A-Level Maths - Exam Style Questions

Section A:	Multiple Choice Questions	90 minutes in total
Section B:	Proof and Geometry	30 minutes per question
Section C:	Algebra 1	30 minutes per question
Section D:	Algebra 2	30 minutes per question
Section E:	Calculus	30 minutes per question
Section F:	Mechanics	30 minutes per question
Section G:	Statistics	30 minutes per question

A1. The correct factorisation of $3x^2 - 2xy - y^2$ is

① $(3x + y)^2$
② $(3x + y)(x - y)$
③ $(3x - y)(x + y)$
④ $(3x + y)^3 - (3x - y)^3$ [1 mark]

A2. A line L passes through the points (5, 1), $(k, 4)$ and (-1, 9). The value of k is

① 2
② 9 / 4
③ 11 / 4
④ 3 [1 mark]

A3. Which techniques, applied one after the other, would be most suitable for

evaluating $\int \frac{\cos x\,(\sin x - 1)}{\sin^2 x + 5\sin x + 6}\,dx$?

① integration by parts then substitution
② substitution then partial fractions
③ quotient rule then integration by parts
④ substitution then integration by parts [1 mark]

A4. Which set contains an element which is smaller than all other elements in that set?

($\mathbb{Q}^+$ represents the set of all positive elements in $\mathbb{Q}$.)

① $\{r \in \mathbb{Q}^+\}$
② $\{r \in \mathbb{Q}^+ \mid r^2 \geq 2\}$
③ $\{r \in \mathbb{Q}^+ \mid r^2 > 4\}$
④ none of these [1 mark]

A5. The nth term of a sequence is denoted a_n.

If $\sum_{k=1}^{n} a_k = \log_{10}\left(\frac{(n+1)(n+2)}{2}\right)$ for all $n \geq 1$ then the value of $\sum_{k=1}^{20} a_{2k}$ is

① $10\log_{10} 2$

② $\log_{10} 21$

③ $\log_{10} \sqrt{42}$

④ $\log_{10} 42$ [2 marks]

A6. The exact value of x such that ln x, ln $2x$ and ln $3x$ form three sides of a right-angled triangle with ln $3x$ as the hypotenuse is

① $x = \frac{3}{2} \exp \sqrt{ln\frac{3}{2}\ ln\ 9}$

② $x = \frac{1}{2} \exp \sqrt{ln\frac{5}{2}\ ln\ 3}$

③ $x = \frac{1}{2} \exp \sqrt{ln\frac{3}{2}\ ln\ 5}$

④ $x = \frac{5}{2} \exp \sqrt{ln\frac{1}{2}\ ln\ 9}$ [2 marks]

A7. $f(x)$ is a quadratic function such that $f(0) = 0$.

$$\int_0^2 |f(x)|\,\mathrm{d}x = -\int_0^2 f(x)\,\mathrm{d}x = 4 \quad \text{and} \quad \int_2^3 |f(x)|\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x.$$

Find the value of $f(5)$.

① 0
② 5
③ 15
④ 45 [4 marks]

***A8.** For any $0 < t < 41$, the curve $y = x^3 + 2x^2 - 15x + 5$ intersects the horizontal line $y = t$ three times. Of these three intersections, let the point with the largest x-coordinate be $(f(t),\ t)$ and the point with the smallest x-coordinate be $(g(t),\ t)$.

If $h(t) = t \times (f(t) - g(t))$, find the value of $h'(5)$.

① $\frac{79}{12}$

② $\frac{91}{12}$

③ $\frac{97}{12}$

④ $\frac{129}{16}$ [4 marks]

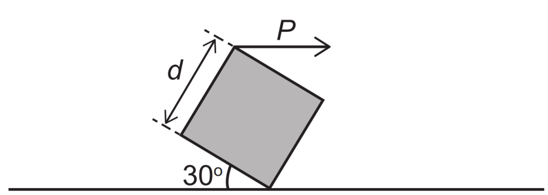
A9. A rower wants to cross a river to a point on the opposite river bank directly opposite from where she will start. The distance to cover is 100 m, and she wants to cross in 10 seconds. The river flows uniformly at 7.5 m/s.

How fast must she row her boat, and at what angle, relative to the velocity of the flowing water? (Round the angle to the nearest degree.)

① speed = 12.5 m/s, angle = 37°
② speed = 12.5 m/s, angle = 127°
③ speed = 6.6 m/s, angle = 37°
④ speed = 6.6 m/s, angle = 127° [1 mark]

A10. The diagram shows a uniform, solid, heavy cube with side d. The cube rests with one of its edges in contact with a table that is perfectly level.

A horizontal force P acts on another edge of the cube, and the cube is stationary.



Below are four statements about the forces on the cube.

A No frictional force acts between the cube and the table.

B A frictional force acts to the left between the cube and the table.

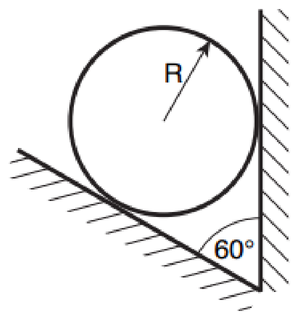
C A frictional force acts to the right between the cube and the table.

D Force P has a clockwise moment about the edge in contact with the table equal to $P \times d$

How many of the above statements must be true?

① 1
② 2
③ 3
④ 4 [1 mark]

A11. A stationary, smooth (i.e. frictionless) sphere of radius R and weight W is in contact with a smooth vertical plane and with a plane inclined at 60° to the vertical.



The magnitudes of the normal contact forces exerted by the inclined plane and the vertical plane are N_1 and N_2 respectively. These values are [1 mark]

	N_1	N_2
①	$\frac{\sqrt{5}}{2}W$	$\frac{1}{2}W$
②	$\frac{\sqrt{5}}{2}W$	W
③	$\frac{\sqrt{3}}{2}W$	$\frac{1}{2}W$
④	$\frac{\sqrt{3}}{2}W$	W

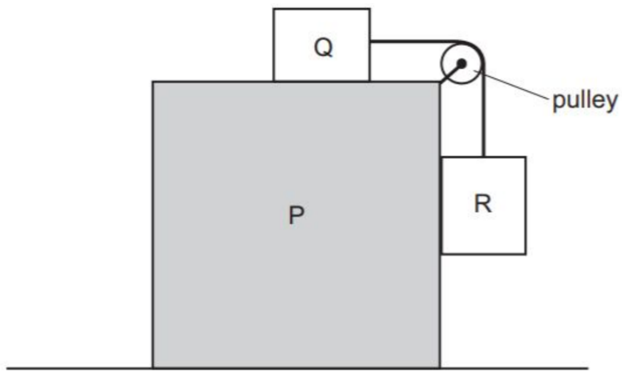
A12. A stone is thrown from the roof of a 10 metre tall building. The stone lands on the surrounding flat ground at a horizontal distance x.

At what angle to the horizontal should the stone be thrown at to maximise x?

(Neglect air resistance. Assume a uniform gravitational field.)

① less than 45°
② exactly 45°
③ more than 45°
④ it could be any of the above, depending on the initial speed [1 mark]

A13. A block P has a smaller block Q resting on its top surface. Q is connected to a hanging block, R, by a light, inextensible string. The string passes over a smooth pulley which is connected to block P, as shown in the diagram.



The masses of blocks P, Q and R are m_P, m_Q and m_R respectively. P is accelerated horizontally to the right by an external force. While this is happening, Q and R do not move relative to P.

What is the acceleration of P? (Gravitational field strength $= g$.)

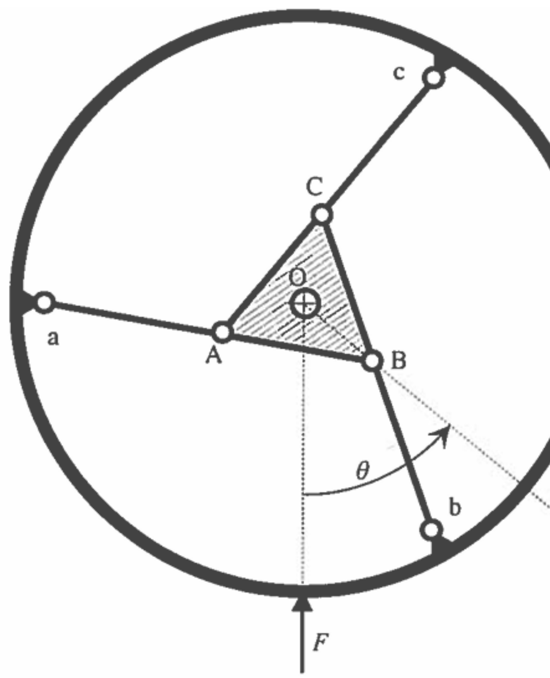
① $\dfrac{m_Q g}{m_R}$

② $\dfrac{m_R g}{m_Q}$

③ $\dfrac{m_R g}{m_R + m_Q}$

④ $\dfrac{(m_Q + m_R)g}{m_P + m_Q + m_R}$ [2 marks]

A14. A bicycle wheel consists of a circular rim with three inextensible spokes connected to the three vertices of a central equilateral triangular hub ABC such that aAB, bBC and cCA are all straight lines. The vertical reaction load F from the road is balanced by an equal and opposite force acting through the centre O.



The angular position of the wheel is defined by the angle θ as shown. The centre point O is located at the centre of mass of the triangle ABC, and is positioned such that $PO : OB = 1 : 2$, where P is the midpoint of AC.

Determine how the force R_C in the spoke cC varies as the wheel turns.

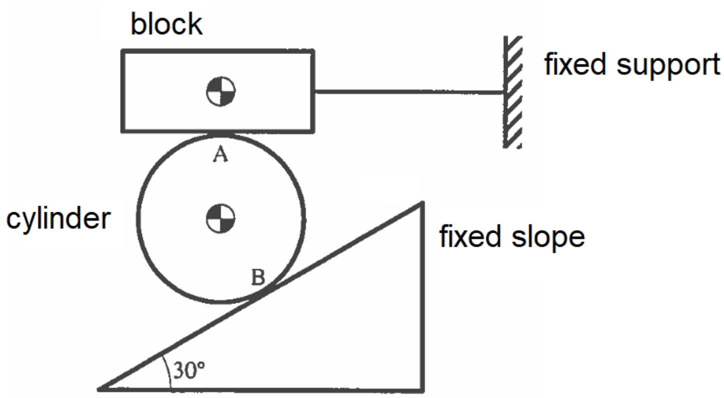
① $R_C = \dfrac{1}{3} F \sin\theta$

② $R_C = \dfrac{2}{3} F \sin\theta$

③ $R_C = \dfrac{1}{3} F \cos\theta$

④ $R_C = \dfrac{2}{3} F \cos\theta$ [2 marks]

***A15.** A cylinder of mass M is on a slope inclined at 30° to the horizontal. The cylinder supports a block, also of mass M. The block is tethered horizontally by a cable to a fixed support so that its centre of mass is directly above that of the cylinder.



The coefficients of friction μ at the contact points A and B are both between 0 and 1, with corresponding angles of friction ψ denoted ψ_A and ψ_B, where μ and ψ are related through $\mu = \tan \psi$.

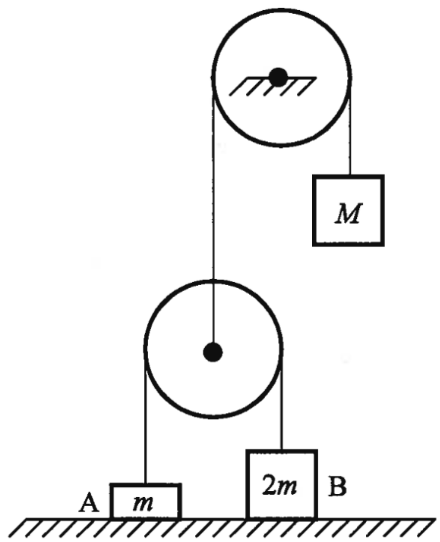
The table shows what happens to the cylinder when the coefficients of friction, and hence the values of ψ, vary.

	$0 < \psi_B < y$	$y < \psi_B < 45°$
$0° < \psi_A < x$	rolls about B	remains stationary
$x < \psi_A < 45°$	slides	rolls about A

What are the values of the limiting angles of friction, x and y? [4 marks]

	x	y
①	30°	36.3°
②	30°	15°
③	28.2°	36.3°
④	28.2°	15°

***A16.** A pulley is pinned to a fixed support and carries a string connecting a hanging mass M kg to a second hanging pulley, which in turn connects two blocks A and B with masses m kg and $2m$ kg respectively. The blocks A and B are resting on horizontal ground. The hanging block is released from rest with all strings taut.



Gravitational acceleration is g. Both strings are vertical and inextensible, and both pulleys are light and frictionless. Which statement is true?

① Both blocks A and B must lift off of the ground if $\frac{M}{m} > 4$.

② If $\frac{M}{m} = 4$ then the vertical acceleration of block A is $\frac{g}{2}$.

③ The system must remain in equilibrium if $\frac{M}{m} \leq 3$.

④ The tension in the lower string is always greater than in the upper string.

[4 marks]

A17. To understand the academic performance of 1,000 students on an exam, the systematic sampling method is adopted to choose 40 samples.

Which statement is most likely to be true?

① The sampling interval should be 50.
② The sample mean will likely be very different from the population mean.
③ Quota sampling at the same sample size would be more representative.
④ Each student in the population has an equal chance of being sampled.

[1 mark]

A18. The random variable Z is distributed with the standard normal distribution. What is the interquartile range of Z?

① 0.1974
② 0.6745
③ 0.6915
④ 1.3490

[1 mark]

A19. A coin of diameter 2 cm is thrown onto a table covered with a 2D grid of lines intersecting at right angles and each separated by 4 cm.

What is the probability that the coin lands in a square without touching any of the lines of the grid?

① 0.2500
② 0.3183
③ 0.3333
④ 0.5000

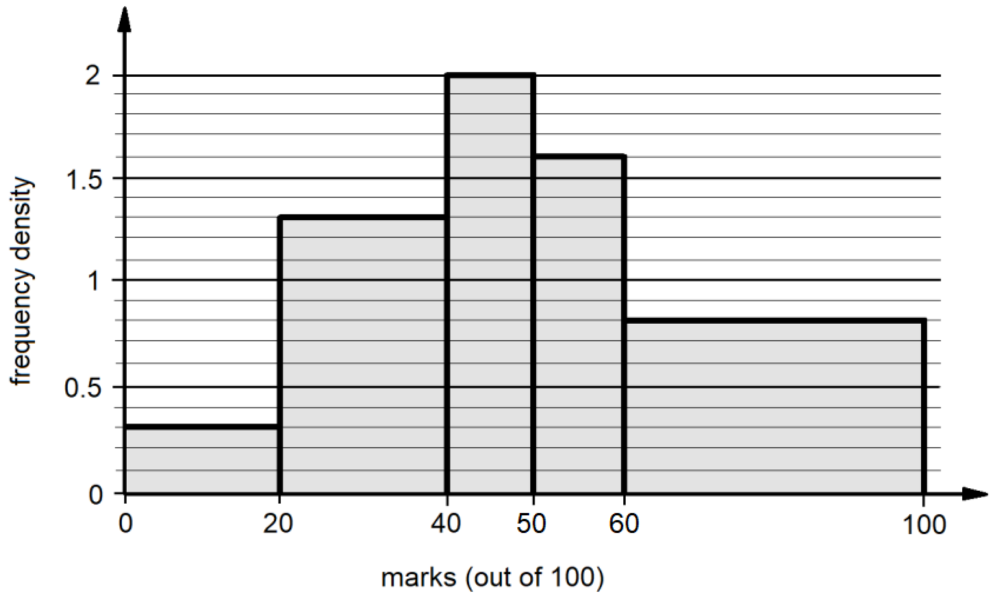
[1 mark]

A20. An appropriate method to test for linear correlation between "proportion of a country's population living in urban areas" and "GDP per capita of a country" is

① Spearman's rank correlation coefficient
② Pearson's product-moment correlation coefficient
③ Hypothesis test on a Normal distribution
④ All of the above would be appropriate

[1 mark]

A21. The histogram below shows the distribution of the marks of a cohort of students who recently sat the same exam.



Assuming the marks are uniformly distributed throughout each bin, which of the following statistics best describe the data shown?

① The students scoring 75 marks or more make up the cohort's top quartile.
② The mark distribution is negatively skewed.
③ The median mark rounds to 49.
④ The mark distribution obeys a binomial model $\mathrm{B}(100, 0.45)$. [2 marks]

A22. A faulty bit in a computer memory changes its state between "0" or "1" in any given clock cycle with probability p.

If P_N is the probability that it is in the same state N cycles later, an inductive sequence relating these probabilities is given by

① $P_{N+1} = 2p + (1 - p)\, P_N$
② $P_{N+1} = p + (1 - p)\, P_N$
③ $P_{N+1} = P_N + (2p - 1)$
④ $P_{N+1} = p + (1 - 2p)\, P_N$ [2 marks]

A23. A component costs £9 to make and can, if perfect, be sold for £10. The machine producing components is not reliable; 5% of components have to be scrapped and 5% can only be sold at half the price of a perfect one.

A replacement machine, that produces a negligible number of faulty components, is purchased for £10,000.

How many components need to be made on the new machine and then sold before there is an increase in profit over that which would have been obtained making the same number on the old machine?

① 6667
② 8001
③ 13334
④ 25001 [4 marks]

***A24.** In Camelot, it never rains on Friday, Saturday, Sunday or Monday. The probability that it rains on a given Tuesday is 1 / 5. For each of the remaining two days, Wednesday and Thursday, the conditional probability that it rains, given that it rained the previous day, is α, and the conditional probability that it rains, given that it did not rain the previous day, is β.

If X is the event that, in a randomly chosen week, it rains on Thursday, Y is the event that it rains on Tuesday, then the value of $P(X \mid Y) - P(X \mid Y')$ is

① $(\alpha - \beta) / 5$
② $(\alpha - \beta)^2$
③ $\alpha^2 - \beta^2$
④ $10\alpha\beta$ [4 marks]

B1.

a. i) Show that the square of any positive integer cannot end in the digit 3.

[3 marks]

ii) Prove algebraically that the product of four consecutive positive integers can never be equal to a perfect square.

[5 marks]

iii) The nth term of a positive integer sequence U is given by

$$U_n = 1! + 2! + 3! + ... + (n-1)! + n! \qquad \text{for } n \in \mathbb{N} : n \geq 1$$

Explain why U_n can never be a square number when $n \geq 4$.

You may use the result(s) shown above to help you. [3 marks]

b. A positive integer m is not divisible by any prime number larger than 20, and is not divisible by the square of any prime number.

i) By considering the prime factorisation of m, show that there are 256 possible values of m. [2 marks]

ii) Show that $0 \leq \log_{10} m < 7$. [1 mark]

c. It is given that x, a and b are positive real numbers, with $a > b$ and $x^2 > ab$.

Use proof by contradiction to show that

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} > 0.$$

Fully justify your answer. [6 marks]

B2.

a. i) Prove that $\sqrt[3]{2}$ is irrational. [6 marks]

ii) Prove that for all real x,

$$\sin x + \sin(\sqrt[3]{2}x) \neq 2.$$

[6 marks]

b. Prove for all $n \in \mathbb{N}$, that $n^3 + 5n$ is always a multiple of 6. [5 marks]

c. a, b, c and d are four real numbers.

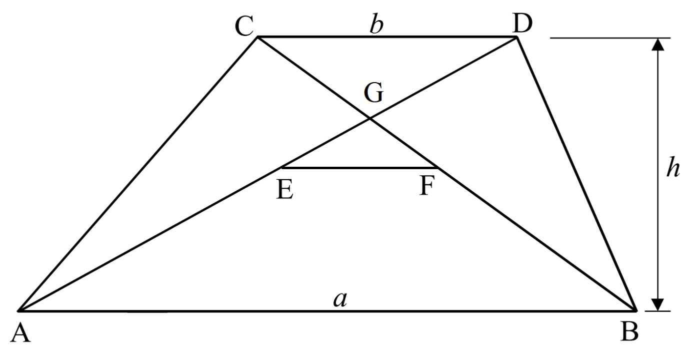
By considering the quadratic function $f(x) = (ax + c)^2 + (bx + d)^2$ for all real x, prove that

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2).$$

[3 marks]

B3.

a. Trapezium $ABCD$ below has parallel sides $|AB| = a$ and $|CD| = b$ with $a > b$. Points E and F are the midpoints of the diagonals AD and BC respectively, which intersect at G, as shown. The perpendicular height from AB to CD is h.



i) Prove that the length of the line segment EF is equal to $\frac{a-b}{2}$.

[3 marks]

ii) Explain why as $b \to 0$, the area of triangle EFG approaches $\frac{ah}{8}$.

[1 mark]

iii) If the perpendicular bisector of side AB also bisects CD, then identify any **two** pairs of congruent triangles in the figure. [2 marks]

iv) Let angle $\angle CGD = \theta$. A circle with radius r and centre O, which lies inside $\triangle EFG$, passes through all three vertices of $\triangle EFG$.

Using a suitable circle theorem, prove that $r = \frac{a-b}{4\sin\theta}$. [4 marks]

b. In the triangle ABC, the point X lies on side AB (length a) such that the line CX bisects $\angle ACB$ with $\angle ACX = \angle XCB = 60°$.

If $|BC| = 2\,|AC|$, prove that the length of line CX is $\frac{2a\sqrt{7}}{21}$. [10 marks]

C1.

a. Find the values of the unknown constants to make the following partial fraction decompositions valid for all appropriate values of t:

i) $$\frac{2t}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1}, \quad \text{for all } |t| \neq 1$$ [1 mark]

ii) $$\frac{1}{t^4+1} = \frac{Pt+Q}{t^2+Ct+D} + \frac{Rt+S}{t^2+Et+F} \quad \text{for all } t \in \mathbb{R}$$ [4 marks]

b. The domain of $f(x)$ is all real numbers and satisfies $2f(x) = f(x+1)$.

In the interval $[0, 1)$, $f(x) = x(x - 1)$.

i) Sketch a graph of $y = f(x)$ for $-1 < x < 3$. [2 marks]

ii) Express $f(x)$ in the form $f(x) = ax^2 + bx + c$ for real coefficients a, b and c, valid for all x in the interval $[2, 3)$. [2 marks]

iii) Find the greatest value of m such that $f(x) \geq -\frac{16}{9}$ for all $x \leq m$.

[2 marks]

c. x and y are variables such that $x + y \geq a$ and $x - y \leq -1$, where $a \neq 0$ is a constant.

i) Sketch, in the x-y coordinate plane, the region satisfied by these inequalities. [1 mark]

*ii) The **smallest** possible value of $x + ay$ is 7.

Find the value(s) of a. Fully justify your answer. [8 marks]

C2.

a. Simplify fully the expression $4 + \frac{4 - x^2}{x^2 - 2x}$. [2 marks]

b. Find the constant term in the expansion of $\left(2x + \frac{3}{x^2}\right)^{12}$. [2 marks]

c. Factorise fully $x^2 + 2xy - 3y^2 + 4x - 4y$. [3 marks]

d. Find the equations of the asymptotes to the curve

$$y = \frac{x^4 - 2x^3}{(x^2 - 3x + 2)\left(x^2 + ax + \frac{1}{2}a^2\right)}, \quad a \in \mathbb{R}.$$

[3 marks]

e. Solve the inequality $|5 - 3x| < \frac{2}{x}$. [4 marks]

f. Express the value of $\sqrt{3 - 2\sqrt{2}}$ in the form $a + b\sqrt{c}$, where *a*, *b* and *c* are integers to be found.

[3 marks]

*g. Rationalise the denominator in $\frac{1}{\sqrt[3]{2} + \sqrt[3]{3}}$. [3 marks]

C3.

a. $\mathbf{u}$ and $\mathbf{v}$ are vectors.

i) Explain why $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$. [1 mark]

ii) If $|\mathbf{u} + \mathbf{v}| = ||\mathbf{u}| - |\mathbf{v}||$, describe the relationship between $\mathbf{u}$ and $\mathbf{v}$. [2 marks]

b. Let $f(x) = 2 - \ln x$ and $g(x) = ke^{-x}$ for some positive constant k.

The domain of both $f(x)$ and $g(x)$ is $\{x \in \mathbb{R} : x > 0\}$.

Find the value of k for which the operation of composition for functions f and g is commutative for all $0 < x < k$.

[3 marks]

*c. A cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ has $a > 0$ and is defined for all real x. $P(x)$ has two real turning points at $x = p$ and $x = q$ where $p < q$.

Let Δx represent the absolute difference in x-coordinate between one of the turning points of $P(x)$ and its inflection point.

With the aid of a sketch, prove that $P(p) = P(p + 3\,\Delta x)$.

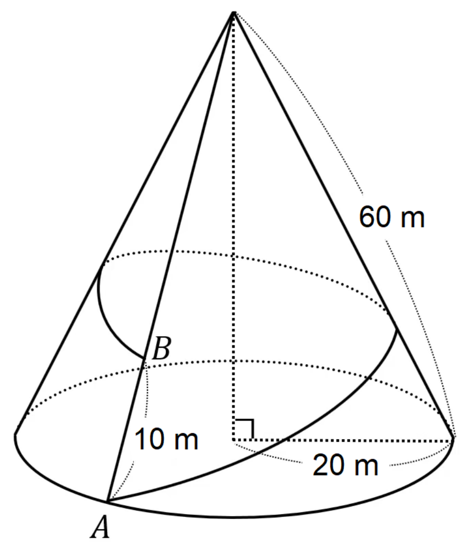
(It is **not** recommended to solve for p and q explicitly.) [7 marks]

*d. Find, in exact form, the real solution(s) to the equation

$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$

[7 marks]

D1. The diagram illustrates a right-circular cone shaped mountain.



A **shortest-distance** track for a sightseeing train is built around the mountain, in which the track starts at a point A at the base of the mountain, and ends at the point B, located 10 metres up the mountain (measured along the slant) above A.

As shown in the diagram, this track first goes uphill then goes downhill.

a. Show that the length of the downhill section of the track is $\frac{400}{\sqrt{91}}$ metres.

[6 marks]

b. Find the maximum **vertical** height reached by the train above the base of the mountain, giving your answer in metres to 3 significant figures. [4 marks]

c. The train completes the journey while moving at a constant speed throughout. Find the fraction of the journey for which the elevation of the train is above *B*.

[5 marks]

d. The track to be redesigned under the condition that there is no downhill section. Calculate the new shortest possible total length of the track. [5 marks]

D2.

a. i) Show that for all real x and y,

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\frac{x+y}{2}.$$ [4 marks]

ii) Hence show that the graph of

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 1$$

is a set of parallel lines, giving their general equations. [4 marks]

b. i) Find the first three terms, in ascending powers of x, of the binomial series expansion of $f(x)$, where

$$f(x) = \sqrt{1 - \frac{3}{4 - x^3}}, \qquad x \leq 1.$$ [6 marks]

ii) Find the interval of validity for the approximation in part b.i). [2 marks]

iii) If the first three terms of the expansion are used to approximate the value of $\int_0^1 f(x)\,dx$, explain whether the value obtained would be an underestimate or an overestimate of the true value. [2 marks]

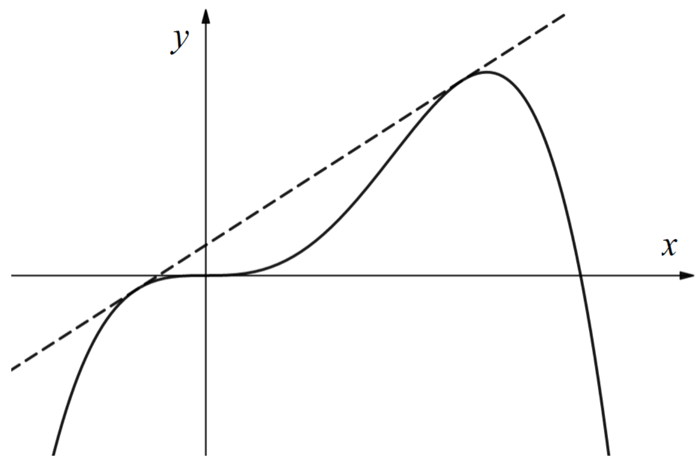
iv) Use the expansion to estimate the solution $0 < x < 1$ to the equation

$$\sin^{-1}\left(\frac{1}{2} - f(x)\right) = 1 - 720x^3.$$

Give your answer to 10 decimal places. [2 marks]

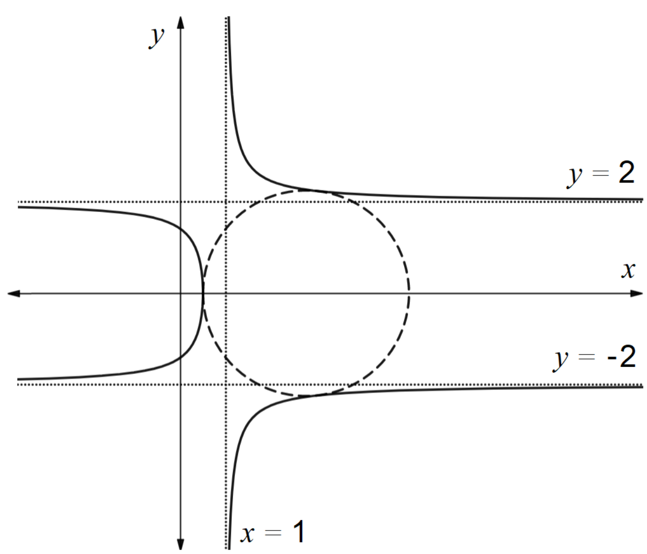
D3. *It is advised (but not required) to use algebra in this question, **not** calculus.*

a. The graph shows the curve $y = x^3 - x^4$ (solid line) and its common tangent line (dashed line), which touches the curve at **two** distinct points.



Find the equation of the tangent line. [10 marks]

b. The graph shows a circle (dashed line) which touches all three branches of the curve with equation $\frac{1}{2}y^2 = \frac{2x-1}{x-1}$ (solid line) at **three** distinct points.
The asymptotes (dotted lines) of the curve, $y = 2$, $y = -2$ and $x = 1$, are shown.



Find the equation of the circle. [10 marks]

E1.

a. i) Show that, for all $|x| < 1$,

$$y = \sin^{-1} x \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}.$$

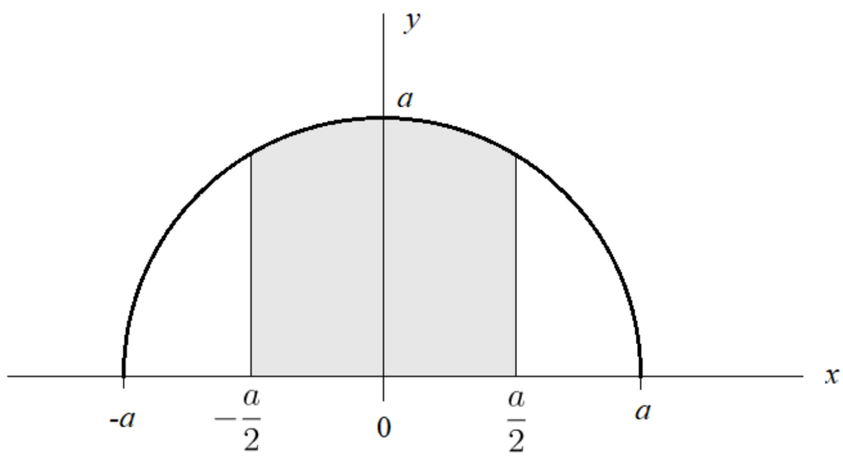
[5 marks]

ii) For some positive constant a, let

$$f(x) = x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}, \qquad |x| \leq a, \quad a > 0.$$

Find $f'(x)$ in its simplest form. [5 marks]

iii) The diagram shows a semicircle of radius a centred at the origin.



By considering your answer to part b.i), show that the area of the shaded region is $f\left(\frac{a}{2}\right)$. [4 marks]

b. Use the substitution $\frac{du}{dx} = \sin x + \cos x$, or otherwise, to show that

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, \mathrm{d}x = \frac{\ln 3}{20}.$$

[6 marks]

E2.

a. For some real constant a, define the function

$$f(x) = \frac{1}{1 + e^{-x^2} + e^{-(x-a)^2}}, \qquad x,\ a \in \mathbb{R}.$$

When $|a| < \sqrt{2}$, $f(x)$ always has only one stationary point, at $x = x_0$.

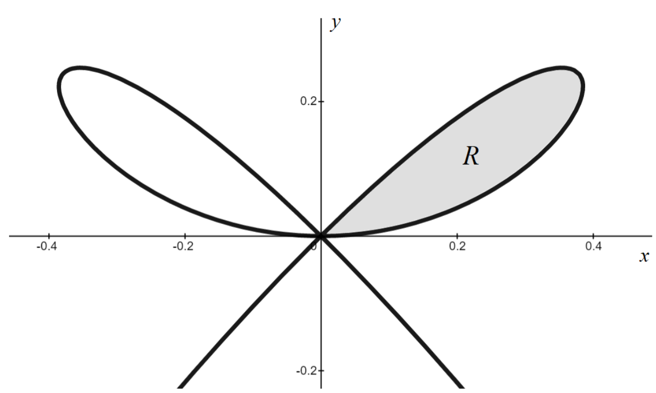
i) By considering the symmetry of $f(x)$, explain why $x_0 = \frac{1}{2}a$. [2 marks]

ii) Use calculus to determine the nature of this stationary point. [6 marks]

b. A water tank has the shape of a horizontal cylinder with radius 1 m and length 6 m. Water is being pumped into the tank at a rate of $\frac{1}{6}$ m^3 per minute.

Find the exact rate at which the water level is rising when the water is at a depth of 50 cm. [7 marks]

c. The graph of $x^4 + y^3 = x^2y$ is shown, with a closed-loop region R shaded.



By transforming the curve to a suitable set of parametric equations, or otherwise, find the area of region R. [4 marks]

E3.

a. Find the exact value of $\int_0^{\pi} e^{-x} \sin x \, dx$. [4 marks]

b. When the trapezium rule with n slices is used to approximate a definite integral $I = \int_a^b f(x)\, dx$ (where $b > a$) the result of the approximation is denoted T_n.

An upper bound for the absolute error E_n, in this approximation, defined as $E_n = | I - T_n |$, can be found using the inequality: $E_n \leq \frac{K(b-a)^3}{12n^2}$, where K is the largest value of $| f''(x) |$ in the interval $a \leq x \leq b$.

i) For the integral in part a), $I = \int_0^{\pi} e^{-x} \sin x \, dx$, find T_4. [3 marks]

ii) Use the formula to find an upper bound for E_4. [4 marks]

iii) Using the true value of I, compare the true error to the maximum possible error. [1 mark]

*c. A function $y(x)$ is positive for all x and has a gradient satisfying $\frac{dy}{dx} = \frac{x-y}{x+y}$.

The function has a stationary point, at which the value of y is equal to 1.

Find a simplified expression for y in terms of x. [8 marks]

F1. A short-barrelled machine gun stands on horizontal ground. The gun fires a near-continuous stream of bullets for a period of one second, from ground level, at the same initial speed, and then stops firing.

During this time period, the angle of elevation of the barrel, $\alpha(t)$, decreases from its initial angle of 45° to 30° to the horizontal at the moment it fires the last bullet.

The variation of the barrel elevation with time *t* since firing the first bullet is such that all the bullets fired by the machine gun land on the ground at the **same time**.

a. Show, for $0 \leq t \leq 1$, that $\alpha(t) = \sin^{-1}(A - Bt)$, where A and B are positive constants to be found in exact form.

Assume that the bullets can be modelled as point particles with no air resistance, and a uniform vertically downward gravitational field strength g ms^{-2}.

[8 marks]

b. Show that the bullets landing a distance r m from the machine gun barrel will do so at times T s after being fired related by

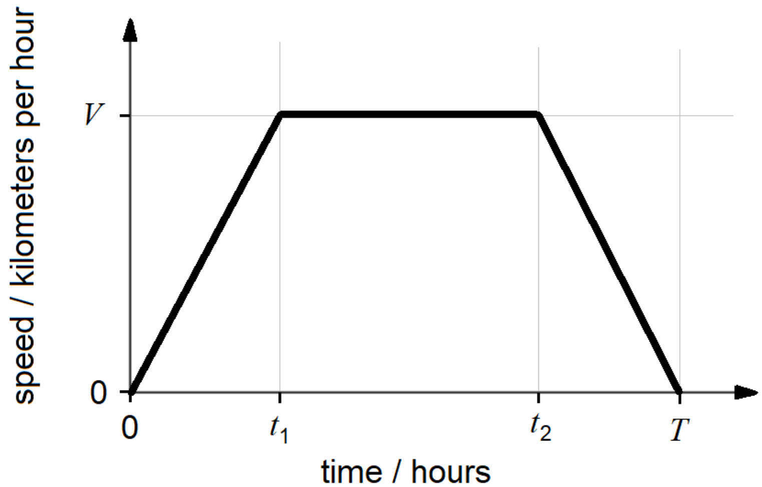
$$\frac{1}{2}g^2T^2 = v^2 - \sqrt{v^4 - g^2r^2},$$

where v is the initial speed of the bullets. [8 marks]

c. Hence find an expression for the greatest height h above the ground reached by a bullet which was fired at speed v ms^{-1} and landed at a distance r m from the machine gun barrel.

Give your answer in terms of v, r and g. [4 marks]

F2. The speed-time graph of a straight-line journey of a hovercraft is shown below.



The hovercraft accelerates uniformly at m km hr $^{-2}$ for an initial period t_1, travels at a constant speed V km hr $^{-1}$, and finally decelerates uniformly at m km hr $^{-2}$. The distance covered in that time is D km and the total time taken is T hours.

a. If the time taken by acceleration and deceleration is $\frac{T}{6}$ in each case, find an equation for D in terms of T and m. [5 marks]

b. The hovercraft is running behind schedule and needs to reduce T to $\frac{7}{9}$ T.

The only way of achieving this is to increase the value of V while the acceleration and deceleration may not be changed from m.

i) Calculate the time, as a fraction of T, that must now be spent accelerating. [8 marks]

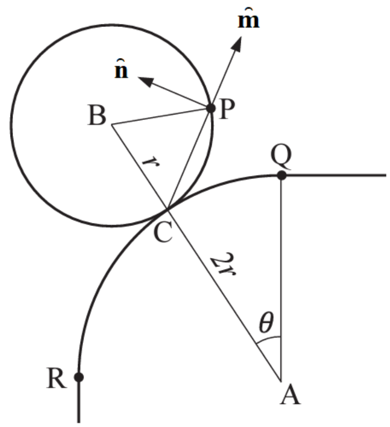
ii) Find the required percentage increase in the top speed. [3 marks]

*c. Consider again the original journey of total time T as described in part a).

Sketch a graph of the **speed** (vertical axis) of the hovercraft against the **distance** from the starting position (horizontal axis), identifying the distances corresponding to travel times t_1, t_2 and T as fractions of D. [4 marks]

F3. A cylinder is rolling over the rounded edge of a table. The rounded edge is a quarter-circle arc between Q and R, centred at A, and having radius $2r$. The cylinder has radius r, centre B, and rolls (without slipping) over the edge.

P is the point on the cylinder that was originally in contact with the table at Q at time $t = 0$. **Unit** vectors $\mathbf{m}$ and $\mathbf{n}$ are defined to always point parallel and perpendicular respectively to the line segment CP as shown. Angle $\angle CAQ$ is $\theta(t)$.



a. i) Using geometry, prove that $\angle CBP : \angle CAQ = 2 : 1$. [2 marks]

ii) Write down an expression for the displacement vector $\mathbf{s}$ of point P relative to A in terms of the vectors $\mathbf{m}$ and $\mathbf{n}$ and in terms of r and θ. [5 marks]

iii) Given that $\frac{\mathrm{d}\hat{\mathbf{m}}}{\mathrm{d}t} = 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{n}}$, explain why $\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}t}$ must be equal to $-2\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{m}}$.

[2 marks]

iv) Using differentiation, show that the velocity vector $\mathbf{v}$ of point P is given by

$$\mathbf{v} = 6r \sin\theta \frac{d\theta}{dt}\,\mathbf{n},$$

and find an expression for the acceleration vector $\mathbf{a}$. [8 marks]

b. The component of acceleration a which is **perpendicular** to the velocity vector $\mathbf{v}$ can be related to the *radius of curvature* R of the path traced out by the point P using the equation $a = \frac{|v|^2}{R}$. Show that R is proportional to $\sin\theta$. [3 marks]

G1. Newborn babies are tested for a mild illness which affects 1 in 500 babies. The result of a test is either positive or negative. A positive test implies the baby has the illness. However, the test is not perfect:

- for babies with the illness, the probability of a positive result is 0.99
- for babies without the illness, the probability of a negative result is 0.95.

a. Find the probability that

i) the test is positive. [2 marks]

ii) the test gives the correct diagnosis. [2 marks]

b. i) Given that the result of a test is positive, show that the probability of the baby having the illness is less than 5%. [3 marks]

ii) Explain why the probability of the test giving a 'true positive' is so low, even though the accuracy of the test in each case is no less than 95%. [2 marks]

c. It is required to raise the reliability of the test by improving the screening process. To achieve this, it is intended to increase the probability of negative results for babies who do not have the illness from its current value of 0.95 to p.

Find the value of p that would raise the probability in part b.i) to 0.50. [4 marks]

d. A particular baby is tested (using the new, improved test) three times.

The tests return one positive result and two negative results.

i) Show that the probability that the baby has the illness is about 1 in 10,000. [6 marks]

ii) Give **one** assumption made in your answer to part d.i). [1 marks]

G2. A manufacturing process produces resistors. Due to the tolerances of the process, the resistors produced have resistance values, in units of ohms, which are Normally distributed about a mean μ ohms and standard deviation σ ohms.

a. A random sample of 12 resistors from the process is (sorted in ascending order):

497.66	497.77	497.81	498.32	498.43	499.53
500.41	500.78	501.55	502.88	504.96	506.41

i) Find the sample median and sample interquartile range. [3 marks]

ii) Find unbiased estimates for the values of μ and σ^2. [2 marks]

iii) Using your estimates of μ and σ, estimate the interquartile range of the population. [3 marks]

b. A large number ($N \gg 1000$) of resistors are manufactured. A quality assurance survey of these resistors shows that $\mu = 501$ ohms and $\sigma = 3$ ohms. Resistors are rejected if their resistance is less than 498 ohms or greater than 508 ohms.

i) Find the expected proportion of manufactured resistors which are rejected. [2 marks]

ii) In order to minimise waste, the value of μ for the process is to be changed while the value of σ is fixed.

Find the minimum expected proportion of resistors which are rejected. [3 marks]

iii) The manufacturing process is improved to use more precise machinery, allowing for less variation in resistance.

For the optimal value of μ found in part b.ii), find the largest possible value of σ for the process such that the expected proportion of rejected resistors is no more than 0.1%. [4 marks]

*iv) Repeat the calculation in part b.iii) to estimate the required value of σ if the original process with $\mu = 501$ ohms is used instead. [3 marks]

G3. A robot is programmed to flip a coin repeatedly. The outcome of the coin flip is either Heads (H) or Tails (T), with probability $\mathrm{P(H)} = p$.

*a. It is initially assumed that the robot flips the coin fairly, so that $p = 0.5$.

Define X_n as the event that there were no consecutive Heads in the first n flips.

i) Let $F(n)$ be the distinct number of ways for the coins to land in a sequence of n flips such that the event X_n occurs.

Explain clearly why $F(n) = F(n - 1) + F(n - 2)$, for $n > 2$. [3 marks]

ii) Hence, or otherwise, show that $\mathrm{P}(X_{10}) = \dfrac{9}{64}$. [3 marks]

b. After the first 13 coin flips, it is observed that 10 of the flips returned Heads.

i) Test, at the 5% significance level, the claim that $p > 0.5$. [4 marks]

ii) Write down the critical region for this hypothesis test. [1 mark]

*c. It is now suspected that the robot may be unintentionally flipping the coins in the direction of Heads more frequently.

The value of the Heads probability p is modelled as random by initially assuming that p can take values with equal likelihood from 0.0 to 1.0 inclusive in discrete steps of 0.1.

i) Explain why the value of

$$\mathrm{P}(p = 0.5 \mid \text{at least 10 Heads in 13 flips})$$

is **not** the p-value of the test conducted in part b). [1 mark]

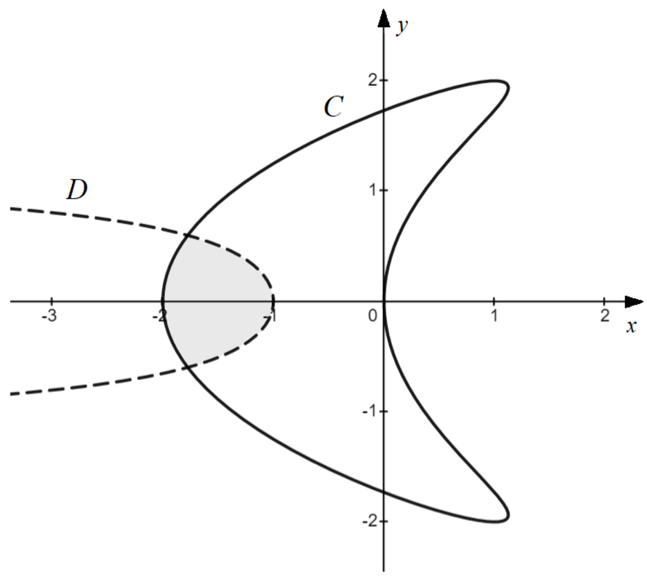
ii) Find the value of $\mathrm{P}(p = 0.5 \mid$ **exactly** 10 Heads in 13 flips$)$. [5 marks]

iii) Given the observation of 10 Heads in 13 flips, estimate the **median** of p. [3 marks]

A30. The curves C and D are are defined by parametric equations

Curve C:	$x = \sin s + \cos 2s$,	$y = 2\cos s$,	for $0 \le s < 2\pi$
Curve D:	$x = \sec^3 t$,	$y = t - \pi$,	for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

The graphs of these curves are plotted, with their intersection region shaded.



Use numerical methods to find the area of the shaded region to 4 d.p.

① 0.8412
② 0.8865
③ 0.9139
④ 0.9514 [4 marks]

A31. Which of the following is a statement of the converse of the factor theorem?

(P is a polynomial function, the symbol | indicates divisibility in the ring of polynomials.)

① $\exists a \in \mathbb{R} : P(a) = 0 \quad \Rightarrow \quad \forall x \in \mathbb{R} : (x - a) \mid P(x)$

② $\forall x \in \mathbb{R},\ \exists a \in \mathbb{R} : (x - a) \mid P(x) \quad \Rightarrow \quad P(a) = 0$

③ $\exists a \in \mathbb{R}, \forall x \in \mathbb{R} : \dfrac{P(a)}{x - a} \in \mathbb{Q} \quad \Rightarrow \quad P(a) = 0$

④ $\exists a \in \mathbb{R} : P(a) \neq 0 \quad \Rightarrow \quad \forall x \in \mathbb{R} : \dfrac{P(a)}{x - a} \notin \mathbb{Q}$

A31. Find the equations of the tangent lines to the parabola $y = x^2$ which pass through the point (1, -1).

① $y = 2(1 + \sqrt{2})x - (1 + \sqrt{2})^2$ and $y = 2(1 - \sqrt{2})x - (1 - \sqrt{2})^2$
② $y = 2(1 - \sqrt{2})x - (1 + \sqrt{2})^2$ and $y = 2(1 + \sqrt{2})x - (1 - \sqrt{2})^2$
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④ $y = 2(1 + \sqrt{2})x + (1 - \sqrt{2})^2$ and $y = 2(1 - \sqrt{2})x + (1 + \sqrt{2})^2$

[2 marks]

A32. The function $f(x)$ satisfies the equation $f(x) + \int_1^x f(t)\,dt = 1$ for all real $x > 1$.

Find the value of $\ln f(2)$.

① -2
② -1
③ 1
④ 2

[2 marks]

A33. A sequence is defined with nth term

$$u_n = \frac{n}{a^n}, \qquad n = 1, 2, 3, \ldots, \qquad a \text{ is a constant.}$$

If the sum to infinity of this sequence, $\sum_{n=1}^{\infty} u_n$, is equal to $\frac{14}{25}$,

then the possible value(s) of a are

① $\frac{2}{7}$ or $\frac{7}{2}$

② $\frac{3}{5}$ or $\frac{5}{3}$

③ $\frac{7}{2}$

④ $-\frac{3}{5}$

[4 marks]

G4. An insurance company launched an automobile insurance business and 40,000 people bought this automobile insurance in 2016. It turns out that out of the 40,000 policy holders, 5% have one claim, 0.5% have two claims, and 94.5% have no claim.

a. A complete dataset obtained indicates the 2400 claims (the number of dollars paid on an automobile insurance claim) made in 2016. The data is summarised as follows:

$$n = 2400 \qquad \sum x = 2,489,313 \qquad \sum x^2 = 8,600,048,675$$

i) Show that the mean payout in 2016 was close to \$62.23 per policyholder. [2 marks]

ii) Calculate the standard deviation of payouts in 2016 per policyholder. [3 marks]

b. The insurance company is considering to re-launch this automobile insurance business in 2018. Based on the economic situation and the company reputation established in 2016, the company forecasts that there are at least 500,000 people who will buy its automobile insurance product, each claim will follow the same distribution as 2016.

i) Explain how and why the central limit theorem can be applied to estimate the distribution of the expected payouts. [2 marks]

ii) Calculate the projected variance of the payouts in 2018 per policyholder. [4 marks]

c. Let P be the premium charged to each policyholder for this automobile insurance over the year of 2018. Suppose that the insurer would like to be 96% confident that total premiums exceed total claims.

If the one-year continuously compounded interest rate on the insurance premium is 5%, calculate P. [5 marks]

*d. The projection of 500,000 new buyers in 2018 is not certain. The actual number of buyers has a Normal distribution with mean 500,000 and standard deviation σ.

Describe how the value of P in the answer to part c) varies with σ. [4 marks]

Question Sources

Sampled from A-Level past papers (AQA/EdExcel/OCR), as well as:

A4	SAT (American High School Exam)
A5, A7, A8, D1a	수학 B 짝수형 (Korean SAT Maths, Track B)
A10, A13	ENGAA (Cambridge Engineering)
A11, A14, A15, G2	Cambridge Engineering Part IA and IB Tripos and Examples Paper
A17, C1b, c	高考 (Chinese SAT)
B3	CUED (Cambridge Engineering) Preparatory Problems
F1	PAT (Oxford Physics)

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