DIFFERENTIAL EQUATIONS

Laplace Transforms

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Preface

Here are my notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self-contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general, I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 4 : Laplace Transforms

In this chapter we will be looking at how to use Laplace transforms to solve differential equations. There are many kinds of transforms out there in the world. Laplace transforms and Fourier transforms are probably the main two kinds of transforms that are used. As we will see in later sections we can use Laplace transforms to reduce a differential equation to an algebra problem. The algebra can be messy on occasion, but it will be simpler than actually solving the differential equation directly in many cases. Laplace transforms can also be used to solve IVP's that we can't use any previous method on.

For "simple" differential equations such as those in the first few sections of the last chapter Laplace transforms will be more complicated than we need. In fact, for most homogeneous differential equations such as those in the last chapter Laplace transforms is significantly longer and not so useful. Also, many of the "simple" nonhomogeneous differential equations that we saw in the Undetermined Coefficients and Variation of Parameters are still simpler (or at the least no more difficult than Laplace transforms) to do as we did them there. However, at this point, the amount of work required for Laplace transforms is starting to equal the amount of work we did in those sections.

Laplace transforms comes into its own when the forcing function in the differential equation starts getting more complicated. In the previous chapter we looked only at nonhomogeneous differential equations in which $g(t)$ was a fairly simple continuous function. In this chapter we will start looking at $g(t)$'s that are not continuous. It is these problems where the reasons for using Laplace transforms start to become clear.

We will also see that, for some of the more complicated nonhomogeneous differential equations from the last chapter, Laplace transforms are actually easier on those problems as well.

Here is a brief rundown of the sections in this chapter.

The Definition – In this section we give the definition of the Laplace transform. We will also compute a couple Laplace transforms using the definition.

Laplace Transforms – In this section we introduce the way we usually compute Laplace transforms that avoids needing to use the definition. We discuss the table of Laplace transforms used in this material and work a variety of examples illustrating the use of the table of Laplace transforms.

Inverse Laplace Transforms – In this section we ask the opposite question from the previous section. In other words, given a Laplace transform, what function did we originally have? We again work a variety of examples illustrating how to use the table of Laplace transforms to do this as well as some of the manipulation of the given Laplace transform that is needed in order to use the table.

Step Functions – In this section we introduce the step or Heaviside function. We illustrate how to write a piecewise function in terms of Heaviside functions. We also work a variety of examples showing how to take Laplace transforms and inverse Laplace transforms that involve Heaviside functions. We also derive the formulas for taking the Laplace transform of functions which involve Heaviside functions.

Solving IVP's with Laplace Transforms – In this section we will examine how to use Laplace transforms to solve IVP's. The examples in this section are restricted to differential equations that could be solved without using Laplace transform. The advantage of starting out with this type of differential equation is that the work tends to be not as involved and we can always check our answers if we wish to.

Nonconstant Coefficient IVP's – In this section we will give a brief overview of using Laplace transforms to solve some nonconstant coefficient IVP's. We do not work a great many examples in this section. We only work a couple to illustrate how the process works with Laplace transforms.

IVP's with Step Functions – This is the section where the reason for using Laplace transforms really becomes apparent. We will use Laplace transforms to solve IVP's that contain Heaviside (or step) functions. Without Laplace transforms solving these would involve quite a bit of work. While we do work one of these examples without Laplace transforms, we do it only to show what would be involved if we did try to solve one of the examples without using Laplace transforms.

Dirac Delta Function – In this section we introduce the Dirac Delta function and derive the Laplace transform of the Dirac Delta function. We work a couple of examples of solving differential equations involving Dirac Delta functions and unlike problems with Heaviside functions our only real option for this kind of differential equation is to use Laplace transforms. We also give a nice relationship between Heaviside and Dirac Delta functions.

Convolution Integral – In this section we give a brief introduction to the convolution integral and how it can be used to take inverse Laplace transforms. We also illustrate its use in solving a differential equation in which the forcing function (*i.e.* the term without any *y*'s in it) is not known.

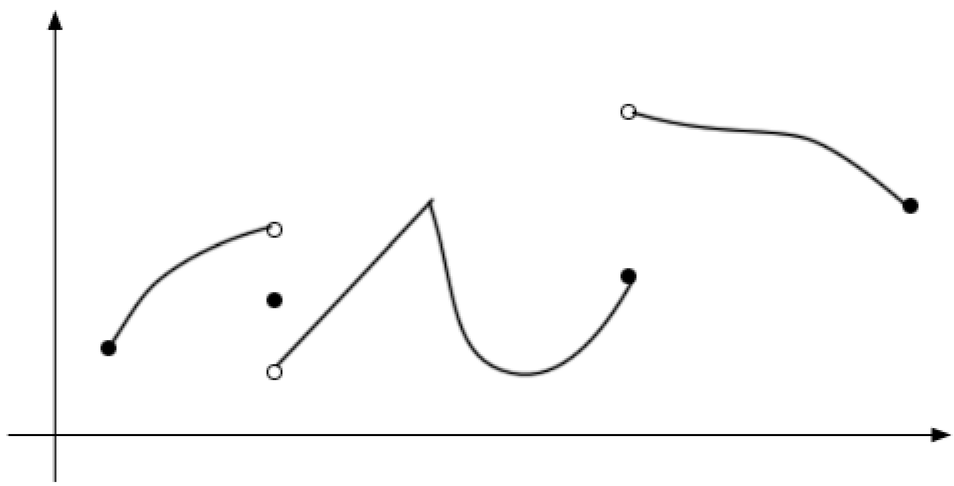
Table of Laplace Transforms – This section is the table of Laplace Transforms that we'll be using in the material. We give as wide a variety of Laplace transforms as possible including some that aren't often given in tables of Laplace transforms.

Section 4-1 : The Definition

You know, it's always a little scary when we devote a whole section just to the definition of something. Laplace transforms (or just transforms) can seem scary when we first start looking at them. However, as we will see, they aren't as bad as they may appear at first.

Before we start with the definition of the Laplace transform we need to get another definition out of the way.

A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.* the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval. Below is a sketch of a piecewise continuous function.



In other words, a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere.

Now, let's take a look at the definition of the Laplace transform.

Definition

Suppose that $f(t)$ is a piecewise continuous function. The Laplace transform of $f(t)$ is denoted $\mathcal{L}\left\{f\left(t\right)\right\}$ and defined as

$$\mathcal{L}\left\{f\left(t\right)\right\} = \int_0^\infty \mathbf{e}^{-st} f\left(t\right)\,dt \qquad \textbf{(1)}$$

There is an alternate notation for Laplace transforms. For the sake of convenience we will often denote Laplace transforms as,

$$\mathcal{L}\left\{f\left(t\right)\right\} = F\left(s\right)$$

With this alternate notation, note that the transform is really a function of a new variable, s, and that all the t's will drop out in the integration process.

Now, the integral in the definition of the transform is called an improper integral and it would probably be best to recall how these kinds of integrals work before we actually jump into computing some transforms.

Example 1 If $c \neq 0$, evaluate the following integral.

$$\int_0^\infty \mathbf{e}^{ct}\, dt$$

Solution

Remember that you need to convert improper integrals to limits as follows,

$$\int_0^\infty \mathbf{e}^{ct}\, dt = \lim_{n\to\infty} \int_0^n \mathbf{e}^{ct}\, dt$$

Now, do the integral, then evaluate the limit.

$$\begin{aligned} \int_0^\infty \mathbf{e}^{ct}\, dt &= \lim_{n\to\infty} \int_0^n \mathbf{e}^{ct}\, dt \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathbf{e}^{ct} \right)\Bigg|_0^n \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathbf{e}^{cn} - \frac{1}{c} \right) \end{aligned}$$

Now, at this point, we've got to be careful. The value of *c* will affect our answer. We've already assumed that *c* was non-zero, now we need to worry about the sign of *c*. If *c* is positive the exponential will go to infinity. On the other hand, if *c* is negative the exponential will go to zero.

So, the integral is only convergent (*i.e.* the limit exists and is finite) provided *c<0*. In this case we get,

$$\int_0^\infty \mathbf{e}^{ct}\, dt = -\frac{1}{c} \qquad \text{provided } c < 0 \tag{2}$$

Now that we remember how to do these, let's compute some Laplace transforms. We'll start off with probably the simplest Laplace transform to compute.

Example 2 Compute $\mathcal{L}\{1\}$.

Solution

There's not really a whole lot do here other than plug the function *f(t)* = 1 into (1)

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\, dt$$

Now, at this point notice that this is nothing more than the integral in the previous example with $c = -s$. Therefore, all we need to do is reuse (2) with the appropriate substitution. Doing this gives,

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\, dt = -\frac{1}{-s} \qquad \text{provided } -s < 0$$

Or, with some simplification we have,

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \text{provided } s > 0$$

Notice that we had to put a restriction on *s* in order to actually compute the transform. All Laplace transforms will have restrictions on *s*. At this stage of the game, this restriction is something that we tend to ignore, but we really shouldn't ever forget that it's there.

Let's do another example.

Example 3 Compute $\mathcal{L}\{\mathbf{e}^{at}\}$

Solution
Plug the function into the definition of the transform and do a little simplification.

$$\mathcal{L}\{\mathbf{e}^{at}\} = \int_0^\infty \mathbf{e}^{-st}\mathbf{e}^{at}\,dt = \int_0^\infty \mathbf{e}^{(a-s)t}\,dt$$

Once again, notice that we can use (2) provided $c = a - s$. So, let's do this.

$$\begin{aligned}\mathcal{L}\{\mathbf{e}^{at}\} &= \int_0^\infty \mathbf{e}^{(a-s)t}\,dt \\ &= -\frac{1}{a-s} \qquad && \text{provided } a - s < 0 \\ &= \frac{1}{s-a} && \text{provided } s > a\end{aligned}$$

Let's do one more example that doesn't come down to an application of (2).

Example 4 Compute $\mathcal{L}\{\sin(at)\}$.

Solution
Note that we're going to leave it to you to check most of the integration here. Plug the function into the definition. This time let's also use the alternate notation.

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= F(s) \\ &= \int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt \\ &= \lim_{n\to\infty}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\end{aligned}$$

Now, if we integrate by parts we will arrive at,

$$F(s) = \lim_{n\to\infty}\left(-\left(\frac{1}{a}\mathbf{e}^{-st}\cos(at)\right)\Bigg|_0^n - \frac{s}{a}\int_0^n \mathbf{e}^{-st}\cos(at)\,dt\right)$$

Now, evaluate the first term to simplify it a little and integrate by parts again on the integral. Doing this arrives at,

$$F(s) = \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathbf{e}^{-st}\sin(at)\right)\Big|_0^n + \frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right)$$

Now, evaluate the second term, take the limit and simplify.

$$\begin{aligned} F(s) &= \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right) - \frac{s}{a}\left(\frac{1}{a}\mathbf{e}^{-sn}\sin(an) + \frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right) \\ &= \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\right) \\ &= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt \end{aligned}$$

Now, notice that in the limits we had to assume that *s>0* in order to do the following two limits.

$$\lim_{n\to\infty}\mathbf{e}^{-sn}\cos(an) = 0$$
$$\lim_{n\to\infty}\mathbf{e}^{-sn}\sin(an) = 0$$

Without this assumption, we get a divergent integral again. Also, note that when we got back to the integral we just converted the upper limit back to infinity. The reason for this is that, if you think about it, this integral is nothing more than the integral that we started with. Therefore, we now get,

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s)$$

Now, simply solve for *F(s)* to get,

$$\mathcal{L}\{\sin(at)\} = F(s) = \frac{a}{s^2+a^2} \qquad \text{provided } s > 0$$

As this example shows, computing Laplace transforms is often messy.

Before moving on to the next section, we need to do a little side note. On occasion you will see the following as the definition of the Laplace transform.

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} \mathbf{e}^{-st} f(t)\,dt$$

Note the change in the lower limit from zero to negative infinity. In these cases there is almost always the assumption that the function *f(t)* is in fact defined as follows,

$$f(t) = \begin{cases} f(t) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

In other words, it is assumed that the function is zero if *t<0*. In this case the first half of the integral will drop out since the function is zero and we will get back to the definition given in (1). A Heaviside function is usually used to make the function zero for *t<0*. We will be looking at these in a later section.

Section 4-2 : Laplace Transforms

As we saw in the last section computing Laplace transforms directly can be fairly complicated. Usually we just use a table of transforms when actually computing Laplace transforms. The table that is provided here is not an all-inclusive table but does include most of the commonly used Laplace transforms and most of the commonly needed formulas pertaining to Laplace transforms.

Before doing a couple of examples to illustrate the use of the table let's get a quick fact out of the way.

Fact

Given *f(t)* and *g(t)* then,

$$\mathcal{L}\left\{af\left(t\right)+bg\left(t\right)\right\}=a\,F\left(s\right)+b\,G\left(s\right)$$

for any constants *a* and *b*.

In other words, we don't worry about constants and we don't worry about sums or differences of functions in taking Laplace transforms. All that we need to do is take the transform of the individual functions, then put any constants back in and add or subtract the results back up.

So, let's do a couple of quick examples.

Example 1 Find the Laplace transforms of the given functions.

(a) $f\left(t\right)=6\mathbf{e}^{-5t}+\mathbf{e}^{3t}+5t^{3}-9$

(b) $g\left(t\right)=4\cos\left(4t\right)-9\sin\left(4t\right)+2\cos\left(10t\right)$

(c) $h\left(t\right)=3\sinh\left(2t\right)+3\sin\left(2t\right)$

(d) $g\left(t\right)=\mathbf{e}^{3t}+\cos\left(6t\right)-\mathbf{e}^{3t}\cos\left(6t\right)$

Solution

Okay, there's not really a whole lot to do here other than go to the table, transform the individual functions up, put any constants back in and then add or subtract the results.

We'll do these examples in a little more detail than is typically used since this is the first time we're using the tables.

(a) $f\left(t\right)=6\mathbf{e}^{-5t}+\mathbf{e}^{3t}+5t^{3}-9$

$$\begin{aligned}F\left(s\right)&=6\frac{1}{s-\left(-5\right)}+\frac{1}{s-3}+5\frac{3!}{s^{3+1}}-9\frac{1}{s}\\&=\frac{6}{s+5}+\frac{1}{s-3}+\frac{30}{s^{4}}-\frac{9}{s}\end{aligned}$$

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$\begin{aligned} G(s) &= 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2} \\ &= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100} \end{aligned}$$

(c) $h(t) = 3\sinh(2t) + 3\sin(2t)$

$$\begin{aligned} H(s) &= 3\frac{2}{s^2-(2)^2} + 3\frac{2}{s^2+(2)^2} \\ &= \frac{6}{s^2-4} + \frac{6}{s^2+4} \end{aligned}$$

(d) $g(t) = \mathbf{e}^{3t} + \cos(6t) - \mathbf{e}^{3t}\cos(6t)$

$$\begin{aligned} G(s) &= \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2} \\ &= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36} \end{aligned}$$

Make sure that you pay attention to the difference between a "normal" trig function and hyperbolic functions. The only difference between them is the "$+a^2$" for the "normal" trig functions becomes a "$-a^2$" in the hyperbolic function! It's very easy to get in a hurry and not pay attention and grab the wrong formula. If you don't recall the definition of the hyperbolic functions see the notes for the table.

Let's do one final set of examples.

Example 2 Find the transform of each of the following functions.

(a) $f(t) = t\cosh(3t)$

(b) $h(t) = t^2\sin(2t)$

(c) $g(t) = t^{\frac{3}{2}}$

(d) $f(t) = (10t)^{\frac{3}{2}}$

(e) $f(t) = tg'(t)$

Solution

(a) $f(t) = t\cosh(3t)$

This function is not in the table of Laplace transforms. However, we can use #30 in the table to compute its transform. This will correspond to #30 if we take *n=1*.

$$F(s) = \mathcal{L}\{tg(t)\} = -G'(s), \quad \text{where } g(t) = \cosh(3t)$$

So, we then have,

$$G(s) = \frac{s}{s^2 - 9} \qquad\qquad G'(s) = -\frac{s^2 + 9}{\left(s^2 - 9\right)^2}$$

Using #30 we then have,

$$F(s) = \frac{s^2 + 9}{\left(s^2 - 9\right)^2}$$

(b) $h(t) = t^2 \sin(2t)$

This part will also use #30 in the table. In fact, we could use #30 in one of two ways. We could use it with $n = 1$.

$$H(s) = \mathcal{L}\{tf(t)\} = -F'(s), \qquad\qquad \text{where } f(t) = t\sin(2t)$$

Or we could use it with $n = 2$.

$$H(s) = \mathcal{L}\{t^2 f(t)\} = F''(s), \qquad\qquad \text{where } f(t) = \sin(2t)$$

Since it's less work to do one derivative, let's do it the first way. So, using #9 we have,

$$F(s) = \frac{4s}{\left(s^2 + 4\right)^2} \qquad\qquad F'(s) = -\frac{12s^2 - 16}{\left(s^2 + 4\right)^3}$$

The transform is then,

$$H(s) = \frac{12s^2 - 16}{\left(s^2 + 4\right)^3}$$

(c) $g(t) = t^{\frac{3}{2}}$

This part can be done using either #6 (with $n = 2$) or #32 (along with #5). We will use #32 so we can see an example of this. In order to use #32 we'll need to notice that

$$\int_0^t \sqrt{v}\, dv = \frac{2}{3} t^{\frac{3}{2}} \qquad \Rightarrow \qquad t^{\frac{3}{2}} = \frac{3}{2}\int_0^t \sqrt{v}\, dv$$

Now, using #5,

$$f(t) = \sqrt{t} \qquad\qquad F(s) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

we get the following.

$$G(s) = \frac{3}{2}\left(\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\right)\left(\frac{1}{s}\right) = \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}$$

This is what we would have gotten had we used #6.

(d) $f\left(t\right)=\left(10t\right)^{\frac{3}{2}}$

For this part we will use #24 along with the answer from the previous part. To see this note that if

$$g\left(t\right)=t^{\frac{3}{2}}$$

then

$$f\left(t\right)=g\left(10t\right)$$

Therefore, the transform is.

$$\begin{aligned}F\left(s\right)&=\frac{1}{10}G\left(\frac{s}{10}\right)\\&=\frac{1}{10}\left(\frac{3\sqrt{\pi}}{4\left(\frac{s}{10}\right)^{\frac{5}{2}}}\right)\\&=10^{\frac{3}{2}}\frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}\end{aligned}$$

(e) $f\left(t\right)=tg'\left(t\right)$

This final part will again use #30 from the table as well as #35.

$$\begin{aligned}\mathcal{L}\left\{tg'\left(t\right)\right\}&=-\frac{d}{ds}\mathcal{L}\left\{g'\right\}\\&=-\frac{d}{ds}\left\{sG\left(s\right)-g\left(0\right)\right\}\\&=-\left(G\left(s\right)+sG'\left(s\right)-0\right)\\&=-G\left(s\right)-sG'\left(s\right)\end{aligned}$$

Remember that *g(0)* is just a constant so when we differentiate it we will get zero!

As this set of examples has shown us we can't forget to use some of the general formulas in the table to derive new Laplace transforms for functions that aren't explicitly listed in the table!

Section 4-3 : Inverse Laplace Transforms

Finding the Laplace transform of a function is not terribly difficult if we've got a table of transforms in front of us to use as we saw in the last section. What we would like to do now is go the other way.

We are going to be given a transform, *F(s)*, and ask what function (or functions) did we have originally. As you will see this can be a more complicated and lengthy process than taking transforms. In these cases we say that we are finding the **Inverse Laplace Transform** of *F(s)* and use the following notation.

$$f\left(t\right)=\mathcal{L}^{-1}\left\{F\left(s\right)\right\}$$

As with Laplace transforms, we've got the following fact to help us take the inverse transform.

Fact

Given the two Laplace transforms *F(s)* and *G(s)* then

$$\mathcal{L}^{-1}\left\{aF\left(s\right)+bG\left(s\right)\right\}=a\mathcal{L}^{-1}\left\{F\left(s\right)\right\}+b\mathcal{L}^{-1}\left\{G\left(s\right)\right\}$$

for any constants *a* and *b*.

So, we take the inverse transform of the individual transforms, put any constants back in and then add or subtract the results back up.

Let's take a look at a couple of fairly simple inverse transforms.

Example 1 Find the inverse transform of each of the following.

(a) $F\left(s\right)=\dfrac{6}{s}-\dfrac{1}{s-8}+\dfrac{4}{s-3}$

(b) $H\left(s\right)=\dfrac{19}{s+2}-\dfrac{1}{3s-5}+\dfrac{7}{s^5}$

(c) $F\left(s\right)=\dfrac{6s}{s^2+25}+\dfrac{3}{s^2+25}$

(d) $G\left(s\right)=\dfrac{8}{3s^2+12}+\dfrac{3}{s^2-49}$

Solution

We've always felt that the key to doing inverse transforms is to look at the denominator and try to identify what you've got based on that. If there is only one entry in the table that has that particular denominator, the next step is to make sure the numerator is correctly set up for the inverse transform process. If it isn't, correct it (this is always easy to do) and then take the inverse transform.

If there is more than one entry in the table that has a particular denominator, then the numerators of each will be different, so go up to the numerator and see which one you've got. If you need to correct the numerator to get it into the correct form and then take the inverse transform.

So, with this advice in mind let's see if we can take some inverse transforms.

(a) $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$

From the denominator of the first term it looks like the first term is just a constant. The correct numerator for this term is a "1" so we'll just factor the 6 out before taking the inverse transform. The second term appears to be an exponential with *a = 8* and the numerator is exactly what it needs to be. The third term also appears to be an exponential, only this time $a = 3$ and we'll need to factor the 4 out before taking the inverse transforms.

So, with a little more detail than we'll usually put into these,

$$\begin{aligned} F(s) &= 6\,\frac{1}{s} - \frac{1}{s-8} + 4\frac{1}{s-3} \\ f(t) &= 6(1) - \mathbf{e}^{8t} + 4\left(\mathbf{e}^{3t}\right) \\ &= 6 - \mathbf{e}^{8t} + 4\mathbf{e}^{3t} \end{aligned}$$

(b) $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$

The first term in this case looks like an exponential with $a = -2$ and we'll need to factor out the 19. Be careful with negative signs in these problems, it's very easy to lose track of them.

The second term almost looks like an exponential, except that it's got a *3s* instead of just an *s* in the denominator. It is an exponential, but in this case, we'll need to factor a 3 out of the denominator before taking the inverse transform.

The denominator of the third term appears to be [#3](#) in the table with $n = 4$. The numerator however, is not correct for this. There is currently a 7 in the numerator and we need a 4! = 24 in the numerator. This is very easy to fix. Whenever a numerator is off by a multiplicative constant, as in this case, all we need to do is put the constant that we need in the numerator. We will just need to remember to take it back out by dividing by the same constant.

So, let's first rewrite the transform.

$$\begin{aligned} H(s) &= \frac{19}{s-(-2)} - \frac{1}{3\left(s - \frac{5}{3}\right)} + \frac{7\,\frac{4!}{4!}}{s^{4+1}} \\ &= 19\frac{1}{s-(-2)} - \frac{1}{3}\frac{1}{s-\frac{5}{3}} + \frac{7}{4!}\frac{4!}{s^{4+1}} \end{aligned}$$

So, what did we do here? We factored the 19 out of the first term. We factored the 3 out of the denominator of the second term since it can't be there for the inverse transform and in the third term we factored everything out of the numerator except the 4! since that is the portion that we need in the numerator for the inverse transform process.

Let's now take the inverse transform.

$$h(t) = 19\mathbf{e}^{-2t} - \frac{1}{3}\mathbf{e}^{\frac{5t}{3}} + \frac{7}{24}t^4$$

(c) $F\left(s\right)=\dfrac{6s}{s^2+25}+\dfrac{3}{s^2+25}$

In this part we've got the same denominator in both terms and our table tells us that we've either got #7 or #8. The numerators will tell us which we've actually got. The first one has an *s* in the numerator and so this means that the first term must be #8 and we'll need to factor the 6 out of the numerator in this case. The second term has only a constant in the numerator and so this term must be #7, however, in order for this to be exactly #7 we'll need to multiply/divide a 5 in the numerator to get it correct for the table.

The transform becomes,

$$\begin{aligned} F\left(s\right) &= 6\frac{s}{s^2+\left(5\right)^2}+\frac{3\frac{5}{5}}{s^2+\left(5\right)^2} \\ &= 6\frac{s}{s^2+\left(5\right)^2}+\frac{3}{5}\frac{5}{s^2+\left(5\right)^2} \end{aligned}$$

Taking the inverse transform gives,

$$f\left(t\right)=6\cos\left(5t\right)+\frac{3}{5}\sin\left(5t\right)$$

(d) $G\left(s\right)=\dfrac{8}{3s^2+12}+\dfrac{3}{s^2-49}$

In this case the first term will be a sine once we factor a 3 out of the denominator, while the second term appears to be a hyperbolic sine (#17). Again, be careful with the difference between these two. Both of the terms will also need to have their numerators fixed up. Here is the transform once we're done rewriting it.

$$\begin{aligned} G\left(s\right) &= \frac{1}{3}\frac{8}{s^2+4}+\frac{3}{s^2-49} \\ &= \frac{1}{3}\frac{\left(4\right)\left(2\right)}{s^2+\left(2\right)^2}+\frac{3\frac{7}{7}}{s^2-\left(7\right)^2} \end{aligned}$$

Notice that in the first term we took advantage of the fact that we could get the 2 in the numerator that we needed by factoring the 8. The inverse transform is then,

$$g\left(t\right)=\frac{4}{3}\sin\left(2t\right)+\frac{3}{7}\sinh\left(7t\right)$$

So, probably the best way to identify the transform is by looking at the denominator. If there is more than one possibility use the numerator to identify the correct one. Fix up the numerator if needed to get it into the form needed for the inverse transform process. Finally, take the inverse transform.

Let's do some slightly harder problems. These are a little more involved than the first set.

Example 2 Find the inverse transform of each of the following.

(a) $F(s) = \dfrac{6s-5}{s^2+7}$

(b) $F(s) = \dfrac{1-3s}{s^2+8s+21}$

(c) $G(s) = \dfrac{3s-2}{2s^2-6s-2}$

(d) $H(s) = \dfrac{s+7}{s^2-3s-10}$

Solution

(a) $F(s) = \dfrac{6s-5}{s^2+7}$

From the denominator of this one it appears that it is either a sine or a cosine. However, the numerator doesn't match up to either of these in the table. A cosine wants just an *s* in the numerator with at most a multiplicative constant, while a sine wants only a constant and no *s* in the numerator.

We've got both in the numerator. This is easy to fix however. We will just split up the transform into two terms and then do inverse transforms.

$$F(s) = \frac{6s}{s^2+7} - \frac{5\frac{\sqrt{7}}{\sqrt{7}}}{s^2+7}$$

$$f(t) = 6\cos\left(\sqrt{7}t\right) - \frac{5}{\sqrt{7}}\sin\left(\sqrt{7}t\right)$$

Do not get too used to always getting the perfect squares in sines and cosines that we saw in the first set of examples. More often than not (at least in my class) they won't be perfect squares!

(b) $F(s) = \dfrac{1-3s}{s^2+8s+21}$

In this case there are no denominators in our table that look like this. We can however make the denominator look like one of the denominators in the table by completing the square on the denominator. So, let's do that first.

$$\begin{aligned} s^2+8s+21 &= s^2+8s+16-16+21 \\ &= s^2+8s+16+5 \\ &= (s+4)^2+5 \end{aligned}$$

Recall that in completing the square you take half the coefficient of the *s*, square this, and then add and subtract the result to the polynomial. After doing this the first three terms should factor as a perfect square.

So, the transform can be written as the following.

$$F\left(s\right)=\frac{1-3s}{\left(s+4\right)^2+5}$$

Okay, with this rewrite it looks like we've got #19 and/or #20's from our table of transforms. However, note that in order for it to be a #19 we want just a constant in the numerator and in order to be a #20 we need an *s – a* in the numerator. We've got neither of these, so we'll have to correct the numerator to get it into proper form.

In correcting the numerator always get the *s – a* first. This is the important part. We will also need to be careful of the 3 that sits in front of the *s*. One way to take care of this is to break the term into two pieces, factor the 3 out of the second and then fix up the numerator of this term. This will work; however, it will put three terms into our answer and there are really only two terms.

So, we will leave the transform as a single term and correct it as follows,

$$\begin{aligned}F\left(s\right)&=\frac{1-3\left(s+4-4\right)}{\left(s+4\right)^2+5}\\&=\frac{1-3\left(s+4\right)+12}{\left(s+4\right)^2+5}\\&=\frac{-3\left(s+4\right)+13}{\left(s+4\right)^2+5}\end{aligned}$$

We needed an *s + 4* in the numerator, so we put that in. We just needed to make sure and take the 4 back out by subtracting it back out. Also, because of the 3 multiplying the *s* we needed to do all this inside a set of parenthesis. Then we partially multiplied the 3 through the second term and combined the constants. With the transform in this form, we can break it up into two transforms each of which are in the tables and so we can do inverse transforms on them,

$$\begin{aligned}F\left(s\right)&=-3\frac{s+4}{\left(s+4\right)^2+5}+\frac{13\frac{\sqrt{5}}{\sqrt{5}}}{\left(s+4\right)^2+5}\\f\left(t\right)&=-3\mathbf{e}^{-4t}\cos\left(\sqrt{5}t\right)+\frac{13}{\sqrt{5}}\mathbf{e}^{-4t}\sin\left(\sqrt{5}t\right)\end{aligned}$$

(c) $G\left(s\right)=\dfrac{3s-2}{2s^2-6s-2}$

This one is similar to the last one. We just need to be careful with the completing the square however. The first thing that we should do is factor a 2 out of the denominator, then complete the square. Remember that when completing the square a coefficient of 1 on the s^2 term is needed! So, here's the work for this transform.

$$G(s) = \frac{3s-2}{2\left(s^2-3s-1\right)}$$
$$= \frac{1}{2}\frac{3s-2}{s^2-3s+\frac{9}{4}-\frac{9}{4}-1}$$
$$= \frac{1}{2}\frac{3s-2}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$

So, it looks like we've got #21 and #22 with a corrected numerator. Here's the work for that and the inverse transform.

$$G(s) = \frac{1}{2}\frac{3\left(s-\frac{3}{2}+\frac{3}{2}\right)-2}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$
$$= \frac{1}{2}\frac{3\left(s-\frac{3}{2}\right)+\frac{5}{2}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$
$$= \frac{1}{2}\left(\frac{3\left(s-\frac{3}{2}\right)}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}+\frac{\frac{5}{2}\frac{\sqrt{13}}{\sqrt{13}}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}\right)$$
$$g(t) = \frac{1}{2}\left(3\mathbf{e}^{\frac{3t}{2}}\cosh\left(\frac{\sqrt{13}}{2}t\right)+\frac{5}{\sqrt{13}}\mathbf{e}^{\frac{3t}{2}}\sinh\left(\frac{\sqrt{13}}{2}t\right)\right)$$

In correcting the numerator of the second term, notice that I only put in the square root since we already had the "over 2" part of the fraction that we needed in the numerator.

(d) $H(s) = \dfrac{s+7}{s^2-3s-10}$

This one appears to be similar to the previous two, but it actually isn't. The denominators in the previous two couldn't be easily factored. In this case the denominator does factor and so we need to deal with it differently. Here is the transform with the factored denominator.

$$H(s) = \frac{s+7}{(s+2)(s-5)}$$

The denominator of this transform seems to suggest that we've got a couple of exponentials, however in order to be exponentials there can only be a single term in the denominator and no *s*'s in the numerator.

To fix this we will need to do partial fractions on this transform. In this case the partial fraction decomposition will be

$$H(s) = \frac{A}{s+2}+\frac{B}{s-5}$$

Don't remember how to do partial fractions? In this example we'll show you one way of getting the values of the constants and after this example we'll review how to get the correct form of the partial fraction decomposition.

Okay, so let's get the constants. There is a method for finding the constants that will always work, however it can lead to more work than is sometimes required. Eventually, we will need that method, however in this case there is an easier way to find the constants.

Regardless of the method used, the first step is to actually add the two terms back up. This gives the following.

$$\frac{s+7}{(s+2)(s-5)} = \frac{A(s-5)+B(s+2)}{(s+2)(s-5)}$$

Now, this needs to be true for any *s* that we should choose to put in. So, since the denominators are the same we just need to get the numerators equal. Therefore, set the numerators equal.

$$s+7 = A(s-5)+B(s+2)$$

Again, this must be true for ANY value of *s* that we want to put in. So, let's take advantage of that. If it must be true for any value of *s* then it must be true for $s=-2$, to pick a value at random. In this case we get,

$$5 = A(-7)+B(0) \qquad \Rightarrow A = -\frac{5}{7}$$

We found *A* by appropriately picking *s*. We can *B* in the same way if we chose $s=5$.

$$12 = A(0)+B(7) \qquad \Rightarrow \qquad B = \frac{12}{7}$$

This will not always work, but when it does it will usually simplify the work considerably.

So, with these constants the transform becomes,

$$H(s) = \frac{-\frac{5}{7}}{s+2} + \frac{\frac{12}{7}}{s-5}$$

We can now easily do the inverse transform to get,

$$h(t) = -\frac{5}{7}\mathbf{e}^{-2t} + \frac{12}{7}\mathbf{e}^{5t}$$

The last part of this example needed partial fractions to get the inverse transform. When we finally get back to differential equations and we start using Laplace transforms to solve them, you will quickly come to understand that partial fractions are a fact of life in these problems. Almost every problem will require partial fractions to one degree or another.

Note that we could have done the last part of this example as we had done the previous two parts. If we had we would have gotten hyperbolic functions. However, recalling the definition of the hyperbolic functions we could have written the result in the form we got from the way we worked our problem. However, most students have a better feel for exponentials than they do for hyperbolic functions and so it's usually best to just use partial fractions and get the answer in terms of exponentials. It may be a little more work, but it will give a nicer (and easier to work with) form of the answer.

Be warned that in my class I've got a rule that if the denominator can be factored with integer coefficients then it must be.

So, let's remind you how to get the correct partial fraction decomposition. The first step is to factor the denominator as much as possible. Then for each term in the denominator we will use the following table to get a term or terms for our partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$\left(ax+b\right)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{\left(ax+b\right)^2}+\cdots+\dfrac{A_k}{\left(ax+b\right)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$\left(ax^2+bx+c\right)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\dfrac{A_kx+B_k}{\left(ax^2+bx+c\right)^k}$

Notice that the first and third cases are really special cases of the second and fourth cases respectively.

So, let's do a couple more examples to remind you how to do partial fractions.

Example 3 Find the inverse transform of each of the following.

(a) $G\left(s\right)=\dfrac{86s-78}{\left(s+3\right)\left(s-4\right)\left(5s-1\right)}$

(b) $F\left(s\right)=\dfrac{2-5s}{\left(s-6\right)\left(s^2+11\right)}$

(c) $G\left(s\right)=\dfrac{25}{s^3\left(s^2+4s+5\right)}$

Solution

(a) $G\left(s\right)=\dfrac{86s-78}{\left(s+3\right)\left(s-4\right)\left(5s-1\right)}$

Here's the partial fraction decomposition for this part.

$$G(s) = \frac{A}{s+3} + \frac{B}{s-4} + \frac{C}{5s-1}$$

Now, this time we won't go into quite the detail as we did in the last example. We are after the numerator of the partial fraction decomposition and this is usually easy enough to do in our heads. Therefore, we will go straight to setting numerators equal.

$$86s - 78 = A(s-4)(5s-1) + B(s+3)(5s-1) + C(s+3)(s-4)$$

As with the last example, we can easily get the constants by correctly picking values of *s*.

$$\begin{array}{llll} s=-3 & -336 = A(-7)(-16) & \Rightarrow & A=-3 \\ s=\frac{1}{5} & -\frac{304}{5} = C\left(\frac{16}{5}\right)\left(-\frac{19}{5}\right) & \Rightarrow & C=5 \\ s=4 & 266 = B(7)(19) & \Rightarrow & B=2 \end{array}$$

So, the partial fraction decomposition for this transform is,

$$G(s) = -\frac{3}{s+3} + \frac{2}{s-4} + \frac{5}{5s-1}$$

Now, in order to actually take the inverse transform we will need to factor a 5 out of the denominator of the last term. The corrected transform as well as its inverse transform is.

$$G(s) = -\frac{3}{s+3} + \frac{2}{s-4} + \frac{1}{s-\frac{1}{5}}$$

$$g(t) = -3\mathbf{e}^{-3t} + 2\mathbf{e}^{4t} + \mathbf{e}^{\frac{t}{5}}$$

(b) $F(s) = \dfrac{2-5s}{(s-6)(s^2+11)}$

So, for the first time we've got a quadratic in the denominator. Here's the decomposition for this part.

$$F(s) = \frac{A}{s-6} + \frac{Bs+C}{s^2+11}$$

Setting numerators equal gives,

$$2 - 5s = A(s^2+11) + (Bs+C)(s-6)$$

Okay, in this case we could use $s=6$ to quickly find *A*, but that's all it would give. In this case we will need to go the "long" way around to getting the constants. Note that this way will always work but is sometimes more work than is required.

The "long" way is to completely multiply out the right side and collect like terms.

$$\begin{aligned} 2-5s &= A\left(s^2+11\right)+\left(Bs+C\right)\left(s-6\right) \\ &= As^2+11A+Bs^2-6B+Cs-6C \\ &= \left(A+B\right)s^2+\left(-6B+C\right)s+11A-6C \end{aligned}$$

In order for these two to be equal the coefficients of the s^2, s and the constants must all be equal. So, setting coefficients equal gives the following system of equations that can be solved.

$$\left.\begin{array}{l} s^2: \quad A+B=0 \\ s^1: \ -6B+C=-5 \\ s^0: 11A-6C=2 \end{array}\right\} \qquad \Rightarrow \qquad A=-\frac{28}{47}, \quad B=\frac{28}{47}, \quad C=-\frac{67}{47}$$

Notice that we used s^0 to denote the constants. This is habit on my part and isn't really required, it's just what I'm used to doing. Also, the coefficients are fairly messy fractions in this case. Get used to that. They will often be like this when we get back into solving differential equations.

There is a way to make our life a little easier as well with this. Since all of the fractions have a denominator of 47 we'll factor that out as we plug them back into the decomposition. This will make dealing with them much easier. The partial fraction decomposition is then,

$$\begin{aligned} F(s) &= \frac{1}{47}\left(-\frac{28}{s-6}+\frac{28s-67}{s^2+11}\right) \\ &= \frac{1}{47}\left(-\frac{28}{s-6}+\frac{28s}{s^2+11}-\frac{67\frac{\sqrt{11}}{\sqrt{11}}}{s^2+11}\right) \end{aligned}$$

The inverse transform is then.

$$f(t)=\frac{1}{47}\left(-28\mathbf{e}^{6t}+28\cos\left(\sqrt{11}t\right)-\frac{67}{\sqrt{11}}\sin\left(\sqrt{11}t\right)\right)$$

(c) $G(s)=\dfrac{25}{s^3\left(s^2+4s+5\right)}$

With this last part do not get excited about the s^3. We can think of this term as

$$s^3=\left(s-0\right)^3$$

and it becomes a linear term to a power. So, the partial fraction decomposition is

$$G(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+4s+5}$$

Setting numerators equal and multiplying out gives.

$$\begin{aligned} 25 &= As^2\left(s^2+4s+5\right)+Bs\left(s^2+4s+5\right)+C\left(s^2+4s+5\right)+\left(Ds+E\right)s^3 \\ &= \left(A+D\right)s^4+\left(4A+B+E\right)s^3+\left(5A+4B+C\right)s^2+\left(5B+4C\right)s+5C \end{aligned}$$

Setting coefficients equal gives the following system.

$$\left.\begin{array}{rr} s^4: & A+D=0 \\ s^3: & 4A+B+E=0 \\ s^2: & 5A+4B+C=0 \\ s^1: & 5B+4C=0 \\ s^0: & 5C=25 \end{array}\right\} \quad \Rightarrow \quad A=\frac{11}{5}, B=-4, C=5, D=-\frac{11}{5}, E=-\frac{24}{5}$$

This system looks messy, but it's easier to solve than it might look. First, we get *C* for free from the last equation. We can then use the fourth equation to find *B*. The third equation will then give *A*, *etc*.

When plugging into the decomposition we'll get everything with a denominator of 5, then factor that out as we did in the previous part in order to make things easier to deal with.

$$G(s)=\frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25}{s^3}-\frac{11s+24}{s^2+4s+5}\right)$$

Note that we also factored a minus sign out of the last two terms. To complete this part we'll need to complete the square on the later term and fix up a couple of numerators. Here's that work.

$$\begin{aligned} G(s) &= \frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25}{s^3}-\frac{11s+24}{s^2+4s+5}\right) \\ &= \frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25}{s^3}-\frac{11(s+2-2)+24}{(s+2)^2+1}\right) \\ &= \frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25\frac{2!}{2!}}{s^3}-\frac{11(s+2)}{(s+2)^2+1}-\frac{2}{(s+2)^2+1}\right) \end{aligned}$$

The inverse transform is then.

$$g(t)=\frac{1}{5}\left(11-20t+\frac{25}{2}t^2-11\mathbf{e}^{-2t}\cos(t)-2\mathbf{e}^{-2t}\sin(t)\right)$$

So, one final time. Partial fractions are a fact of life when using Laplace transforms to solve differential equations. Make sure that you can deal with them.

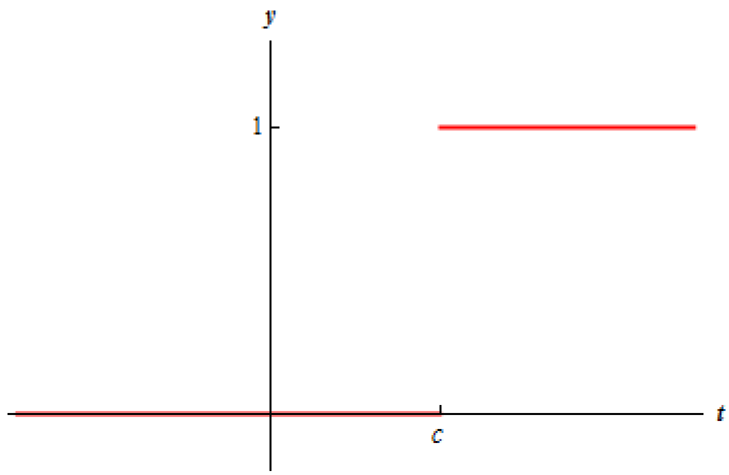
Section 4-4 : Step Functions

Before proceeding into solving differential equations we should take a look at one more function. Without Laplace transforms it would be much more difficult to solve differential equations that involve this function in *g(t)*.

The function is the Heaviside function and is defined as,

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \ge c \end{cases}$$

Here is a graph of the Heaviside function.



Heaviside functions are often called step functions. Here is some alternate notation for Heaviside functions.

$$u_c(t) = u(t-c) = H(t-c)$$

We can think of the Heaviside function as a switch that is off until $t = c$ at which point it turns on and takes a value of 1. So, what if we want a switch that will turn on and takes some other value, say 4, or -7?

Heaviside functions can only take values of 0 or 1, but we can use them to get other kinds of switches. For instance, $4u_c(t)$ is a switch that is off until $t = c$ and then turns on and takes a value of 4. Likewise, $-7u_c(t)$ will be a switch that will take a value of -7 when it turns on.

Now, suppose that we want a switch that is on (with a value of 1) and then turns off at $t = c$. We can use Heaviside functions to represent this as well. The following function will exhibit this kind of behavior.

$$1 - u_c(t) = \begin{cases} 1-0=1 & \text{if } t < c \\ 1-1=0 & \text{if } t \ge c \end{cases}$$

Prior to $t = c$ the Heaviside is off and so has a value of zero. The function as whole then for $t < c$ has a value of 1. When we hit $t = c$ the Heaviside function will turn on and the function will now take a value of 0.

We can also modify this so that it has values other than 1 when it is on. For instance,

$$3-3u_c(t)$$

will be a switch that has a value of 3 until it turns off at $t = c$.

We can also use Heaviside functions to represent much more complicated switches.

Example 1 Write the following function (or switch) in terms of Heaviside functions.

$$f(t)=\begin{cases}-4 & \text{if } t<6\\ 25 & \text{if } 6\le t<8\\ 16 & \text{if } 8\le t<30\\ 10 & \text{if } t\ge 30\end{cases}$$

Solution

There are three sudden shifts in this function and so (hopefully) it's clear that we're going to need three Heaviside functions here, one for each shift in the function. Here's the function in terms of Heaviside functions.

$$f(t)=-4+29u_6(t)-9u_8(t)-6u_{30}(t)$$

It's fairly easy to verify this.

In the first interval, $t < 6$ all three Heaviside functions are off and the function has the value

$$f(t)=-4$$

Notice that when we **know** that Heaviside functions are on or off we tend to not write them at all as we did in this case.

In the next interval, $6\le t<8$ the first Heaviside function is now on while the remaining two are still off. So, in this case the function has the value.

$$f(t)=-4+29=25$$

In the third interval, $8\le t<30$ the first two Heaviside functions are on while the last remains off. Here the function has the value.

$$f(t)=-4+29-9=16$$

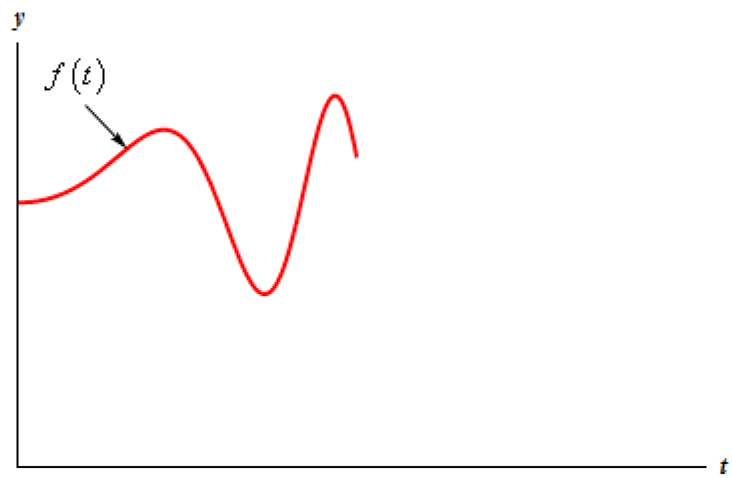
In the last interval, $t\ge 30$ all three Heaviside function are one and the function has the value.

$$f(t)=-4+29-9-6=10$$

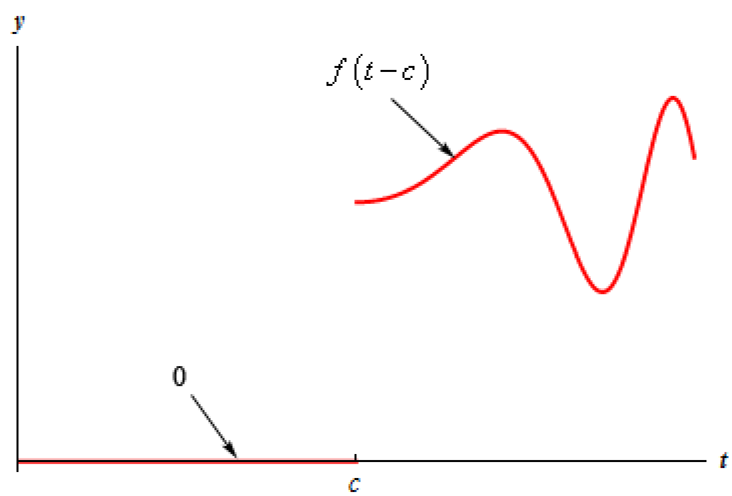
So, the function has the correct value in all the intervals.

All of this is fine, but if we continue the idea of using Heaviside function to represent switches, we really need to acknowledge that most switches will not turn on and take constant values. Most switches will turn on and vary continually with the value of *t.*

So, let's consider the following function.



We would like a switch that is off until *t = c* and then turns on and takes the values above. By this we mean that when *t = c* we want the switch to turn on and take the value of *f(0)* and when *t = c + 4* we want the switch to turn on and take the value of *f(4), etc.* In other words, we want the switch to look like the following,



Notice that in order to take the values that we want the switch to take it needs to turn on and take the values of $f(t-c)$! We can use Heaviside functions to help us represent this switch as well. Using Heaviside functions this switch can be written as

$$g(t) = u_c(t) f(t-c) \qquad \textbf{(1)}$$

Okay, we've talked a lot about Heaviside functions to this point, but we haven't even touched on Laplace transforms yet. So, let's start thinking about that. Let's determine the Laplace transform of (1). This is actually easy enough to derive so let's do that. Plugging (1) into the definition of the Laplace transform gives,

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \int_0^\infty \mathbf{e}^{-st}u_c(t)f(t-c)\,dt$$
$$= \int_c^\infty \mathbf{e}^{-st}f(t-c)\,dt$$

Notice that we took advantage of the fact that the Heaviside function will be zero if $t < c$ and 1 otherwise. This means that we can drop the Heaviside function and start the integral at c instead of 0. Now use the substitution $u = t - c$ and the integral becomes,

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \int_0^\infty \mathbf{e}^{-s(u+c)}f(u)\,du$$
$$= \int_0^\infty \mathbf{e}^{-su}\mathbf{e}^{-cs}f(u)\,du$$

The second exponential has no u's in it and so it can be factored out of the integral. Note as well that in the substitution process the lower limit of integration went back to 0.

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \mathbf{e}^{-cs}\int_0^\infty \mathbf{e}^{-su}f(u)\,du$$

Now, the integral left is nothing more than the integral that we would need to compute if we were going to find the Laplace transform of $f(t)$. Therefore, we get the following formula

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \mathbf{e}^{-cs}F(s) \qquad \textbf{(2)}$$

In order to use (2) the function $f(t)$ must be shifted by c, the same value that is used in the Heaviside function. Also note that we only take the transform of $f(t)$ and not $f(t-c)$! We can also turn this around to get a useful formula for inverse Laplace transforms.

$$\mathcal{L}^{-1}\left\{\mathbf{e}^{-cs}F(s)\right\} = u_c(t)f(t-c) \qquad \textbf{(3)}$$

We can use (2) to get the Laplace transform of a Heaviside function by itself. To do this we will consider the function in (2) to be $f(t) = 1$. Doing this gives us

$$\mathcal{L}\left\{u_c(t)\right\} = \mathcal{L}\left\{u_c(t)\bullet 1\right\} = \mathbf{e}^{-cs}\mathcal{L}\left\{1\right\} = \frac{1}{s}\mathbf{e}^{-cs} = \frac{\mathbf{e}^{-cs}}{s}$$

Putting all of this together leads to the following two formulas.

$$\mathcal{L}\left\{u_c(t)\right\} = \frac{\mathbf{e}^{-cs}}{s} \qquad\qquad \mathcal{L}^{-1}\left\{\frac{\mathbf{e}^{-cs}}{s}\right\} = u_c(t) \qquad \textbf{(4)}$$

Let's do some examples.

Example 2 Find the Laplace transform of each of the following.

(a) $g(t) = 10u_{12}(t) + 2(t-6)^3 u_6(t) - \left(7 - \mathbf{e}^{12-3t}\right)u_4(t)$

(b) $f(t) = -t^2 u_3(t) + \cos(t) u_5(t)$

(c) $h(t) = \begin{cases} t^4 & \text{if } t < 5 \\ t^4 + 3\sin\left(\dfrac{t}{10} - \dfrac{1}{2}\right) & \text{if } t \geq 5 \end{cases}$

(d) $f(t) = \begin{cases} t & \text{if } t < 6 \\ -8 + (t-6)^2 & \text{if } t \geq 6 \end{cases}$

Solution

In all of these problems remember that the function **MUST** be in the form

$$u_c(t) f(t-c)$$

before we start taking transforms. If it isn't in that form we will have to put it into that form!

(a) $g(t) = 10u_{12}(t) + 2(t-6)^3 u_6(t) - \left(7 - \mathbf{e}^{12-3t}\right)u_4(t)$

So, there are three terms in this function. The first is simply a Heaviside function and so we can use (4) on this term. The second and third terms however have functions with them and we need to identify the functions that are shifted for each of these. In the second term it appears that we are using the following function,

$$f(t) = 2t^3 \qquad \Rightarrow \qquad f(t-6) = 2(t-6)^3$$

and this has been shifted by the correct amount.

The third term uses the following function,

$$f(t) = 7 - \mathbf{e}^{-3t} \qquad \Rightarrow \qquad f(t-4) = 7 - \mathbf{e}^{-3(t-4)} = 7 - \mathbf{e}^{12-3t}$$

which has also been shifted by the correct amount.

With these functions identified we can now take the transform of the function.

$$\begin{aligned} G(s) &= \frac{10\mathbf{e}^{-12s}}{s} + \mathbf{e}^{-6s}\frac{2(3!)}{s^{3+1}} - \left(\frac{7}{s} - \frac{1}{s+3}\right)\mathbf{e}^{-4s} \\ &= \frac{10\mathbf{e}^{-12s}}{s} + \frac{12\mathbf{e}^{-6s}}{s^{3+1}} - \left(\frac{7}{s} - \frac{1}{s+3}\right)\mathbf{e}^{-4s} \end{aligned}$$

(b) $f(t) = -t^2 u_3(t) + \cos(t) u_5(t)$

This part is going to cause some problems. There are two terms and neither has been shifted by the proper amount. The first term needs to be shifted by 3 and the second needs to be shifted by 5. So, since they haven't been shifted, we will need to force the issue. We will need to add in the shifts, and then take them back out of course. Here they are.

$$f(t) = -(t-3+3)^2 u_3(t) + \cos(t-5+5) u_5(t)$$

Now we still have some potential problems here. The first function is still not really shifted correctly, so we'll need to use

$$(a+b)^2 = a^2 + 2ab + b^2$$

to get this shifted correctly.

The second term can be dealt with in one of two ways. The first would be to use the formula

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

to break it up into cosines and sines with arguments of t-5 which will be shifted as we expect. There is an easier way to do this one however. From our table of Laplace transforms we have #16 and using that we can see that if

$$g(t) = \cos(t+5) \qquad \Rightarrow \qquad g(t-5) = \cos(t-5+5)$$

This will make our life a little easier so we'll do it this way.

Now, breaking up the first term and leaving the second term alone gives us,

$$f(t) = -\left((t-3)^2 + 6(t-3) + 9\right) u_3(t) + \cos(t-5+5) u_5(t)$$

Okay, so it looks like the two functions that have been shifted here are

$$g(t) = t^2 + 6t + 9$$
$$g(t) = \cos(t+5)$$

Taking the transform then gives,

$$F(s) = -\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)\mathbf{e}^{-3s} + \left(\frac{s\cos(5) - \sin(5)}{s^2+1}\right)\mathbf{e}^{-5s}$$

It's messy, especially the second term, but there it is. Also, do not get excited about the $\cos(5)$ and $\sin(5)$. They are just numbers.

(c) $h(t) = \begin{cases} t^4 & \text{if } t < 5 \\ t^4 + 3\sin\left(\dfrac{t}{10} - \dfrac{1}{2}\right) & \text{if } t \ge 5 \end{cases}$

This one isn't as bad as it might look on the surface. The first thing that we need to do is write it in terms of Heaviside functions.

$$\begin{aligned} h(t) &= t^4 + 3u_5(t)\sin\left(\frac{t}{10} - \frac{1}{2}\right) \\ &= t^4 + 3u_5(t)\sin\left(\frac{1}{10}(t-5)\right) \end{aligned}$$

Since the t^4 is in both terms there isn't anything to do when we add in the Heaviside function. The only thing that gets added in is the sine term. Notice as well that the sine has been shifted by the proper amount.

All we need to do now is to take the transform.

$$\begin{aligned} H(s) &= \frac{4!}{s^5} + \frac{3\left(\frac{1}{10}\right)\mathbf{e}^{-5s}}{s^2 + \left(\frac{1}{10}\right)^2} \\ &= \frac{24}{s^5} + \frac{\frac{3}{10}\mathbf{e}^{-5s}}{s^2 + \frac{1}{100}} \end{aligned}$$

(d) $f(t) = \begin{cases} t & \text{if } t < 6 \\ -8 + (t-6)^2 & \text{if } t \geq 6 \end{cases}$

Again, the first thing that we need to do is write the function in terms of Heaviside functions.

$$f(t) = t + \left(-8 - t + (t-6)^2\right)u_6(t)$$

We had to add in a "-8" in the second term since that appears in the second part and we also had to subtract a t in the second term since the t in the first portion is no longer there. This subtraction of the t adds a problem because the second function is no longer correctly shifted. This is easier to fix than the previous example however.

Here is the corrected function.

$$\begin{aligned} f(t) &= t + \left(-8 - (t-6+6) + (t-6)^2\right)u_6(t) \\ &= t + \left(-8 - (t-6) - 6 + (t-6)^2\right)u_6(t) \\ &= t + \left(-14 - (t-6) + (t-6)^2\right)u_6(t) \end{aligned}$$

So, in the second term it looks like we are shifting

$$g(t) = t^2 - t - 14$$

The transform is then,

$$F(s) = \frac{1}{s^2} + \left(\frac{2}{s^3} - \frac{1}{s^2} - \frac{14}{s}\right)\mathbf{e}^{-6s}$$

Without the Heaviside function taking Laplace transforms is not a terribly difficult process provided we have our trusty table of transforms. However, with the advent of Heaviside functions, taking transforms can become a fairly messy process on occasion.

So, let's do some inverse Laplace transforms to see how they are done.

Example 3 Find the inverse Laplace transform of each of the following.

(a) $H(s) = \dfrac{s\mathbf{e}^{-4s}}{(3s+2)(s-2)}$

(b) $G(s) = \dfrac{5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}}{(s+2)(s^2+9)}$

(c) $F(s) = \dfrac{4s + \mathbf{e}^{-s}}{(s-1)(s+2)}$

(d) $G(s) = \dfrac{3s + 8\mathbf{e}^{-20s} - 2s\mathbf{e}^{-3s} + 6\mathbf{e}^{-7s}}{s^2(s+3)}$

Solution

All of these will use (3) somewhere in the process. Notice that in order to use this formula the exponential doesn't really enter into the mix until the very end. The vast majority of the process is finding the inverse transform of the stuff without the exponential.

In these problems we are not going to go into detail on many of the inverse transforms. If you need a refresher on some of the basics of inverse transforms go back and take a look at the previous section.

(a) $H(s) = \dfrac{s\mathbf{e}^{-4s}}{(3s+2)(s-2)}$

In light of the comments above let's first rewrite the transform in the following way.

$$H(s) = \mathbf{e}^{-4s}\frac{s}{(3s+2)(s-2)} = \mathbf{e}^{-4s}F(s)$$

Now, this problem really comes down to needing *f(t)*. So, let's do that. We'll need to partial fraction *F(s)* up. Here's the partial fraction decomposition.

$$F(s) = \frac{A}{3s+2} + \frac{B}{s-2}$$

Setting numerators equal gives,

$$s = A(s-2) + B(3s+2)$$

We'll find the constants here by selecting values of *s*. Doing this gives,

$$\begin{array}{llll} s = 2 & 2 = 8B & \Rightarrow & B = \dfrac{1}{4} \\ s = -\dfrac{2}{3} & -\dfrac{2}{3} = -\dfrac{8}{3}A & \Rightarrow & A = \dfrac{1}{4} \end{array}$$

So, the partial fraction decomposition becomes,

$$F(s) = \frac{\frac{1}{4}}{3\left(s+\frac{2}{3}\right)} + \frac{\frac{1}{4}}{s-2}$$

Notice that we factored a 3 out of the denominator in order to actually do the inverse transform. The inverse transform of this is then,

$$f(t) = \frac{1}{12}\mathbf{e}^{-\frac{2t}{3}} + \frac{1}{4}\mathbf{e}^{2t}$$

Now, let's go back and do the actual problem. The original transform was,

$$H(s) = \mathbf{e}^{-4s}F(s)$$

Note that we didn't bother to plug in *F(s)*. There really isn't a reason to plug it back in. Let's just use (3) to write down the inverse transform in terms of symbols. The inverse transform is,

$$h(t) = u_4(t)\,f(t-4)$$

where, *f(t)* is,

$$f(t) = \frac{1}{12}\mathbf{e}^{-\frac{2t}{3}} + \frac{1}{4}\mathbf{e}^{2t}$$

This is all the farther that we'll go with the answer. There really isn't any reason to plug in *f(t)* at this point. It would make the function longer and definitely messier. We will give almost all of our answers to these types of inverse transforms in this form.

(b) $G(s) = \dfrac{5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}}{(s+2)(s^2+9)}$

This problem is not as difficult as it might at first appear to be. Because there are two exponentials we will need to deal with them separately eventually. Now, this might lead us to conclude that the best way to deal with this function is to split it up as follows,

$$G(s) = \mathbf{e}^{-6s}\frac{5}{(s+2)(s^2+9)} - \mathbf{e}^{-11s}\frac{3}{(s+2)(s^2+9)}$$

Notice that we factored out the exponential, as we did in the last example, since we would need to do that eventually anyway. This is where a fairly common complication arises. Many people will call the first function *F(s)* and the second function *H(s)* and then partial fraction both of them.

However, if instead of just factoring out the exponential we would also factor out the coefficient we would get,

$$G(s) = 5\mathbf{e}^{-6s}\frac{1}{(s+2)(s^2+9)} - 3\mathbf{e}^{-11s}\frac{1}{(s+2)(s^2+9)}$$

Upon doing this we can see that the two functions are in fact the same function. The only difference is the constant that was in the numerator. So, the way that we'll do these problems is to first notice that both of the exponentials have only constants as coefficients. Instead of breaking things up then, we will simply factor out the whole numerator and get,

$$G(s) = \left(5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}\right)\frac{1}{(s+2)(s^2+9)} = \left(5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}\right)F(s)$$

and now we will just partial fraction *F(s)*.

Here is the partial fraction decomposition.

$$F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}$$

Setting numerators equal and combining gives us,

$$\begin{aligned} 1 &= A(s^2+9) + (s+2)(Bs+C) \\ &= (A+B)s^2 + (2B+C)s + 9A + 2C \end{aligned}$$

Setting coefficient equal and solving gives,

$$\left.\begin{aligned} s^2 &: \quad A+B=0 \\ s^1 &: \quad 2B+C=0 \\ s^0 &: 9A+2C=1 \end{aligned}\right\} \qquad \Rightarrow \qquad A = \frac{1}{13}, \quad B = -\frac{1}{13}, \quad C = \frac{2}{13}$$

Substituting back into the transform gives and fixing up the numerators as needed gives,

$$\begin{aligned} F(s) &= \frac{1}{13}\left(\frac{1}{s+2} + \frac{-s+2}{s^2+9}\right) \\ &= \frac{1}{13}\left(\frac{1}{s+2} - \frac{s}{s^2+9} + \frac{2\frac{3}{3}}{s^2+9}\right) \end{aligned}$$

As we did in the previous section we factored out the common denominator to make our work a little simpler. Taking the inverse transform then gives,

$$f(t) = \frac{1}{13}\left(\mathbf{e}^{-2t} - \cos(3t) + \frac{2}{3}\sin(3t)\right)$$

At this point we can go back and start thinking about the original problem.

$$\begin{aligned} G(s) &= \left(5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}\right)F(s) \\ &= 5\mathbf{e}^{-6s}F(s) - 3\mathbf{e}^{-11s}F(s) \end{aligned}$$

We'll also need to distribute the *F(s)* through as well in order to get the correct inverse transform. Recall that in order to use (3) to take the inverse transform you must have a single exponential times a single transform. This means that we must multiply the *F(s)* through the parenthesis. We can now take the inverse transform,

$$g(t) = 5u_6(t)f(t-6) - 3u_{11}(t)f(t-11)$$

where,

$$f(t) = \frac{1}{13}\left(\mathbf{e}^{-2t} - \cos(3t) + \frac{2}{3}\sin(3t)\right)$$

(c) $F(s) = \dfrac{4s + \mathbf{e}^{-s}}{(s-1)(s+2)}$

In this case, unlike the previous part, we will need to break up the transform since one term has a constant in it and the other has an *s*. Note as well that we don't consider the exponential in this, only its coefficient. Breaking up the transform gives,

$$F(s) = \frac{4s}{(s-1)(s+2)} + \mathbf{e}^{-s}\frac{1}{(s-1)(s+2)} = G(s) + \mathbf{e}^{-s}H(s)$$

We will need to partial fraction both of these terms up. We'll start with *G(s)*.

$$G(s) = \frac{A}{s-1} + \frac{B}{s+2}$$

Setting numerators equal gives,

$$4s = A(s+2) + B(s-1)$$

Now, pick values of *s* to find the constants.

$$s = -2 \qquad -8 = -3B \qquad \Rightarrow \qquad B = \frac{8}{3}$$

$$s = 1 \qquad 4 = 3A \qquad \Rightarrow \qquad A = \frac{4}{3}$$

So *G(s)* and its inverse transform is,

$$G(s) = \frac{\frac{4}{3}}{s-1} + \frac{\frac{8}{3}}{s+2}$$

$$g(t) = \frac{4}{3}\mathbf{e}^{t} + \frac{8}{3}\mathbf{e}^{-2t}$$

Now, repeat the process for *H(s)*.

$$H(s) = \frac{A}{s-1} + \frac{B}{s+2}$$

Setting numerators equal gives,

$$1 = A(s+2) + B(s-1)$$

Now, pick values of *s* to find the constants.

$$s = -2 \qquad 1 = -3B \qquad \Rightarrow \qquad B = -\frac{1}{3}$$

$$s = 1 \qquad 1 = 3A \qquad \Rightarrow \qquad A = \frac{1}{3}$$

So *H(s)* and its inverse transform is,

$$H\left(s\right) = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2}$$
$$h\left(t\right) = \frac{1}{3}\mathbf{e}^{t} - \frac{1}{3}\mathbf{e}^{-2t}$$

Putting all of this together gives the following,

$$F\left(s\right) = G\left(s\right) + \mathbf{e}^{-s}H\left(s\right)$$
$$f\left(t\right) = g\left(t\right) + u_1\left(t\right)h\left(t-1\right)$$

where,

$$g\left(t\right) = \frac{4}{3}\mathbf{e}^{t} + \frac{8}{3}\mathbf{e}^{-2t} \qquad \text{and} \qquad h\left(t\right) = \frac{1}{3}\mathbf{e}^{t} - \frac{1}{3}\mathbf{e}^{-2t}$$

(d) $G\left(s\right) = \dfrac{3s + 8\mathbf{e}^{-20s} - 2s\mathbf{e}^{-3s} + 6\mathbf{e}^{-7s}}{s^2\left(s+3\right)}$

This one looks messier than it actually is. Let's first rearrange the numerator a little.

$$G\left(s\right) = \frac{s\left(3 - 2\mathbf{e}^{-3s}\right) + \left(8\mathbf{e}^{-20s} + 6\mathbf{e}^{-7s}\right)}{s^2\left(s+3\right)}$$

In this form it looks like we can break this up into two pieces that will require partial fractions. When we break these up we should always try and break things up into as few pieces as possible for the partial fractioning. Doing this can save you a great deal of unnecessary work. Breaking up the transform as suggested above gives,

$$\begin{aligned} G\left(s\right) &= \left(3 - 2\mathbf{e}^{-3s}\right)\frac{1}{s\left(s+3\right)} + \left(8\mathbf{e}^{-20s} + 6\mathbf{e}^{-7s}\right)\frac{1}{s^2\left(s+3\right)} \\ &= \left(3 - 2\mathbf{e}^{-3s}\right)F\left(s\right) + \left(8\mathbf{e}^{-20s} + 6\mathbf{e}^{-7s}\right)H\left(s\right) \end{aligned}$$

Note that we canceled an *s* in *F(s)*. You should always simplify as much a possible before doing the partial fractions.

Let's partial fraction up *F(s)* first.

$$F\left(s\right) = \frac{A}{s} + \frac{B}{s+3}$$

Setting numerators equal gives,

$$1 = A(s+3) + Bs$$

Now, pick values of *s* to find the constants.

$$s = -3 \qquad 1 = -3B \qquad \Rightarrow \qquad B = -\frac{1}{3}$$

$$s = 0 \qquad 1 = 3A \qquad \Rightarrow \qquad A = \frac{1}{3}$$

So *F(s)* and its inverse transform is,

$$F(s) = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}}{s+3}$$

$$f(t) = \frac{1}{3} - \frac{1}{3}\mathbf{e}^{-3t}$$

Now partial fraction *H(s)*.

$$H(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Setting numerators equal gives,

$$1 = As(s+3) + B(s+3) + Cs^2$$

Pick values of *s* to find the constants.

$$s = -3 \qquad 1 = 9C \qquad \Rightarrow \qquad C = \frac{1}{9}$$

$$s = 0 \qquad 1 = 3B \qquad \Rightarrow \qquad B = \frac{1}{3}$$

$$s = 1 \qquad 1 = 4A + 4B + C = 4A + \frac{13}{9} \qquad \Rightarrow \qquad A = -\frac{1}{9}$$

So, *H(s)* and its inverse transform is,

$$H(s) = -\frac{\frac{1}{9}}{s} + \frac{\frac{1}{3}}{s^2} + \frac{\frac{1}{9}}{s+3}$$

$$h(t) = -\frac{1}{9} + \frac{1}{3}t + \frac{1}{9}\mathbf{e}^{-3t}$$

Now, let's go back to the original problem, remembering to multiply the transform through the parenthesis.

$$G(s) = 3F(s) - 2\mathbf{e}^{-3s}F(s) + 8\mathbf{e}^{-20s}H(s) + 6\mathbf{e}^{-7s}H(s)$$

Taking the inverse transform gives,

$$g(t) = 3f(t) - 2u_3(t)f(t-3) + 8u_{20}(t)h(t-20) + 6u_7(t)h(t-7)$$

So, as this example has shown, these can be a somewhat messy. However, the mess is really only that of notation and amount of work. The actual partial fraction work was identical to the previous sections work. The main difference in this section is we had to do more of it. As far as the inverse transform process goes. Again, the vast majority of that was identical to the previous section as well.

So, don’t let the apparent messiness of these problems get you to decide that you can’t do them. Generally, they aren’t as bad as they seem initially.

Section 4-5 : Solving IVP's with Laplace Transforms

It's now time to get back to differential equations. We've spent the last three sections learning how to take Laplace transforms and how to take inverse Laplace transforms. These are going to be invaluable skills for the next couple of sections so don't forget what we learned there.

Before proceeding into differential equations we will need one more formula. We will need to know how to take the Laplace transform of a derivative. First recall that $f^{(n)}$ denotes the n^{th} derivative of the function f. We now have the following fact.

Fact

Suppose that $f, f', f'', \ldots f^{(n-1)}$ are all continuous functions and $f^{(n)}$ is a piecewise continuous function. Then,

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Since we are going to be dealing with second order differential equations it will be convenient to have the Laplace transform of the first two derivatives.

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$

Notice that the two function evaluations that appear in these formulas, $y(0)$ and $y'(0)$, are often what we've been using for initial condition in out IVP's. So, this means that if we are to use these formulas to solve an IVP we will need initial conditions at $t = 0$.

While Laplace transforms are particularly useful for nonhomogeneous differential equations which have Heaviside functions in the forcing function we'll start off with a couple of fairly simple problems to illustrate how the process works.

Example 1 Solve the following IVP.

$$y'' - 10y' + 9y = 5t, \qquad y(0) = -1 \quad y'(0) = 2$$

Solution

The first step in using Laplace transforms to solve an IVP is to take the transform of every term in the differential equation.

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using the appropriate formulas from our table of Laplace transforms gives us the following.

$$s^2 Y(s) - sy(0) - y'(0) - 10\left(sY(s) - y(0)\right) + 9Y(s) = \frac{5}{s^2}$$

Plug in the initial conditions and collect all the terms that have a $Y(s)$ in them.

$$\left(s^2 - 10s + 9\right)Y(s) + s - 12 = \frac{5}{s^2}$$

Solve for *Y(s)*.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

At this point it's convenient to recall just what we're trying to do. We are trying to find the solution, *y(t)*, to an IVP. What we've managed to find at this point is not the solution, but its Laplace transform. So, in order to find the solution all that we need to do is to take the inverse transform.

Before doing that let's notice that in its present form we will have to do partial fractions twice. However, if we combine the two terms up we will only be doing partial fractions once. Not only that, but the denominator for the combined term will be identical to the denominator of the first term. This means that we are going to partial fraction up a term with that denominator no matter what so we might as well make the numerator slightly messier and then just partial fraction once.

This is one of those things where we are apparently making the problem messier, but in the process we are going to save ourselves a fair amount of work!

Combining the two terms gives,

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

The partial fraction decomposition for this transform is,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerators equal gives,

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Picking appropriate values of *s* and solving for the constants gives,

$$\begin{array}{llll} s = 0 & 5 = 9B & \Rightarrow & B = \frac{5}{9} \\ s = 1 & 16 = -8D & \Rightarrow & D = -2 \\ s = 9 & 248 = 648C & \Rightarrow & C = \frac{31}{81} \\ s = 2 & 45 = -14A + \frac{4345}{81} & \Rightarrow & A = \frac{50}{81} \end{array}$$

Plugging in the constants gives,

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Finally taking the inverse transform gives us the solution to the IVP.

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}\mathbf{e}^{9t} - 2\mathbf{e}^{t}$$

That was a fair amount of work for a problem that probably could have been solved much quicker using the techniques from the previous chapter. The point of this problem however, was to show how we would use Laplace transforms to solve an IVP.

There are a couple of things to note here about using Laplace transforms to solve an IVP. First, using Laplace transforms reduces a differential equation down to an algebra problem. In the case of the last example the algebra was probably more complicated than the straight forward approach from the last chapter. However, in later problems this will be reversed. The algebra, while still very messy, will often be easier than a straight forward approach.

Second, unlike the approach in the last chapter, we did not need to first find a general solution, differentiate this, plug in the initial conditions and then solve for the constants to get the solution. With Laplace transforms, the initial conditions are applied during the first step and at the end we get the actual solution instead of a general solution.

In many of the later problems Laplace transforms will make the problems significantly easier to work than if we had done the straight forward approach of the last chapter. Also, as we will see, there are some differential equations that simply can't be done using the techniques from the last chapter and so, in those cases, Laplace transforms will be our only solution.

Let's take a look at another fairly simple problem.

Example 2 Solve the following IVP.

$$2y'' + 3y' - 2y = t\mathbf{e}^{-2t}, \qquad y(0) = 0 \quad y'(0) = -2$$

Solution

As with the first example, let's first take the Laplace transform of all the terms in the differential equation. We'll the plug in the initial conditions to get,

$$2\left(s^2 Y(s) - sy(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) - 2Y(s) = \frac{1}{(s+2)^2}$$

$$\left(2s^2 + 3s - 2\right)Y(s) + 4 = \frac{1}{(s+2)^2}$$

Now solve for *Y(s)*.

$$Y(s) = \frac{1}{(2s-1)(s+2)^3} - \frac{4}{(2s-1)(s+2)}$$

Now, as we did in the last example we'll go ahead and combine the two terms together as we will have to partial fraction up the first denominator anyway, so we may as well make the numerator a little more complex and just do a single partial fraction. This will give,

$$Y(s) = \frac{1-4(s+2)^2}{(2s-1)(s+2)^3}$$
$$= \frac{-4s^2-16s-15}{(2s-1)(s+2)^3}$$

The partial fraction decomposition is then,

$$Y(s) = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

Setting numerator equal gives,

$$\begin{aligned}-4s^2-16s-15 &= A(s+2)^3 + B(2s-1)(s+2)^2 + C(2s-1)(s+2) + D(2s-1)\\ &= (A+2B)s^3 + (6A+7B+2C)s^2 + (12A+4B+3C+2D)s\\ &\qquad + 8A-4B-2C-D\end{aligned}$$

In this case it's probably easier to just set coefficients equal and solve the resulting system of equation rather than pick values of *s*. So, here is the system and its solution.

$$\left.\begin{array}{rr} s^3: & A+2B=0\\ s^2: & 6A+7B+2C=-4\\ s^1: & 12A+4B+3C+2D=-16\\ s^0: & 8A-4B-2C-D=-15\end{array}\right\} \Rightarrow \begin{array}{ll} A=-\frac{192}{125} & B=\frac{96}{125}\\ C=-\frac{2}{25} & D=-\frac{1}{5}\end{array}$$

We will get a common denominator of 125 on all these coefficients and factor that out when we go to plug them back into the transform. Doing this gives,

$$Y(s) = \frac{1}{125}\left(\frac{-192}{2\left(s-\frac{1}{2}\right)} + \frac{96}{s+2} - \frac{10}{(s+2)^2} - \frac{25\frac{2!}{2!}}{(s+2)^3}\right)$$

Notice that we also had to factor a 2 out of the denominator of the first term and fix up the numerator of the last term in order to get them to match up to the correct entries in our table of transforms.

Taking the inverse transform then gives,

$$y(t) = \frac{1}{125}\left(-96\mathbf{e}^{\frac{t}{2}} + 96\mathbf{e}^{-2t} - 10t\mathbf{e}^{-2t} - \frac{25}{2}t^2\mathbf{e}^{-2t}\right)$$

Example 3 Solve the following IVP.

$$y'' - 6y' + 15y = 2\sin(3t), \qquad y(0) = -1 \quad y'(0) = -4$$

Solution

Take the Laplace transform of everything and plug in the initial conditions.

$$s^2Y(s) - sy(0) - y'(0) - 6\left(sY(s) - y(0)\right) + 15Y(s) = 2\frac{3}{s^2+9}$$

$$\left(s^2 - 6s + 15\right)Y(s) + s - 2 = \frac{6}{s^2+9}$$

Now solve for *Y(s)* and combine into a single term as we did in the previous two examples.

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{\left(s^2+9\right)\left(s^2 - 6s + 15\right)}$$

Now, do the partial fractions on this. First let's get the partial fraction decomposition.

$$Y(s) = \frac{As + B}{s^2+9} + \frac{Cs + D}{s^2 - 6s + 15}$$

Now, setting numerators equal gives,

$$\begin{aligned} -s^3 + 2s^2 - 9s + 24 &= (As + B)\left(s^2 - 6s + 15\right) + (Cs + D)\left(s^2 + 9\right) \\ &= (A + C)s^3 + (-6A + B + D)s^2 + (15A - 6B + 9C)s + 15B + 9D \end{aligned}$$

Setting coefficients equal and solving for the constants gives,

$$\left.\begin{aligned} s^3 &: \quad A + C = -1 \\ s^2 &: \quad -6A + B + D = 2 \\ s^1 &: 15A - 6B + 9C = -9 \\ s^0 &: \quad 15B + 9D = 24 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{1}{10} & B &= \frac{1}{10} \\ C &= -\frac{11}{10} & D &= \frac{5}{2} \end{aligned}$$

Now, plug these into the decomposition, complete the square on the denominator of the second term and then fix up the numerators for the inverse transform process.

$$\begin{aligned} Y(s) &= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s + 25}{s^2 - 6s + 15}\right) \\ &= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s - 3 + 3) + 25}{(s-3)^2 + 6}\right) \\ &= \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{1\frac{3}{3}}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right) \end{aligned}$$

Finally, take the inverse transform.

$$y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11\mathbf{e}^{3t}\cos\left(\sqrt{6}t\right) - \frac{8}{\sqrt{6}}\mathbf{e}^{3t}\sin\left(\sqrt{6}t\right)\right)$$

To this point we've only looked at IVP's in which the initial values were at *t = 0*. This is because we need the initial values to be at this point in order to take the Laplace transform of the derivatives. The problem with all of this is that there are IVP's out there in the world that have initial values at places other than *t = 0*. Laplace transforms would not be as useful as it is if we couldn't use it on these types of IVP's. So, we need to take a look at an example in which the initial conditions are not at *t = 0* in order to see how to handle these kinds of problems.

Example 4 Solve the following IVP.

$$y'' + 4y' = \cos\left(t-3\right) + 4t, \qquad\qquad y\left(3\right) = 0 \quad y'\left(3\right) = 7$$

Solution

The first thing that we will need to do here is to take care of the fact that initial conditions are not at *t = 0*. The only way that we can take the Laplace transform of the derivatives is to have the initial conditions at *t = 0*.

This means that we will need to formulate the IVP in such a way that the initial conditions are at *t = 0*. This is actually fairly simple to do, however we will need to do a change of variable to make it work. We are going to define

$$\eta = t - 3 \qquad \Rightarrow \qquad t = \eta + 3$$

Let's start with the original differential equation.

$$y''\left(t\right) + 4y'\left(t\right) = \cos\left(t-3\right) + 4t$$

Notice that we put in the $\left(t\right)$ part on the derivatives to make sure that we get things correct here. We will next substitute in for *t*.

$$y''\left(\eta+3\right) + 4y'\left(\eta+3\right) = \cos\left(\eta\right) + 4\left(\eta+3\right)$$

Now, to simplify life a little let's define,

$$u\left(\eta\right) = y\left(\eta+3\right)$$

Then, by the chain rule, we get the following for the first derivative.

$$u'\left(\eta\right) = \frac{du}{d\eta} = \frac{dy}{dt}\frac{dt}{d\eta} = y'\left(\eta+3\right)$$

By a similar argument we get the following for the second derivative.

$$u''\left(\eta\right) = y''\left(\eta+3\right)$$

The initial conditions for $u\left(\eta\right)$ are,

$$u\left(0\right) = y\left(0+3\right) = y\left(3\right) = 0$$

$$u'\left(0\right) = y'\left(0+3\right) = y'\left(3\right) = 7$$

The IVP under these new variables is then,

$$u'' + 4u' = \cos\left(\eta\right) + 4\eta + 12, \qquad\qquad u\left(0\right) = 0 \quad u'\left(0\right) = 7$$

This is an IVP that we can use Laplace transforms on provided we replace all the *t*'s in our table with η's. So, taking the Laplace transform of this new differential equation and plugging in the new initial conditions gives,

$$\begin{aligned} s^2 U(s) - su(0) - u'(0) + 4\left(sU(s) - u(0)\right) &= \frac{s}{s^2+1} + \frac{4}{s^2} + \frac{12}{s} \\ \left(s^2+4s\right)U(s) - 7 &= \frac{s}{s^2+1} + \frac{4+12s}{s^2} \end{aligned}$$

Solving for *U(s)* gives,

$$\begin{aligned} \left(s^2+4s\right)U(s) &= \frac{s}{s^2+1} + \frac{4+12s+7s^2}{s^2} \\ U(s) &= \frac{1}{(s+4)\left(s^2+1\right)} + \frac{4+12s+7s^2}{s^3(s+4)} \end{aligned}$$

Note that unlike the previous examples we did not completely combine all the terms this time. In all the previous examples we did this because the denominator of one of the terms was the common denominator for all the terms. Therefore, upon combining, all we did was make the numerator a little messier and reduced the number of partial fractions required down from two to one. Note that all the terms in this transform that had only powers of *s* in the denominator were combined for exactly this reason.

In this transform however, if we combined both of the remaining terms into a single term we would be left with a fairly involved partial fraction problem. Therefore, in this case, it would probably be easier to just do partial fractions twice. We've done several partial fractions problems in this section and many partial fraction problems in the previous couple of sections so we're going to leave the details of the partial fractioning to you to check. Partial fractioning each of the terms in our transform gives us the following.

$$\begin{aligned} \frac{1}{(s+4)\left(s^2+1\right)} &= \frac{\frac{1}{17}}{s+4} + \frac{1}{17}\left(\frac{-s+4}{s^2+1}\right) \\ \frac{4+12s+7s^2}{s^3(s+4)} &= \frac{1}{s^3} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{17}{16}}{s+4} \end{aligned}$$

Plugging these into our transform and combining like terms gives us

$$\begin{aligned} U(s) &= \frac{1}{s^3} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{273}{272}}{s+4} + \frac{1}{17}\left(\frac{-s+4}{s^2+1}\right) \\ &= \frac{1\frac{2!}{2!}}{s^3} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{273}{272}}{s+4} + \frac{1}{17}\left(\frac{-s}{s^2+1} + \frac{4}{s^2+1}\right) \end{aligned}$$

Now, taking the inverse transform will give the solution to our new IVP. Don't forget to use η's instead of *t*'s!

$$u(\eta) = \frac{1}{2}\eta^2 + \frac{11}{4}\eta + \frac{17}{16} - \frac{273}{272}\mathbf{e}^{-4\eta} + \frac{1}{17}\left(4\sin(\eta) - \cos(\eta)\right)$$

This is not the solution that we are after of course. We are after *y(t)*. However, we can get this by noticing that

$$y(t) = y(\eta + 3) = u(\eta) = u(t-3)$$

So, the solution to the original IVP is,

$$y(t) = \frac{1}{2}(t-3)^2 + \frac{11}{4}(t-3) + \frac{17}{16} - \frac{273}{272}\mathbf{e}^{-4(t-3)} + \frac{1}{17}\left(4\sin(t-3) - \cos(t-3)\right)$$

$$y(t) = \frac{1}{2}t^2 - \frac{1}{4}t - \frac{43}{16} - \frac{273}{272}\mathbf{e}^{-4(t-3)} + \frac{1}{17}\left(4\sin(t-3) - \cos(t-3)\right)$$

So, we can now do IVP's that don't have initial conditions that are at *t = 0*. We also saw in the last example that it isn't always the best to combine all the terms into a single partial fraction problem as we have been doing prior to this example.

The examples worked in this section would have been just as easy, if not easier, if we had used techniques from the previous chapter. They were worked here using Laplace transforms to illustrate the technique and method.

Section 4-6 : Nonconstant Coefficient IVP's

In this section we are going to see how Laplace transforms can be used to solve some differential equations that do not have constant coefficients. This is not always an easy thing to do. However, there are some simple cases that can be done.

To do this we will need a quick fact.

Fact

If *f(t)* is a piecewise continuous function on $\left[0,\infty\right)$ of exponential order then,

$$\lim_{s\to\infty} F\left(s\right) = 0 \qquad \textbf{(1)}$$

A function *f(t)* is said to be of exponential order α if there exists positive constants *T* and *M* such that

$$\left|f\left(t\right)\right| \le M\mathbf{e}^{\alpha t} \qquad \text{for all } t \ge T$$

Put in other words, a function that is of exponential order will grow no faster than

$$M\mathbf{e}^{\alpha t}$$

for some *M* and α and all sufficiently large *t*. One way to check whether a function is of exponential order or not is to compute the following limit.

$$\lim_{t\to\infty}\frac{\left|f\left(t\right)\right|}{\mathbf{e}^{\alpha t}}$$

If this limit is finite for some α then the function will be of exponential order α. Likewise, if the limit is infinite for every α then the function is not of exponential order.

Almost all of the functions that you are liable to deal with in a first course in differential equations are of exponential order. A good example of a function that is not of exponential order is

$$f\left(t\right) = \mathbf{e}^{t^3}$$

We can check this by computing the above limit.

$$\lim_{t\to\infty}\frac{\mathbf{e}^{t^3}}{\mathbf{e}^{\alpha t}} = \lim_{t\to\infty}\mathbf{e}^{t^3-\alpha t} = \lim_{t\to\infty}\mathbf{e}^{t\left(t^2-\alpha\right)} = \infty$$

This is true for any value of α and so the function is not of exponential order.

Do not worry too much about this exponential order stuff. This fact is occasionally needed in using Laplace transforms with non constant coefficients.

So, let's take a look at an example.

Example 1 Solve the following IVP.

$$y'' + 3ty' - 6y = 2, \qquad y(0) = 0 \quad y'(0) = 0$$

Solution

So, for this one we will need to recall that #30 in our table of Laplace transforms tells us that,

$$\begin{aligned}\mathcal{L}\{ty'\} &= -\frac{d}{ds}\left(\mathcal{L}\{y'\}\right)\\ &= -\frac{d}{ds}\left(sY(s) - y(0)\right)\\ &= -sY'(s) - Y(s)\end{aligned}$$

So, upon taking the Laplace transforms of everything and plugging in the initial conditions we get,

$$\begin{aligned}s^2Y(s) - sy(0) - y'(0) + 3\left(-sY'(s) - Y(s)\right) - 6Y(s) &= \frac{2}{s}\\ -3sY'(s) + \left(s^2 - 9\right)Y(s) &= \frac{2}{s}\\ Y'(s) + \left(\frac{3}{s} - \frac{s}{3}\right)Y(s) &= -\frac{2}{3s^2}\end{aligned}$$

Unlike the examples in the previous section where we ended up with a transform for the solution, here we get a linear first order differential equation that must be solved in order to get a transform for the solution.

The integrating factor for this differential equation is,

$$\mu(s) = \mathbf{e}^{\int\left(\frac{3}{s} - \frac{s}{3}\right)ds} = \mathbf{e}^{\ln\left(s^3\right) - \frac{s^2}{6}} = s^3\mathbf{e}^{-\frac{s^2}{6}}$$

Multiplying through, integrating and solving for *Y(s)* gives,

$$\begin{aligned}\int\left(s^3\mathbf{e}^{-\frac{s^2}{6}}Y(s)\right)' ds &= \int -\frac{2}{3}s\mathbf{e}^{-\frac{s^2}{6}}\,ds\\ s^3\mathbf{e}^{-\frac{s^2}{6}}Y(s) &= 2\mathbf{e}^{-\frac{s^2}{6}} + c\\ Y(s) &= \frac{2}{s^3} + c\frac{\mathbf{e}^{\frac{s^2}{6}}}{s^3}\end{aligned}$$

Now, we have a transform for the solution. However, that second term looks unlike anything we've seen to this point. This is where the fact about the transforms of exponential order functions comes into play. We are going to assume that whatever our solution is, it is of exponential order. This means that

$$\lim_{s\to\infty}\left(\frac{2}{s^3} + \frac{c\mathbf{e}^{\frac{s^2}{6}}}{s^3}\right) = 0$$

The first term does go to zero in the limit. The second term however, will only go to zero if $c = 0$. Therefore, we must have $c = 0$ in order for this to be the transform of our solution.

So, the transform of our solution, as well as the solution is,

$$Y(s) = \frac{2}{s^3} \qquad\qquad y(t) = t^2$$

We'll leave it to you to verify that this is in fact a solution if you'd like to.

Now, not all nonconstant differential equations need to use (1). So, let's take a look at one more example.

Example 2 Solve the following IVP.

$$ty'' - ty' + y = 2, \qquad\qquad y(0) = 2 \quad y'(0) = -4$$

Solution

From the first example we have,

$$\mathcal{L}\{ty'\} = -sY'(s) - Y(s)$$

We'll also need,

$$\begin{aligned}\mathcal{L}\{ty''\} &= -\frac{d}{ds}\left(\mathcal{L}\{y''\}\right)\\ &= -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right)\\ &= -s^2Y'(s) - 2sY(s) + y(0)\end{aligned}$$

Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$\begin{aligned}-s^2Y'(s) - 2sY(s) + y(0) - \left(-sY'(s) - Y(s)\right) + Y(s) &= \frac{2}{s}\\ \left(s - s^2\right)Y'(s) + (2 - 2s)Y(s) + 2 &= \frac{2}{s}\\ s(1-s)Y'(s) + 2(1-s)Y(s) &= \frac{2(1-s)}{s}\\ Y'(s) + \frac{2}{s}Y(s) &= \frac{2}{s^2}\end{aligned}$$

Once again we have a linear first order differential equation that we must solve in order to get a transform for the solution. Notice as well that we never used the second initial condition in this work. That is okay, we will use it eventually.

Since this linear differential equation is much easier to solve compared to the first one, we'll leave the details to you. Upon solving the differential equation we get,

$$Y(s) = \frac{2}{s} + \frac{c}{s^2}$$

Now, this transform goes to zero for all values of *c* and we can take the inverse transform of the second term. Therefore, we won't need to use (1) to get rid of the second term as did in the previous example.

Taking the inverse transform gives,

$$y(t) = 2 + ct$$

Now, this is where we will use the second initial condition. Upon differentiating and plugging in the second initial condition we can see that $c = -4$.

So, the solution to this IVP is,

$$y(t) = 2 - 4t$$

So, we've seen how to use Laplace transforms to solve some nonconstant coefficient differential equations. Notice however that all we did was add in an occasional *t* to the coefficients. We couldn't get too complicated with the coefficients. If we had we would not have been able to easily use Laplace transforms to solve them.

Sometimes Laplace transforms can be used to solve nonconstant differential equations, however, in general, nonconstant differential equations are still very difficult to solve.

Section 4-7 : IVP's With Step Functions

In this section we will use Laplace transforms to solve IVP's which contain Heaviside functions in the forcing function. This is where Laplace transform really starts to come into its own as a solution method.

To work these problems we'll just need to remember the following two formulas,

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \mathbf{e}^{-cs}F(s) \qquad \text{where } F(s) = \mathcal{L}\left\{f(t)\right\}$$

$$\mathcal{L}^{-1}\left\{\mathbf{e}^{-cs}F(s)\right\} = u_c(t)f(t-c) \qquad \text{where } f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$$

In other words, we will always need to remember that in order to take the transform of a function that involves a Heaviside we've got to make sure the function has been properly shifted.

Let's work an example.

Example 1 Solve the following IVP.

$$y'' - y' + 5y = 4 + u_2(t)\mathbf{e}^{4-2t}, \qquad y(0) = 2 \qquad y'(0) = -1$$

Solution

First let's rewrite the forcing function to make sure that it's being shifted correctly and to identify the function that is actually being shifted.

$$y'' - y' + 5y = 4 + u_2(t)\mathbf{e}^{-2(t-2)}$$

So, it is being shifted correctly and the function that is being shifted is $\mathbf{e}^{-2t}$. Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$s^2Y(s) - sy(0) - y'(0) - \left(sY(s) - y(0)\right) + 5Y(s) = \frac{4}{s} + \frac{\mathbf{e}^{-2s}}{s+2}$$

$$\left(s^2 - s + 5\right)Y(s) - 2s + 3 = \frac{4}{s} + \frac{\mathbf{e}^{-2s}}{s+2}$$

Now solve for *Y(s)*.

$$\left(s^2 - s + 5\right)Y(s) = \frac{4}{s} + \frac{\mathbf{e}^{-2s}}{s+2} + 2s - 3$$

$$\left(s^2 - s + 5\right)Y(s) = \frac{2s^2 - 3s + 4}{s} + \frac{\mathbf{e}^{-2s}}{s+2}$$

$$Y(s) = \frac{2s^2 - 3s + 4}{s\left(s^2 - s + 5\right)} + \mathbf{e}^{-2s}\frac{1}{(s+2)\left(s^2 - s + 5\right)}$$

$$Y(s) = F(s) + \mathbf{e}^{-2s}G(s)$$

Notice that we combined a couple of terms to simplify things a little. Now we need to partial fraction *F(s)* and *G(s)*. We'll leave it to you to check the details of the partial fractions.

$$F(s)=\frac{2s^2-3s+4}{s(s^2-s+5)}=\frac{1}{5}\left(\frac{4}{s}+\frac{6s-11}{s^2-s+5}\right)$$

$$G(s)=\frac{1}{(s+2)(s^2-s+5)}=\frac{1}{11}\left(\frac{1}{s+2}-\frac{s-3}{s^2-s+5}\right)$$

We now need to do the inverse transforms on each of these. We'll start with *F(s)*.

$$F(s)=\frac{1}{5}\left(\frac{4}{s}+\frac{6\left(s-\frac{1}{2}+\frac{1}{2}\right)-11}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)$$

$$=\frac{1}{5}\left(\frac{4}{s}+\frac{6\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}-\frac{8\frac{\sqrt{19}}{2}\frac{2}{\sqrt{19}}}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)$$

$$f(t)=\frac{1}{5}\left(4+6\mathbf{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{19}}{2}t\right)-\frac{16}{\sqrt{19}}\mathbf{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{19}}{2}t\right)\right)$$

Now *G(s)*.

$$G(s)=\frac{1}{11}\left(\frac{1}{s+2}-\frac{s-\frac{1}{2}+\frac{1}{2}-3}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)$$

$$=\frac{1}{11}\left(\frac{1}{s+2}-\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}+\frac{\frac{5}{2}\frac{\sqrt{19}}{\sqrt{19}}}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)$$

$$g(t)=\frac{1}{11}\left(\mathbf{e}^{-2t}-\mathbf{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{19}}{2}t\right)+\frac{5}{\sqrt{19}}\mathbf{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{19}}{2}t\right)\right)$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y(s)=F(s)+\mathbf{e}^{-2s}G(s)$$

$$y(t)=f(t)+u_2(t)\,g(t-2)$$

where *f(t)* and *g(t)* are the functions shown above.

There can be a fair amount of work involved in solving differential equations that involve Heaviside functions.

Let's take a look at another example or two.

Example 2 Solve the following IVP.

$$y''-y'=\cos(2t)+\cos(2t-12)u_6(t) \qquad y(0)=-4,\ y'(0)=0$$

Solution

Let's rewrite the differential equation so we can identify the function that is actually being shifted.

$$y''-y'=\cos(2t)+\cos\left(2(t-6)\right)u_6(t)$$

So, the function that is being shifted is $\cos(2t)$ and it is being shifted correctly. Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) = \frac{s}{s^2+4} + \frac{s\mathbf{e}^{-6s}}{s^2+4}$$
$$(s^2 - s)Y(s) + 4s - 4 = \frac{s}{s^2+4} + \frac{s\mathbf{e}^{-6s}}{s^2+4}$$

Now solve for *Y(s)*.

$$(s^2 - s)Y(s) = \frac{s + s\mathbf{e}^{-6s}}{s^2+4} - 4s + 4$$
$$Y(s) = \frac{s(1+\mathbf{e}^{-6s})}{s(s-1)(s^2+4)} - 4\frac{s-1}{s(s-1)}$$
$$= \frac{1+\mathbf{e}^{-6s}}{(s-1)(s^2+4)} - \frac{4}{s}$$
$$Y(s) = (1+\mathbf{e}^{-6s})F(s) - \frac{4}{s}$$

Notice that we combined the first two terms to simplify things a little. Also, there was some canceling going on in this one. Do not expect that to happen on a regular basis. We now need to partial fraction *F(s)*. We'll leave the details to you to check.

$$F(s) = \frac{1}{(s-1)(s^2+4)} = \frac{1}{5}\left(\frac{1}{s-1} - \frac{s+1}{s^2+4}\right)$$
$$f(t) = \frac{1}{5}\left(\mathbf{e}^{t} - \cos(2t) - \frac{1}{2}\sin(2t)\right)$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y(s) = F(s) + F(s)\mathbf{e}^{-6s} - \frac{4}{s}$$
$$y(t) = f(t) + u_6(t)f(t-6) - 4$$

where *f(t)* is given above.

Example 3 Solve the following IVP.

$$y'' - 5y' - 14y = 9 + u_3(t) + 4(t-1)u_1(t) \qquad y(0) = 0,\ y'(0) = 10$$

Solution

Let's take the Laplace transform of everything and note that in the third term we are shifting 4*t*.

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) - 14Y(s) = \frac{9}{s} + \frac{\mathbf{e}^{-3s}}{s} + 4\frac{\mathbf{e}^{-s}}{s^2}$$
$$(s^2 - 5s - 14)Y(s) - 10 = \frac{9+\mathbf{e}^{-3s}}{s} + 4\frac{\mathbf{e}^{-s}}{s^2}$$

Now solve for *Y(s)*.

$$\left(s^2-5s-14\right)Y\left(s\right)-10=\frac{9+\mathbf{e}^{-3s}}{s}+4\frac{\mathbf{e}^{-s}}{s^2}$$

$$Y\left(s\right)=\frac{9+\mathbf{e}^{-3s}}{s\left(s-7\right)\left(s+2\right)}+\frac{4\mathbf{e}^{-s}}{s^2\left(s-7\right)\left(s+2\right)}+\frac{10}{\left(s-7\right)\left(s+2\right)}$$

$$Y\left(s\right)=\left(9+\mathbf{e}^{-3s}\right)F\left(s\right)+4\mathbf{e}^{-s}G\left(s\right)+H\left(s\right)$$

So, we have three functions that we'll need to partial fraction for this problem. I'll leave it to you to check the details.

$$F\left(s\right)=\frac{1}{s\left(s-7\right)\left(s+2\right)}=-\frac{1}{14}\frac{1}{s}+\frac{1}{63}\frac{1}{s-7}+\frac{1}{18}\frac{1}{s+2}$$

$$f\left(t\right)=-\frac{1}{14}+\frac{1}{63}\mathbf{e}^{7t}+\frac{1}{18}\mathbf{e}^{-2t}$$

$$G\left(s\right)=\frac{1}{s^2\left(s-7\right)\left(s+2\right)}=\frac{5}{196}\frac{1}{s}-\frac{1}{14}\frac{1}{s^2}+\frac{1}{441}\frac{1}{s-7}-\frac{1}{36}\frac{1}{s+2}$$

$$g\left(t\right)=\frac{5}{196}-\frac{1}{14}t+\frac{1}{441}\mathbf{e}^{7t}-\frac{1}{36}\mathbf{e}^{-2t}$$

$$H\left(s\right)=\frac{10}{\left(s-7\right)\left(s+2\right)}=\frac{10}{9}\frac{1}{s-7}-\frac{10}{9}\frac{1}{s+2}$$

$$h\left(t\right)=\frac{10}{9}\mathbf{e}^{7t}-\frac{10}{9}\mathbf{e}^{-2t}$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y\left(s\right)=9F\left(s\right)+\mathbf{e}^{-3s}F\left(s\right)+4\mathbf{e}^{-s}G\left(s\right)+H\left(s\right)$$

$$y\left(t\right)=9f\left(t\right)+u_3\left(t\right)f\left(t-3\right)+4u_1\left(t\right)g\left(t-1\right)+h\left(t\right)$$

where *f(t), g(t)* and *h(t)* are given above.

Let's work one more example.

Example 4 Solve the following IVP.

$$y''+3y'+2y=g\left(t\right),\qquad y\left(0\right)=0\qquad y'\left(0\right)=-2$$

where,

$$g\left(t\right)=\begin{cases}2 & t<6\\ t & 6\le t<10\\ 4 & t\ge 10\end{cases}$$

Solution

The first step is to get *g(t)* written in terms of Heaviside functions so that we can take the transform.

$$g\left(t\right)=2+\left(t-2\right)u_6\left(t\right)+\left(4-t\right)u_{10}\left(t\right)$$

Now, while this is *g(t)* written in terms of Heaviside functions it is not yet in proper form for us to take the transform. Remember that each function must be shifted by a proper amount. So, getting things set up for the proper shifts gives us,

$$g(t) = 2 + (t - 6 + 6 - 2)u_6(t) + \left(4 - (t - 10 + 10)\right)u_{10}(t)$$
$$g(t) = 2 + (t - 6 + 4)u_6(t) + \left(-6 - (t - 10)\right)u_{10}(t)$$

So, for the first Heaviside it looks like $f(t) = t + 4$ is the function that is being shifted and for the second Heaviside it looks like $f(t) = -6 - t$ is being shifted.

Now take the Laplace transform of everything and plug in the initial conditions.

$$s^2Y(s) - sy(0) - y'(0) + 3\left(sY(s) - y(0)\right) + 2Y(s) = \frac{2}{s} + \mathbf{e}^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - \mathbf{e}^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$
$$\left(s^2 + 3s + 2\right)Y(s) + 2 = \frac{2}{s} + \mathbf{e}^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - \mathbf{e}^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

Solve for *Y(s)*.

$$\left(s^2 + 3s + 2\right)Y(s) = \frac{2}{s} + \mathbf{e}^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - \mathbf{e}^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right) - 2$$
$$\left(s^2 + 3s + 2\right)Y(s) = \frac{2 + 4\mathbf{e}^{-6s} - 6\mathbf{e}^{-10s}}{s} + \frac{\mathbf{e}^{-6s} - \mathbf{e}^{-10s}}{s^2} - 2$$
$$Y(s) = \frac{2 + 4\mathbf{e}^{-6s} - 6\mathbf{e}^{-10s}}{s(s+1)(s+2)} + \frac{\mathbf{e}^{-6s} - \mathbf{e}^{-10s}}{s^2(s+1)(s+2)} - \frac{2}{(s+1)(s+2)}$$

$$Y(s) = \left(2 + 4\mathbf{e}^{-6s} - 6\mathbf{e}^{-10s}\right)F(s) + \left(\mathbf{e}^{-6s} - \mathbf{e}^{-10s}\right)G(s) - H(s)$$

Now, in the solving process we simplified things into as few terms as possible. Even doing this, it looks like we'll still need to do three partial fractions.

We'll leave the details of the partial fractioning to you to verify. The partial fraction form and inverse transform of each of these are.

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$
$$f(t) = \frac{1}{2} - \mathbf{e}^{-t} + \frac{1}{2}\mathbf{e}^{-2t}$$
$$G(s) = \frac{1}{s^2(s+1)(s+2)} = -\frac{\frac{3}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{1}{s+1} - \frac{\frac{1}{4}}{s+2}$$
$$g(t) = -\frac{3}{4} + \frac{1}{2}t + \mathbf{e}^{-t} - \frac{1}{4}\mathbf{e}^{-2t}$$

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$
$$h(t) = 2\mathbf{e}^{-t} - 2\mathbf{e}^{-2t}$$

Putting this all back together is going to be a little messy. First rewrite the transform a little to make the inverse transform process possible.

$$Y(s) = 2F(s) + \mathbf{e}^{-6s}\left(4F(s) + G(s)\right) - \mathbf{e}^{-10s}\left(6F(s) + G(s)\right) - H(s)$$

Now, taking the inverse transform of all the pieces gives us the final solution to the IVP.

$$y(t) = 2f(t) - h(t) + u_6(t)\left(4f(t-6) + g(t-6)\right) - u_{10}(t)\left(6f(t-10) + g(t-10)\right)$$

where *f(t)*, *g(t)*, and *h(t)* are defined above.

So, the answer to this example is a little messy to write down, but overall the work here wasn't too terribly bad.

Before proceeding with the next section let's see how we would have had to solve this IVP if we hadn't had Laplace transforms. To solve this IVP we would have had to solve three separate IVP's. One for each portion of *g(t)*. Here is a list of the IVP's that we would have had to solve.

1. $0 < t < 6$

$$y'' + 3y' + 2y = 2, \qquad y(0) = 0 \qquad y'(0) = -2$$

The solution to this IVP, with some work, can be made to look like,

$$y_1(t) = 2f(t) - h(t)$$

2. $6 \le t < 10$

$$y'' + 3y' + 2y = t, \qquad y(6) = y_1(6) \qquad y'(6) = y_1'(6)$$

where, $y_1(t)$ is the solution to the first IVP. The solution to this IVP, with some work, can be made to look like,

$$y_2(t) = 2f(t) - h(t) + 4f(t-6) + g(t-6)$$

3. $t \ge 10$

$$y'' + 3y' + 2y = 4, \qquad y(10) = y_2(10) \qquad y'(10) = y_2'(10)$$

where, $y_2(t)$ is the solution to the second IVP. The solution to this IVP, with some work, can be made to look like,

$$y_3(t) = 2f(t) - h(t) + 4f(t-6) + g(t-6) - 6f(t-10) - g(t-10)$$

There is a considerable amount of work required to solve all three of these and in each of these the forcing function is not that complicated. Using Laplace transforms saved us a fair amount of work.

Section 4-8 : Dirac Delta Function

When we first introduced Heaviside functions we noted that we could think of them as switches changing the forcing function, *g(t)*, at specified times. However, Heaviside functions are really not suited to forcing functions that exert a “large” force over a “small” time frame.

Examples of this kind of forcing function would be a hammer striking an object or a short in an electrical system. In both of these cases a large force (or voltage) would be exerted on the system over a very short time frame. The Dirac Delta function is used to deal with these kinds of forcing functions.

Dirac Delta Function

There are many ways to actually define the Dirac Delta function. To see some of these definitions visit Wolframs MathWorld. There are three main properties of the Dirac Delta function that we need to be aware of. These are,

1. $\delta\left(t-a\right)=0,\quad t\neq a$

2. $\int_{a-\varepsilon}^{a+\varepsilon}\delta\left(t-a\right)dt=1,\qquad \varepsilon>0$

3. $\int_{a-\varepsilon}^{a+\varepsilon}f\left(t\right)\delta\left(t-a\right)dt=f\left(a\right),\qquad \varepsilon>0$

At $t=a$ the Dirac Delta function is sometimes thought of has having an “infinite” value. So, the Dirac Delta function is a function that is zero everywhere except one point and at that point it can be thought of as either undefined or as having an “infinite” value.

Note that the integrals in the second and third property are actually true for any interval containing $t=a$, provided it’s not one of the endpoints. The limits given here are needed to prove the properties and so they are also given in the properties. We will however use the fact that they are true provided we are integrating over an interval containing $t=a$.

This is a very strange function. It is zero everywhere except one point and yet the integral of any interval containing that one point has a value of 1. The Dirac Delta function is not a real function as we think of them. It is instead an example of something called a **generalized function** or **distribution**.

Despite the strangeness of this “function” it does a very nice job of modeling sudden shocks or large forces to a system.

Before solving an IVP we will need the transform of the Dirac Delta function. We can use the third property above to get this.

$$\mathcal{L}\left\{\delta\left(t-a\right)\right\}=\int_{0}^{\infty}\mathbf{e}^{-st}\delta\left(t-a\right)dt=\mathbf{e}^{-as}\qquad\qquad \text{provided } a>0$$

Note that often the second and third properties are given with limits of infinity and negative infinity, but they are valid for any interval in which $t=a$ is in the interior of the interval.

With this we can now solve an IVP that involves a Dirac Delta function.

Example 1 Solve the following IVP.

$$y'' + 2y' - 15y = 6\delta(t-9), \qquad y(0) = -5 \quad y'(0) = 7$$

Solution

As with all previous problems we'll first take the Laplace transform of everything in the differential equation and apply the initial conditions.

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) - 15Y(s) = 6\mathbf{e}^{-9s}$$
$$\left(s^2 + 2s - 15\right)Y(s) + 5s + 3 = 6\mathbf{e}^{-9s}$$

Now solve for *Y(s)*.

$$\begin{aligned} Y(s) &= \frac{6\mathbf{e}^{-9s}}{(s+5)(s-3)} - \frac{5s+3}{(s+5)(s-3)} \\ &= 6\mathbf{e}^{-9s}F(s) - G(s) \end{aligned}$$

We'll leave it to you to verify the partial fractions and their inverse transforms are,

$$\begin{aligned} F(s) &= \frac{1}{(s+5)(s-3)} = \frac{\frac{1}{8}}{s-3} - \frac{\frac{1}{8}}{s+5} \\ f(t) &= \frac{1}{8}\mathbf{e}^{3t} - \frac{1}{8}\mathbf{e}^{-5t} \\ G(s) &= \frac{5s+3}{(s+5)(s-3)} = \frac{\frac{9}{4}}{s-3} + \frac{\frac{11}{4}}{s+5} \\ g(t) &= \frac{9}{4}\mathbf{e}^{3t} + \frac{11}{4}\mathbf{e}^{-5t} \end{aligned}$$

The solution is then,

$$\begin{aligned} Y(s) &= 6\mathbf{e}^{-9s}F(s) - G(s) \\ y(t) &= 6u_9(t)f(t-9) - g(t) \end{aligned}$$

where, *f(t)* and *g(t)* are defined above.

Example 2 Solve the following IVP.

$$2y'' + 10y = 3u_{12}(t) - 5\delta(t-4), \qquad y(0) = -1 \quad y'(0) = -2$$

Solution

Take the Laplace transform of everything in the differential equation and apply the initial conditions.

$$2\left(s^2Y(s) - sy(0) - y'(0)\right) + 10Y(s) = \frac{3\mathbf{e}^{-12s}}{s} - 5\mathbf{e}^{-4s}$$
$$\left(2s^2 + 10\right)Y(s) + 2s + 4 = \frac{3\mathbf{e}^{-12s}}{s} - 5\mathbf{e}^{-4s}$$

Now solve for *Y(s)*.

$$Y(s) = \frac{3\mathbf{e}^{-12s}}{s(2s^2+10)} - \frac{5\mathbf{e}^{-4s}}{2s^2+10} - \frac{2s+4}{2s^2+10}$$
$$= 3\mathbf{e}^{-12s}F(s) - 5\mathbf{e}^{-4s}G(s) - H(s)$$

We'll need to partial fraction the first function. The remaining two will just need a little work and they'll be ready. I'll leave the details to you to check.

$$F(s) = \frac{1}{s(2s^2+10)} = \frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s}{s^2+5}$$
$$f(t) = \frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}\,t\right)$$
$$g(t) = \frac{1}{2\sqrt{5}}\sin\left(\sqrt{5}\,t\right)$$
$$h(t) = \cos\left(\sqrt{5}\,t\right) + \frac{2}{\sqrt{5}}\sin\left(\sqrt{5}\,t\right)$$

The solution is then,

$$Y(s) = 3\mathbf{e}^{-12s}F(s) - 5\mathbf{e}^{-4s}G(s) - H(s)$$
$$y(t) = 3u_{12}(t)\,f(t-12) - 5u_4(t)\,g(t-4) - h(t)$$

where, *f(t)*,*g(t)* and *h(t)* are defined above.

So, with the exception of the new function these work the same way that all the problems that we've seen to this point work. Note as well that the exponential was introduced into the transform by the Dirac Delta function, but once in the transform it doesn't matter where it came from. In other words, when we went to the inverse transforms it came back out as a Heaviside function.

Before proceeding to the next section let's take a quick side trip and note that we can relate the Heaviside function and the Dirac Delta function. Start with the following integral.

$$\int_{-\infty}^{t} \delta(u-a)\,du = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

However, this is precisely the definition of the Heaviside function. So,

$$\int_{-\infty}^{t} \delta(u-a)\,du = u_a(t)$$

Now, recalling the Fundamental Theorem of Calculus, we get,

$$u_a'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-a)\,du\right) = \delta(t-a)$$

So, the derivative of the Heaviside function is the Dirac Delta function.

Section 4-9 : Convolution Integrals

On occasion we will run across transforms of the form,

$$H(s) = F(s)G(s)$$

that can't be dealt with easily using partial fractions. We would like a way to take the inverse transform of such a transform. We can use a convolution integral to do this.

Convolution Integral

If *f(t)* and *g(t)* are piecewise continuous function on $[0,\infty)$ then the convolution integral of *f(t)* and *g(t)* is,

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

A nice property of convolution integrals is.

$$(f*g)(t) = (g*f)(t)$$

Or,

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

The following fact will allow us to take the inverse transforms of a product of transforms.

Fact

$$\mathcal{L}\{f*g\} = F(s)G(s) \qquad\qquad \mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

Let's work a quick example to see how this can be used.

Example 1 Use a convolution integral to find the inverse transform of the following transform.

$$H(s) = \frac{1}{(s^2+a^2)^2}$$

Solution

First note that we could use #11 from out table to do this one so that will be a nice check against our work here. Also note that using a convolution integral here is one way to derive that formula from our table.

Now, since we are going to use a convolution integral here we will need to write it as a product whose terms are easy to find the inverse transforms of. This is easy to do in this case.

$$H(s) = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right)$$

So, in this case we have,

$$F(s) = G(s) = \frac{1}{s^2+a^2} \qquad \Rightarrow \qquad f(t) = g(t) = \frac{1}{a}\sin(at)$$

Using a convolution integral, along with massive use of trig formulas, *h(t)* is,

$$
\begin{aligned}
h(t) &= (f * g)(t) \\
&= \frac{1}{a^2}\int_0^t \sin(at - a\tau)\sin(a\tau)\,d\tau \\
&= \frac{1}{a^2}\int_0^t \left[\sin(at)\cos(a\tau) - \cos(at)\sin(a\tau)\right]\sin(a\tau)\,d\tau \\
&= \frac{1}{a^2}\int_0^t \sin(at)\cos(a\tau)\sin(a\tau) - \cos(at)\sin^2(a\tau)\,d\tau \\
&= \frac{1}{2a^2}\sin(at)\int_0^t \sin(2a\tau)\,d\tau - \frac{1}{2a^2}\cos(at)\int_0^t 1 - \cos(2a\tau)\,d\tau \\
&= \frac{1}{2a^2}\sin(at)\left(-\frac{1}{2a}\cos(2a\tau)\right)\Bigg|_0^t - \frac{1}{2a^2}\cos(at)\left(\tau - \frac{1}{2a}\sin(2a\tau)\right)\Bigg|_0^t \\
&= \frac{1}{4a^3}\left(\sin(at) - \sin(at)\cos(2at)\right) - \frac{1}{2a^2}\cos(at)\left(t - \frac{1}{2a}\sin(2at)\right) \\
&= \frac{1}{4a^3}\sin(at) - \frac{1}{2a^2}t\cos(at) - \frac{1}{4a^3}\sin(at)\cos(2at) + \frac{1}{4a^3}\cos(at)\sin(2at) \\
&= \frac{1}{4a^3}\sin(at) - \frac{1}{2a^2}t\cos(at) - \frac{1}{8a^3}\left[\sin(3at) + \sin(-at)\right] + \frac{1}{8a^3}\left[\sin(3at) - \sin(-at)\right] \\
&= \frac{1}{4a^3}\sin(at) - \frac{1}{2a^2}t\cos(at) + \frac{1}{8a^3}\sin(at) + \frac{1}{8a^3}\sin(at) \\
&= \frac{1}{2a^3}\sin(at) - \frac{1}{2a^2}t\cos(at) \\
&= \frac{1}{2a^3}\left(\sin(at) - at\cos(at)\right)
\end{aligned}
$$

This is exactly what we would have gotten by using #11 from the table.

Note however, that this did require a massive use of trig formulas that many do not readily recall. Also, while technically the integral was "simple", in reality it was a very long and messy integral and illustrates why convolution integrals are not always done even when they technically can be.

One final note about the integral just to make a point clear. In the fourth step we factored the $\sin(at)$ and $\cos(at)$ out of the integrals. We could do that, in this case, because the integrals are with respect to τ and so, as for as the integrals were concerned, any function of *t* is a constant. We can't, of course, generally factor variables out of integrals. We can only do that when the variables do not, in any way, depend on the variable of integration.

Convolution integrals are very useful in the following kinds of problems.

Example 2 Solve the following IVP

$$4y'' + y = g(t), \qquad y(0) = 3 \qquad y'(0) = -7$$

Solution

First, notice that the forcing function in this case has not been specified. Prior to this section we would not have been able to get a solution to this IVP. With convolution integrals we will be able to get a solution to this kind of IVP. The solution will be in terms of *g(t)* but it will be a solution.

Take the Laplace transform of all the terms and plug in the initial conditions.

$$4\left(s^2Y(s) - sy(0) - y'(0)\right) + Y(s) = G(s)$$
$$\left(4s^2 + 1\right)Y(s) - 12s + 28 = G(s)$$

Notice here that all we could do for the forcing function was to write down *G(s)* for its transform. Now, solve for *Y(s)*.

$$\left(4s^2 + 1\right)Y(s) = G(s) + 12s - 28$$
$$Y(s) = \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)}$$

We factored out a 4 from the denominator in preparation for the inverse transform process. To take inverse transforms we'll need to split up the first term and we'll also rewrite the second term a little.

$$\begin{aligned} Y(s) &= \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \\ &= \frac{3s}{s^2 + \frac{1}{4}} - \frac{7\frac{2}{2}}{s^2 + \frac{1}{4}} + \frac{1}{4}G(s)\frac{\frac{2}{2}}{s^2 + \frac{1}{4}} \end{aligned}$$

Now, the first two terms are easy to inverse transform. We'll need to use a convolution integral on the last term. The two functions that we will be using are,

$$g(t) \qquad f(t) = 2\sin\left(\frac{t}{2}\right)$$

We can shift either of the two functions in the convolution integral. We'll shift *g(t)* in our solution. Taking the inverse transform gives us,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{\tau}{2}\right) g(t - \tau)\, d\tau$$

So, once we decide on a *g(t)* all we need to do is to an integral and we'll have the solution.

As this last example has shown, using convolution integrals will allow us to solve IVP's with general forcing functions. This could be very convenient in cases where we have a variety of possible forcing functions and don't know which one we're going to use. With a convolution integral all that we need to do in these cases is solve the IVP once then go back and evaluate an integral for each possible *g(t)*. This will save us the work of having to solve the IVP for each and every *g(t)*.

Section 4-10 : Table Of Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n, \quad n = 1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$
4.	t^p , *p > -1*	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
6.	$t^{n-\frac{1}{2}}, \quad n = 1,2,3,\ldots$	$\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$
8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$
10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at) - at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$
12.	$\sin(at) + at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at) - at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$
14.	$\cos(at) + at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$
16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$

17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$
22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$
26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$
28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$
30.	$t^nf(t), \quad n=1,2,3,\ldots$	$(-1)^nF^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$
36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.

2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{\mathbf{e}^{t} + \mathbf{e}^{-t}}{2} \qquad \sinh(t) = \frac{\mathbf{e}^{t} - \mathbf{e}^{-t}}{2}$$

3. Be careful when using "normal" trig function vs. hyperbolic functions. The only difference in the formulas is the "+ a^2" for the "normal" trig functions becomes a "- a^2" for the hyperbolic functions!

4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty \mathbf{e}^{-x} x^{t-1}\, dx$$

If n is a positive integer then,

$$\Gamma(n+1) = n!$$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

