CALCULUS I

Applications of Derivatives

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Chapter 4 : Applications of Derivatives

In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus on applications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to determine information about graphs of functions and optimization problems. These will not be the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously. We will also see how derivatives can be used to estimate solutions to equations.

Here is a listing of the topics in this section.

Rates of Change – In this section we review the main application/interpretation of derivatives from the previous chapter (i.e. rates of change) that we will be using in many of the applications in this chapter.

Critical Points – In this section we give the definition of critical points. Critical points will show up in most of the sections in this chapter, so it will be important to understand them and how to find them. We will work a number of examples illustrating how to find them for a wide variety of functions.

Minimum and Maximum Values – In this section we define absolute (or global) minimum and maximum values of a function and relative (or local) minimum and maximum values of a function. It is important to understand the difference between the two types of minimum/maximum (collectively called extrema) values for many of the applications in this chapter and so we use a variety of examples to help with this. We also give the Extreme Value Theorem and Fermat's Theorem, both of which are very important in the many of the applications we'll see in this chapter.

Finding Absolute Extrema – In this section we discuss how to find the absolute (or global) minimum and maximum values of a function. In other words, we will be finding the largest and smallest values that a function will have.

The Shape of a Graph, Part I – In this section we will discuss what the first derivative of a function can tell us about the graph of a function. The first derivative will allow us to identify the relative (or local) minimum and maximum values of a function and where a function will be increasing and decreasing. We will also give the First Derivative test which will allow us to classify critical points as relative minimums, relative maximums or neither a minimum or a maximum.

The Shape of a Graph, Part II – In this section we will discuss what the second derivative of a function can tell us about the graph of a function. The second derivative will allow us to determine where the graph of a function is concave up and concave down. The second derivative will also allow us to identify any inflection points (i.e. where concavity changes) that a function may have. We will also give the Second Derivative Test that will give an alternative method for identifying some critical points (but not all) as relative minimums or relative maximums.

The Mean Value Theorem – In this section we will give Rolle's Theorem and the Mean Value Theorem. With the Mean Value Theorem we will prove a couple of very nice facts, one of which will be very useful in the next chapter.

Optimization Problems – In this section we will be determining the absolute minimum and/or maximum of a function that depends on two variables given some constraint, or relationship, that the two variables must always satisfy. We will discuss several methods for determining the absolute minimum or maximum of the function. Examples in this section tend to center around geometric objects such as squares, boxes, cylinders, etc.

More Optimization Problems – In this section we will continue working optimization problems. The examples in this section tend to be a little more involved and will often involve situations that will be more easily described with a sketch as opposed to the 'simple' geometric objects we looked at in the previous section.

L'Hospital's Rule and Indeterminate Forms – In this section we will revisit indeterminate forms and limits and take a look at L'Hospital's Rule. L'Hospital's Rule will allow us to evaluate some limits we were not able to previously.

Linear Approximations – In this section we discuss using the derivative to compute a linear approximation to a function. We can use the linear approximation to a function to approximate values of the function at certain points. While it might not seem like a useful thing to do with when we have the function there really are reasons that one might want to do this. We give two ways this can be useful in the examples.

Differentials – In this section we will compute the differential for a function. We will give an application of differentials in this section. However, one of the more important uses of differentials will come in the next chapter and unfortunately we will not be able to discuss it until then.

Newton's Method – In this section we will discuss Newton's Method. Newton's Method is an application of derivatives will allow us to approximate solutions to an equation. There are many equations that cannot be solved directly and with this method we can get approximations to the solutions to many of those equations.

Business Applications – In this section we will give a cursory discussion of some basic applications of derivatives to the business field. We will revisit finding the maximum and/or minimum function value and we will define the marginal cost function, the average cost, the revenue function, the marginal revenue function and the marginal profit function. Note that this section is only intended to introduce these concepts and not teach you everything about them.

Section 4-1 : Rates of Change

The purpose of this section is to remind us of one of the more important applications of derivatives. That is the fact that $f'(x)$ represents the rate of change of $f(x)$. This is an application that we repeatedly saw in the previous chapter. Almost every section in the previous chapter contained at least one problem dealing with this application of derivatives. While this application will arise occasionally in this chapter we are going to focus more on other applications in this chapter.

So, to make sure that we don't forget about this application here is a brief set of examples concentrating on the rate of change application of derivatives. Note that the point of these examples is to remind you of material covered in the previous chapter and not to teach you how to do these kinds of problems. If you don't recall how to do these kinds of examples you'll need to go back and review the previous chapter.

Example 1 Determine all the points where the following function is not changing.

$$g(x) = 5 - 6x - 10\cos(2x)$$

Solution

First, we'll need to take the derivative of the function.

$$g'(x) = -6 + 20\sin(2x)$$

Now, the function will not be changing if the rate of change is zero and so to answer this question we need to determine where the derivative is zero. So, let's set this equal to zero and solve.

$$-6 + 20\sin(2x) = 0 \qquad \Rightarrow \qquad \sin(2x) = \frac{6}{20} = 0.3$$

The solution to this is then,

$$2x = 0.3047 + 2\pi n \qquad \text{OR} \qquad 2x = 2.8369 + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$x = 0.1524 + \pi n \qquad \text{OR} \qquad x = 1.4185 + \pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

If you don't recall how to solve trig equations check out the **Solving Trig Equations** sections in the Review Chapter.

Example 2 Determine where the following function is increasing and decreasing.

$$A(t) = 27t^5 - 45t^4 - 130t^3 + 150$$

Solution

As with the first problem we first need to take the derivative of the function.

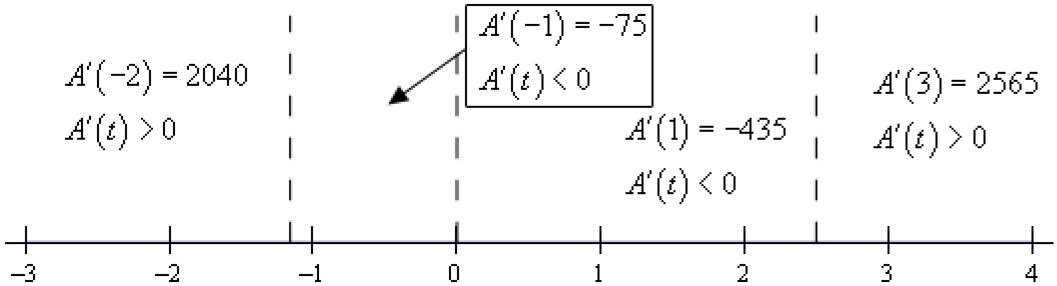
$$A'(t) = 135t^4 - 180t^3 - 390t^2 = 15t^2\left(9t^2 - 12t - 26\right)$$

Next, we need to determine where the function isn't changing. This is at,

$t = 0$

$$t = \frac{12 \pm \sqrt{144 - 4(9)(-26)}}{18} = \frac{12 \pm \sqrt{1080}}{18} = \frac{12 \pm 6\sqrt{30}}{18} = \frac{2 \pm \sqrt{30}}{3} = -1.159, \quad 2.492$$

So, the function is not changing at three values of *t*. Finally, to determine where the function is increasing or decreasing we need to determine where the derivative is positive or negative. Recall that if the derivative is positive then the function must be increasing and if the derivative is negative then the function must be decreasing. The following number line gives this information.



So, from this number line we can see that we have the following increasing and decreasing information.

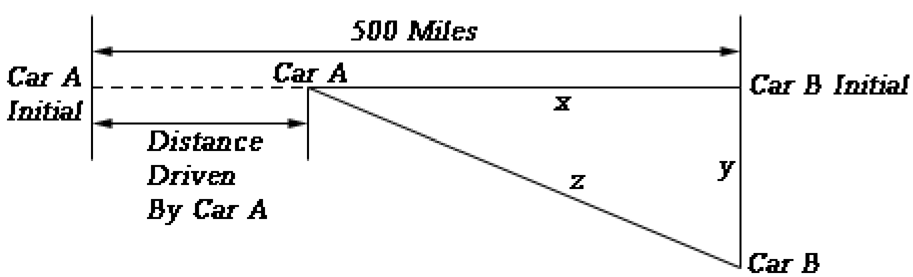
$$\text{Increasing}: -\infty < t < -1.159, \ 2.492 < t < \infty \qquad \text{Decreasing}: \ -1.159 < t < 0, \ 0 < t < 2.492$$

If you don't remember how to solve polynomial and rational inequalities then you should check out the appropriate sections in the Review Chapter.

Finally, we can't forget about **Related Rates** problems.

Example 3 Two cars start out 500 miles apart. Car A is to the west of Car B and starts driving to the east (*i.e.* towards Car B) at 35 mph and at the same time Car B starts driving south at 50 mph. After 3 hours of driving at what rate is the distance between the two cars changing? Is it increasing or decreasing?

Solution
The first thing to do here is to get sketch a figure showing the situation.



In this figure *y* represents the distance driven by Car B and *x* represents the distance separating Car A from Car B's initial position and *z* represents the distance separating the two cars. After 3 hours driving time with have the following values of *x* and *y*.

$$x = 500 - 35(3) = 395 \qquad\qquad y = 50(3) = 150$$

We can use the Pythagorean theorem to find *z* at this time as follows,

$$z^2 = 395^2 + 150^2 = 178525 \qquad \Rightarrow \qquad z = \sqrt{178525} = 422.5222$$

Now, to answer this question we will need to determine z' given that $x' = -35$ and $y' = 50$. Do you agree with the signs on the two given rates? Remember that a rate is negative if the quantity is decreasing and positive if the quantity is increasing.

We can again use the Pythagorean theorem here. First, write it down and the remember that *x*, *y*, and *z* are all changing with time and so differentiate the equation using **Implicit Differentiation**.

$$z^2 = x^2 + y^2 \qquad \Rightarrow \qquad 2zz' = 2xx' + 2yy'$$

Finally, all we need to do is cancel a two from everything, plug in for the known quantities and solve for z'.

$$z'(422.5222) = (395)(-35) + (150)(50) \qquad \Rightarrow \qquad z' = \frac{-6325}{422.5222} = -14.9696$$

So, after three hours the distance between them is decreasing at a rate of 14.9696 mph.

So, in this section we covered three "standard" problems using the idea that the derivative of a function gives the rate of change of the function. As mentioned earlier, this chapter will be focusing more on other applications than the idea of rate of change, however, we can't forget this application as it is a very important one.

Section 4-2 : Critical Points

Critical points will show up throughout a majority of this chapter so we first need to define them and work a few examples before getting into the sections that actually use them.

Definition

We say that $x = c$ is a critical point of the function $f\left(x\right)$ if $f\left(c\right)$ exists and if either of the following are true.

$$f'\left(c\right) = 0 \qquad \text{OR} \qquad f'\left(c\right) \text{ doesn't exist}$$

Note that we require that $f\left(c\right)$ exists in order for $x = c$ to actually be a critical point. This is an important, and often overlooked, point. What this is really saying is that all critical points must be in the domain of the function. If a point is not in the domain of the function then it is not a critical point.

Note as well that, at this point, we only work with real numbers and so any complex numbers that might arise in finding critical points (and they will arise on occasion) will be ignored. There are portions of calculus that work a little differently when working with complex numbers and so in a first calculus class such as this we ignore complex numbers and only work with real numbers. Calculus with complex numbers is beyond the scope of this course and is usually taught in higher level mathematics courses.

The main point of this section is to work some examples finding critical points. So, let’s work some examples.

Example 1 Determine all the critical points for the function.

$$f\left(x\right) = 6x^5 + 33x^4 - 30x^3 + 100$$

Solution

We first need the derivative of the function in order to find the critical points and so let’s get that and notice that we’ll factor it as much as possible to make our life easier when we go to find the critical points.

$$\begin{aligned} f'\left(x\right) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2\left(5x^2 + 22x - 15\right) \\ &= 6x^2\left(5x - 3\right)\left(x + 5\right) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere. Therefore, the only critical points will be those values of *x* which make the derivative zero. So, we must solve.

$$6x^2\left(5x - 3\right)\left(x + 5\right) = 0$$

Because this is the factored form of the derivative it’s pretty easy to identify the three critical points. They are,

$$x = -5, \qquad x = 0, \qquad x = \frac{3}{5}$$

Polynomials are usually fairly simple functions to find critical points for provided the degree doesn't get so large that we have trouble finding the roots of the derivative.

Most of the more "interesting" functions for finding critical points aren't polynomials however. So let's take a look at some functions that require a little more effort on our part.

Example 2 Determine all the critical points for the function.

$$g\left(t\right)=\sqrt[3]{t^2}\left(2t-1\right)$$

Solution

To find the derivative it's probably easiest to do a little simplification before we actually differentiate. Let's multiply the root through the parenthesis and simplify as much as possible. This will allow us to avoid using the product rule when taking the derivative.

$$g\left(t\right)=t^{\frac{2}{3}}\left(2t-1\right)=2t^{\frac{5}{3}}-t^{\frac{2}{3}}$$

Now differentiate.

$$g'\left(t\right)=\frac{10}{3}t^{\frac{2}{3}}-\frac{2}{3}t^{-\frac{1}{3}}=\frac{10t^{\frac{2}{3}}}{3}-\frac{2}{3t^{\frac{1}{3}}}$$

We will need to be careful with this problem. When faced with a negative exponent it is often best to eliminate the minus sign in the exponent as we did above. This isn't really required but it can make our life easier on occasion if we do that.

Notice as well that eliminating the negative exponent in the second term allows us to correctly identify why $t=0$ is a critical point for this function. Once we move the second term to the denominator we can clearly see that the derivative doesn't exist at $t=0$ and so this will be a critical point. If you don't get rid of the negative exponent in the second term many people will incorrectly state that $t=0$ is a critical point because the derivative is zero at $t=0$. While this may seem like a silly point, after all in each case $t=0$ is identified as a critical point, it is sometimes important to know why a point is a critical point. In fact, in a couple of sections we'll see a fact that only works for critical points in which the derivative is zero.

So, we've found one critical point (where the derivative doesn't exist), but we now need to determine where the derivative is zero (provided it is of course…). To help with this it's usually best to combine the two terms into a single rational expression. So, getting a common denominator and combining gives us,

$$g'\left(t\right)=\frac{10t-2}{3t^{\frac{1}{3}}}$$

Notice that we still have $t=0$ as a critical point. Doing this kind of combining should never lose critical points, it's only being done to help us find them. As we can see it's now become much easier to quickly determine where the derivative will be zero. Recall that a rational expression will only be zero if its numerator is zero (and provided the denominator isn't also zero at that point of course).

So, in this case we can see that the numerator will be zero if $t = \frac{1}{5}$ and so there are two critical points for this function.

$$t = 0 \qquad \text{and} \qquad t = \frac{1}{5}$$

Example 3 Determine all the critical points for the function.

$$R(w) = \frac{w^2 + 1}{w^2 - w - 6}$$

Solution

We'll leave it to you to verify that using the quotient rule, along with some simplification, we get that the derivative is,

$$R'(w) = \frac{-w^2 - 14w + 1}{\left(w^2 - w - 6\right)^2} = -\frac{w^2 + 14w - 1}{\left(w^2 - w - 6\right)^2}$$

Notice that we factored a "-1" out of the numerator to help a little with finding the critical points. This negative out in front will not affect the derivative whether or not the derivative is zero or not exist but will make our work a little easier.

Now, we have two issues to deal with. First the derivative will not exist if there is division by zero in the denominator. So we need to solve,

$$w^2 - w - 6 = (w-3)(w+2) = 0$$

We didn't bother squaring this since if this is zero, then zero squared is still zero and if it isn't zero then squaring it won't make it zero.

So, we can see from this that the derivative will not exist at $w = 3$ and $w = -2$. However, these are NOT critical points since the function will also not exist at these points. Recall that in order for a point to be a critical point the function must actually exist at that point.

At this point we need to be careful. The numerator doesn't factor, but that doesn't mean that there aren't any critical points where the derivative is zero. We can use the quadratic formula on the numerator to determine if the fraction as a whole is ever zero.

$$w = \frac{-14 \pm \sqrt{(14)^2 - 4(1)(-1)}}{2(1)} = \frac{-14 \pm \sqrt{200}}{2} = \frac{-14 \pm 10\sqrt{2}}{2} = -7 \pm 5\sqrt{2}$$

So, we get two critical points. Also, these are not "nice" integers or fractions. This will happen on occasion. Don't get too locked into answers always being "nice". Often they aren't.

Note as well that we only use real numbers for critical points. So, if upon solving the quadratic in the numerator, we had gotten complex number these would not have been considered critical points.

Summarizing, we have two critical points. They are,

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$$w = -7 + 5\sqrt{2}, \quad w = -7 - 5\sqrt{2}$$

Again, remember that while the derivative doesn't exist at $w = 3$ and $w = -2$ neither does the function and so these two points are not critical points for this function.

In the previous example we had to use the quadratic formula to determine some potential critical points. We know that sometimes we will get complex numbers out of the quadratic formula. Just remember that, as mentioned at the start of this section, when that happens we will ignore the complex numbers that arise.

So far all the examples have not had any trig functions, exponential functions, *etc*. in them. We shouldn't expect that to always be the case. So, let's take a look at some examples that don't just involve powers of *x*.

Example 4 Determine all the critical points for the function.

$$y = 6x - 4\cos(3x)$$

Solution

First get the derivative and don't forget to use the chain rule on the second term.

$$y' = 6 + 12\sin(3x)$$

Now, this will exist everywhere and so there won't be any critical points for which the derivative doesn't exist. The only critical points will come from points that make the derivative zero. We will need to solve,

$$6 + 12\sin(3x) = 0$$
$$\sin(3x) = -\frac{1}{2}$$

Solving this equation gives the following.

$$3x = 3.6652 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$3x = 5.7596 + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Don't forget the $2\pi n$ on these! There will be problems down the road in which we will miss solutions without this! Also make sure that it gets put on at this stage! Now divide by 3 to get all the critical points for this function.

$$x = 1.2217 + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$
$$x = 1.9199 + \frac{2\pi n}{3}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

Notice that in the previous example we got an infinite number of critical points. That will happen on occasion so don't worry about it when it happens.

Example 5 Determine all the critical points for the function.

$$h(t) = 10t\mathbf{e}^{3-t^2}$$

Solution

Here's the derivative for this function.

$$h'(t) = 10\mathbf{e}^{3-t^2} + 10t\mathbf{e}^{3-t^2}(-2t) = 10\mathbf{e}^{3-t^2} - 20t^2\mathbf{e}^{3-t^2}$$

Now, this looks unpleasant, however with a little factoring we can clean things up a little as follows,

$$h'(t) = 10\mathbf{e}^{3-t^2}(1-2t^2)$$

This function will exist everywhere, so no critical points will come from the derivative not existing. Determining where this is zero is easier than it looks. We know that exponentials are never zero and so the only way the derivative will be zero is if,

$$\begin{aligned} 1 - 2t^2 &= 0 \\ 1 &= 2t^2 \\ \frac{1}{2} &= t^2 \end{aligned}$$

We will have two critical points for this function.

$$t = \pm\frac{1}{\sqrt{2}}$$

Example 6 Determine all the critical points for the function.

$$f(x) = x^2 \ln(3x) + 6$$

Solution

Before getting the derivative let's notice that since we can't take the log of a negative number or zero we will only be able to look at $x > 0$.

The derivative is then,

$$\begin{aligned} f'(x) &= 2x\ln(3x) + x^2\left(\frac{3}{3x}\right) \\ &= 2x\ln(3x) + x \\ &= x\left(2\ln(3x) + 1\right) \end{aligned}$$

Now, this derivative will not exist if *x* is a negative number or if $x = 0$, but then again neither will the function and so these are not critical points. Remember that the function will only exist if $x > 0$ and nicely enough the derivative will also only exist if $x > 0$ and so the only thing we need to worry about is where the derivative is zero.

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First note that, despite appearances, the derivative will not be zero for $x = 0$. As noted above the derivative doesn't exist at $x = 0$ because of the natural logarithm and so the derivative can't be zero there!

So, the derivative will only be zero if,

$$\begin{aligned} 2\ln\left(3x\right) + 1 &= 0 \\ \ln\left(3x\right) &= -\frac{1}{2} \end{aligned}$$

Recall that we can solve this by exponentiating both sides.

$$\begin{aligned} \mathbf{e}^{\ln\left(3x\right)} &= \mathbf{e}^{-\frac{1}{2}} \\ 3x &= \mathbf{e}^{-\frac{1}{2}} \\ x &= \frac{1}{3}\mathbf{e}^{-\frac{1}{2}} = \frac{1}{3\sqrt{\mathbf{e}}} \end{aligned}$$

There is a single critical point for this function.

Let's work one more problem to make a point.

Example 7 Determine all the critical points for the function.

$$f\left(x\right) = x\mathbf{e}^{x^2}$$

Solution

Note that this function is not much different from the function used in Example 5. In this case the derivative is,

$$f'\left(x\right) = \mathbf{e}^{x^2} + x\mathbf{e}^{x^2}\left(2x\right) = \mathbf{e}^{x^2}\left(1 + 2x^2\right)$$

This function will never be zero for any real value of *x*. The exponential is never zero of course and the polynomial will only be zero if *x* is complex and recall that we only want real values of *x* for critical points.

Therefore, this function will not have any critical points.

It is important to note that not all functions will have critical points! In this course most of the functions that we will be looking at do have critical points. That is only because those problems make for more interesting examples. Do not let this fact lead you to always expect that a function will have critical points. Sometimes they don't as this final example has shown.

Section 4-3 : Minimum and Maximum Values

Many of our applications in this chapter will revolve around minimum and maximum values of a function. While we can all visualize the minimum and maximum values of a function we want to be a little more specific in our work here. In particular, we want to differentiate between two types of minimum or maximum values. The following definition gives the types of minimums and/or maximums values that we'll be looking at.

Definition

1. We say that $f\left(x\right)$ has an **absolute (or global) maximum** at $x = c$ if $f\left(x\right) \le f\left(c\right)$ for every *x* in the domain we are working on.
2. We say that $f\left(x\right)$ has a **relative (or local) maximum** at $x = c$ if $f\left(x\right) \le f\left(c\right)$ for every *x* in some open interval around $x = c$.
3. We say that $f\left(x\right)$ has an **absolute (or global) minimum** at $x = c$ if $f\left(x\right) \ge f\left(c\right)$ for every *x* in the domain we are working on.
4. We say that $f\left(x\right)$ has a **relative (or local) minimum** at $x = c$ if $f\left(x\right) \ge f\left(c\right)$ for every *x* in some open interval around $x = c$.

Note that when we say an "open interval around $x = c$" we mean that we can find some interval $\left(a,b\right)$, not including the endpoints, such that $a < c < b$. Or, in other words, *c* will be contained somewhere inside the interval and will not be either of the endpoints.

Also, we will collectively call the minimum and maximum points of a function the **extrema** of the function. So, relative extrema will refer to the relative minimums and maximums while absolute extrema refer to the absolute minimums and maximums.

Now, let's talk a little bit about the subtle difference between the absolute and relative in the definition above.

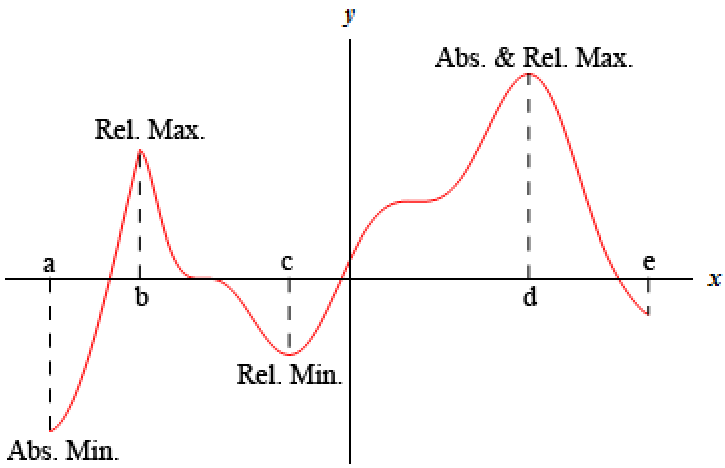
We will have an absolute maximum (or minimum) at $x = c$ provided $f\left(c\right)$ is the largest (or smallest) value that the function will ever take on the domain that we are working on. Also, when we say the "domain we are working on" this simply means the range of *x*'s that we have chosen to work with for a given problem. There may be other values of *x* that we can actually plug into the function but have excluded them for some reason.

A relative maximum or minimum is slightly different. All that's required for a point to be a relative maximum or minimum is for that point to be a maximum or minimum in some interval of *x*'s around $x = c$. There may be larger or smaller values of the function at some other place, but relative to $x = c$, or local to $x = c$, $f\left(c\right)$ is larger or smaller than all the other function values that are near it.

Note as well that in order for a point to be a relative extrema we must be able to look at function values on both sides of $x = c$ to see if it really is a maximum or minimum at that point. This means that relative extrema do not occur at the end points of a domain. They can only occur interior to the domain.

There is actually some debate on the preceding point. Some folks do feel that relative extrema can occur on the end points of a domain. However, in this class we will be using the definition that says that they can't occur at the end points of a domain. This will be discussed in a little more detail at the end of the section once we have a relevant fact taken care of.

It's usually easier to get a feel for the definitions by taking a quick look at a graph.



For the function shown in this graph we have relative maximums at $x = b$ and $x = d$. Both of these points are relative maximums since they are interior to the domain shown and are the largest point on the graph in some interval around the point. We also have a relative minimum at $x = c$ since this point is interior to the domain and is the lowest point on the graph in an interval around it. The far-right end point, $x = e$, will not be a relative minimum since it is an end point.

The function will have an absolute maximum at $x = d$ and an absolute minimum at $x = a$. These two points are the largest and smallest that the function will ever be. We can also notice that the absolute extrema for a function will occur at either the endpoints of the domain or at relative extrema. We will use this idea in later sections so it's more important than it might seem at the present time.

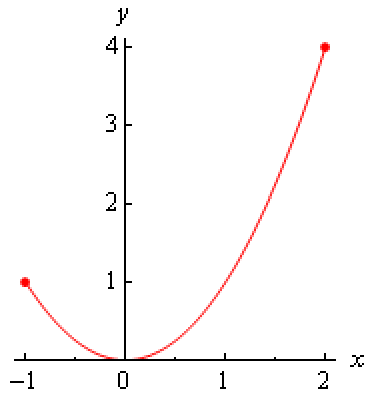
Let's take a quick look at some examples to make sure that we have the definitions of absolute extrema and relative extrema straight.

Example 1 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^2 \qquad \text{on} \qquad [-1, 2]$$

Solution

Since this function is easy enough to graph let's do that. However, we only want the graph on the interval $[-1, 2]$. Here is the graph,



Note that we used dots at the end of the graph to remind us that the graph ends at these points.

We can now identify the extrema from the graph. It looks like we've got a relative and absolute minimum of zero at $x = 0$ and an absolute maximum of four at $x = 2$. Note that $x = -1$ is not a relative maximum since it is at the end point of the interval.

This function doesn't have any relative maximums.

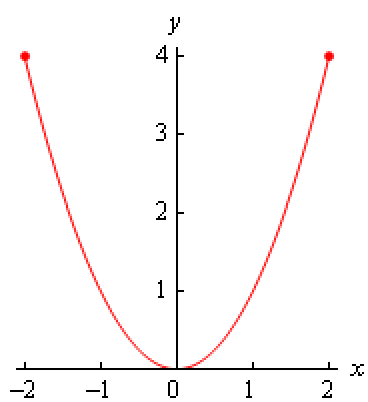
As we saw in the previous example functions do not have to have relative extrema. It is completely possible for a function to not have a relative maximum and/or a relative minimum.

Example 2 Identify the absolute extrema and relative extrema for the following function.

$$f\left(x\right) = x^2 \qquad \text{on} \qquad \left[-2, 2\right]$$

Solution

Here is the graph for this function.



In this case we still have a relative and absolute minimum of zero at $x = 0$. We also still have an absolute maximum of four. However, unlike the first example this will occur at two points, $x = -2$ and $x = 2$.

Again, the function doesn't have any relative maximums.

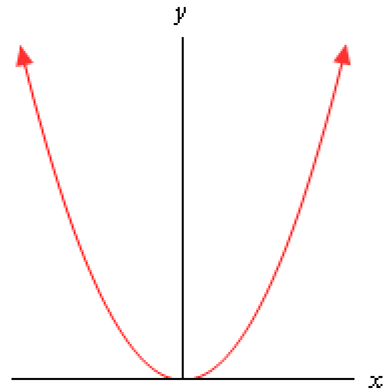
As this example has shown there can only be a single absolute maximum or absolute minimum value, but they can occur at more than one place in the domain.

Example 3 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^2$$

Solution

In this case we've given no domain and so the assumption is that we will take the largest possible domain. For this function that means all the real numbers. Here is the graph.



In this case the graph doesn't stop increasing at either end and so there are no maximums of any kind for this function. No matter which point we pick on the graph there will be points both larger and smaller than it on either side so we can't have any maximums (of any kind, relative or absolute) in a graph.

We still have a relative and absolute minimum value of zero at $x = 0$.

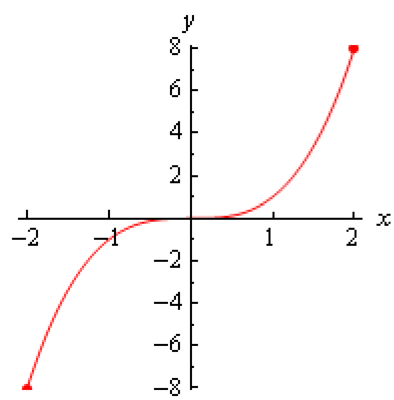
So, some graphs can have minimums but not maximums. Likewise, a graph could have maximums but not minimums.

Example 4 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^3 \qquad \text{on} \qquad [-2, 2]$$

Solution

Here is the graph for this function.



This function has an absolute maximum of eight at $x = 2$ and an absolute minimum of negative eight at $x = -2$. This function has no relative extrema.

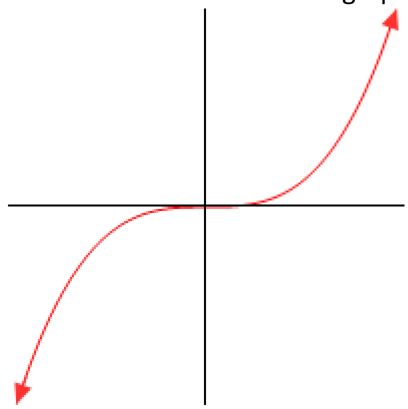
So, a function doesn't have to have relative extrema as this example has shown.

Example 5 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = x^3$$

Solution

Again, we aren't restricting the domain this time so here's the graph.



In this case the function has no relative extrema and no absolute extrema.

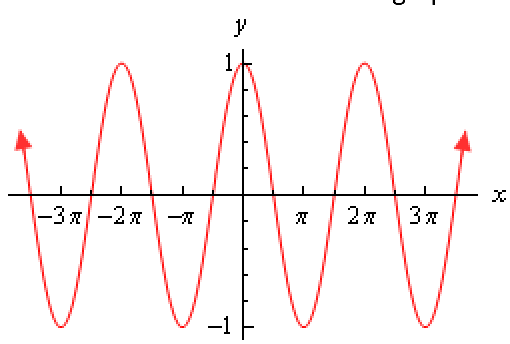
As we've seen in the previous example functions don't have to have any kind of extrema, relative or absolute.

Example 6 Identify the absolute extrema and relative extrema for the following function.

$$f(x) = \cos(x)$$

Solution

We've not restricted the domain for this function. Here is the graph.



Cosine has extrema (relative and absolute) that occur at many points. Cosine has both relative and absolute maximums of 1 at

$$x = \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$$

Cosine also has both relative and absolute minimums of -1 at

$$x = \ldots -3\pi, -\pi, \pi, 3\pi, \ldots$$

As this example has shown a graph can in fact have extrema occurring at a large number (infinite in this case) of points.

We've now worked quite a few examples and we can use these examples to see a nice fact about absolute extrema. First let's notice that all the functions above were **continuous** functions. Next notice that every time we restricted the domain to a closed interval (*i.e.* the interval contains its end points) we got absolute maximums and absolute minimums. Finally, in only one of the three examples in which we did not restrict the domain did we get both an absolute maximum and an absolute minimum.

These observations lead us the following theorem.

Extreme Value Theorem

Suppose that $f(x)$ is continuous on the interval $[a,b]$ then there are two numbers $a \le c,d \le b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

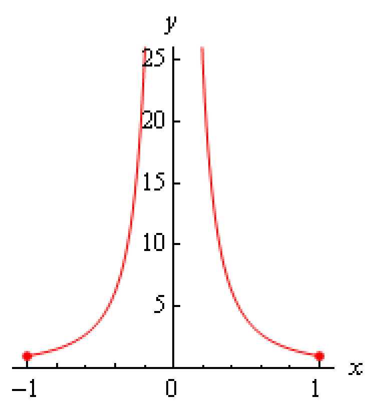
So, if we have a continuous function on an interval $[a,b]$ then we are guaranteed to have both an absolute maximum and an absolute minimum for the function somewhere in the interval. The theorem doesn't tell us where they will occur or if they will occur more than once, but at least it tells us that they do exist somewhere. Sometimes, all that we need to know is that they do exist.

This theorem doesn't say anything about absolute extrema if we aren't working on an interval. We saw examples of functions above that had both absolute extrema, one absolute extrema, and no absolute extrema when we didn't restrict ourselves down to an interval.

The requirement that a function be continuous is also required in order for us to use the theorem. Consider the case of

$$f(x) = \frac{1}{x^2} \qquad \text{on} \qquad [-1,1]$$

Here's the graph.



This function is not continuous at $x = 0$ as we move in towards zero the function is approaching infinity. So, the function does not have an absolute maximum. Note that it does have an absolute minimum however. In fact the absolute minimum occurs twice at both $x = -1$ and $x = 1$.

If we changed the interval a little to say,

$$f(x) = \frac{1}{x^2} \qquad \text{on} \qquad \left[\frac{1}{2},1\right]$$

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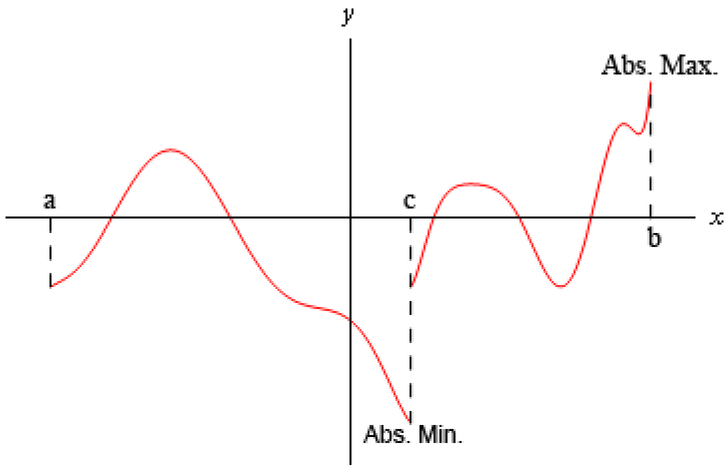
the function would now have both absolute extrema. We may only run into problems if the interval contains the point of discontinuity. If it doesn't then the theorem will hold.

We should also point out that just because a function is not continuous at a point that doesn't mean that it won't have both absolute extrema in an interval that contains that point. Below is the graph of a function that is not continuous at a point in the given interval and yet has both absolute extrema.



This graph is not continuous at $x = c$, yet it does have both an absolute maximum ($x = b$) and an absolute minimum ($x = c$). Also note that, in this case one of the absolute extrema occurred at the point of discontinuity, but it doesn't need to. The absolute minimum could just have easily been at the other end point or at some other point interior to the region. The point here is that this graph is not continuous and yet does have both absolute extrema

The point of all this is that we need to be careful to only use the Extreme Value Theorem when the conditions of the theorem are met and not misinterpret the results if the conditions aren't met.

In order to use the Extreme Value Theorem we must have an interval that includes its endpoints, often called a closed interval, and the function must be continuous on that interval. If we don't have a closed interval and/or the function isn't continuous on the interval then the function may or may not have absolute extrema.

We need to discuss one final topic in this section before moving on to the first major application of the derivative that we're going to be looking at in this chapter.

Fermat's Theorem

If $f\left(x\right)$ has a relative extrema at $x = c$ and $f'\left(c\right)$ exists then $x = c$ is a critical point of $f\left(x\right)$. In fact, it will be a critical point such that $f'\left(c\right) = 0$.

To see the proof of this theorem see the **Proofs From Derivative Applications** section of the Extras chapter.

Also note that we can say that $f'\left(c\right) = 0$ because we are also assuming that $f'\left(c\right)$ exists.

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This theorem tells us that there is a nice relationship between relative extrema and critical points. In fact, it will allow us to get a list of all possible relative extrema. Since a relative extrema must be a critical point the list of all critical points will give us a list of all possible relative extrema.

Consider the case of $f\left(x\right)=x^2$. We saw that this function had a relative minimum at $x=0$ in several earlier examples. So according to Fermat's theorem $x=0$ should be a critical point. The derivative of the function is,

$$f'\left(x\right)=2x$$

Sure enough $x=0$ is a critical point.

Be careful not to misuse this theorem. It doesn't say that a critical point will be a relative extrema. To see this, consider the following case.

$$f\left(x\right)=x^3 \qquad\qquad f'\left(x\right)=3x^2$$

Clearly $x=0$ is a critical point. However, we saw in an earlier example this function has no relative extrema of any kind. So, critical points do not have to be relative extrema.

Also note that this theorem says nothing about absolute extrema. An absolute extrema may or may not be a critical point.

Before we leave this section we need to discuss a couple of issues.

First, Fermat's Theorem only works for critical points in which $f'\left(c\right)=0$. This does not, however, mean that relative extrema won't occur at critical points where the derivative does not exist. To see this consider $f\left(x\right)=\left|x\right|$. This function clearly has a relative minimum at $x=0$ and yet in a previous **section** we showed in an example that $f'\left(0\right)$ does not exist.

What this all means is that if we want to locate relative extrema all we really need to do is look at the critical points as those are the places where relative extrema may exist.

Finally, recall that at that start of the section we stated that relative extrema will not exist at endpoints of the interval we are looking at. The reason for this is that if we allowed relative extrema to occur there it may well (and in fact most of the time) violate Fermat's Theorem. There is no reason to expect end points of intervals to be critical points of any kind. Therefore, we do not allow relative extrema to exist at the endpoints of intervals.

Section 4-4 : Finding Absolute Extrema

It's now time to see our first major application of derivatives in this chapter. Given a continuous function, $f(x)$, on an interval $[a,b]$ we want to determine the absolute extrema of the function. To do this we will need many of the ideas that we looked at in the previous section.

First, since we have a closed interval (*i.e.* and interval that includes the endpoints) and we are assuming that the function is continuous the **Extreme Value Theorem** tells us that we can in fact do this. This is a good thing of course. We don't want to be trying to find something that may not exist.

Next, we saw in the previous section that absolute extrema can occur at endpoints or at relative extrema. Also, from the previous section that we know that the list of critical points is also a list of all possible relative extrema. So, the endpoints along with the list of all critical points will in fact be a list of all possible absolute extrema.

Now we just need to recall that the absolute extrema are nothing more than the largest and smallest values that a function will take so all that we really need to do is get a list of possible absolute extrema, plug these points into our function and then identify the largest and smallest values.

Here is the procedure for finding absolute extrema.

Finding Absolute Extrema of $f(x)$ on $[a,b]$.

0. Verify that the function is continuous on the interval $[a,b]$.
1. Find all critical points of $f(x)$ that are in the interval $[a,b]$. This makes sense if you think about it. Since we are only interested in what the function is doing in this interval we don't care about critical points that fall outside the interval.
2. Evaluate the function at the critical points found in step 1 and the end points.
3. Identify the absolute extrema.

There really isn't a whole lot to this procedure. We called the first step in the process step 0, mostly because all of the functions that we're going to look at here are going to be continuous, but it is something that we do need to be careful with. This process will only work if we have a function that is continuous on the given interval. The most labor intensive step of this process is the second step (step 1) where we find the critical points. It is also important to note that all we want are the critical points that are in the interval.

Let's do some examples.

Example 1 Determine the absolute extrema for the following function and interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \qquad \text{on} \qquad [-4,2]$$

Solution

All we really need to do here is follow the procedure given above. So, first notice that this is a polynomial and so is continuous everywhere and therefore is continuous on the given interval.

Now, we need to get the derivative so that we can find the critical points of the function.

$$g'(t) = 6t^2 + 6t - 12 = 6(t+2)(t-1)$$

It looks like we'll have two critical points, $t = -2$ and $t = 1$. Note that we actually want something more than just the critical points. We only want the critical points of the function that lie in the interval in question. Both of these do fall in the interval as so we will use both of them. That may seem like a silly thing to mention at this point, but it is often forgotten, usually when it becomes important, and so we will mention it at every opportunity to make sure it's not forgotten.

Now we evaluate the function at the critical points and the end points of the interval.

$$g(-2) = 24 \qquad\qquad g(1) = -3$$
$$g(-4) = -28 \qquad\qquad g(2) = 8$$

Absolute extrema are the largest and smallest the function will ever be and these four points represent the only places in the interval where the absolute extrema can occur. So, from this list we see that the absolute maximum of $g(t)$ is 24 and it occurs at $t = -2$ (a critical point) and the absolute minimum of $g(t)$ is -28 which occurs at $t = -4$ (an endpoint).

In this example we saw that absolute extrema can and will occur at both endpoints and critical points. One of the biggest mistakes that students make with these problems is to forget to check the endpoints of the interval.

Example 2 Determine the absolute extrema for the following function and interval.

$$g(t) = 2t^3 + 3t^2 - 12t + 4 \qquad \text{on} \qquad [0,2]$$

Solution

Note that this problem is almost identical to the first problem. The only difference is the interval that we're working on. This small change will completely change our answer however. With this change we have excluded both of the answers from the first example.

The first step is to again find the critical points. From the first example we know these are $t = -2$ and $t = 1$.. At this point it's important to recall that we only want the critical points that actually fall in the interval in question. This means that we only want $t = 1$ since $t = -2$ falls outside the interval.

Now evaluate the function at the single critical point in the interval and the two endpoints.

$$g(1) = -3 \qquad\qquad g(0) = 4 \qquad\qquad g(2) = 8$$

From this list of values we see that the absolute maximum is 8 and will occur at $t = 2$ and the absolute minimum is -3 which occurs at $t = 1$.

As we saw in this example a simple change in the interval can completely change the answer. It also has shown us that we do need to be careful to exclude critical points that aren't in the interval. Had we forgotten this and included $t = -2$ we would have gotten the wrong absolute maximum!

This is the other big mistakes that students make in these problems. All too often they forget to exclude critical points that aren't in the interval. If your instructor is anything like me this will mean that you will get the wrong answer. It's not too hard to make sure that a critical point outside of the interval is larger or smaller than any of the points in the interval.

Example 3 Suppose that the population (in thousands) of a certain kind of insect after *t* months is given by the following formula.

$$P(t) = 3t + \sin(4t) + 100$$

Determine the minimum and maximum population in the first 4 months.

Solution

The question that we're really asking is to find the absolute extrema of $P(t)$ on the interval $[0,4]$. Since this function is continuous everywhere we know we can do this.

Let's start with the derivative.

$$P'(t) = 3 + 4\cos(4t)$$

We need the critical points of the function. The derivative exists everywhere so there are no critical points from that. So, all we need to do is determine where the derivative is zero.

$$3 + 4\cos(4t) = 0$$

$$\cos(4t) = -\frac{3}{4}$$

The solutions to this are,

$$\begin{aligned} 4t &= 2.4189 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \\ 4t &= 3.8643 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned} \quad \Rightarrow \quad \begin{aligned} t &= 0.6047 + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \\ t &= 0.9661 + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

So, these are all the critical points. We need to determine the ones that fall in the interval $[0,4]$. There's nothing to do except plug some *n*'s into the formulas until we get all of them.

$n = 0$:

$$t = 0.6047 \qquad\qquad t = 0.9661$$

We'll need both of these critical points.

$n = 1$:

$$t = 0.6047 + \frac{\pi}{2} = 2.1755 \qquad\qquad t = 0.9661 + \frac{\pi}{2} = 2.5369$$

We'll need these.

$n = 2$:

$$t = 0.6047 + \pi = 3.7463 \qquad\qquad t = 0.9661 + \pi = 4.1077$$

In this case we only need the first one since the second is out of the interval.

There are five critical points that are in the interval. They are,

$$0.6047,\ 0.9661,\ 2.1755,\ 2.5369,\ 3.7463$$

Finally, to determine the absolute minimum and maximum population we only need to plug these values into the function as well as the two end points. Here are the function evaluations.

$$\begin{aligned} P(0) &= 100.0 & P(4) &= 111.7121 \\ P(0.6047) &= 102.4756 & P(0.9661) &= 102.2368 \\ P(2.1755) &= 107.1880 & P(2.5369) &= 106.9492 \\ P(3.7463) &= 111.9004 & & \end{aligned}$$

From these evaluations it appears that the minimum population is 100,000 (remember that *P* is in thousands…) which occurs at $t = 0$ and the maximum population is 111,900 which occurs at $t = 3.7463$.

Make sure that you can correctly solve trig equations. If we had forgotten the $2\pi n$ we would have missed the last three critical points in the interval and hence gotten the wrong answer since the maximum population was at the final critical point.

Also, note that we do really need to be very careful with rounding answers here. If we'd rounded to the nearest integer, for instance, it would appear that the maximum population would have occurred at two different locations instead of only one.

Example 4 Suppose that the amount of money in a bank account after *t* years is given by,

$$A(t) = 2000 - 10t\mathbf{e}^{5-\frac{t^2}{8}}$$

Determine the minimum and maximum amount of money in the account during the first 10 years that it is open.

Solution

Here we are really asking for the absolute extrema of $A(t)$ on the interval $[0,10]$. As with the previous examples this function is continuous everywhere and so we know that this can be done.

We'll first need the derivative so we can find the critical points.

$$\begin{aligned} A'(t) &= -10\mathbf{e}^{5-\frac{t^2}{8}} - 10t\mathbf{e}^{5-\frac{t^2}{8}}\left(-\frac{t}{4}\right) \\ &= 10\mathbf{e}^{5-\frac{t^2}{8}}\left(-1+\frac{t^2}{4}\right) \end{aligned}$$

The derivative exists everywhere and the exponential is never zero. Therefore, the derivative will only be zero where,

$$-1+\frac{t^2}{4}=0 \qquad \Rightarrow \qquad t^2=4 \qquad \Rightarrow \qquad t=\pm 2$$

We've got two critical points, however only $t=2$ is actually in the interval so that is only critical point that we'll use.

Let's now evaluate the function at the lone critical point and the end points of the interval. Here are those function evaluations.

$$A(0)=2000 \qquad A(2)=199.66 \qquad A(10)=1999.94$$

So, the maximum amount in the account will be \$2000 which occurs at $t=0$ and the minimum amount in the account will be \$199.66 which occurs at the 2 year mark.

In this example there are two important things to note. First, if we had included the second critical point we would have gotten an incorrect answer for the maximum amount so it's important to be careful with which critical points to include and which to exclude.

All of the problems that we've worked to this point had derivatives that existed everywhere and so the only critical points that we looked at were those for which the derivative is zero. Do not get too locked into this always happening. Most of the problems that we run into will be like this, but they won't all be like this.

Let's work another example to make this point.

Example 5 Determine the absolute extrema for the following function and interval.

$$Q(y)=3y(y+4)^{\frac{2}{3}} \qquad \text{on} \qquad [-5,-1]$$

Solution

Again, as with all the other examples here, this function is continuous on the given interval and so we know that this can be done.

First, we'll need the derivative and make sure you can do the simplification that we did here to make the work for finding the critical points easier.

$$\begin{aligned} Q'(y) &= 3(y+4)^{\frac{2}{3}}+3y\left(\frac{2}{3}\right)(y+4)^{-\frac{1}{3}} \\ &= 3(y+4)^{\frac{2}{3}}+\frac{2y}{(y+4)^{\frac{1}{3}}} \\ &= \frac{3(y+4)+2y}{(y+4)^{\frac{1}{3}}} \\ &= \frac{5y+12}{(y+4)^{\frac{1}{3}}} \end{aligned}$$

So, it looks like we've got two critical points.

$$y = -4 \qquad \text{Because the derivative doesn't exist here.}$$
$$y = -\frac{12}{5} \qquad \text{Because the derivative is zero here.}$$

Both of these are in the interval so let's evaluate the function at these points and the end points of the interval.

$$Q(-4) = 0 \qquad Q\left(-\frac{12}{5}\right) = -9.849$$
$$Q(-5) = -15 \qquad Q(-1) = -6.241$$

The function has an absolute maximum of zero at $y = -4$ and the function will have an absolute minimum of -15 at $y = -5$.

So, if we had ignored or forgotten about the critical point where the derivative doesn't exist ($y = -4$) we would not have gotten the correct answer.

In this section we've seen how we can use a derivative to identify the absolute extrema of a function. This is an important application of derivatives that will arise from time to time so don't forget about it.

Section 4-5 : The Shape of a Graph, Part I

In the previous section we saw how to use the derivative to determine the absolute minimum and maximum values of a function. However, there is a lot more information about a graph that can be determined from the first derivative of a function. We will start looking at that information in this section. The main idea we'll be looking at in this section we will be identifying all the relative extrema of a function.

Let's start this section off by revisiting a familiar topic from the previous chapter. Let's suppose that we have a function, $f\left(x\right)$. We know from our work in the previous chapter that the first derivative, $f'\left(x\right)$, is the rate of change of the function. We used this idea to identify where a function was increasing, decreasing or not changing.

Before reviewing this idea let's first write down the mathematical definition of increasing and decreasing. We all know what the graph of an increasing/decreasing function looks like but sometimes it is nice to have a mathematical definition as well. Here it is.

Definition

1. Given any x_1 and x_2 from an interval *I* with $x_1 < x_2$ if $f\left(x_1\right) < f\left(x_2\right)$ then $f\left(x\right)$ is ***increasing*** on *I*.
2. Given any x_1 and x_2 from an interval *I* with $x_1 < x_2$ if $f\left(x_1\right) > f\left(x_2\right)$ then $f\left(x\right)$ is ***decreasing*** on *I*.

This definition will actually be used in the proof of the next fact in this section.

Now, recall that in the previous chapter we constantly used the idea that if the derivative of a function was positive at a point then the function was increasing at that point and if the derivative was negative at a point then the function was decreasing at that point. We also used the fact that if the derivative of a function was zero at a point then the function was not changing at that point. We used these ideas to identify the intervals in which a function is increasing and decreasing.

The following fact summarizes up what we were doing in the previous chapter.

Fact

1. If $f'\left(x\right) > 0$ for every *x* on some interval *I*, then $f\left(x\right)$ is increasing on the interval.
2. If $f'\left(x\right) < 0$ for every *x* on some interval *I*, then $f\left(x\right)$ is decreasing on the interval.
3. If $f'\left(x\right) = 0$ for every *x* on some interval *I*, then $f\left(x\right)$ is constant on the interval.

The proof of this fact is in the **Proofs From Derivative Applications** section of the Extras chapter.

Let's take a look at an example. This example has two purposes. First, it will remind us of the increasing/decreasing type of problems that we were doing in the previous chapter. Secondly, and maybe more importantly, it will now incorporate critical points into the solution. We didn't know about critical points in the previous chapter, but if you go back and look at those examples, the first step in almost every increasing/decreasing problem is to find the critical points of the function and so the process we'll be using in the following example should be familiar.

Example 1 Determine all intervals where the following function is increasing or decreasing.

$$f\left(x\right)=-x^5+\frac{5}{2}x^4+\frac{40}{3}x^3+5$$

Solution

To determine if the function is increasing or decreasing we will need the derivative.

$$\begin{aligned} f'\left(x\right) &= -5x^4+10x^3+40x^2 \\ &= -5x^2\left(x^2-2x-8\right) \\ &= -5x^2\left(x-4\right)\left(x+2\right) \end{aligned}$$

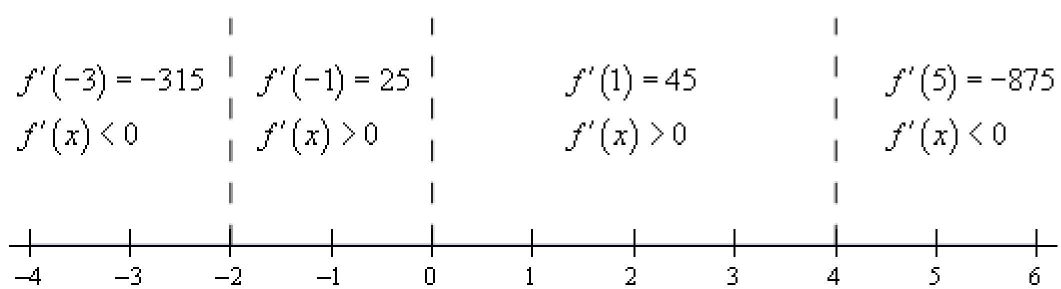
Note that when we factored the derivative we first factored a "-1" out to make the rest of the factoring a little easier.

From the factored form of the derivative we see that we have three critical points : $x=-2$, $x=0$, and $x=4$. We'll need these in a bit.

We now need to determine where the derivative is positive and where it's negative. We've done this several times now in both the Review chapter and the previous chapter. Since the derivative is a polynomial it is continuous and so we know that the only way for it to change signs is to first go through zero.

In other words, the only place that the derivative *may* change signs is at the critical points of the function. We've now got another use for critical points. So, we'll build a number line, graph the critical points and pick test points from each region to see if the derivative is positive or negative in each region.

Here is the number line and the test points for the derivative.



Make sure that you test your points in the derivative. One of the more common mistakes here is to test the points in the function instead! Recall that we know that the derivative will be the same sign in each region. The only place that the derivative can change signs is at the critical points and we've marked the only critical points on the number line.

So, it looks we've got the following intervals of increase and decrease.

$$\begin{array}{ll}\text{Increase}: & -2 < x < 0 \text{ and } 0 < x < 4 \\ \text{Decrease}: & -\infty < x < -2 \text{ and } 4 < x < \infty\end{array}$$

In this example we used the fact that the only place that a derivative can change sign is at the critical points. Also, the critical points for this function were those for which the derivative was zero. However, the same thing can be said for critical points where the derivative doesn't exist. This is nice to know. A function can change signs where it is zero or doesn't exist. In the previous chapter all our examples of this type had only critical points where the derivative was zero. Now, that we know more about critical points we'll also see an example or two later on with critical points where the derivative doesn't exist.

If you aren't sure you believe that functions (they don't have to be derivatives of course) can change sign where they don't exist consider $f(x) = \frac{1}{x}$. This function clearly does not exist at $x = 0$ and is negative if $x < 0$ and positive if $x > 0$ and so does change sign at the point were it does not exist. Be careful to not assume this will always be true however. Take $f(x) = \frac{1}{x^2}$ for example. Again, this clearly does not exist at $x = 0$ and yet is positive on both sides of $x = 0$.

So, just to reiterate one more time. Functions, regardless of whether the are derivatives or not, may (but not guaranteed to) change sign where they are either zero or do not exist.

Now that we have the previous "reminder" example out of the way let's move into some new material. Once we have the intervals of increasing and decreasing for a function we can use this information to get a sketch of the graph. Note that the sketch, at this point, may not be super accurate when it comes to the curvature of the graph, but it will at least have the basic shape correct. To get the curvature of the graph correct we'll need the information from the next section.

Let's attempt to get a sketch of the graph of the function we used in the previous example.

Example 2 Sketch the graph of the following function.

$$f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$$

Solution

There really isn't a whole lot to this example. Whenever we sketch a graph it's nice to have a few points on the graph to give us a starting place. So we'll start by the function at the critical points. These will give us some starting points when we go to sketch the graph. These points are,

$$f(-2) = -\frac{89}{3} = -29.67 \qquad f(0) = 5 \qquad f(4) = \frac{1423}{3} = 474.33$$

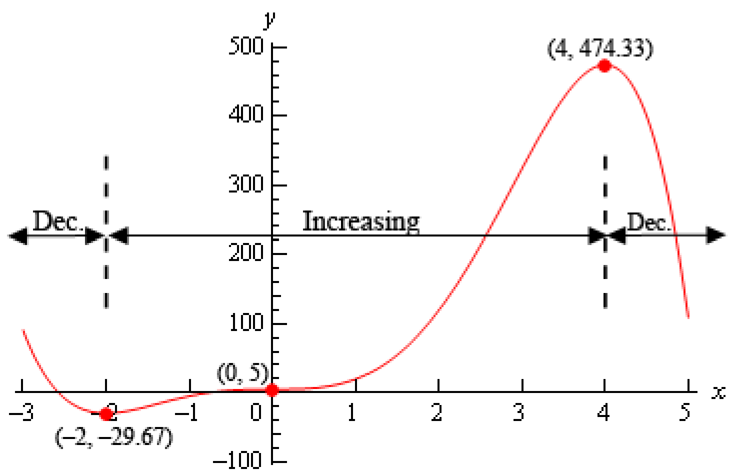
Once these points are graphed we go to the increasing and decreasing information and start sketching. For reference purposes here is the increasing/decreasing information.

$$\text{Increase : } \quad -2 < x < 0 \text{ and } 0 < x < 4$$
$$\text{Decrease : } -\infty < x < -2 \text{ and } 4 < x < \infty$$

Note that we are only after a sketch of the graph. As noted before we started this example we won't be able to accurately predict the curvature of the graph at this point. However, even without this information we will still be able to get a basic idea of what the graph should look like.

To get this sketch we start at the very left of the graph and knowing that the graph must be decreasing and will continue to decrease until we get to $x = -2$. At this point the function will continue to increase until it gets to $x = 4$. However, note that during the increasing phase it does need to go through the point at $x = 0$ and at this point we also know that the derivative is zero here and so the graph goes through $x = 0$ horizontally. Finally, once we hit $x = 4$ the graph starts, and continues, to decrease. Also, note that just like at $x = 0$ the graph will need to be horizontal when it goes through the other two critical points as well.

Here is the graph of the function. We, of course, used a graphical program to generate this graph, however, outside of some potential curvature issues if you followed the increasing/decreasing information and had all the critical points plotted first you should have something similar to this.



Let's use the sketch from this example to give us a very nice test for classifying critical points as relative maximums, relative minimums or neither minimums or maximums.

Recall from the **Minimum and Maximum Values** section that all relative extrema of a function come from the list of critical points. The graph in the previous example has two relative extrema and both occur at critical points as the we predicted in that section. Note as well that we've got a critical point that isn't a relative extrema ($x = 0$). This is okay since there is no reason to think that all critical points will be relative extrema. We only know that relative extrema will come from the list of critical points.

In the sketch of the graph from the previous example we can see that to the left of $x = -2$ the graph is decreasing and to the right of $x = -2$ the graph is increasing and $x = -2$ is a relative minimum. In other words, the graph is behaving around the minimum exactly as it would have to be in order for $x = -2$ to be a minimum. The same thing can be said for the relative maximum at $x = 4$. The graph is increasing on the left and decreasing on the right exactly as it must be in order for $x = 4$ to be a maximum. Finally, the graph is increasing on both sides of $x = 0$ and so this critical point can't be a minimum or a maximum.

These ideas can be generalized to arrive at a nice way to test if a critical point is a relative minimum, relative maximum or neither. If $x = c$ is a critical point and the function is decreasing to the left of $x = c$ and is increasing to the right then $x = c$ must be a relative minimum of the function. Likewise, if the function is increasing to the left of $x = c$ and decreasing to the right then $x = c$ must be a relative maximum of the function. Finally, if the function is increasing on both sides of $x = c$ or decreasing on both sides of $x = c$ then $x = c$ can be neither a relative minimum nor a relative maximum.

These ideas can be summarized up in the following test.

First Derivative Test

Suppose that $x = c$ is a critical point of $f(x)$ then,

1. If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then $x = c$ is a relative maximum.
2. If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then $x = c$ is a relative minimum.
3. If $f'(x)$ is the same sign on both sides of $x = c$ then $x = c$ is neither a relative maximum nor a relative minimum.

It is important to note here that the first derivative test will only classify critical points as relative extrema and not as absolute extrema. As we recall from the **Finding Absolute Extrema** section absolute extrema are largest and smallest function values and may not even exist or be critical points if they do exist.

The first derivative test is exactly that, a test using the first derivative. It doesn't ever use the value of the function and so no conclusions can be drawn from the test about the relative "size" of the function at the critical points (which would be needed to identify absolute extrema) and can't even begin to address the fact that absolute extrema may not occur at critical points.

Let's take at another example.

Example 3 Find and classify all the critical points of the following function. Give the intervals where the function is increasing and decreasing.

$$g(t) = t\sqrt[3]{t^2 - 4}$$

Solution

First, we'll need the derivative so we can get our hands on the critical points. Note as well that we'll do some simplification on the derivative to help us find the critical points.

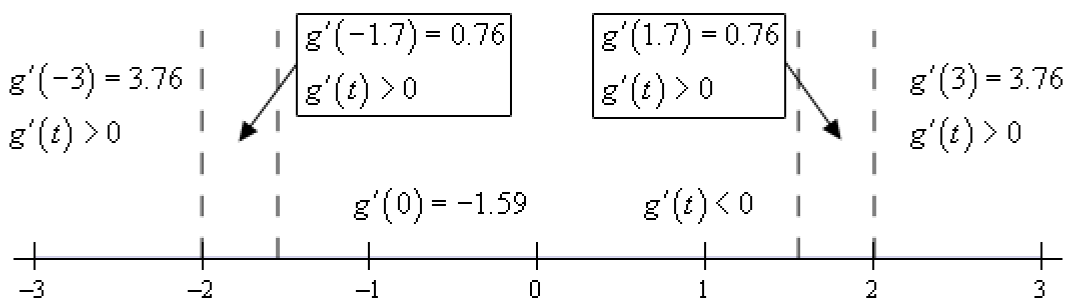
$$\begin{aligned} g'(t) &= \left(t^2-4\right)^{\frac{1}{3}} + \frac{2}{3}t^2\left(t^2-4\right)^{-\frac{2}{3}} \\ &= \left(t^2-4\right)^{\frac{1}{3}} + \frac{2t^2}{3\left(t^2-4\right)^{\frac{2}{3}}} \\ &= \frac{3\left(t^2-4\right)+2t^2}{3\left(t^2-4\right)^{\frac{2}{3}}} \\ &= \frac{5t^2-12}{3\left(t^2-4\right)^{\frac{2}{3}}} \end{aligned}$$

So, it looks like we'll have four critical points here. They are,

$$t = \pm 2 \qquad \text{The derivative doesn't exist here.}$$

$$t = \pm\sqrt{\frac{12}{5}} = \pm 1.549 \qquad \text{The derivative is zero here.}$$

Finding the intervals of increasing and decreasing will also give the classification of the critical points so let's get those first. Here is a number line with the critical points graphed and test points.



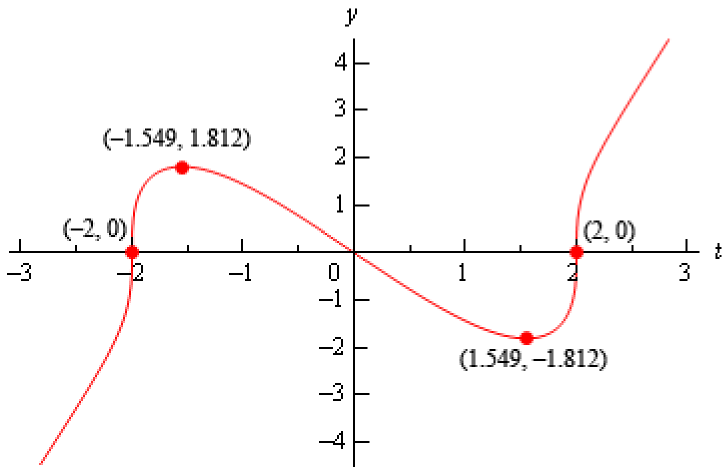
So, it looks like we've got the following intervals of increasing and decreasing.

$$\text{Increase}: \quad -\infty < x < -2, \ -2 < x < -\sqrt{\frac{12}{5}}, \ \sqrt{\frac{12}{5}} < x < 2, \ \& \ 2 < x < \infty$$

$$\text{Decrease}: \ -\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$$

From this it looks like $t = -2$ and $t = 2$ are neither relative minimum or relative maximums since the function is increasing on both side of them. On the other hand, $t = -\sqrt{\frac{12}{5}}$ is a relative maximum and $t = \sqrt{\frac{12}{5}}$ is a relative minimum.

For completeness sake here is the graph of the function. Note that this graph is a little trickier to sketch based only on the increasing and decreasing information. It is only presented here for reference so you can see what it looks like.



In the previous example the two critical points where the derivative didn't exist ended up not being relative extrema. Do not read anything into this. They often will be relative extrema. Check out **Example 5** in the Absolute Extrema to see an example of one such critical point.

Let's work a couple more examples.

Example 4 Suppose that the elevation above sea level of a road is given by the following function.

$$E(x) = 500 + \cos\left(\frac{x}{4}\right) + \sqrt{3}\sin\left(\frac{x}{4}\right)$$

where *x* is in miles. Assume that if *x* is positive we are to the east of the initial point of measurement and if *x* is negative we are to the west of the initial point of measurement.

If we start 25 miles to the west of the initial point of measurement and drive until we are 25 miles east of the initial point how many miles of our drive were we driving up an incline?

Solution

Okay, this is just a really fancy way of asking what the intervals of increasing and decreasing are for the function on the interval $[-25, 25]$. So, we first need the derivative of the function.

$$E'(x) = -\frac{1}{4}\sin\left(\frac{x}{4}\right) + \frac{\sqrt{3}}{4}\cos\left(\frac{x}{4}\right)$$

Setting this equal to zero gives,

$$-\frac{1}{4}\sin\left(\frac{x}{4}\right) + \frac{\sqrt{3}}{4}\cos\left(\frac{x}{4}\right) = 0$$

$$\tan\left(\frac{x}{4}\right) = \sqrt{3}$$

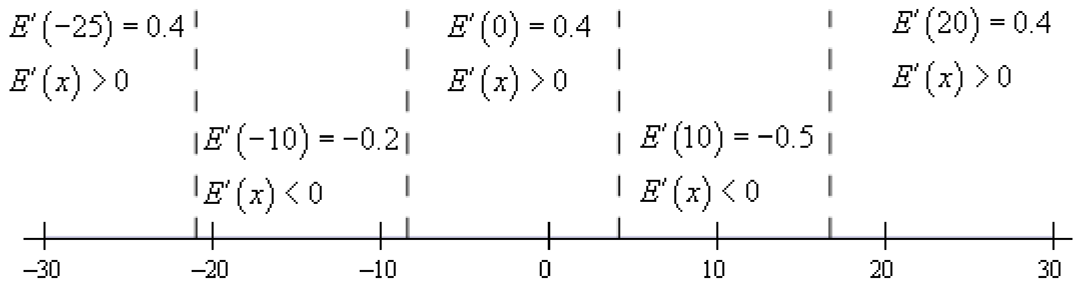
The solutions to this and hence the critical points are,

$$\begin{aligned}\frac{x}{4} &= 1.0472 + 2\pi n, \; n = 0, \pm 1, \pm 2, \ldots \\ \frac{x}{4} &= 4.1888 + 2\pi n, \; n = 0, \pm 1, \pm 2, \ldots\end{aligned} \quad \Rightarrow \quad \begin{aligned}x &= 4.1888 + 8\pi n, \; n = 0, \pm 1, \pm 2, \ldots \\ x &= 16.7552 + 8\pi n, \; n = 0, \pm 1, \pm 2, \ldots\end{aligned}$$

I'll leave it to you to check that the critical points that fall in the interval that we're after are,

$$-20.9439, \; -8.3775, \; 4.1888, \; 16.7552$$

Here is the number line with the critical points and test points.



So, it looks like the intervals of increasing and decreasing are,

$$\begin{aligned}&\text{Increase :} \quad -25 < x < -20.9439, \; -8.3775 < x < 4.1888 \text{ and } 16.7552 < x < 25 \\ &\text{Decrease :} \; -20.9439 < x < -8.3775 \;\text{ and }\; 4.1888 < x < 16.7552\end{aligned}$$

Notice that we had to end our intervals at -25 and 25 since we've done no work outside of these points and so we can't really say anything about the function outside of the interval $\left[-25, 25\right]$.

From the intervals we can actually answer the question. We were driving on an incline during the intervals of increasing and so the total number of miles is,

$$\begin{aligned}\text{Distance } &= \left(-20.9439 - \left(-25\right)\right) + \left(4.1888 - \left(-8.3775\right)\right) + \left(25 - 16.7552\right) \\ &= 24.8652 \text{ miles}\end{aligned}$$

Even though the problem didn't ask for it we can also classify the critical points that are in the interval $\left[-25, 25\right]$.

$$\begin{aligned}\text{Relative Maximums : } &-20.9439, 4.1888 \\ \text{Relative Minimums : } &-8.3775, 16.7552\end{aligned}$$

Example 5 The population of rabbits (in hundreds) after *t* years in a certain area is given by the following function,

$$P\left(t\right) = t^2 \ln\left(3t\right) + 6$$

Determine if the population ever decreases in the first two years.

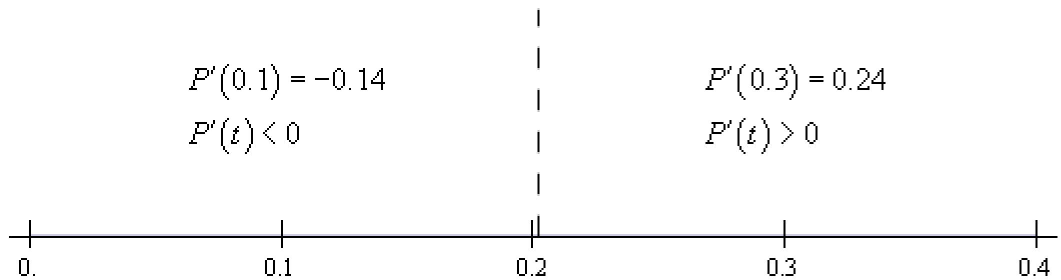
Solution

So, again we are really after the intervals and increasing and decreasing in the interval [0,2].

We found the only critical point to this function back in the **Critical Points** section to be,

$$x = \frac{1}{3\sqrt{\mathbf{e}}} = 0.202$$

Here is a number line for the intervals of increasing and decreasing.



So, it looks like the population will decrease for a short period and then continue to increase forever.

Also, while the problem didn't ask for it we can see that the single critical point is a relative minimum.

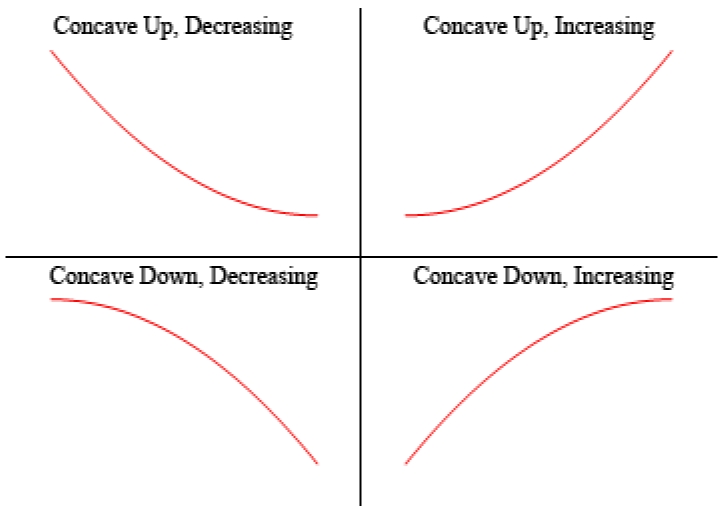
In this section we've seen how we can use the first derivative of a function to give us some information about the shape of a graph and how we can use this information in some applications.

Using the first derivative to give us information about a whether a function is increasing or decreasing is a very important application of derivatives and arises on a fairly regular basis in many areas.

Section 4-6 : The Shape of a Graph, Part II

In the previous section we saw how we could use the first derivative of a function to get some information about the graph of a function. In this section we are going to look at the information that the second derivative of a function can give us a about the graph of a function.

Before we do this we will need a couple of definitions out of the way. The main concept that we'll be discussing in this section is concavity. Concavity is easiest to see with a graph (we'll give the mathematical definition in a bit).



So, a function is **concave up** if it "opens" up and the function is **concave down** if it "opens" down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

It's probably not the best way to define concavity by saying which way it "opens" since this is a somewhat nebulous definition. Here is the mathematical definition of concavity.

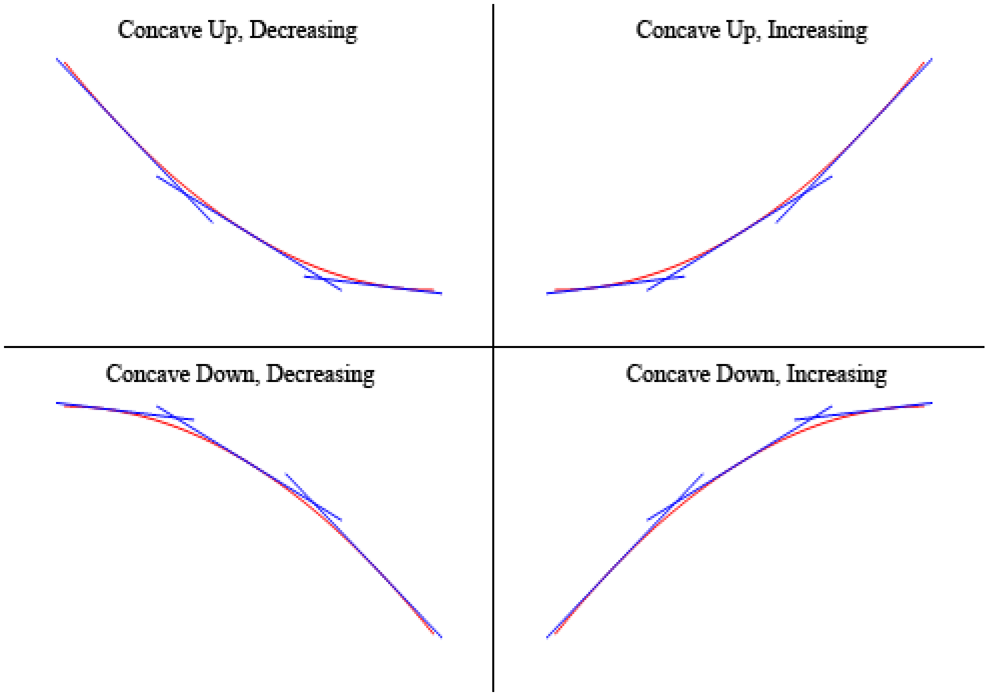
Definition 1

Given the function $f(x)$ then

1. $f(x)$ is **concave up** on an interval *I* if all of the tangents to the curve on *I* are below the graph of $f(x)$.

2. $f(x)$ is **concave down** on an interval *I* if all of the tangents to the curve on *I* are above the graph of $f(x)$.

To show that the graphs above do in fact have concavity claimed above here is the graph again (blown up a little to make things clearer).



So, as you can see, in the two upper graphs all of the tangent lines sketched in are all below the graph of the function and these are concave up. In the lower two graphs all the tangent lines are above the graph of the function and these are concave down.

Again, notice that concavity and the increasing/decreasing aspect of the function is completely separate and do not have anything to do with each other. This is important to note because students often mix these two up and use information about one to get information about the other.

There's one more definition that we need to get out of the way.

Definition 2

A point $x = c$ is called an **inflection point** if the function is continuous at the point and the concavity of the graph changes at that point.

Now that we have all the concavity definitions out of the way we need to bring the second derivative into the mix. We did after all start off this section saying we were going to be using the second derivative to get information about the graph. The following fact relates the second derivative of a function to its concavity. The proof of this fact is in the **Proofs From Derivative Applications** section of the Extras chapter.

Fact

Given the function $f\left(x\right)$ then,

1. If $f''\left(x\right) > 0$ for all *x* in some interval *I* then $f\left(x\right)$ is concave up on *I*.
2. If $f''\left(x\right) < 0$ for all *x* in some interval *I* then $f\left(x\right)$ is concave down on *I*.

So, what this fact tells us is that the inflection points will be all the points were the second derivative changes sign. We saw in the previous chapter that a function may change signs if it is either zero or does not exist. Note that we were working with the first derivative in the previous section but the fact that a function possibly changing signs where it is zero or doesn't exist has nothing to do with the first derivative. It is simply a fact that applies to all functions regardless of whether they are derivatives or not.

This in turn tells us that a list of possible inflection points will be those points where the second derivative is zero or doesn't exist, as these are the only points where the second derivative might change sign.

Be careful however to not make the assumption that just because the second derivative is zero or doesn't exist that the point will be an inflection point. We will only know that it is an inflection point once we determine the concavity on both sides of it. It will only be an inflection point if the concavity is different on both sides of the point.

Now that we know about concavity we can use this information as well as the increasing/decreasing information from the previous section to get a pretty good idea of what a graph should look like. Let's take a look at an example of that.

Example 1 For the following function identify the intervals where the function is increasing and decreasing and the intervals where the function is concave up and concave down. Use this information to sketch the graph.

$$h(x) = 3x^5 - 5x^3 + 3$$

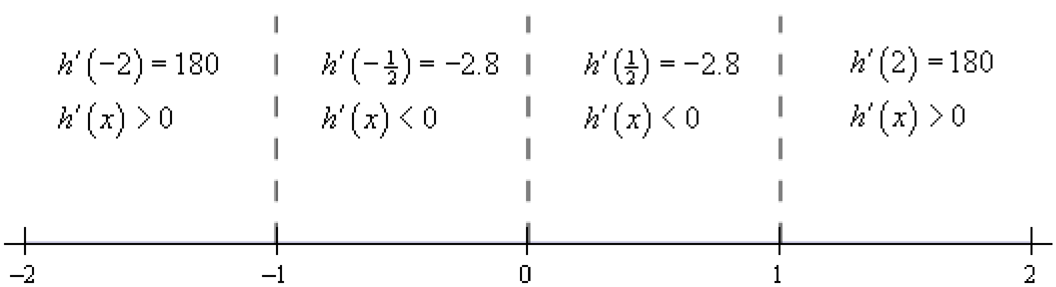
Solution

Okay, we are going to need the first two derivatives so let's get those first.

$$h'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1)$$
$$h''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Let's start with the increasing/decreasing information since we should be fairly comfortable with that after the last section.

There are three critical points for this function : $x = -1$, $x = 0$, and $x = 1$. Below is the number line for the increasing/decreasing information.



So, it looks like we've got the following intervals of increasing and decreasing.

$$\text{Increasing}: \ -\infty < x < -1 \text{ and } 1 < x < \infty$$
$$\text{Decreasing}: \ -1 < x < 0, \ \ 0 < x < 1$$

Note that from the first derivative test we can also say that $x = -1$ is a relative maximum and that $x = 1$ is a relative minimum. Also $x = 0$ is neither a relative minimum or maximum.

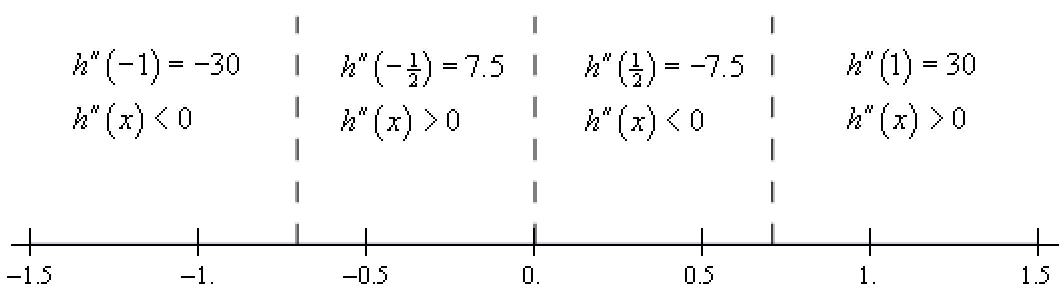
Now let's get the intervals where the function is concave up and concave down. If you think about it this process is almost identical to the process we use to identify the intervals of increasing and decreasing. This only difference is that we will be using the second derivative instead of the first derivative.

The first thing that we need to do is identify the possible inflection points. These will be where the second derivative is zero or doesn't exist. The second derivative in this case is a polynomial and so will exist everywhere. It will be zero at the following points.

$$x = 0, \ x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

As with the increasing and decreasing part we can draw a number line up and use these points to divide the number line into regions. In these regions we know that the second derivative will always have the same sign since these three points are the only places where the function *may* change sign. Therefore, all that we need to do is pick a point from each region and plug it into the second derivative. The second derivative will then have that sign in the whole region from which the point came from

Here is the number line for this second derivative.



So, it looks like we've got the following intervals of concavity.

$$\text{Concave Up}: \ -\frac{1}{\sqrt{2}} < x < 0 \text{ and } \frac{1}{\sqrt{2}} < x < \infty$$
$$\text{Concave Down}: \ -\infty < x < -\frac{1}{\sqrt{2}} \ \text{ and } \ 0 < x < \frac{1}{\sqrt{2}}$$

This also means that

$$x = 0, \ x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

are all inflection points.

All this information can be a little overwhelming when going to sketch the graph. The first thing that we should do is get some starting points. The critical points and inflection points are good starting points. So, first graph these points.

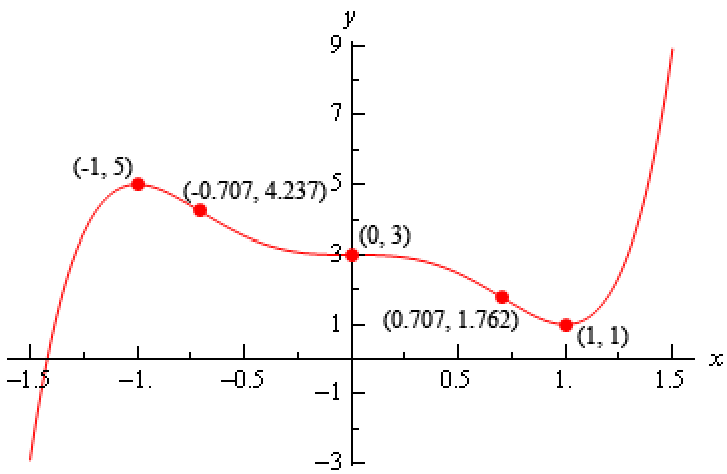
From this point there are several ways to proceed with sketching the graph. The way that we find to be the easiest (although you may not and that is perfectly fine….) is to start with the increasing/decreasing information and start sketching the graph from just that information as we did in the previous section. However, unlike the previous section, this time as we draw an increasing or decreasing portion of the curve we will also pay attention to the concavity of the curve as we are doing this.

So, if we start with $x < -1$ we know that we have an increasing function. At the same time, we know that we also have to be concave down in this range. So, we can start off with sketching an increasing curve that has is also concave down until we reach $x = -1$.

At this point the graph starts to decrease and will continue to decrease until we hit $x = 1$. However, as we decrease the concavity needs to switch to concave up at $x \approx -0.707$ and then switch back to concave down at $x = 0$ with a final switch to concave up at $x \approx 0.707$.

Once we hit $x = 1$ the graph starts to increase and is still concave up and both of these behaviors continue for the rest of the graph.

Putting all this information together will give us the following graph of the function.



We can use the previous example to illustrate another way to classify some of the critical points of a function as relative maximums or relative minimums.

Notice that $x = -1$ is a relative maximum and that the function is concave down at this point. This means that $f''(-1)$ must be negative. Likewise, $x = 1$ is a relative minimum and the function is concave up at this point. This means that $f''(1)$ must be positive.

As we'll see in a bit we will need to be very careful with $x = 0$. In this case the second derivative is zero, but that will not actually mean that $x = 0$ is not a relative minimum or maximum. We'll see some examples of this in a bit, but we need to get some other information taken care of first.

It is also important to note here that all of the critical points in this example were critical points in which the first derivative was zero and this is required for this to work. We will not be able to use this test on critical points where the derivative doesn't exist.

Here is the test that can be used to classify some of the critical points of a function. The proof of this test is in the **Proofs of Derivative Applications** section of the Extras chapter.

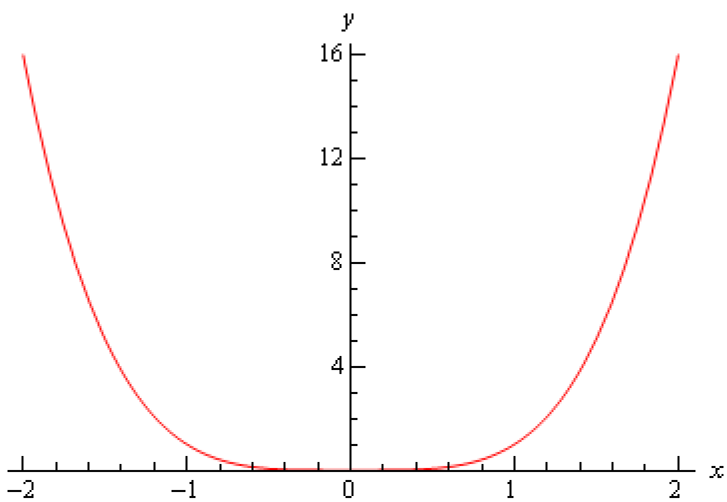
Second Derivative Test

Suppose that $x = c$ is a critical point of $f'(c)$ such that $f'(c) = 0$ and that $f''(x)$ is continuous in a region around $x = c$. Then,

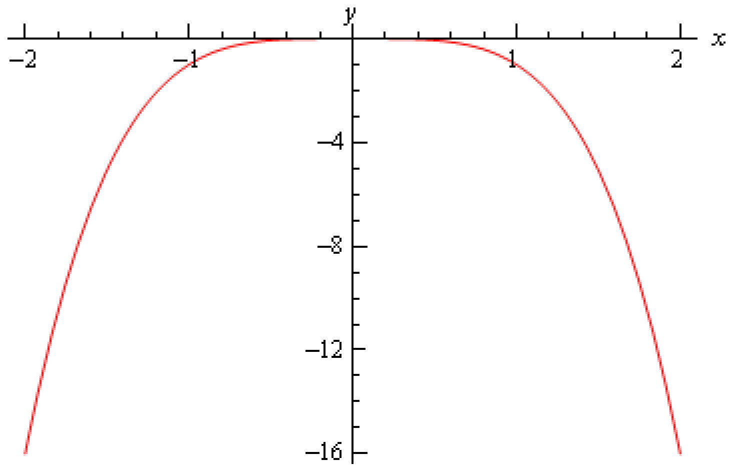
1. If $f''(c) < 0$ then $x = c$ is a relative maximum.
2. If $f''(c) > 0$ then $x = c$ is a relative minimum.
3. If $f''(c) = 0$ then $x = c$ can be a relative maximum, relative minimum or neither.

The third part of the second derivative test is important to notice. If the second derivative is zero then the critical point can be anything. Below are the graphs of three functions all of which have a critical point at $x = 0$, the second derivative of all of the functions is zero at $x = 0$ and yet all three possibilities are exhibited.

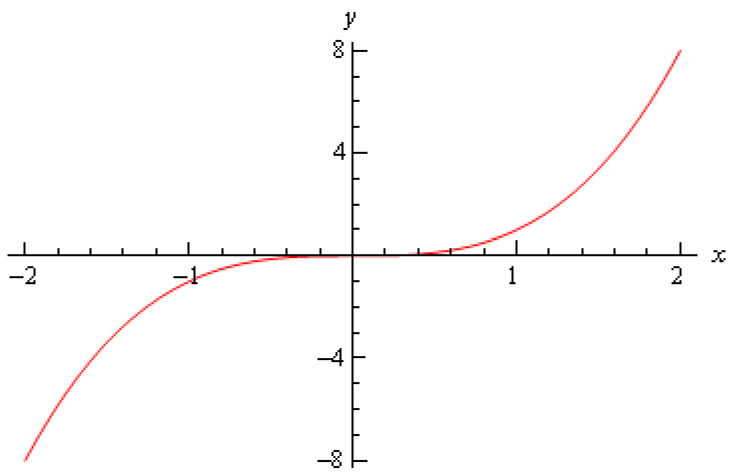
The first is the graph of $f(x) = x^4$. This graph has a relative minimum at $x = 0$.



Next is the graph of $f(x) = -x^4$ which has a relative maximum at $x = 0$.



Finally, there is the graph of $f\left(x\right)=x^3$ and this graph had neither a relative minimum or a relative maximum at $x=0$.



So, we can see that we have to be careful if we fall into the third case. For those times when we do fall into this case we will have to resort to other methods of classifying the critical point. This is usually done with the first derivative test.

Let's go back and take a look at the critical points from the first example and use the Second Derivative Test on them, if possible.

Example 2 Use the second derivative test to classify the critical points of the function,

$$h\left(x\right)=3x^5-5x^3+3$$

Solution

Note that all we're doing here is verifying the results from the first example. The second derivative is,

$$h''\left(x\right)=60x^3-30x$$

The three critical points ($x=-1$, $x=0$, and $x=1$) of this function are all critical points where the first derivative is zero so we know that we at least have a chance that the Second Derivative Test will work. The value of the second derivative for each of these are,

$$h''\left(-1\right)=-30 \qquad h''\left(0\right)=0 \qquad h''\left(1\right)=30$$

The second derivative at $x=-1$ is negative so by the Second Derivative Test this critical point this is a relative maximum as we saw in the first example. The second derivative at $x=1$ is positive and so we have a relative minimum here by the Second Derivative Test as we also saw in the first example.

In the case of $x=0$ the second derivative is zero and so we can't use the Second Derivative Test to classify this critical point. Note however, that we do know from the First Derivative Test we used in the first example that *in this case* the critical point is not a relative extrema.

Let's work one more example.

Example 3 For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points. Also, determine the intervals of increase/decrease and the intervals of concave up/concave down and sketch the graph of the function.

$$f\left(t\right)=t\left(6-t\right)^{\frac{2}{3}}$$

Solution

We'll need the first and second derivatives to get us started. We'll leave it to you to verify these derivatives but be aware that we did a little simplification after taking each derivative.

$$f'\left(t\right)=\frac{18-5t}{3\left(6-t\right)^{\frac{1}{3}}} \qquad f''\left(t\right)=\frac{10t-72}{9\left(6-t\right)^{\frac{4}{3}}}$$

The critical points are,

$$t=\frac{18}{5}=3.6 \qquad t=6$$

Notice as well that we won't be able to use the second derivative test on $t=6$ to classify this critical point since the derivative doesn't exist at this point. To classify this, we'll need the increasing/decreasing information that we'll get to sketch the graph.

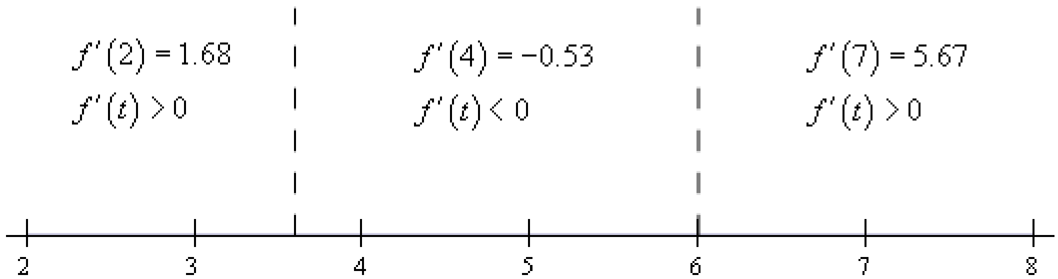
We can however, use the Second Derivative Test to classify the other critical point so let's do that before we proceed with the sketching work. Here is the value of the second derivative at $t=3.6$.

$$f''\left(3.6\right)=-1.245<0$$

So, according to the second derivative test $t=3.6$ is a relative maximum.

Now let's proceed with the work to get the sketch of the graph and notice that once we have the increasing/decreasing information we'll be able to classify $t=6$.

Here is the number line for the first derivative.



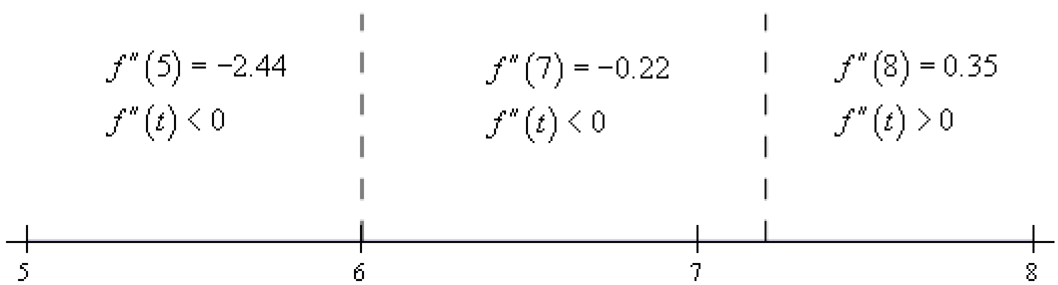
So, according to the first derivative test we can verify that $t = 3.6$ is in fact a relative maximum. We can also see that $t = 6$ is a relative minimum.

Be careful not to assume that a critical point that can't be used in the second derivative test won't be a relative extrema. We've clearly seen now both with this example and in the discussion after we have the test that just because we can't use the Second Derivative Test or the Second Derivative Test doesn't tell us anything about a critical point doesn't mean that the critical point will not be a relative extrema. This is a common mistake that many students make so be careful when using the Second Derivative Test.

Okay, let's finish the problem out. We will need the list of possible inflection points. These are,

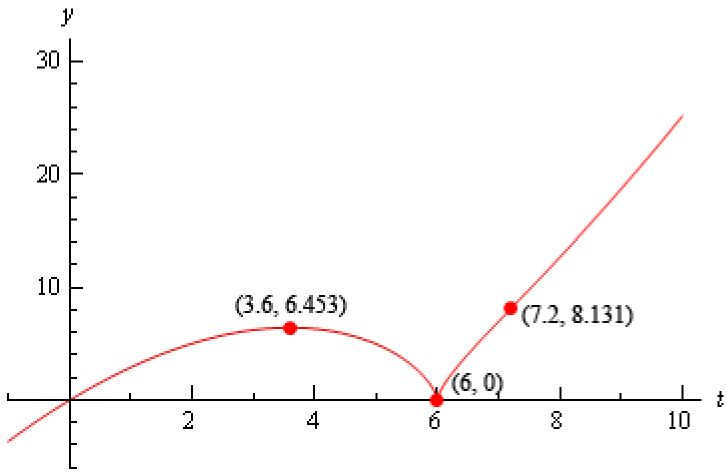
$$t = 6 \qquad\qquad t = \frac{72}{10} = 7.2$$

Here is the number line for the second derivative. Note that we will need this to see if the two points above are in fact inflection points.



So, the concavity only changes at $t = 7.2$ and so this is the only inflection point for this function.

Here is the sketch of the graph.



The change of concavity at $t = 7.2$ is hard to see, but it is there it's just a very subtle change in concavity.

Section 4-7 : The Mean Value Theorem

In this section we want to take a look at the Mean Value Theorem. In most traditional textbooks this section comes before the sections containing the First and Second Derivative Tests because many of the proofs in those sections need the Mean Value Theorem. However, we feel that from a logical point of view it's better to put the Shape of a Graph sections right after the absolute extrema section. So, if you've been following the proofs from the previous two sections you've probably already read through this section.

Before we get to the Mean Value Theorem we need to cover the following theorem.

Rolle's Theorem

Suppose $f\left(x\right)$ is a function that satisfies all of the following.

1. $f\left(x\right)$ is continuous on the closed interval $\left[a,b\right]$.
2. $f\left(x\right)$ is differentiable on the open interval $\left(a,b\right)$.
3. $f\left(a\right)=f\left(b\right)$

Then there is a number *c* such that $a<c<b$ and $f'\left(c\right)=0$. Or, in other words $f\left(x\right)$ has a critical point in $\left(a,b\right)$.

To see the proof of Rolle's Theorem see the **Proofs From Derivative Applications** section of the Extras chapter.

Let's take a look at a quick example that uses Rolle's Theorem.

Example 1 Show that $f\left(x\right)=4x^5+x^3+7x-2$ has exactly one real root.

Solution

From basic Algebra principles we know that since $f\left(x\right)$ is a 5$^{\text{th}}$ degree polynomial it will have five roots. What we're being asked to prove here is that only one of those 5 is a real number and the other 4 must be complex roots.

First, we should show that it does have at least one real root. To do this note that $f\left(0\right)=-2$ and that $f\left(1\right)=10$ and so we can see that $f\left(0\right)<0<f\left(1\right)$. Now, because $f\left(x\right)$ is a polynomial we know that it is continuous everywhere and so by the **Intermediate Value Theorem** there is a number *c* such that $0<c<1$ and $f\left(c\right)=0$. In other words $f\left(x\right)$ has at least one real root.

We now need to show that this is in fact the only real root. To do this we'll use an argument that is called contradiction proof. What we'll do is assume that $f\left(x\right)$ has at least two real roots. This means that we can find real numbers *a* and *b* (there might be more, but all we need for this particular argument is two) such that $f(a) = f(b) = 0$. But if we do this then we know from Rolle's Theorem that there must then be another number *c* such that $f'(c) = 0$.

This is a problem however. The derivative of this function is,

$$f'(x) = 20x^4 + 3x^2 + 7$$

Because the exponents on the first two terms are even we know that the first two terms will always be greater than or equal to zero and we are then going to add a positive number onto that and so we can see that the smallest the derivative will ever be is 7 and this contradicts the statement above that says we MUST have a number *c* such that $f'(c) = 0$.

We reached these contradictory statements by assuming that $f(x)$ has at least two roots. Since this assumption leads to a contradiction the assumption must be false and so we can only have a single real root.

The reason for covering Rolle's Theorem is that it is needed in the proof of the Mean Value Theorem. To see the proof see the **Proofs From Derivative Applications** section of the Extras chapter. Here is the theorem.

Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a,b]$.
2. $f(x)$ is differentiable on the open interval (a,b).

Then there is a number *c* such that *a* < *c* < *b* and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$

Note that the Mean Value Theorem doesn't tell us what *c* is. It only tells us that there is at least one number *c* that will satisfy the conclusion of the theorem.

Also note that if it weren't for the fact that we needed Rolle's Theorem to prove this we could think of Rolle's Theorem as a special case of the Mean Value Theorem. To see that just assume that $f(a) = f(b)$ and then the result of the Mean Value Theorem gives the result of Rolle's Theorem.

Before we take a look at a couple of examples let's think about a geometric interpretation of the Mean Value Theorem. First define $A = (a, f(a))$ and $B = (b, f(b))$ and then we know from the Mean Value theorem that there is a *c* such that $a < c < b$ and that

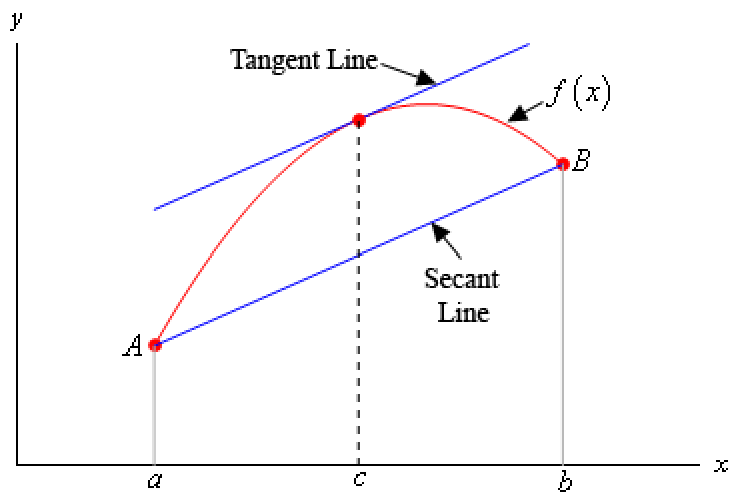
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Now, if we draw in the secant line connecting *A* and *B* then we can know that the slope of the secant line is,

$$\frac{f(b)-f(a)}{b-a}$$

Likewise, if we draw in the tangent line to $f(x)$ at $x=c$ we know that its slope is $f'(c)$.

What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting *A* and *B* and the tangent line at $x=c$ must be parallel. We can see this in the following sketch.



Let's now take a look at a couple of examples using the Mean Value Theorem.

Example 2 Determine all the numbers *c* which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x)=x^3+2x^2-x \qquad \text{on} \qquad [-1,2]$$

Solution

There isn't really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (*i.e.* the derivative exists) on the interval given.

First let's find the derivative.

$$f'(x)=3x^2+4x-1$$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$3c^2 + 4c - 1 = \frac{14 - 2}{3} = \frac{12}{3} = 4$$

Now, this is just a quadratic equation,

$$\begin{aligned} 3c^2 + 4c - 1 &= 4 \\ 3c^2 + 4c - 5 &= 0 \end{aligned}$$

Using the quadratic formula on this we get,

$$c = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{6} = \frac{-4 \pm \sqrt{76}}{6}$$

So, solving gives two values of *c*.

$$c = \frac{-4 + \sqrt{76}}{6} = 0.7863 \qquad\qquad c = \frac{-4 - \sqrt{76}}{6} = -2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means that we will exclude the second one (since it isn't in the interval). The number that we're after in this problem is,

$$c = 0.7863$$

Be careful to not assume that only one of the numbers will work. It is possible for both of them to work.

Example 3 Suppose that we know that $f(x)$ is continuous and differentiable on $[6,15]$. Let's also suppose that we know that $f(6) = -2$ and that we know that $f'(x) \le 10$. What is the largest possible value for $f(15)$?

Solution

Let's start with the conclusion of the Mean Value Theorem.

$$f(15) - f(6) = f'(c)(15 - 6)$$

Plugging in for the known quantities and rewriting this a little gives,

$$f(15) = f(6) + f'(c)(15 - 6) = -2 + 9f'(c)$$

Now we know that $f'(x) \le 10$ so in particular we know that $f'(c) \le 10$. This gives us the following,

$$\begin{aligned} f(15) &= -2 + 9f'(c) \\ &\le -2 + (9)10 \\ &= 88 \end{aligned}$$

All we did was replace $f'(c)$ with its largest possible value.

This means that the largest possible value for $f(15)$ is 88.

Example 4 Suppose that we know that $f(x)$ is continuous and differentiable everywhere. Let's also suppose that we know that $f(x)$ has two roots. Show that $f'(x)$ must have at least one root.

Solution

It is important to note here that all we can say is that $f'(x)$ will have at least one root. We can't say that it will have exactly one root. So don't confuse this problem with the first one we worked.

This is actually a fairly simple thing to prove. Since we know that $f(x)$ has two roots let's suppose that they are *a* and *b*. Now, by assumption we know that $f(x)$ is continuous and differentiable everywhere and so in particular it is continuous on $[a,b]$ and differentiable on (a,b).

Therefore, by the Mean Value Theorem there is a number *c* that is between *a* and *b* (this isn't needed for this problem, but it's true so it should be pointed out) and that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But we now need to recall that *a* and *b* are roots of $f(x)$ and so this is,

$$f'(c) = \frac{0 - 0}{b - a} = 0$$

Or, $f'(x)$ has a root at $x = c$.

Again, it is important to note that we don't have a value of *c*. We have only shown that it exists. We also haven't said anything about *c* being the only root. It is completely possible for $f'(x)$ to have more than one root.

It is completely possible to generalize the previous example significantly. For instance if we know that $f(x)$ is continuous and differentiable everywhere and has three roots we can then show that not only will $f'(x)$ have at least two roots but that $f''(x)$ will have at least one root. We'll leave it to you to verify this, but the ideas involved are identical to those in the previous example.

We'll close this section out with a couple of nice facts that can be proved using the Mean Value Theorem. Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval $[a,b]$.

Fact 1

If $f'(x) = 0$ for all *x* in an interval (a,b) then $f(x)$ is constant on (a,b).

This fact is very easy to prove so let's do that here.

First, notice that because we are assuming the derivative exists on (a,b) we know that $f(x)$ is differentiable on (a,b). In addition, we **know** that if a function is differentiable on an interval then it is also continuous on that interval and so $f(x)$ will also be continuous on (a,b).

Now, take any two *x*'s in the interval (a,b), say x_1 and x_2. Then since $f(x)$ is continuous and differentiable on (a,b) it must also be continuous and differentiable on $[x_1, x_2]$. This means that we can apply the Mean Value Theorem for these two values of *x*. Doing this gives,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

where $x_1 < c < x_2$. But by assumption $f'(x) = 0$ for all *x* in an interval (a,b) and so in particular we must have,

$$f'(c) = 0$$

Putting this into the equation above gives,

$$f(x_2) - f(x_1) = 0 \qquad \Rightarrow \qquad f(x_2) = f(x_1)$$

Now, since x_1 and x_2 where any two values of *x* in the interval (a,b) we can see that we must have $f(x_2) = f(x_1)$ for all x_1 and x_2 in the interval and this is exactly what it means for a function to be constant on the interval and so we've proven the fact.

Fact 2

If $f'(x) = g'(x)$ for all *x* in an interval (a,b) then in this interval we have $f(x) = g(x) + c$ where *c* is some constant.

This fact is a direct result of the previous fact and is also easy to prove.

If we first define,

$$h(x) = f(x) - g(x)$$

Then since both $f(x)$ and $g(x)$ are continuous and differentiable in the interval (a,b) then so must be $h(x)$. Therefore, the derivative of $h(x)$ is,

$$h'(x) = f'(x) - g'(x)$$

However, by assumption $f'(x) = g'(x)$ for all *x* in an interval (a,b) and so we must have that $h'(x) = 0$ for all *x* in an interval (a,b). So, by Fact 1 $h(x)$ must be constant on the interval.

This means that we have,

$$\begin{aligned} h(x) &= c \\ f(x) - g(x) &= c \\ f(x) &= g(x) + c \end{aligned}$$

which is what we were trying to show.

Section 4-8 : Optimization

In this section we are going to look at optimization problems. In optimization problems we are looking for the largest value or the smallest value that a function can take. We saw how to solve one kind of optimization problem in the **Absolute Extrema** section where we found the largest and smallest value that a function would take on an interval.

In this section we are going to look at another type of optimization problem. Here we will be looking for the largest or smallest value of a function subject to some kind of constraint. The constraint will be some condition (that can usually be described by some equation) that must absolutely, positively be true no matter what our solution is. On occasion, the constraint will not be easily described by an equation, but in these problems it will be easy to deal with as we'll see.

This section is generally one of the more difficult for students taking a Calculus course. One of the main reasons for this is that a subtle change of wording can completely change the problem. There is also the problem of identifying the quantity that we'll be optimizing and the quantity that is the constraint and writing down equations for each.

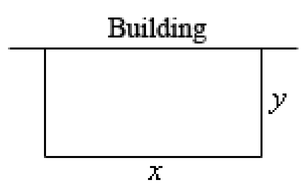
The first step in all of these problems should be to very carefully read the problem. Once you've done that the next step is to identify the quantity to be optimized and the constraint.

In identifying the constraint remember that the constraint is the quantity that must be true regardless of the solution. In almost every one of the problems we'll be looking at here one quantity will be clearly indicated as having a fixed value and so must be the constraint. Once you've got that identified the quantity to be optimized should be fairly simple to get. It is however easy to confuse the two if you just skim the problem so make sure you carefully read the problem first!

Let's start the section off with a simple problem to illustrate the kinds of issues we will be dealing with here.

Example 1 We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Solution
In all of these problems we will have two functions. The first is the function that we are actually trying to optimize and the second will be the constraint. Sketching the situation will often help us to arrive at these equations so let's do that.



In this problem we want to maximize the area of a field and we know that will use 500 ft of fencing material. So, the area will be the function we are trying to optimize and the amount of fencing is the constraint. The two equations for these are,

$$\begin{aligned}\text{Maximize : } & A = xy \\ \text{Constraint : } & 500 = x + 2y\end{aligned}$$

Okay, we know how to find the largest or smallest value of a function provided it's only got a single variable. The area function (as well as the constraint) has two variables in it and so what we know about finding absolute extrema won't work. However, if we solve the constraint for one of the two variables we can substitute this into the area and we will then have a function of a single variable.

So, let's solve the constraint for *x*. Note that we could have just as easily solved for *y* but that would have led to fractions and so, in this case, solving for *x* will probably be best.

$$x = 500 - 2y$$

Substituting this into the area function gives a function of *y*.

$$A(y) = (500 - 2y)\,y = 500y - 2y^2$$

Now we want to find the largest value this will have on the interval $[0, 250]$. The limits in this interval corresponds to taking $y = 0$ (*i.e.* no sides to the fence) and $y = 250$ (*i.e.* only two sides and no width, also if there are two sides each must be 250 ft to use the whole 500ft).

Note that the endpoints of the interval won't make any sense from a physical standpoint if we actually want to enclose some area because they would both give zero area. They do, however, give us a set of limits on *y* and so the **Extreme Value Theorem** tells us that we will have a maximum value of the area somewhere between the two endpoints. Having these limits will also mean that we can use the process we discussed in the **Finding Absolute Extrema** section earlier in the chapter to find the maximum value of the area.

So, recall that the maximum value of a continuous function (which we've got here) on a closed interval (which we also have here) will occur at critical points and/or end points. As we've already pointed out the end points in this case will give zero area and so don't make any sense. That means our only option will be the critical points.

So, let's get the derivative and find the critical points.

$$A'(y) = 500 - 4y$$

Setting this equal to zero and solving gives a lone critical point of $y = 125$. Plugging this into the area gives an area of $A(125) = 31250\,\text{ft}^2$. So according to the method from Absolute Extrema section this must be the largest possible area, since the area at either endpoint is zero.

Finally, let's not forget to get the value of *x* and then we'll have the dimensions since this is what the problem statement asked for. We can get the *x* by plugging in our *y* into the constraint.

$$x = 500 - 2(125) = 250$$

The dimensions of the field that will give the largest area, subject to the fact that we used exactly 500 ft of fencing material, are 250 x 125.

Don't forget to actually read the problem and give the answer that was asked for. These types of problems can take a fair amount of time/effort to solve and it's not hard to sometimes forget what the problem was actually asking for.

In the previous problem we used the method from the Finding Absolute Extrema section to find the maximum value of the function we wanted to optimize. However, as we'll see in later examples it will not always be easy to find endpoints. Also, even if we can find the endpoints we will see that sometimes dealing with the endpoints may not be easy either. Not only that, but this method requires that the function we're optimizing be continuous on the interval we're looking at, including the endpoints, and that may not always be the case.

So, before proceeding with any more examples let's spend a little time discussing some methods for determining if our solution is in fact the absolute minimum/maximum value that we're looking for. In some examples all of these will work while in others one or more won't be all that useful. However, we will always need to use some method for making sure that our answer is in fact that optimal value that we're after.

Method 1 : Use the method used in **Finding Absolute Extrema**.

This is the method used in the first example above. Recall that in order to use this method the interval of possible values of the independent variable in the function we are optimizing, let's call it *I*, must have finite endpoints. Also, the function we're optimizing (once it's down to a single variable) must be continuous on *I*, including the endpoints. If these conditions are met then we know that the optimal value, either the maximum or minimum depending on the problem, will occur at either the endpoints of the range or at a critical point that is inside the range of possible solutions.

There are two main issues that will often prevent this method from being used however. First, not every problem will actually have a range of possible solutions that have finite endpoints at both ends. We'll see at least one example of this as we work through the remaining examples. Also, many of the functions we'll be optimizing will not be continuous once we reduce them down to a single variable and this will prevent us from using this method.

Method 2 : Use a variant of the **First Derivative Test**.

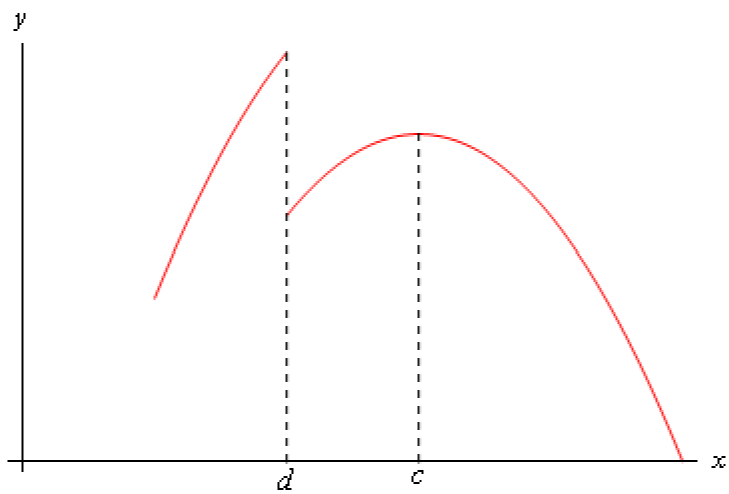
In this method we also will need an interval of possible values of the independent variable in the function we are optimizing, *I*. However, in this case, unlike the previous method the endpoints do not need to be finite. Also, we will need to require that the function be continuous on the interior of the interval *I* and we will only need the function to be continuous at the end points if the endpoint is finite and the function actually exists at the endpoint. We'll see several problems where the function we're optimizing doesn't actually exist at one of the endpoints. This will not prevent this method from being used.

Let's suppose that $x = c$ is a critical point of the function we're trying to optimize, $f\left(x\right)$. We already know from the First Derivative Test that if $f'\left(x\right) > 0$ immediately to the left of $x = c$ (*i.e.* the function is increasing immediately to the left) and if $f'(x) < 0$ immediately to the right of $x = c$ (*i.e.* the function is decreasing immediately to the right) then $x = c$ will be a relative maximum for $f(x)$.

Now, this does not mean that the absolute maximum of $f(x)$ will occur at $x = c$. However, suppose that we knew a little bit more information. Suppose that in fact we knew that $f'(x) > 0$ for all *x* in *I* such that $x < c$. Likewise, suppose that we knew that $f'(x) < 0$ for all *x* in *I* such that $x > c$. In this case we know that to the left of $x = c$, provided we stay in *I* of course, the function is always increasing and to the right of $x = c$, again staying in *I*, we are always decreasing. In this case we can say that the absolute maximum of $f(x)$ in *I* will occur at $x = c$.

Similarly, if we know that to the left of $x = c$ the function is always decreasing and to the right of $x = c$ the function is always increasing then the absolute minimum of $f(x)$ in *I* will occur at $x = c$.

Before we give a summary of this method let's discuss the continuity requirement a little. Nowhere in the above discussion did the continuity requirement apparently come into play. We require that the function we're optimizing to be continuous in *I* to prevent the following situation.



In this case, a relative maximum of the function clearly occurs at $x = c$. Also, the function is always decreasing to the right and is always increasing to the left. However, because of the discontinuity at $x = d$, we can clearly see that $f(d) > f(c)$ and so the absolute maximum of the function does not occur at $x = c$. Had the discontinuity at $x = d$ not been there this would not have happened and the absolute maximum would have occurred at $x = c$.

Here is a summary of this method.

First Derivative Test for Absolute Extrema

Let *I* be the interval of all possible values of *x* in $f(x)$, the function we want to optimize, and further suppose that $f(x)$ is continuous on *I*, except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that *c* is in the interval *I*. If we restrict *x* to values from *I* (*i.e.* we only consider possible optimal values of the function) then,

1. If $f'(x) > 0$ for all $x < c$ and if $f'(x) < 0$ for all $x > c$ then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval *I*.
2. If $f'(x) < 0$ for all $x < c$ and if $f'(x) > 0$ for all $x > c$ then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval *I*.

Method 3 : Use the second derivative.

There are actually two ways to use the second derivative to help us identify the optimal value of a function and both use the **Second Derivative Test** to one extent or another.

The first way to use the second derivative doesn't actually help us to identify the optimal value. What it does do is allow us to potentially exclude values and knowing this can simplify our work somewhat and so is not a bad thing to do.

Suppose that we are looking for the absolute maximum of a function and after finding the critical points we find that we have multiple critical points. Let's also suppose that we run all of them through the second derivative test and determine that some of them are in fact relative minimums of the function. Since we are after the absolute maximum we know that a maximum (of any kind) can't occur at relative minimums and so we immediately know that we can exclude these points from further consideration. We could do a similar check if we were looking for the absolute minimum. Doing this may not seem like all that great of a thing to do, but it can, on occasion, lead to a nice reduction in the amount of work that we need to do in later steps.

The second way of using the second derivative to identify the optimal value of a function is in fact very similar to the second method above. In fact, we will have the same requirements for this method as we did in that method. We need an interval of possible values of the independent variable in function we are optimizing, call it *I* as before, and the endpoint(s) may or may not be finite. We'll also need to require that the function, $f(x)$ be continuous everywhere in *I* except possibly at the endpoints as above.

Now, suppose that $x = c$ is a critical point and that $f''(c) > 0$. The second derivative test tells us that $x = c$ must be a relative minimum of the function. Suppose however that we also knew that $f''(x) > 0$ for all *x* in *I*. In this case we would know that the function was concave up in all of *I* and that would in turn mean that the absolute minimum of $f(x)$ in *I* would in fact have to be at $x = c$.

Likewise, if $x = c$ is a critical point and $f''(x) < 0$ for all *x* in *I* then we would know that the function was concave down in *I* and that the absolute maximum of $f(x)$ in *I* would have to be at $x = c$.

Here is a summary of this method.

Second Derivative Test for Absolute Extrema

Let *I* be the interval of all possible values of *x* in $f(x)$, the function we want to optimize, and suppose that $f(x)$ is continuous on *I* , except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that *c* is in the interval *I*. Then,

1. If $f''(x) > 0$ for all *x* in *I* then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval *I*.
2. If $f''(x) < 0$ for all *x* in *I* then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval *I*.

As we work examples over the next two sections we will use each of these methods as needed in the examples. In some cases, the method we use will be the only method we could use, in others it will be the easiest method to use and in others it will simply be the method we chose to use for that example. It is important to realize that we won't be able to use each of the methods for every example. With some examples one method will be easiest to use or may be the only method that can be used, however, each of the methods described above will be used at least a couple of times through out all of the examples.

It is also important to be aware that some problems don't allow any of the methods discussed above to be used exactly as outlined above. We may need to modify one of them or use a combination of them to fully work the problem. There is an example in the next section where none of the methods above work easily, although we do also present an alternative solution method in which we can use at least one of the methods discussed above.

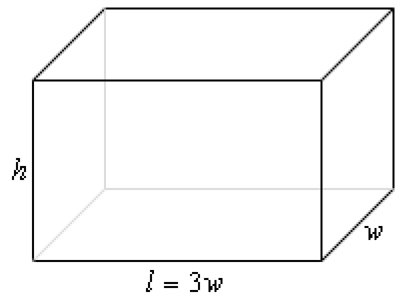
Next, the vast majority of the examples worked over the course of the next section will only have a single critical point. Problems with more than one critical point are often difficult to know which critical point(s) give the optimal value. There are a couple of examples in the next two sections with more than one critical point including one in the next section mentioned above in which none of the methods discussed above easily work. In that example you can see some of the ideas you might need to do in order to find the optimal value.

Finally, in all of the methods above we referenced an interval *I*. This was done to make the discussion a little easier. However, in all of the examples over the next two sections we will never explicitly say "this is the interval *I*". Just remember that the interval *I* is just the largest interval of possible values of the independent variable in the function we are optimizing.

Okay, let's work some more examples.

Example 2 We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box.

Solution
First, a quick figure (probably not to scale…).



We want to minimize the cost of the materials subject to the constraint that the volume must be 50ft^3. Note as well that the cost for each side is just the area of that side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\begin{aligned}&\text{Minimize : } C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh\\&\text{Constraint : } 50 = lwh = 3w^2h\end{aligned}$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for *h* so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}$$

Now, let's get the first and second (we'll be needing this later…) derivatives,

$$C'(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \qquad\qquad C''(w) = 120 + 1600w^{-3}$$

Now we need the critical point(s) for the cost function. First, notice that $w = 0$ is not a critical point. Clearly the derivative does not exist at $w = 0$ but then neither does the function and remember that values of *w* will only be critical points if the function also exists at that point. Note that there is also a physical reason to avoid $w = 0$. We are constructing a box and it would make no sense to have a zero width of the box.

So it looks like the only critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

So, we've got a single critical point and we now have to verify that this is in fact the value that will give the absolute minimum cost.

In this case we can't use Method 1 from above. First, the function is not continuous at one of the endpoints, $w = 0$, of our interval of possible values, *i.e.* $w > 0$. Secondly, there is no theoretical upper limit to the width that will give a box with volume of 50 ft^3. If *w* is very large then we would just need to make *h* very small.

The second method listed above would work here, but that's going to involve some calculations, not difficult calculations, but more work nonetheless.

The third method however, will work quickly and simply here. First, we know that whatever the value of *w* that we get it will have to be positive and we can see second derivative above that provided $w > 0$ we will have $C''(w) > 0$ and so in the interval of possible optimal values the cost function will always be concave up and so $w = 1.8821$ must give the absolute minimum cost.

All we need to do now is to find the remaining dimensions.

$$\begin{aligned} w &= 1.8821 \\ l &= 3w = 3(1.8821) = 5.6463 \\ h &= \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050 \end{aligned}$$

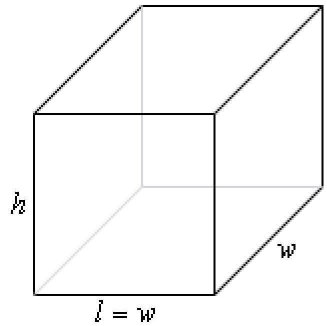
Also, even though it was not asked for, the minimum cost is : $C(1.8821) = \$637.60$.

Example 3 We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

Solution

This example is in many ways the exact opposite of the previous example. In this case we want to optimize the volume and the constraint this time is the amount of material used. We don't have a cost here, but if you think about it the cost is nothing more than the amount of material used times a cost and so the amount of material and cost are pretty much tied together. If you can do one you can do the other as well. Note as well that the amount of material used is really just the surface area of the box.

As always, let's start off with a quick sketch of the box.



Now, as mentioned above we want to maximize the volume and the amount of material is the constraint so here are the equations we'll need.

$$\text{Maximize} : V = lwh = w^2 h$$

$$\text{Constraint} : 10 = 2lw + 2wh + 2lh = 2w^2 + 4wh$$

We'll solve the constraint for *h* and plug this into the equation for the volume.

$$h = \frac{10-2w^2}{4w} = \frac{5-w^2}{2w} \qquad \Rightarrow \qquad V(w) = w^2\left(\frac{5-w^2}{2w}\right) = \frac{1}{2}\left(5w - w^3\right)$$

Here are the first and second derivatives of the volume function.

$$V'(w) = \tfrac{1}{2}\left(5-3w^2\right) \qquad\qquad V''(w) = -3w$$

Note as well here that provided $w > 0$, which from a physical standpoint we know must be true for the width of the box, then the volume function will be concave down and so if we get a single critical point then we know that it will have to be the value that gives the absolute maximum.

Setting the first derivative equal to zero and solving gives us the two critical points,

$$w = \pm\sqrt{\frac{5}{3}} = \pm 1.2910$$

In this case we can exclude the negative critical point since we are dealing with a length of a box and we know that these must be positive. Do not however get into the habit of just excluding any negative critical point. There are problems where negative critical points are perfectly valid possible solutions.

Now, as noted above we got a single critical point, 1.2910, and so this must be the value that gives the maximum volume and since the maximum volume is all that was asked for in the problem statement the answer is then : $V(1.2910) = 2.1517\,\text{m}^3$.

Note that we could also have noted here that if $0 < w < 1.2910$ then $V'(w) > 0$ (using a test point we have $V'(1) = 1 > 0$) and likewise if $w > 1.2910$ then $V'(w) < 0$ (using a test point we have $V'(2) = -\frac{7}{2} < 0$) and so if we are to the left of the critical point the volume is always increasing and if we are to the right of the critical point the volume is always decreasing and so by the Method 2 above we can also see that the single critical point must give the absolute maximum of the volume.

Finally, even though these weren't asked for here are the dimension of the box that gives the maximum volume.

$$l = w = 1.2910 \qquad\qquad h = \frac{5 - 1.2910^2}{2(1.2910)} = 1.2910$$

So, it looks like in this case we actually have a perfect cube.

In the last two examples we've seen that many of these optimization problems can be done in both directions so to speak. In both examples we have essentially the same two equations: volume and surface area. However, in Example 2 the volume was the constraint and the cost (which is directly related to the surface area) was the function we were trying to optimize. In Example 3, on the other hand, we were trying to optimize the volume and the surface area was the constraint.

It is important to not get so locked into one way of doing these problems that we can't do it in the opposite direction as needed as well. This is one of the more common mistakes that students make with these kinds of problems. They see one problem and then try to make every other problem that seems to be the same conform to that one solution even if the problem needs to be worked differently. Keep an open mind with these problems and make sure that you understand what is being optimized and what the constraint is before you jump into the solution.

Also, as seen in the last example we used two different methods of verifying that we did get the optimal value. Do not get too locked into one method of doing this verification that you forget about the other methods.

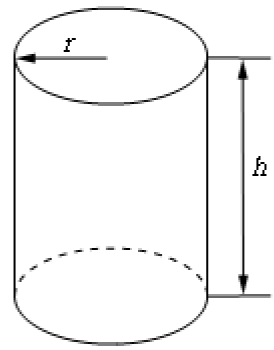
Let's work some another example that this time doesn't involve a rectangle or box.

Example 4 A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Solution

Before starting the solution let's first address the fact that we are using liters for volume. Because we want length measurements for the radius and height we'll also need the volume to in terms of a length measurement. We can easily do this using the fact that 1 Liter = 1000 cm^3 and so we can convert 1.5 liters into 1500 cm^3. This will in turn give a radius and height in terms of centimeters.

In this problem the constraint is the volume and we want to minimize the amount of material used. This means that what we want to minimize is the surface area of the can and we'll need to include both the walls of the can as well as the top and bottom "caps". Here is a quick sketch to get us started off.



We'll need the surface area of this can and that will be the surface area of the walls of the can (which is really just a cylinder) and the area of the top and bottom caps (which are just disks, and don't forget that there are two of them).

Note that if you think of a cylinder of height *h* and radius *r* as just a bunch of disks/circles of radius *r* stacked on top of each other the equations for the surface area and volume are pretty simple to remember. The volume is just the area of each of the disks times the height. Similarly, the surface area of the walls of the cylinder is just the circumference of each circle times the height. We also can't forget to add in the area of the two caps, πr^2, to the total surface area.

So, the equation for the volume and surface area of the walls of a cylinder are then,

$$V = \left(\pi r^2\right)\left(h\right) = \pi r^2 h \qquad\qquad A = \left(2\pi r\right)\left(h\right) = 2\pi r h$$

Adding the surface area of the caps of the cylinder to the surface area the equations that we'll need for this problem are,

$$\begin{aligned}&\text{Minimize : } A = 2\pi r h + 2\pi r^2\\ &\text{Constraint : } 1500 = \pi r^2 h\end{aligned}$$

In this case it looks like our best option is to solve the constraint for *h* and plug this into the area function.

$$h = \frac{1500}{\pi r^2} \qquad \Rightarrow \qquad A\left(r\right) = 2\pi r\left(\frac{1500}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{3000}{r}$$

Notice that this formula will only make sense from a physical standpoint if $r > 0$ which is a good thing as it is not defined at $r = 0$.

Next, let's get the first derivative.

$$A'\left(r\right) = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3 - 3000}{r^2}$$

From this we can see that we have one critical points : $r = \sqrt[3]{\frac{750}{\pi}} = 6.2035$ (where the derivative is zero). Note that $r = 0$ is not a critical point because the area function does not exist there, which makes sense from a physical standpoint as well given that we know that *r* must be positive in order to actually have a can.

So, we only have a single critical point to deal with here and notice that 6.2035 is the only value for which the derivative will be zero and hence the only place (with $r > 0$ of course) that the derivative may change sign. It's not difficult, using test points, to check that if $0 < r < 6.2035$ then $A'(r) < 0$ and likewise if $r > 6.2035$ then $A'(r) > 0$. The variant of the First Derivative Test above then tells us that the absolute minimum value of the area (for $r > 0$) must occur at $r = 6.2035$.

All we need to do this is determine height of the can and we'll be done.

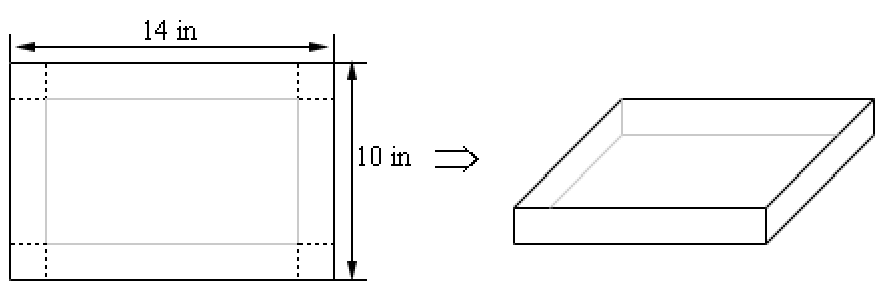
$$h = \frac{1500}{\pi(6.2035)^2} = 12.4070$$

Therefore, if the manufacturer makes the can with a radius of 6.2035 cm and a height of 12.4070 cm the least amount of material will be used to make the can.

As an interesting side problem and extension to the above example you might want to show that for a given volume, *L*, the minimum material will be used if $h = 2r$ regardless of the volume of the can.

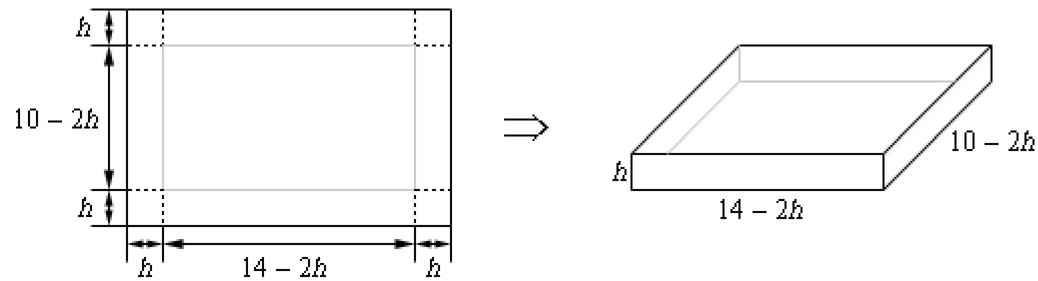
In the examples to this point we've put in quite a bit of discussion in the solution. In the remaining problems we won't be putting in quite as much discussion and leave it to you to fill in any missing details.

Example 5 We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



Solution

Let's let the height of the box be *h*. So, the width/length of the corners being cut out is also *h* and so the vertical side will have a "new" height of $10 - 2h$ and the horizontal side will have a "new" width of $14 - 2h$. Here is a sketch with all this information put in,



In this example, for the first time, we've run into a problem where the constraint doesn't really have an equation. The constraint is simply the size of the piece of cardboard and has already been factored into the figure above. This will happen on occasion and so don't get excited about it when it does. This just means that we have one less equation to worry about. In this case we want to maximize the volume. Here is the volume, in terms of *h* and its first derivative.

$$V\left(h\right)=h\left(14-2h\right)\left(10-2h\right)=140h-48h^2+4h^3 \qquad V'\left(h\right)=140-96h+12h^2$$

Setting the first derivative equal to zero and solving gives the following two critical points,

$$h=\frac{12\pm\sqrt{39}}{3}=1.9183,\quad 6.0817$$

We now have an apparent problem. We have two critical points and we'll need to determine which one is the value we need. The fact that we have two critical points means that neither the first derivative test or the second derivative test can be used here as they both require a single critical point. This isn't a real problem however. Go back to the figure at the start of the solution and notice that we can quite easily find limits on *h*. The smallest *h* can be is $h=0$ even though this doesn't make much sense as we won't get a box in this case. Also, from the 10 inch side we can see that the largest *h* can be is $h=5$ although again, this doesn't make much sense physically.

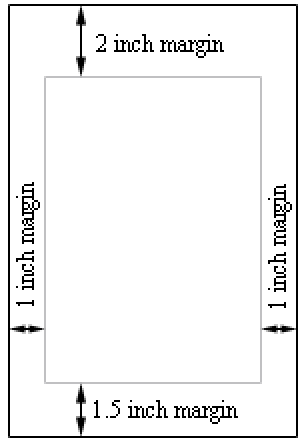
So, knowing that whatever *h* is it must be in the range $0\le h\le 5$ we can see that the second critical point is outside this range and so the only critical point that we need to worry about is 1.9183.

Finally, since the volume is defined and continuous on $0\le h\le 5$ all we need to do is plug in the critical points and endpoints into the volume to determine which gives the largest volume. Here are those function evaluations.

$$V\left(0\right)=0 \qquad V\left(1.9183\right)=120.1644 \qquad V\left(5\right)=0$$

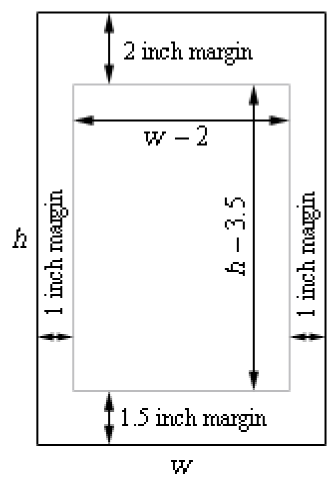
So, if we take $h=1.9183$ we get a maximum volume.

Example 6 A printer need to make a poster that will have a total area of 200 in^2 and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom as shown below. What dimensions will give the largest printed area?



Solution
This problem is a little different from the previous problems. Both the constraint and the function we are going to optimize are areas. The constraint is that the overall area of the poster must be 200 in^2 while we want to optimize the printed area (*i.e.* the area of the poster with the margins taken out).

Let's define the height of the poster to be *h* and the width of the poster to be *w*. Here is a new sketch of the poster and we can see that once we've taken the margins into account the width of the printed area is $w-2$ and the height of the printer area is $h-3.5$.



Here are the equations that we'll be working with.

$$\text{Maximize : } A = (w-2)(h-3.5)$$
$$\text{Constraint : } 200 = wh$$

Solving the constraint for *h* and plugging into the equation for the printed area gives,

$$A\left(w\right)=\left(w-2\right)\left(\frac{200}{w}-3.5\right)=207-3.5w-\frac{400}{w}$$

The first and second derivatives are,

$$A'\left(w\right)=-3.5+\frac{400}{w^2}=\frac{400-3.5w^2}{w^2} \qquad\qquad A''\left(w\right)=-\frac{800}{w^3}$$

From the first derivative we have the following two critical points ($w=0$ is not a critical point because the area function does not exist there).

$$w=\pm\sqrt{\tfrac{400}{3.5}}=\pm 10.6904$$

However, since we're dealing with the dimensions of a piece of paper we know that we must have $w>0$ and so only 10.6904 will make sense.

Also notice that provided $w>0$ the second derivative will always be negative and so in the range of possible optimal values of the width the area function is always concave down and so we know that the maximum printed area will be at $w=10.6904 \text{ inches}$.

The height of the paper that gives the maximum printed area is then,

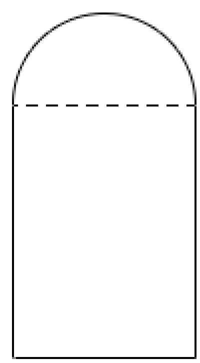
$$h=\frac{200}{10.6904}=18.7084 \text{ inches}$$

We've worked quite a few examples to this point and we have quite a few more to work. However, this section has gotten quite lengthy so let's continue our examples in the next section. This is being done mostly because these notes are also being presented on the web and this will help to keep the load times on the pages down somewhat.

Section 4-9 : More Optimization

Because these notes are also being presented on the web we've broken the optimization examples up into several sections to keep the load times to a minimum. Do not forget the various **methods** for verifying that we have the optimal value that we looked at in the previous section. In this section we'll just use them without acknowledging so make sure you understand them and can use them. So let's get going on some more examples.

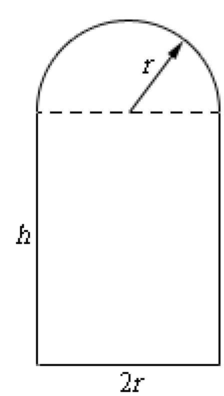
Example 1 A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing materials what must the dimensions of the window be to let in the most light?



Solution

Okay, let's ask this question again in slightly easier to understand terms. We want a window in the shape described above to have a maximum area (and hence let in the most light) and have a perimeter of 12 m (because we have 12 m of framing material). Little bit easier to understand in those terms.

Let the radius of the semicircle on the top be r and the height of the rectangle be h. Now, because the semicircle is on top of the window we can think of the width of the rectangular portion at $2r$ as shown below.



The perimeter (our constraint) is the lengths of the three sides on the rectangular portion plus half the circumference of a circle of radius r. The area (what we want to maximize) is the area of the rectangle plus half the area of a circle of radius r. Here are the equations we'll be working with in this example.

$$\text{Maximize}: A = 2hr + \tfrac{1}{2}\pi r^2$$
$$\text{Constraint}: 12 = 2h + 2r + \pi r$$

In this case we'll solve the constraint for h and plug that into the area equation.

$$h = 6 - r - \tfrac{1}{2}\pi r \qquad \Rightarrow \qquad A(r) = 2r\left(6 - r - \tfrac{1}{2}\pi r\right) + \tfrac{1}{2}\pi r^2 = 12r - 2r^2 - \tfrac{1}{2}\pi r^2$$

The first and second derivatives are,

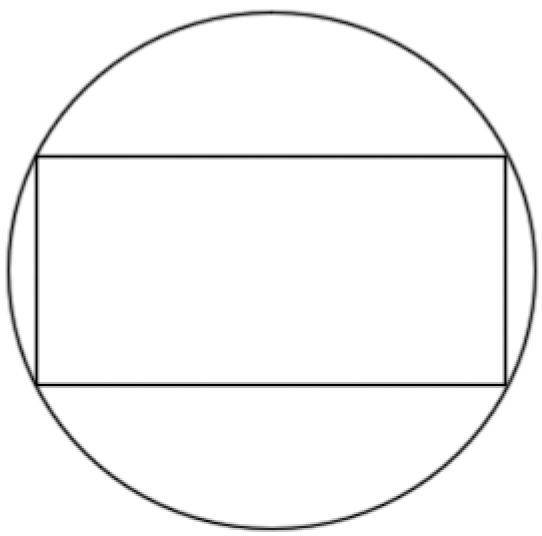
$$A'(r) = 12 - r(4+\pi) \qquad\qquad A''(r) = -4 - \pi$$

We can see that the only critical point is,

$$r = \frac{12}{4+\pi} = 1.6803$$

We can also see that the second derivative is always negative (in fact it's a constant) and so we can see that the maximum area must occur at this point. So, for the maximum area the semicircle on top must have a radius of 1.6803 and the rectangle must have the dimensions 3.3606 x 1.6803 (h x $2r$).

Example 2 Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.



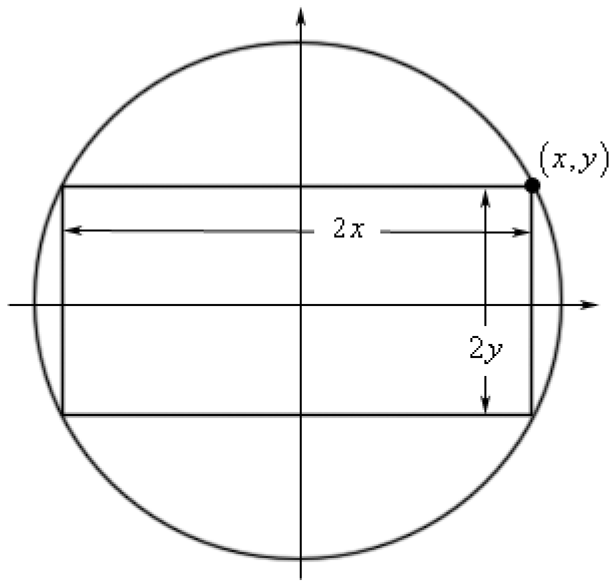
Solution

Huh? This problem type of problem never seems to make sense originally. What we want to do is maximize the area of the largest rectangle that we can fit inside a circle and have all of its corners touching the circle.

To do this problem it's easiest to assume that the circle (and hence the rectangle) is centered at the origin of a standard xy axis system. Doing this we know that the equation of the circle will be

$$x^2 + y^2 = 16$$

and that the right upper corner of the rectangle will have the coordinates (x, y). This means that the width of the rectangle will be 2*x* and the height of the rectangle will be 2*y* as shown below



The area of the rectangle will then be,

$$A = (2x)(2y) = 4xy$$

So, we've got the function we want to maximize (the area), but what is the constraint? Well since the coordinates of the upper right corner must be on the circle we know that *x* and *y* must satisfy the equation of the circle. In other words, the equation of the circle is the constraint.

The first thing to do then is to solve the constraint for one of the variables.

$$y = \pm\sqrt{16 - x^2}$$

Since the point that we're looking at is in the first quadrant we know that *y* must be positive and so we can take the "+" part of this. Plugging this into the area and computing the first derivative gives,

$$A(x) = 4x\sqrt{16 - x^2}$$

$$A'(x) = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16 - x^2}} = \frac{64 - 8x^2}{\sqrt{16 - x^2}}$$

Before getting the critical points let's notice that we can limit *x* to the range $0 \le x \le 4$ since we are assuming that *x* is in the first quadrant and must stay inside the circle. Now the four critical points we get (two from the numerator and two from the denominator) are,

$$16 - x^2 = 0 \qquad \Rightarrow \qquad x = \pm 4$$

$$64 - 8x^2 = 0 \qquad \Rightarrow \qquad x = \pm 2\sqrt{2}$$

We only want critical points that are in the range of possible optimal values so that means that we have two critical points to deal with : $x = 2\sqrt{2}$ and $x = 4$. Notice however that the second critical point is also one of the endpoints of our interval.

Now, area function is continuous and we have an interval of possible solution with finite endpoints so,

$$A(0) = 0 \qquad A\left(2\sqrt{2}\right) = 32 \qquad A(4) = 0$$

So, we can see that we'll get the maximum area if $x = 2\sqrt{2}$ and the corresponding value of *y* is,

$$y = \sqrt{16 - \left(2\sqrt{2}\right)^2} = \sqrt{8} = 2\sqrt{2}$$

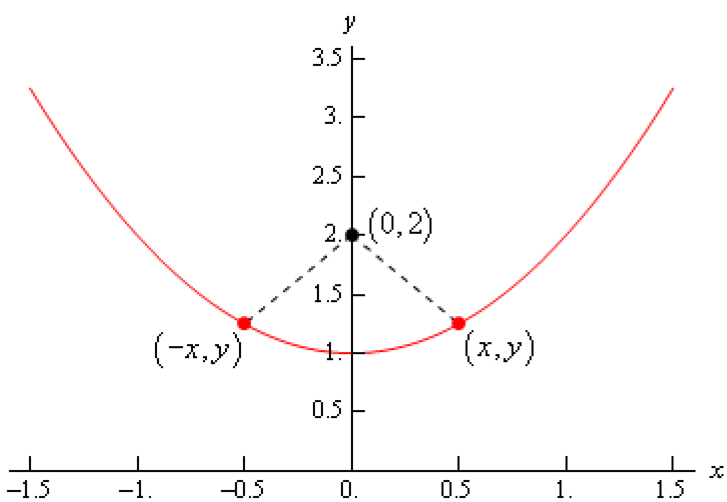
It looks like the maximum area will be found if the inscribed rectangle is in fact a square.

We need to again make a point that was made several times in the previous section. We excluded several critical points in the work above. Do not always expect to do that. There will often be physical reasons to exclude zero and/or negative critical points, however, there will be problems where these are perfectly acceptable values. You should always write down every possible critical point and then exclude any that can't be possible solutions. This keeps you in the habit of finding all the critical points and then deciding which ones you actually need and that in turn will make it less likely that you'll miss one when it is actually needed.

Example 3 Determine the point(s) on $y = x^2 + 1$ that are closest to $(0,2)$.

Solution

Here's a quick sketch of the situation.



So, we're looking for the shortest length of the dashed line. Notice as well that if the shortest distance isn't at $x = 0$ there will be two points on the graph, as we've shown above, that will give the shortest distance. This is because the parabola is symmetric to the *y*-axis and the point in question is on the *y*-axis. This won't always be the case of course so don't always expect two points in these kinds of problems.

In this case we need to minimize the distance between the point $(0,2)$ and any point that is one the graph (x,y). Or,

$$d = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$$

If you think about the situation here it makes sense that the point that minimizes the distance will also minimize the square of the distance and so since it will be easier to work with we will use the square of the distance and minimize that. If you aren't convinced of this we'll take a closer look at this after this problem. So, the function that we're going to minimize is,

$$D = d^2 = x^2 + (y-2)^2$$

The constraint in this case is the function itself since the point must lie on the graph of the function.

At this point there are two methods for proceeding. One of which will require significantly more work than the other. Let's take a look at both of them.

Solution 1

In this case we will use the constraint in probably the most obvious way. We already have the constraint solved for *y* so let's plug that into the square of the distance and get the derivatives.

$$\begin{aligned} D(x) &= x^2 + (x^2+1-2)^2 = x^4 - x^2 + 1 \\ D'(x) &= 4x^3 - 2x = 2x(2x^2-1) \\ D''(x) &= 12x^2 - 2 \end{aligned}$$

So, it looks like there are three critical points for the square of the distance and notice that this time, unlike pretty much every previous example we've worked, we can't exclude zero or negative numbers. They are perfectly valid possible optimal values this time.

$$x = 0, \quad x = \pm\frac{1}{\sqrt{2}}$$

Before going any farther, let's check these in the second derivative to see if they are all relative minimums.

$$D''(0) = -2 < 0 \qquad D''\left(\frac{1}{\sqrt{2}}\right) = 4 \qquad D''\left(-\frac{1}{\sqrt{2}}\right) = 4$$

So, $x = 0$ is a relative maximum and so can't possibly be the minimum distance. That means that we've got two critical points. The question is how we verify that these give the minimum distance and yes we did mean to say that both will give the minimum distance. Recall from our sketch above that if *x* gives the minimum distance then so will –*x* and so if gives the minimum distance then the other should as well.

None of the methods we discussed in the previous section will really work here. We don't have an interval of possible solutions with finite endpoints and both the first and second derivative change sign. In this case however, we can still verify that they are the points that give the minimum distance.

First, let's see what we have if we are working on the interval $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. On this interval we can try to use the first method of finding absolute extrema discussed in the previous section. That says to evaluate the function at the endpoints and the critical points and in this case, even though we've excluded it we'll need to include $x = 0$ since it is a critical point in the region. Doing this gives,

$$D\left(-\tfrac{1}{\sqrt{2}}\right) = \tfrac{3}{4} \qquad D(0) = 1 \qquad D\left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{3}{4}$$

So, we can see that the absolute minimum in the interval must occur at $x = \pm\frac{1}{\sqrt{2}}$.

Next, we can see that if $x < -\frac{1}{\sqrt{2}}$ then $D'(x) < 0$. Or in other words, if $x < -\frac{1}{\sqrt{2}}$ the function is decreasing until it hits $x = -\frac{1}{\sqrt{2}}$ and so must always be larger than the function at $x = -\frac{1}{\sqrt{2}}$.

Similarly, $x > \frac{1}{\sqrt{2}}$ then $D'(x) > 0$ and so the function is always increasing to the right of $x = \frac{1}{\sqrt{2}}$ and so must be larger than the function at $x = \frac{1}{\sqrt{2}}$.

So, putting all of this together tells us that we do in fact have an absolute minimum at $x = \pm\frac{1}{\sqrt{2}}$.

All that we need to do is to find the value of *y* for these points.

$$x = \frac{1}{\sqrt{2}} \quad : \quad y = \frac{3}{2}$$
$$x = -\frac{1}{\sqrt{2}} : \quad y = \frac{3}{2}$$

So, the points on the graph that are closest to $(0,2)$ are,

$$\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \qquad \left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

This solution method shows how tricky it can be to know that we have absolute extrema when there are multiple critical points and none of the methods discussed in the last section will work. Luckily for us, there is another, easier, method we could have done instead.

Solution 2

The first solution that we worked was actually the long solution. There is a much shorter, and easier, solution to this problem. Instead of plugging *y* into the square of the distance let's plug in *x*. From the constraint we get,

$$x^2 = y - 1$$

and notice that the only place *x* show up in the square of the distance it shows up as x^2 and let's just plug this into the square of the distance. Doing this gives,

$$\begin{aligned} D(y) &= y-1+(y-2)^2 = y^2-3y+3 \\ D'(y) &= 2y-3 \\ D''(y) &= 2 \end{aligned}$$

There is now a single critical point, $y=\frac{3}{2}$, and since the second derivative is always positive we know that this point must give the absolute minimum. So, all that we need to do at this point is find the value(s) of *x* that go with this value of *y*.

$$x^2 = \frac{3}{2}-1=\frac{1}{2} \qquad \Rightarrow \qquad x = \pm\frac{1}{\sqrt{2}}$$

The points are then,

$$\left(\frac{1}{\sqrt{2}},\frac{3}{2}\right) \qquad \left(-\frac{1}{\sqrt{2}},\frac{3}{2}\right)$$

So, for significantly less work we got exactly the same answer.

This previous example had a couple of nice points. First, as pointed out in the problem, we couldn't exclude zero or negative critical points this time as we've done in all the previous examples. Again, be careful to not get into the habit of always excluding them as we do many of the examples we'll work.

Next, some of these problems will have multiple solution methods and sometimes one will be significantly easier than the other. The method you use is up to you and often the difficulty of any particular method is dependent upon the person doing the problem. One person may find one way easier and other person may find a different method easier.

Finally, as we saw in the first solution method sometimes we'll need to use a combination of the optimal value verification methods we discussed in the previous section.

Now, before we move onto the next example let's take a look at the claim above that we could find the location of the point that minimizes the distance by finding the point that minimizes the square of the distance. We'll generalize things a little bit,

Fact

Suppose that we have a positive function, $f(x)>0$, that exists everywhere then $f(x)$ and $g(x)=\sqrt{f(x)}$ will have the same critical points and the relative extrema will occur at the same points.

This is simple enough to prove so let's do that here. First let's take the derivative of $g(x)$ and see what we can determine about the critical points of $g(x)$.

$$g'(x) = \tfrac{1}{2}\left[f(x)\right]^{-\frac{1}{2}} f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

Let's plug $x = c$ into this to get,

$$g'(c) = \frac{f'(c)}{2\sqrt{f(c)}}$$

By assumption we know that $f(c)$ exists and $f(c) > 0$ and therefore the denominator of this will always exist and will never be zero. We'll need this in several places so we can't forget this.

If $f'(c) = 0$ then because we know that the denominator will not be zero here we must also have $g'(c) = 0$. Likewise, if $g'(c) = 0$ then we must have $f'(c) = 0$. So, $f(x)$ and $g(x)$ will have the same critical points in which the derivatives will be zero.

Next, if $f'(c)$ doesn't exist then $g'(c)$ will also not exist and likewise if $g'(c)$ doesn't exist then because we know that the denominator will not be zero then this means that $f'(c)$ will also not exist. Therefore, $f(x)$ and $g(x)$ will have the same critical points in which the derivatives does not exist.

So, the upshot of all this is that $f(x)$ and $g(x)$ will have the same critical points.

Next, let's notice that because we know that $2\sqrt{f(x)} > 0$ then $f'(x)$ and $g'(x)$ will have the same sign and so if we apply the first derivative test (and recalling that they have the same critical points) to each of these functions we can see that the results will be the same and so the relative extrema will occur at the same points.

Note that we could also use the second derivative test to verify that the critical points will have the same classification if we wanted to. The second derivative is (and you should see if you can use the quotient rule to verify this),

$$g''(x) = \frac{2\sqrt{f(x)}\,f''(x) - \left[f'(x)\right]^2\left[f(x)\right]^{-\frac{1}{2}}}{4f(x)}$$

Then if $x = c$ is a critical point such that $f'(c) = 0$ (and so we can use the second derivative test) we get,

$$\begin{aligned} g''(c) &= \frac{2\sqrt{f(c)}\,f''(c) - \left[f'(c)\right]^2\left[f(c)\right]^{-\frac{1}{2}}}{4f(c)} \\ &= \frac{2\sqrt{f(c)}\,f''(c) - \left[0\right]^2\left[f(c)\right]^{-\frac{1}{2}}}{4f(c)} = \frac{\sqrt{f(c)}\,f''(c)}{2f(c)} \end{aligned}$$

Now, because we know that $2\sqrt{f(c)} > 0$ and by assumption $f(c) > 0$ we can see that $f''(c)$ and $g''(c)$ will have the same sign and so will have the same conclusion from the second derivative test.

So, now that we have that out of the way let's work some more examples.

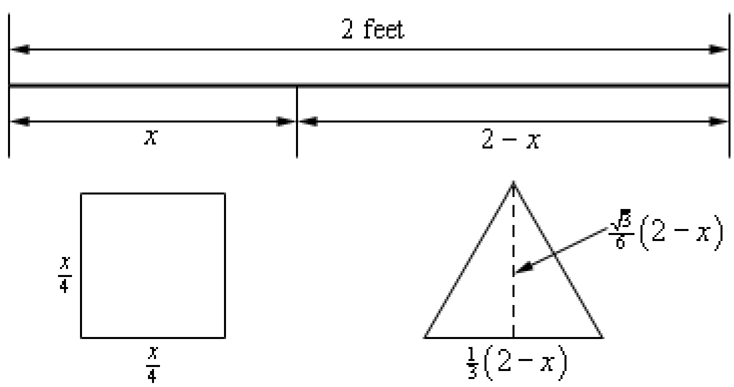
Example 4 A 2 feet piece of wire is cut into two pieces and one piece is bent into a square and the other is bent into an equilateral triangle. Where, if anywhere, should the wire be cut so that the total area enclosed by both is minimum and maximum?

Solution
Before starting the solution recall that an equilateral triangle is a triangle with three equal sides and each of the interior angles are $\frac{\pi}{3}$ (or $60°$).

Also, note the "if anywhere" portion of the problem statement. What this is saying is that it is possible to take the full piece of wire and put all of it into either a square or a triangle. Do not forget about this as it will be important later on in the problem.

Now, this is another problem where the constraint isn't really going to be given by an equation, it is simply that there is 2 ft of wire to work with and this will be taken into account in our work.

So, let's cut the wire into two pieces. The first piece will have length *x* which we'll bend into a square and each side will have length $\frac{x}{4}$. The second piece will then have length $2-x$ (we just used the constraint here…) and we'll bend this into an equilateral triangle and each side will have length $\frac{1}{3}(2-x)$. Here is a sketch of all this.



As noted in the sketch above we also will need the height of the triangle. This is easy to get if you realize that the dashed line divides the equilateral triangle into two other triangles. Let's look at the right one. The hypotenuse is $\frac{1}{3}(2-x)$ while the lower right angle is $\frac{\pi}{3}$. Finally, the height is then the opposite side to the lower right angle so using basic right triangle trig we arrive at the height of the triangle as follows.

$$\sin\left(\tfrac{\pi}{3}\right) = \tfrac{opp}{hyp} \qquad \Rightarrow \qquad opp = \tfrac{1}{3}(2-x)\sin\left(\tfrac{\pi}{3}\right) = \tfrac{1}{3}(2-x)\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{\sqrt{3}}{6}(2-x)$$

So, the total area of both objects is then,

$$A(x) = \left(\tfrac{x}{4}\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{3}(2-x)\right)\left(\tfrac{\sqrt{3}}{6}(2-x)\right) = \tfrac{x^2}{16} + \tfrac{\sqrt{3}}{36}(2-x)^2$$

Here's the first derivative of the area.

$$A'(x) = \tfrac{x}{8} + \tfrac{\sqrt{3}}{36}(2)(2-x)(-1) = \tfrac{x}{8} - \tfrac{\sqrt{3}}{9} + \tfrac{\sqrt{3}}{18}x$$

Setting this equal to zero and solving gives the single critical point of,

$$x = \tfrac{8\sqrt{3}}{9+4\sqrt{3}} = 0.8699$$

Now, let's notice that the problem statement asked for both the minimum and maximum enclosed area and we got a single critical point. This clearly can't be the answer to both, but this is not the problem that it might seem to be.

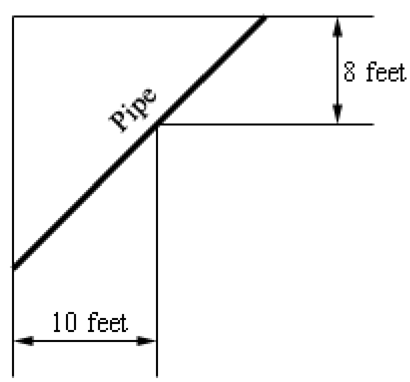
Let's notice that *x* must be in the range $0 \le x \le 2$ and since the area function is continuous we use the basic process for finding absolute extrema of a function.

$$A(0) = 0.1925 \qquad A(0.8699) = 0.1087 \qquad A(2) = 0.25$$

So, it looks like the minimum area will arise if we take $x = 0.8699$ while the maximum area will arise if we take the whole piece of wire and bend it into a square.

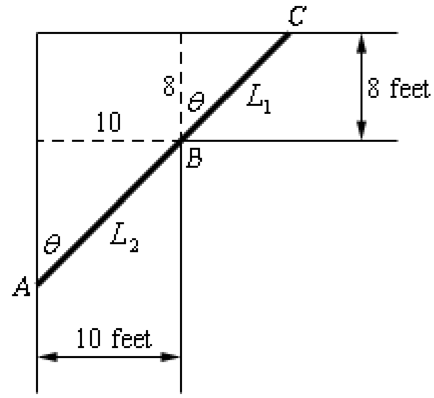
As the previous problem illustrated we can't get too locked into the answers always occurring at the critical points as they have to this point. That will often happen, but one of the extrema in the previous problem was at an endpoint and that will happen on occasion.

Example 5 A piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway the there is a right-angled turn and the hallway narrows down to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway?



Solution

Let's start off with a sketch of the situation adding in some more information so we can get a grip on what's going on and how we're going to have to go about solving this.



The largest pipe that can go around the turn will do so in the position shown above. One end will be touching the outer wall of the hall way at *A* and *C* and the pipe will touch the inner corner at *B*. Let's assume that the length of the pipe in the small hallway is L_1 while L_2 is the length of the pipe in the large hallway. The pipe then has a length of $L = L_1 + L_2$.

Now, if $\theta = 0$ then the pipe is completely in the wider hallway and we can see that as $\theta \to 0$ the point *A* will move down the vertical wall and the point *C* will move along the horizontal wall closer and closer to the corner and as this happens *L* lengthens and so $L \to \infty$ as $\theta \to 0$.

Likewise, if $\theta = \frac{\pi}{2}$ the pipe is completely in the narrow hallway and as $\theta \to \frac{\pi}{2}$ we also have $L \to \infty$ by a similar line of reasoning above for $\theta \to 0$.

So, because $L \to \infty$ as we near the ends of the interval of possible angles somewhere in the interior of the interval, $0 < \theta < \frac{\pi}{2}$, is an angle that will minimize *L* and oddly enough that is the length that we're after. The largest pipe that will fit around the turn will in fact be the minimum value of *L*.

The constraint for this problem is not so obvious and there are actually two of them. The constraints for this problem are the widths of the hallways. We'll use these to get an equation for *L* in terms of θ and then we'll minimize this new equation.

So, using basic right triangle trig we can see that,

$$L_1 = 8\sec\theta \qquad L_2 = 10\csc\theta \qquad \Rightarrow \qquad L = 8\sec\theta + 10\csc\theta$$

So, differentiating *L* gives,

$$L' = 8\sec\theta\tan\theta - 10\csc\theta\cot\theta$$

Setting this equal to zero and solving gives,

$$\begin{aligned} 8\sec\theta\tan\theta &= 10\csc\theta\cot\theta \\ \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} &= \frac{10}{8} \\ \frac{\sin\theta\tan^2\theta}{\cos\theta} &= \frac{5}{4} \qquad \Rightarrow \qquad \tan^3\theta = 1.25 \end{aligned}$$

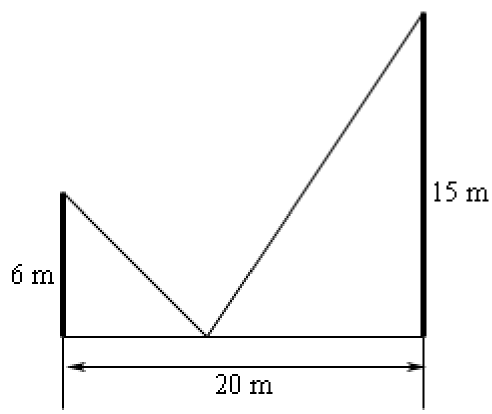
Solving for θ gives,

$$\tan\theta = \sqrt[3]{1.25} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\sqrt[3]{1.25}\right) = 0.8226$$

So, if $\theta = 0.8226$ radians then the pipe will have a minimum length and will just fit around the turn. Anything larger will not fit around the turn and so the largest pipe that can be carried around the turn is,

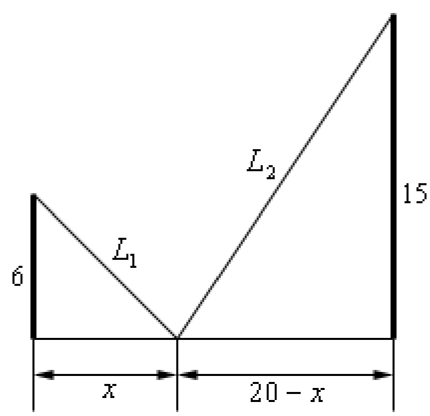
$$L = 8\sec\left(0.8226\right) + 10\csc\left(0.8226\right) = 25.4033 \text{ feet}$$

Example 6 Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?



Solution

As always let's start off with a sketch of this situation with some more information added.



The total length of the wire is $L = L_1 + L_2$ and we need to determine the value of *x* that will minimize this. The constraint in this problem is that the poles must be 20 meters apart and that *x* must be in the range $0 \le x \le 20$. The first thing that we'll need to do here is to get the length of wire in terms of *x*, which is fairly simple to do using the Pythagorean Theorem.

$$L_1 = \sqrt{36 + x^2} \qquad L_2 = \sqrt{225 + \left(20 - x\right)^2} \qquad L = \sqrt{36 + x^2} + \sqrt{625 - 40x + x^2}$$

Not the nicest function we've had to work with but there it is. Note however, that it is a continuous function and we've got an interval with finite endpoints and so finding the absolute minimum won't require much more work than just getting the critical points of this function. So, let's do that. Here's the derivative.

$$L' = \frac{x}{\sqrt{36+x^2}} + \frac{x-20}{\sqrt{625-40x+x^2}}$$

Setting this equal to zero gives,

$$\begin{aligned} \frac{x}{\sqrt{36+x^2}} + \frac{x-20}{\sqrt{625-40x+x^2}} &= 0 \\ x\sqrt{625-40x+x^2} &= -(x-20)\sqrt{36+x^2} \end{aligned}$$

It's probably been quite a while since you've been asked to solve something like this. To solve this, we'll need to square both sides to get rid of the roots, but this will cause problems as well soon see. Let's first just square both sides and solve that equation.

$$\begin{aligned} x^2\left(625-40x+x^2\right) &= (x-20)^2\left(36+x^2\right) \\ 625x^2-40x^3+x^4 &= 14400-1440x+436x^2-40x^3+x^4 \\ 189x^2+1440x-14400 &= 0 \\ 9(3x+40)(7x-40) &= 0 \qquad \Rightarrow \qquad x = -\tfrac{40}{3}, \quad x = \tfrac{40}{7} \end{aligned}$$

Note that if you can't do that factoring don't worry, you can always just use the quadratic formula and you'll get the same answers.

Okay two issues that we need to discuss briefly here. The first solution above (note that we didn't call it a critical point…) doesn't make any sense because it is negative and outside of the range of possible solutions and so we can ignore it.

Secondly, and maybe more importantly, if you were to plug $x = -\frac{40}{3}$ into the derivative you would not get zero and so is not even a critical point. How is this possible? It is a solution after all. We'll recall that we squared both sides of the equation above and it was mentioned at the time that this would cause problems. We'll we've hit those problems. In squaring both sides we've inadvertently introduced a new solution to the equation. When you do something like this you should ALWAYS go back and verify that the solutions that you get are in fact solutions to the original equation. In this case we were lucky and the "bad" solution also happened to be outside the interval of solutions we were interested in but that won't always be the case.

So, if we go back and do a quick verification we can in fact see that the only critical point is $x = \frac{40}{7} = 5.7143$ and this is nicely in our range of acceptable solutions, $0 \le x \le 20$.

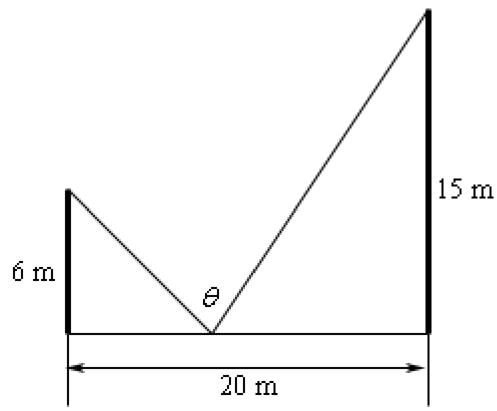
Now all that we need to do is plug this critical point and the endpoints of the wire into the length formula and identify the one that gives the minimum value.

$$L(0) = 31 \qquad L\left(\tfrac{40}{7}\right) = 29 \qquad L(20) = 35.8806$$

So, we will get the minimum length of wire if we stake it to the ground $\frac{40}{7}$ feet from the smaller pole.

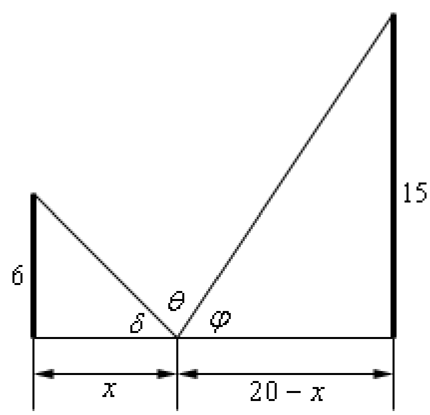
Let's do a modification of the above problem that asks a completely different question.

Example 7 Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the angle formed by the two pieces of wire at the stake is a maximum?



Solution

Here's a sketch for this example with some more information added.



The equation that we're going to need to work with here is not obvious. We can see from the sketch above that,

$$\delta + \theta + \varphi = 180 = \pi$$

Note that we need to make sure that the equation is equal to π because of how we're going to work this problem. Now, basic right triangle trig tells us the following,

$$\tan\delta = \frac{6}{x} \qquad \Rightarrow \qquad \delta = \tan^{-1}\left(\frac{6}{x}\right)$$

$$\tan\varphi = \frac{15}{20-x} \qquad \Rightarrow \qquad \varphi = \tan^{-1}\left(\frac{15}{20-x}\right)$$

Plugging these into the equation above and solving for θ gives,

$$\theta = \pi - \tan^{-1}\left(\frac{6}{x}\right) - \tan^{-1}\left(\frac{15}{20-x}\right)$$

Note that this is the reason for the π in our equation. The inverse tangents give angles that are in radians and so can't use the 180 that we're used to in this kind of equation.

Next, we'll need the derivative so hopefully you'll recall how to **differentiate inverse tangents**.

$$\begin{aligned}\theta' &= -\frac{1}{1+\left(\frac{6}{x}\right)^2}\left(-\frac{6}{x^2}\right) - \frac{1}{1+\left(\frac{15}{20-x}\right)^2}\left(\frac{15}{\left(20-x\right)^2}\right)\\ &= \frac{6}{x^2+36} - \frac{15}{\left(20-x\right)^2+225}\\ &= \frac{6}{x^2+36} - \frac{15}{x^2-40x+625} = \frac{-3\left(3x^2+80x-1070\right)}{\left(x^2+36\right)\left(x^2-40x+625\right)}\end{aligned}$$

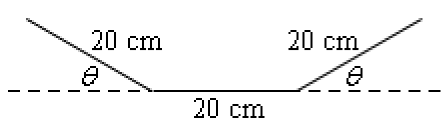
Setting this equal to zero and solving give the following two critical points.

$$x = \tfrac{-40\pm\sqrt{4810}}{3} = -36.4514, \quad 9.7847$$

The first critical point is not in the interval of possible solutions and so we can exclude it.

Clearly x must be in the interval $\left[0,20\right]$ and so using test points it's not difficult to show that if $0 \le x < 9.7847$ we have $\theta' > 0$ (and so θ is increasing) and if $9.7847 < x \le 20$ that $\theta' < 0$ (and so θ is decreasing). So by the first derivative test when $x = 9.7847$ we will get the maximum value of θ.

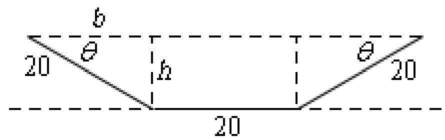
Example 8 A trough for holding water is be formed by taking a piece of sheet metal 60 cm wide and folding the 20 cm on either end up as shown below. Determine the angle θ that will maximize the amount of water that the trough can hold.



Solution

Now, in this case we are being asked to maximize the volume that a trough can hold, but if you think about it the volume of a trough in this shape is nothing more than the cross-sectional area times the length of the trough. So, for a given length in order to maximize the volume all you really need to do is maximize the cross-sectional area.

To get a formula for the cross-sectional area let's redo the sketch above a little.



We can think of the cross-sectional area as a rectangle in the middle with width 20 and height *h* and two identical triangles on either end with height *h*, base *b* and hypotenuse 20. Also note that basic geometry tells us that the angle between the hypotenuse and the base must also be the same angle θ that we had in our original sketch.

Also, basic right triangle trig tells us that the base and height can be written as,

$$b = 20\cos\theta \qquad\qquad h = 20\sin\theta$$

The cross-sectional area for the whole trough, in terms of θ, is then,

$$A = 20h + 2\left(\tfrac{1}{2}bh\right) = 400\sin\theta + \left(20\cos\theta\right)\left(20\sin\theta\right) = 400\left(\sin\theta + \sin\theta\cos\theta\right)$$

The derivative of the area is,

$$\begin{aligned} A'\left(\theta\right) &= 400\left(\cos\theta + \cos^2\theta - \sin^2\theta\right) \\ &= 400\left(\cos\theta + \cos^2\theta - \left(1 - \cos^2\theta\right)\right) \\ &= 400\left(2\cos^2\theta + \cos\theta - 1\right) \\ &= 400\left(2\cos\theta - 1\right)\left(\cos\theta + 1\right) \end{aligned}$$

So, we have either,

$$\begin{array}{ccccc} 2\cos\theta - 1 = 0 & \Rightarrow & \cos\theta = \frac{1}{2} & \Rightarrow & \theta = \frac{\pi}{3} \\ \cos\theta + 1 = 0 & \Rightarrow & \cos\theta = -1 & \Rightarrow & \theta = \pi \end{array}$$

However, we can see that θ must be in the interval $0 \le \theta \le \frac{\pi}{2}$ or we won't get a trough in the proper shape. Therefore, the second critical point makes no sense and also note that we don't need to add on the standard "$+2\pi n$" for the same reason.

Finally, since the equation for the area is continuous all we need to do is plug in the critical point and the end points to find the one that gives the maximum area.

$$A\left(0\right) = 0 \qquad\qquad A\left(\tfrac{\pi}{3}\right) = 519.6152 \qquad\qquad A\left(\tfrac{\pi}{2}\right) = 400$$

So, we will get a maximum cross-sectional area, and hence a maximum volume, when $\theta = \frac{\pi}{3}$.

Section 4-10 : L'Hospital`s Rule and Indeterminate Forms

Back in the chapter on Limits we saw methods for dealing with the following limits.

$$\lim_{x\to 4}\frac{x^2-16}{x-4} \qquad\qquad \lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$$

In the first limit if we plugged in $x=4$ we would get 0/0 and in the second limit if we "plugged" in infinity we would get $\infty/-\infty$ (**recall** that as *x* goes to infinity a polynomial will behave in the same fashion that its largest power behaves). Both of these are called **indeterminate forms**. In both of these cases there are competing interests or rules and it's not clear which will win out.

In the case of 0/0 we typically think of a fraction that has a numerator of zero as being zero. However, we also tend to think of fractions in which the denominator is going to zero, in the limit, as infinity or might not exist at all. Likewise, we tend to think of a fraction in which the numerator and denominator are the same as one. So, which will win out? Or will neither win out and they all "cancel out" and the limit will reach some other value?

In the case of $\infty/-\infty$ we have a similar set of problems. If the numerator of a fraction is going to infinity we tend to think of the whole fraction going to infinity. Also, if the denominator is going to infinity, in the limit, we tend to think of the fraction as going to zero. We also have the case of a fraction in which the numerator and denominator are the same (ignoring the minus sign) and so we might get -1. Again, it's not clear which of these will win out, if any of them will win out.

With the second limit there is the further problem that infinity isn't really a number and so we really shouldn't even treat it like a number. Much of the time it simply won't behave as we would expect it to if it was a number. To look a little more into this, check out the **Types of Infinity** section in the Extras chapter at the end of this document.

This is the problem with indeterminate forms. It's just not clear what is happening in the limit. There are other types of indeterminate forms as well. Some other types are,

$$(0)(\pm\infty) \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad \infty-\infty$$

These all have competing interests or rules that tell us what should happen and it's just not clear which, if any, of the interests or rules will win out. The topic of this section is how to deal with these kinds of limits.

As already pointed out we do know how to deal with some kinds of indeterminate forms already. For the two limits above we work them as follows.

$$\lim_{x\to 4}\frac{x^2-16}{x-4}=\lim_{x\to 4}(x+4)=8$$

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$$\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}=\lim_{x\to\infty}\frac{4-\frac{5}{x}}{\frac{1}{x^2}-3}=-\frac{4}{3}$$

In the first case we simply factored, canceled and took the limit and in the second case we factored out an x^2 from both the numerator and the denominator and took the limit. Notice as well that none of the competing interests or rules in these cases won out! That is often the case.

So, we can deal with some of these. However, what about the following two limits.

$$\lim_{x\to 0}\frac{\sin x}{x} \qquad\qquad \lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$$

This first is a 0/0 indeterminate form, but we can't factor this one. The second is an ∞/∞ indeterminate form, but we can't just factor an x^2 out of the numerator. So, nothing that we've got in our bag of tricks will work with these two limits.

This is where the subject of this section comes into play.

L'Hospital's Rule

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \qquad \text{OR} \qquad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where *a* can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

So, L'Hospital's Rule tells us that if we have an indeterminate form 0/0 or ∞/∞ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Before proceeding with examples let me address the spelling of "L'Hospital". The more modern spelling is "L'Hôpital". However, when I first learned Calculus my teacher used the spelling that I use in these notes and the first text book that I taught Calculus out of also used the spelling that I use here.

Also, as **noted** on the Wikipedia page for **L'Hospital's Rule**,

> "In the 17th and 18th centuries, the name was commonly spelled "l'Hospital", and he himself spelled his name that way. However, French spellings have **been altered**: the silent 's' has been removed and replaced with the **circumflex** over the preceding vowel. The former spelling is still used in English where there is no circumflex."

So, the spelling that I've used here is an acceptable spelling of his name, albeit not the modern spelling, and because I'm used to spelling it as "L'Hospital" that is the spelling that I'm going to use in these notes.

Let's work some examples.

Example 1 Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

(b) $\displaystyle\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}$

(c) $\displaystyle\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$

Solution

(a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

So, we have already established that this is a 0/0 indeterminate form so let's just apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1$$

(b) $\displaystyle\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}$

In this case we also have a 0/0 indeterminate form and if we were really good at factoring we could factor the numerator and denominator, simplify and take the limit. However, that's going to be more work than just using L'Hospital's Rule.

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7}$$

(c) $\displaystyle\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$

This was the other limit that we started off looking at and we know that it's the indeterminate form ∞/∞ so let's apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2x}$$

Now we have a small problem. This new limit is also a ∞/∞ indeterminate form. However, it's not really a problem. We know how to deal with these kinds of limits. Just apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2x}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2}=\infty$$

Sometimes we will need to apply L'Hospital's Rule more than once.

L'Hospital's Rule works great on the two indeterminate forms 0/0 and $\pm\infty/\pm\infty$. However, there are many more indeterminate forms out there as we saw earlier. Let's take a look at some of those and see how we deal with those kinds of indeterminate forms.

We'll start with the indeterminate form $(0)(\pm\infty)$.

Example 2 Evaluate the following limit.

$$\lim_{x\to 0^+} x\ln x$$

Solution

Note that we really do need to do the right-hand limit here. We know that the natural logarithm is only defined for positive *x* and so this is the only limit that makes any sense.

Now, in the limit, we get the indeterminate form $(0)(-\infty)$. L'Hospital's Rule won't work on products, it only works on quotients. However, we can turn this into a fraction if we rewrite things a little.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

The function is the same, just rewritten, and the limit is now in the form $-\infty/\infty$ and we can now use L'Hospital's Rule.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

Now, this is a mess, but it cleans up nicely.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

In the previous example we used the fact that we can always write a product of functions as a quotient by doing one of the following.

$$f(x)g(x) = \frac{g(x)}{1/f(x)} \qquad \text{OR} \qquad f(x)g(x) = \frac{f(x)}{1/g(x)}$$

Using these two facts will allow us to turn any limit in the form $(0)(\pm\infty)$ into a limit in the form 0/0 or $\pm\infty/\pm\infty$. Which one of these two we get after doing the rewrite will depend upon which fact we used to do the rewrite. One of the rewrites will give 0/0 and the other will give $\pm\infty/\pm\infty$. It all depends on which function stays in the numerator and which gets moved down to the denominator.

Let's take a look at another example.

Example 3 Evaluate the following limit.

$$\lim_{x\to-\infty} x\mathbf{e}^x$$

Solution

So, it's in the form $(\infty)(0)$. This means that we'll need to write it as a quotient. Moving the *x* to the denominator worked in the previous example so let's try that with this problem as well.

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{1/x}$$

Writing the product in this way gives us a product that has the form 0/0 in the limit. So, let's use L'Hospital's Rule on the quotient.

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{1/x} = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{-1/x^2} = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{2/x^3} = \lim_{x\to-\infty} \frac{\mathbf{e}^x}{-6/x^4} = \cdots$$

Hummmm…. This doesn't seem to be getting us anywhere. With each application of L'Hospital's Rule we just end up with another 0/0 indeterminate form and in fact the derivatives seem to be getting worse and worse. Also note that if we simplified the quotient back into a product we would just end up with either $(\infty)(0)$ or $(-\infty)(0)$ and so that won't do us any good.

This does not mean however that the limit can't be done. It just means that we moved the wrong function to the denominator. Let's move the exponential function instead.

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{x}{1/\mathbf{e}^x} = \lim_{x\to-\infty} \frac{x}{\mathbf{e}^{-x}}$$

Note that we used the fact that,

$$\frac{1}{\mathbf{e}^x} = \mathbf{e}^{-x}$$

to simplify the quotient up a little. This will help us when it comes time to take some derivatives. The quotient is now an indeterminate form of $-\infty/\infty$ and using L'Hospital's Rule gives,

$$\lim_{x\to-\infty} x\mathbf{e}^x = \lim_{x\to-\infty} \frac{x}{\mathbf{e}^{-x}} = \lim_{x\to-\infty} \frac{1}{-\mathbf{e}^{-x}} = 0$$

So, when faced with a product $(0)(\pm\infty)$ we can turn it into a quotient that will allow us to use L'Hospital's Rule. However, as we saw in the last example we need to be careful with how we do that on occasion. Sometimes we can use either quotient and in other cases only one will work.

Let's now take a look at the indeterminate forms,

$$1^\infty \qquad 0^0 \qquad \infty^0$$

These can all be dealt with in the following way so we'll just work one example.

Example 4 Evaluate the following limit.

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

Solution

In the limit this is the indeterminate form ∞^0. We're actually going to spend most of this problem on a different limit. Let's first define the following.

$$y = x^{\frac{1}{x}}$$

Now, if we take the natural log of both sides we get,

$$\ln(y) = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

Let's now take a look at the following limit.

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

This limit was just a L'Hospital's Rule problem and we know how to do those. So, what did this have to do with our limit? Well first notice that,

$$\mathbf{e}^{\ln(y)} = y$$

and so our limit could be written as,

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty}\mathbf{e}^{\ln(y)}$$

We can now use the limit above to finish this problem.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = \lim_{x\to\infty}\mathbf{e}^{\ln(y)} = \mathbf{e}^{\lim\limits_{x\to\infty}\ln(y)} = \mathbf{e}^0 = 1$$

With L'Hospital's Rule we are now able to take the limit of a wide variety of indeterminate forms that we were unable to deal with prior to this section.

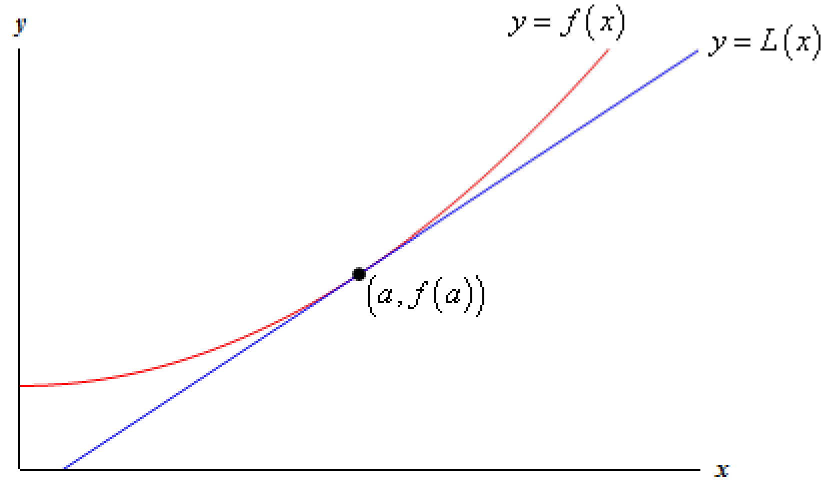
Section 4-11 : Linear Approximations

In this section we're going to take a look at an application not of derivatives but of the tangent line to a function. Of course, to get the tangent line we do need to take derivatives, so in some way this is an application of derivatives as well.

Given a function, $f(x)$, we can find its tangent at $x = a$. The equation of the tangent line, which we'll call $L(x)$ for this discussion, is,

$$L(x) = f(a) + f'(a)(x - a)$$

Take a look at the following graph of a function and its tangent line.



From this graph we can see that near $x = a$ the tangent line and the function have nearly the same graph. On occasion we will use the tangent line, $L(x)$, as an approximation to the function, $f(x)$, near $x = a$. In these cases we call the tangent line the **linear approximation** to the function at $x = a$.

So, why would we do this? Let's take a look at an example.

Example 1 Determine the linear approximation for $f(x) = \sqrt[3]{x}$ at $x = 8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

Solution

Since this is just the tangent line there really isn't a whole lot to finding the linear approximation.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \qquad f(8) = 2 \qquad f'(8) = \frac{1}{12}$$

The linear approximation is then,

$$L(x) = 2 + \frac{1}{12}(x - 8) = \frac{1}{12}x + \frac{4}{3}$$

Now, the approximations are nothing more than plugging the given values of *x* into the linear approximation. For comparison purposes we'll also compute the exact values.

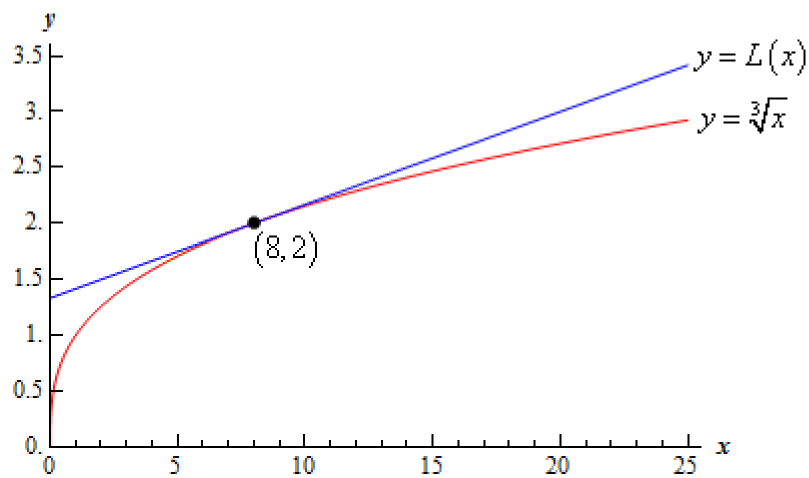
$$L(8.05) = 2.00416667 \qquad \sqrt[3]{8.05} = 2.00415802$$

$$L(25) = 3.41666667 \qquad \sqrt[3]{25} = 2.92401774$$

So, at $x = 8.05$ this linear approximation does a very good job of approximating the actual value. However, at $x = 25$ it doesn't do such a good job.

This shouldn't be too surprising if you think about it. Near $x = 8$ both the function and the linear approximation have nearly the same slope and since they both pass through the point $(8,2)$ they should have nearly the same value as long as we stay close to $x = 8$. However, as we move away from $x = 8$ the linear approximation is a line and so will always have the same slope while the function's slope will change as *x* changes and so the function will, in all likelihood, move away from the linear approximation.

Here's a quick sketch of the function and its linear approximation at $x = 8$.



As noted above, the farther from $x = 8$ we get the more distance separates the function itself and its linear approximation.

Linear approximations do a very good job of approximating values of $f(x)$ as long as we stay "near" $x = a$. However, the farther away from $x = a$ we get the worse the approximation is liable to be. The main problem here is that how near we need to stay to $x = a$ in order to get a good approximation will depend upon both the function we're using and the value of $x = a$ that we're using. Also, there will often be no easy way of predicting how far away from $x = a$ we can get and still have a "good" approximation.

Let's take a look at another example that is actually used fairly heavily in some places.

Example 2 Determine the linear approximation for $\sin\theta$ at $\theta = 0$.

Solution

Again, there really isn't a whole lot to this example. All that we need to do is compute the tangent line to $\sin\theta$ at $\theta = 0$.

$$f(\theta) = \sin\theta \qquad\qquad f'(\theta) = \cos\theta$$
$$f(0) = 0 \qquad\qquad f'(0) = 1$$

The linear approximation is,

$$\begin{aligned} L(\theta) &= f(0) + f'(0)(\theta - a) \\ &= 0 + (1)(\theta - 0) \\ &= \theta \end{aligned}$$

So, as long as θ stays small we can say that $\sin\theta \approx \theta$.

This is actually a somewhat important linear approximation. In optics this linear approximation is often used to simplify formulas. This linear approximation is also used to help describe the motion of a pendulum and vibrations in a string.

Section 4-12 : Differentials

In this section we're going to introduce a notation that we'll be seeing quite a bit in the next chapter. We will also look at an application of this new notation.

Given a function $y = f(x)$ we call dy and dx differentials and the relationship between them is given by,

$$dy = f'(x)\,dx$$

Note that if we are just given $f(x)$ then the differentials are df and dx and we compute them in the same manner.

$$df = f'(x)\,dx$$

Let's compute a couple of differentials.

Example 1 Compute the differential for each of the following.

(a) $y = t^3 - 4t^2 + 7t$

(b) $w = x^2 \sin(2x)$

(c) $f(z) = \mathbf{e}^{3-z^4}$

Solution

Before working any of these we should first discuss just what we're being asked to find here. We defined two differentials earlier and here we're being asked to compute a differential.

So, which differential are we being asked to compute? In this kind of problem we're being asked to compute the differential of the function. In other words, dy for the first problem, dw for the second problem and df for the third problem.

Here are the solutions. Not much to do here other than take a derivative and don't forget to add on the second differential to the derivative.

(a) $dy = \left(3t^2 - 8t + 7\right)dt$

(b) $dw = \left(2x\sin(2x) + 2x^2\cos(2x)\right)dx$

(c) $df = -4z^3\mathbf{e}^{3-z^4}\,dz$

There is a nice application to differentials. If we think of Δx as the change in *x* then $\Delta y = f\left(x+\Delta x\right) - f\left(x\right)$ is the change in *y* corresponding to the change in *x*. Now, if Δx is small we can assume that $\Delta y \approx dy$. Let's see an illustration of this idea.

Example 2 Compute dy and Δy if $y = \cos\left(x^2+1\right) - x$ as *x* changes from $x = 2$ to $x = 2.03$.

Solution
First let's compute actual the change in *y*, Δy .

$$\Delta y = \cos\left(\left(2.03\right)^2+1\right) - 2.03 - \left(\cos\left(2^2+1\right) - 2\right) = 0.083581127$$

Now let's get the formula for *dy*.

$$dy = \left(-2x\sin\left(x^2+1\right) - 1\right)dx$$

Next, the change in *x* from $x = 2$ to $x = 2.03$ is $\Delta x = 0.03$ and so we then assume that $dx \approx \Delta x = 0.03$. This gives an approximate change in *y* of,

$$dy = \left(-2\left(2\right)\sin\left(2^2+1\right) - 1\right)\left(0.03\right) = 0.085070913$$

We can see that in fact we do have that $\Delta y \approx dy$ provided we keep Δx small.

We can use the fact that $\Delta y \approx dy$ in the following way.

Example 3 A sphere was measured and its radius was found to be 45 inches with a possible error of no more that 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

Solution
First, recall the equation for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$

Now, if we start with $r = 45$ and use $dr \approx \Delta r = 0.01$ then $\Delta V \approx dV$ should give us maximum error.

So, first get the formula for the differential.

$$dV = 4\pi r^2 dr$$

Now compute dV .

$$\Delta V \approx dV = 4\pi\left(45\right)^2\left(0.01\right) = 254.47 \text{ in}^3$$

The maximum error in the volume is then approximately 254.47 in^3.

Be careful to not assume this is a large error. On the surface it looks large, however if we compute the actual volume for $r = 45$ we get $V = 381,703.51\text{ in}^3$. So, in comparison the error in the volume is,

$$\frac{254.47}{381703.51} \times 100 = 0.067\%$$

That's not much possible error at all!

Section 4-13 : Newton's Method

The next application that we'll take a look at in this chapter is an important application that is used in many areas. If you've been following along in the chapter to this point it's quite possible that you've gotten the impression that many of the applications that we've looked at are just made up by us to make you work. This is unfortunate because all of the applications that we've looked at to this point are real applications that really are used in real situations. The problem is often that in order to work more meaningful examples of the applications we would need more knowledge than we generally have about the science and/or physics behind the problem. Without that knowledge we're stuck doing some fairly simplistic examples that often don't seem very realistic at all and that makes it hard to understand that the application we're looking at is a real application.

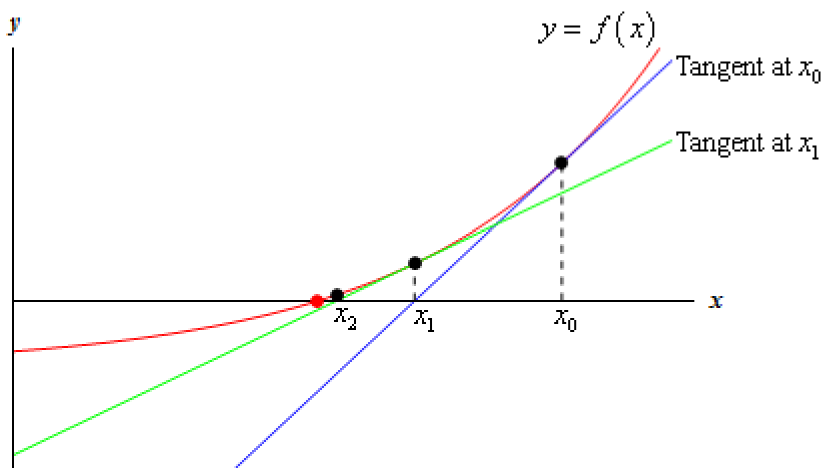
That is going to change in this section. This is an application that we can all understand and we can all understand needs to be done on occasion even if we don't understand the physics/science behind an actual application.

In this section we are going to look at a method for approximating solutions to equations. We all know that equations need to be solved on occasion and in fact we've solved quite a few equations ourselves to this point. In all the examples we've looked at to this point we were able to actually find the solutions, but it's not always possible to do that exactly and/or do the work by hand. That is where this application comes into play. So, let's see what this application is all about.

Let's suppose that we want to approximate the solution to $f\left(x\right)=0$ and let's also suppose that we have somehow found an initial approximation to this solution say, x_0. This initial approximation is probably not all that good, in fact it may be nothing more than a quick guess we made, and so we'd like to find a better approximation. This is easy enough to do. First, we will get the tangent line to $f\left(x\right)$ at x_0.

$$y=f\left(x_0\right)+f'\left(x_0\right)\left(x-x_0\right)$$

Now, take a look at the graph below.



The blue line (if you're reading this in color anyway…) is the tangent line at x_0. We can see that this line will cross the *x*-axis much closer to the actual solution to the equation than x_0 does. Let's call this point where the tangent at x_0 crosses the *x*-axis x_1 and we'll use this point as our new approximation to the solution.

So, how do we find this point? Well we know it's coordinates, $\left(x_1, 0\right)$, and we know that it's on the tangent line so plug this point into the tangent line and solve for x_1 as follows,

$$\begin{aligned} 0 &= f\left(x_0\right) + f'\left(x_0\right)\left(x_1 - x_0\right) \\ x_1 - x_0 &= -\frac{f\left(x_0\right)}{f'\left(x_0\right)} \\ x_1 &= x_0 - \frac{f\left(x_0\right)}{f'\left(x_0\right)} \end{aligned}$$

So, we can find the new approximation provided the derivative isn't zero at the original approximation.

Now we repeat the whole process to find an even better approximation. We form up the tangent line to $f\left(x\right)$ at x_1 and use its root, which we'll call x_2, as a new approximation to the actual solution. If we do this we will arrive at the following formula.

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

This point is also shown on the graph above and we can see from this graph that if we continue following this process will get a sequence of numbers that are getting very close the actual solution. This process is called Newton's Method.

Here is the general Newton's Method

Newton's Method

If x_n is an approximation a solution of $f\left(x\right) = 0$ and if $f'\left(x_n\right) \neq 0$ the next approximation is given by,

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

This should lead to the question of when do we stop? How many times do we go through this process? One of the more common stopping points in the process is to continue until two successive approximations agree to a given number of decimal places.

Before working any examples we should address two issues. First, we really do need to be solving $f(x)=0$ in order for Newton's Method to be applied. This isn't really all that much of an issue but we do need to make sure that the equation is in this form prior to using the method.

Secondly, we do need to somehow get our hands on an initial approximation to the solution (*i.e.* we need x_0 somehow). One of the more common ways of getting our hands on x_0 is to sketch the graph of the function and use that to get an estimate of the solution which we then use as x_0. Another common method is if we know that there is a solution to a function in an interval then we can use the midpoint of the interval as x_0.

Let's work an example of Newton's Method.

Example 1 Use Newton's Method to determine an approximation to the solution to $\cos x = x$ that lies in the interval $[0,2]$. Find the approximation to six decimal places.

Solution

First note that we weren't given an initial guess. We were however, given an interval in which to look. We will use this to get our initial guess. As noted above the general rule of thumb in these cases is to take the initial approximation to be the midpoint of the interval. So, we'll use $x_0 = 1$ as our initial guess.

Next, recall that we **must** have the function in the form $f(x)=0$. Therefore, we first rewrite the equation as,

$$\cos x - x = 0$$

We can now write down the general formula for Newton's Method. Doing this will often simplify up the work a little so it's generally not a bad idea to do this.

$$x_{n+1} = x_n - \frac{\cos x - x}{-\sin x - 1}$$

Let's now get the first approximation.

$$x_1 = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} = 0.7503638679$$

At this point we should point out that the phrase "six decimal places" does not mean just get x_1 to six decimal places and then stop. Instead it means that we continue until two successive approximations agree to six decimal places.

Given that stopping condition we clearly need to go at least one step farther.

$$x_2 = 0.7503638679 - \frac{\cos(0.7503638679) - 0.7503638679}{-\sin(0.7503638679) - 1} = 0.7391128909$$

Alright, we're making progress. We've got the approximation to 1 decimal place. Let's do another one, leaving the details of the computation to you.

$$x_3 = 0.7390851334$$

We've got it to three decimal places. We'll need another one.

$$x_4 = 0.7390851332$$

And now we've got two approximations that agree to 9 decimal places and so we can stop. We will assume that the solution is approximately $x_4 = 0.7390851332$.

In this last example we saw that we didn't have to do too many computations in order for Newton's Method to give us an approximation in the desired range of accuracy. This will not always be the case. Sometimes it will take many iterations through the process to get to the desired accuracy and on occasion it can fail completely.

The following example is a little silly but it makes the point about the method failing.

Example 2 Use $x_0 = 1$ to find the approximation to the solution to $\sqrt[3]{x} = 0$.

Solution

Yes, it's a silly example. Clearly the solution is $x = 0$, but it does make a very important point. Let's get the general formula for Newton's method.

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = x_n - 3x_n = -2x_n$$

In fact, we don't really need to do any computations here. These computations get farther and farther away from the solution, $x = 0$,with each iteration. Here are a couple of computations to make the point.

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 4 \\ x_3 &= -8 \\ x_4 &= 16 \\ &\textit{etc.} \end{aligned}$$

So, in this case the method fails and fails spectacularly.

So, we need to be a little careful with Newton's method. It will usually quickly find an approximation to an equation. However, there are times when it will take a lot of work or when it won't work at all.

Section 4-14 : Business Applications

In the final section of this chapter let's take a look at some applications of derivatives in the business world. For the most part these are really applications that we've already looked at, but they are now going to be approached with an eye towards the business world.

Let's start things out with a couple of optimization problems. We've already looked at more than a few of these in previous sections so there really isn't anything all that new here except for the fact that they are coming out of the business world.

Example 1 An apartment complex has 250 apartments to rent. If they rent *x* apartments then their monthly profit, in dollars, is given by,

$$P\left(x\right) = -8x^2 + 3200x - 80,000$$

How many apartments should they rent in order to maximize their profit?

Solution

All that we're really being asked to do here is to maximize the profit subject to the constraint that *x* must be in the range $0 \le x \le 250$.

First, we'll need the derivative and the critical point(s) that fall in the range $0 \le x \le 250$.

$$P'\left(x\right) = -16x + 3200 \qquad \Rightarrow \qquad 3200 - 16x = 0 \qquad \Rightarrow \qquad x = \tfrac{3200}{16} = 200$$

Since the profit function is continuous and we have an interval with finite bounds we can find the maximum value by simply plugging in the only critical point that we have (which nicely enough in the range of acceptable answers) and the end points of the range.

$$P\left(0\right) = -80,000 \qquad P\left(200\right) = 240,000 \qquad P\left(250\right) = 220,000$$

So, it looks like they will generate the most profit if they only rent out 200 of the apartments instead of all 250 of them.

Note that with these problems you shouldn't just assume that renting all the apartments will generate the most profit. Do not forget that there are all sorts of maintenance costs and that the more tenants renting apartments the more the maintenance costs will be. With this analysis we can see that, for this complex at least, something probably needs to be done to get the maximum profit more towards full capacity. This kind of analysis can help them determine just what they need to do to move towards that goal whether it be raising rent or finding a way to reduce maintenance costs.

Note as well that because most apartment complexes have at least a few units empty after a tenant moves out and the like that it's possible that they would actually like the maximum profit to fall slightly under full capacity to take this into account. Again, another reason to not just assume that maximum profit will always be at the upper limit of the range.

Let's take a quick look at another problem along these lines.

Example 2 A production facility is capable of producing 60,000 widgets in a day and the total daily cost of producing *x* widgets in a day is given by,

$$C\left(x\right) = 250,000 + 0.08x + \frac{200,000,000}{x}$$

How many widgets per day should they produce in order to minimize production costs?

Solution

Here we need to minimize the cost subject to the constraint that *x* must be in the range $0 \le x \le 60,000$. Note that in this case the cost function is not continuous at the left endpoint and so we won't be able to just plug critical points and endpoints into the cost function to find the minimum value.

Let's get the first couple of derivatives of the cost function.

$$C'\left(x\right) = 0.08 - \frac{200,000,000}{x^2} \qquad\qquad C''\left(x\right) = \frac{400,000,000}{x^3}$$

The critical points of the cost function are,

$$\begin{aligned} 0.08 - \frac{200,000,000}{x^2} &= 0 \\ 0.08x^2 &= 200,000,000 \\ x^2 &= 2,500,000,000 \qquad \Rightarrow \qquad x = \pm\sqrt{2,500,000,000} = \pm 50,000 \end{aligned}$$

Now, clearly the negative value doesn't make any sense in this setting and so we have a single critical point in the range of possible solutions : 50,000.

Now, as long as $x > 0$ the second derivative is positive and so, in the range of possible solutions the function is always concave up and so producing 50,000 widgets will yield the absolute minimum production cost.

Recall from the **Optimization** section we discussed how we can use the second derivative to identity the absolute extrema even though all we really get from it is relative extrema.

Now, we shouldn't walk out of the previous two examples with the idea that the only applications to business are just applications we've already looked at but with a business "twist" to them.

There are some very real applications to calculus that are in the business world and at some level that is the point of this section. Note that to really learn these applications and all of their intricacies you'll need to take a business course or two or three. In this section we're just going to scratch the surface and get a feel for some of the actual applications of calculus from the business world and some of the main "buzz" words in the applications.

Let's start off by looking at the following example.

Example 3 The production costs per week for producing *x* widgets is given by,

$$C\left(x\right) = 500 + 350x - 0.09x^2, \quad 0 \le x \le 1000$$

Answer each of the following questions.

(a) What is the cost to produce the 301st widget?
(b) What is the rate of change of the cost at $x = 300$?

Solution

(a) We can't just compute $C\left(301\right)$ as that is the cost of producing 301 widgets while we are looking for the actual cost of producing the 301st widget. In other words, what we're looking for here is,

$$C\left(301\right) - C\left(300\right) = 97,695.91 - 97,400.00 = 295.91$$

So, the cost of producing the 301st widget is $295.91.

(b) In this part all we need to do is get the derivative and then compute $C'\left(300\right)$.

$$C'\left(x\right) = 350 - 0.18x \qquad \Rightarrow \qquad C'\left(300\right) = 296.00$$

Okay, so just what did we learn in this example? The cost to produce an additional item is called the **marginal cost** and as we've seen in the above example the marginal cost is approximated by the rate of change of the **cost function**, $C\left(x\right)$. So, we define the **marginal cost function** to be the derivative of the cost function or, $C'\left(x\right)$. Let's work a quick example of this.

Example 4 The production costs per day for some widget is given by,

$$C\left(x\right) = 2500 - 10x - 0.01x^2 + 0.0002x^3$$

What is the marginal cost when $x = 200$, $x = 300$ and $x = 400$?

Solution

So, we need the derivative and then we'll need to compute some values of the derivative.

$$C'\left(x\right) = -10 - 0.02x + 0.0006x^2$$

$$C'\left(200\right) = 10 \qquad C'\left(300\right) = 38 \qquad C'\left(400\right) = 78$$

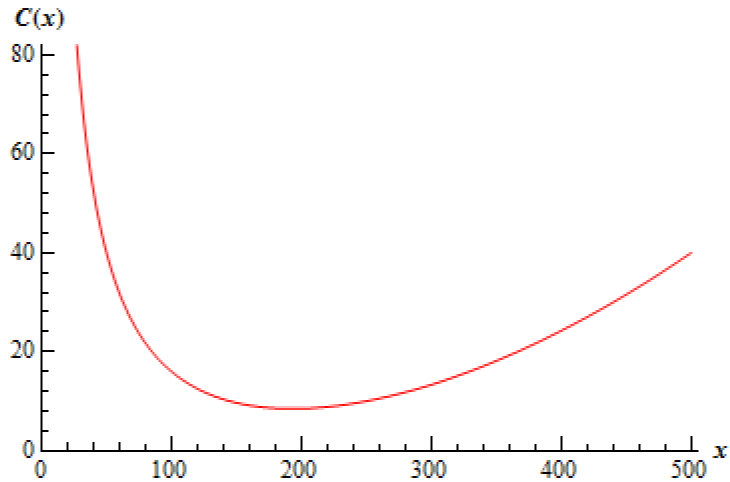
So, in order to produce the 201st widget it will cost approximately $10. To produce the 301st widget will cost around $38. Finally, to product the 401st widget it will cost approximately $78.

Note that it is important to note that $C'\left(n\right)$ is the approximate cost of producing the $\left(n+1\right)^{\text{st}}$ item and NOT the nth item as it may seem to imply!

Let's now turn our attention to the **average cost** function. If $C\left(x\right)$ is the cost function for some item then the average cost function is,

$$\bar{C}(x) = \frac{C(x)}{x}$$

Here is the sketch of the average cost function from Example 4 above.



We can see from this that the average cost function has an absolute minimum. We can also see that this absolute minimum will occur at a critical point when $\bar{C}'(x) = 0$ since it clearly will have a horizontal tangent there.

Now, we could get the average cost function, differentiate that and then find the critical point. However, this average cost function is fairly typical for average cost functions so let's instead differentiate the general formula above using the quotient rule and see what we have.

$$\bar{C}'(x) = \frac{x\,C'(x) - C(x)}{x^2}$$

Now, as we noted above the absolute minimum will occur when $\bar{C}'(x) = 0$ and this will in turn occur when,

$$x\,C'(x) - C(x) = 0 \qquad \Rightarrow \qquad C'(x) = \frac{C(x)}{x} = \bar{C}(x)$$

So, we can see that it looks like for a typical average cost function we will get the minimum average cost when the marginal cost is equal to the average cost.

We should note however that not all average cost functions will look like this and so you shouldn't assume that this will always be the case.

Let's now move onto the revenue and profit functions. First, let's suppose that the price that some item can be sold at if there is a demand for *x* units is given by $p(x)$. This function is typically called either the **demand function** or the **price function**.

The **revenue function** is then how much money is made by selling *x* items and is,

$$R(x) = x\,p(x)$$

The **profit function** is then,

$$P(x) = R(x) - C(x) = x\,p(x) - C(x)$$

Be careful to not confuse the demand function, $p(x)$ - lower case *p*, and the profit function, $P(x)$ - upper case *P*. Bad notation maybe, but there it is.

Finally, the **marginal revenue function** is $R'(x)$ and the **marginal profit function** is $P'(x)$ and these represent the revenue and profit respectively if one more unit is sold.

Let's take a quick look at an example of using these.

Example 5 The weekly cost to produce *x* widgets is given by

$$C(x) = 75{,}000 + 100x - 0.03x^2 + 0.000004x^3 \qquad 0 \le x \le 10000$$

and the demand function for the widgets is given by,

$$p(x) = 200 - 0.005x \qquad 0 \le x \le 10000$$

Determine the marginal cost, marginal revenue and marginal profit when 2500 widgets are sold and when 7500 widgets are sold. Assume that the company sells exactly what they produce.

Solution

Okay, the first thing we need to do is get all the various functions that we'll need. Here are the revenue and profit functions.

$$\begin{aligned} R(x) &= x(200 - 0.005x) = 200x - 0.005x^2 \\ P(x) &= 200x - 0.005x^2 - \left(75{,}000 + 100x - 0.03x^2 + 0.000004x^3\right) \\ &= -75{,}000 + 100x + 0.025x^2 - 0.000004x^3 \end{aligned}$$

Now, all the marginal functions are,

$$\begin{aligned} C'(x) &= 100 - 0.06x + 0.000012x^2 \\ R'(x) &= 200 - 0.01x \\ P'(x) &= 100 + 0.05x - 0.000012x^2 \end{aligned}$$

The marginal functions when 2500 widgets are sold are,

$$C'(2500) = 25 \qquad R'(2500) = 175 \qquad P'(2500) = 150$$

The marginal functions when 7500 are sold are,

$$C'(7500) = 325 \qquad R'(7500) = 125 \qquad P'(7500) = -200$$

So, upon producing and selling the 2501st widget it will cost the company approximately \$25 to produce the widget and they will see an added \$175 in revenue and \$150 in profit.

On the other hand, when they produce and sell the 7501st widget it will cost an additional \$325 and they will receive an extra \$125 in revenue, but lose \$200 in profit.

We'll close this section out with a brief discussion on maximizing the profit. If we assume that the maximum profit will occur at a critical point such that $P'(x) = 0$ we can then say the following,

$$P'(x) = R'(x) - C'(x) = 0 \qquad \Rightarrow \qquad R'(x) = C'(x)$$

We then will know that this will be a maximum we also were to know that the profit was always concave down or,

$$P''(x) = R''(x) - C''(x) < 0 \qquad \Rightarrow \qquad R''(x) < C''(x)$$

So, if we know that $R''(x) < C''(x)$ then we will maximize the profit if $R'(x) = C'(x)$ or if the marginal cost equals the marginal revenue.

In this section we took a brief look at some of the ideas in the business world that involve calculus. Again, it needs to be stressed however that there is a lot more going on here and to really see how these applications are done you should really take some business courses. The point of this section was to just give a few ideas on how calculus is used in a field other than the sciences.