Maths Solutions (Pure 1)

Section A: Multiple Choice

1. **Answer:** $(2x - y)(x + y)$

   Working: $2x^2 - xy - y^2 = (2x + A)(x + B)$
   $AB = -y^2 \rightarrow (A \text{ or } B) = (y \text{ or } -y)$
   $2Ax + Bx = -xy \rightarrow 2A + B = -y \rightarrow A = -y, B = y$

2. **Answer:** -29

   Working: Terms which multiply to x^2 are
   (x^2 * constant), (x * x), and (constant * x^2):
   $\rightarrow = (2 * 1) + (-1 * -1) + (4 * -8) = -29$.

3. **Answer:** $x = -1$ only

   Working: Critical value: $2x - 5 = 0 \rightarrow x = 5/2$:
   If $x > 5/2 \rightarrow 2x - 5 = 1 - 6x \rightarrow 8x = 6 \rightarrow x = 3/4 < 5/2 \rightarrow$ invalid.
   If $x < 5/2 \rightarrow 5 - 2x = 1 - 6x \rightarrow 4x = -4 \rightarrow x = -1 < 5/2 \rightarrow$ valid.

4. **Answer:** $a/b = 3$

   Working: $3x + ay = 2 \rightarrow y = -3/a\, x + 2/a \rightarrow$ gradient $= -3/a$
   $bx - y = a - b \rightarrow y = bx - a + b \rightarrow$ gradient $= b$
   Perpendicular $\rightarrow$ gradients multiply to $-1 \rightarrow -3b/a = -1 \rightarrow a/b = 3$.

5. **Answer:** 3

   Working: $2 \sin x + 1 = 0 \rightarrow \sin x = -1/2 \rightarrow x = 210^o, 330^o, 570^o \ldots$
   $2 \cos 2x = 1 \rightarrow \cos 2x = 1/2 \rightarrow x = 30^o, 150^o, 210^o \ldots$
   $\rightarrow k = 210^o \rightarrow 3$ solutions in $0 \leq x \leq 210$.

6. **Answer:** $h(2x)$

Working: $f'(2x) = g(2x + 1)$. Differentiating both sides,
$2 f''(2x) = 2 g'(2x + 1) = 2 h(2x) \rightarrow f''(2x) = h(2x)$

7. **Answer:** $-m\mathbf{v}$

Working: $\mathbf{v}$ has magnitude $m \rightarrow$ scale by m has magnitude m^2
Opposite direction $\rightarrow$ scale by -1

8. **Answer:** 1/2

Working: Let the geometric sequence have first term a, common ratio r.
Sum of first 5 terms = $S_5 = a(1 - r^5)/(1 - r)$ and infinite sum = $a/(1 - r)$.
$31a/(1 - r) = 32a(1 - r^5)/(1 - r) \rightarrow 31 = 32(1 - r^5)$
$\rightarrow r^5 = 1/32 \rightarrow r = 1/2$

9. **Answer:** 5/2

Working: First integral = $[\, x^{m+1} / (m+1) \,]_0^2 = 2^{m+1} / (m + 1) = (16/7) \sqrt{2}$
Second integral = $[\, x^{m+2} / (m+2) \,]_0^2 = 2^{m+2} / (m + 2) = (32/9) \sqrt{2}$
Divide second by first: $2(m + 1)/(m + 2) = 14/9$
$\rightarrow m + 1 = 7/9m + 14/9 \rightarrow 2/9\, m = 5/9 \rightarrow m = 5/2$

10. **Answer:** 4/45

Working: $f(a) = \int_0^1 x^4 - 2ax^2 + a^2\, dx = [\, x^5/5 - 2ax^3/3 + a^2x \,]_0^1$
$f(a) = 1/5 - 2a/3 + a^2$. This is the equation of a parabola.
Completing the square, $(a - 1/3)^2 - 1/9 + 1/5$
So the minimum value is $1/5 - 1/9 = 4/45$.

Section B: Standard Questions

11.

a. Notice that if $x = 2$ and $y = 0$, we have $0 = t * 0$ which is true for all t.
$\rightarrow$ always passes through (2, 0).

b. Intersections: sub line into circle
$x^2 + t^2(x - 2)^2 = 1$ [1 mark] $\rightarrow x^2 + t^2x^2 - 4xt^2 + 4t^2 - 1 = 0$
$\rightarrow (t^2 + 1)x^2 - 4t^2 x + (4t^2 - 1) = 0$ [1 mark]
Midpoint is the average of the roots, so this is equivalent to finding the coordinates of the vertex of this parabola.
$\rightarrow x^2 + (-4t^2/(t^2 + 1))\, x + ((4t^2 - 1)/(t^2 + 1)) = 0$
Complete the square:
$\rightarrow (x - 2t^2/(t^2 + 1))^2 - (16t^4 + 4t^2 - 1)/(t^2 + 1)^2$
$\rightarrow$ x-coordinate is $2t^2/(t^2 + 1)$ [4 marks for any valid method to find x]
Subbing back into the line, $y = t(x - 2) = t(2t^2/(t^2 + 1) - 2)$ [1 mark]
$\rightarrow y = 2t^3/(t^2 + 1) - 2t(t^2 + 1)/(t^2 + 1) = (2t^3 - 2t(t^2 + 1))/(t^2 + 1)$
$\rightarrow y = -2t/(t^2 + 1)$ [1 mark]
so

$$M = \left(\frac{2t^2}{1+t^2},\ -\frac{2t}{1+t^2}\right)$$

c. Notice that $x/y = -t \rightarrow t = -x/y$ [1 mark]
$y = -2t/(1 + t^2) = -2(-x/y) / (1 + (-x/y)^2) = (2x/y) / (1 + x^2/y^2)$ [1 mark]
Multiply top and bottom by $y^2 \rightarrow y = 2xy / (x^2 + y^2)$ [1 mark]
Divide both sides by y: $1 = 2x / (x^2 + y^2) \rightarrow x^2 + y^2 - 2x = 0$
$\rightarrow (x - 1)^2 + y^2 = 1$ [1 mark] $\rightarrow$ centre (1, 0), radius 1. [1 mark]

12.

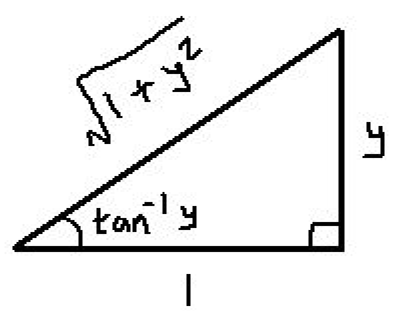
a. Let the sum given be denoted $S_n = \log_{10}((n+1)(n+2)/2)$.
Since $S_n = a_1 + a_2 + \dots + a_{n-1} + a_n$ and $S_{n-1} = a_1 + a_2 + \dots + a_{n-1}$, subtract RHS from LHS to obtain the explicit formula:
$S_n - S_{n-1} = a_n = \log_{10}((n+1)(n+2)/2) - \log_{10}(n(n+1)/2)$ [2 marks]
Using log properties,
$a_n = \log_{10}((n+2)/n)$ [1 mark]

b. Sum of 20 = $a_2 + a_4 + \dots + a_{38} + a_{40}$
$= \log_{10}((2+2)/2) + \log_{10}((4+2)/4) + \dots + \log_{10}((38+2)/38) + \log_{10}((40+2)/40)$
[1 mark]
$= \log_{10}(4/2) + \log_{10}(6/4) + \dots + \log_{10}(40/38) + \log_{10}(42/40)$ [1 mark]
$= \log_{10}(4/2 * 6/4 * \dots * 40/38 * 42/40)$ [1 mark]
$= \log_{10}(42/2) = \log_{10}(21)$. [1 mark]

c. The logarithm function diverges as $n \to \infty$ [1 mark]

13.

a. $y = \tan(\alpha/2) \to \alpha = 2 \tan^{-1} y \to \sin \alpha = \sin (2 \tan^{-1} y)$ [1 mark]
$\sin(2 \tan^{-1} y) = 2 \sin(\tan^{-1} y) \cos(\tan^{-1} y)$ [1 mark]
Setting up a right triangle with angle $\tan^{-1} y$ and using SOHCAHTOA,



$\sin(\tan^{-1} y) = y/\sqrt{1 + y^2}$ and $\cos(\tan^{-1} y) = 1/\sqrt{1 + y^2}$ [1 mark]
$\to \sin \alpha = 2y/\sqrt{1 + y^2}$ [1 mark]

(Alternatively, could rearrange $\csc^2 x = 1 + \cot^2 x$ and/or $\sec^2 x = 1 + \tan^2 x$ then let $x = \tan^{-1} y$).

b. Let $y = \tan(\alpha/2)$, then:

$\cot \alpha = \cos \alpha / \sin \alpha = ((1 - y^2)/(1 + y^2)) / (2y/(1 + y^2)) = (1 - y^2)/(2y)$ [1 mark]

$\sec^2 \alpha = 1/(\cos \alpha)^2 = ((1 + y^2) / (1 - y^2))^2 = (1 + y^2)^2 / (1 - y^2)^2$ [1 mark]

So the equation becomes

$4y + 3(1 - y^2)/(2y) * (1 + y^2)^2 / (1 - y^2)^2 = 0$ [1 mark]

$4y + 3(1 + y^2)^2 / (2y(1 - y^2)) = 0$ [1 mark]

Multiplying both sides by $2y(1 - y^2)$,

$8y^2(1 - y^2) + 3(1 + y^2)^2 = 0$

$\rightarrow 8y^2 - 8y^4 + 3 + 6y^2 + 3y^4 = 0$

$\rightarrow 5y^4 - 14y^2 - 3 = 0$ [1 mark] $\rightarrow y^2 = 3, y^2 = -1/5$ [1 mark]

Reject $-1/5$ since $y^2 > 0 \rightarrow y = \pm \sqrt{3}$ [1 mark]

Since $\alpha = 2 \tan^{-1} y$,

$\rightarrow \alpha/2 = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3 \rightarrow \alpha = 2\pi/3, 4\pi/3$ [1 mark] (since only $0 < \alpha < 2\pi$).

c. Let $x = \cos y$. Since $\cos y = \sin(\pi/2 - y)$ [1 mark], $x = \sin(\pi/2 - y)$

Rearranging for y in each equation, $y = \cos^{-1} x$ and $y = \pi/2 - \sin^{-1} x$ [1 mark]

$\rightarrow \cos^{-1} x = \pi/2 - \sin^{-1} x$. [1 mark]

d. $\sin^{-1} x * \cos^{-1} x = \sin^{-1} x * (\pi/2 - \sin^{-1} x)$ [1 mark]

Let $y = \pi/4 + (1/4)\sqrt{(\pi^2 - 2\sqrt{2})}$, then

LHS $= \sin^{-1} (\sin (y)) * (\pi/2 - \sin^{-1} (\sin (y))) = y(\pi/2 - y)$

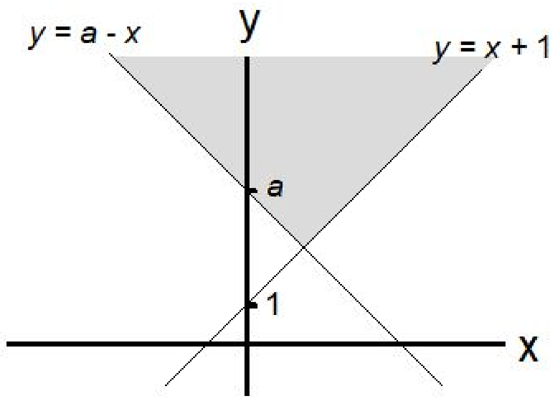
LHS $= ((1/4)\sqrt{(\pi^2 - 2\sqrt{2})} + \pi/4)((1/4)\sqrt{(\pi^2 - 2\sqrt{2})} - \pi/4)$ [1 mark]

Using difference of two squares,

LHS $= ((1/4)\sqrt{(\pi^2 - 2\sqrt{2})})^2 - (\pi/4)^2$ [1 mark] $= (1/16)(\pi^2 - 2\sqrt{2}) - \pi^2/16$

LHS $= (2\sqrt{2})/16 = \sqrt{2} / 8 =$ RHS. [1 mark]

14. Write the bounding inequalities in the standard form of a line:
$x + y \geq a \rightarrow y \geq a - x$ and $x - y \leq -1 \rightarrow -y \leq -1 - x \rightarrow y \geq x + 1$
Sketching the region of interest,



Since the function to be minimised ($x + ay$) is a linear function in terms of x and y, the function is increasing along any given line and therefore attains its maximum/minimum value at the intersection of these lines (the fundamental theorem of linear programming)

Solving for this intersection point,

$$\begin{array}{rl} & y = a - x \\ - & y = x + 1 \\ \hline & 0 = -2x + a - 1 \end{array}$$

[1 mark for showing elimination process]
$\rightarrow x = (a - 1)/2$ [1 mark]

$\rightarrow y = a - (a - 1)/2 \rightarrow y = (a + 1)/2$ [1 mark]
Putting these into the function,
$(a - 1)/2 + a(a + 1)/2 = 7$ [1 mark] $\rightarrow a - 1 + a^2 + a = 14 \rightarrow a^2 + 2a - 15 = 0$
$\rightarrow a = 3, a = -5$ [1 mark]
We do not know if either/both of these correspond to a minimum (they could be a maximum).
When $a = 3$, $x = 1$ and $y = 2$, so considering the region, check a point inside the region (e.g. $x = 1, y = 3$) [1 mark] $\rightarrow x + ay = 10 > 7 \rightarrow a = 3$ is a minimum
[1 mark]
When $a = -5$, $x = -3$ and $y = -2$, so again considering the region, check a point inside the region (e.g. $x = -3, y = -1$) $\rightarrow x + ay = 2 < 7 \rightarrow a = -5$ is a maximum.
[1 mark, must justify rejection of a = -5]
$\rightarrow$ only value that produces a minimum is $a = 3$. [1 mark]

15.

a. Area of triangle: $T = 1/2\ bc \sin A \rightarrow T^2 = 1/4\ b^2c^2 (1 - \cos^2 A)$ [1 mark]
From cosine rule,
$\cos A = (b^2 + c^2 - a^2)/(2bc)$
Subbing into first equation,

$$T = \sqrt{\frac{1}{4}b^2c^2\left(1 - \frac{(b^2+c^2-a^2)^2}{4b^2c^2}\right)}$$ [1 mark]

$$T = \sqrt{\frac{1}{4}b^2c^2 - \frac{1}{16}(b^2+c^2-a^2)^2}$$ [1 mark]

Factoring out 1/16 and writing as the difference of two squares,

$$T = \sqrt{\frac{1}{16}\left((2bc)^2 - (b^2+c^2-a^2)^2\right)}$$ [1 mark]

$$T = \sqrt{\frac{1}{16}(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}$$ [1 mark]

Since $(b + c)^2 = b^2 + 2bc + c^2$ and $(b - c)^2 = b^2 - 2bc + c^2$,

$$T = \sqrt{\frac{1}{16}((b+c)^2-a^2)(a^2-(b-c)^2)}$$ [1 mark]

Using difference of two squares again in each factor,

$$T = \sqrt{\frac{1}{16}(a+b+c)(b+c-a)(a+b-c)(a-b+c)}$$ [2 marks]

Distribute the 1/16 into each factor as 1/2:

$$T = \sqrt{\frac{a+b+c}{2}\cdot\frac{b+c-a}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a-b+c}{2}}$$ [1 mark]

Using the given substitution $s = (a + b + c)/2$, each factor is then

$$T = \sqrt{s(s-a)(s-b)(s-c)}$$. [1 mark]

b. $s = (4 + 6 + 8)/2 = 9$
$T = \sqrt{9(9 - 4)(9 - 6)(9 - 8)} = \sqrt{135} = 3\sqrt{15}$ [1 mark]

16. If the volume of chemical used up in the reaction is 1 - v, then the volume of chemical remaining is 1 - (1 - v) = v.

If the volume of water used up in the reaction is 4(1 - v), then the volume of water remaining is 20 - 4(1 - v) = 16 + 4v.

Therefore, the differential equation is of the form (k > 0):

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -kv(4v+16)$$ [2 marks]

Separating the variables,

$$\int \frac{1}{4v(v+4)}\ \mathrm{d}v = \int -k\ \mathrm{d}t$$ [1 mark]

Using partial fractions on the LHS,

$$\int \frac{1}{4v(v+4)}\ \mathrm{d}v = \int \frac{1/4}{4v} + \frac{-1/16}{v+4}\ \mathrm{d}v = \frac{1}{16}\int \frac{1}{v} - \frac{1}{v+4}\ \mathrm{d}v$$ [2 marks]

Integrating,

$$\frac{1}{16}(\ln v - \ln(v+4)) = -kt + C$$ [1 mark]

Combining logs,

ln(v/(v+4)) = -16kt + 16C

Taking exponentials, and let A = e^{16C},

v/(v+4) = Ae^{-16kt} [1 mark]

Using the initial condition, v(0) = 1:

1/5 = A * e^0 → A = 1/5 → v/(v+4) = 1/5 exp(-16kt) [1 mark]

Using given info, v(2) = 4/19:

(4/19) / (4/19 + 4) = 1/5 exp(-32k) → 1/4 = exp(-32k) → k = -1/32 ln(1/4)

→ 5v/(v+4) = exp(-16 * -1/32 * ln(1/4) t) [1 mark]

→ 5v/(v+4) = exp(1/2 * ln(1/4) t)

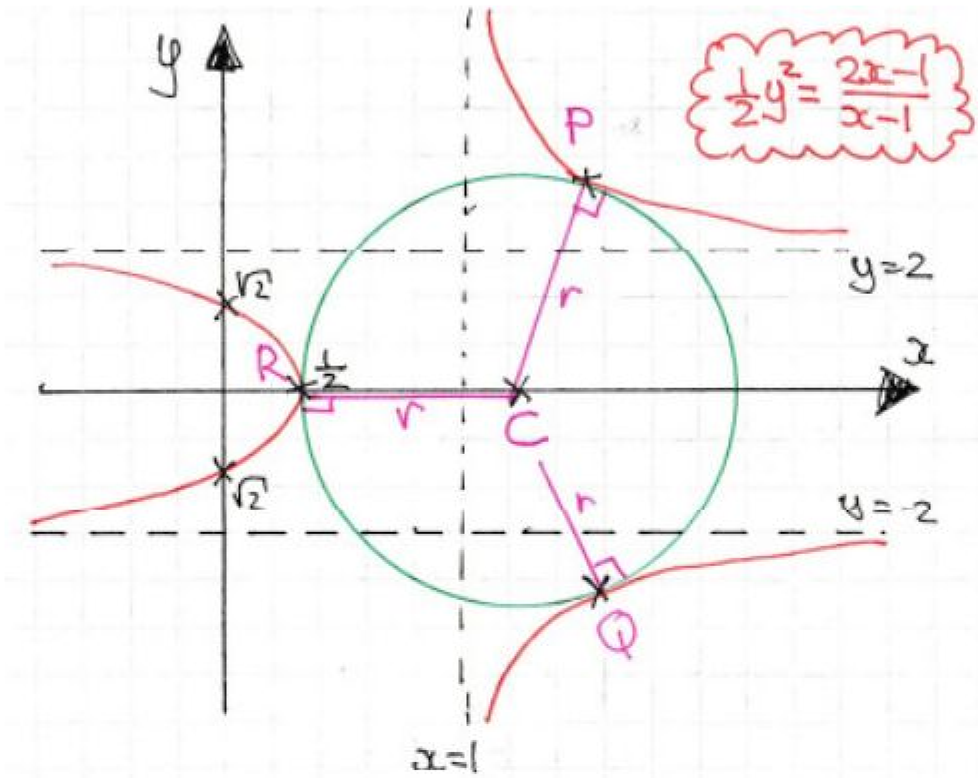
→ 5v/(v+4) = exp(ln(1/2) t)

→ 5v/(v+4) = $(1/2)^t$

→ (v+4)/5v = 2^t [1 mark]

17. **Step 1: Setup**

Let the circle C have centre (a, b) and radius r.
Since the curve function contains a y^2 term, it must be symmetrical in the x-axis and therefore, the circle must also be symmetrical, so its centre lies on the X-axis. Therefore, b = 0 → centre (a, 0), radius r.



The left-most point (which lies on the x-axis) can be found by putting y = 0:
$1/2\ (0)^2 = (2x - 1)/(x - 1) \rightarrow 0 = 2x - 1 \rightarrow x = 1/2$. This will be a helpful solution later.

Step 2: Finding the centre

Differentiating the curve function implicitly, using quotient rule for RHS,

$$\Rightarrow y\frac{dy}{dx} = \frac{2(x-1)-(2x-1)}{(x-1)^2}$$

$$\Rightarrow y\frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{y(x-1)^2}$$

NORMAL GRADIENT FUNCTION

$$\Rightarrow -\frac{dx}{dy} = y(x-1)^2$$

$$\Rightarrow -\frac{dx}{dy} = \sqrt{2}\,\frac{(2x-1)^{\frac{1}{2}}}{(x-1)^{\frac{1}{2}}}(x-1)^2$$

$$\Rightarrow -\frac{dx}{dy} = \sqrt{2}\,(2x-1)^{\frac{1}{2}}(x-1)^{\frac{3}{2}}$$

[4 marks]

Let the x-coordinate of the P be *k*. Then the gradient of the normal at P is

$$\sqrt{2}\,(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}$$

Using point-slope form, the equation of the radius (line CP) is

$$\Rightarrow y - \sqrt{2}\,\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} = \sqrt{2}\,(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}(x-k)$$

[1 mark]

To find the centre of the circle, let y = 0 and solve for x:

$$\Rightarrow -\cancel{\sqrt{2}}^{\,1}\,\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} = \cancel{\sqrt{2}}\,(2k-1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}}(x-k)$$

$$\Rightarrow -\frac{1}{(k-1)^2} = x-k$$

$$\Rightarrow x = k - \frac{1}{(k-1)^2}$$

[2 marks]

Also, |CP| = |CQ| = |CR| = r. Any of these can be used to form the next equation. For simplicity, use $|CP|^2 = |CR|^2 = r^2$.

$$\Rightarrow \underbrace{\left[k - \left(k - \frac{1}{(k-1)^2}\right)\right]^2 + \left[\sqrt{2}\frac{(2k-1)^{\frac{1}{2}}}{(k-1)^{\frac{1}{2}}} - 0\right]^2}_{|CP|^2} = \underbrace{\left[k - \frac{1}{(k-1)^2} - \frac{1}{2}\right]^2}_{|CR|^2}$$

$$\Rightarrow \frac{1}{(k-1)^4} + \frac{2(2k-1)}{k-1} = \left[\frac{2k-1}{2} - \frac{1}{(k-1)^2}\right]^2$$

$$\Rightarrow \cancel{\frac{1}{(k-1)^4}} + \frac{2(2k-1)}{k-1} = \frac{(2k-1)^2}{4} - \frac{2k-1}{(k-1)^2} + \cancel{\frac{1}{(k-1)^4}}$$

$$\Rightarrow \frac{2}{k-1} = \frac{2k-1}{4} - \frac{1}{(k-1)^2}$$

$$\Rightarrow 2(k-1) = \frac{1}{4}(2k-1)(k-1)^2 - 1$$

$$\Rightarrow 8(k-1) = (2k-1)(k^2-2k+1) - 4$$

$$\Rightarrow 8k - 8 = 2k^3 - 4k^2 + 2k - k^2 + 2k - 1 - 4$$

$$\Rightarrow \underline{0 = 2k^3 - 5k^2 - 4k + 3}$$

[7 marks]

We know that k = 1/2 is a solution (since this corresponds to point R) so therefore (2k - 1) will be a factor. By polynomial division, or directly solving, we find the other solutions will be k = 3 and k = -1. [1 mark]

From the graph, P is clearly to the right of the asymptote x = 1 so reject k = -1 → k = 3. [1 mark]

Putting this back in, the centre of the circle is then:
$3 - 1/(3 - 1)^2 = 11/4$. → Centre (11/4, 0). [1 mark]

Step 3: Finding the radius

Since r = |CR|, we have

$$r = k - \frac{1}{(k-1)^2} - \frac{1}{2}$$

$$r = 3 - \frac{1}{(3-1)^2} - \frac{1}{2}$$

$$r = 3 - \frac{1}{4} - \frac{1}{2}$$

$$r = 3 - \frac{3}{4}$$

$$r = \frac{9}{4}$$

[2 marks]

Therefore, the equation of the circle C is:

$$\left(x - \frac{11}{4}\right)^2 + y^2 = \frac{81}{16}$$

. [1 mark]

(Solution by T. Madas at https://madasmaths.com/archive/maths_booklets/standard_topics/various/differeNtiation_ii_exam_questions.pdf, Question 261)