<u>Further Maths Solutions (Multiple Choice)</u>

Section A: Fast

1. **Answer:** undefined; there are no values of *x*

   Working: For 1/2 to be raised to a power and become a larger number, its exponent must be negative. But $|x - 5|$ is never negative so no solutions.

2. **Answer:** **BD**

   Working: To multiply matrices, the length (number of columns) of the first matrix must equal the height (number of rows) of the second matrix.
   ABC = (AB)C: 3 = 3 (result is 2x3) so then 2 = 2.
   CB: 3 = 3.
   CD: 3 = 3.
   BD: 2 is not equal to 3.

3. **Answer:** 10

   Working: $|a \times b|$ is the area of the parallelogram formed by a and b.
   Angle: $\cos \theta = a \cdot b / |a||b| = 30/(10\sqrt{10}) = 3/\sqrt{10} = 3\sqrt{10}/10$
   $\rightarrow \sin \theta = \sqrt{1 - (3/\sqrt{10})^2} = \sqrt{1 - 9/10} = 1/\sqrt{10}$.
   $\rightarrow |a \times b| = |a||b| \sin \theta = 10\sqrt{10} / \sqrt{10} = 10$.

4. **Answer:** $x^2y + y^3 - x = 0$, for all $x \in \mathbb{R}$

   Working: $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$:
   $x^2 + y^2 = \cot(\tan^{-1}(y/x)) \rightarrow x^2 + y^2 = x/y \rightarrow x^2y + y^3 - x = 0$
   Bounds:
   $\theta = 0 \rightarrow r^2 =$ infinity $\rightarrow r =$ plus or minus infinity
   $\rightarrow$ ranges across x-axis $\rightarrow$ all real x.

5. **Answer:** ln $A = x - x^{-1} - 2 \ln x$

Working: $P(x) = (1 - x)^2 / x^2 = (1 - 2x + x^2)/x^2 = x^{-2} - 2x^{-1} + 1$
Integrating, $\ln A = -x^{-1} - 2 \ln x + x = x - x^{-1} - 2 \ln x$

6. **Answer:** **2** only

Working: 1: Eigenvector: does not get rotated. The only vectors which do not get rotated by a rotation matrix are those along the axis, so the only real eigenvector points along the *z*-axis $[0, 0, 1]^T$.
2: Invariant line → all points remain on that line. All points move except z-axis so yes z-axis is only invariant line.
3: M^2 = do transformation twice → rotate 180°. This will not return to the original position (requires 360°).

7. **Answer:** $P(1/x)$

Working: By Vieta's formula, product of roots $\alpha\beta\gamma = -d/a = -(-1)/1 = 1$.
The new roots are of the form
$\alpha\beta\gamma/\alpha, \alpha\beta\gamma/\beta, \alpha\beta\gamma/\gamma = 1/\alpha, 1/\beta, 1/\gamma$ → new roots are $x' = 1/x$
→ $x = 1/x'$

8. **Answer:** $f(x) = e^x \sin x$

Working: Expansion has no constant term → $f(0) = 0$
→ must be either sinh(x) or e^x sin(x)
sinh(x) is an odd function (since sinh x = -sinh -x) so its expansion cannot contain even functions (even powers x^2, x^4 etc).
The given expansion does contain even powers, so it must be e^x sin x.

9. **Answer:** $f(z) = (z - 2 + i)^2$

Working: $z^2 - (4 - 2i)z + (3 - 4i) = (z - (2 - i))^2 - (2 - i)^2 + (3 - 4i)$
$= (z - 2 + i)^2 - (3 - 4i) + (3 - 4i)$
$= (z - 2 + i)^2$

10. **Answer:** $a \leq 2$

Working: sinh(x) passes through (0, 0) and is an concave up (convex) function, so the tangent at (0, 0) will always be less than sinh(x):
dy/dx = 2 cosh(x) : at x = 0, dy/dx = 2 cosh(0) = 2 * 1 = 2
→ y = 2x is the tangent → a ≤ 2.

11. **Answer:** $\cosh(x + y) \equiv \sinh x \sinh y + \cosh x \cosh y$

Working: Using Osborne's rule (in the corresponding trig identity, change the sign of a product of sines):
cos(x + y) = cos x cos y - sin x sin y
cosh(x + y) = cosh x cosh y + sinh x sinh y

12. **Answer:** $3x + 6y - 4z = 49$

Working: Π must contain the midpoint of the points and be normal to the line connecting the points.
Midpoint: (5, 5, -1), vector for line through points: [3; 6; -4]
Using formula (**r** - **a**) • **n** : (**r** - [5; 5; -1]) . [3; 6; -4] = 0
3(x - 5) + 6(y - 5) - 4(z + 1) = 0 → 3x + 6y - 4z = 49

13. **Answer:** Rotation about the *y*-axis by 180°

Working: Let the two transformations be represented by matrices **X** and **Y** such that **M** = **YX** (**X** followed by **Y** gives **M**).
Y is reflection in plane y = 0 → **Y** flips the orientation (sense) of the space → det **Y** = -1 → det **M** = det **X** det **Y** → det **X** = -1 / -1 = 1
So **X** must **not** change the orientation and must **not** scale.
Enlargement with s.f. = -1 changes the orientation.
Rotation matches both requirements.
Reflection changes the orientation.
Projection of 3D space onto a plane (2D) is non-invertible so has determinant = 0 which is not -1.

14. **Answer:** The argument is incomplete since there is no established base case

Working: The inductive step assumes it is true for some *n* = *k*. However, the case *n* = 1 is not tested so it has not been shown that there exists any *k* such that the argument holds.

15. **Answer:** $x = 0$

Working: Vertical asymptotes: $x^2 - 4 = 0 \rightarrow x = 2$ and $x = -2$
Horizontal asymptote: as $x \rightarrow \infty$, $1/(x^2 - 4) \rightarrow 1/\infty \rightarrow 0$ so $y = 0$

16. **Answer:** $\frac{7}{12}$

Working: Adjusting the formula to integrate vertically (instead of the usual horizontal),

$$V = \pi \int_{1}^{2} [x(y)]^2 \, dy$$

where x(y) is the inverse: $y = 18x^2 - 9 \rightarrow x = \sqrt{(y + 9) / 18}$
$\rightarrow x^2 = (y + 9)/18$
Performing the integral on calculator, V = 7/12 π.

17. **Answer:** $$z - \frac{1}{z} = 2i\sin\theta$$

Working: $z = \cos\theta + i\sin\theta$. By De Moivre's theorem,
$z^2 = \cos 2\theta + i \sin 2\theta$
$z + z^{-1} = \cos\theta + i\sin\theta + \cos -\theta + i \sin -\theta$
Since $\cos -\theta = \cos\theta$ (even function) and $\sin -\theta = -\sin\theta$ (odd)
$\rightarrow z + z^{-1} = 2\cos\theta$
$\rightarrow z - z^{-1} = 2i\sin\theta$
$\rightarrow (z - z^{-1}) / (z + z^{-1}) = (2i\sin\theta) / (2\cos\theta)$
$\rightarrow (z^2 - 1) / (z^2 + 1) = i\tan\theta$
(These results could be memorised as they are useful.)

18. **Answer:** Use De Moivre's theorem

Working: The steps of this proof are:
$\sum_{r=1}^{n} \cos(r\theta) = \cos\theta + \cos 2\theta + \ldots + \cos n\theta$
$= \mathrm{Re}(e^{i\theta} + e^{2i\theta} + \ldots + e^{ni\theta})$ [De Moivre's theorem]

$$= \mathrm{Re}\left(e^{i\theta} \cdot \frac{e^{in\theta} - 1}{e^{i\theta} - 1}\right)$$

$$= \mathrm{Re}\left(\frac{e^{i\theta}e^{\frac{in\theta}{2}}}{e^{\frac{i\theta}{2}}} \cdot \frac{e^{\frac{in\theta}{2}} - e^{-\frac{in\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{-\frac{in\theta}{2}}}\right)$$

$$= \mathrm{Re}\left(e^{\frac{1}{2}i(n+1)\theta} \cdot \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right)$$

$$= \frac{\cos\left(\frac{1}{2}(n+1)\theta\right)\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$

(Assuming $e^{i\theta} \neq 1$: case of $\theta = 0$ requires simple alternative)

19. **Answer:** mgh

Working: Potential energy = GPE = mgh
The salmon has gone from high altitude to low altitude
→ Change in GPE < 0
→ 'Loss' in GPE > 0
→ = mgh

20. **Answer:** 2×10^{-7}

Working: $\omega = 2\pi / T = 2\pi / (365 * 24 * 60 * 60) = 2 \times 10^{-7}$ rad s^{-1}

21. **Answer:** 110

Working: Friction = Centripetal force
$\mu mg = mv^2/r \rightarrow r = mv^2 / \mu mg = v^2 / \mu g = 15^2 / (0.2 * 9.8) = 114.8$
= 110 (2 s.f.)

22. **Answer:** None of the above

Working: Angle of limiting friction = $\tan^{-1} \mu$
$\mu > \tan \alpha$ → friction is high enough to prevent sliding → topples
$\mu < \tan \alpha$ → friction too low → slides

23. **Answer:** $F(t) = 1 - e^{-\lambda t}$

Working: For an exponential distribution,
p.d.f: $f(t) = \lambda e^{-\lambda t}$ and c.d.f.: $F(t) = 1 - e^{-\lambda t}$

24. **Answer:** **2** and **4** only

Working: Poisson is positively skewed.
Rectangular is symmetric.
Chi-squared is positively skewed.
t is approximately normal so is symmetric.

25. **Answer:** $\frac{2}{15}$

Working: Mean = 8 buses per hour
→ mean time interval = 7.5 minutes per bus
→ parameter = 1/mean = 1/7.5 = 2/15 buses per minute

26. **Answer:** $y = 2x$

Working: Transforming the general vector [x, y],
$-3x + 2y = x$ and $-10x + 6y = y$
→ $2y = 4x$ and $5y = 10x$
→ $y = 2x$

27. **Answer:** 7

Working: $3^6 = 729$, $6! = 720$ → $3^6 > 6!$
$3^7 = 2187$, $7! = 5040$ → $3^7 < 7!$
and factorials grow faster than exponentials.

28. **Answer:** **2** only

Working: Order: highest number of derivatives (dy/dx → 1st, d^2y/dx^2 → 2nd)
Linear: all terms are constant multiples of *y* and its derivatives
(no *xy*, x*dy/dx, $(dy/dx)^2$, sin *y*, y^2 etc)
Homogeneous: term independent of *y* is zero

29. **Answer:** $\frac{1}{4}\int_0^{\frac{\pi}{2}} 1 - \cos 4\theta \ \mathrm{d}\theta$

Working: Area = 1/2 ∫ r^2 dθ = 1/2 ∫ $\sin^2$ 2θ dθ
Using cos 2θ = 1 - 2 $\sin^2$ θ
→ Area = 1/2 ∫ (1 - cos 4θ) / 2 dθ
= 1/4 ∫ (1 - cos 4θ) dθ

30. **Answer:** the sum of the top-left and bottom-right entries of **M**

Working: Let the matrix be $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:
det(M - λI) = 0
→ (a - λ)(d - λ) - bc = 0
→ λ^2 - dλ - aλ + ad - bc = 0
→ λ^2 - (a + d)λ + (ad - bc) = 0
Roots are eigenvalues, so sum of roots = -(-(a + d)) = a + d = sum of top left and bottom right entries.

31. **Answer:** |**M**| equals the product of the *n* eigenvalues

Working: Let **M** = $\mathbf{UDU}^{-1}$, where **U** is the matrix of eigenvectors and **U** is the diagonal matrix of eigenvalues.
|**M**| = |**U**| * |**D**| * |$\mathbf{U}^{-1}$|
Since |$\mathbf{U}^{-1}$| = 1/|**U**|, we see |**M**| = |**D**|.
The determinant of a diagonal matrix is the product of its diagonal values i.e. the product of the eigenvalues. There is nothing stopping any of these eigenvalues being zero which would make |**M**| = 0.
Coefficient of λ^n in characteristic equation = $(-1)^n$.

32. **Answer:** $\frac{1}{4}$

Working: Since *y* (and hence *x* since $0 \le x \le y$) is small, replace $\sin x = x$:
$\rightarrow 1/y^4 * \int_0^y x^3\,dx$
$\rightarrow 1/y^4 * [x^4/4]_0^y$
$\rightarrow 1/y^4 * y^4/4$
$\rightarrow 1/4$

33. **Answer:**

Trapezium rule	Simpson's rule
overestimate	not enough information

Working: Trapezium rule is an overestimate (lines go above the curve if the curve is concave up (slope increasing; f''(x) > 0).
However, since Simpson's rule uses parabolas with the midpoint also on the curve, the error depends on which side of the parabola is more above or below the curve, which cannot be found by considering f''(x) (we need to know the sign of the third derivative).

34. **Answer:** The number of events occuring in a given interval must be small

Working: Conditions for Poisson distribution are:
- there can only be a whole number of outcomes of the event in the given interval
- outcomes of the event occur at a constant average rate
- outcomes of the event occur independently and at random
- outcomes of the event occur singly

35. **Answer:** **1** only

Working: **1** is valid since the cars are free-flowing so there should be no traffic since cars can overtake (constant mean rate of occurrence).
2 is probably not valid since there will be lots and lots of phone calls, which must be put into a queuing system so when one call finishes, another one is more likely to start (not independent).
3 is probably not valid since there will be traffic forming at irregular intervals (variable mean rate of occurrence).

36. **Answer:** 1 – P(Type II error)

Working: Significance level = P(Type I error)
Confidence level = 1 – P(Type I error)
Power of test = 1 – P(Type II error)

37. **Answer:** $F(x) = \frac{1}{60}(3x^3 - 30x^2 + 99x - 56)$

Working: $F(x) = P(X \leq 1) + \frac{1}{20}\int_1^x (x - 3)(3x - 11)\, dx$

$= \int_0^1 \frac{4}{5}x^2\, dx + \frac{1}{20}\int_1^x 3x^2 - 20x + 33\, dx$

$= \frac{4}{15} + \frac{1}{20}[x^3 - 10x^2 + 33x]_1^x$

$= \frac{1}{20}(x^3 - 10x^2 + 33x - 24) + \frac{4}{15}$

$= \frac{1}{60}(3x^3 - 30x^2 + 99x - 56)$

38. **Answer:** 0.5

Working: Conservation of momentum along dotted line:
$4 * 1 = 2 * \mathbf{v}_{Bi} \rightarrow \mathbf{v}_{Bi} = 2$
Coefficient of restitution along dotted line:
e = (speed of separation) / (speed of approach)
e = 2 / 4 = 0.50

39. **Answer:** 25^{o}

Working: Resolving forces, $g \sin 30 - \mu g \cos 30 = 1$
$\rightarrow \mu = 0.4618$
Angle of friction = $\tan^{-1}(\mu) = \tan^{-1}(0.4618) = 25^{o}$

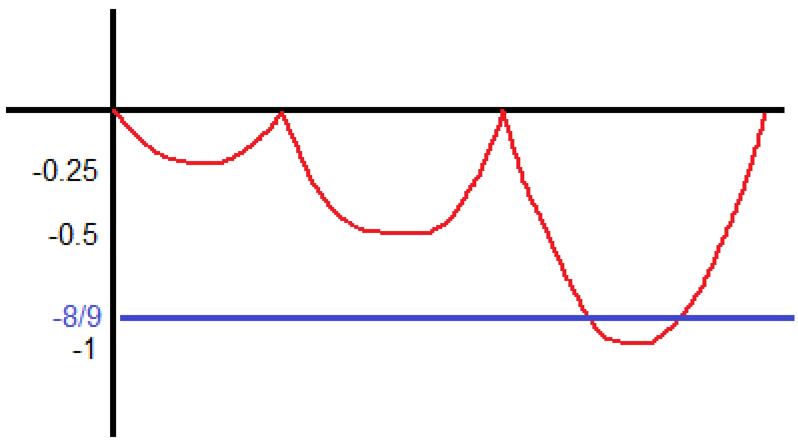
40. **Answer:** **MLT^{-2}**

Working: $\lambda = kl$ (k = force constant; l = natural length)
$[F] = [kx] \rightarrow [k] = \mathbf{MLT^{-2}} / \mathbf{L} = \mathbf{MT^{-2}}$
$\rightarrow [\lambda] = \mathbf{MLT^{-2}}$.

Section B: Standard

41. **Answer:** 7/3

Working: Sketch the function on $0 < x < 1$, it is a parabola with vertex at -0.25. In the interval $1 < x < 2$, the function is scaled up by a factor of 2, so its vertex is -0.5. In $2 < x < 3$, the vertex is at -1:



From observation, the wanted value of m is the lowest intersection. This is in the third part, so in this interval, $y = 4(x - 2)(x - 3)$ (using transformations of the original) → $y = 4x^2 - 20x + 24$
→ $4x^2 - 20x + 24 = -8/9$ → $4x^2 - 20x + 224/9 = 0$
→ $x = 7/3$ or $x = 8/3$
The smallest value is then $m = 7/3$.

42. **Answer:** 0.8185

Working: Normal distribution is symmetric → mean of $X = (1 + 5)/2 = 3$
→ $a = 3$ since Z has mean = 0
→ std.dev of $X = 2$ since $X = \mu + \sigma Z$
→ $P(1 < X < 7) = 0.8185$

43. **Answer:** $f(x) = (3x - 3)^{1/3}$

Working: Differentiating both sides of the equation, (let g(x) be the integrand)
$f'(x) = d/dx\ (\ G(x) - G(a)\ )$
G(a) is a constant so its derivative is zero, and $G'(x) = g(x)$:
$f'(x) = g(x) \rightarrow f'(x) = (1/f(x))^2 \rightarrow dy/dx = y^{-2}$
This DE is separable:
$y^2\ dy = dx \rightarrow \frac{1}{3} y^3 = x + C$
Using initial condition, $y(1) = 0 \rightarrow 0 = 1 + C \rightarrow C = -1$
$\frac{1}{3} y^3 = x - 1 \rightarrow y = (3x - 3)^{1/3}$.

44. **Answer:** $P(x) = x^4 - 2x^2 + 2$

Working: Roots form a square when the roots are roots of unity (1):
$x^4 - 1 = 0 \rightarrow x^4 = 1 \rightarrow$ roots of unity $\rightarrow$ square
$x^4 - 2x^2 + 2 = 0 \rightarrow (x^2 - 1)^2 + 1 = 0 \rightarrow (x^2 - 1)^2 = -1 \rightarrow$ not unity
$x^4 + 4x^3 + 6x^2 + 4x + 2 = 0 \rightarrow (x + 1)^4 + 1 = 0 \rightarrow (x + 1)^4 = -1$
$\rightarrow$ roots of unity
$x^4 - 4ix^3 - 6x^2 + 4ix + 2 = (x + i)^4 + 1 = 0 \rightarrow (x + i)^4 = -1$
$\rightarrow$ roots of unity

45. **Answer:**
$$S = 20\pi \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$

Working: Circle has equation $x^2 + (y - 5)^2 = 1 \rightarrow y = 5 \pm \sqrt{(1 - x^2)}$
$\rightarrow dy/dx = \mp 2x\ /\ (2\ \sqrt{(1 - x^2)}) = \mp x\ /\ \sqrt{(1 - x^2)}$
Total surface area = surface from upper half of circle + surface from lower half of circle
$$S = 2\pi \int_{-1}^{1} \left(5 + \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x + 2\pi \int_{-1}^{1} \left(5 - \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$
$$S = 2\pi \int_{-1}^{1} \left(5 + \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}} + \left(5 - \sqrt{1 - x^2}\right) \sqrt{1 + \frac{x^2}{1 - x^2}}\ \mathrm{d}x$$

$$S = 2\pi \int_{-1}^{1} 10\sqrt{1 + \frac{x^2}{1 - x^2}}\,\mathrm{d}x$$

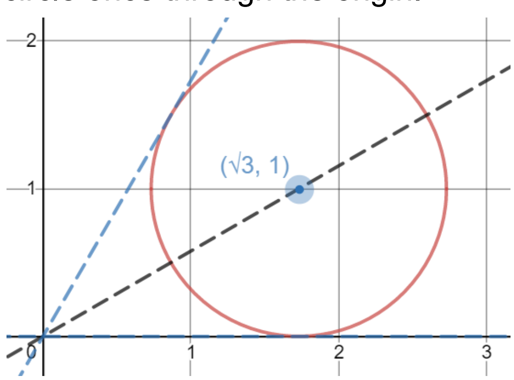
$$S = 20\pi \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\,\mathrm{d}x$$

46. **Answer:** $f''(x)$

Working: f’(x) = 1/x + √(1 + sin x) and f’’(x) = $-1/x^2$ + (cos x) / (2√1 + sin x)
→ asymptote at sin x = -1 → x = 3π/2 → discontinuity in (0, 2π]
Since 1/x and √1 + sin x is always positive (and never both zero),
f’(x) > 0 so f(x) is an increasing function on x > 0 so its inverse $f^{-1}(x)$
exists and is always defined.
ln f(x + 1) must have f(x) > 0 for all x > 1. Since increasing, check
x = 1 → f(1) = 0 + positive = positive > 0 → ln f(x + 1) is defined for
all x > 0.

47. **Answer:** |z| takes its minimum value on *C* when $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

Working: The locus is |z - (√3 + i)| = 1. The extreme values of |z| will be
when z is on the same and opposite sides of the circle to the origin.
The extreme values of arg z will be when the tangents meet the
circle once through the origin.



By trigonometry and simple distances,
z = √3/2 + 1/2 i → |z| = 1 (min), z = 3√3/2 + 3/2 i → |z| = 3 (max)
z = √3 → arg z = 0 (min), z = 1 + √3 i → arg z = π/3 (max)

48. **Answer:** 1

Working: The desired value is

$= (\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta)$

Each factor is equal to $\alpha + \beta + \gamma + \delta$ minus one of the roots.

The sum of the roots is p, so this is equal to

$= (p - \alpha)(p - \beta)(p - \gamma)(p - \delta)$

Letting $x = p - z$ in the original polynomial gives a new polynomial with these roots. The product of these roots (i.e. the constant term = e/a = e) is this value.

$(p - z)^4 - p(p - z)^3 + q(p - z)^2 - pq(p - z) + 1$

Constant term is: $p^4 - p(p^3) + q(p^2) - pq(p) + 1 = 1$

49. **Answer:** $\gamma = 1/2$

Working: $[J] = [x]^\alpha [I]^\beta [g]^\gamma$

$MLT^{-1} = L^\alpha (ML^2)^\beta (LT^{-2})^\gamma$

$MLT^{-1} = M^\beta L^{\alpha + 2\beta + \gamma} T^{-2\gamma}$

M: $\beta = 1$, **T**: $-2\gamma = -1 \rightarrow \gamma = ½$, **L**: $\alpha + 2(1) + (½) = 1 \rightarrow \alpha = -3/2$.

50. **Answer:** The time in contact with the backboard is 0.5 seconds

Working: When not in contact, $F = 0 \rightarrow t^2(1 - 2t) = 0$

$\rightarrow t = 0$ and $t = 0.5 \rightarrow$ between 0 and 0.5 ball is in contact

Maximum force: $dF/dt = 0$

$\rightarrow 2t(1 - 2t) - 2t^2 = 0 \rightarrow 2t - 6t^2 = 0 \rightarrow t = 0, t = 1/3$

At $t = 1/3$, $F = 1480$ N

Total impulse = $\int_0^{0.5} 40000t^2(1 - 2t)\, dt$= 420 Ns

Total work done cannot be determined from the given info, but it cannot be positive since only friction could act so the work done would be a small negative number.

51. **Answer:** $x^2 - 32x - 512 = 0$

Working: Conservation of energy:

	KE	*GPE*	*EPE*
At top	0	0	0
At position x	0	-784(16 + x)	$24.5x^2$

(At max extension, instantaneously at rest → v = 0 → KE = 0)
(h in |GPE| = |mgh| is 16 + x since add natural length + extension)

→ $-784(16 + x) + 24.5x^2 = 0$
→ $784(16 + x) - 24.5x^2 = 0$
→ $-49x^2 + 1568x + 25088 = 0$
→ $x^2 - 32x - 512 = 0$

52. **Answer:** $|\mathbf{M}| = |\mathbf{D}|$

Working: $|\mathbf{M}| = |\mathbf{UDU}^{-1}| = |\mathbf{U}|\,|\mathbf{D}|\,|\mathbf{U}^{-1}| = |\mathbf{U}|\,|\mathbf{D}|\,1/|\mathbf{U}| = |\mathbf{D}|$
$\mathbf{M}^{-1} = (\mathbf{UDU}^{-1})^{-1} = (\mathbf{U}^{-1})^{-1}\,\mathbf{D}^{-1}\,\mathbf{U}^{-1} = \mathbf{UD}^{-1}\mathbf{U}^{-1}$

53. **Answer:** $P(T > s + t \mid T > s) \equiv P(T > t)$ for all $s > 0$ and $t > 0$

Working: Memoryless: probability is independent of previous outcomes.
→ P(T > some value given already greater than some other value) is the same as P(T > some value)
→ $P(T > s + t \mid T > s) \equiv P(T > t)$

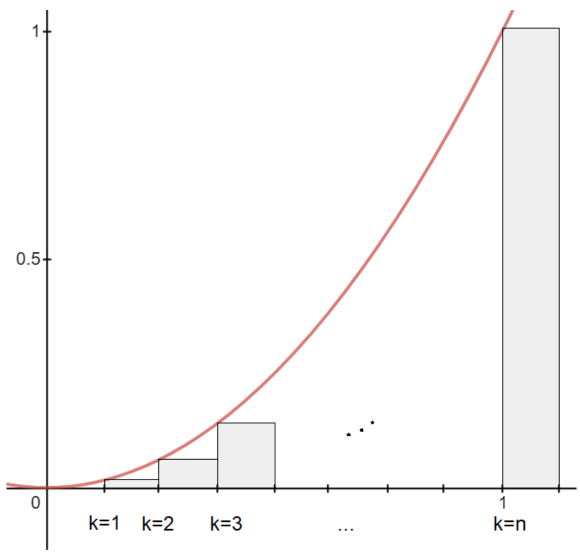
($P(T < t - s) \equiv P(T > t + s)$ for all $0 < s < t$ is false because the exponential distribution is not symmetric about any t.
$P(T < t \mid T > s) \equiv P(T < t - s)$ for all $0 < s < t$ is true but is not a necessary condition for memorylessness.)

54. **Answer:** Replace $\frac{r}{(r+1)!} \equiv \frac{1}{r!} - \frac{1}{(r+1)!}$ and use method of differences

Working: This would work since the sum can be expressed in the form f(r) - f(r + 1), so consecutive terms except the first and last will cancel leaving 1/1! - 1/(n+1)! = 1 - 1/(n+1)!

55. **Answer:** None of the above

Working: This method is the derivation from the rectangular approximation.



Partition the interval into n points evenly spaced between 0 and 1, with n at 1. Clearly the width is 1/n and the height of the kth rectangle is $(k/n)^2$.
No justification needed on step 2 since the index is k, not n.

56. **Answer:** $z = \sqrt{y}$

Working: Considering $y = (Ae^{2x} + Bxe^{2x})^2$ we see that √y = $Ae^{2x} + Bxe^{2x}$ which

is a form of solution for a homogeneous 2nd order DE with constant coefficients (repeated roots case).

So letting z = √y → z = $Ae^{2x} + Bxe^{2x}$→ z is the dependent variable to a homogeneous 2nd order DE with constant coefficients.

57. **Answer:** $qx^2 + p(1 + q)x + p^2 + q^2 - 2q + 1$

Working: New sum of roots = α + 1/α + β + 1/β
= (α + β) + (α + β) / (αβ) = -p + -p/q = (-pq - p)/q = -p(q + 1)/q
New product of roots = (α + 1/α)(β + 1/β)
= αβ + α/β + β/α + 1/(αβ) = αβ + 1/(αβ) + ($\alpha^2 + \beta^2$) / (αβ)
= q + 1/q + ($(-p)^2$ - 2q) / q = ($p^2 + q^2$ - 2q + 1) / q
→ Quadratic is, letting a = 1,
→ x^2 - (α' + β')x + (α'β')
→ x^2 + p(q + 1)/q x + ($p^2 + q^2$ - 2q + 1) / q
Multiplying through by q,
→ q x^2 + p(q + 1) x + ($p^2 + q^2$ - 2q + 1)

58. **Answer:** $\mathrm{P}(T > a) = \mathrm{P}(T < b)$ has no solutions if $a \neq b$

Working: Mean = μ → parameter = λ = 1/μ

Var(T) = E(T^2) - E^2(T) = $\lambda\int_0^\infty t^2 e^{-\lambda t}\,dt$ - $\frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} = \mu^2$

By considering the reduction formula for $I_n = \lambda\int_0^\infty x^n e^{-\lambda x}\,dx$,

u = x^n → du = nx^{n-1}, dv = $e^{-\lambda x}$ → v = -1/λ $e^{-\lambda x}$

$\lambda\int_0^\infty x^n e^{-\lambda x}\,dx = \lambda\left[-\frac{1}{\lambda}x^n e^{-\lambda x}\right]_0^\infty + n\int_0^\infty x^{n-1}e^{-\lambda x}$

→ I_n = n/λ * I_{n-1} → I_n = n(n-1)(n-2)...1 * I_1 / λ^{n-1}

Since I_1 = E(T) = 1/λ → E(T^n) = $\frac{n!}{\lambda^n} = n!\,\mu^n$

$\mathrm{P}(T > a) = \mathrm{P}(T < b)$ implies T is asymmetric about (a+b)/2, which is true unless a = b and P(T > a) = 0.5, but this cannot be since a ≠ b.

59. **Answer:** $\frac{16}{15}$

Working: The desired result has n = 2, a = -1, b = 1. The reduction formula is

$$I_n = \left[\frac{2x^n\sqrt{1-x}}{-(2n+1)}\right]_0^1 + \frac{2n}{2n+1} \times I_{n-1}$$

$$I_n = \frac{2n}{2n+1} \times I_{n-1}$$

So for n = 2,

$$I_2 = \frac{4}{5} \times I_1$$

$$I_2 = \frac{4}{5}\left(\frac{2}{3} \times I_0\right)$$

$$I_2 = \frac{8}{15} \times I_0$$

Calculating I_0,

$$I_0 = \int_0^1 \frac{1}{\sqrt{1-x}}\ \mathrm{d}x = \int_0^1 (1-x)^{-\frac{1}{2}}\ \mathrm{d}x = -2\left[(1-x)^{\frac{1}{2}}\right]_0^1 = 2$$

I_2 = 8/15 * 2 = 16/15

(Calculator does not work because the upper bound is improper (division by zero error).)

60. **Answer:** $2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} + 2^{-n}$

Working: $\alpha = 1 + i \rightarrow \beta = 1 - i$ (real coefficients; conjugate root)

Product of roots = $\alpha\beta\gamma = (1 + i)(1 - i)\gamma = 1 \rightarrow \gamma = ½$

$\alpha^n + \beta^n + \gamma^n = (1 + i)^n + (1 - i)^n + 2^{-n}$

Converting the first two to exponential form,

$\rightarrow (\sqrt{2}\, e^{i\pi/4})^n + (\sqrt{2}\, e^{-i\pi/4})^n + 2^{-n}$

$\rightarrow 2^{n/2}(e^{in\pi/4} + e^{-in\pi/4}) + 2^{-n}$

Using the identity $z^n + z^{-n} = 2 \cos n\theta$ (Euler's identity with $z = e^{i\theta}$)

$\rightarrow 2^{n/2} * 1/2 * \cos(n\pi/4) + 2^{-n}$

$\rightarrow 2^{n/2 - 1} \cos(n\pi/4) + 2^{-n}$

61. **Answer:** If in fact P(heads) = 0.8, then P(Type II error) = 0.1329

Working: H_0: coin is unbiased i.e. P(heads) = 0.5.

$\rightarrow$ under H_0, $X \sim B(18, 0.5)$

Critical region is $X \geq 13$

P(Type I error) = $P(X \geq 13) = 0.0481$

P(Type II error) = $P(X \leq 12$ under $X \sim B(18, 0.8)) = 0.1329$

(type I error = significance level only in continuous distributions)

62. **Answer:** It would be appropriate to apply Yates' correction to the χ^2 statistic.

Working: H_0: no **association** between gender and admittance rate

Yates: yes (use Yates when 2 x 2 table or total number < 20)

Pooling: no (only pool when row/column number ≤ 5 and not 2 x 2)

Degrees of freedom = 1 → critical value = 3.841

2.6601 < 3.841 → not significant enough → accept H_0

(Yates' correction decreases the χ^2 value so always less)

63. **Answer:** A Type I error would be **A**, a Type II error would be **B**

Working: Null: new weedkiller not more effective than old weedkiller

Alt: new weedkiller is more effective than old weedkiller

Type I error: saying alt when in fact null

Type II error: saying null when in fact alt

64. **Answer:** If the significance level is decreased, P(Type II error) decreases.

Working: Critical region at $\lambda = 3$: $X \geq 7$

$P(X \leq 6 \mid \lambda = 5) = 0.762$

→ Power = 1 - P(Type II error) = 0.238 ≠ 0.516 → not actually 5

If α decreases, P(Type I error) decreases since the rejection region decreases in size. However, the acceptance region increases in size so P(Type II error increases).

In Poisson, mean = variance = 3.

65. **Answer:** *L* has a normal distribution with mean 28 mm and variance 9 mm^2.

Working: The population standard deviation is known (3 mm) so the distribution is normal (not *t* despite the small sample size).

The sampled lengths have the same distribution as the population (assuming random sample) → $L \sim N(28, 9)$

The **sample mean** has $\bar{L} \sim N(28, 0.9)$ $(\overline{\sigma^2} = \frac{\sigma^2}{n})$.

If the population std.dev was unknown, the sample would have

$\frac{L-28}{s} \sim$ *t* distribution with 9 d.o.f. (s = sample std.dev)

66. **Answer:** $$P(X = k) = \frac{k^3 - (k-1)^3}{216}$$

Working: First consider P(largest ≤ k) = P(all three rolls ≤ k).

Since each roll is independent and identically distributed (i.i.d.):

P(one roll ≤ k) = k/6

→ $P(X \leq k) = (k/6) * (k/6) * (k/6) = k^3/216$

→ $P(X \leq k - 1) = (k - 1)^3/216$

Subtract,

$P(X \leq k) - P(X \leq k - 1) = k^3/216 - (k - 1)^3/216$

$P(X = k) = (k^3 - (k - 1)^3)/216$.

67. **Answer:** $b = 14$

Working: It is immediately clear that $d = 0$ or else the plane of reflection would not include the origin and therefore could not be inverted. The form of **M** can be found by considering its diagonalisation. Plane of reflection is $2x - y + 5z = 0 \rightarrow$ the normal to this plane is reflected $\rightarrow$ an eigenvector is $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, with eigenvalue -1
Now any two vectors in this plane will also be eigenvectors, and since they are invariant, there will be a repeated eigenvalue of 1.
Let x = 5, y = 0 $\rightarrow$ z = -2 $\rightarrow$ eigenvector $5\mathbf{i} - 2\mathbf{k}$
Let x = 5, z = 0 $\rightarrow$ y = 10 $\rightarrow$ eigenvector $5\mathbf{i} + 10\mathbf{j}$
Then, we have

$$15\mathrm{M} = 15\begin{bmatrix} 2 & 5 & 5 \\ -1 & 0 & 10 \\ 5 & -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & 5 \\ -1 & 0 & 10 \\ 5 & -2 & 0 \end{bmatrix}^{-1}$$

which is evaluated on a calculator to be

$$15\mathrm{M} = \begin{bmatrix} 11 & 2 & -10 \\ 2 & 14 & 5 \\ -10 & 5 & -10 \end{bmatrix}$$

which matches the given form with $a = 11$, $b = 14$ and $c = -10$.
(Alternative method: as a plane of invariant points)

68. **Answer:** $y = x$ and $y = -x$ are invariant lines of **A**.

Working: Eigenvectors point along invariant lines, and when the eigenvector is 1, the eigenvector does not scale, so it is also a line of invariant points. This is not given in this case so we cannot say this.
B can be visualised to be a rotation by 180°, which is
[[-1, 0]; [0, -1]], i.e. a reflection in the line $y = -x$.
|**B**| = -1 * -1 = 1 → |**AB**| = |**A**| → |**A**| - |**AB**| = 0.

69. **Answer:** -1

Working: A and B have swapped, so it is a reflection → sense has flipped → determinant is -1

70. **Answer:** $f(x)$ has a vertical asymptote at $x = 1$

Working: Differentiate both sides:
$f'(x) = \tanh^{-1} x$
$f''(x) = 1/(1 - x^2)$
So $f''(x) \neq 0$ → no inflection points
Since $\tanh^{-1} 1$ is undefined, the integral is improper.
Using limits,

$$\lim_{x\to 1} \int_0^x \tanh^{-1} t \ \mathrm{d}t = \lim_{x\to 1} \left(x \tanh^{-1} x + \frac{1}{2} \ln \left|1 - x^2\right| \right)$$

Writing the definition of $\tanh^{-1} x$ in terms of ln,

$$= \frac{1}{2} \lim_{x\to 1} \left(x \ln \frac{1+x}{1-x} + \ln \left|1 - x^2\right| \right)$$

$$= \frac{1}{2} \lim_{x\to 1} \left(x\ln(1+x) - x\ln(1-x) + \ln(1-x) + \ln(1+x) \right)$$

$$= \frac{1}{2} \lim_{x\to 1} \left((1+x)\ln(1+x) + (1-x)\ln(1-x) \right)$$

Replacing the Maclaurin series for $\ln(1 - x)$ as $-x - x^2/2 - x^3/3 - \ldots$
we see $(1 - x)\ln(1 - x) = (1 - x)(-x - x^2/2 - x^3/3 - \ldots)$
$= -x - x^2/2 - x^3/3 - \ldots + x + x^2/2 + x^3/3 + \ldots = 0$
So the limit is zero for $(1 - x)\ln(1 - x)$.
Continuing, we find

$$= \frac{1}{2} \lim_{x\to 1} \left((1+x)\ln(1+x) \right) = \frac{1}{2} \left(2\ln 2 \right) = \ln 2.$$

So $f(x)$ is finite at $x = 1$ so cannot have an asymptote there.

71. **Answer:** The sum diverges because $f(n) \geq \int_1^n \frac{1}{x}\, dx$, which diverges as $n \to \infty$.

Working: It is true that $f(n) \geq \int_1^n \frac{1}{x}\, dx$, this can be seen by considering a graph

of y = 1/x and filling in rectangles of width 1 and height 1/x, the rectangles will always be above the graph of 1/x.

$\lim_{n \to \infty} \int_1^n \frac{1}{x}\, dx = \lim_{n \to \infty} (ln\, n - ln\, 1) = \lim_{n \to \infty} ln\, n$ which diverges (slowly) to ∞

Since f(n) is more than this, f(n) must also diverge to infinity.

72. **Answer:** $f(y) = \frac{2}{15}y, \text{ if } 1 \leq y \leq 4; \ 0, \text{ otherwise.}$

Working: If X goes from 1 to 16 then Y goes from 1 to 4.

f(x) = 1/15 (for 1 < x < 16)

→ F(x) = $\int_1^x f(x)\, dx$= $[1/15\ x]_1^x$ = 1/15 x - 1/15

Let G(y) be the cdf of Y. Then,

P(Y ≤ y) = P(√X ≤ y) = P(X ≤ y^2)

= F(y^2)

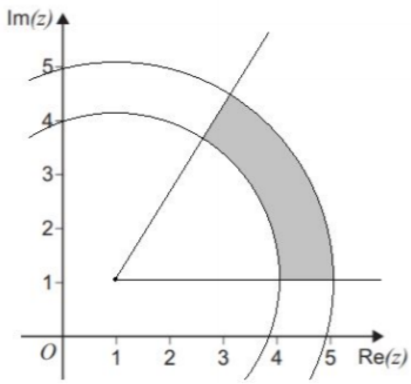
= 1/15 y^2 - 1/15

Differentiating,

g(y) = G'(y) = 2/15 y

73. **Answer:** $2 + 4\sqrt{2}$

Working: The set P ∩ Q is (Argand diagram)



Re(z) + Im(z) = k, for the largest value of k
→ Im(z) = k - Re(z) → in Cartesian coordinates, y = k - x
This line intersects the circle for the largest *k* when it is tangent to the outer circle i.e. at angle 45° = π/4
→ z = 1 + i + 4(cos π/4 + i sin π/4)
→ z = 1 + 2√2 + i(1 + 2√2)
→ Re(z) + Im(z) = 1 + 2√2 + 1 + 2√2
→ Re(z) + Im(z) = 2 + 4√2

74. **Answer:** e^2

Working: Consider the argument of the exp, x / (1 + x^2) * (ln x + 2):
As x → ∞, x^2 / (1 + x^2) → 1 (since highest degrees are same).
So it becomes exp(ln x + 2) = exp(ln x) exp(2) = x * e^2
Then we have

$$= \lim_{x \to \infty} \frac{x+1}{x^2+6x} \cdot xe^2 = e^2 \lim_{x \to \infty} \frac{x(x+1)}{x(x+6)} = e^2 \lim_{x \to \infty} \frac{x+1}{x+6} = e^2 \cdot 1 = e^2.$$

75. **Answer:** $\frac{1}{2\omega} t \sin \omega t$

Working: *There are shortcuts, but in full:* equation for SHM without damping:
x" + ω^2x = 0
When the external force is included,
x" + ω^2x = cos ωt

Complementary solution:
$\lambda^2 + \omega^2 = 0 \rightarrow \lambda = \pm i\omega$
$\rightarrow x_c(t) = e^{0t}(A \cos \omega t + B \sin \omega t) = A \cos \omega t + B \sin \omega t$
Using method of undetermined coefficients,
trial function: $x = P \cos \omega t + Q \sin \omega t$
But this is not linearly independent with the complementary solution, so instead use $x = t(P \cos \omega t + Q \sin \omega t)$
$\rightarrow x' = t(-P\omega \sin \omega t + Q\omega \cos \omega t) + P \cos \omega t + Q \sin \omega t$
$\rightarrow x'' = t(-P\omega^2 \cos \omega t - Q\omega^2 \sin \omega t) - P\omega \sin \omega t + Q\omega \cos \omega t$
$- P\omega \sin \omega t + Q\omega \cos \omega t$
$= [-P\omega^2 t + 2Q\omega] \cos \omega t + [-Q\omega^2 t - 2P\omega] \sin \omega t$
$\rightarrow [-P\omega^2 t + 2Q\omega] \cos \omega t + [-Q\omega^2 t - 2P\omega] \sin \omega t$
$+ P\omega^2 t \cos \omega t + Q\omega^2 t \sin \omega t = \cos \omega t$
$\rightarrow 2Q\omega \cos \omega t - 2P\omega \sin \omega t = \cos \omega t$
$\rightarrow P = 0, Q = 1/(2\omega)$
$\rightarrow I(t) = 1/(2\omega) * \omega * \sin \omega t$
(Could have 'guessed' the only option with t outside as soon as the linear dependence of the trial solution on the complementary solution was seen. Could also guess right from the start - since frequency of driving force matches natural frequency of the system (ω), the amplitude will increase over time (resonance) so the solution must diverge as t → ∞.)

76. **Answer:**

Critical damping	Underdamping	Overdamping
$c = 8\sqrt{3}$	$c < 8\sqrt{3}$	$c > 8\sqrt{3}$

Working: Forces acting: $F = -kx$ from spring, $F = -cv$ from dashpot
$-kx - cv = ma \rightarrow mx'' + cx' + kx = 0$
$\rightarrow 2x'' + cx' + 24x = 0$
Consider the auxiliary equation, $2\lambda^2 + c\lambda + 24 = 0$
Boundary case: repeated root → critical damping (fastest decay)
Discriminant: $c^2 - 4(2)(24) = 0 \rightarrow c = 8\sqrt{3}$
If $c > 8\sqrt{3}$ → real → overdamping (no oscillations; exponential)
If $c < 8\sqrt{3}$ → complex → underdamping (oscillations; sinusoidal)

77. **Answer:** $\ddot{y} - \dot{y} - 6y = 1 - 2t$, $y(0) = 8,\ \dot{y}(0) = 4$

Working: Solve second equation for x: $x = y' + y - t$
Differentiate wrt t: $x' = y'' + y' - 1$
Sub both into first equation:
$(y'' + y' - 1) = 2(y' + y - t) + 4y$
$\rightarrow y'' - y' - 6y = 1 - 2t$.
Initial condition: $y(0) = 8$, $y'(0) = 12 - 8 + 0 = 4$.

78. **Answer:**

Maximum theoretical error	Actual error
0.0416666...	0.0208333...

Working: True value = $\int_0^1 x^5\, dx = \frac{1}{6} = 0.1666666...$
Maximum bound: $f^{(4)}(x) = f''''(x) = 5*4*3*2\, x = 120x$
$\rightarrow$ K = maximum value of 120x on [0, 1] = 120
$\rightarrow$ One parabola $\rightarrow$ 2 intervals $\rightarrow$ n = 2
$\rightarrow$ Theoretical error = $120(1 - 0)^5/(180 * 2^4) = 1/24 = 0.0416666…$
$\rightarrow$ Simpson's value = $(1/2)/3 * (0^5 + 4 * 0.5^5 + 1^5)] = 0.1875$
$\rightarrow$ Actual error = 0.1875 - 0.16666… = 0.0208333…

79. **Answer:** In Step **(5)**, Euler's improved method is used on both *x* and *y*

Working: Step 1: Initialising the table
Step 2: Adding the initial conditions, *n* represents the *n*th point
Step 3: Euler's (standard) method is used. Cannot use improved method since that requires two previous data points and we currently only have one.
Step 4: Defines the variables *t* and *n* to prepare for the loop
From there, we repeat Euler's improved method until the required value of *t* has been obtained (*t* = 60).
Total rows = 60/0.001 + initial condition = 60000 + 1 = 60001.

80. **Answer:** None of these

Working: Derive the relevant Pythagorean identities using Osborne's rule:

$\sin^2 t + \cos^2 t = 1 \rightarrow \cosh^2 t - \sinh^2 t = 1$
$\sec^2 t = 1 + \tan^2 t \rightarrow \text{sech}^2\, t = 1 - \tanh^2 t$
$\csc^2 t = 1 + \cot^2 t \rightarrow -\text{csch}^2\, t = 1 - \coth^2 t \rightarrow \coth^2 t - \text{csch}^2\, t = 1$

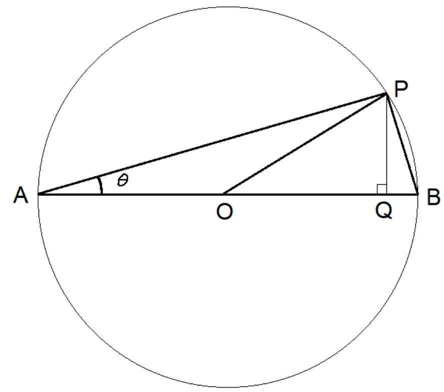
Check each form by substituting in x and y into $y = 1 - x^2$:
$1 - \sinh^2 t = 1 - (\cosh^2 x - 1) = 2 - \cosh^2 t$ (does not match y)
$1 - \tanh^2 t = 1 - (1 - \text{sech}^2\, t) = \text{sech}^2\, t$ (matches y)
$1 - \text{csch}^2\, t = 1 - (\coth^2 t - 1) = 2 - \coth^2 t$ (does not match y)

However, the range of tanh t is $-1 < \tanh t < 1$, so the domain of the resulting curve is $-1 < x < 1$. This does not satisfy the requirement for the curve to be for all real x, so none of them are valid.

Section C: Hard

81. **Answer:** $L = \frac{\pi}{3}$, $S = (2 - \sqrt{3})\pi$

Working: AP . PB = 0 → AP perpendicular to PB.
By the 'angle in semicircle is 90°' circle theorem, P must lie on the circle (case 1) or sphere (case 2) containing AB as diameter.
AB . AP ≥ 2 + √3
→ |AB| |AP| cos θ ≥ 2 + √3
→ 2 |AP| cos θ ≥ 2 + √3
→ |AP| cos θ ≥ 1 + √3/2
Sketching a diagram (2D case)



$|AP| \cos\theta$ represents $|AQ|$, so we must have
$|OQ| \geq \sqrt{3}/2$
→ angle POQ ≥ $\cos^{-1}(\sqrt{3}/2)$ → angle POQ ≥ 30° or π/6
In the 2D case, the locus is **twice** the length of the arc PB (since it could lie on the arc either side)
→ L = rθ = 1 * π/3 = π/3.
For the 3D case, we need the surface area of the arc when revolved 360° about the diameter. Setting its equation centered at O, we get $y = \sqrt{(1 - x^2)}$ → $dy/dx = -x / \sqrt{(1 - x^2)}$,

$$S = 2\pi \int_{\frac{\sqrt{3}}{2}}^{1} \sqrt{1-x^2} \cdot \sqrt{1 + \frac{x^2}{1-x^2}}\ \mathrm{d}x$$

The inside cancels completely, leaving

$$S = 2\pi \int_{\frac{\sqrt{3}}{2}}^{1} 1\ \mathrm{d}x = 2\pi \left(1 - \frac{\sqrt{3}}{2}\right) = (2 - \sqrt{3})\pi.$$

82. **Answer:** **1** and **3** only

Working: Choosing 2 cards from n:
P(first particular card) = 1/n
P(second particular card) = 1/(n - 1)
P(both) = 1/(n(n - 1))
There are 2 possible orders (first then second or second then first), so P(two particular cards) = 2/(n(n - 1)) → **(a)** = n(n - 1).

$(1/2) \sum r^3 - r^2$, using standard results, is
$= (1/2) [1/4\ n^2(n + 1)^2 - 1/6\ n(n + 1)(2n + 1)]$
$= (1/48) [6n^2(n + 1)^2 - 4n(n + 1)(2n + 1)]$
$= n(n + 1)/24 * (3n(n + 1) - 2(2n + 1)$
$= n(n + 1)/24 * (3n^2 - n - 2)$

$= n(n + 1)/24 * (n - 1)(3n + 2) \rightarrow$ **(b)** $= 3n + 2$

Putting S back in, $E[X] = 2/n(n - 1) * n(n + 1)(n - 1)(3n + 2)/24$
$= (n + 1)(3n + 2)/12 \rightarrow$ **(c)** $= (n + 1)(3n + 2)$.

83. **Answer:** The line *L* is a line of invariant points under **M**.

Working: Eigenvalue 2 and eigenvector [1; -2] → line with direction vector [1; -2] is an invariant line (not line of invariant points), scales by factor 2, given by *L*.

Since eigenvalue λ is repeated, any vector on the lines of the eigenvectors (and hence any vector in the plane spanned by them) is scaled by the same amount. Therefore it is an invariant plane.
If $\lambda = 1$ then there is no movement and it is further a plane of invariant points. The normal to this plane is ($\mathbf{u} \times \mathbf{v}$) so Π represents this plane.
If $\lambda = 2$ then all eigenvectors scale equally and since all eigenvectors are linearly independent this applies to any vector i.e. an enlargement → volume sf = $(\text{linear sf})^3 = 2^3 = 8$ (this could be verified by expanding the diagonalised form of **M** to become 2**I**)

84. **Answer:** **1** and **2** only

Working: Starting with the given expression,
$(\alpha + \beta)(\beta + \gamma)(\alpha + \gamma) = (\sum\alpha - \alpha)(\sum\alpha - \beta)(\sum\alpha - \gamma)$
$= (-b/a - \alpha)(-b/a - \beta)(-b/a - \gamma)$
The equation whose roots are each factor is
$z = -b/a - x \rightarrow x = -b/a - z$
$\rightarrow a(-b/a - z)^3 + b(-b/a - z)^2 + c(-b/a - z) + d = 0$
$\rightarrow -a(b/a + z)^3 + b(b/a + z)^2 - c(b/a - z) + d = 0$
$\rightarrow -a\, z^3 + \ldots z^2 + \ldots z + (-b^3/a^2 + b^3/a^2 - bc/a + d) = 0$
→ product of roots $= (-bc/a + d)/a = (ad - bc)/a$.

1: if one root is opposite sign of another i.e. $\alpha = -\beta$ then one of the factors in $(\alpha + \beta)(\beta + \gamma)(\alpha + \gamma)$ is zero, so the product of the roots is zero → (ad - bc)/a = 0 → ad - bc = 0 → true (in determinant form).

2: if roots are in geometric progression then let the roots be α, rα and $r^2\alpha$. Then,
$\sum \alpha = \alpha + r\alpha + r^2\alpha = \alpha(1 + r + r^2) = -b/a$
$\sum \alpha\beta = \alpha^2 r(1 + r + r^2) = c/a$
$\alpha\beta\gamma = r^3\alpha^3 = -d/a$
→ $(\sum\alpha\beta) / (\sum\alpha) = [\alpha^2 r(1 + r + r^2)] / [\alpha(1 + r + r^2)] = r\alpha$
$= (c/a)/(-b/a) = -c/b$
→ $\sum \alpha\beta\gamma = (r\alpha)^3$ → $(-d/a) = (-c/b)^3$ → $-d/a = -c^3/b^3$ → $ac^3 - b^3d = 0$
→ true (in determinant form)

3: if one root is zero then the product of the roots is zero
→ -d/a = 0 → d = 0 → false.

85. **Answer:** $$I(m, n) = \frac{m!n!}{1 + m + n}$$

Working: Integrating by parts with $u = (1 - x)^m$ → $du = -m(1 - x)^{m-1}$
and $dv = x^n$ → $v = 1/(n+1) * x^{n+1}$,

$$\int_0^1 x^n(1 - x)^m dx = \frac{1}{n+1}\left[(1 - x)^m x^{n+1}\right]_0^1 + \frac{m}{n+1}\int_0^1 (1 - x)^{m-1} x^{n+1} dx$$

→ I(m, n) = m/(n+1) * I(m-1, n+1)
Iterating repeatedly,
→ I(m, n) = m/(n + 1) * (m-1)/(n+2) * I(m-2, n+2)
→ I(m, n) = m/(n + 1) * (m-1)/(n+2) * (m-2)/(n+3) * I(m-3, n+3)
Multiplying all fractions down to I(0, n+m),

→ I(m, n) = [m(m-1)(m-2)...1]/[(n+1)(n+2)(n+3)...(n+m) * I(0, n+m)
→ I(m, n) = m!n! / (n + m)! * I(0, m+n)
Calculating I(0, m+n),

$$I(0,\ m\ +\ n)\ =\int_0^1 x^{m+n}dx\ =\frac{1}{m+n+1}\left[x^{m+n+1}\right]_0^1\ =\frac{1}{m+n+1}$$

$$\rightarrow I(m,n)\ =\frac{m!n!}{(m+n)!(m+n+1)}=\frac{m!n!}{(m+n+1)!}.$$

86. **Answer:** $\frac{1}{2}$

Working: Using side length, DB = BE = 1. Using cosine rule in triangle BDE,
→ DE = $\sqrt{(1^2 + 1^2 - 2 \cos \theta)} = \sqrt{(2 - 2 \cos \theta)}$
Let angles BDE = α and EDF = β. In radians,
→ angle DEF = π - 2β
In triangle ADF, 2(π - α - β) + π - 2θ = π
→ π - α - β = θ
In triangle DFE, π - 2α = π - 2β → α = β
→ π - 2α = θ → DEF = θ
Since FE = ED, we have the area of the triangle as
S(θ) = 1/2 * DE * EF * sin DEF
= 1/2 * (2 - 2 cos θ) * sin θ
= sin θ - sin θ cos θ
Using small angle approximations since θ → 0:
$S / \theta^3 = (\theta - \theta(1 - \theta^2/2))/\theta^3 = (\theta^3/2)/\theta^3 = 1/2$.

87. **Answer:** minimum $z = \frac{2}{15}H$, when there is 66 cm^3 of water left.

Working: Let the value of *z* at any height of water *h* be z(*h*). Considering mass-moments on the *z*-axis,
(12 + 495 * h/H)*z* = (12 * H/2) + (495 * h/H * h/2)
Multiplying through by 2H and solving for *z*,
→ (24H + 990h) z = $12H^2 + 495h^2$
→ z = $(12H^2 + 495h^2) / (24H + 990h)$
When z is minimum, dz/dh = 0:
dz/dh = 990h * (24H + 990h) - 990 * $(12H^2 + 495h^2)$ = 0
→ $24Hh + 990h^2 - 12H^2 - 495h^2 = 0$

$\rightarrow$ 495 h^2 + 24H h - 12H^2 = 0

$$\rightarrow h = \frac{-24H \pm \sqrt{576H^2 + 23760H^2}}{990}$$

$$\rightarrow h = \frac{-24H \pm 156H}{990}$$

$$\rightarrow h = \frac{2}{15}H$$

At this height, there is (2/15) * 495 = 66 cm^3 water left.

88. **Answer:** The length of the cable between *P* and *Q* rounds to 351 metres

Working: The curve can be formed by carefully forming a sequence of transformations from *h* = cosh *x* to the desired curve.
Lowest point is at (0, 1), needs to go to (175, 45).
$\rightarrow$ *h* = 44 + cosh(x - 175)
Next we stretch the curve in the *x*-axis - but we must be careful. The curve is no longer symmetric about the *y*-axis so any stretch will translate the curve aswell. This can be undone by multiplying the translation by the same factor, call it 1/b for consistency:
$\rightarrow$ *h* = 44 + cosh(*x* / b - 175 / *b*) = 44 + cosh((*x* - 175) / *b*)
When *x* = 0, *h* = 55:
$\rightarrow$ 55 = 44 + cosh(-175/b) $\rightarrow$ b = 56.653… $\rightarrow$ c = 175/b = 3.089...
$\rightarrow$ a = 44, b = 56.653, c = 3.089

Arc length = $\int_0^{350} \sqrt{1 + (\frac{1}{56.653} sinh(\frac{x}{56.653} - 3.089))^2} dx$= 351.03…

89. **Answer:** 15

Working: Define S_P and S_Q:
$S_P = \{p_i : P(p_i) = 0\}$ (the set of roots of P(x))
$S_Q = \{q_i : Q(q_i) = 0\}$ (the set of roots of Q(x))
The condition “P(x)Q(y) = 0 and Q(x)P(y) = 0” leads to four possible cases - any pair of P(x), P(y), Q(x), Q(y) could be equal to zero.
If P(x) = 0 and Q(x) = 0 $\rightarrow$ y can be anything (it doesn’t have to be

a root) → infinite set.
Also, if P(y) = 0 and Q(y) = 0 → x can be anything → infinite set.
(In the other cases, both x and y must be a root so the set is finite.)

Redefine *B* in terms of *x* only since *x* = *y*, let *y* = x.
By logical simplifications,
→ B = {(x, x) : (x, x) ∈ A, x = x}
→ B = {x : (x, x) ∈ A}
→ B = {x : P(x)Q(x) = 0 and Q(x)P(x) = 0}
→ B = {x : P(x)Q(x) = 0}
→ B = {x : P(x) = 0 or Q(x) = 0}
→ B = {x : x ∈ S_P or x ∈ S_Q}
→ B = {x : x ∈ $S_P \cup S_Q$}

So, the number of elements in B is
→ n(B) = n($S_P \cup S_Q$)
→ n(B) = n(S_P) + n(S_Q) - n($S_P \cap S_Q$)
→ n(B) = 7 + 9 - n($S_P \cap S_Q$)
→ n(B) = 16 - n($S_P \cap S_Q$)
But we know that $S_P \cap S_Q$ is a non-empty set, it must contain at least one *x* (since P(x) = 0 and Q(x) = 0 was a necessary condition)
So the smallest value of n($S_P \cap S_Q$) is 1, so the largest value of n(B) is 16 - 1 = 15.

90. **Answer:** 2

Working: From the identity $\sin^{-1}(x) + \cos^{-1}(x) = \pi/2$
(can be derived from sin(x) = cos(pi/2 - x) or any other reflection)
we have

$$\sum_{r=1}^{\infty} x^{r+1} - x\sum_{r=1}^{\infty}\left(\frac{x}{2}\right)^r = \sum_{r=1}^{\infty}\left(-\frac{x}{2}\right)^r - x\sum_{r=1}^{\infty}(-x)^r$$

The first term is a geometric series, with first term x^2 and common ratio x, so its infinite sum is $x^2 / (1 - x)$.
Similarly, the second term is x * [(x/2) / (1 - x/2)],
the third term is (-x/2) / (1 + x/2),
the last term is x * [(-x) / (1 + x)].

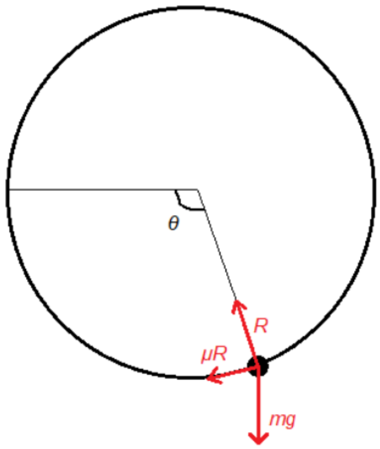
Combining,

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$\longrightarrow x^2(2 - x)(2 + x)(1 + x) - x^2(1 - x)(2 + x)(1 + x) + x(1 - x)(2 - x)(1 + x) - x(1 - x)(2 - x)(2 + x) = 0$
$\rightarrow x[x(4 - x^2)(1 + x) - x(1 - x^2)(2 + x) + (1 - x^2)(2 - x) - (1 - x)(4 - x^2)] = 0$
$\rightarrow x(-x^4 - x^3 + 4x^2 + 4x + x^4 + 2x^3 - x^2 - 2x + x^3 - 2x^2 - x + 2 - x^3 + x^2 + 4x - 4) = 0$
$\rightarrow x(x^3 + 2x^2 + 5x - 2)$
$\rightarrow$ x = 0 is one solution, and $x^3 + 2x^2 + 5x - 2$ has root x = 0.344…
so 2 solutions.

91. **Answer:** $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + \frac{g}{r}\left(\mu\sin\theta - \cos\theta\right) = 0$$

Working: At a general point on the circle,



Resolving forces radially (towards centre of circle):
$R - mg \sin\theta = ma_{centripetal}$
Resolving forces tangentially (in direction of motion):
$mg \cos\theta - \mu R = ma_{tangential}$
But $a_{centripetal} = r\omega^2 = r(d\theta/dt)^2$ and
$a_{tangential} = dv/dt = d(\omega r)/dt = r(d^2\theta/dt^2)$, so putting these in gives

$$R - mg\sin\theta = mr\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$$ and

$$mg\cos\theta - \mu R = mr\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

Eliminating *R*, we have $R = mr(\theta')^2 + mg \sin\theta$
$\rightarrow mg \cos\theta - \mu(mr(\theta')^2 + mg \sin\theta) = mr\,\theta''$
$\rightarrow g \cos\theta - \mu r(\theta')^2 - \mu g \sin\theta = r\theta''$
$\rightarrow \theta'' + \mu(\theta')^2 + (g/r)(\mu \sin\theta - \cos\theta) = 0.$

(This equation is only valid when θ' is positive, otherwise friction would be pointing the wrong way. The complete model is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\left|\frac{\mathrm{d}\theta}{\mathrm{d}t}\right| + \frac{g}{r}\left(\mu\cdot\frac{\frac{\mathrm{d}\theta}{\mathrm{d}t}}{\left|\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|}\sin\theta - \cos\theta\right) = 0$$

i.e. substitute $\mu = \mu * \omega/|\omega|$.)

92. **Answer:** **Alex's claim may not be valid** because he cannot be sure that if he obtained a new set of data instead of re-using the previous data, the *p*-value would still be less than 0.5%.

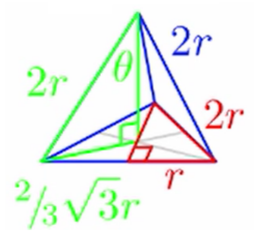
Working: This is a form of the Texas Sharpshooter Fallacy. If the team just report a p-value and let the reader decide if it is significant or not (or even if they want to declare significance or not) and don't make any objective decisions/declarations based on the p-value, then that is reasonable.

But when you indicate that the test was done at a specific significance level, 5%, then you are saying that before collecting the data (before randomisation) there was a 5% chance of declaring significance (rejecting the null hypothesis) if in fact the null hypothesis is true. In Alex's scenario, with a true null hypothesis a p-value of 0.049 would have lead to a declaration of significance, so the probability of rejecting when the null is true is 5%, to say that it was 0.5% is dishonest.

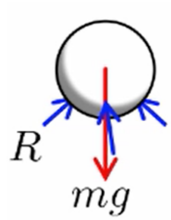
93. **Answer:** $\frac{2\sqrt{6}}{27}\pi r^3 \rho g$

Working: Mass of one sphere = $\rho V = \frac{4}{3}\rho\pi r^3$.

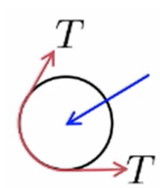
The spheres' centres form the vertices of a tetrahedron, with the edges passing through the points of contact:



The height of each triangular face is $\sqrt{(2r)^2 - r^2} = r\sqrt{3}$.
Since the system remains in equilibrium, the top sphere's weight must act through the centre of mass of the lower spheres. The centre of mass of the lower triangle is 1/3 along this line (a distance of $2r\sqrt{3}/3$ from a vertex).
Let θ be the angle from the vertical to an upper edge. Then we now have: $\sin\theta = (2r\sqrt{3}/3) / (2r) = \sqrt{3}/3 \rightarrow \cos\theta = \sqrt{2} / \sqrt{3} = \sqrt{6} / 3$.
Now consider a free-body diagram of the top sphere:



R: reaction forces from the spheres below, which point along the upper edges of the tetrahedron. Resolving vertically,
$3R\cos\theta = mg \rightarrow R = mg/\sqrt{6}$.



In one of the bottom spheres, resolving in the direction of *R* in the plane of *T*, we get $R\sin\theta = 2T\cos 30$. Solving for *T* then subbing in *R* and *m*, we get

$$\rightarrow T = \frac{R\sin\theta}{2\cos 30^\circ} = \frac{1}{3}R = \frac{mg}{3\sqrt{6}} = \frac{4\rho\pi r^3 g}{9\sqrt{6}} = \frac{2\sqrt{6}}{27}\rho\pi r^3 g.$$

94. **Answer:** None of them

Working: For P_n, we have

$P_n(x) = x * (x * (x * (x * ...(x * \dots)^{1/n} \dots)^{1/4})^{1/3})^{1/2})$
$P_n(x) = x * x^{1/2} * x^{1/2 * 1/3} * x^{1/2 * 1/3 * 1/4} * \dots * x^{1/2 * 1/3 * 1/4 * \dots * 1/n}$
Replacing with factorials,

$$P_n(x) = x^{1/1!} \cdot x^{1/2!} \cdot x^{1/3!} \cdot x^{1/4!} \cdot \dots \cdot x^{1/n!}$$

(**1**) Dividing, $P_n(x) / P_{n-1}(x) = x^{1/n!}$ which is **not** what is given.

$$P_n(x) = x^{\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}\right)}$$

$$P_n(x) = x^{\sum_{r=1}^{n}\frac{1}{r!}} = x^{\left(\sum_{r=0}^{n}\frac{1}{r!}\right)-1}$$

In the limit, as $n \to \infty$, the sum approaches e
(by definition; Maclaurin for e^x, letting $x = 1$).

So we have $\lim_{n \to \infty} P_n(x) = x^{e-1}$.

(**2**) Since this limit is non-zero for all positive x, the infinite sum will never converge.
(**3**) The the limiting integral is

$$\int_0^1 x^{e-1}\,dx = \left[\frac{x^e}{e}\right]_0^1 = \frac{1^e}{e} - \frac{0^e}{e} = \frac{1}{e}.$$

95. **Answer:** $\frac{\sqrt{2}}{12}$

Working: *Method 1: Integration*

Let *a* = 1 so volume will scale. The tetrahedron is the sum of infinitely many short triangular prisms stacked along line *PD*.

The edge length of the triangle at a perpendicular distance *x* down from *D* is proportional to *x*, since triangles *ABC* and the cross-section are similar.

Consider triangle *BPC*. By sine rule, *PC* / sin 30 = 1 / sin 120

→ *PC* = √3/3

Consider triangle *CDP*. By Pythagoras, $PD^2 = 1^2 - (\sqrt{3}/3)^2$

→ *PD* = √6/3

Now let *x* vary from 0 at *D* to √6/3 at *P*.

Then the sum is

$$V = \sum_{\text{prisms}} \text{area of cross section } \times \text{thickness}$$

which approaches (area of equilateral triangle = √3/4 * edge^2):

$$V = \int_0^{\frac{\sqrt{6}}{3}} \frac{\sqrt{3}}{4}\left(\frac{x}{\left(\frac{\sqrt{6}}{3}\right)}\right)^2 \, \mathrm{d}x$$

$$V = \frac{3\sqrt{3}}{8}\int_0^{\frac{\sqrt{6}}{3}} x^2 \, \mathrm{d}x = \frac{3\sqrt{3}}{8} \times \frac{2\sqrt{6}}{27} = \frac{\sqrt{2}}{12}.$$

Method 2: Vectors

The volume of tetrahedron spanned by three vectors equals 1/6 * |scalar triple product| (can be derived from parallelopiped formula)

Define the coordinate axes: x-axis along *BC*, *y*-axis normal in plane *ABC* (*A* has positive *y*-coordinate) and *z*-axis through *B* parallel to *PD.* Let vector **a** be directed along edge *BC* with magnitude 1.

Then **c** = *BC* = **i** and **a** = *BA* = cos 60 **i** + sin 60 **j** = (1/2)**i** + (√3/2)**j**

Next, using various easily derivable results (centre of mass of triangle, height of equilateral triangle)

d = *BD* = 1/2 **i** + 1/3 **j** + √6/3 **k**

Now using the scalar triple product,

$$V=\frac{1}{6}\ |\mathbf{a}\cdot\mathbf{c}\times\mathbf{d}|=\frac{1}{6}\left|\begin{bmatrix}1/2\\ \sqrt{3}/2\\ 0\end{bmatrix}\cdot\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\times\begin{bmatrix}1/2\\1/3\\ \sqrt{6}/3\end{bmatrix}\right)\right|=\frac{\sqrt{2}}{12}.$$

96. **Answer:** 7

Working: Let f(x) = $x^4 + bx^3 + cx^2 + dx + e$. When x = 1 and x = 2, we have

g(1) = 0 → ln f(1) = 0 → f(1) = 1 → b + c + d + e = 0
g(2) = 0 → ln f(2) / 2 = 0 → f(2) = 2 → 8b + 4c + 2d + e = -14

Differentiate g:

$$g'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{f(x)}{x}=\frac{x}{f(x)}\cdot\frac{xf'(x)-f(x)}{x^2}=\frac{f'(x)}{f(x)}-\frac{1}{x}$$

Putting in the expression for f(x), f'(x) = $4x^3 + 3bx^2 + 2cx + d$.
Set g'(x) = 0 at x = 1 and x = 2:

$$\frac{4+3b+2c+d}{1+b+c+d+e}-1=0\to 2b+c-e=-3$$

$$\frac{32+12b+4c+d}{16+8b+4c+2d+e}-\frac{1}{2}=0\to 16b+4c-e=-48$$

We have 4 equations in 4 unknowns, so solve simultaneously (use calculator solver):

→ b = -6, c = 13, d = -11, e = 4
→ f(3) = 3^4 - 6(3^3) + 13(3^2) - 11(3) + 4
→ f(3) = 7.