Maths and Further Maths: Extended Questions

Focus: Extended Questions

Difficulty: Hard

Marks:

Part A (Maths Content): 4 questions

Part B (Further Maths Content): 5 questions

Advice:

1. All questions are far above A-level difficulty.
2. Questions may assess multiple topics across Pure and Applied.
3. Attempt only one question at a time then take a break.
4. It may not be clear how to begin in many questions - consider your options and plan before you write anything down
5. Blank answer space is not provided.
6. Fully justify your answers in every question.
7. Check the fully worked solutions for any questions you missed.

Part A: Maths content. Questions will involve topics from the maths specification.

1. Two spheres of respective radius a and $2a$ are fixed on level horizontal ground. The spheres are touching each other. A uniform rod of length $4a$ is gently placed over the two spheres, touching the two spheres in such a way so that the centre of mass of the rod is equidistant from the two contact points of the rod with the spheres.

   The coefficient of friction at the contact point with the smaller sphere is μ and at the contact point with the larger sphere is 3μ .

   If the rod remains in equilibrium, show that $\mu \geq k\sqrt{2}$, where k is rational.

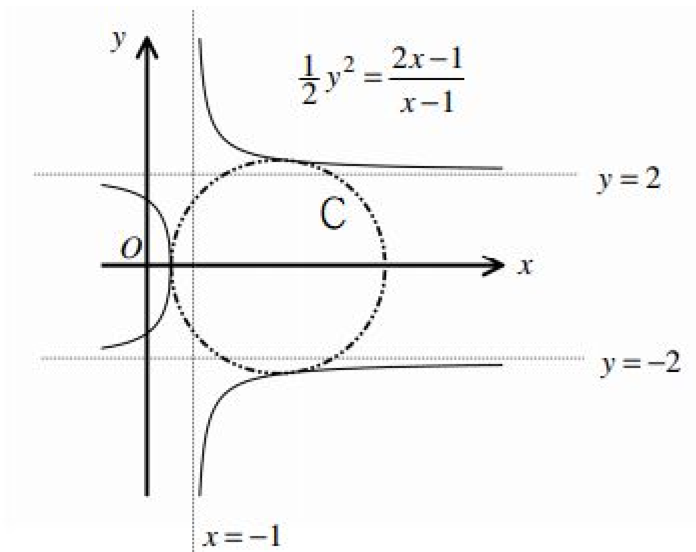
   [10 marks]

2. The figure below shows the curve

$$\frac{1}{2}y^2 = \frac{2x-1}{x-1}$$

whose three asymptotes are shown as the dotted lines x = -1 and y = ± 2.

A circle C is drawn, so that it touches all three branches of the curve, as shown.



Determine the Cartesian equation of C in its simplest form. [20 marks]

3. For a function $f(x)$ defined on some interval $[a, b]$, the value of

$$I = \int_a^b f(x)\ dx$$

is approximated using the trapezium rule, with n intervals. The result of this approximation is S. The error, E, of this approximation is defined as the size of the difference between the approximation and the true value, i.e. $E = |S - I|$.

By considering a single trapezium as the sum of a rectangle and a small triangle, show that the error in the approximation is bounded by

$$E \leq \frac{K(b-a)^3}{12n^2},$$

where K is the maximum value of $|f''(x)|$ in the interval $a \leq x \leq b$. You should include a sketch of the trapezium rule in use to help you.

[20 marks]

4. *Vieta jumping* is a special type of proof by contradiction used to prove certain classes of mathematical statements. The general procedure of Vieta jumping is:

- Assume there exists some solution which violates the given statement (the same foundation as a standard proof by contradiction);
- Define the **smallest possible** such solution according to some sensible choice of definition of ‘smallest’ within the context of the problem;
- Show that this implies the existence of a smaller solution, hence a contradiction.

Let *a* and *b* be positive integers such that $\frac{a^2+b^2}{ab+1} = k$ for some $k \in \mathbb{Z}^+$.

Using the above structure for a Vieta jumping argument, prove that k is always a perfect square.

[20 marks]

Part B: Further Maths content. Questions will involve topics from the FM specification.

1. In the Cartesian plane, circle C has equation $x^2 + y^2 - 8x = 0$ and hyperbola H has equation $4x^2 - 9y^2 = 36$.

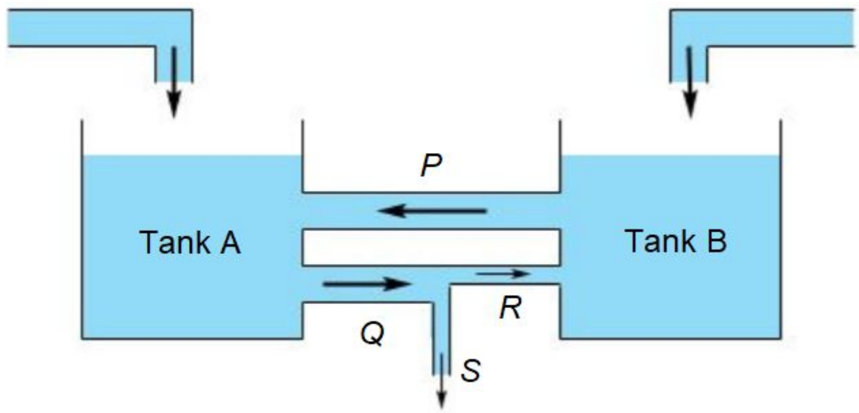
   A line L of positive gradient is tangent to H at point A and tangent to C at point B.

   Find, in any order:

   - an equation for L, in its simplest form
   - the coordinates of the intersections of C and H, in exact form
   - the area of the smaller region bounded by C and H, to 3 significant figures

[20 marks]

2. The diagram below shows two interconnected tanks of water which are being mixed. The water contains a dissolved chemical contaminant.



Initially:

- Tank A contains 50 litres of water and 25 grams of contaminant
- Tank B contains 100 litres of water and 75 grams of contaminant

Contaminated water with a concentration of 1.5 grams of contaminant per litre enters Tank A at a constant rate of 7 litres per hour, while freshwater enters Tank B at a constant rate of 3 litres per hour.

Water flows from Tank B to Tank A through pipe P at a rate of 5 litres per hour. Pipe Q removes 12 litres from Tank A per hour, pumping 2 litres of it into Tank B through pipe R, while the remaining 10 litres is removed from the system through pipe S.

By forming and solving a suitable model for this system, obtain functions to predict the mass of dissolved contaminant in each tank in grams at any time *t*.

You should state any modelling assumptions used in your method.

[20 marks]

3. By integrating with an initial substitution of $u = \tan^{2/3}(x)$, or otherwise, find the exact value of

$$\int_0^{\pi/4} \sqrt[3]{\tan(x)}\ \mathrm{d}x$$
.

Give your answer in terms π, √3 and ln 2. [20 marks]

4. Consider a coordinate system in which the *xy* Cartesian plane is superimposed on a polar coordinate system, with the *x*-axis parallel to the initial line and the origin at the pole.

A straight line *L*, whose gradient in the *xy* plane is -3/11, is a tangent to the curve with polar equation

$$r = 25\cos(2\theta)\ , \quad 0 \le \theta \le \frac{\pi}{4}$$

Find, in exact form, the area of the finite region bounded by the curve, the straight line *L* and the initial line.

Give your answer in the form $A(B + C \tan^{-1} 1/3)$, where A is rational and B and C are integers.

[25 marks]

5. The function $f(x)$ is defined for all real x by

$$f(x) = \begin{cases} 1 - |x-t| \ , & \text{if } |x-t| \leq 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

where t is real. Furthermore, the function $g(t)$ is defined in terms of $f(x)$ as

$$g(t) = \int_{k}^{k+8} f(x) \cos(\pi x) \ \mathrm{d}x$$

where k is an **odd** integer.

Consider all numbers α such that $g(t)$ is a local **minimum** at $t = \alpha$ and $g(\alpha) < 0$. When these m distinct values of α are listed in ascending order as $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$, it is given that

$$\sum_{i=1}^{m} \alpha_i = 45.$$

Find the value of

$$k - \pi^2 \sum_{i=1}^{m} g(\alpha_i).$$

In your working, you should include simple sketches of $f(x)$ and $g(t)$ to assist you.

[25 marks]

Question Sources

Maths

Q1 MadasMaths, Equilibrium of Rigid Bodies, Question 45:
https://madasmaths.com/archive/maths_booklets/mechanics/m2_equilibriuM_of_rigid_bodies.pdf

Q2 MadasMaths, Differentiation Exam Questions, Question 261
https://madasmaths.com/archive/maths_booklets/standard_topics/various/differeNtiation_ii_exam_questions.pdf

Q3 Error bounds for Trapezium rule (proof)
https://hive.blog/mathematics/@mes/approximate-integration-trapezoidal-rule-error-bound-proof)

Q4 IMO, 1988, Australia, Question 6
https://www.sciencealert.com/the-legend-of-question-six-one-of-the-hardest-maths-problems-ever

Further Maths

Q1 Adapted from IIT JEE, 2010, Paper 1, Question 45
https://www.khanacademy.org/test-prep/iit-jee-subject/iit-jee-topic/iit-jee/v/i
i
T-jee-circle-hyperbola-common-tangent-part-1

Q2 Adapted from a problem at
http://tutorial.math.lamar.edu/Classes/DE/SystemsModeling.aspx

Q4 MadasMaths, Polar Coordinates, Question 47
https://madasmaths.com/archive/maths_booklets/further_topics/various/polar_coOrdinates_exam_questions.pdf

Q5 CSAT, 2018, South Korea, Question 30
https://blog.naver.com/dak219324/221147969311