CALCULUS I

Derivatives

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Preface

Here are the notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self-contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 3 : Derivatives

In this chapter we will start looking at the next major topic in a calculus class, derivatives. This chapter is devoted almost exclusively to finding derivatives. We will be looking at one application of them in this chapter. We will be leaving most of the applications of derivatives to the next chapter.

Here is a listing of the topics covered in this chapter.

The Definition of the Derivative **–** In this section we define the derivative, give various notations for the derivative and work a few problems illustrating how to use the definition of the derivative to actually compute the derivative of a function.

Interpretation of the Derivative – In this section we give several of the more important interpretations of the derivative. We discuss the rate of change of a function, the velocity of a moving object and the slope of the tangent line to a graph of a function.

Differentiation Formulas – In this section we give most of the general derivative formulas and properties used when taking the derivative of a function. Examples in this section concentrate mostly on polynomials, roots and more generally variables raised to powers.

Product and Quotient Rule – In this section we will give two of the more important formulas for differentiating functions. We will discuss the Product Rule and the Quotient Rule allowing us to differentiate functions that, up to this point, we were unable to differentiate.

Derivatives of Trig Functions – In this section we will discuss differentiating trig functions. Derivatives of all six trig functions are given and we show the derivation of the derivative of sin(x) and tan(x).

Derivatives of Exponential and Logarithm Functions – In this section we derive the formulas for the derivatives of the exponential and logarithm functions.

Derivatives of Inverse Trig Functions – In this section we give the derivatives of all six inverse trig functions. We show the derivation of the formulas for inverse sine, inverse cosine and inverse tangent.

Derivatives of Hyperbolic Functions – In this section we define the hyperbolic functions, give the relationships between them and some of the basic facts involving hyperbolic functions. We also give the derivatives of each of the six hyperbolic functions and show the derivation of the formula for hyperbolic sine.

Chain Rule – In this section we discuss one of the more useful and important differentiation formulas, The Chain Rule. With the chain rule in hand we will be able to differentiate a much wider variety of functions. As you will see throughout the rest of your Calculus courses a great many of derivatives you take will involve the chain rule!

Implicit Differentiation – In this section we will discuss implicit differentiation. Not every function can be explicitly written in terms of the independent variable, e.g. y = f(x) and yet we will still need to know what f'(x) is. Implicit differentiation will allow us to find the derivative in these cases. Knowing implicit differentiation will allow us to do one of the more important applications of derivatives, Related Rates (the next section).

Related Rates – In this section we will discuss the only application of derivatives in this section, Related Rates. In related rates problems we are give the rate of change of one quantity in a problem and asked to determine the rate of one (or more) quantities in the problem. This is often one of the more difficult sections for students. We work quite a few problems in this section so hopefully by the end of this section you will get a decent understanding on how these problems work.

Higher Order Derivatives – In this section we define the concept of higher order derivatives and give a quick application of the second order derivative and show how implicit differentiation works for higher order derivatives.

Logarithmic Differentiation – In this section we will discuss logarithmic differentiation. Logarithmic differentiation gives an alternative method for differentiating products and quotients (sometimes easier than using product and quotient rule). More importantly, however, is the fact that logarithm differentiation allows us to differentiate functions that are in the form of one function raised to another function, i.e. there are variables in both the base and exponent of the function.

Section 3-1 : The Definition of the Derivative

In the first **section** of the Limits chapter we saw that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

This is such an important limit and it arises in so many places that we give it a name. We call it a **derivative**. Here is the official definition of the derivative.

Definition of the Derivative

The **derivative of** $f(x)$ **with respect to *x*** is the function $f'(x)$ and is defined as,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

Note that we replaced all the *a*'s in (1) with *x*'s to acknowledge the fact that the derivative is really a function as well. We often "read" $f'(x)$ as "*f* prime of *x*".

Let's compute a couple of derivatives using the definition.

Example 1 Find the derivative of the following function using the definition of the derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution

So, all we really need to do is to plug this function into the definition of the derivative, (2), and do some algebra. While, admittedly, the algebra will get somewhat unpleasant at times, but it's just algebra so don't get excited about the fact that we're now computing derivatives.

First plug the function into the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \end{aligned}$$

Be careful and make sure that you properly deal with parenthesis when doing the subtracting.

Now, we know from the previous chapter that we can't just plug in $h = 0$ since this will give us a division by zero error. So, we are going to have to do some work. In this case that means multiplying everything out and distributing the minus sign through on the second term. Doing this gives,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 16h}{h} \end{aligned}$$

Notice that every term in the numerator that didn't have an *h* in it canceled out and we can now factor an *h* out of the numerator which will cancel against the *h* in the denominator. After that we can compute the limit.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{h(4x + 2h - 16)}{h} \\ &= \lim_{h \to 0} 4x + 2h - 16 \\ &= 4x - 16 \end{aligned}$$

So, the derivative is,

$$f'(x) = 4x - 16$$

Example 2 Find the derivative of the following function using the definition of the derivative.

$$g(t) = \frac{t}{t+1}$$

Solution

This one is going to be a little messier as far as the algebra goes. However, outside of that it will work in exactly the same manner as the previous examples. First, we plug the function into the definition of the derivative,

$$\begin{aligned} g'(t) &= \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left(\frac{t+h}{t+h+1} - \frac{t}{t+1} \right) \end{aligned}$$

Note that we changed all the letters in the definition to match up with the given function. Also note that we wrote the fraction a much more compact manner to help us with the work.

As with the first problem we can't just plug in $h = 0$. So, we will need to simplify things a little. In this case we will need to combine the two terms in the numerator into a single rational expression as follows.

$$g'(t) = \lim_{h\to 0}\frac{1}{h}\left(\frac{(t+h)(t+1)-t(t+h+1)}{(t+h+1)(t+1)}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{t^2+t+th+h-(t^2+th+t)}{(t+h+1)(t+1)}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{h}{(t+h+1)(t+1)}\right)$$

Before finishing this let's note a couple of things. First, we didn't multiply out the denominator. Multiplying out the denominator will just overly complicate things so let's keep it simple. Next, as with the first example, after the simplification we only have terms with *h*'s in them left in the numerator and so we can now cancel an *h* out.

So, upon canceling the *h* we can evaluate the limit and get the derivative.

$$g'(t) = \lim_{h\to 0}\frac{1}{(t+h+1)(t+1)}$$
$$= \frac{1}{(t+1)(t+1)}$$
$$= \frac{1}{(t+1)^2}$$

The derivative is then,

$$g'(t) = \frac{1}{(t+1)^2}$$

Example 3 Find the derivative of the following function using the definition of the derivative.

$$R(z) = \sqrt{5z-8}$$

Solution

First plug into the definition of the derivative as we've done with the previous two examples.

$$R'(z) = \lim_{h\to 0}\frac{R(z+h)-R(z)}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{5(z+h)-8}-\sqrt{5z-8}}{h}$$

In this problem we're going to have to rationalize the numerator. You do remember **rationalization** from an Algebra class right? In an Algebra class you probably only rationalized the denominator, but you can also rationalize numerators. Remember that in rationalizing the numerator (in this case) we multiply both the numerator and denominator by the numerator except we change the sign between the two terms. Here's the rationalizing work for this problem,

$$R'(z) = \lim_{h \to 0} \frac{\left(\sqrt{5(z+h)-8} - \sqrt{5z-8}\right)}{h} \frac{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}{\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h \to 0} \frac{5z+5h-8-(5z-8)}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

$$= \lim_{h \to 0} \frac{5h}{h\left(\sqrt{5(z+h)-8} + \sqrt{5z-8}\right)}$$

Again, after the simplification we have only h's left in the numerator. So, cancel the h and evaluate the limit.

$$R'(z) = \lim_{h \to 0} \frac{5}{\sqrt{5(z+h)-8} + \sqrt{5z-8}}$$

$$= \frac{5}{\sqrt{5z-8} + \sqrt{5z-8}}$$

$$= \frac{5}{2\sqrt{5z-8}}$$

And so we get a derivative of,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

Let's work one more example. This one will be a little different, but it's got a point that needs to be made.

Example 4 Determine $f'(0)$ for $f(x) = |x|$

Solution

Since this problem is asking for the derivative at a specific point we'll go ahead and use that in our work. It will make our life easier and that's always a good thing.

So, plug into the definition and simplify.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

We saw a situation like this back when we were looking at **limits at infinity**. As in that section we can't just cancel the h's. We will have to look at the two one sided limits and recall that

$$|h| = \begin{cases} h & \text{if } h \ge 0 \\ -h & \text{if } h < 0 \end{cases}$$

$$\begin{aligned} \lim_{h \to 0^-} \frac{|h|}{h} &= \lim_{h \to 0^-} \frac{-h}{h} \qquad \text{because } h < 0 \text{ in a left-hand limit.} \\ &= \lim_{h \to 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{|h|}{h} &= \lim_{h \to 0^+} \frac{h}{h} \qquad \text{because } h > 0 \text{ in a right-hand limit.} \\ &= \lim_{h \to 0^+} 1 \\ &= 1 \end{aligned}$$

The two one-sided limits are different and so

$$\lim_{h \to 0} \frac{|h|}{h}$$

doesn't exist. However, this is the limit that gives us the derivative that we're after.

If the limit doesn't exist then the derivative doesn't exist either.

In this example we have finally seen a function for which the derivative doesn't exist at a point. This is a fact of life that we've got to be aware of. Derivatives will not always exist. Note as well that this doesn't say anything about whether or not the derivative exists anywhere else. In fact, the derivative of the absolute value function exists at every point except the one we just looked at, $x = 0$.

The preceding discussion leads to the following definition.

Definition

A function $f(x)$ is called **differentiable** at $x = a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval.

The next theorem shows us a very nice relationship between functions that are continuous and those that are differentiable.

Theorem

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

See the **Proof of Various Derivative Formulas** section of the Extras chapter to see the proof of this theorem.

Note that this theorem does not work in reverse. Consider $f(x) = |x|$ and take a look at,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0 = f(0)$$

So, $f(x) = |x|$ is continuous at $x = 0$ but we've just shown above in Example 4 that $f(x) = |x|$ is not differentiable at $x = 0$.

Alternate Notation

Next, we need to discuss some alternate notation for the derivative. The typical derivative notation is the "prime" notation. However, there is another notation that is used on occasion so let's cover that.

Given a function $y = f(x)$ all of the following are equivalent and represent the derivative of $f(x)$ with respect to *x*.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\big(f(x)\big) = \frac{d}{dx}(y)$$

Because we also need to evaluate derivatives on occasion we also need a notation for evaluating derivatives when using the fractional notation. So, if we want to evaluate the derivative at $x = a$ all of the following are equivalent.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

Note as well that on occasion we will drop the (x) part on the function to simplify the notation somewhat. In these cases the following are equivalent.

$$f'(x) = f'$$

As a final note in this section we'll acknowledge that computing most derivatives directly from the definition is a fairly complex (and sometimes painful) process filled with opportunities to make mistakes. In a couple of sections we'll start developing formulas and/or properties that will help us to take the derivative of many of the common functions so we won't need to resort to the definition of the derivative too often.

This does not mean however that it isn't important to know the definition of the derivative! It is an important definition that we should always know and keep in the back of our minds. It is just something that we're not going to be working with all that much.

Section 3-2 : Interpretation of the Derivative

Before moving on to the section where we learn how to compute derivatives by avoiding the limits we were evaluating in the previous section we need to take a quick look at some of the interpretations of the derivative. All of these interpretations arise from recalling how our definition of the derivative came about. The definition came about by noticing that all the problems that we worked in the first **section** in the Limits chapter required us to evaluate the same limit.

Rate of Change

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at in the Limits chapter, but it is the most important interpretation of the derivative. If $f\left(x\right)$ represents a quantity at any *x* then the derivative $f'\left(a\right)$ represents the instantaneous rate of change of $f\left(x\right)$ at $x = a$.

Example 1 Suppose that the amount of water in a holding tank at *t* minutes is given by $V\left(t\right) = 2t^2 - 16t + 35$. Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?

(d) Is the volume of water in the tank ever not changing? If so, when?

Solution

In the solution to this example we will use both notations for the derivative just to get you familiar with the different notations.

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time *t*. Now, notice that the function giving the volume of water in the tank is the same function that we saw in Example 1 in the last **section** except the letters have changed. The change in letters between the function in this example versus the function in the example from the last section won't affect the work and so we can just use the answer from that example with an appropriate change in letters.

The derivative is.

$$V'\left(t\right) = 4t - 16 \qquad \text{OR} \qquad \frac{dV}{dt} = 4t - 16$$

Recall from our work in the first limits section that we determined that if the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

We can now work the problem.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?
In this case all that we need is the rate of change of the volume at $t = 1$ or,

$$V'(1) = -12 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?
Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \qquad \text{OR} \qquad \left.\frac{dV}{dt}\right|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?
To answer this question all that we look at is the size of the rate of change and we don't worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at $t = 1$ than at $t = 5$.

(d) Is the volume of water in the tank ever not changing? If so, when?
The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \qquad \text{OR} \qquad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \qquad \Rightarrow \qquad t = 4$$

So at $t = 4$ the volume isn't changing. Note that all this is saying is that for a brief instant the volume isn't changing. It doesn't say that at this point the volume will quit changing permanently.

If we go back to our answers from parts (a) and (b) we can get an idea about what is going on. At $t = 1$ the volume is decreasing and at $t = 5$ the volume is increasing. So, at some point in time the volume needs to switch from decreasing to increasing. That time is $t = 4$.

This is the time in which the volume goes from decreasing to increasing and so for the briefest instant in time the volume will quit changing as it changes from decreasing to increasing.

Note that one of the more common mistakes that students make in these kinds of problems is to try and determine increasing/decreasing from the function values rather than the derivatives. In this case if we took the function values at $t = 0$, $t = 1$ and $t = 5$ we would get,

$$V(0) = 35 \qquad V(1) = 21 \qquad V(5) = 5$$

Clearly as we go from $t=0$ to $t=1$ the volume has decreased. This might lead us to decide that AT $t=1$ the volume is decreasing. However, we just can't say that. All we can say is that between $t=0$ and $t=1$ the volume has decreased at some point in time. The only way to know what is happening right at $t=1$ is to compute $V'(1)$ and look at its sign to determine increasing/decreasing. In this case $V'(1)$ is negative and so the volume really is decreasing at $t=1$.

Now, if we'd plugged into the function rather than the derivative we would have gotten the correct answer for $t=1$ even though our reasoning would have been wrong. It's important to not let this give you the idea that this will always be the case. It just happened to work out in the case of $t=1$.

To see that this won't always work let's now look at $t=5$. If we plug $t=1$ and $t=5$ into the volume we can see that again as we go from $t=1$ to $t=5$ the volume has decreased. Again, however all this says is that the volume HAS decreased somewhere between $t=1$ and $t=5$. It does NOT say that the volume is decreasing at $t=5$. The only way to know what is going on right at $t=5$ is to compute $V'(5)$ and in this case $V'(5)$ is positive and so the volume is actually increasing at $t=5$.

So, be careful. When asked to determine if a function is increasing or decreasing at a point make sure and look at the derivative. It is the only sure way to get the correct answer. We are not looking to determine is the function has increased/decreased by the time we reach a particular point. We are looking to determine if the function is increasing/decreasing at that point in question.

Slope of Tangent Line

This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x=a$ is $f'(a)$. The tangent line then is given by,

$$y=f(a)+f'(a)(x-a)$$

Example 2 Find the tangent line to the following function at $z=3$.

$$R(z)=\sqrt{5z-8}$$

Solution

We first need the derivative of the function and we found that in Example 3 in the last **section**. The derivative is,

$$R'(z)=\frac{5}{2\sqrt{5z-8}}$$

Now all that we need is the function value and derivative (for the slope) at $z=3$.

$$R(3)=\sqrt{7} \qquad\qquad m=R'(3)=\frac{5}{2\sqrt{7}}$$

The tangent line is then,

$$y=\sqrt{7}+\frac{5}{2\sqrt{7}}(z-3)$$

Velocity

Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after *t* units of time the velocity of the object at $t = a$ is given by $f'(a)$.

Example 3 Suppose that the position of an object after *t* hours is given by,

$$g(t) = \frac{t}{t+1}$$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at $t = 10$ hours?

(b) Does the object ever stop moving?

Solution

Once again, we need the derivative and we found that in Example 2 in the last **section**. The derivative is,

$$g'(t) = \frac{1}{(t+1)^2}$$

(a) Is the object moving to the right or the left at $t = 10$ hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is negative) we need the derivative at $t = 10$.

$$g'(10) = \frac{1}{121}$$

So, the velocity at $t = 10$ is positive and so the object is moving to the right at $t = 10$.

(b) Does the object ever stop moving?

The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can't be zero.

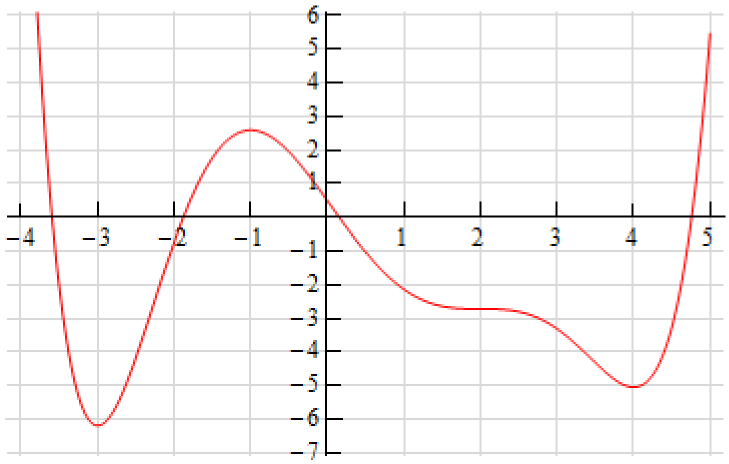
Therefore, the object will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

We've seen three major interpretations of the derivative here. You will need to remember these, especially the rate of change, as they will show up continually throughout this course.

Before we leave this section let's work one more example that encompasses some of the ideas discussed here and is just a nice example to work.

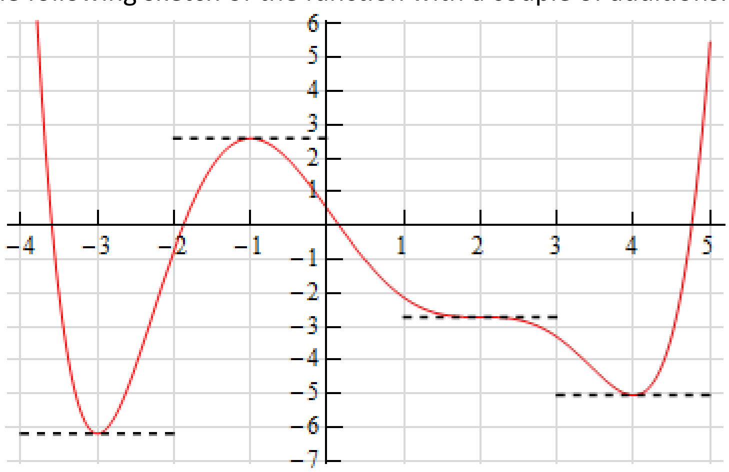
Example 4 Below is the sketch of a function $f(x)$. Sketch the graph of the derivative of this function, $f'(x)$.



Solution

At first glance this seems to an all but impossible task. However, if you have some basic knowledge of the interpretations of the derivative you can get a sketch of the derivative. It will not be a perfect sketch for the most part, but you should be able to get most of the basic features of the derivative in the sketch.

Let's start off with the following sketch of the function with a couple of additions.



Notice that at $x=-3$, $x=-1$, $x=2$ and $x=4$ the tangent line to the function is horizontal. This means that the slope of the tangent line must be zero. Now, we know that the slope of the tangent line at a particular point is also the value of the derivative of the function at that point. Therefore, we now know that,

$$f'(-3)=0 \qquad f'(-1)=0 \qquad f'(2)=0 \qquad f'(4)=0$$

This is a good starting point for us. It gives us a few points on the graph of the derivative. It also breaks the domain of the function up into regions where the function is increasing and decreasing. We know, from our discussions above, that if the function is increasing at a point then the derivative must be positive at that point. Likewise, we know that if the function is decreasing at a point then the derivative must be negative at that point.

We can now give the following information about the derivative.

$$\begin{array}{ll} x < -3 & f'(x) < 0 \\ -3 < x < -1 & f'(x) > 0 \\ -1 < x < 2 & f'(x) < 0 \\ 2 < x < 4 & f'(x) < 0 \\ x > 4 & f'(x) > 0 \end{array}$$

Remember that we are giving the signs of the derivatives here and these are solely a function of whether the function is increasing or decreasing. The sign of the function itself is completely immaterial here and will not in any way effect the sign of the derivative.

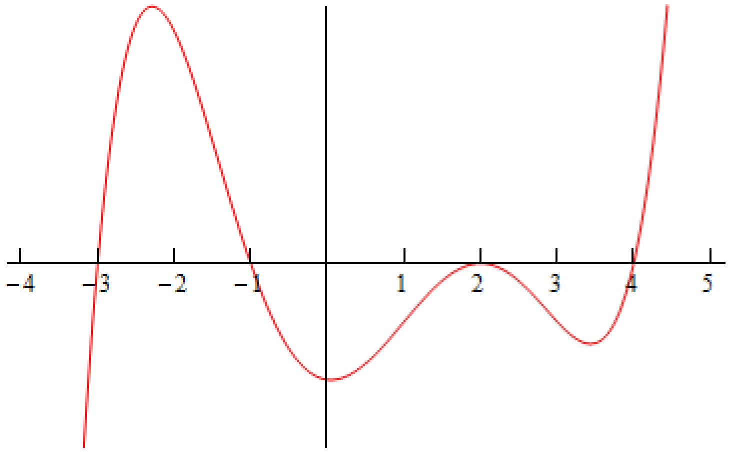
This may still seem like we don't have enough information to get a sketch, but we can get a little bit more information about the derivative from the graph of the function. In the range $x < -3$ we know that the derivative must be negative, however we can also see that the derivative needs to be increasing in this range. It is negative here until we reach $x = -3$ and at this point the derivative must be zero. The only way for the derivative to be negative to the left of $x = -3$ and zero at $x = -3$ is for the derivative to increase as we increase *x* towards $x = -3$.

Now, in the range $-3 < x < -1$ we know that the derivative must be zero at the endpoints and positive in between the two endpoints. Directly to the right of $x = -3$ the derivative must also be increasing (because it starts at zero and then goes positive – therefore it must be increasing). So, the derivative in this range must start out increasing and must eventually get back to zero at $x = -1$. So, at some point in this interval the derivative must start decreasing before it reaches $x = -1$. Now, we have to be careful here because this is just general behavior here at the two endpoints. We won't know where the derivative goes from increasing to decreasing and it may well change between increasing and decreasing several times before we reach $x = -1$. All we can really say is that immediately to the right of $x = -3$ the derivative will be increasing and immediately to the left of $x = -1$ the derivative will be decreasing.

Next, for the ranges $-1 < x < 2$ and $2 < x < 4$ we know the derivative will be zero at the endpoints and negative in between. Also, following the type of reasoning given above we can see in each of these ranges that the derivative will be decreasing just to the right of the left-hand endpoint and increasing just to the left of the right hand endpoint.

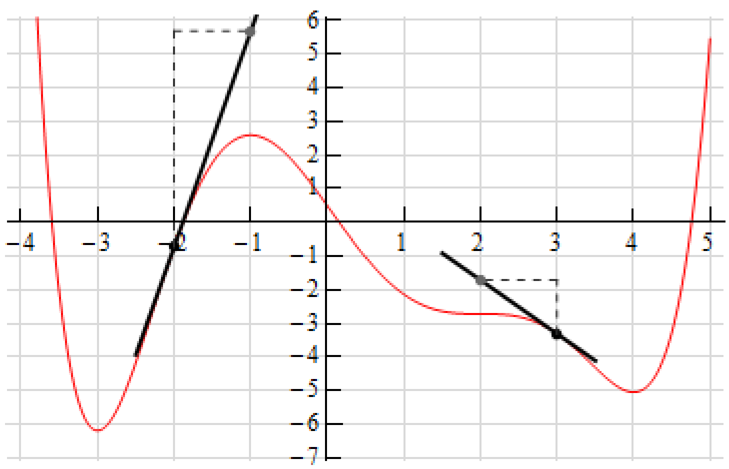
Finally, in the last region $x > 4$ we know that the derivative is zero at $x = 4$ and positive to the right of $x = 4$. Once again, following the reasoning above, the derivative must also be increasing in this range.

Putting all of this material together (and always taking the simplest choices for increasing and/or decreasing information) gives us the following sketch for the derivative.



Note that this was done with the actual derivative and so is in fact accurate. Any sketch you do will probably not look quite the same. The “humps” in each of the regions may be at different places and/or different heights for example. Also, note that we left off the vertical scale because given the information that we’ve got at this point there was no real way to know this information.

That doesn’t mean however that we can’t get some ideas of specific points on the derivative other than where we know the derivative to be zero. To see this let’s check out the following graph of the function (not the derivative, but the function).

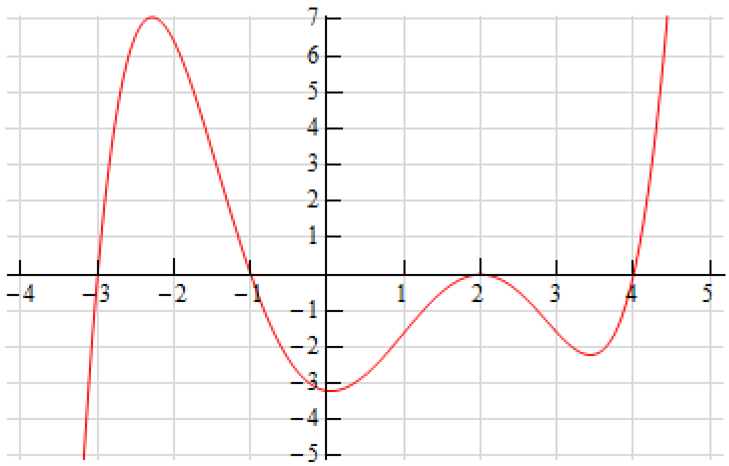


At $x = -2$ and $x = 3$ we’ve sketched in a couple of tangent lines. We can use the basic rise/run slope concept to estimate the value of the derivative at these points.

Let’s start at $x = 3$. We’ve got two points on the line here. We can see that each seem to be about one-quarter of the way off the grid line. So, taking that into account and the fact that we go through one complete grid we can see that the slope of the tangent line, and hence the derivative, is approximately -1.5.

At $x = -2$ it looks like (with some heavy estimation) that the second point is about 6.5 grids above the first point and so the slope of the tangent line here, and hence the derivative, is approximately 6.5.

Here is the sketch of the derivative with the vertical scale included and from this we can see that in fact our estimates are pretty close to reality.



Note that this idea of estimating values of derivatives can be a tricky process and does require a fair amount of (possible bad) approximations so while it can be used, you need to be careful with it.

We'll close out this section by noting that while we're not going to include an example here we could also use the graph of the derivative to give us a sketch of the function itself. In fact, in the next chapter where we discuss some applications of the derivative we will be looking using information the derivative gives us to sketch the graph of a function.

Section 3-3 : Differentiation Formulas

In the first section of this chapter we saw the **definition of the derivative** and we computed a couple of derivatives using the definition. As we saw in those examples there was a fair amount of work involved in computing the limits and the functions that we worked with were not terribly complicated.

For more complex functions using the definition of the derivative would be an almost impossible task. Luckily for us we won't have to use the definition terribly often. We will have to use it on occasion, however we have a large collection of formulas and properties that we can use to simplify our life considerably and will allow us to avoid using the definition whenever possible.

We will introduce most of these formulas over the course of the next several sections. We will start in this section with some of the basic properties and formulas. We will give the properties and formulas in this section in both "prime" notation and "fraction" notation.

Properties

1) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$ **OR** $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$

In other words, to differentiate a sum or difference all we need to do is differentiate the individual terms and then put them back together with the appropriate signs. Note as well that this property is not limited to two functions.

See the **Proof of Various Derivative Formulas** section of the Extras chapter to see the proof of this property. It's a very simple proof using the definition of the derivative.

2) $\left(cf(x)\right)' = cf'(x)$ **OR** $\frac{d}{dx}\left(cf(x)\right) = c\frac{df}{dx}$, *c* is any number

In other words, we can "factor" a multiplicative constant out of a derivative if we need to. See the **Proof of Various Derivative Formulas** section of the Extras chapter to see the proof of this property.

Note that we have not included formulas for the derivative of products or quotients of two functions here. The derivative of a product or quotient of two functions is not the product or quotient of the derivatives of the individual pieces. We will take a look at these in the next section.

Next, let's take a quick look at a couple of basic "computation" formulas that will allow us to actually compute some derivatives.

Formulas

1) If $f(x) = c$ then $f'(x) = 0$ **OR** $\frac{d}{dx}(c) = 0$

The derivative of a constant is zero. See the **Proof of Various Derivative Formulas** section of the Extras chapter to see the proof of this formula.

2) If $f\left(x\right)=x^{n}$ then $f'\left(x\right)=nx^{n-1}$ **OR** $\frac{d}{dx}\left(x^{n}\right)=nx^{n-1}$, *n* is any number.

This formula is sometimes called the **power rule**. All we are doing here is bringing the original exponent down in front and multiplying and then subtracting one from the original exponent.

Note as well that in order to use this formula *n* must be a number, it can't be a variable. Also note that the base, the *x*, must be a variable, it can't be a number. It will be tempting in some later sections to misuse the Power Rule when we run in some functions where the exponent isn't a number and/or the base isn't a variable.

See the **Proof of Various Derivative Formulas** section of the Extras chapter to see the proof of this formula. There are actually three different proofs in this section. The first two restrict the formula to *n* being an integer because at this point that is all that we can do at this point. The third proof is for the general rule but does suppose that you've read most of this chapter.

These are the only properties and formulas that we'll give in this section. Let's compute some derivatives using these properties.

Example 1 Differentiate each of the following functions.

(a) $f\left(x\right)=15x^{100}-3x^{12}+5x-46$

(b) $g\left(t\right)=2t^{6}+7t^{-6}$

(c) $y=8z^{3}-\frac{1}{3z^{5}}+z-23$

(d) $T\left(x\right)=\sqrt{x}+9\sqrt[3]{x^{7}}-\frac{2}{\sqrt[5]{x^{2}}}$

(e) $h\left(x\right)=x^{\pi}-x^{\sqrt{2}}$

Solution

(a) $f\left(x\right)=15x^{100}-3x^{12}+5x-46$

In this case we have the sum and difference of four terms and so we will differentiate each of the terms using the first property from above and then put them back together with the proper sign. Also, for each term with a multiplicative constant remember that all we need to do is "factor" the constant out (using the second property) and then do the derivative.

$$\begin{aligned} f'\left(x\right)&=15\left(100\right)x^{99}-3\left(12\right)x^{11}+5\left(1\right)x^{0}-0\\ &=1500x^{99}-36x^{11}+5\end{aligned}$$

Notice that in the third term the exponent was a one and so upon subtracting 1 from the original exponent we get a new exponent of zero. Now recall that $x^{0}=1$. Don't forget to do any basic arithmetic that needs to be done such as any multiplication and/or division in the coefficients.

(b) $g\left(t\right)=2t^{6}+7t^{-6}$

The point of this problem is to make sure that you deal with negative exponents correctly. Here is the derivative.

$$\begin{aligned} g'(t) &= 2(6)t^5 + 7(-6)t^{-7} \\ &= 12t^5 - 42t^{-7} \end{aligned}$$

Make sure that you correctly deal with the exponents in these cases, especially the negative exponents. It is an easy mistake to "go the other way" when subtracting one off from a negative exponent and get $-6t^{-5}$ instead of the correct $-6t^{-7}$.

(c) $y = 8z^3 - \dfrac{1}{3z^5} + z - 23$

Now in this function the second term is not correctly set up for us to use the power rule. The power rule requires that the term be a variable to a power only and the term must be in the numerator. So, prior to differentiating we first need to rewrite the second term into a form that we can deal with.

$$y = 8z^3 - \frac{1}{3}z^{-5} + z - 23$$

Note that we left the 3 in the denominator and only moved the variable up to the numerator. Remember that the only thing that gets an exponent is the term that is immediately to the left of the exponent. If we'd wanted the three to come up as well we'd have written,

$$\frac{1}{(3z)^5}$$

so be careful with this! It's a very common mistake to bring the 3 up into the numerator as well at this stage.

Now that we've gotten the function rewritten into a proper form that allows us to use the Power Rule we can differentiate the function. Here is the derivative for this part.

$$y' = 24z^2 + \frac{5}{3}z^{-6} + 1$$

(d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

All of the terms in this function have roots in them. In order to use the power rule we need to first convert all the roots to fractional exponents. Again, remember that the Power Rule requires us to have a variable to a number and that it must be in the numerator of the term. Here is the function written in "proper" form.

$$\begin{aligned} T(x) &= x^{\frac{1}{2}} + 9\left(x^7\right)^{\frac{1}{3}} - \frac{2}{\left(x^2\right)^{\frac{1}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - \frac{2}{x^{\frac{2}{5}}} \\ &= x^{\frac{1}{2}} + 9x^{\frac{7}{3}} - 2x^{-\frac{2}{5}} \end{aligned}$$

In the last two terms we combined the exponents. You should always do this with this kind of term. In a later section we will learn of a technique that would allow us to differentiate this term without combining exponents, however it will take significantly more work to do. Also, don't forget to move the term in the denominator of the third term up to the numerator. We can now differentiate the function.

$$\begin{aligned} T'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 9\left(\frac{7}{3}\right)x^{\frac{4}{3}} - 2\left(-\frac{2}{5}\right)x^{-\frac{7}{5}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{63}{3}x^{\frac{4}{3}} + \frac{4}{5}x^{-\frac{7}{5}} \end{aligned}$$

Make sure that you can deal with fractional exponents. You will see a lot of them in this class.

(e) $h(x) = x^{\pi} - x^{\sqrt{2}}$

In all of the previous examples the exponents have been nice integers or fractions. That is usually what we'll see in this class. However, the exponent only needs to be a number so don't get excited about problems like this one. They work exactly the same.

$$h'(x) = \pi x^{\pi-1} - \sqrt{2}x^{\sqrt{2}-1}$$

The answer is a little messy and we won't reduce the exponents down to decimals. However, this problem is not terribly difficult it just looks that way initially.

There is a general rule about derivatives in this class that you will need to get into the habit of using. When you see radicals you should always first convert the radical to a fractional exponent and then simplify exponents as much as possible. Following this rule will save you a lot of grief in the future.

Back when we first put down the properties we noted that we hadn't included a property for products and quotients. That doesn't mean that we can't differentiate any product or quotient at this point. There are some that we can do.

Example 2 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $h(t) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Solution

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

In this function we can't just differentiate the first term, differentiate the second term and then multiply the two back together. That just won't work. We will discuss this in detail in the next section so if you're not sure you believe that hold on for a bit and we'll be looking at that soon as well as showing you an example of why it won't work.

It is still possible to do this derivative however. All that we need to do is convert the radical to fractional exponents (as we should anyway) and then multiply this through the parenthesis.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right) = 2x^{\frac{5}{3}} - x^{\frac{8}{3}}$$

Now we can differentiate the function.

$$y' = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

(b) $h\left(t\right) = \dfrac{2t^5 + t^2 - 5}{t^2}$

As with the first part we can't just differentiate the numerator and the denominator and the put it back together as a fraction. Again, if you're not sure you believe this hold on until the next section and we'll take a more detailed look at this.

We can simplify this rational expression however as follows.

$$h\left(t\right) = \frac{2t^5}{t^2} + \frac{t^2}{t^2} - \frac{5}{t^2} = 2t^3 + 1 - 5t^{-2}$$

This is a function that we can differentiate.

$$h'\left(t\right) = 6t^2 + 10t^{-3}$$

So, as we saw in this example there are a few products and quotients that we can differentiate. If we can first do some simplification the functions will sometimes simplify into a form that can be differentiated using the properties and formulas in this section.

Before moving on to the next section let's work a couple of examples to remind us once again of some of the interpretations of the derivative.

Example 3 Is $f\left(x\right) = 2x^3 + \dfrac{300}{x^3} + 4$ increasing, decreasing or not changing at $x = -2$?

Solution

We know that the rate of change of a function is given by the functions derivative so all we need to do is it rewrite the function (to deal with the second term) and then take the derivative.

$$f\left(x\right) = 2x^3 + 300x^{-3} + 4 \qquad \Rightarrow \qquad f'\left(x\right) = 6x^2 - 900x^{-4} = 6x^2 - \frac{900}{x^4}$$

Note that we rewrote the last term in the derivative back as a fraction. This is not something we've done to this point and is only being done here to help with the evaluation in the next step. It's often easier to do the evaluation with positive exponents.

So, upon evaluating the derivative we get

$$f'\left(-2\right) = 6\left(4\right) - \frac{900}{16} = -\frac{129}{4} = -32.25$$

So, at $x = -2$ the derivative is negative and so the function is decreasing at $x = -2$.

Example 4 Find the equation of the tangent line to $f\left(x\right)=4x-8\sqrt{x}$ at $x=16$.

Solution
We know that the equation of a tangent line is given by,

$$y=f\left(a\right)+f'\left(a\right)\left(x-a\right)$$

So, we will need the derivative of the function (don't forget to get rid of the radical).

$$f\left(x\right)=4x-8x^{\frac{1}{2}} \qquad \Rightarrow \qquad f'\left(x\right)=4-4x^{-\frac{1}{2}}=4-\frac{4}{x^{\frac{1}{2}}}$$

Again, notice that we eliminated the negative exponent in the derivative solely for the sake of the evaluation. All we need to do then is evaluate the function and the derivative at the point in question, $x=16$.

$$f\left(16\right)=64-8\left(4\right)=32 \qquad\qquad f'\left(x\right)=4-\frac{4}{4}=3$$

The tangent line is then,

$$y=32+3\left(x-16\right)=3x-16$$

Example 5 The position of an object at any time *t* (in hours) is given by,

$$s\left(t\right)=2t^3-21t^2+60t-10$$

Determine when the object is moving to the right and when the object is moving to the left.

Solution
The only way that we'll know for sure which direction the object is moving is to have the velocity in hand. Recall that if the velocity is positive the object is moving off to the right and if the velocity is negative then the object is moving to the left.

We need the derivative in order to get the velocity of the object. The derivative, and hence the velocity, is,

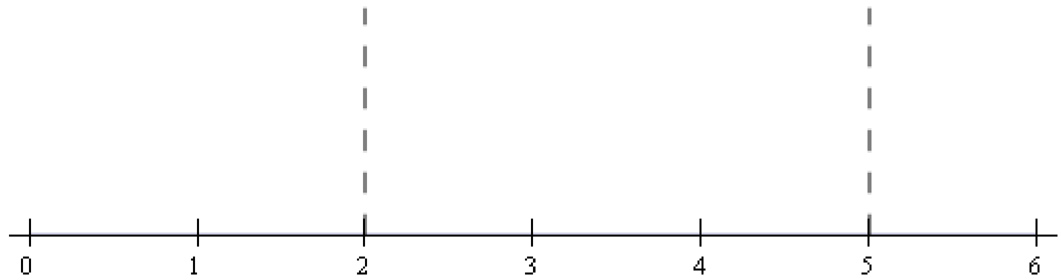
$$s'\left(t\right)=6t^2-42t+60=6\left(t^2-7t+10\right)=6\left(t-2\right)\left(t-5\right)$$

The reason for factoring the derivative will be apparent shortly.

Now, we need to determine where the derivative is positive and where the derivative is negative. There are several ways to do this. The method that we tend to prefer is the following.

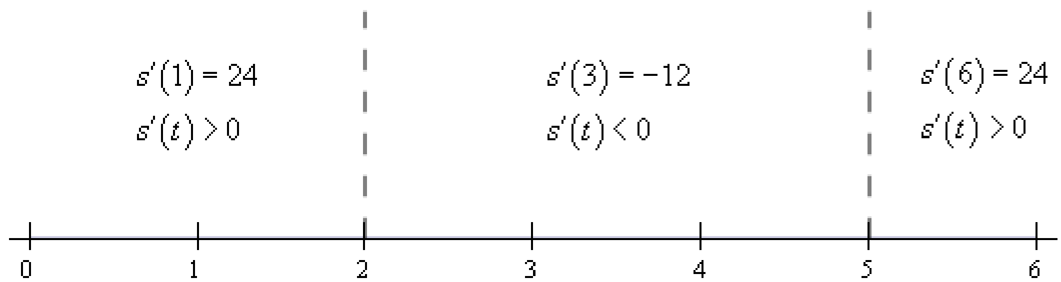
Since polynomials are continuous we know from the **Intermediate Value Theorem** that if the polynomial ever changes sign then it must have first gone through zero. So, if we knew where the derivative was zero we would know the only points where the derivative *might* change sign.

We can see from the factored form of the derivative that the derivative will be zero at $t=2$ and $t=5$. Let's graph these points on a number line.



Now, we can see that these two points divide the number line into three distinct regions. In each of these regions we **know** that the derivative will be the same sign. Recall the derivative can only change sign at the two points that are used to divide the number line up into the regions.

Therefore, all that we need to do is to check the derivative at a test point in each region and the derivative in that region will have the same sign as the test point. Here is the number line with the test points and results shown.



Here are the intervals in which the derivative is positive and negative.

$$\begin{aligned} \text{positive}: &\quad -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{negative}: &\quad 2 < t < 5 \end{aligned}$$

We included negative *t*'s here because we could even though they may not make much sense for this problem. Once we know this we also can answer the question. The object is moving to the right and left in the following intervals.

$$\begin{aligned} \text{moving to the right}: &\quad -\infty < t < 2 \quad \& \quad 5 < t < \infty \\ \text{moving to the left}\ \ : &\quad 2 < t < 5 \end{aligned}$$

Make sure that you can do the kind of work that we just did in this example. You will be asked numerous times over the course of the next two chapters to determine where functions are positive and/or negative. If you need some review or want to practice these kinds of problems you should check out the **Solving Inequalities** section of the **Algebra/Trig Review**.

Section 3-4 : Product and Quotient Rule

In the previous section we noted that we had to be careful when differentiating products or quotients. It's now time to look at products and quotients and see why.

First let's take a look at why we have to be careful with products and quotients. Suppose that we have the two functions $f(x) = x^3$ and $g(x) = x^6$. Let's start by computing the derivative of the product of these two functions. This is easy enough to do directly.

$$(f\,g)' = \left(x^3x^6\right)' = \left(x^9\right)' = 9x^8$$

Remember that on occasion we will drop the (x) part on the functions to simplify notation somewhat. We've done that in the work above.

Now, let's try the following.

$$f'(x)\,g'(x) = \left(3x^2\right)\left(6x^5\right) = 18x^7$$

So, we can very quickly see that.

$$(f\,g)' \neq f'\,g'$$

In other words, the derivative of a product is not the product of the derivatives.

Using the same functions we can do the same thing for quotients.

$$\left(\frac{f}{g}\right)' = \left(\frac{x^3}{x^6}\right)' = \left(\frac{1}{x^3}\right)' = \left(x^{-3}\right)' = -3x^{-4} = -\frac{3}{x^4}$$

$$\frac{f'(x)}{g'(x)} = \frac{3x^2}{6x^5} = \frac{1}{2x^3}$$

So, again we can see that,

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$$

To differentiate products and quotients we have the **Product Rule** and the **Quotient Rule**.

Product Rule

If the two functions $f(x)$ and $g(x)$ are differentiable (*i.e.* the derivative exist) then the product is differentiable and,

$$(f\,g)' = f'\,g + f\,g'$$

The proof of the Product Rule is shown in the **Proof of Various Derivative Formulas** section of the Extras chapter.

Quotient Rule

If the two functions $f(x)$ and $g(x)$ are differentiable (*i.e.* the derivative exist) then the quotient is differentiable and,

$$\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$$

Note that the numerator of the quotient rule is very similar to the product rule so be careful to not mix the two up!

The proof of the Quotient Rule is shown in the **Proof of Various Derivative Formulas** section of the Extras chapter.

Let's do a couple of examples of the product rule.

Example 1 Differentiate each of the following functions.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

(b) $f(x) = \left(6x^3 - x\right)\left(10 - 20x\right)$

Solution

At this point there really aren't a lot of reasons to use the product rule. As we noted in the previous section all we would need to do for either of these is to just multiply out the product and then differentiate.

With that said we will use the product rule on these so we can see an example or two. As we add more functions to our repertoire and as the functions become more complicated the product rule will become more useful and in many cases required.

(a) $y = \sqrt[3]{x^2}\left(2x - x^2\right)$

Note that we took the derivative of this function in the previous **section** and didn't use the product rule at that point. We should however get the same result here as we did then.

Now let's do the problem here. There's not really a lot to do here other than use the product rule. However, before doing that we should convert the radical to a fractional exponent as always.

$$y = x^{\frac{2}{3}}\left(2x - x^2\right)$$

Now let's take the derivative. So, we take the derivative of the first function times the second then add on to that the first function times the derivative of the second function.

$$y' = \frac{2}{3}x^{-\frac{1}{3}}\left(2x - x^2\right) + x^{\frac{2}{3}}\left(2 - 2x\right)$$

This is NOT what we got in the previous section for this derivative. However, with some simplification we can arrive at the same answer.

$$y' = \frac{4}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{5}{3}} + 2x^{\frac{2}{3}} - 2x^{\frac{5}{3}} = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

This is what we got for an answer in the previous section so that is a good check of the product rule.

(b) $f\left(x\right) = \left(6x^3 - x\right)\left(10 - 20x\right)$

This one is actually easier than the previous one. Let's just run it through the product rule.

$$\begin{aligned} f'\left(x\right) &= \left(18x^2 - 1\right)\left(10 - 20x\right) + \left(6x^3 - x\right)\left(-20\right) \\ &= -480x^3 + 180x^2 + 40x - 10 \end{aligned}$$

Since it was easy to do we went ahead and simplified the results a little.

Let's now work an example or two with the quotient rule. In this case, unlike the product rule examples, a couple of these functions will require the quotient rule in order to get the derivative. The last two however, we can avoid the quotient rule if we'd like to as we'll see.

Example 2 Differentiate each of the following functions.

(a) $W\left(z\right) = \dfrac{3z + 9}{2 - z}$

(b) $h\left(x\right) = \dfrac{4\sqrt{x}}{x^2 - 2}$

(c) $f\left(x\right) = \dfrac{4}{x^6}$

(d) $y = \dfrac{w^6}{5}$

Solution

(a) $W\left(z\right) = \dfrac{3z + 9}{2 - z}$

There isn't a lot to do here other than to use the quotient rule. Here is the work for this function.

$$\begin{aligned} W'\left(z\right) &= \frac{3\left(2 - z\right) - \left(3z + 9\right)\left(-1\right)}{\left(2 - z\right)^2} \\ &= \frac{15}{\left(2 - z\right)^2} \end{aligned}$$

(b) $h\left(x\right) = \dfrac{4\sqrt{x}}{x^2 - 2}$

Again, not much to do here other than use the quotient rule. Don't forget to convert the square root into a fractional exponent.

$$h'(x) = \frac{4\left(\frac{1}{2}\right)x^{-\frac{1}{2}}\left(x^2-2\right) - 4x^{\frac{1}{2}}\left(2x\right)}{\left(x^2-2\right)^2}$$

$$= \frac{2x^{\frac{3}{2}} - 4x^{-\frac{1}{2}} - 8x^{\frac{3}{2}}}{\left(x^2-2\right)^2}$$

$$= \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{\left(x^2-2\right)^2}$$

(c) $f(x) = \dfrac{4}{x^6}$

It seems strange to have this one here rather than being the first part of this example given that it definitely appears to be easier than any of the previous two. In fact, it is easier. There is a point to doing it here rather than first. In this case there are two ways to do compute this derivative. There is an easy way and a hard way and in this case the hard way is the quotient rule. That's the point of this example.

Let's do the quotient rule and see what we get.

$$f'(x) = \frac{(0)\left(x^6\right) - 4\left(6x^5\right)}{\left(x^6\right)^2} = \frac{-24x^5}{x^{12}} = -\frac{24}{x^7}$$

Now, that was the "hard" way. So, what was so hard about it? Well actually it wasn't that hard, there is just an easier way to do it that's all. However, having said that, a common mistake here is to do the derivative of the numerator (a constant) incorrectly. For some reason many people will give the derivative of the numerator in these kinds of problems as a 1 instead of 0! Also, there is some simplification that needs to be done in these kinds of problems if you do the quotient rule.

The easy way is to do what we did in the previous section.

$$f'(x) = 4x^{-6} = -24x^{-7} = -\frac{24}{x^7}$$

Either way will work, but I'd rather take the easier route if I had the choice.

(d) $y = \dfrac{w^6}{5}$

This problem also seems a little out of place. However, it is here again to make a point. Do not confuse this with a quotient rule problem. While you can do the quotient rule on this function there is no reason to use the quotient rule on this. Simply rewrite the function as

$$y = \frac{1}{5}w^6$$

and differentiate as always.

$$y' = \frac{6}{5}w^5$$

Finally, let's not forget about our applications of derivatives.

Example 3 Suppose that the amount of air in a balloon at any time *t* is given by

$$V(t) = \frac{6\sqrt[3]{t}}{4t+1}$$

Determine if the balloon is being filled with air or being drained of air at $t = 8$.

Solution

If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing. In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t = 8$.

This will require the quotient rule.

$$\begin{aligned} V'(t) &= \frac{2t^{-\frac{2}{3}}(4t+1) - 6t^{\frac{1}{3}}(4)}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t+1)^2} \\ &= \frac{-16t^{\frac{1}{3}} + \frac{2}{t^{\frac{2}{3}}}}{(4t+1)^2} \end{aligned}$$

Note that we simplified the numerator more than usual here. This was only done to make the derivative easier to evaluate.

The rate of change of the volume at $t = 8$ is then,

$$\begin{aligned} V'(8) &= \frac{-16(2) + \frac{2}{4}}{(33)^2} \qquad (8)^{\frac{1}{3}} = 2 \qquad (8)^{\frac{2}{3}} = \left((8)^{\frac{1}{3}}\right)^2 = (2)^2 = 4 \\ &= -\frac{63}{2178} = -\frac{7}{242} \end{aligned}$$

So, the rate of change of the volume at $t = 8$ is negative and so the volume must be decreasing. Therefore, air is being drained out of the balloon at $t = 8$.

As a final topic let's note that the product rule can be extended to more than two functions, for instance.

$$\left(f\,g\,h\right)' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$
$$\left(f\,g\,h\,w\right)' = f'\,g\,h\,w + f\,g'\,h\,w + f\,g\,h'\,w + f\,g\,h\,w'$$

Deriving these products of more than two functions is actually pretty simple. For example, let's take a look at the three function product rule.

First, we don't think of it as a product of three functions but instead of the product rule of the two functions $f\,g$ and *h* which we can then use the two function product rule on. Doing this gives,

$$\left(f\,g\,h\right)' = \left(\left[f\,g\right]h\right)' = \left[f\,g\right]'h + \left[f\,g\right]h'$$

Note that we put brackets on the $f\,g$ part to make it clear we are thinking of that term as a single function. Now all we need to do is use the two function product rule on the $\left[f\,g\right]'$ term and then do a little simplification.

$$\left(f\,g\,h\right)' = \left[f'\,g + f\,g'\right]h + \left[f\,g\right]h' = f'\,g\,h + f\,g'\,h + f\,g\,h'$$

Any product rule with more functions can be derived in a similar fashion.

With this section and the previous section we are now able to differentiate powers of *x* as well as sums, differences, products and quotients of these kinds of functions. However, there are many more functions out there in the world that are not in this form. The next few sections give many of these functions as well as give their derivatives.

Section 3-5 : Derivatives of Trig Functions

With this section we're going to start looking at the derivatives of functions other than polynomials or roots of polynomials. We'll start this process off by taking a look at the derivatives of the six trig functions. Two of the derivatives will be derived. The remaining four are left to you and will follow similar proofs for the two given here.

Before we actually get into the derivatives of the trig functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

Fact

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad\qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

See the **Proof of Trig Limits** section of the Extras chapter to see the proof of these two limits.

Before proceeding a quick note. Students often ask why we always use radians in a Calculus class. This is the reason why! The proof of the formula involving sine above requires the angles to be in radians. If the angles are in degrees the limit involving sine is not 1 and so the formulas we will derive below would also change. The formulas below would pick up an extra constant that would just get in the way of our work and so we use radians to avoid that. So, remember to always use radians in a Calculus class!

Before we start differentiating trig functions let's work a quick set of limit problems that this fact now allows us to do.

Example 1 Evaluate each of the following limits.

(a) $\displaystyle \lim_{\theta \to 0} \frac{\sin\theta}{6\theta}$

(b) $\displaystyle \lim_{x \to 0} \frac{\sin(6x)}{x}$

(c) $\displaystyle \lim_{x \to 0} \frac{x}{\sin(7x)}$

(d) $\displaystyle \lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)}$

(e) $\displaystyle \lim_{x \to 4} \frac{\sin(x-4)}{x-4}$

(f) $\displaystyle \lim_{z \to 0} \frac{\cos(2z)-1}{z}$

Solution

(a) $\displaystyle \lim_{\theta \to 0} \frac{\sin\theta}{6\theta}$

There really isn't a whole lot to this limit. In fact, it's only here to contrast with the next example so you can see the difference in how these work. In this case since there is only a 6 in the denominator we'll just factor this out and then use the fact.

$$\lim_{\theta\to 0}\frac{\sin\theta}{6\theta}=\frac{1}{6}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{1}{6}(1)=\frac{1}{6}$$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(6x)}{x}$

Now, in this case we can't factor the 6 out of the sine so we're stuck with it there and we'll need to figure out a way to deal with it. To do this problem we need to notice that in the fact the argument of the sine is the same as the denominator (*i.e.* both θ's). So we need to get both of the argument of the sine and the denominator to be the same. We can do this by multiplying the numerator and the denominator by 6 as follows.

$$\lim_{x\to 0}\frac{\sin(6x)}{x}=\lim_{x\to 0}\frac{6\sin(6x)}{6x}=6\lim_{x\to 0}\frac{\sin(6x)}{6x}$$

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the fact above to finish the limit.

To see that we can use the fact on this limit let's do a **change of variables**. A change of variables is really just a renaming of portions of the problem to make something look more like something we know how to deal with. They can't always be done, but sometimes, such as this case, they can simplify the problem. The change of variables here is to let $\theta = 6x$ and then notice that as $x\to 0$ we also have $\theta \to 6(0)=0$. When doing a change of variables in a limit we need to change all the *x*'s into θ's and that includes the one in the limit.

Doing the change of variables on this limit gives,

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(6x)}{x}&=6\lim_{x\to 0}\frac{\sin(6x)}{6x} \qquad\qquad \text{let } \theta=6x\\&=6\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\\&=6(1)\\&=6\end{aligned}$$

And there we are. Note that we didn't really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator and then we can use the fact. A change of variables, in this case, is really only needed to make it clear that the fact does work.

(c) $\displaystyle\lim_{x\to 0}\frac{x}{\sin(7x)}$

In this case we appear to have a small problem in that the function we're taking the limit of here is upside down compared to that in the fact. This is not the problem it appears to be once we notice that,

$$\frac{x}{\sin(7x)} = \frac{1}{\frac{\sin(7x)}{x}}$$

and then all we need to do is recall a nice property of limits that allows us to do ,

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(7x)} &= \lim_{x\to 0}\frac{1}{\frac{\sin(7x)}{x}} \\ &= \frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\frac{\sin(7x)}{x}} \\ &= \frac{1}{\lim\limits_{x\to 0}\frac{\sin(7x)}{x}}\end{aligned}$$

With a little rewriting we can see that we do in fact end up needing to do a limit like the one we did in the previous part. So, let's do the limit here and this time we won't bother with a change of variable to help us out. All we need to do is multiply the numerator and denominator of the fraction in the denominator by 7 to get things set up to use the fact. Here is the work for this limit.

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin(7x)} &= \frac{1}{\lim\limits_{x\to 0}\frac{7\sin(7x)}{7x}} \\ &= \frac{1}{7\lim\limits_{x\to 0}\frac{\sin(7x)}{7x}} \\ &= \frac{1}{(7)(1)} \\ &= \frac{1}{7}\end{aligned}$$

(d) $\displaystyle\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)}$

This limit looks nothing like the limit in the fact, however it can be thought of as a combination of the previous two parts by doing a little rewriting. First, we'll split the fraction up as follows,

$$\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} = \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}$$

Now, the fact wants a *t* in the denominator of the first and in the numerator of the second. This is easy enough to do if we multiply the whole thing by $\frac{t}{t}$ (which is just one after all and so won't change the problem) and then do a little rearranging as follows,

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \lim_{t\to 0}\frac{\sin(3t)}{1}\frac{1}{\sin(8t)}\frac{t}{t}\\ &= \lim_{t\to 0}\frac{\sin(3t)}{t}\frac{t}{\sin(8t)}\\ &= \left(\lim_{t\to 0}\frac{\sin(3t)}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin(8t)}\right)\end{aligned}$$

At this point we can see that this really is two limits that we've seen before. Here is the work for each of these and notice on the second limit that we're going to work it a little differently than we did in the previous part. This time we're going to notice that it doesn't really matter whether the sine is in the numerator or the denominator as long as the argument of the sine is the same as what's in the numerator the limit is still one.

Here is the work for this limit.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(3t)}{\sin(8t)} &= \left(\lim_{t\to 0}\frac{3\sin(3t)}{3t}\right)\left(\lim_{t\to 0}\frac{8t}{8\sin(8t)}\right)\\ &= \left(3\lim_{t\to 0}\frac{\sin(3t)}{3t}\right)\left(\frac{1}{8}\lim_{t\to 0}\frac{8t}{\sin(8t)}\right)\\ &= (3)\left(\frac{1}{8}\right)\\ &= \frac{3}{8}\end{aligned}$$

(e) $\displaystyle\lim_{x\to 4}\frac{\sin(x-4)}{x-4}$

This limit almost looks the same as that in the fact in the sense that the argument of the sine is the same as what is in the denominator. However, notice that, in the limit, *x* is going to 4 and not 0 as the fact requires. However, with a change of variables we can see that this limit is in fact set to use the fact above regardless.

So, let $\theta = x-4$ and then notice that as $x\to 4$ we have $\theta\to 0$. Therefore, after doing the change of variable the limit becomes,

$$\lim_{x\to 4}\frac{\sin(x-4)}{x-4} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

(f) $\displaystyle \lim_{z\to 0}\frac{\cos(2z)-1}{z}$

The previous parts of this example all used the sine portion of the fact. However, we could just have easily used the cosine portion so here is a quick example using the cosine portion to illustrate this. We'll not put in much explanation here as this really does work in the same manner as the sine portion.

$$\begin{aligned}\lim_{z\to 0}\frac{\cos(2z)-1}{z} &= \lim_{z\to 0}\frac{2\left(\cos(2z)-1\right)}{2z}\\ &= 2\lim_{z\to 0}\frac{\cos(2z)-1}{2z}\\ &= 2(0)\\ & 0\end{aligned}$$

All that is required to use the fact is that the argument of the cosine is the same as the denominator.

Okay, now that we've gotten this set of limit examples out of the way let's get back to the main point of this section, differentiating trig functions.

We'll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It's been a while since we've had to use this, but sometimes there just isn't anything we can do about it. Here is the definition of the derivative for the sine function.

$$\frac{d}{dx}\left(\sin(x)\right) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Since we can't just plug in $h=0$ to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

$$\sin(x+h) = \sin(x)\cos(h)+\cos(x)\sin(h)$$

Doing this gives us,

$$\begin{aligned}\frac{d}{dx}\left(\sin(x)\right) &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(x)\left(\cos(h)-1\right)+\cos(x)\sin(h)}{h}\\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h}+\lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}\end{aligned}$$

As you can see upon using the trig formula we can combine the first and third term and then factor a sine out of that. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as *h* approaches zero. In the first limit we have a $\sin\left(x\right)$ and in the second limit we have a $\cos\left(x\right)$. Both of these are only functions of *x* only and as *h* moves in towards zero this has no effect on the value of *x*. Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

$$\frac{d}{dx}\left(\sin\left(x\right)\right) = \sin\left(x\right)\lim_{h \to 0}\frac{\cos\left(h\right)-1}{h} + \cos\left(x\right)\lim_{h \to 0}\frac{\sin\left(h\right)}{h}$$

At this point all we need to do is use the limits in the fact above to finish out this problem.

$$\frac{d}{dx}\left(\sin\left(x\right)\right) = \sin\left(x\right)\left(0\right) + \cos\left(x\right)\left(1\right) = \cos\left(x\right)$$

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

$$\frac{d}{dx}\left(\cos\left(x\right)\right) = -\sin\left(x\right)$$

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions, along with appropriate derivative rules, can be used to get their derivatives.

Let’s take a look at tangent. Tangent is defined as,

$$\tan\left(x\right) = \frac{\sin\left(x\right)}{\cos\left(x\right)}$$

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let’s do that.

$$\begin{aligned} \frac{d}{dx}\left(\tan\left(x\right)\right) &= \frac{d}{dx}\left(\frac{\sin\left(x\right)}{\cos\left(x\right)}\right) \\ &= \frac{\cos\left(x\right)\cos\left(x\right) - \sin\left(x\right)\left(-\sin\left(x\right)\right)}{\left(\cos\left(x\right)\right)^2} \\ &= \frac{\cos^2\left(x\right) + \sin^2\left(x\right)}{\cos^2\left(x\right)} \end{aligned}$$

Now, recall that $\cos^2\left(x\right) + \sin^2\left(x\right) = 1$ and if we also recall the definition of secant in terms of cosine we arrive at,

$$\frac{d}{dx}\left(\tan\left(x\right)\right) = \frac{\cos^2\left(x\right)+\sin^2\left(x\right)}{\cos^2\left(x\right)}$$
$$= \frac{1}{\cos^2\left(x\right)}$$
$$= \sec^2\left(x\right)$$

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

Derivatives of the six trig functions

$$\frac{d}{dx}\left(\sin\left(x\right)\right) = \cos\left(x\right) \qquad \frac{d}{dx}\left(\cos\left(x\right)\right) = -\sin\left(x\right)$$
$$\frac{d}{dx}\left(\tan\left(x\right)\right) = \sec^2\left(x\right) \qquad \frac{d}{dx}\left(\cot\left(x\right)\right) = -\csc^2\left(x\right)$$
$$\frac{d}{dx}\left(\sec\left(x\right)\right) = \sec\left(x\right)\tan\left(x\right) \qquad \frac{d}{dx}\left(\csc\left(x\right)\right) = -\csc\left(x\right)\cot\left(x\right)$$

At this point we should work some examples.

Example 2 Differentiate each of the following functions.

(a) $g\left(x\right) = 3\sec\left(x\right) - 10\cot\left(x\right)$

(b) $h\left(w\right) = 3w^{-4} - w^2\tan\left(w\right)$

(c) $y = 5\sin\left(x\right)\cos\left(x\right) + 4\csc\left(x\right)$

(d) $P\left(t\right) = \dfrac{\sin\left(t\right)}{3-2\cos\left(t\right)}$

Solution

(a) $g\left(x\right) = 3\sec\left(x\right) - 10\cot\left(x\right)$

There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$g'\left(x\right) = 3\sec\left(x\right)\tan\left(x\right) - 10\left(-\csc^2\left(x\right)\right)$$
$$= 3\sec\left(x\right)\tan\left(x\right) + 10\csc^2\left(x\right)$$

(b) $h\left(w\right) = 3w^{-4} - w^2\tan\left(w\right)$

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike what we saw when we first looked at the product rule. When we first looked at the product rule the only functions we knew how to differentiate were polynomials and in those cases all we really needed to do was multiply them out and we could take the derivative without the product rule. We are now getting into the point where we will be forced to do the product rule at times regardless of whether or not we want to.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. There are two ways to deal with this. One way it to make sure that you use a set of parentheses as follows,

$$\begin{aligned} h'(w) &= -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right) \\ &= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w) \end{aligned}$$

Because the second term is being subtracted off of the first term then the whole derivative of the second term must also be subtracted off of the derivative of the first term. The parenthesis make this idea clear.

A potentially easier way to do this is to think of the minus sign as part of the first function in the product. Or, in other words the two functions in the product, using this idea, are $-w^2$ and $\tan(w)$. Doing this gives,

$$h'(w) = -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

So, regardless of how you approach this problem you will get the same derivative.

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$

As with the previous part we'll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly. Alternatively, you could make use of a set of parentheses to make sure the 5 gets dealt with properly. Either way will work, but we'll stick with thinking of the 5 as part of the first term in the product. Here's the derivative of this function.

$$\begin{aligned} y' &= 5\cos(x)\cos(x) + 5\sin(x)\left(-\sin(x)\right) - 4\csc(x)\cot(x) \\ &= 5\cos^2(x) - 5\sin^2(x) - 4\csc(x)\cot(x) \end{aligned}$$

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

In this part we'll need to use the quotient rule to take the derivative.

$$\begin{aligned} P'(t) &= \frac{\cos(t)\left(3 - 2\cos(t)\right) - \sin(t)\left(2\sin(t)\right)}{\left(3 - 2\cos(t)\right)^2} \\ &= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{\left(3 - 2\cos(t)\right)^2} \end{aligned}$$

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a "-2" from the last two terms in the numerator and the make use of the fact that $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$P'(t) = \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{\left(3 - 2\cos(t)\right)^2}$$
$$= \frac{3\cos(t) - 2}{\left(3 - 2\cos(t)\right)^2}$$

As a final problem here let's not forget that we still have our standard interpretations to derivatives.

Example 3 Suppose that the amount of money in a bank account is given by

$$P(t) = 500 + 100\cos(t) - 150\sin(t)$$

where *t* is in years. During the first 10 years in which the account is open when is the amount of money in the account increasing?

Solution

To determine when the amount of money is increasing we need to determine when the rate of change is positive. Since we know that the rate of change is given by the derivative that is the first thing that we need to find.

$$P'(t) = -100\sin(t) - 150\cos(t)$$

Now, we need to determine where in the first 10 years this will be positive. This is equivalent to asking where in the interval $[0,10]$ is the derivative positive. Recall that both sine and cosine are continuous functions and so the derivative is also a continuous function. The **Intermediate Value Theorem** then tells us that the derivative can only change sign if it first goes through zero.

So, we need to solve the following equation.

$$-100\sin(t) - 150\cos(t) = 0$$
$$100\sin(t) = -150\cos(t)$$
$$\frac{\sin(t)}{\cos(t)} = -1.5$$
$$\tan(t) = -1.5$$

The solution to this equation is,

$$t = 2.1588 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$t = 5.3004 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

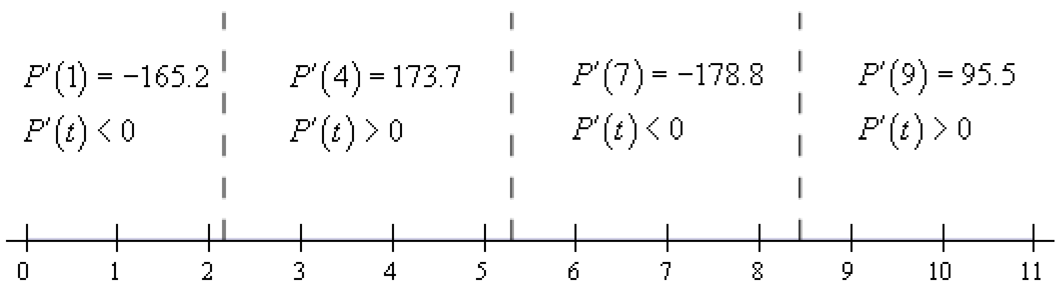
If you don't recall how to solve trig equations go back and take a look at the sections on **solving trig equations** in the Review chapter.

We are only interested in those solutions that fall in the range $\left[0,10\right]$. Plugging in values of *n* into the solutions above we see that the values we need are,

$$t = 2.1588 \qquad\qquad t = 2.1588 + 2\pi = 8.4420$$
$$t = 5.3004$$

So, much like solving polynomial inequalities all that we need to do is sketch in a number line and add in these points. These points will divide the number line into regions in which the derivative must always be the same sign. All that we need to do then is choose a test point from each region to determine the sign of the derivative in that region.

Here is the number line with all the information on it.



So, it looks like the amount of money in the bank account will be increasing during the following intervals.

$$2.1588 < t < 5.3004 \qquad\qquad 8.4420 < t < 10$$

Note that we can't say anything about what is happening after $t = 10$ since we haven't done any work for *t*'s after that point.

In this section we saw how to differentiate trig functions. We also saw in the last example that our interpretations of the derivative are still valid so we can't forget those.

Also, it is important that we be able to solve trig equations as this is something that will arise off and on in this course. It is also important that we can do the kinds of number lines that we used in the last example to determine where a function is positive and where a function is negative. This is something that we will be doing on occasion in both this chapter and the next.

Section 3-6 : Derivatives of Exponential and Logarithm Functions

The next set of functions that we want to take a look at are exponential and logarithm functions. The most common exponential and logarithm functions in a calculus course are the natural exponential function, $\mathbf{e}^x$, and the natural logarithm function, $\ln(x)$. We will take a more general approach however and look at the general exponential and logarithm function.

Exponential Functions
We'll start off by looking at the exponential function,

$$f(x) = a^x$$

We want to differentiate this. The power rule that we looked at a couple of sections ago won't work as that required the exponent to be a fixed number and the base to be a variable. That is exactly the opposite from what we've got with this function. So, we're going to have to start with the definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x\left(a^h - 1\right)}{h} \end{aligned}$$

Now, the a^x is not affected by the limit since it doesn't have any *h*'s in it and so is a constant as far as the limit is concerned. We can therefore factor this out of the limit. This gives,

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Now let's notice that the limit we've got above is exactly the definition of the derivative of $f(x) = a^x$ at $x = 0$, *i.e.* $f'(0)$. Therefore, the derivative becomes,

$$f'(x) = f'(0)\, a^x$$

So, we are kind of stuck. We need to know the derivative in order to get the derivative!

There is one value of *a* that we can deal with at this point. Back in the **Exponential Functions** section of the Review chapter we stated that $\mathbf{e} = 2.71828182845905\ldots$ What we didn't do however is actually define where **e** comes from. There are in fact a variety of ways to define **e**. Here are three of them.

Some Definitions of e.

1. $\mathbf{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
2. **e** is the unique positive number for which $\lim_{h\to 0}\frac{\mathbf{e}^h - 1}{h} = 1$
3. $\mathbf{e} = \sum_{n=0}^{\infty}\frac{1}{n!}$

The second one is the important one for us because that limit is exactly the limit that we're working with above. So, this definition leads to the following fact,

Fact 1

For the natural exponential function, $f(x) = \mathbf{e}^x$ we have $f'(0) = \lim_{h\to 0}\frac{\mathbf{e}^h - 1}{h} = 1$.

So, provided we are using the natural exponential function we get the following.

$$f(x) = \mathbf{e}^x \qquad \Rightarrow \qquad f'(x) = \mathbf{e}^x$$

At this point we're missing some knowledge that will allow us to easily get the derivative for a general function. **Eventually** we will be able to show that for a general exponential function we have,

$$f(x) = a^x \qquad \Rightarrow \qquad f'(x) = a^x \ln(a)$$

Logarithm Functions

Let's now briefly get the derivatives for logarithms. In this case we will need to start with the following fact about functions that are inverses of each other.

Fact 2

If *f(x)* and *g(x)* are inverses of each other then,

$$g'(x) = \frac{1}{f'(g(x))}$$

So, how is this fact useful to us? Well **recall** that the natural exponential function and the natural logarithm function are inverses of each other and we know what the derivative of the natural exponential function is!

So, if we have $f(x) = \mathbf{e}^x$ and $g(x) = \ln x$ then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\mathbf{e}^{g(x)}} = \frac{1}{\mathbf{e}^{\ln x}} = \frac{1}{x}$$

The last step just uses the fact that the two functions are inverses of each other.

Putting this all together gives,

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x} \qquad\qquad x > 0$$

Note that we need to require that $x > 0$ since this is required for the logarithm and so must also be required for its derivative. It can also be shown that,

$$\frac{d}{dx}\left(\ln\left|x\right|\right) = \frac{1}{x} \qquad\qquad x \neq 0$$

Using this all we need to avoid is $x = 0$.

In this case, unlike the exponential function case, we can actually find the derivative of the general logarithm function. All that we need is the derivative of the natural logarithm, which we just found, and the **change of base formula**. Using the change of base formula we can write a general logarithm as,

$$\log_a x = \frac{\ln x}{\ln a}$$

Differentiation is then fairly simple.

$$\begin{aligned}\frac{d}{dx}\left(\log_a x\right) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right)\\ &= \frac{1}{x\ln a}\end{aligned}$$

We took advantage of the fact that *a* was a constant and so $\ln a$ is also a constant and can be factored out of the derivative. Putting all this together gives,

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

Here is a summary of the derivatives in this section.

$$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x \qquad\qquad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$
$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x} \qquad\qquad \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

Okay, now that we have the derivations of the formulas out of the way let's compute a couple of derivatives.

Example 1 Differentiate each of the following functions.

(a) $R(w) = 4^w - 5\log_9 w$

(b) $f(x) = 3\mathbf{e}^x + 10x^3 \ln x$

(c) $y = \dfrac{5\mathbf{e}^x}{3\mathbf{e}^x + 1}$

Solution

(a) This will be the only example that doesn't involve the natural exponential and natural logarithm functions.

$$R'(w) = 4^w \ln 4 - \frac{5}{w \ln 9}$$

(b) Not much to this one. Just remember to use the product rule on the second term.

$$\begin{aligned} f'(x) &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^3 \left(\frac{1}{x}\right) \\ &= 3\mathbf{e}^x + 30x^2 \ln x + 10x^2 \end{aligned}$$

(c) We'll need to use the quotient rule on this one.

$$\begin{aligned} y' &= \frac{5\mathbf{e}^x \left(3\mathbf{e}^x + 1\right) - \left(5\mathbf{e}^x\right)\left(3\mathbf{e}^x\right)}{\left(3\mathbf{e}^x + 1\right)^2} \\ &= \frac{15\mathbf{e}^{2x} + 5\mathbf{e}^x - 15\mathbf{e}^{2x}}{\left(3\mathbf{e}^x + 1\right)^2} \\ &= \frac{5\mathbf{e}^x}{\left(3\mathbf{e}^x + 1\right)^2} \end{aligned}$$

There's really not a lot to differentiating natural logarithms and natural exponential functions at this point as long as you remember the formulas. In later sections as we get more formulas under our belt they will become more complicated.

Next, we need to do our obligatory application/interpretation problem so we don't forget about them.

Example 2 Suppose that the position of an object is given by

$$s(t) = t\mathbf{e}^t$$

Does the object ever stop moving?

Solution

First, we will need the derivative. We need this to determine if the object ever stops moving since at that point (provided there is one) the velocity will be zero and recall that the derivative of the position function is the velocity of the object.

The derivative is,

$$s'(t) = \mathbf{e}^t + t\mathbf{e}^t = (1+t)\mathbf{e}^t$$

So, we need to determine if the derivative is ever zero. To do this we will need to solve,

$$(1+t)\mathbf{e}^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t = -1$. So, if we are going to allow negative values of *t* then the object will stop moving once at $t = -1$. If we aren't going to allow negative values of *t* then the object will never stop moving.

Before moving on to the next section we need to go back over a couple of derivatives to make sure that we don't confuse the two. The two derivatives are,

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \text{Power Rule}$$

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a \qquad \text{Derivative of an exponential function}$$

It is important to note that with the Power rule the exponent MUST be a constant and the base MUST be a variable while we need exactly the opposite for the derivative of an exponential function. For an exponential function the exponent MUST be a variable and the base MUST be a constant.

It is easy to get locked into one of these formulas and just use it for both of these. We also haven't even talked about what to do if both the exponent and the base involve variables. We'll see this situation in a later **section**.

Section 3-7 : Derivatives of Inverse Trig Functions

In this section we are going to look at the derivatives of the inverse trig functions. In order to derive the derivatives of inverse trig functions we'll need the formula from the last section relating the derivatives of inverse functions. If $f\left(x\right)$ and $g\left(x\right)$ are inverse functions then,

$$g'\left(x\right)=\frac{1}{f'\left(g\left(x\right)\right)}$$

Recall as well that two functions are inverses if $f\left(g\left(x\right)\right)=x$ and $g\left(f\left(x\right)\right)=x$.

We'll go through inverse sine, inverse cosine and inverse tangent in detail here and leave the other three to you to derive if you'd like to.

Inverse Sine

Let's start with inverse sine. Here is the definition of the inverse sine.

$$y=\sin^{-1}x \qquad \Leftrightarrow \qquad \sin y=x \qquad \text{for} \qquad -\frac{\pi}{2}\le y\le\frac{\pi}{2}$$

So, evaluating an inverse trig function is the same as asking what angle (*i.e. y*) did we plug into the sine function to get *x*. The restrictions on *y* given above are there to make sure that we get a consistent answer out of the inverse sine. We know that there are in fact an infinite number of angles that will work and we want a consistent value when we work with inverse sine. Using the range of angles above gives all possible values of the sine function exactly once. If you're not sure of that sketch out a unit circle and you'll see that that range of angles (the *y*'s) will cover all possible values of sine.

Note as well that since $-1\le\sin\left(y\right)\le 1$ we also have $-1\le x\le 1$.

Let's work a quick example.

Example 1 Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

Solution

So, we are really asking what angle *y* solves the following equation.

$$\sin\left(y\right)=\frac{1}{2}$$

and we are restricted to the values of *y* above.

From a unit circle we can quickly see that $y=\frac{\pi}{6}$.

We have the following relationship between the inverse sine function and the sine function.

$$\sin\left(\sin^{-1}x\right)=x \qquad\qquad \sin^{-1}\left(\sin x\right)=x$$

In other words they are inverses of each other. This means that we can use the fact above to find the derivative of inverse sine. Let's start with,

$$f(x) = \sin x \qquad\qquad g(x) = \sin^{-1} x$$

Then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos\left(\sin^{-1} x\right)}$$

This is not a very useful formula. Let's see if we can get a better formula. Let's start by recalling the definition of the inverse sine function.

$$y = \sin^{-1}(x) \qquad \Rightarrow \qquad x = \sin(y)$$

Using the first part of this definition the denominator in the derivative becomes,

$$\cos\left(\sin^{-1} x\right) = \cos(y)$$

Now, recall that

$$\cos^2 y + \sin^2 y = 1 \qquad \Rightarrow \qquad \cos y = \sqrt{1 - \sin^2 y}$$

Using this, the denominator is now,

$$\cos\left(\sin^{-1} x\right) = \cos(y) = \sqrt{1 - \sin^2 y}$$

Now, use the second part of the definition of the inverse sine function. The denominator is then,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Putting all of this together gives the following derivative.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

Inverse Cosine

Now let's take a look at the inverse cosine. Here is the definition for the inverse cosine.

$$y = \cos^{-1} x \qquad \Leftrightarrow \qquad \cos y = x \qquad \text{for} \qquad 0 \le y \le \pi$$

As with the inverse sine we've got a restriction on the angles, *y*, that we get out of the inverse cosine function. Again, if you'd like to verify this a quick sketch of a unit circle should convince you that this range will cover all possible values of cosine exactly once. Also, we also have $-1 \le x \le 1$ because $-1 \le \cos(y) \le 1$.

Example 2 Evaluate $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution
As with the inverse sine we are really just asking the following.

$$\cos y = -\frac{\sqrt{2}}{2}$$

where *y* must meet the requirements given above. From a unit circle we can see that we must have $y = \frac{3\pi}{4}$.

The inverse cosine and cosine functions are also inverses of each other and so we have,

$$\cos\left(\cos^{-1} x\right) = x \qquad\qquad \cos^{-1}\left(\cos x\right) = x$$

To find the derivative we'll do the same kind of work that we did with the inverse sine above. If we start with

$$f\left(x\right) = \cos x \qquad\qquad g\left(x\right) = \cos^{-1} x$$

then,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{-\sin\left(\cos^{-1} x\right)}$$

Simplifying the denominator here is almost identical to the work we did for the inverse sine and so isn't shown here. Upon simplifying we get the following derivative.

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

So, the derivative of the inverse cosine is nearly identical to the derivative of the inverse sine. The only difference is the negative sign.

Inverse Tangent
Here is the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Leftrightarrow \qquad \tan y = x \qquad \text{for} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Again, we have a restriction on *y*, but notice that we can't let *y* be either of the two endpoints in the restriction above since tangent isn't even defined at those two points. To convince yourself that this range will cover all possible values of tangent do a quick **sketch** of the tangent function and we can see that in this range we do indeed cover all possible values of tangent. Also, in this case there are no restrictions on *x* because tangent can take on all possible values.

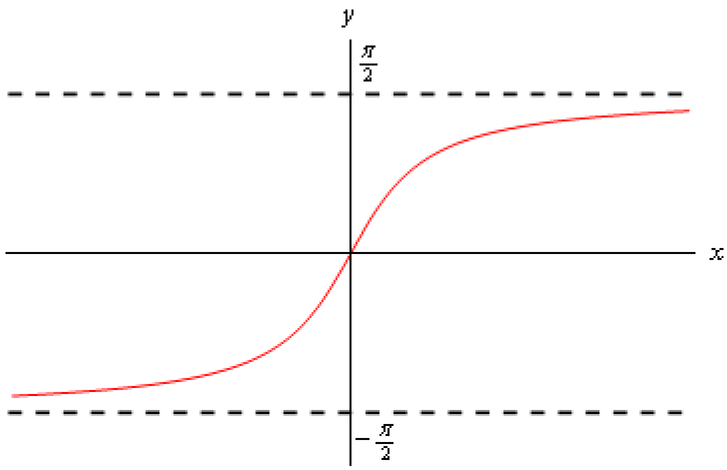
Example 3 Evaluate $\tan^{-1} 1$

Solution
Here we are asking,

$$\tan y = 1$$

where *y* satisfies the restrictions given above. From a unit circle we can see that $y = \frac{\pi}{4}$.

Because there is no restriction on *x* we can ask for the limits of the inverse tangent function as *x* goes to plus or minus infinity. To do this we'll need the graph of the inverse tangent function. This is shown below.



From this graph we can see that

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2} \qquad\qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The tangent and inverse tangent functions are inverse functions so,

$$\tan\left(\tan^{-1} x\right) = x \qquad\qquad \tan^{-1}\left(\tan x\right) = x$$

Therefore, to find the derivative of the inverse tangent function we can start with

$$f\left(x\right) = \tan x \qquad\qquad g\left(x\right) = \tan^{-1} x$$

We then have,

$$g'\left(x\right) = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{\sec^2\left(\tan^{-1} x\right)}$$

Simplifying the denominator is similar to the inverse sine, but different enough to warrant showing the details. We'll start with the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Rightarrow \qquad \tan y = x$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y$$

Now, if we start with the fact that

$$\cos^2 y + \sin^2 y = 1$$

and divide every term by $\cos^2 y$ we will get,

$$1 + \tan^2 y = \sec^2 y$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y = 1 + \tan^2 y$$

Finally using the second portion of the definition of the inverse tangent function gives us,

$$\sec^2\left(\tan^{-1} x\right) = 1 + \tan^2 y = 1 + x^2$$

The derivative of the inverse tangent is then,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

There are three more inverse trig functions but the three shown here the most common ones. Formulas for the remaining three could be derived by a similar process as we did those above. Here are the derivatives of all six inverse trig functions.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

We should probably now do a couple of quick derivatives here before moving on to the next section.

Example 4 Differentiate the following functions.

(a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$

(b) $y = \sqrt{z}\sin^{-1}(z)$

Solution

(a) Not much to do with this one other than differentiate each term.

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b) Don't forget to convert the radical to fractional exponents before using the product rule.

$$y' = \frac{1}{2}z^{-\frac{1}{2}}\sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

$$\sin^{-1} x = \arcsin x \qquad \cos^{-1} x = \arccos x$$

$$\tan^{-1} x = \arctan x \qquad \cot^{-1} x = \operatorname{arccot} x$$

$$\sec^{-1} x = \operatorname{arcsec} x \qquad \csc^{-1} x = \operatorname{arccsc} x$$

Section 3-8 : Derivatives of Hyperbolic Functions

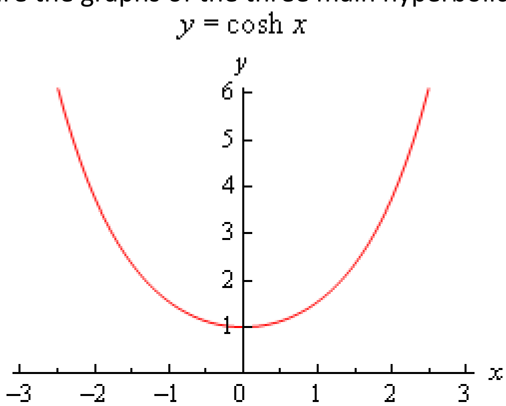
The last set of functions that we're going to be looking in this chapter at are the hyperbolic functions. In many physical situations combinations of $\mathbf{e}^{x}$ and $\mathbf{e}^{-x}$ arise fairly often. Because of this these combinations are given names. There are six hyperbolic functions and they are defined as follows.

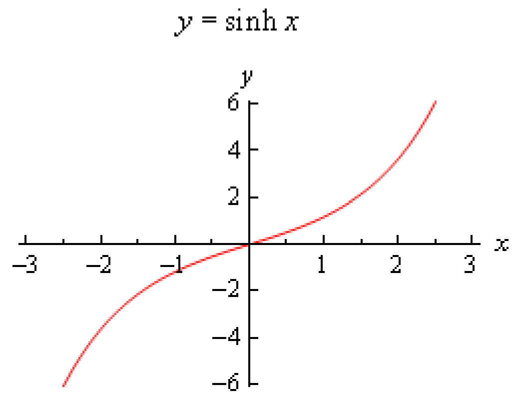
$$\sinh x = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2} \qquad\qquad \cosh x = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2}$$

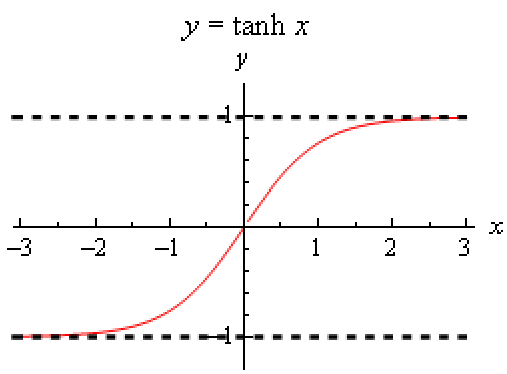
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad\qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad\qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Here are the graphs of the three main hyperbolic functions.







We also have the following facts about the hyperbolic functions.

$$\sinh(-x) = -\sinh(x) \qquad\qquad \cosh(-x) = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad\qquad 1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

You'll note that these are similar, but not quite the same, to some of the more common trig identities so be careful to not confuse the identities here with those of the standard trig functions.

Because the hyperbolic functions are defined in terms of exponential functions finding their derivatives is fairly simple provided you've already read through the next section. We haven't however so we'll need the following formula that can be easily proved after we've covered the next section.

$$\frac{d}{dx}\left(\mathbf{e}^{-x}\right) = -\mathbf{e}^{-x}$$

With this formula we'll do the derivative for hyperbolic sine and leave the rest to you as an exercise.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{\mathbf{e}^{x}-\mathbf{e}^{-x}}{2}\right) = \frac{\mathbf{e}^{x}-\left(-\mathbf{e}^{-x}\right)}{2} = \frac{\mathbf{e}^{x}+\mathbf{e}^{-x}}{2} = \cosh x$$

For the rest we can either use the definition of the hyperbolic function and/or the quotient rule. Here are all six derivatives.

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Here are a couple of quick derivatives using hyperbolic functions.

Example 1 Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \dfrac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

Section 3-9 : Chain Rule

We've taken a lot of derivatives over the course of the last few sections. However, if you look back they have all been functions similar to the following kinds of functions.

$$R(z) = \sqrt{z} \qquad f(t) = t^{50} \qquad y = \tan(x) \qquad h(w) = \mathbf{e}^{w} \qquad g(x) = \ln x$$

These are all fairly simple functions in that wherever the variable appears it is by itself. What about functions like the following,

$$R(z) = \sqrt{5z-8} \qquad f(t) = \left(2t^3 + \cos(t)\right)^{50} \qquad y = \tan\left(\sqrt[3]{3x^2} + \tan(5x)\right)$$
$$h(w) = \mathbf{e}^{w^4 - 3w^2 + 9} \qquad g(x) = \ln\left(x^{-4} + x^4\right)$$

None of our rules will work on these functions and yet some of these functions are closer to the derivatives that we're liable to run into than the functions in the first set.

Let's take the first one for example. Back in the **section** on the definition of the derivative we actually used the definition to compute this derivative. In that section we found that,

$$R'(z) = \frac{5}{2\sqrt{5z-8}}$$

If we were to just use the power rule on this we would get,

$$\frac{1}{2}(5z-8)^{-\frac{1}{2}} = \frac{1}{2\sqrt{5z-8}}$$

which is not the derivative that we computed using the definition. It is close, but it's not the same. So, the power rule alone simply won't work to get the derivative here.

Let's keep looking at this function and note that if we define,

$$f(z) = \sqrt{z} \qquad g(z) = 5z-8$$

then we can write the function as a composition.

$$R(z) = (f \circ g)(z) = f(g(z)) = \sqrt{5z-8}$$

and it turns out that it's actually fairly simple to differentiate a function composition using the **Chain Rule**. There are two forms of the chain rule. Here they are.

Chain Rule

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of $F(x)$ is,
$$F'(x) = f'(g(x)) \; g'(x)$$
2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of *y* is,
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Each of these forms have their uses, however we will work mostly with the first form in this class. To see the proof of the Chain Rule see the **Proof of Various Derivative Formulas** section of the Extras chapter.

Now, let's go back and use the Chain Rule on the function that we used when we opened this section.

Example 1 Use the Chain Rule to differentiate $R(z) = \sqrt{5z-8}$.

Solution

We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$f(z) = \sqrt{z} \qquad\qquad g(z) = 5z - 8$$

$$f'(z) = \frac{1}{2\sqrt{z}} \qquad\qquad g'(z) = 5$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z))\, g'(z) \\ &= f'(5z-8)\, g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}(5) \\ &= \frac{1}{2\sqrt{5z-8}}(5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$

And this is what we got using the definition of the derivative.

In general, we don't really do all the composition stuff in using the Chain Rule. That can get a little complicated and in fact obscures the fact that there is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition.

Let's take the function from the previous example and rewrite it slightly.

$$R(z) = \underbrace{(5z-8)}_{\text{inside function}}{}^{\overbrace{\tfrac{1}{2}}^{}_{\text{outside function}}}$$

This function has an "inside function" and an "outside function". The outside function is the square root or the exponent of $\frac{1}{2}$ depending on how you want to think of it and the inside function is the stuff that we're taking the square root of or raising to the $\frac{1}{2}$, again depending on how you want to look at it.

The derivative is then,

$$R'(z) = \overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

In general, this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}} \underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

We can always identify the "outside function" in the examples below by asking ourselves how we would evaluate the function. For instance in the $R(z)$ case if we were to ask ourselves what $R(2)$ is we would first evaluate the stuff under the radical and then finally take the square root of this result. The square root is the last operation that we perform in the evaluation and this is also the outside function. The outside function will always be the last operation you would perform if you were going to evaluate the function.

Let's take a look at some examples of the Chain Rule.

Example 2 Differentiate each of the following.

(a) $f(x) = \sin(3x^2 + x)$

(b) $f(t) = (2t^3 + \cos(t))^{50}$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

(d) $g(x) = \ln(x^{-4} + x^4)$

(e) $y = \sec(1 - 5x)$

(f) $P(t) = \cos^4(t) + \cos(t^4)$

Solution

(a) $f(x) = \sin(3x^2 + x)$

It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$f'(x) = \underbrace{\cos}_{\substack{\text{derivative of}\\ \text{outside function}}} \underbrace{(3x^2 + x)}_{\substack{\text{leave inside}\\ \text{function alone}}} \underbrace{(6x+1)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

Or with a little rewriting,

$$f'(x) = (6x+1)\cos(3x^2 + x)$$

(b) $f(t) = \left(2t^3 + \cos(t)\right)^{50}$

In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\begin{aligned} f'(t) &= 50\left(2t^3 + \cos(t)\right)^{49}\left(6t^2 - \sin(t)\right) \\ &= 50\left(6t^2 - \sin(t)\right)\left(2t^3 + \cos(t)\right)^{49} \end{aligned}$$

(c) $h(w) = \mathbf{e}^{w^4 - 3w^2 + 9}$

Identifying the outside function in the previous two was fairly simple since it really was the "outside" function in some sense. In this case we need to be a little careful. Recall that the outside function is the last operation that we would perform in an evaluation. In this case if we were to evaluate this function the last operation would be the exponential. Therefore, the outside function is the exponential function and the inside function is its exponent.

Here's the derivative.

$$\begin{aligned} h'(w) &= \mathbf{e}^{w^4 - 3w^2 + 9}\left(4w^3 - 6w\right) \\ &= \left(4w^3 - 6w\right)\mathbf{e}^{w^4 - 3w^2 + 9} \end{aligned}$$

Remember, we leave the inside function alone when we differentiate the outside function. So, the derivative of the exponential function (with the inside left alone) is just the original function.

(d) $g(x) = \ln\left(x^{-4} + x^4\right)$

Here the outside function is the natural logarithm and the inside function is stuff on the inside of the logarithm.

$$g'(x) = \frac{1}{x^{-4} + x^4}\left(-4x^{-5} + 4x^3\right) = \frac{-4x^{-5} + 4x^3}{x^{-4} + x^4}$$

Again, remember to leave the inside function alone when differentiating the outside function. So, upon differentiating the logarithm we end up not with 1/*x* but instead with 1/(inside function).

(e) $y = \sec(1 - 5x)$

In this case the outside function is the secant and the inside is the $1 - 5x$.

$$\begin{aligned} y' &= \sec(1 - 5x)\tan(1 - 5x)(-5) \\ &= -5\sec(1 - 5x)\tan(1 - 5x) \end{aligned}$$

In this case the derivative of the outside function is $\sec(x)\tan(x)$. However, since we leave the inside function alone we don't get *x*'s in both. Instead we get $1 - 5x$ in both.

(f) $P(t)=\cos^4(t)+\cos(t^4)$

There are two points to this problem. First, there are two terms and each will require a different application of the chain rule. That will often be the case so don't expect just a single chain rule when doing these problems. Second, we need to be very careful in choosing the outside and inside function for each term.

Recall that the first term can actually be written as,

$$\cos^4(t)=\left(\cos(t)\right)^4$$

So, in the first term the outside function is the exponent of 4 and the inside function is the cosine. In the second term it's exactly the opposite. In the second term the outside function is the cosine and the inside function is t^4. Here's the derivative for this function.

$$\begin{aligned} P'(t) &= 4\cos^3(t)\left(-\sin(t)\right)-\sin\left(t^4\right)\left(4t^3\right) \\ &= -4\sin(t)\cos^3(t)-4t^3\sin\left(t^4\right) \end{aligned}$$

There are a couple of general formulas that we can get for some special cases of the chain rule. Let's take a quick look at those.

Example 3 Differentiate each of the following.

(a) $f(x)=\left[g(x)\right]^n$

(b) $f(x)=\mathbf{e}^{g(x)}$

(c) $f(x)=\ln\left(g(x)\right)$

Solution

(a) The outside function is the exponent and the inside is $g(x)$.

$$f'(x)=n\left[g(x)\right]^{n-1}g'(x)$$

(b) The outside function is the exponential function and the inside is $g(x)$.

$$f'(x)=g'(x)\mathbf{e}^{g(x)}$$

(c) The outside function is the logarithm and the inside is $g(x)$.

$$f'(x)=\frac{1}{g(x)}g'(x)=\frac{g'(x)}{g(x)}$$

The formulas in this example are really just special cases of the Chain Rule but may be useful to remember in order to quickly do some of these derivatives.

Now, let's also not forget the other rules that we've got for doing derivatives. For the most part we'll not be explicitly identifying the inside and outside functions for the remainder of the problems in this section. We will be assuming that you can see our choices based on the previous examples and the work that we have shown.

Example 4 Differentiate each of the following.

(a) $T\left(x\right)=\tan^{-1}\left(2x\right)\sqrt[3]{1-3x^2}$

(b) $f\left(z\right)=\sin\left(z\mathbf{e}^{z}\right)$

(c) $y=\dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

(d) $h\left(t\right)=\left(\dfrac{2t+3}{6-t^2}\right)^3$

Solution

(a) $T\left(x\right)=\tan^{-1}\left(2x\right)\sqrt[3]{1-3x^2}$

Let's first notice that this problem is first and foremost a product rule problem. This is a product of two functions, the inverse tangent and the root and so the first thing we'll need to do in taking the derivative is use the product rule. However, in using the product rule and each derivative will require a chain rule application as well.

$$\begin{aligned} T'\left(x\right) &= \frac{1}{1+\left(2x\right)^2}\left(2\right)\left(1-3x^2\right)^{\frac{1}{3}}+\tan^{-1}\left(2x\right)\left(\frac{1}{3}\right)\left(1-3x^2\right)^{-\frac{2}{3}}\left(-6x\right) \\ &= \frac{2\left(1-3x^2\right)^{\frac{1}{3}}}{1+\left(2x\right)^2}-2x\left(1-3x^2\right)^{-\frac{2}{3}}\tan^{-1}\left(2x\right) \end{aligned}$$

In this part be careful with the inverse tangent. We know that,

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

When doing the chain rule with this we remember that we've got to leave the inside function alone. That means that where we have the x^2 in the derivative of $\tan^{-1}x$ we will need to have $\left(\text{inside function}\right)^2$.

(b) $f\left(z\right)=\sin\left(z\mathbf{e}^{z}\right)$

Now contrast this with the previous problem. In the previous problem we had a product that required us to use the chain rule in applying the product rule. In this problem we will first need to apply the chain rule and when we go to integrate the inside function we'll need to use the product rule.

Here is the chain rule portion of the problem.

$$f'(z) = \cos\left(z\mathbf{e}^z\right)\frac{d}{dz}\left[z\mathbf{e}^z\right]$$

In this case we did not actually do the derivative of the inside yet. We just left it in the derivative notation to make it clear that in order to do the derivative of the inside function we now have a product rule.

Here is the rest of the work for this problem.

$$f'(z) = \cos\left(z\mathbf{e}^z\right)\left(\mathbf{e}^z + z\mathbf{e}^z\right)$$

(c) $y = \dfrac{\left(x^3+4\right)^5}{\left(1-2x^2\right)^3}$

For this problem we clearly have a rational expression and so the first thing that we'll need to do is apply the quotient rule. In the process of using the quotient rule we'll need to use the chain rule when differentiating the numerator and denominator.

$$y' = \frac{5\left(x^3+4\right)^4\left(3x^2\right)\left(1-2x^2\right)^3 - \left(x^3+4\right)^5(3)\left(1-2x^2\right)^2(-4x)}{\left(\left(1-2x^2\right)^3\right)^2}$$

These tend to be a little messy. Notice that when we go to simplify that we'll be able to a fair amount of factoring in the numerator and this will often greatly simplify the derivative.

$$\begin{aligned} y' &= \frac{\left(x^3+4\right)^4\left(1-2x^2\right)^2\left(5\left(3x^2\right)\left(1-2x^2\right) - \left(x^3+4\right)(3)(-4x)\right)}{\left(1-2x^2\right)^6} \\ &= \frac{3x\left(x^3+4\right)^4\left(5x-6x^3+16\right)}{\left(1-2x^2\right)^4} \end{aligned}$$

After factoring we were able to cancel some of the terms in the numerator against the denominator. So even though the initial chain rule was fairly messy the final answer is significantly simpler because of the factoring.

(d) $h(t) = \left(\dfrac{2t+3}{6-t^2}\right)^3$

Unlike the previous problem the first step for derivative is to use the chain rule and then once we go to differentiate the inside function we'll need to do the quotient rule.

Here is the work for this problem.

$$\begin{aligned} h'(t) &= 3\left(\frac{2t+3}{6-t^2}\right)^2 \frac{d}{dt}\left[\frac{2t+3}{6-t^2}\right] \\ &= 3\left(\frac{2t+3}{6-t^2}\right)^2 \left[\frac{2\left(6-t^2\right)-\left(2t+3\right)\left(-2t\right)}{\left(6-t^2\right)^2}\right] \\ &= 3\left(\frac{2t+3}{6-t^2}\right)^2 \left[\frac{2t^2+6t+12}{\left(6-t^2\right)^2}\right] \end{aligned}$$

As with the second part above we did not initially differentiate the inside function in the first step to make it clear that it would be quotient rule from that point on.

There were several points in the last example. First is to not forget that we've still got other derivatives rules that are still needed on occasion. Just because we now have the chain rule does not mean that the product and quotient rule will no longer be needed.

In addition, as the last example illustrated, the order in which they are done will vary as well. Some problems will be product or quotient rule problems that involve the chain rule. Other problems however, will first require the use the chain rule and in the process of doing that we'll need to use the product and/or quotient rule.

Most of the examples in this section won't involve the product or quotient rule to make the problems a little shorter. However, in practice they will often be in the same problem so you need to be prepared for these kinds of problems.

Now, let's take a look at some more complicated examples.

Example 5 Differentiate each of the following.

(a) $h\left(z\right)=\dfrac{2}{\left(4z+\mathbf{e}^{-9z}\right)^{10}}$

(b) $f\left(y\right)=\sqrt{2y+\left(3y+4y^2\right)^3}$

(c) $y=\tan\left(\sqrt[3]{3x^2}+\ln\left(5x^4\right)\right)$

(d) $g\left(t\right)=\sin^3\left(\mathbf{e}^{1-t}+3\sin\left(6t\right)\right)$

Solution

We're going to be a little more careful in these problems than we were in the previous ones. The reason will be quickly apparent.

(a) $h\left(z\right)=\dfrac{2}{\left(4z+\mathbf{e}^{-9z}\right)^{10}}$

In this case let's first rewrite the function in a form that will be a little easier to deal with.

$$h(z) = 2\left(4z + \mathbf{e}^{-9z}\right)^{-10}$$

Now, let's start the derivative.

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11} \frac{d}{dz}\left(4z + \mathbf{e}^{-9z}\right)$$

Notice that we didn't actually do the derivative of the inside function yet. This is to allow us to notice that when we do differentiate the second term we will require the chain rule again. Notice as well that we will only need the chain rule on the exponential and not the first term. In many functions we will be using the chain rule more than once so don't get excited about this when it happens.

Let's go ahead and finish this example out.

$$h'(z) = -20\left(4z + \mathbf{e}^{-9z}\right)^{-11}\left(4 - 9\mathbf{e}^{-9z}\right)$$

Be careful with the second application of the chain rule. Only the exponential gets multiplied by the "-9" since that's the derivative of the inside function for that term only. One of the more common mistakes in these kinds of problems is to multiply the whole thing by the "-9" and not just the second term.

(b) $f(y) = \sqrt{2y + \left(3y + 4y^2\right)^3}$

We'll not put as many words into this example, but we're still going to be careful with this derivative so make sure you can follow each of the steps here.

$$\begin{aligned} f'(y) &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}} \frac{d}{dy}\left(2y + \left(3y + 4y^2\right)^3\right) \\ &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}}\left(2 + 3\left(3y + 4y^2\right)^2\left(3 + 8y\right)\right) \\ &= \frac{1}{2}\left(2y + \left(3y + 4y^2\right)^3\right)^{-\frac{1}{2}}\left(2 + \left(9 + 24y\right)\left(3y + 4y^2\right)^2\right) \end{aligned}$$

As with the first example the second term of the inside function required the chain rule to differentiate it. Also note that again we need to be careful when multiplying by the derivative of the inside function when doing the chain rule on the second term.

(c) $y = \tan\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)$

Let's jump right into this one.

$$y' = \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\frac{d}{dx}\left(\left(3x^2\right)^{\frac{1}{3}} + \ln\left(5x^4\right)\right)$$
$$= \sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)\left(\frac{1}{3}\left(3x^2\right)^{-\frac{2}{3}}\left(6x\right) + \frac{20x^3}{5x^4}\right)$$
$$= \left(2x\left(3x^2\right)^{-\frac{2}{3}} + \frac{4}{x}\right)\sec^2\left(\sqrt[3]{3x^2} + \ln\left(5x^4\right)\right)$$

In this example both of the terms in the inside function required a separate application of the chain rule.

(d) $g\left(t\right) = \sin^3\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$

We'll need to be a little careful with this one.

$$g'\left(t\right) = 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\frac{d}{dt}\sin\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$$
$$= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\frac{d}{dt}\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$$
$$= 3\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\left(\mathbf{e}^{1-t}\left(-1\right) + 3\cos\left(6t\right)\left(6\right)\right)$$
$$= 3\left(-\mathbf{e}^{1-t} + 18\cos\left(6t\right)\right)\sin^2\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)\cos\left(\mathbf{e}^{1-t} + 3\sin\left(6t\right)\right)$$

This problem required a total of 4 chain rules to complete.

Sometimes these can get quite unpleasant and require many applications of the chain rule. Initially, in these cases it's usually best to be careful as we did in this previous set of examples and write out a couple of extra steps rather than trying to do it all in one step in your head. Once you get better at the chain rule you'll find that you can do these fairly quickly in your head.

Finally, before we move onto the next section there is one more issue that we need to address. In the **Derivatives of Exponential and Logarithm Functions** section we claimed that,

$$f\left(x\right) = a^x \qquad \Rightarrow \qquad f'\left(x\right) = a^x \ln\left(a\right)$$

but at the time we didn't have the knowledge to do this. We now do. What we needed was the chain rule.

First, notice that using a property of logarithms we can write *a* as,

$$a = \mathbf{e}^{\ln a}$$

This may seem kind of silly, but it is needed to compute the derivative. Now, using this we can write the function as,

$$\begin{aligned} f(x) &= a^x \\ &= (a)^x \\ &= \left(\mathbf{e}^{\ln a}\right)^x \\ &= \mathbf{e}^{(\ln a)x} \\ &= \mathbf{e}^{x \ln a} \end{aligned}$$

Okay, now that we've gotten that taken care of all we need to remember is that *a* is a constant and so $\ln a$ is also a constant. Now, differentiating the final version of this function is a (hopefully) fairly simple Chain Rule problem.

$$f'(x) = \mathbf{e}^{x \ln a}(\ln a)$$

Now, all we need to do is rewrite the first term back as a^x to get,

$$f'(x) = a^x \ln(a)$$

So, not too bad if you can see the trick to rewriting the *a* and with using the Chain Rule.

Section 3-10 : Implicit Differentiation

To this point we've done quite a few derivatives, but they have all been derivatives of functions of the form $y = f(x)$. Unfortunately, not all the functions that we're going to look at will fall into this form.

Let's take a look at an example of a function like this.

Example 1 Find y' for $xy = 1$.

Solution
There are actually two solution methods for this problem.

Solution 1 :
This is the simple way of doing the problem. Just solve for *y* to get the function in the form that we're used to dealing with and then differentiate.

$$y = \frac{1}{x} \qquad \Rightarrow \qquad y' = -\frac{1}{x^2}$$

So, that's easy enough to do. However, there are some functions for which this can't be done. That's where the second solution technique comes into play.

Solution 2 :
In this case we're going to leave the function in the form that we were given and work with it in that form. However, let's recall from the first part of this solution that if we could solve for *y* then we will get *y* as a function of *x*. In other words, if we could solve for *y* (as we could in this case but won't always be able to do) we get $y = y(x)$. Let's rewrite the equation to note this.

$$xy = x\,y(x) = 1$$

Be careful here and note that when we write $y(x)$ we don't mean *y* times *x*. What we are noting here is that *y* is some (probably unknown) function of *x*. This is important to recall when doing this solution technique.

The next step in this solution is to differentiate both sides with respect to *x* as follows,

$$\frac{d}{dx}\left(x\,y(x)\right) = \frac{d}{dx}(1)$$

The right side is easy. It's just the derivative of a constant. The left side is also easy, but we've got to recognize that we've actually got a product here, the *x* and the $y(x)$. So, to do the derivative of the left side we'll need to do the product rule. Doing this gives,

$$(1)\,y(x) + x\frac{d}{dx}\left(y(x)\right) = 0$$

Now, recall that we have the following notational way of writing the derivative.

$$\frac{d}{dx}\left(y\left(x\right)\right)=\frac{dy}{dx}=y'$$

Using this we get the following,

$$y+xy'=0$$

Note that we dropped the $\left(x\right)$ on the *y* as it was only there to remind us that the *y* was a function of *x* and now that we've taken the derivative it's no longer really needed. We just wanted it in the equation to recognize the product rule when we took the derivative.

So, let's now recall just what were we after. We were after the derivative, y', and notice that there is now a y' in the equation. So, to get the derivative all that we need to do is solve the equation for y'.

$$y'=-\frac{y}{x}$$

There it is. Using the second solution technique this is our answer. This is not what we got from the first solution however. Or at least it doesn't look like the same derivative that we got from the first solution. Recall however, that we really do know what *y* is in terms of *x* and if we plug that in we will get,

$$y'=-\frac{{}^{1}\!/_{x}}{x}=-\frac{1}{x^2}$$

which is what we got from the first solution. Regardless of the solution technique used we should get the same derivative.

The process that we used in the second solution to the previous example is called **implicit differentiation** and that is the subject of this section. In the previous example we were able to just solve for *y* and avoid implicit differentiation. However, in the remainder of the examples in this section we either won't be able to solve for *y* or, as we'll see in one of the examples below, the answer will not be in a form that we can deal with.

In the second solution above we replaced the *y* with $y\left(x\right)$ and then did the derivative. Recall that we did this to remind us that *y* is in fact a function of *x*. We'll be doing this quite a bit in these problems, although we rarely actually write $y\left(x\right)$. So, before we actually work anymore implicit differentiation problems let's do a quick set of "simple" derivatives that will hopefully help us with doing derivatives of functions that also contain a $y\left(x\right)$.

Example 2 Differentiate each of the following.

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

(b) $\sin(3-6x)$, $\sin\left(y(x)\right)$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

Solution

These are written a little differently from what we're used to seeing here. This is because we want to match up these problems with what we'll be doing in this section. Also, each of these parts has several functions to differentiate starting with a specific function followed by a general function. This again, is to help us with some specific parts of the implicit differentiation process that we'll be doing.

(a) $\left(5x^3-7x+1\right)^5$, $\left[f(x)\right]^5$, $\left[y(x)\right]^5$

With the first function here we're being asked to do the following,

$$\frac{d}{dx}\left[\left(5x^3-7x+1\right)^5\right]=5\left(5x^3-7x+1\right)^4\left(15x^2-7\right)$$

and this is just the chain rule. We differentiated the outside function (the exponent of 5) and then multiplied that by the derivative of the inside function (the stuff inside the parenthesis).

For the second function we're going to do basically the same thing. We're going to need to use the chain rule. The outside function is still the exponent of 5 while the inside function this time is simply $f(x)$. We don't have a specific function here, but that doesn't mean that we can't at least write down the chain rule for this function. Here is the derivative for this function,

$$\frac{d}{dx}\left[f(x)\right]^5=5\left[f(x)\right]^4 f'(x)$$

We don't actually know what $f(x)$ is so when we do the derivative of the inside function all we can do is write down notation for the derivative, *i.e.* $f'(x)$.

With the final function here we simply replaced the *f* in the second function with a *y* since most of our work in this section will involve *y*'s instead of *f*'s. Outside of that this function is identical to the second. So, the derivative is,

$$\frac{d}{dx}\left[y(x)\right]^5=5\left[y(x)\right]^4 y'(x)$$

(b) $\sin(3-6x)$, $\sin\left(y(x)\right)$

The first function to differentiate here is just a quick chain rule problem again so here is it's derivative,

$$\frac{d}{dx}\left[\sin(3-6x)\right]=-6\cos(3-6x)$$

For the second function we didn't bother this time with using $f(x)$ and just jumped straight to $y(x)$ for the general version. This is still just a general version of what we did for the first function.

The outside function is still the sine and the inside is given by $y(x)$ and while we don't have a formula for $y(x)$ and so we can't actually take its derivative we do have a notation for its derivative. Here is the derivative for this function,

$$\frac{d}{dx}\left[\sin\left(y(x)\right)\right] = y'(x)\cos\left(y(x)\right)$$

(c) $\mathbf{e}^{x^2-9x}$, $\mathbf{e}^{y(x)}$

In this part we'll just give the answers for each and leave out the explanation that we had in the first two parts.

$$\frac{d}{dx}\left(\mathbf{e}^{x^2-9x}\right) = (2x-9)\mathbf{e}^{x^2-9x} \qquad\qquad \frac{d}{dx}\left(\mathbf{e}^{y(x)}\right) = y'(x)\mathbf{e}^{y(x)}$$

So, in this set of examples we were just doing some chain rule problems where the inside function was $y(x)$ instead of a specific function. This kind of derivative shows up all the time in doing implicit differentiation so we need to make sure that we can do them. Also note that we only did this for three kinds of functions but there are many more kinds of functions that we could have used here.

So, it's now time to do our first problem where implicit differentiation is required, unlike the first example where we could actually avoid implicit differentiation by solving for *y*.

Example 3 Find y' for the following function.

$$x^2 + y^2 = 9$$

Solution

Now, this is just a circle and we can solve for *y* which would give,

$$y = \pm\sqrt{9 - x^2}$$

Prior to starting this problem, we stated that we had to do implicit differentiation here because we couldn't just solve for *y* and yet that's what we just did. So, why can't we use "normal" differentiation here? The problem is the "$\pm$". With this in the "solution" for *y* we see that *y* is in fact two different functions. Which should we use? Should we use both? We only want a single function for the derivative and at best we have two functions here.

So, in this example we really are going to need to do implicit differentiation so we can avoid this. In this example we'll do the same thing we did in the first example and remind ourselves that *y* is really a function of *x* and write *y* as $y(x)$. Once we've done this all we need to do is differentiate each term with respect to *x*.

$$\frac{d}{dx}\left(x^2 + \left[y(x)\right]^2\right) = \frac{d}{dx}(9)$$

As with the first example the right side is easy. The left side is also pretty easy since all we need to do is take the derivative of each term and note that the second term will be similar the part (a) of the second example. All we need to do for the second term is use the chain rule.

After taking the derivative we have,

$$2x + 2\left[y(x)\right]^1 y'(x) = 0$$

At this point we can drop the (x) part as it was only in the problem to help with the differentiation process. The final step is to simply solve the resulting equation for y'.

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Unlike the first example we can't just plug in for *y* since we wouldn't know which of the two functions to use. Most answers from implicit differentiation will involve both *x* and *y* so don't get excited about that when it happens.

As always, we can't forget our interpretations of derivatives.

Example 4 Find the equation of the tangent line to

$$x^2 + y^2 = 9$$

at the point $\left(2, \sqrt{5}\right)$.

Solution

First note that unlike all the other tangent line problems we've done in previous sections we need to be given both the *x* and the *y* values of the point. Notice as well that this point does lie on the graph of the circle (you can check by plugging the points into the equation) and so it's okay to talk about the tangent line at this point.

Recall that to write down the tangent line all we need is the slope of the tangent line and this is nothing more than the derivative evaluated at the given point. We've got the derivative from the previous example so all we need to do is plug in the given point.

$$m = \left. y' \right|_{x=2,\, y=\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

The tangent line is then.

$$y = \sqrt{5} - \frac{2}{\sqrt{5}}(x-2)$$

Now, let's work some more examples. In the remaining examples we will no longer write $y(x)$ for *y*. This is just something that we were doing to remind ourselves that *y* is really a function of *x* to help with the derivatives. Seeing the $y(x)$ reminded us that we needed to do the chain rule on that portion of the problem. From this point on we'll leave the *y*'s written as *y*'s and in our head we'll need to remember that they really are $y(x)$ and that we'll need to do the chain rule.

There is an easy way to remember how to do the chain rule in these problems. The chain rule really tells us to differentiate the function as we usually would, except we need to add on a derivative of the inside function. In implicit differentiation this means that every time we are differentiating a term with *y* in it the inside function is the *y* and we will need to add a y' onto the term since that will be the derivative of the inside function.

Let's see a couple of examples.

Example 5 Find y' for each of the following.

(a) $x^3y^5 + 3x = 8y^3 + 1$

(b) $x^2\tan(y) + y^{10}\sec(x) = 2x$

(c) $\mathbf{e}^{2x+3y} = x^2 - \ln(xy^3)$

Solution

(a) $x^3y^5 + 3x = 8y^3 + 1$

First differentiate both sides with respect to *x* and remember that each *y* is really $y(x)$ we just aren't going to write it that way anymore. This means that the first term on the left will be a product rule.

We differentiated these kinds of functions involving *y*'s to a power with the chain rule in the **Example 2** above. Also, recall the discussion prior to the start of this problem. When doing this kind of chain rule problem all that we need to do is differentiate the *y*'s as normal and then add on a y', which is nothing more than the derivative of the "inside function".

Here is the differentiation of each side for this function.

$$3x^2y^5 + 5x^3y^4y' + 3 = 24y^2y'$$

Now all that we need to do is solve for the derivative, y'. This is just basic solving algebra that you are capable of doing. The main problem is that it's liable to be messier than what you're used to doing. All we need to do is get all the terms with y' in them on one side and all the terms without y' in them on the other. Then factor y' out of all the terms containing it and divide both sides by the "coefficient" of the y'. Here is the solving work for this one,

$$3x^2y^5 + 3 = 24y^2y' - 5x^3y^4y'$$

$$3x^2y^5 + 3 = \left(24y^2 - 5x^3y^4\right)y'$$

$$y' = \frac{3x^2y^5 + 3}{24y^2 - 5x^3y^4}$$

The algebra in these problems can be quite messy so be careful with that.

(b) $x^2\tan(y) + y^{10}\sec(x) = 2x$

We've got two product rules to deal with this time. Here is the derivative of this function.

$$2x\tan(y) + x^2\sec^2(y)\,y' + 10y^9y'\sec(x) + y^{10}\sec(x)\tan(x) = 2$$

Notice the derivative tacked onto the secant! Again, this is just a chain rule problem similar to the second part of Example 2 above.

Now, solve for the derivative.

$$\begin{aligned}\left(x^2\sec^2\left(y\right)+10y^9\sec\left(x\right)\right)y' &= 2-y^{10}\sec\left(x\right)\tan\left(x\right)-2x\tan\left(y\right)\\ y' &= \frac{2-y^{10}\sec\left(x\right)\tan\left(x\right)-2x\tan\left(y\right)}{x^2\sec^2\left(y\right)+10y^9\sec\left(x\right)}\end{aligned}$$

(c) $\mathbf{e}^{2x+3y}=x^2-\ln\left(xy^3\right)$

We're going to need to be careful with this problem. We've got a couple chain rules that we're going to need to deal with here that are a little different from those that we've dealt with prior to this problem.

In both the exponential and the logarithm we've got a "standard" chain rule in that there is something other than just an *x* or *y* inside the exponential and logarithm. So, this means we'll do the chain rule as usual here and then when we do the derivative of the inside function for each term we'll have to deal with differentiating *y*'s.

Here is the derivative of this equation.

$$\mathbf{e}^{2x+3y}\left(2+3y'\right)=2x-\frac{y^3+3xy^2y'}{xy^3}$$

In both of the chain rules note that the y' didn't get tacked on until we actually differentiated the *y*'s in that term.

Now we need to solve for the derivative and this is liable to be somewhat messy. In order to get the y' on one side we'll need to multiply the exponential through the parenthesis and break up the quotient.

$$\begin{aligned}2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y} &= 2x-\frac{y^3}{xy^3}-\frac{3xy^2y'}{xy^3}\\ 2\mathbf{e}^{2x+3y}+3y'\mathbf{e}^{2x+3y} &= 2x-\frac{1}{x}-\frac{3y'}{y}\\ \left(3\mathbf{e}^{2x+3y}+3y^{-1}\right)y' &= 2x-x^{-1}-2\mathbf{e}^{2x+3y}\\ y' &= \frac{2x-x^{-1}-2\mathbf{e}^{2x+3y}}{3\mathbf{e}^{2x+3y}+3y^{-1}}\end{aligned}$$

Note that to make the derivative at least look a little nicer we converted all the fractions to negative exponents.

Okay, we've seen one application of implicit differentiation in the tangent line example above. However, there is another application that we will be seeing in every problem in the next section.

In some cases we will have two (or more) functions all of which are functions of a third variable. So, we might have $x\left(t\right)$ and $y\left(t\right)$, for example and in these cases, we will be differentiating with respect to *t*. This is just implicit differentiation like we did in the previous examples, but there is a difference however.

In the previous examples we have functions involving *x*'s and *y*'s and thinking of *y* as $y\left(x\right)$. In these problems we differentiated with respect to *x* and so when faced with *x*'s in the function we differentiated as normal and when faced with *y*'s we differentiated as normal except we then added a y' onto that term because we were really doing a chain rule.

In the new example we want to look at we're assuming that $x = x\left(t\right)$ and that $y = y\left(t\right)$ and differentiating with respect to *t*. This means that every time we are faced with an *x* or a *y* we'll be doing the chain rule. This in turn means that when we differentiate an *x* we will need to add on an x' and whenever we differentiate a *y* we will add on a y'.

These new types of problems are really the same kind of problem we've been doing in this section. They are just expanded out a little to include more than one function that will require a chain rule.

Let's take a look at an example of this kind of problem.

Example 6 Assume that $x = x\left(t\right)$ and $y = y\left(t\right)$ and differentiate the following equation with respect to *t*.

$$x^3 y^6 + \mathbf{e}^{1-x} - \cos\left(5y\right) = y^2$$

Solution

So, just differentiate as normal and add on an appropriate derivative at each step. Note as well that the first term will be a product rule since both *x* and *y* are functions of *t*.

$$3x^2 x' y^6 + 6x^3 y^5 y' - x'\mathbf{e}^{1-x} + 5y'\sin\left(5y\right) = 2yy'$$

There really isn't all that much to this problem. Since there are two derivatives in the problem we won't be bothering to solve for one of them. When we do this kind of problem in the next section the problem will imply which one we need to solve for.

At this point there doesn't seem be any real reason for doing this kind of problem, but as we'll see in the next section every problem that we'll be doing there will involve this kind of implicit differentiation.

Section 3-11 : Related Rates

In this section we are going to look at an application of implicit differentiation. Most of the applications of derivatives are in the next chapter however there are a couple of reasons for placing it in this chapter as opposed to putting it into the next chapter with the other applications. The first reason is that it's an application of implicit differentiation and so putting it right after that section means that we won't have forgotten how to do implicit differentiation. The other reason is simply that after doing all these derivatives we need to be reminded that there really are actual applications to derivatives. Sometimes it is easy to forget there really is a reason that we're spending all this time on derivatives.

For these related rates problems, it's usually best to just jump right into some problems and see how they work.

Example 1 Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

Solution

The first thing that we'll need to do here is to identify what information that we've been given and what we want to find. Before we do that let's notice that both the volume of the balloon and the radius of the balloon will vary with time and so are really functions of time, *i.e.* $V(t)$ and $r(t)$.

We know that air is being pumped into the balloon at a rate of 5 cm^3/min. This is the rate at which the volume is increasing. Recall that rates of change are nothing more than derivatives and so we know that,

$$V'(t) = 5$$

We want to determine the rate at which the radius is changing. Again, rates are derivatives and so it looks like we want to determine,

$$r'(t) = ? \qquad \text{when} \quad r(t) = \frac{d}{2} = 10\,\text{cm}$$

Note that we needed to convert the diameter to a radius.

Now that we've identified what we have been given and what we want to find we need to relate these two quantities to each other. In this case we can relate the volume and the radius with the formula for the volume of a sphere.

$$V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$$

As in the previous section when we looked at implicit differentiation, we will typically not use the (t) part of things in the formulas, but since this is the first time through one of these we will do that to remind ourselves that they are really functions of *t*.

Now we don't really want a relationship between the volume and the radius. What we really want is a relationship between their derivatives. We can do this by differentiating both sides with respect to *t*. In other words, we will need to do implicit differentiation on the above formula. Doing this gives,

$$V' = 4\pi r^2 r'$$

Note that at this point we went ahead and dropped the (t) from each of the terms. Now all that we need to do is plug in what we know and solve for what we want to find.

$$5 = 4\pi\left(10^2\right)r' \qquad \Rightarrow \qquad r' = \frac{1}{80\pi} \text{ cm/min}$$

We can get the units of the derivative by recalling that,

$$r' = \frac{dr}{dt}$$

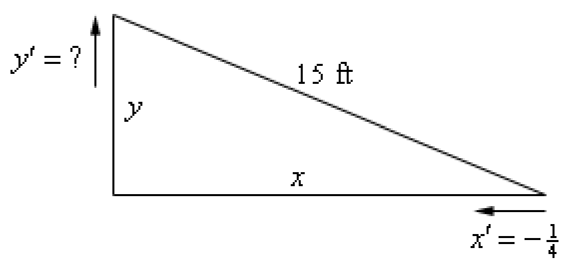
The units of the derivative will be the units of the numerator (cm in the previous example) divided by the units of the denominator (min in the previous example).

Let's work some more examples.

Example 2 A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

Solution

The first thing to do in this case is to sketch picture that shows us what is going on.



We've defined the distance of the bottom of the ladder from the wall to be *x* and the distance of the top of the ladder from the floor to be *y*. Note as well that these are changing with time and so we really should write $x(t)$ and $y(t)$. However, as is often the case with related rates/implicit differentiation problems we don't write the (t) part just try to remember this in our heads as we proceed with the problem.

Next, we need to identify what we know and what we want to find. We know that the rate at which the bottom of the ladder is moving towards the wall. This is,

$$x' = -\frac{1}{4}$$

Note as well that the rate is negative since the distance from the wall, *x*, is decreasing. We always need to be careful with signs with these problems.

We want to find the rate at which the top of the ladder is moving away from the floor. This is y'. Note as well that this quantity should be positive since *y* will be increasing.

As with the first example we first need a relationship between *x* and *y*. We can get this using Pythagorean theorem.

$$x^2 + y^2 = (15)^2 = 225$$

All that we need to do at this point is to differentiate both sides with respect to *t*, remembering that *x* and *y* are really functions of *t* and so we'll need to do implicit differentiation. Doing this gives an equation that shows the relationship between the derivatives.

$$2xx' + 2yy' = 0 \tag{1}$$

Next, let's see which of the various parts of this equation that we know and what we need to find. We know x' and are being asked to determine y' so it's okay that we don't know that. However, we still need to determine *x* and *y*.

Determining *x* and *y* is actually fairly simple. We know that initially $x = 10$ and the end is being pushed in towards the wall at a rate of $\frac{1}{4}$ ft/sec and that we are interested in what has happened after 12 seconds. We know that,

$$\begin{aligned} \text{distance} &= \text{rate} \times \text{time} \\ &= \left(\frac{1}{4}\right)(12) = 3 \end{aligned}$$

So, the end of the ladder has been pushed in 3 feet and so after 12 seconds we must have $x = 7$. Note that we could have computed this in one step as follows,

$$x = 10 - \frac{1}{4}(12) = 7$$

To find *y* (after 12 seconds) all that we need to do is reuse the Pythagorean Theorem with the values of *x* that we just found above.

$$y = \sqrt{225 - x^2} = \sqrt{225 - 49} = \sqrt{176}$$

Now all that we need to do is plug into (1) and solve for y'.

$$2(7)\left(-\frac{1}{4}\right) + 2\left(\sqrt{176}\right)y' = 0 \qquad \Rightarrow \qquad y' = \frac{7/4}{\sqrt{176}} = \frac{7}{4\sqrt{176}} = 0.1319 \text{ ft/sec}$$

Notice that we got the correct sign for y'. If we'd gotten a negative value we'd have known that we had made a mistake and we could go back and look for it.

Before working another example, we need to make a comment about the set up of the previous problem. When we labeled our sketch, we acknowledged that the hypotenuse is constant and so just called it 15 ft. A common mistake that students will sometimes make here is to also label the hypotenuse as a letter, say *z*, in this case.

Well, it's not really a mistake to label with a letter, but it will often lead to problem down the road. Had we labeled the hypotenuse *z* then the Pythagorean theorem and its derivative would have been,

$$x^2 + y^2 = z^2 \qquad \to \qquad 2xx' + 2yy' = 2zz'$$

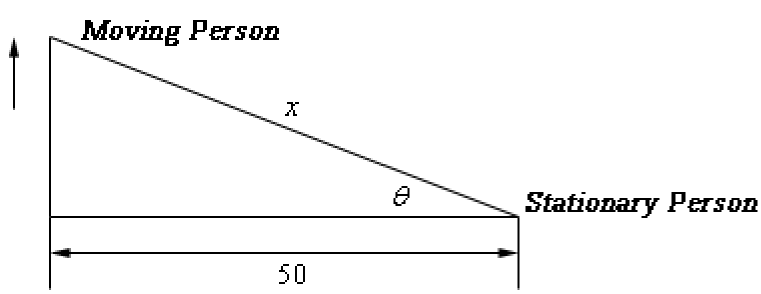
Again, there is nothing wrong with doing this but it does require that we acknowledge the values of two more quantities, *z* and z'. Because *z* is just the hypotenuse that is clearly $z = 15$. The problem that some students then sometimes run into is determining the value of z'. In this case, we have to remember that because the ladder, and hence the hypotenuse has a fixed length, its length can't be changing and so $z' = 0$.

Plugging both of these values into the derivative give us same equation that we got in the example but required a little more effort to get to. It would have been easier to just label the hypotenuse 15 to start off with and not have to worry about remembering that $z' = 0$.

When labeling a fixed quantity (the length of the ladder in this example) with a letter it is sometimes easy to forget that it is a fixed quantity and so it's derivative must be zero. If you don't remember this, the problem becomes impossible to finish as you will have two unknown quantities that you have to deal with. In any problem were a quantity is fixed and will never over the course of the problem change it is always best to just acknowledge that and label it with its value rather than with a letter.

Of course, if we'd had a sliding ladder that was allowed to change then we would have to label it with a letter. However, for that kind of problem we would also need some more information in the problem statement in order to actually do the problem. The practice problems in this section have several problems in which all three sides of a right triangle are changing. You should check them out and see if you can work them.

Example 3 Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. At what rate is distance between the two people changing when $\theta = 0.5$ radians?



Solution

This example is not as tricky as it might at first appear. Let's call the distance between them at any point in time *x* as noted above. We can then relate all the known quantities by one of two trig formulas.

$$\cos\theta = \frac{50}{x} \qquad\qquad \sec\theta = \frac{x}{50}$$

We want to find x' and we could find *x* if we wanted to at the point in question using cosine since we also know the angle at that point in time. However, if we use the second formula we won't need to know *x* as you'll see. So, let's differentiate that formula.

$$\sec\theta\tan\theta\ \theta' = \frac{x'}{50}$$

As noted, there are no *x*'s in this formula. We want to determine x' and we know that $\theta = 0.5$ and $\theta' = 0.01$ (do you agree with it being positive?). So, just plug in and solve.

$$(50)(0.01)\sec(0.5)\tan(0.5) = x' \qquad \Rightarrow \qquad x' = 0.311254 \text{ ft / min}$$

So far we we've seen three related rates problems. While each one was worked in a very different manner the process was essentially the same in each. In each problem we identified what we were given and what we wanted to find. We next wrote down a relationship between all the various quantities and used implicit differentiation to arrive at a relationship between the various derivatives in the problem. Finally, we plugged the known quantities into the equation to find the value we were after.

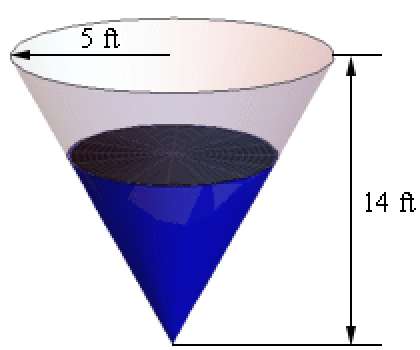
So, in a general sense each problem was worked in pretty much the same manner. The only real difference between them was coming up with the relationship between the known and unknown quantities. This is often the hardest part of the problem. In many problems the best way to come up with the relationship is to sketch a diagram that shows the situation. This often seems like a silly step but can make all the difference in whether we can find the relationship or not.

Let's work another problem that uses some different ideas and shows some of the different kinds of things that can show up in related rates problems.

Example 4 A tank of water in the shape of a cone is leaking water at a constant rate of $2\text{ ft}^3/\text{hour}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

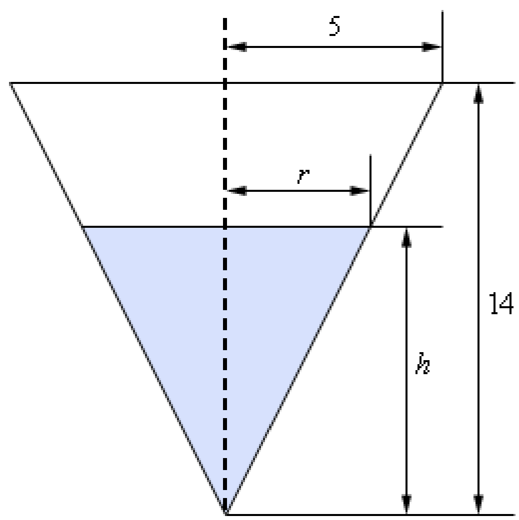
(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?



Solution

Okay, we should probably start off with a quick sketch (probably not to scale) of what is going on here. We'll also be doing the sketch as if we were looking at the tank from directly in front of it (and so the 3D of the tank will not be visible) as this will help a little with seeing what is going on. Showing the 3D nature of the tank is liable to just get in the way. So here is the sketch of the tank with some water in it.



As we can see, the water in the tank actually forms a smaller cone/triangle (depending on which image we are looking at) with the same central angle as the tank itself. The radius of the "water" cone at any time is given by *r* and the height of the "water" cone at any time is given by *h*. The volume of water in the tank at any time *t* is given by,

$$V = \frac{1}{3}\pi r^2 h$$

and we've been given that $V' = -2$.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

For this part we need to determine h' when $h = 6$ and now we have a problem. The only formula that we've got that will relate the volume to the height also includes the radius and so if we were to differentiate this with respect to *t* we would get,

$$V' = \frac{2}{3}\pi r r' h + \frac{1}{3}\pi r^2 h'$$

So, in this equation we know V' and *h* and want to find h', but we don't know *r* and r'. As we'll see finding *r* isn't too bad, but we just don't have enough information, at this point, that will allow us to find r' and h' simultaneously.

To fix this we'll need to eliminate the *r* from the volume formula in some way. This is actually easier than it might at first look. If we go back to our sketch above and look at just the right half of the tank we see that we have two similar triangles and when we say similar we mean similar in the geometric sense. Recall that two triangles are called similar if their angles are identical, which is the case here.

When we have two similar triangles then ratios of any two sides will be equal. For our set this means that we have,

$$\frac{r}{h} = \frac{5}{14} \qquad \Rightarrow \qquad r = \frac{5}{14}h$$

If we take this and plug it into our volume formula we have,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h = \frac{25}{588}\pi h^3$$

This gives us a volume formula that only involved the volume and the height of the water. Note however that this volume formula is only valid for our cone, so don't be tempted to use it for other cones! If we now differentiate this we have,

$$V' = \frac{25}{196}\pi h^2 h'$$

At this point all we need to do is plug in what we know and solve for h'.

$$-2 = \frac{25}{196}\pi\left(6^2\right)h' \qquad \Rightarrow \qquad h' = \frac{-98}{225\pi} = -0.1386$$

So, it looks like the height is decreasing at a rate of 0.1386 ft/hr.

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

In this case we are asking for r' and there is an easy way to do this part and a difficult (well, more difficult than the easy way anyway….) way to do it. The "difficult" way is to redo the work in part (a) above only this time use,

$$\frac{h}{r} = \frac{14}{5} \qquad \Rightarrow \qquad h = \frac{14}{5}r$$

to get the volume in terms of *V* and *r* and then proceed as before.

That's not terribly difficult, but it is more work that we need to so. Recall from the first part that we have,

$$r = \frac{5}{14}h \qquad \Rightarrow \qquad r' = \frac{5}{14}h'$$

So, as we can see if we take the relationship that relates *r* and *h* that we used in the first part and differentiate it we get a relationship between r' and h'. At this point all we need to do here is use the result from the first part to get,

$$r' = \frac{5}{14}\left(\frac{-98}{225\pi}\right) = -\frac{7}{45\pi} = -0.04951$$

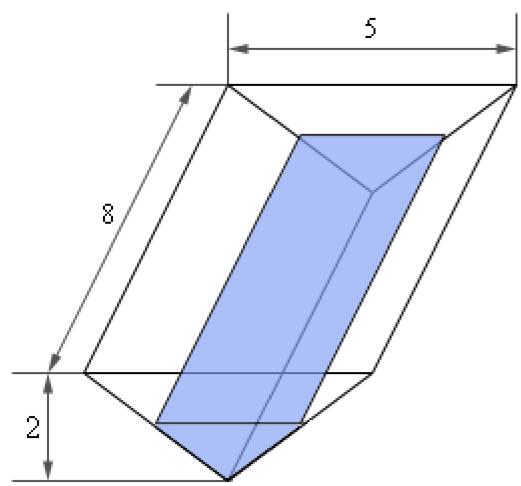
Much easier that redoing all of the first part. Note however, that we were only able to do this the "easier" way because it was asking for r' at exactly the same time that we asked for h' in the first part. If we hadn't been using the same time then we would have had no choice but to do this the "difficult" way.

In the second part of the previous problem we saw an important idea in dealing with related rates. In order to find the asked for rate all we need is an equation that relates the rate we're looking for to a rate that we already know. Sometimes there are multiple equations that we can use and sometimes one will be easier than another.

Also, this problem showed us that we will often have an equation that contains more variables that we have information about and so, in these cases, we will need to eliminate one (or more) of the variables. In this problem we eliminated the extra variable using the idea of similar triangles. This will not always be how we do this, but many of these problems do use similar triangles so make sure you can use that idea.

Let's work some more problems.

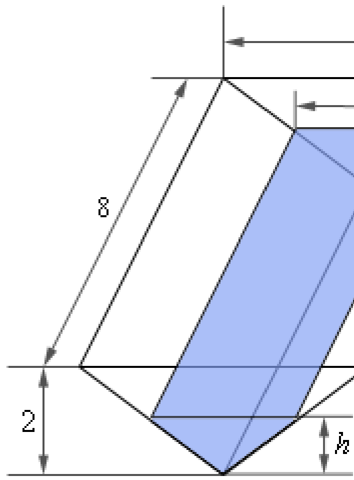
Example 5 A trough of water is 8 meters in length and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of $6\ \mathrm{m}^3/\mathrm{sec}$. At what rate is the height of the water changing when the water has a height of 120 cm? At what rate is the width of the water changing when the water has a height of 120cm?



Solution

Note that an isosceles triangle is just a triangle in which two of the sides are the same length. In our case sides of the tank have the same length.

Let's add in some dimensions for the water to the sketch from above.



Now, in this problem we know that $V' = 6\ \mathrm{m^3/sec}$ and we want to determine h' when $h = 1.2\,\mathrm{m}$. Note that because V' is in terms of meters we need to convert *h* into meters as well. So, we need an equation that will relate these two quantities and the volume of the tank will do it.

The volume of this kind of tank is simple to compute. The volume is the area of the end times the depth. For our case the volume of the water in the tank is,

$$\begin{aligned} V &= (\text{Area of End})(\text{depth}) \\ &= \left(\tfrac{1}{2}\,\text{base} \times \text{height}\right)(\text{depth}) \\ &= \tfrac{1}{2} hw(8) \\ &= 4hw \end{aligned}$$

As with the previous example we've got an extra quantity here, *w*, that is also changing with time and so we need to eliminate it from the problem. To do this we'll again make use of the idea of similar triangles. If we look at the end of the tank we'll see that we again have two similar triangles. One for the tank itself and one formed by the water in the tank. Again, remember that with similar triangles ratios of sides must be equal. In our case we'll use,

$$\frac{w}{5} = \frac{h}{2} \qquad \Rightarrow \qquad w = \frac{5}{2}h$$

Plugging this into the volume gives a formula for the volume (and only for this tank) that only involved the height of the water.

$$V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$$

We can now differentiate this to get,

$$V' = 20hh'$$

Finally, all we need to do is plug in and solve for h'.

$$6 = 20(1.2)h' \qquad \Rightarrow \qquad h' = 0.25\ \mathrm{m/sec}$$

So, the height of the water is rising at a rate of 0.25 m/sec.

In order to answer the second part of this question is not all that difficult.

We will need w' to answer this part and we have the following equation from the similar triangle that relate the width to the height and we can quickly differentiate it to get a relationship between w' and h'.

$$w = \frac{5}{2}h \qquad \Rightarrow \qquad w' = \frac{5}{2}h'$$

From the first part we know the value of h' and so all wee need to do is plug that into this equation and we'll have the answer.

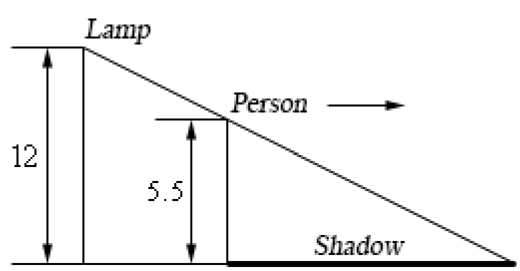
$$w' = \frac{5}{2}(0.25) = 0.625 \text{ m/sec}$$

Therefore the width is increasing at a rate of 0.625 m/sec.

Example 6 A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec.

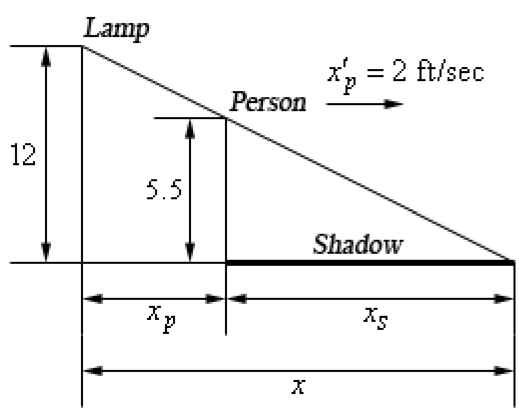
(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?



Solution

Let's start off with putting all the relevant quantities into the sketch from above.



Here *x* is the distance of the tip of the shadow from the pole, x_p is the distance of the person from the pole and x_s is the length of the shadow. Also note that we converted the persons height over to 5.5 feet since all the other measurements are in feet.

The tip of the shadow is defined by the rays of light just getting past the person and so we can see they form a set of similar triangles. This will be useful down the road.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

In this case we want to determine x' when $x_p = 25$ given that $x'_p = 2$.

The equation we'll need here is,

$$x = x_p + x_s$$

but we'll need to eliminate x_s from the equation in order to get an answer. To do this we can again make use of the fact that the two triangles are similar to get,

$$\frac{5.5}{12} = \frac{x_s}{x} \qquad\qquad \text{Note : } \frac{5.5}{12} = \frac{\frac{11}{2}}{12} = \frac{11}{24}$$

From this we can quickly see that,

$$x_s = \frac{11}{24}x$$

We can then plug this into the equation above and solve for *x* as follows.

$$x = x_p + x_s = x_p + \frac{11}{24}x \qquad \Rightarrow \qquad x = \frac{24}{13}x_p$$

Now all that we need to do is differentiate this, plug in and solve for x'.

$$x' = \frac{24}{13}x'_p \qquad\qquad \Rightarrow \qquad\qquad x' = \frac{24}{13}(2) = 3.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the pole at a rate of 3.6923 ft/sec. Notice as well that we never actually had to use the fact that $x_p = 25$ for this problem. That will happen on rare occasions.

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

This part is actually quite simple if we have the answer from (a) in hand, which we do of course. In this case we know that x_s represents the length of the shadow, or the distance of the tip of the shadow from the person so it looks like we want to determine x'_s when $x_p = 25$.

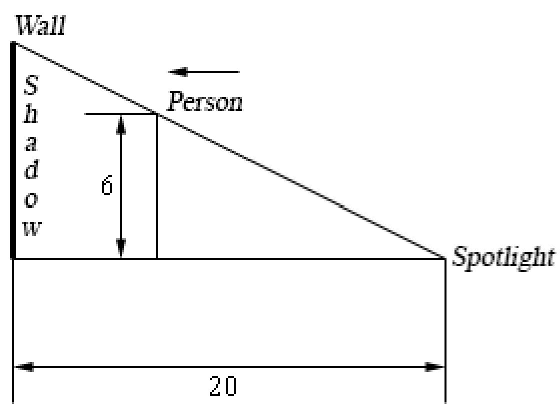
Again, we can use $x = x_p + x_s$, however unlike the first part we now know that $x'_p = 2$ and $x' = 3.6923 \text{ ft/sec}$ so in this case all we need to do is differentiate the equation and plug in for all the known quantities.

$$x' = x'_p + x'_s$$

$$3.6923 = 2 + x'_s \qquad\qquad x'_s = 1.6923 \text{ ft/sec}$$

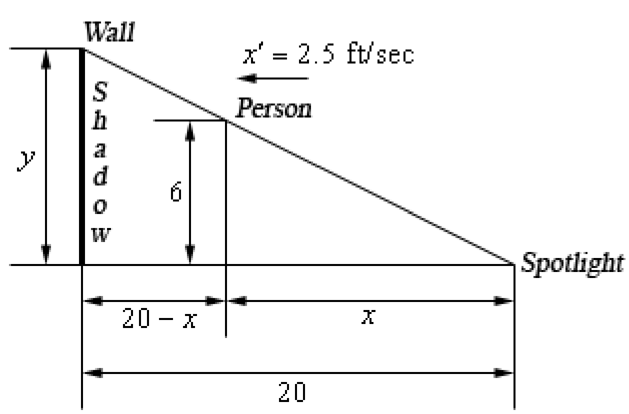
The tip of the shadow is then moving away from the person at a rate of 1.6923 ft/sec.

Example 7 A spot light is on the ground 20 ft away from a wall and a 6 ft tall person is walking towards the wall at a rate of 2.5 ft/sec. How fast is the height of the shadow changing when the person is 8 feet from the wall? Is the shadow increasing or decreasing in height at this time?



Solution

Below is a copy of the sketch in the problem statement with all the relevant quantities added in. The top of the shadow will be defined by the light rays going over the head of the person and so we again get yet another set of similar triangles.



In this case we want to determine y' when the person is 8 ft from wall or $x = 12 \text{ ft}$. Also, if the person is moving towards the wall at 2.5 ft/sec then the person must be moving away from the spotlight at 2.5 ft/sec and so we also know that $x' = 2.5$.

In all the previous problems that used similar triangles we used the similar triangles to eliminate one of the variables from the equation we were working with. In this case however, we can get the equation that relates *x* and *y* directly from the two similar triangles. In this case the equation we're going to work with is,

$$\frac{y}{6} = \frac{20}{x} \qquad \Rightarrow \qquad y = \frac{120}{x}$$

Now all that we need to do is differentiate and plug values into solve to get y'.

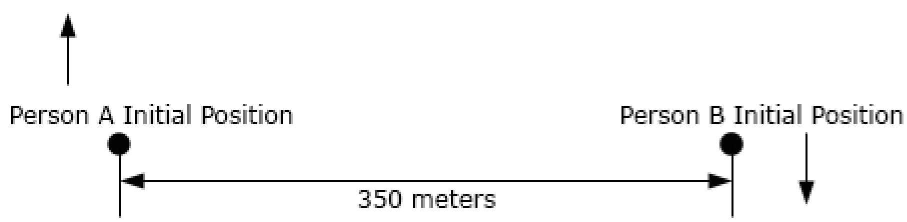
$$y' = -\frac{120}{x^2}x' \qquad \Rightarrow \qquad y' = -\frac{120}{12^2}(2.5) = -2.0833 \text{ ft/sec}$$

The height of the shadow is then decreasing at a rate of 2.0833 ft/sec.

Okay, we've worked quite a few problems now that involved similar triangles in one form or another so make sure you can do these kinds of problems.

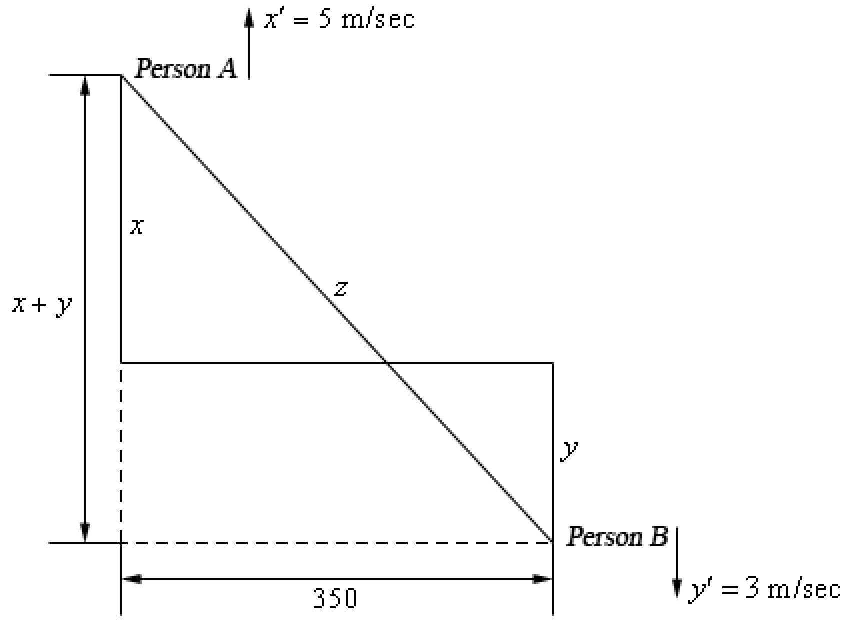
It's now time to do a problem that while similar to some of the problems we've done to this point is also sufficiently different that it can cause problems until you've seen how to do it.

Example 8 Two people on bikes are separated by 350 meters. Person A starts riding north at a rate of 5 m/sec and 7 minutes later Person B starts riding south at 3 m/sec. At what rate is the distance separating the two people changing 25 minutes after Person A starts riding?



Solution

There is a lot to digest here with this problem. Let's start off with a sketch of the situation that shows each person's location sometime after both people start riding.



Now we are after z' and we know that $x' = 5$ and $y' = 3$. We want to know z' after Person A had been riding for 25 minutes and Person B has been riding for $25 - 7 = 18$ minutes. After converting these times to seconds (because our rates are all in m/sec) this means that at the time we're interested in each of the bike riders has rode,

$$x = 5(25 \times 60) = 7500 \text{ m} \qquad\qquad y = 3(18 \times 60) = 3240 \text{ m}$$

Next, the Pythagorean theorem tells us that,

$$z^2 = (x+y)^2 + 350^2 \tag{2}$$

Therefore, 25 minutes after Person A starts riding the two bike riders are

$$z = \sqrt{(x+y)^2 + 350^2} = \sqrt{(7500+3240)^2 + 350^2} = 10745.7015 \text{ m}$$

apart.

To determine the rate at which the two riders are moving apart all we need to do then is differentiate (2) and plug in all the quantities that we know to find z'.

$$\begin{aligned} 2zz' &= 2(x+y)(x'+y') \\ 2(10745.7015)z' &= 2(7500+3240)(5+3) \\ z' &= 7.9958 \text{ m/sec} \end{aligned}$$

So, the two riders are moving apart at a rate of 7.9958 m/sec.

Every problem that we've worked to this point has come down to needing a geometric formula and we should probably work a quick problem that is not geometric in nature.

Example 9 Suppose that we have two resistors connected in parallel with resistances R_1 and R_2 measured in ohms (Ω). The total resistance, *R*, is then given by,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that R_1 is increasing at a rate of 0.4 Ω/min and R_2 is decreasing at a rate of 0.7Ω/min. At what rate is *R* changing when $R_1 = 80\,\Omega$ and $R_2 = 105\,\Omega$?

Solution

Okay, unlike the previous problems there really isn't a whole lot to do here. First, let's note that we're looking for R' and that we know $R_1' = 0.4$ and $R_2' = -0.7$. Be careful with the signs here.

Also, since we'll eventually need it let's determine *R* at the time we're interested in.

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{105} = \frac{37}{1680} \qquad \Rightarrow \qquad R = \frac{1680}{37} = 45.4054\,\Omega$$

Next, we need to differentiate the equation given in the problem statement.

$$-\frac{1}{R^2}R' = -\frac{1}{\left(R_1\right)^2}R_1' - \frac{1}{\left(R_2\right)^2}R_2'$$

$$R' = R^2\left(\frac{1}{\left(R_1\right)^2}R_1' + \frac{1}{\left(R_2\right)^2}R_2'\right)$$

Finally, all we need to do is plug into this and do some quick computations.

$$R' = \left(45.4054\right)^2\left(\frac{1}{80^2}\left(0.4\right) + \frac{1}{105^2}\left(-0.7\right)\right) = -0.002045$$

So, it looks like *R* is decreasing at a rate of 0.002045Ω/min.

We've seen quite a few related rates problems in this section that cover a wide variety of possible problems. There are still many more different kinds of related rates problems out there in the world, but the ones that we've worked here should give you a pretty good idea on how to at least start most of the problems that you're liable to run into.

Section 3-12 : Higher Order Derivatives

Let's start this section with the following function.

$$f\left(x\right)=5x^{3}-3x^{2}+10x-5$$

By this point we should be able to differentiate this function without any problems. Doing this we get,

$$f'\left(x\right)=15x^{2}-6x+10$$

Now, this is a function and so it can be differentiated. Here is the notation that we'll use for that, as well as the derivative.

$$f''\left(x\right)=\left(f'\left(x\right)\right)'=30x-6$$

This is called the **second derivative** and $f'\left(x\right)$ is now called the **first derivative**.

Again, this is a function, so we can differentiate it again. This will be called the **third derivative**. Here is that derivative as well as the notation for the third derivative.

$$f'''\left(x\right)=\left(f''\left(x\right)\right)'=30$$

Continuing, we can differentiate again. This is called, oddly enough, the **fourth derivative**. We're also going to be changing notation at this point. We can keep adding on primes, but that will get cumbersome after a while.

$$f^{(4)}\left(x\right)=\left(f'''\left(x\right)\right)'=0$$

This process can continue but notice that we will get zero for all derivatives after this point. This set of derivatives leads us to the following fact about the differentiation of polynomials.

Fact

If $p\left(x\right)$ is a polynomial of degree *n* (*i.e.* the largest exponent in the polynomial) then,

$$p^{(k)}\left(x\right)=0 \qquad\qquad \text{for } k\geq n+1$$

We will need to be careful with the "non-prime" notation for derivatives. Consider each of the following.

$$f^{(2)}\left(x\right)=f''\left(x\right)$$

$$f^{2}\left(x\right)=\left[f\left(x\right)\right]^{2}$$

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Collectively the second, third, fourth, *etc*. derivatives are called **higher order derivatives**.

Let's take a look at some examples of higher order derivatives.

Example 1 Find the first four derivatives for each of the following.

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

(b) $y = \cos x$

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

Solution

(a) $R(t) = 3t^2 + 8t^{\frac{1}{2}} + \mathbf{e}^t$

There really isn't a lot to do here other than do the derivatives.

$$R'(t) = 6t + 4t^{-\frac{1}{2}} + \mathbf{e}^t$$

$$R''(t) = 6 - 2t^{-\frac{3}{2}} + \mathbf{e}^t$$

$$R'''(t) = 3t^{-\frac{5}{2}} + \mathbf{e}^t$$

$$R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + \mathbf{e}^t$$

Notice that differentiating an exponential function is very simple. It doesn't change with each differentiation.

(b) $y = \cos x$

Again, let's just do some derivatives.

$$y = \cos x$$
$$y' = -\sin x$$
$$y'' = -\cos x$$
$$y''' = \sin x$$
$$y^{(4)} = \cos x$$

Note that cosine (and sine) will repeat every four derivatives. The other four trig functions will not exhibit this behavior. You might want to take a few derivatives to convince yourself of this.

(c) $f(y) = \sin(3y) + \mathbf{e}^{-2y} + \ln(7y)$

In the previous two examples we saw some patterns in the differentiation of exponential functions, cosines and sines. We need to be careful however since they only work if there is just a *t* or an *x* in the argument. This is the point of this example. In this example we will need to use the chain rule on each derivative.

$$\begin{aligned} f'(y) &= 3\cos(3y) - 2\mathbf{e}^{-2y} + \frac{1}{y} = 3\cos(3y) - 2\mathbf{e}^{-2y} + y^{-1} \\ f''(y) &= -9\sin(3y) + 4\mathbf{e}^{-2y} - y^{-2} \\ f'''(y) &= -27\cos(3y) - 8\mathbf{e}^{-2y} + 2y^{-3} \\ f^{(4)}(y) &= 81\sin(3y) + 16\mathbf{e}^{-2y} - 6y^{-4} \end{aligned}$$

So, we can see with slightly more complicated arguments the patterns that we saw for exponential functions, sines and cosines no longer completely hold.

Let's do a couple more examples to make a couple of points.

Example 2 Find the second derivative for each of the following functions.

(a) $Q(t) = \sec(5t)$

(b) $g(w) = \mathbf{e}^{1-2w^3}$

(c) $f(t) = \ln(1+t^2)$

Solution

(a) $Q(t) = \sec(5t)$

Here's the first derivative.

$$Q'(t) = 5\sec(5t)\tan(5t)$$

Notice that the second derivative will now require the product rule.

$$\begin{aligned} Q''(t) &= 25\sec(5t)\tan(5t)\tan(5t) + 25\sec(5t)\sec^2(5t) \\ &= 25\sec(5t)\tan^2(5t) + 25\sec^3(5t) \end{aligned}$$

Notice that each successive derivative will require a product and/or chain rule and that as noted above this will not end up returning back to just a secant after four (or another other number for that matter) derivatives as sine and cosine will.

(b) $g(w) = \mathbf{e}^{1-2w^3}$

Again, let's start with the first derivative.

$$g'(w) = -6w^2\mathbf{e}^{1-2w^3}$$

As with the first example we will need the product rule for the second derivative.

$$\begin{aligned} g''(w) &= -12w\mathbf{e}^{1-2w^3} - 6w^2\left(-6w^2\right)\mathbf{e}^{1-2w^3} \\ &= -12w\mathbf{e}^{1-2w^3} + 36w^4\mathbf{e}^{1-2w^3} \end{aligned}$$

(c) $f(t) = \ln(1+t^2)$

Same thing here.

$$f'(t) = \frac{2t}{1+t^2}$$

The second derivative this time will require the quotient rule.

$$f''(t) = \frac{2(1+t^2)-(2t)(2t)}{(1+t^2)^2}$$
$$= \frac{2-2t^2}{(1+t^2)^2}$$

As we saw in this last set of examples we will often need to use the product or quotient rule for the higher order derivatives, even when the first derivative didn't require these rules.

Let's work one more example that will illustrate how to use implicit differentiation to find higher order derivatives.

Example 3 Find y'' for

$$x^2 + y^4 = 10$$

Solution

Okay, we know that in order to get the second derivative we need the first derivative and in order to get that we'll need to do implicit differentiation. Here is the work for that.

$$2x + 4y^3 y' = 0$$
$$y' = -\frac{x}{2y^3}$$

Now, this is the first derivative. We get the second derivative by differentiating this, which will require implicit differentiation again.

$$y'' = \left(-\frac{x}{2y^3}\right)'$$
$$= -\frac{2y^3 - x(6y^2 y')}{(2y^3)^2}$$
$$= -\frac{2y^3 - 6xy^2 y'}{4y^6}$$
$$= -\frac{y - 3xy'}{2y^4}$$

This is fine as far as it goes. However, we would like there to be no derivatives in the answer. We don't, generally, mind having *x*'s and/or *y*'s in the answer when doing implicit differentiation, but we really don't like derivatives in the answer. We can get rid of the derivative however by acknowledging that we know what the first derivative is and substituting this into the second derivative equation. Doing this gives,

$$\begin{aligned} y'' &= -\frac{y-3xy'}{2y^4} \\ &= -\frac{y-3x\left(-\frac{x}{2y^3}\right)}{2y^4} \\ &= -\frac{y+\frac{3}{2}x^2y^{-3}}{2y^4} \end{aligned}$$

Now that we've found some higher order derivatives we should probably talk about an interpretation of the second derivative.

If the position of an object is given by $s(t)$ we know that the velocity is the first derivative of the position.

$$v(t) = s'(t)$$

The acceleration of the object is the first derivative of the velocity, but since this is the first derivative of the position function we can also think of the acceleration as the second derivative of the position function.

$$a(t) = v'(t) = s''(t)$$

Alternate Notation

There is some alternate notation for higher order derivatives as well. Recall that there was a fractional notation for the first derivative.

$$f'(x) = \frac{df}{dx}$$

We can extend this to higher order derivatives.

$$f''(x) = \frac{d^2y}{dx^2} \qquad\qquad f'''(x) = \frac{d^3y}{dx^3} \qquad\qquad etc.$$

Section 3-13 : Logarithmic Differentiation

There is one last topic to discuss in this section. Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

It's easiest to see how this works in an example.

Example 1 Differentiate the function.

$$y = \frac{x^5}{\left(1-10x\right)\sqrt{x^2+2}}$$

Solution

Differentiating this function could be done with a product rule and a quotient rule. However, that would be a fairly messy process. We can simplify things somewhat by taking logarithms of both sides.

$$\ln y = \ln\left(\frac{x^5}{\left(1-10x\right)\sqrt{x^2+2}}\right)$$

Of course, this isn't really simpler. What we need to do is use the properties of logarithms to expand the right side as follows.

$$\ln y = \ln\left(x^5\right) - \ln\left(\left(1-10x\right)\sqrt{x^2+2}\right)$$
$$\ln y = \ln\left(x^5\right) - \ln\left(1-10x\right) - \ln\left(\sqrt{x^2+2}\right)$$

This doesn't look all that simple. However, the differentiation process will be simpler. What we need to do at this point is differentiate both sides with respect to *x*. Note that this is really **implicit differentiation**.

$$\frac{y'}{y} = \frac{5x^4}{x^5} - \frac{-10}{1-10x} - \frac{\frac{1}{2}\left(x^2+2\right)^{-\frac{1}{2}}\left(2x\right)}{\left(x^2+2\right)^{\frac{1}{2}}}$$
$$\frac{y'}{y} = \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

To finish the problem all that we need to do is multiply both sides by *y* and the plug in for *y* since we do know what that is.

$$y' = y\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$
$$= \frac{x^5}{\left(1-10x\right)\sqrt{x^2+2}}\left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

Depending upon the person, doing this would probably be slightly easier than doing both the product and quotient rule. The answer is almost definitely simpler than what we would have gotten using the product and quotient rule.

So, as the first example has shown we can use logarithmic differentiation to avoid using the product rule and/or quotient rule.

We can also use logarithmic differentiation to differentiate functions in the form.

$$y = \left(f(x)\right)^{g(x)}$$

Let's take a quick look at a simple example of this.

Example 2 Differentiate $y = x^x$

Solution

We've seen two functions similar to this at this point.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

Neither of these two will work here since both require either the base or the exponent to be a constant. In this case both the base and the exponent are variables and so we have no way to differentiate this function using only known rules from previous sections.

With logarithmic differentiation we can do this however. First take the logarithm of both sides as we did in the first example and use the logarithm properties to simplify things a little.

$$\begin{aligned} \ln y &= \ln x^x \\ \ln y &= x \ln x \end{aligned}$$

Differentiate both sides using implicit differentiation.

$$\frac{y'}{y} = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

As with the first example multiply by *y* and substitute back in for *y*.

$$\begin{aligned} y' &= y\left(1 + \ln x\right) \\ &= x^x\left(1 + \ln x\right) \end{aligned}$$

Let's take a look at a more complicated example of this.

Example 3 Differentiate $y = (1-3x)^{\cos(x)}$

Solution

Now, this looks much more complicated than the previous example, but is in fact only slightly more complicated. The process is pretty much identical, so we first take the log of both sides and then simplify the right side.

$$\ln y = \ln\left[(1-3x)^{\cos(x)}\right] = \cos(x)\ln(1-3x)$$

Next, do some implicit differentiation.

$$\frac{y'}{y} = -\sin(x)\ln(1-3x) + \cos(x)\frac{-3}{1-3x} = -\sin(x)\ln(1-3x) - \cos(x)\frac{3}{1-3x}$$

Finally, solve for y' and substitute back in for *y*.

$$\begin{aligned} y' &= -y\left(\sin(x)\ln(1-3x) + \cos(x)\frac{3}{1-3x}\right) \\ &= -(1-3x)^{\cos(x)}\left(\sin(x)\ln(1-3x) + \cos(x)\frac{3}{1-3x}\right) \end{aligned}$$

A messy answer but there it is.

We'll close this section out with a quick recap of all the various ways we've seen of differentiating functions with exponents. It is important to not get all of these confused.

$\frac{d}{dx}\left(a^b\right) = 0$	This is a constant
$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	Power Rule
$\frac{d}{dx}\left(a^x\right) = a^x \ln a$	Derivative of an exponential function
$\frac{d}{dx}\left(x^x\right) = x^x(1+\ln x)$	Logarithmic Differentiation

It is sometimes easy to get these various functions confused and use the wrong rule for differentiation. Always remember that each rule has very specific rules for where the variable and constants must be. For example, the Power Rule requires that the base be a variable and the exponent be a constant, while the exponential function requires exactly the opposite.

If you can keep straight all the rules you can't go wrong with these.