definition of the derivative so define $f\left(x\right)=c$ and the use the definition of the derivative.

$$f'\left(x\right) = \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

Power Rule : $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$

There are actually three proofs that we can give here and we're going to go through all three here so you can see all of them. However, having said that, for the first two we will need to restrict *n* to be a positive integer. At the time that the Power Rule was introduced only enough information has been given to allow the proof for only integers. So, the first two proofs are really to be read at that point.

The third proof will work for any real number *n*. However, it does assume that you've read most of the Derivatives chapter and so should only be read after you've gone through the whole chapter.

Proof 1

In this case as noted above we need to assume that *n* is a positive integer. We'll use the definition of the derivative and the Binomial Theorem in this theorem. The Binomial Theorem tells us that,

$$\begin{aligned}\left(a+b\right)^n &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \\ &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \frac{n\left(n-1\right)}{2!}a^{n-2}b^2 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n\end{aligned}$$

where,

$$\binom{n}{k} = \frac{n!}{k!\left(n-k\right)!}$$

are called the binomial coefficients and $n! = n(n-1)(n-2)\cdots(2)(1)$ is the factorial.

So, let's go through the details of this proof. First, plug $f(x) = x^n$ into the definition of the derivative and use the Binomial Theorem to expand out the first term.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right) - x^n}{h} \end{aligned}$$

Now, notice that we can cancel an x^n and then each term in the numerator will have an *h* in them that can be factored out and then canceled against the *h* in the denominator. At this point we can evaluate the limit.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \\ &= nx^{n-1} \end{aligned}$$

Proof 2

For this proof we'll again need to restrict *n* to be a positive integer. In this case if we define $f(x) = x^n$ we know from the alternate limit form of the definition of the derivative that the derivative $f'(a)$ is given by,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

Now we have the following formula,

$$x^n - a^n = (x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}\right)$$

You can verify this if you'd like by simply multiplying the two factors together. Also, notice that there are a total of *n* terms in the second factor (this will be important in a bit).

If we plug this into the formula for the derivative we see that we can cancel the $x - a$ and then compute the limit.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}\right)}{x - a} \\ &= \lim_{x \to a} x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1} \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-3}a^2 + a^{n-2}a + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

After combining the exponents in each term we can see that we get the same term. So, then recalling that there are *n* terms in second factor we can see that we get what we claimed it would be.

To completely finish this off we simply replace the *a* with an *x* to get,

$$f'(x) = nx^{n-1}$$

Proof 3

In this proof we no longer need to restrict *n* to be a positive integer. It can now be any real number. However, this proof also assumes that you've read all the way through the Derivative chapter. In particular it needs both **Implicit Differentiation** and **Logarithmic Differentiation**. If you've not read, and understand, these sections then this proof will not make any sense to you.

So, to get set up for logarithmic differentiation let's first define $y = x^n$ then take the log of both sides, simplify the right side using logarithm properties and then differentiate using implicit differentiation.

$$\begin{aligned} \ln y &= \ln x^n \\ \ln y &= n \ln x \\ \frac{y'}{y} &= n\frac{1}{x} \end{aligned}$$

Finally, all we need to do is solve for y' and then substitute in for *y*.

$$y' = y\frac{n}{x} = x^n\left(\frac{n}{x}\right) = nx^{n-1}$$

Before moving onto the next proof, let's notice that in all three proofs we did require that the exponent, *n*, be a number (integer in the first two, any real number in the third). In the first proof we couldn't have used the Binomial Theorem if the exponent wasn't a positive integer. In the second proof we couldn't have factored $x^n - a^n$ if the exponent hadn't been a positive integer. Finally, in the third proof we would have gotten a much different derivative if *n* had not been a constant.

This is important because people will often misuse the power rule and use it even when the exponent is not a number and/or the base is not a variable.

Product Rule : $(f\,g)' = f'\,g + f\,g'$

As with the Power Rule above, the Product Rule can be proved either by using the definition of the derivative or it can be proved using Logarithmic Differentiation. We'll show both proofs here.

Proof 1

This proof can be a little tricky when you first see it so let's be a little careful here. We'll first use the definition of the derivative on the product.

$$\left(f\,g\right)' = \lim_{h\to 0}\frac{f\left(x+h\right)g\left(x+h\right)-f\left(x\right)g\left(x\right)}{h}$$

On the surface this appears to do nothing for us. We'll first need to manipulate things a little to get the proof going. What we'll do is subtract out and add in $f\left(x+h\right)g\left(x\right)$ to the numerator. Note that we're really just adding in a zero here since these two terms will cancel. This will give us,

$$\left(f\,g\right)' = \lim_{h\to 0}\frac{f\left(x+h\right)g\left(x+h\right)-f\left(x+h\right)g\left(x\right)+f\left(x+h\right)g\left(x\right)-f\left(x\right)g\left(x\right)}{h}$$

Notice that we added the two terms into the middle of the numerator. As written we can break up the limit into two pieces. From the first piece we can factor a $f\left(x+h\right)$ out and we can factor a $g\left(x\right)$ out of the second piece. Doing this gives,

$$\begin{aligned}\left(f\,g\right)' &= \lim_{h\to 0}\frac{f\left(x+h\right)\left(g\left(x+h\right)-g\left(x\right)\right)}{h}+\lim_{h\to 0}\frac{g\left(x\right)\left(f\left(x+h\right)-f\left(x\right)\right)}{h}\\ &= \lim_{h\to 0}f\left(x+h\right)\frac{g\left(x+h\right)-g\left(x\right)}{h}+\lim_{h\to 0}g\left(x\right)\frac{f\left(x+h\right)-f\left(x\right)}{h}\end{aligned}$$

At this point we can use limit properties to write,

$$\left(f\,g\right)' = \left(\lim_{h\to 0}f\left(x+h\right)\right)\left(\lim_{h\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}\right)+\left(\lim_{h\to 0}g\left(x\right)\right)\left(\lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}\right)$$

The individual limits in here are,

$$\lim_{h\to 0}\frac{g\left(x+h\right)-g\left(x\right)}{h}=g'\left(x\right) \qquad\qquad \lim_{h\to 0}g\left(x\right)=g\left(x\right)$$

$$\lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h}=f'\left(x\right) \qquad\qquad \lim_{h\to 0}f\left(x+h\right)=f\left(x\right)$$

The two limits on the left are nothing more than the definition the derivative for $g\left(x\right)$ and $f\left(x\right)$ respectively. The upper limit on the right seems a little tricky but remember that the limit of a constant is just the constant. In this case since the limit is only concerned with allowing *h* to go to zero. The key here is to recognize that changing *h* will not change *x* and so as far as this limit is concerned $g\left(x\right)$ is a constant. Note that the function is probably not a constant, however as far as the limit is concerned the function can be treated as a constant. We get the lower limit on the right we get simply by plugging $h=0$ into the function

Plugging all these into the last step gives us,

$$(f\,g)' = f(x)\,g'(x) + g(x)\,f'(x)$$

Proof 2

This is a much quicker proof but does presuppose that you've read and understood the **Implicit Differentiation** and **Logarithmic Differentiation** sections. If you haven't then this proof will not make a lot of sense to you.

First write call the product *y* and take the log of both sides and use a property of logarithms on the right side.

$$y = f(x)\,g(x)$$
$$\ln(y) = \ln\big(f(x)\,g(x)\big) = \ln f(x) + \ln g(x)$$

Next, we take the derivative of both sides and solve for y'.

$$\frac{y'}{y} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \qquad \Rightarrow \qquad y' = y\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right)$$

Finally, all we need to do is plug in for *y* and then multiply this through the parenthesis and we get the Product Rule.

$$y' = f(x)\,g(x)\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right) \qquad \Rightarrow \qquad (fg)' = g(x)\,f'(x) + f(x)\,g'(x)$$

Quotient Rule : $\left(\dfrac{f}{g}\right)' = \dfrac{f'\,g - f\,g'}{g^2}$.

Again, we can do this using the definition of the derivative or with Logarithmic Definition.

Proof 1

First plug the quotient into the definition of the derivative and rewrite the quotient a little.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\,\frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}$$

To make our life a little easier we moved the *h* in the denominator of the first step out to the front as a $\frac{1}{h}$. We also wrote the numerator as a single rational expression. This step is required to make this proof work.

Now, for the next step will need to subtract out and add in $f(x)g(x)$ to the numerator.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{h}\,\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

The next step is to rewrite things a little,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\,\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}$$

Note that all we did was interchange the two denominators. Since we are multiplying the fractions we can do this.

Next, the larger fraction can be broken up as follows.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(\frac{f(x+h)g(x) - f(x)g(x)}{h} + \frac{f(x)g(x) - f(x)g(x+h)}{h}\right)$$

In the first fraction we will factor a $g(x)$ out and in the second we will factor a $-f(x)$ out. This gives,

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\left(g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h}\right)$$

We can now use the basic properties of limits to write this as,

$$\left(\frac{f}{g}\right)' = \frac{1}{\lim\limits_{h\to 0} g(x+h)\lim\limits_{h\to 0} g(x)}\left(\left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\right) - \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\right)\right)$$

The individual limits are,

$$\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = g'(x) \qquad \lim_{h\to 0} g(x+h) = g(x) \qquad \lim_{h\to 0} g(x) = g(x)$$

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = f'(x) \qquad \lim_{h\to 0} f(x) = f(x)$$

The first two limits in each row are nothing more than the definition the derivative for $g(x)$ and $f(x)$ respectively. The middle limit in the top row we get simply by plugging in $h = 0$. The final limit in each row may seem a little tricky. Recall that the limit of a constant is just the constant. Well since the limit is only concerned with allowing *h* to go to zero as far as its concerned $g(x)$ and $f(x)$ are constants since changing *h* will not change *x*. Note that the function is probably not a constant, however as far as the limit is concerned the function can be treated as a constant.

Plugging in the limits and doing some rearranging gives,

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \frac{1}{g(x)\,g(x)}\left(g(x)\,f'(x) - f(x)\,g'(x)\right)\\ &= \frac{f'\,g - f\,g'}{g^2}\end{aligned}$$

There's the quotient rule.

Proof 2

Now let's do the proof using Logarithmic Differentiation. We'll first call the quotient *y*, take the log of both sides and use a property of logs on the right side.

$$\begin{aligned} y &= \frac{f(x)}{g(x)}\\ \ln y &= \ln\left(\frac{f(x)}{g(x)}\right) = \ln f(x) - \ln g(x)\end{aligned}$$

Next, we take the derivative of both sides and solve for y'.

$$\frac{y'}{y} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \qquad \Rightarrow \qquad y' = y\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right)$$

Next, plug in *y* and do some simplification to get the quotient rule.

$$\begin{aligned} y' &= \frac{f(x)}{g(x)}\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right)\\ &= \frac{f'(x)}{g(x)} - \frac{g'(x)\,f(x)}{\left(g(x)\right)^2}\\ &= \frac{f'(x)\,g(x)}{\left(g(x)\right)^2} - \frac{f(x)\,g'(x)}{\left(g(x)\right)^2} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}\end{aligned}$$

Chain Rule

If $f(x)$ and $g(x)$ are both differentiable functions and we define $F(x) = (f \circ g)(x)$ then the derivative of *F(x)* is $F'(x) = f'(g(x))\ g'(x)$.

Proof

We'll start off the proof by defining $u = g(x)$ and noticing that in terms of this definition what we're being asked to prove is,

$$\frac{d}{dx}\left[f(u)\right] = f'(u)\frac{du}{dx}$$

Let's take a look at the derivative of $u(x)$ (again, remember we've defined $u = g(x)$ and so *u* really is a function of *x*) which we know exists because we are assuming that $g(x)$ is differentiable. By definition we have,

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

Note as well that,

$$\lim_{h \to 0}\left(\frac{u(x+h) - u(x)}{h} - u'(x)\right) = \lim_{h \to 0}\frac{u(x+h) - u(x)}{h} - \lim_{h \to 0} u'(x) = u'(x) - u'(x) = 0$$

Now, define,

$$v(h) = \begin{cases} \dfrac{u(x+h) - u(x)}{h} - u'(x) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

and notice that $\lim_{h \to 0} v(h) = 0 = v(0)$ and so $v(h)$ is continuous at $h = 0$

Now if we assume that $h \neq 0$ we can rewrite the definition of $v(h)$ to get,

$$u(x+h) = u(x) + h\left(v(h) + u'(x)\right) \qquad (1)$$

Now, notice that (1) is in fact valid even if we let $h = 0$ and so is valid for any value of *h*.

Next, since we also know that $f(x)$ is differentiable we can do something similar. However, we're going to use a different set of letters/variables here for reasons that will be apparent in a bit. So, define,

$$w(k) = \begin{cases} \dfrac{f(z+k) - f(z)}{k} - f'(z) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

we can go through a similar argument that we did above so show that $w(k)$ is continuous at $k = 0$ and that,

$$f(z+k) = f(z) + k\left(w(k) + f'(z)\right) \qquad (2)$$

Do not get excited about the different letters here all we did was use *k* instead of *h* and let $x = z$. Nothing fancy here, but the change of letters will be useful down the road.

Okay, to this point it doesn't look like we've really done anything that gets us even close to proving the chain rule. The work above will turn out to be very important in our proof however so let's get going on the proof.

What we need to do here is use the definition of the derivative and evaluate the following limit.

$$\frac{d}{dx}\Big[f\big[u(x)\big]\Big] = \lim_{h\to 0}\frac{f\big[u(x+h)\big] - f\big[u(x)\big]}{h} \qquad (3)$$

Note that even though the notation is more than a little messy if we use $u(x)$ instead of *u* we need to remind ourselves here that *u* really is a function of *x*.

Let's now use (1) to rewrite the $u(x+h)$ and yes the notation is going to be unpleasant but we're going to have to deal with it. By using (1), the numerator in the limit above becomes,

$$f\big[u(x+h)\big] - f\big[u(x)\big] = f\Big[u(x) + h\big(v(h) + u'(x)\big)\Big] - f\big[u(x)\big]$$

If we then define $z = u(x)$ and $k = h\big(v(h) + u'(x)\big)$ we can use (2) to further write this as,

$$\begin{aligned} f\big[u(x+h)\big] - f\big[u(x)\big] &= f\Big[u(x) + h\big(v(h) + u'(x)\big)\Big] - f\big[u(x)\big] \\ &= f\big[u(x)\big] + h\big(v(h) + u'(x)\big)\Big(w(k) + f'\big[u(x)\big]\Big) - f\big[u(x)\big] \\ &= h\big(v(h) + u'(x)\big)\Big(w(k) + f'\big[u(x)\big]\Big) \end{aligned}$$

Notice that we were able to cancel a $f\big[u(x)\big]$ to simplify things up a little. Also, note that the $w(k)$ was intentionally left that way to keep the mess to a minimum here, just remember that $k = h\big(v(h) + u'(x)\big)$ here as that will be important here in a bit. Let's now go back and remember that all this was the numerator of our limit, (3). Plugging this into (3) gives,

$$\begin{aligned} \frac{d}{dx}\Big[f\big[u(x)\big]\Big] &= \lim_{h\to 0}\frac{h\big(v(h) + u'(x)\big)\Big(w(k) + f'\big[u(x)\big]\Big)}{h} \\ &= \lim_{h\to 0}\big(v(h) + u'(x)\big)\Big(w(k) + f'\big[u(x)\big]\Big) \end{aligned}$$

Notice that the *h*'s canceled out. Next, recall that $k = h\big(v(h) + u'(x)\big)$ and so,

$$\lim_{h\to 0} k = \lim_{h\to 0} h\big(v(h) + u'(x)\big) = 0$$

But, if $\lim\limits_{h \to 0} k = 0$, as we've defined *k* anyway, then by the definition of *w* and the fact that we know $w(k)$ is continuous at $k = 0$ we also know that,

$$\lim_{h \to 0} w(k) = w\left(\lim_{h \to 0} k\right) = w(0) = 0$$

Also, recall that $\lim\limits_{h \to 0} v(h) = 0$. Using all of these facts our limit becomes,

$$\begin{aligned} \frac{d}{dx}\left[f\left[u(x)\right]\right] &= \lim_{h \to 0}\left(v(h) + u'(x)\right)\left(w(k) + f'\left[u(x)\right]\right) \\ &= u'(x) f'\left[u(x)\right] \\ &= f'\left[u(x)\right]\frac{du}{dx} \end{aligned}$$

This is exactly what we needed to prove and so we're done.

Section 7-3 : Proof of Trig Limits

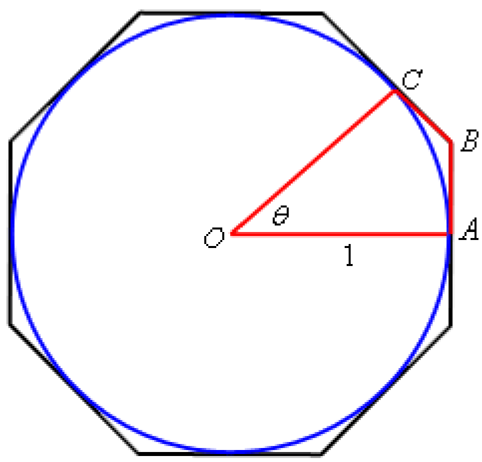
In this section we're going to provide the proof of the two limits that are used in the derivation of the derivative of sine and cosine in the **Derivatives of Trig Functions** section of the Derivatives chapter.

Proof of : $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

This proof of this limit uses the **Squeeze Theorem**. However, getting things set up to use the Squeeze Theorem can be a somewhat complex geometric argument that can be difficult to follow so we'll try to take it fairly slow.

Let's start by assuming that $0 \le \theta \le \frac{\pi}{2}$. Since we are proving a limit that has $\theta \to 0$ it's okay to assume that θ is not too large (*i.e.* $\theta \le \frac{\pi}{2}$). Also, by assuming that θ is positive we're actually going to first prove that the above limit is true if it is the right-hand limit. As you'll see if we can prove this then the proof of the limit will be easy.

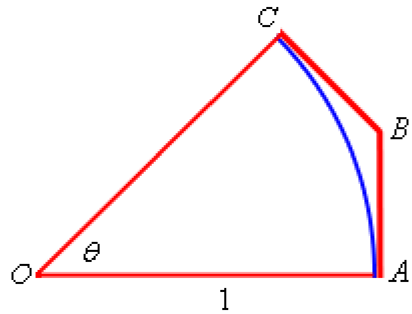
So, now that we've got our assumption on θ taken care of let's start off with the unit circle circumscribed by an octagon with a small slice marked out as shown below.



Points *A* and *C* are the midpoints of their respective sides on the octagon and are in fact tangent to the circle at that point. We'll call the point where these two sides meet *B*.

From this figure we can see that the circumference of the circle is less than the length of the octagon. This also means that if we look at the slice of the figure marked out above then the length of the portion of the circle included in the slice must be less than the length of the portion of the octagon included in the slice.

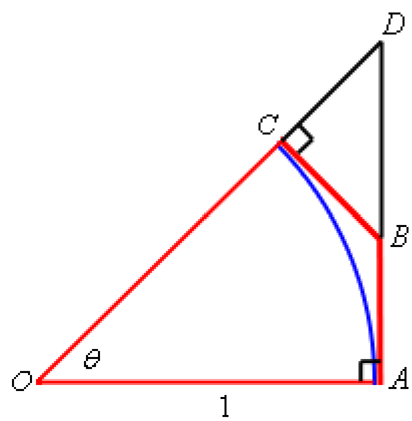
Because we're going to be doing most of our work on just the slice of the figure let's strip that out and look at just it. Here is a sketch of just the slice.



Now denote the portion of the circle by $\text{arc } AC$ and the lengths of the two portion of the octagon shown by $|AB|$ and $|BC|$. Then by the observation about lengths we made above we must have,

$$\text{arc } AC < |AB| + |BC| \tag{1}$$

Next, extend the lines *AB* and *OC* as shown below and call the point that they meet *D*. The triangle now formed by *AOD* is a right triangle. All this is shown in the figure below.



The triangle *BCD* is a right triangle with hypotenuse *BD* and so we know $|BC| < |BD|$. Also notice that $|AB| + |BD| = |AD|$. If we use these two facts in (1) we get,

$$\begin{aligned} \text{arc } AC &< |AB| + |BC| \\ &< |AB| + |BD| \\ &= |AD| \end{aligned} \tag{2}$$

Next, as noted already the triangle *AOD* is a right triangle and so we can use a little right triangle trigonometry to write $|AD| = |AO| \tan\theta$. Also note that $|AO| = 1$ since it is nothing more than the radius of the unit circle. Using this information in (2) gives,

$$\begin{aligned} \text{arc } AC &< |AD| \\ &< |AO| \tan\theta \\ &= \tan\theta \end{aligned} \tag{3}$$

The next thing that we need to recall is that the length of a portion of a circle is given by the radius of the circle times the angle (in radians!) that traces out the portion of the circle we're trying to measure. For our portion this means that,

$$\text{arc } AC = |AO|\theta = \theta$$

Before proceeding a quick note. Students often ask why we always use radians in a Calculus class. This is the reason why! The formula for the length of a portion of a circle used above assumed that the angle is in radians. The formula for angles in degrees is different and if we used that we would get a different answer. So, remember to always use radians.

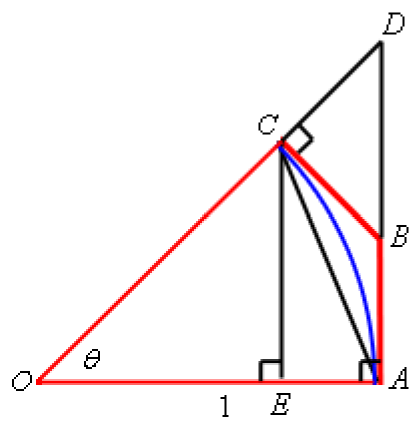
So, putting this into (3) we see that,

$$\theta = \text{arc } AC < \tan\theta = \frac{\sin\theta}{\cos\theta}$$

or, if we do a little rearranging we get,

$$\cos\theta < \frac{\sin\theta}{\theta} \qquad (4)$$

We'll be coming back to (4) in a bit. Let's now add in a couple more lines into our figure above. Let's connect *A* and *C* with a line and drop a line straight down from *C* until it intersects *AO* at a right angle and let's call the intersection point *E*. This is all shown in the figure below.



Okay, the first thing to notice here is that,

$$|CE| < |AC| < \text{arc } AC \qquad (5)$$

Also note that triangle *EOC* is a right triangle with a hypotenuse of $|CO| = 1$. Using some right triangle trig we can see that,

$$|CE| = |CO|\sin\theta = \sin\theta$$

Plugging this into (5) and recalling that $\text{arc } AC = \theta$ we get,

$$\sin\theta = |CE| < \text{arc } AC = \theta$$

and with a little rewriting we get,

$$\frac{\sin\theta}{\theta} < 1 \tag{6}$$

Okay, we're almost done here. Putting (4) and (6) together we see that,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

provided $0 \le \theta \le \frac{\pi}{2}$. Let's also note that,

$$\lim_{\theta \to 0} \cos\theta = 1 \qquad\qquad \lim_{\theta \to 0} 1 = 1$$

We are now set up to use the **Squeeze Theorem**. The only issue that we need to worry about is that we are staying to the right of $\theta = 0$ in our assumptions and so the best that the Squeeze Theorem will tell us is,

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$$

So, we know that the limit is true if we are only working with a right-hand limit. However we know that $\sin\theta$ is an odd function and so,

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

In other words, if we approach zero from the left (*i.e.* negative θ's) then we'll get the same values in the function as if we'd approached zero from the right (*i.e.* positive θ's) and so,

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$$

We have now shown that the two one-sided limits are the same and so we must also have,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

That was a somewhat long proof and if you're not really good at geometric arguments it can be kind of daunting and confusing. Nicely, the second limit is very simple to prove, provided you've already proved the first limit.

Proof of : $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

We'll start by doing the following,

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\cos\theta - 1)(\cos\theta + 1)}{\theta(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} \tag{7}$$

Now, let's recall that,

$$\cos^2\theta + \sin^2\theta = 1 \qquad \Rightarrow \qquad \cos^2\theta - 1 = -\sin^2\theta$$

Using this in (7) gives us,

$$\begin{aligned}\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta\left(\cos\theta+1\right)}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{-\sin\theta}{\cos\theta+1}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\;\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1}\end{aligned}$$

At this point, because we just proved the first limit and the second can be taken directly we're pretty much done. All we need to do is take the limits.

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\;\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta+1} = \left(1\right)\left(0\right) = 0$$

Section 7-4 : Proofs of Derivative Applications Facts

In this section we'll be proving some of the facts and/or theorems from the **Applications of Derivatives** chapter. Not all of the facts and/or theorems will be proved here.

Fermat's Theorem

If $f(x)$ has a relative extrema at $x = c$ and $f'(c)$ exists then $x = c$ is a critical point of $f(x)$. In fact, it will be a critical point such that $f'(c) = 0$.

Proof

This is a fairly simple proof. We'll assume that $f(x)$ has a relative maximum to do the proof. The proof for a relative minimum is nearly identical. So, if we assume that we have a relative maximum at $x = c$ then we know that $f(c) \ge f(x)$ for all *x* that are sufficiently close to $x = c$. In particular for all *h* that are sufficiently close to zero (positive or negative) we must have,

$$f(c) \ge f(c+h)$$

or, with a little rewrite we must have,

$$f(c+h) - f(c) \le 0 \qquad (1)$$

Now, at this point assume that $h > 0$ and divide both sides of (1) by *h*. This gives,

$$\frac{f(c+h) - f(c)}{h} \le 0$$

Because we're assuming that $h > 0$ we can now take the right-hand limit of both sides of this.

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \le \lim_{h \to 0^+} 0 = 0$$

We are also assuming that $f'(c)$ exists and recall that if a normal limit exists then it must be equal to both one-sided limits. We can then say that,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \le 0$$

If we put this together we have now shown that $f'(c) \le 0$.

Okay, now let's turn things around and assume that $h < 0$ and divide both sides of (1) by *h*. This gives,

$$\frac{f(c+h) - f(c)}{h} \ge 0$$

Remember that because we're assuming $h < 0$ we'll need to switch the inequality when we divide by a negative number. We can now do a similar argument as above to get that,

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq \lim_{h \to 0^-} 0 = 0$$

The difference here is that this time we're going to be looking at the left-hand limit since we're assuming that $h < 0$. This argument shows that $f'(c) \geq 0$.

We've now shown that $f'(c) \leq 0$ and $f'(c) \geq 0$. Then only way both of these can be true at the same time is to have $f'(c) = 0$ and this in turn means that $x = c$ must be a critical point.

As noted above, if we assume that $f(x)$ has a relative minimum then the proof is nearly identical and so isn't shown here. The main differences are simply some inequalities need to be switched.

<u>Fact, The Shape of a Graph, Part I</u>

1. If $f'(x) > 0$ for every *x* on some interval *I*, then $f(x)$ is increasing on the interval.
2. If $f'(x) < 0$ for every *x* on some interval *I*, then $f(x)$ is decreasing on the interval.
3. If $f'(x) = 0$ for every *x* on some interval *I*, then $f(x)$ is constant on the interval.

The proof of this fact uses the **Mean Value Theorem** which, if you're following along in my notes has actually not been covered yet. The Mean Value Theorem can be covered at any time and for whatever the reason I decided to put where it is. Before reading through the proof of this fact you should take a quick look at the Mean Value Theorem section. You really just need the conclusion of the Mean Value Theorem for this proof however.

Proof of 1

Let x_1 and x_2 be in *I* and suppose that $x_1 < x_2$. Now, using the Mean Value Theorem on $[x_1, x_2]$ means there is a number *c* such that $x_1 < c < x_2$ and,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Because $x_1 < c < x_2$ we know that *c* must also be in *I* and so we know that $f'(c) > 0$ we also know that $x_2 - x_1 > 0$. So, this means that we have,

$$f(x_2) - f(x_1) > 0$$

Rewriting this gives,

$$f(x_1) < f(x_2)$$

and so, by definition, since x_1 and x_2 were two arbitrary numbers in *I*, $f(x)$ must be increasing on *I*.

Proof of 2

This proof is nearly identical to the previous part.

Let x_1 and x_2 be in *I* and suppose that $x_1 < x_2$. Now, using the Mean Value Theorem on $[x_1, x_2]$ means there is a number *c* such that $x_1 < c < x_2$ and,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Because $x_1 < c < x_2$ we know that *c* must also be in *I* and so we know that $f'(c) < 0$ we also know that $x_2 - x_1 > 0$. So, this means that we have,

$$f(x_2) - f(x_1) < 0$$

Rewriting this gives,

$$f(x_1) > f(x_2)$$

and so, by definition, since x_1 and x_2 were two arbitrary numbers in *I*, $f(x)$ must be decreasing on *I*.

Proof of 3

Again, this proof is nearly identical to the previous two parts, but in this case is actually somewhat easier.

Let x_1 and x_2 be in *I*. Now, using the Mean Value Theorem on $[x_1, x_2]$ there is a number *c* such that *c* is between x_1 and x_2 and,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Note that for this part we didn't need to assume that $x_1 < x_2$ and so all we know is that *c* is between x_1 and x_2 and so, more importantly, *c* is also in *I*. and this means that $f'(c) = 0$. So, this means that we have,

$$f(x_2) - f(x_1) = 0$$

Rewriting this gives,

$$f(x_1) = f(x_2)$$

and so, since x_1 and x_2 were two arbitrary numbers in *I*, $f(x)$ must be constant on *I*.

Fact, The Shape of a Graph, Part II

Given the function $f(x)$ then,

1. If $f''(x) > 0$ for all *x* in some interval *I* then $f(x)$ is concave up on *I*.

2. If $f''(x) < 0$ for all *x* in some interval *I* then $f(x)$ is concave down on *I*.

The proof of this fact uses the **Mean Value Theorem** which, if you're following along in my notes has actually not been covered yet. The Mean Value Theorem can be covered at any time and for whatever the reason I decided to put it after the section this fact is in. Before reading through the proof of this fact you should take a quick look at the Mean Value Theorem section. You really just need the conclusion of the Mean Value Theorem for this proof however.

Proof of 1

Let *a* be any number in the interval *I*. The tangent line to $f(x)$ at $x = a$ is,

$$y = f(a) + f'(a)(x - a)$$

To show that $f(x)$ is concave up on *I* then we need to show that for any *x*, $x \neq a$, in *I* that,

$$f(x) > f(a) + f'(a)(x - a)$$

or in other words, the tangent line is always below the graph of $f(x)$ on *I*. Note that we require $x \neq a$ because at that point we know that $f(x) = f(a)$ since we are talking about the tangent line.

Let's start the proof off by first assuming that $x > a$. Using the Mean Value Theorem on $[a, x]$ means there is a number *c* such that $a < c < x$ and,

$$f(x) - f(a) = f'(c)(x - a)$$

With some rewriting this is,

$$f(x) = f(a) + f'(c)(x - a) \qquad (2)$$

Next, let's use the fact that $f''(x) > 0$ for every *x* on *I*. This means that the first derivative, $f'(x)$, must be increasing (because its derivative, $f''(x)$, is positive). Now, we know from the Mean Value Theorem that $a < c$ and so because $f'(x)$ is increasing we must have,

$$f'(a) < f'(c) \qquad (3)$$

Recall as well that we are assuming $x > a$ and so $x - a > 0$. If we now multiply (3) by $x - a$ (which is positive and so the inequality stays the same) we get,

$$f'(a)(x - a) < f'(c)(x - a)$$

Next, add $f(a)$ to both sides of this to get,

$$f(a)+f'(a)(x-a)<f(a)+f'(c)(x-a)$$

However, by (2), the right side of this is nothing more than $f(x)$ and so we have,

$$f(a)+f'(a)(x-a)<f(x)$$

but this is exactly what we wanted to show.

So, provided $x>a$ the tangent line is in fact below the graph of $f(x)$.

We now need to assume $x<a$. Using the Mean Value Theorem on $[x,a]$ means there is a number *c* such that $x<c<a$ and,

$$f(a)-f(x)=f'(c)(a-x)$$

If we multiply both sides of this by –1 and then adding $f(a)$ to both sides and we again arise at (2).

Now, from the Mean Value Theorem we know that $c<a$ and because $f''(x)>0$ for every *x* on *I* we know that the derivative is still increasing and so we have,

$$f'(c)<f'(a)$$

Let's now multiply this by $x-a$, which is now a negative number since $x<a$. This gives,

$$f'(c)(x-a)>f'(a)(x-a)$$

Notice that we had to switch the direction of the inequality since we were multiplying by a negative number. If we now add $f(a)$ to both sides of this and then substitute (2) into the results we arrive at,

$$\begin{aligned} f(a)+f'(c)(x-a)&>f(a)+f'(a)(x-a) \\ f(x)&>f(a)+f'(a)(x-a) \end{aligned}$$

So, again we've shown that the tangent line is always below the graph of $f(x)$.

We've now shown that if *x* is any number in *I*, with $x \neq a$ the tangent lines are always below the graph of $f(x)$ on *I* and so $f(x)$ is concave up on *I*.

Proof of 2

This proof is nearly identical to the proof of 1 and since that proof is fairly long we're going to just get things started and then leave the rest of it to you to go through.

Let *a* be any number in *I* . To show that $f(x)$ is concave down we need to show that for any *x* in *I*, $x \neq a$, that the tangent line is always above the graph of $f(x)$ or,

$$f(x) < f(a) + f'(a)(x-a)$$

From this point on the proof is almost identical to the proof of 1 except that you'll need to use the fact that the derivative in this case is decreasing since $f''(x) < 0$. We'll leave it to you to fill in the details of this proof.

Second Derivative Test

Suppose that $x = c$ is a critical point of $f'(c)$ such that $f'(c) = 0$ and that $f''(x)$ is continuous in a region around $x = c$. Then,

1. If $f''(c) < 0$ then $x = c$ is a relative maximum.
2. If $f''(c) > 0$ then $x = c$ is a relative minimum.
3. If $f''(c) = 0$ then $x = c$ can be a relative maximum, relative minimum or neither.

The proof of this fact uses the **Mean Value Theorem** which, if you're following along in my notes has actually not been covered yet. The Mean Value Theorem can be covered at any time and for whatever the reason I decided to put it after the section this fact is in. Before reading through the proof of this fact you should take a quick look at the Mean Value Theorem section. You really just need the conclusion of the Mean Value Theorem for this proof however.

Proof of 1

First since we are assuming that $f''(x)$ is continuous in a region around $x = c$ then we can assume that in fact $f''(c) < 0$ is also true in some open region, say (a,b) around $x = c$, *i.e.* $a < c < b$.

Now let *x* be any number such that $a < x < c$, we're going to use the Mean Value Theorem on $[x,c]$. However, instead of using it on the function itself we're going to use it on the first derivative. So, the Mean Value Theorem tells us that there is a number $x < d < c$ such that,

$$f'(c) - f'(x) = f''(d)(c-x)$$

Now, because $a < x < d < c$ we know that $f''(d) < 0$ and we also know that $c - x > 0$ so we then get that,

$$f'(c) - f'(x) < 0$$

However, we also assumed that $f'(c) = 0$ and so we have,

$$-f'(x) < 0 \qquad \Rightarrow \qquad f'(x) > 0$$

Or, in other words to the left of $x = c$ the function is increasing.

Let's now turn things around and let *x* be any number such that $c < x < b$ and use the Mean Value Theorem on $[c, x]$ and the first derivative. The Mean Value Theorem tells us that there is a number $c < d < x$ such that,

$$f'(x) - f'(c) = f''(d)(x - c)$$

Now, because $c < d < x < b$ we know that $f''(d) < 0$ and we also know that $x - c > 0$ so we then get that,

$$f'(x) - f'(c) < 0$$

Again, use the fact that we also assumed that $f'(c) = 0$ to get,

$$f'(x) < 0$$

We now know that to the right of $x = c$ the function is decreasing.

So, to the left of $x = c$ the function is increasing and to the right of $x = c$ the function is decreasing so by the first derivative test this means that $x = c$ must be a relative maximum.

Proof of 2

This proof is nearly identical to the proof of 1 and since that proof is somewhat long we're going to leave the proof to you to do. In this case the only difference is that now we are going to assume that $f''(x) < 0$ and that will give us the opposite signs of the first derivative on either side of $x = c$ which gives us the conclusion we were after. We'll leave it to you to fill in all the details of this.

Proof of 3

There isn't really anything to prove here. All this statement says is that any of the three cases are possible and to "prove" this all one needs to do is provide an example of each of the three cases. This was done in **The Shape of a Graph, Part II** section where this test was presented so we'll leave it to you to go back to that section to see those graphs to verify that all three possibilities really can happen.

Rolle's Theorem

Suppose $f(x)$ is a function that satisfies all of the following.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number *c* such that $a < c < b$ and $f'(c) = 0$. Or, in other words $f(x)$ has a critical point in (a,b).

Proof

We'll need to do this with 3 cases.

Case 1 : $f(x) = k$ on $[a,b]$ where *k* is a constant.
In this case $f'(x) = 0$ for all *x* in $[a,b]$ and so we can take *c* to be any number in $[a,b]$.

Case 2 : There is some number *d* in (a,b) such that $f(d) > f(a)$.
Because $f(x)$ is continuous on $[a,b]$ by the **Extreme Value Theorem** we know that $f(x)$ will have a maximum somewhere in $[a,b]$. Also, because $f(a) = f(b)$ and $f(d) > f(a)$ we know that in fact the maximum value will have to occur at some *c* that is in the open interval (a,b), or $a < c < b$. Because *c* occurs in the interior of the interval this means that $f(x)$ will actually have a relative maximum at $x = c$ and by the second hypothesis above we also know that $f'(c)$ exists. Finally, by **Fermat's Theorem** we then know that in fact $x = c$ must be a critical point and because we know that $f'(c)$ exists we must have $f'(c) = 0$ (as opposed to $f'(c)$ not existing…).

Case 3 : There is some number *d* in (a,b) such that $f(d) < f(a)$.
This is nearly identical to Case 2 so we won't put in quite as much detail. By the Extreme Value Theorem $f(x)$ will have minimum in $[a,b]$ and because $f(a) = f(b)$ and $f(d) < f(a)$ we know that the minimum must occur at $x = c$ where $a < c < b$. Finally, by Fermat's Theorem we know that $f'(c) = 0$.

The Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a,b]$.
2. $f(x)$ is differentiable on the open interval (a,b).

Then there is a number *c* such that *a* < *c* < *b* and

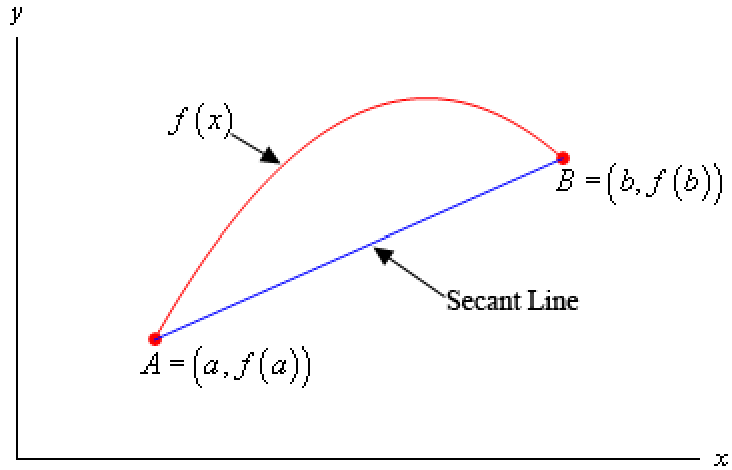
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

Proof

For illustration purposes let's suppose that the graph of $f(x)$ is,



Note of course that it may not look like this, but we just need a quick sketch to make it easier to see what we're talking about here.

The first thing that we need is the equation of the secant line that goes through the two points *A* and *B* as shown above. This is,

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Let's now define a new function, $g(x)$, as to be the difference between $f(x)$ and the equation of the secant line or,

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Next, let's notice that because $g(x)$ is the sum of $f(x)$, which is assumed to be continuous on $[a,b]$, and a linear polynomial, which we know to be continuous everywhere, we know that $g(x)$ must also be continuous on $[a,b]$.

Also, we can see that $g(x)$ must be differentiable on (a,b) because it is the sum of $f(x)$, which is assumed to be differentiable on (a,b), and a linear polynomial, which we know to be differentiable.

We could also have just computed the derivative as follows,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

at which point we can see that it exists on (a,b) because we assumed that $f'(x)$ exists on (a,b) and the last term is just a constant.

Finally, we have,

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b-a}(a-a) = f(a) - f(a) = 0$$

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b-a}(b-a) = f(b) - f(a) - \left(f(b) - f(a)\right) = 0$$

In other words, $g(x)$ satisfies the three conditions of **Rolle's Theorem** and so we know that there must be a number *c* such that $a < c < b$ and that,

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} \qquad \Rightarrow \qquad f'(c) = \frac{f(b) - f(a)}{b-a}$$

Section 7-5 : Proof of Various Integral Properties

In this section we've got the proof of several of the properties we saw in the **Integrals** Chapter as well as a couple from the **Applications of Integrals** Chapter.

Proof of : $\int k\,f(x)\,dx = k\int f(x)\,dx$ where *k* is any number.

This is a very simple proof. Suppose that $F(x)$ is an anti-derivative of $f(x)$, *i.e.* $F'(x) = f(x)$. Then by the basic properties of derivatives we also have that,

$$\left(k\,F(x)\right)' = k\,F'(x) = k\,f(x)$$

and so $k\,F(x)$ is an anti-derivative of $k\,f(x)$, *i.e.* $\left(k\,F(x)\right)' = k\,f(x)$. In other words,

$$\int k\,f(x)\,dx = k\,F(x) + c = k\int f(x)\,dx$$

Proof of : $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

This is also a very simple proof Suppose that $F(x)$ is an anti-derivative of $f(x)$ and that $G(x)$ is an anti-derivative of $g(x)$. So we have that $F'(x) = f(x)$ and $G'(x) = g(x)$. Basic properties of derivatives also tell us that

$$\left(F(x) \pm G(x)\right)' = F'(x) \pm G'(x) = f(x) \pm g(x)$$

and so $F(x) + G(x)$ is an anti-derivative of $f(x) + g(x)$ and $F(x) - G(x)$ is an anti-derivative of $f(x) - g(x)$. In other words,

$$\int f(x) \pm g(x)\,dx = F(x) \pm G(x) + c = \int f(x)\,dx \pm \int g(x)\,dx$$

Proof of : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

From the definition of the definite integral we have,

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \qquad\qquad \Delta x = \frac{b-a}{n}$$

and we also have,

$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \qquad \Delta x = \frac{a-b}{n}$$

Therefore,

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\frac{b-a}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\frac{-(a-b)}{n}\\ &= \lim_{n\to\infty}\left(-\sum_{i=1}^{n} f\left(x_i^*\right)\frac{a-b}{n}\right)\\ &= -\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\frac{a-b}{n} = -\int_b^a f(x)\,dx\end{aligned}$$

Proof of : $\int_a^a f(x)\,dx = 0$

From the definition of the definite integral we have,

$$\begin{aligned}\int_a^a f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \qquad \Delta x = \frac{a-a}{n} = 0\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)(0)\\ &= \lim_{n\to\infty} 0\\ &= 0\end{aligned}$$

Proof of : $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

From the definition of the definite integral we have,

$$\begin{aligned}\int_a^b c\,f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} c\,f\left(x_i^*\right)\Delta x\\ &= \lim_{n\to\infty} c\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x\\ &= c\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x\\ &= c\int_a^b f(x)\,dx\end{aligned}$$

Remember that we can pull constants out of summations and out of limits.

Proof of : $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

First we'll prove the formula for "+". From the definition of the definite integral we have,

$$\begin{aligned}\int_a^b f(x) + g(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(f\left(x_i^*\right) + g\left(x_i^*\right)\right)\Delta x \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x + \sum_{i=1}^{n} g\left(x_i^*\right)\Delta x\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} g\left(x_i^*\right)\Delta x \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

To prove the formula for "-" we can either redo the above work with a minus sign instead of a plus sign or we can use the fact that we now know this is true with a plus and using the properties proved above as follows.

$$\begin{aligned}\int_a^b f(x) - g(x)\,dx &= \int_a^b f(x) + \left(-g(x)\right)dx \\ &= \int_a^b f(x)\,dx + \int_a^b \left(-g(x)\right)dx \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\end{aligned}$$

Proof of : $\int_a^b c\,dx = c\left(b-a\right)$, c is any number.

If we define $f(x) = c$ then from the definition of the definite integral we have,

$$\int_a^b c\,dx = \int_a^b f(x)\,dx$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{b-a}{n}$$
$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} c\right)\frac{b-a}{n}$$
$$= \lim_{n\to\infty}(cn)\frac{b-a}{n}$$
$$= \lim_{n\to\infty} c(b-a)$$
$$= c(b-a)$$

Proof of : If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$.

From the definition of the definite integral we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

Now, by assumption $f(x) \ge 0$ and we also have $\Delta x > 0$ and so we know that

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \ge 0$$

So, from the basic properties of limits we then have,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \ge \lim_{n\to\infty} 0 = 0$$

But the left side is exactly the definition of the integral and so we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \ge 0$$

Proof of : If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Since we have $f(x) \ge g(x)$ then we know that $f(x) - g(x) \ge 0$ on $a \le x \le b$ and so by Property 8 proved above we know that,

$$\int_a^b f(x) - g(x)\,dx \ge 0$$

We also know from Property 4 that,

$$\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

So, we then have,

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \ge 0$$

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Proof of : If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

Given $m \le f(x) \le M$ we can use Property 9 on each inequality to write,

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

Then by Property 7 on the left and right integral to get,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Proof of : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$

First let's note that we can say the following about the function and the absolute value,

$$-\left|f(x)\right| \le f(x) \le \left|f(x)\right|$$

If we now use Property 9 on each inequality we get,

$$\int_a^b -\left|f(x)\right|dx \le \int_a^b f(x)\,dx \le \int_a^b \left|f(x)\right|dx$$

We know that we can factor the minus sign out of the left integral to get,

$$-\int_a^b \left|f(x)\right|dx \le \int_a^b f(x)\,dx \le \int_a^b \left|f(x)\right|dx$$

Finally, recall that if $|p| \le b$ then $-b \le p \le b$ and of course this works in reverse as well so we then must have,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$$

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$ then,

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and it is differentiable on (a,b) and that,

$$g'(x) = f(x)$$

Proof

Suppose that *x* and $x+h$ are in (a,b). We then have,

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

Now, using **Property 5** of the Integral Properties we can rewrite the first integral and then do a little simplification as follows.

$$\begin{aligned} g(x+h) - g(x) &= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

Finally assume that $h \neq 0$ and we get,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \qquad (4)$$

Let's now assume that $h > 0$ and since we are still assuming that $x+h$ are in (a,b) we know that $f(x)$ is continuous on $[x, x+h]$ and so by the **Extreme Value Theorem** we know that there are numbers *c* and *d* in $[x, x+h]$ so that $f(c) = m$ is the absolute minimum of $f(x)$ in $[x, x+h]$ and that $f(d) = M$ is the absolute maximum of $f(x)$ in $[x, x+h]$.

So, by **Property 10** of the Integral Properties we then know that we have,

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

Or,

$$f(c)h \le \int_x^{x+h} f(t)\,dt \le f(d)h$$

Now divide both sides of this by h to get,

$$f(c) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(d)$$

and then use (1) to get,

$$f(c) \le \frac{g(x+h)-g(x)}{h} \le f(d) \qquad (5)$$

Next, if $h < 0$ we can go through the same argument above except we'll be working on $[x+h, x]$ to arrive at exactly the same inequality above. In other words, (2) is true provided $h \ne 0$.

Now, if we take $h \to 0$ we also have $c \to x$ and $d \to x$ because both *c* and *d* are between *x* and $x+h$. This means that we have the following two limits.

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x) \qquad\qquad \lim_{h\to 0} f(d) = \lim_{d\to x} f(d) = f(x)$$

The **Squeeze Theorem** then tells us that,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x) \qquad (6)$$

but the left side of this is exactly the definition of the derivative of $g(x)$ and so we get that,

$$g'(x) = f(x)$$

So, we've shown that $g(x)$ is differentiable on (a,b).

Now, the **Theorem** at the end of the Definition of the Derivative section tells us that $g(x)$ is also continuous on (a,b). Finally, if we take $x = a$ or $x = b$ we can go through a similar argument we used to get (3) using one-sided limits to get the same result and so the theorem at the end of the Definition of the Derivative section will also tell us that $g(x)$ is continuous at $x = a$ or $x = b$ and so in fact $g(x)$ is also continuous on $[a,b]$.

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ is a continuous function on $[a,b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Proof

First let $g\left(x\right)=\int_{a}^{x} f\left(t\right)\,dt$ and then we know from Part I of the Fundamental Theorem of Calculus that $g'\left(x\right)=f\left(x\right)$ and so $g\left(x\right)$ is an anti-derivative of $f\left(x\right)$ on $\left[a,b\right]$. Further suppose that $F\left(x\right)$ is any anti-derivative of $f\left(x\right)$ on $\left[a,b\right]$ that we want to choose. So, this means that we must have,

$$g'\left(x\right)=F'\left(x\right)$$

Then, by **Fact 2** in the Mean Value Theorem section we know that $g\left(x\right)$ and $F\left(x\right)$ can differ by no more than an additive constant on $\left(a,b\right)$. In other words, for $a<x<b$ we have,

$$F\left(x\right)=g\left(x\right)+c$$

Now because $g\left(x\right)$ and $F\left(x\right)$ are continuous on $\left[a,b\right]$, if we take the limit of this as $x\to a^{+}$ and $x\to b^{-}$ we can see that this also holds if $x=a$ and $x=b$.

So, for $a\le x\le b$ we know that $F\left(x\right)=g\left(x\right)+c$. Let's use this and the definition of $g\left(x\right)$ to do the following.

$$\begin{aligned} F\left(b\right)-F\left(a\right) &= \left(g\left(b\right)+c\right)-\left(g\left(a\right)+c\right)\\ &= g\left(b\right)-g\left(a\right)\\ &= \int_{a}^{b} f\left(t\right)\,dt+\int_{a}^{a} f\left(t\right)\,dt\\ &= \int_{a}^{b} f\left(t\right)\,dt+0\\ &= \int_{a}^{b} f\left(x\right)\,dx \end{aligned}$$

Note that in the last step we used the fact that the variable used in the integral does not matter and so we could change the *t*'s to *x*'s.

Average Function Value

The average value of a function $f\left(x\right)$ over the interval $\left[a,b\right]$ is given by,

$$f_{avg}=\frac{1}{b-a}\int_{a}^{b} f\left(x\right)dx$$

Proof

We know that the average value of *n* numbers is simply the sum of all the numbers divided by *n* so let's start off with that. Let's take the interval $\left[a,b\right]$ and divide it into *n* subintervals each of length,

$$\Delta x = \frac{b-a}{n}$$

Now from each of these intervals choose the points $x_1^*, x_2^*, \ldots, x_n^*$ and note that it doesn't really matter how we choose each of these numbers as long as they come from the appropriate interval. We can then compute the average of the function values $f\left(x_1^*\right), f\left(x_2^*\right), \ldots, f\left(x_n^*\right)$ by computing,

$$\frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{n} \qquad (7)$$

Now, from our definition of Δx we can get the following formula for *n*.

$$n = \frac{b-a}{\Delta x}$$

and we can plug this into (4) to get,

$$\begin{aligned}\frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{\frac{b-a}{\Delta x}} &= \frac{\left[f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)\right]\Delta x}{b-a} \\ &= \frac{1}{b-a}\left[f\left(x_1^*\right)\Delta x+f\left(x_2^*\right)\Delta x+\cdots+f\left(x_n^*\right)\Delta x\right] \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x\end{aligned}$$

Let's now increase *n*. Doing this will mean that we're taking the average of more and more function values in the interval and so the larger we chose *n* the better this will approximate the average value of the function.

If we then take the limit as *n* goes to infinity we should get the average function value. Or,

$$f_{avg} = \lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \frac{1}{b-a}\,\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

We can factor the $\frac{1}{b-a}$ out of the limit as we've done and now the limit of the sum should look familiar as that is the definition of the definite integral. So, putting in definite integral we get the formula that we were after.

$$f_{avg} = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

The Mean Value Theorem for Integrals

If $f(x)$ is a continuous function on $[a,b]$ then there is a number *c* in $[a,b]$ such that,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Proof

Let's start off by defining,

$$F(x) = \int_a^x f(t)\,dt$$

Since $f(x)$ is continuous we know from the **Fundamental Theorem of Calculus, Part I** that $F(x)$ is continuous on $[a,b]$, differentiable on (a,b) and that $F'(x) = f(x)$.

Now, from the **Mean Value Theorem** we know that there is a number *c* such that $a < c < b$ and that,

$$F(b) - F(a) = F'(c)(b-a)$$

However, we know that $F'(c) = f(c)$ and,

$$F(b) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx \qquad\qquad F(a) = \int_a^a f(t)\,dt = 0$$

So, we then have,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Work

The work done by the force $F(x)$ (assuming that $F(x)$ is continuous) over the range $a \le x \le b$ is,

$$W = \int_a^b F(x)\,dx$$

Proof

Let's start off by dividing the range $a \le x \le b$ into *n* subintervals of width Δx and from each of these intervals choose the points $x_1^*, x_2^*, \ldots, x_n^*$.

Now, if *n* is large and because $F(x)$ is continuous we can assume that $F(x)$ won't vary by much over each interval and so in the *i*th interval we can assume that the force is approximately constant with a value of $F(x) \approx F(x_i^*)$. The work on each interval is then approximately,

$$W_i \approx F\left(x_i^*\right)\Delta x$$

The total work over $a \le x \le b$ is then approximately,

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=0}^{n} F\left(x_i^*\right)\Delta x$$

Finally, if we take the limit of this as *n* goes to infinity we'll get the exact work done. So,

$$W = \lim_{n\to\infty} \sum_{i=0}^{n} F\left(x_i^*\right)\Delta x$$

This is, however, nothing more than the definition of the definite integral and so the work done by the force $F\left(x\right)$ over $a \le x \le b$ is,

$$W = \int_a^b F\left(x\right)dx$$

Section 7-6 : Area and Volume Formulas

In this section we will derive the formulas used to get the area between two curves and the volume of a solid of revolution.

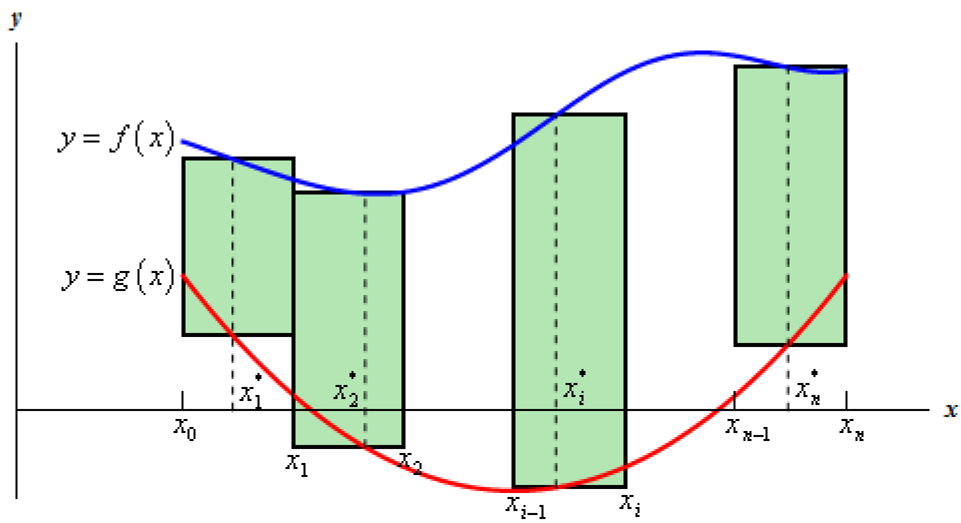
Area Between Two Curves

We will start with the formula for determining the area between $y = f(x)$ and $y = g(x)$ on the interval $[a,b]$. We will also assume that $f(x) \ge g(x)$ on $[a,b]$.

We will now proceed much as we did when we looked that the **Area Problem** in the Integrals Chapter. We will first divide up the interval into *n* equal subintervals each with length,

$$\Delta x = \frac{b-a}{n}$$

Next, pick a point in each subinterval, x_i^*, and we can then use rectangles on each interval as follows.



The height of each of these rectangles is given by,

$$f(x_i^*) - g(x_i^*)$$

and the area of each rectangle is then,

$$\left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

So, the area between the two curves is then approximated by,

$$A \approx \sum_{i=1}^{n}\left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

The exact area is,

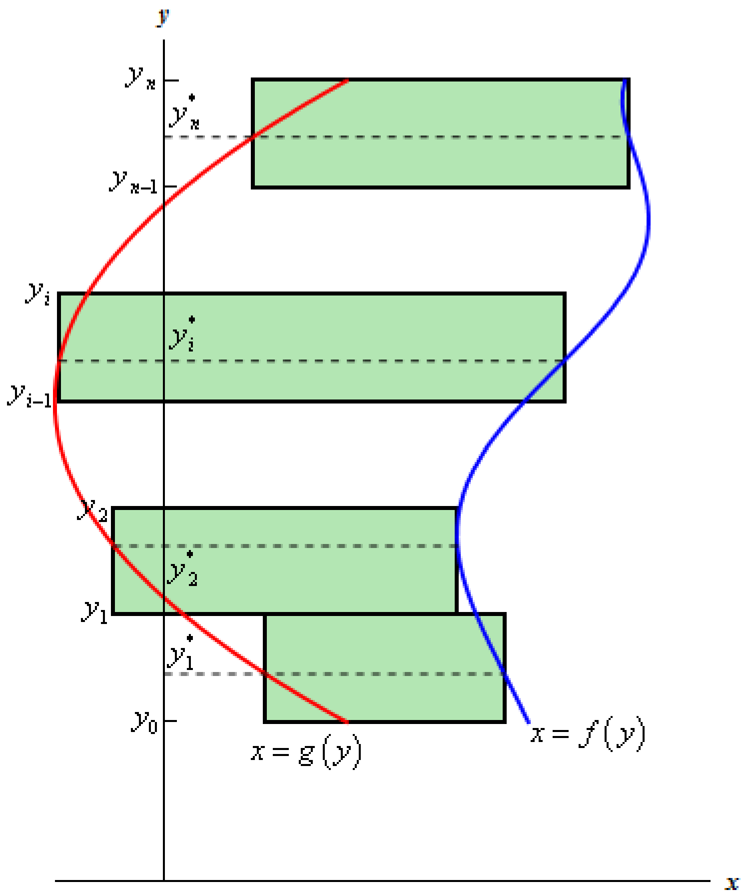
$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(x_i^*) - g(x_i^*)\right)\Delta x$$

Now, recalling the **definition of the definite integral** this is nothing more than,

$$A = \int_a^b f(x) - g(x)\,dx$$

The formula above will work provided the two functions are in the form $y = f(x)$ and $y = g(x)$. However, not all functions are in that form.

Sometimes we will be forced to work with functions in the form between $x = f(y)$ and $x = g(y)$ on the interval $[c,d]$ (an interval of *y* values…). When this happens, the derivation is identical. First we will start by assuming that $f(y) \ge g(y)$ on $[c,d]$. We can then divide up the interval into equal subintervals and build rectangles on each of these intervals. Here is a sketch of this situation.



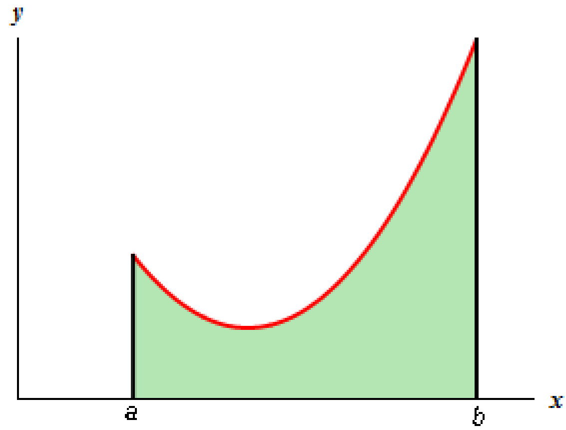
Following the work from above, we will arrive at the following for the area,

$$A = \int_c^d f(y) - g(y)\,dy$$

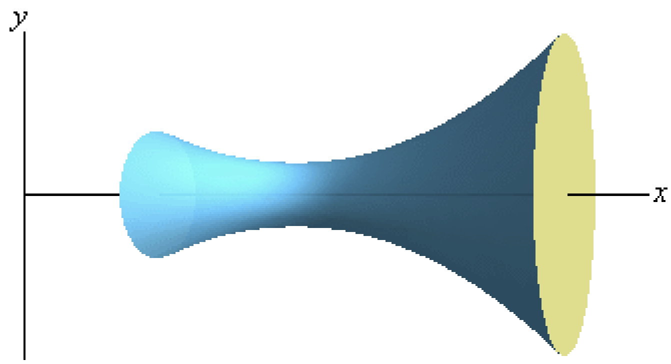
So, regardless of the form that the functions are in we use basically the same formula.

Volumes for Solid of Revolution

Before deriving the formula for this we should probably first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y = f(x)$, on an interval $[a,b]$.



We then rotate this curve about a given axis to get the surface of the solid of revolution. For purposes of this derivation let's rotate the curve about the *x*-axis. Doing this gives the following three dimensional region.



We want to determine the volume of the interior of this object. To do this we will proceed much as we did for the area between two curves case. We will first divide up the interval into *n* subintervals of width,

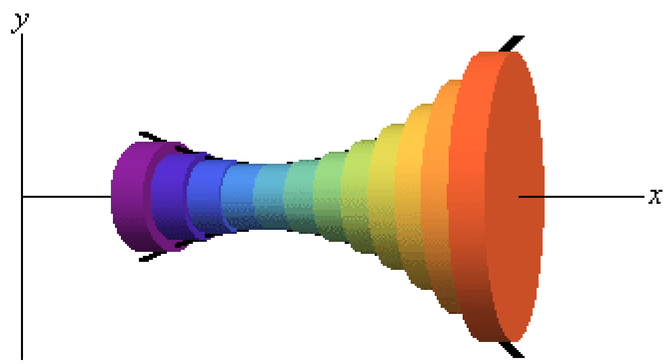
$$\Delta x = \frac{b-a}{n}$$

We will then choose a point from each subinterval, x_i^*.

Now, in the area between two curves case we approximated the area using rectangles on each subinterval. For volumes we will use disks on each subinterval to approximate the area. The area of the face of each disk is given by $A\left(x_i^*\right)$ and the volume of each disk is

$$V_i = A\left(x_i^*\right)\Delta x$$

Here is a sketch of this,



The volume of the region can then be approximated by,

$$V \approx \sum_{i=1}^{n} A\left(x_i^*\right)\Delta x$$

The exact volume is then,

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n} A\left(x_i^*\right)\Delta x \\ &= \int_a^b A\left(x\right)dx \end{aligned}$$

So, in this case the volume will be the integral of the cross-sectional area at any *x*, $A\left(x\right)$. Note as well that, in this case, the cross-sectional area is a circle and we could go farther and get a formula for that as well. However, the formula above is more general and will work for any way of getting a cross section so we will leave it like it is.

In the sections where we actually use this formula we will also see that there are ways of generating the cross section that will actually give a cross-sectional area that is a function of *y* instead of *x*. In these cases the formula will be,

$$V = \int_c^d A\left(y\right)dy, \qquad c \le y \le d$$

In this case we looked at rotating a curve about the *x*-axis, however, we could have just as easily rotated the curve about the *y*-axis. In fact, we could rotate the curve about any vertical or horizontal axis and in all of these, case we can use one or both of the following formulas.

$$V = \int_a^b A\left(x\right)dx \qquad\qquad V = \int_c^d A\left(y\right)dy$$

Section 7-7 : Types of Infinity

Most students have run across infinity at some point in time prior to a calculus class. However, when they have dealt with it, it was just a symbol used to represent a really, really large positive or really, really large negative number and that was the extent of it. Once they get into a calculus class students are asked to do some basic algebra with infinity and this is where they get into trouble. Infinity is NOT a number and for the most part doesn't behave like a number. However, despite that we'll think of infinity in this section as a really, really, really large number that is so large there isn't another number larger than it. This is not correct of course but may help with the discussion in this section. Note as well that everything that we'll be discussing in this section applies only to real numbers. If you move into complex numbers for instance things can and do change.

So, let's start thinking about addition with infinity. When you add two non-zero numbers you get a new number. For example, $4+7=11$. With infinity this is not true. With infinity you have the following.

$$\begin{aligned} \infty + a &= \infty \qquad \text{where } a \neq -\infty \\ \infty + \infty &= \infty \end{aligned}$$

In other words, a really, really large positive number (∞) plus any positive number, regardless of the size, is still a really, really large positive number. Likewise, you can add a negative number (*i.e.* $a<0$) to a really, really large positive number and stay really, really large and positive. So, addition involving infinity can be dealt with in an intuitive way if you're careful. Note as well that the *a* must NOT be negative infinity. If it is, there are some serious issues that we need to deal with as we'll see in a bit.

Subtraction with negative infinity can also be dealt with in an intuitive way in most cases as well. A really, really large negative number minus any positive number, regardless of its size, is still a really, really large negative number. Subtracting a negative number (*i.e.* $a<0$) from a really, really large negative number will still be a really, really large negative number. Or,

$$\begin{aligned} -\infty - a &= -\infty \qquad \text{where } a \neq -\infty \\ -\infty - \infty &= -\infty \end{aligned}$$

Again, *a* must not be negative infinity to avoid some potentially serious difficulties.

Multiplication can be dealt with fairly intuitively as well. A really, really large number (positive, or negative) times any number, regardless of size, is still a really, really large number we'll just need to be careful with signs. In the case of multiplication we have

$$(a)(\infty) = \infty \quad \text{if } a > 0 \qquad\qquad (a)(\infty) = -\infty \quad \text{if } a < 0$$

$$(\infty)(\infty) = \infty \qquad\qquad (-\infty)(-\infty) = \infty \qquad\qquad (-\infty)(\infty) = -\infty$$

What you know about products of positive and negative numbers is still true here.

Some forms of division can be dealt with intuitively as well. A really, really large number divided by a number that isn't too large is still a really, really large number.

$$\frac{\infty}{a} = \infty \qquad \text{if } a > 0, a \neq \infty \qquad\qquad \frac{\infty}{a} = -\infty \qquad \text{if } a < 0, a \neq -\infty$$
$$\frac{-\infty}{a} = -\infty \qquad \text{if } a > 0, a \neq \infty \qquad\qquad \frac{-\infty}{a} = \infty \qquad \text{if } a < 0, a \neq -\infty$$

Division of a number by infinity is somewhat intuitive, but there are a couple of subtleties that you need to be aware of. When we talk about division by infinity we are really talking about a limiting process in which the denominator is going towards infinity. So, a number that isn't too large divided an increasingly large number is an increasingly small number. In other words, in the limit we have,

$$\frac{a}{\infty} = 0 \qquad\qquad \frac{a}{-\infty} = 0$$

So, we've dealt with almost every basic algebraic operation involving infinity. There are two cases that that we haven't dealt with yet. These are

$$\infty - \infty = ? \qquad\qquad \frac{\pm\infty}{\pm\infty} = ?$$

The problem with these two cases is that intuition doesn't really help here. A really, really large number minus a really, really large number can be anything ($-\infty$, a constant, or ∞). Likewise, a really, really large number divided by a really, really large number can also be anything ($\pm\infty$ – this depends on sign issues, 0, or a non-zero constant).

What we've got to remember here is that there are really, really large numbers and then there are really, really, really large numbers. In other words, some infinities are larger than other infinities. With addition, multiplication and the first sets of division we worked this wasn't an issue. The general size of the infinity just doesn't affect the answer in those cases. However, with the subtraction and division cases listed above, it does matter as we will see.

Here is one way to think of this idea that some infinities are larger than others. This is a fairly dry and technical way to think of this and your calculus problems will probably never use this stuff, but it is a nice way of looking at this. Also, please note that I'm not trying to give a precise proof of anything here. I'm just trying to give you a little insight into the problems with infinity and how some infinities can be thought of as larger than others. For a much better (and definitely more precise) discussion see,

http://www.math.vanderbilt.edu/~schectex/courses/infinity.pdf

Let's start by looking at how many integers there are. Clearly, I hope, there are an infinite number of them, but let's try to get a better grasp on the "size" of this infinity. So, pick any two integers completely at random. Start at the smaller of the two and list, in increasing order, all the integers that come after that. Eventually we will reach the larger of the two integers that you picked.

Depending on the relative size of the two integers it might take a very, very long time to list all the integers between them and there isn't really a purpose to doing it. But, it could be done if we wanted to and that's the important part.

Because we could list all these integers between two randomly chosen integers we say that the integers are *countably infinite*. Again, there is no real reason to actually do this, it is simply something that can be done if we should choose to do so.

In general, a set of numbers is called countably infinite if we can find a way to list them all out. In a more precise mathematical setting this is generally done with a special kind of function called a bijection that associates each number in the set with exactly one of the positive integers. To see some more details of this see the pdf given above.

It can also be shown that the set of all fractions are also countably infinite, although this is a little harder to show and is not really the purpose of this discussion. To see a proof of this see the pdf given above. It has a very nice proof of this fact.

Let's contrast this by trying to figure out how many numbers there are in the interval (0,1). By numbers, I mean all possible fractions that lie between zero and one as well as all possible decimals (that aren't fractions) that lie between zero and one. The following is similar to the proof given in the pdf above but was nice enough and easy enough (I hope) that I wanted to include it here.

To start let's assume that all the numbers in the interval (0,1) are countably infinite. This means that there should be a way to list all of them out. We could have something like the following,

$$\begin{aligned} x_1 &= 0.692096\cdots \\ x_2 &= 0.171034\cdots \\ x_3 &= 0.993671\cdots \\ x_4 &= 0.045908\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

Now, select the i^{th} decimal out of x_i as shown below

$$\begin{aligned} x_1 &= 0.\underline{6}92096\cdots \\ x_2 &= 0.1\underline{7}1034\cdots \\ x_3 &= 0.99\underline{3}671\cdots \\ x_4 &= 0.045\underline{9}08\cdots \\ \vdots & \qquad \vdots \end{aligned}$$

and form a new number with these digits. So, for our example we would have the number

$$x = 0.6739\cdots$$

In this new decimal replace all the 3's with a 1 and replace every other numbers with a 3. In the case of our example this would yield the new number

$$\overline{x} = 0.3313\cdots$$

Notice that this number is in the interval (0,1) and also notice that given how we choose the digits of the number this number will not be equal to the first number in our list, x_1, because the first digit of each is guaranteed to not be the same. Likewise, this new number will not get the same number as the second in our list, x_2, because the second digit of each is guaranteed to not be the same. Continuing in this manner we can see that this new number we constructed, $\overline{x}$, is guaranteed to not be in our listing. But this contradicts the initial assumption that we could list out all the numbers in the interval (0,1). Hence, it must not be possible to list out all the numbers in the interval (0,1).

Sets of numbers, such as all the numbers in (0,1), that we can't write down in a list are called *uncountably* infinite.

The reason for going over this is the following. An infinity that is uncountably infinite is significantly larger than an infinity that is only countably infinite. So, if we take the difference of two infinities we have a couple of possibilities.

$$\begin{aligned}\infty\left(\text{uncountable}\right)-\infty\left(\text{countable}\right)&=\infty\\ \infty\left(\text{countable}\right)-\infty\left(\text{uncountable}\right)&=-\infty\end{aligned}$$

Notice that we didn't put down a difference of two infinities of the same type. Depending upon the context there might still have some ambiguity about just what the answer would be in this case, but that is a whole different topic.

We could also do something similar for quotients of infinities.

$$\begin{aligned}\frac{\infty\left(\text{countable}\right)}{\infty\left(\text{uncountable}\right)}&=0\\ \frac{\infty\left(\text{uncountable}\right)}{\infty\left(\text{countable}\right)}&=\infty\end{aligned}$$

Again, we avoided a quotient of two infinities of the same type since, again depending upon the context, there might still be ambiguities about its value.

So, that's it and hopefully you've learned something from this discussion. Infinity simply isn't a number and because there are different kinds of infinity it generally doesn't behave as a number does. Be careful when dealing with infinity.

Section 7-8 : Summation Notation

In this section we need to do a brief review of summation notation or sigma notation. We'll start out with two integers, *n* and *m*, with $n < m$ and a list of numbers denoted as follows,

$$a_n,\ a_{n+1},\ a_{n+2},\ \ldots,\ a_{m-2},\ a_{m-1},\ a_m$$

We want to add them up, in other words we want,

$$a_n + a_{n+1} + a_{n+2} + \ldots + a_{m-2} + a_{m-1} + a_m$$

For large lists this can be a fairly cumbersome notation so we introduce summation notation to denote these kinds of sums. The case above is denoted as follows.

$$\sum_{i=n}^{m} a_i = a_n + a_{n+1} + a_{n+2} + \ldots + a_{m-2} + a_{m-1} + a_m$$

The *i* is called the index of summation. This notation tells us to add all the a_i's up for all integers starting at *n* and ending at *m*.

For instance,

$$\sum_{i=0}^{4} \frac{i}{i+1} = \frac{0}{0+1} + \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} = \frac{163}{60} = 2.71\overline{666}$$

$$\sum_{i=4}^{6} 2^i x^{2i+1} = 2^4 x^9 + 2^5 x^{11} + 2^6 x^{13} = 16x^9 + 32x^{11} + 64x^{13}$$

$$\sum_{i=1}^{4} f\left(x_i^*\right) = f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + f\left(x_4^*\right)$$

Properties

Here are a couple of formulas for summation notation.

1. $\displaystyle\sum_{i=i_0}^{n} c a_i = c\sum_{i=i_0}^{n} a_i$ where *c* is any number. So, we can factor constants out of a summation.

2. $\displaystyle\sum_{i=i_0}^{n} \left(a_i \pm b_i\right) = \sum_{i=i_0}^{n} a_i \pm \sum_{i=i_0}^{n} b_i$ So, we can break up a summation across a sum or difference.

Note that we started the series at i_0 to denote the fact that they can start at any value of *i* that we need them to. Also note that while we can break up sums and differences as we did in **2** above we can't do the same thing for products and quotients. In other words,

$$\sum_{i=i_0}^{n} \left(a_i b_i\right) \neq \left(\sum_{i=i_0}^{n} a_i\right)\left(\sum_{i=i_0}^{n} b_i\right) \qquad\qquad \sum_{i=i_0}^{n} \frac{a_i}{b_i} \neq \frac{\displaystyle\sum_{i=i_0}^{n} a_i}{\displaystyle\sum_{i=i_0}^{n} b_i}$$

Formulas

Here are a couple of nice formulas that we will find useful in a couple of sections. Note that these formulas are only true if starting at $i = 1$. You can, of course, derive other formulas from these for different starting points if you need to.

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Here is a quick example on how to use these properties to quickly evaluate a sum that would not be easy to do by hand.

Example 1 Using the formulas and properties from above determine the value of the following summation.

$$\sum_{i=1}^{100} (3-2i)^2$$

Solution

The first thing that we need to do is square out the stuff being summed and then break up the summation using the properties as follows,

$$\begin{aligned}\sum_{i=1}^{100} (3-2i)^2 &= \sum_{i=1}^{100} 9 - 12i + 4i^2 \\ &= \sum_{i=1}^{100} 9 - \sum_{i=1}^{100} 12i + \sum_{i=1}^{100} 4i^2 \\ &= \sum_{i=1}^{100} 9 - 12\sum_{i=1}^{100} i + 4\sum_{i=1}^{100} i^2\end{aligned}$$

Now, using the formulas, this is easy to compute,

$$\begin{aligned}\sum_{i=1}^{100} (3-2i)^2 &= 9(100) - 12\left(\frac{100(101)}{2}\right) + 4\left(\frac{100(101)(201)}{6}\right) \\ &= 1293700\end{aligned}$$

Doing this by hand would definitely taken some time and there’s a good chance that we might have made a minor mistake somewhere along the line.

Section 7-9 : Constant of Integration

In this section we need to address a couple of topics about the constant of integration. Throughout most calculus classes we play pretty fast and loose with it and because of that many students don't really understand it or how it can be important.

First, let's address how we play fast and loose with it. Recall that technically when we integrate a sum or difference we are actually doing multiple integrals. For instance,

$$\int 15x^4 - 9x^{-2}\,dx = \int 15x^4\,dx - \int 9x^{-2}\,dx$$

Upon evaluating each of these integrals we should get a constant of integration for each integral since we really are doing two integrals.

$$\begin{aligned}\int 15x^4 - 9x^{-2}\,dx &= \int 15x^4\,dx - \int 9x^{-2}\,dx \\ &= 3x^5 + c + 9x^{-1} + k \\ &= 3x^5 + 9x^{-1} + c + k\end{aligned}$$

Since there is no reason to think that the constants of integration will be the same from each integral we use different constants for each integral.

Now, both *c* and *k* are unknown constants and so the sum of two unknown constants is just an unknown constant and we acknowledge that by simply writing the sum as a *c*.

So, the integral is then,

$$\int 15x^4 - 9x^{-2}\,dx = 3x^5 + 9x^{-1} + c$$

We also tend to play fast and loose with constants of integration in some substitution rule problems. Consider the following problem,

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\int \cos u + \sin u\,du \qquad\qquad u = 1+2x$$

Technically when we integrate we should get,

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\left(\sin u - \cos u + c\right)$$

Since the whole integral is multiplied by $\frac{1}{2}$, the whole answer, including the constant of integration, should be multiplied by $\frac{1}{2}$. Upon multiplying the $\frac{1}{2}$ through the answer we get,

$$\int \cos\left(1+2x\right) + \sin\left(1+2x\right)dx = \frac{1}{2}\sin u - \frac{1}{2}\cos u + \frac{c}{2}$$

However, since the constant of integration is an unknown constant dividing it by 2 isn't going to change that fact so we tend to just write the fraction as a *c*.

$$\int \cos(1+2x)+\sin(1+2x)\,dx = \frac{1}{2}\sin u - \frac{1}{2}\cos u + c$$

In general, we don't really need to worry about how we've played fast and loose with the constant of integration in either of the two examples above.

The real problem however is that because we play fast and loose with these constants of integration most students don't really have a good grasp of them and don't understand that there are times where the constants of integration are important and that we need to be careful with them.

To see how a lack of understanding about the constant of integration can cause problems consider the following integral.

$$\int \frac{1}{2x}\,dx$$

This is a really simple integral. However, there are two ways (both simple) to integrate it and that is where the problem arises.

The first integration method is to just break up the fraction and do the integral.

$$\int \frac{1}{2x}\,dx = \int \frac{1}{2}\frac{1}{x}\,dx = \frac{1}{2}\ln|x| + c$$

The second way is to use the following substitution.

$$u = 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x| + c$$

Can you see the problem? We integrated the same function and got very different answers. This doesn't make any sense. Integrating the same function should give us the same answer. We only used different methods to do the integral and both are perfectly legitimate integration methods. So, how can using different methods produce different answer?

The first thing that we should notice is that because we used a different method for each there is no reason to think that the constant of integration will in fact be the same number and so we really should use different letters for each.

More appropriate answers would be,

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|x| + c \qquad\qquad \int \frac{1}{2x}\,dx = \frac{1}{2}\ln|2x| + k$$

Now, let's take another look at the second answer. Using a property of logarithms we can write the answer to the second integral as follows,

$$\begin{aligned}\int \frac{1}{2x}\,dx &= \frac{1}{2}\ln|2x| + k \\ &= \frac{1}{2}\left(\ln 2 + \ln|x|\right) + k \\ &= \frac{1}{2}\ln|x| + \frac{1}{2}\ln 2 + k\end{aligned}$$

Upon doing this we can see that the answers really aren't that different after all. In fact they only differ by a constant and we can even find a relationship between *c* and *k*. It looks like,

$$c = \frac{1}{2}\ln 2 + k$$

So, without a proper understanding of the constant of integration, in particular using different integration techniques on the same integral will likely produce a different constant of integration, we might never figure out why we got "different" answers for the integral.

Note as well that getting answers that differ by a constant doesn't violate any principles of calculus. In fact, we've actually seen a fact that suggested that this might happen. We saw a fact in the **Mean Value Theorem** section that said that if $f'(x) = g'(x)$ then $f(x) = g(x) + c$. In other words, if two functions have the same derivative then they can differ by no more than a constant.

This is exactly what we've got here. The two functions,

$$f(x) = \frac{1}{2}\ln|x| \qquad\qquad g(x) = \frac{1}{2}\ln|2x|$$

have exactly the same derivative,

$$\frac{1}{2x}$$

and as we've shown they really only differ by a constant.

There is another integral that also exhibits this behavior. Consider,

$$\int \sin(x)\cos(x)\,dx$$

There are actually three different methods for doing this integral.

Method 1 :
This method uses a trig formula,

$$\sin(2x) = 2\sin(x)\cos(x)$$

Using this formula (and a quick substitution) the integral becomes,

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\int \sin(2x)\,dx = -\frac{1}{4}\cos(2x) + c_1$$

Method 2 :
This method uses the substitution,

$$u = \cos(x) \qquad\qquad du = -\sin(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + c_2 = -\frac{1}{2}\cos^2(x) + c_2$$

Method 3 :
Here is another substitution that could be done here as well.

$$u = \sin(x) \qquad\qquad du = \cos(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{1}{2}u^2 + c_3 = \frac{1}{2}\sin^2(x) + c_3$$

So, we've got three different answers each with a different constant of integration. However, according to the fact above these three answers should only differ by a constant since they all have the same derivative.

In fact, they do only differ by a constant. We'll need the following trig formulas to prove this.

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad\qquad \cos^2(x) + \sin^2(x) = 1$$

Start with the answer from the first method and use the double angle formula above.

$$-\frac{1}{4}\left(\cos^2(x) - \sin^2(x)\right) + c_1$$

Now, from the second identity above we have,

$$\sin^2(x) = 1 - \cos^2(x)$$

so, plug this in,

$$\begin{aligned} -\frac{1}{4}\left(\cos^2(x) - \left(1 - \cos^2(x)\right)\right) + c_1 &= -\frac{1}{4}\left(2\cos^2(x) - 1\right) + c_1 \\ &= -\frac{1}{2}\cos^2(x) + \frac{1}{4} + c_1 \end{aligned}$$

This is then answer we got from the second method with a slightly different constant. In other words,

$$c_2 = \frac{1}{4} + c_1$$

We can do a similar manipulation to get the answer from the third method. Again, starting with the answer from the first method use the double angle formula and then substitute in for the cosine instead of the sine using,

$$\cos^2(x) = 1 - \sin^2(x)$$

Doing this gives,

$$\begin{aligned} -\frac{1}{4}\left(\left(1 - \sin^2(x)\right) - \sin^2(x)\right) + c_1 &= -\frac{1}{4}\left(1 - 2\sin^2(x)\right) + c_1 \\ &= \frac{1}{2}\sin^2(x) - \frac{1}{4} + c_1 \end{aligned}$$

which is the answer from the third method with a different constant and again we can relate the two constants by,

$$c_3 = -\frac{1}{4} + c_1$$

So, what have we learned here? Hopefully we've seen that constants of integration are important and we can't forget about them. We often don't work with them in a Calculus I course, yet without a good understanding of them we would be hard pressed to understand how different integration methods can apparently produce different answers.