<u>Further Maths Solutions (AS Pure)</u>

Section A: Multiple Choice

1. **Answer:** The argument is incomplete since there is no established base case

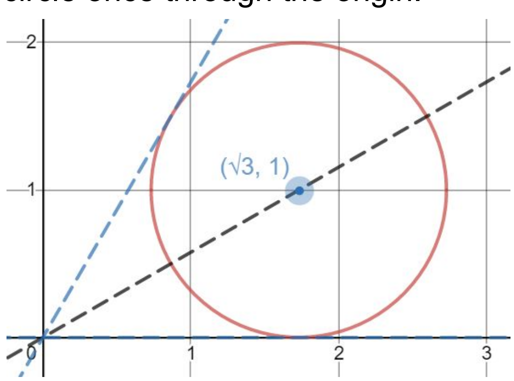
   Working: The inductive step assumes it is true for some $n = k$. However, the case $n = 1$ is not tested so it has not been shown that there exists any k such that the argument holds.

2. **Answer:** 7

   Working: Complex roots → discriminant < 0 → $5^2 - 4n < 0$ → $4n > 25$
   → $n > 25/4$ → $n > 6.25$
   n is an integer so smallest possible value is n = 7

3. **Answer:** |z| takes its minimum value on *C* when $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

   Working: The locus is $|z - (\sqrt{3} + i)| = 1$. The extreme values of |z| will be when z is on the same and opposite sides of the circle to the origin. The extreme values of arg z will be when the tangents meet the circle once through the origin.



By trigonometry and simple distances,
$z = \sqrt{3}/2 + 1/2\,i$ → $|z| = 1$ (min), $z = 3\sqrt{3}/2 + 3/2\,i$ → $|z| = 3$ (max)
$z = \sqrt{3}$ → $\arg z = 0$ (min), $z = 1 + \sqrt{3}\,i$ → $\arg z = \pi/3$ (max)

4. **Answer:** When $t = 7$, **AB** = 4**I**.

Working: det A = 1(-t - 8) - 2(-2 - 12) + 1(4 - 3t) = 20 - 4t (not proportional)

Considering the 4th and 5th elements of AB and 3(A + B), we get
4th: (2 * 15) + (t * -2t) + (4 * 17) = 3(2 - 2t)
$\rightarrow 98 - 2t^2 = 6 - 6t \rightarrow 2t^2 - 6t - 92 = 0$
5th: (2 * -4) + (t * 4) + (4 * -4) = 3(t + 4)
$\rightarrow -24 + 4t = 3t + 12 \rightarrow t = 36$
These roots are not what is given, so cannot be true for any t.

Entering into a calculator, we see that
AB = [[4, 0, 0], [0, 4, 0], [0, 0, 4]] which is 4 times **I** (identity matrix).

Fourth option is always false (det **AB** ≡ det **A** det **B**).

5. **Answer:** $\omega\sigma = -1$

Working: Complex coefficients → roots are **not** complex conjugates
Sum of roots = -b/a = -i
Product of roots = c/a = -1
Sum of squares: $\omega^2 + \sigma^2 = (\omega + \sigma)^2 - 2(\omega\sigma) = (-i)^2 - 2(-1) = 1$

6. **Answer:** 1

Working: The desired value is
$= (\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta)$
Each factor is equal to $\alpha + \beta + \gamma + \delta$ minus one of the roots.
The sum of the roots is p, so this is equal to
$= (p - \alpha)(p - \beta)(p - \gamma)(p - \delta)$
Letting $x = p - z$ in the original polynomial gives a new polynomial with these roots. The product of these roots (i.e. the constant term = e/a = e) is this value.
$(p - z)^4 - p(p - z)^3 + q(p - z)^2 - pq(p - z) + 1$
Constant term is: $p^4 - p(p^3) + q(p^2) - pq(p) + 1 = 1$

7. **Answer:** -1/3

Working: $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$
Let $z = -x \rightarrow \ln(1 - z) = -z - z^2/2 - z^3/3 - z^4/4 - \ldots$
$\rightarrow$ Coefficient of $z^3 = -1/3$

8. **Answer:** $x = 0$

Working: Vertical asymptotes: $x^2 - 4 = 0 \rightarrow x = 2$ and $x = -2$
Horizontal asymptote: as $x \rightarrow \infty$, $1/(x^2 - 4) \rightarrow 1/\infty \rightarrow 0$ so $y = 0$

9. **Answer:** $\frac{7}{12}$

Working: Adjusting the formula to integrate vertically (instead of the usual horizontal),

$$V = \pi \int_1^2 [x(y)]^2 \, dy$$

where x(y) is the inverse: $y = 18x^2 - 9 \rightarrow x = \sqrt{(y + 9) / 18}$
$\rightarrow x^2 = (y + 9)/18$
Performing the integral on calculator, $V = 7/12\ \pi$.

10. **Answer:** 73.2°

Working: *AB* = vector from A to B, **d** = direction vector of line *l*
$\cos\theta = (AB \bullet \mathbf{d}) / (|AB| * |\mathbf{d}|)$
AB = 2**i** - 5**j** - **k** and **d** = **i** - 3**k**
$\cos\theta = (2 + 0 + 3) / (\sqrt{30} * \sqrt{10}) = 5 / \sqrt{300}$
$\theta = 73.2^\circ$

Section B: Standard Questions

11.

a. Identifies the ± sign (or just the negative) as the error. [1 mark]

Matthew has failed to reject the negative solution since the argument to / domain of the natural logarithm must be positive (and $\sqrt{(x^2 + 1)} > x$ for all x) [1 mark]

b. arsinh x = 3 → x = sinh 3
[1 mark; no work needed; accept decimal x = 10.0... or exact form $(e^3 - e^{-3})/2$]

c. Let y = arcosh x → x = cosh y
→ $x = (e^y + e^{-y}) / 2$ [1 mark]
→ $2x = e^y + e^{-y}$
→ $(e^y)^2 - 2x(e^y) + 1 = 0$ [1 mark]
→ $e^y = \dfrac{2x \pm \sqrt{4x^2-4}}{2}$
→ $e^y = x \pm \sqrt{x^2 - 1}$ [1 mark]
→ $y = \ln\ (x \pm \sqrt{x^2 - 1})$ [1 mark]
arcosh *x* is an increasing function for all *x* ≥ 1:

$$\frac{dy}{dx} = \frac{1 \pm \frac{x}{\sqrt{x^2-1}}}{x \pm \sqrt{x^2-1}} > 0$$ [1 mark]

The denominator is always positive, while the numerator is positive when + is chosen and negative when - is chosen. Hence,

arcosh $x = \ \ln\ (x + \sqrt{x^2 - 1})$ [1 mark: must justify]

[Allow any other valid method to justify rejection of -, e.g. domain restriction]
[Do **not** allow “arcosh *x* is always positive” as justification]

12.

a. **Base case:** try $n = 1$

LHS = $1^3 = 1$ and RHS = $1/4 * 1^2 * (1 + 1)^2 = 1$

LHS = RHS, so true for $n = 1$. [1 mark]

Inductive hypothesis: assume true for some integer $n = k$.

Then, for $n = k + 1$,

$$\sum_{r=1}^{k+1} r^3 = (\sum_{r=1}^{k} r^3) + (k+1)^3$$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$ [1 mark]

$= (k+1)^2(\frac{1}{4}k^2 + k + 1)$

$= \frac{1}{4}(k+1)^2(k+2)^2$ (= RHS for $n = k + 1$)

→ true for $n = k + 1$ [1 mark]

Conclusion:

True for $n = 1$, and true for $n = k$ ⇒ true for $n = k + 1$, then by induction it is true for all integers $n \geq 1$. [1 mark]

b. $\sum_{r=1}^{2n} r(r-1)(r+1) = \sum_{r=1}^{2n} r^3 - r = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{2n} r$ [1 mark]

Using previous result and standard result that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$,

$= \frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{2}(2n)(2n+1)$ [1 mark]

$= n^2(2n + 1)^2 - n(2n + 1)$

$= n(2n + 1)(n(2n + 1) - 1)$ [1 mark]

$= n(2n + 1)(2n^2 + n - 1)$

$= n(n + 1)(2n - 1)(2n + 1)$ [1 mark]

13.

a. Rational function with two asymptotes (no obliques) has general form

$y = \frac{ax+b}{cx+d}$

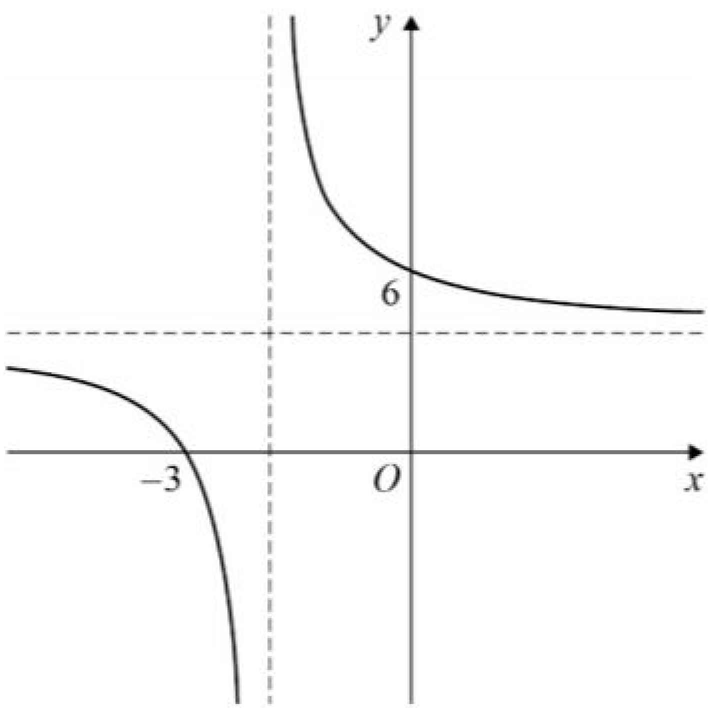
Asymptote x = 1 → c + d = 0

Asymptote y = 2 → a/c = 2 → a = 2c [1 mark; or any general form where a/c = 2]
Passes through (-3, 0) → -3a + b = 0
Solution to system is a = -2d, b = -6d, c = -d
Let d = -1 and then,

$y = \frac{2x+6}{x-1}$ [1 mark]

[Accept alternative form, $y = 2 + \frac{4}{x+1}$]

b.



[1 mark: correct graph shape]
[1 mark: correct asymptotes labelled]
[1 mark: correct intercepts labelled]

c. In limiting case,

$\frac{2x+6}{x+1}$ = 5 [1 mark]

→ 2x + 6 = 5(x + 1)

→ 3x = 1

→ x = 1/3 [1 mark]

Considering the graph, the correct regions are then,

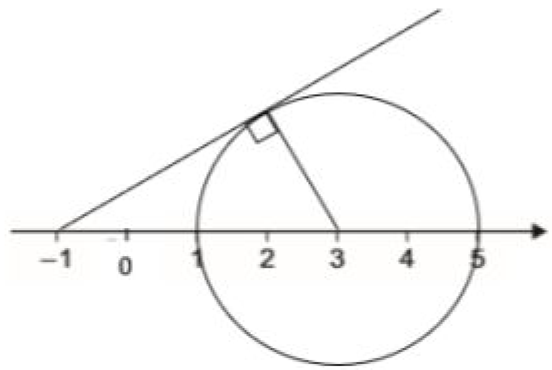
$x \geq 1/3$ or $x < -1$. [2 marks: 1 for each set]

[Do **not** accept $x \leq -1$ since x = -1 is undefined (asymptote)]

[Allow method using inequality sign throughout, but must multiply through by $(x + 1)^2$ and **not** just $(x + 1)$ to get full marks]

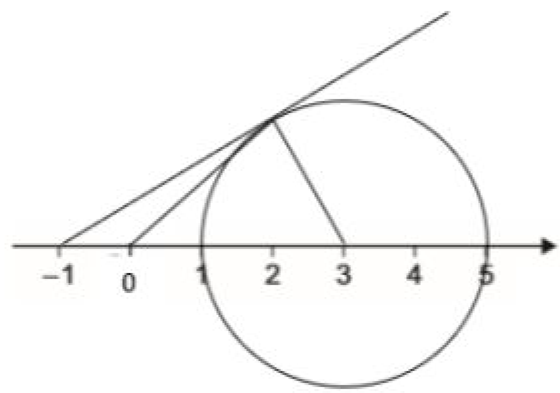
14.

a. i) Sketching a diagram to help, (real axis shown only)



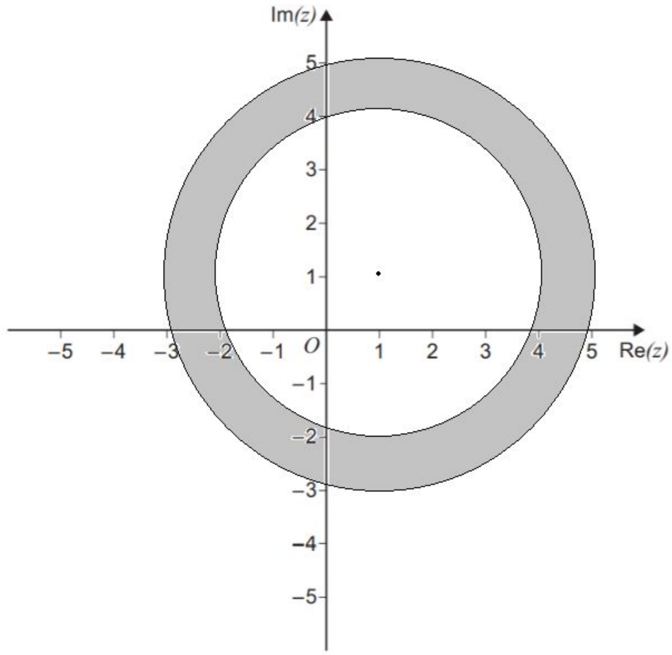
By trigonometry, $\sin \alpha = 2/4 = 1/2$. [1 mark]
$\rightarrow \alpha = \sin^{-1}(1/2) = \pi/6$ [1 mark]

ii) Using cosine rule,

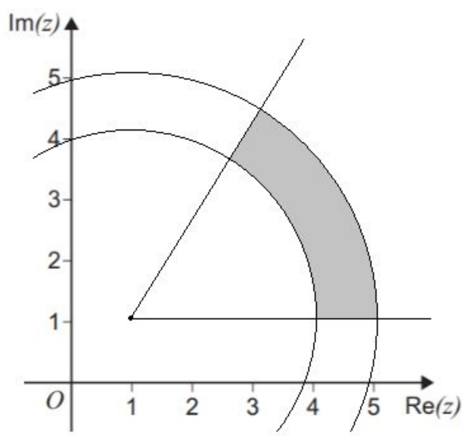


$|w| = 2^2 + 3^2 - 2(2)(3) \cos \pi/3$
$\rightarrow |w| = \sqrt{7}$ [1 mark]
Let $\theta = \arg w$. Using sine rule,
$\sin \theta / 2 = \sin(\pi/3) / \sqrt{7}$ [1 mark]
$\rightarrow \sin \theta = \sqrt{21}/7$
$\rightarrow \theta = 0.714\ldots$ [1 mark]
So the modulus-argument form is
$w = \sqrt{7} (\cos 0.71 + i \sin 0.71)$ [1 mark]

b. i) Two concentric circles, centered at 1 + i [1 mark]
One with radius 3 and the other with radius 4. [1 mark]
Area between the two shaded. [1 mark]



ii) The set *S* is



Re(z) + Im(z) = k, for the largest value of k
→ Im(z) = k - Re(z) → in Cartesian coordinates, y = k - x
This line intersects the circle for the largest *k* when it is tangent to the outer circle i.e. at angle 45° = π/4 [1 mark: no justification needed]
→ z = 1 + i + 4(cos π/4 + i sin π/4) [1 mark]
→ z = 1 + 2√2 + i(1 + 2√2)
→ Re(z) + Im(z) = 1 + 2√2 + 1 + 2√2
→ Re(z) + Im(z) = 2 + 4√2 [1 mark]

15. Starting with the suggested expression,

$(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = \alpha^2 - 2\alpha\beta + \beta^2 + \beta^2 - 2\beta\gamma + \gamma^2 + \gamma^2 - 2\gamma\alpha + \alpha^2$

$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ [1 mark]

$= 2[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 2(\alpha + \beta + \gamma)^2 - 6(\alpha\beta + \beta\gamma + \gamma\alpha)$ [1 mark]

Using Vieta's formulas, $\alpha + \beta + \gamma = -m$ and $\alpha\beta + \beta\gamma + \gamma\alpha = n$. Then,

$(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2(-m)^2 - 6n$

$(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2m^2 - 6n$ [1 mark]

Since the roots are all real, their squares (of differences) are all positive and so the RHS must also be positive. [1 mark]

$\rightarrow 2m^2 - 6n \geq 0$

$\rightarrow m^2 \geq 3n$. [1 mark]

[Allow representing $\alpha + \beta + \gamma = \sum\alpha$; $\alpha\beta + \beta\gamma + \gamma\alpha = \sum\alpha\beta$; $\alpha^2 + \beta^2 + \gamma^2 = \sum\alpha^2$]

16.

a. i)

Method 1: there exists parallel planes containing each line. The normals to these planes will be scalar multiples of each other, and then the distance is $d = |(A - B) \cdot \mathbf{n}|$ where A and B are points on each line and $\mathbf{n}$ is the common unit normal vector:

Direction vector for 2nd wire = $\begin{pmatrix}10\\0\\20\end{pmatrix} - \begin{pmatrix}-10\\100\\-5\end{pmatrix}$ [1 mark]

Let $\begin{pmatrix}x\\y\\z\end{pmatrix}$ be a vector perpendicular to both wires.

$$\therefore \begin{pmatrix}0\\100\\-20\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \quad \textbf{and} \quad \begin{pmatrix}20\\-100\\25\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \quad \text{[1 mark]}$$

$$\Rightarrow 100y - 20z = 0 \quad \textbf{and} \quad 20x - 100y + 25z = 0$$

$$\Rightarrow z = 5y \quad \textbf{and} \quad x = -1.25y \quad \text{[1 mark]}$$

$\therefore$ perpendicular vector is $\begin{pmatrix}-1.25y\\y\\5y\end{pmatrix}$ [1 mark]

$\Rightarrow$ unit perpendicular vector is $\begin{pmatrix}-1.25\\1\\5\end{pmatrix} \div \sqrt{(-1.25)^2 + 1^2 + 5^2}$ [1 mark]

a vector from 1st line to 2nd line is $\begin{pmatrix}10\\0\\20\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}10\\0\\20\end{pmatrix}$ [1 mark]

$\therefore$ distance between lines is $\begin{pmatrix}10\\0\\20\end{pmatrix}.\begin{pmatrix}-1.25\\1\\5\end{pmatrix} \div \frac{21}{4}$ = 16.7 metres [1 mark]

Alternative, Method 2: there exists a vector from one point on the line to the other which is normal to each direction vector, and this vector will be along the shortest distance. This vector is found by finding the generalised vector and dotting with each direction vector equal to zero:

Direction vector for 2[nd] wire = $\begin{pmatrix} 10 \\ 0 \\ 20 \end{pmatrix} - \begin{pmatrix} -10 \\ 100 \\ -5 \end{pmatrix}$ [1 mark]

$\boldsymbol{r_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 100 \\ -20 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} 10 \\ 0 \\ 20 \end{pmatrix} + \mu \begin{pmatrix} 20 \\ -100 \\ 25 \end{pmatrix}$ [1 mark]

$\boldsymbol{r_2} - \boldsymbol{r_1} = \begin{pmatrix} 10 + 20\mu \\ -100\mu - 100\lambda \\ 20 + 25\mu + 20\lambda \end{pmatrix}$ [1 mark]

$\begin{pmatrix} 10 + 20\mu \\ -100\mu - 100\lambda \\ 20 + 25\mu + 20\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 100 \\ -20 \end{pmatrix} = 0$ and $\begin{pmatrix} 10 + 20\mu \\ -100\mu - 100\lambda \\ 20 + 25\mu + 20\lambda \end{pmatrix} . \begin{pmatrix} 20 \\ -100 \\ 25 \end{pmatrix} = 0$

[1 mark]

$-10000\mu - 10000\lambda - 400 - 500\mu - 400\lambda = 0$ and
$200 + 400\mu + 10000\mu + 10000\lambda + 500 + 625\mu + 500\lambda = 0$

$11025\mu + 10500\lambda + 700 = 0$ and $11025\mu + 10920\lambda + 420 = 0$
[1 mark]

$\lambda = \frac{2}{3}$ and $\mu = -\frac{44}{63}$ [1 mark]

$\sqrt{\left(10 + 20\left(-\frac{44}{63}\right)\right)^2 + \left(-100\left(-\frac{44}{63}\right) - 100\left(\frac{2}{3}\right)\right)^2 + \left(20 + 25\left(-\frac{44}{63}\right) + 20\left(\frac{2}{3}\right)\right)^2}$ = 16.7 metres [1 mark]

ii) Model the wires as curves / account for thickness of wires [1 mark]

b. i) The beams of light will intersect if their vector equations have a consistent pair of solutions.

Modelling beams of light as straight lines taking the origin as point A:

$$\mathbf{r}_A = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda\left(\begin{pmatrix} 30 \\ 125 \\ 23 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\right) \quad \text{[1 mark]}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 30 \\ 125 \\ 20 \end{pmatrix} \quad \text{[1 mark]}$$

$$\mathbf{r}_B = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} + \mu\left(\begin{pmatrix} 28 \\ 140 \\ 29 \end{pmatrix} - \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 20 \\ 130 \\ 28 \end{pmatrix} \quad \text{[1 mark]}$$

Equating two components,

$$\begin{aligned} 30\lambda &= 8 + 20\mu \\ 125\lambda &= 10 + 130\mu \end{aligned} \quad \text{[1 mark]}$$

$$\lambda = \frac{3}{5} \text{ and } \mu = \frac{1}{2} \quad \text{[1 mark]}$$

Subbing back in to third component,

$$3 + \frac{3}{5} \times 20 = 15$$

$$1 + \frac{1}{2} \times 28 = 15$$

$\therefore$ Intersect [1 mark]

ii) Using the formula cos θ = (**a** • **b**) / (|**a**| |**b**|):

$$\begin{pmatrix} 30 \\ 125 \\ 20 \end{pmatrix} \bullet \begin{pmatrix} 20 \\ 130 \\ 28 \end{pmatrix} = 17410$$

$$\cos\theta = \frac{17410}{\sqrt{30^2 + 125^2 + 20^2} \times \sqrt{20^2 + 130^2 + 28^2}}$$

$$\cos\theta = \frac{17410}{\sqrt{16925} \times \sqrt{18084}} = 0.9951$$

$$\theta = 5.6°$$

iii) Take into account the width of the beams. [1 mark]

17. **Method 1:** Making the substitution $u = 2x$, we get

$\sinh^2 u - \cosh u = 1$ [1 mark]

$(\cosh^2 u - 1) - \cosh u - 1 = 0$ [1 mark]

$\cosh^2 u - \cosh u - 2 = 0$ [1 mark]

$\cosh u = \frac{1 \pm \sqrt{9}}{2} \rightarrow \cosh u = 2$ or $\cosh u = -1$ [1 mark]

$\cosh u > 1$ for all u so reject -1 $\rightarrow \cosh u = 2$ [1 mark]

$u = \pm \cosh^{-1} 2$ [1 mark]

$u = \pm \ln(2+\sqrt{3})$ [1 mark]

$u = \ln(2+\sqrt{3})$ or $u = \ln(\frac{1}{2+\sqrt{3}})$ [1 mark]

Rationalising the denominator, $u = \ln(2+\sqrt{3})$ or $u = \ln(2-\sqrt{3})$

$\rightarrow x = \frac{1}{2}\ln(2 \pm \sqrt{3})$ [1 mark]

Alternative, method 2: using double-argument identities,

$4\sinh^2 x \cosh^2 x - (\cosh^2 x + \sinh^2 x) = 1$ [1 mark]

$4\sinh^2 x(1+\sinh^2 x) - (1+\sinh^2 x) - \sinh^2 x - 1 = 0$ [1 mark]

$4\sinh^4 x + 2\sinh^2 x - 2 = 0$ [1 mark]

$\sinh^2 x = \frac{-2 \pm \sqrt{36}}{8} \rightarrow \sinh^2 x = \frac{1}{2}$ or -1 [1 mark]

Square is always positive so reject -1 $\rightarrow \sinh^2 x = \frac{1}{2} \rightarrow \sinh x = \pm\frac{\sqrt{2}}{2}$ [1 mark]

$x = \ln(\frac{\sqrt{2}}{2} + \sqrt{(\frac{\sqrt{2}}{2})^2 + 1})$ or $x = \ln(-\frac{\sqrt{2}}{2} + \sqrt{(-\frac{\sqrt{2}}{2})^2 + 1})$

$x = \ln\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{\sqrt{2}}\right)$ or $x = \ln\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{\sqrt{2}}\right)$ [1 mark]

$x = \ln\left(\frac{2+2\sqrt{3}}{2\sqrt{2}}\right)$ or $x = \ln\left(\frac{-2+2\sqrt{3}}{2\sqrt{2}}\right)$ [1 mark]

$x = \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$ or $x = \ln\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)$

$x = \ln\left(\sqrt{\frac{(\sqrt{2}+\sqrt{6})^2}{4}}\right)$ or $x = \ln\left(\sqrt{\frac{(-\sqrt{2}+\sqrt{6})^2}{4}}\right)$ [1 mark]

$x = \frac{1}{2}\ln(2+\sqrt{3})$ or $x = \frac{1}{2}\ln(2-\sqrt{3})$.

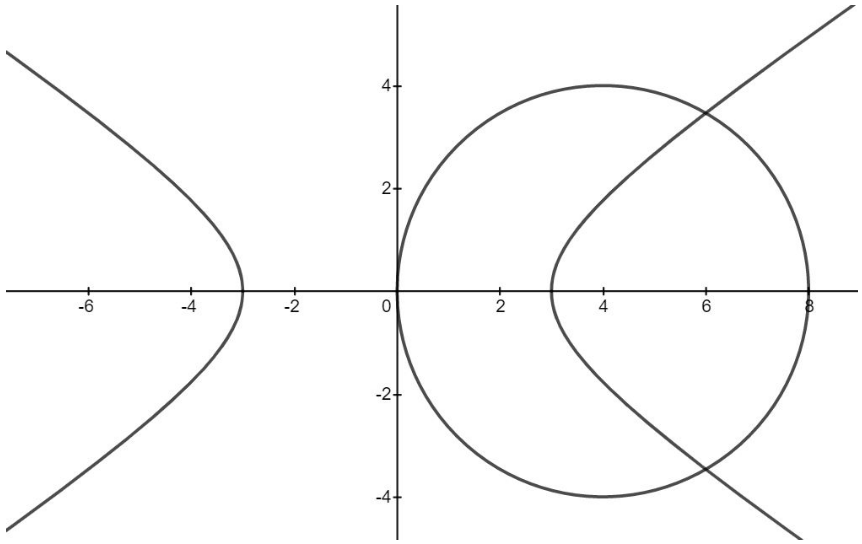
$\rightarrow x = \frac{1}{2}\ln(2 \pm \sqrt{3})$. [1 mark]

[Accept retaining the ± sign throughout the algebraic manipulation instead of splitting into two cases.] [Accept use of exponential forms with either method.]

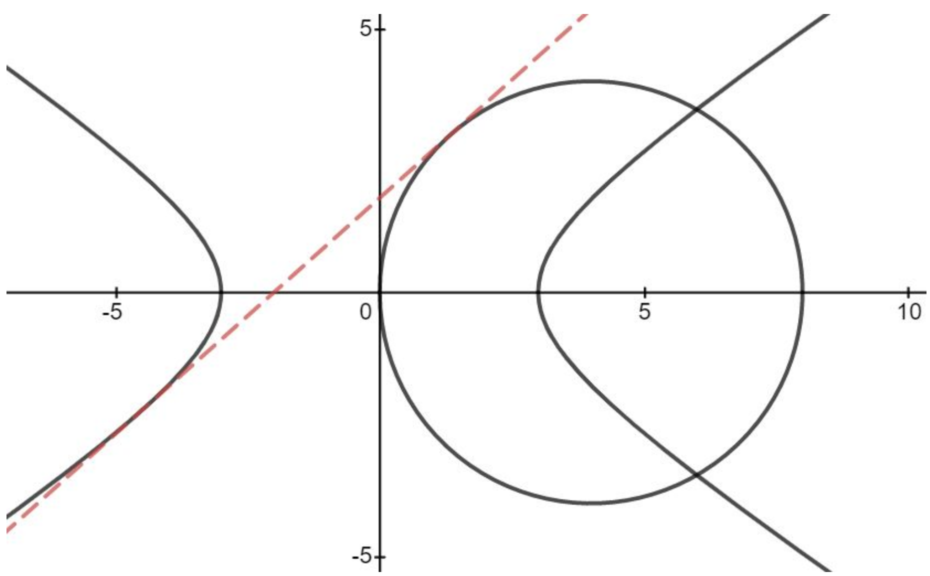
Section C: Extended Question

18. **Planning**

Start with a sketch / plot of these two curves:



It becomes clear that the line (of **positive**) gradient can only be tangent if it touches the bottom-left branch of the hyperbola and the top-left quadrant of the circle, i.e.



We will give this line its general form of $y = mx + c$ and continue.

Finding the equation of the line

If L is tangent, it intersects each curve in one place only.
For its intersection with the circle,
$x^2 + y^2 - 8x = 0 \rightarrow x^2 + (mx + c)^2 - 8x = 0$
$\rightarrow x^2 + m^2x^2 + 2cmx + c^2 - 8x = 0$
$\rightarrow (m^2 + 1)x^2 + (2mc - 8)x + c^2 = 0$ [2 marks]

The discriminant of this quadratic must be 0 since there is only one point:
$\rightarrow (2mc - 8)^2 - 4c^2(m^2 + 1) = 0$
$\rightarrow 4m^2c^2 - 32mc + 64 - 4m^2c^2 - 4c^2 = 0$
$\rightarrow c^2 + 8mc - 16 = 0$ [1 mark]

Repeating the process with the hyperbola,
$4x^2 - 9y^2 = 36 \rightarrow 4x^2 - 9(mx + c)^2 = 36$
$\rightarrow 4x^2 - 9m^2x^2 - 18cmx - 9c^2 - 36 = 0$
$\rightarrow (4 - 9m^2)x^2 + (-18cm)x - 9(c^2 + 4) = 0$ [2 marks]
Discriminant = 0:
$\rightarrow 324c^2m^2 + 36(4 - 9m^2)(c^2 + 4) = 0$
$\rightarrow 324c^2m^2 + 144c^2 - 324c^2m^2 + 576 - 1296m^2 = 0$
$\rightarrow c^2 - 9m^2 + 4 = 0$ [1 mark]

We now have two nonlinear equations in c and m which we solve simultaneously by substitution. Rearrange the first formula for m:
$c^2 + 8mc - 16 = 0 \rightarrow m = (16 - c^2) / 8c$ [1 mark]
Substitute into the second equation,
$$\rightarrow c^2 - \frac{9(16 - c^2)^2}{64c^2} + 4 = 0$$
$\rightarrow 64c^4 - 9(16 - c^2)^2 + 256c^2 = 0$
$\rightarrow 64c^4 - 9c^4 + 288c^2 - 2304 + 256c^2 = 0$
$\rightarrow 55c^4 + 544c^2 - 2304 = 0$
$\rightarrow c^2 = 16/5$ or $c^2 = -144/11$ [1 mark]
Since c^2 must be positive, reject $-144/11 \rightarrow c^2 = 16/5 \rightarrow c = \pm 4/\sqrt{5}$
From the diagram in planning, it is clear that $c > 0 \rightarrow c = 4/\sqrt{5}$. [1 mark]
Subbing back in for m,
$m = (16 - 16/5) / (32/\sqrt{5}) \rightarrow m = 2\sqrt{5}/5$ [1 mark]
So the line is $y = \dfrac{2\sqrt{5}}{5}x + \dfrac{4}{\sqrt{5}}$, or multiplying by $\sqrt{5}$, we get $2x - \sqrt{5}y + 4 = 0$.

Finding the intersection of the curves

We require the solutions to the nonlinear simultaneous equations
$x^2 + y^2 - 8x = 0$ and $4x^2 - 9y^2 = 36$. The approach is similar to before.

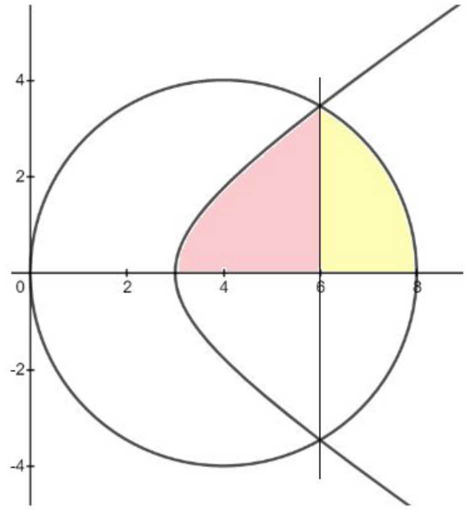
Rearrange first equation for y^2: $y^2 = 8x - x^2$
Substitute into second equation:
$\rightarrow 4x^2 - 9(8x - x^2) = 36$
$\rightarrow 4x^2 - 72x + 9x^2 - 36 = 0$
$\rightarrow 13x^2 - 72x - 36 = 0$
$\rightarrow x = 6$ or $x = -6/13$ [2 marks]

Substituting back in for y,
$\rightarrow y^2 = 8(6) - 6^2$ or $y^2 = 8(-6/13) - (-6/13)^2$
$\rightarrow y^2 = 12$ or $y^2 = -660/169$
$\rightarrow y = \pm\, 2\sqrt{3}$ or $\pm\, (2i\sqrt{165}) / 13$ [2 marks]
(reject complex solutions since coordinates are real, or from graph since clearly no intersections with $x < 0$)

So the intersections are $(6, 2\sqrt{3})$ and $(6, -2\sqrt{3})$. [1 mark]

Finding the area of the bounded region

The half-region of interest can be seen as the sum of two areas under the hyperbola (pink) and circle (yellow), then doubled (since symmetrical):



We will set up two integrals. Since the question asks for a decimal answer, we do not need to worry about how messy the integration is as the calculator will integrate numerically.

Rearranging for y in the hyperbola equation,
$4x^2 - 9y^2 = 36 \rightarrow y = \sqrt{[(4x^2 - 36) / 9]} \rightarrow y = (2/3)\sqrt{(x^2 - 9)}$ [1 mark]
Rearranging for y in the circle equation,
$x^2 + y^2 - 8x = 0 \rightarrow y = \sqrt{(8x - x^2)}$ [1 mark]

The bounds can be easily seen from previous work. The total area is

$$= 2\left(\int_3^6 \frac{2}{3}\sqrt{x^2-9}\ \mathrm{d}x + \int_6^8 \sqrt{8x - x^2}\ \mathrm{d}x\right)$$ [1 mark]

which has a numerical value of 22.7098199…

The exact form is

$$= 8\sqrt{3} + \frac{16\pi}{3} + 6\ln(2-\sqrt{3})$$

which can be derived by lengthy substitutions of x = 3 sec u in the first integral and x = 4 sin u in the second integral. [1 mark]

Either way the area is 22.7 square units to 3 significant figures as required.