STRAIGHT LINES

1. DISTANCE FORMULA

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

2. SECTION FORMULA

The P(x,y) divided the line joining $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m : n, then ;

$$x=\frac{mx_2+nx_1}{+n};\ y=\frac{my_2+ny_1}{+n}$$

Note...

(i) If m/n is positive, the division is internal, but if m/n is negative, the division is external.

(ii) If P divides AB internally in the ratio m:n & Q divides AB externally in the ratio m:n then

P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically, $\frac{2}{AB}=\frac{1}{AP}+\frac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.

3. CENTROID, INCENTRE & EXCENTRE

If $A(x_1, y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the co-ordinates of the special points of triangle ABC are as follows :

Centroid G $\equiv\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

Incentre I $\equiv\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$ and

Excentre (to A) I_1

$\equiv\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$ and so on.

Note...

(i) Incentre divides the angle bisectors in the ratio, (b+c): a; (c+a) : b & (a+b) : c.

(ii) Incentre and excentre are harmonic conjugate of each other w.r.t. the angle bisector on which they lie.

(iii) Orthocentre, Centroid & Circumcentre are always collinear & centroid divides the line joining orthocentre & circumcentre in the ratio 2:1.

(iv) In an isosceles triangle G, O, I & C lie on the same line and in an equilateral traingle, all these four points coincide.

4. AREA OF TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, then its area is equal to

$$\Delta ABC=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix},$$ provided the vertices are considered in the counter clockwise sense.

The above formula will give a (–)ve area if the vertices (x_i, y_i), i = 1, 2, 3 are placed in the clockwise sense.

Note...

Area of n-sided polygon formed by points (x_1, y_1); (x_2, y_2); ....(x_n, y_n) is given by :

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \dots\dots\dots \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix}\right)$$

5. SLOPE FORMULA

If θ is the angle at which a straight line is inclined to the positive direction of x-axis, and $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does n't exist, but the line is parallel to the y-axis, If $\theta = 0$, then $m = 0$ and the line is parallel to the x-axis.

If $A(x_1, y_1)$ & $B(x_2, y_2)$, $x_1 \ne x_2$, are points on a straight line, then the slope m of the line is given by :

$$m = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$

6. CONDITION OF COLLINEARITY OF THREE POINTS

Points A (x_1,y_1), $B(x_2,y_2)$, $C(x_3,y_3)$ are collinear if :

(i) $m_{AB} = m_{BC} = m_{CA}$ i.e. $\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \left(\frac{y_2 - y_3}{x_2 - x_3}\right)$

(ii) $\Delta ABC = 0$ i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(iii) $AC = AB + BC$ or $AB \sim BC$

(iv) A divides the line segment BC in some ratio.

7. EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS

(i) Point-Slope form : $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m and which passes through the point (x_1,y_1).

(ii) Slope-Intercept form : $y = mx + c$ is the equation of a straight line whose slope is m and which makes an intercept c on the y-axis.

(iii) Two point form : $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the point (x_1,y_1) & (x_2,y_2)

(iv) Determinant form : Equation of line passing through (x_1,y_1) and (x_2,y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(v) Intercept form : $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

(vi) Perpendicular/Normal form : $x\cos\alpha + y\sin\alpha = p$ (where $p > 0$, $0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes an angle α with positive x-axis.

(vii) Parametric form : $P(r) = (x,y) = (x_1 + r\cos\theta, y_1 + r\sin\theta)$ or $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ is the equation of the line is parametric form, where 'r' is the parameter whose absolute value is the distance of any point (x,y) on the line from fixed point (x_1,y_1) on the line.

(viii) General Form : $ax + by + c = 0$ is the equation of a straight line in the general form. In this case, slope of line $= -\frac{a}{b}$.

8. POSITION OF THE POINT $(x_1,y_1$ RELATIVE OF THE LINE $ax + by + c = 0$

If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1,y_1) will lie on the non-origin side of $ax + by + c = 0$.

In general two points (x_1,y_1) and (x_2,y_2) will lie on same side or opposite side of $ax+by+c=0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same or opposite sign respectively.

9. THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS

Let the given line ax + by + c = 0 divides the line segment joining $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m:n, then $\frac{m}{n} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A and B are on the same side of the given line then m/n is negative but if A and B are on opposite sides of the given line, then m/n is positive.

10. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of perpendicular from $P(x_1,y_1)$ on

$ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

11. REFLECTION OF A POINT ABOUT A LINE

(i) The image of a point (x_1,y_1) about the line ax + by + c = 0 is :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

(ii) Similarly foot of the perpendicular from a point on the line is :

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$$

12. ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES

If m_1 and m_2 are the slopes of two intersecting straight lines ($m_1m_2 \neq -1$) and θ is the acute angle between them,

then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

Note...

Let m_1, m_2, m_3 are the slopes of three line $L_1=0$; $L_2=0$; $L_3=0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these lines are given by,

$\tan A = \frac{m_1 - m_2}{1 + m_1m_2}$; $\tan B\frac{m_2 - m_3}{1 + m_2m_3}$; and $\tan C = \frac{m_3 - m_1}{1 + m_3m_1}$

13. PARALLEL LINES

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to y = mx + c is of the type y = mx + d, where d is parameter.

(ii) Two lines ax+ by+ c=0 and a′x +b′y +c′ = 0 are parallel

if : $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$

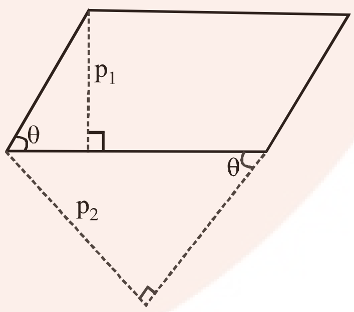
Thus any line parallel to ax+ by+ c =0 is of the type ax+ by+ k =0, where k is a parameter.

(iii) The distance between two parallel lines with equations ax + by+ c_1 =0 and

$ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

Coefficient of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \frac{p_1p_2}{\sin\theta}$, where p_1 and p_2 are



distance between two pairs of opposite sides and θ is the angle between any two adjacent sides. Note that area of the parallogram bounded by the lines $y=m_1x+c_1$, $y=m_1x+c_2$. and $y = m_2x + d_1$, $y= m_2x + d_2$ is given by

$\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$

14. PERPENDICULAR LINES

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slope is -1 i.e., $m_1m_2 = -1$. Thus any line perpendicular to $y = mx + c$ is of the form.

$y = -\frac{1}{m}x + d$, where d is any parameter.

(ii) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

15. STRAIGHT LINES MAKING ANGLE a WITH GIVEN LINE

The equation of lines passing through point (x_1,y_1) and making angle α with the line $y = mx + c$ are given by $(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.

16. BISECTOR OF THE ANGLES BETWEEN TWO LINES

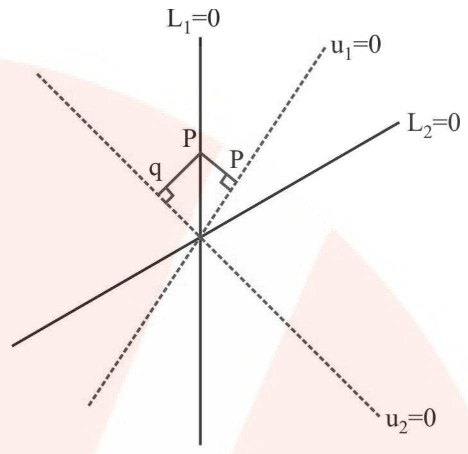
Equations of the bisectors of angles between the lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are :

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \pm\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

Note...

Equation of straight lines through $P(x_1,y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisector between these two lines & passing throught the point P.

17. METHODS TO DISCRIMINATE BETWEEN THE ACUTE BISECTOR AND THE OBTUSE ANGLE BISECTOR



(i) If θ be the angle between one of the lines & one of the bisectors, find $\tan\theta$. if $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector. if $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector

(ii) Let $L_1=0$ & $L_2=0$ are the given lines & $u_1=0$ and $u_2=0$ are bisectors between $L_1=0$ and $L_2=0$. Take a point P on any one of the lines $L_1=0$ or $L_2=0$ and drop perpendicular on $u_1=0$ and $u_2=0$ as shown. If.

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector.

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector.

$|p| = |q| \Rightarrow$ the lines L_1 and L_2 are perpendicular.

(iii) if $aa' + bb' < 0$, while c & c′ are postive, then the angle between the lines is acute and the equation of the bisector of this acute angle is $\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

If, however, $aa' + bb' > 0$, while c and c′ are postive, then the angle between the lines is obtuse & the equation of the bisector of this obtuse angle is :

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

The other equation represents the obtuse angle bisector in both cases.

18. TO DISCRIMINATE BETWEEN THE BISECTOR OF THE ANGLE CONTAINING A POINT

To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equation, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant term c, c' are positive.

Then ; $\frac{ax+by+c}{\sqrt{a^2+b^2}} = +\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing origin and

$\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin. In general equation of the bisector which contains the point (α, β) is.

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \text{ or } \frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$$

according as $a\alpha + b\beta + c$ and $a'\alpha + b'\beta + c'$ having same sign or otherwise.

19. CONDITION OF CONCURRENCY

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Alternatively : If three constants A, B and C (not all zero) can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

20. FAMILY OF STRAIGHT LINES

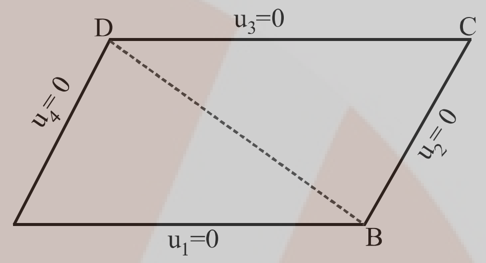
The equation of a family of straight lines passing through the points of intersection of the lines,

$L_1 \equiv a_1x + b_1y + c_1 = 0$ & $L_2 \equiv a_2x + b_2y + c_2 = 0$ is given by $L_1 + kL_2 = 0$ i.e.

$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitray real number.

Note...

(i) If $u_1 = ax+by+c$, $u_2 = a'x+b'y+d$, $u_3 = ax+by+c'$, $u_4 = a'x+b'y+d'$, then $u_1 = 0$; $u_2 = 0$; $u_3 = 0$; $u_4 = 0$; form a parallegoram



The diagonal BD can be given by $u_2u_3 - u_1u_4 = 0$

Proof : Since it is the first degree equation in x & y, it is a straight line. Secondly point B satisfies $u_2 = 0$ and $u_1 = 0$ while point D satisfies $u_3 = 0$ and $u_4 = 0$. Hence the result. Similarly, the diagonal AC can be given by $u_1u_2 - u_3u_4 = 0$

(ii) The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equation are identical for some real λ and μ.

[For getting the values of λ and μ compare the coefficients of x, y & the constant terms.]

21. A PAIR OF STRAIGHT LINES THROUGH ORIGIN

(i) A homogeneous equation of degree two, "$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab \Rightarrow$ lines are real and distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0,0)

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

(iii) If θ is the acute angle between the pair of straight lines represented by,

$$ax^2 + 2hxy + by^2 = 0, \text{ then; } \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

(iv) The condition that these lines are :

(a) At right angles to each other is $a + b = 0$ i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $h = 0$ i.e. coeff. of $xy = 0$.

Note...

A homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angle between the straight lines,

$$ax^2 + 2hxy + by^2 = 0, \text{ is } \frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

22. GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ = represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ i.e. if } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(ii) The angle θ between the two lines representing by a general equation is the same as that between two lines represented by its homogenous part only.

23. HOMOGENIZATION

The equation of a pair of straight lines joining origin to the points of intersection of the line

$L \equiv lx + my + n = 0$ and a second degree curve,

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) +$$

$$2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

The equal is obtained by homogenizing the equation of curve with the help of equation of line.

Note...

Equation of any cure passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$ where λ and μ are parameters.