Maths Solutions (Multiple Choice)

Section A: Fast

1. **Answer:** $\frac{3}{4}$

   Working: Draw 3-4-5 right triangle: $\sec\theta = 5/3 \rightarrow \cos\theta = 3/5$
   $\rightarrow \tan\theta = 4/3 \rightarrow \cot\theta = 3/4$

2. **Answer:** $(3x + y)(x - y)$

   Working: Clearly either $(3x + y)(x - y)$ or $(3x - y)(x + y)$
   Need -2xy from middle terms = -3xy + xy
   → Need -y in x - y term

3. **Answer:** -29

   Working: Terms which multiply to x^2 are
   (x^2 * constant), (x * x), and (constant * x^2):
   → = (2 * 1) + (-1 * -1) + (4 * -8) = -29.

4. **Answer:** C

   Working: For x > 1, |x - 1| > 0 → y = x - (x - 1) = 1
   For x < 1, |x - 1| < 0 → y = x - -(x - 1) = x + (x - 1) = 2x - 1
   → Graph C matches this

5. **Answer:** 49%

   Working: $P = k / Q^2 \rightarrow PQ^2 = k \rightarrow aP * (1.4\ Q)^2 = k \rightarrow aP * 1.96\ Q^2 = k$
   → a = 1 / 1.96 = 0.51
   → decreased by 100(1 - a) = 49%

6. **Answer:** The cube of *x* is inversely proportional to the square of *y*.

Working: Let $x = az^2$, $y = b/z^3$, where a and b are constants.
Eliminating z from y equation: $z = (b / y)^{1/3}$
Subbing into x equation: $x = a (b / y)^{2/3}$
$\rightarrow x = ab^{2/3} * y^{-2/3}$
$\rightarrow x^3 = a^3b^2 * y^{-2}$
$\rightarrow x^3 = \text{constant} / y^2$
$\rightarrow x^3$ is inversely proportional to y^2

7. **Answer:** $y = 4 + \sin x$

Working: Reflection in line $x = \pi$:
$y = \sin x \rightarrow \sin(2\pi - x) = \sin 2\pi \cos x - \sin x \cos 2\pi = - \sin x$
Reflection in line $y = 2$:
$y = - \sin x \rightarrow 4 - (- \sin x) = 4 + \sin x$

8. **Answer:** $y = x^2$

Working: Convex → concave up → gradient increasing → $y'' > 0$
$y = 1$ and $y = x \rightarrow y'' = 0$ → not strictly convex
$y = x^2 \rightarrow y'' = 2 > 0$ → strictly convex
$y = x^3 \rightarrow y'' = 6x$ which can be +ve or -ve → not convex for all x

9. **Answer:** $y = 4 - x$

Working: $x + y = 0 \rightarrow y = -x \rightarrow$ gradient = -1
Using point-slope, $y - 5 = -1 * (x + 1)$
$\rightarrow x + y - 4 = 0 \rightarrow y = 4 - x$

10. **Answer:** 11/4

Working: By similar triangles (or lengths), fraction in y-direction = fraction in x-direction → $(4 - 1) / (9 - 1) = (k - 5) / (-1 - 5)$
$\rightarrow -3/8 = (k - 5) / 6 \rightarrow k = 11/4$

11. **Answer:** $a/b = 3$

Working: $3x + ay = 2 \rightarrow y = -3/a\ x + 2/a \rightarrow$ gradient $= -3/a$
$bx - y = a - b \rightarrow y = bx - a + b \rightarrow$ gradient $= b$
Perpendicular $\rightarrow$ gradients multiply to $-1 \rightarrow -3b/a = -1 \rightarrow a/b = 3$.

12. **Answer:** 8

Working: Period of $\sin x$ is 360^o
$\sin 45x$ is a stretch parallel to x, s.f. $1/45 \rightarrow$ period $= 360 * 1/45 = 8$

13. **Answer:** 1260

Working: Divisible by all primes less than 10
$\rightarrow$ must be divisible by $2 * 3 * 5 * 7 = 210$
Checking each, only $1260/210 = 6$, all others give fractions.

14. **Answer:** $\cot \theta$ is always negative

Working: $90^o < \theta < 180^o \rightarrow \sin x > 0$ and $\cos x < 0$ (by considering graphs)
$\rightarrow \tan x = \sin x / \cos x = +ve * -ve = -ve \rightarrow \cot x < 0$

15. **Answer:** $n(n + 1)$

Working: By standard formula (or sum of arithmetic sequence),
$1 + 2 + 3 + \ldots + n = 1/2 * n(n + 1)$
$\rightarrow 2(1 + 2 + 3 + \ldots + n) = n(n + 1)$

16. **Answer:** $1/x$

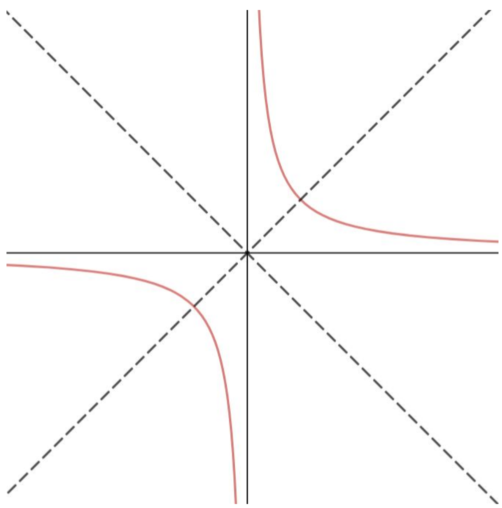
Working: $d/dx \ln(3x) = 3/(3x) = 1/x$

17. **Answer:** $-m\mathbf{v}$

Working: Unit vector of $\mathbf{v}$: $\mathbf{v} / |\mathbf{v}| = \mathbf{v} / m$
Scale by m^2: $m^2 * (\mathbf{v} / m) = m * \mathbf{v}$
Opposite direction: $-1 * m * \mathbf{v} = -m\mathbf{v}$
($-m^2\mathbf{v}$ would have magnitude m^3).

18. **Answer:** None of the above.

Working: Graph of $xy = 4 \rightarrow y = 4/x$ with dashed lines of symmetry shown (reciprocal graph):



$\rightarrow$ symmetrical in lines $y = x$ and $y = -x$, not the axes.

19. **Answer:** $3 - (2 - x)^2$

Working: $4x - 1 - x^2 = -x^2 + 4x - 1 = -[x^2 - 4x + 1] = -[(x - 2)^2 + 1 - 4]$
$= -[(x - 2)^2 - 3] = 3 - (x - 2)^2$
Since $x - 2$ is squared, we can multiply it by -1 and then square it and the result is the same: $(x - 2)^2 = [(-1)(2 - x)]^2 = (2 - x)^2$
$\rightarrow = 3 - (2 - x)^2$

20. **Answer:** $4a^2$

Working: $(x + a)(x - a) = x^2 - a^2 \rightarrow$ discriminant $= (0)^2 - 4(1)(-a^2) = 4a^2$

21. **Answer:** $x < 28$

Working: $-8 < 6 - x/2 \rightarrow -14 < -x/2$
Multiply both sides by -2 (switches inequality direction):
$\rightarrow 28 > x \rightarrow x < 28$

22. **Answer:** $3 - 2/x$

Working: $4 + (4 - x^2) / (x^2 - 2x)$
Difference of two squares in the top and factorise the bottom:
$= 4 + (2 + x)(2 - x) / x(x - 2)$
$= 4 + (2 + x) / (-x)$
$= 4 + (-2/x) + (-x/x) = 4 - 2/x - 1 = 3 - 2/x$

23. **Answer:** $n = 720 \div (180 - x^o)$

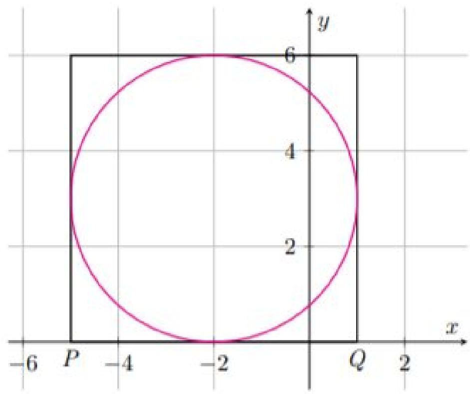
Working: External angle of regular polygon: $\angle RQT = 360 / n$
Angles in triangle TQR: $x + 360/n + 360/n = 180$
$\rightarrow x + 720/n = 180 \rightarrow 720/n = 180 - x \rightarrow n = 720 / (180 - x)$

24. **Answer:** $p = 1$

Working: $f(1) = -3.5 \rightarrow 1 + p + q + p^2 = -3.5 \rightarrow p^2 + p + q = -4.5$
$f(2) = 0 \rightarrow 8 + 4p + 2q + p^2 = 0 \rightarrow p^2 + 4p + 2q = -8$
Multiply first equation by 2 and subtract to eliminate q:
$(2p^2 + 2p + 2q) - (p^2 + 4p + 2q) = (-9) - (-8)$
$\rightarrow p^2 - 2p + 1 = 0 \rightarrow (p - 1)^2 = 0 \rightarrow p = 1$ is a repeated (only) root.

25. **Answer:** $x^2 + y^2 + 4x - 6y + 4 = 0$

Working: Diagram:



Centre is (-2, 3) and radius = 3

$\rightarrow$ equation is $(x + 2)^2 + (y - 3)^2 = 9$

$\rightarrow x^2 + 4x + 4 + y^2 - 6y + 9 = 9$

$\rightarrow x^2 + y^2 + 4x - 6y + 4 = 0$

26. **Answer:** $a < -8$, $a > 4$

Working: $3x^2 - (a + 2)x + 3 = 0$

Two distinct real roots $\rightarrow$ discriminant > 0

$\rightarrow (a + 2)^2 - 4(3)(3) > 0 \rightarrow a^2 + 4a - 32 > 0$

$\rightarrow a < -8$ or $a > 4$

27. **Answer:** $3 \tan x - \cos x + C$

Working: Integral of $\sec^2 x$ is $\tan x$, integral of $\sin x$ is $-\cos x$

28. **Answer:** Integration by substitution then by partial fractions

Working: First few steps are

$$\int \frac{\cos x(\sin x - 1)}{\sin^2 x + 5\sin x + 6}\,\mathrm{d}x \rightarrow \underbrace{\int \frac{u-1}{u^2+5u+6}\,\mathrm{d}u}_{\substack{\text{substitution:} \\ u = \sin x,\ \mathrm{d}u = \cos x}} \rightarrow \underbrace{\int \frac{4}{u+3} + \frac{-3}{u+2}\,\mathrm{d}u}_{\substack{\text{partial fractions:} \\ \text{factor denominator}}}$$

29. **Answer:** $\csc^2 x - \cot^2 x \equiv 1$

Working: Relevant identities are:
$\cos 2x = \cos^2 x - \sin^2 x \rightarrow \cos^2 2x - \sin^2 2x = \cos 4x$
$\sec^2 x = 1 + \tan^2 x \rightarrow \sec^2 x - \tan^2 x = 1$
$\csc^2 x = 1 + \cot^2 x \rightarrow \csc^2 x - \cot^2 x = 1$
$\sin^{-1} x + \cos^{-1} x = \pi/2$

30. **Answer:** 195

Working: $72 = 2 * 36 = 2 * 2 * 18 = 2 * 2 * 2 * 9 = 2^3 * 3^2$
By arranging in a grid it can be seen that,
sum of factors = $(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)$
$= (1 + 2 + 4 + 8)(1 + 3 + 9)$
= 195 (or just by finding all factors explicitly)

31. **Answer:** 23

Working: Square both sides of the given equation:
$(x + 1/x)^2 = 5^2 \rightarrow x^2 + 1/x^2 + 2x/x = 25$
$\rightarrow x^2 + 1/x^2 + 2 = 25$
$\rightarrow x^2 + 1/x^2 = 23$

32. **Answer:** *P* and *Q* are always equal to 1

Working: Sub $b = 1/a$ into P $\rightarrow P = a/(a + 1) + (1/a)/(1/a + 1)$
Sub $a = 1/b$ into Q $\rightarrow Q = 1/(1/b + 1) + 1/(b + 1)$
$P = a/(a + 1) + 1/(a + 1) = (a + 1) / (a + 1) = 1$
$Q = b/(b + 1) + 1/(b + 1) = (b + 1) / (b + 1) = 1$
So $P = Q = 1$.

33. **Answer:** *P* lies on the *y*-axis

Working: P is 1/3 the distance from A $\rightarrow P = (0, 3)$
$\rightarrow$ P lies on the *y*-axis

34. **Answer:** 2

Working: Inflection point → $f''(x) = 0$
Differentiating a polynomial reduces its degree by 1.
$f(x)$ = 4th degree → $f'(x)$ = 3rd degree → $f''(x)$ = 2nd degree
2nd degree polynomial (quadratic) has two solutions
→ max 2 inflection points.
(Could also consider general graph of quartic)

35. **Answer:** ln(0)

Working: tan is undefined at 90, 270...°
cot = 1/tan is undefined when tan = 0 → at 0, π, 2π…
Domain of $\cos^{-1} x$ is (-1, 1)
Domain of $\ln x$ is $x > 0$

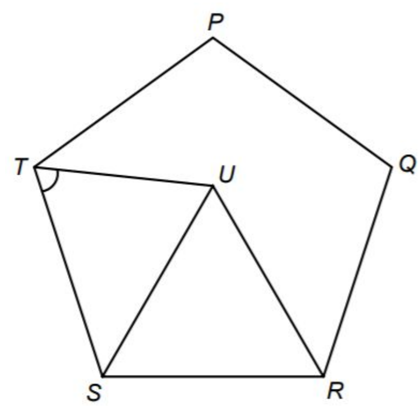
36. **Answer:** 110

Working:

	Passed 1st	*Did not pass 1st*	*Total*
Men	*110*	*190*	300
Women	*57*	143	200
Total	167	*333*	500

37. **Answer:** 66°

Working: Diagram:



TSR = internal angle of pentagon = 180(5-2)/5 = 540/5 = 108°
RSU = internal angle of triangle = 60°
TSU = 108 - 60 = 48
Triangle TSU is isosceles, so
STU = TUS = (180 - 48) / 2 = 66°.

38. **Answer:** $\frac{27}{20}$

Working: Increase by 125% = becomes 225% of its original value
→ increased price = 2.25p
Decreased by 40% → becomes 60% of its original value
→ final price = 2.25p * 0.5 = 1.35p = (27/20)p

39. **Answer:** 4

Working: $\log_2(5/4) + \log_2(6/5) + \log_2(7/6) + \ldots + \log_2(64/63)$
$= \log_2(5/4 * 6/5 * 7/6 * \ldots * 64/63)$
All numerators cancel with denominators except for first and last
$= \log_2(64/4) = \log_2(16) = 4$.

40. **Answer:** 0

Working: $(1 - 2x)^5(1 + 2x)^5$

$= [(1 - 2x)(1 + 2x)]^5$

$= (1 - 4x^2)^5$

This expansion will contain only even powers (1, x^2, x^4, x^6, x^8, x^{10}) so no x^3 term.

Section B: Standard

1. **Answer:** $15 < k < 20$

Working: Finding the turning points, $dy/dx = 0 \rightarrow 12x^3 - 12x^2 - 24x = 0$
$\rightarrow 12x(x^2 - x - 2) = 0 \rightarrow 6x(x - 2)(x + 1) \rightarrow x = 0, -1, 2$
Subbing into the equation to be solved, $y = 3x^4 - 4x^3 - 12x^2 + 20 - k$,
When $x = 0 \rightarrow y = 20 - k$,
when $x = 2 \rightarrow y = 52 - k$,
when $x = -1 \rightarrow y = 15 - k$

For a positive quartic graph to have four distinct real roots, two of the turning points are below the x-axis while the 'middle' turning point is above. The turning point at x = 0 will be positive for 20k > 0 (so k < 20). The other two turning points are both negative as long as 15k < 0 is satisfied. Therefore, k > 15 and the complete inequality is 15 < k < 20.

2. **Answer:** $y = 8x - 21 - x^2$

Working: Translate by 4 to the right and 3 up:
$y = x^2 \rightarrow y = (x - 4)^2 + 3$
Reflect in line $y = -1$:
$y = -2 - [(x - 4)^2 + 3] = -5 - (x - 4)^2 = -5 - x^2 + 8x - 16 = -21 + 8x - x^2$
$\rightarrow y = 8x - 21 - x^2$

3. **Answer:** $\cos\theta = 3/5$ or $-1/2$

Working: $7 \cos\theta - 3 \tan\theta \sin\theta = 1$
$\rightarrow 7 \cos\theta - 3 \sin^2\theta / \cos\theta = 1$
$\rightarrow 7 \cos^2\theta - 3 \sin^2\theta - \cos\theta = 0$
$\rightarrow 7 \cos^2\theta - 3(1 - \cos^2\theta) - \cos\theta = 0$
$\rightarrow 10 \cos^2\theta - \cos\theta - 3 = 0$
Solving as a quadratic, $\cos\theta = 3/5$ or $-1/2$

4. **Answer:** $\cos^{-1} x \approx \sqrt{2 - 2x}, \text{ for } x \text{ close to 1 such that } x \leq 1$

Working: Small angle approximation for cos x is cos x ≈ 1 - $x^2/2$ for x ≈ 0
Let x = $\cos^{-1}$ y i.e. find the inverse function:
→ y ≈ 1 - $(\cos^{-1} y)^2$ / 2 → $(\cos^{-1} y)^2$ ≈ 2(1 - y) → $\cos^{-1}$ y ≈ ±√(2 - 2y)
Reject -ve solution since $\cos^{-1}$ x is always positive (switch variables)
→ $\cos^{-1}$ x ≈√(2 - 2x)
When x ≈ 0, y ≈ cos x ≈ cos 0 ≈ 1 → valid when x ≈ 1.
(Must be less than 1 to remain inside the domain of $\cos^{-1}$ x)

5. **Answer:** 5/6

Working: Perpendicular lines → mp = -1 → p = -1/m
At (M, 0): Mp + 2 = 0 → M = -2/p
At (L, 0): Lm + 3 = 0 → L = -3/m
Since M - L = 5, we have
→ -2/p + 3/m = 5
→ (3p - 2m) / (mp) = 5
→ 2m - 3p = 5
→ 2m + 3/m = 5
→ $2m^2$ - 5m + 3 = 0
→ m = 3/2, m = 1 (reject m = 1 since given m > 1)
Then the value of p are
→ p = -2/3
→ m + p = 3/2 - 2/3 = 5/6

6. **Answer:** *K* = -(*a* - *b*)(*b* - *c*)(*c* - *a*)(*a* + *b* + *c*)

Working: Let a = b: K = $a^2(a^2 - a^2) + ac(a^2 - c^2) + ac(c^2 - a^2) = 0$
Let b = c: K = $ac(a^2 - c^2) + b^2(b^2 - b^2) + ac(c^2 - a^2) = 0$
Let a = c: K = $bc(c^2 - b^2) + bc(b^2 - c^2) + c^2(c^2 - c^2) = 0$
Since a = b → a - b = 0, and b = c → b - c = 0,
and a = c → a - c = 0, by the factor theorem, (a - b), (b - c) and
(a - c) must all be factors of K (since they make it equal to 0).
The only option with these is -(a - b)(b - c)(c - a)(a + b + c).

7. **Answer:** **1** and **2** only

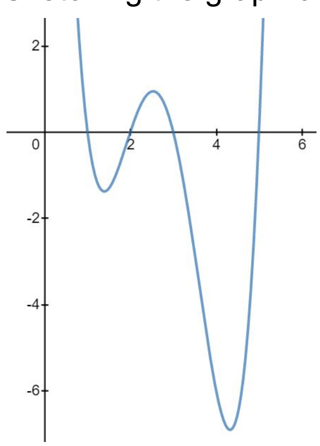
Working: $90^o < \theta < 180^o \rightarrow \sin\theta, \csc\theta > 0$ and all others are < 0
$\rightarrow \sin\theta \cos\theta$ = positive * negative = negative
$\rightarrow \cos\theta \cot\theta$ = negative * negative = positive
$\cos^2\theta > \sin^2\theta \rightarrow \cos^2\theta - \sin^2\theta > 0$
$\rightarrow \cos 2\theta > 0$
→ Not always true ($\cos 2\theta$ crosses the axis at 135^o)

8. **Answer:** 1.3490

Working: Looking for the range where area under curve at centre = 0.5.
InvNorm(0.25) = -0.6745, InvNorm(0.75) = 0.6745
→ Range = 0.6745 - (-0.6745) = 1.3490
(Or use central tail with area = 0.5 if possible)

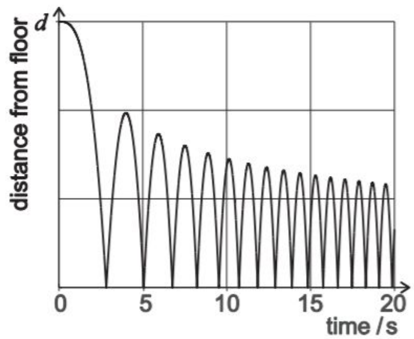
9. **Answer:** The coefficient of x in $F(x)$ is 30

Working: We have $F'(x) = f(x)$, so $F'(x) = 0$ when $x = 1, 2, 3, 5$
Sketching the graph of $f(x)$ as a quartic, we see



When $f(x) > 0$, $F'(x) > 0$ so $F(x)$ is increasing (up to a maximum)
When $f(x) < 0$, $F'(x) < 0$ so $F(x)$ is decreasing (down to a minimum)
→ maximums at 1 and 3, minimums at 2 and 5
$f(x)$ = 4th degree polynomial → $F(x)$ = 5th degree polynomial
Coefficient of x in $F(x)$ = constant term in $f(x) = (-1)(-2)(-3)(-5) = 30$.

10. **Answer:** The distance-time graph of the ball on a timescale showing the ball repeatedly bouncing on the spacecraft floor could be shown by



Working: $F(t) = 0.2t$. $F = mg \rightarrow g(t) = F/m = 0.2t/0.5 = 0.4t$
$\rightarrow$ gravitational acceleration = $0.4t$.
$\rightarrow$ speed = $0.2t^2 + C$ (at rest initially) $\rightarrow$ speed = $0.2t^2$
But graph shows $0.4t^2$.
$\rightarrow$ position = $(0.2/3)\ t^3 + C$ (let $C = 0$) $\rightarrow$ position = $t^3/15$
Time to hit floor $\rightarrow d = t^3/15 \rightarrow t^3 = 15d \rightarrow t = (15d)^{1/3}$
Gravity increases $\rightarrow$ maximum height of each bounce decreases.

11. **Answer:** 5

Working: $2 \sin^3 \theta - \sin \theta = 0$
$\rightarrow \sin \theta\ (2 \sin^2 \theta - 1) = 0$
$\rightarrow \sin \theta \cos 2\theta = 0$
$\rightarrow \sin \theta = 0$ for $\theta = 0, \pi$
$\rightarrow \cos 2\theta = 0$ for $-\pi/4, \pi/4, 3\pi/4$
$\rightarrow$ total 5 solutions

12. **Answer:** **2** and **6**

Working: **1**: cannot be equal magnitude since W = mg and R = mg cos θ (different since θ ≠ 0)
2: two forces (weight and reaction) would result in net acceleration down the slope, so there must be at least one more force to cancel out and make the ball move at constant speed (zero acceleration).
3: false: driving force = friction + mg sin θ (resolving along slope)
4: false: weight acts vertically downward, reaction acts normal
5: may or may not be true, so not 'must' be true
6: true: constant speed = 0 → a = 0 → F = 0

13. **Answer:** 1

Working: Integrate both sides from 2 to 4:

$$\int_2^4 f'(x)\,dx = \int_2^4 ax + g(x)\,dx$$

→ f(4) - f(2) = $\left[\frac{a}{2}x^2\right]_2^4 + \int_2^4 g(x)\,dx$
→ 18 = (8a - 2a) + 12
→ 6 = 6a
→ a = 1

14. **Answer:** 10 s

Working: Let the time interval be *t*.
Distance travelled by man = 9t
Speed of boy at time *t* = 5 + 0.8t
Distance of boy at time *t* = $5t + 0.4t^2$
→ $9t = 5t + 0.4t^2$
→ $0.4t^2 - 4t = 0$
→ t(0.4t - 4) = 0
→ t = 10 (t > 0)

15. **Answer:** $h(2x)$

Working: Using chain rule, f''(x) = g'(x + 1) = h(x) → f''(2x) = h(2x)

16. **Answer:** $\frac{m_R g}{m_R + m_Q}$

Working: Consider all motion relative to *P* i.e. consider *P* stationary.
Resolving forces in the pulley,
R: $m_R g - T = m_R a$
Q: $T = m_Q a$
Eliminating T,
→ $m_R g = m_R a + m_Q a$
→ $m_R g = a(m_R + m_Q)$
→ $a = m_R / (m_R + m_Q)$
If block P moves with this acceleration, Q and R would not move relative to P since they would have the same acceleration.

17. **Answer:** $g(\sin 25^\circ - \cos 25^\circ \tan 20^\circ)$

Working: At 20°, $mg \sin 20^\circ = \mu\, mg \cos 20^\circ$
→ $\mu = \sin 20^\circ / \cos 20^\circ = \tan 20^\circ$
At 25°, $mg \sin 25^\circ - mg \cos 25^\circ \tan 20^\circ = ma$
→ $a = g \sin 25^\circ - g \cos 25^\circ \tan 20^\circ$

18. **Answer:** $\frac{3x-2}{10x}$

Working: Starting from 4 / (3 - 5/(x + 1)), multiply top and bottom by x + 1:
→ 4(x + 1) / (3(x + 1) - 5)
→ (4x + 4) / (3x - 2)
Replace and multiply new top and bottom by 3x - 2:
→ (3x - 2) / (2(3x - 2) + 4x + 4)
→ (3x - 2) / 10x

19. **Answer:** $\frac{x}{|x|}$

Working: Considering the graph, we see that the gradient of |x| is -1 when x < 0 and 1 when x > 0.
Since |x| / x cancels out the x and leaves +1 or -1, we have:
|x| / x = -1 when x < 0 and |x| / x = 1 when x > 0.
This matches the gradient found.

20. **Answer:** P(X ≤ 10) = $\frac{1}{8}$

Working: The distribution of X will be skewed negatively (more likely to be closer to 20) since smaller values are more likely to be replaced with a larger value. So the mean of X will be more than 10.
P(X ≤ k) = P(all three rolls were less than or equal to *k*)
P(one roll less than or equal to *k*) = *k* / 20
Since each roll is independent and identically distributed (i.i.d.),
P(X ≤ k) = $(k / 20)^3$ → P(X ≤ 10) = $(10 / 20)^3$ = 1/8 = 0.125
P(X ≤ k - 1) = $((k - 1) / 20)^3$
Subtracting,
P(X ≤ k) - P(X ≤ k - 1) = $(k / 20)^3 - ((k - 1) / 20)^3$
P(X = k) = $(k^3 - (k - 1)^3) / 20^3 = (k^3 - (k - 1)^3) / 8000$

21. **Answer:** 6

Working: P(More than n scoops bought by n customers)
= P(at least one customer bought more than one scoop)
= 1 - P(all customers bought one scoop)
= $1 - (1/6)^n$
→ $1 - (1/6)^n > 0.9999$
→ n > 5.14…
→ n = 6

22. **Answer:** Both **1** and **2**

Working: From calculator: $Q_1 = 11$, $Q_2 = 18$, $Q_3 = 33$, mean = 21.83
Test for outliers:
Upper bound = $Q_3 + 1.5 * IQR = 33 + 1.5(33 - 11) = 66$
Lower bound = $Q_1 - 1.5 * IQR = 11 - 1.5(33 - 11) = -22$
No values are > 66 or < -22 so no outliers.
Skewness: mean > median (21.83 > 18) → positive skew
(*Alt.*: $Q_3 - Q_2 = 15$ and $Q_2 - Q_1 = 7$ → 15 > 7 → positive skew)

23. **Answer:** Pearson's product-moment correlation coefficient

Working: Spearman rho (rank) used on ranked data to check for association (generalised correlation: nonlinear trends)
Pearson (PMCC) is used for correlation only
Normal distribution would not be appropriate since GDPs are likely not normally distributed around the world, and also the data is bivariate so would need modification

24. **Answer:** $P(99.5 \leq X \leq 400.5)$ under $X \sim N(250, 125)$

Working: $X \sim B(500, 0.5)$. For normal approximation,
Mean = np, Variance = np(1 - p) → $X \sim N(250, 125)$
For the continuity correction, the bounds are extended to round up the lower bound and round down the upper bound:
→ $P(99.5 \leq X \leq 400.5)$.

25. **Answer:** $\frac{1}{\sqrt[3]{100}}$

Working: $\frac{1+x}{1-x^2} = \frac{1+x}{(1-x)(1+x)} = \frac{1}{1-x} = 1 + x + x^2 + ...$

$\rightarrow$ f(x) = 1 + x + x^2

Formula for error: $\frac{|exact - approximation|}{exact}$ $(\times 100\%)$

$\rightarrow$ error = $\frac{\left|\frac{1}{1-x} - (1+x+x^2)\right|}{\frac{1}{1-x}}$

Multiply top and bottom by (1 - x):

$\rightarrow |1 - (1-x)(1+x+x^2)|$

$\rightarrow |1 - (1+x+x^2-x-x^2-x^3)|$

$\rightarrow |x^3|$

If error < 1% (so error fraction < 0.01), then

$|x^3| < 0.01 \rightarrow |x| < \sqrt[3]{0.01}$

$\rightarrow |x| < \frac{1}{\sqrt[3]{100}}$ so the maximum value is $a = \frac{1}{\sqrt[3]{100}}$.

26. **Answer:** 32!

Working: $f(x) = (x^3(1 + x))^8 = x^{24}(1 + x)^8$

This is the product of x^{24} and $(1 + x)^8$, which will be an 8th degree polynomial. By the binomial theorem, we see the coefficient of x^8 will be 1, so overall we have

f(x) = 32nd degree polynomial, coefficient of x^{32} is 1

f(x) = x^{32} + [lots of lower order terms]

Differentiating

$f'(x) = 32x^{31} + \ldots$

$f''(x) = 32 * 31 * x^{30} + \ldots$

$f'''(x) = 32 * 31 * 30 * x^{29} + \ldots$

There is a clear pattern, so

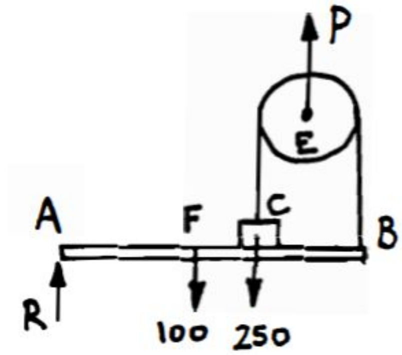
$f^{(32)}(x) = 32 * 31 * 30 * 29 * \ldots * 1 * x^0$

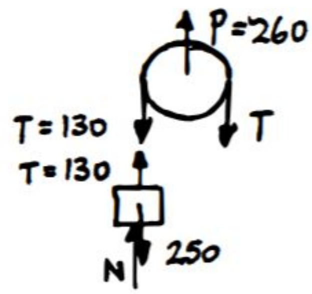
= 32 * 31 * 30 * 29 * … * 1 (= a constant, i.e. *n* = 32)

= 32! (by definition of factorial)

27. **Answer:** R_A = 90 N, R_{bar} = 120 N

Working: Force diagram of whole system and FBD of pulley and block:





Taking moments about *E*,
R_A * 1.25 = 100 * 0.5 + 250 * 0.25
→ R_A = 90 N.
Resolving forces vertically, we must include the force on pulley, *P*:
P + R_A = 100 + 250
→ P = 350 - 90 = 260 N
Considering free-body of pulley, tensions act on both sides, so each tension has 130 N.
Then, considering the block,
130 + R_{bar} = 250 → R_{bar} = 120 N.

28. **Answer:** *c* = 1/2 and *d* = 11/2

Working: The transformations are:
Step **1**: translation by vector (1/8)**j** → $y = e^{-x} + 1/8$
Step **2**: stretch parallel to *y*-axis, scale factor 16 → $y = 16e^{-x} + 2$
Step **3**: stretch parallel to *y*-axis, scale factor 1/2 → $y = 8e^{-x} + 1$
Step **4**: reflect in line $y = 5.5$ → $y = 11 - (8e^{-x} + 1) = 10 - 8e^{-x}$
$a = 1/8$, $b = 16$, $c = 1/2$, $d = 11/2$

29. **Answer:** $-k^2(4k+3)^2$

Working: Rewrite the given sum of sequence in terms of the sum of cubes:

$1^3 - 2^3 + 3^3 - 4^3 + 5^3 - \ldots - (2k)^3$

$= [1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + (2k)^3] - 2[2^3 + 4^3 + \ldots (2k)^3]$

The first series is the sum of the first 2k cubes, the second series is the sum of the first k even cubes:

$$= \sum_{r=1}^{2k} r^3 - 2\sum_{r=1}^{k} (2r)^3$$

$$= \sum_{r=1}^{2k} r^3 - 2\sum_{r=1}^{k} 8r^3$$

$$= \sum_{r=1}^{2k} r^3 - 16\sum_{r=1}^{k} r^3$$

Using the given result,

$$= \frac{(2k)^2}{4}(2k+1)^2 - 16 \times \frac{k^2}{4}(k+1)^2$$

$$= k^2(2k+1)^2 - 4k^2(k+1)^2$$

$$= k^2[(2k+1)^2 - 2^2(k+1)^2]$$

$$= k^2[(2k+1)^2 - (2k+2)^2]$$

Difference of two squares:

$$= k^2(2k+1+2k+2)(2k+1-2k-2)$$

$$= -k^2(4k+3)$$

30. **Answer:** always odd

Working: $a^3 - a + 1 = a(a^2 - 1) + 1 = a(a + 1)(a - 1) + 1$

The first term is the product of three consecutive integers, so it must be both a multiple of 2 and a multiple of 3, so overall a multiple of 6. Then adding 1 → one more than a multiple of 6.

→ always odd

Counterexamples:

a = 3 → $a^3 - a + 1 = 25 = 5^2$ (perfect square)

a = 2 → $a^3 - a + 1 = 7$ (prime)

31. **Answer:** **1** or **3**

Working: *Proof by deduction:*

$\cos \theta + \sin \theta = R \sin(\theta + \alpha)$

$\cos \theta + \sin \theta = \sqrt{2} \sin(\theta + \pi/4)$

Since the range of $\sin \theta$ is $-1 \le \sin \theta \le 1$, and the translation of $\pi/4$ to the left does not affect the range,

$\rightarrow -\sqrt{2} \le \sqrt{2} \sin(\theta + \pi/4) \le \sqrt{2}$

$\rightarrow \cos \theta + \sin \theta \le \sqrt{2}$.

Proof by contradiction:

Assume there exists some θ such that $\cos \theta + \sin \theta > \sqrt{2}$.

$\rightarrow (\cos \theta + \sin \theta)^2 > 2$

$\rightarrow \cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta > 2$

$\rightarrow 2 \sin \theta \cos \theta > 1$

$\rightarrow \sin 2\theta > 1$

But the range of $\sin 2\theta$ is $-1 \le \sin 2\theta \le 1$, so this is a contradiction.

$\rightarrow \cos \theta + \sin \theta \le \sqrt{2}$.

Proof by exhaustion would not work since there are an infinite number of cases ("all real θ") and disproof by counterexample would not work for the same reason.

32. **Answer:** $y = 5^{t+2}$

Working: General form of straight line:

$\ln y = mt + c$

$2 \ln 5 = c$

$4 \ln 5 = 2m + 2 \ln 5$

$\rightarrow m = \ln 5$

$\rightarrow \ln y = (\ln 5)t + (2 \ln 5)$

$\rightarrow \ln y = \ln 5^t + \ln 25$

$\rightarrow \ln y = \ln(5^t * 25)$

$\rightarrow y = 25 * 5^t$

$\rightarrow y = 5^2 * 5^t$

$\rightarrow y = 5^{t+2}$

33. **Answer:** $\dfrac{1}{x^2-1}$

Working: The expression would require quotient rule inside two chain rules, which would be too complicated, so we will simplify first:
Using log properties,

$$\ln\sqrt{\frac{1-x}{1+x}} = \ln\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\frac{1-x}{1+x} = \frac{1}{2}\left(\ln(1-x) - \ln(1+x)\right)$$

Then easily differentiating with chain rule,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\ln(1-x) - \ln(1+x) = \frac{1}{2}\left(\frac{-1}{1-x} - \frac{1}{1+x}\right)$$

$$= \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) = \frac{1}{2}\left(\frac{2}{x^2-1}\right) = \frac{1}{x^2-1}.$$

34. **Answer:** 10,264,320

Working: This term will be when (power of 2x) = 2 * (power of $3/x^2$) so that the sum of the powers is 12 and with the terms included it is x^0 (constant term).
= ${}^{12}C_4 * 2^8 * 3^4$ = 495 * 256 * 81 = 10264320

35. **Answer:** 2,046

Working:

Month:	1	2	3	…
Twigs:	1	2	4	...
Leaves:	2	4	8	…

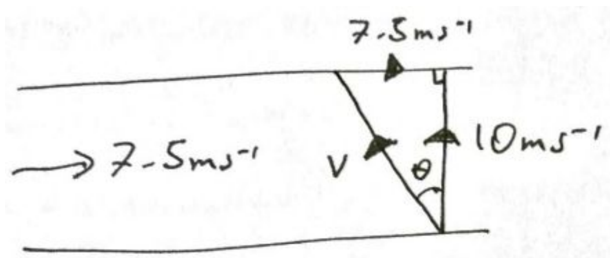
This is geometric sequence, so
$S_{10} = 2(2^{10} - 1) / (2 - 1) = 2(1024 - 1) = 2046$

36. **Answer:** $0 < x < \frac{2}{3}$ or $x > 1$

Working: We cannot multiply by *x* since this would flip the inequality sign if x was negative. Instead multiply by x^2 (since x^2 is always positive, no sign change will occur.)
$\rightarrow 5x^2 - 3x^3 < 2x$
$\rightarrow 3x^3 - 5x^2 + 2x > 0$
$\rightarrow x(3x^2 - 5x + 2) > 0$
$\rightarrow x(3x - 2)(x - 1) > 0$
For a cubic polynomial with positive x^3 term, the interval above the x-axis will be the first closed interval and the last open interval:
$\rightarrow 0 < x < 2/3$ or $x > 1$.

37. **Answer:** speed = 12.5 ms^{-1}, angle = 127°

Working: Drawing a diagram of the velocity vectors,



Magnitude = $|\mathbf{v}| = \sqrt{(7.5^2 + 10^2)} = 12.5\ ms^{-1}$
Angle = $\tan^{-1}(7.5/10) = 37°$
But the angle of the velocity vector is at 90° to the direction of travel, so angle = 37 + 90 = 127°.

38. **Answer:** None of these

Working: Terms are: 4, 5, 9, 14, 23
Differences: 1, 4, 5, 9
The differences are the same as the terms but shifted by 2 places:
$S_n - S_{n-1} = S_{n-2} \rightarrow S_n = S_{n-1} + S_{n-2}$.
Since all terms are positive, the sequence is always increasing.
The next term will be 23 + 14 = 37.

39. **Answer:** left once and right twice

Working: P(LLL) = $(1/3)^3$ = 1/27
P(RRR) = $(2/3)^3$ = 8/27
P(LRR) = ${}^3C_1$ * $(2/3)^2$ * (1/3) = 4/9 = 12/27
P(LLR) = ${}^3C_1$ * (2/3) * $(1/3)^2$ = 2/9 = 6/27
The largest probability is LRR.

40. **Answer:** 0.14%

Working: $f(x) = 2x(4-x^2)^{-\frac{1}{2}} = x(1-x^2/4)^{-1/2}$

$$= x\left(1 + \frac{1}{2}\cdot\frac{x^2}{4} + \frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{(\frac{-x^2}{4})^2}{2} + \cdots\right)$$

$$= x + \frac{1}{8}x^3 + \frac{3}{128}x^5$$

Approximation = $\int_0^1 x + \frac{1}{8}x^3 + \frac{3}{128}x^5\ dx$ = 0.53515625…

Exact = $\int_0^1 \frac{2x}{\sqrt{4-x^2}}\ dx$ = 0.5358983849…

Error = |Exact - Approx| / Exact = 0.00138…
Percentage = 0.00138 * 100% = 0.138% = 0.14%.

41. **Answer:** $\frac{9}{28}$

Working: P(all 3 different) = P(second one not same as first) * P(third one not same as first or second)
= 6/8 * 3/7 = 18/56 = 9/28.

42. **Answer:** $\frac{a-b}{2}$

Working: Using vectors,

EF = -½ AD + AB + ½ BC

EF = ½ AD + DC - ½ BC

Adding together,

2 EF = AB + DC

Since AB = *a* and CD = *b* → DC = -*b*,

2 EF = a - b → EF = (a - b)/2

Alternative method: midpoint theorem

Extend line segment EF to meet line BD at point X.

Then FX = b/2 since triangles BCD and BFX are similar (s.f. ½)

And EX = a/2 since triangles ABD and EXD are similar (s.f. ½)

→ EF = EX - FX = a/2 - b/2 = (a - b)/2.

43. **Answer:** $(x+3y+4)(x-y)$

Working: Factor by grouping:

= $(x^2 + 2xy - 3y^2) + (4x - 4y)$

= $(x + 3y)(x - y) + 4(x - y)$

= $(x + 3y + 4)(x - y)$

Section C: Hard

1. **Answer:** $\frac{2}{e^2}$

Working: Gradient of tangent = $-1/t$

Point-slope for tangent: $y + \ln t = (-1/t)(x - t)$

To find coordinates of R, let $y = 0 \rightarrow \ln t = (t - x)/t$

$\rightarrow x = t - t \ln t \rightarrow x = t(1 - \ln t)$.

Area of triangle = $A(t) = 1/2 * \text{base} * \text{height}$

$A(t) = 1/2 * [t(1 - \ln t) - t] * [-\ln t]$

$= -1/2 * [t \ln t - t \ln^2 t - t \ln t]$

$= 1/2 * (t \ln^2 t)$

Optimise A(t) by finding its maximum:

$A'(t) = 0 \rightarrow 1/2 * (\ln^2 t + t * 2 \ln t / t) = 0$

$\rightarrow \ln^2 t + 2 \ln t = 0 \rightarrow \ln t (\ln t + 2) = 0$

$\rightarrow \ln t = 0$ or $\ln t = -2$

$\rightarrow t = 1$ or $t = e^{-2}$

Since $t < 1$, reject it and take $t = e^{-2}$.

Subbing t back in, we get

$A(e^{-2}) = 1/2 * (e^{-2} * (-2)^2) = 2 e^{-2} = 2/e^2$.

2. **Answer:** **1** only

Working:

1: $S = \left\{ \dfrac{n}{3n-1} : n \in \mathbb{N} \right\}$ means all fractions with natural numerator and denominators forming the sequence $u_n = 3n - 1$. This matches.

2: $S = \left\{ \dfrac{p}{q} : p \in \mathbb{N},\ pq \cup \mathbb{P} = \varnothing \right\}$ means all fractions with natural numerator and any denominator so that their product is never prime. Counterexample - p = 1, q = 2 → pq = 2 which is prime but 1/2 is in the set so this does not match.

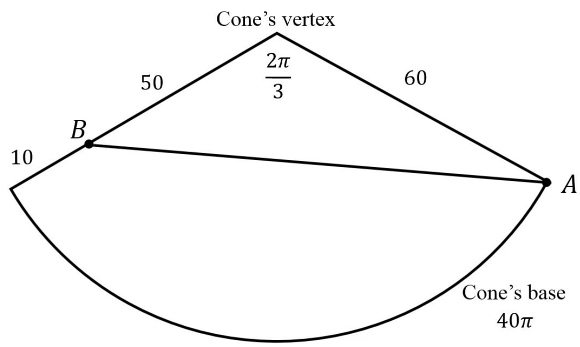
3: $S = \{n \in \mathbb{Q} : n(3k-1) \in \mathbb{N},\ k \in \mathbb{N}\}$ means all rational numbers such that multiplying it by 3k - 1 gives an integer. However, since n can be any rational number, not just those in the set, so there are multiple n for each given k.

E.g. n = 1/5, k = 2 → n(3k - 1) = 1/5 * 5 = 1 = integer, but 1/5 is not in the set.

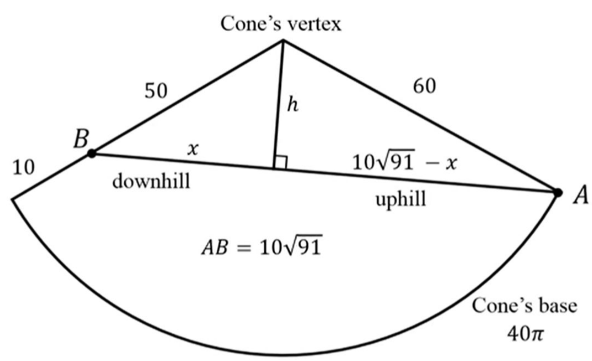
This gives a larger set than **1** so cannot match.

3. **Answer:** $\frac{400}{\sqrt{91}}$

Working: Since the track is curved, it is difficult to work with in its current form. By considering the net of the cone, we can reduce the problem to 2D, and then the shortest line is a straight line.
The cone unwraps to form a sector with central angle
(equating arc length with circumference of base)
$60\theta = 2\pi * 20 \rightarrow \theta = (40/60)\pi = 2\pi/3$:



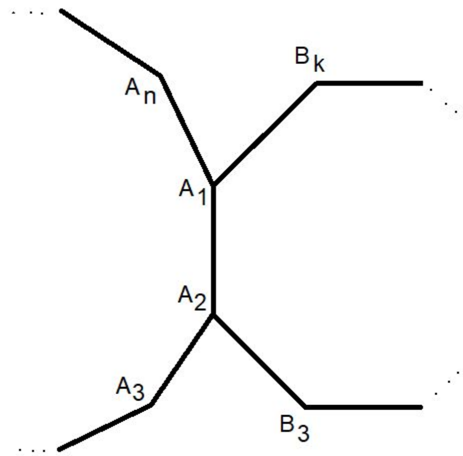
The 'curved' track is now the straight line AB.
Using cosine rule in ABC,
$AB = \sqrt{(50^2 + 60^2 - 2 * 50 * 60 * \cos 2\pi/3)}$
$AB = \sqrt{9100} = 10\sqrt{91}$
The downhill portion is when the distance from a point on the line to the vertex is increasing, i.e. the left-most part (nearest B).
Sketching the perpendicular and labelling relevant sides,



Pythagoras in left and right triangles:
Left: $x^2 + h^2 = 2500$, Right: $h^2 + (10\sqrt{91} - x)^2 = 3600$
Subtract left from right to eliminate h,
$\rightarrow 9100 - 2(10\sqrt{91})x = 3600 - 2500$
$\rightarrow 20\sqrt{91}\, x = 8000$
$\rightarrow x = 400 / \sqrt{91}$

4. **Answer:** Triangle $A_1A_3B_3$ cannot have an obtuse angle.

Working: Drawing a diagram of general polygons,



Angle $B_kA_1A_n$ = exterior angle of polygon A + exterior angle of polygon B

= 360/n + 360/k = 360(n + k)/nk

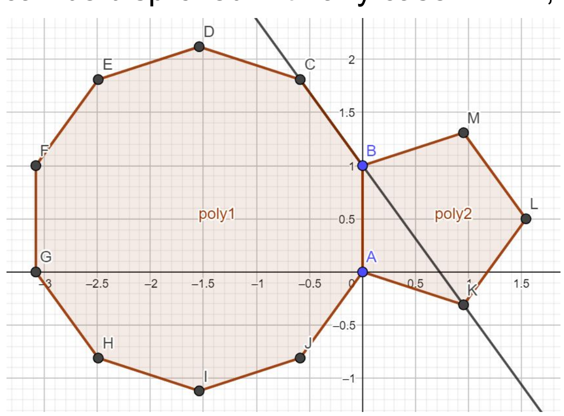
Since n > k ≥ 5, let n = 6 and k = 5. Then $A_1A_2B_3$ = 108° and $A_1A_2A_3$ = 120°, leaving $A_3A_2B_3$ = 132°.

→ angles of $A_1A_3B_3$ are 66°, 54° and 60°.

Since in the limit as n and k grow to infinity, $A_1 \to 0$ while A_3 and $B_3 \to 90°$, all angles are bounded from above and below, so can never be above 90°. (Can also argue same using angle in semicircle theorem with k = 5 and n → infinity.)

The compound shape has one line of symmetry, bisector $\perp A_1A_2$.

Investigating (2) and (3) with full rigor and formality is difficult but can be disproved with any case n = 2k, e.g. n = 10, k = 5:



5. **Answer:** $$a\omega \sin \omega t + \frac{a^2\omega \sin \omega t \cos \omega t}{\sqrt{b^2 - a^2 \sin^2 \omega t}}$$

Working: At any general angle θ,

Position vector *OA*: $\mathbf{r}_A = (a \cos \theta)\mathbf{i} + (a \sin \theta)\mathbf{j}$

Let angle *OBA* = φ. Then the position vector *AB* is

$\mathbf{r}_{B/A} = (b \cos \varphi) - (a \sin \theta)$ [displacement of B with respect to A]

Adding together, vertical components cancel (because *OB* is horizontal) so we get

$\mathbf{r}_B = (a \cos \theta + b \cos \varphi)\mathbf{i}$

But considering triangles made by *OA* and *AB*, we see that

$a \sin \theta = b \sin \varphi \rightarrow \sin \varphi = (a \sin \theta) / b$

$\rightarrow \cos \varphi = \sqrt{1 - \left(\frac{a \sin \theta}{b}\right)^2}$

$\rightarrow \cos \varphi = \sqrt{1 - \frac{a^2 \sin^2 \theta}{b^2}}$

$\rightarrow \cos \varphi = \frac{\sqrt{b^2 - a^2 \sin^2 \theta}}{b}$

Putting this in,

$\mathbf{r}_B = (a \cos \theta + \sqrt{b^2 - a^2 \sin^2 \theta})\mathbf{i}$

Now since ω = dθ/dt (given), we have θ = ωt (since θ = 0 at t = 0).

$\rightarrow \mathbf{r}_B = (a \cos \omega t + \sqrt{b^2 - a^2 \sin^2 \omega t})\mathbf{i}$

Differentiating with respect to t, we get velocity of B:

$\rightarrow \mathbf{v}_B = (-a\omega \sin \omega t + \frac{-2a^2\omega \sin \omega t \cos \omega t}{2\sqrt{b^2 - a^2 \sin^2 \omega t}})\mathbf{i}$

$\rightarrow \mathbf{v}_B = (-a\omega \sin \omega t - \frac{a^2\omega \sin \omega t \cos \omega t}{\sqrt{b^2 - a^2 \sin^2 \omega t}})\mathbf{i}$

But since velocity will be directed left, the speed is the magnitude, so switch the signs:

$\rightarrow \text{speed} = |\mathbf{v}_B| = a\omega \sin \omega t + \frac{a^2\omega \sin \omega t \cos \omega t}{\sqrt{b^2 - a^2 \sin^2 \omega t}}$

6. **Answer:** The maximum vertical velocity of the boat is $\frac{\pi}{5}$ ms^{-1}

Working: Consider the wave at a constant **time** (e.g. $t = 0$) → position y(x).
$y(x) = A \sin(kx)$.
Period of $\sin(x) = 2\pi$ → period of $\sin(kx) = 2\pi/k$
(since stretched by factor 1/k).
Period of water wave (in position basis) = wavelength = 10 m
→ $2\pi/k = 10$ → $k = 2\pi/10 = \pi/5$. (Units are m^{-1}).

Consider the wave at a constant in **space** (position) (e.g. $x = 0$)
→ position = y(t).
$y(t) = A \sin(-\omega t)$
New period = $2\pi/\omega$ = distance/time = 10/2 = 5
→ $2\pi/\omega = 5$ → $\omega = 2\pi/5$, units = s^{-1}.

Vertical velocity = $dy/dt = -A\omega \cos(kx - \omega t)$ [since x is a constant].
Maximum velocity = $A\omega$ [since $|\cos(...)| \leq 1$] = $0.5 * 2\pi/5 = \pi/5$ ms^{-1}.
Vertical acceleration = $d^2y/dt^2 = -A\omega^2 \sin(kx - \omega t)$
Maximum acceleration = $A\omega^2 \neq A$.

7. **Answer:** $$\alpha(t) = \sin^{-1}\left(\frac{1}{2} + \frac{1-t}{2(1+\sqrt{2})}\right)$$

Working: Consider one of these projectiles in particular, launched at time *t* after the first and at angle α. Let *u* be the constant initial speed.
With v = u + at in the vertical direction,
s = 0, *u* = u sin α, *v* = -u sin α, *a* = -g, *t* = t
→ -u sin α = u sin α - gt

$$t = \frac{2u\sin\alpha}{g}$$

Now, for a particular projectile, the key equation is
(time between firing first projectile and firing this projectile)
+ (time of flight of this projectile) = (time of flight of first projectile)

$$t + \frac{2u\sin\alpha}{g} = \frac{2u\sin\frac{\pi}{4}}{g}$$

Solving for α and evaluating sin(π/4) = √2/2 gives

$$2u\ sin\ \alpha = u\sqrt{2} - gt$$

$$\alpha = sin^{-1}\left(\frac{\sqrt{2}}{2} - \frac{gt}{2u}\right)$$

The particular speed, *u*, that makes this possible is
found by verifying the boundary condition of α(1) = $\frac{\pi}{6}$:
1 + (2u sin π/6)/g = (2u sin π/4)/g → $u = (1+\sqrt{2})g.$)
Putting this back in,

$$\alpha = \sin^{-1}\left(\frac{\sqrt{2}}{2} - \frac{t}{2(1+\sqrt{2})}\right)$$

$$\alpha = \sin^{-1}\left(\frac{1}{2} + \frac{\sqrt{2}-1}{2} - \frac{t}{2(1+\sqrt{2})}\right)$$

Divide top and bottom by 1 - √2 and rationalise denominator:

$$\alpha = \sin^{-1}\left(\frac{1}{2} + \frac{1}{2(1+\sqrt{2})} - \frac{t}{2(1+\sqrt{2})}\right)$$

$$\alpha = \sin^{-1}\left(\frac{1}{2} + \frac{1-t}{2(1+\sqrt{2})}\right)$$

Alternative: this form could be reached directly without algebraic manipulation by differentiating the 'key equation' and solving the differential equation, integrating with boundary conditions.

8. **Answer:** **1** and **3** only

Working: **1**: Minimum time → dT/dx = 0 → $\frac{5\times 2x}{2\sqrt{36+x^2}} - 4 = 0$

→ $\frac{5x}{\sqrt{36+x^2}} = 4$ → $5x = 4\sqrt{(36 + x^2)}$ → $25x^2 = 16(36 + x^2)$

→ $9x^2 = 576$ → $x^2 = 64$ → x = 8

→ T = 98 tenths of a second = 9.8 seconds

2: T(x) = total time = time spent in water + time spent on land
= (distance in water / speed in water) + (distance on land / speed on land)
Matching the terms, the $\sqrt{(36 + x^2)}$ clearly represents Pythagoras for the diagonal distance across the water: length = x, width = √36 = 6.

3: So 4(20 - x) is the time on land.
→ distance = 20 - x, speed = ¼ = 0.25
Units of speed are metres per tenths of a second, since units of time are tenths of second.
0.25 m/cs = 2.5 m/s.

9. **Answer:** $\sqrt{\frac{mg}{\alpha}}$

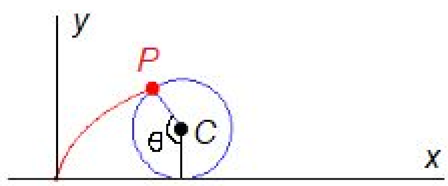
Working: There is no need to solve the differential equation - resolve forces:
$mg - \alpha v^2 = ma$
At terminal velocity, a = 0 → $mg = \alpha v^2$ → $v = \sqrt{(mg/\alpha)}$.

10. **Answer:** $\frac{13}{9}$

Working: sin(x + y) = 11/a → sin x cos y + cos x sin y = 11/a
sin(x - y) = 2/a → sin x cos y - cos x sin y = 2/a
Add and subtract these together:
sin(x + y) + sin(x - y) = 2 sin x cos y = 13/a
sin(x + y) - sin(x - y) = 2 cos x sin y = 9/a
Divide first by second:
(2 sin x cos y) / (2 cos x sin y) = (13/a) / (9/a)
tan x cot y = 13/9
(tan x) / (tan y) = 13/9.

11. **Answer:** 3π

Working: Let the origin be *P* at time *t* = 0 so that the *x*-axis is the ground and the *y*-axis is the vertical passing through the initial centre of the wheel. Let centre of wheel be *C*.



(Position vector of *P*) = (position vector of *C*) + (position vector of *P* relative to *C*)
$\mathbf{r}_P = \mathbf{r}_C + \mathbf{r}_{P/C}$
Distance travelled in one revolution = circumference of wheel = 2π
Time taken = 2π/speed = 2π
Let the angle turned through by the wheel be θ. Then,
θ = t/(2π) * 2π = t
$\mathbf{r}_C = (\theta/2\pi * 2\pi)\mathbf{i} + \mathbf{j} = t\,\mathbf{i} + \mathbf{j}$
Also, by trig definitions,
$\mathbf{r}_{P/C} = (-\sin\theta)\mathbf{i} + (-\cos\theta)\mathbf{j} = (-\sin t)\mathbf{i} + (-\cos t)\mathbf{j}$
Adding, $\mathbf{r}_P = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$
For the area, integrate parametrically,
$$\int_0^{2\pi} y(t)\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = \int_0^{2\pi} (1-\cos t)^2\ \mathrm{d}t = 3\pi.$$

12. **Answer:** $\frac{4}{105}$

Working: $x^4 + y^3 = x^2y \to x^2/y + y^2/x^2 = 1$

Let $u = y/x$

$\to x/u + u^2 = 1$

$\to \quad x = u(1 - u^2) = u - u^3$

and $\quad y = xu = u^2 - u^4$

The interval of integration is:

$x = 0$, $y = 0$. Since the point of interest is the origin, u represents the gradient ($y/x = dy/dx$). From the image, the gradient at the bounds of the loop is 1 and 0 (positive) so $0 < u < 1$.

Then we have, integrating parametrically,

$$\int_0^1 \left(u^2 - u^4\right)\left(1 - 3u^2\right)\ \mathrm{d}u = -\frac{4}{105}$$

The negative sign is due to the opposite orientation of the curve. This doesn't matter since area is magnitude of integral $\to 4/105$.

13. **Answer:** $\sqrt[3]{\tan x}$

Working: Differentiating in its current form would be far too complicated (several chain rules, quotient rules, etc) so simplify first.
Let *y* equal the expression. Using log properties and making the substitution $t = \tan^{2/3} x$, we get

$$y = \frac{1}{4}\ln\left(t^2 - t + 1\right) - \frac{1}{2}\ln\left(t+1\right) + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}t - \frac{1}{\sqrt{3}}\right)$$

Now differentiating w.r.t *t*,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t-1}{4(t^2-t+1)} - \frac{1}{2(t+1)} + \frac{1}{1+\left(\frac{2}{\sqrt{3}}t - \frac{1}{\sqrt{3}}\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t-1}{4(t^2-t+1)} - \frac{1}{2(t+1)} + \frac{3}{4(t^2-t+1)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t+1}{2(t^2-t+1)} - \frac{1}{2(t+1)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2(t+1)^2 - 2(t^2-t+1)}{4(t^2-t+1)(t+1)} = \frac{3t}{2(t^3+1)}$$

Putting $t = \tan^{2/3} x$ back in and using chain rule with
$\mathrm{d}t/\mathrm{d}x = 2/3 \tan^{-1/3} x \sec^2 x$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\tan^{2/3} x}{\tan^2 x + 1} \cdot \tan^{-1/3} x \sec^2 x$$

Replacing $\tan^2 x + 1 = \sec^2$ and cancelling, we get $\mathrm{d}y/\mathrm{d}x = \tan^{1/3} x$.

14. **Answer:** 4

Working: Implicit first derivative: $15x^4 + 10y^9 y' = 15x^2y + 5x^3y'$
Stationary → $y' = 0$ → $15x^4 = 15x^2y$ → $x^2(x^2 - y) = 0$
→ Stationary points when $x = 0$ and when $y = x^2$
Put $x = 0$ back in: $y^{10} = 18$ → 2 solutions $(0, 18^{1/10})$ and $(0, -18^{1/10})$
Put $y = x^2$ back in: $3x^5 + x^{20} = 5x^5 + 18$
→ $x^{20} - 2x^5 - 18 = 0$. Let $t = x^5$ → $t^4 - 2t - 18 = 0$
By graphing (or by considering $d^2/dt^2 = 12t^2 > 0$), this has 2 roots.
Since $t = x^5$ has one real solution, total solutions = 2*1 + 2 = 4.

15. **Answer:** $\frac{3\sqrt{13}}{13}$

Working: Let x = 1 (so then distance = k).

Horizontal distance travelled by ant:

$$\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^5 - \left(\frac{2}{3}\right)^7 + \cdots$$

Vertical distance travelled by ant:

$$1 - \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 - \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 - \cdots$$

Total horizontal distance: geometric series with a = 2/3, r = -4/9:

$$\frac{\frac{2}{3}}{1+\frac{4}{9}} = \frac{6}{13}$$

Total vertical distance: geometric series with a = 1, r = -4/9:

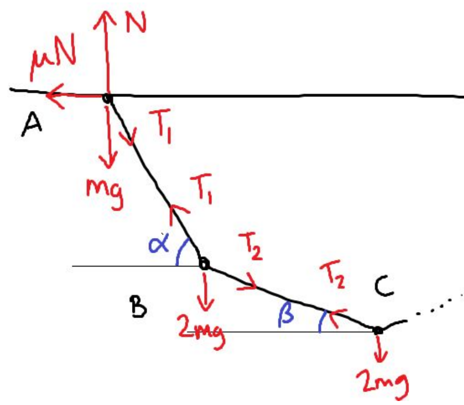
$$\frac{1}{1+\frac{4}{9}} = \frac{9}{13}$$

Total distance, by Pythagoras, is then

$$\sqrt{\left(\frac{6}{13}\right)^2 + \left(\frac{9}{13}\right)^2} = \sqrt{\frac{9}{13}} = \frac{3\sqrt{13}}{13}$$

16. **Answer:** $$2a\left(\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{2}}\right)$$

Working: Since the system is symmetrical, we only need to work with one half of the diagram. Label angles, tensions and external forces:



Resolving vertical forces on the entire system (all tensions and frictions cancel):
2N = 8mg → N = 4mg → friction = 1/4 N = mg.

Resolving forces at A (and at E by symmetry):
Vertical: T_1 sin α + mg = N → T_1 sin α = 3mg
Horizontal: T_1 cos α = mg
Dividing, tan α = 3. Drawing a right triangle, we see that
→ cos α = 1/√10 and sin α = 3/√10.

Resolving forces at B (and at D by symmetry):
Vertical: T_1 sin α = 2mg + T_2 sin β → T_2 sin β = mg
Horizontal: T_1 cos α = T_2 cos β → T_2 cos β = mg
Dividing, tan β = 1. Drawing a right triangle, we see that
→ sin β = cos β = 1/√2.

So the total horizontal distance is 2 * (*a* cos α + *a* cos β)
= 2*a*(1/√10 + 1/√2).