DIFFERENTIAL EQUATIONS

Boundary Value Problems & Fourier Series

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Chapter 8 : Boundary Value Problems & Fourier Series

In this chapter we'll be taking a quick and very brief look at a couple of topics. The two main topics in this chapter are Boundary Value Problems and Fourier Series. We'll also take a look at a couple of other topics in this chapter. The main point of this chapter is to get some of the basics out of the way that we'll need in the next chapter where we'll take a look at one of the more common solution methods for partial differential equations.

It should be pointed out that both of these topics are far more in depth than what we'll be covering here. In fact, you can do whole courses on each of these topics. What we'll be covering here are simply the basics of these topics that well need in order to do the work in the next chapter. There are whole areas of both of these topics that we'll not be even touching on.

Here is a brief listing of the topics in this chapter.

Boundary Value Problems – In this section we'll define boundary conditions (as opposed to initial conditions which we should already be familiar with at this point) and the boundary value problem. We will also work a few examples illustrating some of the interesting differences in using boundary values instead of initial conditions in solving differential equations.

Eigenvalues and Eigenfunctions – In this section we will define eigenvalues and eigenfunctions for boundary value problems. We will work quite a few examples illustrating how to find eigenvalues and eigenfunctions. In one example the best we will be able to do is estimate the eigenvalues as that is something that will happen on a fairly regular basis with these kinds of problems.

Periodic Functions and Orthogonal Functions – In this section we will define periodic functions, orthogonal functions and mutually orthogonal functions. We will also work a couple of examples showing intervals on which $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are mutually orthogonal. The results of these examples will be very useful for the rest of this chapter and most of the next chapter.

Fourier Sine Series – In this section we define the Fourier Sine Series, *i.e.* representing a function with a series in the form $\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$. We will also define the odd extension for a function and work several examples finding the Fourier Sine Series for a function.

Fourier Cosine Series – In this section we define the Fourier Cosine Series, *i.e.* representing a function with a series in the form $\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$. We will also define the even extension for a function and work several examples finding the Fourier Cosine Series for a function.

Fourier Series – In this section we define the Fourier Series, *i.e.* representing a function with a series in the form $\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$. We will also work several examples finding the Fourier Series for a function.

Convergence of Fourier Series – In this section we will define piecewise smooth functions and the periodic extension of a function. In addition, we will give a variety of facts about just what a Fourier series will converge to and when we can expect the derivative or integral of a Fourier series to converge to the derivative or integral of the function it represents.

Section 8-1 : Boundary Value Problems

Before we start off this section we need to make it very clear that we are only going to scratch the surface of the topic of boundary value problems. There is enough material in the topic of boundary value problems that we could devote a whole class to it. The intent of this section is to give a brief (and we mean very brief) look at the idea of boundary value problems and to give enough information to allow us to do some basic partial differential equations in the next chapter.

Now, with that out of the way, the first thing that we need to do is to define just what we mean by a boundary value problem (BVP for short). With initial value problems we had a differential equation and we specified the value of the solution and an appropriate number of derivatives at the same point (collectively called initial conditions). For instance, for a second order differential equation the initial conditions are,

$$y(t_0) = y_0 \qquad\qquad y'(t_0) = y'_0$$

With boundary value problems we will have a differential equation and we will specify the function and/or derivatives at *different* points, which we'll call boundary values. For second order differential equations, which will be looking at pretty much exclusively here, any of the following can, and will, be used for boundary conditions.

$$y(x_0) = y_0 \qquad\qquad y(x_1) = y_1 \tag{1}$$
$$y'(x_0) = y_0 \qquad\qquad y'(x_1) = y_1 \tag{2}$$
$$y'(x_0) = y_0 \qquad\qquad y(x_1) = y_1 \tag{3}$$
$$y(x_0) = y_0 \qquad\qquad y'(x_1) = y_1 \tag{4}$$

As mentioned above we'll be looking pretty much exclusively at second order differential equations. We will also be restricting ourselves down to linear differential equations. So, for the purposes of our discussion here we'll be looking almost exclusively at differential equations in the form,

$$y'' + p(x)y' + q(x)y = g(x) \tag{5}$$

along with one of the sets of boundary conditions given in (1) – (4). We will, on occasion, look at some different boundary conditions but the differential equation will always be on that can be written in this form.

As we'll soon see much of what we know about initial value problems will not hold here. We can, of course, solve (5) provided the coefficients are constant and for a few cases in which they aren't. None of that will change. The changes (and perhaps the problems) arise when we move from initial conditions to boundary conditions.

One of the first changes is a definition that we saw all the time in the earlier chapters. In the earlier chapters we said that a differential equation was homogeneous if $g(x) = 0$ for all *x*. Here we will say that a boundary value problem is **homogeneous** if in addition to $g(x) = 0$ we also have $y_0 = 0$ and

$y_1 = 0$ (regardless of the boundary conditions we use). If any of these are not zero we will call the BVP **nonhomogeneous**.

It is important to now remember that when we say homogeneous (or nonhomogeneous) we are saying something not only about the differential equation itself but also about the boundary conditions as well.

The biggest change that we're going to see here comes when we go to solve the boundary value problem. When solving linear initial value problems a unique solution will be guaranteed under very mild conditions. We only looked at this idea for first order IVP's but the idea does extend to higher order IVP's. In that section we saw that all we needed to guarantee a unique solution was some basic continuity conditions. With boundary value problems we will often have no solution or infinitely many solutions even for very nice differential equations that would yield a unique solution if we had initial conditions instead of boundary conditions.

Before we get into solving some of these let's next address the question of why we're even talking about these in the first place. As we'll see in the next chapter in the process of solving some partial differential equations we will run into boundary value problems that will need to be solved as well. In fact, a large part of the solution process there will be in dealing with the solution to the BVP. In these cases, the boundary conditions will represent things like the temperature at either end of a bar, or the heat flow into/out of either end of a bar. Or maybe they will represent the location of ends of a vibrating string. So, the boundary conditions there will really be conditions on the boundary of some process.

So, with some of basic stuff out of the way let's find some solutions to a few boundary value problems. Note as well that there really isn't anything new here yet. We know how to solve the differential equation and we know how to find the constants by applying the conditions. The only difference is that here we'll be applying boundary conditions instead of initial conditions.

Example 1 Solve the following BVP.

$$y'' + 4y = 0 \qquad y(0) = -2 \qquad y\left(\frac{\pi}{4}\right) = 10$$

Solution

Okay, this is a simple differential equation to solve and so we'll leave it to you to verify that the general solution to this is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Now all that we need to do is apply the boundary conditions.

$$-2 = y(0) = c_1$$
$$10 = y\left(\tfrac{\pi}{4}\right) = c_2$$

The solution is then,

$$y(x) = -2\cos(2x) + 10\sin(2x)$$

We mentioned above that some boundary value problems can have no solutions or infinite solutions we had better do a couple of examples of those as well here. This next set of examples will also show just how small of a change to the BVP it takes to move into these other possibilities.

Example 2 Solve the following BVP.

$$y'' + 4y = 0 \qquad y(0) = -2 \qquad y(2\pi) = -2$$

Solution

We're working with the same differential equation as the first example so we still have,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Upon applying the boundary conditions we get,

$$\begin{aligned} -2 &= y(0) = c_1 \\ -2 &= y(2\pi) = c_1 \end{aligned}$$

So, in this case, unlike previous example, both boundary conditions tell us that we have to have $c_1 = -2$ and neither one of them tell us anything about c_2. Remember however that all we're asking for is a solution to the differential equation that satisfies the two given boundary conditions and the following function will do that,

$$y(x) = -2\cos(2x) + c_2 \sin(2x)$$

In other words, regardless of the value of c_2 we get a solution and so, in this case we get infinitely many solutions to the boundary value problem.

Example 3 Solve the following BVP.

$$y'' + 4y = 0 \qquad y(0) = -2 \qquad y(2\pi) = 3$$

Solution

Again, we have the following general solution,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

This time the boundary conditions give us,

$$\begin{aligned} -2 &= y(0) = c_1 \\ 3 &= y(2\pi) = c_1 \end{aligned}$$

In this case we have a set of boundary conditions each of which requires a different value of c_1 in order to be satisfied. This, however, is not possible and so in this case have **no solution**.

So, with Examples 2 and 3 we can see that only a small change to the boundary conditions, in relation to each other and to Example 1, can completely change the nature of the solution. All three of these examples used the same differential equation and yet a different set of initial conditions yielded, no solutions, one solution, or infinitely many solutions.

Note that this kind of behavior is not always unpredictable however. If we use the conditions $y(0)$ and $y(2\pi)$ the only way we'll ever get a solution to the boundary value problem is if we have,

$$y(0) = a \qquad y(2\pi) = a$$

for any value of *a*. Also, note that if we do have these boundary conditions we'll in fact get infinitely many solutions.

All the examples we've worked to this point involved the same differential equation and the same type of boundary conditions so let's work a couple more just to make sure that we've got some more examples here. Also, note that with each of these we could tweak the boundary conditions a little to get any of the possible solution behaviors to show up (*i.e.* zero, one or infinitely many solutions).

Example 4 Solve the following BVP.

$$y'' + 3y = 0 \qquad y(0) = 7 \qquad y(2\pi) = 0$$

Solution

The general solution for this differential equation is,

$$y(x) = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Applying the boundary conditions gives,

$$7 = y(0) = c_1$$
$$0 = y(2\pi) = c_1 \cos\left(2\sqrt{3}\,\pi\right) + c_2 \sin\left(2\sqrt{3}\,\pi\right) \quad \Rightarrow \quad c_2 = -7\cot\left(2\sqrt{3}\,\pi\right)$$

In this case we get a single solution,

$$y(x) = 7\cos\left(\sqrt{3}\,x\right) - 7\cot\left(2\sqrt{3}\,\pi\right)\sin\left(\sqrt{3}\,x\right)$$

Example 5 Solve the following BVP.

$$y'' + 25y = 0 \qquad y'(0) = 6 \qquad y'(\pi) = -9$$

Solution

Here the general solution is,

$$y(x) = c_1 \cos(5x) + c_2 \sin(5x)$$

and we'll need the derivative to apply the boundary conditions,

$$y'(x) = -5c_1 \sin(5x) + 5c_2 \cos(5x)$$

Applying the boundary conditions gives,

$$6 = y'(0) = 5c_2 \quad \Rightarrow \quad c_2 = \tfrac{6}{5}$$
$$-9 = y'(\pi) = -5c_2 \quad \Rightarrow \quad c_2 = \tfrac{9}{5}$$

This is not possible and so in this case have **no solution**.

All of the examples worked to this point have been nonhomogeneous because at least one of the boundary conditions have been non-zero. Let's work one nonhomogeneous example where the differential equation is also nonhomogeneous before we work a couple of homogeneous examples.

Example 6 Solve the following BVP.

$$y'' + 9y = \cos x \qquad y'(0) = 5 \qquad y\left(\frac{\pi}{2}\right) = -\frac{5}{3}$$

Solution

The complementary solution for this differential equation is,

$$y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

Using Undetermined Coefficients or Variation of Parameters it is easy to show (we'll leave the details to you to verify) that a particular solution is,

$$Y_P(x) = \tfrac{1}{8}\cos x$$

The general solution and its derivative (since we'll need that for the boundary conditions) are,

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) + \tfrac{1}{8}\cos x$$
$$y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \tfrac{1}{8}\sin x$$

Applying the boundary conditions gives,

$$5 = y'(0) = 3c_2 \qquad \Rightarrow \qquad c_2 = \tfrac{5}{3}$$
$$-\tfrac{5}{3} = y\left(\tfrac{\pi}{2}\right) = -c_2 \qquad \Rightarrow \qquad c_2 = \tfrac{5}{3}$$

The boundary conditions then tell us that we must have $c_2 = \frac{5}{3}$ and they don't tell us anything about c_1 and so it is can be arbitrarily chosen. The solution is then,

$$y(x) = c_1 \cos(3x) + \tfrac{5}{3}\sin(3x) + \tfrac{1}{8}\cos x$$

and there will be infinitely many solutions to the BVP.

Let's now work a couple of homogeneous examples that will also be helpful to have worked once we get to the next section.

Example 7 Solve the following BVP.

$$y'' + 4y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0$$

Solution

Here the general solution is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Applying the boundary conditions gives,

$$0 = y(0) = c_1$$
$$0 = y(2\pi) = c_1$$

So c_2 is arbitrary and the solution is,

$$y(x) = c_2 \sin(2x)$$

and in this case we'll get infinitely many solutions.

Example 8 Solve the following BVP.

$$y'' + 3y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0$$

Solution

The general solution in this case is,

$$y(x) = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Applying the boundary conditions gives,

$$0 = y(0) = c_1$$

$$0 = y(2\pi) = c_2 \sin\left(2\sqrt{3}\,\pi\right) \qquad \Rightarrow \qquad c_2 = 0$$

In this case we found both constants to be zero and so the solution is,

$$y(x) = 0$$

In the previous example the solution was $y(x) = 0$. Notice however, that this will always be a solution to any homogenous system given by (5) and any of the (homogeneous) boundary conditions given by (1) – (4). Because of this we usually call this solution the **trivial solution**. Sometimes, as in the case of the last example the trivial solution is the only solution however we generally prefer solutions to be non-trivial. This will be a major idea in the next section.

Before we leave this section an important point needs to be made. In each of the examples, with one exception, the differential equation that we solved was in the form,

$$y'' + \lambda y = 0$$

The one exception to this still solved this differential equation except it was not a homogeneous differential equation and so we were still solving this basic differential equation in some manner.

So, there are probably several natural questions that can arise at this point. Do all BVP's involve this differential equation and if not why did we spend so much time solving this one to the exclusion of all the other possible differential equations?

The answers to these questions are fairly simple. First, this differential equation is most definitely not the only one used in boundary value problems. It does however exhibit all of the behavior that we wanted to talk about here and has the added bonus of being very easy to solve. So, by using this differential equation almost exclusively we can see and discuss the important behavior that we need to discuss and frees us up from lots of potentially messy solution details and or messy solutions. We will, on occasion, look at other differential equations in the rest of this chapter, but we will still be working almost exclusively with this one.

There is another important reason for looking at this differential equation. When we get to the next chapter and take a brief look at solving partial differential equations we will see that almost every one of the examples that we'll work there come down to exactly this differential equation. Also, in those problems we will be working some "real" problems that are actually solved in places and so are not just "made up" problems for the purposes of examples. Admittedly they will have some simplifications in them, but they do come close to realistic problem in some cases.

Section 8-2 : Eigenvalues and Eigenfunctions

As we did in the previous section we need to again note that we are only going to give a brief look at the topic of eigenvalues and eigenfunctions for boundary value problems. There are quite a few ideas that we'll not be looking at here. The intent of this section is simply to give you an idea of the subject and to do enough work to allow us to solve some basic partial differential equations in the next chapter.

Now, before we start talking about the actual subject of this section let's recall a topic from Linear Algebra that we briefly discussed previously in these notes. For a given square matrix, *A*, if we could find values of λ for which we could find nonzero solutions, *i.e.* $\vec{x} \neq \vec{0}$, to,

$$A\vec{x} = \lambda \vec{x}$$

then we called λ an eigenvalue of *A* and $\vec{x}$ was its corresponding eigenvector.

It's important to recall here that in order for λ to be an eigenvalue then we had to be able to find nonzero solutions to the equation.

So, just what does this have to do with boundary value problems? Well go back to the previous section and take a look at Example 7 and Example 8. In those two examples we solved homogeneous (and that's important!) BVP's in the form,

$$y'' + \lambda y = 0 \qquad y(0) = 0 \quad y(2\pi) = 0 \qquad \textbf{(6)}$$

In Example 7 we had $\lambda = 4$ and we found nontrivial (*i.e.* nonzero) solutions to the BVP. In Example 8 we used $\lambda = 3$ and the only solution was the trivial solution (*i.e.* $y(t) = 0$). So, this homogeneous BVP (recall this also means the boundary conditions are zero) seems to exhibit similar behavior to the behavior in the matrix equation above. There are values of λ that will give nontrivial solutions to this BVP and values of λ that will only admit the trivial solution.

So, for those values of λ that give nontrivial solutions we'll call λ an **eigenvalue** for the BVP and the nontrivial solutions will be called **eigenfunctions** for the BVP corresponding to the given eigenvalue.

We now know that for the homogeneous BVP given in (1) $\lambda = 4$ is an eigenvalue (with eigenfunctions $y(x) = c_2 \sin(2x)$) and that $\lambda = 3$ is not an eigenvalue.

Eventually we'll try to determine if there are any other eigenvalues for (1), however before we do that let's comment briefly on why it is so important for the BVP to be homogeneous in this discussion. In Example 2 and Example 3 of the previous section we solved the homogeneous differential equation

$$y'' + 4y = 0$$

with two different nonhomogeneous boundary conditions in the form,

$$y(0) = a \qquad y(2\pi) = b$$

In these two examples we saw that by simply changing the value of *a* and/or *b* we were able to get either nontrivial solutions or to force no solution at all. In the discussion of eigenvalues/eigenfunctions we need solutions to exist and the only way to assure this behavior is to require that the boundary conditions also be homogeneous. In other words, we need for the BVP to be homogeneous.

There is one final topic that we need to discuss before we move into the topic of eigenvalues and eigenfunctions and this is more of a notational issue that will help us with some of the work that we'll need to do.

Let's suppose that we have a second order differential equation and its characteristic polynomial has two real, distinct roots and that they are in the form

$$r_1 = \alpha \qquad\qquad r_2 = -\alpha$$

Then we know that the solution is,

$$y(x) = c_1 \mathbf{e}^{r_1 x} + c_2 \mathbf{e}^{r_2 x} = c_1 \mathbf{e}^{\alpha x} + c_2 \mathbf{e}^{-\alpha x}$$

While there is nothing wrong with this solution let's do a little rewriting of this. We'll start by splitting up the terms as follows,

$$\begin{aligned} y(x) &= c_1 \mathbf{e}^{\alpha x} + c_2 \mathbf{e}^{-\alpha x} \\ &= \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} \end{aligned}$$

Now we'll add/subtract the following terms (note we're "mixing" the c_i and $\pm\alpha$ up in the new terms) to get,

$$y(x) = \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} + \left(\frac{c_1}{2}\mathbf{e}^{-\alpha x} - \frac{c_1}{2}\mathbf{e}^{-\alpha x}\right) + \left(\frac{c_2}{2}\mathbf{e}^{\alpha x} - \frac{c_2}{2}\mathbf{e}^{\alpha x}\right)$$

Next, rearrange terms around a little,

$$y(x) = \frac{1}{2}\left(c_1\mathbf{e}^{\alpha x} + c_1\mathbf{e}^{-\alpha x} + c_2\mathbf{e}^{\alpha x} + c_2\mathbf{e}^{-\alpha x}\right) + \frac{1}{2}\left(c_1\mathbf{e}^{\alpha x} - c_1\mathbf{e}^{-\alpha x} - c_2\mathbf{e}^{\alpha x} + c_2\mathbf{e}^{-\alpha x}\right)$$

Finally, the quantities in parenthesis factor and we'll move the location of the fraction as well. Doing this, as well as renaming the new constants we get,

$$\begin{aligned} y(x) &= (c_1 + c_2)\frac{\mathbf{e}^{\alpha x} + \mathbf{e}^{-\alpha x}}{2} + (c_1 - c_2)\frac{\mathbf{e}^{\alpha x} - \mathbf{e}^{-\alpha x}}{2} \\ &= \overline{c}_1 \frac{\mathbf{e}^{\alpha x} + \mathbf{e}^{-\alpha x}}{2} + \overline{c}_2 \frac{\mathbf{e}^{\alpha x} - \mathbf{e}^{-\alpha x}}{2} \end{aligned}$$

All this work probably seems very mysterious and unnecessary. However there really was a reason for it. In fact, you may have already seen the reason, at least in part. The two "new" functions that we have in our solution are in fact two of the hyperbolic functions. In particular,

$$\cosh(x) = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2} \qquad\qquad \sinh(x) = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2}$$

So, another way to write the solution to a second order differential equation whose characteristic polynomial has two real, distinct roots in the form $r_1 = \alpha,\ r_2 = -\alpha$ is,

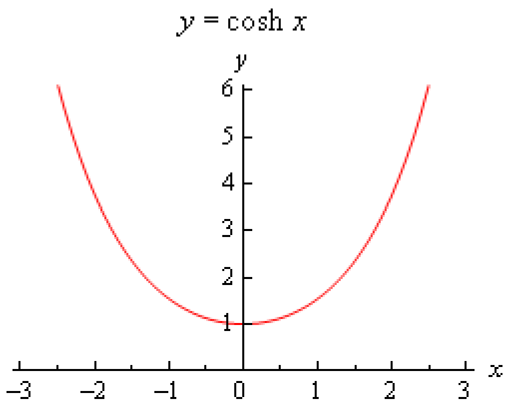
$$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

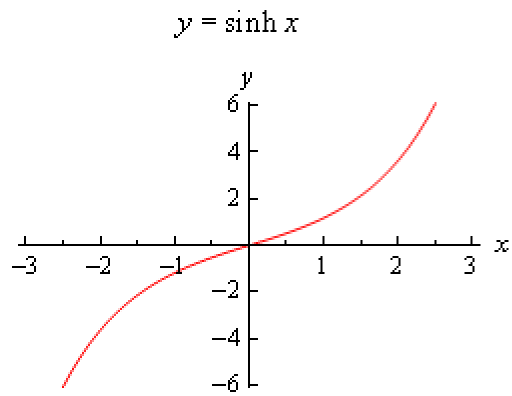
Having the solution in this form for some (actually most) of the problems we'll be looking will make our life a lot easier. The hyperbolic functions have some very nice properties that we can (and will) take advantage of.

First, since we'll be needing them later on, the derivatives are,

$$\frac{d}{dx}\left(\cosh\left(x\right)\right) = \sinh\left(x\right) \qquad\qquad \frac{d}{dx}\left(\sinh\left(x\right)\right) = \cosh\left(x\right)$$

Next let's take a quick look at the graphs of these functions.





Note that $\cosh\left(0\right) = 1$ and $\sinh\left(0\right) = 0$. Because we'll often be working with boundary conditions at $x = 0$ these will be useful evaluations.

Next, and possibly more importantly, let's notice that $\cosh\left(x\right) > 0$ for all *x* and so the hyperbolic cosine will never be zero. Likewise, we can see that $\sinh\left(x\right) = 0$ only if $x = 0$. We will be using both of these facts in some of our work so we shouldn't forget them.

Okay, now that we've got all that out of the way let's work an example to see how we go about finding eigenvalues/eigenfunctions for a BVP.

Example 1 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad\qquad y\left(0\right) = 0 \qquad y\left(2\pi\right) = 0$$

Solution

We started off this section looking at this BVP and we already know one eigenvalue ($\lambda = 4$) and we know one value of λ that is not an eigenvalue ($\lambda = 3$). As we go through the work here we need to remember that we will get an eigenvalue for a particular value of λ if we get non-trivial solutions of the BVP for that particular value of λ.

In order to know that we've found all the eigenvalues we can't just start randomly trying values of λ to see if we get non-trivial solutions or not. Luckily there is a way to do this that's not too bad and will give us all the eigenvalues/eigenfunctions. We are going to have to do some cases however. The three cases that we will need to look at are : $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Each of these cases gives a specific form of the solution to the BVP to which we can then apply the boundary conditions to see if we'll get non-trivial solutions or not. So, let's get started on the cases.

$\underline{\lambda > 0}$

In this case the characteristic polynomial we get from the differential equation is,

$$r^2 + \lambda = 0 \qquad \Rightarrow \qquad r_{1,2} = \pm\sqrt{-\lambda}$$

In this case since we know that $\lambda > 0$ these roots are complex and we can write them instead as,

$$r_{1,2} = \pm\sqrt{\lambda}\, i$$

The general solution to the differential equation is then,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives us,

$$0 = y(0) = c_1$$

So, taking this into account and applying the second boundary condition we get,

$$0 = y(2\pi) = c_2 \sin\left(2\pi\sqrt{\lambda}\right)$$

This means that we have to have one of the following,

$$c_2 = 0 \qquad \text{or} \qquad \sin\left(2\pi\sqrt{\lambda}\right) = 0$$

However, recall that we want non-trivial solutions and if we have the first possibility we will get the trivial solution for all values of $\lambda > 0$. Therefore, let's assume that $c_2 \neq 0$. This means that we have,

$$\sin\left(2\pi\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad 2\pi\sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

In other words, taking advantage of the fact that we know where sine is zero we can arrive at the second equation. Also note that because we are assuming that $\lambda > 0$ we know that $2\pi\sqrt{\lambda} > 0$ and so *n* can only be a positive integer for this case.

Now all we have to do is solve this for λ and we'll have all the positive eigenvalues for this BVP.

The positive eigenvalues are then,

$$\lambda_n = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4} \qquad n = 1, 2, 3, \ldots$$

and the eigenfunctions that correspond to these eigenvalues are,

$$y_n(x) = \sin\left(\frac{n\,x}{2}\right) \qquad n = 1, 2, 3, \ldots$$

Note that we subscripted an *n* on the eigenvalues and eigenfunctions to denote the fact that there is one for each of the given values of *n*. Also note that we dropped the c_2 on the eigenfunctions. For eigenfunctions we are only interested in the function itself and not the constant in front of it and so we generally drop that.

Let's now move into the second case.

$\underline{\lambda = 0}$

In this case the BVP becomes,

$$y'' = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0$$

and integrating the differential equation a couple of times gives us the general solution,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = y(0) = c_1$$

Applying the second boundary condition as well as the results of the first boundary condition gives,

$$0 = y(2\pi) = 2c_2\pi$$

Here, unlike the first case, we don't have a choice on how to make this zero. This will only be zero if $c_2 = 0$.

Therefore, for this BVP (and that's important), if we have $\lambda = 0$ the only solution is the trivial solution and so $\lambda = 0$ cannot be an eigenvalue for this BVP.

Now let's look at the final case.

$\underline{\lambda < 0}$

In this case the characteristic equation and its roots are the same as in the first case. So, we know that,

$$r_{1,2} = \pm\sqrt{-\lambda}$$

However, because we are assuming $\lambda < 0$ here these are now two real distinct roots and so using our work above for these kinds of real, distinct roots we know that the general solution will be,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Note that we could have used the exponential form of the solution here, but our work will be significantly easier if we use the hyperbolic form of the solution here.

Now, applying the first boundary condition gives,

$$0 = y(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1(1) + c_2(0) = c_1 \qquad \Rightarrow \qquad c_1 = 0$$

Applying the second boundary condition gives,

$$0 = y(2\pi) = c_2 \sinh\left(2\pi\sqrt{-\lambda}\right)$$

Because we are assuming $\lambda < 0$ we know that $2\pi\sqrt{-\lambda} \neq 0$ and so we also know that $\sinh\left(2\pi\sqrt{-\lambda}\right) \neq 0$. Therefore, much like the second case, we must have $c_2 = 0$.

So, for this BVP (again that's important), if we have $\lambda < 0$ we only get the trivial solution and so there are no negative eigenvalues.

In summary then we will have the following eigenvalues/eigenfunctions for this BVP.

$$\lambda_n = \frac{n^2}{4} \qquad y_n(x) = \sin\left(\frac{n\,x}{2}\right) \qquad n = 1, 2, 3, \ldots$$

Let's take a look at another example with slightly different boundary conditions.

Example 2 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad y'(0) = 0 \qquad y'(2\pi) = 0$$

Solution

Here we are going to work with derivative boundary conditions. The work is pretty much identical to the previous example however so we won't put in quite as much detail here. We'll need to go through all three cases just as the previous example so let's get started on that.

$\underline{\lambda > 0}$

The general solution to the differential equation is identical to the previous example and so we have,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\,x\right) + c_2 \sin\left(\sqrt{\lambda}\,x\right)$$

Applying the first boundary condition gives us,

$$0 = y'(0) = \sqrt{\lambda}\,c_2 \qquad \Rightarrow \qquad c_2 = 0$$

Recall that we are assuming that $\lambda > 0$ here and so this will only be zero if $c_2 = 0$. Now, the second boundary condition gives us,

$$0 = y'(2\pi) = -\sqrt{\lambda}\,c_1 \sin\left(2\pi\sqrt{\lambda}\right)$$

Recall that we don't want trivial solutions and that $\lambda > 0$ so we will only get non-trivial solution if we require that,

$$\sin\left(2\pi\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad 2\pi\sqrt{\lambda} = n\pi \quad n = 1, 2, 3, \ldots$$

Solving for λ and we see that we get exactly the same positive eigenvalues for this BVP that we got in the previous example.

$$\lambda_n = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4} \qquad\qquad n = 1, 2, 3, \ldots$$

The eigenfunctions that correspond to these eigenvalues however are,

$$y_n(x) = \cos\left(\frac{n\,x}{2}\right) \qquad\qquad n = 1, 2, 3, \ldots$$

So, for this BVP we get cosines for eigenfunctions corresponding to positive eigenvalues.

Now the second case.

$\underline{\lambda = 0}$

The general solution is,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = y'(0) = c_2$$

Using this the general solution is then,

$$y(x) = c_1$$

and note that this will trivially satisfy the second boundary condition,

$$0 = y'(2\pi) = 0$$

Therefore, unlike the first example, $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$y(x) = 1$$

Again, note that we dropped the arbitrary constant for the eigenfunctions.

Finally let's take care of the third case.

$\underline{\lambda < 0}$

The general solution here is,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\,x\right) + c_2 \sinh\left(\sqrt{-\lambda}\,x\right)$$

Applying the first boundary condition gives,

$$0 = y'(0) = \sqrt{-\lambda}\,c_1 \sinh(0) + \sqrt{-\lambda}\,c_2 \cosh(0) = \sqrt{-\lambda}\,c_2 \qquad \Rightarrow \qquad c_2 = 0$$

Applying the second boundary condition gives,

$$0 = y'(2\pi) = \sqrt{-\lambda}\,c_1 \sinh\left(2\pi\sqrt{-\lambda}\right)$$

As with the previous example we again know that $2\pi\sqrt{-\lambda} \neq 0$ and so $\sinh\left(2\pi\sqrt{-\lambda}\right) \neq 0$. Therefore, we must have $c_1 = 0$.

So, for this BVP we again have no negative eigenvalues.

In summary then we will have the following eigenvalues/eigenfunctions for this BVP.

$$\lambda_n = \frac{n^2}{4} \qquad y_n\left(x\right) = \cos\left(\frac{n\,x}{2}\right) \qquad n = 1, 2, 3, \ldots$$
$$\lambda_0 = 0 \qquad y_0\left(x\right) = 1$$

Notice as well that we can actually combine these if we allow the list of *n*'s for the first one to start at zero instead of one. This will often not happen, but when it does we'll take advantage of it. So the "official" list of eigenvalues/eigenfunctions for this BVP is,

$$\lambda_n = \frac{n^2}{4} \qquad y_n\left(x\right) = \cos\left(\frac{n\,x}{2}\right) \qquad n = 0, 1, 2, 3, \ldots$$

So, in the previous two examples we saw that we generally need to consider different cases for λ as different values will often lead to different general solutions. Do not get too locked into the cases we did here. We will mostly be solving this particular differential equation and so it will be tempting to assume that these are always the cases that we'll be looking at, but there are BVP's that will require other/different cases.

Also, as we saw in the two examples sometimes one or more of the cases will not yield any eigenvalues. This will often happen, but again we shouldn't read anything into the fact that we didn't have negative eigenvalues for either of these two BVP's. There are BVP's that will have negative eigenvalues.

Let's take a look at another example with a very different set of boundary conditions. These are not the traditional boundary conditions that we've been looking at to this point, but we'll see in the next chapter how these can arise from certain physical problems.

Example 3 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad y\left(-\pi\right) = y\left(\pi\right) \qquad y'\left(-\pi\right) = y'\left(\pi\right)$$

Solution

So, in this example we aren't actually going to specify the solution or its derivative at the boundaries. Instead we'll simply specify that the solution must be the same at the two boundaries and the derivative of the solution must also be the same at the two boundaries. Also, this type of boundary condition will typically be on an interval of the form [-L,L] instead of [0,L] as we've been working on to this point.

As mentioned above these kind of boundary conditions arise very naturally in certain physical problems and we'll see that in the next chapter.

As with the previous two examples we still have the standard three cases to look at.

$\underline{\lambda > 0}$

The general solution for this case is,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition and using the fact that cosine is an even function (*i.e.* $\cos(-x) = \cos(x)$) and that sine is an odd function (*i.e.* $\sin(-x) = -\sin(x)$). gives us,

$$\begin{aligned} c_1 \cos\left(-\pi\sqrt{\lambda}\right) + c_2 \sin\left(-\pi\sqrt{\lambda}\right) &= c_1 \cos\left(\pi\sqrt{\lambda}\right) + c_2 \sin\left(\pi\sqrt{\lambda}\right) \\ c_1 \cos\left(\pi\sqrt{\lambda}\right) - c_2 \sin\left(\pi\sqrt{\lambda}\right) &= c_1 \cos\left(\pi\sqrt{\lambda}\right) + c_2 \sin\left(\pi\sqrt{\lambda}\right) \\ -c_2 \sin\left(\pi\sqrt{\lambda}\right) &= c_2 \sin\left(\pi\sqrt{\lambda}\right) \\ 0 &= 2c_2 \sin\left(\pi\sqrt{\lambda}\right) \end{aligned}$$

This time, unlike the previous two examples this doesn't really tell us anything. We could have $\sin\left(\pi\sqrt{\lambda}\right) = 0$ but it is also completely possible, at this point in the problem anyway, for us to have $c_2 = 0$ as well.

So, let's go ahead and apply the second boundary condition and see if we get anything out of that.

$$\begin{aligned} -\sqrt{\lambda}\, c_1 \sin\left(-\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(-\pi\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\pi\sqrt{\lambda}\right) \\ \sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\pi\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\pi\sqrt{\lambda}\right) \\ \sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) \\ 2\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) &= 0 \end{aligned}$$

So, we get something very similar to what we got after applying the first boundary condition. Since we are assuming that $\lambda > 0$ this tells us that either $\sin\left(\pi\sqrt{\lambda}\right) = 0$ or $c_1 = 0$.

Note however that if $\sin\left(\pi\sqrt{\lambda}\right) \neq 0$ then we will have to have $c_1 = c_2 = 0$ and we'll get the trivial solution. We therefore need to require that $\sin\left(\pi\sqrt{\lambda}\right) = 0$ and so just as we've done for the previous two examples we can now get the eigenvalues,

$$\pi\sqrt{\lambda} = n\pi \qquad \Rightarrow \qquad \lambda = n^2 \quad n = 1, 2, 3, \ldots$$

Recalling that $\lambda > 0$ and we can see that we do need to start the list of possible *n*'s at one instead of zero.

So, we now know the eigenvalues for this case, but what about the eigenfunctions. The solution for a given eigenvalue is,

$$y(x) = c_1 \cos(n\,x) + c_2 \sin(n\,x)$$

and we've got no reason to believe that either of the two constants are zero or non-zero for that matter. In cases like these we get two sets of eigenfunctions, one corresponding to each constant. The two sets of eigenfunctions for this case are,

$$y_n(x) = \cos(n\,x) \qquad\qquad y_n(x) = \sin(n\,x) \qquad n = 1, 2, 3, \ldots$$

Now the second case.

$\underline{\lambda = 0}$

The general solution is,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$\begin{aligned} c_1 + c_2(-\pi) &= c_1 + c_2(\pi) \\ 2\pi c_2 &= 0 \qquad\qquad \Rightarrow \qquad\qquad c_2 = 0 \end{aligned}$$

Using this the general solution is then,

$$y(x) = c_1$$

and note that this will trivially satisfy the second boundary condition just as we saw in the second example above. Therefore, we again have $\lambda = 0$ as an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$y(x) = 1$$

Finally let's take care of the third case.

$\underline{\lambda < 0}$

The general solution here is,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\,x\right) + c_2 \sinh\left(\sqrt{-\lambda}\,x\right)$$

Applying the first boundary condition and using the fact that hyperbolic cosine is even and hyperbolic sine is odd gives,

$$\begin{aligned} c_1 \cosh\left(-\pi\sqrt{-\lambda}\right) + c_2 \sinh\left(-\pi\sqrt{-\lambda}\right) &= c_1 \cosh\left(\pi\sqrt{-\lambda}\right) + c_2 \sinh\left(\pi\sqrt{-\lambda}\right) \\ -c_2 \sinh\left(-\pi\sqrt{-\lambda}\right) &= c_2 \sinh\left(\pi\sqrt{-\lambda}\right) \\ 2c_2 \sinh\left(\pi\sqrt{-\lambda}\right) &= 0 \end{aligned}$$

Now, in this case we are assuming that $\lambda < 0$ and so we know that $\pi\sqrt{-\lambda} \neq 0$ which in turn tells us that $\sinh\left(\pi\sqrt{-\lambda}\right) \neq 0$. We therefore must have $c_2 = 0$.

Let's now apply the second boundary condition to get,

$$\sqrt{-\lambda}\, c_1 \sinh\left(-\pi\sqrt{-\lambda}\right) = \sqrt{-\lambda}\, c_1 \sinh\left(\pi\sqrt{-\lambda}\right)$$

$$2\sqrt{-\lambda}\, c_1 \sinh\left(\pi\sqrt{-\lambda}\right) = 0 \qquad \Rightarrow \qquad c_1 = 0$$

By our assumption on λ we again have no choice here but to have $c_1 = 0$.

Therefore, in this case the only solution is the trivial solution and so, for this BVP we again have no negative eigenvalues.

In summary then we will have the following eigenvalues/eigenfunctions for this BVP.

$$\begin{array}{lll} \lambda_n = n^2 & y_n(x) = \sin(nx) & n = 1,2,3,\ldots \\ \lambda_n = n^2 & y_n(x) = \cos(nx) & n = 1,2,3,\ldots \\ \lambda_0 = 0 & y_0(x) = 1 & \end{array}$$

Note that we've acknowledged that for $\lambda > 0$ we had two sets of eigenfunctions by listing them each separately. Also, we can again combine the last two into one set of eigenvalues and eigenfunctions. Doing so gives the following set of eigenvalues and eigenfunctions.

$$\begin{array}{lll} \lambda_n = n^2 & y_n(x) = \sin(nx) & n = 1,2,3,\ldots \\ \lambda_n = n^2 & y_n(x) = \cos(nx) & n = 0,1,2,3,\ldots \end{array}$$

Once again, we've got an example with no negative eigenvalues. We can't stress enough that this is more a function of the differential equation we're working with than anything and there will be examples in which we may get negative eigenvalues.

Now, to this point we've only worked with one differential equation so let's work an example with a different differential equation just to make sure that we don't get too locked into this one differential equation.

Before working this example let's note that we will still be working the vast majority of our examples with the one differential equation we've been using to this point. We're working with this other differential equation just to make sure that we don't get too locked into using one single differential equation.

Example 4 Find all the eigenvalues and eigenfunctions for the following BVP.

$$x^2 y'' + 3xy' + \lambda y = 0 \qquad y(1) = 0 \qquad y(2) = 0$$

Solution

This is an Euler differential equation and so we know that we'll need to find the roots of the following quadratic.

$$r(r-1) + 3r + \lambda = r^2 + 2r + \lambda = 0$$

The roots to this quadratic are,

$$r_{1,2} = \frac{-2 \pm \sqrt{4-4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}$$

Now, we are going to again have some cases to work with here, however they won't be the same as the previous examples. The solution will depend on whether or not the roots are real distinct, double or complex and these cases will depend upon the sign/value of $1-\lambda$. So, let's go through the cases.

$\underline{1-\lambda < 0,\ \lambda > 1}$

In this case the roots will be complex and we'll need to write them as follows in order to write down the solution.

$$r_{1,2} = -1 \pm \sqrt{1-\lambda} = -1 \pm \sqrt{-(\lambda-1)} = -1 \pm i\sqrt{\lambda-1}$$

By writing the roots in this fashion we know that $\lambda - 1 > 0$ and so $\sqrt{\lambda-1}$ is now a real number, which we need in order to write the following solution,

$$y(x) = c_1 x^{-1}\cos\left(\ln(x)\sqrt{\lambda-1}\right) + c_2 x^{-1}\sin\left(\ln(x)\sqrt{\lambda-1}\right)$$

Applying the first boundary condition gives us,

$$0 = y(1) = c_1\cos(0) + c_2\sin(0) = c_1 \qquad \Rightarrow \qquad c_1 = 0$$

The second boundary condition gives us,

$$0 = y(2) = \tfrac{1}{2}c_2\sin\left(\ln(2)\sqrt{\lambda-1}\right)$$

In order to avoid the trivial solution for this case we'll require,

$$\sin\left(\ln(2)\sqrt{\lambda-1}\right) = 0 \qquad \Rightarrow \qquad \ln(2)\sqrt{\lambda-1} = n\pi \qquad n = 1,2,3,\ldots$$

This is much more complicated of a condition than we've seen to this point, but other than that we do the same thing. So, solving for λ gives us the following set of eigenvalues for this case.

$$\lambda_n = 1 + \left(\frac{n\pi}{\ln 2}\right)^2 \qquad n = 1,2,3,\ldots$$

Note that we need to start the list of *n*'s off at one and not zero to make sure that we have $\lambda > 1$ as we're assuming for this case.

The eigenfunctions that correspond to these eigenvalues are,

$$y_n(x) = x^{-1}\sin\left(\frac{n\pi}{\ln 2}\ln(x)\right) \qquad n = 1,2,3,\ldots$$

Now the second case.

$\underline{1-\lambda=0,\ \ \lambda=1}$

In this case we get a double root of $r_{1,2}=-1$ and so the solution is,

$$y(x)=c_1x^{-1}+c_2x^{-1}\ln(x)$$

Applying the first boundary condition gives,

$$0=y(1)=c_1$$

The second boundary condition gives,

$$0=y(2)=\tfrac{1}{2}c_2\ln(2)\qquad\Rightarrow\qquad c_2=0$$

We therefore have only the trivial solution for this case and so $\lambda=1$ is not an eigenvalue.

Let's now take care of the third (and final) case.

$\underline{1-\lambda>0,\ \lambda<1}$

This case will have two real distinct roots and the solution is,

$$y(x)=c_1x^{-1+\sqrt{1-\lambda}}+c_2x^{-1-\sqrt{1-\lambda}}$$

Applying the first boundary condition gives,

$$0=y(1)=c_1+c_2\qquad\Rightarrow\qquad c_2=-c_1$$

Using this our solution becomes,

$$y(x)=c_1x^{-1+\sqrt{1-\lambda}}-c_1x^{-1-\sqrt{1-\lambda}}$$

Applying the second boundary condition gives,

$$0=y(2)=c_1 2^{-1+\sqrt{1-\lambda}}-c_1 2^{-1-\sqrt{1-\lambda}}=c_1\left(2^{-1+\sqrt{1-\lambda}}-2^{-1-\sqrt{1-\lambda}}\right)$$

Now, because we know that $\lambda\neq1$ for this case the exponents on the two terms in the parenthesis are not the same and so the term in the parenthesis is not the zero. This means that we can only have,

$$c_1=c_2=0$$

and so in this case we only have the trivial solution and there are no eigenvalues for which $\lambda<1$.

The only eigenvalues for this BVP then come from the first case.

So, we've now worked an example using a differential equation other than the "standard" one we've been using to this point. As we saw in the work however, the basic process was pretty much the same. We determined that there were a number of cases (three here, but it won't always be three) that gave different solutions. We examined each case to determine if non-trivial solutions were possible and if so found the eigenvalues and eigenfunctions corresponding to that case.

We need to work one last example in this section before we leave this section for some new topics. The four examples that we've worked to this point were all fairly simple (with simple being relative of course…), however we don't want to leave without acknowledging that many eigenvalue/eigenfunctions problems are so easy.

In many examples it is not even possible to get a complete list of all possible eigenvalues for a BVP. Often the equations that we need to solve to get the eigenvalues are difficult if not impossible to solve exactly. So, let's take a look at one example like this to see what kinds of things can be done to at least get an idea of what the eigenvalues look like in these kinds of cases.

Example 5 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y'(1) + y(1) = 0$$

Solution

The boundary conditions for this BVP are fairly different from those that we've worked with to this point. However, the basic process is the same. So let's start off with the first case.

$\underline{\lambda > 0}$

The general solution to the differential equation is identical to the first few examples and so we have,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives us,

$$0 = y(0) = c_1 \qquad \Rightarrow \qquad c_1 = 0$$

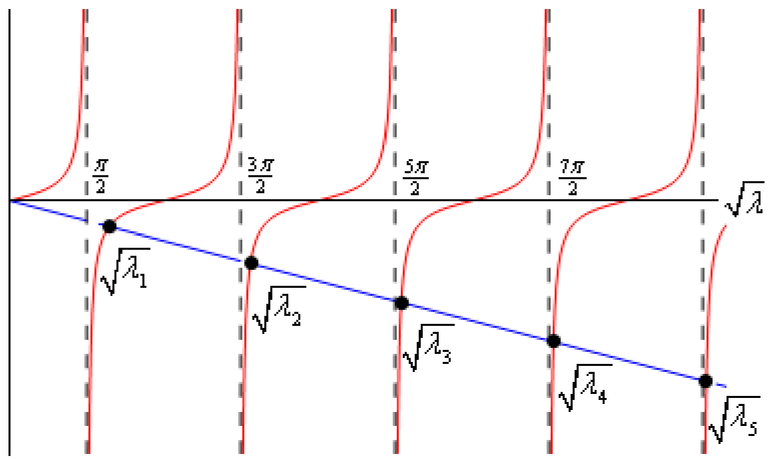
The second boundary condition gives us,

$$\begin{aligned} 0 = y(1) + y'(1) &= c_2 \sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\sqrt{\lambda}\right) \\ &= c_2\left(\sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda} \cos\left(\sqrt{\lambda}\right)\right) \end{aligned}$$

So, if we let $c_2 = 0$ we'll get the trivial solution and so in order to satisfy this boundary condition we'll need to require instead that,

$$\begin{aligned} 0 &= \sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda} \cos\left(\sqrt{\lambda}\right) \\ \sin\left(\sqrt{\lambda}\right) &= -\sqrt{\lambda} \cos\left(\sqrt{\lambda}\right) \\ \tan\left(\sqrt{\lambda}\right) &= -\sqrt{\lambda} \end{aligned}$$

Now, this equation has solutions but we'll need to use some numerical techniques in order to get them. In order to see what's going on here let's graph $\tan\left(\sqrt{\lambda}\right)$ and $-\sqrt{\lambda}$ on the same graph. Here is that graph and note that the horizontal axis really is values of $\sqrt{\lambda}$ as that will make things a little easier to see and relate to values that we're familiar with.



So, eigenvalues for this case will occur where the two curves intersect. We've shown the first five on the graph and again what is showing on the graph is really the square root of the actual eigenvalue as we've noted.

The interesting thing to note here is that the farther out on the graph the closer the eigenvalues come to the asymptotes of tangent and so we'll take advantage of that and say that for large enough *n* we can approximate the eigenvalues with the (very well known) locations of the asymptotes of tangent.

How large the value of *n* is before we start using the approximation will depend on how much accuracy we want, but since we know the location of the asymptotes and as *n* increases the accuracy of the approximation will increase so it will be easy enough to check for a given accuracy.

For the purposes of this example we found the first five numerically and then we'll use the approximation of the remaining eigenvalues. Here are those values/approximations.

$$
\begin{array}{lll}
\sqrt{\lambda_1} = 2.0288 & \lambda_1 = 4.1160 & (2.4674) \\
\sqrt{\lambda_2} = 4.9132 & \lambda_2 = 24.1395 & (22.2066) \\
\sqrt{\lambda_3} = 7.9787 & \lambda_3 = 63.6597 & (61.6850) \\
\sqrt{\lambda_4} = 11.0855 & \lambda_4 = 122.8883 & (120.9027) \\
\sqrt{\lambda_5} = 14.2074 & \lambda_5 = 201.8502 & (199.8595) \\
\sqrt{\lambda_n} \approx \frac{2n-1}{2}\pi & \lambda_n \approx \frac{(2n-1)^2}{4}\pi^2 & n = 6, 7, 8, \ldots
\end{array}
$$

The number in parenthesis after the first five is the approximate value of the asymptote. As we can see they are a little off, but by the time we get to $n = 5$ the error in the approximation is 0.9862%. So less than 1% error by the time we get to $n = 5$ and it will only get better for larger value of *n*.

The eigenfunctions for this case are,

$$y_n(x) = \sin\left(\sqrt{\lambda_n}\, x\right) \qquad n = 1, 2, 3, \ldots$$

where the values of λ_n are given above.

So, now that all that work is out of the way let's take a look at the second case.

$\underline{\lambda = 0}$

The general solution is,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = y(0) = c_1$$

Using this the general solution is then,

$$y(x) = c_2 x$$

Applying the second boundary condition to this gives,

$$0 = y'(1) + y(1) = c_2 + c_2 = 2c_2 \qquad \Rightarrow \qquad c_2 = 0$$

Therefore, for this case we get only the trivial solution and so $\lambda = 0$ is not an eigenvalue. Note however that had the second boundary condition been $y'(1) - y(1) = 0$ then $\lambda = 0$ would have been an eigenvalue (with eigenfunctions $y(x) = x$) and so again we need to be careful about reading too much into our work here.

Finally let's take care of the third case.

$\underline{\lambda < 0}$

The general solution here is,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = y(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1 \qquad \Rightarrow \qquad c_1 = 0$$

Using this the general solution becomes,

$$y(x) = c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the second boundary condition to this gives,

$$\begin{aligned} 0 = y'(1) + y(1) &= \sqrt{-\lambda}\, c_2 \cosh\left(\sqrt{-\lambda}\right) + c_2 \sinh\left(\sqrt{-\lambda}\right) \\ &= c_2\left(\sqrt{-\lambda} \cosh\left(\sqrt{-\lambda}\right) + \sinh\left(\sqrt{-\lambda}\right)\right) \end{aligned}$$

Now, by assumption we know that $\lambda < 0$ and so $\sqrt{-\lambda} > 0$. This in turn tells us that $\sinh\left(\sqrt{-\lambda}\right) > 0$ and we know that $\cosh\left(x\right) > 0$ for all *x*. Therefore,

$$\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\right) + \sinh\left(\sqrt{-\lambda}\right) \neq 0$$

and so we must have $c_2 = 0$ and once again in this third case we get the trivial solution and so this BVP will have no negative eigenvalues.

In summary the only eigenvalues for this BVP come from assuming that $\lambda > 0$ and they are given above.

So, we've worked several eigenvalue/eigenfunctions examples in this section. Before leaving this section we do need to note once again that there are a vast variety of different problems that we can work here and we've really only shown a bare handful of examples and so please do not walk away from this section believing that we've shown you everything.

The whole purpose of this section is to prepare us for the types of problems that we'll be seeing in the next chapter. Also, in the next chapter we will again be restricting ourselves down to some pretty basic and simple problems in order to illustrate one of the more common methods for solving partial differential equations.

Section 8-3 : Periodic Functions & Orthogonal Functions

This is going to be a short section. We just need to have a brief discussion about a couple of ideas that we'll be dealing with on occasion as we move into the next topic of this chapter.

Periodic Function

The first topic we need to discuss is that of a periodic function. A function is said to be **periodic** with **period *T*** if the following is true,

$$f\left(x+T\right)=f\left(x\right) \qquad \text{for all } x$$

The following is a nice little fact about periodic functions.

Fact 1

If *f* and *g* are both periodic functions with period *T* then so is $f+g$ and *fg*.

This is easy enough to prove so let's do that.

$$\begin{aligned}\left(f+g\right)\left(x+T\right)&=f\left(x+T\right)+g\left(x+T\right)=f\left(x\right)+g\left(x\right)=\left(f+g\right)\left(x\right)\\ \left(fg\right)\left(x+T\right)&=f\left(x+T\right)g\left(x+T\right)=f\left(x\right)g\left(x\right)=\left(fg\right)\left(x\right)\end{aligned}$$

The two periodic functions that most of us are familiar are sine and cosine and in fact we'll be using these two functions regularly in the remaining sections of this chapter. So, having said that let's close off this discussion of periodic functions with the following fact,

Fact 2

$\sin\left(\omega x\right)$ and $\cos\left(\omega x\right)$ are periodic functions with period $T=\dfrac{2\pi}{\omega}$.

Even and Odd Functions

The next quick idea that we need to discuss is that of even and odd functions.

Recall that a function is said to be **even** if,

$$f\left(-x\right)=f\left(x\right)$$

and a function is said to be **odd** if,

$$f\left(-x\right)=-f\left(x\right)$$

The standard examples of even functions are $f\left(x\right)=x^2$ and $g\left(x\right)=\cos\left(x\right)$ while the standard examples of odd functions are $f\left(x\right)=x^3$ and $g\left(x\right)=\sin\left(x\right)$. The following fact about certain integrals of even/odd functions will be useful in some of our work.

Fact 3

1. If $f\left(x\right)$ is an even function then,

$$\int_{-L}^{L} f\left(x\right)dx = 2\int_{0}^{L} f\left(x\right)dx$$

2. If $f\left(x\right)$ is an odd function then,

$$\int_{-L}^{L} f\left(x\right)dx = 0$$

Note that this fact is only valid on a "symmetric" interval, *i.e.* an interval in the form $\left[-L,L\right]$. If we aren't integrating on a "symmetric" interval then the fact may or may not be true.

Orthogonal Functions

The final topic that we need to discuss here is that of orthogonal functions. This idea will be integral to what we'll be doing in the remainder of this chapter and in the next chapter as we discuss one of the basic solution methods for partial differential equations.

Let's first get the definition of orthogonal functions out of the way.

Definition

1. Two non-zero functions, $f\left(x\right)$ and $g\left(x\right)$, are said to be **orthogonal** on $a \le x \le b$ if,

$$\int_{a}^{b} f\left(x\right)g\left(x\right)dx = 0$$

2. A set of non-zero functions, $\left\{ f_i\left(x\right)\right\}$, is said to be **mutually orthogonal** on $a \le x \le b$ (or just an **orthogonal set** if we're being lazy) if $f_i\left(x\right)$ and $f_j\left(x\right)$ are orthogonal for every $i \ne j$. In other words,

$$\int_{a}^{b} f_i\left(x\right) f_j\left(x\right)dx = \begin{cases} 0 & i \ne j \\ c > 0 & i = j \end{cases}$$

Note that in the case of $i = j$ for the second definition we know that we'll get a positive value from the integral because,

$$\int_{a}^{b} f_i\left(x\right) f_i\left(x\right)dx = \int_{a}^{b}\left[f_i\left(x\right)\right]^2 dx > 0$$

Recall that when we integrate a positive function we know the result will be positive as well.

Also note that the non-zero requirement is important because otherwise the integral would be trivially zero regardless of the other function we were using.

Before we work some examples there are a nice set of trig formulas that we'll need to help us with some of the integrals.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

Now let's work some examples that we'll need over the course of the next couple of sections.

Example 1 Show that $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ is mutually orthogonal on $-L \le x \le L$.

Solution

This is not too difficult to do. All we really need to do is evaluate the following integral.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

Before we start evaluating this integral let's notice that the integrand is the product of two even functions and so must also be even. This means that we can use Fact 3 above to write the integral as,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

There are two reasons for doing this. First having a limit of zero will often make the evaluation step a little easier and that will be the case here. We'll discuss the second reason after we're done with the example.

Now, in order to do this integral we'll actually need to consider three cases.

$\underline{n = m = 0}$

In this case the integral is very easy and is,

$$\int_{-L}^{L} dx = 2\int_{0}^{L} dx = 2L$$

$\underline{n = m \neq 0}$

This integral is a little harder than the first case, but not by much (provided we recall a simple trig formula). The integral for this case is,

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L} \cos^2\left(\frac{n\pi x}{L}\right)dx = \int_{0}^{L} 1+\cos\left(\frac{2n\pi x}{L}\right)dx$$
$$= \left(x + \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right)\Bigg|_{0}^{L} = L + \frac{L}{2n\pi}\sin(2n\pi)$$

Now, at this point we need to recall that *n* is an integer and so $\sin\left(2n\pi\right)=0$ and our final value for the integral is,

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx = L$$

The first two cases are really just showing that if $n=m$ the integral is not zero (as it shouldn't be) and depending upon the value of *n* (and hence *m*) we get different values of the integral. Now we need to do the third case and this, in some ways, is the important case since we must get zero out of this integral in order to know that the set is an orthogonal set. So, let's take care of the final case.

$\underline{n \ne m}$

This integral is the "messiest" of the three that we've had to do here. Let's just start off by writing the integral down.

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

In this case we can't combine/simplify as we did in the previous two cases. We can however, acknowledge that we've got a product of two cosines with different arguments and so we can use one of the trig formulas above to break up the product as follows,

$$\begin{aligned}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx &= 2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\\ &= \int_{0}^{L}\cos\left(\frac{\left(n-m\right)\pi x}{L}\right)+\cos\left(\frac{\left(n+m\right)\pi x}{L}\right)dx\\ &= \left[\frac{L}{\left(n-m\right)\pi}\sin\left(\frac{\left(n-m\right)\pi x}{L}\right)+\frac{L}{\left(n+m\right)\pi}\sin\left(\frac{\left(n+m\right)\pi x}{L}\right)\right]_{0}^{L}\\ &= \frac{L}{\left(n-m\right)\pi}\sin\left(\left(n-m\right)\pi\right)+\frac{L}{\left(n+m\right)\pi}\sin\left(\left(n+m\right)\pi\right)\end{aligned}$$

Now, we know that *n* and *m* are both integers and so $n-m$ and $n+m$ are also integers and so both of the sines above must be zero and all together we get,

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

So, we've shown that if $n \ne m$ the integral is zero and if $n=m$ the value of the integral is a positive constant and so the set is mutually orthogonal.

In all of the work above we kept both forms of the integral at every step. Let's discuss why we did this a little bit. By keeping both forms of the integral around we were able to show that not only is

$\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ mutually orthogonal on $-L \le x \le L$ but it is also mutually orthogonal on $0 \le x \le L$. The only difference is the value of the integral when $n = m$ and we can get those values from the work above.

Let's take a look at another example.

Example 2 Show that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $-L \le x \le L$ and on $0 \le x \le L$.

Solution
First, we'll acknowledge from the start this time that we'll be showing orthogonality on both of the intervals. Second, we need to start this set at $n = 1$ because if we used $n = 0$ the first function would be zero and we don't want the zero function to show up on our list.

As with the first example all we really need to do is evaluate the following integral.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

In this case integrand is the product of two odd functions and so must be even. This means that we can again use Fact 3 above to write the integral as,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

We only have two cases to do for the integral here.

$\underline{n = m}$
Not much to this integral. It's pretty similar to the previous examples second case.

$$\begin{aligned}\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx &= \int_{0}^{L} 1-\cos\left(\frac{2n\pi x}{L}\right)dx \\ &= \left.\left(x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right)\right|_{0}^{L} = L - \frac{L}{2n\pi}\sin\left(2n\pi\right) = L\end{aligned}$$

Summarizing up we get,

$$\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = L$$

$\underline{n \ne m}$
As with the previous example this can be a little messier but it is also nearly identical to the third case from the previous example so we'll not show a lot of the work.

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{0}^{L}\cos\left(\frac{(n-m)\pi x}{L}\right)-\cos\left(\frac{(n+m)\pi x}{L}\right)dx$$

$$= \left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)-\frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]_{0}^{L}$$

$$= \frac{L}{(n-m)\pi}\sin\left((n-m)\pi\right)-\frac{L}{(n+m)\pi}\sin\left((n+m)\pi\right)$$

As with the previous example we know that *n* and *m* are both integers a and so both of the sines above must be zero and all together we get,

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$

So, we've shown that if $n \neq m$ the integral is zero and if $n = m$ the value of the integral is a positive constant and so the set is mutually orthogonal.

We've now shown that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $-L \le x \le L$ and on $0 \le x \le L$.

We need to work one more example in this section.

Example 3 Show that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ and $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ are mutually orthogonal on $-L \le x \le L$.

Solution

This example is a little different from the previous two examples. Here we want to show that together both sets are mutually orthogonal on $-L \le x \le L$. To show this we need to show three things. First (and second actually) we need to show that individually each set is mutually orthogonal and we've already done that in the previous two examples. The third (and only) thing we need to show here is that if we take one function from one set and another function from the other set and we integrate them we'll get zero.

Also, note that this time we really do only want to do the one interval as the two sets, taken together, are not mutually orthogonal on $0 \le x \le L$. You might want to do the integral on this interval to verify that it won't always be zero.

So, let's take care of the one integral that we need to do here and there isn't a lot to do. Here is the integral.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

The integrand in this case is the product of an odd function (the sine) and an even function (the cosine) and so the integrand is an odd function. Therefore, since the integral is on a symmetric interval, *i.e.* $-L \le x \le L$, and so by Fact 3 above we know the integral must be zero or,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

So, in previous examples we've shown that on the interval $-L \le x \le L$ the two sets are mutually orthogonal individually and here we've shown that integrating a product of a sine and a cosine gives zero. Therefore, as a combined set they are also mutually orthogonal.

We've now worked three examples here dealing with orthogonality and we should note that these were not just pulled out of the air as random examples to work. In the following sections (and following chapter) we'll need the results from these examples. So, let's summarize those results up here.

1. $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are mutually orthogonal on $-L \le x \le L$ as individual sets and as a combined set.

2. $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ is mutually orthogonal on $0 \le x \le L$.

3. $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $0 \le x \le L$.

We will also be needing the results of the integrals themselves, both on $-L \le x \le L$ and on $0 \le x \le L$ so let's also summarize those up here as well so we can refer to them when we need to.

1. $$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

2. $$\int_{0}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m = 0 \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

3. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

4. $$\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

5. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

With this summary we'll leave this section and move off into the second major topic of this chapter : Fourier Series.

Section 8-4 : Fourier Sine Series

In this section we are going to start taking a look at Fourier series. We should point out that this is a subject that can span a whole class and what we'll be doing in this section (as well as the next couple of sections) is intended to be nothing more than a very brief look at the subject. The point here is to do just enough to allow us to do some basic solutions to partial differential equations in the next chapter. There are many topics in the study of Fourier series that we'll not even touch upon here.

So, with that out of the way let's get started, although we're not going to start off with Fourier series. Let's instead think back to our Calculus class where we looked at Taylor Series. With Taylor Series we wrote a series representation of a function, $f(x)$, as a series whose terms were powers of $x-a$ for some $x=a$. With some conditions we were able to show that,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

and that the series will converge to $f(x)$ on $|x-a|<R$ for some *R* that will be dependent upon the function itself.

There is nothing wrong with this, but it does require that derivatives of all orders exist at $x=a$. Or in other words $f^{(n)}(a)$ exists for $n=0,1,2,3,\ldots$ Also for some functions the value of *R* may end up being quite small.

These two issues (along with a couple of others) mean that this is not always the best way of writing a series representation for a function. In many cases it works fine and there will be no reason to need a different kind of series. There are times however where another type of series is either preferable or required.

We're going to build up an alternative series representation for a function over the course of the next couple of sections. The ultimate goal for the rest of this chapter will be to write down a series representation for a function in terms of sines and cosines.

We'll start things off by assuming that the function, $f(x)$, we want to write a series representation for is an odd function (*i.e.* $f(-x)=-f(x)$). Because $f(x)$ is odd it makes some sense that we should be able to write a series representation for this in terms of sines only (since they are also odd functions).

What we'll try to do here is write $f(x)$ as the following series representation, called a **Fourier sine series**, on $-L\le x\le L$.

$$\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

There are a couple of issues to note here. First, at this point, we are going to assume that the series representation will converge to $f(x)$ on $-L\le x\le L$. We will be looking into whether or not it will actually converge in a later section. However, assuming that the series does converge to $f\left(x\right)$ it is interesting to note that, unlike Taylor Series, this representation will always converge on the same interval and that the interval does not depend upon the function.

Second, the series representation will not involve powers of sine (again contrasting this with Taylor Series) but instead will involve sines with different arguments.

Finally, the argument of the sines, $\frac{n\pi x}{L}$, may seem like an odd choice that was arbitrarily chosen and in some ways it was. For Fourier sine series the argument doesn't have to necessarily be this but there are several reasons for the choice here. First, this is the argument that will naturally arise in the next chapter when we use Fourier series (in general and not necessarily Fourier sine series) to help us solve some basic partial differential equations.

The next reason for using this argument is the fact that the set of functions that we chose to work with, $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ in this case, need to be orthogonal on the given interval, $-L \le x \le L$ in this case, and note that in the last section we showed that in fact they are. In other words, the choice of functions we're going to be working with and the interval we're working on will be tied together in some way. We can use a different argument but will need to also choose an interval on which we can prove that the sines (with the different argument) are orthogonal.

So, let's start off by assuming that given an odd function, $f\left(x\right)$, we can in fact find a Fourier sine series, of the form given above, to represent the function on $-L \le x \le L$. This means we will have,

$$f\left(x\right)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

As noted above we'll discuss whether or not this even can be done and if the series representation does in fact converge to the function in later section. At this point we're simply going to assume that it can be done. The question now is how to determine the coefficients, B_n, in the series.

Let's start with the series above and multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ where *m* is a fixed integer in the range $\left\{1,2,3,\ldots\right\}$. In other words, we multiply both sides by any of the sines in the set of sines that we're working with here. Doing this gives,

$$f\left(x\right)\sin\left(\frac{m\pi x}{L}\right)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

Now, let's integrate both sides of this from $x=-L$ to $x=L$.

$$\int_{-L}^{L} f\left(x\right)\sin\left(\frac{m\pi x}{L}\right)dx=\int_{-L}^{L}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

At this point we've got a small issue to deal with. We know from Calculus that an integral of a finite series (more commonly called a finite sum….) is nothing more than the (finite) sum of the integrals of the pieces. In other words, for finite series we can interchange an integral and a series. For infinite series however, we cannot always do this. For some integrals of infinite series we cannot interchange an integral and a series. Luckily enough for us we actually can interchange the integral and the series in this case. Doing this and factoring the constant, B_n, out of the integral gives,

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx &= \sum_{n=1}^{\infty}\int_{-L}^{L} B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \\ &= \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\end{aligned}$$

Now, recall from the last section we proved that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is orthogonal on $-L \le x \le L$ and that,

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \ne m\end{cases}$$

So, what does this mean for us. As we work through the various values of *n* in the series and compute the value of the integrals all but one of the integrals will be zero. The only non-zero integral will come when we have $n = m$, in which case the integral has the value of *L*. Therefore, the only non-zero term in the series will come when we have $n = m$ and our equation becomes,

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m L$$

Finally, all we need to do is divide by *L* and we now have an equation for each of the coefficients.

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Next, note that because we're integrating two odd functions the integrand of this integral is even and so we also know that,

$$B_m = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Summarizing all this work up the Fourier sine series of an odd function $f(x)$ on $-L \le x \le L$ is given by,

$$f(x) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \qquad \begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, 3, \ldots \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, 3, \ldots\end{aligned}$$

Let's take a quick look at an example.

Example 1 Find the Fourier sine series for $f\left(x\right)=x$ on $-L\le x\le L$.

Solution

First note that the function we're working with is in fact an odd function and so this is something we can do. There really isn't much to do here other than to compute the coefficients for $f\left(x\right)=x$.

Here is that work and note that we're going to leave the integration by parts details to you to verify. Don't forget that *n*, *L*, and π are constants!

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right)-n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{n^2\pi^2}\left(L\sin\left(n\pi\right)-n\pi L\cos\left(n\pi\right)\right) \end{aligned}$$

These integrals can, on occasion, be somewhat messy especially when we use a general *L* for the endpoints of the interval instead of a specific number.

Now, taking advantage of the fact that *n* is an integer we know that $\sin\left(n\pi\right)=0$ and that $\cos\left(n\pi\right)=\left(-1\right)^n$. We therefore have,

$$B_n = \frac{2}{n^2\pi^2}\left(-n\pi L\left(-1\right)^n\right)=\frac{\left(-1\right)^{n+1}2L}{n\pi} \qquad n=1,2,3\ldots$$

The Fourier sine series is then,

$$x=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

At this point we should probably point out that we'll be doing most, if not all, of our work here on a general interval ($-L\le x\le L$ or $0\le x\le L$) instead of intervals with specific numbers for the endpoints. There are a couple of reasons for this. First, it gives a much more general formula that will work for any interval of that form which is always nice. Secondly, when we run into this kind of work in the next chapter it will also be on general intervals so we may as well get used to them now.

Now, finding the Fourier sine series of an odd function is fine and good but what if, for some reason, we wanted to find the Fourier sine series for a function that is not odd? To see how to do this we're going to have to make a change. The above work was done on the interval $-L\le x\le L$. In the case of a function that is not odd we'll be working on the interval $0\le x\le L$. The reason for this will be made apparent in a bit.

So, we are now going to do is to try to find a series representation for $f\left(x\right)$ on the interval $0\le x\le L$ that is in the form,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

or in other words,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

As we did with the Fourier sine series on $-L \le x \le L$ we are going to assume that the series will in fact converge to $f(x)$ and we'll hold off discussing the convergence of the series for a later section.

There are two methods of generating formulas for the coefficients, B_n, although we'll see in a bit that they really the same way, just looked at from different perspectives.

The first method is to just ignore the fact that $f(x)$ is not odd and proceed in the same manner that we did above only this time we'll take advantage of the fact that we proved in the previous section that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ also forms an orthogonal set on $0 \le x \le L$ and that,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

So, if we do this then all we need to do is multiply both sides of our series by $\sin\left(\frac{m\pi x}{L}\right)$, integrate from 0 to *L* and interchange the integral and series to get,

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

Now, plugging in for the integral we arrive at,

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m\left(\frac{L}{2}\right)$$

Upon solving for the coefficient we arrive at,

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Note that this is identical to the second form of the coefficients that we arrived at above by assuming $f(x)$ was odd and working on the interval $-L \le x \le L$. The fact that we arrived at essentially the same coefficients is not actually all the surprising as we'll see once we've looked the second method of generating the coefficients.

Before we look at the second method of generating the coefficients we need to take a brief look at another concept. Given a function, $f(x)$, we define the **odd extension** of $f(x)$ to be the new function,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0 \end{cases}$$

It's pretty easy to see that this is an odd function.

$$g(-x) = -f(-(-x)) = -f(x) = -g(x) \qquad \text{for } 0 < x < L$$

and we can also know that on $0 \le x \le L$ we have that $g(x) = f(x)$. Also note that if $f(x)$ is already an odd function then we in fact get $g(x) = f(x)$ on $-L \le x \le L$.

Let's take a quick look at a couple of odd extensions before we proceed any further.

Example 2 Sketch the odd extension of each of the given functions.

(a) $f(x) = L - x$ on $0 \le x \le L$

(b) $f(x) = 1 + x^2$ on $0 \le x \le L$

(c) $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$

Solution

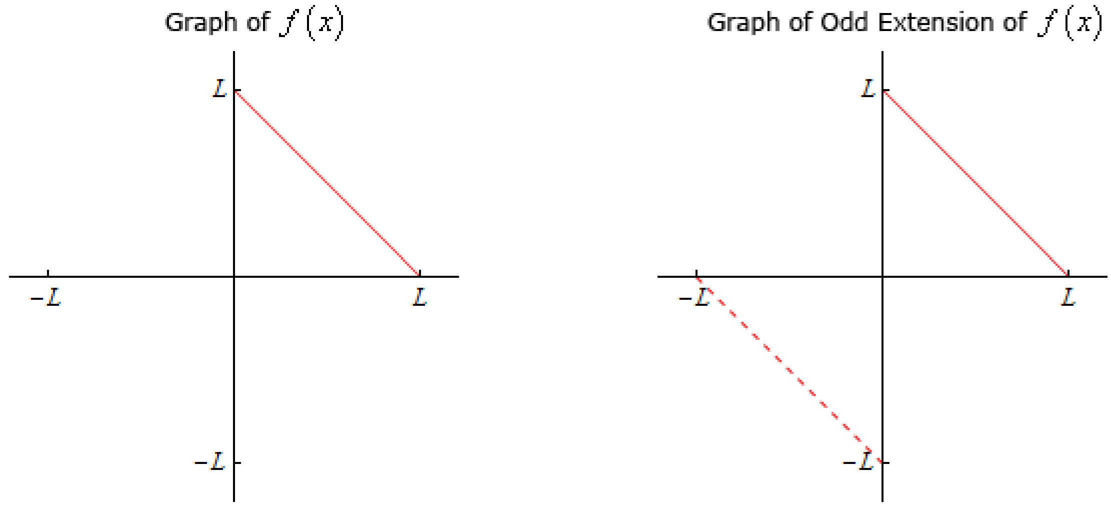
Not much to do with these other than to define the odd extension and then sketch it.

(a) $f(x) = L - x$ on $0 \le x \le L$

Here is the odd extension of this function.

$$\begin{aligned} g(x) &= \begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0 \end{cases} \\ &= \begin{cases} L - x & \text{if } 0 \le x \le L \\ -L - x & \text{if } -L \le x \le 0 \end{cases} \end{aligned}$$

Below is the graph of both the function and its odd extension. Note that we've put the "extension" in with a dashed line to make it clear the portion of the function that is being added to allow us to get the odd extension.

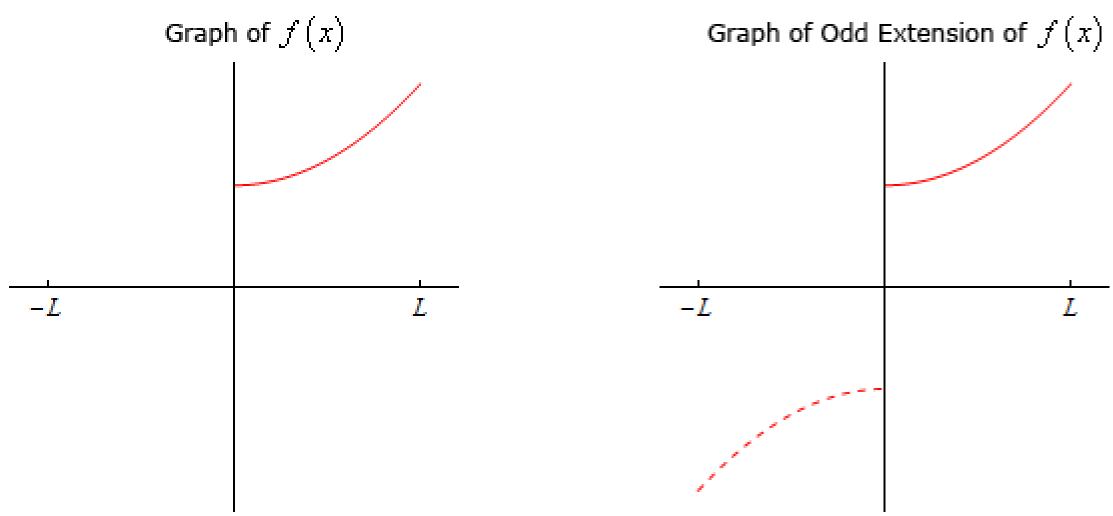


(b) $f(x)=1+x^2$ on $0\le x\le L$

First note that this is clearly an even function. That does not however mean that we can't define the odd extension for it. The odd extension for this function is,

$$\begin{aligned} g(x) &= \begin{cases} f(x) & \text{if } 0\le x\le L \\ -f(-x) & \text{if } -L\le x\le 0 \end{cases} \\ &= \begin{cases} 1+x^2 & \text{if } 0\le x\le L \\ -1-x^2 & \text{if } -L\le x\le 0 \end{cases} \end{aligned}$$

The sketch of the original function and its odd extension are ,

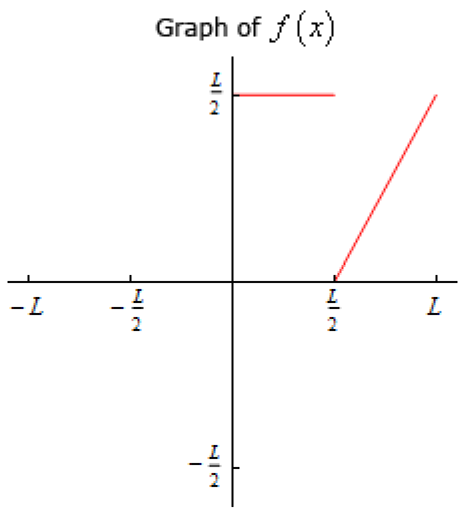


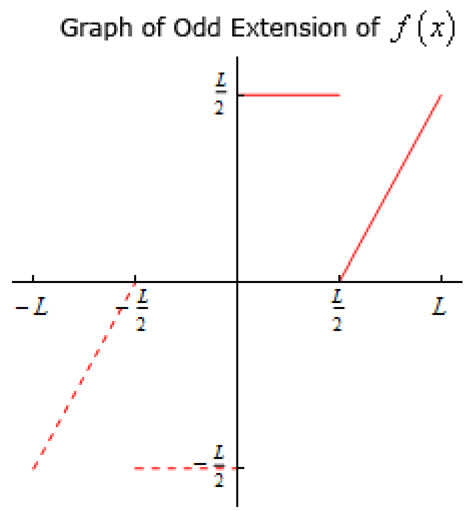
(c) $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$

Let's first write down the odd extension for this function.

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0 \end{cases} = \begin{cases} x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \\ \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ -\frac{L}{2} & \text{if } -\frac{L}{2} \le x \le 0 \\ x + \frac{L}{2} & \text{if } -L \le x \le -\frac{L}{2} \end{cases}$$

The sketch of the original function and its odd extension are,





With the definition of the odd extension (and a couple of examples) out of the way we can now take a look at the second method for getting formulas for the coefficients of the Fourier sine series for a function $f(x)$ on $0 \le x \le L$. First, given such a function define its odd extension as above. At this point, because $g(x)$ is an odd function, we know that on $-L \le x \le L$ the Fourier sine series for $g(x)$ (and NOT $f(x)$ yet) is,

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, 3, \ldots$$

However, because we know that $g(x) = f(x)$ on $0 \le x \le L$ we can also see that as long as we are on $0 \le x \le L$ we have,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, 3, \ldots$$

So, exactly the same formula for the coefficients regardless of how we arrived at the formula and the second method justifies why they are the same here as they were when we derived them for the Fourier sine series for an odd function.

Now, let's find the Fourier sine series for each of the functions that we looked at in Example 2.

Note that again we are working on general intervals here instead of specific numbers for the right endpoint to get a more general formula for any interval of this form and because again this is the kind of work we'll be doing in the next chapter.

Also, we'll again be leaving the actually integration details up to you to verify. In most cases it will involve some fairly simple integration by parts complicated by all the constants (n, L, π , etc.) that show up in the integral.

Example 3 Find the Fourier sine series for $f(x) = L - x$ on $0 \le x \le L$.

Solution

There really isn't much to do here other than computing the coefficients so here they are,

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L (L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L \\ &= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\left(n\pi - \sin(n\pi)\right)\right] = \frac{2L}{n\pi} \end{aligned}$$

In the simplification process don't forget that n is an integer.

So, with the coefficients we get the following Fourier sine series for this function.

$$f(x) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

In the next example it is interesting to note that while we started out this section looking only at odd functions we're now going to be finding the Fourier sine series of an even function on $0 \le x \le L$. Recall however that we're really finding the Fourier sine series of the odd extension of this function and so we're okay.

Example 4 Find the Fourier sine series for $f(x) = 1 + x^2$ on $0 \le x \le L$.

Solution

In this case the coefficients are liable to be somewhat messy given the fact that the integrals will involve integration by parts twice. Here is the work for the coefficients.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \left(1+x^2\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left[\left(2L^2 - n^2\pi^2\left(1+x^2\right)\right)\cos\left(\frac{n\pi x}{L}\right) + 2Ln\pi x\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left[\left(2L^2 - n^2\pi^2\left(1+L^2\right)\right)\cos(n\pi) + 2L^2 n\pi\sin(n\pi) - \left(2L^2 - n^2\pi^2\right)\right]$$

$$= \frac{2}{n^3\pi^3}\left[\left(2L^2 - n^2\pi^2\left(1+L^2\right)\right)(-1)^n - 2L^2 + n^2\pi^2\right]$$

As noted above the coefficients are not the most pleasant ones, but there they are. The Fourier sine series for this function is then,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi^3}\left[\left(2L^2 - n^2\pi^2\left(1+L^2\right)\right)(-1)^n - 2L^2 + n^2\pi^2\right]\sin\left(\frac{n\pi x}{L}\right)$$

In the last example of this section we'll be finding the Fourier sine series of a piecewise function and can definitely complicate the integrals a little but they do show up on occasion and so we need to be able to deal with them.

Example 5 Find the Fourier sine series for $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$ on $0 \le x \le L$.

Solution
Here is the integral for the coefficients.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{L}{2}\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}\left(x - \frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

Note that we need to split the integral up because of the piecewise nature of the original function. Let's do the two integrals separately

$$\int_0^{\frac{L}{2}} \frac{L}{2}\sin\left(\frac{n\pi x}{L}\right)dx = -\left(\frac{L}{2}\right)\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} = \frac{L^2}{2n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\int_{\frac{L}{2}}^{L}\left(x-\frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{n^2\pi^2}\left[L\sin\left(\frac{n\pi x}{L}\right)-n\pi\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{\frac{L}{2}}^{L}$$

$$= \frac{L}{n^2\pi^2}\left[L\sin(n\pi)-\frac{n\pi L}{2}\cos(n\pi)-L\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{L^2}{n^2\pi^2}\left[\frac{n\pi(-1)^n}{2}+\sin\left(\frac{n\pi}{2}\right)\right]$$

Putting all of this together gives,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L^2}{2n\pi}\right)\left[1+(-1)^{n+1}-\cos\left(\frac{n\pi}{2}\right)+\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{L}{n\pi}\left[1+(-1)^{n+1}-\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

So, the Fourier sine series for this function is,

$$f(x) = \sum_{n=1}^{\infty}\frac{L}{n\pi}\left[1+(-1)^{n+1}-\cos\left(\frac{n\pi}{2}\right)-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

As the previous two examples has shown the coefficients for these can be quite messy but that will often be the case and so we shouldn't let that get us too excited.

Section 8-5 : Fourier Cosine Series

In this section we're going to take a look at Fourier cosine series. We'll start off much as we did in the previous section where we looked at Fourier sine series. Let's start by assuming that the function, $f(x)$, we'll be working with initially is an even function (*i.e.* $f(-x) = f(x)$) and that we want to write a series representation for this function on $-L \le x \le L$ in terms of cosines (which are also even). In other words, we are going to look for the following,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

This series is called a **Fourier cosine series** and note that in this case (unlike with Fourier sine series) we're able to start the series representation at $n = 0$ since that term will not be zero as it was with sines. Also, as with Fourier Sine series, the argument of $\frac{n\pi x}{L}$ in the cosines is being used only because it is the argument that we'll be running into in the next chapter. The only real requirement here is that the given set of functions we're using be orthogonal on the interval we're working on.

Note as well that we're assuming that the series will in fact converge to $f(x)$ on $-L \le x \le L$ at this point. In a later section we'll be looking into the convergence of this series in more detail.

So, to determine a formula for the coefficients, A_n, we'll use the fact that $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ do form an orthogonal set on the interval $-L \le x \le L$ as we showed in a previous section. In that section we also derived the following formula that we'll need in a bit.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \ne 0 \\ 0 & \text{if } n \ne m \end{cases}$$

We'll get a formula for the coefficients in almost exactly the same fashion that we did in the previous section. We'll start with the representation above and multiply both sides by $\cos\left(\frac{m\pi x}{L}\right)$ where *m* is a fixed integer in the range $\{0, 1, 2, 3, \ldots\}$. Doing this gives,

$$f(x)\cos\left(\frac{m\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)$$

Next, we integrate both sides from $x = -L$ to $x = L$ and as we were able to do with the Fourier Sine series we can again interchange the integral and the series.

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx &= \int_{-L}^{L} \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx \\ &= \sum_{n=0}^{\infty} A_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\end{aligned}$$

We now know that the all of the integrals on the right side will be zero except when $n = m$ because the set of cosines form an orthogonal set on the interval $-L \le x \le L$. However, we need to be careful about the value of *m* (or *n* depending on the letter you want to use). So, after evaluating all of the integrals we arrive at the following set of formulas for the coefficients.

$m = 0$:

$$\int_{-L}^{L} f(x)\,dx = A_0(2L) \qquad \Rightarrow \qquad A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$m \ne 0$:

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_m(L) \qquad \Rightarrow \qquad A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

Summarizing everything up then, the Fourier cosine series of an even function, $f(x)$ on $-L \le x \le L$ is given by,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \begin{cases} \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx & n = 0 \\ \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

Finally, before we work an example, let’s notice that because both $f(x)$ and the cosines are even the integrand in both of the integrals above is even and so we can write the formulas for the A_n’s as follows,

$$A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_{0}^{L} f(x)\,dx & n = 0 \\ \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

Now let’s take a look at an example.

Example 1 Find the Fourier cosine series for $f(x) = x^2$ on $-L \le x \le L$.

Solution

We clearly have an even function here and so all we really need to do is compute the coefficients and they are liable to be a little messy because we’ll need to do integration by parts twice. We’ll leave most of the actual integration details to you to verify.

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{L}\left(\frac{L^3}{3}\right) = \frac{L^2}{3}$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left(2Ln\pi x\cos\left(\frac{n\pi x}{L}\right) + \left(n^2\pi^2x^2 - 2L^2\right)\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{n^3\pi^3}\left(2L^2n\pi\cos(n\pi) + \left(n^2\pi^2L^2 - 2L^2\right)\sin(n\pi)\right) \\ &= \frac{4L^2(-1)^n}{n^2\pi^2} \qquad\qquad n = 1,2,3,\ldots \end{aligned}$$

The coefficients are then,

$$A_0 = \frac{L^2}{3} \qquad\qquad A_n = \frac{4L^2(-1)^n}{n^2\pi^2},\quad n = 1,2,3,\ldots$$

The Fourier cosine series is then,

$$x^2 = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$

Note that we'll often strip out the $n = 0$ from the series as we've done here because it will almost always be different from the other coefficients and it allows us to actually plug the coefficients into the series.

Now, just as we did in the previous section let's ask what we need to do in order to find the Fourier cosine series of a function that is not even. As with Fourier sine series when we make this change we'll need to move onto the interval $0 \le x \le L$ now instead of $-L \le x \le L$ and again we'll assume that the series will converge to $f(x)$ at this point and leave the discussion of the convergence of this series to a later section.

We could go through the work to find the coefficients here twice as we did with Fourier sine series, however there's no real reason to. So, while we could redo all the work above to get formulas for the coefficients let's instead go straight to the second method of finding the coefficients.

In this case, before we actually proceed with this we'll need to define the even extension of a function, $f(x)$ on $-L \le x \le L$. So, given a function $f(x)$ we'll define the even extension of the function as,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x \le 0 \end{cases}$$

Showing that this is an even function is simple enough.

$$g(-x) = f(-(-x)) = f(x) = g(x) \qquad\qquad \text{for } 0 < x < L$$

and we can see that $g(x) = f(x)$ on $0 \le x \le L$ and if $f(x)$ is already an even function we get $g(x) = f(x)$ on $-L \le x \le L$.

Let's take a look at some functions and sketch the even extensions for the functions.

Example 2 Sketch the even extension of each of the given functions.

(a) $f(x) = L - x$ on $0 \le x \le L$

(b) $f(x) = x^3$ on $0 \le x \le L$

(c) $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$

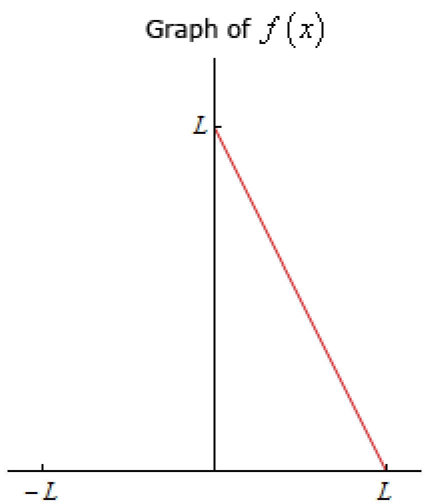
Solution

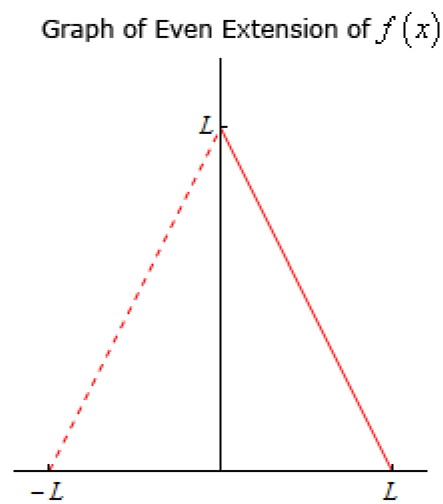
(a) $f(x) = L - x$ on $0 \le x \le L$

Here is the even extension of this function.

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x \le 0 \end{cases}$$
$$= \begin{cases} L - x & \text{if } 0 \le x \le L \\ L + x & \text{if } -L \le x \le 0 \end{cases}$$

Here is the graph of both the original function and its even extension. Note that we've put the "extension" in with a dashed line to make it clear the portion of the function that is being added to allow us to get the even extension





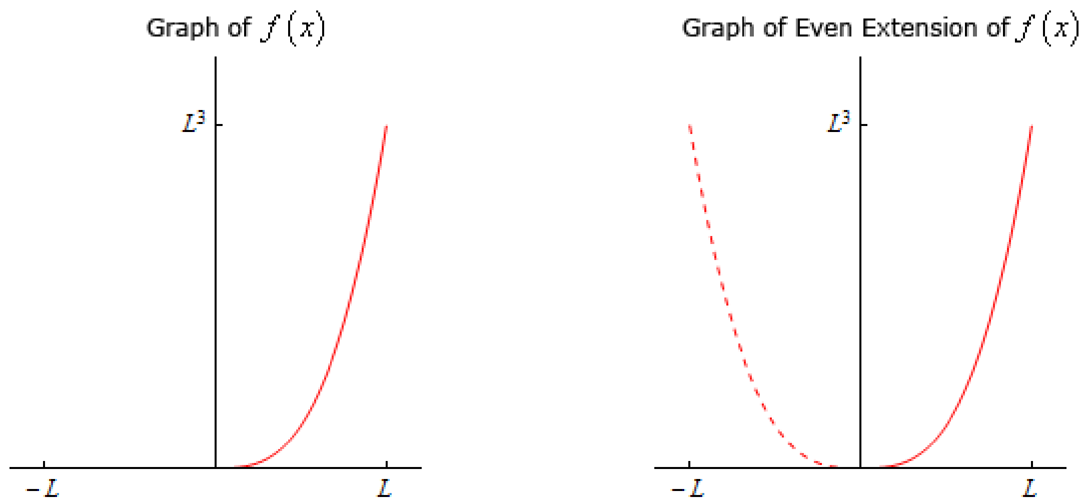
(b) $f\left(x\right)=x^3$ on $0\le x\le L$

The even extension of this function is,

$$g\left(x\right)=\begin{cases} f\left(x\right) & \text{if } 0\le x\le L \\ f\left(-x\right) & \text{if } -L\le x\le 0 \end{cases}$$

$$=\begin{cases} x^3 & \text{if } 0\le x\le L \\ -x^3 & \text{if } -L\le x\le 0 \end{cases}$$

The sketch of the function and the even extension is,



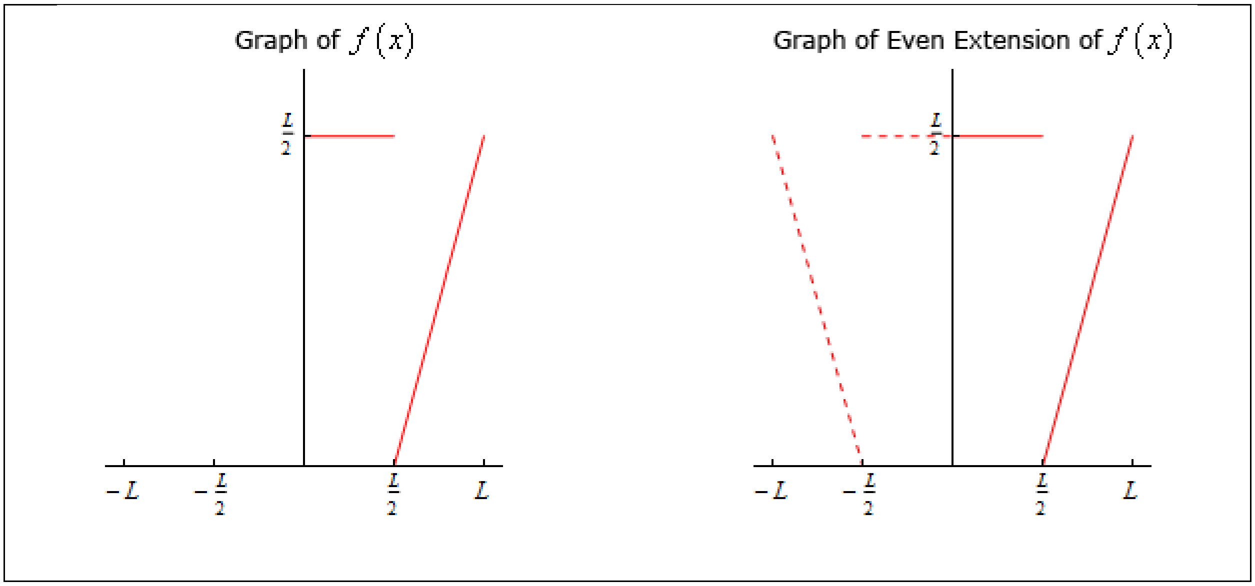
(c) $f\left(x\right)=\begin{cases} \frac{L}{2} & \text{if } 0\le x\le \frac{L}{2} \\ x-\frac{L}{2} & \text{if } \frac{L}{2}\le x\le L \end{cases}$

Here is the even extension of this function,

$$g\left(x\right)=\begin{cases} f\left(x\right) & \text{if } 0\le x\le L \\ f\left(-x\right) & \text{if } -L\le x\le 0 \end{cases}$$

$$=\begin{cases} x-\frac{L}{2} & \text{if } \frac{L}{2}\le x\le L \\ \frac{L}{2} & \text{if } 0\le x\le \frac{L}{2} \\ \frac{L}{2} & \text{if } -\frac{L}{2}\le x\le 0 \\ -x-\frac{L}{2} & \text{if } -L\le x\le -\frac{L}{2} \end{cases}$$

The sketch of the function and the even extension is,



Okay, let's now think about how we can use the even extension of a function to find the Fourier cosine series of any function $f(x)$ on $0 \le x \le L$.

So, given a function $f(x)$ we'll let $g(x)$ be the even extension as defined above. Now, $g(x)$ is an even function on $-L \le x \le L$ and so we can write down its Fourier cosine series. This is,

$$g(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx & n = 0 \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

and note that we'll use the second form of the integrals to compute the constants.

Now, because we know that on $0 \le x \le L$ we have $f(x) = g(x)$ and so the Fourier cosine series of $f(x)$ on $0 \le x \le L$ is also given by,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx & n = 0 \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

Let's take a look at a couple of examples.

Example 3 Find the Fourier cosine series for $f(x) = L - x$ on $0 \le x \le L$.

Solution
All we need to do is compute the coefficients so here is the work for that,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L L - x\,dx = \frac{L}{2}$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(-L\cos(n\pi) + L\right) = \frac{2L}{n^2\pi^2}\left(1 + (-1)^{n+1}\right) \qquad n = 1, 2, 3, \ldots \end{aligned}$$

The Fourier cosine series is then,

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left(1 + (-1)^{n+1}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Note that as we did with the first example in this section we stripped out the A_0 term before we plugged in the coefficients.

Next, let's find the Fourier cosine series of an odd function. Note that this is doable because we are really finding the Fourier cosine series of the even extension of the function.

Example 4 Find the Fourier cosine series for $f(x) = x^3$ on $0 \le x \le L$.

Solution
The integral for A_0 is simple enough but the integral for the rest will be fairly messy as it will require three integration by parts. We'll leave most of the details of the actual integration to you to verify. Here's the work,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x^3\,dx = \frac{L^3}{4}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^3\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left(\frac{L}{n^4\pi^4}\right)\left(n\pi x\left(n^2\pi^2x^2-6L^2\right)\sin\left(\frac{n\pi x}{L}\right)+\left(3Ln^2\pi^2x^2-6L^3\right)\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L$$

$$= \frac{2}{L}\left(\frac{L}{n^4\pi^4}\right)\left(n\pi L\left(n^2\pi^2L^2-6L^2\right)\sin(n\pi)+\left(3L^3n^2\pi^2-6L^3\right)\cos(n\pi)+6L^3\right)$$

$$= \frac{2}{L}\left(\frac{3L^4}{n^4\pi^4}\right)\left(2+\left(n^2\pi^2-2\right)(-1)^n\right) = \frac{6L^3}{n^4\pi^4}\left(2+\left(n^2\pi^2-2\right)(-1)^n\right) \qquad n=1,2,3,\ldots$$

The Fourier cosine series for this function is then,

$$f(x) = \frac{L^3}{4}+\sum_{n=1}^{\infty}\frac{6L^3}{n^4\pi^4}\left(2+\left(n^2\pi^2-2\right)(-1)^n\right)\cos\left(\frac{n\pi x}{L}\right)$$

Finally, let's take a quick look at a piecewise function.

Example 5 Find the Fourier cosine series for $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x-\frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$ on $0 \le x \le L$.

Solution

We'll need to split up the integrals for each of the coefficients here. Here are the coefficients.

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\left[\int_0^{\frac{L}{2}} f(x)\,dx + \int_{\frac{L}{2}}^L f(x)\,dx\right]$$

$$= \frac{1}{L}\left[\int_0^{\frac{L}{2}}\frac{L}{2}dx + \int_{\frac{L}{2}}^L x-\frac{L}{2}dx\right] = \frac{1}{L}\left[\frac{L^2}{4}+\frac{L^2}{8}\right] = \frac{1}{L}\left[\frac{3L^2}{8}\right] = \frac{3L}{8}$$

For the rest of the coefficients here is the integral we'll need to do.

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left[\int_0^{\frac{L}{2}}\frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^L\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

To make life a little easier let's do each of these separately.

$$\int_0^{\frac{L}{2}}\frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} = \frac{L}{2}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) = \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{\frac{L}{2}}^{L}\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{L}{n\pi}\left(\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\left(x-\frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$=\frac{L}{n\pi}\left(\frac{L}{n\pi}\cos(n\pi)+\frac{L}{2}\sin(n\pi)-\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$=\frac{L^2}{n^2\pi^2}\left((-1)^n-\cos\left(\frac{n\pi}{2}\right)\right)$$

Putting these together gives,

$$A_n=\frac{2}{L}\left[\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{L^2}{n^2\pi^2}\left((-1)^n-\cos\left(\frac{n\pi}{2}\right)\right)\right]$$

$$=\frac{2L}{n^2\pi^2}\left[(-1)^n-\cos\left(\frac{n\pi}{2}\right)+\frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right]$$

So, after all that work the Fourier cosine series is then,

$$f(x)=\frac{3L}{8}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[(-1)^n-\cos\left(\frac{n\pi}{2}\right)+\frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

Note that much as we saw with the Fourier sine series many of the coefficients will be quite messy to deal with.

Section 8-6 : Fourier Series

Okay, in the previous two sections we've looked at Fourier sine and Fourier cosine series. It is now time to look at a Fourier series. With a Fourier series we are going to try to write a series representation for $f(x)$ on $-L \le x \le L$ in the form,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

So, a Fourier series is, in some way a combination of the Fourier sine and Fourier cosine series. Also, like the Fourier sine/cosine series we'll not worry about whether or not the series will actually converge to $f(x)$ or not at this point. For now we'll just assume that it will converge and we'll discuss the convergence of the Fourier series in a later section.

Determining formulas for the coefficients, A_n and B_n, will be done in exactly the same manner as we did in the previous two sections. We will take advantage of the fact that $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are mutually orthogonal on $-L \le x \le L$ as we proved earlier. We'll also need the following formulas that we derived when we proved the two sets were mutually orthogonal.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

So, let's start off by multiplying both sides of the series above by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating from $-L$ to L. Doing this gives,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \int_{-L}^{L} \sum_{n=0}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx + \int_{-L}^{L} \sum_{n=1}^{\infty} B_n \sin\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

Now, just as we've been able to do in the last two sections we can interchange the integral and the summation. Doing this gives,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_{-L}^{L}\cos\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L}\sin\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

We can now take advantage of the fact that the sines and cosines are mutually orthogonal. The integral in the second series will always be zero and in the first series the integral will be zero if $n \neq m$ and so this reduces to,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \begin{cases} A_m(2L) & \text{if } n = m = 0 \\ A_m(L) & \text{if } n = m \neq 0 \end{cases}$$

Solving for A_m gives,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$
$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx \qquad m = 1,2,3,\ldots$$

Now, do it all over again only this time multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$, integrate both sides from –*L* to *L* and interchange the integral and summation to get,

$$\int_{-L}^{L} f(x)\sin\left(\tfrac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_{-L}^{L}\cos\left(\tfrac{n\pi x}{L}\right)\sin\left(\tfrac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L}\sin\left(\tfrac{n\pi x}{L}\right)\sin\left(\tfrac{m\pi x}{L}\right)dx$$

In this case the integral in the first series will always be zero and the second will be zero if $n \neq m$ and so we get,

$$\int_{-L}^{L} f(x)\sin\left(\tfrac{m\pi x}{L}\right)dx = B_m(L)$$

Finally, solving for B_m gives,

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{m\pi x}{L}\right)dx \qquad m = 1,2,3,\ldots$$

In the previous two sections we also took advantage of the fact that the integrand was even to give a second form of the coefficients in terms of an integral from 0 to *L*. However, in this case we don’t know anything about whether $f(x)$ will be even, odd, or more likely neither even nor odd. Therefore, this is the only form of the coefficients for the Fourier series.

Before we start examples let’s remind ourselves of a couple of formulas that we’ll make heavy use of here in this section, as we’ve done in the previous two sections as well. Provided *n* in an integer then,

$$\cos(n\pi) = (-1)^n \qquad\qquad \sin(n\pi) = 0$$

In all of the work that we’ll be doing here *n* will be an integer and so we’ll use these without comment in the problems so be prepared for them.

Also, don't forget that sine is an odd function, *i.e.* $\sin(-x) = -\sin(x)$ and that cosine is an even function, *i.e.* $\cos(-x) = \cos(x)$. We'll also be making heavy use of these ideas without comment in many of the integral evaluations so be ready for these as well.

Now let's take a look at an example.

Example 1 Find the Fourier series for $f(x) = L - x$ on $-L \le x \le L$.

Solution

So, let's go ahead and just run through formulas for the coefficients.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} L - x\,dx = L$$

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} (L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} \\
&= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(-2n\pi L\sin(-n\pi)\right) = 0 \qquad\qquad n = 1,2,3,\ldots
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} (L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L} \\
&= \frac{1}{L}\left[\frac{L^2}{n^2\pi^2}\left(2n\pi\cos(n\pi) - 2\sin(n\pi)\right)\right] = \frac{2L(-1)^n}{n\pi} \qquad\qquad n = 1,2,3,\ldots
\end{aligned}$$

Note that in this case we had $A_0 \neq 0$ and $A_n = 0,\ n = 1,2,3,\ldots$ This will happen on occasion so don't get excited about this kind of thing when it happens.

The Fourier series is then,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\
&= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = L + \sum_{n=1}^{\infty}\frac{2L(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

As we saw in the previous example sometimes we'll get $A_0 \neq 0$ and $A_n = 0,\ n = 1,2,3,\ldots$ Whether or not this will happen will depend upon the function $f(x)$ and often won't happen, but when it does don't get excited about it.

Let's take a look at another problem.

Example 2 Find the Fourier series for $f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$ on $-L \le x \le L$.

Solution

Because of the piece-wise nature of the function the work for the coefficients is going to be a little unpleasant but let's get on with it.

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\left[\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\right] \\ &= \frac{1}{2L}\left[\int_{-L}^{0} L\,dx + \int_{0}^{L} 2x\,dx\right] = \frac{1}{2L}\left[L^2 + L^2\right] = L \end{aligned}$$

$$\begin{aligned} A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{1}{L}\left[\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx\right] \end{aligned}$$

At this point it will probably be easier to do each of these individually.

$$\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx = \left.\left(\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\right|_{-L}^{0} = \frac{L^2}{n\pi}\sin(n\pi) = 0$$

$$\begin{aligned} \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx &= \left.\left(\frac{2L}{n^2\pi^2}\right)\left(L\cos\left(\frac{n\pi x}{L}\right) + n\pi x\sin\left(\frac{n\pi x}{L}\right)\right)\right|_{0}^{L} \\ &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\cos(n\pi) + n\pi L\sin(n\pi) - L\cos(0)\right) \\ &= \left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n - 1\right) \end{aligned}$$

So, if we put all of this together we have,

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[0 + \left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n - 1\right)\right] \\ &= \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right), \qquad n = 1,2,3,\ldots \end{aligned}$$

So, we've gotten the coefficients for the cosines taken care of and now we need to take care of the coefficients for the sines.

$$\begin{aligned} B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{L}\left[\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{1}{L}\left[\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx\right] \end{aligned}$$

As with the coefficients for the cosines will probably be easier to do each of these individually.

$$\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx = \left(-\frac{L^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0} = \frac{L^2}{n\pi}\left(-1+\cos(n\pi)\right) = \frac{L^2}{n\pi}\left((-1)^n - 1\right)$$

$$\begin{aligned} \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L} \\ &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin(n\pi) - n\pi L\cos(n\pi)\right) \\ &= \left(\frac{2L^2}{n^2\pi^2}\right)\left(-n\pi(-1)^n\right) = -\frac{2L^2}{n\pi}(-1)^n \end{aligned}$$

So, if we put all of this together we have,

$$\begin{aligned} B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{L}\left[\frac{L^2}{n\pi}\left((-1)^n - 1\right) - \frac{2L^2}{n\pi}(-1)^n\right] \\ &= \frac{L}{n\pi}\left[-1-(-1)^n\right] = -\frac{L}{n\pi}\left(1+(-1)^n\right) \qquad n = 1,2,3,\ldots \end{aligned}$$

So, after all that work the Fourier series is,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\ &= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\ &= L + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty} \frac{L}{n\pi}\left(1+(-1)^n\right)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

As we saw in the previous example there is often quite a bit of work involved in computing the integrals involved here.

The next couple of examples are here so we can make a nice observation about some Fourier series and their relation to Fourier sine/cosine series

Example 3 Find the Fourier series for $f\left(x\right)=x$ on $-L \le x \le L$.

Solution

Let's start with the integrals for A_n.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f\left(x\right)dx = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$$
$$A_n = \frac{1}{L}\int_{-L}^{L} f\left(x\right)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

In both cases note that we are integrating an odd function (*x* is odd and cosine is even so the product is odd) over the interval $\left[-L,L\right]$ and so we know that both of these integrals will be zero.

Next here is the integral for B_n

$$B_n = \frac{1}{L}\int_{-L}^{L} f\left(x\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

In this case we're integrating an even function (*x* and sine are both odd so the product is even) on the interval $\left[-L,L\right]$ and so we can "simplify" the integral as shown above. The reason for doing this here is not actually to simplify the integral however. It is instead done so that we can note that we did this integral back in the Fourier sine series section and so don't need to redo it in this section. Using the previous result we get,

$$B_n = \frac{\left(-1\right)^{n+1}2L}{n\pi} \qquad\qquad n = 1,2,3,\ldots$$

In this case the Fourier series is,

$$f\left(x\right) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

If you go back and take a look at Example 1 in the Fourier sine series section, the same example we used to get the integral out of, you will see that in that example we were finding the Fourier sine series for $f\left(x\right)=x$ on $-L \le x \le L$. The important thing to note here is that the answer that we got in that example is identical to the answer we got here.

If you think about it however, this should not be too surprising. In both cases we were using an odd function on $-L \le x \le L$ and because we know that we had an odd function the coefficients of the cosines in the Fourier series, A_n, will involve integrating and odd function over a symmetric interval, $-L \le x \le L$, and so will be zero. So, in these cases the Fourier sine series of an odd function on $-L \le x \le L$ is really just a special case of a Fourier series.

Note however that when we moved over to doing the Fourier sine series of any function on $0 \le x \le L$ we should no longer expect to get the same results. You can see this by comparing Example 1 above with Example 3 in the Fourier sine series section. In both examples we are finding the series for $f(x) = x - L$ and yet got very different answers.

So, why did we get different answers in this case? Recall that when we find the Fourier sine series of a function on $0 \le x \le L$ we are really finding the Fourier sine series of the odd extension of the function on $-L \le x \le L$ and then just restricting the result down to $0 \le x \le L$. For a Fourier series we are actually using the whole function on $-L \le x \le L$ instead of its odd extension. We should therefore not expect to get the same results since we are really using different functions (at least on part of the interval) in each case.

So, if the Fourier sine series of an odd function is just a special case of a Fourier series it makes some sense that the Fourier cosine series of an even function should also be a special case of a Fourier series. Let's do a quick example to verify this.

Example 4 Find the Fourier series for $f(x) = x^2$ on $-L \le x \le L$.

Solution

Here are the integrals for the A_n and in this case because both the function and cosine are even we'll be integrating an even function and so can "simplify" the integral.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\int_{0}^{L} x^2\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx$$

As with the previous example both of these integrals were done in Example 1 in the Fourier cosine series section and so we'll not bother redoing them here. The coefficients are,

$$A_0 = \frac{L^2}{3} \qquad\qquad A_n = \frac{4L^2(-1)^n}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

Next here is the integral for the B_n

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

In this case the function is even and sine is odd so the product is odd and we're integrating over $-L \le x \le L$ and so the integral is zero.

The Fourier series is then,

$$f\left(x\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2\left(-1\right)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)$$

As suggested before we started this example the result here is identical to the result from Example 1 in the Fourier cosine series section and so we can see that the Fourier cosine series of an even function is just a special case a Fourier series.

Section 8-7 : Convergence of Fourier Series

Over the last few sections we've spent a fair amount of time to computing Fourier series, but we've avoided discussing the topic of convergence of the series. In other words, will the Fourier series converge to the function on the given interval?

In this section we're going to address this issue as well as a couple of other issues about Fourier series. We'll be giving a fair number of theorems in this section but are not going to be proving any of them. We'll also not be doing a whole lot of in the way of examples in this section.

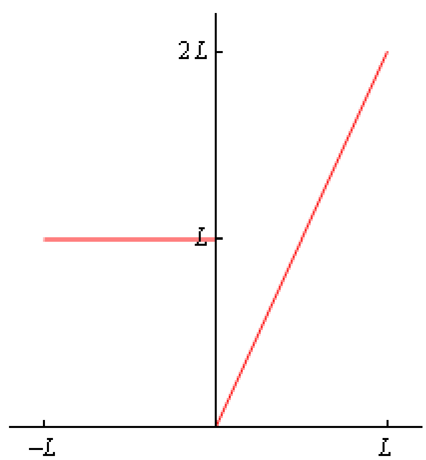
Before we get into the topic of convergence we need to first define a couple of terms that we'll run into in the rest of the section. First, we say that $f\left(x\right)$ has a **jump discontinuity** at $x = a$ if the limit of the function from the left, denoted $f\left(a^-\right)$, and the limit of the function from the right, denoted $f\left(a^+\right)$, both exist and $f\left(a^-\right) \ne f\left(a^+\right)$.

Next, we say that $f\left(x\right)$ is **piecewise smooth** if the function can be broken into distinct pieces and on each piece both the function and its derivative, $f'\left(x\right)$, are continuous. A piecewise smooth function may not be continuous everywhere however the only discontinuities that are allowed are a finite number of jump discontinuities.

Let's consider the function,

$$f\left(x\right) = \begin{cases} L & \text{if } \; -L \le x \le 0 \\ 2x & \text{if } \; 0 \le x \le L \end{cases}$$

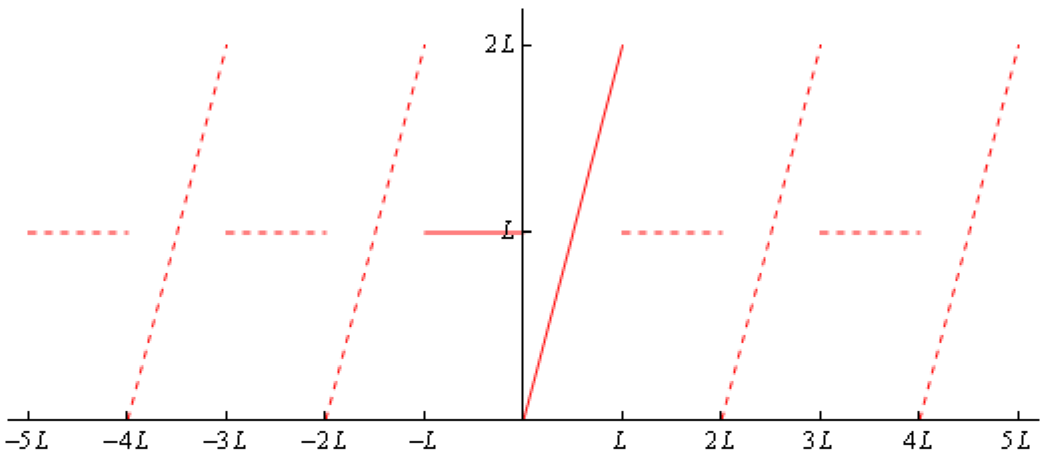
We found the Fourier series for this function in Example 2 of the previous section. Here is a sketch of this function on the interval on which it is defined, *i.e.* $-L \le x \le L$.



This function has a jump discontinuity at $x = 0$ because $f\left(0^-\right) = L \ne 0 = f\left(0^+\right)$ and note that on the intervals $-L \le x \le 0$ and $0 \le x \le L$ both the function and its derivative are continuous. This is therefore an example of a piecewise smooth function. Note that the function itself is not continuous at $x = 0$ but because this point of discontinuity is a jump discontinuity the function is still piecewise smooth.

The last term we need to define is that of **periodic extension**. Given a function, $f(x)$, defined on some interval, we'll be using $-L \le x \le L$ exclusively here, the periodic extension of this function is the new function we get by taking the graph of the function on the given interval and then repeating that graph to the right and left of the graph of the original function on the given interval.

It is probably best to see an example of a periodic extension at this point to help make the words above a little clearer. Here is a sketch of the period extension of the function we looked at above,



The original function is the solid line in the range $-L \le x \le L$. We then got the periodic extension of this by picking this piece up and copying it every interval of length 2L to the right and left of the original graph. This is shown with the two sets of dashed lines to either side of the original graph.

Note that the resulting function that we get from defining the periodic extension is in fact a new periodic function that is equal to the original function on $-L \le x \le L$.

With these definitions out of the way we can now proceed to talk a little bit about the convergence of Fourier series. We will start off with the convergence of a Fourier series and once we have that taken care of the convergence of Fourier Sine/Cosine series will follow as a direct consequence. Here then is the theorem giving the convergence of a Fourier series.

Convergence of Fourier series

Suppose $f(x)$ is a piecewise smooth on the interval $-L \le x \le L$. The Fourier series of $f(x)$ will then converge to,

1. the periodic extension of $f(x)$ if the periodic extension is continuous.
2. the average of the two one-sided limits, $\frac{1}{2}\left[f(a^-) + f(a^+)\right]$, if the periodic extension has a jump discontinuity at $x = a$.

The first thing to note about this is that on the interval $-L \le x \le L$ both the function and the periodic extension are equal and so where the function is continuous on $-L \le x \le L$ the periodic extension will also be continuous and hence at these points the Fourier series will in fact converge to the function. The only points in the interval $-L \le x \le L$ where the Fourier series will not converge to the function is where the function has a jump discontinuity.

Let's again consider Example 2 of the previous section. In that section we found that the Fourier series of,

$$f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$$

on $-L \le x \le L$ to be,

$$f(x) = L + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty} \frac{L}{n\pi}\left(1 + (-1)^n\right)\sin\left(\frac{n\pi x}{L}\right)$$

We now know that in the intervals $-L < x < 0$ and $0 < x < L$ the function and hence the periodic extension are both continuous and so on these two intervals the Fourier series will converge to the periodic extension and hence will converge to the function itself.

At the point $x = 0$ the function has a jump discontinuity and so the periodic extension will also have a jump discontinuity at this point. That means that at $x = 0$ the Fourier series will converge to,

$$\frac{1}{2}\left[f\left(0^-\right) + f\left(0^+\right)\right] = \frac{1}{2}[L + 0] = \frac{L}{2}$$

At the two endpoints of the interval, $x = -L$ and $x = L$, we can see from the sketch of the periodic extension above that the periodic extension has a jump discontinuity here and so the Fourier series will not converge to the function there but instead the averages of the limits.

So, at $x = -L$ the Fourier series will converge to,

$$\frac{1}{2}\left[f\left(-L^-\right) + f\left(-L^+\right)\right] = \frac{1}{2}[2L + L] = \frac{3L}{2}$$

and at $x = L$ the Fourier series will converge to,

$$\frac{1}{2}\left[f\left(L^-\right) + f\left(L^+\right)\right] = \frac{1}{2}[2L + L] = \frac{3L}{2}$$

Now that we have addressed the convergence of a Fourier series we can briefly turn our attention to the convergence of Fourier sine/cosine series. First, as noted in the previous section the Fourier sine series of an odd function on $-L \le x \le L$ and the Fourier cosine series of an even function on $-L \le x \le L$ are both just special cases of a Fourier series we now know that both of these will have the same convergence as a Fourier series.

Next, if we look at the Fourier sine series of any function, $g(x)$, on $0 \le x \le L$ then we know that this is just the Fourier series of the odd extension of $g(x)$ restricted down to the interval $0 \le x \le L$. Therefore, we know that the Fourier series will converge to the odd extension on $-L \le x \le L$ where it is continuous and the average of the limits where the odd extension has a jump discontinuity. However, on $0 \le x \le L$ we know that $g(x)$ and the odd extension are equal and so we can again see that the Fourier sine series will have the same convergence as the Fourier series.

Likewise, we can go through a similar argument for the Fourier cosine series using even extensions to see that Fourier cosine series for a function on $0 \le x \le L$ will also have the same convergence as a Fourier series.

The next topic that we want to briefly discuss here is when will a Fourier series be continuous. From the theorem on the convergence of Fourier series we know that where the function is continuous the Fourier series will converge to the function and hence be continuous at these points. The only places where the Fourier series may not be continuous is if there is a jump discontinuity on the interval $-L \le x \le L$ and potentially at the endpoints as we saw that the periodic extension may introduce a jump discontinuity there.

So, if we're going to want the Fourier series to be continuous everywhere we'll need to make sure that the function does not have any discontinuities in $-L \le x \le L$. Also, in order to avoid having the periodic extension introduce a jump discontinuity we'll need to require that $f(-L) = f(L)$. By doing this the two ends of the graph will match up when we form the periodic extension and hence we will avoid a jump discontinuity at the end points.

Here is a summary of these ideas for a Fourier series.

Suppose $f(x)$ is a piecewise smooth on the interval $-L \le x \le L$. The Fourier series of $f(x)$ will be continuous and will converge to $f(x)$ on $-L \le x \le L$ provided $f(x)$ is continuous on $-L \le x \le L$ and $f(-L) = f(L)$.

Now, how can we use this to get similar statements about Fourier sine/cosine series on $0 \le x \le L$? Let's start with a Fourier cosine series. The first thing that we do is form the even extension of $f(x)$ on $-L \le x \le L$. For the purposes of this discussion let's call the even extension $g(x)$ As we saw when we sketched several even extensions in the Fourier cosine series section that in order for the sketch to be the even extension of the function we must have both,

$$g(0^-) = g(0^+) \qquad\qquad g(-L) = g(L)$$

If one or both of these aren't true then $g(x)$ will not be an even extension of $f(x)$.

So, in forming the even extension we do not introduce any jump discontinuities at $x = 0$ and we get for free that $g(-L) = g(L)$. If we now apply the above theorem to the even extension we see that the Fourier series of the even extension is continuous on $-L \le x \le L$. However, because the even extension and the function itself are the same on $0 \le x \le L$ then the Fourier cosine series of $f(x)$ must also be continuous on $0 \le x \le L$.

Here is a summary of this discussion for the Fourier cosine series.

Suppose $f\left(x\right)$ is a piecewise smooth on the interval $0 \le x \le L$. The Fourier cosine series of $f\left(x\right)$ will be continuous and will converge to $f\left(x\right)$ on $0 \le x \le L$ provided $f\left(x\right)$ is continuous on $0 \le x \le L$.

Note that we don't need any requirements on the end points here because they are trivially satisfied when we convert over to the even extension.

For a Fourier sine series we need to be a little more careful. Again, the first thing that we need to do is form the odd extension on $-L \le x \le L$ and let's call it $g\left(x\right)$. We know that in order for it to be the odd extension then we know that at all points in $-L \le x \le L$ it must satisfy $g\left(-x\right) = -g\left(x\right)$ and that is what can lead to problems.

As we saw in the Fourier sine series section it is very easy to introduce a jump discontinuity at $x = 0$ when we form the odd extension. In fact, the only way to avoid forming a jump discontinuity at this point is to require that $f\left(0\right) = 0$.

Next, the requirement that at the endpoints we must have $g\left(-L\right) = -g\left(L\right)$ will practically guarantee that we'll introduce a jump discontinuity here as well when we form the odd extension. Again, the only way to avoid doing this is to require $f\left(L\right) = 0$.

So, with these two requirements we will get an odd extension that is continuous and so we know that the Fourier series of the odd extension on $-L \le x \le L$ will be continuous and hence the Fourier sine series will be continuous on $0 \le x \le L$.

Here is a summary of all this for the Fourier sine series.

Suppose $f\left(x\right)$ is a piecewise smooth on the interval $0 \le x \le L$. The Fourier sine series of $f\left(x\right)$ will be continuous and will converge to $f\left(x\right)$ on $0 \le x \le L$ provided $f\left(x\right)$ is continuous on $0 \le x \le L$, $f\left(0\right) = 0$ and $f\left(L\right) = 0$.

The next topic of discussion here is differentiation and integration of Fourier series. In particular, we want to know if we can differentiate a Fourier series term by term and have the result be the Fourier series of the derivative of the function. Likewise, we want to know if we can integrate a Fourier series term by term and arrive at the Fourier series of the integral of the function.

Note that we'll not be doing much discussion of the details here. All we're really going to be doing is giving the theorems that govern the ideas here so that we can say we've given them.

Let's start off with the theorem for term by term differentiation of a Fourier series.

Given a function $f(x)$ if the derivative, $f'(x)$, is piecewise smooth and the Fourier series of $f(x)$ is continuous then the Fourier series can be differentiated term by term. The result of the differentiation is the Fourier series of the derivative, $f'(x)$.

One of the main condition of this theorem is that the Fourier series be continuous and from above we also know the conditions on the function that will give this. So, if we add this into the theorem to get this form of the theorem,

Suppose $f(x)$ is a continuous function, its derivative $f'(x)$ is piecewise smooth and $f(-L) = f(L)$ then the Fourier series of the function can be differentiated term by term and the result is the Fourier series of the derivative.

For Fourier cosine/sine series the basic theorem is the same as for Fourier series. All that's required is that the Fourier cosine/sine series be continuous and then you can differentiate term by term. The theorems that we'll give here will merge the conditions for the Fourier cosine/sine series to be continuous into the theorem.

Let's start with the Fourier cosine series.

Suppose $f(x)$ is a continuous function and its derivative $f'(x)$ is piecewise smooth then the Fourier cosine series of the function can be differentiated term by term and the result is the Fourier sine series of the derivative.

Next the theorem for Fourier sine series.

Suppose $f(x)$ is a continuous function, its derivative $f'(x)$ is piecewise smooth, $f(0) = 0$ and $f(L) = 0$ then the Fourier sine series of the function can be differentiated term by term and the result is the Fourier cosine series of the derivative.

The theorem for integration of Fourier series term by term is simple so there it is.

Suppose $f(x)$ is piecewise smooth then the Fourier sine series of the function can be integrated term by term and the result is a convergent infinite series that will converge to the integral of $f(x)$.

Note however that the new series that results from term by term integration may not be the Fourier series for the integral of the function.