ALGEBRA

Exponential and Logarithm Functions

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Chapter 6 : Exponential and Logarithm Functions

In this chapter we are going to look at exponential and logarithm functions. Both of these functions are very important and need to be understood by anyone who is going on to later math courses. These functions also have applications in science, engineering, and business to name a few areas. In fact, these functions can show up in just about any field that uses even a small degree of mathematics.

Many students find both of these functions, especially logarithm functions, difficult to deal with. This is probably because they are so different from any of the other functions that they've looked at to this point and logarithms use a notation that will be new to almost everyone in an algebra class. However, you will find that once you get past the notation and start to understand some of their properties they really aren't too bad.

Here is a listing of the topics covered in this chapter.

Exponential Functions – In this section we will introduce exponential functions. We will give some of the basic properties and graphs of exponential functions. We will also discuss what many people consider to be the exponential function, $f\left(x\right)=\mathbf{e}^{x}$.

Logarithm Functions – In this section we will introduce logarithm functions. We give the basic properties and graphs of logarithm functions. In addition, we discuss how to evaluate some basic logarithms including the use of the change of base formula. We will also discuss the common logarithm, $\log\left(x\right)$, and the natural logarithm, $\ln\left(x\right)$.

Solving Exponential Equations – In this section we will discuss a couple of methods for solving equations that contain exponentials.

Solving Logarithm Equations – In this section we will discuss a couple of methods for solving equations that contain logarithms. Also, as we'll see, with one of the methods we will need to be careful of the results of the method as it is always possible that the method gives values that are, in fact, not solutions to the equation.

Applications – In this section we will look at a couple of applications of exponential functions and an application of logarithms. We look at compound interest, exponential growth and decay and earthquake intensity.

Section 6-1 : Exponential Functions

Let's start off this section with the definition of an exponential function.

If *b* is any number such that $b > 0$ and $b \neq 1$ then an **exponential function** is a function in the form,

$$f\left(x\right) = b^x$$

where *b* is called the **base** and *x* can be any real number.

Notice that the *x* is now in the exponent and the base is a fixed number. This is exactly the opposite from what we've seen to this point. To this point the base has been the variable, *x* in most cases, and the exponent was a fixed number. However, despite these differences these functions evaluate in exactly the same way as those that we are used to. We will see some examples of exponential functions shortly.

Before we get too far into this section we should address the restrictions on *b*. We avoid one and zero because in this case the function would be,

$$f\left(x\right) = 0^x = 0 \qquad \text{and} \qquad f\left(x\right) = 1^x = 1$$

and these are constant functions and won't have many of the same properties that general exponential functions have.

Next, we avoid negative numbers so that we don't get any complex values out of the function evaluation. For instance, if we allowed $b = -4$ the function would be,

$$f\left(x\right) = \left(-4\right)^x \qquad \Rightarrow \qquad f\left(\frac{1}{2}\right) = \left(-4\right)^{\frac{1}{2}} = \sqrt{-4}$$

and as you can see there are some function evaluations that will give complex numbers. We only want real numbers to arise from function evaluation and so to make sure of this we require that *b* not be a negative number.

Now, let's take a look at a couple of graphs. We will be able to get most of the properties of exponential functions from these graphs.

Example 1 Sketch the graph of $f\left(x\right) = 2^x$ and $g\left(x\right) = \left(\frac{1}{2}\right)^x$ on the same axis system.

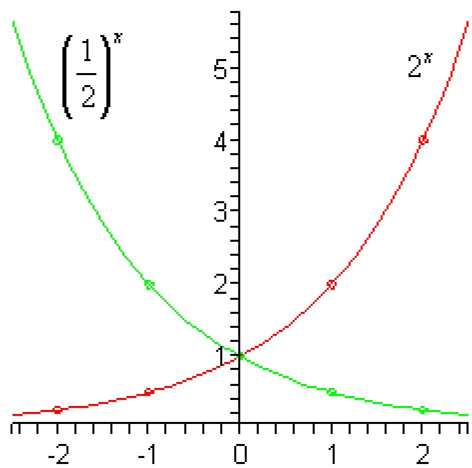
Solution

Okay, since we don't have any knowledge on what these graphs look like we're going to have to pick some values of *x* and do some function evaluations. Function evaluation with exponential functions works in exactly the same manner that all function evaluation has worked to this point. Whatever is in the parenthesis on the left we substitute into all the *x*'s on the right side.

Here are some evaluations for these two functions,

x	$f(x)=2^x$	$g(x)=\left(\frac{1}{2}\right)^x$
-2	$f(-2)=2^{-2}=\frac{1}{2^2}=\frac{1}{4}$	$g(-2)=\left(\frac{1}{2}\right)^{-2}=\left(\frac{2}{1}\right)^2=4$
-1	$f(-1)=2^{-1}=\frac{1}{2^1}=\frac{1}{2}$	$g(-1)=\left(\frac{1}{2}\right)^{-1}=\left(\frac{2}{1}\right)^1=2$
0	$f(0)=2^0=1$	$g(0)=\left(\frac{1}{2}\right)^0=1$
1	$f(1)=2^1=2$	$g(1)=\left(\frac{1}{2}\right)^1=\frac{1}{2}$
2	$f(2)=2^2=4$	$g(2)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$

Here is the sketch of the two graphs.



Note as well that we could have written $g(x)$ in the following way,

$$g(x)=\left(\frac{1}{2}\right)^x=\frac{1}{2^x}=2^{-x}$$

Sometimes we'll see this kind of exponential function and so it's important to be able to go between these two forms.

Now, let's talk about some of the properties of exponential functions.

Properties of $f(x) = b^x$

1. The graph of $f(x)$ will always contain the point $(0,1)$. Or put another way, $f(0) = 1$ regardless of the value of *b*.
2. For every possible *b* we have $b^x > 0$. Note that this implies that $b^x \neq 0$.
3. If $0 < b < 1$ then the graph of b^x will decrease as we move from left to right. Check out the graph of $\left(\frac{1}{2}\right)^x$ above for verification of this property.
4. If $b > 1$ then the graph of b^x will increase as we move from left to right. Check out the graph of 2^x above for verification of this property.
5. If $b^x = b^y$ then $x = y$

All of these properties except the final one can be verified easily from the graphs in the first example. We will hold off discussing the final property for a couple of sections where we will actually be using it.

As a final topic in this section we need to discuss a special exponential function. In fact this is so special that for many people this is THE exponential function. Here it is,

$$f(x) = \mathbf{e}^x$$

where $\mathbf{e} = 2.718281828\ldots$. Note the difference between $f(x) = b^x$ and $f(x) = \mathbf{e}^x$. In the first case *b* is any number that meets the restrictions given above while **e** is a very specific number. Also note that **e** is not a terminating decimal.

This special exponential function is very important and arises naturally in many areas. As noted above, this function arises so often that many people will think of this function if you talk about exponential functions. We will see some of the applications of this function in the final section of this chapter.

Let's get a quick graph of this function.

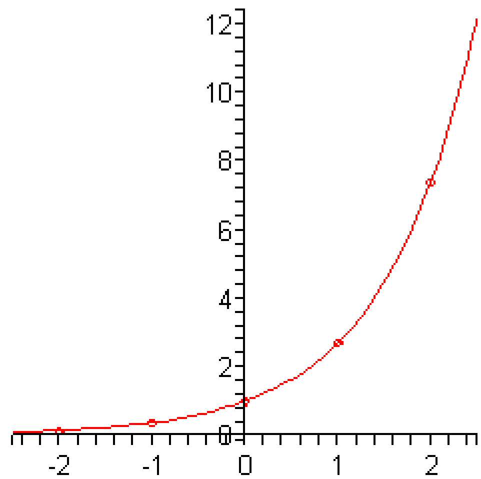
Example 2 Sketch the graph of $f(x) = \mathbf{e}^x$.

Solution
Let's first build up a table of values for this function.

x	-2	-1	0	1	2
f(x)	0.1353…	0.3679…	1	2.718…	7.389…

To get these evaluation (with the exception of $x = 0$) you will need to use a calculator. In fact, that is part of the point of this example. Make sure that you can run your calculator and verify these numbers.

Here is a sketch of this graph.



Notice that this is an increasing graph as we should expect since $\mathbf{e} = 2.718281827\ldots > 1$.

There is one final example that we need to work before moving onto the next section. This example is more about the evaluation process for exponential functions than the graphing process. We need to be very careful with the evaluation of exponential functions.

Example 3 Sketch the graph of $g\left(x\right) = 5\mathbf{e}^{1-x} - 4$.

Solution

Here is a quick table of values for this function.

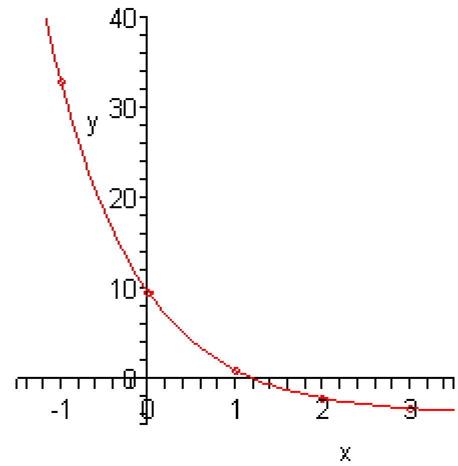
x	-1	0	1	2	3
g(x)	32.945…	9.591…	1	-2.161…	-3.323…

Now, as we stated above this example was more about the evaluation process than the graph so let's go through the first one to make sure that you can do these.

$$\begin{aligned} g\left(-1\right) &= 5\mathbf{e}^{1-(-1)} - 4 \\ &= 5\mathbf{e}^{2} - 4 \\ &= 5\left(7.389\right) - 4 \end{aligned}$$

Notice that when evaluating exponential functions we first need to actually do the exponentiation before we multiply by any coefficients (5 in this case). Also, we used only 3 decimal places here since we are only graphing. In many applications we will want to use far more decimal places in these computations.

Here is a sketch of the graph.



Notice that this graph violates all the properties we listed above. That is okay. Those properties are only valid for functions in the form $f\left(x\right)=b^x$ or $f\left(x\right)=\mathbf{e}^x$. We've got a lot more going on in this function and so the properties, as written above, won't hold for this function.

Section 6-2 : Logarithm Functions

In this section we now need to move into logarithm functions. This can be a tricky function to graph right away. There is going to be some different notation that you aren't used to and some of the properties may not be all that intuitive. Do not get discouraged however. Once you figure these out you will find that they really aren't that bad and it usually just takes a little working with them to get them figured out.

Here is the definition of the logarithm function.

If *b* is any number such that $b > 0$ and $b \neq 1$ and $x > 0$ then,

$$y = \log_b x \qquad \text{is equivalent to} \qquad b^y = x$$

We usually read this as "log base *b* of *x*".

In this definition $y = \log_b x$ is called the **logarithm form** and $b^y = x$ is called the **exponential form**.

Note that the requirement that $x > 0$ is really a result of the fact that we are also requiring $b > 0$. If you think about it, it will make sense. We are raising a positive number to an exponent and so there is no way that the result can possibly be anything other than another positive number. It is very important to remember that we can't take the logarithm of zero or a negative number.

Now, let's address the notation used here as that is usually the biggest hurdle that students need to overcome before starting to understand logarithms. First, the "log" part of the function is simply three letters that are used to denote the fact that we are dealing with a logarithm. They are not variables and they aren't signifying multiplication. They are just there to tell us we are dealing with a logarithm.

Next, the *b* that is subscripted on the "log" part is there to tell us what the base is as this is an important piece of information. Also, despite what it might look like there is no exponentiation in the logarithm form above. It might look like we've got b^x in that form, but it isn't. It just looks like that might be what's happening.

It is important to keep the notation with logarithms straight, if you don't you will find it very difficult to understand them and to work with them.

Now, let's take a quick look at how we evaluate logarithms.

Example 1 Evaluate each of the following logarithms.

(a) $\log_4 16$

(b) $\log_2 16$

(c) $\log_6 216$

(d) $\log_5 \dfrac{1}{125}$

(e) $\log_{\frac{1}{3}} 81$

(f) $\log_{\frac{3}{2}} \dfrac{27}{8}$

Solution

Now, the reality is that evaluating logarithms directly can be a very difficult process, even for those who really understand them. It is usually much easier to first convert the logarithm form into exponential form. In that form we can usually get the answer pretty quickly.

(a) $\log_4 16$

Okay what we are really asking here is the following.

$$\log_4 16 = ?$$

As suggested above, let's convert this to exponential form.

$$\log_4 16 = ? \qquad \Rightarrow \qquad 4^? = 16$$

Most people cannot evaluate the logarithm $\log_4 16$ right off the top of their head. However, most people can determine the exponent that we need on 4 to get 16 once we do the exponentiation. So, since,

$$4^2 = 16$$

we must have the following value of the logarithm.

$$\log_4 16 = 2$$

(b) $\log_2 16$

This one is similar to the previous part. Let's first convert to exponential form.

$$\log_2 16 = ? \qquad \Rightarrow \qquad 2^? = 16$$

If you don't know this answer right off the top of your head, start trying numbers. In other words, compute 2^2, 2^3, 2^4, *etc* until you get 16. In this case we need an exponent of 4. Therefore, the value of this logarithm is,

$$\log_2 16 = 4$$

Before moving on to the next part notice that the base on these is a very important piece of notation. Changing the base will change the answer and so we always need to keep track of the base.

(c) $\log_6 216$

We'll do this one without any real explanation to see how well you've got the evaluation of logarithms down.

$$\log_6 216 = 3 \qquad \text{because} \qquad 6^3 = 216$$

(d) $\log_5 \frac{1}{125}$

Now, this one looks different from the previous parts, but it really isn't any different. As always let's first convert to exponential form.

$$\log_5 \frac{1}{125} = ? \qquad \Rightarrow \qquad 5^? = \frac{1}{125}$$

First, notice that the only way that we can raise an integer to an integer power and get a fraction as an answer is for the exponent to be negative. So, we know that the exponent has to be negative.

Now, let's ignore the fraction for a second and ask $5^? = 125$. In this case if we cube 5 we will get 125.

So, it looks like we have the following,

$$\log_5 \frac{1}{125} = -3 \qquad \text{because} \qquad 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

(e) $\log_{\frac{1}{3}} 81$

Converting this logarithm to exponential form gives,

$$\log_{\frac{1}{3}} 81 = ? \qquad \Rightarrow \qquad \left(\frac{1}{3}\right)^? = 81$$

Now, just like the previous part, the only way that this is going to work out is if the exponent is negative. Then all we need to do is recognize that $3^4 = 81$ and we can see that,

$$\log_{\frac{1}{3}} 81 = -4 \qquad \text{because} \qquad \left(\frac{1}{3}\right)^{-4} = \left(\frac{3}{1}\right)^4 = 3^4 = 81$$

(f) $\log_{\frac{3}{2}} \frac{27}{8}$

Here is the answer to this one.

$$\log_{\frac{3}{2}} \frac{27}{8} = 3 \qquad \text{because} \qquad \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$$

Hopefully, you now have an idea on how to evaluate logarithms and are starting to get a grasp on the notation. There are a few more evaluations that we want to do however, we need to introduce some special logarithms that occur on a very regular basis. They are the **common logarithm** and the **natural logarithm**. Here are the definitions and notations that we will be using for these two logarithms.

$$\text{common logarithm}: \qquad \log x = \log_{10} x$$
$$\text{natural logarithm}: \qquad \ln x = \log_{\mathbf{e}} x$$

So, the common logarithm is simply the log base 10, except we drop the "base 10" part of the notation. Similarly, the natural logarithm is simply the log base **e** with a different notation and where **e** is the same number that we saw in the previous section and is defined to be $\mathbf{e} = 2.718281828\ldots$.

Let's take a look at a couple more evaluations.

Example 2 Evaluate each of the following logarithms.

(a) $\log 1000$

(b) $\log \frac{1}{100}$

(c) $\ln \frac{1}{\mathbf{e}}$

(d) $\ln \sqrt{\mathbf{e}}$

(e) $\log_{34} 34$

(f) $\log_8 1$

Solution

To do the first four evaluations we just need to remember what the notation for these are and what base is implied by the notation. The final two evaluations are to illustrate some of the properties of all logarithms that we'll be looking at eventually.

(a) $\log 1000 = 3$ because $10^3 = 1000$.

(b) $\log \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

(c) $\ln \frac{1}{\mathbf{e}} = -1$ because $\mathbf{e}^{-1} = \frac{1}{\mathbf{e}}$.

(d) $\ln \sqrt{\mathbf{e}} = \frac{1}{2}$ because $\mathbf{e}^{\frac{1}{2}} = \sqrt{\mathbf{e}}$. Notice that with this one we are really just acknowledging a change of notation from fractional exponent into radical form.

(e) $\log_{34} 34 = 1$ because $34^1 = 34$. Notice that this one will work regardless of the base that we're using.

(f) $\log_8 1 = 0$ because $8^0 = 1$. Again, note that the base that we're using here won't change the answer.

So, when evaluating logarithms all that we're really asking is what exponent did we put onto the base to get the number in the logarithm.

Now, before we get into some of the properties of logarithms let's first do a couple of quick graphs.

Example 3 Sketch the graph of the common logarithm and the natural logarithm on the same axis system.

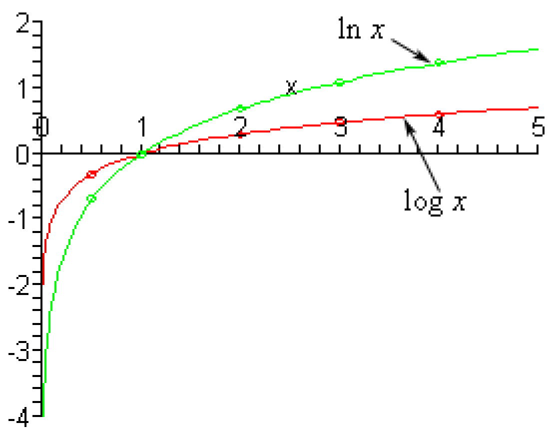
Solution

This example has two points. First, it will familiarize us with the graphs of the two logarithms that we are most likely to see in other classes. Also, it will give us some practice using our calculator to evaluate these logarithms because the reality is that is how we will need to do most of these evaluations.

Here is a table of values for the two logarithms.

x	$\log x$	$\ln x$
$\frac{1}{2}$	-0.3010	-0.6931
1	0	0
2	0.3010	0.6931
3	0.4771	1.0986
4	0.6021	1.3863

Here is a sketch of the graphs of these two functions.



Now let's start looking at some properties of logarithms. We'll start off with some basic evaluation properties.

Properties of Logarithms

1. $\log_b 1 = 0$. This follows from the fact that $b^0 = 1$.
2. $\log_b b = 1$. This follows from the fact that $b^1 = b$.
3. $\log_b b^x = x$. This can be generalized out to $\log_b b^{f(x)} = f(x)$.
4. $b^{\log_b x} = x$. This can be generalized out to $b^{\log_b f(x)} = f(x)$.

Properties 3 and 4 leads to a nice relationship between the logarithm and exponential function. Let's first compute the following function compositions for $f(x) = b^x$ and $g(x) = \log_b x$.

$$(f \circ g)(x) = f[g(x)] = f(\log_b x) = b^{\log_b x} = x$$
$$(g \circ f)(x) = g[f(x)] = g[b^x] = \log_b b^x = x$$

Recall from the section on inverse functions that this means that the exponential and logarithm functions are inverses of each other. This is a nice fact to remember on occasion.

We should also give the generalized version of Properties 3 and 4 in terms of both the natural and common logarithm as we'll be seeing those in the next couple of sections on occasion.

$$\ln \mathbf{e}^{f(x)} = f(x) \qquad \log 10^{f(x)} = f(x)$$
$$\mathbf{e}^{\ln f(x)} = f(x) \qquad 10^{\log f(x)} = f(x)$$

Now, let's take a look at some manipulation properties of the logarithm.

More Properties of Logarithms

For these properties we will assume that $x > 0$ and $y > 0$.

5. $\log_b(xy) = \log_b x + \log_b y$
6. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
7. $\log_b(x^r) = r \log_b x$
8. If $\log_b x = \log_b y$ then $x = y$.

We won't be doing anything with the final property in this section; it is here only for the sake of completeness. We will be looking at this property in detail in a couple of sections.

The first two properties listed here can be a little confusing at first since on one side we've got a product or a quotient inside the logarithm and on the other side we've got a sum or difference of two logarithms. We will just need to be careful with these properties and make sure to use them correctly.

Also, note that there are no rules on how to break up the logarithm of the sum or difference of two terms. To be clear about this let's note the following,

$$\log_b(x+y) \neq \log_b x + \log_b y$$
$$\log_b(x-y) \neq \log_b x - \log_b y$$

Be careful with these and do not try to use these as they simply aren't true.

Note that all of the properties given to this point are valid for both the common and natural logarithms. We just didn't write them out explicitly using the notation for these two logarithms, the properties do hold for them nonetheless

Now, let's see some examples of how to use these properties.

Example 4 Simplify each of the following logarithms.

(a) $\log_4\left(x^3y^5\right)$

(b) $\log\left(\dfrac{x^9y^5}{z^3}\right)$

(c) $\ln\sqrt{xy}$

(d) $\log_3\left(\dfrac{\left(x+y\right)^2}{x^2+y^2}\right)$

Solution

The instructions here may be a little misleading. When we say simplify we really mean to say that we want to use as many of the logarithm properties as we can.

(a) $\log_4\left(x^3y^5\right)$

Note that we can't use Property 7 to bring the 3 and the 5 down into the front of the logarithm at this point. In order to use Property 7 the whole term in the logarithm needs to be raised to the power. In this case the two exponents are only on individual terms in the logarithm and so Property 7 can't be used here.

We do, however, have a product inside the logarithm so we can use Property 5 on this logarithm.

$$\log_4\left(x^3y^5\right)=\log_4\left(x^3\right)+\log_4\left(y^5\right)$$

Now that we've done this we can use Property 7 on each of these individual logarithms to get the final simplified answer.

$$\log_4\left(x^3y^5\right)=3\log_4 x+5\log_4 y$$

(b) $\log\left(\dfrac{x^9y^5}{z^3}\right)$

In this case we've got a product and a quotient in the logarithm. In these cases it is almost always best to deal with the quotient before dealing with the product. Here is the first step in this part.

$$\log\left(\frac{x^9y^5}{z^3}\right)=\log\left(x^9y^5\right)-\log z^3$$

Now, we'll break up the product in the first term and once we've done that we'll take care of the exponents on the terms.

$$\log\left(\frac{x^9 y^5}{z^3}\right) = \log\left(x^9 y^5\right) - \log z^3$$
$$= \log x^9 + \log y^5 - \log z^3$$
$$= 9\log x + 5\log y - 3\log z$$

(c) $\ln\sqrt{xy}$

For this part let's first rewrite the logarithm a little so that we can see the first step.

$$\ln\sqrt{xy} = \ln\left(xy\right)^{\frac{1}{2}}$$

Written in this form we can see that there is a single exponent on the whole term and so we'll take care of that first.

$$\ln\sqrt{xy} = \frac{1}{2}\ln\left(xy\right)$$

Now, we will take care of the product.

$$\ln\sqrt{xy} = \frac{1}{2}\left(\ln x + \ln y\right)$$

Notice the parenthesis in this the answer. The $\frac{1}{2}$ multiplies the original logarithm and so it will also need to multiply the whole "simplified" logarithm. Therefore, we need to have a set of parenthesis there to make sure that this is taken care of correctly.

(d) $\log_3\left(\frac{\left(x+y\right)^2}{x^2+y^2}\right)$

We'll first take care of the quotient in this logarithm.

$$\log_3\left(\frac{\left(x+y\right)^2}{x^2+y^2}\right) = \log_3\left(x+y\right)^2 - \log_3\left(x^2+y^2\right)$$

We now reach the real point to this problem. The second logarithm is as simplified as we can make it. Remember that we can't break up a log of a sum or difference and so this can't be broken up any farther. Also, we can only deal with exponents if the term as a whole is raised to the exponent. The fact that both pieces of this term are squared doesn't matter. It needs to be the whole term squared, as in the first logarithm.

So, we can further simplify the first logarithm, but the second logarithm can't be simplified any more. Here is the final answer for this problem.

$$\log_3\left(\frac{\left(x+y\right)^2}{x^2+y^2}\right) = 2\log_3\left(x+y\right) - \log_3\left(x^2+y^2\right)$$

Now, we need to work some examples that go the other way. This next set of examples is probably more important than the previous set. We will be doing this kind of logarithm work in a couple of sections.

Example 5 Write each of the following as a single logarithm with a coefficient of 1.

(a) $7\log_{12} x + 2\log_{12} y$

(b) $3\log x - 6\log y$

(c) $5\ln\left(x+y\right) - 2\ln y - 8\ln x$

Solution

The instruction requiring a coefficient of 1 means that the when we get down to a final logarithm there shouldn't be any number in front of the logarithm.

Note as well that these examples are going to be using Properties 5 – 7 only we'll be using them in reverse. We will have expressions that look like the right side of the property and use the property to write it so it looks like the left side of the property.

(a) The first step here is to get rid of the coefficients on the logarithms. This will use Property 7 in reverse. In this direction, Property 7 says that we can move the coefficient of a logarithm up to become a power on the term inside the logarithm.

Here is that step for this part.

$$7\log_{12} x + 2\log_{12} y = \log_{12} x^7 + \log_{12} y^2$$

We've now got a sum of two logarithms both with coefficients of 1 and both with the same base. This means that we can use Property 5 in reverse. Here is the answer for this part.

$$7\log_{12} x + 2\log_{12} y = \log_{12}\left(x^7 y^2\right)$$

(b) Again, we will first take care of the coefficients on the logarithms.

$$3\log x - 6\log y = \log x^3 - \log y^6$$

We now have a difference of two logarithms and so we can use Property 6 in reverse. When using Property 6 in reverse remember that the term from the logarithm that is subtracted off goes in the denominator of the quotient. Here is the answer to this part.

$$3\log x - 6\log y = \log\left(\frac{x^3}{y^6}\right)$$

(c) In this case we've got three terms to deal with and none of the properties have three terms in them. That isn't a problem. Let's first take care of the coefficients and at the same time we'll factor a minus sign out of the last two terms. The reason for this will be apparent in the next step.

$$5\ln\left(x+y\right) - 2\ln y - 8\ln x = \ln\left(x+y\right)^5 - \left(\ln y^2 + \ln x^8\right)$$

Now, notice that the quantity in the parenthesis is a sum of two logarithms and so can be combined into a single logarithm with a product as follows,

$$5\ln\left(x+y\right) - 2\ln y - 8\ln x = \ln\left(x+y\right)^5 - \ln\left(y^2 x^8\right)$$

Now we are down to two logarithms and they are a difference of logarithms and so we can write it as a single logarithm with a quotient.

$$5\ln\left(x+y\right)-2\ln y-8\ln x=\ln\left(\frac{\left(x+y\right)^5}{y^2x^8}\right)$$

The final topic that we need to discuss in this section is the **change of base** formula.

Most calculators these days are capable of evaluating common logarithms and natural logarithms. However, that is about it, so what do we do if we need to evaluate another logarithm that can't be done easily as we did in the first set of examples that we looked at?

To do this we have the change of base formula. Here is the change of base formula.

$$\log_a x=\frac{\log_b x}{\log_b a}$$

where we can choose *b* to be anything we want it to be. In order to use this to help us evaluate logarithms this is usually the common or natural logarithm. Here is the change of base formula using both the common logarithm and the natural logarithm.

$$\log_a x=\frac{\log x}{\log a} \qquad\qquad \log_a x=\frac{\ln x}{\ln a}$$

Let's see how this works with an example.

Example 6 Evaluate $\log_5 7$.

Solution

First, notice that we can't use the same method to do this evaluation that we did in the first set of examples. This would require us to look at the following exponential form,

$$5^? = 7$$

and that's just not something that anyone can answer off the top of their head. If the 7 had been a 5, or a 25, or a 125, *etc*. we could do this, but it's not. Therefore, we have to use the change of base formula.

Now, we can use either one and we'll get the same answer. So, let's use both and verify that. We'll start with the common logarithm form of the change of base.

$$\log_5 7=\frac{\log 7}{\log 5}=\frac{0.845098040014}{0.698970004336}=1.20906195512$$

Now, let's try the natural logarithm form of the change of base formula.

$$\log_5 7=\frac{\ln 7}{\ln 5}=\frac{1.94591014906}{1.60943791243}=1.20906195512$$

So, we got the same answer despite the fact that the fractions involved different answers.

Section 6-3 : Solving Exponential Equations

Now that we've seen the definitions of exponential and logarithm functions we need to start thinking about how to solve equations involving them. In this section we will look at solving exponential equations and we will look at solving logarithm equations in the next section.

There are two methods for solving exponential equations. One method is fairly simple but requires a very special form of the exponential equation. The other will work on more complicated exponential equations but can be a little messy at times.

Let's start off by looking at the simpler method. This method will use the following fact about exponential functions.

$$\text{If} \quad b^x = b^y \quad \text{then} \quad x = y$$

Note that this fact does require that the base in both exponentials to be the same. If it isn't then this fact will do us no good.

Let's take a look at a couple of examples.

Example 1 Solve each of the following.

(a) $5^{3x} = 5^{7x-2}$

(b) $4^{t^2} = 4^{6-t}$

(c) $3^z = 9^{z+5}$

(d) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

Solution

(a) $5^{3x} = 5^{7x-2}$

In this first part we have the same base on both exponentials so there really isn't much to do other than to set the two exponents equal to each other and solve for *x*.

$$\begin{aligned} 3x &= 7x - 2 \\ 2 &= 4x \\ \frac{1}{2} &= x \end{aligned}$$

So, if we were to plug $x = \dfrac{1}{2}$ into the equation then we would get the same number on both sides of the equal sign.

(b) $4^{t^2} = 4^{6-t}$

Again, there really isn't much to do here other than set the exponents equal since the base is the same in both exponentials.

$$\begin{aligned} t^2 &= 6-t \\ t^2+t-6 &= 0 \\ (t+3)(t-2) &= 0 \qquad\qquad \Rightarrow \qquad t=-3,\ t=2 \end{aligned}$$

In this case we get two solutions to the equation. That is perfectly acceptable so don't worry about it when it happens.

(c) $3^z = 9^{z+5}$

Now, in this case we don't have the same base so we can't just set exponents equal. However, with a little manipulation of the right side we can get the same base on both exponents. To do this all we need to notice is that $9 = 3^2$. Here's what we get when we use this fact.

$$3^z = \left(3^2\right)^{z+5}$$

Now, we still can't just set exponents equal since the right side now has two exponents. If we recall our exponent properties we can fix this however.

$$3^z = 3^{2(z+5)}$$

We now have the same base and a single exponent on each base so we can use the property and set the exponents equal. Doing this gives,

$$\begin{aligned} z &= 2(z+5) \\ z &= 2z+10 \\ z &= -10 \end{aligned}$$

So, after all that work we get a solution of $z = -10$.

(d) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

In this part we've got some issues with both sides. First the right side is a fraction and the left side isn't. That is not the problem that it might appear to be however, so for a second let's ignore that. The real issue here is that we can't write 8 as a power of 4 and we can't write 4 as a power of 8 as we did in the previous part.

The first thing to do in this problem is to get the same base on both sides and to so that we'll have to note that we can write both 4 and 8 as a power of 2. So let's do that.

$$\begin{aligned} \left(2^2\right)^{5-9x} &= \frac{1}{\left(2^3\right)^{x-2}} \\ 2^{2(5-9x)} &= \frac{1}{2^{3(x-2)}} \end{aligned}$$

It's now time to take care of the fraction on the right side. To do this we simply need to remember the following exponent property.

$$\frac{1}{a^n} = a^{-n}$$

Using this gives,

$$2^{2(5-9x)} = 2^{-3(x-2)}$$

So, we now have the same base and each base has a single exponent on it so we can set the exponents equal.

$$\begin{aligned} 2(5-9x) &= -3(x-2) \\ 10-18x &= -3x+6 \\ 4 &= 15x \\ x &= \frac{4}{15} \end{aligned}$$

And there is the answer to this part.

Now, the equations in the previous set of examples all relied upon the fact that we were able to get the same base on both exponentials, but that just isn't always possible. Consider the following equation.

$$7^x = 9$$

This is a fairly simple equation however the method we used in the previous examples just won't work because we don't know how to write 9 as a power of 7. In fact, if you think about it that is exactly what this equation is asking us to find.

So, the method we used in the first set of examples won't work. The problem here is that the *x* is in the exponent. Because of that all our knowledge about solving equations won't do us any good. We need a way to get the *x* out of the exponent and luckily for us we have a way to do that. Recall the following logarithm property from the last section.

$$\log_b a^r = r \log_b a$$

Note that to avoid confusion with *x*'s we replaced the *x* in this property with an *a*. The important part of this property is that we can take an exponent and move it into the front of the term.

So, if we had,

$$\log_b 7^x$$

we could use this property as follows.

$$x \log_b 7$$

The *x* in now out of the exponent! Of course, we are now stuck with a logarithm in the problem and not only that but we haven't specified the base of the logarithm.

The reality is that we can use any logarithm to do this so we should pick one that we can deal with. This usually means that we'll work with the common logarithm or the natural logarithm.

So, let's work a set of examples to see how we actually use this idea to solve these equations.

Example 2 Solve each of the following equations.

(a) $7^x = 9$

(b) $2^{4y+1} - 3^y = 0$

(c) $\mathbf{e}^{t+6} = 2$

(d) $10^{5-x} = 8$

(e) $5\mathbf{e}^{2z+4} - 8 = 0$

Solution

(a) $7^x = 9$

Okay, so we say above that if we had a logarithm in front the left side we could get the *x* out of the exponent. That's easy enough to do. We'll just put a logarithm in front of the left side. However, if we put a logarithm there we also must put a logarithm in front of the right side. This is commonly referred to as **taking the logarithm of both sides**.

We can use any logarithm that we'd like to so let's try the natural logarithm.

$$\begin{aligned} \ln 7^x &= \ln 9 \\ x \ln 7 &= \ln 9 \end{aligned}$$

Now, we need to solve for *x*. This is easier than it looks. If we had $7x = 9$ then we could all solve for *x* simply by dividing both sides by 7. It works in exactly the same manner here. Both ln7 and ln9 are just numbers. Admittedly, it would take a calculator to determine just what those numbers are, but they are numbers and so we can do the same thing here.

$$\begin{aligned} \frac{x \ln 7}{\ln 7} &= \frac{\ln 9}{\ln 7} \\ x &= \frac{\ln 9}{\ln 7} \end{aligned}$$

Now, that is technically the exact answer. However, in this case it's usually best to get a decimal answer so let's go one step further.

$$x = \frac{\ln 9}{\ln 7} = \frac{2.19722458}{1.94591015} = 1.12915007$$

Note that the answers to these are decimal answers more often than not.

Also, be careful here to not make the following mistake.

$$1.12915007 = \frac{\ln 9}{\ln 7} \neq \ln\left(\frac{9}{7}\right) = 0.2513144283$$

The two are clearly different numbers.

Finally, let's also use the common logarithm to make sure that we get the same answer.

$$\log 7^x = \log 9$$
$$x \log 7 = \log 9$$
$$x = \frac{\log 9}{\log 7} = \frac{0.954242509}{0.845098040} = 1.12915007$$

So, sure enough the same answer. We can use either logarithm, although there are times when it is more convenient to use one over the other.

(b) $2^{4y+1} - 3^y = 0$

In this case we can't just put a logarithm in front of both sides. There are two reasons for this. First on the right side we've got a zero and we know from the previous section that we can't take the logarithm of zero. Next, in order to move the exponent down it has to be on the whole term inside the logarithm and that just won't be the case with this equation in its present form.

So, the first step is to move on of the terms to the other side of the equal sign, then we will take the logarithm of both sides using the natural logarithm.

$$2^{4y+1} = 3^y$$
$$\ln 2^{4y+1} = \ln 3^y$$
$$(4y+1)\ln 2 = y \ln 3$$

Okay, this looks messy, but again, it's really not that bad. Let's look at the following equation first.

$$2(4y+1) = 3y$$
$$8y + 2 = 3y$$
$$5y = -2$$
$$y = -\frac{2}{5}$$

We can all solve this equation and so that means that we can solve the one that we've got. Again, the ln2 and ln3 are just numbers and so the process is exactly the same. The answer will be messier than this equation, but the process is identical. Here is the work for this one.

$$(4y+1)\ln 2 = y \ln 3$$
$$4y \ln 2 + \ln 2 = y \ln 3$$
$$4y \ln 2 - y \ln 3 = -\ln 2$$
$$y(4\ln 2 - \ln 3) = -\ln 2$$
$$y = -\frac{\ln 2}{4\ln 2 - \ln 3}$$

So, we get all the terms with *y* in them on one side and all the other terms on the other side. Once this is done we then factor out a *y* and divide by the coefficient. Again, we would prefer a decimal answer so let's get that.

$$y = -\frac{\ln 2}{4\ln 2 - \ln 3} = -\frac{0.693147181}{4\left(0.693147181\right) - 1.098612289} = -0.414072245$$

(c) $\mathbf{e}^{t+6} = 2$
Now, this one is a little easier than the previous one. Again, we'll take the natural logarithm of both sides.

$$\ln \mathbf{e}^{t+6} = \ln 2$$

Notice that we didn't take the exponent out of this one. That is because we want to use the following property with this one.

$$\ln \mathbf{e}^{f(x)} = f\left(x\right)$$

We saw this in the previous section (in more general form) and by using this here we will make our life significantly easier. Using this property gives,

$$\begin{aligned} t + 6 &= \ln 2 \\ t &= \ln\left(2\right) - 6 = 0.69314718 - 6 = -5.30685202 \end{aligned}$$

Notice the parenthesis around the 2 in the logarithm this time. They are there to make sure that we don't make the following mistake.

$$-5.30685202 = \ln\left(2\right) - 6 \neq \ln\left(2-6\right) = \ln\left(-4\right) \text{ can't be done}$$

Be very careful with this mistake. It is easy to make when you aren't paying attention to what you're doing or are in a hurry.

(d) $10^{5-x} = 8$
The equation in this part is similar to the previous part except this time we've got a base of 10 and so recalling the fact that,

$$\log 10^{f(x)} = f\left(x\right)$$

it makes more sense to use common logarithms this time around.

Here is the work for this equation.

$$\begin{aligned} \log 10^{5-x} &= \log 8 \\ 5 - x &= \log 8 \\ 5 - \log 8 &= x \qquad \Rightarrow \qquad x = 5 - 0.903089987 = 4.096910013 \end{aligned}$$

This could have been done with natural logarithms but the work would have been messier.

(e) $5\mathbf{e}^{2z+4} - 8 = 0$
With this final equation we've got a couple of issues. First, we'll need to move the number over to the other side. In order to take the logarithm of both sides we need to have the exponential on one side by itself. Doing this gives,

$$5\mathbf{e}^{2z+4} = 8$$

Next, we've got to get a coefficient of 1 on the exponential. We can only use the facts to simplify this if there isn't a coefficient on the exponential. So, divide both sides by 5 to get,

$$\mathbf{e}^{2z+4} = \frac{8}{5}$$

At this point we will take the logarithm of both sides using the natural logarithm since there is an **e** in the equation.

$$\begin{aligned}
\ln \mathbf{e}^{2z+4} &= \ln\left(\frac{8}{5}\right) \\
2z+4 &= \ln\left(\frac{8}{5}\right) \\
2z &= \ln\left(\frac{8}{5}\right) - 4 \\
z &= \frac{1}{2}\left(\ln\left(\frac{8}{5}\right) - 4\right) = \frac{1}{2}\left(0.470003629 - 4\right) = -1.76499819
\end{aligned}$$

Note that we could have used this second method on the first set of examples as well if we'd wanted to although the work would have been more complicated and prone to mistakes if we'd done that.

Section 6-4 : Solving Logarithm Equations

In this section we will now take a look at solving logarithmic equations, or equations with logarithms in them. We will be looking at two specific types of equations here. In particular we will look at equations in which every term is a logarithm and we also look at equations in which all but one term in the equation is a logarithm and the term without the logarithm will be a constant. Also, we will be assuming that the logarithms in each equation will have the same base. If there is more than one base in the logarithms in the equation the solution process becomes much more difficult.

Before we get into the solution process we will need to remember that we can only plug positive numbers into a logarithm. This will be important down the road and so we can't forget that.

Now, let's start off by looking at equations in which each term is a logarithm and all the bases on the logarithms are the same. In this case we will use the fact that,

$$\text{If } \log_b x = \log_b y \text{ then } x = y$$

In other words, if we've got two logs in the problem, one on either side of an equal sign and both with a coefficient of one, then we can just drop the logarithms.

Let's take a look at a couple of examples.

Example 1 Solve each of the following equations.

(a) $2\log_9\left(\sqrt{x}\right) - \log_9\left(6x-1\right) = 0$

(b) $\log x + \log\left(x-1\right) = \log\left(3x+12\right)$

(c) $\ln 10 - \ln\left(7-x\right) = \ln x$

Solution

(a) $2\log_9\left(\sqrt{x}\right) - \log_9\left(6x-1\right) = 0$

With this equation there are only two logarithms in the equation so it's easy to get on one either side of the equal sign. We will also need to deal with the coefficient in front of the first term.

$$\log_9\left(\sqrt{x}\right)^2 = \log_9\left(6x-1\right)$$
$$\log_9 x = \log_9\left(6x-1\right)$$

Now that we've got two logarithms with the same base and coefficients of 1 on either side of the equal sign we can drop the logs and solve.

$$x = 6x-1$$
$$1 = 5x \qquad \Rightarrow \qquad x = \frac{1}{5}$$

Now, we do need to worry if this solution will produce any negative numbers or zeroes in the logarithms so the next step is to plug this into the **original** equation and see if it does.

$$2\log_9\left(\sqrt{\frac{1}{5}}\right)-\log_9\left(6\left(\frac{1}{5}\right)-1\right)=2\log_9\left(\sqrt{\frac{1}{5}}\right)-\log_9\left(\frac{1}{5}\right)=0$$

Note that we don't need to go all the way out with the check here. We just need to make sure that once we plug in the *x* we don't have any negative numbers or zeroes in the logarithms. Since we don't in this case we have the solution, it is $x=\frac{1}{5}$.

(b) $\log x+\log\left(x-1\right)=\log\left(3x+12\right)$

Okay, in this equation we've got three logarithms and we can only have two. So, we saw how to do this kind of work in a set of examples in the previous section so we just need to do the same thing here. It doesn't really matter how we do this, but since one side already has one logarithm on it we might as well combine the logs on the other side.

$$\log\left(x\left(x-1\right)\right)=\log\left(3x+12\right)$$

Now we've got one logarithm on either side of the equal sign, they are the same base and have coefficients of one so we can drop the logarithms and solve.

$$\begin{aligned} x\left(x-1\right)&=3x+12\\ x^2-x-3x-12&=0\\ x^2-4x-12&=0\\ \left(x-6\right)\left(x+2\right)&=0 \qquad \Rightarrow \qquad x=-2,\ x=6\end{aligned}$$

Now, before we declare these to be solutions we MUST check them in the original equation.

$x=6$:

$$\begin{aligned}\log 6+\log\left(6-1\right)&=\log\left(3\left(6\right)+12\right)\\ \log 6+\log 5&=\log 30\end{aligned}$$

No logarithms of negative numbers and no logarithms of zero so this is a solution.

$x=-2$:

$$\log\left(-2\right)+\log\left(-2-1\right)=\log\left(3\left(-2\right)+12\right)$$

We don't need to go any farther, there is a logarithm of a negative number in the first term (the others are also negative) and that's all we need in order to exclude this as a solution.

Be careful here. We are not excluding $x=-2$ because it is negative, that's not the problem. We are excluding it because once we plug it into the original equation we end up with logarithms of negative numbers. It is possible to have negative values of *x* be solutions to these problems, so don't mistake the reason for excluding this value.

Also, along those lines we didn't take $x = 6$ as a solution because it was positive, but because it didn't produce any negative numbers or zero in the logarithms upon substitution. It is possible for positive numbers to not be solutions.

So, with all that out of the way, we've got a single solution to this equation, $x = 6$.

(c) $\ln 10 - \ln(7 - x) = \ln x$

We will work this equation in the same manner that we worked the previous one. We've got two logarithms on one side so we'll combine those, drop the logarithms and then solve.

$$\begin{aligned} \ln\left(\frac{10}{7-x}\right) &= \ln x \\ \frac{10}{7-x} &= x \\ 10 &= x(7-x) \\ 10 &= 7x - x^2 \\ x^2 - 7x + 10 &= 0 \\ (x-5)(x-2) &= 0 \qquad \Rightarrow \qquad x = 2,\ x = 5 \end{aligned}$$

We've got two possible solutions to check here.

$x = 2$:

$$\begin{aligned} \ln 10 - \ln(7-2) &= \ln 2 \\ \ln 10 - \ln 5 &= \ln 2 \end{aligned}$$

This one is okay.

$x = 5$:

$$\begin{aligned} \ln 10 - \ln(7-5) &= \ln 5 \\ \ln 10 - \ln 2 &= \ln 5 \end{aligned}$$

This one is also okay.

In this case both possible solutions, $x = 2$ and $x = 5$, end up actually being solutions. There is no reason to expect to always have to throw one of the two out as a solution.

Now we need to take a look at the second kind of logarithmic equation that we'll be solving here. This equation will have all the terms but one be a logarithm and the one term that doesn't have a logarithm will be a constant.

In order to solve these kinds of equations we will need to remember the exponential form of the logarithm. Here it is if you don't remember.

$$y = \log_b x \qquad \Rightarrow \qquad b^y = x$$

We will be using this conversion to exponential form in all of these equations so it's important that you can do it. Let's work some examples so we can see how these kinds of equations can be solved.

Example 2 Solve each of the following equations.

(a) $\log_5(2x+4)=2$

(b) $\log x = 1-\log(x-3)$

(c) $\log_2(x^2-6x)=3+\log_2(1-x)$

Solution

(a) $\log_5(2x+4)=2$

To solve these we need to get the equation into exactly the form that this one is in. We need a single log in the equation with a coefficient of one and a constant on the other side of the equal sign. Once we have the equation in this form we simply convert to exponential form.

So, let's do that with this equation. The exponential form of this equation is,

$$2x+4=5^2=25$$

Notice that this is an equation that we can easily solve.

$$2x=21 \qquad \Rightarrow \qquad x=\frac{21}{2}$$

Now, just as with the first set of examples we need to plug this back into the **original** equation and see if it will produce negative numbers or zeroes in the logarithms. If it does it can't be a solution and if it doesn't then it is a solution.

$$\log_5\left(2\left(\frac{21}{2}\right)+4\right)=2$$
$$\log_5(25)=2$$

Only positive numbers in the logarithm and so $x=\frac{21}{2}$ is in fact a solution.

(b) $\log x = 1-\log(x-3)$

In this case we've got two logarithms in the problem so we are going to have to combine them into a single logarithm as we did in the first set of examples. Doing this for this equation gives,

$$\log x+\log(x-3)=1$$
$$\log(x(x-3))=1$$

Now, that we've got the equation into the proper form we convert to exponential form. Recall as well that we're dealing with the common logarithm here and so the base is 10.

Here is the exponential form of this equation.

$$x(x-3) = 10^1$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0 \qquad \Rightarrow \qquad x = -2,\ x = 5$$

So, we've got two potential solutions. Let's check them both.

$x = -2$:

$$\log(-2) = 1 - \log(-2-3)$$

We've got negative numbers in the logarithms and so this can't be a solution.

$x = 5$:

$$\log 5 = 1 - \log(5-3)$$
$$\log 5 = 1 - \log 2$$

No negative numbers or zeroes in the logarithms and so this is a solution.

Therefore, we have a single solution to this equation, $x = 5$.

Again, remember that we don't exclude a potential solution because it's negative or include a potential solution because it's positive. We exclude a potential solution if it produces negative numbers or zeroes in the logarithms upon substituting it into the equation and we include a potential solution if it doesn't.

(c) $\log_2\left(x^2 - 6x\right) = 3 + \log_2\left(1-x\right)$

Again, let's get the logarithms onto one side and combined into a single logarithm.

$$\log_2\left(x^2 - 6x\right) - \log_2\left(1-x\right) = 3$$
$$\log_2\left(\frac{x^2 - 6x}{1-x}\right) = 3$$

Now, convert it to exponential form.

$$\frac{x^2 - 6x}{1-x} = 2^3 = 8$$

Now, let's solve this equation.

$$x^2 - 6x = 8(1-x)$$
$$x^2 - 6x = 8 - 8x$$
$$x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0 \qquad \Rightarrow \qquad x = -4,\ x = 2$$

Now, let's check both of these solutions in the original equation.

$x = -4$:

$$\log_2\left(\left(-4\right)^2 - 6\left(-4\right)\right) = 3 + \log_2\left(1 - \left(-4\right)\right)$$
$$\log_2\left(16 + 24\right) = 3 + \log_2\left(5\right)$$

So, upon substituting this solution in we see that all the numbers in the logarithms are positive and so this IS a solution. Note again that it doesn't matter that the solution is negative, it just can't produce negative numbers or zeroes in the logarithms.

$x = 2$:

$$\log_2\left(2^2 - 6\left(2\right)\right) = 3 + \log_2\left(1 - 2\right)$$
$$\log_2\left(4 - 12\right) = 3 + \log_2\left(-1\right)$$

In this case, despite the fact that the potential solution is positive we get negative numbers in the logarithms and so it can't possibly be a solution.

Therefore, we get a single solution for this equation, $x = -4$.

Section 6-5 : Applications

In this final section of this chapter we need to look at some applications of exponential and logarithm functions.

Compound Interest

This first application is compounding interest and there are actually two separate formulas that we'll be looking at here. Let's first get those out of the way.

If we were to put *P* dollars into an account that earns interest at a rate of *r* (written as a decimal) for *t* years (yes, it must be years) then,

1. if interest is compounded *m* times per year we will have
$$A = P\left(1+\frac{r}{m}\right)^{t\,m}$$
dollars after *t* years.

2. if interest is compounded continuously then we will have
$$A = P\mathbf{e}^{r\,t}$$
dollars after *t* years.

Let's take a look at a couple of examples.

Example 1 We are going to invest \$100,000 in an account that earns interest at a rate of 7.5% for 54 months. Determine how much money will be in the account if,

(a) interest is compounded quarterly.
(b) interest is compounded monthly.
(c) interest is compounded continuously.

Solution

Before getting into each part let's identify the quantities that we will need in all the parts and won't change.

$$P = 100,000 \qquad\qquad r = \frac{7.5}{100} = 0.075 \qquad\qquad t = \frac{54}{12} = 4.5$$

Remember that interest rates must be decimals for these computations and *t* must be in years! Now, let's work the problems.

(a) Interest is compounded quarterly.

In this part the interest is compounded quarterly and that means it is compounded 4 times a year. After 54 months we then have,

$$\begin{aligned} A &= 100000\left(1+\frac{0.075}{4}\right)^{(4)(4.5)} \\ &= 100000(1.01875)^{18} \\ &= 100000(1.39706686207) \\ &= 139706.686207 = \$139,706.69 \end{aligned}$$

Notice the amount of decimal places used here. We didn't do any rounding until the very last step. It is important to not do too much rounding in intermediate steps with these problems.

(b) Interest is compounded monthly.
Here we are compounding monthly and so that means we are compounding 12 times a year. Here is how much we'll have after 54 months.

$$\begin{aligned} A &= 100000\left(1+\frac{0.075}{12}\right)^{(12)(4.5)} \\ &= 100000(1.00625)^{54} \\ &= 100000(1.39996843023) \\ &= 139996.843023 = \$139,996.84 \end{aligned}$$

So, compounding more times per year will yield more money.

(c) Interest is compounded continuously.
Finally, if we compound continuously then after 54 months we will have,

$$\begin{aligned} A &= 100000\mathbf{e}^{(0.075)(4.5)} \\ &= 100000(1.40143960839) \\ &= 140143.960839 = \$140,143.96 \end{aligned}$$

Now, as pointed out in the first part of this example it is important to not round too much before the final answer. Let's go back and work the first part again and this time let's round to three decimal places at each step.

$$\begin{aligned} A &= 100000\left(1+\frac{0.075}{4}\right)^{(4)(4.5)} \\ &= 100000(1.019)^{18} \\ &= 100000(1.403) \\ &= \$140,300.00 \end{aligned}$$

This answer is off from the correct answer by $593.31 and that's a fairly large difference. So, how many decimal places should we keep in these? Well, unfortunately the answer is that it depends. The larger the initial amount the more decimal places we will need to keep around. As a general rule of thumb, set your calculator to the maximum number of decimal places it can handle and take all of them until the final answer and then round at that point.

Let's now look at a different kind of example with compounding interest.

Example 2 We are going to put \$2500 into an account that earns interest at a rate of 12%. If we want to have \$4000 in the account when we close it how long should we keep the money in the account if,

(a) we compound interest continuously.

(b) we compound interest 6 times a year.

Solution

Again, let's identify the quantities that won't change with each part.

$$A = 4000 \qquad P = 2500 \qquad r = \frac{12}{100} = 0.12$$

Notice that this time we've been given *A* and are asking to find *t*. This means that we are going to have to solve an exponential equation to get at the answer.

(a) Compound interest continuously.

Let's first set up the equation that we'll need to solve.

$$4000 = 2500\mathbf{e}^{0.12t}$$

Now, we saw how to solve these kinds of equations a couple of sections ago. In that section we saw that we need to get the exponential on one side by itself with a coefficient of 1 and then take the natural logarithm of both sides. Let's do that.

$$\begin{aligned}\frac{4000}{2500} &= \mathbf{e}^{0.12t}\\ 1.6 &= \mathbf{e}^{0.12t}\\ \ln 1.6 &= \ln \mathbf{e}^{0.12t}\\ \ln 1.6 &= 0.12t \qquad \Rightarrow \qquad t = \frac{\ln 1.6}{0.12} = 3.917\end{aligned}$$

We need to keep the amount in the account for 3.917 years to get \$4000.

(b) Compound interest 6 times a year.

Again, let's first set up the equation that we need to solve.

$$\begin{aligned}4000 &= 2500\left(1 + \frac{0.12}{6}\right)^{6t}\\ 4000 &= 2500\left(1.02\right)^{6t}\end{aligned}$$

We will solve this the same way that we solved the previous part. The work will be a little messier, but for the most part it will be the same.

$$\frac{4000}{2500} = (1.02)^{6t}$$
$$1.6 = (1.02)^{6t}$$
$$\ln 1.6 = \ln(1.02)^{6t}$$
$$\ln 1.6 = 6t\ln(1.02)$$
$$t = \frac{\ln 1.6}{6\ln(1.02)} = \frac{0.470003629246}{6(0.019802627296)} = 3.956$$

In this case we need to keep the amount slightly longer to reach $4000.

Exponential Growth and Decay

There are many quantities out there in the world that are governed (at least for a short time period) by the equation,

$$Q = Q_0\mathbf{e}^{kt}$$

where Q_0 is positive and is the amount initially present at $t = 0$ and *k* is a non-zero constant. If *k* is positive then the equation will grow without bound and is called the **exponential growth** equation. Likewise, if *k* is negative the equation will die down to zero and is called the **exponential decay** equation.

Short term population growth is often modeled by the exponential growth equation and the decay of a radioactive element is governed the exponential decay equation.

Example 3 The growth of a colony of bacteria is given by the equation,

$$Q = Q_0\mathbf{e}^{0.195t}$$

If there are initially 500 bacteria present and *t* is given in hours determine each of the following.

(a) How many bacteria are there after a half of a day?

(b) How long will it take before there are 10000 bacteria in the colony?

Solution

Here is the equation for this starting amount of bacteria.

$$Q = 500\,\mathbf{e}^{0.195t}$$

(a) How many bacteria are there after a half of a day?

In this case if we want the number of bacteria after half of a day we will need to use $t = 12$ since *t* is in hours. So, to get the answer to this part we just need to plug *t* into the equation.

$$Q = 500\,\mathbf{e}^{0.195(12)} = 500(10.3812365627) = 5190.618$$

So, since a fractional population doesn't make much sense we'll say that after half of a day there are 5190 of the bacteria present.

(b) How long will it take before there are 10000 bacteria in the colony?

Do NOT make the mistake of assuming that it will be approximately 1 day for this answer based on the answer to the previous part. With exponential growth things just don't work that way as we'll see. In order to answer this part we will need to solve the following exponential equation.

$$10000 = 500\,\mathbf{e}^{0.195\,t}$$

Let's do that.

$$\begin{aligned} \frac{10000}{500} &= \mathbf{e}^{0.195\,t} \\ 20 &= \mathbf{e}^{0.195\,t} \\ \ln 20 &= \ln \mathbf{e}^{0.195\,t} \\ \ln 20 &= 0.195t \qquad \Rightarrow \qquad t = \frac{\ln 20}{0.195} = 15.3627 \end{aligned}$$

So, it only takes approximately 15.4 hours to reach 10000 bacteria and NOT 24 hours if we just double the time from the first part. In other words, be careful!

Example 4 Carbon 14 dating works by measuring the amount of Carbon 14 (a radioactive element) that is in a fossil. All living things have a constant level of Carbon 14 in them and once they die it starts to decay according to the formula,

$$Q = Q_0\mathbf{e}^{-0.000124t}$$

where *t* is in years and Q_0 is the amount of Carbon 14 present at death and for this example let's assume that there will be 100 milligrams present at death.

(a) How much Carbon 14 will there be after 1000 years?
(b) How long will it take for half of the Carbon 14 to decay?

Solution

(a) How much Carbon 14 will there be after 1000 years?

In this case all we need to do is plug in *t=1000* into the equation.

$$Q = 100\mathbf{e}^{-0.000124(1000)} = 100(0.883379840883) = 88.338 \text{ milligrams}$$

So, it looks like we will have around 88.338 milligrams left after 1000 years.

(b) How long will it take for half of the Carbon 14 to decay?

So, we want to know how long it will take until there is 50 milligrams of the Carbon 14 left. That means we will have to solve the following equation,

$$50 = 100\mathbf{e}^{-0.000124t}$$

Here is that work.

$$\frac{50}{100} = \mathbf{e}^{-0.000124t}$$
$$\frac{1}{2} = \mathbf{e}^{-0.000124t}$$
$$\ln\frac{1}{2} = \ln\mathbf{e}^{-0.000124t}$$
$$\ln\frac{1}{2} = -0.000124t$$
$$t = \frac{\ln\frac{1}{2}}{-0.000124} = \frac{-0.69314718056}{-0.000124} = 5589.89661742$$

So, it looks like it will take about 5589.897 years for half of the Carbon 14 to decay. This number is called the **half-life** of Carbon 14.

We've now looked at a couple of applications of exponential equations and we should now look at a quick application of a logarithm.

Earthquake Intensity

The **Richter scale** is commonly used to measure the intensity of an earthquake. There are many different ways of computing this based on a variety of different quantities. We are going to take a quick look at the formula that uses the energy released during an earthquake.

If *E* is the energy released, measured in joules, during an earthquake then the magnitude of the earthquake is given by,

$$M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$$

where $E_0 = 10^{4.4}$ joules.

Example 5 If 8×10^{14} joules of energy is released during an earthquake what was the magnitude of the earthquake?

Solution

There really isn't much to do here other than to plug into the formula.

$$M = \frac{2}{3}\log\left(\frac{8\times 10^{14}}{10^{4.4}}\right) = \frac{2}{3}\log\left(8\times 10^{9.6}\right) = \frac{2}{3}\left(10.50308999\right) = 7.002$$

So, it looks like we'll have a magnitude of about 7.

Example 6 How much energy will be released in an earthquake with a magnitude of 5.9?

Solution

In this case we will need to solve the following equation.

$$5.9 = \frac{2}{3}\log\left(\frac{E}{10^{4.4}}\right)$$

We saw how solve these kinds of equations in the previous section. First we need the logarithm on one side by itself with a coefficient of one. Once we have it in that form we convert to exponential form and solve.

$$\begin{aligned} 5.9 &= \frac{2}{3}\log\left(\frac{E}{10^{4.4}}\right) \\ 8.85 &= \log\left(\frac{E}{10^{4.4}}\right) \\ 10^{8.85} &= \frac{E}{10^{4.4}} \\ E &= 10^{8.85}\left(10^{4.4}\right) = 10^{13.25} \end{aligned}$$

So, it looks like there would be a release of $10^{13.25}$ joules of energy in an earthquake with a magnitude of 5.9.