DIFFERENTIAL EQUATIONS

Second Order Differential Equations

Paul Dawkins

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Preface

Here are my notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I've tried to make these notes as self-contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general, I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 3 : Second Order Differential Equations

In the previous chapter we looked at first order differential equations. In this chapter we will move on to second order differential equations. Just as we did in the last chapter we will look at some special cases of second order differential equations that we can solve. Unlike the previous chapter however, we are going to have to be even more restrictive as to the kinds of differential equations that we'll look at. This will be required in order for us to actually be able to solve them.

Here is a list of topics that will be covered in this chapter.

Basic Concepts – In this section give an in depth discussion on the process used to solve homogeneous, linear, second order differential equations, $ay'' + by' + cy = 0$. We derive the characteristic polynomial and discuss how the Principle of Superposition is used to get the general solution.

Real Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, $ay'' + by' + cy = 0$, in which the roots of the characteristic polynomial, $ar^2 + br + c = 0$, are real distinct roots.

Complex Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, $ay'' + by' + cy = 0$, in which the roots of the characteristic polynomial, $ar^2 + br + c = 0$, are complex roots. We will also derive from the complex roots the standard solution that is typically used in this case that will not involve complex numbers.

Repeated Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, $ay'' + by' + cy = 0$, in which the roots of the characteristic polynomial, $ar^2 + br + c = 0$, are repeated, *i.e.* double, roots. We will use reduction of order to derive the second solution needed to get a general solution in this case.

Reduction of Order – In this section we will discuss reduction of order, the process used to derive the solution to the repeated roots case for homogeneous linear second order differential equations, in greater detail. This will be one of the few times in this chapter that non-constant coefficient differential equation will be looked at.

Fundamental Sets of Solutions – In this section we will a look at some of the theory behind the solution to second order differential equations. We define fundamental sets of solutions and discuss how they can be used to get a general solution to a homogeneous second order differential equation. We will also define the Wronskian and show how it can be used to determine if a pair of solutions are a fundamental set of solutions.

More on the Wronskian – In this section we will examine how the Wronskian, introduced in the previous section, can be used to determine if two functions are linearly independent or linearly dependent. We will also give and an alternate method for finding the Wronskian.

Nonhomogeneous Differential Equations – In this section we will discuss the basics of solving nonhomogeneous differential equations. We define the complimentary and particular solution and give the form of the general solution to a nonhomogeneous differential equation.

Undetermined Coefficients – In this section we introduce the method of undetermined coefficients to find particular solutions to nonhomogeneous differential equation. We work a wide variety of examples illustrating the many guidelines for making the initial guess of the form of the particular solution that is needed for the method.

Variation of Parameters – In this section we introduce the method of variation of parameters to find particular solutions to nonhomogeneous differential equation. We give a detailed examination of the method as well as derive a formula that can be used to find particular solutions.

Mechanical Vibrations – In this section we will examine mechanical vibrations. In particular we will model an object connected to a spring and moving up and down. We also allow for the introduction of a damper to the system and for general external forces to act on the object. Note as well that while we example mechanical vibrations in this section a simple change of notation (and corresponding change in what the quantities represent) can move this into almost any other engineering field.

Section 3-1 : Basic Concepts

In this chapter we will be looking exclusively at linear second order differential equations. The most general linear second order differential equation is in the form.

$$p(t)y'' + q(t)y' + r(t)y = g(t) \tag{1}$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$ay'' + by' + cy = g(t) \tag{2}$$

Where possible we will use (1) just to make the point that certain facts, theorems, properties, and/or techniques can be used with the non-constant form. However, most of the time we will be using (2) as it can be fairly difficult to solve second order non-constant coefficient differential equations.

Initially we will make our life easier by looking at differential equations with *g(t)* = 0. When *g(t)* = 0 we call the differential equation **homogeneous** and when $g(t) \neq 0$ we call the differential equation **nonhomogeneous**.

So, let's start thinking about how to go about solving a constant coefficient, homogeneous, linear, second order differential equation. Here is the general constant coefficient, homogeneous, linear, second order differential equation.

$$ay'' + by' + cy = 0$$

It's probably best to start off with an example. This example will lead us to a very important fact that we will use in every problem from this point on. The example will also give us clues into how to go about solving these in general.

Example 1 Determine some solutions to
$$y'' - 9y = 0$$

Solution
We can get some solutions here simply by inspection. We need functions whose second derivative is 9 times the original function. One of the first functions that I can think of that comes back to itself after two derivatives is an exponential function and with proper exponents the 9 will get taken care of as well.

So, it looks like the following two functions are solutions.
$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$

We'll leave it to you to verify that these are in fact solutions.

These two functions are not the only solutions to the differential equation however. Any of the following are also solutions to the differential equation.

$$y(t) = -9\mathbf{e}^{3t} \qquad\qquad\qquad y(t) = 123\mathbf{e}^{3t}$$

$$y(t) = 56\mathbf{e}^{-3t} \qquad\qquad\qquad y(t) = \frac{14}{9}\mathbf{e}^{-3t}$$

$$y(t) = 7\mathbf{e}^{3t} - 6\mathbf{e}^{-3t} \qquad\qquad y(t) = -92\mathbf{e}^{3t} - 16\mathbf{e}^{-3t}$$

In fact if you think about it any function that is in the form
$$y(t) = c_1\mathbf{e}^{3t} + c_2\mathbf{e}^{-3t}$$
will be a solution to the differential equation.

This example leads us to a very important fact that we will use in practically every problem in this chapter.

Principle of Superposition

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, homogeneous differential equation then so is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \qquad\qquad\qquad \textbf{(3)}$$

Note that we didn't include the restriction of constant coefficient or second order in this. This will work for any linear homogeneous differential equation.

If we further assume second order and one other condition (which we'll give in a second) we can go a step further.

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, second order homogeneous differential equation and they are "nice enough" then the general solution to the linear, second order homogeneous differential equation is given by (3).

So, just what do we mean by "nice enough"? We'll hold off on that until a later section. At this point you'll hopefully believe it when we say that specific functions are "nice enough".

So, if we now make the assumption that we are dealing with a linear, second order homogeneous differential equation, we now know that (3) will be its general solution. The next question that we can ask is how to find the constants c_1 and c_2. Since we have two constants it makes sense, hopefully, that we will need two equations, or conditions, to find them.

One way to do this is to specify the value of the solution at two distinct points, or,
$$y(t_0) = y_0 \qquad\qquad\qquad y(t_1) = y_1$$

These are typically called boundary values and are not really the focus of this course so we won't be working with them here. We do give a brief introduction to boundary values in a later chapter if you are interested in seeing how they work and some of the issues that arise when working with boundary values.

Another way to find the constants would be to specify the value of the solution and its derivative at a particular point. Or,

$$y(t_0) = y_0 \qquad\qquad y'(t_0) = y'_0$$

These are the two conditions that we'll be using here. As with the first order differential equations these will be called initial conditions.

Example 2 Solve the following IVP.
$$y'' - 9y = 0 \qquad y(0) = 2 \qquad y'(0) = -1$$

Solution
First, the two functions
$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$
are "nice enough" for us to form the general solution to the differential equation. At this point, please just believe this. You will be able to verify this for yourself in a couple of sections.

The general solution to our differential equation is then
$$y(t) = c_1 \mathbf{e}^{-3t} + c_2 \mathbf{e}^{3t}$$

Now all we need to do is apply the initial conditions. This means that we need the derivative of the solution.
$$y'(t) = -3c_1 \mathbf{e}^{-3t} + 3c_2 \mathbf{e}^{3t}$$

Plug in the initial conditions
$$2 = y(0) = c_1 + c_2$$
$$-1 = y'(0) = -3c_1 + 3c_2$$

This gives us a system of two equations and two unknowns that can be solved. Doing this yields
$$c_1 = \frac{7}{6} \qquad\qquad c_2 = \frac{5}{6}$$

The solution to the IVP is then,
$$y(t) = \frac{7}{6}\mathbf{e}^{-3t} + \frac{5}{6}\mathbf{e}^{3t}$$

Up to this point we've only looked at a single differential equation and we got its solution by inspection. For a rare few differential equations we can do this. However, for the vast majority of the second order differential equations out there we will be unable to do this.

So, we would like a method for arriving at the two solutions we will need in order to form a general solution that will work for any linear, constant coefficient, second order homogeneous differential equation. This is easier than it might initially look.

We will use the solutions we found in the first example as a guide. All of the solutions in this example were in the form
$$y(t) = \mathbf{e}^{rt}$$

Note, that we didn't include a constant in front of it since we can literally include any constant that we want and still get a solution. The important idea here is to get the exponential function. Once we have that we can add on constants to our hearts content.

So, let's assume that all solutions to

$$ay'' + by' + cy = 0 \tag{4}$$

will be of the form

$$y(t) = \mathbf{e}^{rt} \tag{5}$$

To see if we are correct all we need to do is plug this into the differential equation and see what happens. So, let's get some derivatives and then plug in.

$$y'(t) = r\mathbf{e}^{rt} \qquad\qquad y''(t) = r^2\mathbf{e}^{rt}$$

$$a\left(r^2\mathbf{e}^{rt}\right) + b\left(r\mathbf{e}^{rt}\right) + c\left(\mathbf{e}^{rt}\right) = 0$$

$$\mathbf{e}^{rt}\left(ar^2 + br + c\right) = 0$$

So, if (5) is to be a solution to (4) then the following must be true

$$\mathbf{e}^{rt}\left(ar^2 + br + c\right) = 0$$

This can be reduced further by noting that exponentials are never zero. Therefore, (5) will be a solution to (4) provided r is a solution to

$$ar^2 + br + c = 0 \tag{6}$$

This equation is typically called the **characteristic equation** for (4).

Okay, so how do we use this to find solutions to a linear, constant coefficient, second order homogeneous differential equation? First write down the characteristic equation, (6), for the differential equation, (4). This will be a quadratic equation and so we should expect two roots, r_1 and r_2. Once we have these two roots we have two solutions to the differential equation.

$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t} \tag{7}$$

Let's take a look at a quick example.

Example 3 Find two solutions to

$$y'' - 9y = 0$$

Solution
This is the same differential equation that we looked at in the first example. This time however, let's not just guess. Let's go through the process as outlined above to see the functions that we guess above are the same as the functions the process gives us.

First write down the characteristic equation for this differential equation and solve it.

$$r^2 - 9 = 0 \qquad \Rightarrow \qquad r = \pm 3$$

The two roots are 3 and -3. Therefore, two solutions are

$$y_1(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{-3t}$$

These match up with the first guesses that we made in the first example.

You'll notice that we neglected to mention whether or not the two solutions listed in (7) are in fact "nice enough" to form the general solution to (4). This was intentional. We have three cases that we need to look at and this will be addressed differently in each of these cases.

So, what are the cases? As we previously noted the characteristic equation is quadratic and so will have two roots, r_1 and r_2. The roots will have three possible forms. These are

1. Real, distinct roots, $r_1 \neq r_2$.
2. Complex root, $r_{1,2} = \lambda \pm \mu i$.
3. Double roots, $r_1 = r_2 = r$.

The next three sections will look at each of these in some more depth, including giving forms for the solution that will be "nice enough" to get a general solution.

Section 3-2 : Real & Distinct Roots

It's time to start solving constant coefficient, homogeneous, linear, second order differential equations. So, let's recap how we do this from the last section. We start with the differential equation.
$$ay'' + by' + cy = 0$$

Write down the characteristic equation.
$$ar^2 + br + c = 0$$

Solve the characteristic equation for the two roots, r_1 and r_2. This gives the two solutions
$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t}$$

Now, if the two roots are real and distinct (*i.e.* $r_1 \neq r_2$) it will turn out that these two solutions are "nice enough" to form the general solution
$$y(t) = c_1 \mathbf{e}^{r_1 t} + c_2 \mathbf{e}^{r_2 t}$$

As with the last section, we'll ask that you believe us when we say that these are "nice enough". You will be able to prove this easily enough once we reach a later section.

With real, distinct roots there really isn't a whole lot to do other than work a couple of examples so let's do that.

Example 1 Solve the following IVP.
$$y'' + 11y' + 24y = 0 \qquad y(0) = 0 \qquad y'(0) = -7$$

Solution
The characteristic equation is
$$r^2 + 11r + 24 = 0$$
$$(r + 8)(r + 3) = 0$$

Its roots are r_1 = - 8 and r_2 = -3 and so the general solution and its derivative is.
$$y(t) = c_1 \mathbf{e}^{-8t} + c_2 \mathbf{e}^{-3t}$$
$$y'(t) = -8c_1 \mathbf{e}^{-8t} - 3c_2 \mathbf{e}^{-3t}$$

Now, plug in the initial conditions to get the following system of equations.
$$0 = y(0) = c_1 + c_2$$
$$-7 = y'(0) = -8c_1 - 3c_2$$

Solving this system gives $c_1 = \frac{7}{5}$ and $c_2 = -\frac{7}{5}$. The actual solution to the differential equation is then
$$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$$

Example 2 Solve the following IVP
$$y'' + 3y' - 10y = 0 \qquad y(0) = 4 \qquad y'(0) = -2$$

Solution
The characteristic equation is
$$r^2 + 3r - 10 = 0$$
$$(r+5)(r-2) = 0$$

Its roots are r_1 = - 5 and r_2 = 2 and so the general solution and its derivative is.
$$y(t) = c_1 \mathbf{e}^{-5t} + c_2 \mathbf{e}^{2t}$$
$$y'(t) = -5c_1 \mathbf{e}^{-5t} + 2c_2 \mathbf{e}^{2t}$$

Now, plug in the initial conditions to get the following system of equations.
$$4 = y(0) = c_1 + c_2$$
$$-2 = y'(0) = -5c_1 + 2c_2$$

Solving this system gives $c_1 = \frac{10}{7}$ and $c_2 = \frac{18}{7}$. The actual solution to the differential equation is then
$$y(t) = \frac{10}{7} \mathbf{e}^{-5t} + \frac{18}{7} \mathbf{e}^{2t}$$

Example 3 Solve the following IVP.
$$3y'' + 2y' - 8y = 0 \qquad y(0) = -6 \qquad y'(0) = -18$$

Solution
The characteristic equation is
$$3r^2 + 2r - 8 = 0$$
$$(3r-4)(r+2) = 0$$

Its roots are $r_1 = \frac{4}{3}$ and r_2 = -2 and so the general solution and its derivative is.
$$y(t) = c_1 \mathbf{e}^{\frac{4t}{3}} + c_2 \mathbf{e}^{-2t}$$
$$y'(t) = \frac{4}{3} c_1 \mathbf{e}^{\frac{4t}{3}} - 2c_2 \mathbf{e}^{-2t}$$

Now, plug in the initial conditions to get the following system of equations.
$$-6 = y(0) = c_1 + c_2$$
$$-18 = y'(0) = \frac{4}{3}c_1 - 2c_2$$

Solving this system gives c_1 = -9 and c_2 = 3. The actual solution to the differential equation is then.
$$y(t) = -9\mathbf{e}^{\frac{4t}{3}} + 3\mathbf{e}^{-2t}$$

Example 4 Solve the following IVP
$$4y'' - 5y' = 0 \qquad y(-2) = 0 \qquad y'(-2) = 7$$

Solution

The characteristic equation is
$$4r^2 - 5r = 0$$
$$r(4r - 5) = 0$$

The roots of this equation are $r_1 = 0$ and $r_2 = \frac{5}{4}$. Here is the general solution as well as its derivative.
$$y(t) = c_1 \mathbf{e}^0 + c_2 \mathbf{e}^{\frac{5t}{4}} = c_1 + c_2 \mathbf{e}^{\frac{5t}{4}}$$
$$y'(t) = \frac{5}{4} c_2 \mathbf{e}^{\frac{5t}{4}}$$

Up to this point all of the initial conditions have been at $t = 0$ and this one isn't. Don't get too locked into initial conditions always being at $t = 0$ and you just automatically use that instead of the actual value for a given problem.

So, plugging in the initial conditions gives the following system of equations to solve.
$$0 = y(-2) = c_1 + c_2 \mathbf{e}^{-\frac{5}{2}}$$
$$7 = y'(-2) = \frac{5}{4} c_2 \mathbf{e}^{-\frac{5}{2}}$$

Solving this gives.
$$c_1 = -\frac{28}{5} \qquad c_2 = \frac{28}{5} \mathbf{e}^{\frac{5}{2}}$$

The solution to the differential equation is then.
$$y(t) = -\frac{28}{5} + \frac{28}{5} \mathbf{e}^{\frac{5}{2}} \mathbf{e}^{\frac{5t}{4}} = -\frac{28}{5} + \frac{28}{5} \mathbf{e}^{\frac{5t}{4} + \frac{5}{2}}$$

In a differential equations class most instructors (including me….) tend to use initial conditions at $t = 0$ because it makes the work a little easier for the students as they are trying to learn the subject. However, there is no reason to always expect that this will be the case, so do not start to always expect initial conditions at $t = 0$!

Let's do one final example to make another point that you need to be made aware of.

Example 5 Find the general solution to the following differential equation.
$$y'' - 6y' - 2y = 0$$

Solution

The characteristic equation is.
$$r^2 - 6r - 2 = 0$$

The roots of this equation are.

$$r_{1,2} = 3 \pm \sqrt{11}$$

Now, do NOT get excited about these roots they are just two real numbers.

$$r_1 = 3 + \sqrt{11} \qquad \text{and} \qquad r_1 = 3 - \sqrt{11}$$

Admittedly they are not as nice looking as we may be used to, but they are just real numbers. Therefore, the general solution is

$$y(t) = c_1 \mathbf{e}^{\left(3 + \sqrt{11}\right)t} + c_2 \mathbf{e}^{\left(3 - \sqrt{11}\right)t}$$

If we had initial conditions we could proceed as we did in the previous two examples although the work would be somewhat messy and so we aren't going to do that for this example.

The point of the last example is make sure that you don't get to used to "nice", simple roots. In practice roots of the characteristic equation will generally not be nice, simple integers or fractions so don't get too used to them!

Section 3-3 : Complex Roots

In this section we will be looking at solutions to the differential equation

$$ay'' + by' + cy = 0$$

in which roots of the characteristic equation,

$$ar^2 + br + c = 0$$

are complex roots in the form $r_{1,2} = \lambda \pm \mu i$.

Now, recall that we arrived at the characteristic equation by assuming that all solutions to the differential equation will be of the form

$$y(t) = \mathbf{e}^{rt}$$

Plugging our two roots into the general form of the solution gives the following solutions to the differential equation.

$$y_1(t) = \mathbf{e}^{(\lambda + \mu i)t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{(\lambda - \mu i)t}$$

Now, these two functions are "nice enough" (there's those words again… we'll get around to defining them eventually) to form the general solution. We do have a problem however. Since we started with only real numbers in our differential equation we would like our solution to only involve real numbers. The two solutions above are complex and so we would like to get our hands on a couple of solutions ("nice enough" of course…) that are real.

To do this we'll need Euler's Formula.

$$\mathbf{e}^{i\theta} = \cos\theta + i\sin\theta$$

A nice variant of Euler's Formula that we'll need is.

$$\mathbf{e}^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Now, split up our two solutions into exponentials that only have real exponents and exponentials that only have imaginary exponents. Then use Euler's formula, or its variant, to rewrite the second exponential.

$$y_1(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)$$
$$y_2(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{-i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right)$$

This doesn't eliminate the complex nature of the solutions, but it does put the two solutions into a form that we can eliminate the complex parts.

Recall from the basics section that if two solutions are "nice enough" then any solution can be written as a combination of the two solutions. In other words,

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

will also be a solution.

Using this let's notice that if we add the two solutions together we will arrive at.

$$y_1(t) + y_2(t) = 2\mathbf{e}^{\lambda t} \cos(\mu t)$$

This is a real solution and just to eliminate the extraneous 2 let's divide everything by a 2. This gives the first real solution that we're after.

$$u(t) = \frac{1}{2} y_1(t) + \frac{1}{2} y_2(t) = \mathbf{e}^{\lambda t} \cos(\mu t)$$

Note that this is just equivalent to taking

$$c_1 = c_2 = \frac{1}{2}$$

Now, we can arrive at a second solution in a similar manner. This time let's subtract the two original solutions to arrive at.

$$y_1(t) - y_2(t) = 2i\,\mathbf{e}^{\lambda t} \sin(\mu t)$$

On the surface this doesn't appear to fix the problem as the solution is still complex. However, upon learning that the two constants, c_1 and c_2 can be complex numbers we can arrive at a real solution by dividing this by $2i$. This is equivalent to taking

$$c_1 = \frac{1}{2i} \qquad \text{and} \qquad c_2 = -\frac{1}{2i}$$

Our second solution will then be

$$v(t) = \frac{1}{2i} y_1(t) - \frac{1}{2i} y_2(t) = \mathbf{e}^{\lambda t} \sin(\mu t)$$

We now have two solutions (we'll leave it to you to check that they are in fact solutions) to the differential equation.

$$u(t) = \mathbf{e}^{\lambda t} \cos(\mu t) \qquad \text{and} \qquad v(t) = \mathbf{e}^{\lambda t} \sin(\mu t)$$

It also turns out that these two solutions are "nice enough" to form a general solution.

So, if the roots of the characteristic equation happen to be $r_{1,2} = \lambda \pm \mu i$ the general solution to the differential equation is.

$$y(t) = c_1 \mathbf{e}^{\lambda t} \cos(\mu t) + c_2 \mathbf{e}^{\lambda t} \sin(\mu t)$$

Let's take a look at a couple of examples now.

Example 1 Solve the following IVP.
$$y'' - 4y' + 9y = 0 \qquad y(0) = 0 \qquad y'(0) = -8$$

Solution
The characteristic equation for this differential equation is.
$$r^2 - 4r + 9 = 0$$

The roots of this equation are $r_{1,2} = 2 \pm \sqrt{5}\,i$. The general solution to the differential equation is then.

$$y(t) = c_1 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right) + c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Now, you'll note that we didn't differentiate this right away as we did in the last section. The reason for this is simple. While the differentiation is not terribly difficult, it can get a little messy. So, first looking at the initial conditions we can see from the first one that if we just applied it we would get the following.

$$0 = y(0) = c_1$$

In other words, the first term will drop out in order to meet the first condition. This makes the solution, along with its derivative

$$y(t) = c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$
$$y'(t) = 2c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right) + \sqrt{5}c_2 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right)$$

A much nicer derivative than if we'd done the original solution. Now, apply the second initial condition to the derivative to get.

$$-8 = y'(0) = \sqrt{5}c_2 \qquad \Rightarrow \qquad c_2 = -\frac{8}{\sqrt{5}}$$

The actual solution is then.

$$y(t) = -\frac{8}{\sqrt{5}} \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Example 2 Solve the following IVP.
$$y'' - 8y' + 17y = 0 \qquad y(0) = -4 \quad y'(0) = -1$$

Solution
The characteristic equation this time is.
$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$y(t) = c_1 \mathbf{e}^{4t} \cos(t) + c_2 \mathbf{e}^{4t} \sin(t)$$
$$y'(t) = 4c_1 \mathbf{e}^{4t} \cos(t) - c_1 \mathbf{e}^{4t} \sin(t) + 4c_2 \mathbf{e}^{4t} \sin(t) + c_2 \mathbf{e}^{4t} \cos(t)$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$-4 = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + c_2$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.
$$y(t) = -4e^{4t} \cos(t) + 15e^{4t} \sin(t)$$

Example 3 Solve the following IVP.
$$4y'' + 24y' + 37y = 0 \qquad y(\pi) = 1 \qquad y'(\pi) = 0$$

Solution
The characteristic equation this time is.
$$4r^2 + 24r + 37 = 0$$

The roots of this are $r_{1,2} = -3 \pm \frac{1}{2}i$. The general solution as well as its derivative is

$$y(t) = c_1 e^{-3t} \cos\left(\frac{t}{2}\right) + c_2 e^{-3t} \sin\left(\frac{t}{2}\right)$$

$$y'(t) = -3c_1 e^{-3t} \cos\left(\frac{t}{2}\right) - \frac{c_1}{2} e^{-3t} \sin\left(\frac{t}{2}\right) - 3c_2 e^{-3t} \sin\left(\frac{t}{2}\right) + \frac{c_2}{2} e^{-3t} \cos\left(\frac{t}{2}\right)$$

Applying the initial conditions gives the following system.

$$1 = y(\pi) = c_1 e^{-3\pi} \cos\left(\frac{\pi}{2}\right) + c_2 e^{-3\pi} \sin\left(\frac{\pi}{2}\right) = c_2 e^{-3\pi}$$

$$0 = y'(\pi) = -\frac{c_1}{2} e^{-3\pi} - 3c_2 e^{-3\pi}$$

Do not forget to plug the $t = \pi$ into the exponential! This is one of the more common mistakes that students make on these problems. Also, make sure that you evaluate the trig functions as much as possible in these cases. It will only make your life simpler. Solving this system gives
$$c_1 = -6e^{3\pi} \qquad\qquad c_2 = e^{3\pi}$$

The actual solution to the IVP is then.

$$y(t) = -6e^{3\pi}e^{-3t} \cos\left(\frac{t}{2}\right) + e^{3\pi}e^{-3t} \sin\left(\frac{t}{2}\right)$$

$$y(t) = -6e^{-3(t-\pi)} \cos\left(\frac{t}{2}\right) + e^{-3(t-\pi)} \sin\left(\frac{t}{2}\right)$$

Let's do one final example before moving on to the next topic.

Example 4 Solve the following IVP.
$$y'' + 16y = 0 \qquad y\left(\frac{\pi}{2}\right) = -10 \qquad y'\left(\frac{\pi}{2}\right) = 3$$

Solution
The characteristic equation for this differential equation and its roots are.
$$r^2 + 16 = 0 \qquad \Rightarrow \qquad r = \pm 4i$$

Be careful with this characteristic polynomial. One of the biggest mistakes students make here is to write it as,

$$r^2 + 16r = 0$$

The problem is that the second term will only have an *r* if the second term in the differential equation has a y' in it and this one clearly does not. Students however, tend to just start at r^2 and write times down until they run out of terms in the differential equation. That can, and often does mean, they write down the wrong characteristic polynomial so be careful.

Okay, back to the problem.

The general solution to this differential equation and its derivative is.

$$y(t) = c_1 \cos(4t) + c_2 \sin(4t)$$
$$y'(t) = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

Plugging in the initial conditions gives the following system.

$$-10 = y\left(\frac{\pi}{2}\right) = c_1 \qquad\qquad c_1 = -10$$

$$3 = y'\left(\frac{\pi}{2}\right) = 4c_2 \qquad\qquad c_2 = \frac{3}{4}$$

So, the constants drop right out with this system and the actual solution is.

$$y(t) = -10\cos(4t) + \frac{3}{4}\sin(4t)$$

Section 3-4 : Repeated Roots

In this section we will be looking at the last case for the constant coefficient, linear, homogeneous second order differential equations. In this case we want solutions to

$$ay'' + by' + cy = 0$$

where solutions to the characteristic equation

$$ar^2 + br + c = 0$$

are double roots $r_1 = r_2 = r$.

This leads to a problem however. Recall that the solutions are

$$y_1(t) = e^{r_1 t} = e^{rt} \qquad\qquad y_2(t) = e^{r_2 t} = e^{rt}$$

These are the same solution and will NOT be "nice enough" to form a general solution. We do promise that we'll define "nice enough" eventually! So, we can use the first solution, but we're going to need a second solution.

Before finding this second solution let's take a little side trip. The reason for the side trip will be clear eventually. From the quadratic formula we know that the roots to the characteristic equation are,

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, since we have double roots we must have

$$b^2 - 4ac = 0$$

This is the only way that we can get double roots and in this case the roots will be

$$r_{1,2} = \frac{-b}{2a}$$

So, the one solution that we've got is

$$y_1(t) = e^{-\frac{bt}{2a}}$$

To find a second solution we will use the fact that a constant times a solution to a linear homogeneous differential equation is also a solution. If this is true then *maybe* we'll get lucky and the following will also be a solution

$$y_2(t) = v(t) y_1(t) = v(t) e^{-\frac{bt}{2a}} \qquad\qquad (1)$$

with a proper choice of *v(t)*. To determine if this in fact can be done, let's plug this back into the differential equation and see what we get. We'll first need a couple of derivatives.

$$y_2'(t) = v' e^{-\frac{bt}{2a}} - \frac{b}{2a} v e^{-\frac{bt}{2a}}$$

$$y_2''(t) = v'' e^{-\frac{bt}{2a}} - \frac{b}{2a} v' e^{-\frac{bt}{2a}} - \frac{b}{2a} v' e^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v e^{-\frac{bt}{2a}}$$

$$= v'' e^{-\frac{bt}{2a}} - \frac{b}{a} v' e^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v e^{-\frac{bt}{2a}}$$

We dropped the (t) part on the *v* to simplify things a little for the writing out of the derivatives. Now, plug these into the differential equation.

$$a\left(v''\,\mathbf{e}^{-\frac{bt}{2a}}-\frac{b}{a}v'\,\mathbf{e}^{-\frac{bt}{2a}}+\frac{b^2}{4a^2}v\,\mathbf{e}^{-\frac{bt}{2a}}\right)+b\left(v'\,\mathbf{e}^{-\frac{bt}{2a}}-\frac{b}{2a}v\,\mathbf{e}^{-\frac{bt}{2a}}\right)+c\left(v\,\mathbf{e}^{-\frac{bt}{2a}}\right)=0$$

We can factor an exponential out of all the terms so let's do that. We'll also collect all the coefficients of *v* and its derivatives.

$$\mathbf{e}^{-\frac{bt}{2a}}\left(av''+\left(-b+b\right)v'+\left(\frac{b^2}{4a}-\frac{b^2}{2a}+c\right)v\right)=0$$

$$\mathbf{e}^{-\frac{bt}{2a}}\left(av''+\left(-\frac{b^2}{4a}+c\right)v\right)=0$$

$$\mathbf{e}^{-\frac{bt}{2a}}\left(av''-\frac{1}{4a}\left(b^2-4ac\right)v\right)=0$$

Now, because we are working with a double root we know that that the second term will be zero. Also exponentials are never zero. Therefore, (1) will be a solution to the differential equation provided *v(t)* is a function that satisfies the following differential equation.

$$av''=0 \qquad \text{OR} \qquad v''=0$$

We can drop the *a* because we know that it can't be zero. If it were we wouldn't have a second order differential equation! So, we can now determine the most general possible form that is allowable for *v(t)*.

$$v'=\int v''\,dt=c \qquad\qquad v(t)=\int v'\,dt=ct+k$$

This is actually more complicated than we need and in fact we can drop both of the constants from this. To see why this is let's go ahead and use this to get the second solution. The two solutions are then

$$y_1(t)=\mathbf{e}^{-\frac{bt}{2a}} \qquad\qquad y_2(t)=\left(ct+k\right)\mathbf{e}^{-\frac{bt}{2a}}$$

Eventually you will be able to show that these two solutions are "nice enough" to form a general solution. The general solution would then be the following.

$$y(t)=c_1\mathbf{e}^{-\frac{bt}{2a}}+c_2\left(ct+k\right)\mathbf{e}^{-\frac{bt}{2a}}$$

$$=c_1\mathbf{e}^{-\frac{bt}{2a}}+\left(c_2ct+c_2k\right)\mathbf{e}^{-\frac{bt}{2a}}$$

$$=\left(c_1+c_2k\right)\mathbf{e}^{-\frac{bt}{2a}}+c_2ct\,\mathbf{e}^{-\frac{bt}{2a}}$$

Notice that we rearranged things a little. Now, *c*, *k*, c_1, and c_2 are all unknown constants so any combination of them will also be unknown constants. In particular, $c_1+c_2 k$ and $c_2 c$ are unknown constants so we'll just rewrite them as follows.

$$y(t)=c_1\mathbf{e}^{-\frac{bt}{2a}}+c_2 t\,\mathbf{e}^{-\frac{bt}{2a}}$$

So, if we go back to the most general form for *v(t)* we can take *c=1* and *k=0* and we will arrive at the same general solution.

Let's recap. If the roots of the characteristic equation are $r_1 = r_2 = r$, then the general solution is then
$$y(t) = c_1 \mathbf{e}^{rt} + c_2 t \mathbf{e}^{rt}$$

Now, let's work a couple of examples.

Example 1 Solve the following IVP.
$$y'' - 4y' + 4y = 0 \qquad y(0) = 12 \qquad y'(0) = -3$$

Solution
The characteristic equation and its roots are.
$$r^2 - 4r + 4 = (r-2)^2 = 0 \qquad\qquad r_{1,2} = 2$$

The general solution and its derivative are
$$y(t) = c_1 \mathbf{e}^{2t} + c_2 t \mathbf{e}^{2t}$$
$$y'(t) = 2c_1 \mathbf{e}^{2t} + c_2 \mathbf{e}^{2t} + 2c_2 t \mathbf{e}^{2t}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.
$$12 = y(0) = c_1$$
$$-3 = y'(0) = 2c_1 + c_2$$

This system is easily solved to get c_1 = 12 and c_2 = -27. The actual solution to the IVP is then.
$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

Example 2 Solve the following IVP.
$$16y'' - 40y' + 25y = 0 \qquad y(0) = 3 \qquad y'(0) = -\frac{9}{4}$$

Solution
The characteristic equation and its roots are.
$$16r^2 - 40r + 25 = (4r - 5)^2 = 0 \qquad r_{1,2} = \frac{5}{4}$$

The general solution and its derivative are
$$y(t) = c_1 \mathbf{e}^{\frac{5t}{4}} + c_2 t \mathbf{e}^{\frac{5t}{4}}$$
$$y'(t) = \frac{5}{4} c_1 \mathbf{e}^{\frac{5t}{4}} + c_2 \mathbf{e}^{\frac{5t}{4}} + \frac{5}{4} c_2 t \mathbf{e}^{\frac{5t}{4}}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$3 = y(0) = c_1$$

$$-\frac{9}{4} = y'(0) = \frac{5}{4}c_1 + c_2$$

This system is easily solve to get c_1 = 3 and c_2 = -6. The actual solution to the IVP is then.

$$y(t) = 3e^{\frac{5t}{4}} - 6te^{\frac{5t}{4}}$$

Example 3 Solve the following IVP

$$y'' + 14y' + 49y = 0 \qquad y(-4) = -1 \quad y'(-4) = 5$$

Solution

The characteristic equation and its roots are.

$$r^2 + 14r + 49 = (r+7)^2 = 0 \qquad\qquad r_{1,2} = -7$$

The general solution and its derivative are

$$y(t) = c_1 e^{-7t} + c_2 t e^{-7t}$$

$$y'(t) = -7c_1 e^{-7t} + c_2 e^{-7t} - 7c_2 t e^{-7t}$$

Plugging in the initial conditions gives the following system of equations.

$$-1 = y(-4) = c_1 e^{28} - 4c_2 e^{28}$$

$$5 = y'(-4) = -7c_1 e^{28} + c_2 e^{28} + 28c_2 e^{28} = -7c_1 e^{28} + 29c_2 e^{28}$$

Solving this system gives the following constants.

$$c_1 = -9e^{-28} \qquad\qquad c_2 = -2e^{-28}$$

The actual solution to the IVP is then.

$$y(t) = -9e^{-28}e^{-7t} - 2te^{-28}e^{-7t}$$

$$y(t) = -9e^{-7(t+4)} - 2te^{-7(t+4)}$$

Section 3-5 : Reduction of Order

We're now going to take a brief detour and look at solutions to non-constant coefficient, second order differential equations of the form.

$$p(t)y'' + q(t)y' + r(t)y = 0$$

In general, finding solutions to these kinds of differential equations can be much more difficult than finding solutions to constant coefficient differential equations. However, if we already know one solution to the differential equation we can use the method that we used in the last section to find a second solution. This method is called **reduction of order**.

Let's take a quick look at an example to see how this is done.

Example 1 Find the general solution to

$$2t^2 y'' + ty' - 3y = 0, \quad t > 0$$

given that $y_1(t) = t^{-1}$ is a solution.

Solution
Reduction of order requires that a solution already be known. Without this known solution we won't be able to do reduction of order.

Once we have this first solution we will then assume that a second solution will have the form

$$y_2(t) = v(t) y_1(t) \tag{1}$$

for a proper choice of *v(t)*. To determine the proper choice, we plug the guess into the differential equation and get a new differential equation that can be solved for *v(t)*.

So, let's do that for this problem. Here is the form of the second solution as well as the derivatives that we'll need.

$$y_2(t) = t^{-1}v \qquad\qquad y_2'(t) = -t^{-2}v + t^{-1}v' \qquad\qquad y_2''(t) = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$

Plugging these into the differential equation gives

$$2t^2\left(2t^{-3}v - 2t^{-2}v' + t^{-1}v''\right) + t\left(-t^{-2}v + t^{-1}v'\right) - 3\left(t^{-1}v\right) = 0$$

Rearranging and simplifying gives

$$2tv'' + (-4+1)v' + \left(4t^{-1} - t^{-1} - 3t^{-1}\right)v = 0$$
$$2tv'' - 3v' = 0$$

Note that upon simplifying the only terms remaining are those involving the derivatives of *v*. The term involving *v* drops out. If you've done all of your work correctly this should always happen. Sometimes, as in the repeated roots case, the first derivative term will also drop out.

So, in order for (1) to be a solution then *v* must satisfy

$$2tv'' - 3v' = 0 \qquad\qquad\qquad (2)$$

This appears to be a problem. In order to find a solution to a second order non-constant coefficient differential equation we need to solve a different second order non-constant coefficient differential equation.

However, this isn't the problem that it appears to be. Because the term involving the *v* drops out we can actually solve (2) and we can do it with the knowledge that we already have at this point. We will solve this by making the following **change of variable**.

$$w = v' \qquad\qquad \Rightarrow \qquad\qquad w' = v''$$

With this change of variable (2) becomes

$$2tw' - 3w = 0$$

and this is a linear, first order differential equation that we can solve. This also explains the name of this method. We've managed to reduce a second order differential equation down to a first order differential equation.

This is a fairly simple first order differential equation so I'll leave the details of the solving to you. If you need a refresher on solving linear, first order differential equations go back to the second chapter and check out that section. The solution to this differential equation is

$$w(t) = c\, t^{\frac{3}{2}}$$

Now, this is not quite what we were after.   We are after a solution to (2). However, we can now find this. Recall our change of variable.

$$v' = w$$

With this we can easily solve for *v(t)*.

$$v(t) = \int w\,dt = \int c t^{\frac{3}{2}}\,dt = \frac{2}{5}c t^{\frac{5}{2}} + k$$

This is the most general possible *v(t)* that we can use to get a second solution. So, just as we did in the repeated roots section, we can choose the constants to be anything we want so choose them to clear out all the extraneous constants. In this case we can use

$$c = \frac{5}{2} \qquad\qquad k = 0$$

Using these gives the following for *v(t)* and for the second solution.

$$v(t) = t^{\frac{5}{2}} \qquad \Rightarrow \qquad y_2(t) = t^{-1}\left(t^{\frac{5}{2}}\right) = t^{\frac{3}{2}}$$

Then general solution will then be,

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$$

If we had been given initial conditions we could then differentiate, apply the initial conditions and solve for the constants.

Reduction of order, the method used in the previous example can be used to find second solutions to differential equations. However, this does require that we already have a solution and often finding that first solution is a very difficult task and often in the process of finding the first solution you will also get the second solution without needing to resort to reduction of order. So, for those cases when we do have a first solution this is a nice method for getting a second solution.

Let's do one more example.

Example 2 Find the general solution to
$$t^2 y'' + 2ty' - 2y = 0$$
given that $y_1(t) = t$ is a solution.

Solution
The form for the second solution as well as its derivatives are,
$$y_2(t) = tv \qquad\qquad y_2'(t) = v + tv' \qquad\qquad y_2''(t) = 2v' + tv''$$

Plugging these into the differential equation gives,
$$t^2\left(2v' + tv''\right) + 2t\left(v + tv'\right) - 2\left(tv\right) = 0 = 0$$

Rearranging and simplifying gives the differential equation that we'll need to solve in order to determine the correct *v* that we'll need for the second solution.
$$t^3 v'' + 4t^2 v' = 0$$

Next use the variable transformation as we did in the previous example.
$$w = v' \qquad\qquad \Rightarrow \qquad\qquad w' = v''$$

With this change of variable the differential equation becomes
$$t^3 w' + 4t^2 w = 0$$
and this is a linear, first order differential equation that we can solve. We'll leave the details of the solution process to you.
$$w(t) = c\,t^{-4}$$

Now solve for *v(t)*.
$$v(t) = \int w\,dt = \int ct^{-4}\,dt = -\frac{1}{3}ct^{-3} + k$$

As with the first example we'll drop the constants and use the following *v(t)*
$$v(t) = t^{-3} \qquad\qquad \Rightarrow \qquad\qquad y_2(t) = t\left(t^{-3}\right) = t^{-2}$$

Then general solution will then be,
$$y(t) = c_1 t + \frac{c_2}{t^2}$$

On a side note, both of the differential equations in this section were of the form,

$$t^2 y'' + \alpha\, t\, y' + \beta y = 0$$

These are called Euler differential equations and are fairly simple to solve directly for both solutions. To see how to solve these directly take a look at the Euler Differential Equation section.

Section 3-6 : Fundamental Sets of Solutions

The time has finally come to define "nice enough". We've been using this term throughout the last few sections to describe those solutions that could be used to form a general solution and it is now time to officially define it.

First, because everything that we're going to be doing here only requires linear and homogeneous we won't require constant coefficients in our differential equation. So, let's start with the following IVP.

$$p(t)y'' + q(t)y' + r(t)y = 0$$
$$y(t_0) = y_0 \qquad y'(t_0) = y_0' \tag{1}$$

Let's also suppose that we have already found two solutions to this differential equation, $y_1(t)$ and $y_2(t)$. We know from the Principle of Superposition that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2}$$

will also be a solution to the differential equation. What we want to know is whether or not it will be a general solution. In order for (2) to be considered a general solution it must satisfy the general initial conditions in (1).

$$y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

This will also imply that any solution to the differential equation can be written in this form.

So, let's see if we can find constants that will satisfy these conditions. First differentiate (2) and plug in the initial conditions.

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0)$$
$$y_0' = y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) \tag{3}$$

Since we are assuming that we've already got the two solutions everything in this system is technically known and so this is a system that can be solved for c_1 and c_2. This can be done in general using Cramer's Rule. Using Cramer's Rule gives the following solution.

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \tag{4}$$

where,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is the determinant of a 2x2 matrix. If you don't know about determinants that is okay, just use the formula that we've provided above.

Now, (4) will give the solution to the system (3). Note that in practice we generally don't use Cramer's Rule to solve systems, we just proceed in a straightforward manner and solve the system using basic algebra techniques. So, why did we use Cramer's Rule here then?

We used Cramer's Rule because we can use (4) to develop a condition that will allow us to determine when we can solve for the constants. All three (yes three, the denominators are the same!) of the quantities in (4) are just numbers and the only thing that will prevent us from actually getting a solution will be when the denominator is zero.

The quantity in the denominator is called the **Wronskian** and is denoted as

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - g(t)f'(t)$$

When it is clear what the functions and/or *t* are we often just denote the Wronskian by *W*.

Let's recall what we were after here. We wanted to determine when two solutions to (1) would be nice enough to form a general solution. The two solutions will form a general solution to (1) if they satisfy the general initial conditions given in (1) and we can see from Cramer's Rule that they will satisfy the initial conditions provided the Wronskian isn't zero. Or,

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0$$

So, suppose that $y_1(t)$ and $y_2(t)$ are two solutions to (1) and that $W(y_1,y_2)(t) \neq 0$. Then the two solutions are called a **fundamental set of solutions** and the general solution to (1) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

We know now what "nice enough" means. Two solutions are "nice enough" if they are a fundamental set of solutions.

So, let's check one of the claims that we made in a previous section. We'll leave the other two to you to check if you'd like to.

Example 1 Back in the complex root section we made the claim that
$$y_1(t) = \mathbf{e}^{\lambda t} \cos(\mu t) \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{\lambda t} \sin(\mu t)$$
were a fundamental set of solutions. Prove that they in fact are.

Solution
So, to prove this we will need to take the Wronskian for these two solutions and show that it isn't zero.

$$W = \begin{vmatrix} \mathbf{e}^{\lambda t}\cos(\mu t) & \mathbf{e}^{\lambda t}\sin(\mu t) \\ \lambda\mathbf{e}^{\lambda t}\cos(\mu t) - \mu\mathbf{e}^{\lambda t}\sin(\mu t) & \lambda\mathbf{e}^{\lambda t}\sin(\mu t) + \mu\mathbf{e}^{\lambda t}\cos(\mu t) \end{vmatrix}$$

$$= \mathbf{e}^{\lambda t}\cos(\mu t)\left(\lambda\mathbf{e}^{\lambda t}\sin(\mu t) + \mu\mathbf{e}^{\lambda t}\cos(\mu t)\right) -$$

$$\mathbf{e}^{\lambda t}\sin(\mu t)\left(\lambda\mathbf{e}^{\lambda t}\cos(\mu t) - \mu\mathbf{e}^{\lambda t}\sin(\mu t)\right)$$

$$= \mu\mathbf{e}^{2\lambda t}\cos^2(\mu t) + \mu\mathbf{e}^{2\lambda t}\sin^2(\mu t)$$

$$= \mu\mathbf{e}^{2\lambda t}\left(\cos^2(\mu t) + \sin^2(\mu t)\right)$$

$$= \mu\mathbf{e}^{2\lambda t}$$

Now, the exponential will never be zero and $\mu \neq 0$ (if it were we wouldn't have complex roots!) and so $W \neq 0$. Therefore, these two solutions are in fact a fundamental set of solutions and so the general solution in this case is.

$$y(t) = c_1\mathbf{e}^{\lambda t}\cos(\mu t) + c_2\mathbf{e}^{\lambda t}\sin(\mu t)$$

Example 2 In the first example that we worked in the <ins>Reduction of Order</ins> section we found a second solution to

$$2t^2 y'' + ty' - 3y = 0$$

Show that this second solution, along with the given solution, form a fundamental set of solutions for the differential equation.

Solution
The two solutions from that example are

$$y_1(t) = t^{-1} \qquad\qquad y_2(t) = t^{\frac{3}{2}}$$

Let's compute the Wronskian of these two solutions.

$$W = \begin{vmatrix} t^{-1} & t^{\frac{3}{2}} \\ -t^{-2} & \frac{3}{2}t^{\frac{1}{2}} \end{vmatrix} = \frac{3}{2}t^{-\frac{1}{2}} - \left(-t^{-\frac{1}{2}}\right) = \frac{5}{2}t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}}$$

So, the Wronskian will never be zero. Note that we can't plug t = 0 into the Wronskian. This would be a problem in finding the constants in the general solution, except that we also can't plug t = 0 into the solution either and so this isn't the problem that it might appear to be.

So, since the Wronskian isn't zero for any t the two solutions form a fundamental set of solutions and the general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$$

as we claimed in that example.

To this point we've found a set of solutions then we've claimed that they are in fact a fundamental set of solutions. Of course, you can now verify all those claims that we've made, however this does bring up a

question. How do we know that for a given differential equation a set of fundamental solutions will exist? The following theorem answers this question.

Theorem

Consider the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

where *p(t)* and *q(t)* are continuous functions on some interval I. Choose t_0 to be any point in the interval I. Let $y_1(t)$ be a solution to the differential equation that satisfies the initial conditions.

$$y(t_0) = 1 \qquad y'(t_0) = 0$$

Let $y_2(t)$ be a solution to the differential equation that satisfies the initial conditions.

$$y(t_0) = 0 \qquad y'(t_0) = 1$$

Then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the differential equation.

It is easy enough to show that these two solutions form a fundamental set of solutions. Just compute the Wronskian.

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

So, fundamental sets of solutions will exist provided we can solve the two IVP's given in the theorem.

Example 3 Use the theorem to find a fundamental set of solutions for
$$y'' + 4y' + 3y = 0$$
using t_0 = 0.

Solution
Using the techniques from the first part of this chapter we can find the two solutions that we've been using to this point.

$$y(t) = \mathbf{e}^{-3t} \qquad\qquad y(t) = \mathbf{e}^{-t}$$

These do form a fundamental set of solutions as we can easily verify. However, they are NOT the set that will be given by the theorem. Neither of these solutions will satisfy either of the two sets of initial conditions given in the theorem. We will have to use these to find the fundamental set of solutions that is given by the theorem.

We know that the following is also a solution to the differential equation.

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t}$$

So, let's apply the first set of initial conditions and see if we can find constants that will work.

$$y(0) = 1 \qquad y'(0) = 0$$

We'll leave it to you to verify that we get the following solution upon doing this.

$$y_1(t) = -\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t}$$

Likewise, if we apply the second set of initial conditions,

$$y(0) = 0 \qquad y'(0) = 1$$

we will get

$$y_2(t) = -\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t}$$

According to the theorem these should form a fundament set of solutions. This is easy enough to check.

$$W = \begin{vmatrix} -\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t} & -\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t} \\ \frac{3}{2}\mathbf{e}^{-3t} - \frac{3}{2}\mathbf{e}^{-t} & \frac{3}{2}\mathbf{e}^{-3t} - \frac{1}{2}\mathbf{e}^{-t} \end{vmatrix}$$

$$= \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t} \right)\left(\frac{3}{2}\mathbf{e}^{-3t} - \frac{1}{2}\mathbf{e}^{-t} \right) - \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t} \right)\left(\frac{3}{2}\mathbf{e}^{-3t} - \frac{3}{2}\mathbf{e}^{-t} \right)$$

$$= \mathbf{e}^{-4t} \neq 0$$

So, we got a completely different set of fundamental solutions from the theorem than what we've been using up to this point. This is not a problem. There are an infinite number of pairs of functions that we could use as a fundamental set of solutions for this problem.

So, which set of fundamental solutions should we use? Well, if we use the ones that we originally found, the general solution would be,

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t}$$

Whereas, if we used the set from the theorem the general solution would be,

$$y(t) = c_1\left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t} \right) + c_2\left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t} \right)$$

This would not be very fun to work with when it came to determining the coefficients to satisfy a general set of initial conditions.

So, which set of fundamental solutions should we use? We should always try to use the set that is the most convenient to use for a given problem.

Section 3-7 : More on the Wronskian

In the previous section we introduced the Wronskian to help us determine whether two solutions were a fundamental set of solutions. In this section we will look at another application of the Wronskian as well as an alternate method of computing the Wronskian.

Let's start with the application. We need to introduce a couple of new concepts first.

Given two non-zero functions $f(x)$ and $g(x)$ write down the following equation.

$$c\,f(x) + k\,g(x) = 0 \qquad\qquad\qquad (1)$$

Notice that $c = 0$ and $k = 0$ will make (1) true for all x regardless of the functions that we use.

Now, if we can find non-zero constants c and k for which (1) will also be true for all x then we call the two functions **linearly dependent**. On the other hand if the only two constants for which (1) is true are $c = 0$ and $k = 0$ then we call the functions **linearly independent.**

Example 1 Determine if the following sets of functions are linearly dependent or linearly independent.

(a) $f(x) = 9\cos(2x)$ $g(x) = 2\cos^2(x) - 2\sin^2(x)$

(b) $f(t) = 2t^2$ $g(t) = t^4$

Solution

(a) $f(x) = 9\cos(2x)$ $g(x) = 2\cos^2(x) - 2\sin^2(x)$

We'll start by writing down (1) for these two functions.

$$c\left(9\cos(2x)\right) + k\left(2\cos^2(x) - 2\sin^2(x)\right) = 0$$

We need to determine if we can find non-zero constants c and k that will make this true for all x or if $c = 0$ and $k = 0$ are the only constants that will make this true for all x. This is often a fairly difficult process. The process can be simplified with a good intuition for this kind of thing, but that's hard to come by, especially if you haven't done many of these kinds of problems.

In this case the problem can be simplified by recalling

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

Using this fact our equation becomes.

$$9c\cos(2x) + 2k\cos(2x) = 0$$
$$(9c + 2k)\cos(2x) = 0$$

With this simplification we can see that this will be zero for any pair of constants c and k that satisfy

$$9c + 2k = 0$$

Among the possible pairs on constants that we could use are the following pairs.

$$c = 1, \qquad k = -\frac{9}{2}$$

$$c = \frac{2}{9}, \qquad k = -1$$

$$c = -2 \qquad k = 9$$

$$c = -\frac{7}{6} \qquad k = \frac{21}{4}$$

etc.

As we're sure you can see there are literally thousands of possible pairs and they can be made as "simple" or as "complicated" as you want them to be.

So, we've managed to find a pair of non-zero constants that will make the equation true for all *x* and so the two functions are linearly dependent.

(b) $f(t) = 2t^2 \qquad\qquad g(t) = t^4$

As with the last part, we'll start by writing down (1) for these functions.

$$2ct^2 + kt^4 = 0$$

In this case there isn't any quick and simple formula to write one of the functions in terms of the other as we did in the first part. So, we're just going to have to see if we can find constants. We'll start by noticing that if the original equation is true, then if we differentiate everything we get a new equation that must also be true. In other words, we've got the following system of two equations in two unknowns.

$$2ct^2 + kt^4 = 0$$

$$4ct + 4kt^3 = 0$$

We can solve this system for *c* and *k* and see what we get. We'll start by solving the second equation for *c*.

$$c = -kt^2$$

Now, plug this into the first equation.

$$2\left(-kt^2\right)t^2 + kt^4 = 0$$

$$-kt^4 = 0$$

Recall that we are after constants that will make this true for all *t*. The only way that this will ever be zero for all *t* is if *k* = 0! So, if *k* = 0 we must also have *c* = 0.

Therefore, we've shown that the only way that

$$2ct^2 + kt^4 = 0$$

will be true for all *t* is to require that *c* = 0 and *k* = 0. The two functions therefore, are linearly independent.

As we saw in the previous examples determining whether two functions are linearly independent or linearly dependent can be a fairly involved process. This is where the Wronskian can help.

Fact

Given two functions *f(x)* and *g(x)* that are differentiable on some interval I.

    (1) If $W(f,g)(x_0) \neq 0$ for some x_0 in I, then *f(x)* and *g(x)* are linearly independent on the interval I.

    (2) If *f(x)* and *g(x)* are linearly dependent on I then *W(f,g)(x)* = 0 for all *x* in the interval I.

Be very careful with this fact. It DOES NOT say that if *W(f,g)(x)* = 0 then *f(x)* and *g(x)* are linearly dependent! In fact, it is possible for two linearly independent functions to have a zero Wronskian!

This fact is used to quickly identify linearly independent functions and functions that are liable to be linearly dependent.

Example 2 Verify the fact using the functions from the previous example.

Solution

(a) $f(x) = 9\cos(2x)$ $g(x) = 2\cos^2(x) - 2\sin^2(x)$

In this case if we compute the Wronskian of the two functions we should get zero since we have already determined that these functions are linearly dependent.

$$W = \begin{vmatrix} 9\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \\ -18\sin(2x) & -4\cos(x)\sin(x) - 4\sin(x)\cos(x) \end{vmatrix}$$

$$= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -2\sin(2x) - 2\sin(2x) \end{vmatrix}$$

$$= \begin{vmatrix} 9\cos(2x) & 2\cos(2x) \\ -18\sin(2x) & -4\sin(2x) \end{vmatrix}$$

$$= -36\cos(2x)\sin(2x) - \left(-36\cos(2x)\sin(2x)\right) = 0$$

So, we get zero as we should have. Notice the heavy use of trig formulas to simplify the work!

(b) $f(t) = 2t^2$ $g(t) = t^4$

Here we know that the two functions are linearly independent and so we should get a non-zero Wronskian.

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} = 8t^5 - 4t^5 = 4t^5$$

The Wronskian is non-zero as we expected provided $t \neq 0$. This is not a problem. As long as the Wronskian is not identically zero for all t we are okay.

Example 3 Determine if the following functions are linearly dependent or linearly independent.

   (a) $f(t) = \cos t$ $g(t) = \sin t$

   (b) $f(x) = 6^x$ $g(x) = 6^{x+2}$

Solution

(a) Now that we have the Wronskian to use here let's first check that. If its non-zero then we will know that the two functions are linearly independent and if its zero then we can be pretty sure that they are linearly dependent.

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

So, by the fact these two functions are linearly independent. Much easier this time around!

(b) We'll do the same thing here as we did in the first part. Recall that

$$\left(a^x \right)' = a^x \ln a$$

Now compute the Wronskian.

$$W = \begin{vmatrix} 6^x & 6^{x+2} \\ 6^x \ln 6 & 6^{x+2} \ln 6 \end{vmatrix} = 6^x 6^{x+2} \ln 6 - 6^{x+2} 6^x \ln 6 = 0$$

Now, this does not say that the two functions are linearly dependent! However, we can guess that they probably are linearly dependent. To prove that they are in fact linearly dependent we'll need to write down (1) and see if we can find non-zero c and k that will make it true for all x.

$$c\, 6^x + k\, 6^{x+2} = 0$$
$$c\, 6^x + k\, 6^x 6^2 = 0$$
$$c\, 6^x + 36k\, 6^x = 0$$
$$(c + 36k) 6^x = 0$$

So, it looks like we could use any constants that satisfy
$$c + 36k = 0$$
to make this zero for all x. In particular we could use

$$c = 36 \qquad\qquad k = -1$$
$$c = -36 \qquad\qquad k = 1$$
$$c = 9 \qquad\qquad k = -\frac{1}{4}$$
$$etc.$$

We have non-zero constants that will make the equation true for all x. Therefore, the functions are linearly dependent.

Before proceeding to the next topic in this section let's talk a little more about linearly independent and linearly dependent functions. Let's start off by assuming that $f(x)$ and $g(x)$ are linearly dependent. So, that means there are non-zero constants c and k so that

$$c\,f(x) + k\,g(x) = 0$$

is true for all x.

Now, we can solve this in either of the following two ways.

$$f(x) = -\frac{k}{c}g(x) \qquad \text{OR} \qquad g(x) = -\frac{c}{k}f(x)$$

Note that this can be done because we know that c and k are non-zero and hence the divisions can be done without worrying about division by zero.

So, this means that two linearly dependent functions can be written in such a way that one is nothing more than a constants time the other. Go back and look at both of the sets of linearly dependent functions that we wrote down and you will see that this is true for both of them.

Two functions that are linearly independent can't be written in this manner and so we can't get from one to the other simply by multiplying by a constant.

Next, we don't want to leave you with the impression that linear independence and linear dependence is only for two functions. We can easily extend the idea to as many functions as we'd like.

Let's suppose that we have n non-zero functions, $f_1(x)$, $f_2(x)$,…, $f_n(x)$. Write down the following equation.

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad\qquad \textbf{(2)}$$

If we can find constants c_1, c_2, …, c_n with at least two non-zero so that (2) is true for all x then we call the functions linearly dependent. If, on the other hand, the only constants that make (2) true for x are $c_1 = 0$, $c_2 = 0$, …, $c_n = 0$ then we call the functions linearly independent.

Note that unlike the two function case we can have some of the constants be zero and still have the functions be linearly dependent.

In this case just what does it mean for the functions to be linearly dependent? Well, let's suppose that they are. So, this means that we can find constants, with at least two non-zero so that (2) is true for all x. For the sake of argument let's suppose that c_1 is one of the non-zero constants. This means that we can do the following.

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$
$$c_1 f_1(x) = -\left(c_2 f_2(x) + \cdots + c_n f_n(x)\right)$$
$$f_1(x) = -\frac{1}{c_1}\left(c_2 f_2(x) + \cdots + c_n f_n(x)\right)$$

In other words, if the functions are linearly dependent then we can write at least one of them in terms of the other functions.

Okay, let's move on to the other topic of this section. There is an alternate method of computing the Wronskian. The following theorem gives this alternate method.

Abel's Theorem

If $y_1(t)$ and $y_2(t)$ are two solutions to
$$y'' + p(t) y' + q(t) y = 0$$
then the Wronskian of the two solutions is
$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \, \mathbf{e}^{-\int_{t_0}^{t} p(x)\,dx}$$
for some t_0.

Because we don't know the Wronskian and we don't know t_0 this won't do us a lot of good apparently. However, we can rewrite this as
$$W(y_1, y_2)(t) = c \, \mathbf{e}^{-\int p(t)\,dt} \tag{3}$$
where the original Wronskian sitting in front of the exponential is absorbed into the c and the evaluation of the integral at t_0 will put a constant in the exponential that can also be brought out and absorbed into the constant c. If you don't recall how to do this go back and take a look at the linear, first order differential equation [section](#) as we did something similar there.

With this rewrite we can compute the Wronskian up to a multiplicative constant, which isn't too bad. Notice as well that we don't actually need the two solutions to do this. All we need is the coefficient of the first derivative from the differential equation (provided the coefficient of the second derivative is one of course…).

Let's take a look at a quick example of this.

Example 4 Without solving, determine the Wronskian of two solutions to the following differential equation.
$$t^4 y'' - 2t^3 y' - t^8 y = 0$$

Solution
The first thing that we need to do is divide the differential equation by the coefficient of the second derivative as that needs to be a one. This gives us
$$y'' - \frac{2}{t} y' - t^4 y = 0$$
Now, using (3) the Wronskian is
$$W = c \, \mathbf{e}^{-\int -\frac{2}{t}\,dt} = c \, \mathbf{e}^{2\ln t} = c \, \mathbf{e}^{\ln t^2} = c t^2$$

Section 3-8 : Nonhomogeneous Differential Equations

It's now time to start thinking about how to solve nonhomogeneous differential equations. A second order, linear nonhomogeneous differential equation is

$$y'' + p(t)y' + q(t)y = g(t) \qquad\qquad (1)$$

where *g(t)* is a non-zero function. Note that we didn't go with constant coefficients here because everything that we're going to do in this section doesn't require it. Also, we're using a coefficient of 1 on the second derivative just to make some of the work a little easier to write down. It is not required to be a 1.

Before talking about how to solve one of these we need to get some basics out of the way, which is the point of this section.

First, we will call

$$y'' + p(t)y' + q(t)y = 0 \qquad\qquad (2)$$

the associated homogeneous differential equation to (1).

Now, let's take a look at the following theorem.

Theorem

Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to (1) and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to the associated homogeneous differential equation (2) then,

$$Y_1(t) - Y_2(t)$$

is a solution to (2) and it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Note the notation used here. Capital letters referred to solutions to (1) while lower case letters referred to solutions to (2). This is a fairly common convention when dealing with nonhomogeneous differential equations.

This theorem is easy enough to prove so let's do that. To prove that $Y_1(t)$ - $Y_2(t)$ is a solution to (2) all we need to do is plug this into the differential equation and check it.

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) = 0$$

$$Y_1'' + p(t)Y_1' + q(t)Y_1 - \left(Y_2'' + p(t)Y_2' + q(t)Y_2\right) = 0$$

$$g(t) - g(t) = 0$$

$$0 = 0$$

We used the fact that $Y_1(t)$ and $Y_2(t)$ are two solutions to (1) in the third step. Because they are solutions to (1) we know that

$$Y_1'' + p(t)Y_1' + q(t)Y_1 = g(t)$$

$$Y_2'' + p(t)Y_2' + q(t)Y_2 = g(t)$$

So, we were able to prove that the difference of the two solutions is a solution to (2).

Proving that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

is even easier. Since *y₁(t)* and *y₂(t)* are a fundamental set of solutions to (2) we know that they form a general solution and so any solution to (2) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Well, *Y₁(t) - Y₂(t)* is a solution to (2), as we've shown above, therefore it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

So, what does this theorem do for us? We can use this theorem to write down the form of the general solution to (1). Let's suppose that *y(t)* is the general solution to (1) and that *YP(t)* is any solution to (1) that we can get our hands on. Then using the second part of our theorem we know that

$$y(t) - Y_P(t) = c_1 y_1(t) + c_2 y_2(t)$$

where *y₁(t)* and *y₂(t)* are a fundamental set of solutions for (2). Solving for *y(t)* gives,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_P(t)$$

We will call

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

the complementary solution and *YP(t)* a particular solution. The general solution to a differential equation can then be written as.

$$y(t) = y_c(t) + Y_P(t)$$

So, to solve a nonhomogeneous differential equation, we will need to solve the homogeneous differential equation, (2), which for constant coefficient differential equations is pretty easy to do, and we'll need a solution to (1).

This seems to be a circular argument. In order to write down a solution to (1) we need a solution. However, this isn't the problem that it seems to be. There are ways to find a solution to (1). They just won't, in general, be the general solution. In fact, the next two sections are devoted to exactly that, finding a particular solution to a nonhomogeneous differential equation.

There are two common methods for finding particular solutions : Undetermined Coefficients and Variation of Parameters. Both have their advantages and disadvantages as you will see in the next couple of sections.

Section 3-9 : Undetermined Coefficients

In this section we will take a look at the first method that can be used to find a particular solution to a nonhomogeneous differential equation.

$$y'' + p(t)y' + q(t)y = g(t)$$

One of the main advantages of this method is that it reduces the problem down to an algebra problem. The algebra can get messy on occasion, but for most of the problems it will not be terribly difficult. Another nice thing about this method is that the complementary solution will not be explicitly required, although as we will see knowledge of the complementary solution will be needed in some cases and so we'll generally find that as well.

There are two disadvantages to this method. First, it will only work for a fairly small class of *g(t)*'s. The class of *g(t)*'s for which the method works, does include some of the more common functions, however, there are many functions out there for which undetermined coefficients simply won't work. Second, it is generally only useful for constant coefficient differential equations.

The method is quite simple. All that we need to do is look at *g(t)* and make a guess as to the form of $Y_P(t)$ leaving the coefficient(s) undetermined (and hence the name of the method). Plug the guess into the differential equation and see if we can determine values of the coefficients. If we can determine values for the coefficients then we guessed correctly, if we can't find values for the coefficients then we guessed incorrectly.

It's usually easier to see this method in action rather than to try and describe it, so let's jump into some examples.

Example 1 Determine a particular solution to
$$y'' - 4y' - 12y = 3e^{5t}$$

Solution
The point here is to find a particular solution, however the first thing that we're going to do is find the complementary solution to this differential equation. Recall that the complementary solution comes from solving,
$$y'' - 4y' - 12y = 0$$

The characteristic equation for this differential equation and its roots are.
$$r^2 - 4r - 12 = (r-6)(r+2) = 0 \qquad \Rightarrow \qquad r_1 = -2, \ r_2 = 6$$

The complementary solution is then,
$$y_c(t) = c_1 e^{-2t} + c_2 e^{6t}$$

At this point the reason for doing this first will not be apparent, however we want you in the habit of finding it before we start the work to find a particular solution. Eventually, as we'll see, having the complementary solution in hand will be helpful and so it's best to be in the habit of finding it first prior to doing the work for undetermined coefficients.

Now, let's proceed with finding a particular solution. As mentioned prior to the start of this example we need to make a guess as to the form of a particular solution to this differential equation. Since *g(t)* is an exponential and we know that exponentials never just appear or disappear in the differentiation process it seems that a likely form of the particular solution would be

$$Y_P(t) = A\mathbf{e}^{5t}$$

Now, all that we need to do is do a couple of derivatives, plug this into the differential equation and see if we can determine what *A* needs to be.

Plugging into the differential equation gives

$$25A\mathbf{e}^{5t} - 4\left(5A\mathbf{e}^{5t}\right) - 12\left(A\mathbf{e}^{5t}\right) = 3\mathbf{e}^{5t}$$
$$-7A\mathbf{e}^{5t} = 3\mathbf{e}^{5t}$$

So, in order for our guess to be a solution we will need to choose *A* so that the coefficients of the exponentials on either side of the equal sign are the same. In other words we need to choose *A* so that,

$$-7A = 3 \qquad \Rightarrow \qquad A = -\frac{3}{7}$$

Okay, we found a value for the coefficient. This means that we guessed correctly. A particular solution to the differential equation is then,

$$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t}$$

Before proceeding any further let's again note that we started off the solution above by finding the complementary solution. This is not technically part the method of Undetermined Coefficients however, as we'll eventually see, having this in hand before we make our guess for the particular solution can save us a lot of work and/or headache. Finding the complementary solution first is simply a good habit to have so we'll try to get you in the habit over the course of the next few examples. At this point do not worry about why it is a good habit. We'll eventually see why it is a good habit.

Now, back to the work at hand. Notice in the last example that we kept saying "a" particular solution, not "the" particular solution. This is because there are other possibilities out there for the particular solution we've just managed to find one of them. Any of them will work when it comes to writing down the general solution to the differential equation.

Speaking of which… This section is devoted to finding particular solutions and most of the examples will be finding only the particular solution. However, we should do at least one full blown IVP to make sure that we can say that we've done one.

Example 2 Solve the following IVP

$$y'' - 4y' - 12y = 3e^{5t} \qquad y(0) = \frac{18}{7} \qquad y'(0) = -\frac{1}{7}$$

Solution

We know that the general solution will be of the form,

$$y(t) = y_c(t) + Y_P(t)$$

and we already have both the complementary and particular solution from the first example so we don't really need to do any extra work for this problem.

One of the more common mistakes in these problems is to find the complementary solution and then, because we're probably in the habit of doing it, apply the initial conditions to the complementary solution to find the constants. This however, is incorrect. The complementary solution is only the solution to the homogeneous differential equation and we are after a solution to the nonhomogeneous differential equation and the initial conditions must satisfy that solution instead of the complementary solution.

So, we need the general solution to the nonhomogeneous differential equation. Taking the complementary solution and the particular solution that we found in the previous example we get the following for a general solution and its derivative.

$$y(t) = c_1 e^{-2t} + c_2 e^{6t} - \frac{3}{7} e^{5t}$$

$$y'(t) = -2c_1 e^{-2t} + 6c_2 e^{6t} - \frac{15}{7} e^{5t}$$

Now, apply the initial conditions to these.

$$\frac{18}{7} = y(0) = c_1 + c_2 - \frac{3}{7}$$

$$-\frac{1}{7} = y'(0) = -2c_1 + 6c_2 - \frac{15}{7}$$

Solving this system gives $c_1 = 2$ and $c_2 = 1$. The actual solution is then.

$$y(t) = 2e^{-2t} + e^{6t} - \frac{3}{7} e^{5t}$$

This will be the only IVP in this section so don't forget how these are done for nonhomogeneous differential equations!

Let's take a look at another example that will give the second type of *g(t)* for which undetermined coefficients will work.

Example 3 Find a particular solution for the following differential equation.
$$y'' - 4y' - 12y = \sin(2t)$$

Solution

Again, let's note that we should probably find the complementary solution before we proceed onto the guess for a particular solution. However, because the homogeneous differential equation for this example is the same as that for the first example we won't bother with that here.

Now, let's take our experience from the first example and apply that here. The first example had an exponential function in the *g(t)* and our guess was an exponential. This differential equation has a sine so let's try the following guess for the particular solution.
$$Y_P(t) = A\sin(2t)$$

Differentiating and plugging into the differential equation gives,
$$-4A\sin(2t) - 4(2A\cos(2t)) - 12(A\sin(2t)) = \sin(2t)$$

Collecting like terms yields
$$-16A\sin(2t) - 8A\cos(2t) = \sin(2t)$$

We need to pick *A* so that we get the same function on both sides of the equal sign. This means that the coefficients of the sines and cosines must be equal. Or,

$$\cos(2t): \qquad -8A = 0 \qquad \Rightarrow \qquad A = 0$$

$$\sin(2t): \qquad -16A = 1 \qquad \Rightarrow \qquad A = -\frac{1}{16}$$

Notice two things. First, since there is no cosine on the right hand side this means that the coefficient must be zero on that side. More importantly we have a serious problem here. In order for the cosine to drop out, as it must in order for the guess to satisfy the differential equation, we need to set *A* = 0, but if *A* = 0, the sine will also drop out and that can't happen. Likewise, choosing *A* to keep the sine around will also keep the cosine around.

What this means is that our initial guess was wrong. If we get multiple values of the same constant or are unable to find the value of a constant then we have guessed wrong.

One of the nicer aspects of this method is that when we guess wrong our work will often suggest a fix. In this case the problem was the cosine that cropped up. So, to counter this let's add a cosine to our guess. Our new guess is
$$Y_P(t) = A\cos(2t) + B\sin(2t)$$

Plugging this into the differential equation and collecting like terms gives,

$$-4A\cos(2t)-4B\sin(2t)-4\big(-2A\sin(2t)+2B\cos(2t)\big)-$$
$$12\big(A\cos(2t)+B\sin(2t)\big)=\sin(2t)$$
$$\big(-4A-8B-12A\big)\cos(2t)+\big(-4B+8A-12B\big)\sin(2t)=\sin(2t)$$
$$\big(-16A-8B\big)\cos(2t)+\big(8A-16B\big)\sin(2t)=\sin(2t)$$

Now, set the coefficients equal

$$\cos(2t):\qquad -16A-8B=0$$
$$\sin(2t):\qquad 8A-16B=1$$

Solving this system gives us

$$A=\frac{1}{40}\qquad B=-\frac{1}{20}$$

We found constants and this time we guessed correctly. A particular solution to the differential equation is then,

$$Y_P(t)=\frac{1}{40}\cos(2t)-\frac{1}{20}\sin(2t)$$

Notice that if we had had a cosine instead of a sine in the last example then our guess would have been the same. In fact, if both a sine and a cosine had shown up we will see that the same guess will also work.

Let's take a look at the third and final type of basic *g(t)* that we can have. There are other types of *g(t)* that we can have, but as we will see they will all come back to two types that we've already done as well as the next one.

Example 4 Find a particular solution for the following differential equation.
$$y''-4y'-12y=2t^3-t+3$$

Solution
Once, again we will generally want the complementary solution in hand first, but again we're working with the same homogeneous differential equation (you'll eventually see why we keep working with the same homogeneous problem) so we'll again just refer to the first example.

For this example, *g(t)* is a cubic polynomial. For this we will need the following guess for the particular solution.
$$Y_P(t)=At^3+Bt^2+Ct+D$$

Notice that even though *g(t)* doesn't have a t^2 in it our guess will still need one! So, differentiate and plug into the differential equation.
$$6At+2B-4\big(3At^2+2Bt+C\big)-12\big(At^3+Bt^2+Ct+D\big)=2t^3-t+3$$
$$-12At^3+\big(-12A-12B\big)t^2+\big(6A-8B-12C\big)t+2B-4C-12D=2t^3-t+3$$

Now, as we've done in the previous examples we will need the coefficients of the terms on both sides of the equal sign to be the same so set coefficients equal and solve.

$$t^3: \qquad\qquad -12A = 2 \qquad \Rightarrow \quad A = -\frac{1}{6}$$

$$t^2: \qquad\quad -12A - 12B = 0 \qquad \Rightarrow \quad B = \frac{1}{6}$$

$$t^1: \qquad 6A - 8B - 12C = -1 \qquad \Rightarrow \quad C = -\frac{1}{9}$$

$$t^0: \qquad 2B - 4C - 12D = 3 \qquad \Rightarrow \quad D = -\frac{5}{27}$$

Notice that in this case it was very easy to solve for the constants. The first equation gave *A*. Then once we knew *A* the second equation gave *B*, *etc*. A particular solution for this differential equation is then

$$Y_P(t) = -\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

Now that we've gone over the three basic kinds of functions that we can use undetermined coefficients on let's summarize.

g(t)	$Y_P(t)$ guess
$a\mathbf{e}^{\beta t}$	$A\mathbf{e}^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1} t^{n-1} + \cdots A_1 t + A_0$

Notice that there are really only three kinds of functions given above. If you think about it the single cosine and single sine functions are really special cases of the case where both the sine and cosine are present. Also, we have not yet justified the guess for the case where both a sine and a cosine show up. We will justify this later.

We now need move on to some more complicated functions. The more complicated functions arise by taking products and sums of the basic kinds of functions. Let's first look at products.

Example 5 Find a particular solution for the following differential equation.
$$y'' - 4y' - 12y = t\mathbf{e}^{4t}$$

Solution
You're probably getting tired of the opening comment, but again finding the complementary solution first really a good idea but again we've already done the work in the first example so we won't do it again here. We promise that eventually you'll see why we keep using the same homogeneous problem and why we say it's a good idea to have the complementary solution in hand first. At this point all we're trying to do is reinforce the habit of finding the complementary solution first.

Okay, let's start off by writing down the guesses for the individual pieces of the function. The guess for the t would be

$$At + B$$

while the guess for the exponential would be

$$C\mathbf{e}^{4t}$$

Now, since we've got a product of two functions it seems like taking a product of the guesses for the individual pieces might work. Doing this would give

$$C\mathbf{e}^{4t}\left(At + B\right)$$

However, we will have problems with this. As we will see, when we plug our guess into the differential equation we will only get two equations out of this. The problem is that with this guess we've got three unknown constants. With only two equations we won't be able to solve for all the constants.

This is easy to fix however. Let's notice that we could do the following

$$C\mathbf{e}^{4t}\left(At + B\right) = \mathbf{e}^{4t}\left(ACt + BC\right)$$

If we multiply the C through, we can see that the guess can be written in such a way that there are really only two constants. So, we will use the following for our guess.

$$Y_P\left(t\right) = \mathbf{e}^{4t}\left(At + B\right)$$

Notice that this is nothing more than the guess for the t with an exponential tacked on for good measure.

Now that we've got our guess, let's differentiate, plug into the differential equation and collect like terms.

$$\mathbf{e}^{4t}\left(16At + 16B + 8A\right) - 4\left(\mathbf{e}^{4t}\left(4At + 4B + A\right)\right) - 12\left(\mathbf{e}^{4t}\left(At + B\right)\right) = t\mathbf{e}^{4t}$$

$$\left(16A - 16A - 12A\right)t\mathbf{e}^{4t} + \left(16B + 8A - 16B - 4A - 12B\right)\mathbf{e}^{4t} = t\mathbf{e}^{4t}$$

$$-12At\mathbf{e}^{4t} + \left(4A - 12B\right)\mathbf{e}^{4t} = t\mathbf{e}^{4t}$$

Note that when we're collecting like terms we want the coefficient of each term to have only constants in it. Following this rule we will get two terms when we collect like terms. Now, set coefficients equal.

$$t\mathbf{e}^{4t}: \qquad\qquad -12A = 1 \qquad \Rightarrow \quad A = -\frac{1}{12}$$

$$\mathbf{e}^{4t}: \qquad 4A - 12B = 0 \qquad \Rightarrow \quad B = -\frac{1}{36}$$

A particular solution for this differential equation is then

$$Y_P\left(t\right) = \mathbf{e}^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right) = -\frac{1}{36}\left(3t + 1\right)\mathbf{e}^{4t}$$

This last example illustrated the general rule that we will follow when products involve an exponential. When a product involves an exponential we will first strip out the exponential and write down the guess for the portion of the function without the exponential, then we will go back and tack on the exponential without any leading coefficient.

Let's take a look at some more products. In the interest of brevity we will just write down the guess for a particular solution and not go through all the details of finding the constants. Also, because we aren't going to give an actual differential equation we can't deal with finding the complementary solution first.

Example 6 Write down the form of the particular solution to
$$y'' + p(t)y' + q(t)y = g(t)$$
for the following *g(t)*'s.

    (a) $g(t) = 16e^{7t}\sin(10t)$

    (b) $g(t) = (9t^2 - 103t)\cos t$

    (c) $g(t) = -e^{-2t}(3 - 5t)\cos(9t)$

Solution

(a) $g(t) = 16e^{7t}\sin(10t)$

So, we have an exponential in the function. Remember the rule. We will ignore the exponential and write down a guess for $16\sin(10t)$ then put the exponential back in.

The guess for the sine is
$$A\cos(10t) + B\sin(10t)$$

Now, for the actual guess for the particular solution we'll take the above guess and tack an exponential onto it. This gives,
$$Y_P(t) = e^{7t}\left(A\cos(10t) + B\sin(10t)\right)$$

One final note before we move onto the next part. The 16 in front of the function has absolutely no bearing on our guess. Any constants multiplying the whole function are ignored.

(b) $g(t) = (9t^2 - 103t)\cos t$

We will start this one the same way that we initially started the previous example. The guess for the polynomial is
$$At^2 + Bt + C$$
and the guess for the cosine is
$$D\cos t + E\sin t$$

If we multiply the two guesses we get.
$$(At^2 + Bt + C)(D\cos t + E\sin t)$$

Let's simplify things up a little. First multiply the polynomial through as follows.

$$\left(At^2 + Bt + C\right)\left(D\cos t\right) + \left(At^2 + Bt + C\right)\left(E\sin t\right)$$
$$\left(ADt^2 + BDt + CD\right)\cos t + \left(AEt^2 + BEt + CE\right)\sin t$$

Notice that everywhere one of the unknown constants occurs it is in a product of unknown constants. This means that if we went through and used this as our guess the system of equations that we would need to solve for the unknown constants would have products of the unknowns in them. These types of systems are generally very difficult to solve.

So, to avoid this we will do the same thing that we did in the previous example. Everywhere we see a product of constants we will rename it and call it a single constant. The guess that we'll use for this function will be.

$$Y_P\left(t\right) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t$$

This is a general rule that we will use when faced with a product of a polynomial and a trig function. We write down the guess for the polynomial and then multiply that by a cosine. We then write down the guess for the polynomial again, using different coefficients, and multiply this by a sine.

(c) $g\left(t\right) = -\mathbf{e}^{-2t}\left(3 - 5t\right)\cos\left(9t\right)$

This final part has all three parts to it. First, we will ignore the exponential and write down a guess for.

$$-\left(3 - 5t\right)\cos\left(9t\right)$$

The minus sign can also be ignored. The guess for this is
$$\left(At + B\right)\cos\left(9t\right) + \left(Ct + D\right)\sin\left(9t\right)$$

Now, tack an exponential back on and we're done.
$$Y_P\left(t\right) = \mathbf{e}^{-2t}\left(At + B\right)\cos\left(9t\right) + \mathbf{e}^{-2t}\left(Ct + D\right)\sin\left(9t\right)$$

Notice that we put the exponential on both terms.

There a couple of general rules that you need to remember for products.

1. If *g(t)* contains an exponential, ignore it and write down the guess for the remainder. Then tack the exponential back on without any leading coefficient.

2. For products of polynomials and trig functions you first write down the guess for just the polynomial and multiply that by the appropriate cosine. Then add on a new guess for the polynomial with different coefficients and multiply that by the appropriate sine.

If you can remember these two rules you can't go wrong with products. Writing down the guesses for products is usually not that difficult. The difficulty arises when you need to actually find the constants.

Now, let's take a look at sums of the basic components and/or products of the basic components. To do this we'll need the following fact.

Fact

If $Y_{P1}(t)$ is a particular solution for
$$y'' + p(t)y' + q(t)y = g_1(t)$$

and if $Y_{P2}(t)$ is a particular solution for
$$y'' + p(t)y' + q(t)y = g_2(t)$$

then $Y_{P1}(t) + Y_{P2}(t)$ is a particular solution for
$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$$

This fact can be used to both find particular solutions to differential equations that have sums in them and to write down guess for functions that have sums in them.

Example 7 Find a particular solution for the following differential equation.
$$y'' - 4y' - 12y = 3e^{5t} + \sin(2t) + te^{4t}$$

Solution
This example is the reason that we've been using the same homogeneous differential equation for all the previous examples. There is nothing to do with this problem. All that we need to do it go back to the appropriate examples above and get the particular solution from that example and add them all together.

Doing this gives
$$Y_P(t) = -\frac{3}{7}e^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)e^{4t}$$

Let's take a look at a couple of other examples. As with the products we'll just get guesses here and not worry about actually finding the coefficients.

Example 8 Write down the form of the particular solution to
$$y'' + p(t)y' + q(t)y = g(t)$$
for the following $g(t)$'s.

    (a) $g(t) = 4\cos(6t) - 9\sin(6t)$

    (b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

    (c) $g(t) = e^{7t} + 6$

    (d) $g(t) = 6t^2 - 7\sin(3t) + 9$

    (e) $g(t) = 10e^t - 5te^{-8t} + 2e^{-8t}$

    (f) $g(t) = t^2\cos t - 5t\sin t$

    (g) $g(t) = 5e^{-3t} + e^{-3t}\cos(6t) - \sin(6t)$

Solution

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

This first one we've actually already told you how to do. This is in the table of the basic functions. However, we wanted to justify the guess that we put down there. Using the fact on sums of function we would be tempted to write down a guess for the cosine and a guess for the sine. This would give.

$$\underbrace{A\cos(6t) + B\sin(6t)}_{\text{guess for the cosine}} + \underbrace{C\cos(6t) + D\sin(6t)}_{\text{guess for the sine}}$$

So, we would get a cosine from each guess and a sine from each guess. The problem with this as a guess is that we are only going to get two equations to solve after plugging into the differential equation and yet we have 4 unknowns. We will never be able to solve for each of the constants.

To fix this notice that we can combine some terms as follows.

$$(A+C)\cos(6t) + (B+D)\sin(6t)$$

Upon doing this we can see that we've really got a single cosine with a coefficient and a single sine with a coefficient and so we may as well just use

$$Y_P(t) = A\cos(6t) + B\sin(6t)$$

The general rule of thumb for writing down guesses for functions that involve sums is to always combine like terms into single terms with single coefficients. This will greatly simplify the work required to find the coefficients.

(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

For this one we will get two sets of sines and cosines. This will arise because we have two different arguments in them. We will get one set for the sine with just a *t* as its argument and we'll get another set for the sine and cosine with the 14*t* as their arguments.

The guess for this function is

$$Y_P(t) = A\cos t + B\sin t + C\cos(14t) + D\sin(14t)$$

(c) $g(t) = \mathbf{e}^{7t} + 6$

The main point of this problem is dealing with the constant. But that isn't too bad. We just wanted to make sure that an example of that is somewhere in the notes. If you recall that a constant is nothing more than a zeroth degree polynomial the guess becomes clear.

The guess for this function is

$$Y_p(t) = A\mathbf{e}^{7t} + B$$

(d) $g(t) = 6t^2 - 7\sin(3t) + 9$

This one can be a little tricky if you aren't paying attention. Let's first rewrite the function
$$g(t) = 6t^2 - 7\sin(3t) + 9 \qquad \text{as}$$
$$g(t) = 6t^2 + 9 - 7\sin(3t)$$

All we did was move the 9. However, upon doing that we see that the function is really a sum of a quadratic polynomial and a sine. The guess for this is then
$$Y_P(t) = At^2 + Bt + C + D\cos(3t) + E\sin(3t)$$

If we don't do this and treat the function as the sum of three terms we would get
$$At^2 + Bt + C + D\cos(3t) + E\sin(3t) + G$$
and as with the first part in this example we would end up with two terms that are essentially the same (the *C* and the *G*) and so would need to be combined. An added step that isn't really necessary if we first rewrite the function.

Look for problems where rearranging the function can simplify the initial guess.

(e) $g(t) = 10\mathbf{e}^t - 5t\mathbf{e}^{-8t} + 2\mathbf{e}^{-8t}$

 So, this look like we've got a sum of three terms here. Let's write down a guess for that.
$$A\mathbf{e}^t + (Bt + C)\mathbf{e}^{-8t} + D\mathbf{e}^{-8t}$$

Notice however that if we were to multiply the exponential in the second term through we would end up with two terms that are essentially the same and would need to be combined. This is a case where the guess for one term is completely contained in the guess for a different term. When this happens we just drop the guess that's already included in the other term.

So, the guess here is actually.
$$Y_P(t) = A\mathbf{e}^t + (Bt + C)\mathbf{e}^{-8t}$$

Notice that this arose because we had two terms in our *g(t)* whose only difference was the polynomial that sat in front of them. When this happens we look at the term that contains the largest degree polynomial, write down the guess for that and don't bother writing down the guess for the other term as that guess will be completely contained in the first guess.

(f) $g(t) = t^2 \cos t - 5t\sin t$

In this case we've got two terms whose guess without the polynomials in front of them would be the same. Therefore, we will take the one with the largest degree polynomial in front of it and write down the guess for that one and ignore the other term. So, the guess for the function is
$$Y_P(t) = (At^2 + Bt + C)\cos t + (Dt^2 + Et + F)\sin t$$

(g) $g(t) = 5\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\cos(6t) - \sin(6t)$

This last part is designed to make sure you understand the general rule that we used in the last two parts. This time there really are three terms and we will need a guess for each term. The guess here is

$$Y_P(t) = A\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\left(B\cos(6t) + C\sin(6t)\right) + D\cos(6t) + E\sin(6t)$$

We can only combine guesses if they are identical up to the constant. So, we can't combine the first exponential with the second because the second is really multiplied by a cosine and a sine and so the two exponentials are in fact different functions. Likewise, the last sine and cosine can't be combined with those in the middle term because the sine and cosine in the middle term are in fact multiplied by an exponential and so are different.

So, when dealing with sums of functions make sure that you look for identical guesses that may or may not be contained in other guesses and combine them. This will simplify your work later on.

We have one last topic in this section that needs to be dealt with. In the first few examples we were constantly harping on the usefulness of having the complementary solution in hand before making the guess for a particular solution. We never gave any reason for this other that "trust us". It is now time to see why having the complementary solution in hand first is useful. This is best shown with an example so let's jump into one.

Example 9 Find a particular solution for the following differential equation.
$$y'' - 4y' - 12y = \mathbf{e}^{6t}$$

Solution
This problem seems almost too simple to be given this late in the section. This is especially true given the ease of finding a particular solution for $g(t)$'s that are just exponential functions. Also, because the point of this example is to illustrate why it is generally a good idea to have the complementary solution in hand first we'll let's go ahead and recall the complementary solution first. Here it is,
$$y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t}$$

Now, without worrying about the complementary solution for a couple more seconds let's go ahead and get to work on the particular solution. There is not much to the guess here. From our previous work we know that the guess for the particular solution should be,
$$Y_P(t) = A\mathbf{e}^{6t}$$

Plugging this into the differential equation gives,
$$36A\mathbf{e}^{6t} - 24A\mathbf{e}^{6t} - 12A\mathbf{e}^{6t} = \mathbf{e}^{6t}$$
$$0 = \mathbf{e}^{6t}$$

Hmmmm…. Something seems wrong here. Clearly an exponential can't be zero. So, what went wrong? We finally need the complementary solution. Notice that the second term in the complementary solution (listed above) is exactly our guess for the form of the particular solution and now recall that both portions of the complementary solution are solutions to the homogeneous differential equation,

$$y'' - 4y' - 12y = 0$$

In other words, we had better have gotten zero by plugging our guess into the differential equation, it is a solution to the homogeneous differential equation!

So, how do we fix this? The way that we fix this is to add a *t* to our guess as follows.

$$Y_P(t) = At\mathbf{e}^{6t}$$

Plugging this into our differential equation gives,

$$\left(12A\mathbf{e}^{6t} + 36At\mathbf{e}^{6t}\right) - 4\left(A\mathbf{e}^{6t} + 6At\mathbf{e}^{6t}\right) - 12At\mathbf{e}^{6t} = \mathbf{e}^{6t}$$

$$\left(36A - 24A - 12A\right)t\mathbf{e}^{6t} + \left(12A - 4A\right)\mathbf{e}^{6t} = \mathbf{e}^{6t}$$

$$8A\mathbf{e}^{6t} = \mathbf{e}^{6t}$$

Now, we can set coefficients equal.

$$8A = 1 \qquad \Rightarrow \qquad A = \frac{1}{8}$$

So, the particular solution in this case is,

$$Y_P(t) = \frac{t}{8}\mathbf{e}^{6t}$$

So, what did we learn from this last example. While technically we don't need the complementary solution to do undetermined coefficients, you can go through a lot of work only to figure out at the end that you needed to add in a *t* to the guess because it appeared in the complementary solution. This work is avoidable if we first find the complementary solution and comparing our guess to the complementary solution and seeing if any portion of your guess shows up in the complementary solution.

If a portion of your guess does show up in the complementary solution then we'll need to modify that portion of the guess by adding in a *t* to the portion of the guess that is causing the problems. We do need to be a little careful and make sure that we add the *t* in the correct place however. The following set of examples will show you how to do this.

Example 10 Write down the guess for the particular solution to the given differential equation. Do not find the coefficients.

   (a) $y'' + 3y' - 28y = 7t + \mathbf{e}^{-7t} - 1$

   (b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$

   (c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\dfrac{t}{2}\right) + 6t\cos\left(\dfrac{t}{2}\right)$

   (d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\dfrac{t}{2}\right) + 6t\cos\left(\dfrac{t}{2}\right)$

   (e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + \left(t^2 + 5\right)\mathbf{e}^{-4t}$

Solution
In these solutions we'll leave the details of checking the complementary solution to you.

(a) $y'' + 3y' - 28y = 7t + \mathbf{e}^{-7t} - 1$
The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{4t} + c_2\mathbf{e}^{-7t}$$

Remembering to put the "-1" with the 7t gives a first guess for the particular solution.

$$Y_P(t) = At + B + C\mathbf{e}^{-7t}$$

Notice that the last term in the guess is the last term in the complementary solution. The first two terms however aren't a problem and don't appear in the complementary solution. Therefore, we will only add a *t* onto the last term.

The correct guess for the form of the particular solution is.

$$Y_P(t) = At + B + Ct\mathbf{e}^{-7t}$$

(b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$
The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{10t} + c_2\mathbf{e}^{-10t}$$

A first guess for the particular solution is

$$Y_P(t) = \left(At^2 + Bt + C\right)\mathbf{e}^{10t} + \left(Et + F\right)\cos t + \left(Gt + H\right)\sin t$$

Notice that if we multiplied the exponential term through the parenthesis that we would end up getting part of the complementary solution showing up. Since the problem part arises from the first term the *whole* first term will get multiplied by *t*. The second and third terms are okay as they are.

The correct guess for the form of the particular solution in this case is.

$$Y_P(t) = t\left(At^2 + Bt + C\right)\mathbf{e}^{10t} + \left(Et + F\right)\cos t + \left(Gt + H\right)\sin t$$

So, in general, if you were to multiply out a guess and if any term in the result shows up in the complementary solution, then the whole term will get a *t* not just the problem portion of the term.

(c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\dfrac{t}{2}\right) + 6t\cos\left(\dfrac{t}{2}\right)$
The complementary solution is

$$y_c(t) = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)$$

A first guess for the particular solution is

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct+D)\cos\left(\frac{t}{2}\right) + (Et+F)\sin\left(\frac{t}{2}\right)$$

In this case both the second and third terms contain portions of the complementary solution. The first term doesn't however, since upon multiplying out, both the sine and the cosine would have an exponential with them and that isn't part of the complementary solution. We only need to worry about terms showing up in the complementary solution if the only difference between the complementary solution term and the particular guess term is the constant in front of them.

So, in this case the second and third terms will get a *t* while the first won't

The correct guess for the form of the particular solution is.

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + t(Ct+D)\cos\left(\frac{t}{2}\right) + t(Et+F)\sin\left(\frac{t}{2}\right)$$

(d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

To get this problem we changed the differential equation from the last example and left the *g(t)* alone. The complementary solution this time is

$$y_c(t) = c_1\mathbf{e}^{-2t}\cos\left(\frac{t}{2}\right) + c_2\mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right)$$

As with the last part, a first guess for the particular solution is

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct+D)\cos\left(\frac{t}{2}\right) + (Et+F)\sin\left(\frac{t}{2}\right)$$

This time however it is the first term that causes problems and not the second or third. In fact, the first term is exactly the complementary solution and so it will need a *t*. Recall that we will only have a problem with a term in our guess if it only differs from the complementary solution by a constant. The second and third terms in our guess don't have the exponential in them and so they don't differ from the complementary solution by only a constant.

The correct guess for the form of the particular solution is.

$$Y_P(t) = t\mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct+D)\cos\left(\frac{t}{2}\right) + (Et+F)\sin\left(\frac{t}{2}\right)$$

(e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + (t^2+5)\mathbf{e}^{-4t}$

The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{-4t} + c_2 t\mathbf{e}^{-4t}$$

The two terms in *g(t)* are identical with the exception of a polynomial in front of them. So this means that we only need to look at the term with the highest degree polynomial in front of it. A first guess for the particular solution is

$$Y_P(t) = \left(At^2 + Bt + C\right)e^{-4t}$$

Notice that if we multiplied the exponential term through the parenthesis the last two terms would be the complementary solution. Therefore, we will need to multiply this whole thing by a *t*.

The next guess for the particular solution is then.

$$Y_P(t) = t\left(At^2 + Bt + C\right)e^{-4t}$$

This still causes problems however. If we multiplied the *t* and the exponential through, the last term will still be in the complementary solution. In this case, unlike the previous ones, a *t* wasn't sufficient to fix the problem. So, we will add in another *t* to our guess.

The correct guess for the form of the particular solution is.

$$Y_P(t) = t^2\left(At^2 + Bt + C\right)e^{-4t}$$

Upon multiplying this out none of the terms are in the complementary solution and so it will be okay.

As this last set of examples has shown, we really should have the complementary solution in hand before even writing down the first guess for the particular solution. By doing this we can compare our guess to the complementary solution and if any of the terms from your particular solution show up we will know that we'll have problems. Once the problem is identified we can add a *t* to the problem term(s) and compare our new guess to the complementary solution. If there are no problems we can proceed with the problem, if there are problems add in another *t* and compare again.

Can you see a general rule as to when a *t* will be needed and when a t^2 will be needed for second order differential equations?

Section 3-10 : Variation of Parameters

In the last section we looked at the method of undetermined coefficients for finding a particular solution to

$$p(t)y'' + q(t)y' + r(t)y = g(t) \tag{1}$$

and we saw that while it reduced things down to just an algebra problem, the algebra could become quite messy. On top of that undetermined coefficients will only work for a fairly small class of functions.

The method of Variation of Parameters is a much more general method that can be used in many more cases. However, there are two disadvantages to the method. First, the complementary solution is absolutely required to do the problem. This is in contrast to the method of undetermined coefficients where it was advisable to have the complementary solution on hand but was not required. Second, as we will see, in order to complete the method we will be doing a couple of integrals and there is no guarantee that we will be able to do the integrals. So, while it will always be possible to write down a formula to get the particular solution, we may not be able to actually find it if the integrals are too difficult or if we are unable to find the complementary solution.

We're going to derive the formula for variation of parameters. We'll start off by acknowledging that the complementary solution to (1) is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Remember as well that this is the general solution to the homogeneous differential equation.

$$p(t)y'' + q(t)y' + r(t)y = 0 \tag{2}$$

Also recall that in order to write down the complementary solution we know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions.

What we're going to do is see if we can find a pair of functions, $u_1(t)$ and $u_2(t)$ so that

$$Y_P(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

will be a solution to (1). We have two unknowns here and so we'll need two equations eventually. One equation is easy. Our proposed solution must satisfy the differential equation, so we'll get the first equation by plugging our proposed solution into (1). The second equation can come from a variety of places. We are going to get our second equation simply by making an assumption that will make our work easier. We'll say more about this shortly.

So, let's start. If we're going to plug our proposed solution into the differential equation we're going to need some derivatives so let's get those. The first derivative is

$$Y_P'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

Here's the assumption. Simply to make the first derivative easier to deal with we are going to assume that whatever $u_1(t)$ and $u_2(t)$ are they will satisfy the following.

$$u_1' y_1 + u_2' y_2 = 0 \tag{3}$$

Now, there is no reason ahead of time to believe that this can be done. However, we will see that this will work out. We simply make this assumption on the hope that it won't cause problems down the road and to make the first derivative easier so don't get excited about it.

With this assumption the first derivative becomes.
$$Y_P'(t) = u_1 y_1' + u_2 y_2'$$

The second derivative is then,
$$Y_P''(t) = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Plug the solution and its derivatives into (1).
$$p(t)\left(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''\right) + q(t)\left(u_1 y_1' + u_2 y_2'\right) + r(t)\left(u_1 y_1 + u_2 y_2\right) = g(t)$$

Rearranging a little gives the following.
$$p(t)\left(u_1' y_1' + u_2' y_2'\right) + u_1(t)\left(p(t) y_1'' + q(t) y_1' + r(t) y_1\right) +$$
$$u_2(t)\left(p(t) y_2'' + q(t) y_2' + r(t) y_2\right) = g(t)$$

Now, both $y_1(t)$ and $y_2(t)$ are solutions to (2) and so the second and third terms are zero. Acknowledging this and rearranging a little gives us,
$$p(t)\left(u_1' y_1' + u_2' y_2'\right) + u_1(t)(0) + u_2(t)(0) = g(t)$$
$$u_1' y_1' + u_2' y_2' = \frac{g(t)}{p(t)} \qquad\qquad\qquad \textbf{(4)}$$

We've almost got the two equations that we need. Before proceeding we're going to go back and make a further assumption. The last equation, (4), is actually the one that we want, however, in order to make things simpler for us we are going to assume that the function $p(t) = 1$.

In other words, we are going to go back and start working with the differential equation,
$$y'' + q(t) y' + r(t) y = g(t)$$

If the coefficient of the second derivative isn't one divide it out so that it becomes a one. The formula that we're going to be getting will assume this! Upon doing this the two equations that we want to solve for the unknown functions are
$$u_1' y_1 + u_2' y_2 = 0 \qquad\qquad\qquad \textbf{(5)}$$
$$u_1' y_1' + u_2' y_2' = g(t) \qquad\qquad\qquad \textbf{(6)}$$

Note that in this system we know the two solutions and so the only two unknowns here are u_1' and u_2'.

Solving this system is actually quite simple. First, solve (5) for u_1' and plug this into (6) and do some simplification.

$$u_1' = -\frac{u_2' y_2}{y_1} \tag{7}$$

$$\left(-\frac{u_2' y_2}{y_1}\right) y_1' + u_2' y_2' = g(t)$$

$$u_2' \left(y_2' - \frac{y_2 y_1'}{y_1}\right) = g(t)$$

$$u_2' \left(\frac{y_1 y_2' - y_2 y_1'}{y_1}\right) = g(t)$$

$$u_2' = \frac{y_1 g(t)}{y_1 y_2' - y_2 y_1'} \tag{8}$$

So, we now have an expression for u_2'. Plugging this into (7) will give us an expression for u_1'.

$$u_1' = -\frac{y_2 g(t)}{y_1 y_2' - y_2 y_1'} \tag{9}$$

Next, let's notice that

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0$$

Recall that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions and so we know that the Wronskian won't be zero!

Finally, all that we need to do is integrate (8) and (9) in order to determine what $u_1(t)$ and $u_2(t)$ are. Doing this gives,

$$u_1(t) = -\int \frac{y_2 g(t)}{W(y_1, y_2)} dt \qquad\qquad u_2(t) = \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

So, provided we can do these integrals, a particular solution to the differential equation is

$$Y_P(t) = y_1 u_1 + y_2 u_2$$

$$= -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

So, let's summarize up what we've determined here.

Variation of Parameters

> Consider the differential equation,
> $$y'' + q(t)y' + r(t)y = g(t)$$
> Assume that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for
> $$y'' + q(t)y' + r(t)y = 0$$
> Then a particular solution to the nonhomogeneous differential equation is,
> $$Y_P(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

Depending on the person and the problem, some will find the formula easier to memorize and use, while others will find the process used to get the formula easier. The examples in this section will be done using the formula.

Before proceeding with a couple of examples let's first address the issues involving the constants of integration that will arise out of the integrals. Putting in the constants of integration will give the following.

$$Y_P(t) = -y_1 \left(\int \frac{y_2 g(t)}{W(y_1, y_2)} dt + c \right) + y_2 \left(\int \frac{y_1 g(t)}{W(y_1, y_2)} dt + k \right)$$

$$= -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt + (-cy_1 + ky_2)$$

The final quantity in the parenthesis is nothing more than the complementary solution with $c_1 = -c$ and $c_2 = k$ and we know that if we plug this into the differential equation it will simplify out to zero since it is the solution to the homogeneous differential equation. In other words, these terms add nothing to the particular solution and so we will go ahead and assume that $c = 0$ and $k = 0$ in all the examples.

One final note before we proceed with examples. Do not worry about which of your two solutions in the complementary solution is $y_1(t)$ and which one is $y_2(t)$. It doesn't matter. You will get the same answer no matter which one you choose to be $y_1(t)$ and which one you choose to be $y_2(t)$.

Let's work a couple of examples now.

> ***Example 1*** Find a general solution to the following differential equation.
> $$2y'' + 18y = 6\tan(3t)$$
>
> **Solution**
> First, since the formula for variation of parameters requires a coefficient of a one in front of the second derivative let's take care of that before we forget. The differential equation that we'll actually be solving is
> $$y'' + 9y = 3\tan(3t)$$
>
> We'll leave it to you to verify that the complementary solution for this differential equation is
> $$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t)$$

So, we have
$$y_1(t) = \cos(3t) \qquad\qquad y_2(t) = \sin(3t)$$

The Wronskian of these two functions is
$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3$$

The particular solution is then,
$$Y_P(t) = -\cos(3t)\int \frac{3\sin(3t)\tan(3t)}{3}\,dt + \sin(3t)\int \frac{3\cos(3t)\tan(3t)}{3}\,dt$$

$$= -\cos(3t)\int \frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt$$

$$= -\cos(3t)\int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt$$

$$= -\cos(3t)\int \sec(3t) - \cos(3t)\,dt + \sin(3t)\int \sin(3t)\,dt$$

$$= -\frac{\cos(3t)}{3}\big(\ln|\sec(3t)+\tan(3t)| - \sin(3t)\big) + \frac{\sin(3t)}{3}\big(-\cos(3t)\big)$$

$$= -\frac{\cos(3t)}{3}\ln|\sec(3t)+\tan(3t)|$$

The general solution is,
$$y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(3t)}{3}\ln|\sec(3t)+\tan(3t)|$$

Example 2 Find a general solution to the following differential equation.
$$y'' - 2y' + y = \frac{\mathbf{e}^t}{t^2+1}$$

Solution

We first need the complementary solution for this differential equation. We'll leave it to you to verify that the complementary solution is,
$$y_c(t) = c_1\mathbf{e}^t + c_2 t\mathbf{e}^t$$

So, we have
$$y_1(t) = \mathbf{e}^t \qquad\qquad y_2(t) = t\mathbf{e}^t$$

The Wronskian of these two functions is
$$W = \begin{vmatrix} \mathbf{e}^t & t\mathbf{e}^t \\ \mathbf{e}^t & \mathbf{e}^t + t\mathbf{e}^t \end{vmatrix} = \mathbf{e}^t\big(\mathbf{e}^t + t\mathbf{e}^t\big) - \mathbf{e}^t\big(t\mathbf{e}^t\big) = \mathbf{e}^{2t}$$

The particular solution is then,

$$Y_P(t) = -\mathbf{e}^t \int \frac{t\mathbf{e}^t \, \mathbf{e}^t}{\mathbf{e}^{2t}(t^2+1)} \, dt + t\mathbf{e}^t \int \frac{\mathbf{e}^t \, \mathbf{e}^t}{\mathbf{e}^{2t}(t^2+1)} \, dt$$

$$= -\mathbf{e}^t \int \frac{t}{t^2+1} \, dt + t\mathbf{e}^t \int \frac{1}{t^2+1} \, dt$$

$$= -\frac{1}{2}\mathbf{e}^t \ln(1+t^2) + t\mathbf{e}^t \tan^{-1}(t)$$

The general solution is,

$$y(t) = c_1\mathbf{e}^t + c_2 t\mathbf{e}^t - \frac{1}{2}\mathbf{e}^t \ln(1+t^2) + t\mathbf{e}^t \tan^{-1}(t)$$

This method can also be used on non-constant coefficient differential equations, provided we know a fundamental set of solutions for the associated homogeneous differential equation.

Example 3 Find the general solution to
$$ty'' - (t+1)y' + y = t^2$$

given that

$$y_1(t) = \mathbf{e}^t \qquad\qquad y_2(t) = t+1$$

form a fundamental set of solutions for the homogeneous differential equation.

Solution
As with the first example, we first need to divide out by a *t*.
$$y'' - \left(1 + \frac{1}{t}\right)y' + \frac{1}{t}y = t$$

The Wronskian for the fundamental set of solutions is

$$W = \begin{vmatrix} \mathbf{e}^t & t+1 \\ \mathbf{e}^t & 1 \end{vmatrix} = \mathbf{e}^t - \mathbf{e}^t(t+1) = -t\mathbf{e}^t$$

The particular solution is.

$$Y_P(t) = -\mathbf{e}^t \int \frac{(t+1)t}{-t\mathbf{e}^t} \, dt + (t+1) \int \frac{\mathbf{e}^t(t)}{-t\mathbf{e}^t} \, dt$$

$$= \mathbf{e}^t \int (t+1)\mathbf{e}^{-t} \, dt - (t+1) \int \, dt$$

$$= \mathbf{e}^t \left(-\mathbf{e}^{-t}(t+2)\right) - (t+1)t$$

$$= -t^2 - 2t - 2$$

The general solution for this differential equation is.

$$y(t) = c_1\mathbf{e}^t + c_2(t+1) - t^2 - 2t - 2$$

We need to address one more topic about the solution to the previous example. The solution can be simplified down somewhat if we do the following.

$$y(t) = c_1 \mathbf{e}^t + c_2(t+1) - t^2 - 2t - 2$$
$$= c_1 \mathbf{e}^t + c_2(t+1) - t^2 - 2(t+1)$$
$$= c_1 \mathbf{e}^t + (c_2 - 2)(t+1) - t^2$$

Now, since c_2 is an unknown constant subtracting 2 from it won't change that fact. So we can just write the $c_2 - 2$ as c_2 and be done with it. Here is a simplified version of the solution for this example.

$$y(t) = c_1 \mathbf{e}^t + c_2(t+1) - t^2$$

This isn't always possible to do, but when it is you can simplify future work.

Section 3-11 : Mechanical Vibrations

It's now time to take a look at an application of second order differential equations. We're going to take a look at mechanical vibrations. In particular we are going to look at a mass that is hanging from a spring.

Vibrations can occur in pretty much all branches of engineering and so what we're going to be doing here can be easily adapted to other situations, usually with just a change in notation.

Let's get the situation setup. We are going to start with a spring of length *l*, called the natural length, and we're going to hook an object with mass *m* up to it. When the object is attached to the spring the spring will stretch a length of *L*. We will call the equilibrium position the position of the center of gravity for the object as it hangs on the spring with no movement.

Below is sketch of the spring with and without the object attached to it.
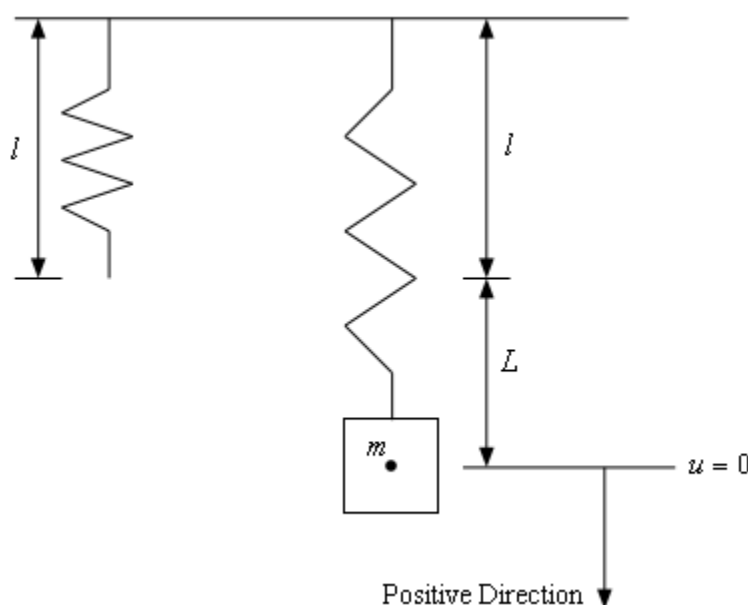
As denoted in the sketch we are going to assume that all forces, velocities, and displacements in the downward direction will be positive. All forces, velocities, and displacements in the upward direction will be negative.

Also, as shown in the sketch above, we will measure all displacement of the mass from its equilibrium position. Therefore, the *u* = 0 position will correspond to the center of gravity for the mass as it hangs on the spring and is at rest (*i.e.* no movement).

Now, we need to develop a differential equation that will give the displacement of the object at any time *t*. First, recall Newton's Second Law of Motion.

$$ma = F$$

In this case we will use the second derivative of the displacement, *u*, for the acceleration and so Newton's Second Law becomes,

$$mu'' = F\left(t, u, u'\right)$$

We now need to determine all the forces that will act upon the object. There are four forces that we will assume act upon the object. Two that will always act on the object and two that may or may not act upon the object.

Here is a list of the forces that will act upon the object.

1. **Gravity, F_g**
    The force due to gravity will always act upon the object of course. This force is
    $$F_g = mg$$

2. **Spring, F_s**
    We are going to assume that Hooke's Law will govern the force that the spring exerts on the object. This force will always be present as well and is
    $$F_s = -k\left(L + u\right)$$

    Hooke's Law tells us that the force exerted by a spring will be the spring constant, $k > 0$, times the displacement of the spring from its natural length. For our set up the displacement from the spring's natural length is *L + u* and the minus sign is in there to make sure that the force always has the correct direction.

    Let's make sure that this force does what we expect it to. If the object is at rest in its equilibrium position the displacement is *L* and the force is simply $F_s = -kL$ which will act in the upward position as it should since the spring has been stretched from its natural length.

    If the spring has been stretched further down from the equilibrium position then $L + u$ will be positive and F_s will be negative acting to pull the object back up as it should be.

    Next, if the object has been moved up past its equilibrium point, but not yet to its natural length then *u* will be negative, but still less than *L* and so *L + u* will be positive and once again F_s will be negative acting to pull the object up.

    Finally, if the object has been moved upwards so that the spring is now compressed, then *u* will be negative and greater than *L*. Therefore, *L + u* will be negative and now F_s will be positive acting to push the object down.

    So, it looks like this force will act as we expect that it should.

3. **Damping, F_d**
    The next force that we need to consider is damping. This force may or may not be present for any given problem.

Dampers work to counteract any movement. There are several ways to define a damping force. The one that we'll use is the following.

$$F_d = -\gamma u'$$

where, $\gamma > 0$ is the damping coefficient. Let's think for a minute about how this force will act. If the object is moving downward, then the velocity (u') will be positive and so F_d will be negative and acting to pull the object back up. Likewise, if the object is moving upward, the velocity (u') will be negative and so F_d will be positive and acting to push the object back down.

In other words, the damping force as we've defined it will always act to counter the current motion of the object and so will act to damp out any motion in the object.

4. **External Forces, *F(t)***
    This is the catch all force. If there are any other forces that we decide we want to act on our object we lump them in here and call it good. We typically call *F(t)* the forcing function.

Putting all of these together gives us the following for Newton's Second Law.

$$mu'' = mg - k(L+u) - \gamma u' + F(t)$$

Or, upon rewriting, we get,

$$mu'' + \gamma u' + ku = mg - kL + F(t)$$

Now, when the object is at rest in its equilibrium position there are exactly two forces acting on the object, the force due to gravity and the force due to the spring. Also, since the object is at rest (*i.e.* not moving) these two forces must be canceling each other out. This means that we must have,

$$mg = kL \qquad \qquad \textbf{(1)}$$

Using this in Newton's Second Law gives us the final version of the differential equation that we'll work with.

$$mu'' + \gamma u' + ku = F(t) \qquad \qquad \textbf{(2)}$$

Along with this differential equation we will have the following initial conditions.

$$u(0) = u_0 \qquad \qquad \text{Initial displacement from the equilibrium position.}$$
$$u'(0) = u'_0 \qquad \qquad \text{Initial velocity.} \qquad \qquad \textbf{(3)}$$

Note that we'll also be using (1) to determine the spring constant, *k*.

Okay. Let's start looking at some specific cases.

Free, Undamped Vibrations
This is the simplest case that we can consider. Free or unforced vibrations means that *F(t) = 0* and undamped vibrations means that $\gamma = 0$. In this case the differential equation becomes,

$$mu'' + ku = 0$$

This is easy enough to solve in general. The characteristic equation has the roots,

$$r = \pm\, i \sqrt{\frac{k}{m}}$$

This is usually reduced to,

$$r = \pm\, \omega_0\, i$$

where,

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and ω_0 is called the natural frequency. Recall as well that $m > 0$ and $k > 0$ and so we can guarantee that this quantity will not be complex. The solution in this case is then

$$u(t) = c_1 \cos\left(\omega_0\, t\right) + c_2 \sin\left(\omega_0\, t\right) \tag{4}$$

We can write (4) in the following form,

$$u(t) = R \cos\left(\omega_0 t - \delta\right) \tag{5}$$

where R is the amplitude of the displacement and δ is the phase shift or phase angle of the displacement.

When the displacement is in the form of (5) it is usually easier to work with. However, it's easier to find the constants in (4) from the initial conditions than it is to find the amplitude and phase shift in (5) from the initial conditions. So, in order to get the equation into the form in (5) we will first put the equation in the form in (4), find the constants, c_1 and c_2 and then convert this into the form in (5).

So, assuming that we have c_1 and c_2 how do we determine R and δ? Let's start with (5) and use a trig identity to write it as

$$u(t) = R \cos\left(\delta\right) \cos\left(\omega_0 t\right) + R \sin\left(\delta\right) \sin\left(\omega_0 t\right) \tag{6}$$

Now, R and δ are constants and so if we compare (6) to (4) we can see that

$$c_1 = R \cos\delta \qquad\qquad c_2 = R \sin\delta$$

We can find R in the following way.

$$c_1^2 + c_2^2 = R^2 \cos^2\delta + R^2 \sin^2\delta = R^2$$

Taking the square root of both sides and assuming that R is positive will give

$$R = \sqrt{c_1^2 + c_2^2} \tag{7}$$

Finding δ is just as easy. We'll start with

$$\frac{c_2}{c_1} = \frac{R \sin\delta}{R \cos\delta} = \tan\delta$$

Taking the inverse tangent of both sides gives,

$$\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right)$$ **(8)**

Before we work any examples let's talk a little bit about units of mass and the Imperial vs. metric system differences.

Recall that the weight of the object is given by
$$W = mg$$
where m is the mass of the object and g is the gravitational acceleration. For the examples in this problem we'll be using the following values for g.

$$\text{Imperial} : g = 32 \ ft/s^2$$
$$\text{Metric} : g = 9.8 \ m/s^2$$

This is not the standard 32.2 ft/s^2 or 9.81 m/s^2, but using these will make some of the numbers come out a little nicer.

In the metric system the mass of objects is given in kilograms (*kg*) and there is nothing for us to do. However, in the British system we tend to be given the weight of an object in pounds (yes, pounds are the units of weight not mass…) and so we'll need to compute the mass for these problems.

At this point we should probably work an example of all this to see how this stuff works.

Example 1 A 16 lb object stretches a spring $\frac{8}{9}$ ft by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 inches upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement at any time *t*, *u(t)*.

Solution
We first need to set up the IVP for the problem. This requires us to get our hands on *m* and *k*.

This is the Imperial system so we'll need to compute the mass.
$$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$$

Now, let's get *k*. We can use the fact that $mg = kL$ to find *k*. Don't forget that we'll need all of our length units the same. We'll use feet for the unit of measurement for this problem.
$$k = \frac{mg}{L} = \frac{16}{8/9} = 18$$

We can now set up the IVP.
$$\frac{1}{2}u'' + 18u = 0 \qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

For the initial conditions recall that upward displacement/motion is negative while downward displacement/motion is positive. Also, since we decided to do everything in feet we had to convert the initial displacement to feet.

Now, to solve this we can either go through the characteristic equation or we can just jump straight to the formula that we derived above. We'll do it that way. First, we need the natural frequency,

$$\omega_0 = \sqrt{\frac{18}{1/2}} = \sqrt{36} = 6$$

The general solution, along with its derivative, is then,

$$u(t) = c_1 \cos(6t) + c_2 \sin(6t)$$
$$u'(t) = -6c_1 \sin(6t) + 6c_2 \cos(6t)$$

Applying the initial conditions gives

$$-\frac{1}{2} = u(0) = c_1 \qquad\qquad c_1 = -\frac{1}{2}$$

$$1 = u'(0) = 6c_2 \cos(6t) \qquad c_2 = \frac{1}{6}$$

The displacement at any time t is then

$$u(t) = -\frac{1}{2}\cos(6t) + \frac{1}{6}\sin(6t)$$

Now, let's convert this to a single cosine. First let's get the amplitude, R.

$$R = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{\sqrt{10}}{6} = 0.52705$$

You can use either the exact value here or a decimal approximation. Often the decimal approximation will be easier.

Now let's get the phase shift.

$$\delta = \tan^{-1}\left(\frac{1/6}{-1/2}\right) = -0.32175$$

We need to be careful with this part. The phase angle found above is in Quadrant IV, but there is also an angle in Quadrant II that would work as well. We get this second angle by adding π onto the first angle. So, we actually have two angles. They are

$$\delta_1 = -0.32175$$
$$\delta_2 = \delta_1 + \pi = 2.81984$$

We need to decide which of these phase shifts is correct, because only one will be correct. To do this recall that

$$c_1 = R\cos\delta$$
$$c_2 = R\sin\delta$$

Now, since we are assuming that R is positive this means that the sign of $\cos\delta$ will be the same as the sign of c_1 and the sign of $\sin\delta$ will be the same as the sign of c_2. So, for this particular case we must have $\cos\delta < 0$ and $\sin\delta > 0$. This means that the phase shift must be in Quadrant II and so the second angle is the one that we need.

So, after all of this the displacement at any time t is.

$$u(t) = 0.52705\cos(6t - 2.81984)$$

Here is a sketch of the displacement for the first 5 seconds.
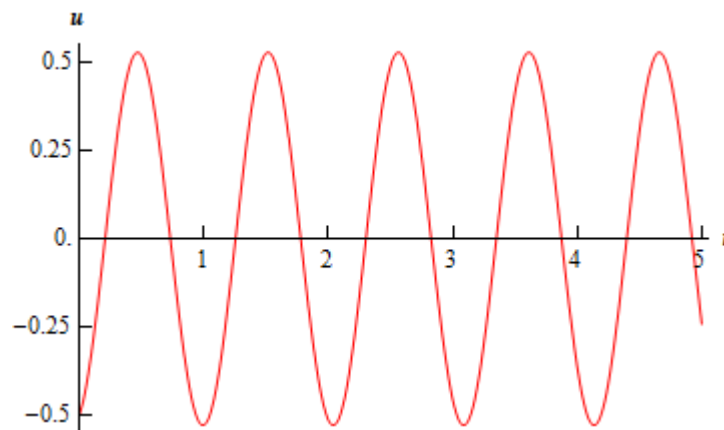
Now, let's take a look at a slightly more realistic situation. No vibration will go on forever. So, let's add in a damper and see what happens now.

Free, Damped Vibrations
We are still going to assume that there will be no external forces acting on the system, with the exception of damping of course. In this case the differential equation will be.

$$mu'' + \gamma u' + ku = 0$$

where m, γ, and k are all positive constants. Upon solving for the roots of the characteristic equation we get the following.

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

We will have three cases here.

1. $\gamma^2 - 4mk = 0$

In this case we will get a double root out of the characteristic equation and the displacement at any time t will be.

$$u(t) = c_1 \mathbf{e}^{-\frac{\gamma t}{2m}} + c_2 t \mathbf{e}^{-\frac{\gamma t}{2m}}$$

Notice that as $t \rightarrow \infty$ the displacement will approach zero and so the damping in this case will do what it's supposed to do.

This case is called **critical damping** and will happen when the damping coefficient is,

$$\gamma^2 - 4mk = 0$$
$$\gamma^2 = 4mk$$
$$\gamma = 2\sqrt{mk} = \gamma_{CR}$$

The value of the damping coefficient that gives critical damping is called the critical damping coefficient and denoted by γ_{CR}.

2. $\gamma^2 - 4mk > 0$

In this case let's rewrite the roots a little.

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$
$$= \frac{-\gamma \pm \gamma\sqrt{1 - \frac{4mk}{\gamma^2}}}{2m}$$
$$= -\frac{\gamma}{2m}\left(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right)$$

Also notice that from our initial assumption that we have,

$$\gamma^2 > 4mk$$
$$1 > \frac{4mk}{\gamma^2}$$

Using this we can see that the fraction under the square root above is less than one. Then if the quantity under the square root is less than one, this means that the square root of this quantity is also going to be less than one. In other words,

$$\sqrt{1 - \frac{4mk}{\gamma^2}} < 1$$

Why is this important? Well, the quantity in the parenthesis is now one plus/minus a number that is less than one. This means that the quantity in the parenthesis is guaranteed to be positive and so the two roots in this case are guaranteed to be negative. Therefore, the displacement at any time t is,

$$u(t) = c_1 \mathbf{e}^{r_1 t} + c_2 \mathbf{e}^{r_2 t}$$

and will approach zero as $t \to \infty$. So, once again the damper does what it is supposed to do.

This case will occur when

$$\gamma^2 > 4mk$$

$$\gamma > 2\sqrt{mk}$$

$$\gamma > \gamma_{CR}$$

and is called **over damping.**

3. $\gamma^2 - 4mk < 0$

In this case we will get complex roots out of the characteristic equation.

$$r_{1,2} = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m} = \lambda \pm \mu i$$

where the real part is guaranteed to be negative and so the displacement is

$$u(t) = c_1 \mathbf{e}^{\lambda t} \cos(\mu t) + c_2 \mathbf{e}^{\lambda t} \sin(\mu t)$$

$$= \mathbf{e}^{\lambda t} \left(c_1 \cos(\mu t) + c_2 \sin(\mu t) \right)$$

$$= R \mathbf{e}^{\lambda t} \cos(\mu t - \delta)$$

Notice that we reduced the sine and cosine down to a single cosine in this case as we did in the undamped case. Also, since $\lambda < 0$ the displacement will approach zero as $t \to \infty$ and the damper will also work as it's supposed to in this case.

We will get this case will occur when

$$\gamma^2 < 4mk$$

$$\gamma < 2\sqrt{mk}$$

$$\gamma < \gamma_{CR}$$

and is called **under damping.**

Let's take a look at a couple of examples here with damping.

Example 2 Take the spring and mass system from the first example and attach a damper to it that will exert a force of 12 lbs when the velocity is 2 ft/s. Find the displacement at any time *t*, *u(t)*.

Solution
The mass and spring constant were already found in the first example so we won't do the work here. We do need to find the damping coefficient however. To do this we will use the formula for the damping force given above with one modification. The original damping force formula is,

$$F_d = -\gamma u'$$

However, remember that the force and the velocity are always acting in opposite directions. So, if the velocity is upward (*i.e.* negative) the force will be downward (*i.e.* positive) and so the minus in the formula will cancel against the minus in the velocity. Likewise, if the velocity is downward (*i.e.*

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positive) the force will be upwards (*i.e.* negative) and in this case the minus sign in the formula will cancel against the minus in the force. In other words, we can drop the minus sign in the formula and use

$$F_d = \gamma u'$$

and then just ignore any signs for the force and velocity.

Doing this gives us the following for the damping coefficient

$$12 = \gamma(2) \qquad \Rightarrow \qquad \gamma = 6$$

The IVP for this example is then,

$$\frac{1}{2}u'' + 6u' + 18u = 0 \qquad\qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

Before solving let's check to see what kind of damping we've got. To do this all we need is the critical damping coefficient.

$$\gamma_{CR} = 2\sqrt{km} = 2\sqrt{(18)\left(\frac{1}{2}\right)} = 2\sqrt{9} = 6$$

So, it looks like we've got critical damping. Note that this means that when we go to solve the differential equation we should get a double root.

Speaking of solving, let's do that. I'll leave the details to you to check that the displacement at any time *t* is.

$$u(t) = -\frac{1}{2}e^{-6t} - 2te^{-6t}$$

Here is a sketch of the displacement during the first 3 seconds.
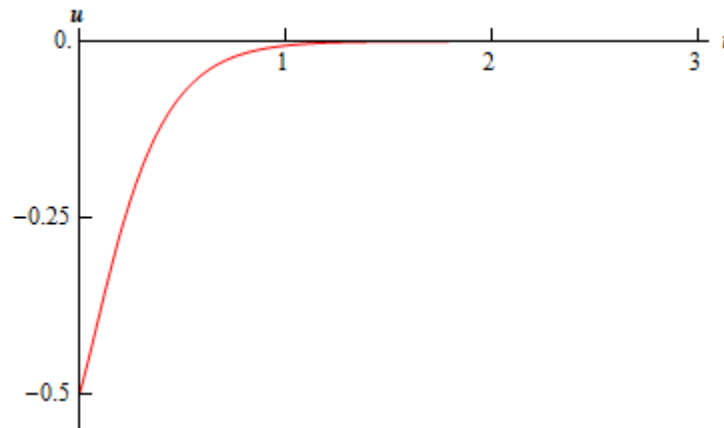
Notice that the "vibration" in the system is not really a true vibration as we tend to think of them. In the critical damping case there isn't going to be a real oscillation about the equilibrium point that we tend to associate with vibrations. The damping in this system is strong enough to force the "vibration" to die out before it ever really gets a chance to do much in the way of oscillation.

Example 3 Take the spring and mass system from the first example and this time let's attach a damper to it that will exert a force of 17 lbs when the velocity is 2 ft/s. Find the displacement at any time *t*, *u(t)*.

Solution
So, the only difference between this example and the previous example is damping force. So, let's find the damping coefficient

$$17 = \gamma(2) \qquad \Rightarrow \qquad \gamma = \frac{17}{2} = 8.5 > \gamma_{CR}$$

So, it looks like we've got over damping this time around so we should expect to get two real distinct roots from the characteristic equation and they should both be negative. The IVP for this example is,

$$\frac{1}{2}u'' + \frac{17}{2}u' + 18u = 0 \qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

This one's a little messier than the previous example so we'll do a couple of the steps, leaving it to you to fill in the blanks. The roots of the characteristic equation are
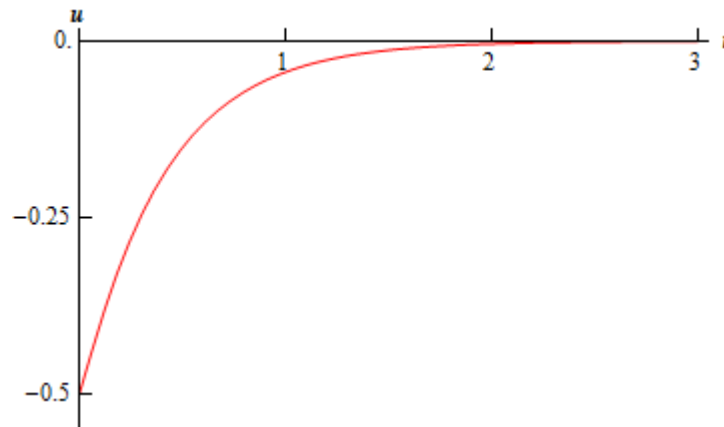
$$r_{1,2} = \frac{-17 \pm \sqrt{145}}{2} = -2.4792, \ -14.5208$$

In this case it will be easier to just convert to decimals and go that route. Note that, as predicted we got two real, distinct and negative roots. The general and actual solution for this example are then,

$$u(t) = c_1 e^{-2.4792\,t} + c_2 e^{-14.5208\,t}$$
$$u(t) = -0.5198 e^{-2.4792\,t} + 0.0199 e^{-14.5208\,t}$$

Here's a sketch of the displacement for this example.

Notice an interesting thing here about the displacement here. Even though we are "over" damped in this case, it actually takes longer for the vibration to die out than in the critical damping case. Sometimes this happens, although it will not always be the case that over damping will allow the vibration to continue longer than the critical damping case.

Also notice that, as with the critical damping case, we don't get a vibration in the sense that we usually think of them. Again, the damping is strong enough to force the vibration do die out quick enough so that we don't see much, if any, of the oscillation that we typically associate with vibrations.

Let's take a look at one more example before moving on the next type of vibrations.

Example 4 Take the spring and mass system from the first example and for this example let's attach a damper to it that will exert a force of 5 lbs when the velocity is 2 ft/s. Find the displacement at any time *t*, u(t).

Solution
So, let's get the damping coefficient.

$$5 = \gamma(2) \qquad \Rightarrow \qquad \gamma = \frac{5}{2} = 2.5 < \gamma_{CR}$$

So, it's under damping this time. That shouldn't be too surprising given the first two examples. The IVP for this example is,

$$\frac{1}{2}u'' + \frac{5}{2}u' + 18u = 0 \qquad\qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

In this case the roots of the characteristic equation are

$$r_{1,2} = \frac{-5 \pm \sqrt{119}\, i}{2}$$

They are complex as we expected to get since we are in the under damped case. The general solution and actual solution are

$$u(t) = \mathbf{e}^{-\frac{5t}{2}}\left(c_1 \cos\left(\frac{\sqrt{119}}{2}t \right) + c_2 \sin\left(\frac{\sqrt{119}}{2}t \right) \right)$$

$$u(t) = \mathbf{e}^{-\frac{5t}{2}}\left(-0.5 \cos\left(\frac{\sqrt{119}}{2}t \right) - 0.04583 \sin\left(\frac{\sqrt{119}}{2}t \right) \right)$$

Let's convert this to a single cosine as we did in the undamped case.

$$R = \sqrt{(-0.5)^2 + (-0.04583)^2} = 0.502096$$

$$\delta_1 = \tan^{-1}\left(\frac{-0.04583}{-0.5} \right) = 0.09051 \qquad \text{OR} \qquad \delta_2 = \delta_1 + \pi = 3.2321$$

As with the undamped case we can use the coefficients of the cosine and the sine to determine which phase shift that we should use. The coefficient of the cosine (c_1) is negative and so $\cos\delta$ must also be negative. Likewise, the coefficient of the sine (c_2) is also negative and so $\sin\delta$ must also be negative. This means that δ must be in the Quadrant III and so the second angle is the one that we want.

The displacement is then

$$u(t) = 0.502096\mathbf{e}^{-\frac{5t}{2}}\cos\left(\frac{\sqrt{119}}{2}t - 3.2321\right)$$

Here is a sketch of this displacement.
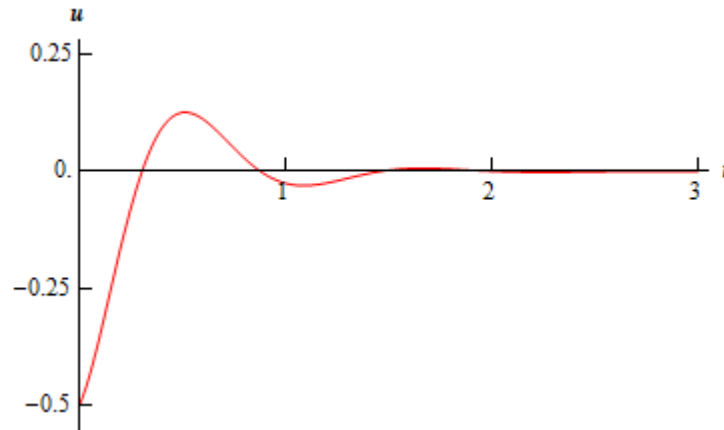
In this case we finally got what we usually consider to be a true vibration. In fact, that is the point of critical damping. As we increase the damping coefficient, the critical damping coefficient will be the first one in which a true oscillation in the displacement will not occur. For all values of the damping coefficient larger than this (*i.e.* over damping) we will also not see a true oscillation in the displacement.

From a physical standpoint critical (and over) damping is usually preferred to under damping. Think of the shock absorbers in your car. When you hit a bump you don't want to spend the next few minutes bouncing up and down while the vibration set up by the bump die out. You would like there to be as little movement as possible. In other words, you will want to set up the shock absorbers in your car so get at the least critical damping so that you can avoid the oscillations that will arise from an under damped case.

It's now time to look at systems in which we allow other external forces to act on the object in the system.

Undamped, Forced Vibrations
 We will first take a look at the undamped case. The differential equation in this case is

$$mu'' + ku = F(t)$$

This is just a nonhomogeneous differential equation and we know how to solve these. The general solution will be

$$u(t) = u_c(t) + U_P(t)$$

where the complementary solution is the solution to the free, undamped vibration case. To get the particular solution we can use either undetermined coefficients or variation of parameters depending on which we find easier for a given forcing function.

There is a particular type of forcing function that we should take a look at since it leads to some interesting results. Let's suppose that the forcing function is a simple periodic function of the form

$$F(t) = F_0 \cos(\omega t) \qquad \text{OR} \qquad F(t) = F_0 \sin(\omega t)$$

For the purposes of this discussion we'll use the first one. Using this, the IVP becomes,

$$mu'' + ku = F_0 \cos(\omega t)$$

The complementary solution, as pointed out above, is just

$$u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where ω_0 is the natural frequency.

We will need to be careful in finding a particular solution. The reason for this will be clear if we use undetermined coefficients. With undetermined coefficients our guess for the form of the particular solution would be,

$$U_P(t) = A \cos(\omega t) + B \sin(\omega t)$$

Now, this guess will be problems if $\omega_0 = \omega$. If this were to happen the guess for the particular solution is exactly the complementary solution and so we'd need to add in a t. Of course, if we don't have $\omega_0 = \omega$ then there will be nothing wrong with the guess.

So, we will need to look at this in two cases.

1. $\omega_0 \neq \omega$

In this case our initial guess is okay since it won't be the complementary solution. Upon differentiating the guess and plugging it into the differential equation and simplifying we get,

$$(-m\omega^2 A + kA)\cos(wt) + (-m\omega^2 B + kB)\sin(wt) = F_0 \cos(wt)$$

Setting coefficients equal gives us,

$$\cos(\omega t): \qquad (-m\omega^2 + k)A = F_0 \qquad \Rightarrow \qquad A = \frac{F_0}{k - m\omega^2}$$

$$\sin(\omega t): \qquad (-m\omega^2 + k)B = 0 \qquad \Rightarrow \qquad B = 0$$

The particular solution is then

$$U_P(t) = \frac{F_0}{k - m\omega^2}\cos(\omega t)$$

$$= \frac{F_0}{m\left(\dfrac{k}{m} - \omega^2\right)}\cos(\omega t)$$

$$= \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t)$$

Note that we rearranged things a little. Depending on the form that you'd like the displacement to be in we can have either of the following.

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$u(t) = R \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

If we used the sine form of the forcing function we could get a similar formula.

2. $\omega_0 = \omega$

In this case we will need to add in a *t* to the guess for the particular solution.

$$U_P(t) = At \cos(\omega_0 t) + Bt \sin(\omega_0 t)$$

Note that we went ahead and acknowledge that $\omega_0 = \omega$ in our guess. Acknowledging this will help with some simplification that we'll need to do later on. Differentiating our guess, plugging it into the differential equation and simplifying gives us the following.

$$(-m\omega_0^2 + k) At \cos(wt) + (-m\omega_0^2 + k) Bt \sin(wt) +$$
$$2m\omega_0 B \cos(\omega t) - 2m\omega_0 A \sin(wt) = F_0 \cos(wt)$$

Before setting coefficients equal, let's remember the definition of the natural frequency and note that

$$-m\omega_0^2 + k = -m\left(\sqrt{\frac{k}{m}}\right)^2 + k = -m\left(\frac{k}{m}\right) + k = 0$$

So, the first two terms actually drop out (which is a very good thing...) and this gives us,

$$2m\omega_0 B \cos(\omega t) - 2m\omega_0 A \sin(wt) = F_0 \cos(wt)$$

Now let's set coefficient equal.

$$\cos(\omega t): \qquad 2m\omega_0 B = F_0 \quad \Rightarrow \quad B = \frac{F_0}{2m\omega_0}$$

$$\sin(\omega t): \qquad 2m\omega_0 A = 0 \quad \Rightarrow \quad A = 0$$

In this case the particular will be,

$$U_P(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

The displacement for this case is then

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

$$u(t) = R \cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

depending on the form that you prefer for the displacement.

So, what was the point of the two cases here? Well in the first case, $\omega_0 \neq \omega$ our displacement function consists of two cosines and is nice and well behaved for all time.

In contrast, the second case, $\omega_0 = \omega$ will have some serious issues at t increases. The addition of the t in the particular solution will mean that we are going to see an oscillation that grows in amplitude as t increases. This case is called **resonance** and we would generally like to avoid this at all costs.

In this case resonance arose by assuming that the forcing function was,
$$F(t) = F_0 \cos(\omega_0 t)$$

We would also have the possibility of resonance if we assumed a forcing function of the form.
$$F(t) = F_0 \sin(\omega_0 t)$$

We should also take care to not assume that a forcing function will be in one of these two forms. Forcing functions can come in a wide variety of forms. If we do run into a forcing function different from the one that used here you will have to go through undetermined coefficients or variation of parameters to determine the particular solution.

Example 5 A 3 kg object is attached to spring and will stretch the spring 392 mm by itself. There is no damping in the system and a forcing function of the form
$$F(t) = 10\cos(\omega t)$$
is attached to the object and the system will experience resonance. If the object is initially displaced 20 cm downward from its equilibrium position and given a velocity of 10 cm/sec upward find the displacement at any time t.

Solution
Since we are in the metric system we won't need to find mass as it's been given to us. Also, for all calculations we'll be converting all lengths over to meters.

The first thing we need to do is find k.
$$k = \frac{mg}{L} = \frac{(3)(9.8)}{0.392} = 75$$

Now, we are told that the system experiences resonance so let's go ahead and get the natural frequency so we can completely set up the IVP.
$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5$$

The IVP for this is then
$$3u'' + 75u = 10\cos(5t) \qquad u(0) = 0.2 \qquad u'(0) = -0.1$$

Solution wise there isn't a whole lot to do here. The complementary solution is the free undamped solution which is easy to get and for the particular solution we can just use the formula that we derived above.

The general solution is then,

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{10}{2(3)(5)} t \sin(5t)$$

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{3} t \sin(5t)$$

Applying the initial conditions gives the displacement at any time t. We'll leave the details to you to check.

$$u(t) = \frac{1}{5} \cos(5t) - \frac{1}{50} \sin(5t) + \frac{1}{3} t \sin(5t)$$

The last thing that we'll do is combine the first two terms into a single cosine.

$$R = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{50}\right)^2} = 0.200998$$

$$\delta_1 = \tan^{-1}\left(\frac{-\frac{1}{50}}{\frac{1}{5}}\right) = -0.099669 \qquad\qquad \delta_2 = \delta_1 + \pi = 3.041924$$

In this case the coefficient of the cosine is positive and the coefficient of the sine is negative. This forces $\cos\delta$ to be positive and $\sin\delta$ to be negative. This means that the phase shift needs to be in Quadrant IV and so the first one is the correct phase shift this time.

The displacement then becomes,

$$u(t) = 0.200998 \cos(5t + 0.099669) + \frac{1}{3} t \sin(5t)$$

Here is a sketch of the displacement for this example.
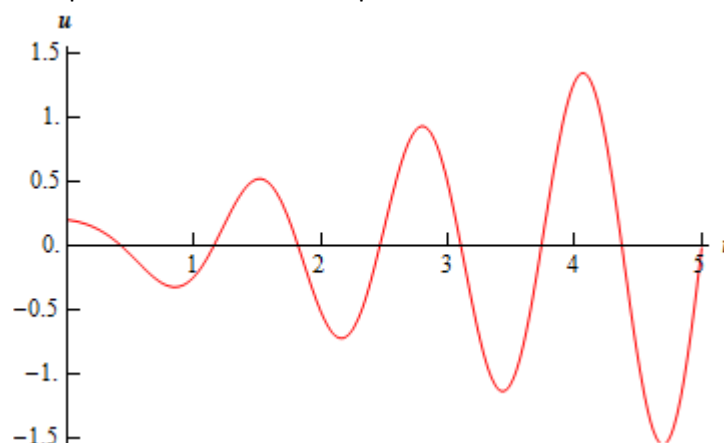
It's now time to look at the final vibration case.

Forced, Damped Vibrations
This is the full blown case where we consider every last possible force that can act upon the system. The differential equation for this case is,

$$mu'' + \gamma u' + ku = F(t)$$

The displacement function this time will be,

$$u(t) = u_c(t) + U_P(t)$$

where the complementary solution will be the solution to the free, damped case and the particular solution will be found using undetermined coefficients or variation of parameter, whichever is most convenient to use.

There are a couple of things to note here about this case. First, from our work back in the free, damped case we know that the complementary solution will approach zero as *t* increases. Because of this the complementary solution is often called the **transient solution** in this case.

Also, because of this behavior the displacement will start to look more and more like the particular solution as *t* increases and so the particular solution is often called the **steady state solution** or **forced response**.

Let's work one final example before leaving this section. As with the previous examples, we're going to leave most of the details out for you to check.

Example 6 Take the system from the last example and add in a damper that will exert a force of 45 Newtons when then velocity is 50 cm/sec.

Solution
So, all we need to do is compute the damping coefficient for this problem then pull everything else down from the previous problem. The damping coefficient is

$$F_d = \gamma u'$$
$$45 = \gamma(0.5)$$
$$\gamma = 90$$

The IVP for this problem is.

$$3u'' + 90u' + 75u = 10\cos(5t) \qquad u(0) = 0.2 \qquad u'(0) = -0.1$$

The complementary solution for this example is

$$u_c(t) = c_1 e^{\left(-15+10\sqrt{2}\right)t} + c_2 e^{\left(-15-10\sqrt{2}\right)t}$$
$$u_c(t) = c_1 e^{-0.8579t} + c_2 e^{-29.1421t}$$

For the particular solution we the form will be,

$$U_P(t) = A\cos(5t) + B\sin(5t)$$

Plugging this into the differential equation and simplifying gives us,

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$$450B\cos(5t) - 450A\sin(5t) = 10\cos(5t)$$

Setting coefficient equal gives,

$$U_P(t) = \frac{1}{45}\sin(5t)$$

The general solution is then

$$u(t) = c_1 e^{-0.8579t} + c_2 e^{-29.1421t} + \frac{1}{45}\sin(5t)$$

Applying the initial condition gives

$$u(t) = 0.1986 e^{-0.8579t} + 0.001398 e^{-29.1421t} + \frac{1}{45}\sin(5t)$$

Here is a sketch of the displacement for this example.