## Section A - Quick Reference Answer Key

| Q | A | Marks | Q | A | Marks | Q | A | Marks |
|---|---|---|---|---|---|---|---|---|
| 1 | ② | 1 | 9 | ② | 1 | 17 | ④ | 1 |
| 2 | ③ | 1 | 10 | ① | 1 | 18 | ④ | 1 |
| 3 | ② | 1 | 11 | ① | 1 | 19 | ① | 1 |
| 4 | ④ | 1 | 12 | ① | 1 | 20 | ② | 1 |
| 5 | ② | 2 | 13 | ③ | 2 | 21 | ③ | 2 |
| 6 | ① | 2 | 14 | ② | 2 | 22 | ④ | 2 |
| 7 | ④ | 4 | 15 | ④ | 4 | 23 | ③ | 4 |
| 8 | ③ | 4 | 16 | ② | 4 | 24 | ② | 4 |

**A1. Answer:** ②

$3x^2 - 2xy - y^2 = 3x^2 - 3xy + xy - y^2 = 3x(x - y) + y(x - y) = (3x + y)(x - y)$. (②)

**A2. Answer:** ③

Gradient of line $= \frac{9-1}{-1-5} = \frac{8}{-6} = \frac{-4}{3}$.

Therefore $-\frac{4}{3} = \frac{4-1}{k-5} \Rightarrow \frac{-4}{3} = \frac{3}{k-5} \Rightarrow 4(5-k) = 9 \Rightarrow k = \frac{11}{4}$. (③)

**A3. Answer:** ②

Substitute $u = \sin x \rightarrow du = \cos x\, dx$ to convert $\int \frac{\cos x\,(\sin x - 1)}{\sin^2 x + 5\sin x + 6}\, dx$ into

$\int \frac{u-1}{u^2 + 5u + 6}\, du$. The denominator can be factored, and so integration by partial fractions is most appropriate. (②)

**A4. Answer:** ④

$\{r \in \mathbb{Q}^+\}$ is the set of positive rational numbers. There is no 'smallest' rational: assume that any rational is the 'smallest', then a new, smaller rational can be found by dividing by e.g. 2. Therefore there are an infinite number of rational numbers between any rational and zero.

$\{r \in \mathbb{Q}^+ \mid r^2 \geq 2\}$ is the set of rationals larger than or equal to $\sqrt{2}$. Since $\sqrt{2}$ is not rational, there are an infinite number of rationals between $\sqrt{2}$ and any other rational, so for the same reason there is no smallest here either.

$\{r \in \mathbb{Q}^+ \mid r^2 > 4\}$ is the set of rationals larger than 2, which has no smallest for the same reason as $\{r \in \mathbb{Q}^+\}$. Therefore, none of them have a smallest. (④)

**A5. Answer:** ②

$$\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = (a_1 + a_2 + \ ... + a_{n-1} + a_n) - (a_1 + a_2 + \ ... + a_{n-1}) = a_n.$$

$$\Rightarrow \ a_n = log_{10}\left(\frac{(n+1)(n+2)}{2}\right) - log_{10}\left(\frac{n(n+1)}{2}\right) = log_{10}\left(\frac{n+2}{n}\right)$$

$$\sum_{k=1}^{20} a_{2k} = \sum_{k=1}^{20} log_{10}\left(\frac{2k+2}{2k}\right)$$

$$= log_{10}\left(\frac{4}{2}\right) + log_{10}\left(\frac{6}{4}\right) + log_{10}\left(\frac{8}{6}\right) + ... \ log_{10}\left(\frac{42}{40}\right)$$

$$= log_{10}\left(\frac{4 \times 6 \times 8 \times ... \times 42}{2 \times 4 \times 6 \times ... \times 40}\right)$$ (telescoping product - all middle terms cancel)

$$= log_{10}\left(\frac{42}{2}\right) = log_{10}(21).$$ (②)

**A6. Answer:** ①

$(\ln x)^2 + (\ln 2x)^2 = (\ln 3x)^3$

$\rightarrow (\ln x)^2 + (\ln 2 + \ln x)^2 = (\ln 3 + \ln x)^2$

$\rightarrow (\ln x)^2 + (2 \ln 2 - 2 \ln 3) \ln x + ((\ln 2)^2 - (\ln 3)^2)$

$\rightarrow (\ln x)^2 + (\ln \frac{4}{9}) \ln x + (\ln 6)(\ln \frac{2}{3})$

$\rightarrow \ln x = \frac{1}{2} (\ln \frac{9}{4} \pm \sqrt{(ln \ \frac{4}{9})^2 - 4(ln \ 6)(ln \ \frac{2}{3})}) \rightarrow x = \frac{3}{2} \exp \sqrt{ln \ \frac{3}{2} \ ln \ 9}.$ (①)

**A7. Answer: ④**

$f(0) = 0 \rightarrow f(x) = ax^2 + bx = kx(x - r)$ (no constant term; factorise for convenience)

Since $\int_0^2 f(x)\,dx < 0$, the curve must be (at least partly) below the $x$-axis in this interval. Additionally since $\int_0^2 |f(x)|\,dx = -\int_0^2 f(x)\,dx$, this implies that $|f(x)| = -f(x)$ for all $x$ in this interval, so $f(x) < 0$ for all $0 < x < 2$, so $f(x) = kx(x - r)$ for some $r \geq 2$ and $k > 0$. Next, since $\int_2^3 |f(x)|\,dx = \int_2^3 f(x)\,dx$, there cannot be any $f(x) < 0$ in the interval $2 < x < 3$. Combining these conditions, we must have $r = 2$ and so $f(x) = kx(x - 2)$.

Using the second part of the first equation, $-k\int_0^2 x(x-2)\,dx = 4$

$\Rightarrow \quad -\frac{-4}{3}k = 4 \;\Rightarrow\; k = 3 \;\Rightarrow\; f(x) = 3x(x-2)$

The required value is then

$f(5) = 3 \times 5 \times (5-2) = 45.$ (④)

**A8. Answer: ③**

Let $t = 5$: $\quad x^3 + 2x^2 - 15x + 5 = 5 \;\Rightarrow\; x(x-3)(x+5) = 0$

$\Rightarrow x = 0,\ 3,\ -5$

$\Rightarrow f(5) = 3,\ g(5) = -5$

Consider the general case: $\quad x^3 + 2x^2 - 15x + 5 = t$

$\Rightarrow \frac{dt}{dx} = 3x^2 + 4x - 15$ (differentiate both sides wrt $x$)

$\Rightarrow \frac{dx}{dt} = \frac{1}{3x^2 + 4x - 15}$ (reciprocal both sides)

$\Rightarrow f'(t) = \frac{1}{3f(t)^2 + 4f(t) - 15}$ and $g'(t) = \frac{1}{3g(t)^2 + 4g(t) - 15}$ (let $x = f(t)$ and $x = g(t)$)

When $t = 5$, $f(5) = 3$ and $g(5) = -5$, so

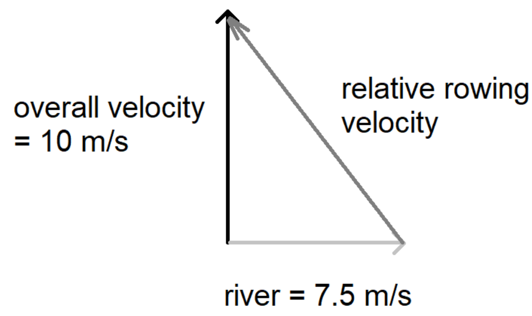
$\Rightarrow f'(5) = \frac{1}{3(3)^2 + 4(3) - 15} = \frac{1}{24}$ and $g'(5) = \frac{1}{3(-5)^2 + 4(-5) - 15} = \frac{1}{40}$

$\Rightarrow h'(5) = (f(5) - g(5)) + t \times (f'(5) - g'(5))$ (product rule)

$\Rightarrow h'(5) = 8 + 5 \times \frac{1}{60} = \frac{97}{12}.$ (③)

**A9.** **Answer:** ②

The velocity diagram is




Speed = $\sqrt{10^2 + 7.5^2} = 12.5$ m/s, Direction = 180° - tan$^{-1}$ $\frac{10}{7.5}$ = 127°. (②)

**A10.** **Answer:** ①

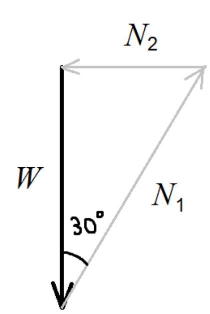
Applied force acts to the right + object is in equilibrium → there must be a friction force acting to the left.

Moment about bottom = $(sin\,30 + sin\,60)d \times P = \frac{1+\sqrt{3}}{2}Pd$.

→ Only statement A is correct. (①)

**A11.** **Answer:** ①

Since the ball is in equilibrium, we can draw a triangle of forces as the force vectors must sum to zero. Since both planes are frictionless, forces act perpendicular to the planes:




By trigonometry / Pythagoras, $N_2 = \frac{1}{2}W$ and $N_1 = \frac{\sqrt{5}}{2}W$. (①)

**A12. Answer:** ①

When a projectile is launched with some initial vertical displacement, the optimum angle for maximum range is less than $45^o$. This is because the stone has already been provided with some vertical height, so does not need to put as much into its vertical velocity component to achieve the ‘equivalent’ to a flat $45^o$ launch angle. This means the horizontal initial speed should exceed the vertical component, giving a smaller launch angle. (①)

**A13. Answer:** ③

Consider all motion relative to P i.e. consider P stationary.
Resolving forces in the pulley,

R: $m_R g - T = m_R a$
Q: $T = m_Q a$

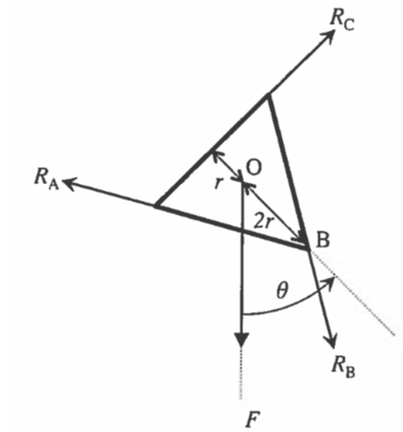
Eliminating $T$,

$\rightarrow m_R g = m_R a + m_Q a \rightarrow m_R g = a(m_R + m_Q) \rightarrow a = m_R / (m_R + m_Q)$.

If block P moves with this acceleration, Q and R would not move relative to P since they would have the same acceleration. (③)

**A14. Answer:** ②

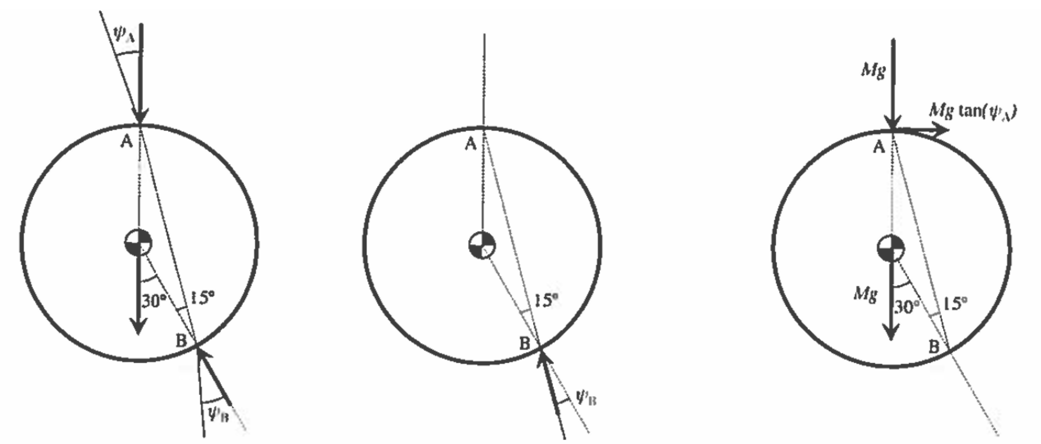
Free-body diagram, taking moments about point $B$ to eliminate $R_A$ and $R_B$:




Therefore, $3r \times R_C = 2r \sin\theta \times F$, so $R_C = \frac{2}{3} F \sin\theta$. (②)

**A15. Answer:** ④

The angle of friction indicates the maximum angle through which the reaction force can turn from the normal direction to sustain a friction force $F \le \mu R$. Since there are three forces acting on the cylinder (combining normal contact force and friction into a single force), their lines of action must all pass through the same point, so with some geometry, and taking moments about the contacts, we get:




Free body diagram (reaction forces shown for frictionless contact)

Moments about A: the cylinder can only be in equilibrium if $\psi_B \ge 15°$. Otherwise it will slip at B.

Moments about B: the cylinder can only be in equilibrium if $2MgR\sin 30 \le Mg\tan\psi_A R(1+\cos 30)$ *i.e.* $\psi_A \ge 28.2°$. Otherwise it will slip at A.

Therefore $x = 28.2°$ and $y = 15°$. (④)

**A16. Answer:** ②

Let tension in the upper and lower string be $T_1$ and $T_2$ respectively.
Since two tensions pull down either side on the hanging pulley, $T_1 = 2T_2$.

Both blocks A and B will lift up if $T_2 > 2mg \Rightarrow T_1 > 4mg$.
In this case, block A has a resultant force of $mg$ upwards, and so accelerates upwards at $g$, while block B remains stationary on the ground.
Therefore the pulley must accelerate upwards at $\frac{g}{2}$ to maintain a taut string.
The hanging mass must therefore accelerate downwards at $\frac{g}{2}$.
Balancing forces on the hanging mass, $Mg - T_1 = \frac{Mg}{2} \Rightarrow \frac{1}{2}Mg > 4mg$
$\Rightarrow M > 8m$.

If $M = 4m$, block B will not move (by condition above). Let upwards acceleration of block A be $a$, then $T_2 = m(g + a) \Rightarrow T_1 = 2m(g + a)$. The acceleration of the pulley (and hence the hanging mass) is $\frac{a}{2}$, so $Mg - 2m(g + a) = \frac{1}{2}Ma$
$\Rightarrow 2mg - 2ma = 2ma \Rightarrow 4ma = 2mg \Rightarrow a = \frac{g}{2}$. (②)

If the system is in equilibrium: $T_2 \leq mg$ or else block A will lift up.
Since $T_1 = 2T_2$ (2 tensions pull down either side on the pulley), $T_1 \leq 2mg$.
By equilibrium of the hanging mass, $T_1 = Mg \Rightarrow M \leq 2m$.

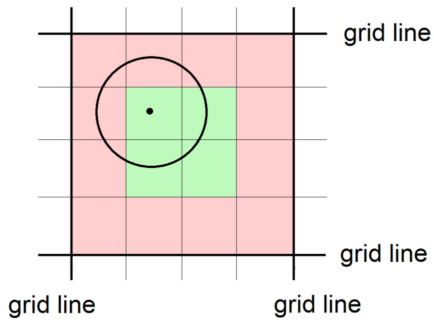
**A17. Answer:** ④

Sampling interval = separation between samples = $\frac{population\ size}{sample\ size} = \frac{1000}{40} = 25$.
The sample size is larger than 30, so the population mean should match the sample mean fairly well, assuming the sampling is done properly. Quota sampling would split the sample of 40 into small groups, which would not be sufficiently large to analyse as a representative of the population, so should not be used. Systematic sampling is uniform. (④)

**A18. Answer:** ④

We are looking for the range where area under curve at centre = 0.5.
InvNorm(0.25) = -0.6745, InvNorm(0.75) = 0.6745
→ Range = 0.6745 - (-0.6745) = 1.3490 (④)
(Or use central tail with area = 0.5 if possible)

**A19. Answer:** ①




The centre of the coin must lie between 1 cm and 3 cm from each line, covering half the area. However, there are two 'degrees of freedom' for the centre of the coin, so this must be true in both the horizontal and vertical directions, so the probability is $1/4 =$ 0.2500. (①)

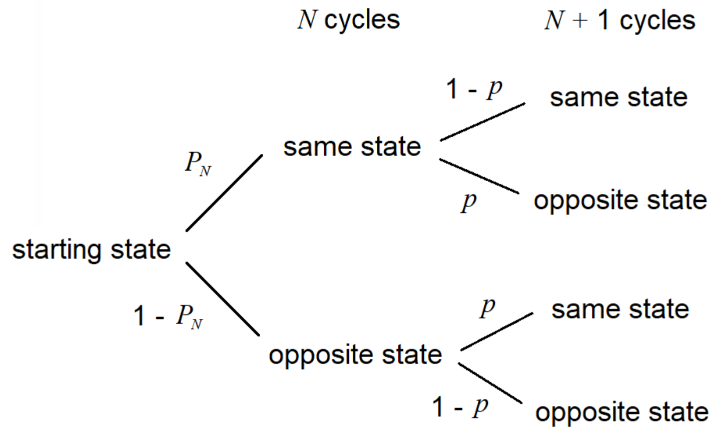
**A20. Answer:** ②

Let the total population be $N$. When the sample is collected and released, we have a proportion of $30/N$ marked frogs per frog, evenly distributed in the wild. Both the marked and unmarked frogs grow at the same rate, so the net proportion does not change. A marked frog therefore has a probability $30/N$ of being chosen again. Equating to the sample (assuming a representative sample) we have $\frac{30}{N} = \frac{3}{30}$ so $N = 300$. (②).

**A21. Answer:** ③

Total number of students
= 20 × 0.3 + 20 × 1.3 + 10 × 2 + 10 × 1.6 + 40 × 0.8 = 100.
Upper quartile: (100 - $x$) × 0.8 = 25 → $x = 68.75$
Mean = 1/100 × (20 × 0.3 × 10 + 20 × 1.3 × 30 + 10 × 2 × 45 + …
Mode = 40 to 50 … + 10 × 1.6 × 55 + 40 × 0.8 × 80) = 51.8
Since mean > mode, the data is positively skewed (or by guessing)
Median: uppermost bin contains 32 students, 2nd uppermost bin contains 16 students (total: 48). Middle bin contains 20 students, need 2 more to get the median → median = 50 - 2/20 = 49. (③)
Mean of binomial $= np = 45$, which does not match the found mean.

**A22. Answer:** ④




Evaluating P(same state after $N + 1$),

$P_{N+1} = P_N(1 - p) + p(1 - P_N) = p + (1 - 2p)P_N$. (④)

**A23. Answer:** ③

With the old machine, 90% of the time the profit is £1; 5% of the time the profit is -£4 (loss), 5% of the time the profit is -£9 (loss), so the weighted average profit per part is $(0.9 \times 1 + 0.05 \times -4 + 0.05 \times -9) =$ £0.25 $\times$ number of parts.

With the new machine, we have full profit with an upfront cost, so that the profit is £1 $\times$ number of parts - £10000.

Solving the inequality $n - 10000 > 0.25n$ gives $n > 13333.33\ldots$ (③)

**A24. Answer:** ②

$$\begin{aligned}
P(X \mid Y) &= P(X \mid r_{\text{Wed}})\, P(r_{\text{Wed}} \mid Y) + P(X \mid r_{\text{Wed}}')\, P(r_{\text{Wed}}' \mid Y) \\
&= \alpha \times \alpha + \beta \times (1 - \alpha) \\
&= \alpha^2 + \beta - \alpha\beta
\end{aligned}$$

$$\begin{aligned}
P(X \mid Y') &= P(X \mid r_{\text{Wed}})\, P(r_{\text{Wed}} \mid Y') + P(X \mid r_{\text{Wed}}')\, P(r_{\text{Wed}}' \mid Y') \\
&= \alpha \times \beta + \beta \times (1 - \beta) \\
&= \alpha\beta + \beta - \beta^2
\end{aligned}$$

Therefore, $P(X \mid Y) - P(X \mid Y') = \alpha^2 + \beta^2 - 2\alpha\beta = (\alpha - \beta)^2$. (②)

## Section B

### Question B1

a. i) The first ten squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 - none of which end in a digit 3.

Adding a multiple of 10 to a single-digit number and squaring does not affect the units digit.

Therefore, all positive integers square to a number that does not end in the digit 3.

ii) Let the four consecutive integers be $n(n + 1)(n + 2)(n + 3)$.

$$= (n(n + 3))((n + 1)(n + 2))$$
$$= (n^2 + 3n)(n^2 + 3n + 2)$$
$$= (n^2 + 3n)(n^2 + 3n) + 2(n^2 + 3n)$$
$$= (n^2 + 3n)^2 + 2(n^2 + 3n)$$
$$= (n^2 + 3n + 1)^2 - 1$$

Since no two square numbers differ by 1 (other than 0 and 1 but $n^2 + 3n + 1 > 1$ for $n > 0$), this cannot be equal to a square number.

iii) $U_4 = 1 + 2 + 6 + 24 = 33$ which ends in the digit 3.

When $n \geq 5$, the value of $n!$ has a factor of 10 due to 5! having factors of 5 and 2, so adding them to 33 does not change the units digit.

Therefore by the result in part a.i), $U_n$ for $n \geq 5$ ends in the digit 3 and so cannot be a square number.

b. Let $(m + 1) = (n + 1)^2$, so that $m + 1$ and $n + 1$ have the same prime factors.
It follows that $m = (n + 1)^2 - 1 = n^2 + 2n = n(n + 2)$.
Suppose that $n + 2 = 2^k$, so that $m = 2^k n$. Then $m$ and $n$ have the same prime factors since $m$ is already even and has 2 as a prime factor.
Since any such $m$ and $k$ can be chosen, the set of such pairs $(m, n)$ is infinite.

c. Assume initially that

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} \leq 0.$$

It follows algebraically that

$$\frac{x+a}{\sqrt{x^2+a^2}} \leq \frac{x+b}{\sqrt{x^2+b^2}}$$

$$\frac{x^2+2ax+a^2}{x^2+a^2} \leq \frac{x^2+2bx+b^2}{x^2+b^2}$$ (both sides positive → can square both sides)

$$\frac{a}{x^2+a^2} \leq \frac{b}{x^2+b^2}$$ (can divide both sides by $2x$ as $x > 0$)

$$a(x^2 + b^2) \leq b(x^2 + a^2) \Rightarrow ax^2 - bx^2 + ab^2 - a^2b \leq 0 \Rightarrow (a - b)(x^2 - ab) \leq 0$$

However, since $a > b$ and $x^2 > ab$, the factors $a$ - $b$ and $x^2$ - $ab$ are both positive and therefore there is a contradiction with the RHS. It follows that the original assertion is false, and so $\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} > 0$.

**Question B2**

a. i) Assume that $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2} = \frac{p}{q}$ for some integers $p$ and $q$ with no common factors. It follows that $2q^3 = p^3$.

$\rightarrow p$ has a factor of 2 - let $p = 2m$ for some integer $m$
$\rightarrow q^3 = p^3/2 = 4m^3$
$\rightarrow q$ has a factor of 2
$\rightarrow p$ and $q$ have a common factor of 2 - this is a contradiction
$\rightarrow \sqrt[3]{2}$ must be irrational.

ii) Assume that there exists some real $x$ such that $\sin x + \sin \sqrt[3]{2}\, x = 2$.
Since both $-1 \leq \sin x \leq 1$ and $-1 \leq \sin \sqrt[3]{2}\, x \leq 1$, the only way for $\sin x + \sin \sqrt[3]{2}\, x = 2$ is if $\sin x = 1$ and $\sin \sqrt[3]{2}\, x = 1$ for the same value of $x$.

$\Rightarrow x = sin^{-1}1 + 2\pi n = \frac{\pi}{2} + 2\pi n = (2n + \frac{1}{2})\pi$ for any integer $n$, and

$\Rightarrow x = \frac{1}{\sqrt[3]{2}}(sin^{-1}1 + 2\pi m) = \frac{1}{\sqrt[3]{2}}(2m + \frac{1}{2})\pi$ for any integer $m$.

Therefore, there must be integers $m$ and $n$ such that
$\sqrt[3]{2}(2n + \frac{1}{2}) = 2m + \frac{1}{2} \quad \Rightarrow \quad \sqrt[3]{2} = \frac{2m+\frac{1}{2}}{2n+\frac{1}{2}} = \frac{4m+1}{4n+1}$
which implies that $\sqrt[3]{2}$ is rational, which is a known contradiction.
Therefore, $\sin x + \sin \sqrt[3]{2}\, x \neq 2$.

b. $n^3 + 5n = n^3 - n + 6n = n(n^2 - 1) + 6n = (n - 1)n(n + 1) + 6n.$

Since $(n - 1)n(n + 1)$ is a product of three consecutive integers, at least one of the factors must be a multiple of 3, and at least one of the factors must have a factor of 2.

Therefore the product must be a multiple of 6, i.e. let $(n - 1)n(n + 1) = 6k$.
Since $n^3 + 5n = 6k + 6n = 6(k + n)$ for integers $k$ and $n$, this is a multiple of 6.

c. $f(x) = (ax + c)^2 + (bx + d)^2$ is the sum of two squares, which can never be negative. Therefore, $f(x) \geq 0$ for all $x$ and so the discriminant of $f(x)$ in quadratic form cannot be positive (since $f(x)$ cannot have two roots).

$$f(x) = (ax + c)^2 + (bx + d)^2 = a^2x^2 + 2acx + c^2 + b^2x^2 + 2bdx + d^2$$
$$f(x) = (a^2 + b^2)x^2 + 2(ac + bd)x + (c^2 + d^2)$$

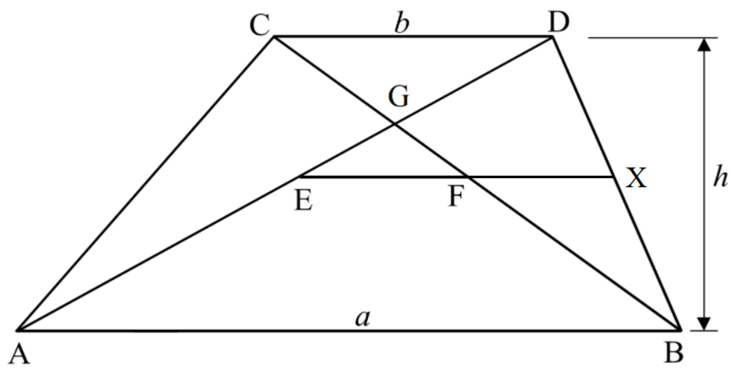
Discriminant $= 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2) \leq 0$
$\Rightarrow (ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$.

(This is a two-dimensional version of the ‘Cauchy-Schwarz inequality’.)

**Question B3**

a. i) ***Method 1 - Using similar triangles***

Extend line *EF* to intersect side *BD* at the point *X* as shown:




By the definition of the midpoint, $|ED| = \frac{1}{2}|AD|$ and $|BF| = \frac{1}{2}|BC|$.

$\triangle DAB$ and $\triangle DEX$ are similar, so $|EX| = \frac{a}{2}$.

$\triangle CDB$ and $\triangle FXB$ are similar, so $|FX| = \frac{b}{2}$.

$|EF| = |EX| - |FX| = \frac{a}{2} - \frac{b}{2} = \frac{a-b}{2}$.

***Method 2 - Using vectors***

Write the vector $\overrightarrow{EF}$ in two ways by considering the paths $E \to A \to B \to F$ and $E \to D \to C \to F$ separately:

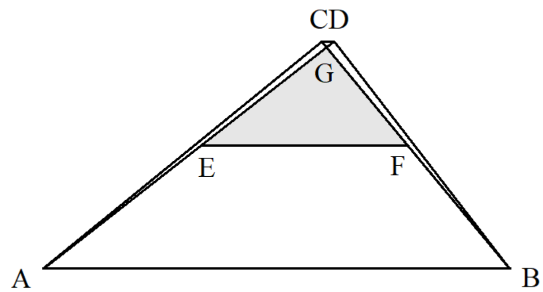
$$\overrightarrow{EF} = -\frac{1}{2}\overrightarrow{AD} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \quad \text{and} \quad \overrightarrow{EF} = \frac{1}{2}\overrightarrow{AD} - \overrightarrow{CD} - \frac{1}{2}\overrightarrow{BC}$$

Adding these two equations together, $2\overrightarrow{EF} = \overrightarrow{AB} - \overrightarrow{CD}$.

Taking the magnitudes of both sides since $AB \parallel CD$:

$$\left|\overrightarrow{EF}\right| = \frac{1}{2}\left|\overrightarrow{AB} - \overrightarrow{CD}\right| = \frac{1}{2}(a-b).$$

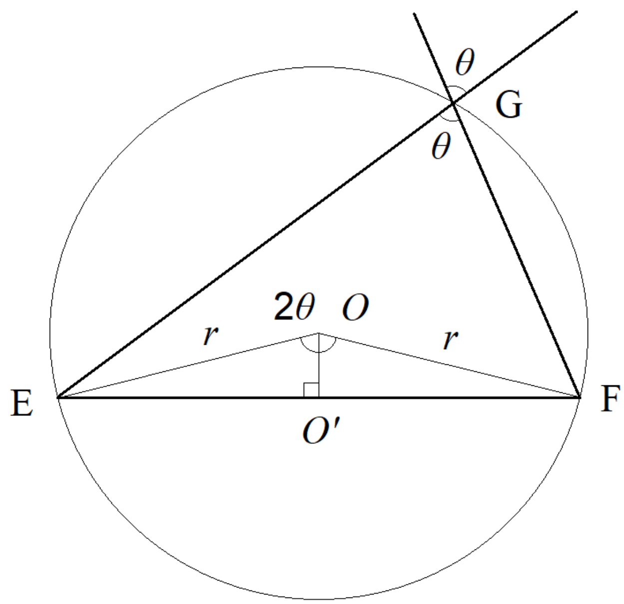
ii) When $b$ is very small, the trapezium collapses to a triangle:




As $b \rightarrow 0$, $\triangle EFG$ has base $EF = \frac{a-b}{2} = \frac{a}{2}$ and height $\frac{h}{2}$.
Therefore its area is approximately $\frac{1}{2} \cdot \frac{a}{2} \cdot \frac{h}{2} = \frac{ah}{8}$.

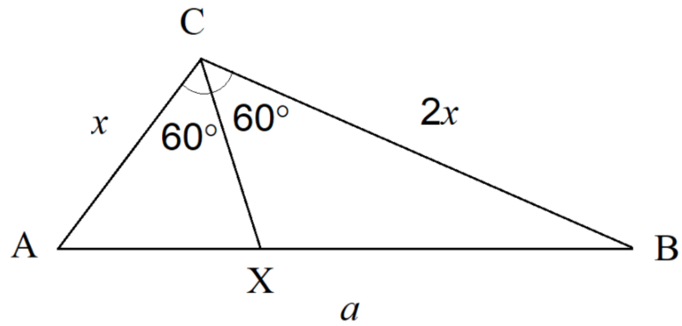
iii) When the trapezium is symmetric about the vertical line through *G*, the congruent triangle pairs are (any two from)
{*ACG* and *BDG*}, {*ACD* and *BCD*}, and {*ABC* and *ABD*}.

iv) Consider triangle *EFG* and the circle:




Opposite angles at $G \rightarrow \angle EGF = \theta$ ($= \angle CGD$). The angle at the circumference is twice the angle at the centre $\rightarrow \angle EOF = 2\angle EGF = 2\theta$.
Let $O'$ be the midpoint of chord $EF \rightarrow EO' = ½ EF = \frac{1}{2} \cdot \frac{a-b}{2} = \frac{a-b}{4}$.
$OO'$ bisects chord $EF$, $\angle EO'O = 90°$ and $\angle EOO' = ½ \angle EOF = \theta$.
From trigonometry in $\triangle OEO'$, $\sin\theta = \frac{|EO'|}{|EO|} = \frac{a-b}{4r} \Rightarrow r = \frac{a-b}{4\sin\theta}$.

b. Consider triangle *ABC*. Let $|CB| = x$ and $|AC| = 2x$:




Sine rule in $\triangle BCX$:

$\frac{2x}{\sin BXC} = \frac{|BX|}{\sin 60^o}$

Sine rule in $\triangle AXC$:

$\frac{x}{\sin AXC} = \frac{|AX|}{\sin 60^o}$

$\frac{2x}{\sin(180^o - AXC)} = \frac{|BX|}{\sin 60^o}$ ($\angle BXC = 180^o - \angle AXC$)

$\frac{2x}{\sin AXC} = \frac{|BX|}{\sin 60^o}$ ($\sin(180^o - \theta) = \sin\theta$)

$\Rightarrow \frac{|BX|}{|AX|} = 2 \Rightarrow |BX| = 2|AX|$ (divide first eqn. by second eqn.)

$|AX| + |BX| = |AB| = a \Rightarrow |AX| = \frac{a}{3}$ and $|BX| = \frac{2a}{3}$. (line segment ratios)

Cosine rule in $\triangle ABC$:

$a^2 = (2x)^2 + x^2 - 2(2x)x\cos 120^\circ$

$a^2 = 7x^2 \Rightarrow x^2 = \frac{1}{7}a^2$

Cosine rule in $\triangle ACX$:

$\frac{a^2}{9} = x^2 + |CX|^2 - 2x\,|CX|\cos 60^o$

$\frac{a^2}{9} = \frac{a^2}{7} + |CX|^2 - \frac{a}{\sqrt{7}}|CX|$

$a^2 = \frac{63}{2}|CX|(\frac{a}{\sqrt{7}} - |CX|)$

Cosine rule in $\triangle BXC$:

$\frac{4a^2}{9} = (2x)^2 + |CX|^2 - 2(2x)\,|CX|\cos 60^o$

$\frac{4a^2}{9} = \frac{4a^2}{7} + |CX|^2 - \frac{2a}{\sqrt{7}}|CX|$

$a^2 = \frac{63}{8}|CX|(\frac{2a}{\sqrt{7}} - |CX|)$

Equating these two:

$\frac{63}{8}|CX|(\frac{2a}{\sqrt{7}} - |CX|) = \frac{63}{2}|CX|(\frac{a}{\sqrt{7}} - |CX|)$ (divide by $|CX| \neq 0$)

$\frac{2a}{\sqrt{7}} - |CX| = \frac{4a}{\sqrt{7}} - 4|CX| \Rightarrow 3\,|CX| = \frac{2a}{\sqrt{7}} \Rightarrow |CX| = \frac{2\sqrt{7}a}{21}$.

**Section C**

**Question C1**

a. i) Use difference of two squares, or by inspection:

$$\frac{2t}{t^2-1} = \frac{2t}{(t+1)(t-1)} = \frac{1}{t+1} + \frac{1}{t-1}$$

$\Rightarrow A = B = 1.$

ii) Factorise the denominator:

$= t^4 + 1 = t^4 + 2t^2 + 1 - 2t^2 = (t^2 + 1)^2 - (\sqrt{2}t)^2$

$= (t^2 + \sqrt{2}t + 1)(t^2 - \sqrt{2}t + 1)$

$\Rightarrow C = \sqrt{2},\ D = 1,\ E = \ -\sqrt{2},\ F = 1.$

Cross-multiply and equate numerators:

$(Pt + Q)(t^2 - \sqrt{2}t + 1) + (Rt + S)(t^2 + \sqrt{2}t + 1) = 1$

$(P + R)t^3 + (Q - P\sqrt{2} + S + R\sqrt{2})t^2 + (P - Q\sqrt{2} + R + S\sqrt{2})t + (Q + S) = 1.$

Equating coefficients:

$P + R = 0$

$Q + S + \sqrt{2}(R - P) = 0$

$P + R + \sqrt{2}(S - Q) = 0$

$Q + S = 1$

Combining the (1st and 3rd) and (2nd and 4th) equations,

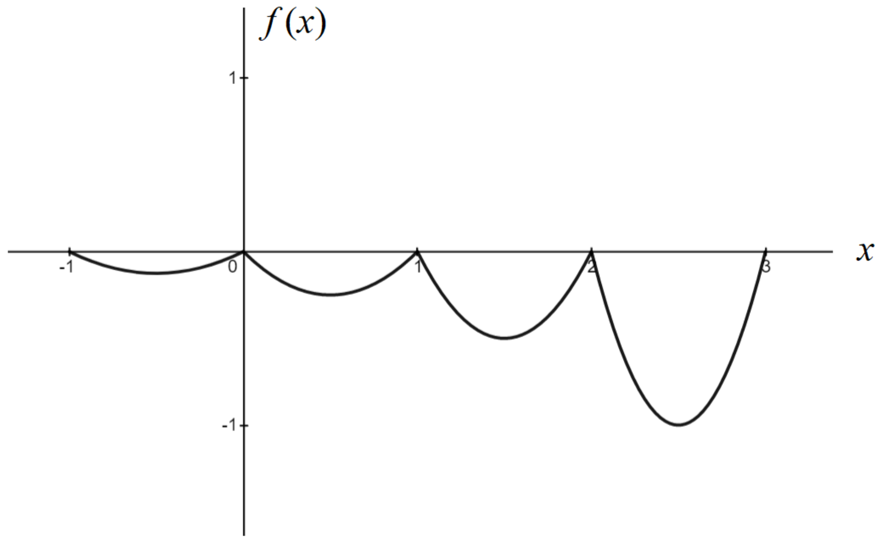
$Q = S$ and $R = P$

$\Rightarrow Q = S = \frac{1}{2}$ and $R = P = 0.$

Or, in the original form,

$$\frac{1}{t^4+1} = \frac{\frac{1}{2}}{t^2+\sqrt{2}t+1} + \frac{\frac{1}{2}}{t^2-\sqrt{2}t+1}.$$

b. i) For the interval $1 \le x < 2$, the value of the function is twice the value of the corresponding value in the interval $0 \le x < 1$, which repeats for every integer interval. Sketching the graph of $f(x) = x(x - 1) = (x - \frac{1}{2})^2 - \frac{1}{4}$ the curve can be translated and scaled to get the graph elsewhere:




(Open circles are not required at the integers because the function is always defined in any given interval.)

**Alternative interpretation:** imagine the given relation as an inductive sequence, i.e. $u_{n+1} = 2u_n$. It is then clear that the function doubles at every integer step.

ii) The function has been translated by 2 units to the right, then scaled by a factor of 4, i.e. $4f(x - 2) = 4(x - 2)(x - 3) = 4x^2 - 20x + 24$.

iii) The next interval of the curve will intersect the line $y = -\frac{16}{9}$.

The equation of the curve in this interval is $f(x) = 8(x - 3)(x - 4)$

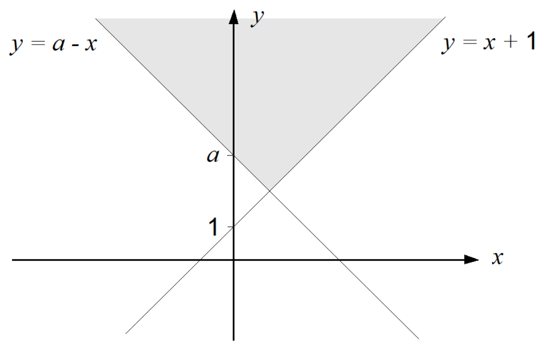
$\Rightarrow 8(x - 3)(x - 4) = -\frac{16}{9}$

$\Rightarrow 8x^2 - 56x + \frac{880}{9} = 0$

$\Rightarrow x = \frac{10}{3}$ and $x = \frac{11}{3}$.

The lesser of these values will ensure $f(x) \geq -\frac{16}{9}$ for all $x \leq m$, so $m = \frac{10}{3}$.

c. i) Writing the equations as $y \geq -x + a$ and $y \geq x + 1$, the region satisfying both inequalities lies above these lines:




ii) If $x + ay = c \Rightarrow y = \frac{c-x}{a}$ for the smallest possible $c$, then the values of $x + ay$ are constant along lines of gradient $-\frac{1}{a}$. It follows that the most extreme value of $x + ay$ will be located at the bottom-most point of the shaded region, where there is only one intersection with this line.

Solving for the intersection: $y = a - x$ and $y = x + 1$
$\Rightarrow x = \frac{a-1}{2}$ and $y = \frac{a+1}{2}$.

Substituting these into the given condition,
$x + ay = 7 \Rightarrow \left(\frac{a-1}{2}\right) + a\left(\frac{a+1}{2}\right) = 7 \Rightarrow a^2 + 2a - 15 = 0$
$\Rightarrow a = 3$ or $a = -5$.

If $a > 0$, then the line $y = \frac{7-x}{a}$ has a negative gradient. If the value of 7 were replaced with a larger number, the line would move upwards (into the shaded region), therefore making the value of 7 the **smallest** possible value of $x + ay$. However, if $a < 0$, then the line $y = \frac{7-x}{a}$ has a positive gradient. If the value of 7 were replaced with a larger number, the line would move downwards (away from the shaded region), therefore making the value of 7 the **largest** possible value of $x + ay$.

Since it is given that 7 is the **smallest** value, conclude that $a > 0$ and hence the value of $a$ must be **only** $a = 3$.

**Question C2**

a. $4 + \frac{4-x^2}{x^2-2x} = 4 + \frac{(2-x)(2+x)}{x(x-2)} = 4 - \frac{2+x}{x} = 4 - \frac{2}{x} - 1 = 3 - \frac{2}{x}$ or $\frac{3x-2}{x}$.

b. Binomial expansion → powers of $(2x)^m$ and $(3x^{-2})^n$ must add up to 12
Constant term → product $(2x)^m\,(3x^{-2})^n$ must have degree 0 in $x$
By inspection, constant is ${}^{12}C_8 \times (2x)^8 \times (3x^{-2})^4 = 495 \times 256 \times 81 = 10{,}264{,}320$.

c. The easier to spot factor is $x - y$ from the last two terms, so divide this out from the rest (use synthetic division in $x$, treating $y$ as a constant):

$$\begin{array}{c|ccc} y & 1 & 2y & -3y^2 \\ & & y & 3y^2 \\ \hline & 1 & 3y & 0 \end{array}$$

Therefore the remaining factor is $x + 3y$ and so
$x^2 + 2xy - 3y^2 + 4x - 4y = (x - y)(x + 3y) + 4(x - y)$
$= (x - y)(x + 3y + 4)$.

d. Factorise: $y = \frac{x^3(x-2)}{(x-2)(x-1)(x^2+ax+\frac{1}{2}a^2)} = \frac{x^3}{(x-1)(x^2+ax+\frac{1}{2}a^2)}$

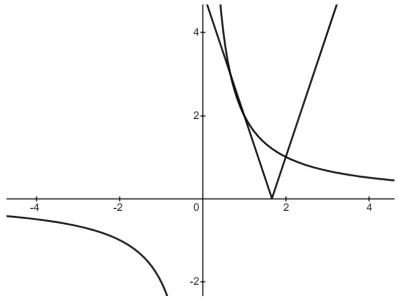
Vertical asymptotes: The obvious asymptote is $x = 1$. Check if any asymptotes could arise from the quadratic factor by checking the discriminant:
$a^2 - 4(1)(\frac{1}{2}a^2) = -a^2 < 0$. Since the discriminant is negative, this factor is never zero, so cannot produce an asymptote.

Horizontal asymptotes: Both top and bottom have the same degree, both with coefficient 1, so the horizontal asymptote is $y = 1$.

Therefore the asymptotes are $x = 1$ and $y = 1$ only.

**Note:** $x = 2$ is **not** an asymptote as it is a removable discontinuity which cancels with the numerator.

e. Sketch graphs of the two functions. The solution is if the line is below the curve.




By inspection, $x$ cannot be negative since $\frac{2}{x} < 0 < |5 - 3x|$.

Check for solutions in the first part of the line i.e. $0 < x < \frac{5}{3}$:

$5 - 3x < \frac{2}{x} \Rightarrow -3x^2 + 5x - 2 < 0 \Rightarrow 0 < x < \frac{2}{3}$ or $1 < x < \frac{5}{3}$.

Check for solutions in the second part of the line i.e. $x > \frac{5}{3}$:

$3x - 5 < \frac{2}{x} \Rightarrow 3x^2 - 5x - 2 < 0 \Rightarrow \frac{5}{3} < x < 2$.

The union of these intervals solves the full inequality, so
$0 < x < \frac{2}{3}$ or $1 < x < 2$.

***Alternative method:*** Solving without the aid of a sketch requires splitting into the absolute value part into two cases, as well as two further cases to account for the fact that if $x < 0$ then multiplying by $x$ will flip the inequality. Combining the intervals correctly is tricky and leaves lots of room for errors.

f. Observe that $\sqrt{3 - 2\sqrt{2}} = \sqrt{(\sqrt{2})^2 + 1^2 - 2(\sqrt{2})(1)} = \sqrt{(\sqrt{2} - 1)^2} = \sqrt{2} - 1$.

***Alternative methods:*** assume that the answer will contain $\sqrt{2}$, so let
$3 - 2\sqrt{2} = (a + b\sqrt{2})^2 = (a^2 + 2b^2) + 2ab\sqrt{2}$ and equate components; or simply evaluate on a calculator and notice the decimal pattern of $\sqrt{2}$.

g. To rationalise, use the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
Let $a = \sqrt[3]{2}$ and $b = \sqrt[3]{3}$ and multiply top and bottom by the second factor:

$$\frac{1}{\sqrt[3]{2} + \sqrt[3]{3}} \times \frac{(\sqrt[3]{2})^2 - (\sqrt[3]{2})(\sqrt[3]{3}) + (\sqrt[3]{3})^2}{(\sqrt[3]{2})^2 - (\sqrt[3]{2})(\sqrt[3]{3}) + (\sqrt[3]{3})^2} = \frac{\sqrt[3]{4} - \sqrt[3]{6} + \sqrt[3]{9}}{2 + 3} = \frac{\sqrt[3]{4} - \sqrt[3]{6} + \sqrt[3]{9}}{5}.$$

**Question C3**

a. i) Vector $\mathbf{u} + \mathbf{v}$ represents the third side of a triangle formed by $\mathbf{u}$ and $\mathbf{v}$ joined tip to tail. Since the side length of any side in a triangle is never more than the sum of its other two side lengths (triangle inequality), the inequality follows.

ii) $\mathbf{u}$ and $\mathbf{v}$ must be parallel / collinear, and $\mathbf{u}$ must be in the opposite direction to $\mathbf{v}$

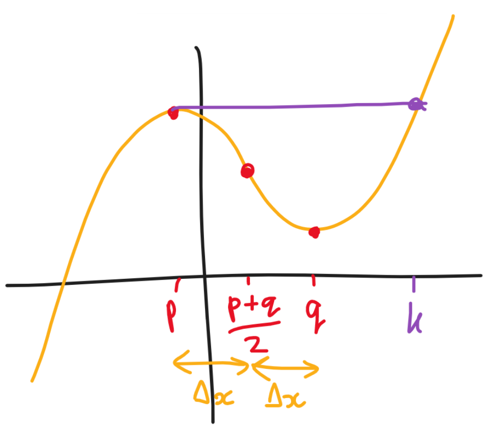
$\rightarrow$ $\mathbf{u}$ and $\mathbf{v}$ must be antiparallel.

b. If composition is commutative then $f(g(x)) = g(f(x))$ for all $x > 0$.

$2 - ln(ke^{-x}) = ke^{-(2 - ln\,x)} \Rightarrow 2 - ln\,k + x = ke^{-2}x$

By inspection, $k = e^{2}$ as then both sides are equal to $x$ for all $0 < x < e^2$..

c. Since $a > 0$, the general shape of the cubic is as shown,




By considering the symmetry of the parabola $P'(x)$, it is clear that $\Delta x = \frac{q-p}{2}$.

Consider translating the cubic curve vertically by $-P(p)$, so that the maximum point becomes a repeated root. There exists another root $x = k$ such that $P(p) = P(k)$ (purple line).

The cubic can therefore be written (by undoing the translation and then using factor theorem) as $P(x) = a(x - p)^{2}(x - k) + P(k)$.

$\rightarrow P'(x) = 2a(x - p)(x - k) + a(x - p)^{2} = a(x - p)(3x - (2k + p))$

The second factor must correspond to the minimum point $x = q$, so

$\rightarrow 3q - (2k + p) = 0 \Rightarrow k = \frac{3q-p}{2} = p + \frac{3}{2}(q - p) = p + 3\,\Delta x.$

From the definition, $P(p) = P(k) = P(p + 3\,\Delta x)$.

d. Square both sides and rearrange for the remaining square root term:

$$2\sqrt{\left(x - \frac{1}{x}\right)\left(1 - \frac{1}{x}\right)} = x^2 - x - 1 + \frac{2}{x}$$

Square both sides again, grouping the quadratic factor on the RHS:

$$4\left(x - \frac{1}{x}\right)\left(1 - \frac{1}{x}\right) = (x^2 - x - 1)^2 + \frac{4}{x^2} + 4x - 4 - \frac{4}{x}$$

$$4\left(x - 1 - \frac{1}{x} + \frac{1}{x^2}\right) = (x^2 - x - 1)^2 + \frac{4}{x}(x^2 - x - 1) + \frac{4}{x^2}$$

$$\frac{4}{x}(x^2 - x - 1) + \frac{4}{x^2} = (x^2 - x - 1)^2 + \frac{4}{x}(x^2 - x - 1) + \frac{4}{x^2}$$

$$(x^2 - x - 1)^2 = 0$$

$$x^2 - x - 1 = 0$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{5}}{2}.$$

The negative solution produces a positive LHS and a negative RHS, so cannot be a solution, so reject it, leaving the only solution as $x = \frac{1+\sqrt{5}}{2}$.

(Interestingly, the numerical solver on the CASIO fx-991EX (the standard UK exam calculator) almost always fails to solve this equation in its original form, Alternatively, expand to a quartic and use the solver and recognise the decimal pattern as the golden ratio $\phi = 1.618... = \frac{1+\sqrt{5}}{2}$.)

**Alternative method:** manipulate before expanding.

Multiply both sides by $\sqrt{x - \frac{1}{x}} - \sqrt{1 - \frac{1}{x}}$ (conjugate of the LHS):

$$\left(x - \frac{1}{x}\right) - \left(1 - \frac{1}{x}\right) = x\left(\sqrt{x - \frac{1}{x}} - \sqrt{1 - \frac{1}{x}}\right)$$ (difference of two squares)

$$\sqrt{x - \frac{1}{x}} - \sqrt{1 - \frac{1}{x}} = 1 - \frac{1}{x}$$

Add this equation together with the original to cancel one of the radicals:

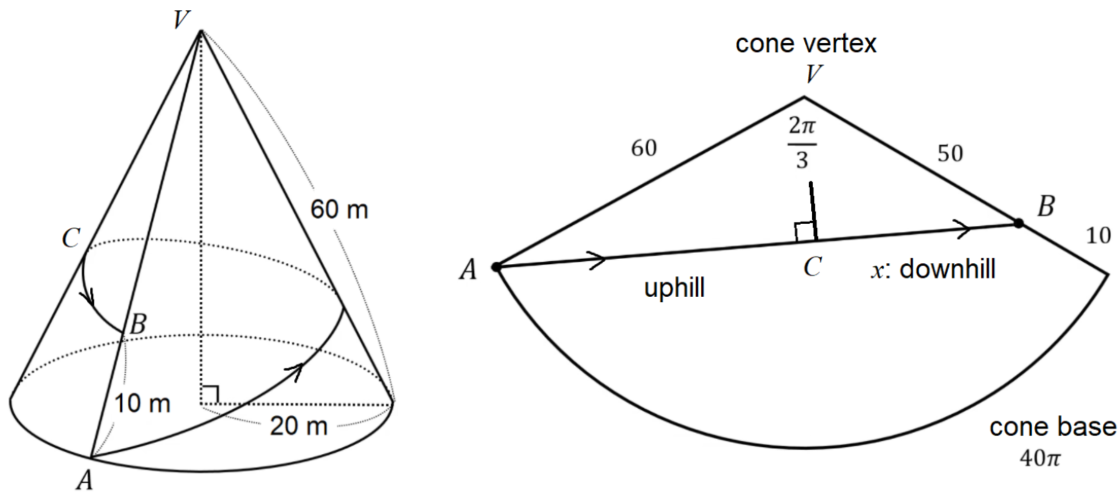
$$2\sqrt{x - \frac{1}{x}} = 1 + x - \frac{1}{x}$$

Substitute $u = \sqrt{x - \frac{1}{x}}$ and this equation becomes $2u = 1 + u^2$, which is easily solved for $u = 1$ and then $x = \frac{1+\sqrt{5}}{2}$, rejecting the negative solution as before.

**Section D**

**Question D1**

a. Consider the net of the cone, which unwraps into a circular sector - the shortest distance between two points becomes a straight line.




The arc length of the sector is the circumference of the base $= 2\pi \times 20 = 40\pi$.

The angle of the sector satisfies $60\,\theta = 40\pi \Rightarrow \theta = \frac{2}{3}\pi = 120^o$.

Cosine rule: $AB = \sqrt{50^2 + 60^2 - 2 \times 50 \times 60 \times cos(120)} = 10\sqrt{91}$.

The downhill section begins at the point on the line for which the radius passing through the point is perpendicular to the line AB. Let this point be C, and the downhill distance BC $= x$, so AC $= 10\sqrt{91} - x$. Let the distance between C and the vertex be $h$.

Pythagoras in triangle ACV: $(10\sqrt{91} - x)^2 + h^2 = 60^2$

Pythagoras in triangle BCV: $x^2 + h^2 = 50^2$

$\Rightarrow (10\sqrt{91} - x)^2 - x^2 = 1100$ (subtract 2nd eqn from 1st)

$\Rightarrow 20\sqrt{91}x = 8000$

$\Rightarrow x = \frac{400}{\sqrt{91}}$.

***Alternative Solution:*** other ways of finding the lengths using e.g. similar triangles and extending lines are possible.

b. The slant length from the highest point on the track *C* to the vertex *V* is

$$VC = \sqrt{50^2 - \left(\frac{400}{\sqrt{91}}\right)^2} = \frac{150\sqrt{273}}{91} = 27.24 \text{ m}$$

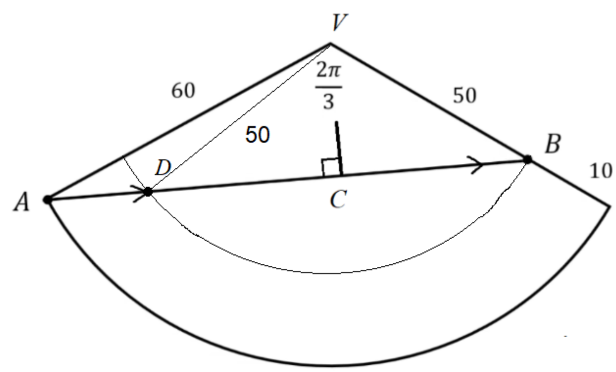
Using trigonometry in the cone, the vertical distance of $C$ from $V$ is

$VC\, cos(sin^{-1} \frac{1}{3}) = 25.6776$ m

The full vertical height of the cone is $\sqrt{60^2 - 20^2} = 40\sqrt{2}$ m

Therefore the vertical distance of $C$ from the base is $40\sqrt{2} - 25.6776$ m
= 30.9 m (3 s.f.)

c. The train will be 'above' $B$ in height when the line representing its journey on the flat sector passes the circle of radius 50, centred at the peak. Let this point be $D$.




Angle $VAC = \cos^{-1} \frac{60^2 + (10\sqrt{91})^2 - 50^2}{2 \times 60 \times 10\sqrt{91}} = \cos^{-1} \frac{17}{2\sqrt{91}} = 0.4712$ rad (cosine rule)

Angle $ADV = \pi - \sin^{-1} (\frac{60}{50} \times \sin (\cos^{-1} \frac{17}{2\sqrt{91}})) = 2.5655$ rad (sine rule, obtuse)

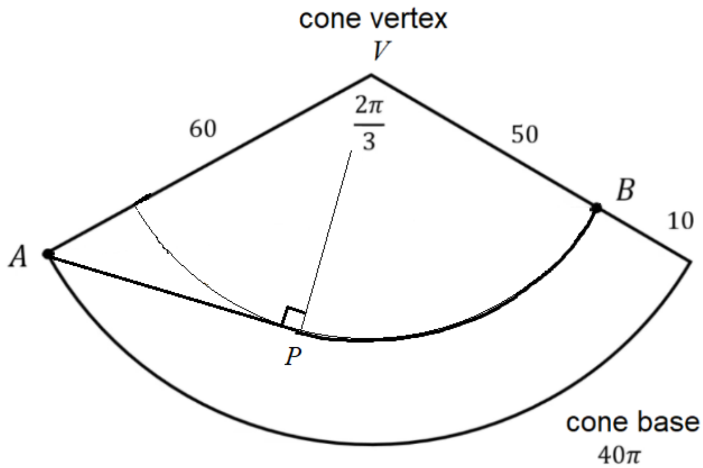
Angle $AVD = \pi - \angle ADV - \angle VAC = 0.10487$ rad (angles in triangle sum to $\pi$ rad)

Length $AD = \sqrt{60^2 + 50^2 - 2 \times 60 \times 50 \times cos(0.10487...)} = 11.5311...$ m (cosine rule)

Therefore, the proportion of the track beyond this point is

Fraction = $\frac{|DB|}{10\sqrt{91}} = \frac{10\sqrt{91} - |AD|}{10\sqrt{91}} = \frac{80}{91} \approx 0.879... \approx 87.9\%$.

d. In this case, the path cannot pass the circle defined by the height of $B$. The shortest path will therefore be a straight line to the point of tangency *P* with the circle, then follows the 'horizontal' (circular arc) to *B*.




Length $AP = \sqrt{60^2 - 50^2} = 10\sqrt{11}$

Angle $PVB = \frac{2\pi}{3} - \cos^{-1}\frac{50}{60} = 1.5087\ldots$ rad

Arc length $PB = 50\ \angle PVB = 75.434\ldots$ m

Total track length $= |AP| + |PB| \approx 108.6$ m.

*(**Sense check:** this is about 14% longer than shortest possible length $10\sqrt{91}$.)*

**Question D2**

a. i) $= (\cos x + \cos y)^2 + (\sin x - \sin y)^2$

$= \cos^2 x + \cos^2 y + 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y$

$= 2 + 2(\cos x \cos y - \sin x \sin y)$ (Pythagorean identity)

$= 2(1 + \cos(x + y))$ (cosine addition formula)

$= 4(\frac{1+\cos(x+y)}{2})$

$= 4 \cos^2\left(\frac{x+y}{2}\right)$. (cosine double-angle formula:

let $2\theta = x + y \rightarrow \cos 2\theta = 2 \cos^2 \theta - 1 \rightarrow \cos^2 \theta = \frac{1 + \cos 2\theta}{2}$)

***Alternative method:*** prove from right to left instead - however this requires spotting to replace 2 with $\sin^2 x + \cos^2 x + \sin^2 y + \cos^2 y$: the easiest way to deduce this is a ‘meet-in-the-middle’ method.

ii) $\Rightarrow\ 4 \cos^2\left(\frac{x+y}{2}\right) = 1$

$\Rightarrow\ \cos\left(\frac{x+y}{2}\right) = \pm \frac{1}{2}$

$\Rightarrow\ \frac{x+y}{2} = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\} + 2\pi n$

$\Rightarrow\ x + y = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} + 2\pi n$

$\Rightarrow\ y = \frac{(6n \pm 2)\pi}{3} - x$ for all $n \in \mathbb{Z}$.

All gradients are -1 $\rightarrow$ parallel lines.

b. i) $\sqrt{1-\frac{3}{4-x^3}} = \sqrt{\frac{1-x^3}{4-x^3}} = \frac{\sqrt{1-x^3}}{2\sqrt{1-\frac{1}{4}x^3}} = \frac{1}{2}\left(1-x^3\right)^{1/2}\left(1-\frac{1}{4}x^3\right)^{-1/2}$

$\approx \frac{1}{2}\left(1-\frac{1}{2}x^3-\frac{1}{8}x^6-\ ...\right)\left(1+\frac{1}{8}x^3+\frac{3}{128}x^6+...\right)$

$\approx \frac{1}{2}\left(1-\frac{3}{8}x^3-\frac{21}{128}x^6-...\right)$

$\approx \frac{1}{2}-\frac{3}{16}x^3-\frac{21}{256}x^6-...$

***Note:*** Expanding directly from $\sqrt{1-u}$ with $u=\frac{3}{4-x^3}$ (or similar) is **not** the correct approach, because the series for $u$ contains a nonzero constant term, which will generate a new infinite series when substituted back in.

ii) The approximations used were $\sqrt{1-x^3}$ ($|x^3| < 1 \rightarrow |x| < 1$) and $\sqrt{1-\frac{1}{4}x^3}$ ($|x^3/4| < 1 \rightarrow |x| < \sqrt[3]{4}$).

Since $1 < \sqrt[3]{4}$, the first approximation sets the combined interval so $|x| < 1 \rightarrow -1 < x < 1$.

iii) When evaluating the binomial approximation, it can be seen that all terms in the first factor (after the 1) are negative, and all terms in the second factor are positive. It appears that all terms in the product will therefore be negative. The terms therefore form a decreasing sequence and so the approximation must be converging from above:
→ the integral of the approximation will be an **overestimate**.

(Proving that all terms are negative rigorously is difficult, but it is verifiable for the next individual term, which always dominates when $|x| < 1$.)

iv) Since $f(x) \approx \frac{1}{2}-\frac{3}{16}x^3-\frac{21}{256}x^6 \rightarrow \frac{1}{2}-f(x) \approx \frac{3}{16}x^3+\frac{21}{256}x^6$
This value is small when $x$ is small, therefore it is safe to assume $\sin x = x$ (small angle approximation) → $\sin^{-1} x = x$ in the region of the solution:

$\Rightarrow \frac{3}{16}x^3+\frac{21}{256}x^6 = 1-720x^3$

$\Rightarrow \frac{21}{256}x^6+\frac{11523}{16}x^3-1=0$

$\Rightarrow x^3 = 0.001388527074$ (reject larger solution; outside interval)

$\Rightarrow x = 0.1115624691$ (taking cube root preserves precision)

(True solution: exact to 11 d.p. - the approximation is very good!)

**Question D3**

a. Let the equation of the curve be $y = mx + c$.

Intersections: $x^3 - x^4 = mx + c \Rightarrow x^4 - x^3 + mx + c = 0$

As $m$ and $c$ vary, the line will intersect the curve either 0, 1, 2, 3 or 4 times, corresponding to the number of real solutions to this quartic.
In the case of the common tangent, there will be 2 solutions, requiring two repeated roots. Therefore, the quartic must factorise as $(x - A)^2(x - B)^2$, where $A$ and $B$ are the $x$-coordinates of the tangent points.

$\Rightarrow (x^2 - 2Ax + A^2)(x^2 - 2Bx + B^2) = x^4 - x^3 + mx + c$

$\Rightarrow x^4 - (2A + 2B)x^3 + (A^2 + B^2 + 4AB)x^2 - (2A^2B + 2AB^2)x + (A^2B^2)$
$= x^4 - x^3 + mx + c$

Equate coefficients:

$x^3$: $-2A - 2B = -1 \Rightarrow A + B = \frac{1}{2}$

$x$: $-2A^2B - 2AB^2 = -2AB(A + B) = -AB = m \Rightarrow AB = -m$

$x^2$: $A^2 + B^2 + 4AB = (A + B)^2 + 2AB = \frac{1}{4} - 2m = 0 \Rightarrow m = \frac{1}{8}$

1: $A^2B^2 = (AB)^2 = m^2 = c \Rightarrow c = \frac{1}{64}$

Therefore the equation of the common tangent line is $y = \frac{1}{8}x + \frac{1}{64}$.

b. The curve intersects the circle at some point on the $x$-axis and two additional points symmetric about the $x$-axis (by symmetry in $y$).

Intersection on $x$-axis: $y = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Let the equation of the circle be $(x - A)^2 + y^2 = R^2 \Rightarrow y^2 = R^2 - (x - A)^2$.
Using the intersection found, $R = A - \frac{1}{2} \Rightarrow y^2 = -A + \frac{1}{4} - x^2 + 2Ax$

Substituting in, the intersections with the curve satisfy the equation

$$\Rightarrow \frac{1}{2}(-A + \frac{1}{4} - x^2 + 2Ax) = \frac{2x-1}{x-1}$$

$$\Rightarrow (-A + \frac{1}{4} - x^2 + 2Ax)(x - 1) = 4x - 2$$

$$\Rightarrow -x^3 + (2A + 1)x^2 + (-3A - \frac{15}{4})x + (A + \frac{7}{4}) = 0$$

The roots of this are $x = \frac{1}{2}$ and a repeated root at the other intersection.
Use synthetic division to factor out $x - \frac{1}{2}$ from the cubic:

$$\begin{array}{r|cccc} & -1 & 2A+1 & -3A-\frac{15}{4} & A+\frac{7}{4} \\ 1/2 & & -\frac{1}{2} & A+\frac{1}{4} & -A-\frac{7}{4} \\ \hline & -1 & 2A+\frac{1}{2} & -2A-\frac{7}{2} & 0 \end{array}$$

so the remaining roots are of the form $-x^2 + (2A + \frac{1}{2})x - 2A - \frac{7}{2} = 0$.

Since the roots must be repeated, the discriminant must be zero, so

$$\Rightarrow (2A + \frac{1}{2})^2 - 4(2A + \frac{7}{2}) = 0 \Rightarrow 4A^2 - 6A - \frac{55}{4} = 0 \Rightarrow A = \frac{11}{4} \text{ or } -\frac{5}{4}$$

Checking with the given graph, clearly $A > 0$, so $A = \frac{11}{4}$ and so $R = A - \frac{1}{2} = \frac{9}{4}$.

Therefore, the equation of the circle is

$(x - \frac{11}{4})^2 + y^2 = \frac{81}{16}$ or equivalently $\left(\frac{4x-11}{9}\right)^2 + \left(\frac{4y}{9}\right)^2 = 1$.

## Section E

### Question E1

a. i) Use implicit differentiation. Let $y = sin^{-1}x \Rightarrow x = sin\, y$.

Differentiate both sides with respect to $x$: $1 = \frac{d}{dx} sin\, y = cos\, y \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{cos\, y}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{cos(sin^{-1}x)}$ (replace $y = sin^{-1}x$)

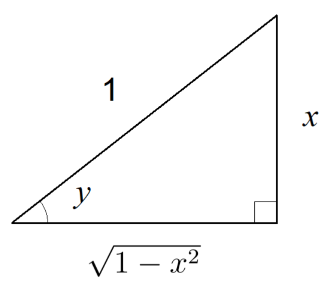
$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-sin^2(sin^{-1}x)}}$ (use $\sin^2 y + \cos^2 y = 1$)

$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-x^2}}$. (function-inverse cancels)

$y = \sin^{-1} x$ is an increasing function → reject negative gradient solution:

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

***Alternative method:*** use right-triangle to find $\cos(\sin^{-1} x)$. Justify rejecting -ve solution: $|y| \leq \frac{\pi}{2} \rightarrow \cos y > 0$.



a. ii) Using chain and product rules,

$$f'(x) = \sqrt{a^2 - x^2} + x \cdot \frac{d}{dx}\sqrt{a^2 - x^2} + a^2 \cdot \frac{d}{dx} sin^{-1} \frac{x}{a}$$

Using the result for $\frac{d}{dx}$ $\sin^{-1} x$ from above, chain rule due to factor of $\frac{1}{a}$:

$$f'(x) = \sqrt{a^2 - x^2} + x \cdot \frac{-2x}{2\sqrt{a^2-x^2}} + a \cdot \frac{1}{\sqrt{1-(\frac{x}{a})^2}}$$

Simplifying,

$$f'(x) = \sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2-x^2}} + \frac{a^2}{\sqrt{a^2-x^2}}$$

$$f'(x) = \sqrt{a^2 - x^2} + \frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$f'(x) = \sqrt{a^2 - x^2} + \sqrt{a^2 - x^2}$$

$$f'(x) = 2\sqrt{a^2 - x^2}.$$

iii) Equation of semicircle: $x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$
(positive solution only since the semi-circle is above the $x$-axis).

area $= 2\int_0^{a/2} \sqrt{a^2 - x^2}\, dx = \int_0^{a/2} 2\sqrt{a^2 - x^2}\, dx$

From part a.i), since the integrand is exactly $f'(x)$, by the fundamental theorem of calculus, the value of this integral is $f(\frac{a}{2}) - f(0)$.

From the original function, $f(0) = 0$, so the area is $f(\frac{a}{2})$.

b. Let $du = (\sin x + \cos x)\, dx \iff u = \sin x - \cos x$.

The denominator is

$= 9 + 16 \sin 2x = 9 + 32 \sin x \cos x$ (sin $2x = 2$ sin $x$ cos $x$)

$= 9 - 16(\sin x - \cos x)^2 + 16(\sin^2 x + \cos^2 x)$ (rearrange expansion)

$= 25 - 16u^2$. (sub in $u$ and $\sin^2 x + \cos^2 x = 1$)

The bounds of the integral become $x = 0 \Rightarrow u = -1$ and $x = \frac{\pi}{4} \Rightarrow u = 0$:

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, \mathrm{d}x = \int_{-1}^{0} \frac{1}{25 - 16u^2}\, \mathrm{d}u = \int_{-1}^{0} \frac{1}{(5-4u)(5+4u)}\, \mathrm{d}u$$

Applying partial fractions,

$$= \int_{-1}^{0} \frac{1}{(5-4u)(5+4u)}\, \mathrm{d}u$$

$$= \int_{-1}^{0} \frac{\frac{1}{10}}{5-4u} + \frac{\frac{1}{10}}{5+4u} \mathrm{d}u$$

$$= \frac{1}{10}\left[-\frac{1}{4}\ln|5-4u| + \frac{1}{4}\ln|5+4u|\right]_{-1}^{0}$$

$$= \frac{1}{40}(-\ln 9 + \ln 1 + \ln 5 - \ln 5)$$

$$= \frac{\ln 9}{40} = \frac{\ln 3}{20}.$$

***Alternative method:*** use the symmetric property $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$, then simplify using symmetry of trigonometric functions, integrate with a substitution $u = \sin x$ and finally use partial fractions.

**Question E2**

a.

i) $e^{-x^2}$ and $e^{-(a-x)^2} = e^{-(x-a)^2}$ are the same function, reflected in the line $x = \frac{1}{2}a$. Therefore, their gradients are the negative of each other and so the derivative of $e^{-x^2} + e^{-(x-a)^2}$ adds to zero at $x = \frac{1}{2}a$, implying a stationary point.
The function $f(x) = \frac{1}{1+e^{-x^2}+e^{-(x-a)^2}}$ and so shares its critical value, at $x = \frac{1}{2}a$.

ii) First derivative:

$$f'(x) = \frac{2xe^{-x^2} + 2(x-a)e^{-(x-a)^2}}{(1+e^{-x^2}+e^{-(x-a)^2})^2}$$

(chain rule)

Second derivative: $f''(x) = \frac{u'v-v'u}{v^2}$ where $u$ = top and $v$ = bottom

$u = 2xe^{-x^2} + 2(x-a)e^{-(x-a)^2}$ and $v = \left(1 + e^{-x^2} + e^{-(x-a)^2}\right)^2$

$u' = 2\left(1 - 2x^2\right)e^{-x^2} + 2\left(1 - 2(x-a)^2\right)e^{-(x-a)^2}$ (product rule)

$v' = 2\left(1 + e^{-x^2} + e^{-(x-a)^2}\right)\left(-2xe^{-x^2} - 2(x-a)e^{-(x-a)^2}\right)$ (chain rule)

Substituting in $x = \frac{1}{2}a$ to these expressions,

$u(\frac{a}{2}) = ae^{-\frac{1}{4}a^2} - ae^{-\frac{1}{4}a^2} = 0$ and $v(\frac{a}{2}) = (1 + 2e^{-\frac{1}{4}a^2})^2$

$u'(\frac{a}{2}) = 2(1 - \frac{1}{2}a^2)e^{-\frac{1}{4}a^2} + 2(1 - \frac{1}{2}a^2)e^{-\frac{1}{4}a^2} = 4(1 - \frac{1}{2}a^2)e^{-\frac{1}{4}a^2}$

$v'(\frac{a}{2}) = 2\left(1 + 2e^{-\frac{1}{4}a^2}\right)\left(-u(\frac{a}{2})\right) = 0$

Therefore the value of the second derivative at the stationary point is

$$f''\left(\frac{a}{2}\right) = \frac{4\left(1-\frac{1}{2}a^2\right)e^{-\frac{1}{4}a^2}\cdot\left(1+2e^{-\frac{1}{4}a^2}\right)^2}{\left(1+2e^{-\frac{1}{4}a^2}\right)^4}$$

(quotient rule)

Using the given condition,
$|a| < \sqrt{2} \Rightarrow a^2 < 2 \Rightarrow \frac{1}{2}a^2 < 1 \Rightarrow 1 - \frac{1}{2}a^2 > 0$, and all exponentials, squares and 4th powers are positive, so the value of $f''(\frac{a}{2}) > 0$.
→ gradient increasing → this stationary point is a **minimum** point.

b. The water forms a prism-like shape with the cross-section of a circular segment:

$V = 6\left(\frac{1}{2}r^2(\theta - \sin\theta)\right) = 3(\theta - \sin\theta)$ where $\theta$ is the central angle.

By geometry, the depth is $x = 1 - \cos\frac{\theta}{2}$.

By relating the rates (chain rule), $\frac{dV}{dt} = \frac{dV}{d\theta} \times \frac{d\theta}{dx} \times \frac{dx}{dt}$.

$x = 1 - \cos\frac{\theta}{2} \;\Rightarrow\; 1 = \frac{1}{2}\sin\frac{\theta}{2}\,\frac{d\theta}{dx} \;\Rightarrow\; \frac{d\theta}{dx} = 2\csc\frac{\theta}{2}$.

When $x = \frac{1}{2}$, $\cos\frac{\theta}{2} = \frac{1}{2} \;\Rightarrow\; \theta = 120^o = \frac{2}{3}\pi$ radians.

The derivatives are $\frac{dV}{d\theta}(\theta = \frac{2}{3}\pi) = \frac{9}{2}$ and $\frac{d\theta}{dx}(\theta = \frac{2}{3}\pi) = \frac{4\sqrt{3}}{3}$, so

$\frac{dx}{dt} = \frac{1/6}{9/2 \times \frac{4\sqrt{3}}{3}} = \frac{1}{36\sqrt{3}} = \frac{\sqrt{3}}{108} \approx$ 0.0160 (3 s.f.) metres per minute.

c. $x^4 + y^3 = x^2y \;\Rightarrow\; x^2y^{-1} + x^{-2}y^2 = 1 \;\Rightarrow\; (\frac{y}{x})^{-2}y + (\frac{y}{x})^2 = 1$.

Let $t = \frac{y}{x} \;\Rightarrow\; t^{-2}y + t^2 = 1 \;\Rightarrow\; y = t^2(1 - t^2) \;\Rightarrow\; y = t^2 - t^4$.

Rearranging for $x$: $x = \frac{y}{t} = t - t^3$.

Bounds for *t* for integrating around the region *R*: $x = y = 0$

$\Rightarrow\; t(t - 1) = 0$ and $t^2(1 - t^2) = 0$

$\Rightarrow$ $\{t = 0$ or $t = 1\}$ and $\{t = 0$ or $t = 1$ or $t = -1\}$

$\Rightarrow$ bounds of parametric integration are $0 \le t \le 1$.

Integrating the region parametrically,

$$R = \left|\int_0^1 y\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t\right| = \left|\int_0^1 (t^2 - t^4)(1 - 3t^2)\,\mathrm{d}t\right| = \left|-\frac{4}{105}\right| = \frac{4}{105}.$$

(The integral is negative because the orientation of the curve is anticlockwise, meaning the integral represents the area under the lower (smaller values) branch minus the area under the upper (larger values) branch. This does not matter - as there are clearly no overlapping sections, the area is simply the absolute value.)

***Alternative method:*** if the integral was computed as $\int_0^1 x\,\frac{dy}{dt}\,dt$ instead, the area would come out positive without needing to take the absolute value.

**Question E3**

a. $\int_0^{\pi} e^{-x} \sin x \, dx = \left[- e^{-x}\cos x - e^{-x} \sin x\right]_0^{\pi} - \int_0^{\pi} e^{-x} \sin x \, dx$

(integration by parts, twice; the choice of $u$ and d$v$ does not matter)

$2 \int_0^{\pi} e^{-x} \sin x \, dx = \left(e^{-\pi} + 1\right)$ (add $\int_0^{\pi} e^{-x} \sin x \, dx$ to both sides, sub in limits)

$\int_0^{\pi} e^{-x} \sin x \, dx = \frac{1}{2}\left(e^{-\pi} + 1\right).$

b. i) Let $f(x) = e^{-x} \sin x$. For $n = 4$, the points of the trapezium rule will be at $f(0) = 0,\ f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}e^{-\pi/4},\ f(\frac{\pi}{2}) = e^{-\pi/2},\ f(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}e^{-3\pi/4},\ f(\pi) = 0.$

The approximation is then

$T_4 = \frac{1}{2} \times \frac{\pi}{4} \times \left(0 + 2 \times \left(\frac{\sqrt{2}}{2}e^{-\pi/4} + e^{-\pi/2} + \frac{\sqrt{2}}{2}e^{-3\pi/4}\right) + 0\right)$

$T_4 = 0.469115384...$

ii) $f(x) = e^{-x} \sin x$

$f'(x) = e^{-x} \cos x - e^{-x} \sin x = e^{-x}(\cos x - \sin x)$ (product rule twice)

$f''(x) = e^{-x}(- \sin x - \cos x) - e^{-x}(\cos x - \sin x) = - 2e^{-x} \cos x$

$K$ will be the largest value of $2 \mid e^{-x} \cos x \mid$ for $x$ in $[0, \pi]$. By inspection or sketching the graph, this occurs at $x = 0$, so $K = 2$.

(Differentiating and checking for maxima gives a local maximum, but not the overall maximum.)

Therefore, $E_4 \leq \frac{2\pi^3}{12(4^2)} = \frac{\pi^3}{96} = 0.3229820488...$

iii) The true error is $E_4 = \left|\frac{1}{2}\left(e^{-\pi} + 1\right) - T_4\right| = 0.1986249104...$, which is within the upper bound found in part b.ii).

c. Divide the numerator and denominator of the given RHS by $x$:

$$\Rightarrow \frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

Use the homogeneous substitution, $u = \frac{y}{x} \;\Rightarrow\; y = ux \;\Rightarrow\; \frac{dy}{dx} = u + x\frac{du}{dx}$:

$$\Rightarrow u + x\frac{du}{dx} = \frac{1-u}{1+u}$$

This equation is separable:

$$\Rightarrow x\frac{du}{dx} = \frac{1-u}{1+u} - u = \frac{1-2u-u^2}{1+u}$$

$$\Rightarrow \int \frac{1+u}{1-2u-u^2}\, du = \int \frac{1}{x}\, dx$$

$$\Rightarrow \; -\frac{1}{2} ln\left|1 - 2u - u^2\right| = A_0\, ln|x|$$ (integration by substitution)

$$\Rightarrow 1 - 2u - u^2 = \frac{A_1}{x^2}$$ (replace arbitrary integration constants)

Replace $u = \frac{y}{x}$:

$$\Rightarrow \frac{y^2}{x^2} + \frac{2y}{x} - 1 = \frac{A_2}{x^2}$$

$$\Rightarrow y^2 + 2xy - x^2 = A_2$$

Apply given initial condition in original differential equation:

let $\frac{dy}{dx} = 0$ (stationary) and $y = 1 \rightarrow x = 1$.

Therefore $A_2 = 2$.

$$\Rightarrow y^2 + 2xy - x^2 = 2$$

$$\Rightarrow (x + y)^2 - 2x^2 = 2$$ (alternative: quadratic formula)

$$\Rightarrow y = \sqrt{2 + 2x^2} - x.$$ (take +ve root since $y > 0$)

**Section F**

**Question F1**

a. Consider one of the bullets fired at time $t$. For this bullet to land at the same time as the first bullet (or any other),

flight time of first bullet = $t$ + flight time of bullet at $\alpha$

Let the initial speed be $U$.
Vertical kinematics: $s = 0$, $u = U\sin\theta$, $v = -U\sin\theta$, $a = -g$, $t = T$

$v = u + at \rightarrow T(\theta) = \frac{2U\sin\theta}{g}$,

Comparing the first and last bullet, $T(45^o) = 1 + T(30^o)$

$\rightarrow \frac{U\sqrt{2}}{g} = 1 + \frac{U}{g} \Rightarrow U = (1+\sqrt{2})g.$

Forming the equation, $T(45^o) = t + T(\alpha) \Rightarrow \frac{(1+\sqrt{2})g\sqrt{2}}{g} = t + \frac{2(1+\sqrt{2})g\sin\alpha}{g}$

$\rightarrow \sqrt{2} = \frac{t}{(1+\sqrt{2})\sqrt{2}} + 2\sin\alpha$

Solving for the angle, $\alpha(t) = \sin^{-1}\left(\frac{\sqrt{2}}{2} - \frac{t}{2(1+\sqrt{2})}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}-1}{2}\right)t\right)$.

b. By kinematics in both component directions,
Horizontal: $s = r$, $u = v\cos\theta$, $v = v\cos\theta$, $a = 0$, $t = T$
Vertical: $s = 0$, $u = v\sin\theta$, $v = -v\sin\theta$, $a = -g$, $t = T$
Using $s = vt$ in the horizontal direction,

$r = vT\cos\theta \Rightarrow r^2 = v^2T^2\cos^2\theta = v^2T^2(1-\sin^2\theta) \Rightarrow \sin^2\theta = 1 - \frac{r^2}{v^2T^2}$

Using $s = ut + \frac{1}{2}at^2$ in the vertical direction,

$vT\sin\theta - \frac{1}{2}gT^2 = 0 \Rightarrow 2v\sin\theta = gT \Rightarrow 4v^2\sin^2\theta = g^2T^2$ (sub above)

$\Rightarrow 4v^2\left(1 - \frac{r^2}{v^2T^2}\right) = g^2T^2 \Rightarrow g^2T^4 - 4v^2T^2 + 4r^2 = 0$ (quadratic formula)

$\Rightarrow T^2 = \frac{4v^2 \pm \sqrt{16v^4 - 16g^2r^2}}{2g^2} \Rightarrow \frac{1}{2}g^2T^2 = \frac{4v^2 \pm 4\sqrt{v^4 - g^2r^2}}{4} = v^2 \pm \sqrt{v^4 - g^2r^2}$

Observe that the positive and negative solutions correspond to launch angles above and below 45° respectively. Since in this case, all angles are below 45°, reject the positive solution:

$\Rightarrow \frac{1}{2}g^2T^2 = v^2 - \sqrt{v^4 - g^2r^2}$.

c. In the vertical direction, use $v^2 - u^2 = 2as$:

$\Rightarrow \; v^2 sin^2 \theta = 2gh$

Also, using $v = u + at$ (time at maximum = half of flight time = $\frac{1}{2}T$)

$\Rightarrow \; v \, sin \, \theta = \frac{1}{2}gT \;\; \Rightarrow \;\; 2v^2 sin^2 \theta = \frac{1}{2}g^2 T^2$

Equating the LHS with the first equation, and the RHS with the equation in part b)

$4gh = v^2 - \sqrt{v^4 - g^2 r^2}$

Therefore the maximum height is $h = \dfrac{v^2 - \sqrt{v^4 - g^2 r^2}}{4g}$.

**Question F2**

a. $D$ is the area of the graph, in the shape of a trapezium, so

$D = \frac{1}{2}V(T + \frac{2}{3}T) = \frac{5}{6}VT$ $(t_1 = \frac{1}{6}T,\ t_2 = \frac{5}{6}T)$

$m$ is the gradient of the accelerating section, so

$m = \frac{V}{T/6} = \frac{6V}{T} \Rightarrow V = \frac{1}{6}mT$

Substituting in gives $D = \frac{5}{6}\left(\frac{1}{6}mT\right)T = \frac{5}{36}mT^2$.

b.

i) The areas under the new velocity-time graphs must be the same as before since $D$ is constant. Let $U$ be the new top speed. Let $kT$ be the time spent accelerating.

New time spent at top speed $= \frac{7}{9}T - 2kT = \left(\frac{7}{9} - 2k\right)T$.

Area under new graph $= 2 \times \left(\frac{1}{2} \times kT \times U\right) + \left(\frac{7}{9} - 2k\right)TU = \left(\frac{7}{9} - k\right)TU$.

Equate with $D$ and substitute the result from part a):

$\left(\frac{7}{9} - k\right)TU = \frac{5}{36}mT^2$

To eliminate $U$, reuse the definition of acceleration as $m = \frac{U}{kT} \Rightarrow U = mkT$:

$\left(\frac{7}{9} - k\right)mkT^2 = \frac{5}{36}mT^2 \Rightarrow \frac{5}{36} = \left(\frac{7}{9} - k\right)k \Rightarrow k^2 - \frac{7}{9}k + \frac{5}{36} = 0$

Solving gives $k = \frac{1}{2}$ and $k = \frac{5}{18}$. Since we must have $2k \leq \frac{7}{9}$ otherwise the acceleration would never be completed in time, reject the positive solution and take $k = \frac{5}{18}$.

$\Rightarrow$ time spent accelerating $= kT = \frac{5}{18}T$.

ii) Old top speed $= V = \frac{1}{6}mT$, New top speed $= U = \frac{5}{18}mT$.

Proportion increased $= \frac{5/18 - 1/6}{1/6} = \frac{2}{3}$

$\Rightarrow$ 66.66…% increase in top speed.

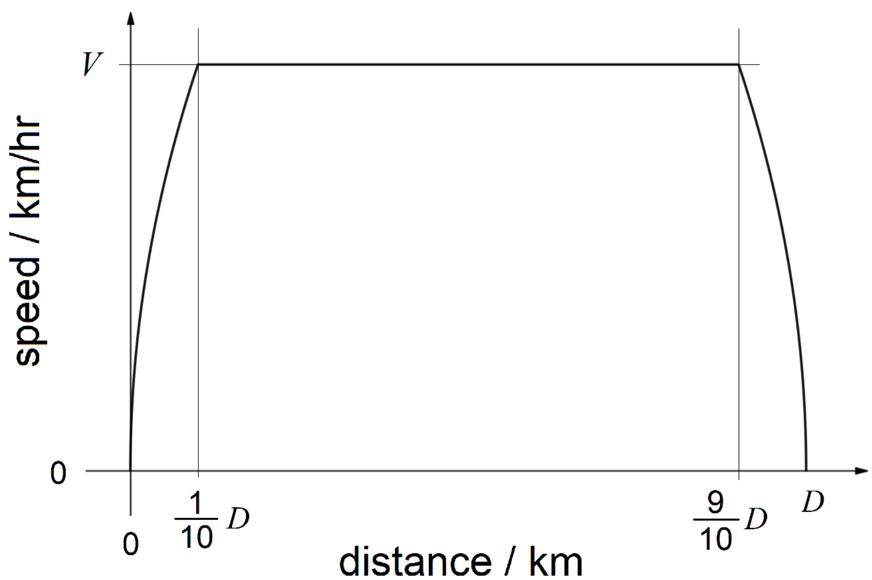
c. Displacement at $t = t_1$: $s = ut + \frac{1}{2}at^2 = \frac{1}{2}mt_1^{\ 2} = \frac{1}{72}mT^2 = \frac{1}{10}D$.

Using $v^2 - u^2 = 2as$ at a general position, $v(x) = \sqrt{2mx}$ i.e. **curved** path.

Between $t = t_1$ and $t = t_2$, velocity is constant. The displacement is given by $s = ut = Vt$ so the displacement at $t_2$ is

$\frac{1}{10}D + \frac{1}{9}mT^2 = \frac{1}{10}D + \frac{4}{5}D = \frac{9}{10}D$.

The last section is curved in the opposite direction by symmetry, ending at $D$.

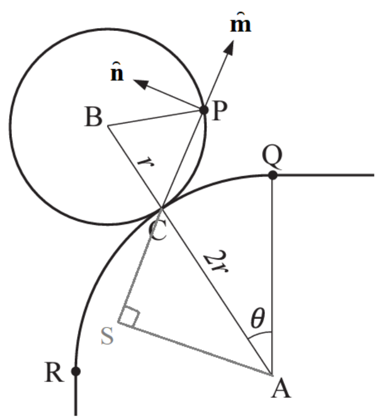


## Question F3

a. i) Since the cylinder rolls without slipping, the rolling arc lengths $CP$ and $CQ$ must be equal. We have $CQ = 2r\ \angle CAQ$ and $CP = r\ \angle CBP$. Since $CP = CQ$,

it follows that $2\ \angle CAQ = \angle CBP$ and so $\angle CBP : \angle CAQ = 2 : 1$.

ii) Extend line $CP$ to point $S$ such that $PS \perp AS$, and therefore $SP \parallel \mathbf{m}$ and $AS \parallel \mathbf{n}$.




Since we know that $\angle CBP = 2\theta$, and $\triangle BCP$ is isosceles, it follows that $\angle BCP = \frac{\pi}{2} - \theta$. By opposite angles, $\angle ACS = \angle BCP = \frac{\pi}{2} - \theta$, and by adding angles in right triangle $\triangle ACS$, we get $\angle CAS = \theta$.

Using trigonometry:

$|AS| = 2r \cos\theta$ (Soh-Cah-Toa in $\triangle ACS$)

$|CS| = 2r \sin\theta$ (Soh-Cah-Toa in $\triangle ACS$)

$|CP| = \sqrt{r^2 + r^2 - 2r^2 \cos 2\theta} = 2r \sin\theta$ (cosine rule in $\triangle BCP$, then $\cos 2\theta = 1 - 2\sin^2\theta$)

Therefore the displacement vector $AP$ is

$\mathbf{s}_{P/A} = |AS|\ \mathbf{n} + (|CS| + |CP|)\mathbf{m} = 4r \sin\theta\ \mathbf{m} + 2r \cos\theta\ \mathbf{n}$.

iii) It is given that the direction of change of $\mathbf{m}$ is towards $\mathbf{n}$ (90° rotation) and scaled by a factor of 2. Therefore similarly, the direction of change of $\mathbf{n}$ is towards $-\mathbf{m}$ (90° rotation to maintain perpendicular relationship), also by factor 2. Therefore $\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}t} = -2\frac{d\theta}{dt}\hat{\mathbf{m}}$.

iv) To find velocity, differentiate the expression for displacement:

$\mathbf{v} = \frac{d}{dt}\left(4r\, sin\,\theta\, \widehat{m} \; + \; 2r\, cos\,\theta\, \widehat{n}\right) = 4r\frac{d}{dt}(\sin\theta\,\mathbf{m}) + 2r\frac{d}{dt}(\cos\theta\,\mathbf{n})$

Use product rule and chain rule twice. Note that the vectors must also be differentiated as they depend on $t$.

$\frac{d}{dt}(\sin\theta\,\mathbf{m}) = (\frac{d}{dt}\sin\theta)\,\mathbf{m} + \sin\theta\frac{d\widehat{m}}{dt} = \cos\theta\frac{d\theta}{dt}\,\mathbf{m} + 2\sin\theta\frac{d\theta}{dt}\,\mathbf{n}$

$\frac{d}{dt}(\cos\theta\,\mathbf{n}) = (\frac{d}{dt}\cos\theta)\,\mathbf{n} + \cos\theta\frac{d\widehat{n}}{dt} = -\sin\theta\frac{d\theta}{dt}\,\mathbf{n} - 2\cos\theta\frac{d\theta}{dt}\,\mathbf{m}$

Therefore

$\mathbf{v} = 4r(\cos\theta\frac{d\theta}{dt}\,\mathbf{m} + 2\sin\theta\frac{d\theta}{dt}\,\mathbf{n}) + 2r(-\sin\theta\frac{d\theta}{dt}\,\mathbf{n} - 2\cos\theta\frac{d\theta}{dt}\,\mathbf{m})$

$\mathbf{v} = 6r\sin\theta\frac{d\theta}{dt}\,\mathbf{n}.$

For the acceleration, differentiate this velocity. Use the 'triple product rule':

$\mathbf{a} = 6r\frac{d}{dt}(\sin\theta\frac{d\theta}{dt}\,\mathbf{n}) = 6r(\frac{d\theta}{dt}\,\mathbf{n}\frac{d}{dt}\sin\theta + \sin\theta\frac{d}{dt}\frac{d\theta}{dt}\,\mathbf{n} + \sin\theta\frac{d\theta}{dt}\frac{d}{dt}\mathbf{n})$

$\mathbf{a} = 6r(\left(\frac{d\theta}{dt}\right)^2\cos\theta\,\mathbf{n} + \frac{d^2\theta}{dt^2}\sin\theta\,\mathbf{n} - 2\sin\theta\frac{d\theta}{dt}\,\mathbf{m})$

$\mathbf{a} = 6r(\left(\frac{d\theta}{dt}\right)^2\cos\theta + \frac{d^2\theta}{dt^2}\sin\theta)\,\mathbf{n} - 12r\left(\frac{d\theta}{dt}\right)^2\sin\theta\,\mathbf{m}.$

b. Velocity only has a component in the $\mathbf{n}$ direction, so we want to get the (absolute value) component of acceleration in the $\mathbf{m}$ direction, which is

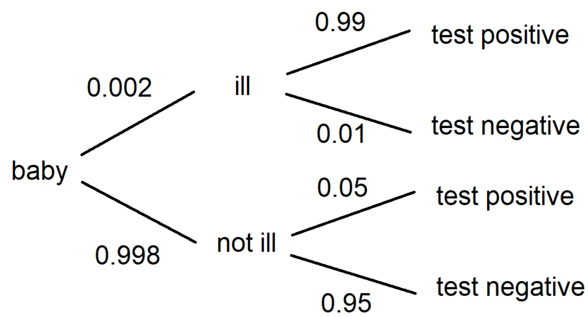
$12r\left(\frac{d\theta}{dt}\right)^2\sin\theta.$

Therefore $12r\left(\frac{d\theta}{dt}\right)^2 sin\,\theta \;=\; \frac{\left(6r\, sin\,\theta\,\frac{d\theta}{dt}\right)^2}{R} \;\Rightarrow\; R = 3r\, sin\,\theta$ which is proportional to $\sin\theta$ since $r$ is constant.

## Section G

### Question G1

a. The tree diagram is




i) P(test positive) = $0.99 \times 0.002 + 0.05 \times 0.998 = 0.05188$

ii) P(test correct) = $0.99 \times 0.002 + 0.95 \times 0.998 = 0.9501$

b. i) P(ill | test positive) = P(ill ∩ test positive) ÷ P(test positive)

$= \frac{0.002 \times 0.99}{0.05188} = 0.03816$

ii) Since the illness is so rare (only 0.2%), the healthy babies vastly outnumber the ill babies, so the proportion of false positives from the healthy population is in fact more than the true positives.

c. The new probabilities are

P(test positive) = $0.99 \times 0.002 + (1 - p) \times 0.998 = 0.99998 - 0.998p$

P(ill | test positive) = $\frac{0.002 \times 0.99}{0.99998 - 0.998p} = 0.50$ (required)

Solving for $p$ gives $p = 0.998016\ldots = 0.9980$.

d. i) Let $X$ be the number of positive test results. Then $X \sim B(3, p)$, where $p = 0.99$ if ill and $p = 0.001984$ (from $1 - p$ above) if not ill.

P(illness | $X = 1$) = $\frac{P(illness)}{P(X=1)} P(X = 1 \mid illness)$

P(illness) = 0.002 and P($X = 1 \mid p = 0.99$) = 0.000297 (binomial calculator)

P($X = 1$) = P($X = 1$ | illness) P(illness) + P($X = 1$ | no illness) P(no illness)

$= 0.000297 \times 0.002 + 0.0059283 \times 0.998 = 0.00591705$

P(illness | $X = 1$) = $\frac{0.002}{0.00591705} \times 0.000297 = 1.0039 \times 10^{-4} \approx \frac{1}{10000}$.

ii) Assumes independent tests / always either positive or negative / const $p$

**Question G2**

a. Entering the data into the calculator statistics menu,

i) sample median = "Med" = 499.97 ohms
sample interquartile range = $Q_3$ - $Q_1$ = 502.215 - 498.065 = 4.15 ohms

ii) $\mu$ = pop. mean $\approx \overline{x}$ = 500.5425 ohms,
$\sigma^2$ = pop. variance $\approx s^2_x$ = 8.58 ohms$^2$
(estimating using $\sigma^2_x$ = 7.87 ohms$^2$ is biassed as the sample size is small.)

iii) $Q_1$ = InvNorm(0.25, $\overline{x}$, $s_x$) = 498.573, $Q_3$ = InvNorm(0.75, $\overline{x}$, $s_x$) = 502.512,
$\Rightarrow$ $IQR$ of population $= Q_3 - Q_1 = 3.939$ ohms

b. i) Let $R$ be the resistance of a resistor produced, with $R \sim N(501, 3^2)$.
P(rejected) = 1 - P(accepted) = 1 - P(498 < $R$ < 508)
= 1 - 0.83153 = 0.1685 (16.85% of resistors expected to be rejected).

ii) The wastage will be minimised when the tails of the distribution are as close to the rejection regions as possible, i.e. when it is symmetric about the rejection region. Therefore the new optimum $\mu = \frac{498+508}{2} = 503$ ohms.
Now, P(rejected) = 2 × P($R$ < 498) = 2 × 0.04779 = 0.09558 = 9.56%.

iii) The standard (normalised $z$-score) of the lower limit is $z = \frac{498-503}{\sigma}$.
From the inverse normal distribution (calculator), $z$ must be more extreme than the half-tail area (0.1% = 0.05% per tail = area 0.0005):
$z < \Phi^{-1}(0.0005) = -3.2905268\ldots$
Equating these two $z$-values,
$\Rightarrow\ 503 - 498 < 3.2905\sigma$
$\Rightarrow\ \sigma < 1.520$ ohms

iv) When $\mu$ = 501 ohms, the left tail is much closer than the right tail, and so the probability for the right tail of the normal distribution can be neglected.
New $z$-value: $z < \Phi^{-1}(0.001) = -3.0902323\ldots$
$\Rightarrow\ 501 - 498 = 3.0902\sigma\ \Rightarrow\ \sigma < 0.9708$ ohms.

(Check that the right tail was in fact negligible:
P($R$ > 508) = $2.786 \times 10^{-13} \ll 0.1\%$. True value: $\sigma < 0.9708008013\ldots$)

**Question G3**

a. i) Consider a sequence of length $n$ flips. Each flip is either $\mathrm{H}$ or $\mathrm{T}$.

If the sequence starts with $\mathrm{T}$, the remaining $n$ - 1 flips only have to avoid having two consecutive $\mathrm{H}$'s: number of ways of this happening is $F(n - 1)$.

If this sequence starts with $\mathrm{H}$, there is no choice in what the second flips must be: it must be a $\mathrm{T}$. The remaining $n$ - 2 flips then have to avoid having two consecutive $\mathrm{H}$'s: number of ways of this happening is $F(n - 2)$.

All sequences must start with either $\mathrm{H}$ or $\mathrm{T}$, and therefore $F(n) = F(n - 1) + F(n - 2)$, which is a Fibonacci-type sequence.

ii) When $n = 1$: sample space is $\{\{\mathrm{H}\}, \{\mathrm{T}\}\} \rightarrow F(1) = 2$
(does not matter whether it is H or T, both of them avoid consecutive H)

When $n = 2$: sample space is $\{\{\mathrm{H, H}\}, \{\mathrm{H, T}\}, \{\mathrm{T, H}\}, \{\mathrm{T, T}\}\} \rightarrow F(2) = 3$
($\{\mathrm{H, T}\}$, $\{\mathrm{T, H}\}$, $\{\mathrm{T, T}\}$ are valid, $\{\mathrm{H, H}\}$ is not)

Applying the Fibonacci series, add the two previous numbers:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $F(n)$ | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |

There are 2 outcomes per trial, so after $n$ trials there are $2^n$ possible arrangements of the $n$ flips.

Using the combinatorial definition of probability as $\frac{number\ of\ successful\ outcomes}{total\ number\ of\ outcomes}$

the probability of any matching arrangement is

$P(X_{10}) = \frac{F(10)}{2^{10}} = \frac{144}{1024} = \frac{9}{64}$.

b. i) $H_0$: the coin lands heads and tails with equal probability ($p = 0.5$)

$H_1$: the coin lands with heads more frequently ($p > 0.5$)

Let $N$ be the number of heads in a sequence of 13 flips.

Under $H_0$, $N \sim B(13,\ 0.5)$

$\Rightarrow\ P(N \geq 10) = 1 - P(N \leq 9) = 1 - 0.953857 = 0.0461.$

This is less than the significance level of 0.05, so the test rejects $H_0$.

There is sufficient evidence to suggest that the coin lands Heads with an average probability greater than 0.5.

ii) Listing the values of $P(N \leq n)$, the value is greater than 0.95 for $n \geq 9$.

$\rightarrow$ critical region is $9 \leq N \leq 13$.

c. i) The $p$-value represents the value of observing the data (or more extreme) given the null hypothesis, i.e. P(at least 10 Heads in 13 flips | $p = 0.5$).

Since P(at least 10 Heads in 13 flips) $\neq 1$, and these two conditional probabilities are the 'reverses' of each other, they are not the same.

ii) Using conditional probability for $\mathrm{P}(p = 0.5 \cap N_{10})$:
("$N_{10}$" = "exactly 10 Heads in 13 flips", for short)

$\mathrm{P}(p = 0.5 \mid N_{10}) \times \mathrm{P}(N_{10}) = \mathrm{P}(N_{10} \mid p = 0.5) \times \mathrm{P}(p = 0.5)$

$$P(p\ =\ 0.5 \mid N_{10})\ =\frac{P(N_{10} \mid p = 0.5)\ P(p = 0.5)}{P(N_{10})}$$

Since $p$ can take 11 different values (0.0, 0.1, 0.2, … 0.9, 1.0),
$\mathrm{P}(p = 0.5) = \frac{1}{11} = 0.09090909...$

From the binomial distribution, $\mathrm{P}(N_{10} \mid p = 0.5) = 0.03491211\ldots$

From the law of total probability (this is a sum over all *probabilities* - **not** a cumulative probability, which sums over *values*):

$$\mathrm{P}(N_{10}) = \sum_{x=0,\, 0.1,\, 0.2,\, ...,\, 1} \mathrm{P}(N_{10} \mid p = x)\ \mathrm{P}(p = x)$$ ($x$: probability to check)

$$P(N_{10})\ = \frac{1}{11}\sum_{i=0}^{10}\ P(N_{10} \mid p\ = \frac{i}{10})$$ (let $x = \frac{i}{10}$ to use integer indices)

$$P(N_{10})\ = \frac{1}{11}\sum_{i=0}^{10}\ {}^{13}C_{10} \times \left(\frac{i}{10}\right)^{10} \times \left(1\ -\frac{i}{10}\right)^{3}$$

$$P(N_{10})\ =\ 0.06510754...$$ (using calculator summation)

Therefore,
$$P(p\ =\ 0.5 \mid N_{10})\ = \frac{0.03491211 \times 0.09090909}{0.06510754}\ =\ 0.048747477...=\ 0.0487.$$

iii) The median will be when the sum of the values for increasing (or decreasing) values of $p$ found in part c.ii) is closest to 0.5. Since the given observation implies that the mean of $p$ is roughly $\frac{10}{13}$, we expect the median to be close to this i.e. median ~ 0.7 - 0.8, so it will be faster to count down from $p$ = 1.0.

From the conditional probability formula and the values found in part c.ii), we can now easily 'invert' the conditional probabilities:

$\Rightarrow\ P(p = x \mid N_{10}) = 1.39629 \times P(N_{10} \mid p = x)$

Evaluating the binomial probabilities with decreasing probabilities $x$,

$P(p = 1.0 \mid N_{10}) = 0$ (if $p$ was 1.0, there could never be any 3 Tails)

$P(p = 0.9 \mid N_{10}) = 1.39629 \times P(N_{10} \mid p = 0.9) = 0.13924$

$P(p = 0.8 \mid N_{10}) = 1.39629 \times P(N_{10} \mid p = 0.8) = 0.34303$

$P(p = 0.7 \mid N_{10}) = 1.39629 \times P(N_{10} \mid p = 0.7) = 0.30457$

The cumulative sum reaches 0.48227 (very close to 0.5) at $p = 0.8$, and reaches 0.78684 at $p = 0.7$.

Using linear interpolation and the continuity approximation:

$P(0.75 \leq p \leq 1) = 0.48227$ and $P(0.65 \leq p \leq 1) = 0.78684$

$\Rightarrow\ 0.5 = 0.78684 + (p_{median} - 0.65) \times \frac{0.48227 - 0.78684}{0.75 - 0.65}$

$\Rightarrow\ p_{median} = 0.7442.$

(Comparing this to the other statistics, $p_{mean} = 0.7339$, $p_{mode} = 0.8$, and so these probabilities are all close to each other but different, which seems reasonable. Note that due to the asymmetry of the distribution of $p$, the expected value (mean) of $p$ is **not** $\frac{10}{13}$. Since $p_{mean} < p_{mode}$, the distribution of $p$ is negatively skewed. The true median can be found using the Beta distribution (not A-level) with $\alpha = 10 + 1$ and $\beta = 3 + 1$ and approaches 0.7439… as $p$ becomes more and more continuous. In this case, the mode then approaches $\frac{10}{13}$ as expected).

**A32. Answer:** ②

Starting with $f(x) + \int_1^x f(t)\,dt = 1$, differentiate both sides with respect to *x*:

$\Rightarrow \frac{d}{dx}\left(f(x) + \int_1^x f(t)\,dt\right) = 0 \quad \Rightarrow \quad f'(x) + f(x) = 0$ (FTC: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$)

$\Rightarrow \frac{df}{dx} = -f \Rightarrow f(x) = A\,e^{-x}$

Putting this back into the original equation,

$f(x) + \int_1^x f(t)\,dt = 1 \quad \Rightarrow \quad A\,e^{-x} + \int_1^x A\,e^{-t}\,dt = 1 \quad \Rightarrow \quad Ae^{-x} + A\left(-e^{-x} + e^{-1}\right) = 1$

$\Rightarrow Ae^{-1} = 1 \quad \Rightarrow \quad A = e$

$\Rightarrow \quad f(x) = e\,e^{-x} = e^{1-x}$

$\Rightarrow \quad \ln f(2) = \ln(e^{1-2}) = \ln(e^{-1}) = -1.$ (②)

**A33. Answer:** ③

$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n}{a^n} = \frac{1}{a^1} + \frac{2}{a^2} + \frac{3}{a^3} + \ldots$

$= \left(\frac{1}{a^1} + \frac{1}{a^2} + \frac{1}{a^3} + \ldots\right) + \left(\frac{1}{a^2} + \frac{1}{a^3} + \ldots\right) + \left(\frac{1}{a^3} + \ldots\right) + \ldots$

$= \left(\frac{1/a}{1-1/a}\right) + \left(\frac{1/a^2}{1-1/a}\right) + \left(\frac{1/a^3}{1-1/a}\right) + \ldots$

$= \frac{1}{1-1/a}\left(\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \ldots\right)$

$= \frac{1}{1-1/a} \times \frac{1/a}{1-1/a} = \frac{1/a}{(1-1/a)^2} = \frac{14}{25}$

Let $u = 1/a \rightarrow \frac{u}{(1-u)^2} = \frac{14}{25} \quad \Rightarrow \quad 14(1-u)^2 = 25u \quad \Rightarrow \quad 14u^2 - 53u + 14 = 0$

$\Rightarrow \quad u = \frac{7}{2}$ or $u = \frac{2}{7} \quad \Rightarrow \quad a = \frac{2}{7}$ or $a = \frac{7}{2}$.

However, notice that if $0 < a < 1$ then $a^n \rightarrow 0$ as $n \rightarrow \infty$, so the sum diverges.

Therefore the only correct answer is $a = \frac{7}{2}$. (③)

**A34. Answer:** ④

$$\sum_{k=1}^{10}(a_k+1)^2=\sum_{k=1}^{10}(a_k^2+2a_k+1)=28$$
$$\sum_{k=1}^{10}a_k(a_k+1)=\sum_{k=1}^{10}(a_k^2+a_k)=16$$

Subtracting the second equation from the first: $\sum_{k=1}^{10}(a_k+1)=12$

Since $\sum_{k=1}^{10}1=10$, we have $\left(\sum_{k=1}^{10}a_k\right)+10=12 \Rightarrow \sum_{k=1}^{10}a_k=2.$

So $\sum_{k=1}^{10}(a_k^2+a_k)-\sum_{k=1}^{10}a_k=\sum_{k=1}^{10}a_k^2=16-2=14.$ (④)

**A35. Answer:** ④

If the difference is equal to the first term then $a_k=ka_1=ka$.

Therefore $\sum_{k=1}^{15}\frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}}=\sum_{k=1}^{15}\frac{1}{\sqrt{ka}+\sqrt{(k+1)a}}$.

Rationalising the denominator and expanding:

$$=\sum_{k=1}^{15}\frac{\sqrt{ka}-\sqrt{(k+1)a}}{ka-(k+1)a}=\frac{1}{a}\sum_{k=1}^{15}\sqrt{(k+1)a}-\sqrt{ka}$$
$$=\frac{1}{a}\left(-\sqrt{a}+\sqrt{2a}-\sqrt{2a}+\sqrt{3a}-\sqrt{3a}+\sqrt{4a}-\ldots-\sqrt{15a}+\sqrt{16a}\right)$$
$$=\frac{1}{a}\left(\sqrt{16a}-\sqrt{a}\right)=2$$ (telescoping series)

$$\Rightarrow \sqrt{16a}-\sqrt{a}=2a \Rightarrow 16a+a-2\sqrt{16a^2}=4a^2 \Rightarrow 9a=4a^2 \Rightarrow 4a=a_4=9.$$ (④)

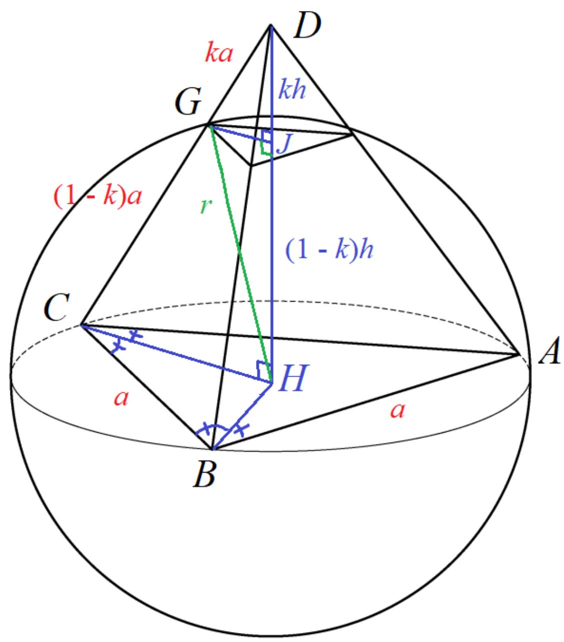
**A36. Answer:** ①

$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta=\frac{1}{\sqrt{a}}\rightarrow\sin\theta=\frac{-1}{\sqrt{a}}$ so both $\sin\theta$ and $\tan\theta$ are negative, so $\cos\theta$ must be positive. Then,

$$\cos\theta = \cos\left(\sin^{-1}\frac{-1}{\sqrt{a}}\right) = \sqrt{1 - \sin^2\left(\sin^{-1}\frac{-1}{\sqrt{a}}\right)} = \sqrt{1 - \left(\frac{-1}{\sqrt{a}}\right)^2} = \sqrt{1 - \frac{1}{a}} = \sqrt{\frac{a-1}{a}}.\ (①)$$

**A37. Answer:** ②

Let the tetrahedron edge length be $a$, the sphere radius be $r$, and the perpendicular height of the tetrahedron be $h$. By inspection, $EFG$ is similar to $ABC$ so let $k$ be the scale factor of enlargement. Let $H$ be the projection of $D$ onto $ABC$ and let $J$ be the projection of $D$ onto $EFG$.




Cosine rule in $\triangle BCH$: $a^2 = r^2 + r^2 - 2rr\cos 120^o$

$\Rightarrow r = |CH| = |BH| = \frac{\sqrt{3}}{3}a.$

Similarity: $|GJ| = k\,|CH| = \frac{\sqrt{3}ka}{3}.$

Pythagoras in $\triangle CHD$: $h^2 = a^2 - r^2 \Rightarrow h = \frac{\sqrt{6}a}{3}.$

Pythagoras in $\triangle GJH$: $[(1-k)h]^2 + \left(\frac{\sqrt{3}ka}{3}\right)^2 = r^2$

$\Rightarrow (1-k)^2\frac{2}{3}a^2 + \frac{1}{3}k^2a^2 = \frac{1}{3}a^2$

$\Rightarrow 2(1-k)^2 + k^2 = 1$

$\Rightarrow k = 1/3 \text{ or } k = 1$

Take $k = 1/3$ as the scale factor by inspection.

Then the ratio of areas scales with $k^2$ so the required answer is $1/9$. (②)

**B4.**

a. The roots of $f(x) = \cos(\ln x)$ are when $\ln x = \left\{..., \ -\frac{3\pi}{2}, \ -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, ...\right\}$.

Therefore let $ln\ \alpha_i = \left(\frac{2i-1}{2}\right)\pi$.

Choose $ln\ \alpha_0 = \ -\frac{\pi}{2}$: arbitrary choice, but useful for part b), so that

$\alpha_i = e^{\left(\frac{2i-1}{2}\right)\pi}$ and $\alpha_{i+1} = e^{\left(\frac{2i+1}{2}\right)\pi}$. The required series is then $u_i = \int_{e^{\left(\frac{2i-1}{2}\right)\pi}}^{e^{\left(\frac{2i+1}{2}\right)\pi}} cos(ln\ x)\ dx$.

We will find the indefinite integral first.

To find $I = \int\ cos(ln\ x)\ dx$, let $t = ln\ x$ so $dt = \frac{1}{x}\ dx$ and $dx = x\ dt = e^t\ dt$.

Then $I = \int e^t\ cos(t)\ dt$. Next, use integration parts. Let $u = cos\ t,\ dv = e^t\ dt$.

Then $I = e^t\ cos\ t + \int e^t\ sin\ t\ dt$. Use integration by parts **again.** Let $u = sin\ t,\ dv = e^t\ dt$.

Then $I = e^t\ cos\ t + e^t\ sin\ t - \int e^t\ cos\ t\ dt$. Observe that the integral is identical to $I$.

So $I = e^t\ cos\ t + e^t\ sin\ t - I \ \Rightarrow\ 2I = e^t(cos\ t + sin\ t) \ \Rightarrow\ I = \frac{e^t}{2}(sin\ t + cos\ t)$.

For simplicity, we will not undo the substitution $t = \ln x$ and immediately use the bounds.

We have $u_i = \int_{e^{\left(\frac{2i-1}{2}\right)\pi}}^{e^{\left(\frac{2i+1}{2}\right)\pi}} cos(ln\ x)\ dx = I\left(\left(\frac{2i+1}{2}\right)\pi\right) - I\left(\left(\frac{2i-1}{2}\right)\pi\right)$

$= \frac{e^{\left(\frac{2i+1}{2}\right)\pi}}{2}\left(sin\left(\frac{2i+1}{2}\right)\pi + cos\left(\frac{2i+1}{2}\right)\pi\right) - \frac{e^{\left(\frac{2i-1}{2}\right)\pi}}{2}\left(sin\left(\frac{2i-1}{2}\right)\pi + cos\left(\frac{2i-1}{2}\right)\pi\right)$

Carefully evaluate the trig terms, noticing they are always either -1, 1 or 0:

$u_i = \frac{e^{\left(\frac{2i+1}{2}\right)\pi}}{2}((-1)^i + 0) - \frac{e^{\left(\frac{2i-1}{2}\right)\pi}}{2}((-1)^{i+1} + 0)$

$= \frac{1}{2}\left((-1)^i e^{\left(\frac{2i+1}{2}\right)\pi} - (-1)^{i+1} e^{\left(\frac{2i-1}{2}\right)\pi}\right)$

$= \frac{1}{2}(-1)^i\left(e^{\left(\frac{2i+1}{2}\right)\pi} + e^{\left(\frac{2i-1}{2}\right)\pi}\right)$

$= \frac{1}{2}(-1)^i(e^{i\pi})(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}})$

$= \frac{e^{\pi/2} + e^{-\pi/2}}{2} \times \left(-e^{\pi}\right)^{i}$ which is a geometric series with common ratio $-e^{\pi}$.

b. We have $u_i = \frac{e^{\pi/2} + e^{-\pi/2}}{2} \times (-e^{\pi})^{i}$. Let $k = \frac{e^{\pi/2} + e^{-\pi/2}}{2}$ so that $u_i = k(-e^{\pi})^{i}$.

Then $\sum_{i=-\infty}^{\infty} \left|\frac{i}{u_{|i|}}\right| = \sum_{i=-\infty}^{\infty} \left|\frac{i}{k(-e^{\pi})^{|i|}}\right| = \frac{1}{k} \sum_{i=-\infty}^{\infty} \left|\frac{(-1)^{-|i|}\, i}{e^{\pi|i|}}\right| = \frac{1}{k} \sum_{i=-\infty}^{\infty} \frac{|i|}{e^{\pi|i|}}$

Split the summation over the domains of the piecewise absolute value function:

$$= \frac{1}{k}\left(\sum_{i=-\infty}^{-1} \frac{-i}{e^{-i\pi}} + \sum_{i=1}^{\infty} \frac{i}{e^{i\pi}}\right) = \frac{1}{k}\left(\sum_{i=1}^{\infty} \frac{i}{e^{i\pi}} + \sum_{i=1}^{\infty} \frac{i}{e^{i\pi}}\right) = \frac{2}{k} \sum_{i=1}^{\infty} \frac{i}{e^{i\pi}} = \frac{2}{k} \sum_{i=1}^{\infty} i\, e^{-i\pi}$$

Expand the summation as follows:

$$\sum_{i=1}^{\infty} i\, e^{-i\pi} = 1\, e^{-\pi} + 2\, e^{-2\pi} + 3\, e^{-3\pi} + \ldots$$

$$= (e^{-\pi} + e^{-2\pi} + e^{-3\pi} + \ldots) + (e^{-2\pi} + e^{-3\pi} + \ldots) + (e^{-3\pi} + \ldots) + \ldots$$

$$= \frac{e^{-\pi}}{1 - e^{-\pi}} + \frac{e^{-2\pi}}{1 - e^{-\pi}} + \frac{e^{-3\pi}}{1 - e^{-\pi}} + \ldots$$

$$= \frac{1}{1 - e^{-\pi}}\left(e^{-\pi} + e^{-2\pi} + e^{-3\pi} + \ldots\right)$$

$$= \frac{e^{-\pi}}{(1 - e^{-\pi})^2}$$

Therefore the summation is $\frac{2}{k} \times \frac{e^{-\pi}}{(1 - e^{-\pi})^2} = \frac{4\, e^{-\pi}}{(e^{\pi/2} + e^{-\pi/2})(1 - e^{-\pi})^2}$

$$= \frac{4}{(e^{\pi/2} + e^{-\pi/2})\, e^{\pi}(1 - e^{-\pi})^2} = \frac{4}{(e^{\pi/2} + e^{-\pi/2})\, e^{\pi}\,(1 - 2e^{-\pi} + e^{-2\pi})} = \frac{4}{(e^{\pi/2} + e^{-\pi/2})\,(e^{\pi} + e^{-\pi} - 2)}$$

To make required factorisation easier to spot, substitute $x = e^{\pi/2}$. We have

$$= \frac{4}{(x + x^{-1})(x^2 + x^{-2} - 2)} = \frac{4x^3}{(x^2 + 1)(x^4 - 2x^2 + 1)} = \frac{4x^3}{x^6 - x^4 - x^2 + 1}$$

$$= \frac{4x^3}{(x^4 - 1)(x^2 - 1)} = \frac{4}{(x^2 - x^{-2})(x - x^{-1})} = \frac{4}{\left(e^{\pi} - e^{-\pi}\right)\left(e^{\pi/2} - e^{-\pi/2}\right)}.$$