A-Level Maths - Exam Style Questions

Section A:	Multiple Choice Questions	90 minutes in total
Section B:	Proof and Geometry	30 minutes per question
Section C:	Algebra 1	30 minutes per question
Section D:	Algebra 2	30 minutes per question
Section E:	Calculus	30 minutes per question
Section F:	Mechanics	30 minutes per question
Section G:	Statistics	30 minutes per question

A1. The correct factorisation of $3x^2 - 2xy - y^2$ is

① $(3x + y)^2$
② $(3x + y)(x - y)$
③ $(3x - y)(x + y)$
④ $(3x + y)^3 - (3x - y)^3$ [1 mark]

A2. A line L passes through the points (5, 1), $(k, 4)$ and (-1, 9). The value of k is

① 2
② 9 / 4
③ 11 / 4
④ 3 [1 mark]

A3. Which techniques, applied one after the other, would be most suitable for

evaluating $\int \frac{\cos x\,(\sin x - 1)}{\sin^2 x + 5 \sin x + 6}\,dx$?

① integration by parts then substitution
② substitution then partial fractions
③ quotient rule then integration by parts
④ substitution then integration by parts [1 mark]

A4. Which set contains an element which is smaller than all other elements in that set?

($\mathbb{Q}^+$ represents the set of all positive elements in $\mathbb{Q}$.)

① $\{r \in \mathbb{Q}^+\}$
② $\{r \in \mathbb{Q}^+ \mid r^2 \geq 2\}$
③ $\{r \in \mathbb{Q}^+ \mid r^2 > 4\}$
④ none of these [1 mark]

A5. The nth term of a sequence is denoted a_n.

If $\sum_{k=1}^{n} a_k = \log_{10}\left(\frac{(n+1)(n+2)}{2}\right)$ for all $n \geq 1$ then the value of $\sum_{k=1}^{20} a_{2k}$ is

① $10 \log_{10} 2$

② $\log_{10} 21$

③ $\log_{10} \sqrt{42}$

④ $\log_{10} 42$ [2 marks]

A6. The exact value of x such that ln x, ln $2x$ and ln $3x$ form three sides of a right-angled triangle with ln $3x$ as the hypotenuse is

① $x = \frac{3}{2} \exp \sqrt{ln \frac{3}{2}\ ln\ 9}$

② $x = \frac{1}{2} \exp \sqrt{ln \frac{5}{2}\ ln\ 3}$

③ $x = \frac{1}{2} \exp \sqrt{ln \frac{3}{2}\ ln\ 5}$

④ $x = \frac{5}{2} \exp \sqrt{ln \frac{1}{2}\ ln\ 9}$ [2 marks]

A7. $f(x)$ is a quadratic function such that $f(0) = 0$. Furthermore, $f(x)$ satisfies

$$\int_0^2 |f(x)|\,\mathrm{d}x = -\int_0^2 f(x)\,\mathrm{d}x = 4 \qquad \text{and} \qquad \int_2^3 |f(x)|\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x.$$

Find the value of $f(5)$.

① 0
② 5
③ 15
④ 45 [4 marks]

***A8.** For any $0 < t < 41$, the curve $y = x^3 + 2x^2 - 15x + 5$ intersects the horizontal line $y = t$ three times. Of these three intersections, let the point with the largest x-coordinate be $(f(t),\ t)$ and the point with the smallest x-coordinate be $(g(t),\ t)$.

If $h(t) = t \times (f(t) - g(t))$, find the value of $h'(5)$.

① $\frac{79}{12}$

② $\frac{91}{12}$

③ $\frac{97}{12}$

④ $\frac{129}{16}$ [4 marks]

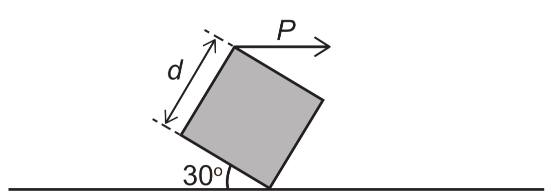
A9. A rower wants to cross a river to a point on the opposite river bank directly opposite from where she will start. The distance to cover is 100 m, and she wants to cross in 10 seconds. The river flows uniformly at 7.5 m/s.

How fast must she row her boat, and at what angle, relative to the velocity of the flowing water? (Round the angle to the nearest degree.)

① speed = 12.5 m/s, angle = 37°
② speed = 12.5 m/s, angle = 127°
③ speed = 6.6 m/s, angle = 37°
④ speed = 6.6 m/s, angle = 127° [1 mark]

A10. The diagram shows a uniform, solid, heavy cube with side d. The cube rests with one of its edges in contact with a table that is perfectly level.

A horizontal force P acts on another edge of the cube, and the cube is stationary.



Below are four statements about the forces on the cube.

A No frictional force acts between the cube and the table.

B A frictional force acts to the left between the cube and the table.

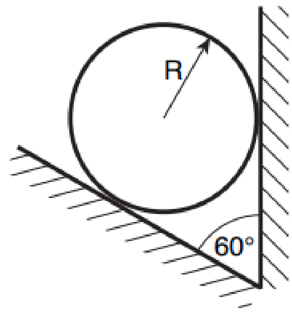
C A frictional force acts to the right between the cube and the table.

D Force *P* has a clockwise moment about the edge in contact with the table equal to $P \times d$

How many of the above statements must be true?

① 1
② 2
③ 3
④ 4 [1 mark]

A11. A stationary, smooth (i.e. frictionless) sphere of radius R and weight W is in contact with a smooth vertical plane and with a plane inclined at 60° to the vertical.



The magnitudes of the normal contact forces exerted by the inclined plane and the vertical plane are N_1 and N_2 respectively. These values are [1 mark]

	N_1	N_2
①	$\frac{\sqrt{5}}{2}W$	$\frac{1}{2}W$
②	$\frac{\sqrt{5}}{2}W$	W
③	$\frac{\sqrt{3}}{2}W$	$\frac{1}{2}W$
④	$\frac{\sqrt{3}}{2}W$	W

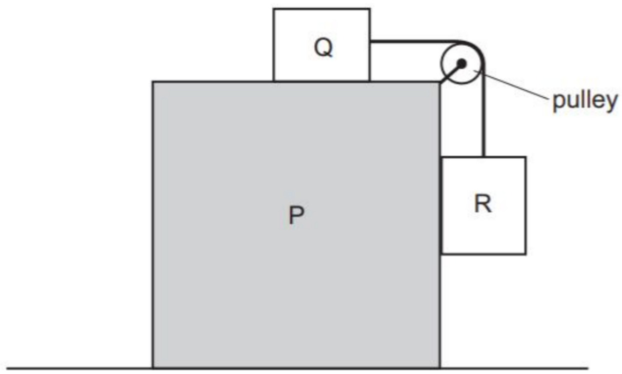
A12. A stone is thrown from the roof of a 10 metre tall building. The stone lands on the surrounding flat ground at a horizontal distance x.

At what angle to the horizontal should the stone be thrown at to maximise x?

(Neglect air resistance. Assume a uniform gravitational field.)

① less than 45°
② exactly 45°
③ more than 45°
④ it could be any of the above, depending on the initial speed [1 mark]

A13. A block P has a smaller block Q resting on its top surface. Q is connected to a hanging block, R, by a light, inextensible string. The string passes over a smooth pulley which is connected to block P, as shown in the diagram.



The masses of blocks P, Q and R are m_P, m_Q and m_R respectively. P is accelerated horizontally to the right by an external force. While this is happening, Q and R do not move relative to P.

What is the acceleration of P? (Gravitational field strength $= g$.)

① $\dfrac{m_Q g}{m_R}$

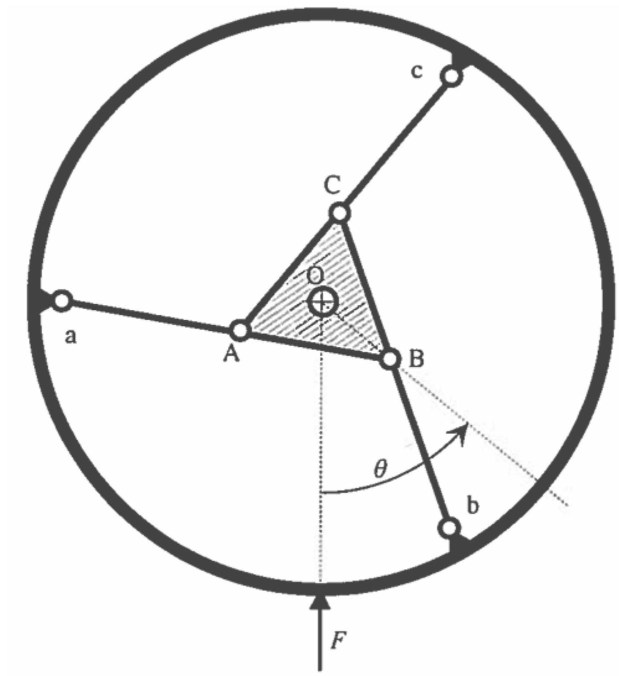
② $\dfrac{m_R g}{m_Q}$

③ $\dfrac{m_R g}{m_R + m_Q}$

④ $\dfrac{(m_Q + m_R)g}{m_P + m_Q + m_R}$

[2 marks]

A14. A bicycle wheel consists of a circular rim with three inextensible spokes connected to the three vertices of a central equilateral triangular hub ABC such that aAB, bBC and cCA are all straight lines. The vertical reaction load F from the road is balanced by an equal and opposite force acting through the centre O.



The angular position of the wheel is defined by the angle θ as shown. The centre point O is located at the centre of mass of the triangle ABC, and is positioned such that $PO : OB = 1 : 2$, where P is the midpoint of AC.

Determine how the force R_C in the spoke cC varies as the wheel turns.

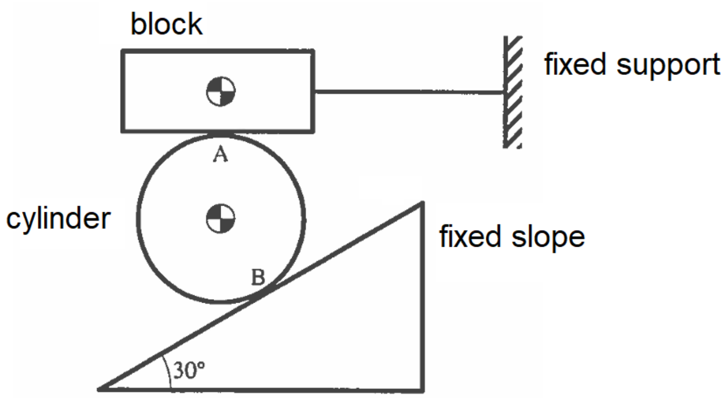
① $R_C = \dfrac{1}{3} F \sin\theta$

② $R_C = \dfrac{2}{3} F \sin\theta$

③ $R_C = \dfrac{1}{3} F \cos\theta$

④ $R_C = \dfrac{2}{3} F \cos\theta$ [2 marks]

***A15.** A cylinder of mass M is on a slope inclined at 30° to the horizontal. The cylinder supports a block, also of mass M. The block is tethered horizontally by a cable to a fixed support so that its centre of mass is directly above that of the cylinder.



The coefficients of friction μ at the contact points A and B are both between 0 and 1, with corresponding angles of friction ψ denoted ψ_A and ψ_B, where μ and ψ are related through $\mu = \tan \psi$.

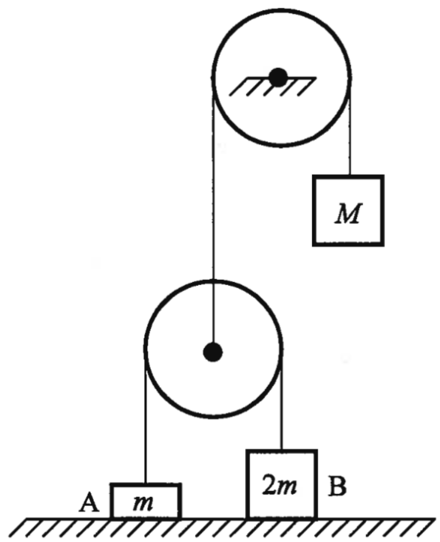
The table shows what happens to the cylinder when the coefficients of friction, and hence the values of ψ, vary.

	$0 < \psi_B < y$	$y < \psi_B < 45°$
$0° < \psi_A < x$	rolls about B	remains stationary
$x < \psi_A < 45°$	slides	rolls about A

What are the values of the limiting angles of friction, x and y? [4 marks]

	x	y
①	30°	36.3°
②	30°	15°
③	28.2°	36.3°
④	28.2°	15°

***A16.** A pulley is pinned to a fixed support and carries a string connecting a hanging mass M kg to a second hanging pulley, which in turn connects two blocks A and B with masses m kg and $2m$ kg respectively. The blocks A and B are resting on horizontal ground. The hanging block is released from rest with all strings taut.



Gravitational acceleration is g. Both strings are vertical and inextensible, and both pulleys are light and frictionless. Which statement is true?

① Both blocks A and B must lift off of the ground if $\frac{M}{m} > 4$.

② If $\frac{M}{m} = 4$ then the vertical acceleration of block A is $\frac{g}{2}$.

③ The system must remain in equilibrium if $\frac{M}{m} \leq 3$.

④ The tension in the lower string is always greater than in the upper string.

[4 marks]

A17. To understand the academic performance of 1,000 students on an exam, the systematic sampling method is adopted to choose 40 samples.

Which statement is most likely to be true?

① The sampling interval should be 50.
② The sample mean will likely be very different from the population mean.
③ Quota sampling at the same sample size would be more representative.
④ Each student in the population has an equal chance of being sampled.

[1 mark]

A18. The random variable Z is distributed with the standard normal distribution. What is the interquartile range of Z?

① 0.1974
② 0.6745
③ 0.6915
④ 1.3490

[1 mark]

A19. A coin of diameter 2 cm is thrown onto a table covered with a 2D grid of lines intersecting at right angles and each separated by 4 cm.

What is the probability that the coin lands in a square without touching any of the lines of the grid?

① 0.2500
② 0.3183
③ 0.3333
④ 0.5000

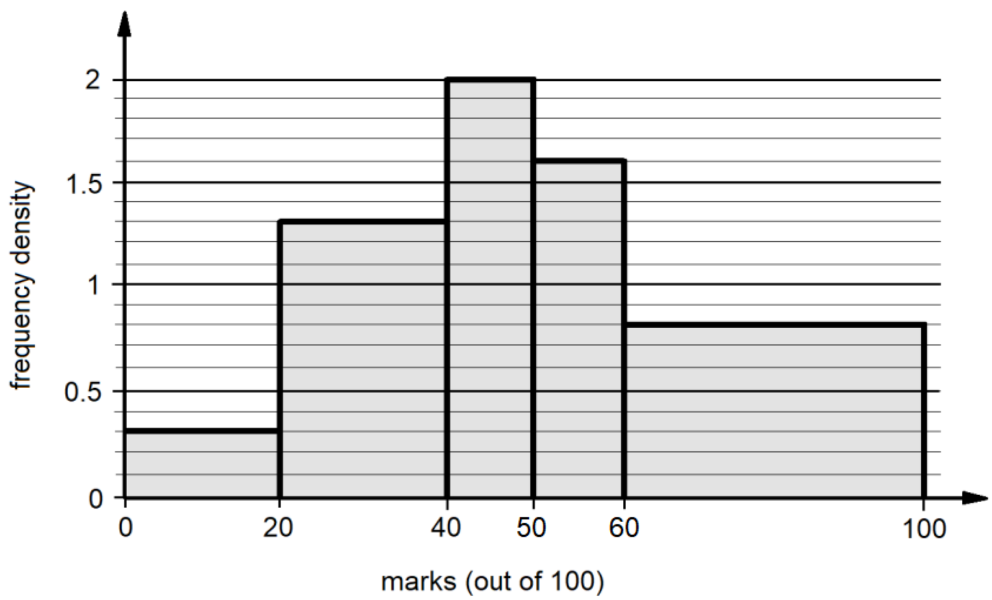
[1 mark]

A20. A biologist is studying a wild population of frogs in a forest. One day, the biologist captures 30 frogs, marks them by attaching small markers on their legs, and releases them back into the forest.

A week later, the biologist returns to the forest and collects another random sample of 30 frogs, of which they find 3 are marked. Assuming a net population change of $100x\%$, what is the best estimate for the original total population of frogs?

① $300(1 - x)$
② 300
③ $300(1 + x)$
④ $300 / (1 + x)$ [1 mark]

A21. The histogram below shows the distribution of the marks of a cohort of students who recently sat the same exam.



Assuming the marks are uniformly distributed throughout each bin, which of the following statistics best describe the data shown?

① The students scoring 75 marks or more make up the cohort's top quartile.
② The mark distribution is negatively skewed.
③ The median mark rounds to 49.
④ The mark distribution obeys a binomial model $\mathrm{B}(100, 0.45)$. [2 marks]

A22. A faulty bit in a computer memory changes its state between "0" or "1" in any given clock cycle with probability p. If P_N is the probability that it is in the same state N cycles later, an inductive sequence relating these probabilities is given by

① $P_{N+1} = 2p + (1 - p)\, P_N$
② $P_{N+1} = p + (1 - p)\, P_N$
③ $P_{N+1} = P_N + (2p - 1)$
④ $P_{N+1} = p + (1 - 2p)\, P_N$ [2 marks]

A23. A component costs £9 to make and can, if perfect, be sold for £10. The machine producing components is not reliable; 5% of components have to be scrapped and 5% can only be sold at half the price of a perfect one.

A replacement machine, that produces a negligible number of faulty components, is purchased for £10,000. How many components need to be made on the new machine and then sold before there is an increase in profit over that which would have been obtained making the same number on the old machine?

① 6667
② 8001
③ 13334
④ 25001 [4 marks]

***A24.** In Camelot, it never rains on Friday, Saturday, Sunday or Monday. The probability that it rains on a given Tuesday is 1 / 5. For each of the remaining two days, Wednesday and Thursday, the conditional probability that it rains, given that it rained the previous day, is α, and the conditional probability that it rains, given that it did not rain the previous day, is β.

If X is the event that, in a randomly chosen week, it rains on Thursday, Y is the event that it rains on Tuesday, then the value of $\mathrm{P}(X \mid Y) - \mathrm{P}(X \mid Y')$ is

① $(\alpha - \beta) / 5$
② $(\alpha - \beta)^2$
③ $\alpha^2 - \beta^2$
④ $10\alpha\beta$ [4 marks]

B1.

a. i) Show that the square of any positive integer cannot end in the digit 3.

[3 marks]

ii) Prove algebraically that the product of four consecutive positive integers can never be equal to a perfect square.

[5 marks]

iii) The nth term of a positive integer sequence U is given by

$$U_n = 1! + 2! + 3! + ... + (n-1)! + n! \qquad \text{for } n \in \mathbb{N} : n \geq 1$$

Explain why U_n can never be a square number when $n \geq 4$.

You may use the result(s) shown above to help you. [3 marks]

*b. Let m and n be different integers. Let $S(x)$ denote the set containing the distinct prime factors in the prime factorisation of an integer x.

Prove that the set of all pairs (m, n) such that $S(m) = S(n)$ and $S(m+1) = S(n+1)$ is infinite.

[3 marks]

c. It is given that x, a and b are positive real numbers, with $a > b$ and $x^2 > ab$.

Use proof by contradiction to show that

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} > 0.$$

Fully justify your answer. [6 marks]

B2.

a. i) Prove that $\sqrt[3]{2}$ is irrational. [6 marks]

ii) Prove that for all real x,

$$\sin x + \sin(\sqrt[3]{2}x) \neq 2.$$

[6 marks]

b. Prove for all $n \in \mathbb{N}$, that $n^3 + 5n$ is always a multiple of 6. [5 marks]

c. a, b, c and d are four real numbers.

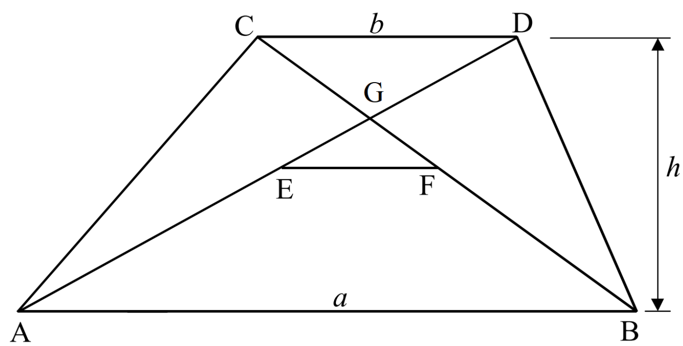
By considering the quadratic function $f(x) = (ax + c)^2 + (bx + d)^2$ for all real x, prove that

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2).$$

[3 marks]

B3.

a. Trapezium $ABCD$ below has parallel sides $|AB| = a$ and $|CD| = b$ with $a > b$. Points E and F are the midpoints of the diagonals AD and BC respectively, which intersect at G, as shown. The perpendicular height from AB to CD is h.



i) Prove that the length of the line segment EF is equal to $\frac{a-b}{2}$.

[3 marks]

ii) Explain why as $b \to 0$, the area of triangle EFG approaches $\frac{ah}{8}$.

[1 mark]

iii) If the perpendicular bisector of side AB also bisects CD, then identify any **two** pairs of congruent triangles in the figure. [2 marks]

iv) Let angle $\angle CGD = \theta$. A circle with radius r and centre O, which lies inside $\triangle EFG$, passes through all three vertices of $\triangle EFG$.

Using a suitable circle theorem, prove that $r = \frac{a-b}{4 \sin \theta}$. [4 marks]

b. In the triangle ABC, the point X lies on side AB (length a) such that the line CX bisects $\angle ACB$ with $\angle ACX = \angle XCB = 60°$.

If $|BC| = 2\,|AC|$, prove that the length of line CX is $\frac{2a\sqrt{7}}{21}$. [10 marks]

C1.

a. Find the values of the unknown constants to make the following partial fraction decompositions valid for all appropriate values of t:

i) $$\frac{2t}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1}, \quad \text{for all } |t| \neq 1$$ [1 mark]

ii) $$\frac{1}{t^4+1} = \frac{Pt+Q}{t^2+Ct+D} + \frac{Rt+S}{t^2+Et+F} \quad \text{for all } t \in \mathbb{R}$$ [4 marks]

b. The domain of $f(x)$ is all real numbers and satisfies $2f(x) = f(x + 1)$.

In the interval [0, 1), $f(x) = x(x - 1)$.

i) Sketch a graph of $y = f(x)$ for $-1 < x < 3$. [2 marks]

ii) Express $f(x)$ in the form $f(x) = ax^2 + bx + c$ for real coefficients a, b and c, valid for all x in the interval [2, 3). [2 marks]

iii) Find the greatest value of m such that $f(x) \geq -\frac{16}{9}$ for all $x \leq m$.

[2 marks]

c. x and y are variables such that $x + y \geq a$ and $x - y \leq -1$, where $a \neq 0$ is a constant.

i) Sketch, in the x-y coordinate plane, the region satisfied by these inequalities. [1 mark]

*ii) The **smallest** possible value of $x + ay$ is 7.

Find the value(s) of a. Fully justify your answer. [8 marks]

C2.

a. Simplify fully the expression $4 + \frac{4 - x^2}{x^2 - 2x}$. [2 marks]

b. Find the constant term in the expansion of $\left(2x + \frac{3}{x^2}\right)^{12}$. [2 marks]

c. Factorise fully $x^2 + 2xy - 3y^2 + 4x - 4y$. [3 marks]

d. Find the equations of the asymptotes to the curve

$$y = \frac{x^4 - 2x^3}{(x^2 - 3x + 2)\left(x^2 + ax + \frac{1}{2}a^2\right)}, \quad a \in \mathbb{R}.$$

[3 marks]

e. Solve the inequality $|5 - 3x| < \frac{2}{x}$. [4 marks]

f. Express the value of $\sqrt{3 - 2\sqrt{2}}$ in the form $a + b\sqrt{c}$, where *a*, *b* and *c* are integers to be found.

[3 marks]

*g. Rationalise the denominator in $\frac{1}{\sqrt[3]{2} + \sqrt[3]{3}}$. [3 marks]

C3.

a. $\mathbf{u}$ and $\mathbf{v}$ are vectors.

i) Explain why $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$. [1 mark]

ii) If $|\mathbf{u} + \mathbf{v}| = ||\mathbf{u}| - |\mathbf{v}||$, describe the relationship between $\mathbf{u}$ and $\mathbf{v}$. [2 marks]

b. Let $f(x) = 2 - \ln x$ and $g(x) = ke^{-x}$ for some positive constant k.

The domain of both $f(x)$ and $g(x)$ is $\{x \in \mathbb{R}: x > 0\}$.

Find the value of k for which the operation of composition for functions f and g is commutative for all $0 < x < k$.

[3 marks]

*c. A cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ has $a > 0$ and is defined for all real x. $P(x)$ has two real turning points at $x = p$ and $x = q$ where $p < q$.

Let Δx represent the absolute difference in x-coordinate between one of the turning points of $P(x)$ and its inflection point.

With the aid of a sketch, prove that $P(p) = P(p + 3\,\Delta x)$.

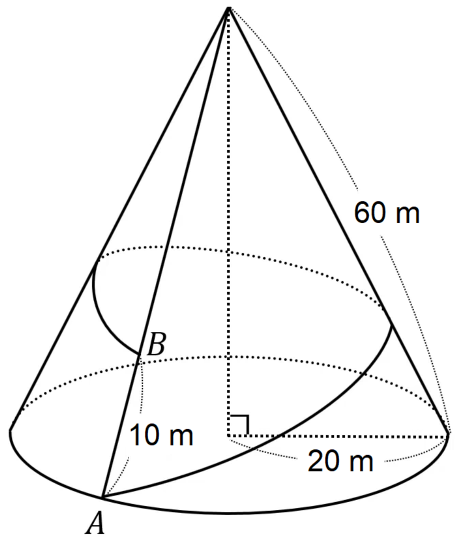
(It is **not** recommended to solve for p and q explicitly.) [7 marks]

*d. Find, in exact form, the real solution(s) to the equation

$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$

[7 marks]

D1. The diagram illustrates a right-circular cone shaped mountain.



A **shortest-distance** track for a sightseeing train is built around the mountain, in which the track starts at a point A at the base of the mountain, and ends at the point B, located 10 metres up the mountain (measured along the slant) above A.

As shown in the diagram, this track first goes uphill then goes downhill.

a. Show that the length of the downhill section of the track is $\frac{400}{\sqrt{91}}$ metres.

[6 marks]

b. Find the maximum **vertical** height reached by the train above the base of the mountain, giving your answer in metres to 3 significant figures. [4 marks]

c. The train completes the journey while moving at a constant speed throughout. Find the fraction of the journey for which the elevation of the train is above B.

[5 marks]

d. The track to be redesigned under the condition that there is no downhill section. Calculate the new shortest possible total length of the track. [5 marks]

D2.

a. i) Show that for all real x and y,

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\frac{x+y}{2}.$$ [4 marks]

ii) Hence show that the graph of

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 1$$

is a set of parallel lines, giving their general equations. [4 marks]

b. i) Find the first three terms, in ascending powers of x, of the binomial series expansion of $f(x)$, where

$$f(x) = \sqrt{1 - \frac{3}{4 - x^3}}, \qquad x \leq 1.$$ [6 marks]

ii) Find the interval of validity for the approximation in part b.i). [2 marks]

iii) If the first three terms of the expansion are used to approximate the value of $\int_0^1 f(x)\,dx$, explain whether the value obtained would be an underestimate or an overestimate of the true value. [2 marks]

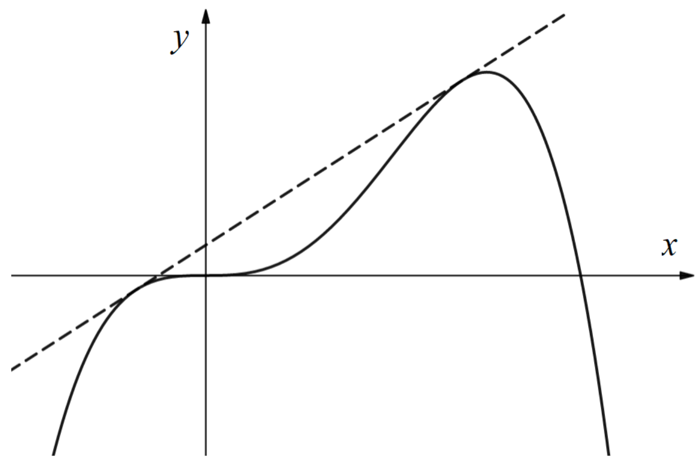
iv) Use the expansion to estimate the solution $0 < x < 1$ to the equation

$$\sin^{-1}\left(\frac{1}{2} - f(x)\right) = 1 - 720x^3.$$

Give your answer to 10 decimal places. [2 marks]

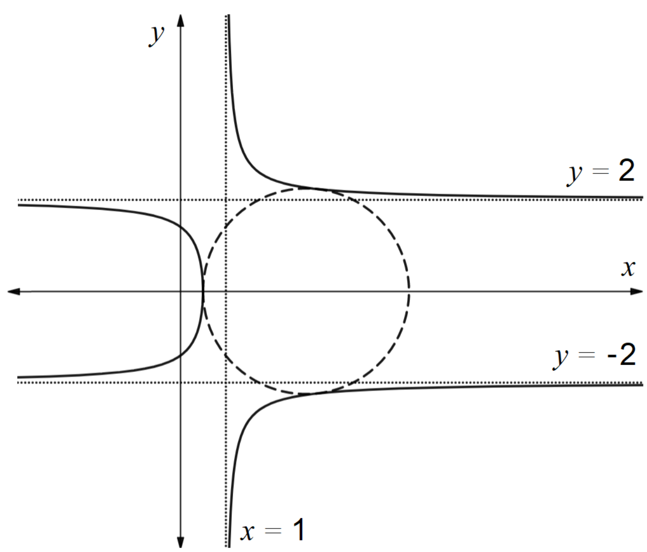
D3. *It is advised (but not required) to use algebra in this question,* ***not*** *calculus.*

a. The graph shows the curve $y = x^3 - x^4$ (solid line) and its common tangent line (dashed line), which touches the curve at **two** distinct points.



Find the equation of the tangent line. [10 marks]

b. The graph shows a circle (dashed line) which touches all three branches of the curve with equation $\frac{1}{2}y^2 = \frac{2x-1}{x-1}$ (solid line) at **three** distinct points.
The asymptotes (dotted lines) of the curve, $y = 2$, $y = -2$ and $x = 1$, are shown.



Find the equation of the circle. [10 marks]

E1.

a. i) Show that, for all $|x| < 1$,

$$y = \sin^{-1} x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

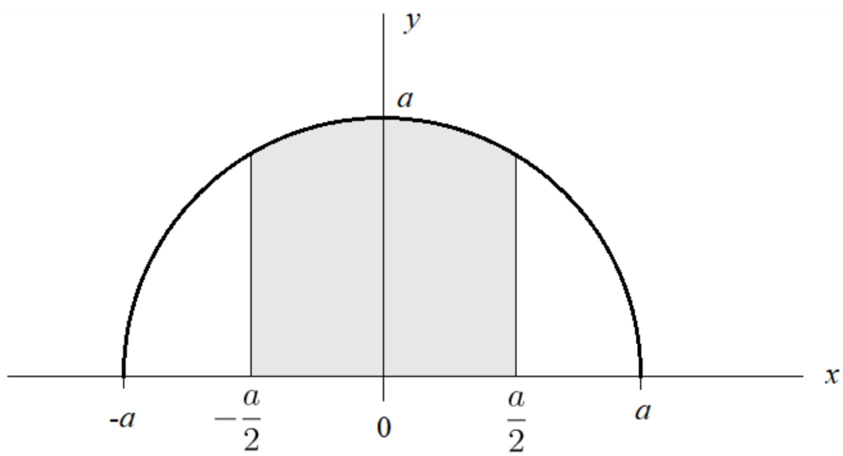
[5 marks]

ii) For some positive constant a, let

$$f(x) = x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}, \qquad |x| \leq a, \quad a > 0.$$

Find $f'(x)$ in its simplest form. [5 marks]

iii) The diagram shows a semicircle of radius a centred at the origin.



By considering your answer to part b.i), show that the area of the shaded region is $f\left(\frac{a}{2}\right)$. [4 marks]

b. Use the substitution $\frac{du}{dx} = \sin x + \cos x$, or otherwise, to show that

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx = \frac{\ln 3}{20}.$$

[6 marks]

E2.

a. For some real constant a, define the function

$$f(x) = \frac{1}{1 + e^{-x^2} + e^{-(x-a)^2}}, \qquad x,\ a \in \mathbb{R}.$$

When $|a| < \sqrt{2}$, $f(x)$ always has only one stationary point, at $x = x_0$.

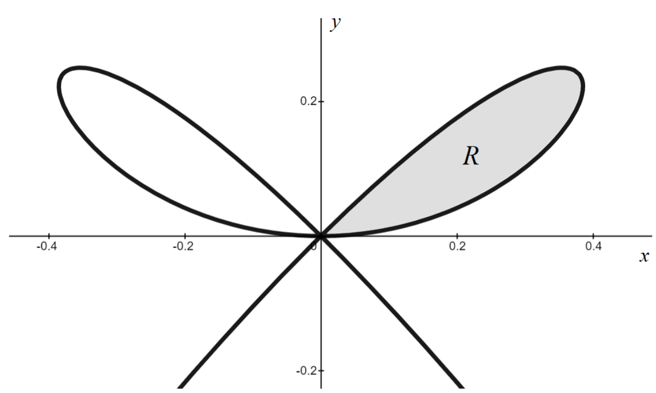
i) By considering the symmetry of $f(x)$, explain why $x_0 = \frac{1}{2}a$. [2 marks]

ii) Use calculus to determine the nature of this stationary point. [6 marks]

b. A water tank has the shape of a horizontal cylinder with radius 1 m and length 6 m. Water is being pumped into the tank at a rate of $\frac{1}{6}$ m^3 per minute.

Find the exact rate at which the water level is rising when the water is at a depth of 50 cm. [7 marks]

c. The graph of $x^4 + y^3 = x^2y$ is shown, with a closed-loop region R shaded.



By transforming the curve to a suitable set of parametric equations, or otherwise, find the area of region R. [4 marks]

E3.

a. Find the exact value of $\int_0^{\pi} e^{-x} \sin x \, dx$. [4 marks]

b. When the trapezium rule with n slices is used to approximate a definite integral $I = \int_a^b f(x)\, dx$ (where $b > a$) the result of the approximation is denoted T_n.

An upper bound for the absolute error E_n, in this approximation, defined as $E_n = | I - T_n |$, can be found using the inequality: $E_n \le \frac{K\,(b-a)^3}{12n^2}$, where K is the largest value of $| f''(x) |$ in the interval $a \le x \le b$.

i) For the integral in part a), $I = \int_0^{\pi} e^{-x} \sin x \, dx$, find T_4. [3 marks]

ii) Use the formula to find an upper bound for E_4. [4 marks]

iii) Using the true value of I, compare the true error to the maximum possible error. [1 mark]

*c. A function $y(x)$ is positive for all x and has a gradient satisfying $\frac{dy}{dx} = \frac{x-y}{x+y}$.

The function has a stationary point, at which the value of y is equal to 1.

Find a simplified expression for y in terms of x. [8 marks]

F1. A short-barrelled machine gun stands on horizontal ground. The gun fires a near-continuous stream of bullets for a period of one second, from ground level, at the same initial speed, and then stops firing.

During this time period, the angle of elevation of the barrel, $\alpha(t)$, decreases from its initial angle of 45° to 30° to the horizontal at the moment it fires the last bullet.

The variation of the barrel elevation with time *t* since firing the first bullet is such that all the bullets fired by the machine gun land on the ground at the **same time**.

a. Show, for $0 \leq t \leq 1$, that $\alpha(t) = \sin^{-1}(A - Bt)$, where A and B are positive constants to be found in exact form.

Assume that the bullets can be modelled as point particles with no air resistance, and a uniform vertically downward gravitational field strength g ms^{-2}.

[8 marks]

b. Show that the bullets landing a distance r m from the machine gun barrel will do so at times T s after being fired related by

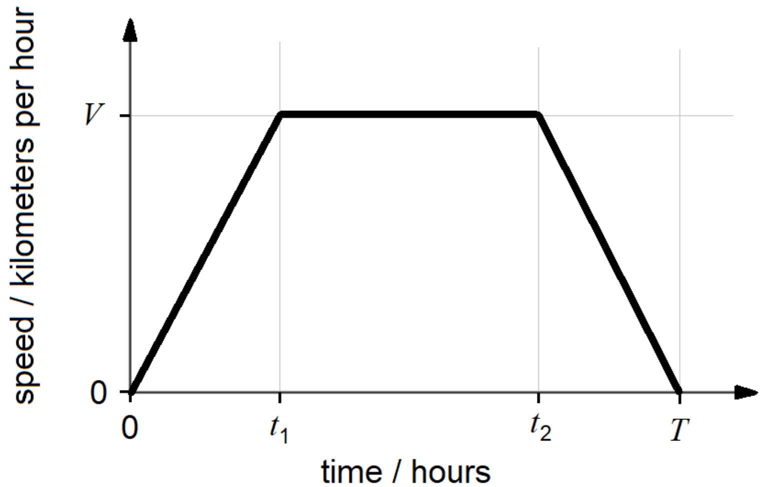
$$\frac{1}{2}g^2T^2 = v^2 - \sqrt{v^4 - g^2r^2},$$

where v is the initial speed of the bullets. [8 marks]

c. Hence find an expression for the greatest height h above the ground reached by a bullet which was fired at speed v ms^{-1} and landed at a distance r m from the machine gun barrel.

Give your answer in terms of v, r and g. [4 marks]

F2. The speed-time graph of a straight-line journey of a hovercraft is shown below.



The hovercraft accelerates uniformly at m km hr $^{-2}$ for an initial period t_1, travels at a constant speed V km hr $^{-1}$, and finally decelerates uniformly at m km hr $^{-2}$. The distance covered in that time is D km and the total time taken is T hours.

a. If the time taken by acceleration and deceleration is $\frac{T}{6}$ in each case, find an equation for D in terms of T and m. [5 marks]

b. The hovercraft is running behind schedule and needs to reduce T to $\frac{7}{9}$ T.

The only way of achieving this is to increase the value of V while the acceleration and deceleration may not be changed from m.

i) Calculate the time, as a fraction of T, that must now be spent accelerating. [8 marks]

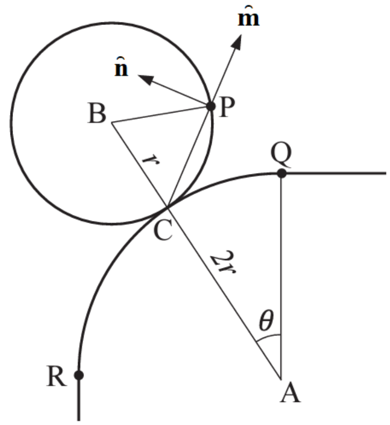
ii) Find the required percentage increase in the top speed. [3 marks]

*c. Consider again the original journey of total time T as described in part a).

Sketch a graph of the **speed** (vertical axis) of the hovercraft against the **distance** from the starting position (horizontal axis), identifying the distances corresponding to travel times t_1, t_2 and T as fractions of D. [4 marks]

F3. A cylinder is rolling over the rounded edge of a table. The rounded edge is a quarter-circle arc between Q and R, centred at A, and having radius $2r$. The cylinder has radius r, centre B, and rolls (without slipping) over the edge.

P is the point on the cylinder that was originally in contact with the table at Q at time $t = 0$. **Unit** vectors $\mathbf{m}$ and $\mathbf{n}$ are defined to always point parallel and perpendicular respectively to the line segment CP as shown. Angle $\angle CAQ$ is $\theta(t)$.



a. i) Using geometry, prove that $\angle CBP : \angle CAQ = 2 : 1$. [2 marks]

ii) Write down an expression for the displacement vector $\mathbf{s}$ of point P relative to A in terms of the vectors $\mathbf{m}$ and $\mathbf{n}$ and in terms of r and θ. [5 marks]

iii) Given that $\dfrac{\mathrm{d}\hat{\mathbf{m}}}{\mathrm{d}t} = 2\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{n}}$, explain why $\dfrac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}t}$ must be equal to $-2\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{m}}$.

[2 marks]

iv) Using differentiation, show that the velocity vector $\mathbf{v}$ of point P is given by

$$\mathbf{v} = 6r \sin\theta \frac{d\theta}{dt}\,\mathbf{n},$$

and find an expression for the acceleration vector $\mathbf{a}$. [8 marks]

b. The component of acceleration a which is **perpendicular** to the velocity vector $\mathbf{v}$ can be related to the *radius of curvature* R of the path traced out by the point P using the equation $a = \frac{|v|^2}{R}$. Show that R is proportional to $\sin\theta$. [3 marks]

G1. Newborn babies are tested for a mild illness which affects 1 in 500 babies. The result of a test is either positive or negative. A positive test implies the baby has the illness. However, the test is not perfect:

- for babies with the illness, the probability of a positive result is 0.99
- for babies without the illness, the probability of a negative result is 0.95.

a. Find the probability that

i) the test is positive. [2 marks]

ii) the test gives the correct diagnosis. [2 marks]

b. i) Given that the result of a test is positive, show that the probability of the baby having the illness is less than 5%. [3 marks]

ii) Explain why the probability of the test giving a 'true positive' is so low, even though the accuracy of the test in each case is no less than 95%. [2 marks]

c. It is required to raise the reliability of the test by improving the screening process. To achieve this, it is intended to increase the probability of negative results for babies who do not have the illness from its current value of 0.95 to p.

Find the value of p that would raise the probability in part b.i) to 0.50. [4 marks]

d. A particular baby is tested (using the new, improved test) three times.

The tests return one positive result and two negative results.

i) Show that the probability that the baby has the illness is about 1 in 10,000. [6 marks]

ii) Give **one** assumption made in your answer to part d.i). [1 marks]

G2. A manufacturing process produces resistors. Due to the tolerances of the process, the resistors produced have resistance values, in units of ohms, which are Normally distributed about a mean μ ohms and standard deviation σ ohms.

a. A random sample of 12 resistors from the process is (sorted in ascending order):

497.66	497.77	497.81	498.32	498.43	499.53
500.41	500.78	501.55	502.88	504.96	506.41

i) Find the sample median and sample interquartile range. [3 marks]

ii) Find unbiased estimates for the values of μ and σ^2. [2 marks]

iii) Using your estimates of μ and σ, estimate the interquartile range of the population. [3 marks]

b. A large number ($N \gg 1000$) of resistors are manufactured. A quality assurance survey of these resistors shows that $\mu = 501$ ohms and $\sigma = 3$ ohms. Resistors are rejected if their resistance is less than 498 ohms or greater than 508 ohms.

i) Find the expected proportion of manufactured resistors which are rejected. [2 marks]

ii) In order to minimise waste, the value of μ for the process is to be changed while the value of σ is fixed.

Find the minimum expected proportion of resistors which are rejected. [3 marks]

iii) The manufacturing process is improved to use more precise machinery, allowing for less variation in resistance.

For the optimal value of μ found in part b.ii), find the largest possible value of σ for the process such that the expected proportion of rejected resistors is no more than 0.1%. [4 marks]

*iv) Repeat the calculation in part b.iii) to estimate the required value of σ if the original process with $\mu = 501$ ohms is used instead. [3 marks]

G3. A robot is programmed to flip a coin repeatedly. The outcome of the coin flip is either Heads (H) or Tails (T), with probability $\mathrm{P(H)} = p$.

*a. It is initially assumed that the robot flips the coin fairly, so that $p = 0.5$.

Define X_n as the event that there were no consecutive Heads in the first n flips.

i) Let $F(n)$ be the distinct number of ways for the coins to land in a sequence of n flips such that the event X_n occurs.

Explain clearly why $F(n) = F(n - 1) + F(n - 2)$, for $n > 2$. [3 marks]

ii) Hence, or otherwise, show that $\mathrm{P}(X_{10}) = \dfrac{9}{64}$. [3 marks]

b. After the first 13 coin flips, it is observed that 10 of the flips returned Heads.

i) Test, at the 5% significance level, the claim that $p > 0.5$. [4 marks]

ii) Write down the critical region for this hypothesis test. [1 mark]

*c. It is now suspected that the robot may be unintentionally flipping the coins in the direction of Heads more frequently.

The value of the Heads probability p is modelled as random by initially assuming that p can take values with equal likelihood from 0.0 to 1.0 inclusive in discrete steps of 0.1.

i) A student claims that $\mathrm{P}(p = 0.5 \mid \text{at least 10 Heads in 13 flips})$ is the p-value of the test conducted in part b). Explain whether you agree. [1 mark]

ii) Find the value of $\mathrm{P}(p = 0.5 \mid$ **exactly** 10 Heads in 13 flips$)$. [5 marks]

iii) Given the observation of 10 Heads in 13 flips, estimate the **median** of p. [3 marks]

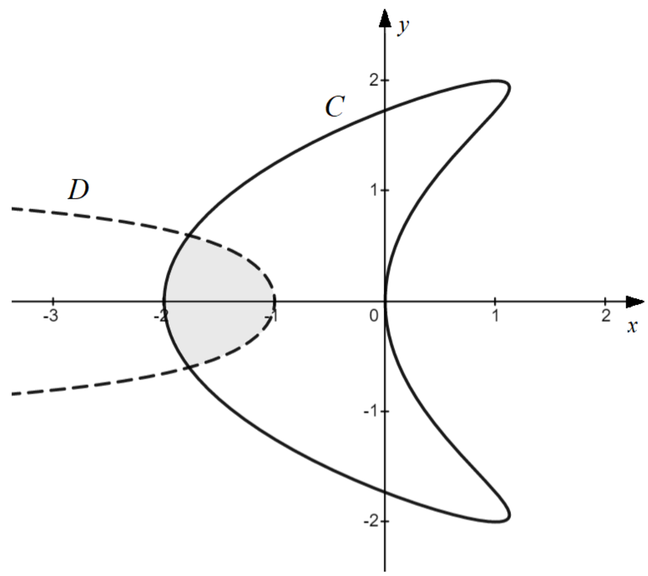
Additional Questions

A30. The curves C and D are are defined by parametric equations

Curve C: $x = \sin s + \cos 2s,\quad y = 2\cos s,\quad$ for $0 \leq s < 2\pi$

Curve D: $x = \sec^3 t,\quad y = t - \pi,\quad$ for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

The graphs of these curves are plotted, with their intersection region shaded.



Use numerical methods to find the area of the shaded region to 4 d.p.

① 0.8412

② 0.8865

③ 0.9139

④ 0.9514 [4 marks]

A31. Find the equations of the tangent lines to the parabola $y = x^2$ which pass through the point (1, -1).

① $y = 2(1+\sqrt{2})x - (1+\sqrt{2})^2$ and $y = 2(1-\sqrt{2})x - (1-\sqrt{2})^2$
② $y = 2(1-\sqrt{2})x - (1+\sqrt{2})^2$ and $y = 2(1+\sqrt{2})x - (1-\sqrt{2})^2$
③ $y = 2(1+\sqrt{2})x + (1+\sqrt{2})^2$ and $y = 2(1-\sqrt{2})x + (1-\sqrt{2})^2$
④ $y = 2(1+\sqrt{2})x + (1-\sqrt{2})^2$ and $y = 2(1-\sqrt{2})x + (1+\sqrt{2})^2$

[2 marks]

A32. The function $f(x)$ satisfies the equation $f(x) + \int_1^x f(t)\,dt = 1$ for all real $x > 1$.

Find the value of $\ln f(2)$.

① -2
② -1
③ 1
④ 2

[2 marks]

A33. A sequence is defined with nth term

$$u_n = \frac{n}{a^n}, \qquad n = 1, 2, 3, \ldots, \qquad a \text{ is a constant.}$$

If the sum to infinity of this sequence, $\sum_{n=1}^{\infty} u_n$, is equal to $\frac{14}{25}$,

then the possible value(s) of a are

① $\frac{2}{7}$ or $\frac{7}{2}$
② $\frac{3}{5}$ or $\frac{5}{3}$
③ $\frac{7}{2}$
④ $-\frac{3}{5}$

[4 marks]

A34. Let a_k be a sequence for positive integers k.

Given that $\sum_{k=1}^{10} (a_k + 1)^2 = 28$ and $\sum_{k=1}^{10} a_k(a_k + 1) = 16$, find the value of $\sum_{k=1}^{10} (a_k)^2$.

① 4
② 6
③ 12
④ 14 [2 marks]

A35. Let a_k be an arithmetic sequence of positive numbers, in which the difference between consecutive terms is equal to the first term a_1.

Given that $\sum_{k=1}^{15} \frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} = 2$, find the value of a_4.

① 6
② 7
③ 8
④ 9 [2 marks]

A36. If $\tan\theta < 0$ and $\cos\left(\frac{\pi}{2} + \theta\right) = \frac{1}{\sqrt{a}}$ for some $a > 0$, find the value of $\cos\theta$.

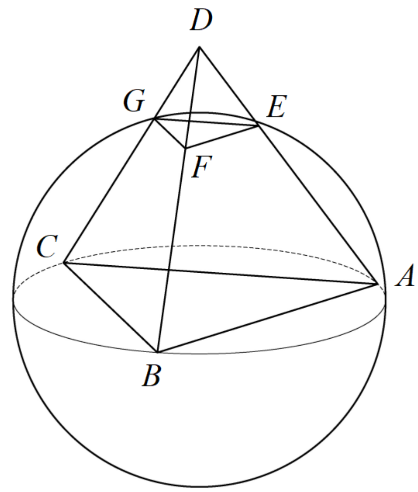
① $\sqrt{\frac{a-1}{a}}$

② $\sqrt{\frac{a+1}{a}}$

③ $-\sqrt{\frac{a+1}{a}}$

④ $-\sqrt{\frac{a-1}{a}}$ [1 mark]

A37. $ABCD$ is a regular tetrahedron. Points A, B and C lie on the surface of a sphere such that the plane containing $\triangle ABC$ exactly divides the sphere into two equal halves. The edges AD, BD and CD intersect the sphere at E, F and G respectively.



What is the value of $\dfrac{\text{area of } \triangle EFG}{\text{area of } \triangle ABC}$?

① $\dfrac{1}{8}$

② $\dfrac{1}{9}$

③ $\dfrac{1}{12}$

④ $\dfrac{1}{16}$ [4 marks]

A36. Let the functions $f(x)$ and $g(t)$ be defined as

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 + f(x-1) & \text{for } x \ge 1. \end{cases} \qquad \text{and} \qquad g(t) = \int_0^t e^{x-2f(x)}\,\mathrm{d}x.$$

Find the equation of the horizontal asymptote of $y = g(t)$. [4 marks]

A37. Find the value(s) of k such that the graph of $y = \frac{x-3}{1-3x}$ can be transformed into the graph of $xy = k$ using only a translation.

① -1/3 or 3
② -8/3
③ -1/3 or 1/3
④ 8/9 [2 marks]

A38. The solutions to the equation $\tan \frac{1}{x} = \cot x$ satisfy, for any integer n,

① $x^2 + \frac{n\pi}{2}x - 1 = 0$
② $x^2 + n\pi x + 1 = 0$
③ $x^2 + 2n\pi x - 1 = 0$
④ $x^2 + \frac{(2n+1)\pi}{2}x + 1 = 0$ [2 marks]

A39. If A and B are independent events with $P(A) = \frac{2}{3}$ and $P(A \cup B) = \frac{5}{6}$, find $P(B)$.

① $\frac{5}{12}$
② $\frac{1}{2}$
③ $\frac{7}{12}$
④ $\frac{2}{3}$ [2 marks]

A40. The terms of the integer sequence $\{a_n\}$ satisfy for $n \geq 1$,

$$a_{n+1} = \begin{cases} a_n - 1, & \text{if } a_n \text{ is even} \\ a_n + n, & \text{if } a_n \text{ is odd} \end{cases}$$

If $a_4 = 12$, what is/are the possible value(s) of a_1?

① 6 only
② 6 or 9
③ 8 or 9
④ 9 only [2 marks]

A41. Factorise the expression $(x^2 - 2x + 5)(x^2 - 2x + 3) - 3$.

① $(x - 1)^2(x - 4)(x - 3)$
② $(x^2 - x + 1)(x + 2)(x + 6)$
③ $(x - 2)(x - 1)(x^2 + 6)$
④ $(x^2 - 2x + 2)(x^2 - 2x + 6)$ [1 mark]

A42. If $(x^2 + x)\, dy = \dfrac{dx}{y}$ then the value of e^{y^2} for all $(x, y) > 0$ is

① proportional to the square of $\dfrac{x}{1 + x}$

② equal to $(x + \dfrac{1}{x})^2$ plus a constant

③ proportional to $\dfrac{y}{\sqrt{x}}$

④ inversely proportional to xy [2 marks]

A43. The position of a particle relative to the origin is $\mathbf{r}(t) = \sqrt{3}\sin t\,\mathbf{i} + (2\cos t - 5)\,\mathbf{j}$.

At time $t = \alpha$ for some $0 < \alpha < \pi$, the particle is moving parallel to its displacement.

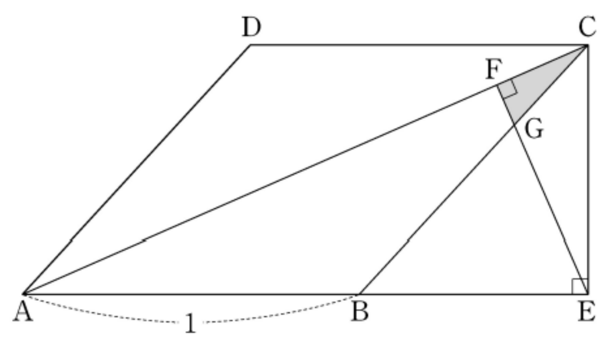
Find the value of $10\cos\alpha$.

① 1
② 2
③ 3
④ 4 [2 marks]

***A43.** If $f(x) = \int_0^x \frac{1}{1+e^{-t}}\,dt$ for all $x > 0$ and $(f \circ f)(a) = \ln 5$, what is the value of e^a?

① 11
② 13
③ 15
④ 17 [4 marks]

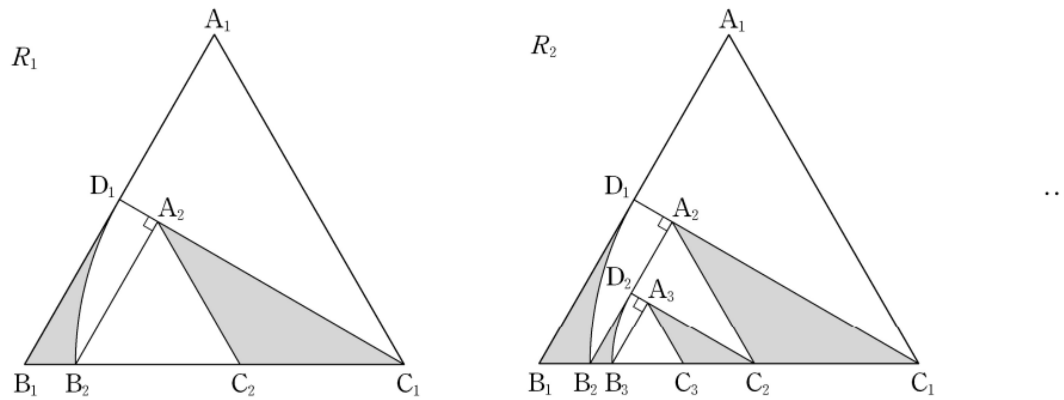
A44. $ABCD$ is a rhombus with $|AB| = 1$. Extend edge AB and let E be the point on this line such that $AE \perp CE$. Let F be the point on AC such that $CF \perp EF$ and let G be the intersection of line segments EF and BC.



If $\angle DAB = \theta$ is a small positive acute angle (in radians), then the area of the shaded region $\triangle CFG$ is approximately equal to $k\theta^5$, where the value of k is

① 1 / 8
② 1 / 12
③ 1 / 16
④ 1 / 20 [2 marks]

***A45.** A sequence of geometric figures $\{R_n\}$ is generated as shown in the diagram. Each figure R_n shows an equilateral triangle $\triangle A_nB_nC_n$ with a point D_n at the midpoint of edge A_nB_n. Then, a circular arc is constructed with centre C_n which passes through D_n, intersecting edge B_nC_n at a point B_{n+1}. Point A_{n+1} lies on line C_nD_n such that $A_{n+1}B_{n+1} \perp C_nD_n$. Point C_{n+1} lies on edge B_nC_n such that $\triangle A_{n+1}B_{n+1}C_{n+1}$ forms the next equilateral triangle in the sequence, and this figure is denoted R_{n+1}.



The regions in figure R_n enclosed by $B_nD_nB_{n+1}$ (with D_nB_{n+1} as the circular arc) and $C_nA_{n+1}C_{n+1}$ are shaded, and the total area of these regions is denoted S_n.

If $\triangle A_1B_1C_1$ has side length 1, what is the limiting value of S_n as $n \to \infty$?

① $\dfrac{11\sqrt{3} - 4\pi}{64}$

② $\dfrac{11\sqrt{3} - 4\pi}{52}$

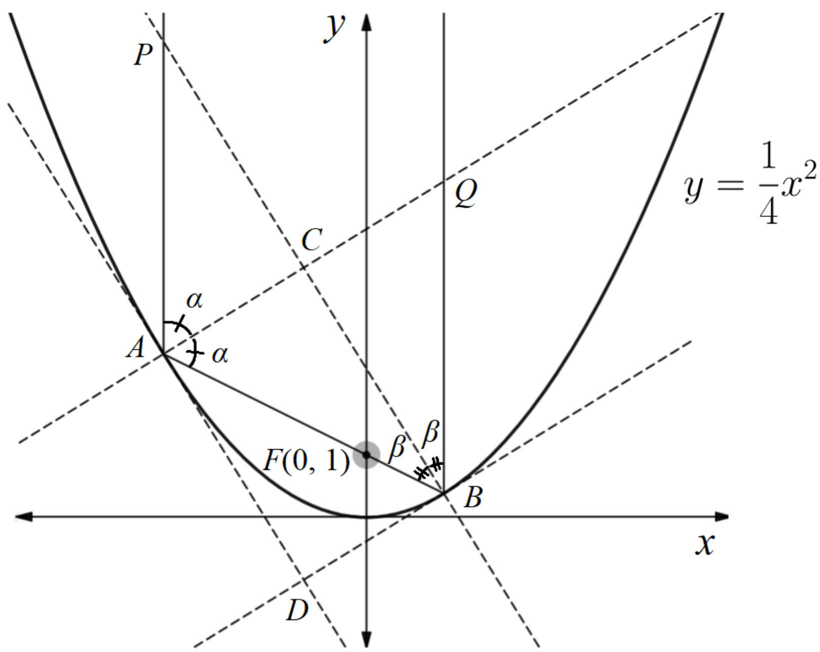
③ $\dfrac{15\sqrt{3} - 6\pi}{64}$

④ $\dfrac{15\sqrt{3} - 6\pi}{52}$

[4 marks]

A46. The diagram shows a parabola with equation $y = \frac{1}{4}x^2$. Consider the set of straight lines passing through the point $F(0, 1)$, any of which has variable gradient m, and intersects the parabola at distinct points A and B.

The normal lines and tangent lines (shown as dashed lines) to the parabola at points A and B intersect each once at points C and D, respectively. Construct rays projecting along AP and BQ such that lines AC and BC each bisect angles $\angle PAB$ and $\angle ABQ$ into half-angles α and β, respectively, where P is the intersection of the ray from A with BC, and Q is the intersection of the ray from B with AC.



The value of $\dfrac{\sin^2\alpha - \sin^2\beta}{\cos^2\alpha - \cos^2\beta}$, as m varies,

① is proportional to $\cot^{-1} m$ for $m > 0$, and proportional to $\tan^{-1} m$ for $m < 0$

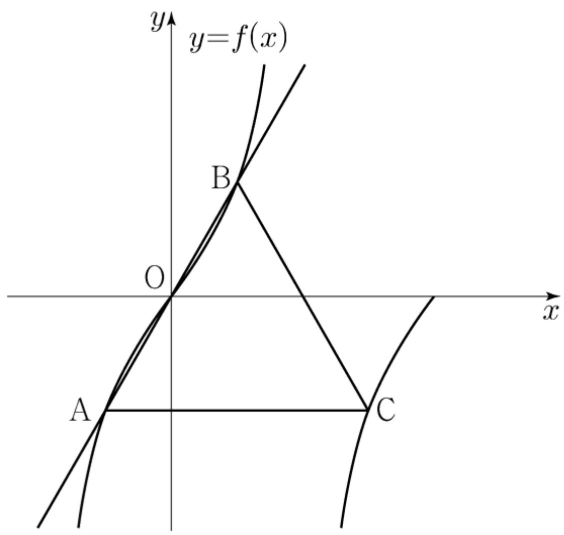
② attains its maximum value of $\frac{\sqrt{2}}{6}$ when $m = 0$

③ attains its minimum value of -1 when $m = \pm\frac{1}{2}$, and maximum value of $\frac{\sqrt{2}}{6}$ when $m = 0$

④ is independent of m, and always equal to -1 [4 marks]

A47. The graph of $y = f(x)$ is shown, where $f(x) = \tan \frac{\pi x}{a}$ for $\left\{x : \frac{-a}{2} < x \leq a,\ x \neq \frac{a}{2}\right\}$.

The equilateral triangle $\triangle ABC$ has all three vertices lying on the curve $y = f(x)$, and the edge AC is parallel to the x-axis.



What is the area of $\triangle ABC$?

① $\frac{3\sqrt{3}}{2}$

② $\frac{17\sqrt{3}}{12}$

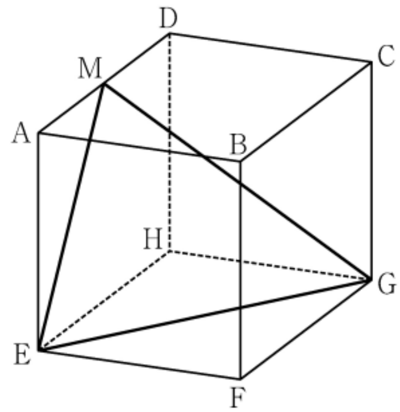
③ $\frac{4\sqrt{3}}{3}$

④ $\frac{5\sqrt{3}}{4}$ [2 marks]

A48. Find the value of $log_{10}\left[log_{(2^{10})}\left(2^{10^{100}}\right)\right]$.

① 2
② 10
③ 99
④ 1024 [2 marks]

A49. Cube $ABCD$-$EFGH$ is shown, with point M at the midpoint of edge AD.



What is the value of $\sec^2 \angle EMG$?

① 2
② 3
③ 4
④ 5 [2 marks]

A50. The line $2x - y + 1 = 0$ intersects the curve $x^2 + 3y = 19$ at points A and B.

The perpendicular bisector of line segment AB meets the x-axis at C.

What is the area of $\triangle ABC$?

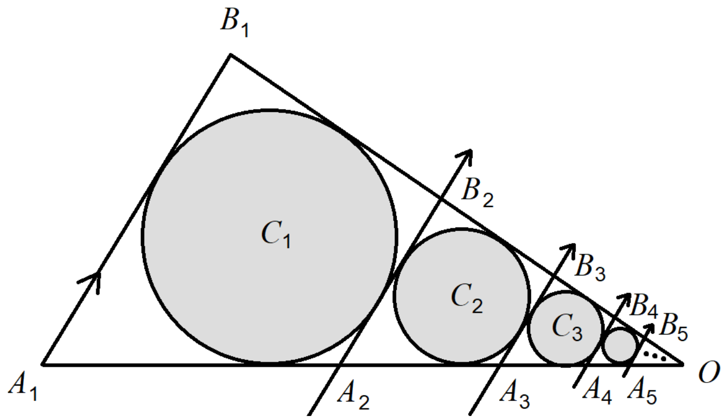
① 125
② 150
③ 196
④ 216 [4 marks]

A51. The four real roots of the polynomial $y = (x^2 + ax + 2)(x^2 + bx)$, where a and b are positive, form an arithmetic sequence.

What is the largest possible value of $a + b$?

① -6
② 0
③ 6
④ 9 [4 marks]

A52. For any positive integer n, the three edges of triangle $\triangle OA_nB_n$ contain and are tangent to a circle C_n. Points A_{n+1} and B_{n+1} are then chosen on edges A_nO and B_nO such that $A_{n+1}B_{n+1} \parallel A_nB_n$ and $A_{n+1}B_{n+1}$ is tangent to C_n. This process is repeated to generate the figure below.



Which statement, if any, is true?

① If $\triangle OA_nB_n$ is equilateral, then the shaded area is $\frac{\sqrt{3}\pi}{6}$ times the area of $\triangle OA_1B_1$.

② The centres of all circles $\{C_n\}$ are collinear.

③ If $OA_n \perp OB_n$ and $|OA_n| = |OB_n|$ then $\frac{radius\ of\ C_n}{radius\ of\ C_{n+1}} = 6$.

④ None of the above. [4 marks]

A51. Let a sequence of functions $\{u_n(x)\}$ for positive integers n be defined as

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ f(f(f(f(x)))),\ \ldots$$

where $f(x) = x^3 - 6x^2 + \frac{25}{2}x - 7$ for all real x.

For all $x \in [a, b]$, the sequence $\{u_n(x)\}$ converges as $n \to \infty$.

Find the largest possible value of $b - a$.

① 2
② $\sqrt{2}$
③ $\sqrt{6} / 3$
④ $2\sqrt{6} / 3$ [4 marks]

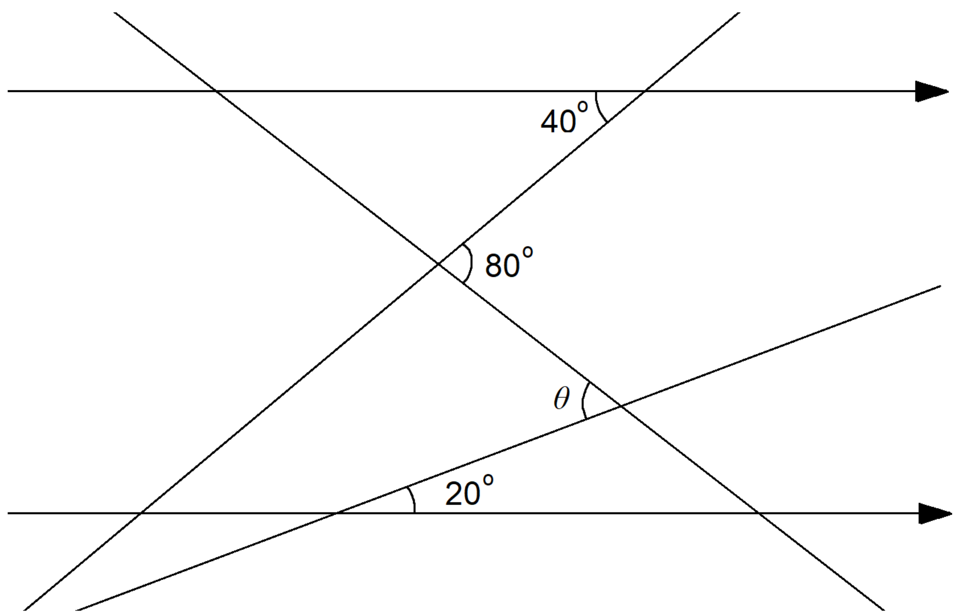
A52. For any $n \in \mathrm{N}$, let $f(n)$ be the number of distinct positive integers k such that $\frac{n}{k} \in \mathrm{N}$.

Which value of n satisfies $f(n) > f(k)$ for all integers $0 < k < n$?

① 8
② 18
③ 36
④ 90

[2 marks]

A53. The diagram shows two parallel rays and three intersecting lines.



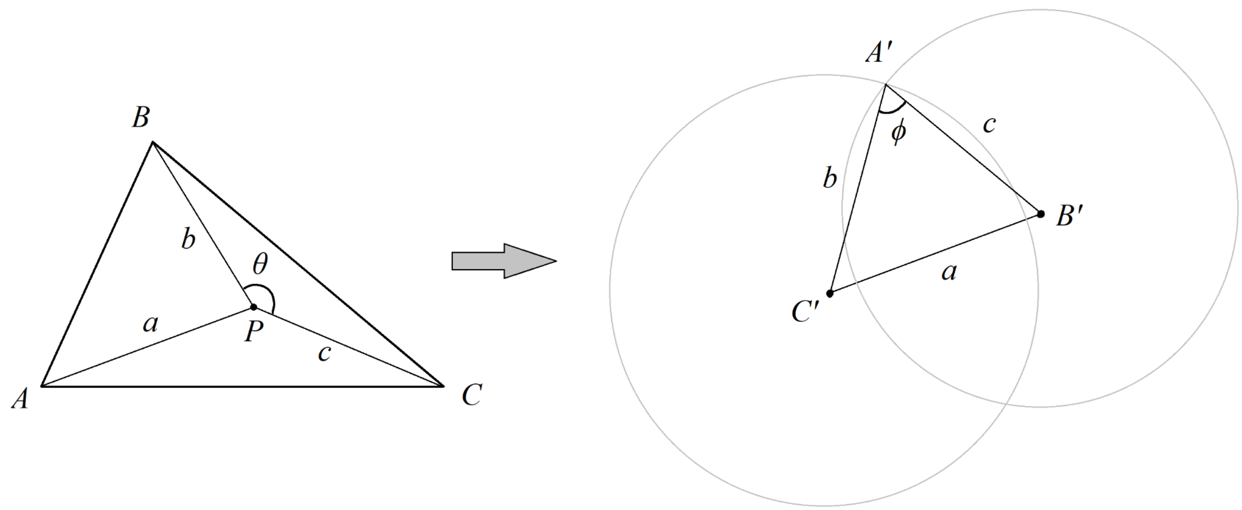
The value of θ is

① 40°
② 50°
③ 60°
④ 70°

[2 marks]

***A54.** Consider the geometric construction process below.

1. Draw any triangle, $\triangle ABC$.
2. Choose a point P within $\triangle ABC$.
3. Measure the distances between P and vertices A, B and C as a, b and c respectively.
4. Construct a single separate line segment labelled $B'C'$ with length a.
5. Using a compass, draw circles centred at C' and B' with radii b and c respectively.
6. If the circles intersect, mark either one of the intersection points as A'. If they do not intersect, return to Step 2 and choose a different point P.
7. Measure non-reflex angles $\theta = \angle BPC$ and $\phi = \angle B'A'C'$.



Which statement is true?

① ϕ is minimum when θ is maximum.

② ϕ is always 60° less than θ.

③ ϕ is always half of θ.

④ "$\triangle ABC$ is isosceles" ⇔ "$\triangle A'B'C'$ is equilateral". [4 marks]

***A55.** Let $a, b \in \mathrm{N}$ such that $a^2 + b^2$ is divisible by $ab + 1$. A valid proof that $\frac{a^2 + b^2}{ab + 1}$ is always a perfect square is given below, but some statements are missing.

Proof:

1. Let $k = \frac{a^2 + b^2}{ab + 1}$, which is a natural number. Assume that k is **not** a perfect square.
2. Let $\{a = A, b = B\}$ be a solution to the above equation such that $A + B$ is minimised. Without loss of generality, assume that $A \geq B$.
3. By substituting this solution into the equation, it is evident that $x = A$ is one root of the quadratic polynomial _______①_______ .
4. Let the other root to this quadratic be $x = A_1$. By considering the factored form of this equation, it can be seen that $A + A_1 = kB$ and $AA_1 =$ _______②_______ .
5. Therefore, $A_1 = kB - A = \frac{②}{A}$ which is a non-zero integer.
6. Substituting back in, we see that $\frac{A_1 + B^2}{A_1 B + 1} = k > 0 \Rightarrow A_1 B + 1 > 0 \Rightarrow A_1 > 0.$
7. Therefore, A_1 is a natural number and $\{a = A_1, b = B\}$ is a valid solution to the equation.
8. From our assumption that $A \geq B$, it can be seen that
$$\frac{B^2 - k}{A} \leq \frac{A^2 - k}{A} \Rightarrow \frac{B^2 - k}{A} < A \Rightarrow \text{\_\_\_\_\_\_\_③\_\_\_\_\_\_\_} .$$
9. However, this means that $A + B > A_1 + B$. This violates our condition that _______④_______ .
10. This is a contradiction.
    Our initial assumption must be false: thus, $k = \frac{a^2 + b^2}{ab + 1}$ must be a perfect square.

Which statement correctly completes one of the indicated gaps in the proof?

① $x^2 + kB\,x - B^2 + k$

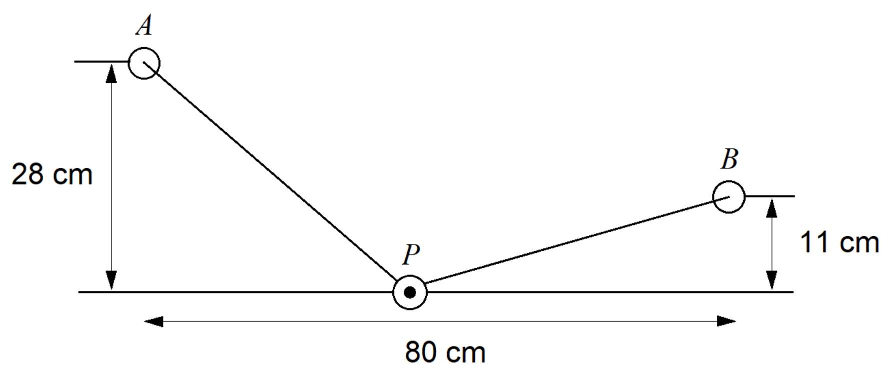
② $k - B^2$

③ $A_1 > A$

④ $A \geq B$

[4 marks]

A56. A mechanical system is shown below. A taut cable runs from pulley A to B through a third pulley P located at some point on the horizontal ground. Pulleys A and B are located 28 cm and 11 cm above ground respectively, and are separated by a fixed horizontal distance of 80 cm.



By moving P along the ground, what is the **shortest** possible length of cable that can be used?

① 82 cm
② 89 cm
③ 91 cm
④ 109 cm [1 mark]

A57. The line $2y - x = 1$ is reflected in the line $2x - 3y = 2$ to create the image line L'.

An equation for L' is

① $17x - 20y = 39$
② $19x - 22y = 45$
③ $8x - 9y = 20$
④ $5x - 6y = 11$ [2 marks]

A58. Find the minimum value of $f(x) = \ln x - \ln(\ln x)$.

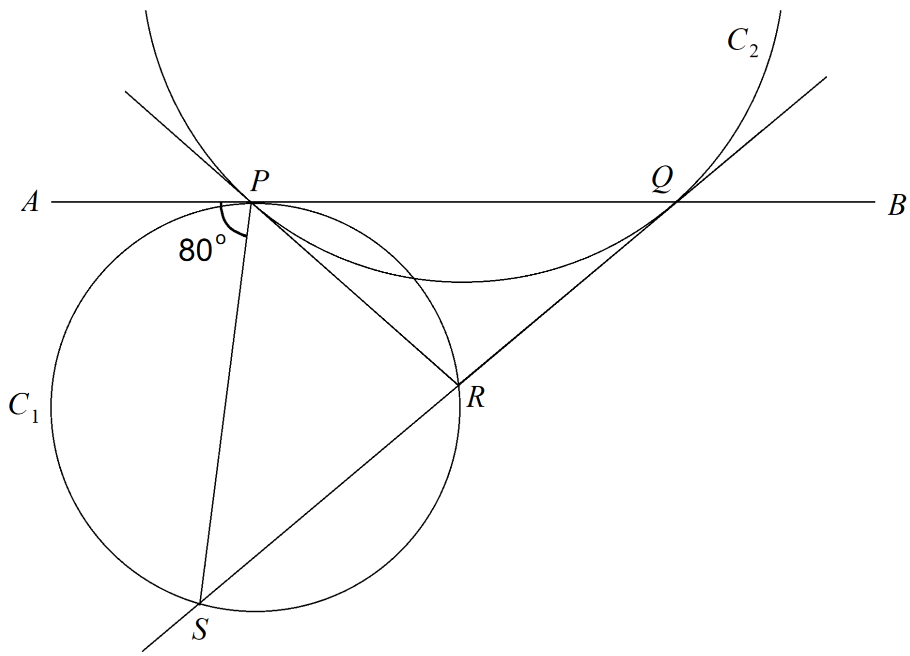
① 1
② e
③ e^e
④ $1/e$ [1 mark]

A59. The graph of $y = \dfrac{a \times 6^{x+1} - 5^x}{6^x + 5^x}$ has a horizontal asymptote at $y = 4$.

The value of a is

① $1/2$
② $1/3$
③ $2/3$
④ $4/3$ [2 marks]

A60. A line AB is tangent to a circle C_1 at point P. Points R and S lie on C_1 such that acute angle $\angle APS = 80°$. An arc of another circle C_2 is drawn such that C_2 is tangent to both PR and QR where Q is the other intersection of C_2 with AB.



Which statements are true?

I $|RS| = |QR|$

II $\angle PSR = 35°$

III $|PS| = |PQ|$

IV $\triangle PQR$ and $\triangle PQS$ are similar triangles

① **I** and **II**

② **II** and **III**

③ **I** and **IV**

④ **III** and **IV** [2 marks]

***B4.** Let $f(x) = \cos(\ln x)$ for all positive real x.

The infinitely many distinct roots of $f(x)$ are denoted α_i, arranged in ascending order so that $\alpha_{i+1} > \alpha_i$ for all integers i.

Define a sequence $u_i = \int_{\alpha_i}^{\alpha_{i+1}} f(x)\, dx$ for all integers i.

*a. Prove that u_i is a geometric sequence and calculate its common ratio.

[10 marks]

*b. Given that $\alpha_0 = e^{-\pi/2}$, prove that $\sum_{i=-\infty}^{\infty} \left| \frac{i}{u_{|i|}} \right| = \frac{4}{\left(e^{\pi} - e^{-\pi}\right)\left(e^{\pi/2} - e^{-\pi/2}\right)}$.

[10 marks]

G4. An insurance company launched an automobile insurance business and 40,000 people bought this automobile insurance in 2016. It turns out that out of the 40,000 policy holders, 5% have one claim, 0.5% have two claims, and 94.5% have no claim.

a. A complete dataset obtained indicates the 2400 claims (the number of dollars paid on an automobile insurance claim) made in 2016. The data is summarised as follows:

$$n = 2400 \qquad \sum x = 2,489,313 \qquad \sum x^2 = 8,600,048,675$$

i) Show that the mean payout in 2016 was close to $62.23 per policyholder.
[2 marks]

ii) Calculate the standard deviation of payouts in 2016 per policyholder.
[3 marks]

b. The insurance company is considering to re-launch this automobile insurance business in 2018. Based on the economic situation and the company reputation established in 2016, the company forecasts that there are at least 500,000 people who will buy its automobile insurance product, each claim will follow the same distribution as 2016.

i) Explain how and why the central limit theorem can be applied to estimate the distribution of the expected payouts. [2 marks]

ii) Calculate the projected variance of the payouts in 2018 per policyholder. [4 marks]

c. Let P be the premium charged to each policyholder for this automobile insurance over the year of 2018. Suppose that the insurer would like to be 96% confident that total premiums exceed total claims.

If the one-year continuously compounded interest rate on the insurance premium is 5%, calculate P. [5 marks]

*d. The projection of 500,000 new buyers in 2018 is not certain. The actual number of buyers has a Normal distribution with mean 500,000 and standard deviation σ.

Describe how the value of P in the answer to part c) varies with σ. [4 marks]

G5. A test has been devised for assessing whether the level of a drug, X, in the bloodstream is above a particular threshold. The mechanism associated with the test, results in a Normally distributed noise (uncertainty) being added to the true value, to give a reading, Y. The uncertainty is known to have zero mean and variance σ^2.

a. If the threshold for the drug is τ, and the true level is x, show that the probability that the measured level is above τ, is given by

$$P(Y > \tau) = \Phi\left(\frac{x-\tau}{\sigma}\right)$$

where Φ is the cumulative distribution function of the standard Normal distribution.

[5 marks]

b. The threshold is set at 0.8. If the true value is 0.7 and the value of the variance of the uncertainty is 0.1, calculate the probability that the test incorrectly classifies the level as being above the threshold. [2 marks]

c. To improve the accuracy of the result, the test is repeated five times. The absolute uncertainty for each test is independent of the uncertainty for any other test.

i) If the measured values from the five tests are averaged, calculate the probability of an incorrect classification. Use the same values as part b).

[3 marks]

ii) Instead of averaging the values, the most frequent classification results from the five tests is used as the final classification result. Calculate the probability of the final classification being incorrect. Use the same values as in part b).

[5 marks]

iii) Comment on the performance of the two approaches for combining the tests. How do you expect the difference in performance to change as the number of tests is decreased? [5 marks]

Question Sources

Sampled from A-Level past papers (AQA/EdExcel/OCR), as well as:

A4	SAT (American High School Exam)
A5, A7, A8, D1a	수학 B 짝수형 (Korean SAT Maths, Track B)
A10, A13	ENGAA (Cambridge Engineering)
A11, A14, A15, G2	Cambridge Engineering Part IA and IB Tripos and Examples Paper
A17, C1b, c	高考 (Chinese SAT)
B3	CUED (Cambridge Engineering) Preparatory Problems
F1	PAT (Oxford Physics)

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