

# **CALCULUS II**

## **Integration Techniques**

Paul Dawkins

# Table of Contents

---

<b>Preface .....</b>	<b>ii</b>
<b>Chapter 1 : Integration Techniques .....</b>	<b>1</b>
Section 1-1 : Integration by Parts .....	3
Section 1-2 : Integrals Involving Trig Functions .....	14
Section 1-3 : Trig Substitutions .....	25
Section 1-4 : Partial Fractions .....	39
Section 1-5 : Integrals Involving Roots .....	48
Section 1-6 : Integrals Involving Quadratics .....	50
Section 1-7 : Integration Strategy .....	59
Section 1-8 : Improper Integrals .....	66
Section 1-9 : Comparison Test for Improper Integrals .....	74
Section 1-10 : Approximating Definite Integrals .....	82

## Preface

---

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.
2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than

are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.
4. This is somewhat related to the previous three items, but is important enough to merit its own item. **THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!!** Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

# Chapter 1 : Integration Techniques

---

In this chapter we are going to be looking at various integration techniques. There are a fair number of them and some will be easier than others. The point of the chapter is to teach you these new techniques and so this chapter assumes that you've got a fairly good working knowledge of basic integration as well as substitutions with integrals. In fact, most integrals involving "simple" substitutions will not have any of the substitution work shown. It is going to be assumed that you can verify the substitution portion of the integration yourself.

Also, most of the integrals done in this chapter will be indefinite integrals. It is also assumed that once you can do the indefinite integrals you can also do the definite integrals and so to conserve space we concentrate mostly on indefinite integrals. There is one exception to this and that is the Trig Substitution section and in this case there are some subtleties involved with definite integrals that we're going to have to watch out for. Outside of that however, most sections will have at most one definite integral example and some sections will not have any definite integral examples.

Here is a list of topics that are covered in this chapter.

**Integration by Parts** – In this section we will be looking at Integration by Parts. Of all the techniques we'll be looking at in this class this is the technique that students are most likely to run into down the road in other classes. We also give a derivation of the integration by parts formula.

**Integrals Involving Trig Functions** – In this section we look at integrals that involve trig functions. In particular we concentrate integrating products of sines and cosines as well as products of secants and tangents. We will also briefly look at how to modify the work for products of these trig functions for some quotients of trig functions.

**Trig Substitutions** – In this section we will look at integrals (both indefinite and definite) that require the use of a substitutions involving trig functions and how they can be used to simplify certain integrals.

**Partial Fractions** – In this section we will use partial fractions to rewrite integrands into a form that will allow us to do integrals involving some rational functions.

**Integrals Involving Roots** – In this section we will take a look at a substitution that can, on occasion, be used with integrals involving roots.

**Integrals Involving Quadratics** – In this section we are going to look at some integrals that involve quadratics for which the previous techniques won't work right away. In some cases, manipulation of the quadratic needs to be done before we can do the integral. We will see several cases where this is needed in this section.

**Integration Strategy** – In this section we give a general set of guidelines for determining how to evaluate an integral. The guidelines give here involve a mix of both Calculus I and Calculus II techniques to be as general as possible. Also note that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines.

**Improper Integrals** – In this section we will look at integrals with infinite intervals of integration and integrals with discontinuous integrands in this section. Collectively, they are called improper integrals and as we will see they may or may not have a finite (*i.e.* not infinite) value. Determining if they have finite values will, in fact, be one of the major topics of this section.

**Comparison Test for Improper Integrals** – It will not always be possible to evaluate improper integrals and yet we still need to determine if they converge or diverge (*i.e.* if they have a finite value or not). So, in this section we will use the Comparison Test to determine if improper integrals converge or diverge.

**Approximating Definite Integrals** – In this section we will look at several fairly simple methods of approximating the value of a definite integral. It is not possible to evaluate every definite integral (*i.e.* because it is not possible to do the indefinite integral) and yet we may need to know the value of the definite integral anyway. These methods allow us to at least get an approximate value which may be enough in a lot of cases.

## Section 1-1 : Integration by Parts

---

Let's start off with this section with a couple of integrals that we should already be able to do to get us started. First let's take a look at the following.

$$\int e^x dx = e^x + c$$

So, that was simple enough. Now, let's take a look at,

$$\int x e^{x^2} dx$$

To do this integral we'll use the following substitution.

$$u = x^2 \quad du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du$$

$$\int x e^{x^2} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c$$

Again, simple enough to do provided you remember how to do [substitutions](#). By the way make sure that you can do these kinds of substitutions quickly and easily. From this point on we are going to be doing these kinds of substitutions in our head. If you have to stop and write these out with every problem you will find that it will take you significantly longer to do these problems.

Now, let's look at the integral that we really want to do.

$$\int x e^{6x} dx$$

If we just had an  $x$  by itself or  $e^{6x}$  by itself we could do the integral easily enough. Likewise, if the integrand was  $x e^{6x^2}$  we could do the integral with a substitution. Unfortunately, however, neither of these are options. So, at this point we don't have the knowledge to do this integral.

To do this integral we will need to use integration by parts so let's derive the integration by parts formula. We'll start with the product rule.

$$(f g)' = f' g + f g'$$

Now, integrate both sides of this.

$$\int (f g)' dx = \int f' g + f g' dx$$

The left side is easy enough to integrate (we know that integrating a derivative just "undoes" the derivative) and we'll split up the right side of the integral.

$$f g = \int f' g dx + \int f g' dx$$

Note that technically we should have had a constant of integration show up on the left side after doing the integration. We can drop it at this point since other constants of integration will be showing up down the road and they would just end up absorbing this one.

Finally, rewrite the formula as follows and we arrive at the integration by parts formula.

$$\int f g' dx = fg - \int f' g dx$$

This is not the easiest formula to use however. So, let's do a couple of substitutions.

$$\begin{aligned} u &= f(x) & v &= g(x) \\ du &= f'(x)dx & dv &= g'(x)dx \end{aligned}$$

Both of these are just the standard Calculus I substitutions that hopefully you are used to by now. Don't get excited by the fact that we are using two substitutions here. They will work the same way.

Using these substitutions gives us the formula that most people think of as the integration by parts formula.

$$\int u dv = uv - \int v du$$

To use this formula, we will need to identify  $u$  and  $dv$ , compute  $du$  and  $v$  and then use the formula. Note as well that computing  $v$  is very easy. All we need to do is integrate  $dv$ .

$$v = \int dv$$

One of the more complicated things about using this formula is you need to be able to correctly identify both the  $u$  and the  $dv$ . It won't always be clear what the correct choices are and we will, on occasion, make the wrong choice. This is not something to worry about. If we make the wrong choice, we can always go back and try a different set of choices.

This does lead to the obvious question of how do we know if we made the correct choice for  $u$  and  $dv$ ? The answer is actually pretty simple. We made the correct choices for  $u$  and  $dv$  if, after using the integration by parts formula the new integral (the one on the right of the formula) is one we can actually integrate.

So, let's take a look at the integral above that we mentioned we wanted to do.

**Example 1** Evaluate the following integral.

$$\int x e^{6x} dx$$

**Solution**

So, on some level, the problem here is the  $x$  that is in front of the exponential. If that wasn't there we could do the integral. Notice as well that in doing integration by parts anything that we choose for  $u$  will be differentiated. So, it seems that choosing  $u = x$  will be a good choice since upon differentiating the  $x$  will drop out.

Now that we've chosen  $u$  we know that  $dv$  will be everything else that remains. So, here are the choices for  $u$  and  $dv$  as well as  $du$  and  $v$ .



$$u = x$$

$$dv = e^{6x} dx$$

$$du = dx$$

$$v = \int e^{6x} dx = \frac{1}{6} e^{6x}$$

The integral is then,

$$\begin{aligned} \int x e^{6x} dx &= \frac{x}{6} e^{6x} - \int \frac{1}{6} e^{6x} dx \\ &= \frac{x}{6} e^{6x} - \frac{1}{36} e^{6x} + c \end{aligned}$$

Once we have done the last integral in the problem we will add in the constant of integration to get our final answer.

Note as well that, as noted above, we know we had made a correct choice for  $u$  and  $dv$  when we got a new integral that we actually evaluate after applying the integration by parts formula.

Next, let's take a look at integration by parts for definite integrals. The integration by parts formula for definite integrals is,

### Integration by Parts, Definite Integrals

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Note that the  $uv \Big|_a^b$  in the first term is just the standard integral evaluation notation that you should be familiar with at this point. All we do is evaluate the term,  $uv$  in this case, at  $b$  then subtract off the evaluation of the term at  $a$ .

At some level we don't really need a formula here because we know that when doing definite integrals all we need to do is evaluate the indefinite integral and then do the evaluation. In fact, this is probably going to be slightly easier as we don't need to track evaluating each term this way.

Let's take a quick look at a definite integral using integration by parts.

**Example 2** Evaluate the following integral.

$$\int_{-1}^2 x e^{6x} dx$$

#### **Solution**

This is the same integral that we looked at in the first example so we'll use the same  $u$  and  $dv$  to get,

$$\begin{aligned} \int_{-1}^2 x e^{6x} dx &= \frac{x}{6} e^{6x} \Big|_{-1}^2 - \frac{1}{6} \int_{-1}^2 e^{6x} dx \\ &= \frac{x}{6} e^{6x} \Big|_{-1}^2 - \frac{1}{36} e^{6x} \Big|_{-1}^2 \\ &= \frac{11}{36} e^{12} + \frac{7}{36} e^{-6} \end{aligned}$$

As noted above we could just as easily use the result from the first example to do the evaluation. We know, from the first example that,

$$\int x e^{6x} dx = \frac{x}{6} e^{6x} - \frac{1}{36} e^{6x} + c$$

Using this we can quickly proceed to the evaluate of the definite integral as follows,

$$\begin{aligned} \int_{-1}^2 x e^{6x} dx &= \left( \frac{x}{6} e^{6x} - \frac{1}{36} e^{6x} \right) \Big|_{-1}^2 \\ &= \left( \frac{1}{3} e^{12} - \frac{1}{36} e^{12} \right) - \left( -\frac{1}{6} e^{-6} - \frac{1}{36} e^{-6} \right) \\ &= \frac{11}{36} e^{12} + \frac{7}{36} e^{-6} \end{aligned}$$

Either method of evaluating definite integrals with integration by part are pretty simple so which one you choose to use is pretty much up to you.

Since we need to be able to do the indefinite integral in order to do the definite integral and doing the definite integral amounts to nothing more than evaluating the indefinite integral at a couple of points we will concentrate on doing indefinite integrals in the rest of this section. In fact, throughout most of this chapter this will be the case. We will be doing far more indefinite integrals than definite integrals.

Let's take a look at some more examples.

**Example 3** Evaluate the following integral.

$$\int (3t + 5) \cos\left(\frac{t}{4}\right) dt$$

**Solution**

There are two ways to proceed with this example. For many, the first thing that they try is multiplying the cosine through the parenthesis, splitting up the integral and then doing integration by parts on the first integral.

While that is a perfectly acceptable way of doing the problem it's more work than we really need to do. Instead of splitting the integral up let's instead use the following choices for  $u$  and  $dv$ .

$$\begin{aligned} u &= 3t + 5 & dv &= \cos\left(\frac{t}{4}\right) dt \\ du &= 3 dt & v &= 4 \sin\left(\frac{t}{4}\right) \end{aligned}$$

The integral is then,

$$\begin{aligned}\int (3t+5)\cos\left(\frac{t}{4}\right)dt &= 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)dt \\ &= 4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + c\end{aligned}$$

Notice that we pulled any constants out of the integral when we used the integration by parts formula. We will usually do this in order to simplify the integral a little.

**Example 4** Evaluate the following integral.

$$\int w^2 \sin(10w) dw$$

**Solution**

For this example, we'll use the following choices for  $u$  and  $dv$ .

$$\begin{aligned}u &= w^2 & dv &= \sin(10w) dw \\ du &= 2w dw & v &= -\frac{1}{10}\cos(10w)\end{aligned}$$

The integral is then,

$$\int w^2 \sin(10w) dw = -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\int w\cos(10w) dw$$

In this example, unlike the previous examples, the new integral will also require integration by parts. For this second integral we will use the following choices.

$$\begin{aligned}u &= w & dv &= \cos(10w) dw \\ du &= dw & v &= \frac{1}{10}\sin(10w)\end{aligned}$$

So, the integral becomes,

$$\begin{aligned}\int w^2 \sin(10w) dw &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) - \frac{1}{10}\int \sin(10w) dw\right) \\ &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) + \frac{1}{100}\cos(10w)\right) + c \\ &= -\frac{w^2}{10}\cos(10w) + \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + c\end{aligned}$$

Be careful with the coefficient on the integral for the second application of integration by parts. Since the integral is multiplied by  $\frac{1}{5}$  we need to make sure that the results of actually doing the integral are also multiplied by  $\frac{1}{5}$ . Forgetting to do this is one of the more common mistakes with integration by parts problems.

As this last example has shown us, we will sometimes need more than one application of integration by parts to completely evaluate an integral. This is something that will happen so don't get excited about it when it does.

In this next example we need to acknowledge an important point about integration techniques. Some integrals can be done in using several different techniques. That is the case with the integral in the next example.

**Example 5** Evaluate the following integral

$$\int x\sqrt{x+1} \, dx$$

(a) Using Integration by Parts.

(b) Using a standard Calculus I substitution.

**Solution**

**(a) Evaluate using Integration by Parts.**

First notice that there are no trig functions or exponentials in this integral. While a good many integration by parts integrals will involve trig functions and/or exponentials not all of them will so don't get too locked into the idea of expecting them to show up.

In this case we'll use the following choices for  $u$  and  $dv$ .

$$\begin{aligned} u &= x & dv &= \sqrt{x+1} \, dx \\ du &= dx & v &= \frac{2}{3}(x+1)^{\frac{3}{2}} \end{aligned}$$

The integral is then,

$$\begin{aligned} \int x\sqrt{x+1} \, dx &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3} \int (x+1)^{\frac{3}{2}} \, dx \\ &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c \end{aligned}$$

**(b) Evaluate Using a standard Calculus I substitution.**

Now let's do the integral with a substitution. We can use the following substitution.

$$\begin{aligned} u &= x+1 & x &= u-1 & du &= dx \end{aligned}$$

Notice that we'll actually use the substitution twice, once for the quantity under the square root and once for the  $x$  in front of the square root. The integral is then,

$$\begin{aligned} \int x\sqrt{x+1} \, dx &= \int (u-1)\sqrt{u} \, du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c \end{aligned}$$

So, we used two different integration techniques in this example and we got two different answers. The obvious question then should be : Did we do something wrong?

Actually, we didn't do anything wrong. We need to [remember](#) the following fact from Calculus I.

$$\text{If } f'(x) = g'(x) \text{ then } f(x) = g(x) + c$$

In other words, if two functions have the same derivative then they will differ by no more than a constant. So, how does this apply to the above problem? First define the following,

$$f'(x) = g'(x) = x\sqrt{x+1}$$

Then we can compute  $f(x)$  and  $g(x)$  by integrating as follows,

$$f(x) = \int f'(x) dx \qquad g(x) = \int g'(x) dx$$

We'll use integration by parts for the first integral and the substitution for the second integral. Then according to the fact  $f(x)$  and  $g(x)$  should differ by no more than a constant. Let's verify this and see if this is the case. We can verify that they differ by no more than a constant if we take a look at the difference of the two and do a little algebraic manipulation and simplification.

$$\begin{aligned} & \left( \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} \right) - \left( \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} \right) \\ &= (x+1)^{\frac{3}{2}} \left( \frac{2}{3}x - \frac{4}{15}(x+1) - \frac{2}{5}(x+1) + \frac{2}{3} \right) \\ &= (x+1)^{\frac{3}{2}} (0) \\ &= 0 \end{aligned}$$

So, in this case it turns out the two functions are exactly the same function since the difference is zero. Note that this won't always happen. Sometimes the difference will yield a nonzero constant. For an example of this check out the [Constant of Integration](#) section in the Calculus I notes.

So just what have we learned? First, there will, on occasion, be more than one method for evaluating an integral. Secondly, we saw that different methods will often lead to different answers. Last, even though the answers are different it can be shown, sometimes with a lot of work, that they differ by no more than a constant.

When we are faced with an integral the first thing that we'll need to decide is if there is more than one way to do the integral. If there is more than one way we'll then need to determine which method we should use. The general rule of thumb that I use in my classes is that you should use the method that *you* find easiest. This may not be the method that others find easiest, but that doesn't make it the wrong method.

One of the more common mistakes with integration by parts is for people to get too locked into perceived patterns. For instance, all of the previous examples used the basic pattern of taking  $u$  to be the polynomial that sat in front of another function and then letting  $dv$  be the other function. This will not always happen so we need to be careful and not get locked into any patterns that we think we see.

Let's take a look at some integrals that don't fit into the above pattern.

**Example 6** Evaluate the following integral.

$$\int \ln x \, dx$$

**Solution**

So, unlike any of the other integrals we've done to this point there is only a single function in the integral and no polynomial sitting in front of the logarithm.

The first choice of many people here is to try and fit this into the pattern from above and make the following choices for  $u$  and  $dv$ .

$$u = 1 \qquad dv = \ln x \, dx$$

This leads to a real problem however since that means  $v$  must be,

$$v = \int \ln x \, dx$$

In other words, we would need to know the answer ahead of time in order to actually do the problem. So, this choice simply won't work.

Therefore, if the logarithm doesn't belong in the  $dv$  it must belong instead in the  $u$ . So, let's use the following choices instead

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

The integral is then,

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int \frac{1}{x} x \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + c \end{aligned}$$

**Example 7** Evaluate the following integral.

$$\int x^5 \sqrt{x^3 + 1} \, dx$$

**Solution**

So, if we again try to use the pattern from the first few examples for this integral our choices for  $u$  and  $dv$  would probably be the following.

$$u = x^5 \qquad dv = \sqrt{x^3 + 1} \, dx$$

However, as with the previous example this won't work since we can't easily compute  $v$ .

$$v = \int \sqrt{x^3 + 1} \, dx$$

This is not an easy integral to do. However, notice that if we had an  $x^2$  in the integral along with the root we could very easily do the integral with a substitution. Also notice that we do have a lot of  $x$ 's floating around in the original integral. So instead of putting all the  $x$ 's (outside of the root) in the  $u$  let's split them up as follows.

$$\begin{aligned}u &= x^3 & dv &= x^2 \sqrt{x^3 + 1} \, dx \\ du &= 3x^2 \, dx & v &= \frac{2}{9} (x^3 + 1)^{\frac{3}{2}}\end{aligned}$$

We can now easily compute  $v$  and after using integration by parts we get,

$$\begin{aligned}\int x^5 \sqrt{x^3 + 1} \, dx &= \frac{2}{9} x^3 (x^3 + 1)^{\frac{3}{2}} - \frac{2}{3} \int x^2 (x^3 + 1)^{\frac{3}{2}} \, dx \\ &= \frac{2}{9} x^3 (x^3 + 1)^{\frac{3}{2}} - \frac{4}{45} (x^3 + 1)^{\frac{5}{2}} + c\end{aligned}$$

So, in the previous two examples we saw cases that didn't quite fit into any perceived pattern that we might have gotten from the first couple of examples. This is always something that we need to be on the lookout for with integration by parts.

Let's take a look at another example that also illustrates another integration technique that sometimes arises out of integration by parts problems.

**Example 8** Evaluate the following integral.

$$\int e^{\theta} \cos \theta \, d\theta$$

**Solution**

Okay, to this point we've always picked  $u$  in such a way that upon differentiating it would make that portion go away or at the very least put it the integral into a form that would make it easier to deal with. In this case no matter which part we make  $u$  it will never go away in the differentiation process.

It doesn't much matter which we choose to be  $u$  so we'll choose in the following way. Note however that we could choose the other way as well and we'll get the same result in the end.

$$\begin{aligned}u &= \cos \theta & dv &= e^{\theta} \, d\theta \\ du &= -\sin \theta \, d\theta & v &= e^{\theta}\end{aligned}$$

The integral is then,

$$\int e^{\theta} \cos \theta \, d\theta = e^{\theta} \cos \theta + \int e^{\theta} \sin \theta \, d\theta$$

So, it looks like we'll do integration by parts again. Here are our choices this time.

$$\begin{aligned}u &= \sin \theta & dv &= e^{\theta} \, d\theta \\ du &= \cos \theta \, d\theta & v &= e^{\theta}\end{aligned}$$

The integral is now,

$$\int e^{\theta} \cos \theta \, d\theta = e^{\theta} \cos \theta + e^{\theta} \sin \theta - \int e^{\theta} \cos \theta \, d\theta$$

Now, at this point it looks like we're just running in circles. However, notice that we now have the same integral on both sides and on the right side it's got a minus sign in front of it. This means that we can add the integral to both sides to get,

$$2 \int e^{\theta} \cos \theta d\theta = e^{\theta} \cos \theta + e^{\theta} \sin \theta$$

All we need to do now is divide by 2 and we're done. The integral is,

$$\int e^{\theta} \cos \theta d\theta = \frac{1}{2} (e^{\theta} \cos \theta + e^{\theta} \sin \theta) + c$$

Notice that after dividing by the two we add in the constant of integration at that point.

This idea of integrating until you get the same integral on both sides of the equal sign and then simply solving for the integral is kind of nice to remember. It doesn't show up all that often, but when it does it may be the only way to actually do the integral.

Note as well that this is really just Algebra, admittedly done in a way that you may not be used to seeing it, but it is really just Algebra.

At this stage of your mathematical career everyone can solve,

$$x = 3 - x \quad \rightarrow \quad x = \frac{3}{2}$$

We are still solving an "equation". The only difference is that instead of solving for an  $x$  in we are solving for an integral and instead of a nice constant, "3" in the above Algebra problem, we've got a "messier" function.

We've got one more example to do. As we will see some problems could require us to do integration by parts numerous times and there is a short hand method that will allow us to do multiple applications of integration by parts quickly and easily.

**Example 9** Evaluate the following integral.

$$\int x^4 e^{\frac{x}{2}} dx$$

**Solution**

We start off by choosing  $u$  and  $dv$  as we always would. However, instead of computing  $du$  and  $v$  we put these into the following table. We then differentiate down the column corresponding to  $u$  until we hit zero. In the column corresponding to  $dv$  we integrate once for each entry in the first column. There is also a third column which we will explain in a bit and it always starts with a "+" and then alternates signs as shown.



$x^4$	$e^{\frac{x}{2}}$	+
$4x^3$	$2e^{\frac{x}{2}}$	-
$12x^2$	$4e^{\frac{x}{2}}$	+
$24x$	$8e^{\frac{x}{2}}$	-
$24$	$16e^{\frac{x}{2}}$	+
$0$	$32e^{\frac{x}{2}}$	-

Now, multiply along the diagonals shown in the table. In front of each product put the sign in the third column that corresponds to the “ $u$ ” term for that product. In this case this would give,

$$\begin{aligned}\int x^4 e^{\frac{x}{2}} dx &= (x^4) \left( 2e^{\frac{x}{2}} \right) - (4x^3) \left( 4e^{\frac{x}{2}} \right) + (12x^2) \left( 8e^{\frac{x}{2}} \right) - (24x) \left( 16e^{\frac{x}{2}} \right) + (24) \left( 32e^{\frac{x}{2}} \right) \\ &= 2x^4 e^{\frac{x}{2}} - 16x^3 e^{\frac{x}{2}} + 96x^2 e^{\frac{x}{2}} - 384x e^{\frac{x}{2}} + 768e^{\frac{x}{2}} + c\end{aligned}$$

We’ve got the integral. This is much easier than writing down all the various  $u$ ’s and  $dv$ ’s that we’d have to do otherwise.

So, in this section we’ve seen how to do integration by parts. In your later math classes this is liable to be one of the more frequent integration techniques that you’ll encounter.

It is important to not get too locked into patterns that you may think you’ve seen. In most cases any pattern that you think you’ve seen can (and will be) violated at some point in time. Be careful!

## Section 1-2 : Integrals Involving Trig Functions

In this section we are going to look at quite a few integrals involving trig functions and some of the techniques we can use to help us evaluate them. Let's start off with an integral that we should already be able to do.

$$\begin{aligned}\int \cos x \sin^5 x \, dx &= \int u^5 \, du \quad \text{using the substitution } u = \sin x \\ &= \frac{1}{6} \sin^6 x + c\end{aligned}$$

This integral is easy to do with a substitution because the presence of the cosine, however, what about the following integral.

**Example 1** Evaluate the following integral.

$$\int \sin^5 x \, dx$$

**Solution**

This integral no longer has the cosine in it that would allow us to use the substitution that we used above. Therefore, that substitution won't work and we are going to have to find another way of doing this integral.

Let's first notice that we could write the integral as follows,

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int (\sin^2 x)^2 \sin x \, dx$$

Now recall the trig identity,

$$\cos^2 x + \sin^2 x = 1 \quad \Rightarrow \quad \sin^2 x = 1 - \cos^2 x$$

With this identity the integral can be written as,

$$\int \sin^5 x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx$$

and we can now use the substitution  $u = \cos x$ . Doing this gives us,

$$\begin{aligned}\int \sin^5 x \, dx &= -\int (1 - u^2)^2 \, du \\ &= -\int 1 - 2u^2 + u^4 \, du \\ &= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c\end{aligned}$$

So, with a little rewriting on the integrand we were able to reduce this to a fairly simple substitution.

Notice that we were able to do the rewrite that we did in the previous example because the exponent on the sine was odd. In these cases all that we need to do is strip out one of the sines. The exponent on the remaining sines will then be even and we can easily convert the remaining sines to cosines using the identity,

$$\cos^2 x + \sin^2 x = 1 \quad (1)$$

If the exponent on the sines had been even this would have been difficult to do. We could strip out a sine, but the remaining sines would then have an odd exponent and while we could convert them to cosines the resulting integral would often be even more difficult than the original integral in most cases.

Let's take a look at another example.

**Example 2** Evaluate the following integral.

$$\int \sin^6 x \cos^3 x \, dx$$

**Solution**

So, in this case we've got both sines and cosines in the problem and in this case the exponent on the sine is even while the exponent on the cosine is odd. So, we can use a similar technique in this integral. This time we'll strip out a cosine and convert the rest to sines.

$$\begin{aligned} \int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x \cos^2 x \cos x \, dx \\ &= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx & u = \sin x \\ &= \int u^6 (1 - u^2) \, du \\ &= \int u^6 - u^8 \, du \\ &= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + c \end{aligned}$$

At this point let's pause for a second to summarize what we've learned so far about integrating powers of sine and cosine.

$$\int \sin^n x \cos^m x \, dx \quad (2)$$

In this integral if the exponent on the sines ( $n$ ) is odd we can strip out one sine, convert the rest to cosines using (1) and then use the substitution  $u = \cos x$ . Likewise, if the exponent on the cosines ( $m$ ) is odd we can strip out one cosine and convert the rest to sines and then use the substitution  $u = \sin x$ .

Of course, if both exponents are odd then we can use either method. However, in these cases it's usually easier to convert the term with the smaller exponent.

The one case we haven't looked at is what happens if both of the exponents are even? In this case the technique we used in the first couple of examples simply won't work and in fact there really isn't any one set method for doing these integrals. Each integral is different and in some cases there will be more than one way to do the integral.

With that being said most, if not all, of integrals involving products of sines and cosines in which both exponents are even can be done using one or more of the following formulas to rewrite the integrand.

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\sin x \cos x = \frac{1}{2} \sin(2x)$$

The first two formulas are the standard half angle formula from a trig class written in a form that will be more convenient for us to use. The last is the standard double angle formula for sine, again with a small rewrite.

Let's take a look at an example.

**Example 3** Evaluate the following integral.

$$\int \sin^2 x \cos^2 x \, dx$$

**Solution**

As noted above there are often more than one way to do integrals in which both of the exponents are even. This integral is an example of that. There are at least two solution techniques for this problem. We will do both solutions starting with what is probably the longer of the two, but it's also the one that many people see first.

**Solution 1**

In this solution we will use the two half angle formulas above and just substitute them into the integral.

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{2}(1 - \cos(2x)) \left( \frac{1}{2} \right) (1 + \cos(2x)) \, dx \\ &= \frac{1}{4} \int 1 - \cos^2(2x) \, dx \end{aligned}$$

So, we still have an integral that can't be completely done, however notice that we have managed to reduce the integral down to just one term causing problems (a cosine with an even power) rather than two terms causing problems.

In fact to eliminate the remaining problem term all that we need to do is reuse the first half angle formula given above.

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \frac{1}{4} \int 1 - \frac{1}{2}(1 + \cos(4x)) \, dx \\ &= \frac{1}{4} \int \frac{1}{2} - \frac{1}{2} \cos(4x) \, dx \\ &= \frac{1}{4} \left( \frac{1}{2} x - \frac{1}{8} \sin(4x) \right) + c \\ &= \frac{1}{8} x - \frac{1}{32} \sin(4x) + c \end{aligned}$$

So, this solution required a total of three trig identities to complete.

*Solution 2*

In this solution we will use the half angle formula to help simplify the integral as follows.

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int (\sin x \cos x)^2 \, dx \\ &= \int \left( \frac{1}{2} \sin(2x) \right)^2 \, dx \\ &= \frac{1}{4} \int \sin^2(2x) \, dx\end{aligned}$$

Now, we use the double angle formula for sine to reduce to an integral that we can do.

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \frac{1}{8} \int 1 - \cos(4x) \, dx \\ &= \frac{1}{8} x - \frac{1}{32} \sin(4x) + c\end{aligned}$$

This method required only two trig identities to complete.

Notice that the difference between these two methods is more one of “messiness”. The second method is not appreciably easier (other than needing one less trig identity) it is just not as messy and that will often translate into an “easier” process.

In the previous example we saw two different solution methods that gave the same answer. Note that this will not always happen. In fact, more often than not we will get different answers. However, as we discussed in the [Integration by Parts](#) section, the two answers will differ by no more than a constant.

In general, when we have products of sines and cosines in which both exponents are even we will need to use a series of half angle and/or double angle formulas to reduce the integral into a form that we can integrate. Also, the larger the exponents the more we’ll need to use these formulas and hence the messier the problem.

Sometimes in the process of reducing integrals in which both exponents are even we will run across products of sine and cosine in which the arguments are different. These will require one of the following formulas to reduce the products to integrals that we can do.

$$\begin{aligned}\sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)] \\ \sin \alpha \sin \beta &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]\end{aligned}$$

Let’s take a look at an example of one of these kinds of integrals.

**Example 4** Evaluate the following integral.

$$\int \cos(15x) \cos(4x) dx$$

**Solution**

This integral requires the last formula listed above.

$$\begin{aligned} \int \cos(15x) \cos(4x) dx &= \frac{1}{2} \int \cos(11x) + \cos(19x) dx \\ &= \frac{1}{2} \left( \frac{1}{11} \sin(11x) + \frac{1}{19} \sin(19x) \right) + c \end{aligned}$$

Okay, at this point we've covered pretty much all the possible cases involving products of sines and cosines. It's now time to look at integrals that involve products of secants and tangents.

This time, let's do a little analysis of the possibilities before we just jump into examples. The general integral will be,

$$\int \sec^n x \tan^m x dx \quad (3)$$

The first thing to notice is that we can easily convert even powers of secants to tangents and even powers of tangents to secants by using a formula similar to (1). In fact, the formula can be derived from (1) so let's do that.

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\ \tan^2 x + 1 &= \sec^2 x \end{aligned} \quad (4)$$

Now, we're going to want to deal with (3) similarly to how we dealt with (2). We'll want to eventually use one of the following substitutions.

$$\begin{array}{ll} u = \tan x & du = \sec^2 x dx \\ u = \sec x & du = \sec x \tan x dx \end{array}$$

So, if we use the substitution  $u = \tan x$  we will need two secants left for the substitution to work. This means that if the exponent on the secant ( $n$ ) is even we can strip two out and then convert the remaining secants to tangents using (4).

Next, if we want to use the substitution  $u = \sec x$  we will need one secant and one tangent left over in order to use the substitution. This means that if the exponent on the tangent ( $m$ ) is odd and we have at least one secant in the integrand we can strip out one of the tangents along with one of the secants of course. The tangent will then have an even exponent and so we can use (4) to convert the rest of the tangents to secants. Note that this method does require that we have at least one secant in the integral as well. If there aren't any secants then we'll need to do something different.

If the exponent on the secant is even and the exponent on the tangent is odd then we can use either case. Again, it will be easier to convert the term with the smallest exponent.

Let's take a look at a couple of examples.

**Example 5** Evaluate the following integral.

$$\int \sec^9 x \tan^5 x \, dx$$

**Solution**

First note that since the exponent on the secant isn't even we can't use the substitution  $u = \tan x$ . However, the exponent on the tangent is odd and we've got a secant in the integral and so we will be able to use the substitution  $u = \sec x$ . This means stripping out a single tangent (along with a secant) and converting the remaining tangents to secants using (4).

Here's the work for this integral.

$$\begin{aligned} \int \sec^9 x \tan^5 x \, dx &= \int \sec^8 x \tan^4 x \tan x \sec x \, dx \\ &= \int \sec^8 x (\sec^2 x - 1)^2 \tan x \sec x \, dx & u = \sec x \\ &= \int u^8 (u^2 - 1)^2 \, du \\ &= \int u^{12} - 2u^{10} + u^8 \, du \\ &= \frac{1}{13} \sec^{13} x - \frac{2}{11} \sec^{11} x + \frac{1}{9} \sec^9 x + c \end{aligned}$$

**Example 6** Evaluate the following integral.

$$\int \sec^4 x \tan^6 x \, dx$$

**Solution**

So, in this example the exponent on the tangent is even so the substitution  $u = \sec x$  won't work. The exponent on the secant is even and so we can use the substitution  $u = \tan x$  for this integral. That means that we need to strip out two secants and convert the rest to tangents. Here is the work for this integral.

$$\begin{aligned} \int \sec^4 x \tan^6 x \, dx &= \int \sec^2 x \tan^6 x \sec^2 x \, dx \\ &= \int (\tan^2 x + 1) \tan^6 x \sec^2 x \, dx & u = \tan x \\ &= \int (u^2 + 1) u^6 \, du \\ &= \int u^8 + u^6 \, du \\ &= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + c \end{aligned}$$

Both of the previous examples fit very nicely into the patterns discussed above and so were not all that difficult to work. However, there are a couple of exceptions to the patterns above and in these cases there is no single method that will work for every problem. Each integral will be different and may require different solution methods in order to evaluate the integral.

Let's first take a look at a couple of integrals that have odd exponents on the tangents, but no secants. In these cases we can't use the substitution  $u = \sec x$  since it requires there to be at least one secant in the integral.

**Example 7** Evaluate the following integral.

$$\int \tan x \, dx$$

**Solution**

To do this integral all we need to do is recall the definition of tangent in terms of sine and cosine and then this integral is nothing more than a Calculus I substitution.

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx & u &= \cos x \\ &= - \int \frac{1}{u} \, du \\ &= -\ln|\cos x| + c & r \ln x &= \ln x^r \\ &= \ln|\cos x|^{-1} + c \\ &= \ln|\sec x| + c \end{aligned}$$

Note that for many folks,

$$\int \tan x \, dx = -\ln|\cos x| + c$$

We went a step or two further with some simplification. The simplification was done solely to eliminate the minus sign that was in front of the logarithm. This does not have to be done in general, but it is always easy to lose minus signs and in this case it was easy to eliminate it without introducing any real complexity to the answer and so we did.

**Example 8** Evaluate the following integral.

$$\int \tan^3 x \, dx$$

**Solution**

The trick to this one is do the following manipulation of the integrand.

$$\begin{aligned} \int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\ &= \int \tan x (\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \end{aligned}$$

We can now use the substitution  $u = \tan x$  on the first integral and the results from the previous example on the second integral.

The integral is then,

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \ln|\sec x| + c$$



Note that all odd powers of tangent (with the exception of the first power) can be integrated using the same method we used in the previous example. For instance,

$$\int \tan^5 x \, dx = \int \tan^3 x (\sec^2 x - 1) \, dx = \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

So, a quick substitution ( $u = \tan x$ ) will give us the first integral and the second integral will always be the previous odd power.

Now let's take a look at a couple of examples in which the exponent on the secant is odd and the exponent on the tangent is even. In these cases the substitutions used above won't work.

It should also be noted that both of the following two integrals are integrals that we'll be seeing on occasion in later sections of this chapter and in later chapters. Because of this it wouldn't be a bad idea to make a note of these results so you'll have them ready when you need them later.

**Example 9** Evaluate the following integral.

$$\int \sec x \, dx$$

**Solution**

This one isn't too bad once you see what you've got to do. By itself the integral can't be done. However, if we manipulate the integrand as follows we can do it.

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} \, dx \end{aligned}$$

In this form we can do the integral using the substitution  $u = \sec x + \tan x$ . Doing this gives,

$$\int \sec x \, dx = \ln |\sec x + \tan x| + c$$

The idea used in the above example is a nice idea to keep in mind. Multiplying the numerator and denominator of a term by the same term above can, on occasion, put the integral into a form that can be integrated. Note that this method won't always work and even when it does it won't always be clear what you need to multiply the numerator and denominator by. However, when it does work and you can figure out what term you need it can greatly simplify the integral.

Here's the next example.

**Example 10** Evaluate the following integral.

$$\int \sec^3 x \, dx$$

**Solution**

This one is different from any of the other integrals that we've done in this section. The first step to doing this integral is to perform integration by parts using the following choices for  $u$  and  $dv$ .

$$\begin{array}{ll} u = \sec x & dv = \sec^2 x \, dx \\ du = \sec x \tan x \, dx & v = \tan x \end{array}$$

Note that using integration by parts on this problem is not an obvious choice, but it does work very nicely here. After doing integration by parts we have,

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

Now the new integral also has an odd exponent on the secant and an even exponent on the tangent and so the previous examples of products of secants and tangents still won't do us any good.

To do this integral we'll first write the tangents in the integral in terms of secants. Again, this is not necessarily an obvious choice but it's what we need to do in this case.

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \end{aligned}$$

Now, we can use the results from the previous example to do the second integral and notice that the first integral is exactly the integral we're being asked to evaluate with a minus sign in front. So, add it to both sides to get,

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x|$$

Finally divide by two and we're done.

$$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + c$$

Again, note that we've again used the idea of integrating the right side until the original integral shows up and then moving this to the left side and dividing by its coefficient to complete the evaluation. We first saw this in the [Integration by Parts](#) section and noted at the time that this was a nice technique to remember. Here is another example of this technique.

Now that we've looked at products of secants and tangents let's also acknowledge that because we can relate cosecants and cotangents by

$$1 + \cot^2 x = \csc^2 x$$

all of the work that we did for products of secants and tangents will also work for products of cosecants and cotangents. We'll leave it to you to verify that.

There is one final topic to be discussed in this section before moving on.

To this point we've looked only at products of sines and cosines and products of secants and tangents. However, the methods used to do these integrals can also be used on some quotients involving sines and cosines and quotients involving secants and tangents (and hence quotients involving cosecants and cotangents).

Let's take a quick look at an example of this.

**Example 11** Evaluate the following integral.

$$\int \frac{\sin^7 x}{\cos^4 x} dx$$

**Solution**

If this were a product of sines and cosines we would know what to do. We would strip out a sine (since the exponent on the sine is odd) and convert the rest of the sines to cosines. The same idea will work in this case. We'll strip out a sine from the numerator and convert the rest to cosines as follows,

$$\begin{aligned} \int \frac{\sin^7 x}{\cos^4 x} dx &= \int \frac{\sin^6 x}{\cos^4 x} \sin x dx \\ &= \int \frac{(\sin^2 x)^3}{\cos^4 x} \sin x dx \\ &= \int \frac{(1 - \cos^2 x)^3}{\cos^4 x} \sin x dx \end{aligned}$$

At this point all we need to do is use the substitution  $u = \cos x$  and we're done.

$$\begin{aligned} \int \frac{\sin^7 x}{\cos^4 x} dx &= - \int \frac{(1 - u^2)^3}{u^4} du \\ &= - \int u^{-4} - 3u^{-2} + 3 - u^2 du \\ &= - \left( -\frac{1}{3} \frac{1}{u^3} + 3 \frac{1}{u} + 3u - \frac{1}{3} u^3 \right) + c \\ &= \frac{1}{3 \cos^3 x} - \frac{3}{\cos x} - 3 \cos x + \frac{1}{3} \cos^3 x + c \end{aligned}$$

So, under the right circumstances, we can use the ideas developed to help us deal with products of trig functions to deal with quotients of trig functions. The natural question then, is just what are the right circumstances?

First notice that if the quotient had been reversed as in this integral,

$$\int \frac{\cos^4 x}{\sin^7 x} dx$$

we wouldn't have been able to strip out a sine.

$$\int \frac{\cos^4 x}{\sin^7 x} dx = \int \frac{\cos^4 x}{\sin^6 x} \frac{1}{\sin x} dx$$

In this case the "stripped out" sine remains in the denominator and it won't do us any good for the substitution  $u = \cos x$  since this substitution requires a sine in the numerator of the quotient. Also note that, while we could convert the sines to cosines, the resulting integral would still be a fairly difficult integral.

So, we can use the methods we applied to products of trig functions to quotients of trig functions provided the term that needs parts stripped out in is the numerator of the quotient.

## Section 1-3 : Trig Substitutions

As we have done in the last couple of sections, let's start off with a couple of integrals that we should already be able to do with a standard substitution.

$$\int x\sqrt{25x^2 - 4} \, dx = \frac{1}{75}(25x^2 - 4)^{\frac{3}{2}} + c \qquad \int \frac{x}{\sqrt{25x^2 - 4}} \, dx = \frac{1}{25}\sqrt{25x^2 - 4} + c$$

Both of these used the substitution  $u = 25x^2 - 4$  and at this point should be pretty easy for you to do. However, let's take a look at the following integral.

**Example 1** Evaluate the following integral.

$$\int \frac{\sqrt{25x^2 - 4}}{x} \, dx$$

**Solution**

In this case the substitution  $u = 25x^2 - 4$  will not work (we don't have the  $x \, dx$  in the numerator the substitution needs) and so we're going to have to do something different for this integral.

It would be nice if we could reduce the two terms in the root down to a single term somehow. The following substitution will do that for us.

$$x = \frac{2}{5} \sec \theta$$

Do not worry about where this came from at this point. As we work the problem you will see that it works and that if we have a similar type of square root in the problem we can always use a similar substitution.

Before we actually do the substitution however let's verify the claim that this will allow us to reduce the two terms in the root to a single term.

$$\sqrt{25x^2 - 4} = \sqrt{25\left(\frac{4}{25}\right)\sec^2 \theta - 4} = \sqrt{4(\sec^2 \theta - 1)} = 2\sqrt{\sec^2 \theta - 1}$$

Now reduce the two terms to a single term all we need to do is recall the relationship,

$$\tan^2 \theta + 1 = \sec^2 \theta \qquad \Rightarrow \qquad \sec^2 \theta - 1 = \tan^2 \theta$$

Using this fact the square root becomes,

$$\sqrt{25x^2 - 4} = 2\sqrt{\tan^2 \theta} = 2|\tan \theta|$$

So, not only were we able to reduce the two terms to a single term in the process we were able to easily eliminate the root as well!

Note, however, the presence of the absolute value bars. These are important. Recall that

$$\sqrt{x^2} = |x|$$

There should always be absolute value bars at this stage. If we knew that  $\tan \theta$  was always positive or always negative we could eliminate the absolute value bars using,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Without limits we won't be able to determine if  $\tan \theta$  is positive or negative, however, we will need to eliminate them in order to do the integral. Therefore, since we are doing an indefinite integral we will assume that  $\tan \theta$  will be positive and so we can drop the absolute value bars. This gives,

$$\sqrt{25x^2 - 4} = 2 \tan \theta$$

So, we were able to reduce the two terms under the root to a single term with this substitution and in the process eliminate the root as well. Eliminating the root is a nice side effect of this substitution as the problem will now become somewhat easier to do.

Let's now do the substitution and see what we get. In doing the substitution don't forget that we'll also need to substitute for the  $dx$ . This is easy enough to get from the substitution.

$$x = \frac{2}{5} \sec \theta \quad \Rightarrow \quad dx = \frac{2}{5} \sec \theta \tan \theta d\theta$$

Using this substitution the integral becomes,

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x} dx &= \int \frac{2 \tan \theta}{\frac{2}{5} \sec \theta} \left( \frac{2}{5} \sec \theta \tan \theta \right) d\theta \\ &= 2 \int \tan^2 \theta d\theta \end{aligned}$$

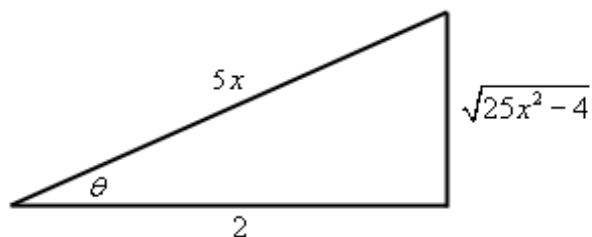
With this substitution we were able to reduce the given integral to an integral involving trig functions and we saw how to do these problems in the previous [section](#). Let's finish the integral.

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x} dx &= 2 \int \sec^2 \theta - 1 d\theta \\ &= 2(\tan \theta - \theta) + c \end{aligned}$$

So, we've got an answer for the integral. Unfortunately, the answer isn't given in  $x$ 's as it should be. So, we need to write our answer in terms of  $x$ . We can do this with some right triangle trig. From our original substitution we have,

$$\sec \theta = \frac{5x}{2} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

This gives the following right triangle.



From this we can see that,

$$\tan \theta = \frac{\sqrt{25x^2 - 4}}{2}$$

We can deal with the  $\theta$  in one of any variety of ways. From our substitution we can see that,

$$\theta = \sec^{-1}\left(\frac{5x}{2}\right)$$

While this is a perfectly acceptable method of dealing with the  $\theta$  we can use any of the possible six inverse trig functions and since sine and cosine are the two trig functions most people are familiar with we will usually use the inverse sine or inverse cosine. In this case we'll use the inverse cosine.

$$\theta = \cos^{-1}\left(\frac{2}{5x}\right)$$

So, with all of this the integral becomes,

$$\begin{aligned} \int \frac{\sqrt{25x^2 - 4}}{x} dx &= 2 \left( \frac{\sqrt{25x^2 - 4}}{2} - \cos^{-1}\left(\frac{2}{5x}\right) \right) + c \\ &= \sqrt{25x^2 - 4} - 2 \cos^{-1}\left(\frac{2}{5x}\right) + c \end{aligned}$$

We now have the answer back in terms of  $x$ .

Wow! That was a lot of work. Most of these won't take as long to work however. This first one needed lots of explanation since it was the first one. The remaining examples won't need quite as much explanation and so won't take as long to work.

However, before we move onto more problems let's first address the issue of definite integrals and how the process differs in these cases.

**Example 2** Evaluate the following integral.

$$\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2 - 4}}{x} dx$$

**Solution**

The limits here won't change the substitution so that will remain the same.

$$x = \frac{2}{5} \sec \theta$$

Using this substitution the square root still reduces down to,

$$\sqrt{25x^2 - 4} = 2|\tan \theta|$$

However, unlike the previous example we can't just drop the absolute value bars. In this case we've got limits on the integral and so we can use the limits as well as the substitution to determine the range of  $\theta$  that we're in. Once we've got that we can determine how to drop the absolute value bars.

Here's the limits of  $\theta$  and note that if you aren't good at solving trig equations in terms of secant you can always convert to cosine as we do below.

$$x = \frac{2}{5} : \frac{2}{5} = \frac{2}{5} \sec \theta = \frac{2}{5} \frac{1}{\cos \theta} \rightarrow \cos \theta = 1 \Rightarrow \theta = \cos^{-1}(1) = 0$$

$$x = \frac{4}{5} : \frac{4}{5} = \frac{2}{5} \sec \theta = \frac{2}{5} \frac{1}{\cos \theta} \rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Now, we know from solving trig equations, that there are in fact an infinite number of possible answers we could use. In fact, the more "correct" answer for the above work is,

$$\theta = 0 + 2\pi n = 2\pi n \quad \& \quad \theta = \frac{\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \pm 3, \dots$$

So, which ones should we use? The answer is simple. When using a secant trig substitution and converting the limits we always assume that  $\theta$  is in the range of inverse secant. Or,

$$\text{If } \theta = \sec^{-1}(x) \text{ then } 0 \leq \theta < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < \theta \leq \pi$$

Note that we have to avoid  $\theta = \frac{\pi}{2}$  because secant will not exist at that point. Also note that the range of  $\theta$  was given in terms secant even though we actually used inverse cosine to get the answers. This will not be a problem because even though inverse cosine can give  $\theta = \frac{\pi}{2}$  we'll never get it in our work above because that would require that we started with the secant being undefined and that will not happen when converting the limits as that would in turn require one of the limits to also be undefined!

So, in finding the new limits we didn't need all possible values of  $\theta$  we just need the inverse cosine answers we got when we converted the limits. Therefore, if we are in the range  $\frac{2}{5} \leq x \leq \frac{4}{5}$  then  $\theta$  is in the range of  $0 \leq \theta \leq \frac{\pi}{3}$  and in this range of  $\theta$ 's tangent is positive and so we can just drop the absolute value bars.



Let's do the substitution. Note that the work is identical to the previous example and so most of it is left out. We'll pick up at the final integral and then do the substitution.

$$\begin{aligned}\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2 - 4}}{x} dx &= 2 \int_0^{\frac{\pi}{3}} \sec^2 \theta - 1 d\theta \\ &= 2(\tan \theta - \theta) \Big|_0^{\pi/3} \\ &= 2\sqrt{3} - \frac{2\pi}{3}\end{aligned}$$

Note that because of the limits we didn't need to resort to a right triangle to complete the problem.

Let's take a look at a different set of limits for this integral.

**Example 3** Evaluate the following integral.

$$\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2 - 4}}{x} dx$$

**Solution**

Again, the substitution and square root are the same as the first two examples.

$$x = \frac{2}{5} \sec \theta \qquad \sqrt{25x^2 - 4} = 2|\tan \theta|$$

Let's next see the limits  $\theta$  for this problem.

$$x = -\frac{2}{5} : -\frac{2}{5} = \frac{2}{5} \sec \theta = \frac{2}{5} \frac{1}{\cos \theta} \rightarrow \cos \theta = -1 \Rightarrow \theta = \cos^{-1}(-1) = \pi$$

$$x = -\frac{4}{5} : -\frac{4}{5} = \frac{2}{5} \sec \theta = \frac{2}{5} \frac{1}{\cos \theta} \rightarrow \cos \theta = -\frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Remember that in converting the limits we use the results from the inverse secant/cosine. So, for this range of  $x$ 's we have  $\frac{2\pi}{3} \leq \theta \leq \pi$  and in this range of  $\theta$  tangent is negative and so in this case we can drop the absolute value bars, but will need to add in a minus sign upon doing so. In other words,

$$\sqrt{25x^2 - 4} = -2 \tan \theta$$

So, the only change this will make in the integration process is to put a minus sign in front of the integral. The integral is then,

$$\begin{aligned}\int_{-\frac{4}{5}}^{-\frac{2}{5}} \frac{\sqrt{25x^2 - 4}}{x} dx &= -2 \int_{\frac{2\pi}{3}}^{\pi} \sec^2 \theta - 1 d\theta \\ &= -2(\tan \theta - \theta) \Big|_{2\pi/3}^{\pi} \\ &= \frac{2\pi}{3} - 2\sqrt{3}\end{aligned}$$

In the last two examples we saw that we have to be very careful with definite integrals. We need to make sure that we determine the limits on  $\theta$  and whether or not this will mean that we can just drop the absolute value bars or if we need to add in a minus sign when we drop them.

Before moving on to the next example let's get the general form for the secant trig substitution that we used in the previous set of examples and the assumed limits on  $\theta$ .

$$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b} \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$$

Let's work a new and different type of example.

**Example 4** Evaluate the following integral.

$$\int \frac{1}{x^4 \sqrt{9-x^2}} dx$$

**Solution**

Now, the terms under the root in this problem looks to be (almost) the same as the previous ones so let's try the same type of substitution and see if it will work here as well.

$$x = 3 \sec \theta$$

Using this substitution, the square root becomes,

$$\sqrt{9-x^2} = \sqrt{9-9\sec^2 \theta} = 3\sqrt{1-\sec^2 \theta} = 3\sqrt{-\tan^2 \theta}$$

So, using this substitution we will end up with a negative quantity (the tangent squared is always positive of course) under the square root and this will be trouble. Using this substitution will give complex values and we don't want that. So, using secant for the substitution won't work.

However, the following substitution (and differential) will work.

$$x = 3 \sin \theta \qquad dx = 3 \cos \theta d\theta$$

With this substitution the square root is,

$$\sqrt{9-x^2} = 3\sqrt{1-\sin^2 \theta} = 3\sqrt{\cos^2 \theta} = 3|\cos \theta| = 3 \cos \theta$$

We were able to drop the absolute value bars because we are doing an indefinite integral and so we'll assume that everything is positive.

The integral is now,

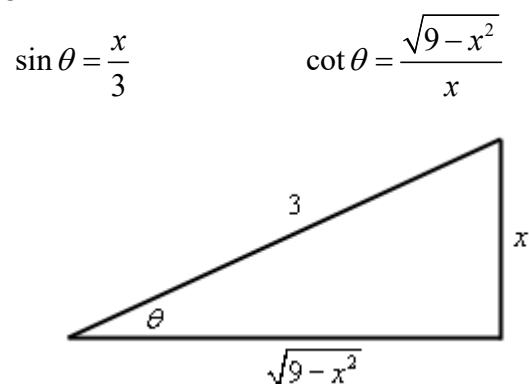
$$\begin{aligned} \int \frac{1}{x^4 \sqrt{9-x^2}} dx &= \int \frac{1}{81 \sin^4 \theta (3 \cos \theta)} 3 \cos \theta d\theta \\ &= \frac{1}{81} \int \frac{1}{\sin^4 \theta} d\theta \\ &= \frac{1}{81} \int \csc^4 \theta d\theta \end{aligned}$$

In the previous section we saw how to deal with integrals in which the exponent on the secant was even and since cosecants behave an awful lot like secants we should be able to do something similar with this.

Here is the integral.

$$\begin{aligned}
 \int \frac{1}{x^4 \sqrt{9-x^2}} dx &= \frac{1}{81} \int \csc^2 \theta \csc^2 \theta d\theta \\
 &= \frac{1}{81} \int (\cot^2 \theta + 1) \csc^2 \theta d\theta & u = \cot \theta \\
 &= -\frac{1}{81} \int u^2 + 1 du \\
 &= -\frac{1}{81} \left( \frac{1}{3} \cot^3 \theta + \cot \theta \right) + c
 \end{aligned}$$

Now we need to go back to  $x$ 's using a right triangle. Here is the right triangle for this problem and trig functions for this problem.



The integral is then,

$$\begin{aligned}
 \int \frac{1}{x^4 \sqrt{9-x^2}} dx &= -\frac{1}{81} \left( \frac{1}{3} \left( \frac{\sqrt{9-x^2}}{x} \right)^3 + \frac{\sqrt{9-x^2}}{x} \right) + c \\
 &= -\frac{(9-x^2)^{\frac{3}{2}}}{243x^3} - \frac{\sqrt{9-x^2}}{81x} + c
 \end{aligned}$$

We aren't going to be doing a definite integral example with a sine trig substitution. However, if we had we would need to convert the limits and that would mean eventually needing to evaluate an inverse sine. So, much like with the secant trig substitution, the values of  $\theta$  that we'll use will be those from the inverse sine or,

$$\text{If } \theta = \sin^{-1}(x) \text{ then } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Here is a summary for the sine trig substitution.

$$\sqrt{a^2 - b^2 x^2} \Rightarrow x = \frac{a}{b} \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

There is one final case that we need to look at. The next integral will also contain something that we need to make sure we can deal with.

**Example 5** Evaluate the following integral.

$$\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2 + 1)^{\frac{3}{2}}} dx$$

**Solution**

First, notice that there really is a square root in this problem even though it isn't explicitly written out. To see the root let's rewrite things a little.

$$(36x^2 + 1)^{\frac{3}{2}} = \left( (36x^2 + 1)^{\frac{1}{2}} \right)^3 = \left( \sqrt{36x^2 + 1} \right)^3$$

This terms under the root are not in the form we saw in the previous examples. Here we will use the substitution for this root.

$$x = \frac{1}{6} \tan \theta \qquad dx = \frac{1}{6} \sec^2 \theta d\theta$$

With this substitution the denominator becomes,

$$\left( \sqrt{36x^2 + 1} \right)^3 = \left( \sqrt{\tan^2 \theta + 1} \right)^3 = \left( \sqrt{\sec^2 \theta} \right)^3 = |\sec \theta|^3$$

Now, because we have limits we'll need to convert them to  $\theta$  so we can determine how to drop the absolute value bars.

$$x = 0 : 0 = \frac{1}{6} \tan \theta \Rightarrow \theta = \tan^{-1}(0) = 0$$

$$x = \frac{1}{6} : \frac{1}{6} = \frac{1}{6} \tan \theta \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

As with the previous two cases when converting limits here we will use the results of the inverse tangent or,

$$\text{If } \theta = \tan^{-1}(x) \text{ then } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

So, in this range of  $\theta$  secant is positive and so we can drop the absolute value bars.

Here is the integral,

$$\begin{aligned}\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}} dx &= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{7776} \tan^5 \theta}{\sec^3 \theta} \left( \frac{1}{6} \sec^2 \theta \right) d\theta \\ &= \frac{1}{46656} \int_0^{\frac{\pi}{4}} \frac{\tan^5 \theta}{\sec \theta} d\theta\end{aligned}$$

There are several ways to proceed from this point. Normally with an odd exponent on the tangent we would strip one of them out and convert to secants. However, that would require that we also have a secant in the numerator which we don't have. Therefore, it seems like the best way to do this one would be to convert the integrand to sines and cosines.

$$\begin{aligned}\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}} dx &= \frac{1}{46656} \int_0^{\frac{\pi}{4}} \frac{\sin^5 \theta}{\cos^4 \theta} d\theta \\ &= \frac{1}{46656} \int_0^{\frac{\pi}{4}} \frac{(1-\cos^2 \theta)^2}{\cos^4 \theta} \sin \theta d\theta\end{aligned}$$

We can now use the substitution  $u = \cos \theta$  and we might as well convert the limits as well.

$$\begin{aligned}\theta = 0 & \quad u = \cos 0 = 1 \\ \theta = \frac{\pi}{4} & \quad u = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}\end{aligned}$$

The integral is then,

$$\begin{aligned}\int_0^{\frac{1}{6}} \frac{x^5}{(36x^2+1)^{\frac{3}{2}}} dx &= -\frac{1}{46656} \int_1^{\frac{\sqrt{2}}{2}} u^{-4} - 2u^{-2} + 1 du \\ &= -\frac{1}{46656} \left( -\frac{1}{3u^3} + \frac{2}{u} + u \right) \Bigg|_1^{\frac{\sqrt{2}}{2}} \\ &= \frac{1}{17496} - \frac{11\sqrt{2}}{279936}\end{aligned}$$

Here is a summary for this final type of trig substitution.

$$\sqrt{a^2 + b^2 x^2} \quad \Rightarrow \quad x = \frac{a}{b} \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Before proceeding with some more examples let's discuss just how we knew to use the substitutions that we did in the previous examples.

The main idea was to determine a substitution that would allow us to reduce the two terms under the root that was always in the problem (more on this in a bit) into a single term and in doing so we were also able to easily eliminate the root. To do this we made use of the following formulas.

$$25x^2 - 4 \quad \Rightarrow \quad \sec^2 \theta - 1 = \tan^2 \theta$$

$$9 - x^2 \quad \Rightarrow \quad 1 - \sin^2 \theta = \cos^2 \theta$$

$$36x^2 + 1 \quad \Rightarrow \quad \tan^2 \theta + 1 = \sec^2 \theta$$

If we step back a bit we can notice that the terms we reduced look like the trig identities we used to reduce them in a vague way.

For instance,  $25x^2 - 4$  is something squared (*i.e.* the  $25x^2$ ) minus a number (*i.e.* the 4) and the left side of formula we used,  $\sec^2 \theta - 1$ , also follows this basic form. So, because the two look alike in a very vague way that suggests using a secant substitution for that problem. We can notice similar vague similarities in the other two cases as well.

If we keep this idea in mind we don't need the "formulas" listed after each example to tell us which trig substitution to use and since we have to know the trig identities anyway to do the problems keeping this idea in mind doesn't really add anything to what we need to know for the problems.

Once we've identified the trig function to use in the substitution the coefficient, the  $\frac{a}{b}$  in the formulas, is also easy to get. Just remember that in order to use the trig identities the coefficient of the trig function and the number in the identity must be the same, *i.e.* both 4 or 9, so that the trig identity can be used after we factor the common number out. What this means is that we need to "turn" the coefficient of the squared term into the constant number through our substitution.

So, in the first example we needed to "turn" the 25 into a 4 through our substitution. Remembering that we are eventually going to square the substitution that means we need to divide out by a 5 so the 25 will cancel out, upon squaring. Likewise, we'll need to add a 2 to the substitution so the coefficient will "turn" into a 4 upon squaring. In other words, we would need to use the substitution that we did in the problem.

The same idea holds for the other two trig substitutions.

Notice as well that we could have used cosecant in the first case, cosine in the second case and cotangent in the third case. So, why didn't we? Simply because of the differential work. Had we used these trig functions instead we would have picked up a minus sign in the differential that we'd need to keep track of. So, while these could be used they generally aren't to avoid extra minus signs that we need to keep track of.

Next, let's quickly address the fact that a root was in all of these problems. Note that the root is not required in order to use a trig substitution. Instead, the trig substitution gave us a really nice of eliminating the root from the problem. In this section we will always be having roots in the problems, and in fact our summaries above all assumed roots, roots are not actually required in order use a trig substitution. We will be seeing an example or two of trig substitutions in integrals that do not have roots in the [Integrals Involving Quadratics](#) section.

Finally, let's summarize up all the ideas with the trig substitutions we've discussed and again we will be using roots in the summary simply because all the integrals in this section will have roots and those tend to be the most likely places for using trig substitutions but again, are not required in order to use a trig substitution.

Form	Looks Like	Substitution	Limit Assumptions
$\sqrt{b^2x^2 - a^2}$	$\sec^2 \theta - 1 = \tan^2 \theta$	$x = \frac{a}{b} \sec \theta$	$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$
$\sqrt{a^2 - b^2x^2}$	$1 - \sin^2 \theta = \cos^2 \theta$	$x = \frac{a}{b} \sin \theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + b^2x^2}$	$\tan^2 \theta + 1 = \sec^2 \theta$	$x = \frac{a}{b} \tan \theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Now, we have a couple of final examples to work in this section. Not all trig substitutions will just jump right out at us. Sometimes we need to do a little work on the integrand first to get it into the correct form and that is the point of the remaining examples.

**Example 6** Evaluate the following integral.

$$\int \frac{x}{\sqrt{2x^2 - 4x - 7}} dx$$

**Solution**

In this case the quantity under the root doesn't obviously fit into any of the cases we looked at above and in fact isn't in any of the forms we saw in the previous examples. Note however that if we complete the square on the quadratic we can make it look somewhat like the above integrals.

Remember that completing the square requires a coefficient of one in front of the  $x^2$ . Once we have that we take half the coefficient of the  $x$ , square it, and then add and subtract it to the quantity. Here is the completing the square for this problem.

$$2\left(x^2 - 2x - \frac{7}{2}\right) = 2\left(x^2 - 2x + 1 - 1 - \frac{7}{2}\right) = 2\left((x-1)^2 - \frac{9}{2}\right) = 2(x-1)^2 - 9$$

So, the root becomes,

$$\sqrt{2x^2 - 4x - 7} = \sqrt{2(x-1)^2 - 9}$$

Now, this looks (very) vaguely like  $\sec^2 \theta - 1$  (i.e. something squared minus a number) except we've got something more complicated in the squared term. That is okay we'll still be able to do a secant substitution and it will work in pretty much the same way.

$$x - 1 = \frac{3}{\sqrt{2}} \sec \theta \quad x = 1 + \frac{3}{\sqrt{2}} \sec \theta \quad dx = \frac{3}{\sqrt{2}} \sec \theta \tan \theta d\theta$$

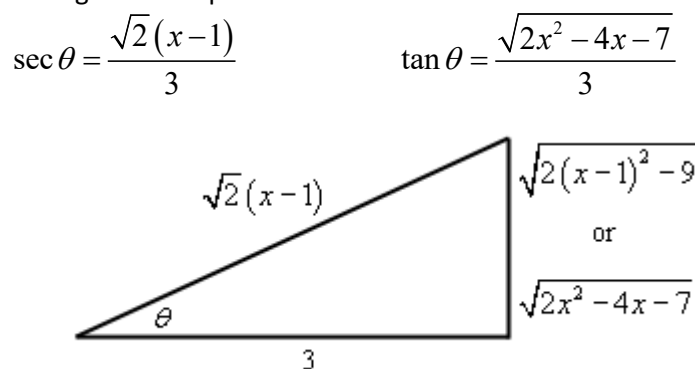
Using this substitution the root reduces to,

$$\sqrt{2x^2 - 4x - 7} = \sqrt{2(x-1)^2 - 9} = \sqrt{9\sec^2 \theta - 9} = 3\sqrt{\tan^2 \theta} = 3|\tan \theta| = 3 \tan \theta$$

Note we could drop the absolute value bars since we are doing an indefinite integral. Here is the integral.

$$\begin{aligned} \int \frac{x}{\sqrt{2x^2 - 4x - 7}} dx &= \int \frac{1 + \frac{3}{\sqrt{2}} \sec \theta}{3 \tan \theta} \left( \frac{3}{\sqrt{2}} \sec \theta \tan \theta \right) d\theta \\ &= \int \frac{1}{\sqrt{2}} \sec \theta + \frac{3}{2} \sec^2 \theta d\theta \\ &= \frac{1}{\sqrt{2}} \ln |\sec \theta + \tan \theta| + \frac{3}{2} \tan \theta + c \end{aligned}$$

And here is the right triangle for this problem.



The integral is then,

$$\int \frac{x}{\sqrt{2x^2 - 4x - 7}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}(x-1)}{3} + \frac{\sqrt{2x^2 - 4x - 7}}{3} \right| + \frac{\sqrt{2x^2 - 4x - 7}}{2} + c$$

**Example 7** Evaluate the following integral.

$$\int e^{4x} \sqrt{1 + e^{2x}} dx$$

**Solution**

This doesn't look to be anything like the other problems in this section. However it is. To see this we first need to notice that,

$$e^{2x} = (e^x)^2$$

Upon noticing this we can use the following standard Calculus I substitution.

$$u = e^x \qquad du = e^x dx$$

We do need to be a little careful with the differential work however. We don't have just an  $e^x$  out in front of the root. Instead we have an  $e^{4x}$ . So, we'll need to strip one of those out for the differential and then use the substitution on the rest. Here is the substitution work.



$$\begin{aligned}\int e^{4x} \sqrt{1+e^{2x}} dx &= \int e^{3x} e^x \sqrt{1+e^{2x}} dx \\ &= \int (e^x)^3 \sqrt{1+(e^x)^2} e^x dx = \int u^3 \sqrt{1+u^2} du\end{aligned}$$

This is now a fairly obvious trig substitution (hopefully). The quantity under the root looks almost exactly like  $1 + \tan^2 \theta$  and so we can use a tangent substitution. Here is that work.

$$u = \tan \theta \quad du = \sec^2 \theta d\theta \quad \sqrt{1+u^2} = \sqrt{1+\tan^2 \theta} = \sqrt{\sec^2 \theta} = |\sec \theta|$$

Because we are doing an indefinite integral we can assume secant is positive and drop the absolute value bars. Applying this substitution to the integral gives,

$$\int e^{4x} \sqrt{1+e^{2x}} dx = \int \tan^3 \theta (\sec \theta) (\sec^2 \theta) d\theta = \int \tan^3 \theta \sec^3 \theta d\theta$$

We'll finish this integral off in a bit. Before we get to that there is a "quicker" (although not super obvious) way of doing the substitutions above. Let's cover that first then we'll come back and finish working the integral.

We can notice that the  $u$  in the Calculus I substitution and the trig substitution are the same  $u$  and so we can combine them into the following substitution.

$$e^x = \tan \theta$$

We can then compute the differential. Just remember that all we do is differentiate both sides and then tack on  $dx$  or  $d\theta$  onto the appropriate side. Doing this gives,

$$e^x dx = \sec^2 \theta d\theta$$

With this substitution the square root becomes,

$$\sqrt{1+e^{2x}} = \sqrt{1+(e^x)^2} = \sqrt{1+\tan^2 \theta} = \sqrt{\sec^2 \theta} = |\sec \theta| = \sec \theta$$

Again, we can drop the absolute value bars because we are doing an indefinite integral. The integral then becomes,

$$\begin{aligned}\int e^{4x} \sqrt{1+e^{2x}} dx &= \int e^{3x} e^x \sqrt{1+e^{2x}} dx \\ &= \int (e^x)^3 \sqrt{1+e^{2x}} (e^x) dx \\ &= \int \tan^3 \theta (\sec \theta) (\sec^2 \theta) d\theta = \int \tan^3 \theta \sec^3 \theta d\theta\end{aligned}$$

So, the same integral with less work. However, it does require that you be able to combine the two substitutions in to a single substitution. How you do this type of problem is up to you but if you don't

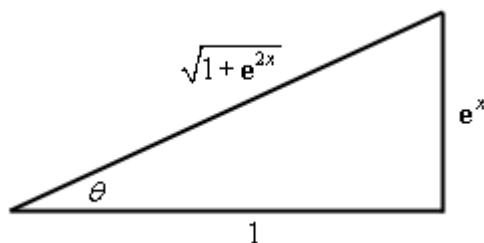
feel comfortable with the single substitution (and there's nothing wrong if you don't!) then just do the two individual substitutions. The single substitution method was given only to show you that it can be done so that those that are really comfortable with both kinds of substitutions can do the work a little quicker.

Now, let's finish the integral work.

$$\begin{aligned}
 \int e^{4x} \sqrt{1+e^{2x}} \, dx &= \int \tan^3 \theta \sec^3 \theta \, d\theta \\
 &= \int (\sec^2 \theta - 1) \sec^2 \theta \sec \theta \tan \theta \, d\theta & v = \sec \theta \\
 &= \int v^4 - v^2 \, du \\
 &= \frac{1}{5} \sec^5 \theta - \frac{1}{3} \sec^3 \theta + c
 \end{aligned}$$

Here is the right triangle for this integral.

$$\tan \theta = \frac{e^x}{1} \qquad \sec \theta = \frac{\sqrt{1+e^{2x}}}{1} = \sqrt{1+e^{2x}}$$



The integral is then,

$$\int e^{4x} \sqrt{1+e^{2x}} \, dx = \frac{1}{5} (1+e^{2x})^{\frac{5}{2}} - \frac{1}{3} (1+e^{2x})^{\frac{3}{2}} + c$$

This was a messy problem, but we will be seeing some of this type of integral in later sections on occasion so we needed to make sure you'd seen at least one like it.

So, as we've seen in the final two examples in this section some integrals that look nothing like the first few examples can in fact be turned into a trig substitution problem with a little work.

## Section 1-4 : Partial Fractions

---

In this section we are going to take a look at integrals of rational expressions of polynomials and once again let's start this section out with an integral that we can already do so we can contrast it with the integrals that we'll be doing in this section.

$$\begin{aligned}\int \frac{2x-1}{x^2-x-6} dx &= \int \frac{1}{u} du \quad \text{using } u = x^2 - x - 6 \text{ and } du = (2x-1)dx \\ &= \ln|x^2 - x - 6| + c\end{aligned}$$

So, if the numerator is the derivative of the denominator (or a constant multiple of the derivative of the denominator) doing this kind of integral is fairly simple. However, often the numerator isn't the derivative of the denominator (or a constant multiple). For example, consider the following integral.

$$\int \frac{3x+11}{x^2-x-6} dx$$

In this case the numerator is definitely not the derivative of the denominator nor is it a constant multiple of the derivative of the denominator. Therefore, the simple substitution that we used above won't work. However, if we notice that the integrand can be broken up as follows,

$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

then the integral is actually quite simple.

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6} dx &= \int \frac{4}{x-3} - \frac{1}{x+2} dx \\ &= 4 \ln|x-3| - \ln|x+2| + c\end{aligned}$$

This process of taking a rational expression and decomposing it into simpler rational expressions that we can add or subtract to get the original rational expression is called **partial fraction decomposition**. Many integrals involving rational expressions can be done if we first do partial fractions on the integrand.

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both  $P(x)$  and  $Q(x)$  are polynomials and the degree of  $P(x)$  is smaller than the degree of  $Q(x)$ . Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}, \quad k = 1, 2, 3, \dots$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}, \quad k = 1, 2, 3, \dots$

Notice that the first and third cases are really special cases of the second and fourth cases respectively.

There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.

Let's start the examples by doing the integral above.

**Example 1** Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6} dx$$

**Solution**

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose  $A$  and  $B$  so that the numerators of these two are equal for every  $x$ . To do this we'll need to set the numerators equal.

$$3x+11 = A(x+2) + B(x-3)$$

Note that in most problems we will go straight from the general form of the decomposition to this step and not bother with actually adding the terms back up. The only point to adding the terms is to get the numerator and we can get that without actually writing down the results of the addition.

At this point we have one of two ways to proceed. One way will always work but is often more work. The other, while it won't always work, is often quicker when it does work. In this case both will work

and so we'll use the quicker way for this example. We'll take a look at the other method in a later example.

What we're going to do here is to notice that the numerators must be equal for *any*  $x$  that we would choose to use. In particular the numerators must be equal for  $x = -2$  and  $x = 3$ . So, let's plug these in and see what we get.

$$\begin{aligned}x = -2 : \quad 5 &= A(0) + B(-5) &\Rightarrow B &= -1 \\x = 3 : 20 &= A(5) + B(0) &\Rightarrow A &= 4\end{aligned}$$

So, by carefully picking the  $x$ 's we got the unknown constants to quickly drop out. Note that these are the values we claimed they would be above.

At this point there really isn't a whole lot to do other than the integral.

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6} dx &= \int \frac{4}{x-3} - \frac{1}{x+2} dx \\&= \int \frac{4}{x-3} dx - \int \frac{1}{x+2} dx \\&= 4 \ln|x-3| - \ln|x+2| + c\end{aligned}$$

Recall that to do this integral we first split it up into two integrals and then used the substitutions,

$$u = x - 3 \qquad v = x + 2$$

on the integrals to get the final answer.

Before moving onto the next example a couple of quick notes are in order here. First, many of the integrals in partial fractions problems come down to the type of integral seen above. Make sure that you can do those integrals.

There is also another integral that often shows up in these kinds of problems so we may as well give the formula for it here since we are already on the subject.

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

It will be an example or two before we use this so don't forget about it.

Now, let's work some more examples.

**Example 2** Evaluate the following integral.

$$\int \frac{x^2 + 4}{3x^3 + 4x^2 - 4x} dx$$

**Solution**

We won't be putting as much detail into this solution as we did in the previous example. The first thing is to factor the denominator and get the form of the partial fraction decomposition.

$$\frac{x^2 + 4}{x(x+2)(3x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{3x-2}$$

The next step is to set numerators equal. If you need to actually add the right side together to get the numerator for that side then you should do so, however, it will definitely make the problem quicker if you can do the addition in your head to get,

$$x^2 + 4 = A(x+2)(3x-2) + Bx(3x-2) + Cx(x+2)$$

As with the previous example it looks like we can just pick a few values of  $x$  and find the constants so let's do that.

$$x = 0 : \quad 4 = A(2)(-2) \quad \Rightarrow \quad A = -1$$

$$x = -2 : \quad 8 = B(-2)(-8) \quad \Rightarrow \quad B = \frac{1}{2}$$

$$x = \frac{2}{3} : \quad \frac{40}{9} = C\left(\frac{2}{3}\right)\left(\frac{8}{3}\right) \quad \Rightarrow \quad C = \frac{40}{16} = \frac{5}{2}$$

Note that unlike the first example most of the coefficients here are fractions. That is not unusual so don't get excited about it when it happens.

Now, let's do the integral.

$$\begin{aligned} \int \frac{x^2 + 4}{3x^3 + 4x^2 - 4x} dx &= \int -\frac{1}{x} + \frac{\frac{1}{2}}{x+2} + \frac{\frac{5}{2}}{3x-2} dx \\ &= -\ln|x| + \frac{1}{2} \ln|x+2| + \frac{5}{6} \ln|3x-2| + c \end{aligned}$$

Again, as noted above, integrals that generate natural logarithms are very common in these problems so make sure you can do them. Also, you were able to correctly do the last integral right? The coefficient of  $\frac{5}{6}$  is correct. Make sure that you do the substitution required for the term properly.

**Example 3** Evaluate the following integral.

$$\int \frac{x^2 - 29x + 5}{(x-4)^2(x^2+3)} dx$$

**Solution**

This time the denominator is already factored so let's just jump right to the partial fraction decomposition.

$$\frac{x^2 - 29x + 5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

Setting numerators gives,

$$x^2 - 29x + 5 = A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2$$

In this case we aren't going to be able to just pick values of  $x$  that will give us all the constants. Therefore, we will need to work this the second (and often longer) way. The first step is to multiply out the right side and collect all the like terms together. Doing this gives,

$$x^2 - 29x + 5 = (A+C)x^3 + (-4A+B-8C+D)x^2 + (3A+16C-8D)x - 12A+3B+16D$$

Now we need to choose  $A$ ,  $B$ ,  $C$ , and  $D$  so that these two are equal. In other words, we will need to set the coefficients of like powers of  $x$  equal. This will give a system of equations that can be solved.

$$\left. \begin{array}{l} x^3: \quad A+C=0 \\ x^2: \quad -4A+B-8C+D=1 \\ x^1: \quad 3A+16C-8D=-29 \\ x^0: \quad -12A+3B+16D=5 \end{array} \right\} \Rightarrow A=1, B=-5, C=-1, D=2$$

Note that we used  $x^0$  to represent the constants. Also note that these systems can often be quite large and have a fair amount of work involved in solving them. The best way to deal with these is to use some form of computer aided solving techniques.

Now, let's take a look at the integral.

$$\begin{aligned} \int \frac{x^2 - 29x + 5}{(x-4)^2(x^2+3)} dx &= \int \frac{1}{x-4} - \frac{5}{(x-4)^2} + \frac{-x+2}{x^2+3} dx \\ &= \int \frac{1}{x-4} - \frac{5}{(x-4)^2} - \frac{x}{x^2+3} + \frac{2}{x^2+3} dx \\ &= \ln|x-4| + \frac{5}{x-4} - \frac{1}{2} \ln|x^2+3| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c \end{aligned}$$

In order to take care of the third term we needed to split it up into two separate terms. Once we've done this we can do all the integrals in the problem. The first two use the substitution  $u = x-4$ , the third uses the substitution  $v = x^2+3$  and the fourth term uses the formula given above for inverse tangents.

**Example 4** Evaluate the following integral.

$$\int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)(x^2+4)^2} dx$$

**Solution**

Let's first get the general form of the partial fraction decomposition.

$$\frac{x^3 + 10x^2 + 3x + 36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Now, set numerators equal, expand the right side and collect like terms.

$$\begin{aligned} x^3 + 10x^2 + 3x + 36 &= A(x^2 + 4)^2 + (Bx + C)(x-1)(x^2 + 4) + (Dx + E)(x-1) \\ &= (A+B)x^4 + (C-B)x^3 + (8A+4B-C+D)x^2 + \\ &\quad (-4B+4C-D+E)x + 16A-4C-E \end{aligned}$$

Setting coefficient equal gives the following system.

$$\left. \begin{array}{lcl} x^4 : & A+B=0 \\ x^3 : & C-B=1 \\ x^2 : & 8A+4B-C+D=10 \\ x^1 : & -4B+4C-D+E=3 \\ x^0 : & 16A-4C-E=36 \end{array} \right\} \Rightarrow A=2, B=-2, C=-1, D=1, E=0$$

Don't get excited if some of the coefficients end up being zero. It happens on occasion.

Here's the integral.

$$\begin{aligned} \int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)(x^2+4)^2} dx &= \int \frac{2}{x-1} + \frac{-2x-1}{x^2+4} + \frac{x}{(x^2+4)^2} dx \\ &= \int \frac{2}{x-1} - \frac{2x}{x^2+4} - \frac{1}{x^2+4} + \frac{x}{(x^2+4)^2} dx \\ &= 2 \ln|x-1| - \ln|x^2+4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2} \frac{1}{x^2+4} + c \end{aligned}$$

To this point we've only looked at rational expressions where the degree of the numerator was strictly less than the degree of the denominator. Of course, not all rational expressions will fit into this form and so we need to take a look at a couple of examples where this isn't the case.

**Example 5** Evaluate the following integral.

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2} dx$$

**Solution**

So, in this case the degree of the numerator is 4 and the degree of the denominator is 3. Therefore, partial fractions can't be done on this rational expression.



To fix this up we'll need to do long division on this to get it into a form that we can deal with. Here is the work for that.

$$\begin{array}{r}
 x-2 \\
 x^3-3x^2 \overline{) x^4-5x^3+6x^2-18} \\
 \underline{-(x^4-3x^3)} \phantom{-18} \\
 -2x^3+6x^2-18 \\
 \underline{-(-2x^3+6x^2)} \\
 -18
 \end{array}$$

So, from the long division we see that,

$$\frac{x^4-5x^3+6x^2-18}{x^3-3x^2} = x-2 - \frac{18}{x^3-3x^2}$$

and the integral becomes,

$$\begin{aligned}
 \int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2} dx &= \int x-2 - \frac{18}{x^3-3x^2} dx \\
 &= \int x-2 dx - \int \frac{18}{x^3-3x^2} dx
 \end{aligned}$$

The first integral we can do easily enough and the second integral is now in a form that allows us to do partial fractions. So, let's get the general form of the partial fractions for the second integrand.

$$\frac{18}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

Setting numerators equal gives us,

$$18 = Ax(x-3) + B(x-3) + Cx^2$$

Now, there is a variation of the method we used in the first couple of examples that will work here. There are a couple of values of  $x$  that will allow us to quickly get two of the three constants, but there is no value of  $x$  that will just hand us the third.

What we'll do in this example is pick  $x$ 's to get the two constants that we can easily get and then we'll just pick another value of  $x$  that will be easy to work with (*i.e.* it won't give large/messy numbers anywhere) and then we'll use the fact that we also know the other two constants to find the third.

$$x=0: 18 = B(-3) \quad \Rightarrow \quad B = -6$$

$$x=3: 18 = C(9) \quad \Rightarrow \quad C = 2$$

$$x=1: 18 = A(-2) + B(-2) + C = -2A + 14 \quad \Rightarrow \quad A = -2$$

The integral is then,

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2} dx = \int x - 2 dx - \int \frac{2}{x} - \frac{6}{x^2} + \frac{2}{x-3} dx$$

$$= \frac{1}{2}x^2 - 2x + 2 \ln|x| - \frac{6}{x} - 2 \ln|x-3| + c$$

In the previous example there were actually two different ways of dealing with the  $x^2$  in the denominator. One is to treat it as a quadratic which would give the following term in the decomposition

$$\frac{Ax + B}{x^2}$$

and the other is to treat it as a linear term in the following way,

$$x^2 = (x-0)^2$$

which gives the following two terms in the decomposition,

$$\frac{A}{x} + \frac{B}{x^2}$$

We used the second way of thinking about it in our example. Notice however that the two will give identical partial fraction decompositions. So, why talk about this? Simple. This will work for  $x^2$ , but what about  $x^3$  or  $x^4$ ? In these cases, we really will need to use the second way of thinking about these kinds of terms.

$$x^3 \Rightarrow \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} \qquad x^4 \Rightarrow \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4}$$

Let's take a look at one more example.

**Example 6** Evaluate the following integral.

$$\int \frac{x^2}{x^2 - 1} dx$$

**Solution**

In this case the numerator and denominator have the same degree. As with the last example we'll need to do long division to get this into the correct form. We'll leave the details of that to you to check.

$$\int \frac{x^2}{x^2 - 1} dx = \int 1 + \frac{1}{x^2 - 1} dx = \int dx + \int \frac{1}{x^2 - 1} dx$$

So, we'll need to partial fraction the second integral. Here's the decomposition.

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Setting numerator equal gives,

$$1 = A(x+1) + B(x-1)$$

Picking value of  $x$  gives us the following coefficients.

$$x = -1: 1 = B(-2) \quad \Rightarrow \quad B = -\frac{1}{2}$$

$$x = 1 : 1 = A(2) \quad \Rightarrow \quad A = \frac{1}{2}$$

The integral is then,

$$\begin{aligned} \int \frac{x^2}{x^2-1} dx &= \int dx + \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} dx \\ &= x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + c \end{aligned}$$

## Section 1-5 : Integrals Involving Roots

In this section we're going to look at an integration technique that can be useful for *some* integrals with roots in them. We've already seen some integrals with roots in them. Some can be done quickly with a simple Calculus I substitution and some can be done with trig substitutions.

However, not all integrals with roots will allow us to use one of these methods. Let's look at a couple of examples to see another technique that can be used on occasion to help with these integrals.

**Example 1** Evaluate the following integral.

$$\int \frac{x+2}{\sqrt[3]{x-3}} dx$$

**Solution**

Sometimes when faced with an integral that contains a root we can use the following substitution to simplify the integral into a form that can be easily worked with.

$$u = \sqrt[3]{x-3}$$

So, instead of letting  $u$  be the stuff under the radical as we often did in Calculus I we let  $u$  be the whole radical. Now, there will be a little more work here since we will also need to know what  $x$  is so we can substitute in for that in the numerator and so we can compute the differential,  $dx$ . This is easy enough to get however. Just solve the substitution for  $x$  as follows,

$$x = u^3 + 3 \qquad dx = 3u^2 du$$

Using this substitution the integral is now,

$$\begin{aligned} \int \frac{(u^3 + 3) + 2}{u} 3u^2 du &= \int 3u^4 + 15u du \\ &= \frac{3}{5}u^5 + \frac{15}{2}u^2 + c \\ &= \frac{3}{5}(x-3)^{\frac{5}{3}} + \frac{15}{2}(x-3)^{\frac{2}{3}} + c \end{aligned}$$

So, sometimes, when an integral contains the root  $\sqrt[n]{g(x)}$  the substitution,

$$u = \sqrt[n]{g(x)}$$

can be used to simplify the integral into a form that we can deal with.

Let's take a look at another example real quick.

**Example 2** Evaluate the following integral.

$$\int \frac{2}{x-3\sqrt{x+10}} dx$$

**Solution**

We'll do the same thing we did in the previous example. Here's the substitution and the extra work we'll need to do to get  $x$  in terms of  $u$ .

$$u = \sqrt{x+10} \qquad x = u^2 - 10 \qquad dx = 2u \, du$$

With this substitution the integral is,

$$\int \frac{2}{x-3\sqrt{x+10}} dx = \int \frac{2}{u^2-10-3u} (2u) du = \int \frac{4u}{u^2-3u-10} du$$

This integral can now be done with partial fractions.

$$\frac{4u}{(u-5)(u+2)} = \frac{A}{u-5} + \frac{B}{u+2}$$

Setting numerators equal gives,

$$4u = A(u+2) + B(u-5)$$

Picking value of  $u$  gives the coefficients.

$$\begin{array}{lll} u = -2 & -8 = B(-7) & B = \frac{8}{7} \\ u = 5 & 20 = A(7) & A = \frac{20}{7} \end{array}$$

The integral is then,

$$\begin{aligned} \int \frac{2}{x-3\sqrt{x+10}} dx &= \int \frac{\frac{20}{7}}{u-5} + \frac{\frac{8}{7}}{u+2} du \\ &= \frac{20}{7} \ln|u-5| + \frac{8}{7} \ln|u+2| + c \\ &= \frac{20}{7} \ln|\sqrt{x+10}-5| + \frac{8}{7} \ln|\sqrt{x+10}+2| + c \end{aligned}$$

So, we've seen a nice method to eliminate roots from the integral and put it into a form that we can deal with. Note however, that this won't always work and sometimes the new integral will be just as difficult to do.

## Section 1-6 : Integrals Involving Quadratics

To this point we've seen quite a few integrals that involve quadratics. A couple of examples are,

$$\int \frac{x}{x^2 \pm a} dx = \frac{1}{2} \ln|x^2 \pm a| + c \qquad \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

We also saw that integrals involving  $\sqrt{b^2x^2 - a^2}$ ,  $\sqrt{a^2 - b^2x^2}$  and  $\sqrt{a^2 + b^2x^2}$  could be done with a trig substitution.

Notice however that all of these integrals were missing an  $x$  term. They all consist of only a quadratic term and a constant.

Some integrals involving general quadratics are easy enough to do. For instance, the following integral can be done with a quick substitution.

$$\begin{aligned} \int \frac{2x+3}{4x^2+12x-1} dx &= \frac{1}{4} \int \frac{1}{u} du & u &= 4x^2 + 12x - 1 \quad du = 4(2x+3) dx \\ &= \frac{1}{4} \ln|4x^2 + 12x - 1| + c \end{aligned}$$

Some integrals with quadratics can be done with partial fractions. For instance,

$$\int \frac{10x-6}{3x^2+16x+5} dx = \int \frac{4}{x+5} - \frac{2}{3x+1} dx = 4 \ln|x+5| - \frac{2}{3} \ln|3x+1| + c$$

Unfortunately, these methods won't work on a lot of integrals. A simple substitution will only work if the numerator is a constant multiple of the derivative of the denominator and partial fractions will only work if the denominator can be factored.

The topic of this section is how to deal with integrals involving quadratics when the techniques that we've looked at to this point simply won't work.

Back in the [Trig Substitution](#) section we saw how to deal with square roots that had a general quadratic in them. Let's take a quick look at another one like that since the idea involved in doing that kind of integral is exactly what we are going to need for the other integrals in this section.

**Example 1** Evaluate the following integral.

$$\int \sqrt{x^2 + 4x + 5} dx$$

**Solution**

Recall from the Trig Substitution section that in order to do a trig substitution here we first needed to complete the square on the quadratic. This gives,

$$x^2 + 4x + 5 = x^2 + 4x + 4 - 4 + 5 = (x+2)^2 + 1$$

After completing the square the integral becomes,

$$\int \sqrt{x^2 + 4x + 5} \, dx = \int \sqrt{(x+2)^2 + 1} \, dx$$

Upon doing this we can identify the trig substitution that we need. Here it is,

$$x + 2 = \tan \theta \quad x = \tan \theta - 2 \quad dx = \sec^2 \theta \, d\theta$$

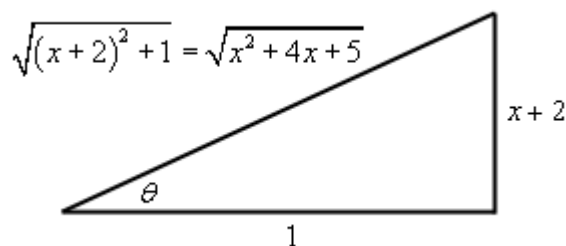
$$\sqrt{(x+2)^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sqrt{\sec^2 \theta} = |\sec \theta| = \sec \theta$$

Recall that since we are doing an indefinite integral we can drop the absolute value bars. Using this substitution the integral becomes,

$$\begin{aligned} \int \sqrt{x^2 + 4x + 5} \, dx &= \int \sec^3 \theta \, d\theta \\ &= \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + c \end{aligned}$$

We can finish the integral out with the following right triangle.

$$\tan \theta = \frac{x+2}{1} \quad \sec \theta = \frac{\sqrt{x^2 + 4x + 5}}{1} = \sqrt{x^2 + 4x + 5}$$



$$\int \sqrt{x^2 + 4x + 5} \, dx = \frac{1}{2} \left( (x+2) \sqrt{x^2 + 4x + 5} + \ln |x+2 + \sqrt{x^2 + 4x + 5}| \right) + c$$

So, by completing the square we were able to take an integral that had a general quadratic in it and convert it into a form that allowed us to use a known integration technique.

Let's do a quick review of completing the square before proceeding. Here is the general completing the square formula that we'll use.

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

This will always take a general quadratic and write it in terms of a squared term and a constant term.

Recall as well that in order to do this we must have a coefficient of one in front of the  $x^2$ . If not, we'll need to factor out the coefficient before completing the square. In other words,

$$ax^2 + bx + c = a \underbrace{\left( x^2 + \frac{b}{a}x + \frac{c}{a} \right)}_{\text{complete the square on this!}}$$

Now, let's see how completing the square can be used to do integrals that we aren't able to do at this point.

**Example 2** Evaluate the following integral.

$$\int \frac{1}{2x^2 - 3x + 2} dx$$

**Solution**

Okay, this doesn't factor so partial fractions just won't work on this. Likewise, since the numerator is just "1" we can't use the substitution  $u = 2x^2 - 3x + 8$ . So, let's see what happens if we complete the square on the denominator.

$$\begin{aligned} 2x^2 - 3x + 2 &= 2\left(x^2 - \frac{3}{2}x + 1\right) \\ &= 2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} + 1\right) \\ &= 2\left(\left(x - \frac{3}{4}\right)^2 + \frac{7}{16}\right) \end{aligned}$$

With this the integral is,

$$\int \frac{1}{2x^2 - 3x + 2} dx = \frac{1}{2} \int \frac{1}{\left(x - \frac{3}{4}\right)^2 + \frac{7}{16}} dx$$

Now this may not seem like all that great of a change. However, notice that we can now use the following substitution.

$$u = x - \frac{3}{4} \quad du = dx$$

and the integral is now,

$$\int \frac{1}{2x^2 - 3x + 2} dx = \frac{1}{2} \int \frac{1}{u^2 + \frac{7}{16}} du$$

We can now see that this is an inverse tangent! So, using the formula from the start of the section we get,

$$\begin{aligned} \int \frac{1}{2x^2 - 3x + 2} dx &= \frac{1}{2} \left( \frac{4}{\sqrt{7}} \right) \tan^{-1} \left( \frac{4u}{\sqrt{7}} \right) + c \\ &= \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{4x - 3}{\sqrt{7}} \right) + c \end{aligned}$$



**Example 3** Evaluate the following integral.

$$\int \frac{3x-1}{x^2+10x+28} dx$$

**Solution**

This example is a little different from the previous one. In this case we do have an  $x$  in the numerator however the numerator still isn't a multiple of the derivative of the denominator and so a simple Calculus I substitution won't work.

So, let's again complete the square on the denominator and see what we get,

$$x^2 + 10x + 28 = x^2 + 10x + 25 - 25 + 28 = (x+5)^2 + 3$$

Upon completing the square the integral becomes,

$$\int \frac{3x-1}{x^2+10x+28} dx = \int \frac{3x-1}{(x+5)^2+3} dx$$

At this point we can use the same type of substitution that we did in the previous example. The only real difference is that we'll need to make sure that we plug the substitution back into the numerator as well.

$$u = x + 5$$

$$x = u - 5$$

$$dx = du$$

$$\begin{aligned} \int \frac{3x-1}{x^2+10x+28} dx &= \int \frac{3(u-5)-1}{u^2+3} du \\ &= \int \frac{3u}{u^2+3} - \frac{16}{u^2+3} du \\ &= \frac{3}{2} \ln|u^2+3| - \frac{16}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + c \\ &= \frac{3}{2} \ln|(x+5)^2+3| - \frac{16}{\sqrt{3}} \tan^{-1}\left(\frac{x+5}{\sqrt{3}}\right) + c \end{aligned}$$

So, in general when dealing with an integral in the form,

$$\int \frac{Ax+B}{ax^2+bx+c} dx \tag{1}$$

Here we are going to assume that the denominator doesn't factor and the numerator isn't a constant multiple of the derivative of the denominator. In these cases, we complete the square on the denominator and then do a substitution that will yield an inverse tangent and/or a logarithm depending on the exact form of the numerator.

Let's now take a look at a couple of integrals that are in the same general form as (1) except the denominator will also be raised to a power. In other words, let's look at integrals in the form,

$$\int \frac{Ax + B}{(ax^2 + bx + c)^n} dx \quad (2)$$

**Example 4** Evaluate the following integral.

$$\int \frac{x}{(x^2 - 6x + 11)^3} dx$$

**Solution**

For the most part this integral will work the same as the previous two with one exception that will occur down the road. So, let's start by completing the square on the quadratic in the denominator.

$$x^2 - 6x + 11 = x^2 - 6x + 9 - 9 + 11 = (x - 3)^2 + 2$$

The integral is then,

$$\int \frac{x}{(x^2 - 6x + 11)^3} dx = \int \frac{x}{[(x - 3)^2 + 2]^3} dx$$

Now, we will use the same substitution that we've used to this point in the previous two examples.

$$u = x - 3 \qquad x = u + 3 \qquad dx = du$$

$$\begin{aligned} \int \frac{x}{(x^2 - 6x + 11)^3} dx &= \int \frac{u + 3}{(u^2 + 2)^3} du \\ &= \int \frac{u}{(u^2 + 2)^3} du + \int \frac{3}{(u^2 + 2)^3} du \end{aligned}$$

Now, here is where the differences start cropping up. The first integral can be done with the substitution  $v = u^2 + 2$  and isn't too difficult. The second integral however, can't be done with the substitution used on the first integral and it isn't an inverse tangent.

It turns out that a trig substitution will work nicely on the second integral and it will be the same as we did [when](#) we had square roots in the problem.

$$u = \sqrt{2} \tan \theta \qquad du = \sqrt{2} \sec^2 \theta d\theta$$

With these two substitutions the integrals become,

$$\begin{aligned}
 \int \frac{x}{(x^2 - 6x + 11)^3} dx &= \frac{1}{2} \int \frac{1}{v^3} dv + \int \frac{3}{(2 \tan^2 \theta + 2)^3} (\sqrt{2} \sec^2 \theta) d\theta \\
 &= -\frac{1}{4} \frac{1}{v^2} + \int \frac{3\sqrt{2} \sec^2 \theta}{8(\tan^2 \theta + 1)^3} d\theta \\
 &= -\frac{1}{4} \frac{1}{(u^2 + 2)^2} + \frac{3\sqrt{2}}{8} \int \frac{\sec^2 \theta}{(\sec^2 \theta)^3} d\theta \\
 &= -\frac{1}{4} \frac{1}{((x-3)^2 + 2)^2} + \frac{3\sqrt{2}}{8} \int \frac{1}{\sec^4 \theta} d\theta \\
 &= -\frac{1}{4} \frac{1}{((x-3)^2 + 2)^2} + \frac{3\sqrt{2}}{8} \int \cos^4 \theta d\theta
 \end{aligned}$$

Okay, at this point we've got two options for the remaining integral. We can either use the ideas we learned in the [section](#) about integrals involving trig integrals or we could use the following formula.

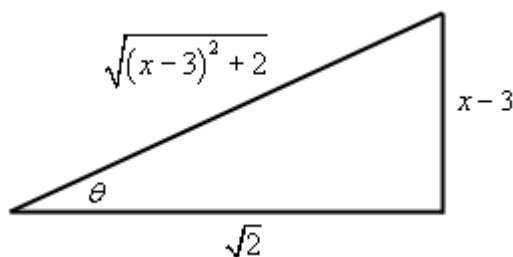
$$\int \cos^m \theta d\theta = \frac{1}{m} \sin \theta \cos^{m-1} \theta + \frac{m-1}{m} \int \cos^{m-2} \theta d\theta$$

Let's use this formula to do the integral.

$$\begin{aligned}
 \int \cos^4 \theta d\theta &= \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{4} \int \cos^2 \theta d\theta \\
 &= \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{4} \left( \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \int \cos^0 \theta d\theta \right) \quad \cos^0 \theta = 1! \\
 &= \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta
 \end{aligned}$$

Next, let's use the following right triangle to get this back to  $x$ 's.

$$\tan \theta = \frac{u}{\sqrt{2}} = \frac{x-3}{\sqrt{2}} \quad \sin \theta = \frac{x-3}{\sqrt{(x-3)^2 + 2}} \quad \cos \theta = \frac{\sqrt{2}}{\sqrt{(x-3)^2 + 2}}$$



The cosine integral is then,

$$\begin{aligned}\int \cos^4 \theta d\theta &= \frac{1}{4} \frac{2\sqrt{2}(x-3)}{\left((x-3)^2 + 2\right)^2} + \frac{3}{8} \frac{\sqrt{2}(x-3)}{(x-3)^2 + 2} + \frac{3}{8} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) \\ &= \frac{\sqrt{2}}{2} \frac{x-3}{\left((x-3)^2 + 2\right)^2} + \frac{3\sqrt{2}}{8} \frac{x-3}{(x-3)^2 + 2} + \frac{3}{8} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right)\end{aligned}$$

All told then the original integral is,

$$\begin{aligned}\int \frac{x}{(x^2 - 6x + 11)^3} dx &= -\frac{1}{4} \frac{1}{\left((x-3)^2 + 2\right)^2} + \\ &\quad \frac{3\sqrt{2}}{8} \left( \frac{\sqrt{2}}{2} \frac{x-3}{\left((x-3)^2 + 2\right)^2} + \frac{3\sqrt{2}}{8} \frac{x-3}{(x-3)^2 + 2} + \frac{3}{8} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) \right) \\ &= \frac{1}{8} \frac{3x-9}{\left((x-3)^2 + 2\right)^2} + \frac{9}{32} \frac{x-3}{(x-3)^2 + 2} + \frac{9\sqrt{2}}{64} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + c\end{aligned}$$

It's a long and messy answer, but there it is.

**Example 5** Evaluate the following integral.

$$\int \frac{x-3}{(4-2x-x^2)^2} dx$$

**Solution**

As with the other problems we'll first complete the square on the denominator.

$$4 - 2x - x^2 = -(x^2 + 2x - 4) = -(x^2 + 2x + 1 - 1 - 4) = -((x+1)^2 - 5) = 5 - (x+1)^2$$

The integral is,

$$\int \frac{x-3}{(4-2x-x^2)^2} dx = \int \frac{x-3}{[5-(x+1)^2]^2} dx$$

Now, let's do the substitution.

$$u = x + 1$$

$$x = u - 1$$

$$dx = du$$

and the integral is now,

$$\begin{aligned}\int \frac{x-3}{(4-2x-x^2)^2} dx &= \int \frac{u-4}{(5-u^2)^2} du \\ &= \int \frac{u}{(5-u^2)^2} du - \int \frac{4}{(5-u^2)^2} du\end{aligned}$$

In the first integral we'll use the substitution

$$v = 5 - u^2$$

and in the second integral we'll use the following trig substitution

$$u = \sqrt{5} \sin \theta \quad du = \sqrt{5} \cos \theta d\theta$$

Using these substitutions the integral becomes,

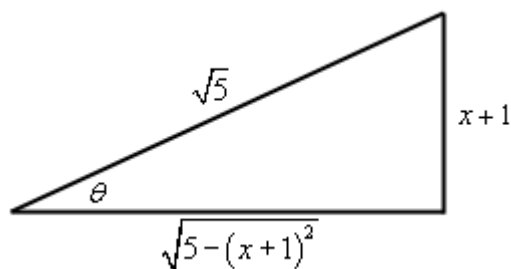
$$\begin{aligned} \int \frac{x-3}{(4-2x-x^2)^2} dx &= -\frac{1}{2} \int \frac{1}{v^2} dv - \int \frac{4}{(5-5\sin^2 \theta)^2} (\sqrt{5} \cos \theta) d\theta \\ &= \frac{1}{2} \frac{1}{v} - \frac{4\sqrt{5}}{25} \int \frac{\cos \theta}{(1-\sin^2 \theta)^2} d\theta \\ &= \frac{1}{2} \frac{1}{v} - \frac{4\sqrt{5}}{25} \int \frac{\cos \theta}{\cos^4 \theta} d\theta \\ &= \frac{1}{2} \frac{1}{v} - \frac{4\sqrt{5}}{25} \int \sec^3 \theta d\theta \\ &= \frac{1}{2} \frac{1}{v} - \frac{2\sqrt{5}}{25} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + c \end{aligned}$$

We'll need the following right triangle to finish this integral out.

$$\sin \theta = \frac{u}{\sqrt{5}} = \frac{x+1}{\sqrt{5}}$$

$$\sec \theta = \frac{\sqrt{5}}{\sqrt{5-(x+1)^2}}$$

$$\tan \theta = \frac{x+1}{\sqrt{5-(x+1)^2}}$$



So, going back to  $x$ 's the integral becomes,

$$\begin{aligned} \int \frac{x-3}{(4-2x-x^2)^2} dx &= \frac{1}{2} \frac{1}{5-u^2} - \frac{2\sqrt{5}}{25} \left( \frac{\sqrt{5}(x+1)}{5-(x+1)^2} + \ln \left| \frac{\sqrt{5}}{\sqrt{5-(x+1)^2}} + \frac{x+1}{\sqrt{5-(x+1)^2}} \right| \right) + c \\ &= \frac{1}{10} \frac{1-4x}{5-(x+1)^2} - \frac{2\sqrt{5}}{25} \ln \left| \frac{x+1+\sqrt{5}}{\sqrt{5-(x+1)^2}} \right| + c \end{aligned}$$

Often the following formula is needed when using the trig substitution that we used in the previous example.

$$\int \sec^m \theta d\theta = \frac{1}{m-1} \tan \theta \sec^{m-2} \theta + \frac{m-2}{m-1} \int \sec^{m-2} \theta d\theta$$

Note that we'll only need the two trig substitutions (sine and tangent) that we used here. The third trig substitution (secant) that we used will not be needed here. Any quadratic that could use a secant substitution can be turned into a sine substitution simply by factoring a minus sign out of the quadratic. Note that we can do that for these types of problems because we don't have a root and so the minus sign can be completely factored out of the integrand while we couldn't do that with the roots we had in the problems back in the Trig Substitution section.

## Section 1-7 : Integration Strategy

---

We've now seen a fair number of different integration techniques and so we should probably pause at this point and talk a little bit about a strategy to use for determining the correct technique to use when faced with an integral.

There are a couple of points that need to be made about this strategy. First, it isn't a hard and fast set of rules for determining the method that should be used. It is really nothing more than a general set of guidelines that will help us to identify techniques that may work. Some integrals can be done in more than one way and so depending on the path you take through the strategy you may end up with a different technique than somebody else who also went through this strategy.

Second, while the strategy is presented as a way to identify the technique that could be used on an integral also keep in mind that, for many integrals, it can also automatically exclude certain techniques as well. When going through the strategy keep two lists in mind. The first list is integration techniques that simply won't work and the second list is techniques that look like they might work. After going through the strategy and the second list has only one entry then that is the technique to use. If, on the other hand, there is more than one possible technique to use we will then have to decide on which is liable to be the best for us to use. Unfortunately, there is no way to teach which technique is the best as that usually depends upon the person and which technique they find to be the easiest.

Third, don't forget that many integrals can be evaluated in multiple ways and so more than one technique may be used on it. This has already been mentioned in each of the previous points but is important enough to warrant a separate mention. Sometimes one technique will be significantly easier than the others and so don't just stop at the first technique that appears to work. Always identify all possible techniques and then go back and determine which you feel will be the easiest for you to use.

Next, it's entirely possible that you will need to use more than one method to completely do an integral. For instance, a substitution may lead to using integration by parts or partial fractions integral.

Finally, in my class I will accept any valid integration technique as a solution. As already noted there is often more than one way to do an integral and just because I find one technique to be the easiest doesn't mean that you will as well. So, in my class, there is no one right way of doing an integral. You may use any integration technique that I've taught you in this class or you learned in Calculus I to evaluate integrals in this class. In other words, always take the approach that you find to be the easiest.

Note that this final point is more geared towards my class and it's completely possible that your instructor may not agree with this and so be careful in applying this point if you aren't in my class.

Okay, let's get on with the strategy.

1. **Simplify the integrand, if possible.** This step is very important in the integration process. Many integrals can be taken from impossible or very difficult to very easy with a little simplification or manipulation. Don't forget basic trig and algebraic identities as these can often be used to simplify the integral.

We used this idea when we were looking at integrals involving trig functions. For example, consider the following integral.

$$\int \cos^2 x \, dx$$

This integral can't be done as is, however simply by recalling the identity,

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

the integral becomes very easy to do.

Note that this example also shows that simplification does not necessarily mean that we'll write the integrand in a "simpler" form. It only means that we'll write the integrand into a form that we can deal with and this is often longer and/or "messier" than the original integral.

2. **See if a "simple" substitution will work.** Look to see if a simple substitution can be used instead of the often more complicated methods from Calculus II. For example, consider both of the following integrals.

$$\int \frac{x}{x^2 - 1} \, dx \qquad \int x\sqrt{x^2 - 1} \, dx$$

The first integral can be done with partial fractions and the second could be done with a trig substitution.

However, both could also be evaluated using the substitution  $u = x^2 - 1$  and the work involved in the substitution would be significantly less than the work involved in either partial fractions or trig substitution.

So, always look for quick, simple substitutions before moving on to the more complicated Calculus II techniques.

3. **Identify the type of integral.** Note that any integral may fall into more than one of these types. Because of this fact it's usually best to go all the way through the list and identify all possible types since one may be easier than the other and it's entirely possible that the easier type is listed lower in the list.
  - a. Is the integrand a rational expression (*i.e* is the integrand a polynomial divided by a polynomial)? If so, then partial fractions may work on the integral.
  - b. Is the integrand a polynomial times a trig function, exponential, or logarithm? If so, then integration by parts may work.



- c. Is the integrand a product of sines and cosines, secant and tangents, or cosecants and cotangents? If so, then the topics from the second section may work. Likewise, don't forget that some quotients involving these functions can also be done using these techniques.
  - d. Does the integrand involve  $\sqrt{b^2x^2 + a^2}$ ,  $\sqrt{b^2x^2 - a^2}$ , or  $\sqrt{a^2 - b^2x^2}$ ? If so, then a trig substitution might work nicely.
  - e. Does the integrand have roots other than those listed above in it? If so, then the substitution  $u = \sqrt[n]{g(x)}$  might work.
  - f. Does the integrand have a quadratic in it? If so, then completing the square on the quadratic might put it into a form that we can deal with.
4. **Can we relate the integral to an integral we already know how to do?** In other words, can we use a substitution or manipulation to write the integrand into a form that does fit into the forms we've looked at previously in this chapter.

A typical example here is the following integral.

$$\int \cos x \sqrt{1 + \sin^2 x} \, dx$$

This integral doesn't obviously fit into any of the forms we looked at in this chapter. However, with the substitution  $u = \sin x$  we can reduce the integral to the form,

$$\int \sqrt{1 + u^2} \, du$$

which is a trig substitution problem.

5. **Do we need to use multiple techniques?** In this step we need to ask ourselves if it is possible that we'll need to use multiple techniques. The example in the previous part is a good example. Using a substitution didn't allow us to actually do the integral. All it did was put the integral and put it into a form that we could use a different technique on.
- Don't ever get locked into the idea that an integral will only require one step to completely evaluate it. Many will require more than one step.
6. **Try again.** If everything that you've tried to this point doesn't work then go back through the process and try again. This time try a technique that you didn't use the first time around.

As noted above this strategy is not a hard and fast set of rules. It is only intended to guide you through the process of best determining how to do any given integral. Note as well that the only place Calculus II actually arises is in the third step. Steps 1, 2 and 4 involve nothing more than manipulation of the integrand either through direct manipulation of the integrand or by using a substitution. The last two steps are simply ideas to think about in going through this strategy.

Many students go through this process and concentrate almost exclusively on Step 3 (after all this is Calculus II, so it's easy to see why they might do that....) to the exclusion of the other steps. One very large consequence of that exclusion is that often a simple manipulation or substitution is overlooked that could make the integral very easy to do.

Before moving on to the next section we should work a couple of quick problems illustrating a couple of not so obvious simplifications/manipulations and a not so obvious substitution.

**Example 1** Evaluate the following integral.

$$\int \frac{\tan x}{\sec^4 x} dx$$

**Solution**

This integral almost falls into the form given in **3c**. It is a quotient of tangent and secant and we know that sometimes we can use the same methods for products of tangents and secants on quotients.

The process from that [section](#) tells us that if we have even powers of secant to strip two of them off and convert the rest to tangents. That won't work here. We can split two secants off, but they would be in the denominator and they won't do us any good there. Remember that the point of splitting them off is so they would be there for the substitution  $u = \tan x$ . That requires them to be in the numerator. So, that won't work and so we'll have to find another solution method.

There are in fact two solution methods to this integral depending on how you want to go about it. We'll take a look at both.

**Solution 1**

In this solution method we could just convert everything to sines and cosines and see if that gives us an integral we can deal with.

$$\begin{aligned} \int \frac{\tan x}{\sec^4 x} dx &= \int \frac{\sin x}{\cos x} \cos^4 x dx \\ &= \int \sin x \cos^3 x dx & u = \cos x \\ &= -\int u^3 du \\ &= -\frac{1}{4} \cos^4 x + c \end{aligned}$$

Note that just converting to sines and cosines won't always work and if it does it won't always work this nicely. Often there will be a lot more work that would need to be done to complete the integral.

**Solution 2**

This solution method goes back to dealing with secants and tangents. Let's notice that if we had a secant in the numerator we could just use  $u = \sec x$  as a substitution and it would be a fairly quick and simple substitution to use. We don't have a secant in the numerator. However, we could very easily get a secant in the numerator simply by multiplying the numerator and denominator by secant.

$$\begin{aligned}
 \int \frac{\tan x}{\sec^4 x} dx &= \int \frac{\tan x \sec x}{\sec^5 x} dx & u &= \sec x \\
 &= \int \frac{1}{u^5} du \\
 &= -\frac{1}{4} \frac{1}{\sec^4 x} + c \\
 &= -\frac{1}{4} \cos^4 x + c
 \end{aligned}$$

In the previous example we saw two “simplifications” that allowed us to do the integral. The first was using identities to rewrite the integral into terms we could deal with and the second involved multiplying the numerator and the denominator by something to again put the integral into terms we could deal with.

Using identities to rewrite an integral is an important “simplification” and we should not forget about it. Integrals can often be greatly simplified or at least put into a form that can be dealt with by using an identity.

The second “simplification” is not used as often, but does show up on occasion so again, it’s best to not forget about it. In fact, let’s take another look at an example in which multiplying the numerator and denominator by something will allow us to do an integral.

**Example 2** Evaluate the following integral.

$$\int \frac{1}{1 + \sin x} dx$$

**Solution**

This is an integral in which if we just concentrate on the third step we won’t get anywhere. This integral doesn’t appear to be any of the kinds of integrals that we worked in this chapter.

We can do the integral however, if we do the following,

$$\begin{aligned}
 \int \frac{1}{1 + \sin x} dx &= \int \frac{1}{1 + \sin x} \frac{1 - \sin x}{1 - \sin x} dx \\
 &= \int \frac{1 - \sin x}{1 - \sin^2 x} dx
 \end{aligned}$$

This does not appear to have done anything for us. However, if we now remember the first “simplification” we looked at above we will notice that we can use an identity to rewrite the denominator. Once we do that we can further reduce the integral into something we can deal with.

$$\begin{aligned}
 \int \frac{1}{1 + \sin x} dx &= \int \frac{1 - \sin x}{\cos^2 x} dx \\
 &= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \frac{1}{\cos x} dx \\
 &= \int \sec^2 x - \tan x \sec x dx \\
 &= \tan x - \sec x + c
 \end{aligned}$$

So, we've seen once again that multiplying the numerator and denominator by something can put the integral into a form that we can integrate. Notice as well that this example also showed that "simplifications" do not necessarily put an integral into a simpler form. They only put the integral into a form that is easier to integrate.

Let's now take a quick look at an example of a substitution that is not so obvious.

**Example 3** Evaluate the following integral.

$$\int \cos(\sqrt{x}) dx$$

**Solution**

We introduced this example saying that the substitution was not so obvious. However, this is really an integral that falls into the form given by 3e in our strategy above. However, many people miss that form and so don't think about it. So, let's try the following substitution.

$$u = \sqrt{x} \quad x = u^2 \quad dx = 2u du$$

With this substitution the integral becomes,

$$\int \cos(\sqrt{x}) dx = 2 \int u \cos u du$$

This is now an integration by parts integral. Remember that often we will need to use more than one technique to completely do the integral. This is a fairly simple integration by parts problem so we'll leave the remainder of the details to you to check.

$$\int \cos(\sqrt{x}) dx = 2 \left( \cos(\sqrt{x}) + \sqrt{x} \sin(\sqrt{x}) \right) + c$$

Before leaving this section we should also point out that there are integrals out there in the world that just can't be done in terms of functions that we know. Some examples of these are.

$$\int e^{-x^2} dx \quad \int \cos(x^2) dx \quad \int \frac{\sin(x)}{x} dx \quad \int \cos(e^x) dx$$

That doesn't mean that these integrals can't be done at some level. If you go to a computer algebra system such as [Maple](#) or [Mathematica](#) and have it do these integrals it will return the following.

$$\int e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$$

$$\int \cos(x^2) dx = \sqrt{\frac{\pi}{2}} \operatorname{FresnelC}\left(x\sqrt{\frac{2}{\pi}}\right)$$

$$\int \frac{\sin(x)}{x} dx = \operatorname{Si}(x)$$

$$\int \cos(e^x) dx = \operatorname{Ci}(e^x)$$

So, it appears that these integrals can in fact be done. However, this is a little misleading. Here are the definitions of each of the functions given above.

### Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

### The Sine Integral

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt$$

### The Fresnel Cosine Integral

$$\operatorname{FresnelC}(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$$

### The Cosine Integral

$$\operatorname{Ci}(x) = \gamma + \ln(x) + \int_0^x \frac{\cos t - 1}{t} dt$$

Where  $\gamma$  is the [Euler-Mascheroni constant](#).

Note that the first three are simply defined in terms of themselves and so when we say we can integrate them all we are really doing is renaming the integral. The fourth one is a little different and yet it is still defined in terms of an integral that can't be done in practice.

It will be possible to integrate every integral given in this class, but it is important to note that there are integrals that just can't be done. We should also note that after we look at [Series](#) we will be able to write down series representations of each of the integrals above.

## Section 1-8 : Improper Integrals

In this section we need to take a look at a couple of different kinds of integrals. Both of these are examples of integrals that are called Improper Integrals.

Let's start with the first kind of improper integrals that we're going to take a look at.

### Infinite Interval

In this kind of integral one or both of the limits of integration are infinity. In these cases, the interval of integration is said to be over an infinite interval.

Let's take a look at an example that will also show us how we are going to deal with these integrals.

**Example 1** Evaluate the following integral.

$$\int_1^{\infty} \frac{1}{x^2} dx$$

### Solution

This is an innocent enough looking integral. However, because infinity is not a real number we can't just integrate as normal and then "plug in" the infinity to get an answer.

To see how we're going to do this integral let's think of this as an area problem. So instead of asking what the integral is, let's instead ask what the area under  $f(x) = \frac{1}{x^2}$  on the interval  $[1, \infty)$  is.

We still aren't able to do this, however, let's step back a little and instead ask what the area under  $f(x)$  is on the interval  $[1, t]$  where  $t > 1$  and  $t$  is finite. This is a problem that we can do.

$$A_t = \int_1^t \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^t = 1 - \frac{1}{t}$$

Now, we can get the area under  $f(x)$  on  $[1, \infty)$  simply by taking the limit of  $A_t$  as  $t$  goes to infinity.

$$A = \lim_{t \rightarrow \infty} A_t = \lim_{t \rightarrow \infty} \left( 1 - \frac{1}{t} \right) = 1$$

This is then how we will do the integral itself.

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{x} \right) \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left( 1 - \frac{1}{t} \right) = 1 \end{aligned}$$

So, this is how we will deal with these kinds of integrals in general. We will replace the infinity with a variable (usually  $t$ ), do the integral and then take the limit of the result as  $t$  goes to infinity.

On a side note, notice that the area under a curve on an infinite interval was not infinity as we might have suspected it to be. In fact, it was a surprisingly small number. Of course, this won't always be the case, but it is important enough to point out that not all areas on an infinite interval will yield infinite areas.

Let's now get some definitions out of the way. We will call these integrals **convergent** if the associated limit exists and is a finite number (*i.e.* it's not plus or minus infinity) and **divergent** if the associated limit either doesn't exist or is (plus or minus) infinity.

Let's now formalize up the method for dealing with infinite intervals. There are essentially three cases that we'll need to look at.

1. If  $\int_a^t f(x) dx$  exists for every  $t > a$  then,

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided the limit exists and is finite.

2. If  $\int_t^b f(x) dx$  exists for every  $t < b$  then,

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

provided the limits exists and is finite.

3. If  $\int_{-\infty}^c f(x) dx$  and  $\int_c^\infty f(x) dx$  are both convergent then,

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$$

Where  $c$  is any number. Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

Let's take a look at a couple more examples.

**Example 2** Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_1^\infty \frac{1}{x} dx$$

**Solution**

So, the first thing we do is convert the integral to a limit.

$$\int_1^\infty \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx$$

Now, do the integral and the limit.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \ln(x) \Big|_1^t \\ &= \lim_{t \rightarrow \infty} (\ln(t) - \ln(1)) \\ &= \infty\end{aligned}$$

So, the limit is infinite and so the integral is divergent.

If we go back to thinking in terms of area notice that the area under  $g(x) = \frac{1}{x}$  on the interval  $[1, \infty)$  is infinite. This is in contrast to the area under  $f(x) = \frac{1}{x^2}$  which was quite small. There really isn't all that much difference between these two functions and yet there is a large difference in the area under them. We can actually extend this out to the following fact.

### Fact

If  $a > 0$  then

$$\int_a^{\infty} \frac{1}{x^p} dx$$

is convergent if  $p > 1$  and divergent if  $p \leq 1$ .

One thing to note about this fact is that it's in essence saying that if an integrand goes to zero fast enough then the integral will converge. How fast is fast enough? If we use this fact as a guide it looks like integrands that go to zero faster than  $\frac{1}{x}$  goes to zero will probably converge.

Let's take a look at a couple more examples.

**Example 3** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx$$

### Solution

There really isn't much to do with these problems once you know how to do them. We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$$\begin{aligned}\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{\sqrt{3-x}} dx \\ &= \lim_{t \rightarrow -\infty} -2\sqrt{3-x} \Big|_t^0 \\ &= \lim_{t \rightarrow -\infty} (-2\sqrt{3} + 2\sqrt{3-t}) \\ &= -2\sqrt{3} + \infty \\ &= \infty\end{aligned}$$

So, the limit is infinite and so this integral is divergent.



**Example 4** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} x e^{-x^2} dx$$

**Solution**

In this case we've got infinities in both limits. The process we are using to deal with the infinite limits requires only one infinite limit in the integral and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose  $x = 0$  since this will be a convenient point for the evaluation process. The integral is then,

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx$$

We've now got to look at each of the individual limits.

$$\begin{aligned} \int_{-\infty}^0 x e^{-x^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 x e^{-x^2} dx \\ &= \lim_{t \rightarrow -\infty} \left( -\frac{1}{2} e^{-x^2} \right) \Big|_t^0 \\ &= \lim_{t \rightarrow -\infty} \left( -\frac{1}{2} + \frac{1}{2} e^{-t^2} \right) \\ &= -\frac{1}{2} \end{aligned}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\begin{aligned} \int_0^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} e^{-x^2} \right) \Big|_0^t \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} e^{-t^2} + \frac{1}{2} \right) \\ &= \frac{1}{2} \end{aligned}$$

This integral is convergent and so since they are both convergent the integral we were actually asked to deal with is also convergent and its value is,

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = 0$$

**Example 5** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{\infty} \sin x \, dx$$

**Solution**

First convert to a limit.

$$\begin{aligned} \int_{-2}^{\infty} \sin x \, dx &= \lim_{t \rightarrow \infty} \int_{-2}^t \sin x \, dx \\ &= \lim_{t \rightarrow \infty} (-\cos x) \Big|_{-2}^t \\ &= \lim_{t \rightarrow \infty} (\cos 2 - \cos t) \end{aligned}$$

This limit doesn't exist and so the integral is divergent.

In most examples in a Calculus II class that are worked over infinite intervals the limit either exists or is infinite. However, there are limits that don't exist, as the previous example showed, so don't forget about those.

**Discontinuous Integrand**

We now need to look at the second type of improper integrals that we'll be looking at in this section. These are integrals that have discontinuous integrands. The process here is basically the same with one subtle difference. Here are the general cases that we'll look at for these integrals.

1. If  $f(x)$  is continuous on the interval  $[a, b)$  and not continuous at  $x = b$  then,

$$\int_a^b f(x) \, dx = \lim_{t \rightarrow b^-} \int_a^t f(x) \, dx$$

provided the limit exists and is finite. Note as well that we do need to use a left-hand limit here since the interval of integration is entirely on the left side of the upper limit.

2. If  $f(x)$  is continuous on the interval  $(a, b]$  and not continuous at  $x = a$  then,

$$\int_a^b f(x) \, dx = \lim_{t \rightarrow a^+} \int_t^b f(x) \, dx$$

provided the limit exists and is finite. In this case we need to use a right-hand limit here since the interval of integration is entirely on the right side of the lower limit.

3. If  $f(x)$  is not continuous at  $x = c$  where  $a < c < b$  and  $\int_a^c f(x) \, dx$  and  $\int_c^b f(x) \, dx$  are both convergent then,

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

As with the infinite interval case this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

4. If  $f(x)$  is not continuous at  $x = a$  and  $x = b$  and if  $\int_a^c f(x) dx$  and  $\int_c^b f(x) dx$  are both convergent then,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Where  $c$  is any number. Again, this requires BOTH of the integrals to be convergent in order for this integral to also be convergent.

Note that the limits in these cases really do need to be right or left-handed limits. Since we will be working inside the interval of integration we will need to make sure that we stay inside that interval. This means that we'll use one-sided limits to make sure we stay inside the interval.

Let's do a couple of examples of these kinds of integrals.

**Example 6** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx$$

**Solution**

The problem point is the upper limit so we are in the first case above.

$$\begin{aligned} \int_0^3 \frac{1}{\sqrt{3-x}} dx &= \lim_{t \rightarrow 3^-} \int_0^t \frac{1}{\sqrt{3-x}} dx \\ &= \lim_{t \rightarrow 3^-} \left( -2\sqrt{3-x} \right) \Big|_0^t \\ &= \lim_{t \rightarrow 3^-} \left( 2\sqrt{3} - 2\sqrt{3-t} \right) \\ &= 2\sqrt{3} \end{aligned}$$

The limit exists and is finite and so the integral converges and the integral's value is  $2\sqrt{3}$ .

**Example 7** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^3 \frac{1}{x^3} dx$$

**Solution**

This integrand is not continuous at  $x = 0$  and so we'll need to split the integral up at that point.

$$\int_{-2}^3 \frac{1}{x^3} dx = \int_{-2}^0 \frac{1}{x^3} dx + \int_0^3 \frac{1}{x^3} dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\begin{aligned}
 \int_{-2}^0 \frac{1}{x^3} dx &= \lim_{t \rightarrow 0^-} \int_{-2}^t \frac{1}{x^3} dx \\
 &= \lim_{t \rightarrow 0^-} \left( -\frac{1}{2x^2} \right) \Big|_{-2}^t \\
 &= \lim_{t \rightarrow 0^-} \left( -\frac{1}{2t^2} + \frac{1}{8} \right) \\
 &= -\infty
 \end{aligned}$$

At this point we're done. One of the integrals is divergent that means the integral that we were asked to look at is divergent. We don't even need to bother with the second integral.

Before leaving this section let's note that we can also have integrals that involve both of these cases. Consider the following integral.

**Example 8** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{\infty} \frac{1}{x^2} dx$$

**Solution**

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals so each integral contains only one point of discontinuity. It is important to remember that all of the processes we are working with in this section so that each integral only contains one problem point.

We can split it up anywhere but pick a value that will be convenient for evaluation purposes.

$$\int_0^{\infty} \frac{1}{x^2} dx = \int_0^1 \frac{1}{x^2} dx + \int_1^{\infty} \frac{1}{x^2} dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent.

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\begin{aligned}
 \int_0^1 \frac{1}{x^2} dx &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x^2} dx \\
 &= \lim_{t \rightarrow 0^+} \left( -\frac{1}{x} \right) \Big|_t^1 \\
 &= \lim_{t \rightarrow 0^+} \left( -1 + \frac{1}{t} \right) \\
 &= \infty
 \end{aligned}$$

So, the first integral is divergent and so the whole integral is divergent.



## Section 1-9 : Comparison Test for Improper Integrals

---

Now that we've seen how to actually compute improper integrals we need to address one more topic about them. Often we aren't concerned with the actual value of these integrals. Instead we might only be interested in whether the integral is convergent or divergent. Also, there will be some integrals that we simply won't be able to integrate and yet we would still like to know if they converge or diverge.

To deal with this we've got a test for convergence or divergence that we can use to help us answer the question of convergence for an improper integral.

We will give this test only for a sub-case of the infinite interval integral, however versions of the test exist for the other sub-cases of the infinite interval integrals as well as integrals with discontinuous integrands.

### Comparison Test

If  $f(x) \geq g(x) \geq 0$  on the interval  $[a, \infty)$  then,

1. If  $\int_a^\infty f(x) dx$  converges then so does  $\int_a^\infty g(x) dx$ .
2. If  $\int_a^\infty g(x) dx$  diverges then so does  $\int_a^\infty f(x) dx$ .

Note that if you think in terms of area the Comparison Test makes a lot of sense. If  $f(x)$  is larger than  $g(x)$  then the area under  $f(x)$  must also be larger than the area under  $g(x)$ .

So, if the area under the larger function is finite (i.e.  $\int_a^\infty f(x) dx$  converges) then the area under the smaller function must also be finite (i.e.  $\int_a^\infty g(x) dx$  converges). Likewise, if the area under the smaller function is infinite (i.e.  $\int_a^\infty g(x) dx$  diverges) then the area under the larger function must also be infinite (i.e.  $\int_a^\infty f(x) dx$  diverges).

Be careful not to misuse this test. If the smaller function converges there is no reason to believe that the larger will also converge (after all infinity is larger than a finite number...) and if the larger function diverges there is no reason to believe that the smaller function will also diverge.

Let's work a couple of examples using the comparison test. Note that all we'll be able to do is determine the convergence of the integral. We won't be able to determine the value of the integrals and so won't even bother with that.

**Example 1** Determine if the following integral is convergent or divergent.

$$\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$$

**Solution**

Let's take a second and think about how the Comparison Test works. If this integral is convergent then we'll need to find a larger function that also converges on the same interval. Likewise, if this integral is divergent then we'll need to find a smaller function that also diverges.

So, it seems like it would be nice to have some idea as to whether the integral converges or diverges ahead of time so we will know whether we will need to look for a larger (and convergent) function or a smaller (and divergent) function.

To get the guess for this function let's notice that the numerator is nice and bounded because we know that,

$$0 \leq \cos^2 x \leq 1$$

Therefore, the numerator simply won't get too large.

So, it seems likely that the denominator will determine the convergence/divergence of this integral and we know that

$$\int_2^{\infty} \frac{1}{x^2} dx$$

converges since  $p = 2 > 1$  by the fact in the previous [section](#). So, let's guess that this integral will converge.

So we now know that we need to find a function that is larger than

$$\frac{\cos^2 x}{x^2}$$

and also converges. Making a fraction larger is actually a fairly simple process. We can either make the numerator larger or we can make the denominator smaller. In this case we can't do a lot about the denominator in a way that will help. However, we can use the fact that  $0 \leq \cos^2 x \leq 1$  to make the numerator larger (*i.e.* we'll replace the cosine with something we know to be larger, namely 1). So,

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

Now, as we've already noted

$$\int_2^{\infty} \frac{1}{x^2} dx$$

converges and so by the Comparison Test we know that

$$\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$$

must also converge.

**Example 2** Determine if the following integral is convergent or divergent.

$$\int_3^{\infty} \frac{1}{x + e^x} dx$$

**Solution**

Let's first take a guess about the convergence of this integral. As noted after the fact in the last section about

$$\int_a^{\infty} \frac{1}{x^p} dx$$

if the integrand goes to zero faster than  $\frac{1}{x}$  then the integral will probably converge. Now, we've got an exponential in the denominator which is approaching infinity much faster than the  $x$  and so it looks like this integral should probably converge.

So, we need a larger function that will also converge. In this case we can't really make the numerator larger and so we'll need to make the denominator smaller in order to make the function larger as a whole. We will need to be careful however. There are two ways to do this and only one, in this case only one, of them will work for us.

First, notice that since the lower limit of integration is 3 we can say that  $x \geq 3 > 0$  and we know that exponentials are always positive. So, the denominator is the sum of two positive terms and if we were to drop one of them the denominator would get smaller. This would in turn make the function larger.

The question then is which one to drop? Let's first drop the exponential. Doing this gives,

$$\frac{1}{x + e^x} < \frac{1}{x}$$

This is a problem however, since

$$\int_3^{\infty} \frac{1}{x} dx$$

diverges by the [fact](#). We've got a larger function that is divergent. This doesn't say anything about the smaller function. Therefore, we chose the wrong one to drop.

Let's try it again and this time let's drop the  $x$ .

$$\frac{1}{x + e^x} < \frac{1}{e^x} = e^{-x}$$

Also,



$$\begin{aligned}\int_3^{\infty} e^{-x} dx &= \lim_{t \rightarrow \infty} \int_3^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} (-e^{-t} + e^{-3}) \\ &= e^{-3}\end{aligned}$$

So,  $\int_3^{\infty} e^{-x} dx$  is convergent. Therefore, by the Comparison test

$$\int_3^{\infty} \frac{1}{x + e^x} dx$$

is also convergent.

**Example 3** Determine if the following integral is convergent or divergent.

$$\int_3^{\infty} \frac{1}{x - e^{-x}} dx$$

**Solution**

This is very similar to the previous example with a couple of very important differences. First, notice that the exponential now goes to zero as  $x$  increases instead of growing larger as it did in the previous example (because of the negative in the exponent). Also note that the exponential is now subtracted off the  $x$  instead of added onto it.

The fact that the exponential goes to zero means that this time the  $x$  in the denominator will probably dominate the term and that means that the integral probably diverges. We will therefore need to find a smaller function that also diverges.

Making fractions smaller is pretty much the same as making fractions larger. In this case we'll need to either make the numerator smaller or the denominator larger.

This is where the second change will come into play. As before we know that both  $x$  and the exponential are positive. However, this time since we are subtracting the exponential from the  $x$  if we were to drop the exponential the denominator will become larger (we will no longer be subtracting a positive number off the  $x$ ) and so the fraction will become smaller. In other words,

$$\frac{1}{x - e^{-x}} > \frac{1}{x}$$

and we know that

$$\int_3^{\infty} \frac{1}{x} dx$$

diverges and so by the Comparison Test we know that

$$\int_3^{\infty} \frac{1}{x - e^{-x}} dx$$

must also diverge.

**Example 4** Determine if the following integral is convergent or divergent.

$$\int_1^{\infty} \frac{1 + 3 \sin^4(2x)}{\sqrt{x}} dx$$

**Solution**

First notice that as with the first example, the numerator in this function is going to be bounded since the sine is never larger than 1. Therefore, since the exponent on the denominator is less than 1 we can guess that the integral will probably diverge. We will need a smaller function that also diverges.

We know that  $0 \leq 3 \sin^4(2x) \leq 3$ . In particular, this term is positive and so if we drop it from the numerator the numerator will get smaller. This gives,

$$\frac{1 + 3 \sin^4(2x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$$

and

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

diverges so by the Comparison Test

$$\int_1^{\infty} \frac{1 + 3 \sin^4(2x)}{\sqrt{x}} dx$$

also diverges.

Up to this point all the examples used on manipulation of either the numerator or the denominator in order to use the Comparison Test. Don't get so locked into that idea that you decide that is all you will ever have to do. Sometimes you will need to manipulate both the numerator and the denominator.

Let's do an example like that.

**Example 5** Determine if the following integral is convergent or divergent.

$$\int_2^{\infty} \frac{1 + \cos^2(x)}{\sqrt{x} [2 - \sin^4(x)]} dx$$

**Solution**

In this case we can notice that because the cosine in the numerator is bounded the numerator will never get too large. Likewise, the sine in the denominator is bounded and so again that term will not get too large or too small.

That leaves only the square root in the denominator and because the exponent is less than one we can guess that the integral will probably diverge. Therefore, we will need a smaller function that also diverges.

We know that  $0 \leq \cos^2(x) \leq 1$ . In particular, this term is positive and so if we drop it from the numerator the numerator will get smaller. This gives,

$$\frac{1 + \cos^2(x)}{\sqrt{x}[2 - \sin^4(x)]} > \frac{1}{\sqrt{x}[2 - \sin^4(x)]}$$

Next, we also know that  $0 \leq \sin^4(x) \leq 1$ . Again, this is a positive term and so if we no longer subtract this off from the 2 the term in the brackets will get larger and so the rational expression will get smaller. This gives,

$$\frac{1 + \cos^2(x)}{\sqrt{x}[2 - \sin^4(x)]} > \frac{1}{\sqrt{x}[2 - \sin^4(x)]} > \frac{1}{2\sqrt{x}}$$

Finally, we know that

$$\int_2^{\infty} \frac{1}{2\sqrt{x}} dx$$

Diverges (the 2 in the denominator will not affect this) so by the Comparison Test

$$\int_2^{\infty} \frac{1 + \cos^2(x)}{\sqrt{x}[2 - \sin^4(x)]} dx$$

also diverges.

Okay, we've seen a few examples of the Comparison Test now. However, most of them worked pretty much the same way. All the functions were rational and all we did for most of them was add or subtract something from the numerator and/or the denominator to get what we want.

Let's take a look at an example that works a little differently so we don't get too locked into these ideas.

**Example 6** Determine if the following integral is convergent or divergent.

$$\int_1^{\infty} \frac{e^{-x}}{x} dx$$

**Solution**

Normally, the presence of just an  $x$  in the denominator would lead us to guess divergent for this integral. However, the exponential in the numerator will approach zero so fast that instead we'll need to guess that this integral converges.

To get a larger function we'll use the fact that we know from the limits of integration that  $x > 1$ . This means that if we just replace the  $x$  in the denominator with 1 (which is always smaller than  $x$ ) we will make the denominator smaller and so the function will get larger.

$$\frac{e^{-x}}{x} < \frac{e^{-x}}{1} = e^{-x}$$

and we can show that

$$\int_1^{\infty} e^{-x} dx$$

converges. In fact, we've already done this for a lower limit of 3 and changing that to a 1 won't change the convergence of the integral. Therefore, by the Comparison Test

$$\int_1^{\infty} \frac{e^{-x}}{x} dx$$

also converges.

We should also really work an example that doesn't involve a rational function since there is no reason to assume that we'll always be working with rational functions.

**Example 7** Determine if the following integral is convergent or divergent.

$$\int_1^{\infty} e^{-x^2} dx$$

**Solution**

We know that exponentials with negative exponents die down to zero very fast so it makes sense to guess that this integral will be convergent. We need a larger function, but this time we don't have a fraction to work with so we'll need to do something different.

We'll take advantage of the fact that  $e^{-x}$  is a decreasing function. This means that

$$x_1 > x_2 \quad \Rightarrow \quad e^{-x_1} < e^{-x_2}$$

In other words, plug in a larger number and the function gets smaller.

From the limits of integration we know that  $x > 1$  and this means that if we square  $x$  it will get larger. Or,

$$x^2 > x \quad \text{provided } x > 1$$

Note that we can only say this since  $x > 1$ . This won't be true if  $x \leq 1$ ! We can now use the fact that  $e^{-x}$  is a decreasing function to get,

$$e^{-x^2} < e^{-x}$$

So,  $e^{-x}$  is a larger function than  $e^{-x^2}$  and we know that

$$\int_1^{\infty} e^{-x} dx$$

converges so by the Comparison Test we also know that

$$\int_1^{\infty} e^{-x^2} dx$$

is convergent.

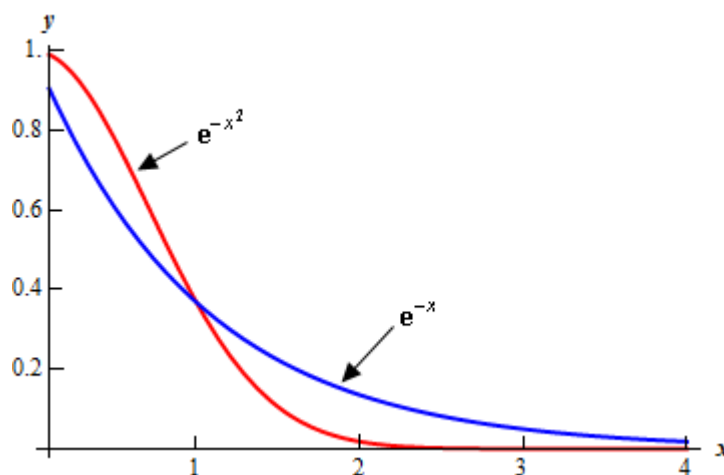
The last two examples made use of the fact that  $x > 1$ . Let's take a look at an example to see how we would have to go about these if the lower limit had been smaller than 1.

**Example 8** Determine if the following integral is convergent or divergent.

$$\int_{\frac{1}{2}}^{\infty} e^{-x^2} dx$$

**Solution**

First, we need to note that  $e^{-x^2} \leq e^{-x}$  is only true on the interval  $[1, \infty)$  as is illustrated in the graph below.



So, we can't just proceed as we did in the previous example with the Comparison Test on the interval  $[\frac{1}{2}, \infty)$ . However, this isn't the problem it might at first appear to be. We can always write the integral as follows,

$$\begin{aligned} \int_{\frac{1}{2}}^{\infty} e^{-x^2} dx &= \int_{\frac{1}{2}}^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx \\ &= 0.28554 + \int_1^{\infty} e^{-x^2} dx \end{aligned}$$

We used [Mathematica](#) to get the value of the first integral. Now, if the second integral converges it will have a finite value and so the sum of two finite values will also be finite and so the original integral will converge. Likewise, if the second integral diverges it will either be infinite or not have a value at all and adding a finite number onto this will not all of a sudden make it finite or exist and so the original integral will diverge. Therefore, this integral will converge or diverge depending only on the convergence of the second integral.

As we saw in Example 7 the second integral does converge and so the whole integral must also converge.

As we saw in this example, if we need to, we can split the integral up into one that doesn't involve any problems and can be computed and one that may contain a problem that we can use the Comparison Test on to determine its convergence.

## Section 1-10 : Approximating Definite Integrals

In this chapter we've spent quite a bit of time on computing the values of integrals. However, not all integrals can be computed. A perfect example is the following definite integral.

$$\int_0^2 e^{x^2} dx$$

We now need to talk a little bit about estimating values of definite integrals. We will look at three different methods, although one should already be familiar to you from your Calculus I days. We will develop all three methods for estimating

$$\int_a^b f(x) dx$$

by thinking of the integral as an area problem and using known shapes to estimate the area under the curve.

Let's get first develop the methods and then we'll try to estimate the integral shown above.

### Midpoint Rule

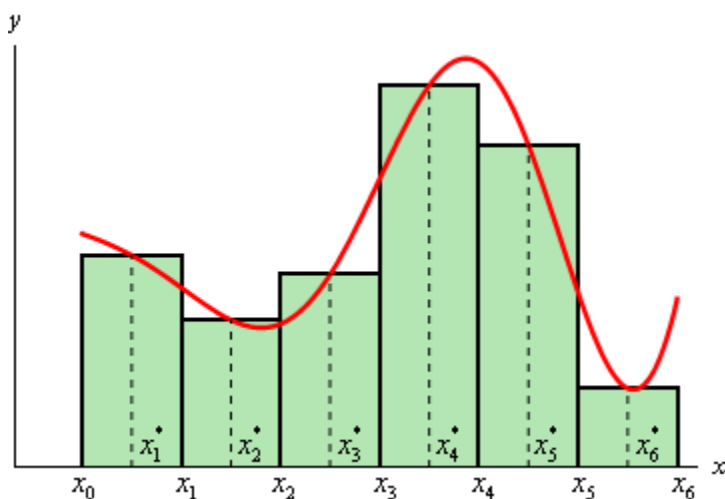
This is the rule that should be somewhat familiar to you. We will divide the interval  $[a, b]$  into  $n$  subintervals of equal width,

$$\Delta x = \frac{b-a}{n}$$

We will denote each of the intervals as follows,

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n] \quad \text{where } x_0 = a \text{ and } x_n = b$$

Then for each interval let  $x_i^*$  be the midpoint of the interval. We then sketch in rectangles for each subinterval with a height of  $f(x_i^*)$ . Here is a graph showing the set up using  $n = 6$ .



We can easily find the area for each of these rectangles and so for a general  $n$  we get that,

$$\int_a^b f(x) dx \approx \Delta x f(x_1^*) + \Delta x f(x_2^*) + \dots + \Delta x f(x_n^*)$$

Or, upon factoring out a  $\Delta x$  we get the general Midpoint Rule.

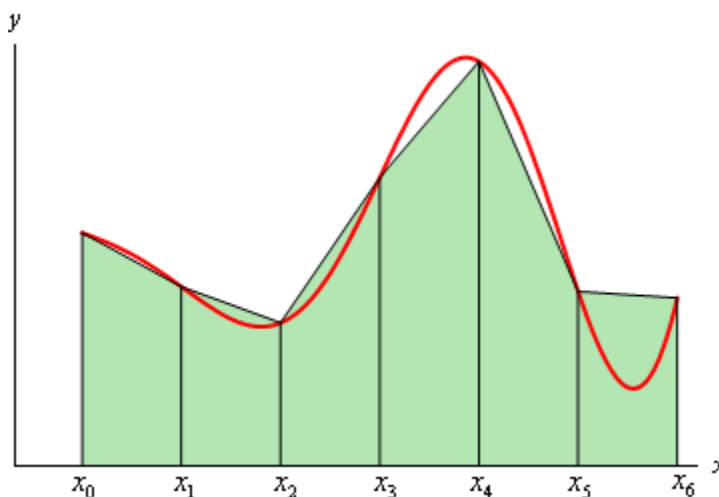
$$\int_a^b f(x) dx \approx \Delta x [f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

### Trapezoid Rule

For this rule we will do the same set up as for the Midpoint Rule. We will break up the interval  $[a, b]$  into  $n$  subintervals of width,

$$\Delta x = \frac{b-a}{n}$$

Then on each subinterval we will approximate the function with a straight line that is equal to the function values at either endpoint of the interval. Here is a sketch of this case for  $n = 6$ .



Each of these objects is a trapezoid (hence the rule's name...) and as we can see some of them do a very good job of approximating the actual area under the curve and others don't do such a good job.

The area of the trapezoid in the interval  $[x_{i-1}, x_i]$  is given by,

$$A_i = \frac{\Delta x}{2} (f(x_{i-1}) + f(x_i))$$

So, if we use  $n$  subintervals the integral is approximately,

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} (f(x_0) + f(x_1)) + \frac{\Delta x}{2} (f(x_1) + f(x_2)) + \cdots + \frac{\Delta x}{2} (f(x_{n-1}) + f(x_n))$$

Upon doing a little simplification we arrive at the general Trapezoid Rule.

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

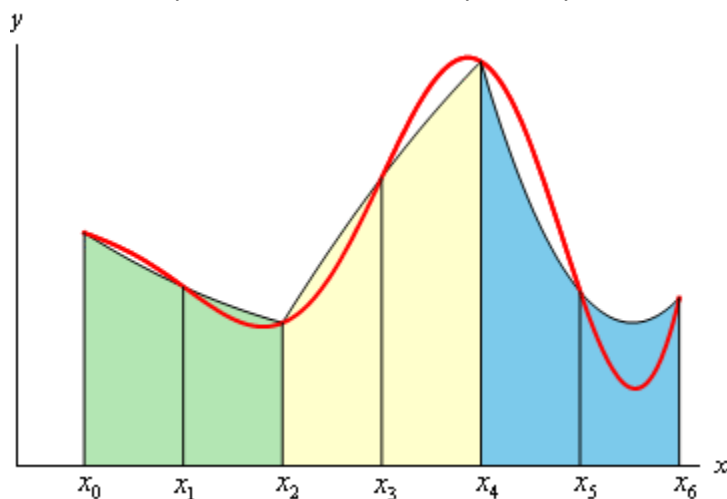
Note that all the function evaluations, with the exception of the first and last, are multiplied by 2.

### Simpson's Rule

This is the final method we're going to take a look at and in this case we will again divide up the interval  $[a, b]$  into  $n$  subintervals. However, unlike the previous two methods we need to require that  $n$  be even. The reason for this will be evident in a bit. The width of each subinterval is,

$$\Delta x = \frac{b-a}{n}$$

In the Trapezoid Rule we approximated the curve with a straight line. For Simpson's Rule we are going to approximate the function with a quadratic and we're going to require that the quadratic agree with three of the points from our subintervals. Below is a sketch of this using  $n = 6$ . Each of the approximations is colored differently so we can see how they actually work.



Notice that each approximation actually covers two of the subintervals. This is the reason for requiring  $n$  to be even. Some of the approximations look more like a line than a quadratic, but they really are quadratics. Also note that some of the approximations do a better job than others. It can be shown that the area under the approximation on the intervals  $[x_{i-1}, x_i]$  and  $[x_i, x_{i+1}]$  is,

$$A_i = \frac{\Delta x}{3} (f(x_{i-1}) + 4f(x_i) + f(x_{i+1}))$$

If we use  $n$  subintervals the integral is then approximately,

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + f(x_2)) + \frac{\Delta x}{3} (f(x_2) + 4f(x_3) + f(x_4)) \\ &\quad + \cdots + \frac{\Delta x}{3} (f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) \end{aligned}$$

Upon simplifying we arrive at the general Simpson's Rule.

$$\boxed{\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]}$$



In this case notice that all the function evaluations at points with odd subscripts are multiplied by 4 and all the function evaluations at points with even subscripts (except for the first and last) are multiplied by 2. If you can remember this, this is a fairly easy rule to remember.

Okay, it's time to work an example and see how these rules work.

**Example 1** Using  $n = 4$  and all three rules to approximate the value of the following integral.

$$\int_0^2 e^{x^2} dx$$

**Solution**

First, for reference purposes, Mathematica gives the following value for this integral.

$$\int_0^2 e^{x^2} dx = 16.45262776$$

In each case the width of the subintervals will be,

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

and so the subintervals will be,

$$[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$$

Let's go through each of the methods.

**Midpoint Rule**

$$\int_0^2 e^{x^2} dx \approx \frac{1}{2} \left( e^{(0.25)^2} + e^{(0.75)^2} + e^{(1.25)^2} + e^{(1.75)^2} \right) = 14.48561253$$

Remember that we evaluate at the midpoints of each of the subintervals here! The Midpoint Rule has an error of 1.96701523.

**Trapezoid Rule**

$$\int_0^2 e^{x^2} dx \approx \frac{1/2}{2} \left( e^{(0)^2} + 2e^{(0.5)^2} + 2e^{(1)^2} + 2e^{(1.5)^2} + e^{(2)^2} \right) = 20.64455905$$

The Trapezoid Rule has an error of 4.19193129

**Simpson's Rule**

$$\int_0^2 e^{x^2} dx \approx \frac{1/2}{3} \left( e^{(0)^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2} \right) = 17.35362645$$

The Simpson's Rule has an error of 0.90099869.

None of the estimations in the previous example are all that good. The best approximation in this case is from the Simpson's Rule and yet it still had an error of almost 1. To get a better estimation we would need to use a larger  $n$ . So, for completeness sake here are the estimates for some larger value of  $n$ .

$n$	Midpoint		Trapezoid		Simpson's	
	Approx.	Error	Approx.	Error	Approx.	Error
8	15.9056767	0.5469511	17.5650858	1.1124580	16.5385947	0.0859669
16	16.3118539	0.1407739	16.7353812	0.2827535	16.4588131	0.0061853
32	16.4171709	0.0354568	16.5236176	0.0709898	16.4530297	0.0004019
64	16.4437469	0.0088809	16.4703942	0.0177665	16.4526531	0.0000254
128	16.4504065	0.0022212	16.4570706	0.0044428	16.4526294	0.0000016

In this case we were able to determine the error for each estimate because we could get our hands on the exact value. Often this won't be the case and so we'd next like to look at error bounds for each estimate.

These bounds will give the largest possible error in the estimate, but it should also be pointed out that the actual error may be significantly smaller than the bound. The bound is only there so we can say that we know the actual error will be less than the bound.

So, suppose that  $|f''(x)| \leq K$  and  $|f^{(4)}(x)| \leq M$  for  $a \leq x \leq b$  then if  $E_M$ ,  $E_T$ , and  $E_S$  are the actual errors for the Midpoint, Trapezoid and Simpson's Rule we have the following bounds,

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2} \quad |E_S| \leq \frac{M(b-a)^5}{180n^4}$$

**Example 2** Determine the error bounds for the estimations in the last example.

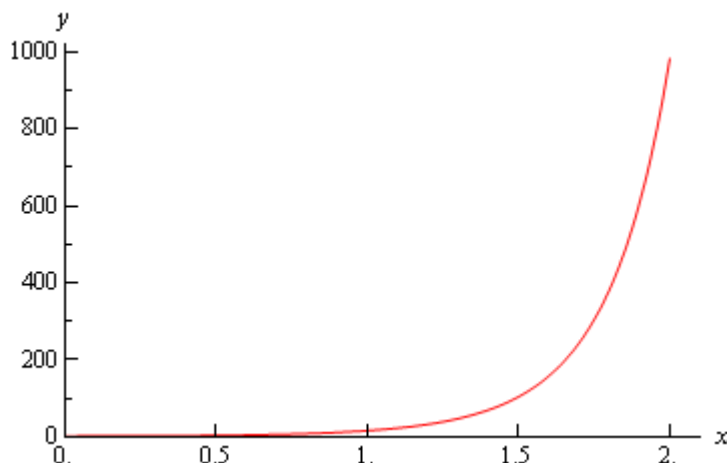
**Solution**

We already know that  $n = 4$ ,  $a = 0$ , and  $b = 2$  so we just need to compute  $K$  (the largest value of the second derivative) and  $M$  (the largest value of the fourth derivative). This means that we'll need the second and fourth derivative of  $f(x)$ .

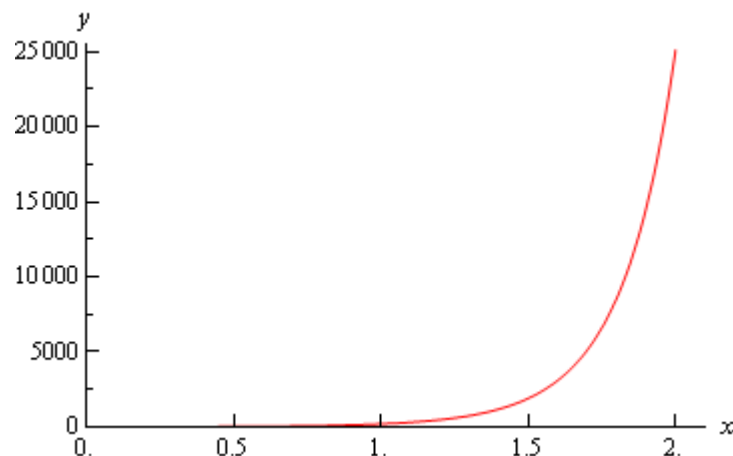
$$f''(x) = 2e^{x^2}(1 + 2x^2)$$

$$f^{(4)}(x) = 4e^{x^2}(3 + 12x^2 + 4x^4)$$

Here is a graph of the second derivative.



Here is a graph of the fourth derivative.



So, from these graphs it's clear that the largest value of both of these are at  $x = 2$ . So,

$$f''(2) = 982.7667 \quad \Rightarrow \quad K = 983$$

$$f^{(4)}(2) = 25115.14901 \quad \Rightarrow \quad M = 25116$$

We rounded to make the computations simpler. Note however, that this does not need to be done.

Here are the bounds for each rule.

$$|E_M| \leq \frac{983(2-0)^3}{24(4)^2} = 20.4791666667$$

$$|E_T| \leq \frac{983(2-0)^3}{12(4)^2} = 40.9583333333$$

$$|E_S| \leq \frac{25116(2-0)^5}{180(4)^4} = 17.4416666667$$

In each case we can see that the errors are significantly smaller than the actual bounds.