Further Maths Solutions (Pure 2)

Section A: Multiple Choice

1. **Answer:** $a \leq 2$

Working: sinh(x) passes through (0, 0) and is an concave up (convex) function, so the tangent at (0, 0) will always be less than sinh(x):
dy/dx = 2 cosh(x) : at x = 0, dy/dx = 2 cosh(0) = 2 * 1 = 2
→ y = 2x is the tangent → a ≤ 2

2. **Answer:** $P(x) = x^4 - 2x^2 + 2$

Working: Roots form a square when the roots are roots of unity (1):
$x^4 - 1 = 0 \rightarrow x^4 = 1 \rightarrow$ roots of unity → square
$x^4 - 2x^2 + 2 = 0 \rightarrow (x^2 - 1)^2 + 1 = 0 \rightarrow (x^2 - 1)^2 = -1 \rightarrow$ not unity
$x^4 + 4x^3 + 6x^2 + 4x + 2 = 0 \rightarrow (x + 1)^4 + 1 = 0 \rightarrow (x + 1)^4 = -1$
→ roots of unity
$x^4 - 4ix^3 - 6x^2 + 4ix + 2 = (x + i)^4 + 1 = 0 \rightarrow (x + i)^4 = -1$
→ roots of unity

3. **Answer:** $8xy(x^2 + y^2)$

Working: Starting with $(x + y)^4 - (x - y)^4$
$= (x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4) - (x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4)$
$= 8x^3y + 8xy^3$
$= 8xy(x^2 + y^2)$

4. **Answer:** $\cosh(x + y) \equiv \sinh x \sinh y + \cosh x \cosh y$

Working: Using Osborne's rule (in the corresponding trig identity, change the sign of a product of sines):
cos(x + y) = cos x cos y - sin x sin y
cosh(x + y) = cosh x cosh y + sinh x sinh y

5. **Answer:** The iterative formula for x_n is $x_n = x_{n+1} + 0.2$ for all $\{n \in \mathbb{Z} : n \geq 0\}$

Working: Standard formulae are (Euler's methods)

$x_n = x_{n-1} + h$

$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$

$y_n = y_{n-2} + 2h\, f(x_{n-1}, y_{n-1})$

We see that both formulae are used. By equating terms, h = 0.1 and $f(x, y) = x^2 + y^2 + 2xy = (x + y)^2$, and y(1) = 0.

So (B) is false since it assumed h = 0.2.

6. **Answer:**

$$S = 20\pi \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

Working: Circle has equation $x^2 + (y - 5)^2 = 1 \rightarrow y = 5 \pm \sqrt{1 - x^2}$

$\rightarrow dy/dx = \mp 2x / (2\sqrt{1 - x^2}) = \mp x / \sqrt{1 - x^2}$

Total surface area = surface from upper half of circle + surface from lower half of circle

$$S = 2\pi \int_{-1}^{1} \left(5 + \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x + 2\pi \int_{-1}^{1} \left(5 - \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

$$S = 2\pi \int_{-1}^{1} \left(5 + \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}} + \left(5 - \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

$$S = 2\pi \int_{-1}^{1} 10 \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

$$S = 20\pi \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

7. **Answer:** $3x + 6y - 4z = 49$

Working: Π must contain the midpoint of the points and be normal to the line connecting the points.

Midpoint: (5, 5, -1), vector for line through points: [3; 6; -4]

Using formula (**r** - **a**) • **n** : (**r** - [5; 5; -1]) . [3; 6; -4] = 0

3(x - 5) + 6(y - 5) - 4(z + 1) = 0 → 3x + 6y - 4z = 49

8. **Answer:** Rotation about the *y*-axis by 180°

Working: Let the two transformations be represented by matrices **X** and **Y** such that **M** = **YX** (**X** followed by **Y** gives **M**).
Y is reflection in plane y = 0 → **Y** flips the orientation (sense) of the space → det **Y** = -1 → det **M** = det **X** det **Y** → det **X** = -1 / -1 = 1
So **X** must **not** change the orientation and must **not** scale.
Enlargement with s.f. = -1 changes the orientation.
Rotation matches both requirements.
Reflection changes the orientation.
Projection of 3D space onto a plane (2D) is non-invertible so has determinant = 0 which is not -1.

9. **Answer:** The line *L* is a line of invariant points under **M**.

Working: Eigenvalue 2 and eigenvector [1; -2] → line with direction vector [1; -2] is an invariant line (not line of invariant points), scales by factor 2, given by *L*.

Since eigenvalue λ is repeated, any vector on the lines of the eigenvectors (and hence any vector in the plane spanned by them) is scaled by the same amount. Therefore it is an invariant plane.
If λ = 1 then there is no movement and it is further a plane of invariant points. The normal to this plane is (**u** × **v**) so Π represents this plane.

10. **Answer:** $f''(x)$

Working: f'(x) = 1/x + √(1 + sin x) and f''(x) = $-1/x^2$ + (cos x) / (2√1 + sin x)
→ asymptote at sin x = -1 → x = 3π/2 → discontinuity in (0, 2π]
Since 1/x and √1 + sin x is always positive (and never both zero), f'(x) > 0 so f(x) is an increasing function on x > 0 so its inverse $f^{-1}(x)$ exists and is always defined.
ln f(x + 1) must have f(x) > 0 for all x > 1. Since increasing, check x = 1 → f(1) = 0 + positive = positive > 0 → ln f(x + 1) is defined for all x > 0.

Section B: Standard Questions

11.

a. Base case: $n = 1$

$(\cos\theta + i\sin\theta)^1 = \cos(1\theta) + i\sin(1\theta) \rightarrow$ LHS = RHS $\rightarrow$ valid for $n = 1$. [1 mark]

Induction hypothesis: let $n = k$, $k \in \mathbb{Z}^+ \rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.
[1 mark]

Then, for $n = k + 1$,

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ [1 mark]

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i\sin k\theta\cos\theta + i\sin\theta\cos k\theta$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$ [1 mark]

$= \cos((k + 1)\theta) + \sin((k + 1)\theta)$ = RHS for $n = k + 1$. [1 mark]

Since valid for $n = 1$, $n = k$ and $n = k + 1$, the result is valid for all $n \in \mathbb{Z}^+$. [1 mark]

b. Let $z = \cos\theta + i\sin\theta \rightarrow z^7 = \cos 7\theta + i\sin 7\theta$ [1 mark]

$\rightarrow \sin 7\theta = \mathrm{Im}(z^7)$. Let $s = \sin\theta$ and $c = \cos\theta$ to save space $\rightarrow z = c + is$

$\rightarrow \sin 7\theta$ = imaginary part of:

$(c + is)^7 = c^7 + 7i\,c^6s - 21\,c^5s^2 - 35i\,c^4s^3 + 35\,c^3s^4 + 21i\,c^2s^5 - 7\,cs^6 - i\,s^7$ [1 mark]

$\rightarrow \sin 7\theta = 7\,c^6s - 35\,c^4s^3 + 21\,c^2s^5 - s^7$ [1 mark]

$\rightarrow \sin 7\theta / \sin\theta = (7\,c^6s - 35\,c^4s^3 + 21\,c^2s^5 - s^7) / s = 7c^6 - 35c^4s^2 + 21c^2s^4 - s^6$

[1 mark]. Convert cos to sin by $s^2 + c^2 = 1 \rightarrow c^2 = 1 - s^2$:

$= 7(1 - s^2)^3 - 35s^2(1 - s^2)^2 + 21s^4(1 - s^2) - s^6$ [1 mark]

$= 7 - 21s^2 + 21s^4 - 7s^6 - 35s^2 + 70s^4 - 35s^6 + 21s^4 - 21s^6 - s^6$ [1 mark]

$= -64\,s^6 + 112s^4 - 56s^2 + 7$

$\rightarrow \sin 7\theta / \sin\theta = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$ [1 mark]

c. i) The equation is then

$64\sin^6(\pi/7) - 112\sin^4(\pi/7) + 56\sin^2(\pi/7) - 7 = 0$

$\rightarrow -\sin(7\pi/7) = 0$ [1 mark] $\rightarrow \sin(\pi) = 0 \rightarrow$ solution. [1 mark]

Other roots will be $x = \sin^2(2\pi/7)$ and $x = \sin^2(3\pi/7)$ [1 mark]

(since $\sin(2\pi) = \sin(3\pi) = 0$. Other roots are repeated.).

ii) Let the roots $\sin^2(\pi/7)$, $\sin^2(2\pi/7)$ and $\sin^2(3\pi/7)$ be α, β and γ.

$\csc^2(\pi/7) + \csc^2(2\pi/7) + \csc^2(3\pi/7)$

$= 1/\sin^2(\pi/7) + 1/\sin^2(2\pi/7) + 1/\sin^2(3\pi/7)$ [1 mark]

$= 1/\alpha + 1/\beta + 1/\gamma$

$= (\alpha\beta + \beta\gamma + \gamma\alpha) / (\alpha\beta\gamma)$ [1 mark]

= (sum of product pairs of roots) / (product of roots)

By Vieta's formulas, this is

$= (56/64) / (7/64) = 8$ (integer). [1 mark]

d. i)

$C + iS = 1 + a(\cos\theta + i\sin\theta) + a^2(\cos 2\theta + i\sin 2\theta) + a^3(\cos 3\theta + i\sin 3\theta) + \ldots$

$= 1 + a(\cos\theta + i\sin\theta) + a^2(\cos\theta + i\sin\theta)^2 + a^3(\cos\theta + i\sin\theta)^3 + \ldots$ [2 marks]

= geometric series with first term 1 and common ratio $a(\cos\theta + i\sin\theta)$

$= 1 / [1 - a(\cos\theta + i\sin\theta)]$ [1 mark]

$= 1 / ((1 - a\cos\theta) - ai\sin\theta)$

$= (1 - a\cos\theta + ai\sin\theta) / (((1 - a\cos\theta) - ai\sin\theta)((1 - a\cos\theta) + ai\sin\theta))$

$= (1 - a\cos\theta + ai\sin\theta) / ((1 - a\cos\theta)^2 - (ai\sin\theta)^2)$ [2 marks]

$= (1 - a\cos\theta + ai\sin\theta) / (1 - 2a\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta)$

$= (1 - a\cos\theta + ai\sin\theta) / (1 + a^2 - 2a\cos\theta)$ [1 mark]

Real part: $C = (1 - a\cos\theta) / (1 + a^2 - 2a\cos\theta)$ [1 mark]

ii)

Imaginary part: $S = (a\sin\theta) / (1 + a^2 - 2a\cos\theta)$ [1 mark]

12.

a. Using definition of $\tanh^{-1}$:

$$\tanh^{-1}(x^2) = \frac{1}{2}\ln\left(\frac{1+x^2}{1-x^2}\right)$$

$= 1/2\ [\ \ln(1 + x^2) - \ln(1 - x^2)\ ]$ [1 mark]

$= 1/2\ [(x^2 - x^4/2 + x^6/3 + \ldots + (-1)^{r+1}(x^{2r}/r)) - (-x^2 - x^4/2 - x^6/3 + \ldots + (-1)^1(x^{2r}/r)]$

$= 1/2\ [2x^2 + 2x^6/3 + 2x^{10}/5 + \ldots]$

$= x^2 + x^6/3 + x^{10}/5 + \ldots$ [2 marks]

Exponent: 2, 6, 10… → nth term is $4n - 2$ [1 mark]

Denominator: 1, 3, 5… → nth term is $2n - 1$ [1 mark]

→ expansion is

$$\tanh^{-1}(x^2) = \sum_{n=1}^{\infty}\left(\frac{x^{4n-2}}{2n-1}\right)\ ,\ |x| < 1$$

[1 mark]

b. Let $x = (\sqrt{p} - \sqrt{q}) / (\sqrt{p} + \sqrt{q})$. [1 mark]

Then, LHS $= 2\tanh^{-1}(x^2) = \ln((1 + x^2) / (1 - x^2))$ [1 mark]

→ LHS is:

$$= \ln\left[\left(1 + \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^2\right) \div \left(1 - \left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^2\right)\right]$$

[1 mark]

Multiply top and bottom of full fraction by $(\sqrt{p} + \sqrt{q})^2$:

$$= \ln\left[\frac{(\sqrt{p}+\sqrt{q})^2 + (\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2 - (\sqrt{p}-\sqrt{q})^2}\right]$$

[1 mark]

Expand and simplify:

$$= \ln\left[\frac{p+q}{2\sqrt{pq}}\right]$$

[2 marks]

$= \ln(p + q) - \ln(2\sqrt{pq})$ [1 mark]

$= \ln(p + q) - \ln(2) - \ln(\sqrt{pq})$

$= \ln((p + q)/2) - 1/2 \ln(pq)$

$= \ln((p + q)/2) - (\ln p + \ln q)/2$ [1 mark]

$= A - B$. [1 mark]

13.

a. Definitions of polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$ [1 mark]
Using chain rule, $dy/dx = dy/d\theta \cdot d\theta/dx = dy/d\theta \div dx/d\theta$ [1 mark]
Using product rule on definitions,
$dy/d\theta = (dr/d\theta)\sin\theta + r\cos\theta$
$dx/d\theta = (dr/d\theta)\cos\theta - r\sin\theta$ [1 mark]
$\rightarrow dy/dx = [(dr/d\theta)\sin\theta + r\cos\theta] / [(dr/d\theta)\cos\theta - r\sin\theta]$ [1 mark]

b. $r = e^{(\cot b)\theta} \rightarrow dr/d\theta = (\cot b)\, e^{(\cot b)\theta}$
Gradient = dy/dx, using formula, this is

$$= \frac{(\cot b)e^{(\cot b)\theta}\sin\theta + e^{(\cot b)\theta}\cos\theta}{(\cot b)e^{(\cot b)\theta}\cos\theta - e^{(\cot b)\theta}\sin\theta}$$

Divide both sides by $e^{(\cot b)\theta}$,

$$= \frac{\cot b\sin\theta + \cos\theta}{\cot b\cos\theta - \sin\theta}$$ [1 mark]

{*Method 1: Heavy algebra/trig manipulation*

Multiply top and bottom by $(\cot b\cos\theta + \sin\theta)$:

$$= \frac{\cot^2 b\sin\theta\cos\theta + \cot b(\sin^2\theta + \cos^2\theta) + \sin\theta\cos\theta}{\cot^2 b\cos^2\theta - \sin^2\theta}$$ [1 mark]

Factorising and simplifying with $\sin^2\theta + \cos^2\theta = 1$,

$$= \frac{\cot b + (1 + \cot^2 b)\sin\theta\cos\theta}{\cot^2 b\cos^2\theta - 1 + \cos^2\theta}$$ [1 mark]

In terms of double angles, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$:

$$= \frac{2\cot b + \sin 2\theta\csc^2 b}{\cot^2 b\,(\cos 2\theta + 1) + \cos 2\theta - 1}$$ [1 mark]

Convert denominator using $\cot^2 b = \csc^2 b - 1$:

$$= \frac{2\cot b + \sin 2\theta\csc^2 b}{\csc^2 b + \cos 2\theta\csc^2 b - 2}$$ [1 mark]

Factoring out $\csc^2 b$ from the denominator and using $\cos 2b = 1 - 2\sin^2 b$,

$$= \frac{\csc^2 b(\sin 2\theta + \sin 2b)}{\csc^2 b(\cos 2\theta + \cos 2b)} = \frac{\sin 2\theta + \sin 2b}{\cos 2\theta + \cos 2b}$$ [1 mark]

Using the sum-to-product formulas for sine and cosine,

$$= \frac{2\sin(\theta+b)\cos(\theta-b)}{2\cos(\theta+b)\cos(\theta-b)} = \frac{\sin(\theta+b)}{\cos(\theta+b)} = \tan(\theta+b)$$

. [1 mark]

Method 2: Converting with sum and difference identities

Convert the numerator to the form $P\sin(\theta + \alpha)$ and convert the denominator to the form $Q\cos(\theta + \beta)$:

$$\frac{\cot b\sin\theta + \cos\theta}{\cot b\cos\theta - \sin\theta} = \frac{P\sin(\theta+\alpha)}{Q\cos(\theta+\beta)} = \frac{P\sin\theta\cos\alpha + P\sin\alpha\cos\theta}{Q\cos\theta\cos\beta - Q\sin\theta\sin\beta}$$ [1 mark]

Equating terms:

$P\cos\alpha = \cot b$, $P\sin\alpha = 1$; $Q\cos\beta = \cot b$, $Q\sin\beta = 1$ [1 mark]

$\rightarrow \tan\alpha = 1/\cot b = \tan b \rightarrow \alpha = b$ [1 mark]

$\rightarrow \tan\beta = 1/\cot b = \tan b \rightarrow \beta = b$ [1 mark]

$\rightarrow P^2 = Q^2 = 1 + \cot^2 b = \csc^2 b$ [1 mark]

Putting all these back in,

$$= \frac{\csc^2 b\sin(\theta+b)}{\csc^2 b\cos(\theta+b)} = \tan(\theta+b)$$

. [1 mark] *}*

Consider the diagram again. By trig definitions, angle PTS = $\tan^{-1}(dy/dx)$

$\rightarrow$ PTS = $\tan^{-1}(\tan(\theta + b)) = \theta + b$ [1 mark]

$\rightarrow$ OTP = $\pi - \theta - b$

Angles in triangle OPT add to π radians:

$\rightarrow$ OPT = $\pi - (\pi - \theta - b + \theta) = b$, which is independent of θ. [1 mark]

14.

a. Find all substitutions in terms of y and t:

$t = \sqrt{x} \rightarrow x = t^2 \rightarrow dx/dt = 2t$: [1 mark]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2t}\frac{\mathrm{d}y}{\mathrm{d}t}$$ [1 mark]

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) \div 2t$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2t}\left(\frac{2t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t}}{4t^2}\right) = \frac{1}{4t^3}\left(t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right)$$ [2 marks]

Making these substitutions, the differential equation is

$$\frac{1}{t}\left(t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right) + 4t^2\left(\frac{1}{2t}\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} - 1 = 0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} - 1 = 0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - 1 = 0$$ [2 marks]

Since there is no *y* term, apply another substitution. Let *u* = dy/dt:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u^2 - 1 = 0 \rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 1 - u^2$$ [1 mark]

$$\int \frac{1}{1-u^2}\,\mathrm{d}u = \int\,\mathrm{d}t \rightarrow t = \tanh^{-1}(u) + C_1 \rightarrow u = \tanh(t - C_1)$$ [1 mark]

$$y = \int \tanh(t - C_1)\,\mathrm{d}t \rightarrow y = \ln(\cosh(t - C_1)) + C_2$$

$$y = \ln(\cosh(\sqrt{x} - C_1)) + C_2$$ [1 mark]

$$y = \ln\left(e^{\sqrt{x}-C_1} + e^{C_1-\sqrt{x}}\right) + C_2 - \ln 2$$

Let $C_3 = C_2 - \ln 2$:

$$y = C_3 + \ln\left(\frac{e^{\sqrt{x}}}{e^{C_1}} + \frac{e^{C_1}}{e^{\sqrt{x}}}\right)$$

Let $C_4 = e^{C1}$:

$$y = C_3 + \ln\left(\frac{e^{2\sqrt{x}} + C_4^2}{C_4 e^{\sqrt{x}}}\right)$$

Let $C_5 = (C_4)^2$:

$$y = C_3 + \ln\left(e^{2\sqrt{x}} + C_5\right) - \ln\left(C_4 e^{\sqrt{x}}\right)$$

$$y = C_3 + \ln\left(C_5(1 + \frac{1}{C_5}e^{2\sqrt{x}})\right) - \ln C_4 - \ln e^{\sqrt{x}}$$

$$y = C_3 + \ln C_5 + \ln\left(1 + \frac{1}{C_5}e^{2\sqrt{x}}\right) - \ln C_4 - \ln e^{\sqrt{x}}$$

Finally, let $A = C_3 + \ln C_5 - \ln C_4$ and let $B = 1/C_5$, and simplify $\ln e^{\sqrt{x}} = \sqrt{x}$:

$$y = A - \sqrt{x} + \ln\left(1 + Be^{2\sqrt{x}}\right)$$ [3 marks]

$A = C_2 - \ln 2 + C_1$ and $B = \exp(-2\ C_1)$

b. The domain of *y* is *x* > 0 (due to the $\sqrt{x}$).
(Alternatively by considering the unmanipulated form of the solution, $y = \ln(\cosh(\sqrt{x} - C_1)) + C_2$, the domain is clearly x > 0.) [1 mark]

If B was negative, the expression $1 + Be^{2\sqrt{x}}$ would become negative or zero for large x [1 mark].

This means the domain of the solution would be 0 < x < some finite value, which is not the required domain, so the solution is not considered valid. [1 mark]

Section C: Extended Questions

15. **Part 1: Making the substitution**

Starting with $u = (\tan x)^{2/3} \rightarrow \tan x = u^{3/2}$:
$du/dx = (2/3)\,(\tan x)^{-1/3} \sec^2 x$ [1 mark] $\rightarrow du = (2/3)\,(\tan x)^{-1/3} \sec^2 x\, dx$
$\rightarrow dx = (3/2) \tan^{1/3} x \cos^2 x\, du$ [1 mark]
Expressing this in terms of u,
$\rightarrow dx = (3/2)\,(\tan^{1/3} x)\,(1/\sec^2 x) = (3/2)\,(\tan^{1/3} x)\,(1/(1 + \tan^2 x))$
$= (3/2)\,(u)\,(1 + u^3)^{-1}$ [2 marks]
The bounds become $[\tan 0]^{2/3} = 0$ and $[\tan \pi/4]^{2/3} = 1$
So the integral is

$$\frac{3}{2}\int_0^1 \frac{u}{1+u^3}\,\mathrm{d}u$$ [1 mark]

Part 2: Partial fractions

Using partial fractions. Factorise the bottom: $u^3 + 1 = (u + 1)(u^2 - u + 1)$:

$$\frac{3}{2}\int_0^1 \frac{u}{(u+1)(u^2-u+1)}\,\mathrm{d}u = \frac{3}{2}\int_0^1 \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}\,\mathrm{d}u$$ [1 mark]

Let $u = -1$: $A = (-1) / (1 + 1 + 1) = -1/3$.
Let $u = 0$: $0 = -1/3 + C \rightarrow C = 1/3$.
Let $u = 1$: $1/2 = -1/6 + B + 1/3 \rightarrow B = 1/3$. Putting these in and factoring out 1/3,

$$= \frac{1}{2}\int_0^1 \frac{u+1}{u^2-u+1} - \frac{1}{u+1}\,\mathrm{d}u$$ [3 marks]

Part 3: The first integral

Focussing only on the first fraction (since the second is much simpler), complete the square in the denominator:

$$\int_0^1 \frac{u+1}{u^2-u+1}\,\mathrm{d}u = \int_0^1 \frac{u+1}{(u-\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}u$$ [1 mark]

Substitute $t = u - 1/2 \rightarrow$ du = dt. Bounds are now -1/2 to 1/2:

$$= \int_{-1/2}^{1/2} \frac{t+\frac{3}{2}}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t \;\; = \int_{-1/2}^{1/2} \frac{t}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t + \frac{3}{2}\int_{-1/2}^{1/2} \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t$$

[2 marks]

In the first fraction, integrate by substitution, giving

$$\int_{-1/2}^{1/2} \frac{t}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t = \frac{1}{2}\left[\ln\left|t^2+\frac{3}{4}\right|\right]_{-1/2}^{1/2} = 0$$ [2 marks]

In the second fraction, integrate by the standard result, using $a = \sqrt{3}/2$:

$$\frac{3}{2}\int_{-1/2}^{1/2} \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}t = \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)\right]_{-1/2}^{1/2}$$
$$= \sqrt{3}\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) = \frac{\pi}{\sqrt{3}}$$ [3 marks]

Part 4: Overall integral

Now, the second integral is

$$\int_0^1 \frac{1}{u+1}\,\mathrm{d}u = [\ln|u+1|]_0^1 = \ln 2$$ [1 mark]

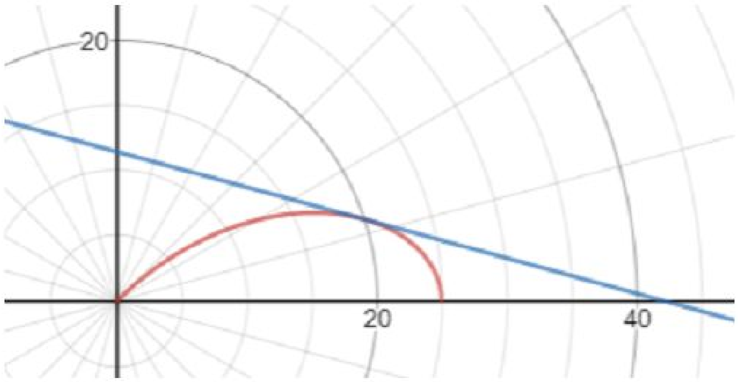
So overall, the integral is then,

$$\frac{1}{2}\left(\frac{\pi}{\sqrt{3}} - \ln 2\right) = \frac{\pi\sqrt{3}}{6} - \frac{1}{2}\ln 2$$. [2 marks]

(Solution by blackpenredpen at https://www.youtube.com/watch?v=NO693oP7nHQ)

16. **Part 1: Finding the gradient of the curve**

Start with a sketch:



From Q13 we know that the gradient of a polar curve is given by

$$\frac{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

, where r = 25 cos 2θ → dr/dθ = -50 sin 2θ

[2 marks; or derive again]

So in this case the gradient is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{25\cos 2\theta\cos\theta - 50\sin 2\theta\sin\theta}{-25\cos 2\theta\sin\theta - 50\sin 2\theta\cos\theta} = \frac{2\sin 2\theta\sin\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta}$$

[1 mark]

Part 2: Finding the point where the curve meets the tangent

So the point at which the curve meets the tangent is when

$$\frac{2\sin 2\theta\sin\theta - \cos 2\theta\cos\theta}{2\sin 2\theta\cos\theta + \cos 2\theta\sin\theta} = -\frac{3}{11}$$ [1 mark]

$$22\sin 2\theta\sin\theta - 11\cos 2\theta\cos\theta = -6\sin 2\theta\cos\theta - 3\cos 2\theta\sin\theta$$

Factor out/divide by cos θ cos 2θ. The lost solutions will be θ = π/2 and θ = π/4, but these cannot be correct since the domain of the curve is 0 ≤ θ ≤ π/4 and θ = π/4 is at the pole. [2 marks]

The remaining equation is $22 \tan 2\theta \tan \theta - 11 = -6 \tan 2\theta - 3 \tan \theta$.
$\rightarrow (22 \tan \theta + 6) \tan 2\theta = 11 - 3 \tan \theta$ [2 marks]
Using double angle identity,

$$(22\tan\theta + 6)\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = 11 - 3\tan\theta$$

$$44\tan^2\theta + 12\tan\theta = (11 - 3\tan\theta)(1-\tan^2\theta)$$

$$3\tan^3\theta - 55\tan^2\theta - 15\tan\theta + 11 = 0$$ [2 marks]

Solving on a calculator, we find a root at $\tan \theta = 1/3$, and the other two roots can be found by factoring as $\tan \theta = 9 \pm \sqrt{92}$. [2 marks]

Part 3: Justifying which root is correct

The point of tangency is to the right of the turning point of the curve. This fact can be used to identify which root is correct. Turning point $\rightarrow dy/dx = 0$, so
$2 \sin 2\theta \sin \theta - \cos 2\theta \cos \theta = 0$ [1 mark]
$\rightarrow 4 \sin^2 \theta \cos \theta - \cos \theta + 2 \sin^2 \theta \cos \theta = 0$
$\rightarrow \cos \theta (6 \sin^2 \theta - 1) = 0 \rightarrow \cos \theta = 0, \sin \theta = \pm 1/\sqrt{6}$,
$\rightarrow \theta = 90^o, 24.1^o, -24.1^o, 204.1^o\ldots$ Only solution in range is $\theta = 24.1^o$, so the correct root must have $\theta < 24.1^o$ [1 mark]
(since the curve is traced out right to left).

Taking inverse tangents on all roots gives $\theta = 18.4^o$, -30.6^o and 86.9^o. So the correct root is $\tan \theta = 1/3 \rightarrow \theta = \tan^{-1}(1/3)$. [1 mark]

Part 4: Finding Cartesian coordinates of points of interest

$\tan \theta = 1/3 \rightarrow \sin \theta = 1/\sqrt{10}$ and $\cos \theta = 3/\sqrt{10}$ (by Pythagorean identities)
$\rightarrow \cos 2\theta = 2(9/10) - 1 = 4/5$ and $\sin 2\theta = 2(1/\sqrt{10})(3/\sqrt{10}) = 3/5$ [1 mark]
$\rightarrow r = 25 \cos 2\theta = 25(4/5) = 20$.
$\rightarrow x = r \cos \theta = 20(3/\sqrt{10}) = 6\sqrt{10}$
$\rightarrow y = r \sin \theta = 20(1/\sqrt{10}) = 2\sqrt{10}$ [2 marks]
So the point of tangency is $(2\sqrt{10}, 6\sqrt{10})$.
Using the point-slope formula for line L,
$y - 6\sqrt{10} = -3/11(x - 2\sqrt{10}) \rightarrow 3x + 11y = 40\sqrt{10}$. [1 mark]
Intercept with initial line: $y = 0 \rightarrow x = (40/3)\sqrt{10}$
So the area of the triangle is $1/2 * x * y = (1/2)(40/3)(2\sqrt{10})(\sqrt{10}) = 400/3$.
[1 mark]

Part 5: Area of the region

The desired area is the area of the triangle minus the area of the polar curve up to the point of tangency. The bounds are θ = 0 and θ = $\tan^{-1}(1/3)$. The area is,

$$= \frac{400}{3} - \frac{625}{2}\int_0^{\tan^{-1}\left(\frac{1}{3}\right)} \cos^2 2\theta \ d\theta$$ [1 mark]

$$= \frac{400}{3} - \frac{625}{4}\int_0^{\tan^{-1}\left(\frac{1}{3}\right)} \cos 4\theta + 1 \ d\theta$$

$$= \frac{400}{3} - \frac{625}{4}\left[\frac{1}{4}\sin 4\theta + \theta\right]_0^{\tan^{-1}\left(\frac{1}{3}\right)}$$ [1 mark]

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{1}{4}\sin\left[4\tan^{-1}\left(\frac{1}{3}\right)\right] + \tan^{-1}\left(\frac{1}{3}\right)\right)$$

Using double-angle identity,

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{1}{2}\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]\cos\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] + \tan^{-1}\left(\frac{1}{3}\right)\right)$$ [1 mark]

Recall from earlier calculations, $\tan^{-1}(1/3) = \theta$ and $\sin 2\theta = 3/5$ and $\cos 2\theta = 4/5$,

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{4}{5} + \tan^{-1}\left(\frac{1}{3}\right)\right)$$ [1 mark]

$$= \frac{400}{3} - \frac{625}{4}\left(\frac{6}{25} + \tan^{-1}\left(\frac{1}{3}\right)\right)$$

$$= \frac{575}{6} - \frac{625}{4}\tan^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{25}{12}\left[46 - 75\tan^{-1}\left(\frac{1}{3}\right)\right]$$. [1 mark]

(Solution by T. Madas at https://madasmaths.com/archive/maths_booklets/further_topics/various/polar_coOrdinates_exam_questions.pdf, Question 47)