Maths Solutions (Pure 2)

Section A: Multiple Choice

1. **Answer:** $13 - (x - 3)^2$

   Working: $4 + 6x - x^2 = -[x^2 - 6x - 4] = -[(x - 3)^2 - 9 - 4] = -[(x - 3)^2 - 13]$
   $= 13 - (x - 3)^2$

2. **Answer:** $x^3 \div (x^3 - a^2y)$

   Working: The denominator is a difference of two squares:
   $(x^6 + a^2x^3y) \div (x^6 - a^4y^2) = x^3(x^3 + a^2y) \div (x^3 + a^2y)(x^3 - a^2y)$
   Cancel common factor: $= x^3 \div (x^3 - a^2y)$

3. **Answer:** C

   Working: For $x > 1$, $|x - 1| > 0 \rightarrow y = x - (x - 1) = 1$
   For $x < 1$, $|x - 1| < 0 \rightarrow y = x - -(x - 1) = x + (x - 1) = 2x - 1$
   $\rightarrow$ Graph C matches this

4. **Answer:** 11/4

   Working: Gradient $= (9 - 1)/(-1 - 5) = 8/-6 = -4/3$
   Point-slope form: $y - 1 = -4/3(x - 5)$, at $y = 4$:
   $4 - 1 = -4/3(x - 5) \rightarrow 3 = -4/3(x - 5) \rightarrow x - 5 = -9/4 \rightarrow x = 11/4$

5. **Answer:** 19440

   Working: $y = (3x + 2)^6 \rightarrow dy/dx = 6(3x + 2)^5 * 3 = 18(3x + 2)^5$
   Coefficient of $x^3 = 18 * ({}^5C_3 * 3^3 * 2^2) = 18 * 10 * 27 * 4 = 19440$

6. **Answer:** 0

Working: Since e^{4x} and e^{2x} are always positive, it is immediately clear that there are no solutions. Could also justify by forming a quadratic in e^{2x} and considering the discriminant.

7. **Answer:** For all $\{x_0 \in (0.5, 1) : x_0 \neq \alpha\}$, the iterative formula diverges.

Working: Let the iterative function be $g(x) = 1 - 4x^5$.
Then $g'(x) = -20x^4$.
If $|g'(\alpha)| < 1$ then it converges, but if $|g'(\alpha)| > 1$ it diverges.
If $g'(\alpha) > 0$ then it forms a staircase and if $g'(\alpha) < 0$ it forms a cobweb.
$g'(\alpha) = g'(0.623) = -3.01 < -1$ and $-3.01 < 0$
$\rightarrow$ diverges and forms a cobweb

8. **Answer:** 19.8°

Working: Starting with, $\sin^2 2\theta + 8 \sin\theta \cos\theta \cos 2\theta - 4\cos^2 2\theta = 0$
Use sine double angle identity in middle term:
$\rightarrow \sin^2 2\theta + 4 \sin 2\theta \cos 2\theta - 4 \cos^2 2\theta = 0$
Divide both sides by $\cos^2 2\theta$:
$\rightarrow \tan^2 2\theta + 4 \tan 2\theta - 4 = 0$
$\rightarrow \tan 2\theta = -2 \pm \sqrt{2} \rightarrow \tan 2\theta = 0.8284, -4.828$
$\rightarrow 2\theta = 39.6^{\circ}, 101.7^{\circ}$
$\rightarrow \theta = 19.8^{\circ}, 50.9^{\circ}$

9. **Answer:** 1/12

Working: Find the intersections of the curves:
$x^2 + x = x - x^3 \rightarrow x^2(x + 1) = 0 \rightarrow x = 0$ and $x = -1$
Integral of $(x^2 + x) - (x - x^3)$ from -1 to 0 = 1/12

10. **Answer:** When the equation $f(x) = 0$ has three real roots, the product of these roots is independent of k.

Working: $f'(x) = 12x^2 + 6kx - k$ and $f''(x) = 24x + 6k$

Increasing → $f'(x) > 0$ → $12x^2 + 6kx - k$ has no roots

→ discriminant < 0 → $36k^2 + 48k < 0$ → $12k(3k + 4) < 0$

→ $-4/3 < k < 0$ → $-4 < 3k < 0$ → (1) is false.

Stationary point of inflection: check $f'(x) = 0$ → $12x^2 + 6kx - k = 0$

Since cubic, if it has a stationary point of inflection this must be the only stationary point → one solution → discriminant = 0

→ $36k^2 + 48k = 0$ → $k = 0$, $k = -4/3$ (from above)

Graphing with $k = 0$ shows it is an inflection point → (2) is false.

Choose a large $k = 10$ → it is clear there will be a difference > 12.

→ (3) is false

By process of elimination, (4) is correct.

Proof of (4): Let the roots be a, b, c:

By factor theorem, → $f(x) = 4(x - a)(x - b)(x - c)$

The final term in this expansion will be $4abc$ which is independent of x → it is the constant term in f(x).

→ $4abc = -2$ → product of roots = $-2/4 = -1/2$ (independent of k).

Section B: Standard Questions

11.

a. Let the consecutive numbers be $n(n + 1)(n + 2)(n + 3)$ [1 mark]
$= n(n^2 + 3n + 2)(n + 3)$
$= (n^2 + 3n)(n^2 + 3n + 2)$ [1 mark for any pair expanded]
$= (n^2 + 3n)(n^2 + 3n) + 2(n^2 + 3n)$
$= (n^2 + 3n)^2 + 2(n^2 + 3n)$ [1 mark for any correct similar form]
Notice this is almost a perfect square:
$= (n^2 + 3n)^2 + 2(n^2 + 3n) + 1 - 1$
$= (n^2 + 3n + 1)^2 - 1$ [1 mark]
(Alternatively, could substitute: let $x = n^2 + 3n \rightarrow x^2 + 2x = (x + 1)^2 - 1$)
Since no two square numbers differ by 1 (other than 0 and 1 but $n^2 + 3n + 1 > 1$ for $n > 0$), this cannot be equal to a square number.
[1 mark]

(It is also possible to use similar techniques to prove it starting from different forms such as $(n - 1)n(n + 1)(n + 2) = (n^2 + n)(n^2 + n - 2) \rightarrow (n^2 + n)^2 - 2(n^2 + n) \rightarrow (n^2 + n - 1)^2 - 1$.)

b. i) Assume $\sqrt{p}$ is rational $\rightarrow \sqrt{p} = a/b$ where a and b are integers with no common factors. [1 mark]
$\rightarrow p = a^2/b^2 \rightarrow pb^2 = a^2$ [1 mark]
$\rightarrow a^2$ has a factor of $b^2 \rightarrow$ a has a factor of b [1 mark]
$\rightarrow$ this is a contradiction since we assumed a and b have no common factors
$\rightarrow \sqrt{p}$ is irrational [1 mark]

ii) Assume the set of all primes is finite and can be written as $\{p_1, p_2, \dots, p_n\}$. [1 mark]
Define q as the product of all primes + 1 $\rightarrow q = p_1p_2...p_n + 1$ [1 mark]
Since consecutive integers have no common factors, q must have a factor(s) which does not exist in the original set [1 mark]
$\rightarrow$ there exist other primes
$\rightarrow$ the original set did not contain all the primes [1 mark]
$\rightarrow$ there exists an integer k such that $p + k$ is prime. [1 mark]

iii) The prime p + k can be added to the set and the proof can be repeated infinitely many times to obtain infinitely many primes. [1 mark]

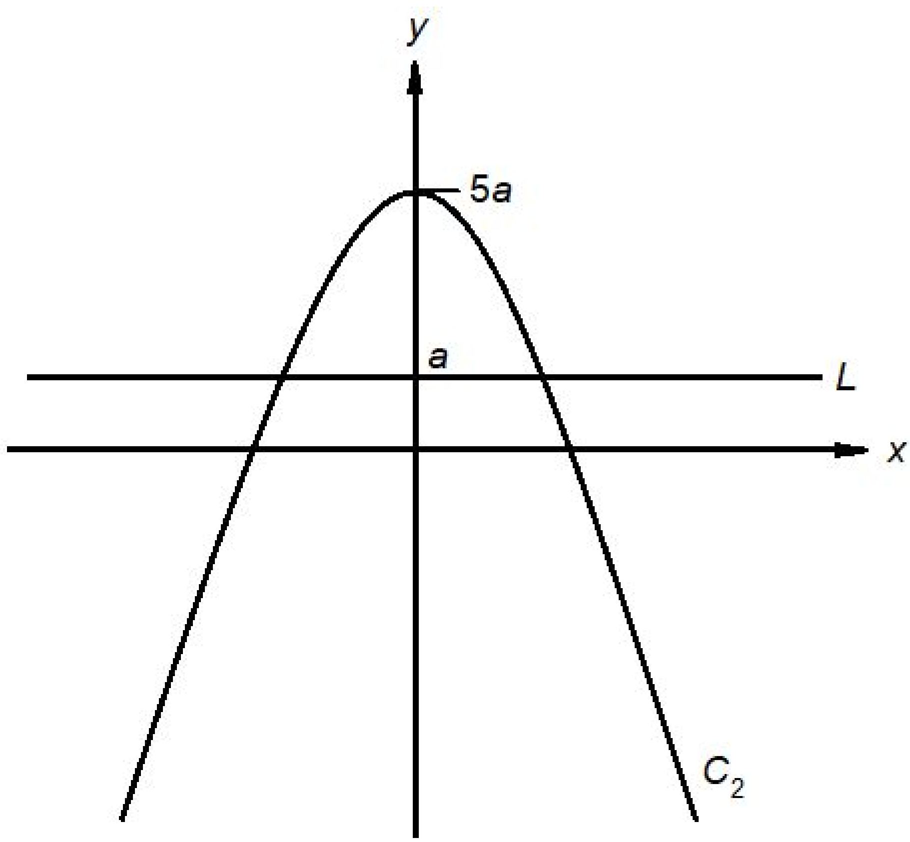
12. Equation 1: $3 \log_8(xy) = 4 \log_2 x$
$\rightarrow 3 \log_2(xy) / \log_2(8) = 4 \log_2 x$ [1 mark]
$\rightarrow \log_2 x + \log_2 y = 4 \log_2 x$ [1 mark] $\rightarrow -3 \log_2 x + \log_2 y = 0$ [1 mark]

Equation 2: $\log_2 y = a + \log_2 x$
$\rightarrow -\log_2 x + \log_2 y = a$ [1 mark]

Eqn 1 - Eqn 2 $\rightarrow -2 \log_2 x = -a$ [1 mark] $\rightarrow x = 2^{a/2} = (\sqrt{2})^a$ [1 mark]
$\rightarrow \log_2 y = a + a/2 = 3a/2$ [1 mark] $\rightarrow y = 2^{3a/2} = (2\sqrt{2})^a$ [1 mark]

13.

a. i) [1 mark for correct shapes, 1 mark for labelled intersect with y-axis]



ii) Equation of C_2 : $y = 5a - x^2$. [1 mark]
Intersects: $5a - x^2 = a \rightarrow x^2 - 4a = 0 \rightarrow (x - 2\sqrt{a})(x + 2\sqrt{a}) \rightarrow x = \pm 2\sqrt{a}$ [1 mark]
$\int_{-2\sqrt{a}}^{2\sqrt{a}} 5a - x^2 - a \, dx = \int_{-2\sqrt{a}}^{2\sqrt{a}} 4a - x^2 \, dx$ [1 mark]
$= [4ax - x^3/3]_{-2\sqrt{a}}^{2\sqrt{a}}$ [1 mark]
$= (8a\sqrt{a} - (2\sqrt{a})^3/3) - (-8a\sqrt{a} - (-2\sqrt{a})^3/3)$ [1 mark]
$= 16a^{3/2} - 8/3\ a^{3/2} - 8/3\ a^{3/2} = 32/3\ a^{3/2}$. [1 mark]

iii) $32/3\ a^{3/2} = 108 \rightarrow a^{3/2} = 81/8$ [1 mark]
$a = (81/8)^{2/3} = (3^4/2^3)^{2/3} = 3^{8/3} / 4$ or $9/4 \times 3^{2/3}$ [1 mark]

b. C_2 : $y = 5a - x^2 \rightarrow x = \pm \sqrt{(5a - y)}$. Since region is symmetrical in y-axis, can consider only one part $\rightarrow x = \sqrt{(5a - y)}$.[1 mark]

$$\int_a^k \sqrt{5a - y}\, \mathrm{d}y = \int_k^{5a} \sqrt{5a - y}\, \mathrm{d}y$$ [1 mark]

By substitution $u = 5a - y \rightarrow du = - dy$,

$$\left[-\frac{2}{3}(5a-y)^{\frac{3}{2}}\right]_a^k = \left[-\frac{2}{3}(5a-y)^{\frac{3}{2}}\right]_k^{5a}$$ [1 mark, or transforming bounds]

$\rightarrow -2/3\ ((5a - k)^{3/2} - (5a - a)^{3/2}) = -2/3\ ((5a - 5a)^{3/2} - (5a - k)^{3/2})$ [1 mark]
$\rightarrow (5a - k)^{3/2} - (4a)^{3/2} = - (5a - k)^{3/2} \rightarrow 2(5a - k)^{3/2} = (4a)^{3/2}$ [1 mark]
Squaring both sides $\rightarrow (5a - k)^3 = 16a^3$
Cube root of both sides $\rightarrow 5a - k = (2^{4/3})a$ [1 mark]
$\rightarrow k = (5 - 2^{4/3})a$ [1 mark] $\rightarrow k/a = 5 - 2^{4/3}$ [1 mark]

14.

a. $u = a + b - x \rightarrow x = a + b - u \rightarrow dx = - du$ [1 mark]

Limits of integration are (a + b - a) = b to (a + b - b) = a, so

$$I = -\int_b^a \frac{f(a+b-u)}{f(u)+f(a+b-u)}\ \mathrm{d}u$$ [1 mark]

Using the negative sign to reverse the limits and let x = u:

$$I = \int_a^b \frac{f(a+b-u)}{f(u)+f(a+b-u)}\ \mathrm{d}u = \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)}\ \mathrm{d}x$$ [1 mark]

Adding this and the original definition together,

$$2I = \int_a^b \frac{f(a+b-x)}{f(x)+f(a+b-x)} + \frac{f(x)}{f(x)+f(a+b-x)}\ \mathrm{d}x$$

$$2I = \int_a^b \frac{f(a+b-x)+f(x)}{f(x)+f(a+b-x)}\ \mathrm{d}x$$

$$2I = \int_a^b 1\ \mathrm{d}x$$

[1 mark]

$\rightarrow 2I = b - a \rightarrow I = (b - a)/2$. [1 mark]

b. Let $u = x^3 \rightarrow du = 3x^2\ dx$. Bounds are ln 3 to ln 4 and the integral is

$$\frac{1}{3}\int_{\ln 3}^{\ln 4} \frac{\sin u}{\sin u + \sin(\ln 12 - u)}\ \mathrm{d}u$$ [1 mark]

Since ln 12 = ln 3 + ln 4 (by log properties), this is

$$\frac{1}{3}\int_{\ln 3}^{\ln 4} \frac{\sin u}{\sin u + \sin(\ln 3 + \ln 4 - u)}\ \mathrm{d}u$$ [1 mark]

Using the formula in part a), the integral is

$$\frac{1}{3}\left(\frac{\ln 4 - \ln 3}{2}\right) = \frac{1}{6}\ln\frac{4}{3}$$ [1 mark]

15.

a. OR = OA + AR

= **a** + h AB [1 mark]

AB = OB - OA = **b** - **a** [1 mark]

→ OR = **a** + h(**b** - **a**) = (1 - h)**a** + **b** [1 mark]

b. OR = OP + PR

= 3/5 **a** + k PQ [1 mark]

PQ = PO + OQ = -3/5 **a** + 3 **b** [1 mark]

→ OR = 3/5 **a** + k (-3/5 **a** + 3 **b**)

→ OR = 3/5 (1 - k) **a** + 3k **b** [1 mark]

c. i) Components of a) and b) must be equal:

3/5 (1 - k) = 1 - h [1 mark] and 3k = 1 → k = 1/3 [1 mark]

→ 3/5 * 2/3 = 1 - h [1 mark] → h = 3/5 [1 mark]

ii) PR : PQ = 1 : 2 [1 mark]

16. Separating the variables,

$$\int \frac{5}{2y^2 - 7y + 3}\, \mathrm{d}y = \int\, \mathrm{d}x$$ [1 mark]

On the LHS, factorise and use partial fractions:

$$= \int \frac{5}{2y^2 - 7y + 3}\, \mathrm{d}y$$
$$= \int \frac{5}{(2y-1)(y-3)}\, \mathrm{d}y$$
$$= \int \frac{-2}{2y-1} + \frac{1}{y-3}\, \mathrm{d}y$$
$$= -\ln(2y-1) + \ln(y-3) + C$$ [4 marks]

Combining logs and equating with the RHS,

$$\ln \frac{y-3}{2y-1} = x + C$$ [1 mark]

Take exponentials on both sides and let A = e^C,

$$\frac{y-3}{2y-1} = Ae^x$$ [2 marks]

Rearranging for *y*,

$y - 3 = Ae^x(2y - 1) \rightarrow y - 2Ae^x y = 3 - Ae^x \rightarrow y(1 - 2Ae^x) = 3 - Ae^x$ [1 mark]

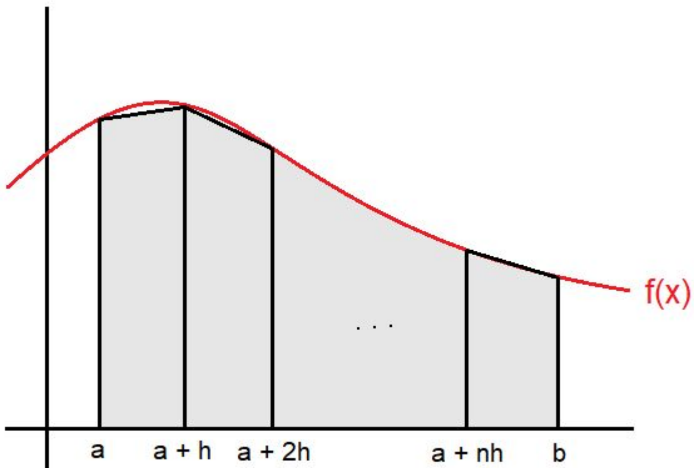
$\rightarrow$ $$y = \frac{3 - Ae^x}{1 - 2Ae^x} = \frac{Ae^x - 3}{2Ae^x - 1}.$$ [1 mark]

Section C: Extended Question

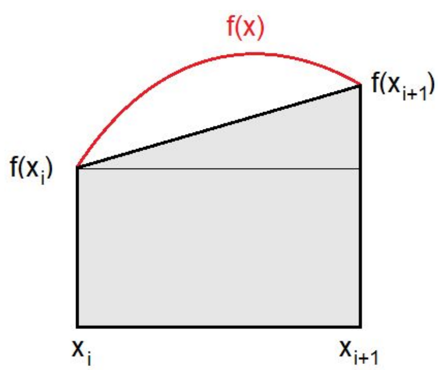
17.

a. **Step 1: Setup**

Begin by sketching the trapezium rule in use on a function f(x) between a and b. Let h = (b - a)/n = step size:



Label each x-coordinate x_i, with x_0 = a and x_n = b. Then,
Now, zooming in on one particular trapezium,



[1 mark]

Let the area of this trapezium be S_i. Then, considering it as the sum of the rectangle and the small triangle as advised,

$S_i = h\, f(x_i) + h(f(x_{i+1}) - f(x_i))/2 = (h/2)(f(x_{i+1}) - f(x_i))$ [1 mark]

This is as far as we can go with the approximation.

Step 2: Finding an expression for I

Let us consider the actual area, I_i:

$$I_i = \int_{x_i}^{x_{i+1}} f(x)\ \mathrm{d}x$$

To simplify the algebra, substitute t = x - x_i → x = t + x_i,
then the bounds become 0 to t:

$$I_i = \int_0^h f(t + x_i)\ \mathrm{d}t$$

Since we cannot integrate the function, but the final result contains derivatives, it makes sense to integrate by parts twice to obtain an expression in terms of second derivatives.

First time:

↳ by parts:
$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

↳ let $u = f(t+x_i)$ $\quad dv = dt$

$du = f'(t+x_i)$ $\quad v = t + C_1 \leftarrow$ constant

↳ $dv = dt + 0$

$$\int_0^h f(t+x_i)dt = (t+C_1)f(t+x_i)\Big]_0^h - \int_0^h (t+C_1)f'(t+x_i)dt$$

[3 marks]

Second time (on the RHS integral):

$$u = f'(t + x_i) \qquad dv = (t + C_1)\,dt$$
$$du = f''(t + x_i) \qquad v = \frac{t^2}{2} + \frac{2C_1 t}{2} + C_*$$

complete square: $(t + C_1)^2 = t^2 + 2C_1 t + C_1^2$

$$= \frac{t^2 + 2C_1 t}{2} + C_*$$
$$= \frac{(t + C_1)^2 - C_1^2}{2} + C_*$$
$$= \frac{(t + C_1)^2}{2} \underbrace{- \frac{C_1^2}{2} + C_*}_{= C_2}$$
$$= \frac{(t + C_1)^2}{2} + C_2$$

$$\int_0^h f(t + x_i)\,dt = \overbrace{(t + C_1) f(t + x_i)\Big]_0^h}^{T_i} - \left(\frac{(t + C_1)^2}{2} + C_2\right) f'(t + x_i)\Big|_0^h$$
$$+ \int_0^h \left(\frac{(t + C_1)^2}{2} + C_2\right) f''(t + x_i)\,dt$$

[4 marks]

This result consists of three terms. Notice that the first one is
$(t + C_1)\, f(t + x_i)\, |_0^h = (h + C_1)\, f(x_{i+1}) - C_1\, f(x_i) = h\, f(x_{i+1}) + C_1\, f(x_{i+1}) - C_1\, f(x_i)$
This contains all terms present in the formula for S_i, and for some constant C_1 it will be equal to it.

It is easy to see that when $C_1 = -h/2$ [1 mark],
this first term is $h\, f(x_{i+1}) - h/2\, f(x_{i+1}) + h/2\, f(x_i)$
$= h/2\, (\, f(x_i) + f(x_{i+1})\,)$ which is exactly S_i. We can choose this constant C_1 freely since it will cancel out when the bounds of integration are applied.

This will mean our result will be $I_i = S_i$ + two other terms.

Since we know therefore, $I_i = S_i + E_i$, the two other terms must add to E_i. Since we are given the formula for E_i as containing $f''(x)$ terms, we must have the term containing $f'(x)$ equal to zero.

Step 3: Finding C_2

Considering the second term (containing first derivative):

we want:

$$\left(\frac{(t+c_1)^2}{2}+c_2\right)f'(t+x_i)\Big]_0^h = 0$$

$$\hookrightarrow \underbrace{\left(\frac{(h-\frac{h}{2})^2}{2}+c_2\right)}f'(h+x_i)-\underbrace{\left(\frac{(-\frac{h}{2})^2}{2}+c_2\right)}f'(x_i)=0$$

$\hookrightarrow$ notice: $\longrightarrow = \longleftarrow$

$$\hookrightarrow \frac{(\frac{h}{2})^2}{2}+c_2=0$$

$$\hookrightarrow c_2 = -\frac{h^2}{4\cdot 2} = \boxed{\frac{-h^2}{8}=c_2}$$

[2 marks]

So putting it back together we have $I_i = S_i + E_i$ as required:

$$\int_0^h f(t+x_i)\,dt = \underbrace{\frac{h}{2}\left[f(x_i)+f(x_{i+1})\right]}_{S_i} + \underbrace{\int_0^h\left(\frac{(t-\frac{h}{2})^2}{2}-\frac{h^2}{8}\right)f''(t+x_i)\,dt}_{=\text{Error}=E_i}$$

Step 4: Maximising the error function

The total error E is the sum of each small error: $E = |E_1 + E_2 + \dots + E_{n-1}|$. Writing these as integrals using the form above,

Total Error

$$= \int_0^h \left(\frac{(t - \frac{h}{2})^2}{2} - \frac{h^2}{8} \right) f''(t + \underline{x_0})\, dt$$

$$+ \dots + \dots$$

$$+ \int_0^h \underbrace{\left(\frac{(t - \frac{h}{2})^2}{2} - \frac{h^2}{8} \right)}_{= \text{same}} f''(t + \underline{x_{n-1}})\, dt$$

$$= \int_0^h \left(\frac{(t - \frac{h}{2})^2}{2} - \frac{h^2}{8} \right) \left[f''(t + x_0) + \dots + f''(t + x_{n-1}) \right] dt$$

[1 mark]

We need the maximum possible value of this integral to correspond to the result we want to prove. The greatest possible value of f''(x) has been denoted K, so the maximum value of all of them added up will be less than nK. Since these errors must be positive, this matches the definition that $| f''(x) | \le K$. Applying this,

$$\hookrightarrow \quad |E| = \left| \int_0^h \left[\frac{(t - \frac{h}{2})^2}{2} - \frac{h^2}{8} \right] \cdot \left[f''(t + x_0) + \dots + f''(t + x_{n-1}) \right] dt \right|$$

$$\le \int_0^h \left| \frac{(t - \frac{h}{2})^2}{2} - \frac{h^2}{8} \right| \overbrace{\left[|f''(t + x_0)| + \dots + |f''(t + x_{n-1})| \right]}^{\le nK} dt$$

$$\le nK \int_0^h \left| \frac{(t - \frac{h}{2})^2}{2} - \frac{h^2}{8} \right| dt$$

[1 mark]

Part 5: Obtaining the formula

All we have left to do is evaluate the remaining integral. To deal with the absolute value bars, we must consider the shape of the graph. Let it equal g(t):

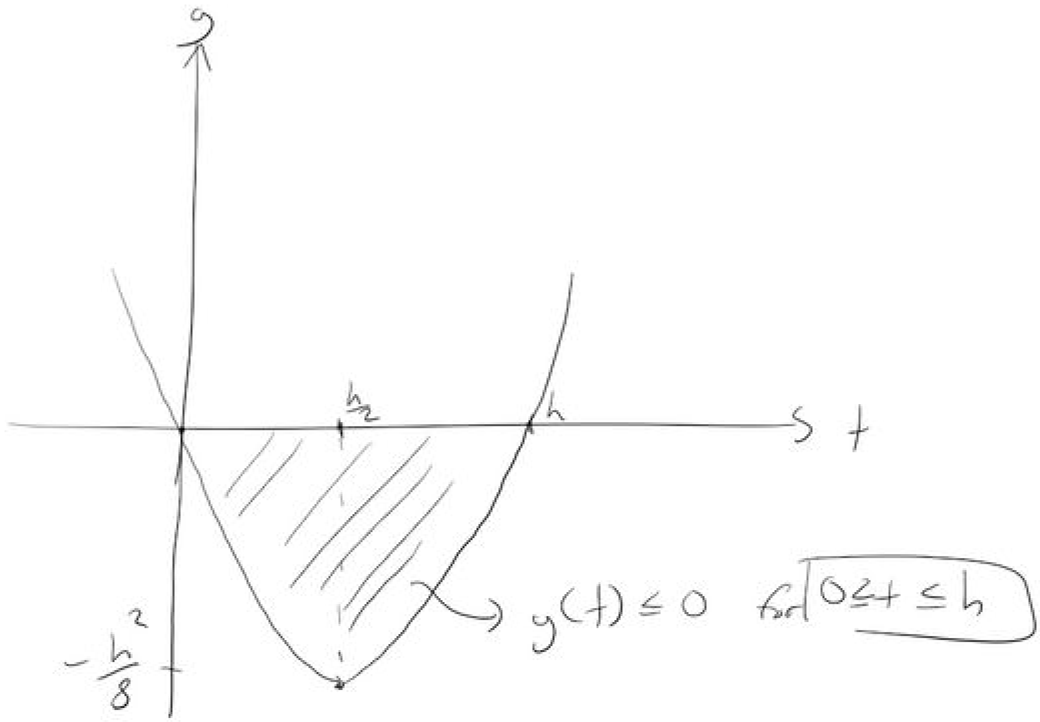
$$\hookrightarrow \quad g(t) = \frac{\left(t - \frac{h}{2}\right)^2}{2} - \frac{h^2}{8} \rightarrow \text{parabola}$$

@ $t = 0, h$

$$\hookrightarrow \quad g(0) = \frac{\left(-\frac{h}{2}\right)^2}{2} - \frac{h^2}{8} = \frac{h^2}{4 \cdot 2} - \frac{h^2}{8} = 0$$

$$g(h) = 0$$

It is clear to see that in the interval, g(t) is negative:



So the absolute value will reflect it in the x-axis, i.e. multiply by -1: [1 mark]

$$\int_0^h |g(t)|\,dt = -\int_0^h g(t)\,dt = \int_0^h \left(\frac{h^2}{8} - \frac{\left(t - \frac{h}{2}\right)^2}{2}\right) dt$$

This is a relatively simple integral - using a substitution,

$$= \left[\frac{h^2}{8}t - \frac{(t-\frac{h}{2})^3}{6}\right]_0^h$$

$$= \frac{h^3}{8} - \frac{(\frac{h}{2})^3}{6} - \left[-\frac{(-\frac{h}{2})^3}{6}\right]$$

$$= \frac{6h^3}{6\cdot 8} - \frac{h^3}{8\cdot 6} - \frac{h^3}{8\cdot 6}$$

$$= \frac{(6-2)h^3}{6\cdot 8} = \frac{4h^3}{6\cdot 2\cdot 4} = \boxed{\frac{h^3}{12}}$$ [2 marks]

So, putting this as the result for the integral we now have:

$$\Rightarrow |E_T| \le nk\frac{h^3}{12} = \frac{nk\left(\frac{b-a}{n}\right)^3}{12}$$

$$\Rightarrow \boxed{|E_T| \le \frac{k(b-a)^3}{12n^2}} \leftarrow$$ [1 mark]

as required.

(Solution by MathEasySolutions at https://hive.blog/mathematics/@mes/approximate-integration-trapezoidal-rule-error-bound-proof)

b. The true value is I = 1/4 and the approximation is S = 17/64
so the actual error is S - I = 1/64. [1 mark]

$f(x) = x^3 \rightarrow f''(x) = 6x$. Maximum value of 6x on interval (0, 1) is at x = 1 → K = 6.
So, using the formula, maximum error = 1/32.

So the difference in these values is 1/32. [1 mark]