CALCULUS I

Integrals

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Chapter 5 : Integrals

In this chapter we will be looking at integrals. Integrals are the third and final major topic that will be covered in this class. As with derivatives this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the last half is devoted to definite integrals. As we will see in the last half of the chapter if we don't know indefinite integrals we will not be able to do definite integrals.

Here is a quick listing of the material that is in this chapter.

Indefinite Integrals – In this section we will start off the chapter with the definition and properties of indefinite integrals. We will not be computing many indefinite integrals in this section. This section is devoted to simply defining what an indefinite integral is and to give many of the properties of the indefinite integral. Actually computing indefinite integrals will start in the next section.

Computing Indefinite Integrals – In this section we will compute some indefinite integrals. The integrals in this section will tend to be those that do not require a lot of manipulation of the function we are integrating in order to actually compute the integral. As we will see starting in the next section many integrals do require some manipulation of the function before we can actually do the integral. We will also take a quick look at an application of indefinite integrals.

Substitution Rule for Indefinite Integrals – In this section we will start using one of the more common and useful integration techniques – The Substitution Rule. With the substitution rule we will be able integrate a wider variety of functions. The integrals in this section will all require some manipulation of the function prior to integrating unlike most of the integrals from the previous section where all we really needed were the basic integration formulas.

More Substitution Rule – In this section we will continue to look at the substitution rule. The problems in this section will tend to be a little more involved than those in the previous section.

Area Problem – In this section we start off with the motivation for definite integrals and give one of the interpretations of definite integrals. We will be approximating the amount of area that lies between a function and the \(x\)-axis. As we will see in the next section this problem will lead us to the definition of the definite integral and will be one of the main interpretations of the definite integral that we'll be looking at in this material.

Definition of the Definite Integral – In this section we will formally define the definite integral, give many of its properties and discuss a couple of interpretations of the definite integral. We will also look at the first part of the Fundamental Theorem of Calculus which shows the very close relationship between derivatives and integrals.

Computing Definite Integrals – In this section we will take a look at the second part of the Fundamental Theorem of Calculus. This will show us how we compute definite integrals without using (the often very unpleasant) definition. The examples in this section can all be done with a basic knowledge of indefinite integrals and will not require the use of the substitution rule. Included in the examples in this section are computing definite integrals of piecewise and absolute value functions.

Substitution Rule for Definite Integrals – In this section we will revisit the substitution rule as it applies to definite integrals. The only real requirements to being able to do the examples in this section are being able to do the substitution rule for indefinite integrals and understanding how to compute definite integrals in general.

Section 5-1 : Indefinite Integrals

In the past two chapters we've been given a function, $f(x)$, and asking what the derivative of this function was. Starting with this section we are now going to turn things around. We now want to ask what function we differentiated to get the function $f(x)$.

Let's take a quick look at an example to get us started.

Example 1 What function did we differentiate to get the following function.

$$f(x) = x^4 + 3x - 9$$

Solution

Let's actually start by getting the derivative of this function to help us see how we're going to have to approach this problem. The derivative of this function is,

$$f'(x) = 4x^3 + 3$$

The point of this was to remind us of how differentiation works. When differentiating powers of *x* we multiply the term by the original exponent and then drop the exponent by one.

Now, let's go back and work the problem. In fact, let's just start with the first term. We got x^4 by differentiating a function and since we drop the exponent by one it looks like we must have differentiated x^5. However, if we had differentiated x^5 we would have $5x^4$ and we don't have a 5 in front our first term, so the 5 needs to cancel out after we've differentiated. It looks then like we would have to differentiate $\frac{1}{5}x^5$ in order to get x^4.

Likewise, for the second term, in order to get *3x* after differentiating we would have to differentiate $\frac{3}{2}x^2$. Again, the fraction is there to cancel out the 2 we pick up in the differentiation.

The third term is just a constant and we know that if we differentiate *x* we get 1. So, it looks like we had to differentiate -9*x* to get the last term.

Putting all of this together gives the following function,

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x$$

Our answer is easy enough to check. Simply differentiate $F(x)$.

$$F'(x) = x^4 + 3x - 9 = f(x)$$

So, it looks like we got the correct function. Or did we? We know that the derivative of a constant is zero and so any of the following will also give $f(x)$ upon differentiating.

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + 10$$
$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x - 1954$$
$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + \frac{3469}{123}$$
$$etc.$$

In fact, any function of the form,

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c, \qquad c \text{ is a constant}$$

will give $f(x)$ upon differentiating.

There were two points to this last example. The first point was to get you thinking about how to do these problems. It is important initially to remember that we are really just asking what we differentiated to get the given function.

The other point is to recognize that there are actually an infinite number of functions that we could use and they will all differ by a constant.

Now that we've worked an example let's get some of the definitions and terminology out of the way.

Definitions

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\,dx = F(x) + c, \qquad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, *x* is called the **integration variable** and the "*c*" is called the **constant of integration**.

Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the context of the problem that we are talking about an indefinite integral (or definite integral).

The process of finding the indefinite integral is called **integration** or **integrating** $f(x)$. If we need to be specific about the integration variable we will say that we are **integrating** $f(x)$ **with respect to *x***.

Let's rework the first problem in light of the new terminology.

Example 2 Evaluate the following indefinite integral.

$$\int x^4 + 3x - 9\,dx$$

Solution

Since this is really asking for the most general anti-derivative we just need to reuse the final answer from the first example.

The indefinite integral is,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

A couple of warnings are now in order. One of the more common mistakes that students make with integrals (both indefinite and definite) is to drop the *dx* at the end of the integral. This is required! Think of the integral sign and the *dx* as a set of parentheses. You already know and are probably quite comfortable with the idea that every time you open a parenthesis you must close it. With integrals, think of the integral sign as an "open parenthesis" and the *dx* as a "close parenthesis".

If you drop the *dx* it won't be clear where the integrand ends. Consider the following variations of the above example.

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

$$\int x^4 + 3x\,dx - 9 = \frac{1}{5}x^5 + \frac{3}{2}x^2 + c - 9$$

$$\int x^4\,dx + 3x - 9 = \frac{1}{5}x^5 + c + 3x - 9$$

You only integrate what is between the integral sign and the *dx*. Each of the above integrals end in a different place and so we get different answers because we integrate a different number of terms each time. In the second integral the "-9" is outside the integral and so is left alone and not integrated. Likewise, in the third integral the "$3x - 9$" is outside the integral and so is left alone.

Knowing which terms to integrate is not the only reason for writing the dx down. In the **Substitution Rule** section we will actually be working with the dx in the problem and if we aren't in the habit of writing it down it will be easy to forget about it and then we will get the wrong answer at that stage.

The moral of this is to make sure and put in the dx! At this stage it may seem like a silly thing to do, but it just needs to be there, if for no other reason than knowing where the integral stops.

On a side note, the dx notation should seem a little familiar to you. We saw things like this a couple of sections ago. We called the dx a **differential** in that section and yes that is exactly what it is. The dx that ends the integral is nothing more than a differential.

The next topic that we should discuss here is the integration variable used in the integral. Actually, there isn't really a lot to discuss here other than to note that the integration variable doesn't really matter. For instance,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$
$$\int t^4 + 3t - 9\,dt = \frac{1}{5}t^5 + \frac{3}{2}t^2 - 9t + c$$
$$\int w^4 + 3w - 9\,dw = \frac{1}{5}w^5 + \frac{3}{2}w^2 - 9w + c$$

Changing the integration variable in the integral simply changes the variable in the answer. It is important to notice however that when we change the integration variable in the integral we also changed the differential (dx, dt, or dw) to match the new variable. This is more important than we might realize at this point.

Another use of the differential at the end of integral is to tell us what variable we are integrating with respect to. At this stage that may seem unimportant since most of the integrals that we're going to be working with here will only involve a single variable. However, if you are on a degree track that will take you into multi-variable calculus this will be very important at that stage since there will be more than one variable in the problem. You need to get into the habit of writing the correct differential at the end of the integral so when it becomes important in those classes you will already be in the habit of writing it down.

To see why this is important take a look at the following two integrals.

$$\int 2x\,dx \qquad\qquad \int 2t\,dx$$

The first integral is simple enough.

$$\int 2x\,dx = x^2 + c$$

The second integral is also fairly simple, but we need to be careful. The *dx* tells us that we are integrating *x*'s. That means that we only integrate *x*'s that are in the integrand and all other variables in the integrand are considered to be constants. The second integral is then,

$$\int 2t\,dx = 2tx + c$$

So, it may seem silly to always put in the *dx*, but it is a vital bit of notation that can cause us to get the incorrect answer if we neglect to put it in.

Now, there are some important properties of integrals that we should take a look at.

Properties of the Indefinite Integral

1. $\int k\, f\left(x\right) dx = k\int f\left(x\right) dx$ where *k* is any number. So, we can factor multiplicative constants out of indefinite integrals.

   See the **Proof of Various Integral Formulas** section of the Extras chapter to see the proof of this property.

2. $\int -f\left(x\right) dx = -\int f\left(x\right) dx$. This is really the first property with $k = -1$ and so no proof of this property will be given.

3. $\int f\left(x\right) \pm g\left(x\right) dx = \int f\left(x\right) dx \pm \int g\left(x\right) dx$. In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.

   See the **Proof of Various Integral Formulas** section of the Extras chapter to see the proof of this property.

Notice that when we worked the first example above we used the first and third property in the discussion. We integrated each term individually, put any constants back in and then put everything back together with the appropriate sign.

Not listed in the properties above were integrals of products and quotients. The reason for this is simple. Just like with derivatives each of the following will NOT work.

$$\int f\left(x\right) g\left(x\right) dx \neq \int f\left(x\right) dx \int g\left(x\right) dx \qquad\qquad \int \frac{f\left(x\right)}{g\left(x\right)} dx \neq \frac{\int f\left(x\right) dx}{\int g\left(x\right) dx}$$

With derivatives we had a product rule and a quotient rule to deal with these cases. However, with integrals there are no such rules. When faced with a product and quotient in an integral we will have a variety of ways of dealing with it depending on just what the integrand is.

There is one final topic to be discussed briefly in this section. On occasion we will be given $f'\left(x\right)$ and will ask what $f\left(x\right)$ was. We can now answer this question easily with an indefinite integral.

$$f\left(x\right) = \int f'\left(x\right) dx$$

Example 3 If $f'\left(x\right) = x^4 + 3x - 9$ what was $f\left(x\right)$?

Solution
By this point in this section this is a simple question to answer.

$$f\left(x\right) = \int f'\left(x\right) dx = \int x^4 + 3x - 9\, dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

In this section we kept evaluating the same indefinite integral in all of our examples. The point of this section was not to do indefinite integrals, but instead to get us familiar with the notation and some of the basic ideas and properties of indefinite integrals. The next couple of sections are devoted to actually evaluating indefinite integrals.

Section 5-2 : Computing Indefinite Integrals

In the previous section we started looking at indefinite integrals and in that section we concentrated almost exclusively on notation, concepts and properties of the indefinite integral. In this section we need to start thinking about how we actually compute indefinite integrals. We'll start off with some of the basic indefinite integrals.

The first integral that we'll look at is the integral of a power of *x*.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

The general rule when integrating a power of *x* we add one onto the exponent and then divide by the new exponent. It is clear (hopefully) that we will need to avoid $n = -1$ in this formula. If we allow $n = -1$ in this formula we will end up with division by zero. We will take care of this case in a bit.

Next is one of the easier integrals but always seems to cause problems for people.

$$\int k \, dx = kx + c, \quad c \text{ and } k \text{ are constants}$$

If you remember that all we're asking is what did we differentiate to get the integrand this is pretty simple, but it does seem to cause problems on occasion.

Let's now take a look at the trig functions.

$$\int \sin x \, dx = -\cos x + c \qquad \int \cos x \, dx = \sin x + c$$

$$\int \sec^2 x \, dx = \tan x + c \qquad \int \sec x \tan x \, dx = \sec x + c$$

$$\int \csc^2 x \, dx = -\cot x + c \qquad \int \csc x \cot x \, dx = -\csc x + c$$

Notice that we only integrated two of the six trig functions here. The remaining four integrals are really integrals that give the remaining four trig functions. Also, be careful with signs here. It is easy to get the signs for derivatives and integrals mixed up. Again, remember that we're asking what function we differentiated to get the integrand.

We will be able to integrate the remaining four trig functions in a couple of sections, but they all require the **Substitution Rule**.

Now, let's take care of exponential and logarithm functions.

$$\int \mathbf{e}^x \, dx = \mathbf{e}^x + c \qquad \int a^x \, dx = \frac{a^x}{\ln a} + c \qquad \int \frac{1}{x} dx = \int x^{-1} \, dx = \ln|x| + c$$

Integrating logarithms requires a topic that is usually taught in Calculus II and so we won't be integrating a logarithm in this class. Also note the third integrand can be written in a couple of ways and don't forget the absolute value bars in the *x* in the answer to the third integral.

Finally, let's take care of the inverse trig and hyperbolic functions.

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + c \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \sinh x\,dx = \cosh x + c \qquad \int \cosh x\,dx = \sinh x + c$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + c \qquad \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$$

$$\int \operatorname{csch}^2 x\,dx = -\coth x + c \qquad \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + c$$

As with logarithms integrating inverse trig functions requires a topic usually taught in Calculus II and so we won't be integrating them in this class. There is also a different answer for the second integral above. Recalling that since all we are asking here is what function did we differentiate to get the integrand the second integral could also be,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1} x + c$$

Traditionally we use the first form of this integral.

Okay, now that we've got most of the basic integrals out of the way let's do some indefinite integrals. In all of these problems remember that we can always check our answer by differentiating and making sure that we get the integrand.

Example 1 Evaluate each of the following indefinite integrals.

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

(b) $\int x^8 + x^{-8}\,dx$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Solution

Okay, in all of these remember the basic rules of indefinite integrals. First, to integrate sums and differences all we really do is integrate the individual terms and then put the terms back together with the appropriate signs. Next, we can ignore any coefficients until we are done with integrating that particular term and then put the coefficient back in. Also, do not forget the "+*c*" at the end it is important and must be there.

So, let's evaluate some integrals.

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

There's not really a whole lot to do here other than use the first two formulas from the beginning of this section. Remember that when integrating powers (that aren't -1 of course) we just add one onto the exponent and then divide by the new exponent.

$$\begin{aligned}\int 5t^3 - 10t^{-6} + 4\,dt &= 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c \\ &= \frac{5}{4}t^4 + 2t^{-5} + 4t + c\end{aligned}$$

Be careful when integrating negative exponents. Remember to add one onto the exponent. One of the more common mistakes that students make when integrating negative exponents is to "add one" and end up with an exponent of "-7" instead of the correct exponent of "-5".

(b) $\int x^8 + x^{-8}\,dx$

This is here just to make sure we get the point about integrating negative exponents.

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

In this case there isn't a formula for explicitly dealing with radicals or rational expressions. However, just like with derivatives we can write all these terms so they are in the numerator and they all have an exponent. This should always be your first step when faced with this kind of integral just as it was when differentiating.

$$\begin{aligned}\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx &= \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}\,dx \\ &= 3\frac{1}{7/4}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\left(\frac{1}{1/2}\right)x^{\frac{1}{2}} + c \\ &= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + c\end{aligned}$$

When dealing with fractional exponents we usually don't "divide by the new exponent". Doing this is equivalent to multiplying by the reciprocal of the new exponent and so that is what we will usually do.

(d) $\int dy$

Don't make this one harder than it is…

$$\int dy = \int 1\,dy = y + c$$

In this case we are really just integrating a one!

(e) $\int \left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw$

We've got a product here and as we noted in the previous section there is no rule for dealing with products. However, in this case we don't need a rule. All that we need to do is multiply things out (taking care of the radicals at the same time of course) and then we will be able to integrate.

$$\begin{aligned}\int \left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw &= \int 4w - w^3 + 4w^{\frac{1}{3}} - w^{\frac{7}{3}}\,dw \\ &= 2w^2 - \frac{1}{4}w^4 + 3w^{\frac{4}{3}} - \frac{3}{10}w^{\frac{10}{3}} + c\end{aligned}$$

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

As with the previous part it's not really a problem that we don't have a rule for quotients for this integral. In this case all we need to do is break up the quotient and then integrate the individual terms.

$$\begin{aligned}\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx &= \int \frac{4x^{10}}{x^3} - \frac{2x^4}{x^3} + \frac{15x^2}{x^3}\,dx \\ &= \int 4x^7 - 2x + \frac{15}{x}\,dx \\ &= \frac{1}{2}x^8 - x^2 + 15\ln\left|x\right| + c\end{aligned}$$

Be careful to not think of the third term as *x* to a power for the purposes of integration. Using that rule on the third term will NOT work. The third term is simply a logarithm. Also, don't get excited about the 15. The 15 is just a constant and so it can be factored out of the integral. In other words, here is what we did to integrate the third term.

$$\int \frac{15}{x}\,dx = 15\int \frac{1}{x}\,dx = 15\ln\left|x\right| + c$$

Always remember that you can't integrate products and quotients in the same way that we integrate sums and differences. At this point the only way to integrate products and quotients is to multiply the product out or break up the quotient. Eventually we'll see some other products and quotients that can be dealt with in other ways. However, there will never be a single rule that will work for all products and there will never be a single rule that will work for all quotients. Every product and quotient is different and will need to be worked on a case by case basis.

The first set of examples focused almost exclusively on powers of *x* (or whatever variable we used in each example). It's time to do some examples that involve other functions.

Example 2 Evaluate each of the following integrals.

(a) $\int 3\mathbf{e}^{x} + 5\cos x - 10\sec^2 x \, dx$

(b) $\int 2\sec w \tan w + \frac{1}{6w} \, dw$

(c) $\int \frac{23}{y^2+1} + 6\csc y \cot y + \frac{9}{y} \, dy$

(d) $\int \frac{3}{\sqrt{1-x^2}} + 6\sin x + 10\sinh x \, dx$

(e) $\int \frac{7-6\sin^2\theta}{\sin^2\theta} \, d\theta$

Solution

Most of the problems in this example will simply use the formulas from the beginning of this section. More complicated problems involving most of these functions will need to wait until we reach the Substitution Rule.

(a) $\int 3\mathbf{e}^{x} + 5\cos x - 10\sec^2 x \, dx$

There isn't anything to this one other than using the formulas.

$$\int 3\mathbf{e}^{x} + 5\cos x - 10\sec^2 x \, dx = 3\mathbf{e}^{x} + 5\sin x - 10\tan x + c$$

(b) $\int 2\sec w \tan w + \frac{1}{6w} \, dw$

Let's be a little careful with this one. First break it up into two integrals and note the rewritten integrand on the second integral.

$$\begin{aligned}\int 2\sec w \tan w + \frac{1}{6w} \, dw &= \int 2\sec w \tan w \, dw + \int \frac{1}{6}\frac{1}{w} \, dw \\ &= \int 2\sec w \tan w \, dw + \frac{1}{6}\int \frac{1}{w} \, dw\end{aligned}$$

Rewriting the second integrand will help a little with the integration at this early stage. We can think of the 6 in the denominator as a 1/6 out in front of the term and then since this is a constant it can be factored out of the integral. The answer is then,

$$\int 2\sec w \tan w + \frac{1}{6w} \, dw = 2\sec w + \frac{1}{6}\ln\left|w\right| + c$$

Note that we didn't factor the 2 out of the first integral as we factored the 1/6 out of the second. In fact, we will generally not factor the 1/6 out either in later problems. It was only done here to make sure that you could follow what we were doing.

(c) $\int \frac{23}{y^2+1} + 6\csc y \cot y + \frac{9}{y} \, dy$

In this one we'll just use the formulas from above and don't get excited about the coefficients. They are just multiplicative constants and so can be ignored while we integrate each term and then once we're done integrating a given term we simply put the coefficients back in.

$$\int \frac{23}{y^2+1} + 6\csc y\cot y + \frac{9}{y}\,dy = 23\tan^{-1} y - 6\csc y + 9\ln\left|y\right| + c$$

(d) $\displaystyle\int \frac{3}{\sqrt{1-x^2}} + 6\sin x + 10\sinh x\,dx$

Again, there really isn't a whole lot to do with this one other than to use the appropriate formula from above while taking care of coefficients.

$$\int \frac{3}{\sqrt{1-x^2}} + 6\sin x + 10\sinh x\,dx = 3\sin^{-1} x - 6\cos x + 10\cosh x + c$$

(e) $\displaystyle\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta$

This one can be a little tricky if you aren't ready for it. As discussed previously, at this point the only way we have of dealing with quotients is to break it up.

$$\begin{aligned}\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta &= \int \frac{7}{\sin^2\theta} - 6\,d\theta \\ &= \int 7\csc^2\theta - 6\,d\theta\end{aligned}$$

Notice that upon breaking the integral up we further simplified the integrand by recalling the definition of cosecant. With this simplification we can do the integral.

$$\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta = -7\cot\theta - 6\theta + c$$

As shown in the last part of this example we can do some fairly complicated looking quotients at this point if we remember to do simplifications when we see them. In fact, this is something that you should always keep in mind. In almost any problem that we're doing here don't forget to simplify where possible. In almost every case this can only help the problem and will rarely complicate the problem.

In the next problem we're going to take a look at a product and this time we're not going to be able to just multiply the product out. However, if we recall the comment about simplifying a little this problem becomes fairly simple.

Example 3 Integrate $\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt$.

Solution
There are several ways to do this integral and most of them require the next section. However, there is a way to do this integral using only the material from this section. All that is required is to remember the trig formula that we can use to simplify the integrand up a little. Recall the following double angle formula.

$$\sin\left(2t\right) = 2\sin t\cos t$$

A small rewrite of this formula gives,

$$\sin t\cos t = \tfrac{1}{2}\sin\left(2t\right)$$

If we now replace all the *t*'s with $\frac{t}{2}$ we get,

$$\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = \frac{1}{2}\sin\left(t\right)$$

Using this formula, the integral becomes something we can do.

$$\begin{aligned}\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt &= \int \frac{1}{2}\sin\left(t\right)dt\\ &= -\frac{1}{2}\cos\left(t\right)+c\end{aligned}$$

As noted earlier there is another method for doing this integral. In fact, there are two alternate methods. To see all three check out the section on **Constant of Integration** in the Extras chapter but be aware that the other two do require the material covered in the next section.

The formula/simplification in the previous problem is a nice "trick" to remember. It can be used on occasion to greatly simplify some problems.

There is one more set of examples that we should do before moving out of this section.

Example 4 Given the following information determine the function $f\left(x\right)$.

(a) $f'\left(x\right) = 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x,\ f\left(0\right) = 15$

(b) $f''\left(x\right) = 15\sqrt{x} + 5x^3 + 6,\quad f\left(1\right) = -\frac{5}{4},\ f\left(4\right) = 404$

Solution
In both of these we will need to remember that

$$f\left(x\right) = \int f'\left(x\right)dx$$

Also note that because we are giving values of the function at specific points we are also going to be determining what the constant of integration will be in these problems.

(a) $f'(x) = 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x$, $f(0) = 15$

The first step here is integrating to determine the most general possible $f(x)$.

$$\begin{aligned} f(x) &= \int 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x \, dx \\ &= x^4 - 9x - 2\cos x + 7\mathbf{e}^x + c \end{aligned}$$

Now we have a value of the function so let's plug in $x = 0$ and determine the value of the constant of integration *c*.

$$\begin{aligned} 15 = f(0) &= 0^4 - 9(0) - 2\cos(0) + 7\mathbf{e}^0 + c \\ &= -2 + 7 + c \\ &= 5 + c \end{aligned}$$

So, from this it looks like $c = 10$. This means that the function is,

$$f(x) = x^4 - 9x - 2\cos x + 7\mathbf{e}^x + 10$$

(b) $f''(x) = 15\sqrt{x} + 5x^3 + 6$, $f(1) = -\frac{5}{4}$, $f(4) = 404$

This one is a little different form the first one. In order to get the function we will need the first derivative and we have the second derivative. We can however, use an integral to get the first derivative from the second derivative, just as we used an integral to get the function from the first derivative.

So, let's first get the most general possible first derivative by integrating the second derivative.

$$\begin{aligned} f'(x) &= \int f''(x) \, dx \\ &= \int 15x^{\frac{1}{2}} + 5x^3 + 6 \, dx \\ &= 15\left(\frac{2}{3}\right)x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \\ &= 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \end{aligned}$$

Don't forget the constant of integration!

We can now find the most general possible function by integrating the first derivative which we found above.

$$\begin{aligned} f(x) &= \int 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \, dx \\ &= 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 + cx + d \end{aligned}$$

Do not get excited about integrating the *c*. It's just a constant and we know how to integrate constants. Also, there will be no reason to think the constants of integration from the integration in each step will be the same and so we'll need to call each constant of integration something different, *d* in this case.

Now, plug in the two values of the function that we've got.

$$-\frac{5}{4} = f\left(1\right) = 4 + \frac{1}{4} + 3 + c + d = \frac{29}{4} + c + d$$

$$404 = f\left(4\right) = 4\left(32\right) + \frac{1}{4}\left(1024\right) + 3\left(16\right) + c\left(4\right) + d = 432 + 4c + d$$

This gives us a system of two equations in two unknowns that we can solve.

$$\begin{aligned} -\frac{5}{4} &= \frac{29}{4} + c + d \\ 404 &= 432 + 4c + d \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c &= -\frac{13}{2} \\ d &= -2 \end{aligned}$$

The function is then,

$$f\left(x\right) = 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 - \frac{13}{2}x - 2$$

Don't remember how to solve systems? Check out the **Solving Systems** portion of the Algebra/Trig Review.

In this section we've started the process of integration. We've seen how to do quite a few basic integrals and we also saw a quick application of integrals in the last example.

There are many new formulas in this section that we'll now have to know. However, if you think about it, they aren't really new formulas. They are really nothing more than derivative formulas that we should already know written in terms of integrals. If you remember that you should find it easier to remember the formulas in this section.

Always remember that integration is asking nothing more than what function did we differentiate to get the integrand. If you can remember that many of the basic integrals that we saw in this section and many of the integrals in the coming sections aren't too bad.

Section 5-3 : Substitution Rule for Indefinite Integrals

After the last section we now know how to do the following integrals.

$$\int \sqrt[4]{x}\, dx \qquad \int \frac{1}{t^3}\, dt \qquad \int \cos w\, dw \qquad \int \mathbf{e}^y\, dy$$

All of the integrals we've done to this point have required that we just had an *x*, or a *t*, or a *w*, *etc*. and not more complicated terms such as,

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx \qquad \int \frac{2t^3+1}{\left(t^4+2t\right)^3}\, dt$$

$$\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right) dw \qquad \int \left(8y-1\right)\mathbf{e}^{4y^2-y}\, dy$$

All of these look considerably more difficult than the first set. However, they aren't too bad once you see how to do them. Let's start with the first one.

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx$$

In this case let's notice that if we let

$$u = 6x^3+5$$

and we compute the **differential** (you remember how to compute these right?) for this we get,

$$du = 18x^2 dx$$

Now, let's go back to our integral and notice that we can eliminate every *x* that exists in the integral and write the integral completely in terms of *u* using both the definition of *u* and its differential.

$$\begin{aligned}\int 18x^2 \sqrt[4]{6x^3+5}\, dx &= \int \left(6x^3+5\right)^{\frac{1}{4}}\left(18x^2 dx\right)\\ &= \int u^{\frac{1}{4}}\, du\end{aligned}$$

In the process of doing this we've taken an integral that looked very difficult and with a quick substitution we were able to rewrite the integral into a very simple integral that we can do.

Evaluating the integral gives,

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx = \int u^{\frac{1}{4}}\, du = \frac{4}{5}u^{\frac{5}{4}} + c = \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}} + c$$

As always, we can check our answer with a quick derivative if we'd like to and don't forget to "back substitute" and get the integral back into terms of the original variable.

What we've done in the work above is called the **Substitution Rule**. Here is the substitution rule in general.

Substitution Rule

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx = \int f\left(u\right)du, \qquad \text{where, } u = g\left(x\right)$$

A natural question at this stage is how to identify the correct substitution. Unfortunately, the answer is it depends on the integral. However, there is a general rule of thumb that will work for many of the integrals that we're going to be running across.

When faced with an integral we'll ask ourselves what we know how to integrate. With the integral above we can quickly recognize that we know how to integrate

$$\int \sqrt[4]{x}\,dx$$

However, we didn't have just the root we also had stuff in front of the root and (more importantly in this case) stuff under the root. Since we can only integrate roots if there is just an *x* under the root a good first guess for the substitution is then to make *u* be the stuff under the root.

Another way to think of this is to ask yourself if you were to differentiate the integrand (we're not of course, but just for a second pretend that we were) is there a chain rule and what is the inside function for the chain rule. If there is a chain rule (for a derivative) then there is a pretty good chance that the inside function will be the substitution that will allow us to do the integral.

We will have to be careful however. There are times when using this general rule can get us in trouble or overly complicate the problem. We'll eventually see problems where there are more than one "inside function" and/or integrals that will look very similar and yet use completely different substitutions. The reality is that the only way to really learn how to do substitutions is to just work lots of problems and eventually you'll start to get a feel for how these work and you'll find it easier and easier to identify the proper substitutions.

Now, with that out of the way we should ask the following question. How, do we know if we got the correct substitution? Well, upon computing the differential and actually performing the substitution every *x* in the integral (including the *x* in the dx) must disappear in the substitution process and the only letters left should be *u*'s (including a du) and we should be left with an integral that we can do.

If there are *x*'s left over or we have an integral that cannot be evaluated then there is a pretty good chance that we chose the wrong substitution. Unfortunately, however there is at least one case (we'll be seeing an example of this in the next section) where the correct substitution will actually leave some *x*'s and we'll need to do a little more work to get it to work.

Again, it cannot be stressed enough at this point that the only way to really learn how to do substitutions is to just work lots of problems. There are lots of different kinds of problems and after working many problems you'll start to get a real feel for these problems and after you work enough of them you'll reach the point where you'll be able to do simple substitutions in your head without having to actually write anything down.

As a final note we should point out that often (in fact in almost every case) the differential will not appear exactly in the integrand as it did in the example above and sometimes we'll need to do some manipulation of the integrand and/or the differential to get all the *x*'s to disappear in the substitution.

Let's work some examples so we can get a better idea on how the substitution rule works.

Example 1 Evaluate each of the following integrals.

(a) $\displaystyle\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw$

(b) $\displaystyle\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$

(c) $\displaystyle\int x^2\left(3-10x^3\right)^4\,dx$

(d) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Solution

(a) $\displaystyle\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw$

In this case it looks like we have a cosine with an inside function and so let's use that as the substitution.

$$u = w-\ln w \qquad\qquad du = \left(1-\frac{1}{w}\right)dw$$

So, as with the first example we worked the stuff in front of the cosine appears exactly in the differential. The integral is then,

$$\begin{aligned}\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw &= \int\cos\left(u\right)du\\ &= \sin\left(u\right)+c\\ &= \sin\left(w-\ln w\right)+c\end{aligned}$$

Don't forget to go back to the original variable in the problem.

(b) $\displaystyle\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$

Again, it looks like we have an exponential function with an inside function (*i.e.* the exponent) and it looks like the substitution should be,

$$u = 4y^2-y \qquad\qquad du = \left(8y-1\right)dy$$

Now, with the exception of the 3 the stuff in front of the exponential appears exactly in the differential. Recall however that we can factor the 3 out of the integral and so it won't cause any problems. The integral is then,

$$\begin{aligned}\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy &= 3\int\mathbf{e}^{u}\,du\\ &= 3\mathbf{e}^{u}+c\\ &= 3\mathbf{e}^{4y^2-y}+c\end{aligned}$$

(c) $\displaystyle\int x^2\left(3-10x^3\right)^4\,dx$

In this case it looks like the following should be the substitution.

$$u = 3 - 10x^3 \qquad du = -30x^2\,dx$$

Okay, now we have a small problem. We've got an x^2 out in front of the parenthesis but we don't have a "-30". This is not really the problem it might appear to be at first. We will simply rewrite the differential as follows.

$$x^2\,dx = -\frac{1}{30}du$$

With this we can now substitute the $x^2\,dx$ away. In the process we will pick up a constant, but that isn't a problem since it can always be factored out of the integral.

We can now do the integral.

$$\begin{aligned}\int x^2\left(3-10x^3\right)^4 dx &= \int \left(3-10x^3\right)^4 x^2\,dx \\ &= \int u^4\left(-\frac{1}{30}\right)du \\ &= -\frac{1}{30}\left(\frac{1}{5}\right)u^5 + c \\ &= -\frac{1}{150}\left(3-10x^3\right)^5 + c\end{aligned}$$

Note that in most problems when we pick up a constant as we did in this example we will generally factor it out of the integral in the same step that we substitute it in.

(d) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

In this example don't forget to bring the root up to the numerator and change it into fractional exponent form. Upon doing this we can see that the substitution is,

$$u = 1-4x^2 \qquad du = -8x\,dx \qquad \Rightarrow \qquad xdx = -\frac{1}{8}du$$

The integral is then,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}\,dx &= \int x\left(1-4x^2\right)^{-\frac{1}{2}} dx \\ &= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du \\ &= -\frac{1}{4}u^{\frac{1}{2}} + c \\ &= -\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}} + c\end{aligned}$$

In the previous set of examples the substitution was generally pretty clear. There was exactly one term that had an "inside function" and so there wasn't really much in the way of options for the substitution.

Let's take a look at some more complicated problems to make sure we don't come to expect all substitutions are like those in the previous set of examples.

Example 2 Evaluate each of the following integrals.

(a) $\int \sin(1-x)\left(2-\cos(1-x)\right)^4\,dx$

(b) $\int \cos(3z)\sin^{10}(3z)\,dz$

(c) $\int \sec^2(4t)\left(3-\tan(4t)\right)^3\,dt$

Solution

(a) $\int \sin(1-x)\left(2-\cos(1-x)\right)^4\,dx$

In this problem there are two "inside functions". There is the $1-x$ that is inside the two trig functions and there is also the term that is raised to the 4th power.

There are two ways to proceed with this problem. The first idea that many students have is substitute the $1-x$ away. There is nothing wrong with doing this but it doesn't eliminate the problem of the term to the 4th power. That's still there and if we used this idea we would then need to do a second substitution to deal with that.

The second (and much easier) way of doing this problem is to just deal with the stuff raised to the 4th power and see what we get. The substitution in this case would be,

$$u = 2-\cos(1-x) \qquad du = -\sin(1-x)\,dx \qquad \Rightarrow \qquad \sin(1-x)\,dx = -du$$

Two things to note here. First, don't forget to correctly deal with the "-". A common mistake at this point is to lose it. Secondly, notice that the $1-x$ turns out to not really be a problem after all. Because the $1-x$ was "buried" in the substitution that we actually used it was also taken care of at the same time. The integral is then,

$$\begin{aligned}\int \sin(1-x)\left(2-\cos(1-x)\right)^4\,dx &= -\int u^4\,du \\ &= -\frac{1}{5}u^5+c \\ &= -\frac{1}{5}\left(2-\cos(1-x)\right)^5+c\end{aligned}$$

As seen in this example sometimes there will seem to be two substitutions that will need to be done however, if one of them is buried inside of another substitution then we'll only really need to do one. Recognizing this can save a lot of time in working some of these problems.

(b) $\int \cos(3z)\sin^{10}(3z)\,dz$

This one is a little tricky at first. We can see the correct substitution by recalling that,

$$\sin^{10}(3z) = \left(\sin(3z)\right)^{10}$$

Using this it looks like the correct substitution is,

$$u = \sin(3z) \qquad du = 3\cos(3z)\,dz \qquad \Rightarrow \qquad \cos(3z)\,dz = \frac{1}{3}du$$

Notice that we again had two apparent substitutions in this integral but again the 3*z* is buried in the substitution we're using and so we didn't need to worry about it.
Here is the integral.

$$\begin{aligned}\int \cos(3z)\sin^{10}(3z)\,dz &= \frac{1}{3}\int u^{10}\,du \\ &= \frac{1}{3}\left(\frac{1}{11}\right)u^{11} + c \\ &= \frac{1}{33}\sin^{11}(3z) + c\end{aligned}$$

Note that the one third in front of the integral came about from the substitution on the differential and we just factored it out to the front of the integral. This is what we will usually do with these constants.

(c) $\int \sec^2(4t)\left(3 - \tan(4t)\right)^3\,dt$

In this case we've got a $4t$, a secant squared as well as a term cubed. However, it looks like if we use the following substitution the first two issues are going to be taken care of for us.

$$u = 3 - \tan(4t) \qquad du = -4\sec^2(4t)\,dt \qquad \Rightarrow \qquad \sec^2(4t)\,dt = -\frac{1}{4}du$$

The integral is now,

$$\begin{aligned}\int \sec^2(4t)\left(3 - \tan(4t)\right)^3\,dt &= -\frac{1}{4}\int u^3\,du \\ &= -\frac{1}{16}u^4 + c \\ &= -\frac{1}{16}\left(3 - \tan(4t)\right)^4 + c\end{aligned}$$

The most important thing to remember in substitution problems is that after the substitution all the original variables need to disappear from the integral. After the substitution the only variables that should be present in the integral should be the new variable from the substitution (usually *u*). Note as well that this includes the variables in the differential!

This next set of examples, while not particularly difficult, can cause trouble if we aren't paying attention to what we're doing.

Example 3 Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{3}{5y+4}\,dy$

(b) $\displaystyle\int \frac{3y}{5y^2+4}\,dy$

(c) $\displaystyle\int \frac{3y}{\left(5y^2+4\right)^2}\,dy$

(d) $\displaystyle\int \frac{3}{5y^2+4}\,dy$

Solution

(a) $\displaystyle\int \frac{3}{5y+4}\,dy$

We haven't seen a problem quite like this one yet. Let's notice that if we take the denominator and differentiate it we get just a constant and the only thing that we have in the numerator is also a constant. This is a pretty good indication that we can use the denominator for our substitution so,

$$u = 5y+4 \qquad\qquad du = 5\,dy \qquad \Rightarrow \qquad dy = \frac{1}{5}\,du$$

The integral is now,

$$\begin{aligned}\int \frac{3}{5y+4}\,dy &= \frac{3}{5}\int \frac{1}{u}\,du\\ &= \frac{3}{5}\ln\left|u\right| + c\\ &= \frac{3}{5}\ln\left|5y+4\right| + c\end{aligned}$$

Remember that we can just factor the 3 in the numerator out of the integral and that makes the integral a little clearer in this case.

(b) $\displaystyle\int \frac{3y}{5y^2+4}\,dy$

The integral is very similar to the previous one with a couple of minor differences but notice that again if we differentiate the denominator we get something that is different from the numerator by only a multiplicative constant. Therefore, we'll again take the denominator as our substitution.

$$u = 5y^2+4 \qquad\qquad du = 10y\,dy \qquad \Rightarrow \qquad y\,dy = \frac{1}{10}\,du$$

The integral is,

$$\int \frac{3y}{5y^2+4}\,dy = \frac{3}{10}\int \frac{1}{u}\,du$$
$$= \frac{3}{10}\ln|u| + c$$
$$= \frac{3}{10}\ln\left|5y^2+4\right| + c$$

(c) $\displaystyle\int \frac{3y}{\left(5y^2+4\right)^2}\,dy$

Now, this one is almost identical to the previous part except we added a power onto the denominator. Notice however that if we ignore the power and differentiate what's left we get the same thing as the previous example so we'll use the same substitution here.

$$u = 5y^2+4 \qquad\qquad du = 10y\,dy \quad \Rightarrow \qquad y\,dy = \frac{1}{10}du$$

The integral in this case is,

$$\int \frac{3y}{\left(5y^2+4\right)^2}\,dy = \frac{3}{10}\int u^{-2}\,du$$
$$= -\frac{3}{10}u^{-1} + c$$
$$= -\frac{3}{10}\left(5y^2+4\right)^{-1} + c = -\frac{3}{10\left(5y^2+4\right)} + c$$

Be careful in this case to not turn this into a logarithm. After working problems like the first two in this set a common error is to turn every rational expression into a logarithm. If there is a power on the whole denominator then there is a good chance that it isn't a logarithm.

The idea that we used in the last three parts to determine the substitution is not a bad idea to remember. If we've got a rational expression try differentiating the denominator (ignoring any powers that are on the whole denominator) and if the result is the numerator or only differs from the numerator by a multiplicative constant then we can usually use that as our substitution.

(d) $\displaystyle\int \frac{3}{5y^2+4}\,dy$

Now, this part is completely different from the first three and yet seems similar to them as well. In this case if we differentiate the denominator we get a *y* that is not in the numerator and so we can't use the denominator as our substitution.

In fact, because we have y^2 in the denominator and no *y* in the numerator is an indication of how to work this problem. This integral is going to be an inverse tangent when we are done. The key to seeing this is to recall the following formula,

$$\int \frac{1}{1+u^2}\,du = \tan^{-1} u + c$$

We clearly don't have exactly this but we do have something that is similar. The denominator has a squared term plus a constant and the numerator is just a constant. So, with a little work and the proper substitution we should be able to get our integral into a form that will allow us to use this formula.

There is one part of this formula that is really important and that is the "1+" in the denominator. The "1+" must be there and we've got a "4+" but it is easy enough to take care of that. We'll just factor a 4 out of the denominator and at the same time we'll factor the 3 in the numerator out of the integral as well. Doing this gives,

$$\begin{aligned}\int \frac{3}{5y^2+4}\,dy &= \int \frac{3}{4\left(\frac{5y^2}{4}+1\right)}\,dy\\ &= \frac{3}{4}\int \frac{1}{\frac{5y^2}{4}+1}\,dy\\ &= \frac{3}{4}\int \frac{1}{\left(\frac{\sqrt{5}\,y}{2}\right)^2+1}\,dy\end{aligned}$$

Notice that in the last step we rewrote things a little in the denominator. This will help us to see what the substitution needs to be. In order to get this integral into the formula above we need to end up with a u^2 in the denominator. Our substitution will then need to be something that upon squaring gives us $\frac{5y^2}{4}$. With the rewrite we can see what that we'll need to use the following substitution.

$$u = \frac{\sqrt{5}y}{2} \qquad du = \frac{\sqrt{5}}{2}dy \qquad \Rightarrow \qquad dy = \frac{2}{\sqrt{5}}du$$

Don't get excited about the root in the substitution, these will show up on occasion. Upon plugging our substitution in we get,

$$\int \frac{3}{5y^2+4}\,dy = \frac{3}{4}\left(\frac{2}{\sqrt{5}}\right)\int \frac{1}{u^2+1}\,du$$

After doing the substitution, and factoring any constants out, we get exactly the integral that gives an inverse tangent and so we know that we did the correct substitution for this integral. The integral is then,

$$\int \frac{3}{5y^2+4}\,dy = \frac{3}{2\sqrt{5}}\int \frac{1}{u^2+1}\,du$$
$$= \frac{3}{2\sqrt{5}}\tan^{-1}(u)+c$$
$$= \frac{3}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}\,y}{2}\right)+c$$

In this last set of integrals we had four integrals that were similar to each other in many ways and yet all either yielded different answer using the same substitution or used a completely different substitution than one that was similar to it.

This is a fairly common occurrence and so you will need to be able to deal with these kinds of issues. There are many integrals that on the surface look very similar and yet will use a completely different substitution or will yield a completely different answer when using the same substitution.

Let's take a look at another set of examples to give us more practice in recognizing these kinds of issues. Note however that we won't be putting as much detail into these as we did with the previous examples.

Example 4 Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$

(b) $\displaystyle\int \frac{2t^3+1}{t^4+2t}\,dt$

(c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

(d) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

Solution

(a) $\displaystyle\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt$

Clearly the derivative of the denominator, ignoring the exponent, differs from the numerator only by a multiplicative constant and so the substitution is,

$$u = t^4+2t \qquad du = \left(4t^3+2\right)dt = 2\left(2t^3+1\right)dt \qquad \Rightarrow \qquad \left(2t^3+1\right)dt = \frac{1}{2}du$$

After a little manipulation of the differential we get the following integral.

$$\int \frac{2t^3+1}{\left(t^4+2t\right)^3}\,dt = \frac{1}{2}\int \frac{1}{u^3}\,du$$
$$= \frac{1}{2}\int u^{-3}\,du$$
$$= \frac{1}{2}\left(-\frac{1}{2}\right)u^{-2} + c$$
$$= -\frac{1}{4}\left(t^4+2t\right)^{-2} + c$$

(b) $\displaystyle\int \frac{2t^3+1}{t^4+2t}\,dt$

The only difference between this problem and the previous one is the denominator. In the previous problem the whole denominator is cubed and in this problem the denominator has no power on it. The same substitution will work in this problem but because we no longer have the power the problem will be different.

So, using the substitution from the previous example the integral is,

$$\int \frac{2t^3+1}{t^4+2t}\,dt = \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln\left|u\right| + c$$
$$= \frac{1}{2}\ln\left|t^4+2t\right| + c$$

So, in this case we get a logarithm from the integral.

(c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Here, if we ignore the root we can again see that the derivative of the stuff under the radical differs from the numerator by only a multiplicative constant and so we'll use that as the substitution.

$$u = 1-4x^2 \qquad du = -8x\,dx \qquad \Rightarrow \qquad x\,dx = -\frac{1}{8}\,du$$

The integral is then,

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$$
$$= -\frac{1}{8}\left(2\right)u^{\frac{1}{2}} + c$$
$$= -\frac{1}{4}\sqrt{1-4x^2} + c$$

(d) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

In this case we are missing the *x* in the numerator and so the substitution from the last part will do us no good here. This integral is another inverse trig function integral that is similar to the last part of the previous set of problems. In this case we need to following formula.

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + c$$

The integral in this problem is nearly this. The only difference is the presence of the coefficient of 4 on the x^2. With the correct substitution this can be dealt with however. To see what this substitution should be let's rewrite the integral a little. We need to figure out what we squared to get $4x^2$ and that will be our substitution.

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

With this rewrite it looks like we can use the following substitution.

$$u = 2x \qquad du = 2dx \qquad \Rightarrow \qquad dx = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\sin^{-1} u + c \\ &= \frac{1}{2}\sin^{-1}(2x) + c\end{aligned}$$

Since this document is also being presented on the web we're going to put the rest of the substitution rule examples in the next section. With all the examples in one section the section was becoming too large for web presentation.

Section 5-4 : More Substitution Rule

In order to allow these pages to be displayed on the web we've broken the substitution rule examples into two sections. The previous section contains the introduction to the substitution rule and some fairly basic examples. The examples in this section tend towards the slightly more difficult side. Also, we'll not be putting quite as much explanation into the solutions here as we did in the previous section.

In the first couple of sets of problems in this section the difficulty is not with the actual integration itself, but with the set up for the integration. Most of the integrals are fairly simple and most of the substitutions are fairly simple. The problems arise in getting the integral set up properly for the substitution(s) to be done. Once you see how these are done it's easy to see what you have to do, but the first time through these can cause problems if you aren't on the lookout for potential problems.

Example 1 Evaluate each of the following integrals.

(a) $\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt$

(b) $\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)dt$

(c) $\int x\cos\left(x^2+1\right) + \dfrac{x}{x^2+1}\,dx$

Solution

(a) $\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt$

This first integral has two terms in it and both will require the same substitution. This means that we won't have to do anything special to the integral. One of the more common "mistakes" here is to break the integral up and do a separate substitution on each part. This isn't really mistake but will definitely increase the amount of work we'll need to do. So, since both terms in the integral use the same substitution we'll just do everything as a single integral using the following substitution.

$$u = 2t \qquad du = 2dt \qquad \Rightarrow \qquad dt = \frac{1}{2}du$$

The integral is then,

$$\begin{aligned}\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt &= \frac{1}{2}\int \mathbf{e}^{u} + \sec(u)\tan(u)\,du \\ &= \frac{1}{2}\left(\mathbf{e}^{u} + \sec(u)\right) + c \\ &= \frac{1}{2}\left(\mathbf{e}^{2t} + \sec(2t)\right) + c\end{aligned}$$

Often a substitution can be used multiple times in an integral so don't get excited about that if it happens. Also note that since there was a $\frac{1}{2}$ in front of the whole integral there must also be a $\frac{1}{2}$ in front of the answer from the integral.

(b) $\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)dt$

This integral is similar to the previous one, but it might not look like it at first glance. Here is the substitution for this problem,

$$u = \cos(t) \qquad du = -\sin(t)\,dt \qquad \Rightarrow \qquad \sin(t)\,dt = -du$$

We'll plug the substitution into the problem twice (since there are two cosines) and will only work because there is a sine multiplying everything. Without that sine in front we would not be able to use this substitution.

The integral in this case is,

$$\begin{aligned}\int \sin(t)\left(4\cos^3(t)+6\cos^2(t)-8\right)dt &= -\int 4u^3+6u^2-8\,du\\ &= -\left(u^4+2u^3-8u\right)+c\\ &= -\left(\cos^4(t)+2\cos^3(t)-8\cos(t)\right)+c\end{aligned}$$

Again, be careful with the minus sign in front of the whole integral.

(c) $\displaystyle\int x\cos\left(x^2+1\right)+\frac{x}{x^2+1}\,dx$

It should be fairly clear that each term in this integral will use the same substitution, but let's rewrite things a little to make things really clear.

$$\int x\cos\left(x^2+1\right)+\frac{x}{x^2+1}\,dx=\int x\left(\cos\left(x^2+1\right)+\frac{1}{x^2+1}\right)dx$$

Since each term had an *x* in it and we'll need that for the differential we factored that out of both terms to get it into the front. This integral is now very similar to the previous one. Here's the substitution.

$$u=x^2+1 \qquad\qquad du=2x\,dx \qquad\qquad \Rightarrow \qquad\qquad x\,dx=\frac{1}{2}du$$

The integral is,

$$\begin{aligned}\int x\cos\left(x^2+1\right)+\frac{x}{x^2+1}\,dx &= \frac{1}{2}\int \cos(u)+\frac{1}{u}\,du\\ &= \frac{1}{2}\left(\sin(u)+\ln|u|\right)+c\\ &= \frac{1}{2}\left(\sin\left(x^2+1\right)+\ln\left|x^2+1\right|\right)+c\end{aligned}$$

So, as we've seen in the previous set of examples sometimes we can use the same substitution more than once in an integral and doing so will simplify the work.

Example 2 Evaluate each of the following integrals.

(a) $\int x^2 + \mathbf{e}^{1-x}\,dx$

(b) $\int x\cos\left(x^2+1\right) + \dfrac{1}{x^2+1}\,dx$

Solution

(a) $\int x^2 + \mathbf{e}^{1-x}\,dx$

In this integral the first term does not need any substitution while the second term does need a substitution. So, to deal with that we'll need to split the integral up as follows,

$$\int x^2 + \mathbf{e}^{1-x}\,dx = \int x^2\,dx + \int \mathbf{e}^{1-x}\,dx$$

The substitution for the second integral is then,

$$u = 1 - x \qquad du = -dx \qquad \Rightarrow \qquad dx = -du$$

The integral is,

$$\begin{aligned}\int x^2 + \mathbf{e}^{1-x}\,dx &= \int x^2\,dx - \int \mathbf{e}^{u}\,du \\ &= \frac{1}{3}x^3 - \mathbf{e}^{u} + c \\ &= \frac{1}{3}x^3 - \mathbf{e}^{1-x} + c\end{aligned}$$

Be careful with this kind of integral. One of the more common mistakes here is do the following "shortcut".

$$\int x^2 + \mathbf{e}^{1-x}\,dx = -\int x^2 + \mathbf{e}^{u}\,du$$

In other words, some students will try do the substitution just the second term without breaking up the integral. There are two issues with this. First, there is a "-" in front of the whole integral that shouldn't be there. It should only be on the second term because that is the term getting the substitution. Secondly, and probably more importantly, there are *x*'s in the integral and we have a *du* for the differential. We can't mix variables like this. When we do integrals all the variables in the integrand must match the variable in the differential.

(b) $\int x\cos\left(x^2+1\right) + \dfrac{1}{x^2+1}\,dx$

This integral looks very similar to Example 1c above, but it is different. In this integral we no longer have the x in the numerator of the second term and that means that the substitution we'll use for the first term will no longer work for the second term. In fact, the second term doesn't need a substitution at all since it is just an inverse tangent.

The substitution for the first term is then,

$$u = x^2 + 1 \qquad du = 2x\,dx \qquad \Rightarrow \qquad x\,dx = \frac{1}{2}du$$

Now let's do the integral. Remember to first break it up into two terms before using the substitution.

$$\begin{aligned}\int x\cos\left(x^2+1\right)+\frac{1}{x^2+1}\,dx &= \int x\cos\left(x^2+1\right)dx+\int\frac{1}{x^2+1}\,dx\\ &= \frac{1}{2}\int\cos\left(u\right)du+\int\frac{1}{x^2+1}\,dx\\ &= \frac{1}{2}\sin\left(u\right)+\tan^{-1}\left(x\right)+c\\ &= \frac{1}{2}\sin\left(x^2+1\right)+\tan^{-1}\left(x\right)+c\end{aligned}$$

In this set of examples we saw that sometimes one (or potentially more than one) term in the integrand will not require a substitution. In these cases we'll need to break up the integral into two integrals, one involving the terms that don't need a substitution and another with the term(s) that do need a substitution.

Example 3 Evaluate each of the following integrals.

(a) $\displaystyle\int \mathbf{e}^{-z}+\sec^2\left(\frac{z}{10}\right)dz$

(b) $\displaystyle\int \sin w\sqrt{1-2\cos w}+\frac{1}{7w+2}\,dw$

(c) $\displaystyle\int \frac{10x+3}{x^2+16}\,dx$

Solution

(a) $\displaystyle\int \mathbf{e}^{-z}+\sec^2\left(\frac{z}{10}\right)dz$

In this integral, unlike any integrals that we've yet done, there are two terms and each will require a different substitution. So, to do this integral we'll first need to split up the integral as follows,

$$\int \mathbf{e}^{-z}+\sec^2\left(\frac{z}{10}\right)dz=\int \mathbf{e}^{-z}dz+\int\sec^2\left(\frac{z}{10}\right)dz$$

Here are the substitutions for each integral.

$$u=-z \qquad du=-dz \qquad \Rightarrow \qquad dz=-du$$

$$v=\frac{z}{10} \qquad dv=\frac{1}{10}dz \qquad \Rightarrow \qquad dz=10dv$$

Notice that we used different letters for each substitution to avoid confusion when we go to plug back in for *u* and *v*.

Here is the integral.

$$\int \mathbf{e}^{-z} + \sec^2\left(\frac{z}{10}\right)dz = -\int \mathbf{e}^u\,du + 10\int \sec^2(v)\,dv$$
$$= -\mathbf{e}^u + 10\tan(v) + c$$
$$= -\mathbf{e}^{-z} + 10\tan\left(\frac{z}{10}\right) + c$$

(b) $\displaystyle\int \sin w\sqrt{1-2\cos w} + \frac{1}{7w+2}\,dw$

As with the last problem this integral will require two separate substitutions. Let's first break up the integral.

$$\int \sin w\sqrt{1-2\cos w} + \frac{1}{7w+2}\,dw = \int \sin w\left(1-2\cos w\right)^{\frac{1}{2}}\,dw + \int \frac{1}{7w+2}\,dw$$

Here are the substitutions for this integral.

$$u = 1-2\cos(w) \qquad du = 2\sin(w)\,dw \qquad \Rightarrow \qquad \sin(w)\,dw = \frac{1}{2}du$$

$$v = 7w+2 \qquad dv = 7dw \qquad \Rightarrow \qquad dw = \frac{1}{7}dv$$

The integral is then,

$$\int \sin w\sqrt{1-2\cos w} + \frac{1}{7w+2}\,dw = \frac{1}{2}\int u^{\frac{1}{2}}\,du + \frac{1}{7}\int \frac{1}{v}\,dv$$
$$= \frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + \frac{1}{7}\ln|v| + c$$
$$= \frac{1}{3}\left(1-2\cos w\right)^{\frac{3}{2}} + \frac{1}{7}\ln|7w+2| + c$$

(c) $\displaystyle\int \frac{10x+3}{x^2+16}\,dx$

The last problem in this set can be tricky. If there was just an *x* in the numerator we could do a quick substitution to get a natural logarithm. Likewise, if there wasn't an *x* in the numerator we would get an inverse tangent after a quick substitution.

To get this integral into a form that we can work with we will first need to break it up as follows.

$$\int \frac{10x+3}{x^2+16}\,dx = \int \frac{10x}{x^2+16}\,dx + \int \frac{3}{x^2+16}\,dx$$
$$= \int \frac{10x}{x^2+16}\,dx + \frac{1}{16}\int \frac{3}{\frac{x^2}{16}+1}\,dx$$

We now have two integrals each requiring a different substitution. The substitutions for each of the integrals above are,

$$u = x^2 + 16 \qquad du = 2x\,dx \qquad \Rightarrow \qquad x\,dx = \frac{1}{2}du$$
$$v = \frac{x}{4} \qquad dv = \frac{1}{4}dx \qquad \Rightarrow \qquad dx = 4dv$$

The integral is then,

$$\begin{aligned}\int \frac{10x+3}{x^2+16}\,dx &= 5\int \frac{1}{u}\,du + \frac{3}{4}\int \frac{1}{v^2+1}\,dv \\ &= 5\ln\left|u\right| + \frac{3}{4}\tan^{-1}\left(v\right) + c \\ &= 5\ln\left|x^2+16\right| + \frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right) + c\end{aligned}$$

We've now seen a set of integrals in which we need to do more than one substitution. In these cases we will need to break up the integral into separate integrals and do separate substitutions for each.

We now need to move onto a different set of examples that can be a little tricky. Once you've seen how to do these they aren't too bad but doing them for the first time can be difficult if you aren't ready for them.

Example 4 Evaluate each of the following integrals.

(a) $\displaystyle\int \tan x\,dx$

(b) $\displaystyle\int \sec y\,dy$

(c) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

(d) $\displaystyle\int \mathbf{e}^{t+\mathbf{e}^t}\,dt$

(e) $\displaystyle\int 2x^3\sqrt{x^2+1}\,dx$

Solution

(a) $\int \tan x\,dx$

The first question about this problem is probably why is it here? Substitution rule problems generally require more than a single function. The key to this problem is to realize that there really are two functions here. All we need to do is remember the definition of tangent and we can write the integral as,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Written in this way we can see that the following substitution will work for us,

$$u = \cos x \qquad du = -\sin x\,dx \qquad \Rightarrow \qquad \sin x\,dx = -du$$

The integral is then,

$$\begin{aligned}\int \tan x\,dx &= -\int \frac{1}{u}\,du\\ &= -\ln\left|u\right| + c\\ &= -\ln\left|\cos x\right| + c\end{aligned}$$

Now, while this is a perfectly serviceable answer that minus sign in front is liable to cause problems if we aren't careful. So, let's rewrite things a little. Recalling a **property** of logarithms we can move the minus sign in front to an exponent on the cosine and then do a little simplification.

$$\begin{aligned}\int \tan x\,dx &= -\ln\left|\cos x\right| + c\\ &= \ln\left|\cos x\right|^{-1} + c\\ &= \ln\frac{1}{\left|\cos x\right|} + c\\ &= \ln\left|\sec x\right| + c\end{aligned}$$

This is the formula that is typically given for the integral of tangent.

Note that we could integrate cotangent in a similar manner.

(b) $\int \sec y\,dy$

This problem also at first appears to not belong in the substitution rule problems. This is even more of a problem upon noticing that we can't just use the definition of the secant function to write this in a form that will allow the use of the substitution rule.

This problem is going to require a technique that isn't used terribly often at this level but is a useful technique to be aware of. Sometimes we can make an integral doable by multiplying the top and bottom by a common term. This will not always work and even when it does it is not always clear what we should multiply by but when it works it is very useful.

Here is how we'll use this idea for this problem.

$$\int \sec y\,dy = \int \frac{\sec y}{1}\frac{\left(\sec y + \tan y\right)}{\left(\sec y + \tan y\right)}\,dy$$

First, we will think of the secant as a fraction and then multiply the top and bottom of the fraction by the same term. It is probably not clear why one would want to do this here but doing this will actually allow us to use the substitution rule. To see how this will work let's simplify the integrand somewhat.

$$\int \sec y\,dy = \int \frac{\sec^2 y + \tan y \sec y}{\sec y + \tan y}\,dy$$

We can now use the following substitution.

$$u = \sec y + \tan y \qquad du = \left(\sec y \tan y + \sec^2 y\right) dy$$

The integral is then,

$$\begin{aligned}\int \sec y \, dy &= \int \frac{1}{u} du \\ &= \ln\left|u\right| + c \\ &= \ln\left|\sec y + \tan y\right| + c\end{aligned}$$

Sometimes multiplying the top and bottom of a fraction by a carefully chosen term will allow us to work a problem. It does however take some thought sometimes to determine just what the term should be.

We can use a similar process for integrating cosecant.

(c) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}} dx$

This next problem has a subtlety to it that can get us in trouble if we aren't paying attention. Because of the root in the cosine it makes some sense to use the following substitution.

$$u = x^{\frac{1}{2}} \qquad du = \frac{1}{2} x^{-\frac{1}{2}} dx$$

This is where we need to be careful. Upon rewriting the differential we get,

$$2du = \frac{1}{\sqrt{x}} dx$$

The root that is in the denominator will not become a *u* as we might have been tempted to do. Instead it will get taken care of in the differential.

The integral is,

$$\begin{aligned}\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}} dx &= 2\int \cos\left(u\right) du \\ &= 2\sin\left(u\right) + c \\ &= 2\sin\left(\sqrt{x}\right) + c\end{aligned}$$

(d) $\displaystyle\int \mathbf{e}^{t+\mathbf{e}^t} \, dt$

With this problem we need to very carefully pick our substitution. As the problem is written we might be tempted to use the following substitution,

$$u = t + \mathbf{e}^t \qquad du = \left(1 + \mathbf{e}^t\right) dt$$

However, this won't work as you can probably see. The differential doesn't show up anywhere in the integrand and we just wouldn't be able to eliminate all the *t*'s with this substitution.

In order to work this problem we will need to rewrite the integrand as follows,

$$\int \mathbf{e}^{t+\mathbf{e}^t}\,dt = \int \mathbf{e}^t \mathbf{e}^{\mathbf{e}^t}\,dt$$

We will now use the substitution,

$$u = \mathbf{e}^t \qquad\qquad du = \mathbf{e}^t dt$$

The integral is,

$$\begin{aligned}\int \mathbf{e}^{t+\mathbf{e}^t}\,dt &= \int \mathbf{e}^u\,du\\ &= \mathbf{e}^u + c\\ &= \mathbf{e}^{\mathbf{e}^t} + c\end{aligned}$$

Some substitutions can be really tricky to see and it's not unusual that you'll need to do some simplification and/or rewriting to get a substitution to work.

(e) $\int 2x^3\sqrt{x^2+1}\,dx$

This last problem in this set is different from all the other substitution problems that we've worked to this point. Given the fact that we've got more than an *x* under the root it makes sense that the substitution pretty much has to be,

$$u = x^2 + 1 \qquad\qquad du = 2x\,dx$$

At first glance it looks like this might not work for the substitution because we have an x^3 in front of the root. However, if we first rewrite $2x^3 = x^2\left(2x\right)$ we could then move the $2x$ to the end of the integral so at least the du will show up explicitly in the integral. Doing this gives the following,

$$\begin{aligned}\int 2x^3\sqrt{x^2+1}\,dx &= \int x^2\sqrt{x^2+1}\left(2x\right)dx\\ &= \int x^2 u^{\frac{1}{2}}\,du\end{aligned}$$

This is a real problem. Our integrals can't have two variables in them. Normally this would mean that we chose our substitution incorrectly. However, in this case we can rewrite the substitution as follows,

$$x^2 = u - 1$$

and now, we can eliminate the remaining *x*'s from our integral. Doing this gives,

$$\begin{aligned}\int 2x^3\sqrt{x^2+1}\,dx &= \int (u-1)u^{\frac{1}{2}}\,du\\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c\\ &= \frac{2}{5}\left(x^2+1\right)^{\frac{5}{2}} - \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + c\end{aligned}$$

Sometimes, we will need to use a substitution more than once.

This kind of problem doesn't arise all that often and when it does there will sometimes be alternate methods of doing the integral. However, it will often work out that the easiest method of doing the integral is to do what we just did here.

This final set of examples isn't too bad once you see the substitutions and that is the point with this set of problems. These all involve substitutions that we've not seen prior to this and so we need to see some of these kinds of problems.

Example 5 Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{1}{x\ln x}\,dx$

(b) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\,dt$

(c) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt$

(d) $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Solution

(a) $\displaystyle\int \frac{1}{x\ln x}\,dx$

In this case we know that we can't integrate a logarithm by itself and so it makes some sense (hopefully) that the logarithm will need to be in the substitution. Here is the substitution for this problem.

$$u = \ln x \qquad\qquad du = \frac{1}{x}\,dx$$

So, the *x* in the denominator of the integrand will get substituted away with the differential. Here is the integral for this problem.

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du$$
$$= \ln\left|u\right| + c$$
$$= \ln\left|\ln x\right| + c$$

(b) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\, dt$

Again, the substitution here may seem a little tricky. In this case the substitution is,

$$u = 1 + \mathbf{e}^{2t} \qquad du = 2\mathbf{e}^{2t}dt \qquad \Rightarrow \qquad \mathbf{e}^{2t}dt = \frac{1}{2}du$$

The integral is then,

$$\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\, dt = \frac{1}{2}\int \frac{1}{u}\, du$$
$$= \frac{1}{2}\ln\left|1+\mathbf{e}^{2t}\right| + c$$

(c) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\, dt$

In this case we can't use the same type of substitution that we used in the previous problem. In order to use the substitution in the previous example the exponential in the numerator and the denominator need to be the same and in this case they aren't.

To see the correct substitution for this problem note that,

$$\mathbf{e}^{4t} = \left(\mathbf{e}^{2t}\right)^2$$

Using this, the integral can be written as follows,

$$\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\, dt = \int \frac{\mathbf{e}^{2t}}{1+\left(\mathbf{e}^{2t}\right)^2}\, dt$$

We can now use the following substitution.

$$u = \mathbf{e}^{2t} \qquad du = 2\mathbf{e}^{2t}dt \qquad \Rightarrow \qquad \mathbf{e}^{2t}dt = \frac{1}{2}du$$

The integral is then,

$$\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt = \frac{1}{2}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\tan^{-1}\left(u\right) + c$$

$$= \frac{1}{2}\tan^{-1}\left(\mathbf{e}^{2t}\right) + c$$

(d) $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

This integral is similar to the first problem in this set. Since we don't know how to integrate inverse sine functions it seems likely that this will be our substitution. If we use this as our substitution we get,

$$u = \sin^{-1}\left(x\right) \qquad\qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$$

So, the root in the integral will get taken care of in the substitution process and this will eliminate all the *x*'s from the integral. Therefore, this was the correct substitution.

The integral is,

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int u\,du$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}\left(\sin^{-1} x\right)^2 + c$$

Over the last couple of sections we've seen a lot of substitution rule examples. There are a couple of general rules that we will need to remember when doing these problems. First, when doing a substitution remember that when the substitution is done all the *x*'s in the integral (or whatever variable is being used for that particular integral) should all be substituted away. This includes the *x* in the *dx*. After the substitution only *u*'s should be left in the integral. Also, sometimes the correct substitution is a little tricky to find and more often than not there will need to be some manipulation of the differential or integrand in order to actually do the substitution.

Also, many integrals will require us to break them up so we can do multiple substitutions so be on the lookout for those kinds of integrals/substitutions.

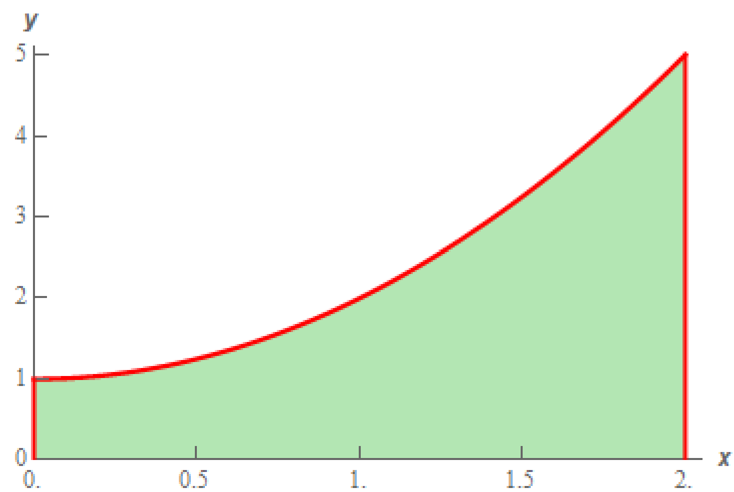
Section 5-5 : Area Problem

As noted in the first section of this section there are two kinds of integrals and to this point we've looked at indefinite integrals. It is now time to start thinking about the second kind of integral : Definite Integrals. However, before we do that we're going to take a look at the Area Problem. The area problem is to definite integrals what the tangent and rate of change problems are to derivatives.

The area problem will give us one of the interpretations of a definite integral and it will lead us to the definition of the definite integral.

To start off we are going to assume that we've got a function $f\left(x\right)$ that is positive on some interval $\left[a,b\right]$. What we want to do is determine the area of the region between the function and the *x*-axis.

It's probably easiest to see how we do this with an example. So, let's determine the area between $f\left(x\right)=x^2+1$ on $\left[0,2\right]$. In other words, we want to determine the area of the shaded region below.

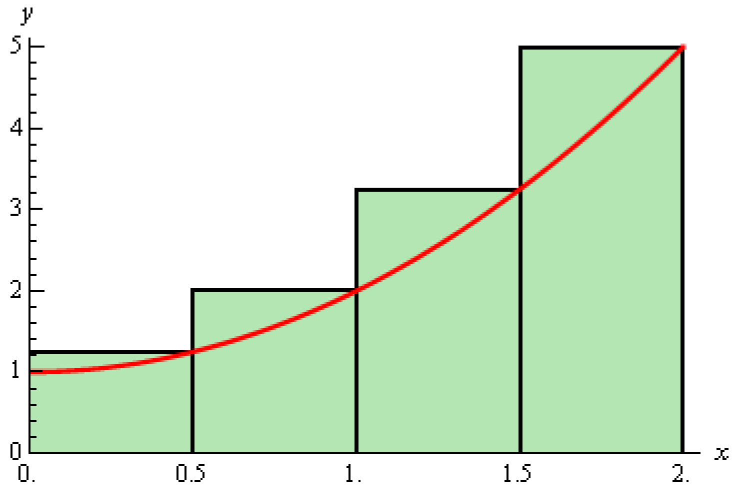


Now, at this point, we can't do this exactly. However, we can estimate the area. We will estimate the area by dividing up the interval into *n* subintervals each of width,

$$\Delta x=\frac{b-a}{n}$$

Then in each interval we can form a rectangle whose height is given by the function value at a specific point in the interval. We can then find the area of each of these rectangles, add them up and this will be an estimate of the area.

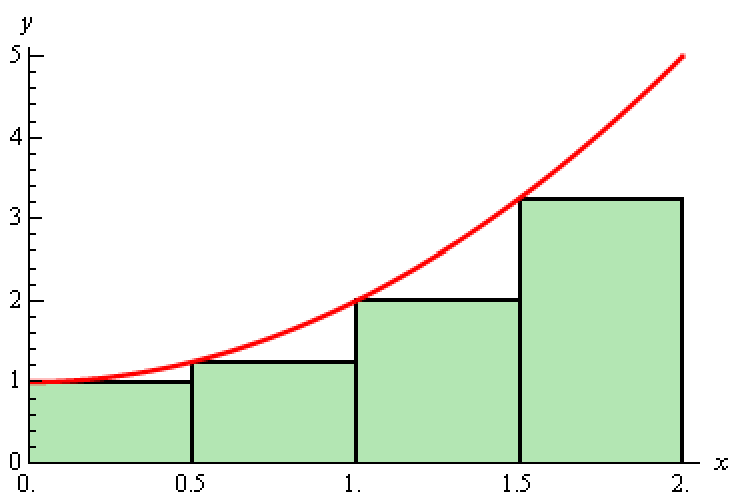
It's probably easier to see this with a sketch of the situation. So, let's divide up the interval into 4 subintervals and use the function value at the right endpoint of each interval to define the height of the rectangle. This gives,



Note that by choosing the height as we did each of the rectangles will over estimate the area since each rectangle takes in more area than the graph each time. Now let's estimate the area. First, the width of each of the rectangles is $\frac{1}{2}$. The height of each rectangle is determined by the function value at the right endpoint and so the height of each rectangle is nothing more that the function value at the right endpoint. Here is the estimated area.

$$\begin{aligned} A_r &= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2) \\ &= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) \\ &= 5.75 \end{aligned}$$

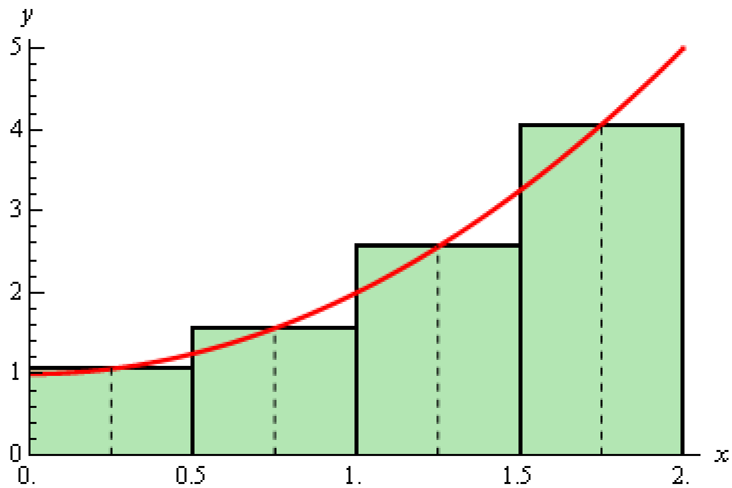
Of course, taking the rectangle heights to be the function value at the right endpoint is not our only option. We could have taken the rectangle heights to be the function value at the left endpoint. Using the left endpoints as the heights of the rectangles will give the following graph and estimated area.



$$\begin{aligned} A_l &= \frac{1}{2} f(0) + \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) \\ &= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) \\ &= 3.75 \end{aligned}$$

In this case we can see that the estimation will be an underestimation since each rectangle misses some of the area each time.

There is one more common point for getting the heights of the rectangles that is often more accurate. Instead of using the right or left endpoints of each sub interval we could take the midpoint of each subinterval as the height of each rectangle. Here is the graph for this case.



So, it looks like each rectangle will over and under estimate the area. This means that the approximation this time should be much better than the previous two choices of points. Here is the estimation for this case.

$$\begin{aligned} A_m &= \frac{1}{2} f\left(\frac{1}{4}\right) + \frac{1}{2} f\left(\frac{3}{4}\right) + \frac{1}{2} f\left(\frac{5}{4}\right) + \frac{1}{2} f\left(\frac{7}{4}\right) \\ &= \frac{1}{2}\left(\frac{17}{16}\right) + \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{2}\left(\frac{41}{16}\right) + \frac{1}{2}\left(\frac{65}{16}\right) \\ &= 4.625 \end{aligned}$$

We've now got three estimates. For comparison's sake the exact area is

$$A = \frac{14}{3} = 4.66\overline{6}$$

So, both the right and left endpoint estimation did not do all that great of a job at the estimation. The midpoint estimation however did quite well.

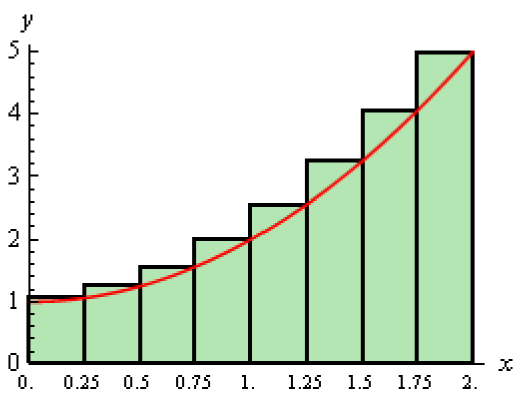
Be careful to not draw any conclusion about how choosing each of the points will affect our estimation. In this case, because we are working with an increasing function choosing the right endpoints will overestimate and choosing left endpoint will underestimate.

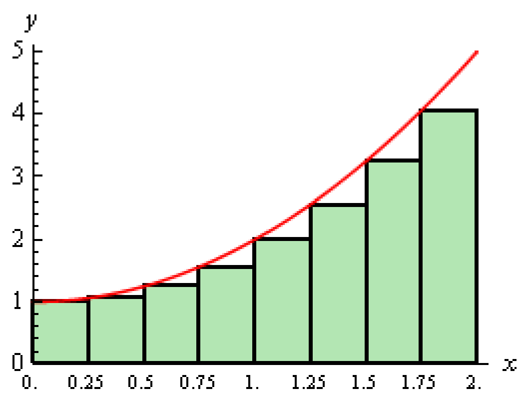
If we were to work with a decreasing function we would get the opposite results. For decreasing functions the right endpoints will underestimate and the left endpoints will overestimate.

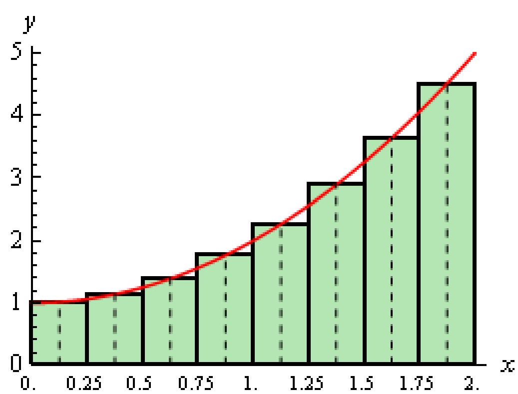
Also, if we had a function that both increased and decreased in the interval we would, in all likelihood, not even be able to determine if we would get an overestimation or underestimation.

Now, let's suppose that we want a better estimation, because none of the estimations above really did all that great of a job at estimating the area. We could try to find a different point to use for the height of each rectangle but that would be cumbersome and there wouldn't be any guarantee that the estimation would in fact be better. Also, we would like a method for getting better approximations that would work for any function we would chose to work with and if we just pick new points that may not work for other functions.

The easiest way to get a better approximation is to take more rectangles (*i.e.* increase *n*). Let's double the number of rectangles that we used and see what happens. Here are the graphs showing the eight rectangles and the estimations for each of the three choices for rectangle heights that we used above.







Here are the area estimations for each of these cases.

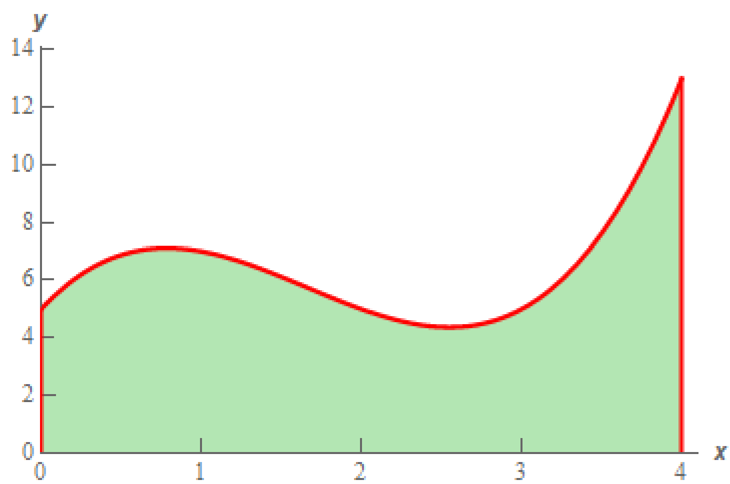
$$A_r = 5.1875 \qquad A_l = 4.1875 \qquad A_m = 4.65625$$

So, increasing the number of rectangles did improve the accuracy of the estimation as we'd guessed that it would.

Let's work a slightly more complicated example.

Example 1 Estimate the area between $f(x) = x^3 - 5x^2 + 6x + 5$ and the *x*-axis on $[0,4]$ using $n = 5$ subintervals and all three cases above for the heights of each rectangle.

Solution
First, let's get the graph to make sure that the function is positive.



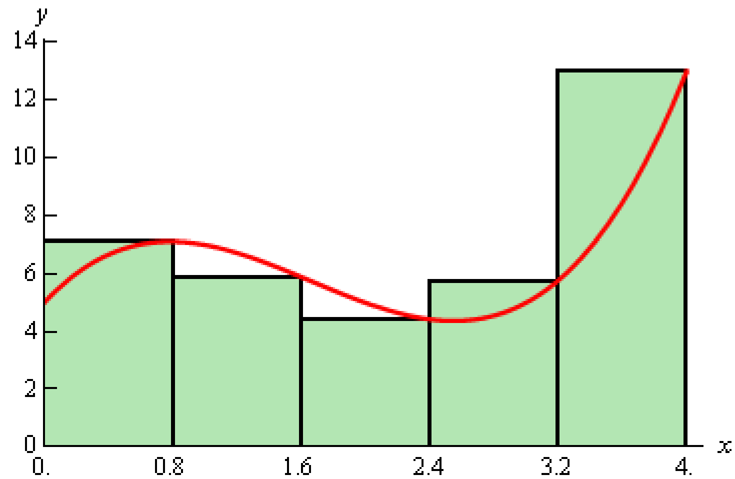
So, the graph is positive and the width of each subinterval will be,

$$\Delta x = \frac{4}{5} = 0.8$$

This means that the endpoints of the subintervals are,

$$0, \ 0.8, \ 1.6, \ 2.4, \ 3.2, \ 4$$

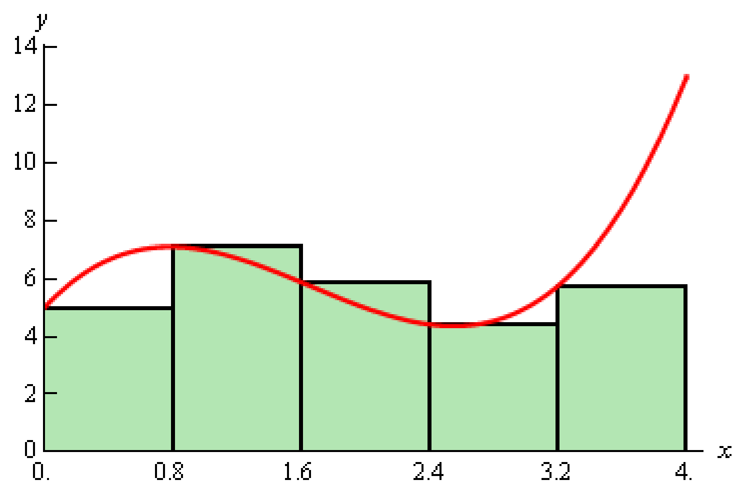
Let's first look at using the right endpoints for the function height. Here is the graph for this case.



Notice, that unlike the first area we looked at, the choosing the right endpoints here will both over and underestimate the area depending on where we are on the curve. This will often be the case with a more general curve that the one we initially looked at. The area estimation using the right endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_r &= 0.8f\left(0.8\right)+0.8f\left(1.6\right)+0.8f\left(2.4\right)+0.8f\left(3.2\right)+0.8f\left(4\right) \\ &= 28.96 \end{aligned}$$

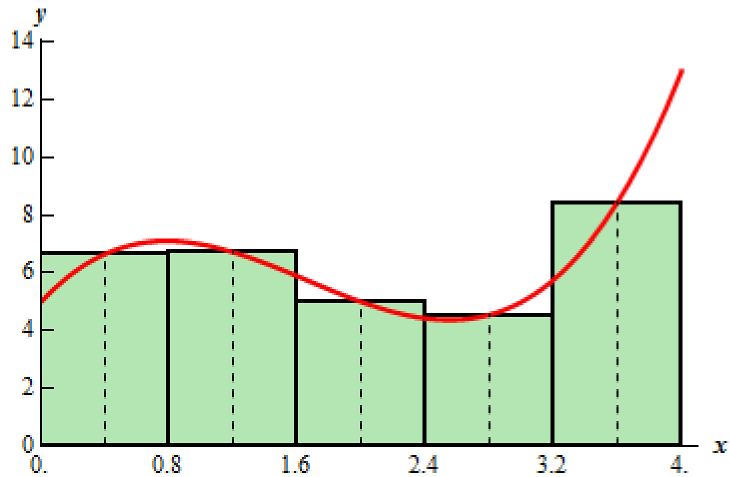
Now let's take a look at left endpoints for the function height. Here is the graph.



The area estimation using the left endpoints of each interval for the rectangle height is,

$$\begin{aligned} A_r &= 0.8f\left(0\right)+0.8f\left(0.8\right)+0.8f\left(1.6\right)+0.8f\left(2.4\right)+0.8f\left(3.2\right) \\ &= 22.56 \end{aligned}$$

Finally, let's take a look at the midpoints for the heights of each rectangle. Here is the graph,



The area estimation using the midpoint is then,

$$\begin{aligned} A_r &= 0.8f(0.4) + 0.8f(1.2) + 0.8f(2) + 0.8f(2.8) + 0.8f(3.6) \\ &= 25.12 \end{aligned}$$

For comparison purposes the exact area is,

$$A = \frac{76}{3} = 25.33\overline{3}$$

So, again the midpoint did a better job than the other two. While this will be the case more often than not, it won't always be the case and so don't expect this to always happen.

Now, let's move on to the general case. Let's start out with $f(x) \ge 0$ on $[a,b]$ and we'll divide the interval into *n* subintervals each of length,

$$\Delta x = \frac{b-a}{n}$$

Note that the subintervals don't have to be equal length, but it will make our work significantly easier. The endpoints of each subinterval are,

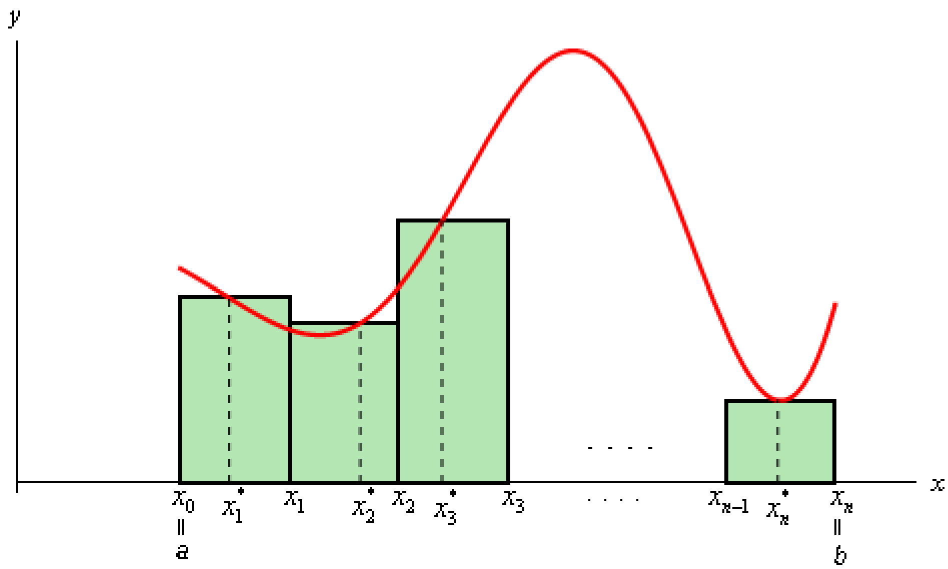
$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \\ &\vdots \\ x_{n-1} &= a + (n-1)\Delta x \\ x_n &= a + n\Delta x = b \end{aligned}$$

Next in each interval,

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$$

we choose a point $x_1^*, x_2^*, \ldots, x_i^*, \ldots x_n^*$. These points will define the height of the rectangle in each subinterval. Note as well that these points do not have to occur at the same point in each subinterval. However, they are usually the left end point of the interval, right end point of the interval or the midpoint of the interval.

Here is a sketch of this situation.



The area under the curve on the given interval is then approximately,

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_i^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

We will use **summation notation** or **sigma notation** at this point to simplify up our notation a little. If you need a refresher on summation notation check out the **section** devoted to this in the Extras chapter.

Using summation notation the area estimation is,

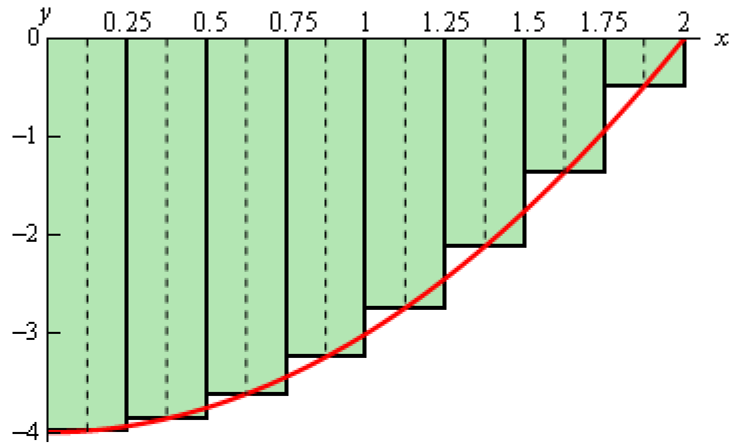
$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The summation in the above equation is called a **Riemann Sum**.

To get a better estimation we will take *n* larger and larger. In fact, if we let *n* go out to infinity we will get the exact area. In other words,

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Before leaving this section let's address one more issue. To this point we've required the function to be positive in our work. Many functions are not positive however. Consider the case of $f(x) = x^2 - 4$ on [0,2]. If we use $n = 8$ and the midpoints for the rectangle height we get the following graph,

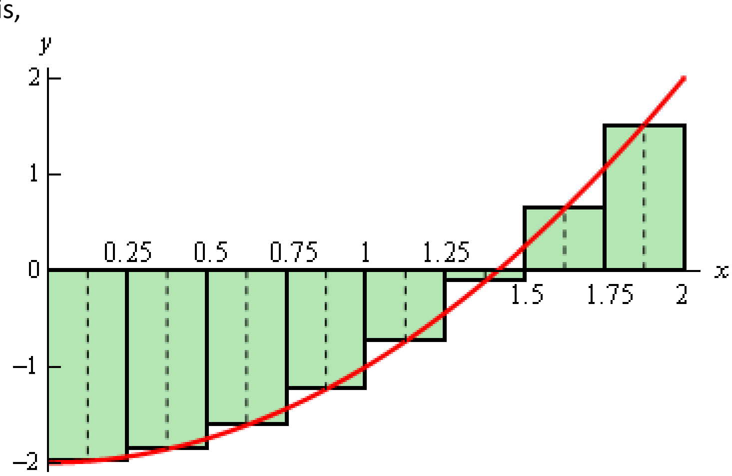


In this case let's notice that the function lies completely below the *x*-axis and hence is always negative. If we ignore the fact that the function is always negative and use the same ideas above to estimate the area between the graph and the *x*-axis we get,

$$\begin{aligned} A_m &= \frac{1}{4}f\left(\frac{1}{8}\right)+\frac{1}{4}f\left(\frac{3}{8}\right)+\frac{1}{4}f\left(\frac{5}{8}\right)+\frac{1}{4}f\left(\frac{7}{8}\right)+\frac{1}{4}f\left(\frac{9}{8}\right)+ \\ &\qquad \frac{1}{4}f\left(\frac{11}{8}\right)+\frac{1}{4}f\left(\frac{13}{8}\right)+\frac{1}{4}f\left(\frac{15}{8}\right) \\ &= -5.34375 \end{aligned}$$

Our answer is negative as we might have expected given that all the function evaluations are negative.

So, using the technique in this section it looks like if the function is above the *x*-axis we will get a positive area and if the function is below the *x*-axis we will get a negative area. Now, what about a function that is both positive and negative in the interval? For example, $f\left(x\right)=x^2-2$ on [0,2]. Using $n=8$ and midpoints the graph is,



Some of the rectangles are below the *x*-axis and so will give negative areas while some are above the *x*-axis and will give positive areas. Since more rectangles are below the *x*-axis than above it looks like we should probably get a negative area estimation for this case. In fact that is correct. Here the area estimation for this case.

$$\begin{aligned}A_m &= \frac{1}{4}f\left(\frac{1}{8}\right)+\frac{1}{4}f\left(\frac{3}{8}\right)+\frac{1}{4}f\left(\frac{5}{8}\right)\frac{1}{4}+f\left(\frac{7}{8}\right)+\frac{1}{4}f\left(\frac{9}{8}\right)+\\ &\qquad \frac{1}{4}f\left(\frac{11}{8}\right)+\frac{1}{4}f\left(\frac{13}{8}\right)+\frac{1}{4}f\left(\frac{15}{8}\right)\\ &= -1.34375\end{aligned}$$

In cases where the function is both above and below the *x*-axis the technique given in the section will give the *net* area between the function and the *x*-axis with areas below the *x*-axis negative and areas above the *x*-axis positive. So, if the net area is negative then there is more area under the *x*-axis than above while a positive net area will mean that more of the area is above the *x*-axis.

Section 5-6 : Definition of the Definite Integral

In this section we will formally define the definite integral and give many of the properties of definite integrals. Let's start off with the definition of a definite integral.

Definite Integral

Given a function $f\left(x\right)$ that is continuous on the interval $\left[a,b\right]$ we divide the interval into *n* subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of $f\left(x\right)$ from *a* to *b*** is

$$\int_a^b f\left(x\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

The definite integral is defined to be exactly the limit and summation that we looked at in the last section to find the net area between a function and the *x*-axis. Also note that the notation for the definite integral is very similar to the notation for an indefinite integral. The reason for this will be apparent eventually.

There is also a little bit of terminology that we should get out of the way here. The number "*a*" that is at the bottom of the integral sign is called the **lower limit** of the integral and the number "*b*" at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that *a* and *b* were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call *a* and *b* the **interval of integration**.

Let's work a quick example. This example will use many of the properties and facts from the brief review of **summation notation** in the Extras chapter.

Example 1 Using the definition of the definite integral compute the following.

$$\int_0^2 x^2+1\,dx$$

Solution

First, we can't actually use the definition unless we determine which points in each interval that well use for x_i^*. In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general *n* the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0,\frac{2}{n}\right],\ \left[\frac{2}{n},\frac{4}{n}\right],\ \left[\frac{4}{n},\frac{6}{n}\right],\ \ldots,\ \left[\frac{2\left(i-1\right)}{n},\frac{2i}{n}\right],\ \ldots,\ \left[\frac{2\left(n-1\right)}{n},2\right]$$

As we can see the right endpoint of the i^{th} subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)\\ &= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2+1\right)\left(\frac{2}{n}\right)\\ &= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3}+\frac{2}{n}\right)\end{aligned}$$

Now, we are going to have to take a limit of this. That means that we are going to need to "evaluate" this summation. In other words, we are going to have to use the formulas given in the **summation notation** review to eliminate the actual summation and get a formula for this for a general n.

To do this we will need to recognize that n is a constant as far as the summation notation is concerned. As we cycle through the integers from 1 to n in the summation only i changes and so anything that isn't an i will be a constant and can be factored out of the summation. In particular any n that is in the summation can be factored out if we need to.

Here is the summation "evaluation".

$$\begin{aligned}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x &= \sum_{i=1}^{n}\frac{8i^2}{n^3}+\sum_{i=1}^{n}\frac{2}{n}\\ &= \frac{8}{n^3}\sum_{i=1}^{n} i^2+\frac{1}{n}\sum_{i=1}^{n} 2\\ &= \frac{8}{n^3}\left(\frac{n\left(n+1\right)\left(2n+1\right)}{6}\right)+\frac{1}{n}\left(2n\right)\\ &= \frac{4\left(n+1\right)\left(2n+1\right)}{3n^2}+2\\ &= \frac{14n^2+12n+4}{3n^2}\end{aligned}$$

We can now compute the definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x\\ &= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2}\\ &= \frac{14}{3}\end{aligned}$$

We've seen several methods for dealing with the limit in this problem so we'll leave it to you to verify the results.

Wow, that was a lot of work for a fairly simple function. There is a much simpler way of evaluating these and we will get to it eventually. The main purpose to this section is to get the main properties and facts about the definite integral out of the way. We'll discuss how we compute these in practice starting with the next section.

So, let's start taking a look at some of the properties of the definite integral.

Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where *c* is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where *c* is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a,c]$ and $[c,b]$. Note however that *c* doesn't need to be between *a* and *b*.

6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

See the **Proof of Various Integral Properties** section of the Extras chapter for the proof of properties 1 – 4. Property 5 is not easy to prove and so is not shown there. Property 6 is not really a property in the full sense of the word. It is only here to acknowledge that as long as the function and limits are the same it doesn't matter what letter we use for the variable. The answer will be the same.

Let's do a couple of examples dealing with these properties.

Example 2 Use the results from the first example to evaluate each of the following.

(a) $\int_{2}^{0} x^2 + 1\,dx$

(b) $\int_{0}^{2} 10x^2 + 10\,dx$

(c) $\int_{0}^{2} t^2 + 1\,dt$

Solution

All of the solutions to these problems will rely on the fact we proved in the first example. Namely that,

$$\int_{0}^{2} x^2 + 1\,dx = \frac{14}{3}$$

(a) $\int_{2}^{0} x^2 + 1\,dx$

In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\begin{aligned}\int_{2}^{0} x^2 + 1\,dx &= -\int_{0}^{2} x^2 + 1\,dx \\ &= -\frac{14}{3}\end{aligned}$$

(b) $\int_{0}^{2} 10x^2 + 10\,dx$

For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\begin{aligned}\int_{0}^{2} 10x^2 + 10\,dx &= \int_{0}^{2} 10\left(x^2 + 1\right)dx \\ &= 10\int_{0}^{2} x^2 + 1\,dx \\ &= 10\left(\frac{14}{3}\right) \\ &= \frac{140}{3}\end{aligned}$$

(c) $\int_{0}^{2} t^2 + 1\,dt$

In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_{0}^{2} t^2 + 1\,dt = \int_{0}^{2} x^2 + 1\,dx = \frac{14}{3}$$

Here are a couple of examples using the other properties.

Example 3 Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2 + 1}\,dx$$

Solution

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2 + 1}\,dx = 0$$

Example 4 Given that $\int_{6}^{-10} f(x)\,dx = 23$ and $\int_{-10}^{6} g(x)\,dx = -9$ determine the value of

$$\int_{-10}^{6} 2f(x) - 10g(x)\,dx$$

Solution

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\,dx &= \int_{-10}^{6} 2f(x)\,dx - \int_{-10}^{6} 10g(x)\,dx \\ &= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx\end{aligned}$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\begin{aligned}\int_{-10}^{6} 2f(x) - 10g(x)\,dx &= -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx \\ &= -2(23) - 10(-9) \\ &= 44\end{aligned}$$

Example 5 Given that $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$, and $\int_{100}^{-5} f(x)\,dx = 4$ determine the value of $\int_{-5}^{12} f(x)\,dx$.

Solution

This example is mostly an example of property 5 although there are a couple of uses of property 1 in the solution as well.

We need to figure out how to correctly break up the integral using property 5 to allow us to use the given pieces of information. First, we'll note that there is an integral that has a "-5" in one of the limits. It's not the lower limit, but we can use property 1 to correct that eventually. The other limit is 100 so this is the number *c* that we'll use in property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{12} f(x)\,dx$$

We'll be able to get the value of the first integral, but the second still isn't in the list of know integrals. However, we do have second integral that has a limit of 100 in it. The other limit for this second integral is -10 and this will be *c* in this application of property 5.

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx$$

At this point all that we need to do is use the property 1 on the first and third integral to get the limits to match up with the known integrals. After that we can plug in for the known integrals.

$$\begin{aligned}\int_{-5}^{12} f(x)\,dx &= -\int_{100}^{-5} f(x)\,dx + \int_{100}^{-10} f(x)\,dx - \int_{12}^{-10} f(x)\,dx \\ &= -4-2-6 \\ &= -12\end{aligned}$$

There are also some nice properties that we can use in comparing the general size of definite integrals. Here they are.

More Properties

7. $\int_a^b c\,dx = c(b-a)$, *c* is any number.
8. If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$.
9. If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
10. If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.
11. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right|dx$

See the **Proof of Various Integral Properties** section of the Extras chapter for the proof of these properties.

Interpretations of Definite Integral

There are a couple of quick interpretations of the definite integral that we can give here.

First, as we alluded to in the previous section one possible interpretation of the definite integral is to give the net area between the graph of $f(x)$ and the *x*-axis on the interval $[a,b]$. So, the net area between the graph of $f(x) = x^2 + 1$ and the *x*-axis on $[0,2]$ is,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

If you look back in the last section this was the exact area that was given for the initial set of problems that we looked at in this area.

Another interpretation is sometimes called the Net Change Theorem. This interpretation says that if $f(x)$ is some quantity (so $f'(x)$ is the rate of change of $f(x)$, then,

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

is the net change in $f(x)$ on the interval $[a,b]$. In other words, compute the definite integral of a rate of change and you'll get the net change in the quantity. We can see that the value of the definite integral, $f(b)-f(a)$, does in fact give us the net change in $f(x)$ and so there really isn't anything to prove with this statement. This is really just an acknowledgment of what the definite integral of a rate of change tells us.

So as a quick example, if $V(t)$ is the volume of water in a tank then,

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the net change in the volume as we go from time t_1 to time t_2.

Likewise, if $s(t)$ is the function giving the position of some object at time *t* we know that the velocity of the object at any time *t* is : $v(t) = s'(t)$. Therefore, the displacement of the object time t_1 to time t_2 is,

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

Note that in this case if $v(t)$ is both positive and negative (*i.e.* the object moves to both the right and left) in the time frame this will NOT give the total distance traveled. It will only give the displacement, *i.e.* the difference between where the object started and where it ended up. To get the total distance traveled by an object we'd have to compute,

$$\int_{t_1}^{t_2} \left| v(t) \right| dt$$

It is important to note here that the Net Change Theorem only really makes sense if we're integrating a derivative of a function.

Fundamental Theorem of Calculus, Part I

As noted by the title above this is only the first part to the Fundamental Theorem of Calculus. We will give the second part in the next section as it is the key to easily computing definite integrals and that is the subject of the next section.

The first part of the Fundamental Theorem of Calculus tells us how to differentiate certain types of definite integrals and it also tells us about the very close relationship between integrals and derivatives.

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$ then,

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and it is differentiable on (a,b) and that,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

To see the proof of this see the **Proof of Various Integral Properties** section of the Extras chapter.

Let's check out a couple of quick examples using this.

Example 6 Differentiate each of the following.

(a) $g(x) = \int_{-4}^{x} \mathbf{e}^{2t}\cos^2(1-5t)\,dt$

(b) $\displaystyle\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt$

Solution

(a) $g(x) = \int_{-4}^{x} \mathbf{e}^{2t}\cos^2(1-5t)\,dt$

This one is nothing more than a quick application of the Fundamental Theorem of Calculus.

$$g'(x) = \mathbf{e}^{2x}\cos^2(1-5x)$$

(b) $\displaystyle\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt$

This one needs a little work before we can use the Fundamental Theorem of Calculus. The first thing to notice is that the Fundamental Theorem of Calculus requires the lower limit to be a constant and the upper limit to be the variable. So, using a property of definite integrals we can interchange the limits of the integral we just need to remember to add in a minus sign after we do that. Doing this gives,

$$\frac{d}{dx}\int_{x^2}^{1} \frac{t^4+1}{t^2+1}\,dt = \frac{d}{dx}\left(-\int_{1}^{x^2} \frac{t^4+1}{t^2+1}\,dt\right) = -\frac{d}{dx}\int_{1}^{x^2} \frac{t^4+1}{t^2+1}\,dt$$

The next thing to notice is that the Fundamental Theorem of Calculus also requires an *x* in the upper limit of integration and we've got x^2. To do this derivative we're going to need the following version of the **chain rule**.

$$\frac{d}{dx}\big(g(u)\big) = \frac{d}{du}\big(g(u)\big)\,\frac{du}{dx} \qquad \text{where } u = f(x)$$

So, if we let $u = x^2$ we use the chain rule to get,

$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}\,dt \\
&= -\frac{d}{du}\int_{1}^{u}\frac{t^4+1}{t^2+1}\,dt\ \frac{du}{dx} \qquad \text{where } u = x^2 \\
&= -\frac{u^4+1}{u^2+1}\left(2x\right) \\
&= -2x\frac{u^4+1}{u^2+1}
\end{aligned}$$

The final step is to get everything back in terms of *x*.

$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1} \\
&= -2x\frac{x^8+1}{x^4+1}
\end{aligned}$$

Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First,

$$\frac{d}{dx}\int_{a}^{u(x)} f\left(t\right)dt = u'\left(x\right)f\left(u\left(x\right)\right)$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of *x*. All we need to do here is interchange the limits on the integral (adding in a minus sign of course) and then using the formula above to get,

$$\frac{d}{dx}\int_{v(x)}^{b} f\left(t\right)dt = -\frac{d}{dx}\int_{b}^{v(x)} f\left(t\right)dt = -v'\left(x\right)f\left(v\left(x\right)\right)$$

Finally, we can also get a version for both limits being functions of *x*. In this case we'll need to use Property 5 above to break up the integral as follows,

$$\int_{v(x)}^{u(x)} f\left(t\right)dt = \int_{v(x)}^{a} f\left(t\right)dt + \int_{a}^{u(x)} f\left(t\right)dt$$

We can use pretty much any value of *a* when we break up the integral. The only thing that we need to avoid is to make sure that $f\left(a\right)$ exists. So, assuming that $f\left(a\right)$ exists after we break up the integral we can then differentiate and use the two formulas above to get,

$$\begin{aligned}\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt &= \frac{d}{dx}\left(\int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt\right)\\ &= -v'(x)\,f\left(v(x)\right) + u'(x)\,f\left(u(x)\right)\end{aligned}$$

Let's work a quick example.

Example 7 Differentiate the following integral.

$$\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)dt$$

Solution

This will use the final formula that we derived above.

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)dt &= -\frac{1}{2}x^{-\frac{1}{2}}\left(\sqrt{x}\right)^2 \sin\left(1+\left(\sqrt{x}\right)^2\right) + (3)(3x)^2 \sin\left(1+(3x)^2\right)\\ &= -\frac{1}{2}\sqrt{x}\sin\left(1+x\right) + 27x^2\sin\left(1+9x^2\right)\end{aligned}$$

Section 5-7 : Computing Definite Integrals

In this section we are going to concentrate on how we actually evaluate definite integrals in practice. To do this we will need the Fundamental Theorem of Calculus, Part II.

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ is a continuous function on $[a,b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

To see the proof of this see the **Proof of Various Integral Properties** section of the Extras chapter.

Recall that when we talk about an anti-derivative for a function we are really talking about the indefinite integral for the function. So, to evaluate a definite integral the first thing that we're going to do is evaluate the indefinite integral for the function. This should explain the similarity in the notations for the indefinite and definite integrals.

Also notice that we require the function to be continuous in the interval of integration. This was also a requirement in the definition of the definite integral. We didn't make a big deal about this in the last section. In this section however, we will need to keep this condition in mind as we do our evaluations.

Next let's address the fact that we can use any anti-derivative of $f(x)$ in the evaluation. Let's take a final look at the following integral.

$$\int_0^2 x^2 + 1\,dx$$

Both of the following are anti-derivatives of the integrand.

$$F(x) = \frac{1}{3}x^3 + x \qquad \text{and} \qquad F(x) = \frac{1}{3}x^3 + x - \frac{18}{31}$$

Using the Fundamental Theorem of Calculus to evaluate this integral with the first anti-derivatives gives,

$$\begin{aligned}\int_0^2 x^2 + 1\,dx &= \left(\frac{1}{3}x^3 + x\right)\Big|_0^2 \\ &= \frac{1}{3}(2)^3 + 2 - \left(\frac{1}{3}(0)^3 + 0\right) \\ &= \frac{14}{3}\end{aligned}$$

Much easier than **using the definition** wasn't it? Let's now use the second anti-derivative to evaluate this definite integral.

$$\begin{aligned}\int_0^2 x^2+1\,dx &= \left(\frac{1}{3}x^3+x-\frac{18}{31}\right)\Bigg|_0^2\\ &=\frac{1}{3}(2)^3+2-\frac{18}{31}-\left(\frac{1}{3}(0)^3+0-\frac{18}{31}\right)\\ &=\frac{14}{3}-\frac{18}{31}+\frac{18}{31}\\ &=\frac{14}{3}\end{aligned}$$

The constant that we tacked onto the second anti-derivative canceled in the evaluation step. So, when choosing the anti-derivative to use in the evaluation process make your life easier and don't bother with the constant as it will only end up canceling in the long run.

Also, note that we're going to have to be very careful with minus signs and parentheses with these problems. It's very easy to get in a hurry and mess them up.

Let's start our examples with the following set designed to make a couple of quick points that are very important.

Example 1 Evaluate each of the following.

(a) $\int y^2+y^{-2}\,dy$

(b) $\int_1^2 y^2+y^{-2}\,dy$

(c) $\int_{-1}^2 y^2+y^{-2}\,dy$

Solution

(a) $\int y^2+y^{-2}\,dy$

This is the only indefinite integral in this section and by now we should be getting pretty good with these so we won't spend a lot of time on this part. This is here only to make sure that we understand the difference between an indefinite and a definite integral. The integral is,

$$\int y^2+y^{-2}\,dy=\frac{1}{3}y^3-y^{-1}+c$$

(b) $\int_1^2 y^2+y^{-2}\,dy$

Recall from our first example above that all we really need here is any anti-derivative of the integrand. We just computed the most general anti-derivative in the first part so we can use that if we want to. However, recall that as we noted above any constants we tack on will just cancel in the long run and so we'll use the answer from (a) without the "+*c*".

Here's the integral,

$$
\begin{aligned}
\int_1^2 y^2 + y^{-2}\,dy &= \left(\frac{1}{3}y^3 - \frac{1}{y}\right)\Bigg|_1^2 \\
&= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right) \\
&= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \\
&= \frac{17}{6}
\end{aligned}
$$

Remember that the evaluation is always done in the order of evaluation at the upper limit minus evaluation at the lower limit. Also, be very careful with minus signs and parenthesis. It's very easy to forget them or mishandle them and get the wrong answer.

Notice as well that, in order to help with the evaluation, we rewrote the indefinite integral a little. In particular we got rid of the negative exponent on the second term. It's generally easier to evaluate the term with positive exponents.

(c) $\int_{-1}^{2} y^2 + y^{-2}\,dy$

This integral is here to make a point. Recall that in order for us to do an integral the integrand must be continuous in the range of the limits. In this case the second term will have division by zero at $y = 0$ and since $y = 0$ is in the interval of integration, *i.e.* it is between the lower and upper limit, this integrand is not continuous in the interval of integration and so we can't do this integral.

Note that this problem will not prevent us from doing the integral in (b) since $y = 0$ is not in the interval of integration.

So, what have we learned from this example?

First, in order to do a definite integral the first thing that we need to do is the indefinite integral. So, we aren't going to get out of doing indefinite integrals, they will be in every integral that we'll be doing in the rest of this course so make sure that you're getting good at computing them.

Second, we need to be on the lookout for functions that aren't continuous at any point between the limits of integration. Also, it's important to note that this will only be a problem if the point(s) of discontinuity occur between the limits of integration or at the limits themselves. If the point of discontinuity occurs outside of the limits of integration the integral can still be evaluated.

In the following sets of examples we won't make too much of an issue with continuity problems, or lack of continuity problems, unless it affects the evaluation of the integral. Do not let this convince you that you don't need to worry about this idea. It arises often enough that it can cause real problems if you aren't on the lookout for it.

Finally, note the difference between indefinite and definite integrals. Indefinite integrals are functions while definite integrals are numbers.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$

(c) $\displaystyle\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$

(d) $\int_{25}^{-10} dR$

Solution

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

There isn't a lot to this one other than simply doing the work.

$$\begin{aligned}\int_{-3}^{1} 6x^2 - 5x + 2\,dx &= \left(2x^3 - \frac{5}{2}x^2 + 2x\right)\Bigg|_{-3}^{1} \\ &= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right) \\ &= 84\end{aligned}$$

(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$

Recall that we can't integrate products as a product of integrals and so we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.

$$\begin{aligned}\int_{4}^{0} \sqrt{t}\left(t-2\right)dt &= \int_{4}^{0} t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\,dt \\ &= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Bigg|_{4}^{0} \\ &= 0 - \left(\frac{2}{5}\left(4\right)^{\frac{5}{2}} - \frac{4}{3}\left(4\right)^{\frac{3}{2}}\right) \\ &= -\frac{32}{15}\end{aligned}$$

In the evaluation process recall that,

$$\left(4\right)^{\frac{5}{2}} = \left(\left(4\right)^{\frac{1}{2}}\right)^5 = \left(2\right)^5 = 32$$

$$\left(4\right)^{\frac{3}{2}} = \left(\left(4\right)^{\frac{1}{2}}\right)^3 = \left(2\right)^3 = 8$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion and there is absolutely nothing wrong with this.

(c) $\displaystyle\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$

First, notice that we will have a division by zero issue at $w = 0$, but since this isn't in the interval of integration we won't have to worry about it.

Next again recall that we can't integrate quotients as a quotient of integrals and so the first step that we'll need to do is break up the quotient so we can integrate the function.

$$\begin{aligned}\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw &= \int_{1}^{2} 2w^3 - \frac{1}{w} + 3w^{-2}\,dw \\ &= \left.\left(\frac{1}{2}w^4 - \ln|w| - \frac{3}{w}\right)\right|_{1}^{2} \\ &= \left(8 - \ln 2 - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right) \\ &= 9 - \ln 2\end{aligned}$$

Don't get excited about answers that don't come down to a simple integer or fraction. Often times they won't. Also, don't forget that $\ln(1) = 0$.

(d) $\displaystyle\int_{25}^{-10} dR$

This one is actually pretty easy. Recall that we're just integrating 1.

$$\begin{aligned}\int_{25}^{-10} dR &= \left.R\right|_{25}^{-10} \\ &= -10 - 25 \\ &= -35\end{aligned}$$

The last set of examples dealt exclusively with integrating powers of *x*. Let's work a couple of examples that involve other functions.

Example 3 Evaluate each of the following.

(a) $\displaystyle\int_0^1 4x - 6\sqrt[3]{x^2}\,dx$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\,d\theta$

(c) $\displaystyle\int_{\pi/6}^{\pi/4} 5 - 2\sec z\tan z\,dz$

(d) $\displaystyle\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz$

(e) $\displaystyle\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

Solution

(a) $\displaystyle\int_0^1 4x - 6\sqrt[3]{x^2}\,dx$.

This one is here mostly here to contrast with the next example.

$$\begin{aligned}\int_0^1 4x - 6\sqrt[3]{x^2}\,dx &= \int_0^1 4x - 6x^{\frac{2}{3}}\,dx\\ &= \left.\left(2x^2 - \frac{18}{5}x^{\frac{5}{3}}\right)\right|_0^1\\ &= 2 - \frac{18}{5} - (0)\\ &= -\frac{8}{5}\end{aligned}$$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\,d\theta$

Be careful with signs with this one. Recall from the indefinite integral sections that it's easy to mess up the signs when integrating sine and cosine.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\,d\theta &= \left.\left(-2\cos\theta - 5\sin\theta\right)\right|_0^{\pi/3}\\ &= -2\cos\left(\frac{\pi}{3}\right) - 5\sin\left(\frac{\pi}{3}\right) - \left(-2\cos 0 - 5\sin 0\right)\\ &= -1 - \frac{5\sqrt{3}}{2} + 2\\ &= 1 - \frac{5\sqrt{3}}{2}\end{aligned}$$

Compare this answer to the previous answer, especially the evaluation at zero. It's very easy to get into the habit of just writing down zero when evaluating a function at zero. This is especially a problem when many of the functions that we integrate involve only *x*'s raised to positive integers; these evaluate is zero of course. After evaluating many of these kinds of definite integrals it's easy to get into the habit of just writing down zero when you evaluate at zero. However, there are many functions out there that aren't zero when evaluated at zero so be careful.

(c) $\int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z \, dz$

Not much to do other than do the integral.

$$\begin{aligned}\int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z \, dz &= \left(5z - 2\sec z\right)\Big|_{\pi/6}^{\pi/4} \\ &= 5\left(\frac{\pi}{4}\right) - 2\sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2\sec\left(\frac{\pi}{6}\right)\right) \\ &= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}\end{aligned}$$

For the evaluation, recall that

$$\sec z = \frac{1}{\cos z}$$

and so if we can evaluate cosine at these angles we can evaluate secant at these angles.

(d) $\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z} \, dz$

In order to do this one will need to rewrite both of the terms in the integral a little as follows,

$$\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z} \, dz = \int_{-20}^{-1} 3\mathbf{e}^{z} - \frac{1}{3}\frac{1}{z} \, dz$$

For the first term recall we used the following fact about exponents.

$$x^{-a} = \frac{1}{x^{a}} \qquad\qquad \frac{1}{x^{-a}} = x^{a}$$

In the second term, taking the 3 out of the denominator will just make integrating that term easier.

Now the integral.

$$\begin{aligned}\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z} \, dz &= \left(3\mathbf{e}^{z} - \frac{1}{3}\ln|z|\right)\Big|_{-20}^{-1} \\ &= 3\mathbf{e}^{-1} - \frac{1}{3}\ln|-1| - \left(3\mathbf{e}^{-20} - \frac{1}{3}\ln|-20|\right) \\ &= 3\mathbf{e}^{-1} - 3\mathbf{e}^{-20} + \frac{1}{3}\ln|20|\end{aligned}$$

Just leave the answer like this. It's messy, but it's also exact.

Note that the absolute value bars on the logarithm are required here. Without them we couldn't have done the evaluation.

(e) $\displaystyle\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

This integral can't be done. There is division by zero in the third term at $t = 0$ and $t = 0$ lies in the interval of integration. The fact that the first two terms can be integrated doesn't matter. If even one term in the integral can't be integrated then the whole integral can't be done.

So, we've computed a fair number of definite integrals at this point. Remember that the vast majority of the work in computing them is first finding the indefinite integral. Once we've found that the rest is just some number crunching.

There are a couple of particularly tricky definite integrals that we need to take a look at next. Actually they are only tricky until you see how to do them, so don't get too excited about them. The first one involves integrating a piecewise function.

Example 4 Given,

$$f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^2 & \text{if } x \le 1 \end{cases}$$

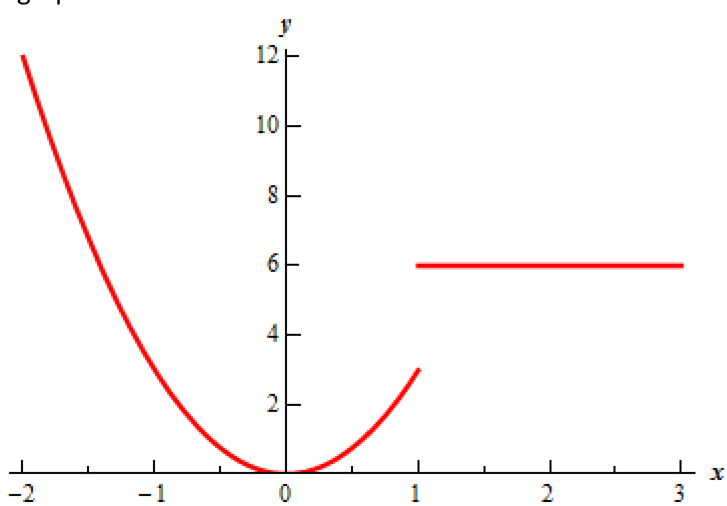
Evaluate each of the following integrals.

(a) $\displaystyle\int_{10}^{22} f(x)\,dx$

(b) $\displaystyle\int_{-2}^{3} f(x)\,dx$

Solution

Let's first start with a graph of this function.



The graph reveals a problem. This function is not continuous at $x = 1$ and we're going to have to watch out for that.

(a) $\displaystyle\int_{10}^{22} f(x)\,dx$

For this integral notice that $x=1$ is not in the interval of integration and so that is something that we'll not need to worry about in this part.

Also note the limits for the integral lie entirely in the range for the first function. What this means for us is that when we do the integral all we need to do is plug in the first function into the integral.

Here is the integral.

$$\begin{aligned}\int_{10}^{22} f(x)\,dx &= \int_{10}^{22} 6\,dx \\ &= 6x\Big|_{10}^{22} \\ &= 132-60 \\ &= 72\end{aligned}$$

(b) $\int_{-2}^{3} f(x)\,dx$

In this part $x=1$ is between the limits of integration. This means that the integrand is no longer continuous in the interval of integration and that is a show stopper as far we're concerned. As noted above we simply can't integrate functions that aren't continuous in the interval of integration.

Also, even if the function was continuous at $x=1$ we would still have the problem that the function is actually two different equations depending where we are in the interval of integration.

Let's first address the problem of the function not being continuous at $x=1$. As we'll see, in this case, if we can find a way around this problem the second problem will also get taken care of at the same time.

In the previous examples where we had functions that weren't continuous we had division by zero and no matter how hard we try we can't get rid of that problem. Division by zero is a real problem and we can't really avoid it. In this case the discontinuity does not stem from problems with the function not existing at $x=1$. Instead the function is not continuous because it takes on different values on either sides of $x=1$. We can "remove" this problem by recalling **Property 5** from the previous section. This property tells us that we can write the integral as follows,

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

On each of these intervals the function is continuous. In fact we can say more. In the first integral we will have *x* between -2 and 1 and this means that we can use the second equation for $f(x)$ and likewise for the second integral *x* will be between 1 and 3 and so we can use the first function for $f(x)$. The integral in this case is then,

$$\begin{aligned}\int_{-2}^{3} f(x)\,dx &= \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx \\ &= \int_{-2}^{1} 3x^2\,dx + \int_{1}^{3} 6\,dx \\ &= x^3\Big|_{-2}^{1} + 6x\Big|_{1}^{3} \\ &= 1-(-8)+(18-6) \\ &= 21\end{aligned}$$

So, to integrate a piecewise function, all we need to do is break up the integral at the break point(s) that happen to occur in the interval of integration and then integrate each piece.

Next, we need to look at is how to integrate an absolute value function.

Example 5 Evaluate the following integral.

$$\int_{0}^{3} |3t-5|\,dt$$

Solution

Recall that the point behind indefinite integration (which we'll need to do in this problem) is to determine what function we differentiated to get the integrand. To this point we've not seen any functions that will differentiate to get an absolute value nor will we ever see a function that will differentiate to get an absolute value.

The only way that we can do this problem is to get rid of the absolute value. To do this we need to recall the definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Once we remember that we can define absolute value as a piecewise function we can use the work from Example 4 as a guide for doing this integral.

What we need to do is determine where the quantity on the inside of the absolute value bars is negative and where it is positive. It looks like if $t > \frac{5}{3}$ the quantity inside the absolute value is positive and if $t < \frac{5}{3}$ the quantity inside the absolute value is negative.

Next, note that $t = \frac{5}{3}$ is in the interval of integration and so, if we break up the integral at this point we get,

$$\int_{0}^{3} |3t-5|\,dt = \int_{0}^{\frac{5}{3}} |3t-5|\,dt + \int_{\frac{5}{3}}^{3} |3t-5|\,dt$$

Now, in the first integrals we have $t < \frac{5}{3}$ and so $3t-5<0$ in this interval of integration. That means we can drop the absolute value bars if we put in a minus sign. Likewise, in the second integral we have $t > \frac{5}{3}$ which means that in this interval of integration we have $3t-5>0$ and so we can just drop the absolute value bars in this integral.

After getting rid of the absolute value bars in each integral we can do each integral. So, doing the integration gives,

$$\begin{aligned}\int_0^3 \left|3t-5\right| dt &= \int_0^{\frac{5}{3}} -\left(3t-5\right) dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\ &= \int_0^{\frac{5}{3}} -3t+5\,dt + \int_{\frac{5}{3}}^3 3t-5\,dt \\ &= \left(-\frac{3}{2}t^2+5t\right)\Big|_0^{\frac{5}{3}} + \left(\frac{3}{2}t^2-5t\right)\Big|_{\frac{5}{3}}^3 \\ &= -\frac{3}{2}\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - \left(0\right) + \left(\frac{3}{2}\left(3\right)^2 - 5\left(3\right) - \left(\frac{3}{2}\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right)\right)\right) \\ &= \frac{25}{6} + \frac{8}{3} \\ &= \frac{41}{6}\end{aligned}$$

Integrating absolute value functions isn't too bad. It's a little more work than the "standard" definite integral, but it's not really all that much more work. First, determine where the quantity inside the absolute value bars is negative and where it is positive. When we've determined that point all we need to do is break up the integral so that in each range of limits the quantity inside the absolute value bars is always positive or always negative. Once this is done we can drop the absolute value bars (adding negative signs when the quantity is negative) and then we can do the integral as we've always done.

Even and Odd Functions

This is the last topic that we need to discuss in this section.

First, recall that an even function is any function which satisfies,

$$f\left(-x\right) = f\left(x\right)$$

Typical examples of even functions are,

$$f\left(x\right) = x^2 \qquad\qquad f\left(x\right) = \cos\left(x\right)$$

An odd function is any function which satisfies,

$$f\left(-x\right) = -f\left(x\right)$$

The typical examples of odd functions are,

$$f\left(x\right) = x^3 \qquad\qquad f\left(x\right) = \sin\left(x\right)$$

There are a couple of nice facts about integrating even and odd functions over the interval $\left[-a,a\right]$. If $f\left(x\right)$ is an even function then,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Likewise, if $f(x)$ is an odd function then,

$$\int_{-a}^{a} f(x)\,dx = 0$$

Note that in order to use these facts the limit of integration must be the same number, but opposite signs!

Example 6 Integrate each of the following.

(a) $\displaystyle\int_{-2}^{2} 4x^4 - x^2 + 1\,dx$

(b) $\displaystyle\int_{-10}^{10} x^5 + \sin(x)\,dx$

Solution

Neither of these are terribly difficult integrals, but we can use the facts on them anyway.

(a) $\displaystyle\int_{-2}^{2} 4x^4 - x^2 + 1\,dx$

In this case the integrand is even and the interval is correct so,

$$\begin{aligned}\int_{-2}^{2} 4x^4 - x^2 + 1\,dx &= 2\int_{0}^{2} 4x^4 - x^2 + 1\,dx \\ &= 2\left(\frac{4}{5}x^5 - \frac{1}{3}x^3 + x\right)\Bigg|_{0}^{2} \\ &= \frac{748}{15}\end{aligned}$$

So, using the fact cut the evaluation in half (in essence since one of the new limits was zero).

(b) $\displaystyle\int_{-10}^{10} x^5 + \sin(x)\,dx$

The integrand in this case is odd and the interval is in the correct form and so we don't even need to integrate. Just use the fact.

$$\int_{-10}^{10} x^5 + \sin(x)\,dx = 0$$

Note that the limits of integration are important here. Take the last integral as an example. A small change to the limits will not give us zero.

$$\int_{-10}^{9} x^5 + \sin(x)\,dx = \cos(10) - \cos(9) - \frac{468559}{6} = -78093.09461$$

The moral here is to be careful and not misuse these facts.

Section 5-8 : Substitution Rule for Definite Integrals

We now need to go back and revisit the substitution rule as it applies to definite integrals. At some level there really isn't a lot to do in this section. Recall that the first step in doing a definite integral is to compute the indefinite integral and that hasn't changed. We will still compute the indefinite integral first. This means that we already know how to do these. We use the substitution rule to find the indefinite integral and then do the evaluation.

There are however, two ways to deal with the evaluation step. One of the ways of doing the evaluation is the probably the most obvious at this point, but also has a point in the process where we can get in trouble if we aren't paying attention.

Let's work an example illustrating both ways of doing the evaluation step.

Example 1 Evaluate the following definite integral.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$$

Solution

Let's start off looking at the first way of dealing with the evaluation step. We'll need to be careful with this method as there is a point in the process where if we aren't paying attention we'll get the wrong answer.

Solution 1 :

We'll first need to compute the indefinite integral using the substitution rule. Note however, that we will constantly remind ourselves that this is a definite integral by putting the limits on the integral at each step. Without the limits it's easy to forget that we had a definite integral when we've gotten the indefinite integral computed.

In this case the substitution is,

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

Plugging this into the integral gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\,du \\ &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{-2}^{0}\end{aligned}$$

Notice that we didn't do the evaluation yet. This is where the potential problem arises with this solution method. The limits given here are from the original integral and hence are values of *t*. We have *u*'s in our solution. We can't plug values of *t* in for *u*.

Therefore, we will have to go back to *t*'s before we do the substitution. This is the standard step in the substitution process, but it is often forgotten when doing definite integrals. Note as well that in this case, if we don't go back to *t*'s we will have a small problem in that one of the evaluations will end up giving us a complex number.

So, finishing this problem gives,

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}}\Big|_{-2}^{0}$$
$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right)$$
$$= \frac{1}{9}\left(33\sqrt{33}-1\right)$$

So, that was the first solution method. Let's take a look at the second method.

Solution 2 :
Note that this solution method isn't really all that different from the first method. In this method we are going to remember that when doing a substitution we want to eliminate all the *t*'s in the integral and write everything in terms of *u*.

When we say all here we really mean all. In other words, remember that the limits on the integral are also values of *t* and we're going to convert the limits into *u* values. Converting the limits is pretty simple since our substitution will tell us how to relate *t* and *u* so all we need to do is plug in the original *t* limits into the substitution and we'll get the new *u* limits.

Here is the substitution (it's the same as the first method) as well as the limit conversions.

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$
$$t = -2 \qquad \Rightarrow \qquad u = 1-4(-2)^3 = 33$$
$$t = 0 \qquad \Rightarrow \qquad u = 1-4(0)^3 = 1$$

The integral is now,

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\,du$$
$$= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}$$

As with the first method let's pause here a moment to remind us what we're doing. In this case, we've converted the limits to *u*'s and we've also got our integral in terms of *u*'s and so here we can just plug the limits directly into our integral. Note that in this case we won't plug our substitution back in. Doing this here would cause problems as we would have *t*'s in the integral and our limits would be *u*'s. Here's the rest of this problem.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}$$
$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33}-1\right)$$

We got exactly the same answer and this time didn't have to worry about going back to t's in our answer.

So, we've seen two solution techniques for computing definite integrals that require the substitution rule. Both are valid solution methods and each have their uses. We will be using the second almost exclusively however since it makes the evaluation step a little easier.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-1}^{5}(1+w)\left(2w+w^2\right)^5\,dw$

(b) $\displaystyle\int_{-2}^{-6}\frac{4}{(1+2x)^3}-\frac{5}{1+2x}\,dx$

(c) $\int_{0}^{\frac{1}{2}}\mathbf{e}^y+2\cos(\pi y)\,dy$

(d) $\displaystyle\int_{\frac{\pi}{3}}^{0}3\sin\left(\frac{z}{2}\right)-5\cos(\pi-z)\,dz$

Solution

Since we've done quite a few substitution rule integrals to this time we aren't going to put a lot of effort into explaining the substitution part of things here.

(a) $\int_{-1}^{5}(1+w)\left(2w+w^2\right)^5\,dw$

The substitution and converted limits are,

$$u=2w+w^2 \qquad du=(2+2w)\,dw \qquad \Rightarrow \qquad (1+w)\,dw=\frac{1}{2}du$$
$$w=-1 \quad \Rightarrow \quad u=-1 \qquad\qquad w=5 \quad \Rightarrow \quad u=35$$

Sometimes a limit will remain the same after the substitution. Don't get excited when it happens and don't expect it to happen all the time.

Here is the integral,

$$\int_{-1}^{5}(1+w)\left(2w+w^2\right)^5 dw=\frac{1}{2}\int_{-1}^{35}u^5\,du$$
$$=\frac{1}{12}u^6\Big|_{-1}^{35}=153188802$$

Don’t get excited about large numbers for answers here. Sometimes they are. That’s life.

(b) $\displaystyle\int_{-2}^{-6}\frac{4}{\left(1+2x\right)^3}-\frac{5}{1+2x}dx$

Here is the substitution and converted limits for this problem,

$$u=1+2x \qquad du=2dx \qquad \Rightarrow \qquad dx=\frac{1}{2}du$$
$$x=-2 \quad \Rightarrow \quad u=-3 \qquad\qquad x=-6 \quad \Rightarrow \quad u=-11$$

The integral is then,

$$\int_{-2}^{-6}\frac{4}{\left(1+2x\right)^3}-\frac{5}{1+2x}dx=\frac{1}{2}\int_{-3}^{-11}4u^{-3}-\frac{5}{u}du$$
$$=\frac{1}{2}\left(-2u^{-2}-5\ln\left|u\right|\right)\Big|_{-3}^{-11}$$
$$=\frac{1}{2}\left(-\frac{2}{121}-5\ln 11\right)-\frac{1}{2}\left(-\frac{2}{9}-5\ln 3\right)$$
$$=\frac{112}{1089}-\frac{5}{2}\ln 11+\frac{5}{2}\ln 3$$

(c) $\displaystyle\int_{0}^{\frac{1}{2}}\mathbf{e}^{y}+2\cos\left(\pi y\right)dy$

This integral needs to be split into two integrals since the first term doesn’t require a substitution and the second does.

$$\int_{0}^{\frac{1}{2}}\mathbf{e}^{y}+2\cos\left(\pi y\right)dy=\int_{0}^{\frac{1}{2}}\mathbf{e}^{y}\,dy+\int_{0}^{\frac{1}{2}}2\cos\left(\pi y\right)dy$$

Here is the substitution and converted limits for the second term.

$$u=\pi y \qquad du=\pi dy \qquad \Rightarrow \qquad dy=\frac{1}{\pi}du$$
$$y=0 \quad \Rightarrow \quad u=0 \qquad\qquad y=\frac{1}{2} \quad \Rightarrow \quad u=\frac{\pi}{2}$$

Here is the integral.

$$\int_0^{\frac{1}{2}} \mathbf{e}^y + 2\cos(\pi y)\,dy = \int_0^{\frac{1}{2}} \mathbf{e}^y\,dy + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u)\,du$$

$$= \mathbf{e}^y\Big|_0^{\frac{1}{2}} + \frac{2}{\pi}\sin u\Big|_0^{\frac{\pi}{2}}$$

$$= \mathbf{e}^{\frac{1}{2}} - \mathbf{e}^0 + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\sin 0$$

$$= \mathbf{e}^{\frac{1}{2}} - 1 + \frac{2}{\pi}$$

(d) $\displaystyle\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz$

This integral will require two substitutions. So first split up the integral so we can do a substitution on each term.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = \int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right)dz - \int_{\frac{\pi}{3}}^{0} 5\cos(\pi - z)\,dz$$

There are the two substitutions for these integrals.

$$u = \frac{z}{2} \qquad du = \frac{1}{2}dz \qquad \Rightarrow \qquad dz = 2du$$

$$z = \frac{\pi}{3} \quad \Rightarrow \quad u = \frac{\pi}{6} \qquad\qquad z = 0 \quad \Rightarrow \quad u = 0$$

$$v = \pi - z \qquad dv = -dz \qquad \Rightarrow \qquad dz = -dv$$

$$z = \frac{\pi}{3} \quad \Rightarrow \quad v = \frac{2\pi}{3} \qquad\qquad z = 0 \quad \Rightarrow \quad v = \pi$$

Here is the integral for this problem.

$$\int_{\frac{\pi}{3}}^{0} 3\sin\left(\frac{z}{2}\right) - 5\cos(\pi - z)\,dz = 6\int_{\frac{\pi}{6}}^{0} \sin(u)\,du + 5\int_{\frac{2\pi}{3}}^{\pi} \cos(v)\,dv$$

$$= -6\cos(u)\Big|_{\frac{\pi}{6}}^{0} + 5\sin(v)\Big|_{\frac{2\pi}{3}}^{\pi}$$

$$= 3\sqrt{3} - 6 + \left(-\frac{5\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} - 6$$

The next set of examples is designed to make sure that we don't forget about a very important point about definite integrals.

Example 3 Evaluate each of the following.

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$

Solution

(a) $\displaystyle\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$

Be careful with this integral. The denominator is zero at $t = \pm\frac{1}{2}$ and both of these are in the interval of integration. Therefore, this integrand is not continuous in the interval and so the integral can't be done.

Be careful with definite integrals and be on the lookout for division by zero problems. In the previous section they were easy to spot since all the division by zero problems that we had there were where the variable was itself zero. Once we move into substitution problems however they will not always be so easy to spot so make sure that you first take a quick look at the integrand and see if there are any continuity problems with the integrand and if they occur in the interval of integration.

(b) $\displaystyle\int_{3}^{5} \frac{4t}{2-8t^2}\,dt$

Now, in this case the integral can be done because the two points of discontinuity, $t = \pm\frac{1}{2}$, are both outside of the interval of integration. The substitution and converted limits in this case are,

$$u = 2-8t^2 \qquad du = -16t\,dt \qquad \Rightarrow \qquad t\,dt = -\frac{1}{16}du$$

$$t = 3 \quad \Rightarrow \quad u = -70 \qquad\qquad t = 5 \quad \Rightarrow \quad u = -198$$

The integral is then,

$$\begin{aligned}\int_{3}^{5} \frac{4t}{2-8t^2}\,dt &= -\frac{4}{16}\int_{-70}^{-198} \frac{1}{u}\,du \\ &= -\frac{1}{4}\ln|u|\Big|_{-70}^{-198} \\ &= -\frac{1}{4}\left(\ln(198) - \ln(70)\right)\end{aligned}$$

Let's work another set of examples. These are a little tougher (at least in appearance) than the previous sets.

Example 4 Evaluate each of the following.

(a) $\displaystyle\int_0^{\ln(1+\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx$

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{\left[\ln t\right]^4}{t} dt$

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}} dP$

(d) $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right) dx$

(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2} dw$

Solution

(a) $\displaystyle\int_0^{\ln(1+\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx$

The limits are a little unusual in this case, but that will happen sometimes so don't get too excited about it. Here is the substitution.

$$u = 1-\mathbf{e}^x \qquad du = -\mathbf{e}^x dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1-\mathbf{e}^0 = 1-1 = 0$$

$$x = \ln(1+\pi) \qquad \Rightarrow \qquad u = 1-\mathbf{e}^{\ln(1+\pi)} = 1-(1+\pi) = -\pi$$

The integral is then,

$$\begin{aligned}\int_0^{\ln(1+\pi)} \mathbf{e}^x \cos\left(1-\mathbf{e}^x\right) dx &= -\int_0^{-\pi} \cos u \, du \\ &= -\sin(u)\Big|_0^{-\pi} \\ &= -\left(\sin(-\pi) - \sin 0\right) = 0\end{aligned}$$

(b) $\displaystyle\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{\left[\ln t\right]^4}{t} dt$

Here is the substitution and converted limits for this problem.

$$u = \ln t \qquad du = \frac{1}{t} dt$$

$$t = \mathbf{e}^2 \quad \Rightarrow \quad u = \ln \mathbf{e}^2 = 2 \qquad\qquad t = \mathbf{e}^6 \quad \Rightarrow \quad u = \ln \mathbf{e}^6 = 6$$

The integral is,

$$\int_{\mathbf{e}^2}^{\mathbf{e}^6} \frac{[\ln t]^4}{t}\,dt = \int_2^6 u^4\,du$$

$$= \frac{1}{5}u^5\Big|_2^6$$

$$= \frac{7744}{5}$$

(c) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP$

Here is the substitution and converted limits and don't get too excited about the substitution. It's a little messy in the case, but that can happen on occasion.

$$u = 2+\sec(3P) \qquad du = 3\sec(3P)\tan(3P)\,dP \quad \Rightarrow \quad \sec(3P)\tan(3P)\,dP = \frac{1}{3}du$$

$$P = \frac{\pi}{12} \qquad \Rightarrow \qquad u = 2+\sec\left(\frac{\pi}{4}\right) = 2+\sqrt{2}$$

$$P = \frac{\pi}{9} \qquad \Rightarrow \qquad u = 2+\sec\left(\frac{\pi}{3}\right) = 4$$

Here is the integral,

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \frac{\sec(3P)\tan(3P)}{\sqrt[3]{2+\sec(3P)}}\,dP = \frac{1}{3}\int_{2+\sqrt{2}}^{4} u^{-\frac{1}{3}}\,du$$

$$= \frac{1}{2}u^{\frac{2}{3}}\Big|_{2+\sqrt{2}}^{4}$$

$$= \frac{1}{2}\left(4^{\frac{2}{3}} - \left(2+\sqrt{2}\right)^{\frac{2}{3}}\right)$$

So, not only was the substitution messy, but we also have a messy answer, but again that's life on occasion.

(d) $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

This problem not as bad as it looks. Here is the substitution and converted limits.

$$u = \sin x \qquad du = \cos x\,dx$$

$$x = \frac{\pi}{2} \quad \Rightarrow \quad u = \sin\frac{\pi}{2} = 1 \qquad\qquad x = -\pi \quad \Rightarrow \quad u = \sin(-\pi) = 0$$

The cosine in the very front of the integrand will get substituted away in the differential and so this integrand actually simplifies down significantly. Here is the integral.

$$\int_{-\pi}^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx = \int_0^1 \cos u\,du$$
$$= \sin(u)\Big|_0^1$$
$$= \sin(1) - \sin(0)$$
$$= \sin(1)$$

Don't get excited about these kinds of answers. On occasion we will end up with trig function evaluations like this.

(e) $\displaystyle\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw$

This is also a tricky substitution (at least until you see it). Here it is,

$$u = \frac{2}{w} \qquad du = -\frac{2}{w^2}\,dw \qquad \Rightarrow \qquad \frac{1}{w^2}\,dw = -\frac{1}{2}\,du$$
$$w = 2 \quad \Rightarrow \quad u = 1 \qquad w = \frac{1}{50} \quad \Rightarrow \quad u = 100$$

Here is the integral.

$$\int_{\frac{1}{50}}^{2} \frac{\mathbf{e}^{\frac{2}{w}}}{w^2}\,dw = -\frac{1}{2}\int_{100}^{1} \mathbf{e}^u\,du$$
$$= -\frac{1}{2}\mathbf{e}^u\Big|_{100}^{1}$$
$$= -\frac{1}{2}\left(\mathbf{e}^1 - \mathbf{e}^{100}\right)$$

In this last set of examples we saw some tricky substitutions and messy limits, but these are a fact of life with some substitution problems and so we need to be prepared for dealing with them when they happen.