ALGEBRA

Solving Equations and Inequalities

Paul Dawkins

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Preface

Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it's been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Algebra or needing a refresher for Algebra. I've tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you've had some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don't always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 2 : Solving Equations and Inequalities

In this chapter we will look at one of the standard topics in any Algebra class. The ability to solve equations and/or inequalities is very important and is used time and again both in this class and in later classes. We will cover a wide variety of solving topics in this chapter that should cover most of the basic equations/inequalities/techniques that are involved in solving.

Here is a brief listing of the material covered in this chapter.

Solutions and Solution Sets – In this section we introduce some of the basic notation and ideas involved in solving equations and inequalities. We define solutions for equations and inequalities and solution sets.

Linear Equations – In this section we give a process for solving linear equations, including equations with rational expressions, and we illustrate the process with several examples. In addition, we discuss a subtlety involved in solving equations that students often overlook.

Applications of Linear Equations – In this section we discuss a process for solving applications in general although we will focus only on linear equations here. We will work applications in pricing, distance/rate problems, work rate problems and mixing problems.

Equations With More Than One Variable – In this section we will look at solving equations with more than one variable in them. These equations will have multiple variables in them and we will be asked to solve the equation for one of the variables. This is something that we will be asked to do on a fairly regular basis.

Quadratic Equations, Part I – In this section we will start looking at solving quadratic equations. Specifically, we will concentrate on solving quadratic equations by factoring and the square root property in this section.

Quadratic Equations, Part II – In this section we will continue solving quadratic equations. We will use completing the square to solve quadratic equations in this section and use that to derive the quadratic formula. The quadratic formula is a quick way that will allow us to quickly solve any quadratic equation.

Quadratic Equations : A Summary – In this section we will summarize the topics from the last two sections. We will give a procedure for determining which method to use in solving quadratic equations and we will define the discriminant which will allow us to quickly determine what kind of solutions we will get from solving a quadratic equation.

Applications of Quadratic Equations – In this section we will revisit some of the applications we saw in the linear application section, only this time they will involve solving a quadratic equation. Included are examples in distance/rate problems and work rate problems.

Equations Reducible to Quadratic Form – Not all equations are in what we generally consider quadratic equations. However, some equations, with a proper substitution can be turned into a quadratic equation. These types of equations are called quadratic in form. In this section we will solve this type of equation.

Equations with Radicals – In this section we will discuss how to solve equations with square roots in them. As we will see we will need to be very careful with the potential solutions we get as the process used in solving these equations can lead to values that are not, in fact, solutions to the equation.

Linear Inequalities – In this section we will start solving inequalities. We will concentrate on solving linear inequalities in this section (both single and double inequalities). We will also introduce interval notation.

Polynomial Inequalities – In this section we will continue solving inequalities. However, in this section we move away from linear inequalities and move on to solving inequalities that involve polynomials of degree at least 2.

Rational Inequalities – We continue solving inequalities in this section. We now will solve inequalities that involve rational expressions, although as we'll see the process here is pretty much identical to the process used when solving inequalities with polynomials.

Absolute Value Equations – In this section we will give a geometric as well as a mathematical definition of absolute value. We will then proceed to solve equations that involve an absolute value. We will also work an example that involved two absolute values.

Absolute Value Inequalities – In this final section of the Solving chapter we will solve inequalities that involve absolute value. As we will see the process for solving inequalities with a < (*i.e.* a less than) is very different from solving an inequality with a > (*i.e.* greater than).

Section 2-1 : Solutions and Solution Sets

We will start off this chapter with a fairly short section with some basic terminology that we use on a fairly regular basis in solving equations and inequalities.

First, a **solution** to an equation or inequality is any number that, when plugged into the equation/inequality, will satisfy the equation/inequality. So, just what do we mean by satisfy? Let's work an example or two to illustrate this.

Example 1 Show that each of the following numbers are solutions to the given equation or inequality.

      (a) $x = 3$ in $x^2 - 9 = 0$

      (b) $y = 8$ in $3(y+1) = 4y - 5$

      (c) $z = 1$ in $2(z-5) \le 4z$

      (d) $z = -5$ in $2(z-5) \le 4z$

Solution
(a) We first plug the proposed solution into the equation.

$$3^2 - 9 \overset{?}{=} 0$$
$$9 - 9 = 0$$
$$0 = 0 \quad \text{OK}$$

So, what we are asking here is does the right side equal the left side after we plug in the proposed solution. That is the meaning of the "?" above the equal sign in the first line.

Since the right side and the left side are the same we say that $x = 3$ **satisfies** the equation.

(b) So, we want to see if $y = 8$ satisfies the equation. First plug the value into the equation.

$$3(8+1) \overset{?}{=} 4(8) - 5$$
$$27 = 27 \qquad \text{OK}$$

So, $y = 8$ satisfies the equation and so is a solution.

(c) In this case we've got an inequality and in this case "satisfy" means something slightly different. In this case we will say that a number will satisfy the inequality if, after plugging it in, we get a true inequality as a result.

Let's check $z = 1$.

$$2(1-5) \overset{?}{\le} 4(1)$$
$$-8 \le 4 \quad \text{OK}$$

So, -8 is less than or equal to 4 (in fact it's less than) and so we have a true inequality. Therefore $z = 1$ will satisfy the inequality and hence is a solution

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(d) This is the same inequality with a different value so let's check that.

$$2(-5-5)\overset{?}{\leq}4(-5)$$
$$-20\leq-20 \qquad \text{OK}$$

In this case -20 is less than or equal to -20 (in this case it's equal) and so again we get a true inequality and so $z=-5$ satisfies the inequality and so will be a solution.

We should also do a quick example of numbers that aren't solution so we can see how these will work as well.

Example 2 Show that the following numbers aren't solutions to the given equation or inequality.

  (a) $y=-2$ in $3(y+1)=4y-5$

  (b) $z=-12$ in $2(z-5)\leq4z$

Solution

(a) In this case we do essentially the same thing that we did in the previous example. Plug the number in and show that this time it doesn't satisfy the equation. For equations that will mean that the right side of the equation will not equal the left side of the equation.

$$3(-2+1)\overset{?}{=}4(-2)-5$$
$$-3\neq-13 \qquad \text{NOT OK}$$

So, -3 is not the same as -13 and so the equation isn't satisfied. Therefore $y=-2$ isn't a solution to the equation.

(b) This time we've got an inequality. A number will not satisfy an inequality if we get an inequality that isn't true after plugging the number in.

$$2(-12-5)\overset{?}{\leq}4(-12)$$
$$-34\not\leq-48 \qquad \text{NOT OK}$$

In this case -34 is NOT less than or equal to -48 and so the inequality isn't satisfied. Therefore $z=-12$ is not a solution to the inequality.

Now, there is no reason to think that a given equation or inequality will only have a single solution. In fact, as the first example showed the inequality $2(z-5)\leq4z$ has at least two solutions. Also, you might have noticed that $x=3$ is not the only solution to $x^2-9=0$. In this case $x=-3$ is also a solution.

We call the complete set of all solutions the **solution set** for the equation or inequality. There is also some formal notation for solution sets although we won't be using it all that often in this course. Regardless of that fact we should still acknowledge it.

For equations we denote the solution set by enclosing all the solutions in a set of braces, $\{\ \}$. For the two equations we looked at above here are the solution sets.

$$3(y+1) = 4y - 5 \qquad\qquad \text{Solution Set} : \{8\}$$
$$x^2 - 9 = 0 \qquad\qquad \text{Solution Set} : \{-3, 3\}$$

For inequalities we have a similar notation. Depending on the complexity of the inequality the solution set may be a single number or it may be a range of numbers. If it is a single number then we use the same notation as we used for equations. If the solution set is a range of numbers, as the one we looked at above is, we will use something called **set builder notation**. Here is the solution set for the inequality we looked at above.

$$\{z \mid z \geq -5\}$$

This is read as : "The set of all *z* such that *z* is greater than or equal to -5".

Most of the inequalities that we will be looking at will have simple enough solution sets that we often just shorthand this as,

$$z \geq -5$$

There is one final topic that we need to address as far as solution sets go before leaving this section. Consider the following equation and inequality.

$$x^2 + 1 = 0$$
$$x^2 < 0$$

If we restrict ourselves to only real solutions (which we won't always do) then there is no solution to the equation. Squaring *x* makes *x* greater than equal to zero, then adding 1 onto that means that the left side is guaranteed to be at least 1. In other words, there is no real solution to this equation. For the same basic reason there is no solution to the inequality. Squaring any real *x* makes it positive or zero and so will never be negative.

We need a way to denote the fact that there are no solutions here. In solution set notation we say that the solution set is **empty** and denote it with the symbol : $\varnothing$. This symbol is often called the **empty set**.

We now need to make a couple of final comments before leaving this section.

In the above discussion of empty sets we assumed that we were only looking for real solutions. While that is what we will be doing for inequalities, we won't be restricting ourselves to real solutions with equations. Once we get around to solving quadratic equations (which $x^2 + 1 = 0$ is) we will allow solutions to be complex numbers and in the case looked at above there are complex solutions to $x^2 + 1 = 0$. If you don't know how to find these at this point that is fine we will be covering that material in a couple of sections. At this point just accept that $x^2 + 1 = 0$ does have complex solutions.

Finally, as noted above we won't be using the solution set notation much in this course. It is a nice notation and does have some use on occasion especially for complicated solutions. However, for the

vast majority of the equations and inequalities that we will be looking at will have simple enough solution sets that it's just easier to write down the solutions and let it go at that. Therefore, that is what we will not be using the notation for our solution sets. However, you should be aware of the notation and know what it means.

Section 2-2 : Linear Equations

We'll start off the solving portion of this chapter by solving linear equations. A **linear equation** is any equation that can be written in the form

$$ax + b = 0$$

where *a* and *b* are real numbers and *x* is a variable. This form is sometimes called the **standard form** of a linear equation. Note that most linear equations will not start off in this form. Also, the variable may or may not be an *x* so don't get too locked into always seeing an *x* there.

To solve linear equations we will make heavy use of the following facts.

1. If $a = b$ then $a + c = b + c$ for any *c*. All this is saying is that we can add a number, *c*, to both sides of the equation and not change the equation.

2. If $a = b$ then $a - c = b - c$ for any *c*. As with the last property we can subtract a number, *c*, from both sides of an equation.

3. If $a = b$ then $ac = bc$ for any *c*. Like addition and subtraction, we can multiply both sides of an equation by a number, *c*, without changing the equation.

4. If $a = b$ then $\dfrac{a}{c} = \dfrac{b}{c}$ for any non-zero *c*. We can divide both sides of an equation by a non-zero number, *c*, without changing the equation.

These facts form the basis of almost all the solving techniques that we'll be looking at in this chapter so it's very important that you know them and don't forget about them. One way to think of these rules is the following. What we do to one side of an equation we have to do to the other side of the equation. If you remember that then you will always get these facts correct.

In this section we will be solving linear equations and there is a nice simple process for solving linear equations. Let's first summarize the process and then we will work some examples.

Process for Solving Linear Equations

1. If the equation contains any fractions use the least common denominator to clear the fractions. We will do this by multiplying both sides of the equation by the LCD.

   Also, if there are variables in the denominators of the fractions identify values of the variable which will give division by zero as we will need to avoid these values in our solution.

2. Simplify both sides of the equation. This means clearing out any parenthesis and combining like terms.

3. Use the first two facts above to get all terms with the variable in them on one side of the equations (combining into a single term of course) and all constants on the other side.

4. If the coefficient of the variable is not a one use the third or fourth fact above (this will depend on just what the number is) to make the coefficient a one.

   Note that we usually just divide both sides of the equation by the coefficient if it is an integer or multiply both sides of the equation by the reciprocal of the coefficient if it is a fraction.

5. **VERIFY YOUR ANSWER!** This is the final step and the most often skipped step, yet it is probably the most important step in the process. With this step you can know whether or not you got the correct answer long before your instructor ever looks at it. We verify the answer by plugging the results from the previous steps into the **original** equation. It is very important to plug into the original equation since you may have made a mistake in the very first step that led you to an incorrect answer.

   Also, if there were fractions in the problem and there were values of the variable that give division by zero (recall the first step...) it is important to make sure that one of these values didn't end up in the solution set. It is possible, as we'll see in an example, to have these values show up in the solution set.

Let's take a look at some examples.

Example 1 Solve each of the following equations.

(a) $3(x+5)=2(-6-x)-2x$

(b) $\dfrac{m-2}{3}+1=\dfrac{2m}{7}$

(c) $\dfrac{5}{2y-6}=\dfrac{10-y}{y^2-6y+9}$

(d) $\dfrac{2z}{z+3}=\dfrac{3}{z-10}+2$

Solution

In the following problems we will describe in detail the first problem and the leave most of the explanation out of the following problems.

(a) $3(x+5)=2(-6-x)-2x$

For this problem there are no fractions so we don't need to worry about the first step in the process. The next step tells to simplify both sides. So, we will clear out any parenthesis by multiplying the numbers through and then combine like terms.

$$3(x+5)=2(-6-x)-2x$$
$$3x+15=-12-2x-2x$$
$$3x+15=-12-4x$$

The next step is to get all the *x*'s on one side and all the numbers on the other side. Which side the *x*'s go on is up to you and will probably vary with the problem. As a rule of thumb, we will usually put the variables on the side that will give a positive coefficient. This is done simply because it is often easy to lose track of the minus sign on the coefficient and so if we make sure it is positive we won't need to worry about it.

So, for our case this will mean adding 4*x* to both sides and subtracting 15 from both sides. Note as well that while we will actually put those operations in this time we normally do these operations in our head.

$$3x + 15 = -12 - 4x$$
$$3x + 15 - 15 + 4x = -12 - 4x + 4x - 15$$
$$7x = -27$$

The next step says to get a coefficient of 1 in front of the *x*. In this case we can do this by dividing both sides by a 7.

$$\frac{7x}{7} = \frac{-27}{7}$$
$$x = -\frac{27}{7}$$

Now, if we've done all of our work correct $x = -\dfrac{27}{7}$ is the solution to the equation.

The last and final step is to then check the solution. As pointed out in the process outline we need to check the solution in the **original** equation. This is important, because we may have made a mistake in the very first step and if we did and then checked the answer in the results from that step it may seem to indicate that the solution is correct when the reality will be that we don't have the correct answer because of the mistake that we originally made.

The problem of course is that, with this solution, that checking might be a little messy. Let's do it anyway.

$$3\left(-\frac{27}{7} + 5\right) \overset{?}{=} 2\left(-6 - \left(-\frac{27}{7}\right)\right) - 2\left(-\frac{27}{7}\right)$$
$$3\left(\frac{8}{7}\right) \overset{?}{=} 2\left(-\frac{15}{7}\right) + \frac{54}{7}$$
$$\frac{24}{7} = \frac{24}{7} \qquad\qquad \text{OK}$$

So, we did our work correctly and the solution to the equation is,
$$x = -\frac{27}{7}$$

Note that we didn't use the solution set notation here. For single solutions we will rarely do that in this class. However, if we had wanted to the solution set notation for this problem would be,

$$\left\{ -\frac{27}{7} \right\}$$

Before proceeding to the next problem let's first make a quick comment about the "messiness' of this answer. Do NOT expect all answers to be nice simple integers. While we do try to keep most answer simple often they won't be so do NOT get so locked into the idea that an answer must be a simple integer that you immediately assume that you've made a mistake because of the "messiness" of the answer.

(b) $\dfrac{m-2}{3} + 1 = \dfrac{2m}{7}$

Okay, with this one we won't be putting quite as much explanation into the problem.

In this case we have fractions so to make our life easier we will multiply both sides by the LCD, which is 21 in this case. After doing that the problem will be very similar to the previous problem. Note as well that the denominators are only numbers and so we won't need to worry about division by zero issues.

Let's first multiply both sides by the LCD.

$$21\left(\frac{m-2}{3} + 1 \right) = \left(\frac{2m}{7} \right)21$$

$$21\left(\frac{m-2}{3} \right) + 21(1) = \left(\frac{2m}{7} \right)21$$

$$7(m-2) + 21 = (2m)(3)$$

Be careful to correctly distribute the 21 through the parenthesis on the left side. Everything inside the parenthesis needs to be multiplied by the 21 before we simplify. At this point we've got a problem that is similar the previous problem and we won't bother with all the explanation this time.

$$7(m-2) + 21 = (2m)(3)$$

$$7m - 14 + 21 = 6m$$

$$7m + 7 = 6m$$

$$m = -7$$

So, it looks like $m = -7$ is the solution. Let's verify it to make sure.

$$\frac{-7-2}{3} + 1 \overset{?}{=} \frac{2(-7)}{7}$$

$$\frac{-9}{3} + 1 \overset{?}{=} -\frac{14}{7}$$

$$-3 + 1 \overset{?}{=} -2$$

$$-2 = -2 \qquad \text{OK}$$

So, it is the solution.

(c) $\dfrac{5}{2y-6}=\dfrac{10-y}{y^2-6y+9}$

This one is similar to the previous one except now we've got variables in the denominator. So, to get the LCD we'll first need to completely factor the denominators of each rational expression.

$$\frac{5}{2(y-3)}=\frac{10-y}{(y-3)^2}$$

So, it looks like the LCD is $2(y-3)^2$. Also note that we will need to avoid $y=3$ since if we plugged that into the equation we would get division by zero.

Now, outside of the *y*'s in the denominator this problem works identical to the previous one so let's do the work.

$$(2)(y-3)^2\left(\frac{5}{2(y-3)}\right)=\left(\frac{10-y}{(y-3)^2}\right)(2)(y-3)^2$$
$$5(y-3)=2(10-y)$$
$$5y-15=20-2y$$
$$7y=35$$
$$y=5$$

Now the solution is not $y=3$ so we won't get division by zero with the solution which is a good thing. Finally, let's do a quick verification.

$$\frac{5}{2(5)-6}\overset{?}{=}\frac{10-5}{5^2-6(5)+9}$$
$$\frac{5}{4}=\frac{5}{4}\qquad\qquad\text{OK}$$

So we did the work correctly.

(d) $\dfrac{2z}{z+3}=\dfrac{3}{z-10}+2$

In this case it looks like the LCD is $(z+3)(z-10)$ and it also looks like we will need to avoid $z=-3$ and $z=10$ to make sure that we don't get division by zero.

Let's get started on the work for this problem.

$$(z+3)(z-10)\left(\frac{2z}{z+3}\right)=\left(\frac{3}{z-10}+2\right)(z+3)(z-10)$$
$$2z(z-10)=3(z+3)+2(z+3)(z-10)$$
$$2z^2-20z=3z+9+2(z^2-7z-30)$$

At this point let's pause and acknowledge that we've got a z^2 in the work here. Do not get excited about that. Sometimes these will show up temporarily in these problems. You should only worry about it if it is still there after we finish the simplification work.

So, let's finish the problem.

$$2z^2 - 20z = 3z + 9 + 2z^2 - 14z - 60$$
$$-20z = -11z - 51$$
$$51 = 9z$$
$$\frac{51}{9} = z$$
$$\frac{17}{3} = z$$

Notice that the z^2 did in fact cancel out. Now, if we did our work correctly $z = \dfrac{17}{3}$ should be the solution since it is not either of the two values that will give division by zero. Let's verify this.

$$\frac{2\left(\dfrac{17}{3}\right)}{\dfrac{17}{3} + 3} \overset{?}{=} \frac{3}{\dfrac{17}{3} - 10} + 2$$

$$\frac{\dfrac{34}{3}}{\dfrac{26}{3}} \overset{?}{=} \frac{3}{-\dfrac{13}{3}} + 2$$

$$\frac{34}{3}\left(\frac{3}{26}\right) \overset{?}{=} 3\left(-\frac{3}{13}\right) + 2$$

$$\frac{17}{13} = \frac{17}{13} \qquad\qquad \text{OK}$$

The checking can be a little messy at times, but it does mean that we KNOW the solution is correct.

Okay, in the last couple of parts of the previous example we kept going on about watching out for division by zero problems and yet we never did get a solution where that was an issue. So, we should now do a couple of those problems to see how they work.

Example 2 Solve each of the following equations.

    (a) $\dfrac{2}{x+2} = \dfrac{-x}{x^2+5x+6}$

    (b) $\dfrac{2}{x+1} = 4 - \dfrac{2x}{x+1}$

Solution

(a) $\dfrac{2}{x+2} = \dfrac{-x}{x^2+5x+6}$

The first step is to factor the denominators to get the LCD.

$$\frac{2}{x+2} = \frac{-x}{(x+2)(x+3)}$$

So, the LCD is $(x+2)(x+3)$ and we will need to avoid $x=-2$ and $x=-3$ so we don't get division by zero.

Here is the work for this problem.

$$(x+2)(x+3)\left(\frac{2}{x+2}\right) = \left(\frac{-x}{(x+2)(x+3)}\right)(x+2)(x+3)$$
$$2(x+3) = -x$$
$$2x+6 = -x$$
$$3x = -6$$
$$x = -2$$

So, we get a "solution" that is in the list of numbers that we need to avoid so we don't get division by zero and so we can't use it as a solution. However, this is also the only possible solution. That is okay. This just means that this equation has **no solution**.

(b) $\dfrac{2}{x+1} = 4 - \dfrac{2x}{x+1}$

The LCD for this equation is $x+1$ and we will need to avoid $x=-1$ so we don't get division by zero. Here is the work for this equation.

$$\left(\frac{2}{x+1}\right)(x+1) = \left(4 - \frac{2x}{x+1}\right)(x+1)$$
$$2 = 4(x+1) - 2x$$
$$2 = 4x+4-2x$$
$$2 = 2x+4$$
$$-2 = 2x$$
$$-1 = x$$

So, we once again arrive at the single value of x that we needed to avoid so we didn't get division by zero. Therefore, this equation has **no solution**.

So, as we've seen we do need to be careful with division by zero issues when we start off with equations that contain rational expressions.

At this point we should probably also acknowledge that provided we don't have any division by zero issues (such as those in the last set of examples) linear equations will have exactly one solution. We will never get more than one solution and the only time that we won't get any solutions is if we run across a division by zero problems with the "solution".

Before leaving this section we should note that many of the techniques for solving linear equations will show up time and again as we cover different kinds of equations so it very important that you understand this process.

Section 2- 3 : Applications of Linear Equations

We now need to discuss the section that most students hate. We need to talk about applications to linear equations. Or, put in other words, we will now start looking at story problems or word problems. Throughout history students have hated these. It is my belief however that the main reason for this is that students really don't know how to work them. Once you understand how to work them, you'll probably find that they aren't as bad as they may seem on occasion. So, we'll start this section off with a process for working applications.

Process for Working Story/Word Problems

1. **READ THE PROBLEM.**

2. **READ THE PROBLEM AGAIN.** Okay, this may be a little bit of overkill here. However, the point of these first two steps is that you must read the problem. This step is the MOST important step, but it is also the step that most people don't do properly.

    You need to read the problem very carefully and as many times as it takes. You are only done with this step when you have completely understood what the problem is asking you to do. This includes identifying all the given information and identifying what you are being asked to find.

    Again, it can't be stressed enough that you've got to carefully read the problem. Sometimes a single word can completely change how the problem is worked. If you just skim the problem you may well miss that very important word.

3. Represent one of the unknown quantities with a variable and try to relate all the other unknown quantities (if there are any of course) to this variable.

4. If applicable, sketch a figure illustrating the situation. This may seem like a silly step, but it can be incredibly helpful with the next step on occasion.

5. Form an equation that will relate known quantities to the unknown quantities. To do this make use of known formulas and often the figure sketched in the previous step can be used to determine the equation.

6. Solve the equation formed in the previous step and write down the answer to all the questions. It is important to answer all the questions that you were asked. Often you will be asked for several quantities in the answer and the equation will only give one of them.

7. Check your answer. Do this by plugging into the equation, but also use intuition to make sure that the answer makes sense. Mistakes can often be identified by acknowledging that the answer just doesn't make sense.

Let's start things off with a couple of fairly basic examples to illustrate the process. Note as well that at this point it is assumed that you are capable of solving fairly simple linear equations and so not a lot of

detail will be given for the actual solution stage. The point of this section is more on the set up of the equation than the solving of the equation.

Example 1 In a certain Algebra class there is a total of 350 possible points. These points come from 5 homework sets that are worth 10 points each and 3 hour exams that are worth 100 points each. A student has received homework scores of 4, 8, 7, 7, and 9 and the first two exam scores are 78 and 83. Assuming that grades are assigned according to the standard scale and there are no weights assigned to any of the grades is it possible for the student to receive an A in the class and if so what is the minimum score on the third exam that will give an A? What about a B?

Solution
Okay, let's start off by defining *p* to be the minimum required score on the third exam.

Now, let's recall how grades are set. Since there are no weights or anything on the grades, the grade will be set by first computing the following percentage.

$$\frac{\text{actual points}}{\text{total possible points}} = \text{ grade percentage}$$

Since we are using the standard scale if the grade percentage is 0.9 or higher the student will get an A. Likewise, if the grade percentage is between 0.8 and 0.9 the student will get a B.

We know that the total possible points is 350 and the student has a total points (including the third exam) of,
$$4+8+7+7+9+78+83+p = 196+p$$

The smallest possible percentage for an A is 0.9 and so if *p* is the minimum required score on the third exam for an A we will have the following equation.
$$\frac{196+p}{350} = 0.9$$
This is a linear equation that we will need to solve for *p*.
$$196+p = 0.9(350) = 315 \qquad \Rightarrow \qquad p = 315-196 = 119$$

So, the minimum required score on the third exam is 119. This is a problem since the exam is worth only 100 points. In other words, the student will not be getting an A in the Algebra class.

Now let's check if the student will get a B. In this case the minimum percentage is 0.8. So, to find the minimum required score on the third exam for a B we will need to solve,
$$\frac{196+p}{350} = 0.8$$
Solving this for *p* gives,
$$196+p = 0.8(350) = 280 \qquad \Rightarrow \qquad p = 280-196 = 84$$

So, it is possible for the student to get a B in the class. All that the student will need to do is get at least an 84 on the third exam.

Example 2 We want to build a set of shelves. The width of the set of shelves needs to be 4 times the height of the set of shelves and the set of shelves must have three shelves in it. If there are 72 feet of wood to use to build the set of shelves what should the dimensions of the set of shelves be?

Solution
We will first define *x* to be the height of the set of shelves. This means that 4*x* is width of the set of shelves. In this case we definitely need to sketch a figure so we can correctly set up the equation. Here it is,
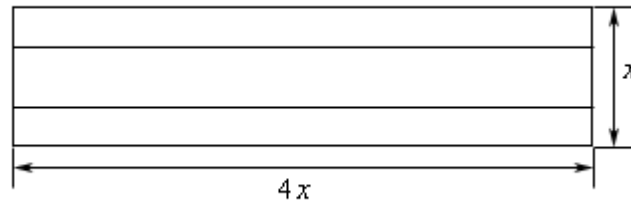
Now we know that there are 72 feet of wood to be used and we will assume that all of it will be used. So, we can set up the following word equation.

$$\left(\begin{array}{c} \text{length of} \\ \text{vertical pieces} \end{array} \right) + \left(\begin{array}{c} \text{length of} \\ \text{horizontal pieces} \end{array} \right) = 72$$

It is often a good idea to first put the equation in words before actually writing down the equation as we did here. At this point, we can see from the figure there are two vertical pieces; each one has a length of *x*. Also, there are 4 horizontal pieces, each with a length of 4*x*. So, the equation is then,

$$4(4x) + 2(x) = 72$$
$$16x + 2x = 72$$
$$18x = 72$$
$$x = 4$$

So, it looks like the height of the set of shelves should be 4 feet. Note however that we haven't actually answered the question however. The problem asked us to find the dimensions. This means that we also need the width of the set of shelves. The width is 4(4)=16 feet. So the dimensions will need to be 4x16 feet.

Pricing Problems
The next couple of problems deal with some basic principles of pricing.

Example 3 A calculator has been marked up 15% and is being sold for $78.50. How much did the store pay the manufacturer of the calculator?

Solution
First, let's define *p* to be the cost that the store paid for the calculator. The stores markup on the calculator is 15%. This means that 0.15*p* has been added on to the original price (*p*) to get the amount the calculator is being sold for. In other words, we have the following equation

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$$p + 0.15p = 78.50$$

that we need to solve for *p*. Doing this gives,

$$1.15p = 78.50 \qquad \Rightarrow \qquad p = \frac{78.50}{1.15} = 68.26087$$

The store paid $68.26 for the calculator. Note that since we are dealing with money we rounded the answer down to two decimal places.

Example 4 A shirt is on sale for $15.00 and has been marked down 35%. How much was the shirt being sold for before the sale?

Solution
This problem is pretty much the opposite of the previous example. Let's start with defining *p* to be the price of the shirt before the sale. It has been marked down by 35%. This means that 0.35*p* has been subtracted off from the original price. Therefore, the equation (and solution) is,

$$p - 0.35p = 15.00$$
$$0.65p = 15.00$$
$$p = \frac{15.00}{0.65} = 23.0769$$

So, with rounding it looks like the shirt was originally sold for $23.08.

Distance/Rate Problems
These are some of the standard problems that most people think about when they think about Algebra word problems. The standard formula that we will be using here is

$$\text{Distance} = \text{Rate} \times \text{Time}$$

All of the problems that we'll be doing in this set of examples will use this to one degree or another and often more than once as we will see.

Example 5 Two cars are 500 miles apart and moving directly towards each other. One car is moving at a speed of 100 mph and the other is moving at 70 mph. Assuming that the cars start moving at the same time how long does it take for the two cars to meet?

Solution
Let's let *t* represent the amount of time that the cars are traveling before they meet. Now, we need to sketch a figure for this one. This figure will help us to write down the equation that we'll need to solve.
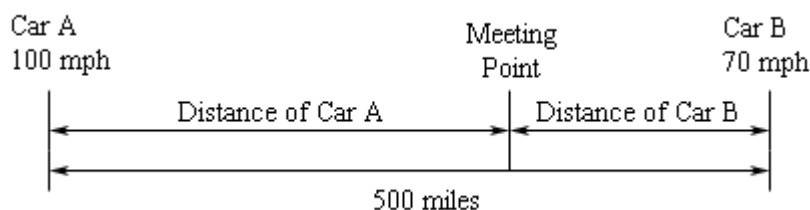
From this figure we can see that the Distance Car A travels plus the Distance Car B travels must equal the total distance separating the two cars, 500 miles.

Here is the word equation for this problem in two separate forms.

$$\left(\begin{array}{c}\text{Distance}\\\text{of Car A}\end{array}\right)+\left(\begin{array}{c}\text{Distance}\\\text{of Car B}\end{array}\right)=500$$

$$\left(\begin{array}{c}\text{Rate of}\\\text{Car A}\end{array}\right)\left(\begin{array}{c}\text{Time of}\\\text{Car A}\end{array}\right)+\left(\begin{array}{c}\text{Rate of}\\\text{Car B}\end{array}\right)\left(\begin{array}{c}\text{Time of}\\\text{Car B}\end{array}\right)=500$$

We used the standard formula here twice, once for each car. We know that the distance a car travels is the rate of the car times the time traveled by the car. In this case we know that Car A travels at 100 mph for t hours and that Car B travels at 70 mph for t hours as well. Plugging these into the word equation and solving gives us,

$$100t+70t=500$$
$$170t=500$$
$$t=\frac{500}{170}=2.941176\text{ hrs}$$

So, they will travel for approximately 2.94 hours before meeting.

Example 6 Repeat the previous example except this time assume that the faster car will start 1 hour after slower car starts.

Solution
For this problem we are going to need to be careful with the time traveled by each car. Let's let t be the amount of time that the slower travel car travels. Now, since the faster car starts out 1 hour after the slower car it will only travel for $t-1$ hours.

Now, since we are repeating the problem from above the figure and word equation will remain identical and so we won't bother repeating them here. The only difference is what we substitute for the time traveled for the faster car. Instead of t as we used in the previous example we will use $t-1$ since it travels for one hour less that the slower car.

Here is the equation and solution for this example.
$$100(t-1)+70t=500$$
$$100t-100+70t=500$$
$$170t=600$$
$$t=\frac{600}{170}=3.529412\text{ hrs}$$

In this case the slower car will travel for 3.53 hours before meeting while the faster car will travel for 2.53 hrs (1 hour less than the slower car…).

Example 7 Two boats start out 100 miles apart and start moving to the right at the same time. The boat on the left is moving at twice the speed as the boat on the right. Five hours after starting the boat on the left catches up with the boat on the right. How fast was each boat moving?

Solution
Let's start off by letting *r* be the speed of the boat on the right (the slower boat). This means that the boat to the left (the faster boat) is moving at a speed of 2*r*. Here is the figure for this situation.
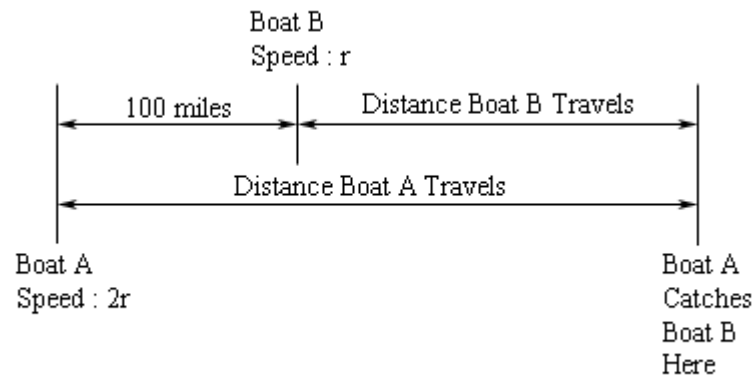
From the figure it looks like we've got the following word equation.

$$100 + \begin{pmatrix} \text{Distance} \\ \text{of Boat B} \end{pmatrix} = \begin{pmatrix} \text{Distance} \\ \text{of Boat A} \end{pmatrix}$$

Upon plugging in the standard formula for the distance gives,

$$100 + \begin{pmatrix} \text{Rate of} \\ \text{Boat B} \end{pmatrix} \begin{pmatrix} \text{Time of} \\ \text{Boat B} \end{pmatrix} = \begin{pmatrix} \text{Rate of} \\ \text{Boat A} \end{pmatrix} \begin{pmatrix} \text{Time of} \\ \text{Boat A} \end{pmatrix}$$

For this problem we know that the time each is 5 hours and we know that the rate of Boat A is 2*r* and the rate of Boat B is *r*. Plugging these into the work equation and solving gives,

$$100 + (r)(5) = (2r)(5)$$
$$100 + 5r = 10r$$
$$100 = 5r$$
$$20 = r$$

So, the slower boat is moving at 20 mph and the faster boat is moving at 40 mph (twice as fast).

Work/Rate Problems

These problems are actually variants of the Distance/Rate problems that we just got done working. The standard equation that will be needed for these problems is,

$$\begin{pmatrix} \text{Portion of job} \\ \text{done in given time} \end{pmatrix} = \begin{pmatrix} \text{Work} \\ \text{Rate} \end{pmatrix} \times \begin{pmatrix} \text{Time Spent} \\ \text{Working} \end{pmatrix}$$

As you can see this formula is very similar to the formula we used above.

Example 8 An office has two envelope stuffing machines. Machine A can stuff a batch of envelopes in 5 hours, while Machine B can stuff a batch of envelopes in 3 hours. How long would it take the two machines working together to stuff a batch of envelopes?

Solution

Let t be the time that it takes both machines, working together, to stuff a batch of envelopes. The word equation for this problem is,

$$\begin{pmatrix} \text{Portion of job} \\ \text{done by Machine A} \end{pmatrix} + \begin{pmatrix} \text{Portion of job} \\ \text{done by Machine B} \end{pmatrix} = 1 \text{ Job}$$

$$\begin{pmatrix} \text{Work Rate} \\ \text{of Machine A} \end{pmatrix}\begin{pmatrix} \text{Time Spent} \\ \text{Working} \end{pmatrix} + \begin{pmatrix} \text{Work Rate} \\ \text{of Machine B} \end{pmatrix}\begin{pmatrix} \text{Time Spent} \\ \text{Working} \end{pmatrix} = 1$$

We know that the time spent working is t however we don't know the work rate of each machine. To get these we'll need to use the initial information given about how long it takes each machine to do the job individually. We can use the following equation to get these rates.

$$1 \text{ Job} = \begin{pmatrix} \text{Work} \\ \text{Rate} \end{pmatrix} \times \begin{pmatrix} \text{Time Spent} \\ \text{Working} \end{pmatrix}$$

Let's start with Machine A.

$$1 \text{ Job} = (\text{Work Rate of A}) \times (5) \qquad \Rightarrow \qquad \text{Work Rate of A} = \frac{1}{5}$$

Now, Machine B.

$$1 \text{ Job} = (\text{Work Rate of B}) \times (3) \qquad \Rightarrow \qquad \text{Work Rate of B} = \frac{1}{3}$$

Plugging these quantities into the main equation above gives the following equation that we need to solve.

$$\frac{1}{5}t + \frac{1}{3}t = 1 \qquad\qquad \text{Multiplying both sides by 15}$$

$$3t + 5t = 15$$

$$8t = 15$$

$$t = \frac{15}{8} = 1.875 \text{ hours}$$

So, it looks like it will take the two machines, working together, 1.875 hours to stuff a batch of envelopes.

Example 9 Mary can clean an office complex in 5 hours. Working together John and Mary can clean the office complex in 3.5 hours. How long would it take John to clean the office complex by himself?

Solution
Let *t* be the amount of time it would take John to clean the office complex by himself. The basic word equation for this problem is,

$$\left(\begin{array}{c}\text{Portion of job}\\ \text{done by Mary}\end{array}\right)+\left(\begin{array}{c}\text{Portion of job}\\ \text{done by John}\end{array}\right)=1\text{ Job}$$

$$\left(\begin{array}{c}\text{Work Rate}\\ \text{of Mary}\end{array}\right)\left(\begin{array}{c}\text{Time Spent}\\ \text{Working}\end{array}\right)+\left(\begin{array}{c}\text{Work Rate}\\ \text{of John}\end{array}\right)\left(\begin{array}{c}\text{Time Spent}\\ \text{Working}\end{array}\right)=1$$

This time we know that the time spent working together is 3.5 hours. We now need to find the work rates for each person. We'll start with Mary.

$$1\text{ Job}=\left(\text{Work Rate of Mary}\right)\times\left(5\right)\quad\Rightarrow\quad\text{Work Rate of Mary}=\frac{1}{5}$$

Now we'll find the work rate of John. Notice however, that since we don't know how long it will take him to do the job by himself we aren't going to be able to get a single number for this. That is not a problem as we'll see in a second.

$$1\text{ Job}=\left(\text{Work Rate of John}\right)\times\left(t\right)\quad\Rightarrow\quad\text{Work Rate of John}=\frac{1}{t}$$

Notice that we've managed to get the work rate of John in terms of the time it would take him to do the job himself. This means that once we solve the equation above we'll have the answer that we want. So, let's plug into the work equation and solve for the time it would take John to do the job by himself.

$$\frac{1}{5}(3.5)+\frac{1}{t}(3.5)=1\qquad\qquad\text{Multiplying both sides by }5t$$
$$3.5t+(3.5)(5)=5t$$
$$17.5=1.5t$$
$$\frac{17.5}{1.5}=t\qquad\qquad\Rightarrow\qquad\qquad t=11.67\text{ hrs}$$

So, it looks like it would take John 11.67 hours to clean the complex by himself.

Mixing Problems
This is the final type of problems that we'll be looking at in this section. We are going to be looking at mixing solutions of different percentages to get a new percentage. The solution will consist of a secondary liquid mixed in with water. The secondary liquid can be alcohol or acid for instance.

The standard equation that we'll use here will be the following.

$$\left(\begin{array}{c}\text{Amount of secondary}\\ \text{liquid in the water}\end{array}\right)=\left(\begin{array}{c}\text{Percentage of}\\ \text{Solution}\end{array}\right)\times\left(\begin{array}{c}\text{Volume of}\\ \text{Solution}\end{array}\right)$$

Note as well that the percentage needs to be a decimal. So if we have an 80% solution we will need to use 0.80.

Example 10 How much of a 50% alcohol solution should we mix with 10 gallons of a 35% solution to get a 40% solution?

Solution
Okay, let *x* be the amount of 50% solution that we need. This means that there will be $x + 10$ gallons of the 40% solution once we're done mixing the two.

Here is the basic work equation for this problem.
$$\left(\begin{array}{c} \text{Amount of alcohol} \\ \text{in 50\% Solution} \end{array}\right) + \left(\begin{array}{c} \text{Amount of alcohol} \\ \text{in 35\% Solution} \end{array}\right) = \left(\begin{array}{c} \text{Amount of alcohol} \\ \text{in 40\% Solution} \end{array}\right)$$

$$(0.5)\left(\begin{array}{c} \text{Volume of} \\ \text{50\% Solution} \end{array}\right) + (0.35)\left(\begin{array}{c} \text{Volume of} \\ \text{35\% Solution} \end{array}\right) = (0.4)\left(\begin{array}{c} \text{Volume of} \\ \text{40\% Solution} \end{array}\right)$$

Now, plug in the volumes and solve for *x*.
$$0.5x + 0.35(10) = 0.4(x + 10)$$
$$0.5x + 3.5 = 0.4x + 4$$
$$0.1x = 0.5$$
$$x = \frac{0.5}{0.1} = 5 \text{ gallons}$$

So, we need 5 gallons of the 50% solution to get a 40% solution.

Example 11 We have a 40% acid solution and we want 75 liters of a 15% acid solution. How much water should we put into the 40% solution to do this?

Solution
Let *x* be the amount of water we need to add to the 40% solution. Now, we also don't how much of the 40% solution we'll need. However, since we know the final volume (75 liters) we will know that we will need $75 - x$ liters of the 40% solution.

Here is the word equation for this problem.

$$\left(\begin{array}{c} \text{Amount of acid} \\ \text{in the water} \end{array}\right) + \left(\begin{array}{c} \text{Amount of acid} \\ \text{in 40\% Solution} \end{array}\right) = \left(\begin{array}{c} \text{Amount of acid} \\ \text{in 15\% Solution} \end{array}\right)$$

Notice that in the first term we used the "Amount of acid in the water". This might look a little weird to you because there shouldn't be any acid in the water. However, this is exactly what we want. The basic equation tells us to look at how much of the secondary liquid is in the water. So, this is the

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correct wording. When we plug in the percentages and volumes we will think of the water as a 0% percent solution since that is in fact what it is. So, the new word equation is,

$$(0)\begin{pmatrix} \text{Volume} \\ \text{of Water} \end{pmatrix} + (0.4)\begin{pmatrix} \text{Volume of} \\ \text{40\% Solution} \end{pmatrix} = (0.15)\begin{pmatrix} \text{Volume of} \\ \text{15\% Solution} \end{pmatrix}$$

Do not get excited about the zero in the first term. This is okay and will not be a problem. Let's now plug in the volumes and solve for *x*.

$$(0)(x) + (0.4)(75 - x) = (0.15)(75)$$
$$30 - 0.4x = 11.25$$
$$18.75 = 0.4x$$
$$x = \frac{18.75}{0.4} = 46.875 \text{ liters}$$

So, we need to add in 46.875 liters of water to 28.125 liters of a 40% solution to get 75 liters of a 15% solution.

Section 2-4 : Equations With More Than One Variable

In this section we are going to take a look at a topic that often doesn't get the coverage that it deserves in an Algebra class. This is probably because it isn't used in more than a couple of sections in an Algebra class. However, this is a topic that can, and often is, used extensively in other classes.

What we'll be doing here is solving equations that have more than one variable in them. The process that we'll be going through here is very similar to solving linear equations, which is one of the reasons why this is being introduced at this point. There is however one exception to that. Sometimes, as we will see, the ordering of the process will be different for some problems. Here is the process in the standard order.

1. Multiply both sides by the LCD to clear out any fractions.
2. Simplify both sides as much as possible. This will often mean clearing out parenthesis and the like.
3. Move all terms containing the variable we're solving for to one side and all terms that don't contain the variable to the opposite side.
4. Get a single instance of the variable we're solving for in the equation. For the types of problems that we'll be looking at here this will almost always be accomplished by simply factoring the variable out of each of the terms.
5. Divide by the coefficient of the variable. This step will make sense as we work problems. Note as well that in these problems the "coefficient" will probably contain things other than numbers.

It is usually easiest to see just what we're going to be working with and just how they work with an example. We will also give the basic process for solving these inside the first example.

Example 1 Solve $A = P(1 + rt)$ for r.

Solution
What we're looking for here is an expression in the form,

$$r = \underline{\text{Equation involving numbers, } A, P, \text{and } t}$$

In other words, the only place that we want to see an r is on the left side of the equal sign all by itself. There should be no other r's anywhere in the equation. The process given above should do that for us.

Okay, let's do this problem. We don't have any fractions so we don't need to worry about that. To simplify we will multiply the P through the parenthesis. Doing this gives,
$$A = P + Prt$$

Now, we need to get all the terms with an r on them on one side. This equation already has that set up for us which is nice. Next, we need to get all terms that don't have an r in them to the other side. This means subtracting a P from both sides.
$$A - P = Prt$$

As a final step we will divide both sides by the coefficient of *r*. Also, as noted in the process listed above the "coefficient" is not a number. In this case it is *Pt*. At this stage the coefficient of a variable is simply all the stuff that multiplies the variable.

$$\frac{A-P}{Pt} = r \qquad \Rightarrow \qquad r = \frac{A-P}{Pt}$$

To get a final answer we went ahead and flipped the order to get the answer into a more "standard" form.

We will work more examples in a bit. However, let's note a couple things first. These problems tend to seem fairly difficult at first, but if you think about it all we really did was use exactly the same process we used to solve linear equations. The main difference of course, is that there is more "mess" in this process. That brings us to the second point. Do not get excited about the mess in these problems. The problems will, on occasion, be a little messy, but the steps involved are steps that you can do! Finally, the answer will not be a simple number, but again it will be a little messy, often messier than the original equation. That is okay and expected.

Let's work some more examples.

Example 2 Solve $V = m\left(\dfrac{1}{b} - \dfrac{5aR}{m}\right)$ for *R*.

Solution
This one is fairly similar to the first example. However, it does work a little differently. Recall from the first example that we made the comment that sometimes the ordering of the steps in the process needs to be changed? Well, that's what we're going to do here.

The first step in the process tells us to clear fractions. However, since the fraction is inside a set of parentheses let's first multiply the *m* through the parenthesis. Notice as well that if we multiply the *m* through first we will in fact clear one of the fractions out automatically. This will make our work a little easier when we do clear the fractions out.

$$V = \frac{m}{b} - 5aR$$

Now, clear fractions by multiplying both sides by *b*. We'll also go ahead move all terms that don't have an *R* in them to the other side.

$$Vb = m - 5abR$$

$$Vb - m = -5abR$$

Be careful to not lose the minus sign in front of the 5! It's very easy to lose track of that. The final step is to then divide both sides by the coefficient of the *R*, in this case *-5ab*.

$$R = \frac{Vb-m}{-5ab} = -\frac{Vb-m}{5ab} = \frac{-(Vb-m)}{5ab} = \frac{-Vb+m}{5ab} = \frac{m-Vb}{5ab}$$

Notice as well that we did some manipulation of the minus sign that was in the denominator so that we could simplify the answer somewhat.

In the previous example we solved for *R*, but there is no reason for not solving for one of the other variables in the problems. For instance, consider the following example.

Example 3 Solve $V = m\left(\dfrac{1}{b} - \dfrac{5aR}{m}\right)$ for *b*.

Solution
The first couple of steps are identical to the previous example. First, we will multiply the *m* through the parenthesis and then we will multiply both sides by *b* to clear the fractions. We've already done this work so from the previous example we have,
$$Vb - m = -5abR$$

In this case we've got *b*'s on both sides of the equal sign and we need all terms with *b*'s in them on one side of the equation and all other terms on the other side of the equation. In this case we can eliminate the minus signs if we collect the *b*'s on the left side and the other terms on the right side. Doing this gives,
$$Vb + 5abR = m$$

Now, both terms on the right side have a *b* in them so if we factor that out of both terms we arrive at,
$$b(V + 5aR) = m$$

Finally, divide by the coefficient of *b*. Recall as well that the "coefficient" is all the stuff that multiplies the *b*. Doing this gives,
$$b = \frac{m}{V + 5aR}$$

Example 4 Solve $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$ for *c*.

Solution
First, multiply by the LCD, which is abc for this problem.
$$\frac{1}{a}(abc) = \left(\frac{1}{b} + \frac{1}{c}\right)(abc)$$
$$bc = ac + ab$$

Next, collect all the *c*'s on one side (the left will probably be easiest here), factor a *c* out of the terms and divide by the coefficient.
$$bc - ac = ab$$
$$c(b - a) = ab$$
$$c = \frac{ab}{b - a}$$

Example 5 Solve $y = \dfrac{4}{5x-9}$ for *x*.

Solution

First, we'll need to clear the denominator. To do this we will multiply both sides by $5x-9$. We'll also clear out any parenthesis in the problem after we do the multiplication.

$$y(5x-9) = 4$$
$$5xy - 9y = 4$$

Now, we want to solve for *x* so that means that we need to get all terms without a *y* in them to the other side. So, add 9*y* to both sides and the divide by the coefficient of *x*.

$$5xy = 9y + 4$$
$$x = \frac{9y+4}{5y}$$

Example 6 Solve $y = \dfrac{4-3x}{1+8x}$ for *x*.

Solution

This one is very similar to the previous example. Here is the work for this problem.

$$y(1+8x) = 4 - 3x$$
$$y + 8xy = 4 - 3x$$
$$8xy + 3x = 4 - y$$
$$x(8y+3) = 4 - y$$
$$x = \frac{4-y}{8y+3}$$

As mentioned at the start of this section we won't be seeing this kind of problem all that often in this class. However, outside of this class (a Calculus class for example) this kind of problem shows up with surprising regularity.

Section 2-5 : Quadratic Equations - Part I

Before proceeding with this section we should note that the topic of solving quadratic equations will be covered in two sections. This is done for the benefit of those viewing the material on the web. This is a long topic and to keep page load times down to a minimum the material was split into two sections.

So, we are now going to solve quadratic equations. First, the **standard form** of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

The only requirement here is that we have an x^2 in the equation. We guarantee that this term will be present in the equation by requiring $a \neq 0$. Note however, that it is okay if *b* and/or *c* are zero.

There are many ways to solve quadratic equations. We will look at four of them over the course of the next two sections. The first two methods won't always work yet are probably a little simpler to use when they work. This section will cover these two methods. The last two methods will always work, but often require a little more work or attention to get correct. We will cover these methods in the next section.

So, let's get started.

Solving by Factoring
As the heading suggests we will be solving quadratic equations here by factoring them. To do this we will need the following fact.

$$\text{If } ab = 0 \text{ then either } a = 0 \text{ and/or } b = 0$$

This fact is called the **zero factor property** or **zero factor principle**. All the fact says is that if a product of two terms is zero then at least one of the terms had to be zero to start off with.

Notice that this fact will ONLY work if the product is equal to zero. Consider the following product.
$$ab = 6$$
In this case there is no reason to believe that either *a* or *b* will be 6. We could have $a = 2$ and $b = 3$ for instance. So, do not misuse this fact!

To solve a quadratic equation by factoring we first must move all the terms over to one side of the equation. Doing this serves two purposes. First, it puts the quadratics into a form that can be factored. Secondly, and probably more importantly, in order to use the zero factor property we MUST have a zero on one side of the equation. If we don't have a zero on one side of the equation we won't be able to use the zero factor property.

Let's take a look at a couple of examples. Note that it is assumed that you can do the factoring at this point and so we won't be giving any details on the factoring. If you need a review of factoring you should go back and take a look at the Factoring section of the previous chapter.

Example 1 Solve each of the following equations by factoring.

(a) $x^2 - x = 12$

(b) $x^2 + 40 = -14x$

(c) $y^2 + 12y + 36 = 0$

(d) $4m^2 - 1 = 0$

(e) $3x^2 = 2x + 8$

(f) $10z^2 + 19z + 6 = 0$

(g) $5x^2 = 2x$

Solution

Now, as noted earlier, we won't be putting any detail into the factoring process, so make sure that you can do the factoring here.

(a) $x^2 - x = 12$

First, get everything on side of the equation and then factor.

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0$$

Now at this point we've got a product of two terms that is equal to zero. This means that at least one of the following must be true.

$$x - 4 = 0 \qquad\qquad \text{OR} \qquad\qquad x + 3 = 0$$

$$x = 4 \qquad\qquad \text{OR} \qquad\qquad x = -3$$

Note that each of these is a linear equation that is easy enough to solve. What this tell us is that we have two solutions to the equation, $x = 4$ and $x = -3$. As with linear equations we can always check our solutions by plugging the solution back into the equation. We will check $x = -3$ and leave the other to you to check.

$$(-3)^2 - (-3) \overset{?}{=} 12$$

$$9 + 3 \overset{?}{=} 12$$

$$12 = 12 \qquad \text{OK}$$

So, this was in fact a solution.

(b) $x^2 + 40 = -14x$

As with the first one we first get everything on side of the equal sign and then factor.

$$x^2 + 40 + 14x = 0$$

$$(x + 4)(x + 10) = 0$$

Now, we once again have a product of two terms that equals zero so we know that one or both of them have to be zero. So, technically we need to set each one equal to zero and solve. However, this is usually easy enough to do in our heads and so from now on we will be doing this solving in our head.

The solutions to this equation are,
$$x = -4 \quad \text{AND} \quad x = -10$$
To save space we won't be checking any more of the solutions here, but you should do so to make sure we didn't make any mistakes.

(c) $y^2 + 12y + 36 = 0$

In this case we already have zero on one side and so we don't need to do any manipulation to the equation all that we need to do is factor. Also, don't get excited about the fact that we now have y's in the equation. We won't always be dealing with x's so don't expect to always see them.

So, let's factor this equation.
$$y^2 + 12y + 36 = 0$$
$$(y+6)^2 = 0$$
$$(y+6)(y+6) = 0$$

In this case we've got a perfect square. We broke up the square to denote that we really do have an application of the zero factor property. However, we usually don't do that. We usually will go straight to the answer from the squared part.

The solution to the equation in this case is,
$$y = -6$$
We only have a single value here as opposed to the two solutions we've been getting to this point. We will often call this solution a **double root** or say that it has **multiplicity of 2** because it came from a term that was squared.

(d) $4m^2 - 1 = 0$

As always let's first factor the equation.
$$4m^2 - 1 = 0$$
$$(2m-1)(2m+1) = 0$$
Now apply the zero factor property. The zero factor property tells us that,

$2m - 1 = 0$	OR	$2m + 1 = 0$
$2m = 1$	OR	$2m = -1$
$m = \dfrac{1}{2}$	OR	$m = -\dfrac{1}{2}$

Again, we will typically solve these in our head, but we needed to do at least one in complete detail. So, we have two solutions to the equation.
$$m = \frac{1}{2} \quad \text{AND} \quad m = -\frac{1}{2}$$

(e) $3x^2 = 2x + 8$

Now that we've done quite a few of these, we won't be putting in as much detail for the next two problems. Here is the work for this equation.

$$3x^2 - 2x - 8 = 0$$

$$(3x+4)(x-2) = 0 \quad \Rightarrow \quad x = -\frac{4}{3} \quad \text{and} \quad x = 2$$

(f) $10z^2 + 19z + 6 = 0$

Again, factor and use the zero factor property for this one.

$$10z^2 + 19z + 6 = 0$$

$$(5z+2)(2z+3) = 0 \quad \Rightarrow \quad z = -\frac{2}{5} \quad \text{and} \quad z = -\frac{3}{2}$$

(g) $5x^2 = 2x$

This one always seems to cause trouble for students even though it's really not too bad.

First off. DO NOT CANCEL AN *x* FROM BOTH SIDES!!!! Do you get the idea that might be bad? It is. If you cancel an *x* from both sides, you WILL miss a solution so don't do it. Remember we are solving by factoring here so let's first get everything on one side of the equal sign.

$$5x^2 - 2x = 0$$

Now, notice that all we can do for factoring is to factor an *x* out of everything. Doing this gives,

$$x(5x-2) = 0$$

From the first factor we get that $x = 0$ and from the second we get that $x = \frac{2}{5}$. These are the two solutions to this equation. Note that is we'd canceled an *x* in the first step we would NOT have gotten $x = 0$ as an answer!

Let's work another type of problem here. We saw some of these back in the Solving Linear Equations section and since they can also occur with quadratic equations we should go ahead and work on to make sure that we can do them here as well.

Example 2 Solve each of the following equations.

    (a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

    (b) $x + 3 + \dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

Solution

Okay, just like with the linear equations the first thing that we're going to need to do here is to clear the denominators out by multiplying by the LCD. Recall that we will also need to note value(s) of *x* that will give division by zero so that we can make sure that these aren't included in the solution.

(a) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$

The LCD for this problem is $(x+1)(2x-4)$ and we will need to avoid $x = -1$ and $x = 2$ to make sure we don't get division by zero. Here is the work for this equation.

$$(x+1)(2x-4)\left(\frac{1}{x+1}\right) = (x+1)(2x-4)\left(1 - \frac{5}{2x-4}\right)$$
$$2x-4 = (x+1)(2x-4) - 5(x+1)$$
$$2x-4 = 2x^2 - 2x - 4 - 5x - 5$$
$$0 = 2x^2 - 9x - 5$$
$$0 = (2x+1)(x-5)$$

So, it looks like the two solutions to this equation are,

$$x = -\frac{1}{2} \quad \text{and} \quad x = 5$$

Notice as well that neither of these are the values of *x* that we needed to avoid and so both are solutions.

(b) $x + 3 + \dfrac{3}{x-1} = \dfrac{4-x}{x-1}$

In this case the LCD is $x-1$ and we will need to avoid $x = 1$ so we don't get division by zero. Here is the work for this problem.

$$(x-1)\left(x + 3 + \frac{3}{x-1}\right) = \left(\frac{4-x}{x-1}\right)(x-1)$$
$$(x-1)(x+3) + 3 = 4 - x$$
$$x^2 + 2x - 3 + 3 = 4 - x$$
$$x^2 + 3x - 4 = 0$$
$$(x-1)(x+4) = 0$$

So, the quadratic that we factored and solved has two solutions, $x = 1$ and $x = -4$. However, when we found the LCD we also saw that we needed to avoid $x = 1$ so we didn't get division by zero. Therefore, this equation has a single solution,

$$x = -4$$

Before proceeding to the next topic we should address that this idea of factoring can be used to solve equations with degree larger than two as well. Consider the following example.

Example 3 Solve $5x^3 - 5x^2 - 10x = 0$.

Solution
The first thing to do is factor this equation as much as possible. In this case that means factoring out the greatest common factor first. Here is the factored form of this equation.

$$5x\left(x^2 - x - 2\right) = 0$$
$$5x(x-2)(x+1) = 0$$

Now, the zero factor property will still hold here. In this case we have a product of three terms that is zero. The only way this product can be zero is if one of the terms is zero. This means that,

$$5x = 0 \qquad \Rightarrow \qquad x = 0$$
$$x - 2 = 0 \qquad \Rightarrow \qquad x = 2$$
$$x + 1 = 0 \qquad \Rightarrow \qquad x = -1$$

So, we have three solutions to this equation.

So, provided we can factor a polynomial we can always use this as a solution technique. The problem is, of course, that it is sometimes not easy to do the factoring.

Square Root Property
The second method of solving quadratics we'll be looking at uses the **square root property**,

$$\text{If } p^2 = d \quad \text{then} \quad p = \pm\sqrt{d}$$

There is a (potentially) new symbol here that we should define first in case you haven't seen it yet. The symbol "$\pm$" is read as : "plus or minus" and that is exactly what it tells us. This symbol is shorthand that tells us that we really have two numbers here. One is $p = \sqrt{d}$ and the other is $p = -\sqrt{d}$. Get used to this notation as it will be used frequently in the next couple of sections as we discuss the remaining solution techniques. It will also arise in other sections of this chapter and even in other chapters.

This is a fairly simple property to use, however it can only be used on a small portion of the equations that we're ever likely to encounter. Let's see some examples of this property.

Example 4 Solve each of the following equations.

      (a) $x^2 - 100 = 0$
      (b) $25y^2 - 3 = 0$
      (c) $4z^2 + 49 = 0$
      (d) $\left(2t - 9\right)^2 = 5$
      (e) $\left(3x + 10\right)^2 + 81 = 0$

Solution
There really isn't all that much to these problems. In order to use the square root property all that we need to do is get the squared quantity on the left side by itself with a coefficient of 1 and the number on the other side. Once this is done we can use the square root property.

(a) $x^2 - 100 = 0$

This is a fairly simple problem so here is the work for this equation.

$$x^2 = 100 \qquad\qquad x = \pm\sqrt{100} = \pm 10$$

So, there are two solutions to this equation, $x = \pm 10$. Remember this means that there are really two solutions here, $x = -10$ and $x = 10$.

(b) $25y^2 - 3 = 0$

Okay, the main difference between this one and the previous one is the 25 in front of the squared term. The square root property wants a coefficient of one there. That's easy enough to deal with however; we'll just divide both sides by 25. Here is the work for this equation.

$$25y^2 = 3$$

$$y^2 = \frac{3}{25} \qquad\Rightarrow\qquad y = \pm\sqrt{\frac{3}{25}} = \pm\frac{\sqrt{3}}{5}$$

In this case the solutions are a little messy, but many of these will do so don't worry about that. Also note that since we knew what the square root of 25 was we went ahead and split the square root of the fraction up as shown. Again, remember that there are really two solutions here, one positive and one negative.

(c) $4z^2 + 49 = 0$

This one is nearly identical to the previous part with one difference that we'll see at the end of the example. Here is the work for this equation.

$$4z^2 = -49$$

$$z^2 = -\frac{49}{4} \qquad\Rightarrow\qquad z = \pm\sqrt{-\frac{49}{4}} = \pm i\sqrt{\frac{49}{4}} = \pm\frac{7}{2}i$$

So, there are two solutions to this equation : $z = \pm\frac{7}{2}i$. Notice as well that they are complex solutions. This will happen with the solution to many quadratic equations so make sure that you can deal with them.

(d) $(2t - 9)^2 = 5$

This one looks different from the previous parts, however it works the same way. The square root property can be used anytime we have *something* squared equals a number. That is what we have here. The main difference of course is that the something that is squared isn't a single variable it is something else. So, here is the application of the square root property for this equation.

$$2t - 9 = \pm\sqrt{5}$$

Now, we just need to solve for *t* and despite the "plus or minus" in the equation it works the same way we would solve any linear equation. We will add 9 to both sides and then divide by a 2.

$$2t = 9 \pm\sqrt{5}$$

$$t = \frac{1}{2}\left(9 \pm\sqrt{5}\right) = \frac{9}{2} \pm\frac{\sqrt{5}}{2}$$

Note that we multiplied the fraction through the parenthesis for the final answer. We will usually do this in these problems. Also, do NOT convert these to decimals unless you are asked to. This is the standard form for these answers. With that being said we should convert them to decimals just to make sure that you can. Here are the decimal values of the two solutions.

$$t = \frac{9}{2} + \frac{\sqrt{5}}{2} = 5.61803 \qquad \text{and} \qquad t = \frac{9}{2} - \frac{\sqrt{5}}{2} = 3.38197$$

(e) $(3x+10)^2 + 81 = 0$

In this final part we'll not put much in the way of details into the work.

$$(3x+10)^2 = -81$$
$$3x+10 = \pm 9i$$
$$3x = -10 \pm 9i$$
$$x = -\frac{10}{3} \pm 3i$$

So we got two complex solutions again and notice as well that with both of the previous part we put the "plus or minus" part last. This is usually the way these are written.

As mentioned at the start of this section we are going to break this topic up into two sections for the benefit of those viewing this on the web. The next two methods of solving quadratic equations, completing the square and quadratic formula, are given in the next section.

Section 2-6 : Quadratic Equations - Part II

The topic of solving quadratic equations has been broken into two sections for the benefit of those viewing this on the web. As a single section the load time for the page would have been quite long. This is the second section on solving quadratic equations.

In the previous section we looked at using factoring and the square root property to solve quadratic equations. The problem is that both of these solution methods will not always work. Not every quadratic is factorable and not every quadratic is in the form required for the square root property.

It is now time to start looking into methods that will work for all quadratic equations. So, in this section we will look at completing the square and the quadratic formula for solving the quadratic equation,

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

Completing the Square

The first method we'll look at in this section is completing the square. It is called this because it uses a process called completing the square in the solution process. So, we should first define just what completing the square is.

Let's start with

$$x^2 + bx$$

and notice that the x^2 has a coefficient of one. That is required in order to do this. Now, to this lets add $\left(\dfrac{b}{2}\right)^2$. Doing this gives the following **factorable** quadratic equation.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

This process is called **completing the square** and if we do all the arithmetic correctly we can guarantee that the quadratic will factor as a perfect square.

Let's do a couple of examples for just completing the square before looking at how we use this to solve quadratic equations.

Example 1 Complete the square on each of the following.

      (a) $x^2 - 16x$

      (b) $y^2 + 7y$

Solution

(a) $x^2 - 16x$

Here's the number that we'll add to the equation.

$$\left(\frac{-16}{2}\right)^2 = (-8)^2 = 64$$

Notice that we kept the minus sign here even though it will always drop out after we square things. The reason for this will be apparent in a second. Let's now complete the square.

$$x^2 - 16x + 64 = (x - 8)^2$$

Now, this is a quadratic that hopefully you can factor fairly quickly. However notice that it will always factor as *x* plus the blue number we computed above that is in the parenthesis (in our case that is -8). This is the reason for leaving the minus sign. It makes sure that we don't make any mistakes in the factoring process.

(b) $y^2 + 7y$

Here's the number we'll need this time.

$$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

It's a fraction and that will happen fairly often with these so don't get excited about it. Also, leave it as a fraction. Don't convert to a decimal. Now complete the square.

$$y^2 + 7y + \frac{49}{4} = \left(y + \frac{7}{2}\right)^2$$

This one is not so easy to factor. However, if you again recall that this will ALWAYS factor as *y* plus the blue number above we don't have to worry about the factoring process.

It's now time to see how we use completing the square to solve a quadratic equation. The process is best seen as we work an example so let's do that.

Example 2 Use completing the square to solve each of the following quadratic equations.

        (a) $x^2 - 6x + 1 = 0$
        (b) $2x^2 + 6x + 7 = 0$
        (c) $3x^2 - 2x - 1 = 0$

Solution
We will do the first problem in detail explicitly giving each step. In the remaining problems we will just do the work without as much explanation.

(a) $x^2 - 6x + 1 = 0$
So, let's get started.

Step 1 : Divide the equation by the coefficient of the x^2 term. Recall that completing the square required a coefficient of one on this term and this will guarantee that we will get that. We don't need to do that for this equation however.

Step 2 : Set the equation up so that the *x*'s are on the left side and the constant is on the right side.

$$x^2 - 6x = -1$$

Step 3 : Complete the square on the left side. However, this time we will need to add the number to both sides of the equal sign instead of just the left side. This is because we have to remember the rule that what we do to one side of an equation we need to do to the other side of the equation.

First, here is the number we add to both sides.

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Now, complete the square.

$$x^2 - 6x + 9 = -1 + 9$$

$$(x-3)^2 = 8$$

Step 4 : Now, at this point notice that we can use the square root property on this equation. That was the purpose of the first three steps. Doing this will give us the solution to the equation.

$$x - 3 = \pm\sqrt{8} \qquad \Rightarrow \qquad x = 3 \pm \sqrt{8}$$

And that is the process. Let's do the remaining parts now.

(b) $2x^2 + 6x + 7 = 0$

We will not explicitly put in the steps this time nor will we put in a lot of explanation for this equation. This that being said, notice that we will have to do the first step this time. We don't have a coefficient of one on the x^2 term and so we will need to divide the equation by that first.

Here is the work for this equation.

$$x^2 + 3x + \frac{7}{2} = 0$$

$$x^2 + 3x = -\frac{7}{2} \qquad\qquad\qquad \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$x^2 + 3x + \frac{9}{4} = -\frac{7}{2} + \frac{9}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = -\frac{5}{4}$$

$$x + \frac{3}{2} = \pm\sqrt{-\frac{5}{4}} \qquad \Rightarrow \qquad x = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$$

Don't forget to convert square roots of negative numbers to complex numbers!

(c) $3x^2 - 2x - 1 = 0$

Again, we won't put a lot of explanation for this problem.

$$x^2 - \frac{2}{3}x - \frac{1}{3} = 0$$

$$x^2 - \frac{2}{3}x = \frac{1}{3}$$

At this point we should be careful about computing the number for completing the square since *b* is now a fraction for the first time.

$$\left(\dfrac{-\dfrac{2}{3}}{2}\right)^{2} = \left(-\dfrac{2}{3}\cdot\dfrac{1}{2}\right)^{2} = \left(-\dfrac{1}{3}\right)^{2} = \dfrac{1}{9}$$

Now finish the problem.

$$x^{2} - \dfrac{2}{3}x + \dfrac{1}{9} = \dfrac{1}{3} + \dfrac{1}{9}$$

$$\left(x - \dfrac{1}{3}\right)^{2} = \dfrac{4}{9}$$

$$x - \dfrac{1}{3} = \pm\sqrt{\dfrac{4}{9}} \qquad \Rightarrow \qquad x = \dfrac{1}{3} \pm \dfrac{2}{3}$$

In this case notice that we can actually do the arithmetic here to get two integer and/or fractional solutions. We should always do this when there are only integers and/or fractions in our solution. Here are the two solutions.

$$x = \dfrac{1}{3} + \dfrac{2}{3} = \dfrac{3}{3} = 1 \qquad \text{and} \qquad x = \dfrac{1}{3} - \dfrac{2}{3} = -\dfrac{1}{3}$$

A quick comment about the last equation that we solved in the previous example is in order. Since we received integer and fractions as solutions, we could have just factored this equation from the start rather than used completing the square. In cases like this we could use either method and we will get the same result.

Now, the reality is that completing the square is a fairly long process and it's easy to make mistakes. So, we rarely actually use it to solve equations. That doesn't mean that it isn't important to know the process however. We will be using it in several sections in later chapters and is often used in other classes.

Quadratic Formula
This is the final method for solving quadratic equations and will always work. Not only that, but if you can remember the formula it's a fairly simple process as well.

We can derive the quadratic formula by completing the square on the general quadratic formula in standard form. Let's do that and we'll take it kind of slow to make sure all the steps are clear.

First, we MUST have the quadratic equation in standard form as already noted. Next, we need to divide both sides by a to get a coefficient of one on the x^2 term.

$$ax^{2} + bx + c = 0$$

$$x^{2} + \dfrac{b}{a}x + \dfrac{c}{a} = 0$$

Next, move the constant to the right side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now, we need to compute the number we'll need to complete the square. Again, this is one-half the coefficient of *x*, squared.

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Now, add this to both sides, complete the square and get common denominators on the right side to simplify things up a little.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now we can use the square root property on this.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Solve for *x* and we'll also simplify the square root a little.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

As a last step we will notice that we've got common denominators on the two terms and so we'll add them up. Doing this gives,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, summarizing up, provided that we start off in standard form,

$$ax^2 + bx + c = 0$$

and that's very important, then the solution to any quadratic equation is,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's work a couple of examples.

Example 3 Use the quadratic formula to solve each of the following equations.

   (a) $x^2 + 2x = 7$

   (b) $3q^2 + 11 = 5q$

   (c) $7t^2 = 6 - 19t$

   (d) $\dfrac{3}{y-2} = \dfrac{1}{y} + 1$

   (e) $16x - x^2 = 0$

Solution

The important part here is to make sure that before we start using the quadratic formula that we have the equation in standard form first.

(a) $x^2 + 2x = 7$

So, the first thing that we need to do here is to put the equation in standard form.

$$x^2 + 2x - 7 = 0$$

At this point we can identify the values for use in the quadratic formula. For this equation we have.

$$a = 1 \qquad\qquad b = 2 \qquad\qquad c = -7$$

Notice the "-" with *c*. It is important to make sure that we carry any minus signs along with the constants.

At this point there really isn't anything more to do other than plug into the formula.

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{32}}{2}$$

There are the two solutions for this equation. There is also some simplification that we can do. We need to be careful however. One of the larger mistakes at this point is to "cancel" two 2's in the numerator and denominator. Remember that in order to cancel anything from the numerator or denominator then it must be multiplied by the whole numerator or denominator. Since the 2 in the numerator isn't multiplied by the whole denominator it can't be canceled.

In order to do any simplification here we will first need to reduce the square root. At that point we can do some canceling.

$$x = \frac{-2 \pm \sqrt{(16)2}}{2} = \frac{-2 \pm 4\sqrt{2}}{2} = \frac{2\left(-1 \pm 2\sqrt{2}\right)}{2} = -1 \pm 2\sqrt{2}$$

That's a much nicer answer to deal with and so we will almost always do this kind of simplification when it can be done.

(b) $3q^2 + 11 = 5q$

Now, in this case don't get excited about the fact that the variable isn't an *x*. Everything works the same regardless of the letter used for the variable. So, let's first get the equation into standard form.

$$3q^2 + 11 - 5q = 0$$

Now, this isn't quite in the typical standard form. However, we need to make a point here so that we don't make a very common mistake that many student make when first learning the quadratic formula.

Many students will just get everything on one side as we've done here and then get the values of *a*, *b*, and *c* based upon position. In other words, often students will just let *a* be the first number listed, *b* be the second number listed and then *c* be the final number listed. This is not correct however. For the quadratic formula *a* is the coefficient of the squared term, *b* is the coefficient of the term with just the variable in it (not squared) and *c* is the constant term. So, to avoid making this mistake we should always put the quadratic equation into the official standard form.

$$3q^2 - 5q + 11 = 0$$

Now we can identify the value of *a*, *b*, and *c*.
$$a = 3 \qquad\qquad b = -5 \qquad\qquad c = 11$$
Again, be careful with minus signs. They need to get carried along with the values.

Finally, plug into the quadratic formula to get the solution.

$$q = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(11)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25 - 132}}{6}$$

$$= \frac{5 \pm \sqrt{-107}}{6}$$

$$= \frac{5 \pm \sqrt{107}\ i}{6}$$

As with all the other methods we've looked at for solving quadratic equations, don't forget to convert square roots of negative numbers into complex numbers. Also, when *b* is negative be very careful with the substitution. This is particularly true for the squared portion under the radical. Remember that when you square a negative number it will become positive. One of the more common mistakes here is to get in a hurry and forget to drop the minus sign after you square *b*, so be careful.

(c) $7t^2 = 6 - 19t$
We won't put in quite the detail with this one that we've done for the first two. Here is the standard form of this equation.

$$7t^2 + 19t - 6 = 0$$

Here are the values for the quadratic formula as well as the quadratic formula itself.
$$a = 7 \qquad\qquad b = 19 \qquad\qquad c = -6$$

$$t = \frac{-19 \pm \sqrt{(19)^2 - 4(7)(-6)}}{2(7)}$$

$$= \frac{-19 \pm \sqrt{361 + 168}}{14}$$

$$= \frac{-19 \pm \sqrt{529}}{14}$$

$$= \frac{-19 \pm 23}{14}$$

Now, recall that when we get solutions like this we need to go the extra step and actually determine the integer and/or fractional solutions. In this case they are,

$$t = \frac{-19 + 23}{14} = \frac{2}{7} \qquad\qquad\qquad t = \frac{-19 - 23}{14} = -3$$

Now, as with completing the square, the fact that we got integer and/or fractional solutions means that we could have factored this quadratic equation as well.

(d) $\dfrac{3}{y-2} = \dfrac{1}{y} + 1$

So, an equation with fractions in it. The first step then is to identify the LCD.
$$\text{LCD} : y(y-2)$$

So, it looks like we'll need to make sure that neither $y = 0$ or $y = 2$ is in our answers so that we don't get division by zero.

Multiply both sides by the LCD and then put the result in standard form.
$$(y)(y-2)\left(\frac{3}{y-2}\right) = \left(\frac{1}{y} + 1\right)(y)(y-2)$$

$$3y = y - 2 + y(y-2)$$

$$3y = y - 2 + y^2 - 2y$$

$$0 = y^2 - 4y - 2$$

Okay, it looks like we've got the following values for the quadratic formula.
$$a = 1 \qquad\qquad b = -4 \qquad\qquad c = -2$$

Plugging into the quadratic formula gives,

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{24}}{2}$$

$$= \frac{4 \pm 2\sqrt{6}}{2}$$

$$= 2 \pm \sqrt{6}$$

Note that both of these are going to be solutions since neither of them are the values that we need to avoid.

(e) $16x - x^2 = 0$

We saw an equation similar to this in the previous section when we were looking at factoring equations and it would definitely be easier to solve this by factoring. However, we are going to use the quadratic formula anyway to make a couple of points.

First, let's rearrange the order a little bit just to make it look more like the standard form.

$$-x^2 + 16x = 0$$

Here are the constants for use in the quadratic formula.

$$a = -1 \qquad\qquad b = 16 \qquad\qquad c = 0$$

There are two things to note about these values. First, we've got a negative *a* for the first time. Not a big deal, but it is the first time we've seen one. Secondly, and more importantly, one of the values is zero. This is fine. It will happen on occasion and in fact, having one of the values zero will make the work much simpler.

Here is the quadratic formula for this equation.

$$x = \frac{-16 \pm \sqrt{(16)^2 - 4(-1)(0)}}{2(-1)}$$

$$= \frac{-16 \pm \sqrt{256}}{-2}$$

$$= \frac{-16 \pm 16}{-2}$$

Reducing these to integers/fractions gives,

$$x = \frac{-16 + 16}{-2} = \frac{0}{2} = 0 \qquad\qquad\qquad x = \frac{-16 - 16}{-2} = \frac{-32}{-2} = 16$$

So we get the two solutions, $x = 0$ and $x = 16$. These are exactly the solutions we would have gotten by factoring the equation.

To this point in both this section and the previous section we have only looked at equations with integer coefficients. However, this doesn't have to be the case. We could have coefficient that are fractions or decimals. So, let's work a couple of examples so we can say that we've seen something like that as well.

Example 4 Solve each of the following equations.

(a) $\dfrac{1}{2}x^2 + x - \dfrac{1}{10} = 0$

(b) $0.04x^2 - 0.23x + 0.09 = 0$

Solution

(a) There are two ways to work this one. We can either leave the fractions in or multiply by the LCD (10 in this case) and solve that equation. Either way will give the same answer. We will only do the fractional case here since that is the point of this problem. You should try the other way to verify that you get the same solution.

In this case here are the values for the quadratic formula as well as the quadratic formula work for this equation.

$$a = \frac{1}{2} \qquad\qquad b = 1 \qquad\qquad c = -\frac{1}{10}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\frac{1}{2}\right)\left(-\frac{1}{10}\right)}}{2\left(\frac{1}{2}\right)} = \frac{-1 \pm \sqrt{1 + \frac{1}{5}}}{1} = -1 \pm \sqrt{\frac{6}{5}}$$

In these cases we usually go the extra step of eliminating the square root from the denominator so let's also do that,

$$x = -1 \pm \frac{\sqrt{6}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = -1 \pm \frac{\sqrt{(6)(5)}}{5} = -1 \pm \frac{\sqrt{30}}{5}$$

If you do clear the fractions out and run through the quadratic formula then you should get exactly the same result. For the practice you really should try that.

(b) In this case do not get excited about the decimals. The quadratic formula works in exactly the same manner. Here are the values and the quadratic formula work for this problem.

$$a = 0.04 \qquad\qquad b = -0.23 \qquad\qquad c = 0.09$$

$$x = \frac{-(-0.23) \pm \sqrt{(-0.23)^2 - 4(0.04)(0.09)}}{2(0.04)}$$

$$= \frac{0.23 \pm \sqrt{0.0529 - 0.0144}}{0.08}$$

$$= \frac{0.23 \pm \sqrt{0.0385}}{0.08}$$

Now, to this will be the one difference between these problems and those with integer or fractional coefficients. When we have decimal coefficients we usually go ahead and figure the two individual numbers. So, let's do that,

$$x = \frac{0.23 \pm \sqrt{0.0385}}{0.08} = \frac{0.23 \pm 0.19621}{0.08}$$

$$x = \frac{0.23 + 0.19621}{0.08} \qquad \text{and} \qquad x = \frac{0.23 - 0.19621}{0.08}$$
$$= 5.327625 \qquad\qquad \text{and} \qquad\qquad = 0.422375$$

Notice that we did use some rounding on the square root.

Over the course of the last two sections we've done quite a bit of solving. It is important that you understand most, if not all, of what we did in these sections as you will be asked to do this kind of work in some later sections.

Section 2-7 : Quadratic Equations : A Summary

In the previous two sections we've talked quite a bit about solving quadratic equations. A logical question to ask at this point is which method should we use to solve a given quadratic equation? Unfortunately, the answer is, it depends.

If your instructor has specified the method to use then that, of course, is the method you should use. However, if your instructor had NOT specified the method to use then we will have to make the decision ourselves. Here is a general set of guidelines that *may* be helpful in determining which method to use.

1. Is it clearly a square root property problem? In other words, does the equation consist ONLY of something squared and a constant. If this is true then the square root property is probably the easiest method for use.
2. Does it factor? If so, that is probably the way to go. Note that you shouldn't spend a lot of time trying to determine if the quadratic equation factors. Look at the equation and if you can quickly determine that it factors then go with that. If you can't quickly determine that it factors then don't worry about it.
3. If you've reached this point then you've determined that the equation is not in the correct form for the square root property and that it doesn't factor (or that you can't quickly see that it factors). So, at this point you're only real option is the quadratic formula.

Once you've solved enough quadratic equations the above set of guidelines will become almost second nature to you and you will find yourself going through them almost without thinking.

Notice as well that nowhere in the set of guidelines was completing the square mentioned. The reason for this is simply that it's a long method that is prone to mistakes when you get in a hurry. The quadratic formula will also always work and is much shorter of a method to use. In general, you should only use completing the square if your instructor has required you to use it.

As a solving technique completing the square should always be your last choice. This doesn't mean however that it isn't an important method. We will see the completing the square process arise in several sections in later chapters. Interestingly enough when we do see this process in later sections we won't be solving equations! This process is very useful in many situations of which solving is only one.

Before leaving this section we have one more topic to discuss. In the previous couple of sections we saw that solving a quadratic equation in standard form,

$$ax^2 + bx + c = 0$$

we will get one of the following three possible solution sets.

1. Two real distinct (*i.e.* not equal) solutions.
2. A double root. Recall this arises when we can factor the equation into a perfect square.
3. Two complex solutions.

These are the ONLY possibilities for solving quadratic equations in standard form. Note however, that if we start with rational expression in the equation we may get different solution sets because we may need avoid one of the possible solutions so we don't get division by zero errors.

Now, it turns out that all we need to do is look at the quadratic equation (in standard form of course) to determine which of the three cases that we'll get. To see how this works let's start off by recalling the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ in the quadratic formula is called the **discriminant**.   It is the value of the discriminant that will determine which solution set we will get. Let's go through the cases one at a time.

1. Two real distinct solutions. We will get this solution set if $b^2 - 4ac > 0$. In this case we will be taking the square root of a positive number and so the square root will be a real number. Therefore, the numerator in the quadratic formula will be $-b$ plus or minus a real number. This means that the numerator will be two different real numbers. Dividing either one by 2a won't change the fact that they are real, nor will it change the fact that they are different.

2. A double root. We will get this solution set if $b^2 - 4ac = 0$. Here we will be taking the square root of zero, which is zero. However, this means that the "plus or minus" part of the numerator will be zero and so the numerator in the quadratic formula will be $-b$. In other words, we will get a single real number out of the quadratic formula, which is what we get when we get a double root.

3. Two complex solutions. We will get this solution set if $b^2 - 4ac < 0$. If the discriminant is negative we will be taking the square root of negative numbers in the quadratic formula which means that we will get complex solutions. Also, we will get two since they have a "plus or minus" in front of the square root.

So, let's summarize up the results here.

1. If $b^2 - 4ac > 0$ then we will get two real solutions to the quadratic equation.
2. If $b^2 - 4ac = 0$ then we will get a double root to the quadratic equation.
3. If $b^2 - 4ac < 0$ then we will get two complex solutions to the quadratic equation.

Example 1 Using the discriminant determine which solution set we get for each of the following quadratic equations.
>   **(a)** $13x^2 + 1 = 5x$
>   **(b)** $6q^2 + 20q = 3$
>   **(c)** $49t^2 + 126t + 81 = 0$

Solution
All we need to do here is make sure the equation is in standard form, determine the value of a, b, and c, then plug them into the discriminant.

(a) $13x^2 + 1 = 5x$

First get the equation in standard form.

$$13x^2 - 5x + 1 = 0$$

We then have,

$$a = 13 \qquad b = -5 \qquad c = 1$$

Plugging into the discriminant gives,

$$b^2 - 4ac = (-5)^2 - 4(13)(1) = -27$$

The discriminant is negative and so we will have two complex solutions. For reference purposes the actual solutions are,

$$x = \frac{5 \pm 3\sqrt{3}\, i}{26}$$

(b) $6q^2 + 20q = 3$

Again, we first need to get the equation in standard form.

$$6q^2 + 20q - 3 = 0$$

This gives,

$$a = 6 \qquad b = 20 \qquad c = -3$$

The discriminant is then,

$$b^2 - 4ac = (20)^2 - 4(6)(-3) = 472$$

The discriminant is positive we will get two real distinct solutions. Here they are,

$$x = \frac{-20 \pm \sqrt{472}}{12} = \frac{-10 \pm \sqrt{118}}{6}$$

(c) $49t^2 + 126t + 81 = 0$

This equation is already in standard form so let's jump straight in.

$$a = 49 \qquad b = 126 \qquad c = 81$$

The discriminant is then,

$$b^2 - 4ac = (126)^2 - 4(49)(81) = 0$$

In this case we'll get a double root since the discriminant is zero. Here it is,

$$x = -\frac{9}{7}$$

For practice you should verify the solutions in each of these examples.

Section 2-8 : Applications of Quadratic Equations

In this section we're going to go back and revisit some of the applications that we saw in the Linear Applications section and see some examples that will require us to solve a quadratic equation to get the answer.

Note that the solutions in these cases will almost always require the quadratic formula so expect to use it and don't get excited about it. Also, we are going to assume that you can do the quadratic formula work and so we won't be showing that work. We will give the results of the quadratic formula, we just won't be showing the work.

Also, as we will see, we will need to get decimal answer to these and so as a general rule here we will round all answers to 4 decimal places.

Example 1 We are going to fence in a rectangular field and we know that for some reason we want the field to have an enclosed area of 75 ft². We also know that we want the width of the field to be 3 feet longer than the length of the field. What are the dimensions of the field?

Solution

So, we'll let *x* be the length of the field and so we know that $x + 3$ will be the width of the field. Now, we also know that area of a rectangle is length times width and so we know that,

$$x(x + 3) = 75$$

Now, this is a quadratic equation so let's first write it in standard form.

$$x^2 + 3x = 75$$
$$x^2 + 3x - 75 = 0$$

Using the quadratic formula gives,

$$x = \frac{-3 \pm \sqrt{309}}{2}$$

Now, at this point, we've got to deal with the fact that there are two solutions here and we only want a single answer. So, let's convert to decimals and see what the solutions actually are.

$$x = \frac{-3 + \sqrt{309}}{2} = 7.2892 \qquad \text{and} \qquad x = \frac{-3 - \sqrt{309}}{2} = -10.2892$$

So, we have one positive and one negative. From the stand point of needing the dimensions of a field the negative solution doesn't make any sense so we will ignore it.

Therefore, the length of the field is 7.2892 feet. The width is 3 feet longer than this and so is 10.2892 feet.

Notice that the width is almost the second solution to the quadratic equation. The only difference is the minus sign. Do NOT expect this to always happen. In this case this is more of a function of the problem. For a more complicated set up this will NOT happen.

Now, from a physical standpoint we can see that we should expect to NOT get complex solutions to these problems. Upon solving the quadratic equation we should get either two real distinct solutions or a double root. Also, as the previous example has shown, when we get two real distinct solutions we will be able to eliminate one of them for physical reasons.

Let's work another example or two.

Example 2 Two cars start out at the same point. One car starts out driving north at 25 mph. Two hours later the second car starts driving east at 20 mph. How long after the first car starts traveling does it take for the two cars to be 300 miles apart?

Solution
We'll start off by letting t be the amount of time that the first car, let's call it car A, travels. Since the second car, let's call that car B, starts out two hours later then we know that it will travel for $t - 2$ hours.

Now, we know that the distance traveled by an object (or car since that's what we're dealing with here) is its speed times time traveled. So we have the following distances traveled for each car.

$$\text{distance of car A}: 25t$$

$$\text{distance of car B}: 20(t-2)$$

At this point a quick sketch of the situation is probably in order so we can see just what is going on. In the sketch we will assume that the two cars have traveled long enough so that they are 300 miles apart.
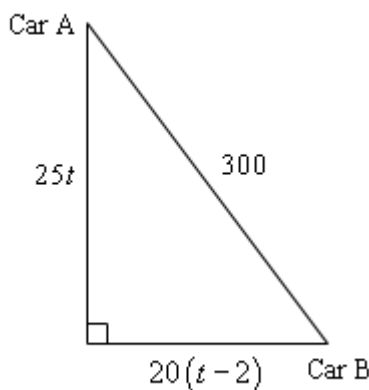
So, we have a right triangle here. That means that we can use the Pythagorean Theorem to say,

$$(25t)^2 + (20(t-2))^2 = (300)^2$$

This is a quadratic equation, but it is going to need some fairly heavy simplification before we can solve it so let's do that.

$$625t^2 + (20t - 40)^2 = 90000$$
$$625t^2 + 400t^2 - 1600t + 1600 = 90000$$
$$1025t^2 - 1600t - 88400 = 0$$

Now, the coefficients here are quite large, but that is just something that will happen fairly often with these problems so don't worry about that. Using the quadratic formula (and simplifying that answer) gives,

$$t = \frac{1600 \pm \sqrt{365000000}}{2050} = \frac{1600 \pm 1000\sqrt{365}}{2050} = \frac{32 \pm 20\sqrt{365}}{41}$$

Again, we have two solutions and we're going to need to determine which one is the correct one, so let's convert them to decimals.

$$t = \frac{32 + 20\sqrt{365}}{41} = 10.09998 \qquad \text{and} \qquad t = \frac{32 - 20\sqrt{365}}{41} = -8.539011$$

As with the previous example the negative answer just doesn't make any sense. So, it looks like the car A traveled for 10.09998 hours when they were finally 300 miles apart.

Also, even though the problem didn't ask for it, the second car will have traveled for 8.09998 hours before they are 300 miles apart. Notice as well that this is NOT the second solution without the negative this time, unlike the first example.

Example 3 An office has two envelope stuffing machines. Working together they can stuff a batch of envelopes in 2 hours. Working separately, it will take the second machine 1 hour longer than the first machine to stuff a batch of envelopes. How long would it take each machine to stuff a batch of envelopes by themselves?

Solution
Let *t* be the amount of time it takes the first machine (Machine A) to stuff a batch of envelopes by itself. That means that it will take the second machine (Machine B) $t + 1$ hours to stuff a batch of envelopes by itself.

The word equation for this problem is then,

$$\begin{pmatrix} \text{Portion of job} \\ \text{done by Machine A} \end{pmatrix} + \begin{pmatrix} \text{Portion of job} \\ \text{done by Machine B} \end{pmatrix} = 1 \text{ Job}$$

$$\begin{pmatrix} \text{Work Rate} \\ \text{of Machine A} \end{pmatrix} \begin{pmatrix} \text{Time Spent} \\ \text{Working} \end{pmatrix} + \begin{pmatrix} \text{Work Rate} \\ \text{of Machine B} \end{pmatrix} \begin{pmatrix} \text{Time Spent} \\ \text{Working} \end{pmatrix} = 1$$

We know the time spent working together (2 hours) so we need to work rates of each machine. Here are those computations.

$$1 \text{ Job} = (\text{Work Rate of Machine A}) \times (t) \quad \Rightarrow \quad \text{Machine A} = \frac{1}{t}$$

$$1 \text{ Job} = (\text{Work Rate of Machine B}) \times (t+1) \quad \Rightarrow \quad \text{Machine B} = \frac{1}{t+1}$$

Note that it's okay that the work rates contain *t*. In fact, they will need to so we can solve for it! Plugging into the word equation gives,

$$\left(\frac{1}{t}\right)(2) + \left(\frac{1}{t+1}\right)(2) = 1$$

$$\frac{2}{t} + \frac{2}{t+1} = 1$$

So, to solve we'll first need to clear denominators and get the equation in standard form.

$$\left(\frac{2}{t} + \frac{2}{t+1}\right)(t)(t+1) = (1)(t)(t+1)$$

$$2(t+1) + 2t = t^2 + t$$

$$4t + 2 = t^2 + t$$

$$0 = t^2 - 3t - 2$$

Using the quadratic formula gives,

$$t = \frac{3 \pm \sqrt{17}}{2}$$

Converting to decimals gives,

$$t = \frac{3 + \sqrt{17}}{2} = 3.5616 \qquad \text{and} \qquad t = \frac{3 - \sqrt{17}}{2} = -0.5616$$

Again, the negative doesn't make any sense and so Machine A will work for 3.5616 hours to stuff a batch of envelopes by itself. Machine B will need 4.5616 hours to stuff a batch of envelopes by itself. Again, unlike the first example, note that the time for Machine B was NOT the second solution from the quadratic without the minus sign.

Section 2-9 : Equations Reducible to Quadratic in Form

In this section we are going to look at equations that are called **quadratic in form** or **reducible to quadratic in form**. What this means is that we will be looking at equations that if we look at them in the correct light we can make them look like quadratic equations. At that point we can use the techniques we developed for quadratic equations to help us with the solution of the actual equation.

It is usually best with these to show the process with an example so let's do that.

Example 1 Solve $x^4 - 7x^2 + 12 = 0$

Solution
Now, let's start off here by noticing that
$$x^4 = \left(x^2\right)^2$$
In other words, we can notice here that the variable portion of the first term (*i.e.* ignore the coefficient) is nothing more than the variable portion of the second term squared. Note as well that all we really needed to notice here is that the exponent on the first term was twice the exponent on the second term.

This, along with the fact that third term is a constant, means that this equation is reducible to quadratic in form. We will solve this by first defining,
$$u = x^2$$
Now, this means that
$$u^2 = \left(x^2\right)^2 = x^4$$

Therefore, we can write the equation in terms of u's instead of x's as follows,

$$x^4 - 7x^2 + 12 = 0 \qquad \Rightarrow \qquad u^2 - 7u + 12 = 0$$

The new equation (the one with the u's) is a quadratic equation and we can solve that. In fact, this equation is factorable, so the solution is,
$$u^2 - 7u + 12 = (u - 4)(u - 3) = 0 \qquad \Rightarrow \qquad u = 3, \ u = 4$$

So, we get the two solutions shown above. These aren't the solutions that we're looking for. We want values of *x*, not values of *u*. That isn't really a problem once we recall that we've defined
$$u = x^2$$

To get values of *x* for the solution all we need to do is plug in *u* into this equation and solve that for *x*. Let's do that.

$$\begin{aligned} u = 3: \qquad & 3 = x^2 \qquad && \Rightarrow \qquad && x = \pm\sqrt{3} \\ u = 4: \qquad & 4 = x^2 \qquad && \Rightarrow \qquad && x = \pm\sqrt{4} = \pm 2 \end{aligned}$$

So, we have four solutions to the original equation, $x = \pm 2$ and $x = \pm\sqrt{3}$.

So, the basic process is to check that the equation is reducible to quadratic in form then make a quick substitution to turn it into a quadratic equation. We solve the new equation for *u*, the variable from the substitution, and then use these solutions and the substitution definition to get the solutions to the equation that we really want.

In most cases to make the check that it's reducible to quadratic in form all that we really need to do is to check that one of the exponents is twice the other. There is one exception to this that we'll see here once we get into a set of examples.

Also, once you get "good" at these you often don't really need to do the substitution either. We will do them to make sure that the work is clear. However, these problems can be done without the substitution in many cases.

Example 2 Solve each of the following equations.

     (a) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

     (b) $y^{-6} - 9y^{-3} + 8 = 0$

     (c) $z - 9\sqrt{z} + 14 = 0$

     (d) $t^4 - 4 = 0$

Solution

(a) $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

Okay, in this case we can see that,

$$\frac{2}{3} = 2\left(\frac{1}{3}\right)$$

and so one of the exponents is twice the other so it looks like we've got an equation that is reducible to quadratic in form. The substitution will then be,

$$u = x^{\frac{1}{3}} \qquad\qquad u^2 = \left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$$

Substituting this into the equation gives,

$$u^2 - 2u - 15 = 0$$
$$(u-5)(u+3) = 0 \qquad\Rightarrow\qquad u = -3, \ u = 5$$

Now that we've gotten the solutions for *u* we can find values of *x*.

$$u = -3: \qquad\qquad x^{\frac{1}{3}} = -3 \qquad\Rightarrow\qquad x = (-3)^3 = -27$$

$$u = 5: \qquad\qquad x^{\frac{1}{3}} = 5 \qquad\Rightarrow\qquad x = (5)^3 = 125$$

So, we have two solutions here $x = -27$ and $x = 125$.

(b) $y^{-6} - 9y^{-3} + 8 = 0$

For this part notice that,

$$-6 = 2(-3)$$

and so we do have an equation that is reducible to quadratic form. The substitution is,

$$u = y^{-3} \qquad\qquad u^2 = \left(y^{-3}\right)^2 = y^{-6}$$

The equation becomes,

$$u^2 - 9u + 8 = 0$$
$$(u - 8)(u - 1) = 0 \qquad\qquad u = 1,\ u = 8$$

Now, going back to y's is going to take a little more work here, but shouldn't be too bad.

$$u = 1: \quad \Rightarrow \quad y^{-3} = \frac{1}{y^3} = 1 \quad \Rightarrow \quad y^3 = \frac{1}{1} = 1 \quad \Rightarrow \quad y = (1)^{\frac{1}{3}} = 1$$

$$u = 8: \quad \Rightarrow \quad y^{-3} = \frac{1}{y^3} = 8 \quad \Rightarrow \quad y^3 = \frac{1}{8} \qquad \Rightarrow \quad y = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$$

The two solutions to this equation are $y = 1$ and $y = \dfrac{1}{2}$.

(c) $z - 9\sqrt{z} + 14 = 0$

This one is a little trickier to see that it's quadratic in form, yet it is. To see this recall that the exponent on the square root is one-half, then we can notice that the exponent on the first term is twice the exponent on the second term. So, this equation is in fact reducible to quadratic in form.

Here is the substitution.

$$u = \sqrt{z} \qquad\qquad u^2 = \left(\sqrt{z}\right)^2 = z$$

The equation then becomes,

$$u^2 - 9u + 14 = 0$$
$$(u - 7)(u - 2) = 0 \qquad\qquad u = 2,\ u = 7$$

Now go back to z's.

$$u = 2: \qquad \Rightarrow \qquad \sqrt{z} = 2 \qquad \Rightarrow \qquad z = (2)^2 = 4$$
$$u = 7: \qquad \Rightarrow \qquad \sqrt{z} = 7 \qquad \Rightarrow \qquad z = (7)^2 = 49$$

The two solutions for this equation are $z = 4$ and $z = 49$

(d) $t^4 - 4 = 0$

Now, this part is the exception to the rule that we've been using to identify equations that are reducible to quadratic in form. There is only one term with a t in it. However, notice that we can write the equation as,

$$\left(t^2\right)^2 - 4 = 0$$

So, if we use the substitution,

$$u = t^2 \qquad\qquad u^2 = \left(t^2\right)^2 = t^4$$

the equation becomes,

$$u^2 - 4 = 0$$

and so it is reducible to quadratic in form.

Now, we can solve this using the square root property. Doing that gives,

$$u = \pm\sqrt{4} = \pm 2$$

Now, going back to t's gives us,

$$u = 2: \qquad \Rightarrow \qquad t^2 = 2 \qquad \Rightarrow \qquad t = \pm\sqrt{2}$$
$$u = -2: \qquad \Rightarrow \qquad t^2 = -2 \qquad \Rightarrow \qquad t = \pm\sqrt{-2} = \pm\sqrt{2}\ i$$

In this case we get four solutions and two of them are complex solutions. Getting complex solutions out of these are actually more common that this set of examples might suggest. The problem is that to get some of the complex solutions requires knowledge that we haven't (and won't) cover in this course. So, they don't show up all that often.

All of the examples to this point gave quadratic equations that were factorable or in the case of the last part of the previous example was an equation that we could use the square root property on. That need not always be the case however. It is more than possible that we would need the quadratic formula to do some of these. We should do an example of one of these just to make the point.

Example 3 Solve $2x^{10} - x^5 - 4 = 0$.

Solution
In this case we can reduce this to quadratic in form by using the substitution,

$$u = x^5 \qquad\qquad u^2 = x^{10}$$

Using this substitution the equation becomes,

$$2u^2 - u - 4 = 0$$

This doesn't factor and so we'll need to use the quadratic formula on it. From the quadratic formula the solutions are,

$$u = \frac{1 \pm \sqrt{33}}{4}$$

Now, in order to get back to x's we are going to need decimals values for these so,

$$u = \frac{1 + \sqrt{33}}{4} = 1.68614 \qquad\qquad u = \frac{1 - \sqrt{33}}{4} = -1.18614$$

Now, using the substitution to get back to x's gives the following,

$$u = 1.68614 \qquad\qquad x^5 = 1.68614 \qquad\qquad x = (1.68614)^{\frac{1}{5}} = 1.11014$$

$$u = -1.18614 \qquad\qquad x^5 = -1.18614 \qquad\qquad x = (-1.18614)^{\frac{1}{5}} = -1.03473$$

We had to use a calculator to get the final answer for these. This is one of the reasons that you don't tend to see too many of these done in an Algebra class. The work and/or answers tend to be a little messy.

Section 2-10 : Equations with Radicals

The title of this section is maybe a little misleading. The title seems to imply that we're going to look at equations that involve any radicals. However, we are going to restrict ourselves to equations involving square roots. The techniques we are going to apply here can be used to solve equations with other radicals, however the work is usually significantly messier than when dealing with square roots. Therefore, we will work only with square roots in this section.

Before proceeding it should be mentioned as well that in some Algebra textbooks you will find this section in with the equations reducible to quadratic form material. The reason is that we will in fact end up solving a quadratic equation in most cases. However, the approach is significantly different and so we're going to separate the two topics into different sections in this course.

It is usually best to see how these work with an example.

Example 1 Solve $x = \sqrt{x+6}$.

Solution
In this equation the basic problem is the square root. If that weren't there we could do the problem. The whole process that we're going to go through here is set up to eliminate the square root. However, as we will see, the steps that we're going to take can actually cause problems for us. So, let's see how this all works.

Let's notice that if we just square both sides we can make the square root go away. Let's do that and see what happens.

$$(x)^2 = \left(\sqrt{x+6}\right)^2$$
$$x^2 = x+6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0 \qquad \Rightarrow \qquad x = 3, \ x = -2$$

Upon squaring both sides we see that we get a factorable quadratic equation that gives us two solutions $x = 3$ and $x = -2$.

Now, for no apparent reason, let's do something that we haven't actually done since the section on solving linear equations. Let's check our answers. Remember as well that we need to check the answers in the original equation! That is very important.

Let's first check $x = 3$

$$3 \overset{?}{=} \sqrt{3+6}$$
$$3 = \sqrt{9} \qquad \text{OK}$$

So $x = 3$ is a solution. Now let's check $x = -2$.

$$-2 \overset{?}{=} \sqrt{-2+6}$$

$$-2 \neq \sqrt{4} = 2 \qquad\qquad \text{NOT OK}$$

We have a problem. Recall that square roots are ALWAYS positive and so $x = -2$ does not work in the original equation. One possibility here is that we made a mistake somewhere. We can go back and look however, and we'll quickly see that we haven't made a mistake.

So, what is the deal? Remember that our first step in the solution process was to square both sides. Notice that if we plug $x = -2$ into the quadratic we solved it would in fact be a solution to that. When we squared both sides of the equation we actually changed the equation and, in the process, introduced a solution that is not a solution to the original equation.

With these problems it is vitally important that you check your solutions as this will often happen. When this does we only take the values that are actual solutions to the original equation.

So, the original equation had a single solution $x = 3$.

Now, as this example has shown us, we have to be very careful in solving these equations. When we solve the quadratic we will get two solutions and it is possible both of these, one of these, or none of these values to be solutions to the original equation. The only way to know is to check your solutions!

Let's work a couple more examples that are a little more difficult.

Example 2 Solve each of the following equations.

(a) $y + \sqrt{y-4} = 4$

(b) $1 = t + \sqrt{2t - 3}$

(c) $\sqrt{5z + 6} - 2 = z$

Solution

(a) $y + \sqrt{y - 4} = 4$

In this case let's notice that if we just square both sides we're going to have problems.

$$\left(y + \sqrt{y-4}\right)^2 = (4)^2$$

$$y^2 + 2y\sqrt{y-4} + y - 4 = 16$$

Before discussing the problem we've got here let's make sure you can do the squaring that we did above since it will show up on occasion. All that we did here was use the formula

$$(a+b)^2 = a^2 + 2ab + b^2$$

with $a = y$ and $b = \sqrt{y-4}$. You will need to be able to do these because while this may not have worked here we will need to this kind of work in the next set of problems.

Now, just what is the problem with this? Well recall that the point behind squaring both sides in the first problem was to eliminate the square root. We haven't done that. There is still a square root in the problem and we've made the remainder of the problem messier as well.

So, what we're going to need to do here is make sure that we've got a square root all by itself on one side of the equation before squaring. Once that is done we can square both sides and the square root really will disappear.

Here is the correct way to do this problem.

$$\sqrt{y-4} = 4 - y \qquad \text{now square both sides}$$

$$\left(\sqrt{y-4}\right)^2 = (4-y)^2$$

$$y - 4 = 16 - 8y + y^2$$

$$0 = y^2 - 9y + 20$$

$$0 = (y-5)(y-4) \qquad \Rightarrow \qquad y = 4, \ y = 5$$

As with the first example we will need to make sure and check both of these solutions. Again, make sure that you check in the original equation. Once we've square both sides we've changed the problem and so checking there won't do us any good. In fact, checking there could well lead us into trouble.

First $y = 4$.

$$4 + \sqrt{4-4} \overset{?}{=} 4$$

$$4 = 4 \qquad \text{OK}$$

So, that is a solution. Now $y = 5$.

$$5 + \sqrt{5-4} \overset{?}{=} 4$$

$$5 + \sqrt{1} \overset{?}{=} 4$$

$$6 \neq 4 \qquad \text{NOT OK}$$

So, as with the first example we worked there is in fact a single solution to the original equation, $y = 4$.

(b) $1 = t + \sqrt{2t-3}$

Okay, so we will again need to get the square root on one side by itself before squaring both sides.

$$1 - t = \sqrt{2t-3}$$

$$(1-t)^2 = \left(\sqrt{2t-3}\right)^2$$

$$1 - 2t + t^2 = 2t - 3$$

$$t^2 - 4t + 4 = 0$$

$$(t-2)^2 = 0 \qquad \Rightarrow \qquad t = 2$$

So, we have a double root this time. Let's check it to see if it really is a solution to the original equation.

$$1 \overset{?}{=} 2 + \sqrt{2(2) - 3}$$

$$1 \overset{?}{=} 2 + \sqrt{1}$$

$$1 \neq 3$$

So, $t = 2$ isn't a solution to the original equation. Since this was the only possible solution, this means that there are **no solutions** to the original equation. This doesn't happen too often, but it does happen so don't be surprised by it when it does.

(c) $\sqrt{5z + 6} - 2 = z$

This one will work the same as the previous two.

$$\sqrt{5z + 6} = z + 2$$

$$\left(\sqrt{5z + 6}\right)^2 = (z + 2)^2$$

$$5z + 6 = z^2 + 4z + 4$$

$$0 = z^2 - z - 2$$

$$0 = (z - 2)(z + 1) \qquad \Rightarrow \qquad z = -1, \ z = 2$$

Let's check these possible solutions start with $z = -1$.

$$\sqrt{5(-1) + 6} - 2 \overset{?}{=} -1$$

$$\sqrt{1} - 2 \overset{?}{=} -1$$

$$-1 = -1 \qquad\qquad \text{OK}$$

So, that's was a solution. Now let's check $z = 2$.

$$\sqrt{5(2) + 6} - 2 \overset{?}{=} 2$$

$$\sqrt{16} - 2 \overset{?}{=} 2$$

$$4 - 2 = 2 \qquad\qquad \text{OK}$$

This was also a solution.

So, in this case we've now seen an example where both possible solutions are in fact solutions to the original equation as well.

So, as we've seen in the previous set of examples once we get our list of possible solutions anywhere from none to all of them can be solutions to the original equation. Always remember to check your answers!

Okay, let's work one more set of examples that have an added complexity to them. To this point all the equations that we've looked at have had a single square root in them. However, there can be more

than one square root in these equations. The next set of examples is designed to show us how to deal with these kinds of problems.

Example 3 Solve each of the following equations.

      (a) $\sqrt{2x-1} - \sqrt{x-4} = 2$

      (b) $\sqrt{t+7} + 2 = \sqrt{3-t}$

Solution

In both of these there are two square roots in the problem. We will work these in basically the same manner however. The first step is to get one of the square roots by itself on one side of the equation then square both sides. At this point the process is different so we'll see how to proceed from this point once we reach it in the first example.

(a) $\sqrt{2x-1} - \sqrt{x-4} = 2$

So, the first thing to do is get one of the square roots by itself. It doesn't matter which one we get by itself. We'll end up the same solution(s) in the end.

$$\sqrt{2x-1} = 2 + \sqrt{x-4}$$

$$\left(\sqrt{2x-1}\right)^2 = \left(2 + \sqrt{x-4}\right)^2$$

$$2x - 1 = 4 + 4\sqrt{x-4} + x - 4$$

$$2x - 1 = 4\sqrt{x-4} + x$$

Now, we still have a square root in the problem, but we have managed to eliminate one of them. Not only that, but what we've got left here is identical to the examples we worked in the first part of this section. Therefore, we will continue now work this problem as we did in the previous sets of examples.

$$(x-1)^2 = \left(4\sqrt{x-4}\right)^2$$

$$x^2 - 2x + 1 = 16(x-4)$$

$$x^2 - 2x + 1 = 16x - 64$$

$$x^2 - 18x + 65 = 0$$

$$(x-13)(x-5) = 0 \qquad\qquad \Rightarrow \qquad\qquad x = 13, \ x = 5$$

Now, let's check both possible solutions in the original equation. We'll start with $x = 13$

$$\sqrt{2(13)-1} - \sqrt{13-4} \overset{?}{=} 2$$

$$\sqrt{25} - \sqrt{9} \overset{?}{=} 2$$

$$5 - 3 = 2 \qquad\qquad \text{OK}$$

So, the one is a solution. Now let's check $x = 5$.

$$\sqrt{2(5)-1}-\sqrt{5-4}\overset{?}{=}2$$

$$\sqrt{9}-\sqrt{1}\overset{?}{=}2$$

$$3-1=2 \qquad \text{OK}$$

So, they are both solutions to the original equation.

(b) $\sqrt{t+7}+2=\sqrt{3-t}$

In this case we've already got a square root on one side by itself so we can go straight to squaring both sides.

$$\left(\sqrt{t+7}+2\right)^2=\left(\sqrt{3-t}\right)^2$$

$$t+7+4\sqrt{t+7}+4=3-t$$

$$t+11+4\sqrt{t+7}=3-t$$

Next, get the remaining square root back on one side by itself and square both sides again.

$$4\sqrt{t+7}=-8-2t$$

$$\left(4\sqrt{t+7}\right)^2=\left(-8-2t\right)^2$$

$$16(t+7)=64+32t+4t^2$$

$$16t+112=64+32t+4t^2$$

$$0=4t^2+16t-48$$

$$0=4\left(t^2+4t-12\right)$$

$$0=4(t+6)(t-2) \qquad \Rightarrow \qquad t=-6,\ \ t=2$$

Now check both possible solutions starting with $t=2$.

$$\sqrt{2+7}+2\overset{?}{=}\sqrt{3-2}$$

$$\sqrt{9}+2\overset{?}{=}\sqrt{1}$$

$$3+2\neq1 \qquad \text{NOT OK}$$

So, that wasn't a solution. Now let's check $t=-6$.

$$\sqrt{-6+7}+2\overset{?}{=}\sqrt{3-(-6)}$$

$$\sqrt{1}+2\overset{?}{=}\sqrt{9}$$

$$1+2=3 \qquad \text{OK}$$

It looks like in this case we've got a single solution, $t=-6$.

So, when there is more than one square root in the problem we are again faced with the task of checking our possible solutions. It is possible that anywhere from none to all of the possible solutions will in fact be solutions and the only way to know for sure is to check them in the original equation.

Section 2-11 : Linear Inequalities

To this point in this chapter we've concentrated on solving equations. It is now time to switch gears a little and start thinking about solving inequalities. Before we get into solving inequalities we should go over a couple of the basics first.

At this stage of your mathematical career it is assumed that you know that
$$a < b$$
means that a is some number that is strictly less that b. It is also assumed that you know that
$$a \geq b$$
means that a is some number that is either strictly bigger than b or is exactly equal to b. Likewise, it is assumed that you know how to deal with the remaining two inequalities. > (greater than) and $\leq$ (less than or equal to).

What we want to discuss is some notational issues and some subtleties that sometimes get students when they really start working with inequalities.

First, remember that when we say that a is less than b we mean that a is to the left of b on a number line. So,
$$-1000 < 0$$
is a true inequality.

Next, don't forget how to correctly interpret $\leq$ and $\geq$. Both of the following are true inequalities.
$$4 \leq 4 \qquad\qquad -6 \leq 4$$

In the first case 4 is equal to 4 and so it is "less than or equal" to 4. In the second case -6 is strictly less than 4 and so it is "less than or equal" to 4. The most common mistake is to decide that the first inequality is not a true inequality. Also be careful to not take this interpretation and translate it to < and/or >. For instance,
$$4 < 4$$
is not a true inequality since 4 is equal to 4 and not strictly less than 4.

Finally, we will be seeing many **double inequalities** throughout this section and later sections so we can't forget about those. The following is a double inequality.
$$-9 < 5 \leq 6$$

In a double inequality we are saying that both inequalities must be simultaneously true. In this case 5 is definitely greater than -9 and at the same time is less than or equal to 6. Therefore, this double inequality is a true inequality.
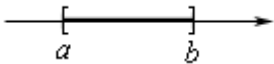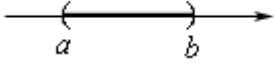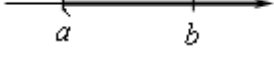
On the other hand,
$$10 \leq 5 < 20$$
is not a true inequality. While it is true that 5 is less than 20 (so the second inequality is true) it is not true that 5 is greater than or equal to 10 (so the first inequality is not true). If even one of the inequalities in a double inequality is not true then the whole inequality is not true. This point is more important than you might realize at this point. In a later section we will run across situations where

many students try to combine two inequalities into a double inequality that simply can't be combined, so be careful.

The next topic that we need to discuss is the idea of **interval notation**. Interval notation is some very nice shorthand for inequalities and will be used extensively in the next few sections of this chapter.

The best way to define interval notation is the following table. There are three columns to the table. Each row contains an inequality, a graph representing the inequality and finally the interval notation for the given inequality.

Inequality	Graph	Interval Notation
$a \leq x \leq b$		$[a,b]$
$a < x < b$		(a,b)
$a \leq x < b$		$[a,b)$
$a < x \leq b$		$(a,b]$
$x > a$		(a,∞)
$x \geq a$		$[a,\infty)$
$x < b$		$(-\infty,b)$
$x \leq b$		$(-\infty,b]$

Remember that a bracket, "[" or "]", means that we include the endpoint while a parenthesis, "(" or ")", means we don't include the endpoint.

Now, with the first four inequalities in the table the interval notation is really nothing more than the graph without the number line on it. With the final four inequalities the interval notation is almost the graph, except we need to add in an appropriate infinity to make sure we get the correct portion of the number line. Also note that infinities NEVER get a bracket. They only get a parenthesis.

We need to give one final note on interval notation before moving on to solving inequalities. Always remember that when we are writing down an interval notation for an inequality that the number on the left must be the smaller of the two.

It's now time to start thinking about solving linear inequalities. We will use the following set of facts in our solving of inequalities. Note that the facts are given for <. We can however, write down an equivalent set of facts for the remaining three inequalities.

1. If $a < b$ then $a + c < b + c$ and $a - c < b - c$ for any number c. In other words, we can add or subtract a number to both sides of the inequality and we don't change the inequality itself.

2. If $a < b$ and $c > 0$ then $ac < bc$ and $\dfrac{a}{c} < \dfrac{b}{c}$. So, provided c is a positive number we can multiply or divide both sides of an inequality by the number without changing the inequality.

3. If $a < b$ and $c < 0$ then $ac > bc$ and $\dfrac{a}{c} > \dfrac{b}{c}$. In this case, unlike the previous fact, if c is negative we need to flip the direction of the inequality when we multiply or divide both sides by the inequality by c.

These are nearly the same facts that we used to solve linear equations. The only real exception is the third fact. This is the important fact as it is often the most misused and/or forgotten fact in solving inequalities.

If you aren't sure that you believe that the sign of c matters for the second and third fact consider the following number example.

$$-3 < 5$$

I hope that we would all agree that this is a true inequality. Now multiply both sides by 2 and by -2.

$$
\begin{array}{cc}
-3 < 5 & -3 < 5 \\
-3(2) < 5(2) & -3(-2) > 5(-2) \\
-6 < 10 & 6 > -10
\end{array}
$$

Sure enough, when multiplying by a positive number the direction of the inequality remains the same, however when multiplying by a negative number the direction of the inequality does change.

Okay, let's solve some inequalities. We will start off with inequalities that only have a single inequality in them. In other words, we'll hold off on solving double inequalities for the next set of examples.

The thing that we've got to remember here is that we're asking to determine all the values of the variable that we can substitute into the inequality and get a true inequality. This means that our solutions will, in most cases, be inequalities themselves.

Example 1 Solving the following inequalities. Give both inequality and interval notation forms of the solution.

    (a) $-2(m - 3) < 5(m + 1) - 12$

    (b) $2(1 - x) + 5 \le 3(2x - 1)$

Solution

Solving single linear inequalities follow pretty much the same process for solving linear equations. We will simplify both sides, get all the terms with the variable on one side and the numbers on the other side, and then multiply/divide both sides by the coefficient of the variable to get the solution. The one thing that you've got to remember is that if you multiply/divide by a negative number then switch the direction of the inequality.

(a) $-2(m-3) < 5(m+1) - 12$

There really isn't much to do here other than follow the process outlined above.

$$-2(m-3) < 5(m+1) - 12$$
$$-2m + 6 < 5m + 5 - 12$$
$$-7m < -13$$
$$m > \frac{13}{7}$$

You did catch the fact that the direction of the inequality changed here didn't you? We divided by a

"-7" and so we had to change the direction. The inequality form of the solution is $m > \dfrac{13}{7}$. The

interval notation for this solution is, $\left(\dfrac{13}{7}, \infty\right)$.

(b) $2(1-x) + 5 \le 3(2x-1)$

Again, not much to do here.

$$2(1-x) + 5 \le 3(2x-1)$$
$$2 - 2x + 5 \le 6x - 3$$
$$10 \le 8x$$
$$\frac{10}{8} \le x$$
$$\frac{5}{4} \le x$$

Now, with this inequality we ended up with the variable on the right side when it more traditionally on the left side. So, let's switch things around to get the variable onto the left side. Note however, that we're going to need also switch the direction of the inequality to make sure that we don't change the answer. So, here is the inequality notation for the inequality.

$$x \ge \frac{5}{4}$$

The interval notation for the solution is $\left[\dfrac{5}{4}, \infty\right)$.

Now, let's solve some double inequalities. The process here is similar in some ways to solving single inequalities and yet very different in other ways. Since there are two inequalities there isn't any way to get the variables on "one side" of the inequality and the numbers on the other. It is easier to see how these work if we do an example or two so let's do that.

Example 2 Solve each of the following inequalities. Give both inequality and interval notation forms for the solution.

    (a) $-6 \le 2(x-5) < 7$

    (b) $-3 < \dfrac{3}{2}(2-x) \le 5$

    (c) $-14 < -7(3x+2) < 1$

Solution

(a) $-6 \le 2(x-5) < 7$

The process here is fairly similar to the process for single inequalities, but we will first need to be careful in a couple of places. Our first step in this case will be to clear any parenthesis in the middle term.

$$-6 \le 2x - 10 < 7$$

Now, we want the *x* all by itself in the middle term and only numbers in the two outer terms. To do this we will add/subtract/multiply/divide as needed. The only thing that we need to remember here is that if we do something to middle term we need to do the same thing to BOTH of the out terms. One of the more common mistakes at this point is to add something, for example, to the middle and only add it to one of the two sides.

Okay, we'll add 10 to all three parts and then divide all three parts by two.

$$4 \le 2x < 17$$

$$2 \le x < \frac{17}{2}$$

That is the inequality form of the answer. The interval notation form of the answer is $\left[2, \dfrac{17}{2} \right)$.

(b) $-3 < \dfrac{3}{2}(2-x) \le 5$

In this case the first thing that we need to do is clear fractions out by multiplying all three parts by 2. We will then proceed as we did in the first part.

$$-6 < 3(2-x) \le 10$$

$$-6 < 6 - 3x \le 10$$

$$-12 < -3x \le 4$$

Now, we're not quite done here, but we need to be very careful with the next step. In this step we need to divide all three parts by -3. However, recall that whenever we divide both sides of an inequality by a negative number we need to switch the direction of the inequality. For us, this means that both of the inequalities will need to switch direction here.

$$4 > x \ge -\frac{4}{3}$$

So, there is the inequality form of the solution. We will need to be careful with the interval notation for the solution. First, the interval notation is NOT $\left(4, -\dfrac{4}{3} \right]$. Remember that in interval notation the smaller number must always go on the left side! Therefore, the correct interval notation for the solution is $\left[-\dfrac{4}{3}, 4 \right)$.

Note as well that this does match up with the inequality form of the solution as well. The inequality is telling us that x is any number between 4 and $-\dfrac{4}{3}$ or possibly $-\dfrac{4}{3}$ itself and this is exactly what the interval notation is telling us.

Also, the inequality could be flipped around to get the smaller number on the left if we'd like to. Here is that form,

$$-\frac{4}{3} \le x < 4$$

When doing this make sure to correctly deal with the inequalities as well.

(c) $-14 < -7\left(3x+2\right) < 1$

Not much to this one. We'll proceed as we've done the previous two.

$$-14 < -21x - 14 < 1$$
$$0 < -21x < 15$$

Don't get excited about the fact that one of the sides is now zero. This isn't a problem. Again, as with the last part, we'll be dividing by a negative number and so don't forget to switch the direction of the inequalities.

$$0 > x > -\frac{15}{21}$$

$$0 > x > -\frac{5}{7} \qquad\qquad \text{OR} \qquad\qquad -\frac{5}{7} < x < 0$$

Either of the inequalities in the second row will work for the solution. The interval notation of the solution is $\left(-\dfrac{5}{7}, 0 \right)$.

When solving double inequalities make sure to pay attention to the inequalities that are in the original problem. One of the more common mistakes here is to start with a problem in which one of the inequalities is < or > and the other is $\le$ or $\ge$, as we had in the first two parts of the previous example, and then by the final answer they are both < or > or they are both $\le$ or $\ge$. In other words, it is easy to all of a sudden make both of the inequalities the same. Be careful with this.

There is one final example that we want to work here.

Example 3 If $-1 < x < 4$ then determine *a* and *b* in $a < 2x + 3 < b$.

Solution
This is easier than it may appear at first. All we are really going to do is start with the given inequality and then manipulate the middle term to look like the second inequality. Again, we'll need to remember that whatever we do to the middle term we'll also need to do to the two outer terms.

So, first we'll multiply everything by 2.
$$-2 < 2x < 8$$
Now add 3 to everything.
$$1 < 2x + 3 < 11$$

We've now got the middle term identical to the second inequality in the problems statement and so all we need to do is pick off *a* and *b*. From this inequality we can see that $a = 1$ and $b = 11$.

Section 2-12 : Polynomial Inequalities

It is now time to look at solving some more difficult inequalities. In this section we will be solving (single) inequalities that involve polynomials of degree at least two. Or, to put it in other words, the polynomials won't be linear any more. Just as we saw when solving equations the process that we have for solving linear inequalities just won't work here.

Since it's easier to see the process as we work an example let's do that. As with the linear inequalities, we are looking for all the values of the variable that will make the inequality true. This means that our solution will almost certainly involve inequalities as well. The process that we're going to go through will give the answers in that form.

Example 1 Solve $x^2 - 10 < 3x$.

Solution
There is a fairly simple process to solving these. If you can remember it you'll always be able to solve these kinds of inequalities.

Step 1 : Get a zero on one side of the inequality. It doesn't matter which side has the zero, however, we're going to be factoring in the next step so keep that in mind as you do this step. Make sure that you've got something that's going to be easy to factor.

$$x^2 - 3x - 10 < 0$$

Step 2 : If possible, factor the polynomial. Note that it won't always be possible to factor this, but that won't change things. This step is really here to simplify the process more than anything. Almost all of the problems that we're going to look at will be factorable.

$$(x - 5)(x + 2) < 0$$

Step 3 : Determine where the polynomial is zero. Notice that these points won't make the inequality true (in this case) because $0 < 0$ is NOT a true inequality. That isn't a problem. These points are going to allow us to find the actual solution.

In our case the polynomial will be zero at $x = -2$ and $x = 5$.

Now, before moving on to the next step let's address why we want these points.

We haven't discussed graphing polynomials yet, however, the graphs of polynomials are nice smooth functions that have no breaks in them. This means that as we are moving across the number line (in any direction) if the value of the polynomial changes sign (say from positive to negative) then it MUST go through zero!

So, that means that these two numbers ($x = 5$ and $x = -2$) are the ONLY places where the polynomial can change sign. The number line is then divided into three regions. In each region if the inequality is satisfied by one point from that region then it is satisfied for ALL points in that region. If

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this wasn't true (*i.e* it was positive at one point in the region and negative at another) then it must also be zero somewhere in that region, but that can't happen as we've already determined all the places where the polynomial can be zero! Likewise, if the inequality isn't satisfied for some point in that region then it isn't satisfied for ANY point in that region.

This leads us into the next step.

Step 4 : Graph the points where the polynomial is zero (*i.e.* the points from the previous step) on a number line and pick a **test point** from each of the regions. Plug each of these test points into the polynomial and determine the sign of the polynomial at that point.

This is the step in the process that has all the work, although it isn't too bad. Here is the number line for this problem.
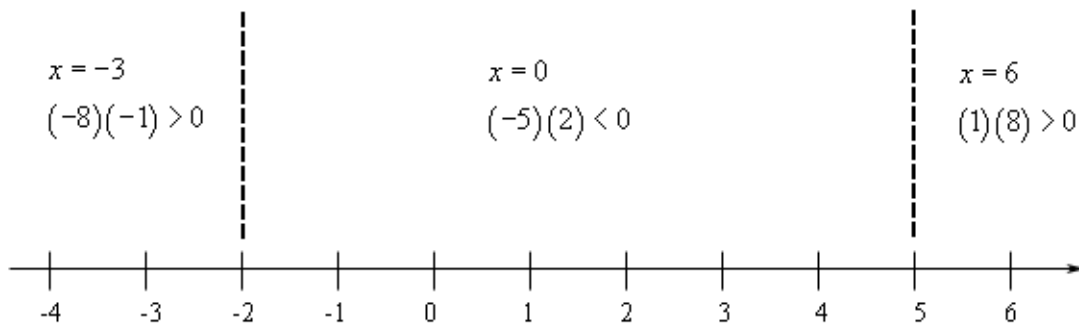
Now, let's talk about this a little. When we pick test points make sure that you pick easy numbers to work with. So, don't choose large numbers or fractions unless you are forced to by the problem.

Also, note that we plugged the test points into the factored from of the polynomial and all we're really after here is whether or not the polynomial is positive or negative. Therefore, we didn't actually bother with values of the polynomial just the sign and we can get that from the product shown. The product of two negatives is a positive, *etc*.

We are now ready for the final step in the process.

Step 5 : Write down the answer. Recall that we discussed earlier that if any point from a region satisfied the inequality then ALL points in that region satisfied the inequality and likewise if any point from a region did not satisfy the inequality then NONE of the points in that region would satisfy the inequality.

This means that all we need to do is look up at the number line above. If the test point from a region satisfies the inequality then that region is part of the solution. If the test point doesn't satisfy the inequality then that region isn't part of the solution.

Now, also notice that any value of *x* that will satisfy the original inequality will also satisfy the inequality from Step 2 and likewise, if an *x* satisfies the inequality from Step 2 then it will satisfy the original inequality.

So, that means that all we need to do is determine the regions in which the polynomial from Step 2 is negative. For this problem that is only the middle region. The inequality and interval notation for the solution to this inequality are,

$$-2 < x < 5 \qquad\qquad (-2, 5)$$

Notice that we do need to exclude the endpoints since we have a strict inequality (< in this case) in the inequality.

Okay, that seems like a long process, however, it really isn't. There was lots of explanation in the previous example. The remaining examples won't be as long because we won't need quite as much explanation in them.

***Example 2*** Solve $x^2 - 5x \geq -6$.

Solution
Okay, this time we'll just go through the process without all the explanations and steps. The first thing to do is get a zero on one side and factor the polynomial if possible.
$$x^2 - 5x + 6 \geq 0$$
$$(x-3)(x-2) \geq 0$$
So, the polynomial will be zero at $x = 2$ and $x = 3$. Notice as well that unlike the previous example, these will be solutions to the inequality since we've got a "greater than or equal to" in the inequality.

Here is the number line for this example.
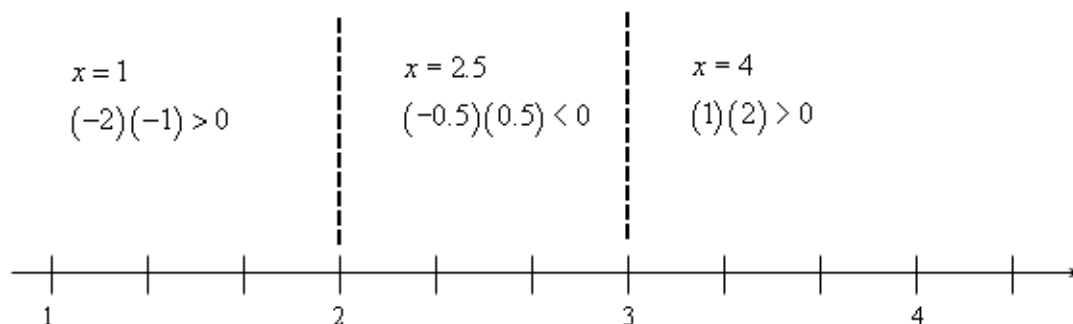
Notice that in this case we were forced to choose a decimal for one of the test points.

Now, we want regions where the polynomial will be positive. So, the first and last regions will be part of the solution. Also, in this case, we've got an "or equal to" in the inequality and so we'll need to include the endpoints in our solution since at this points we get zero for the inequality and $0 \geq 0$ is a true inequality.

Here is the solution in both inequality and interval notation form.
$$-\infty < x \leq 2 \quad \text{and} \quad 3 \leq x < \infty$$
$$(-\infty, 2] \quad \text{and} \quad [3, \infty)$$

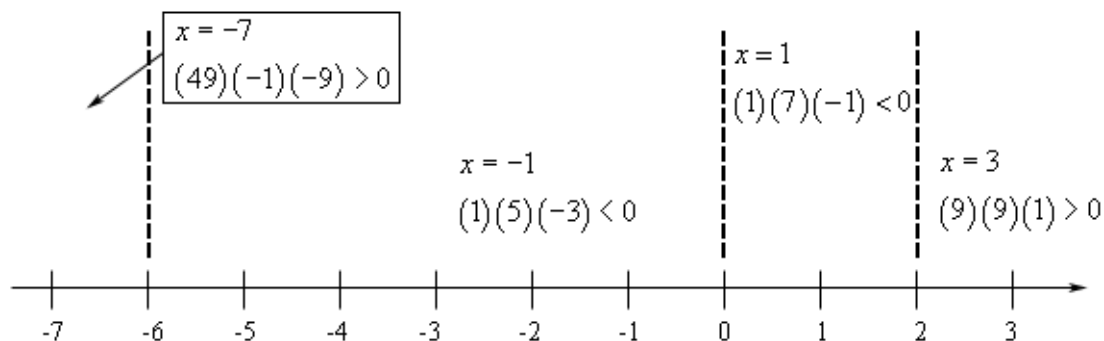**Example 3** Solve $x^4 + 4x^3 - 12x^2 \le 0$.

Solution
Again, we'll just jump right into the problem. We've already got zero on one side so we can go straight to factoring.

$$x^4 + 4x^3 - 12x^2 \le 0$$
$$x^2\left(x^2 + 4x - 12\right) \le 0$$
$$x^2\left(x + 6\right)\left(x - 2\right) \le 0$$

So, this polynomial is zero at $x = -6$, $x = 0$ and $x = 2$. Here is the number line for this problem.

First, notice that unlike the first two examples these regions do NOT alternate between positive and negative. This is a common mistake that students make. You really do need to plug in test points from each region. Don't ever just plug in for the first region and then assume that the other regions will alternate from that point.

Now, for our solution we want regions where the polynomial will be negative (that's the middle two here) or zero (that's all three points that divide the regions). So, we can combine up the middle two regions and the three points into a single inequality in this case. The solution, in both inequality and interval notation form, is.
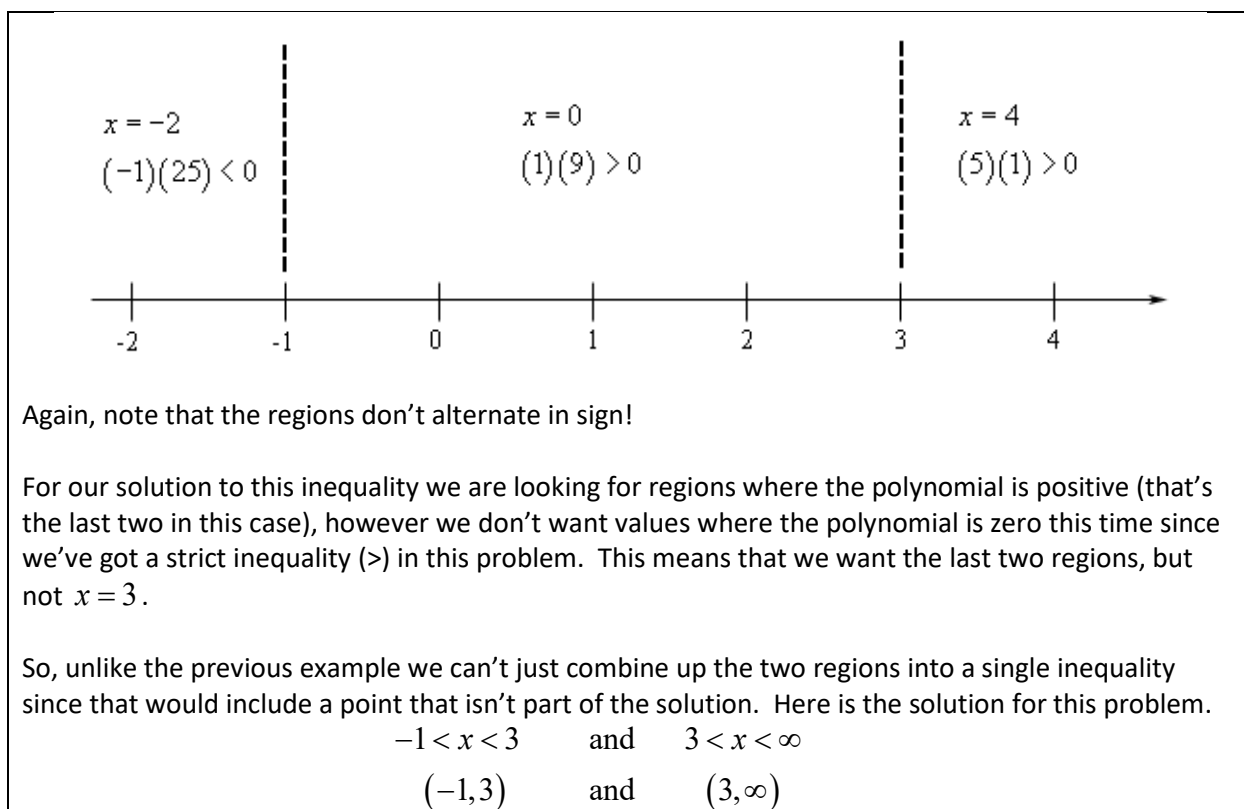
$$-6 \le x \le 2 \qquad\qquad \left[-6, 2\right]$$

Example 4 Solve $(x+1)(x-3)^2 > 0$.

Solution
The first couple of steps have already been done for us here. So, we can just straight into the work. This polynomial will be zero at $x = -1$ and $x = 3$. Here is the number line for this problem.

Again, note that the regions don't alternate in sign!

For our solution to this inequality we are looking for regions where the polynomial is positive (that's the last two in this case), however we don't want values where the polynomial is zero this time since we've got a strict inequality (>) in this problem. This means that we want the last two regions, but not $x = 3$.

So, unlike the previous example we can't just combine up the two regions into a single inequality since that would include a point that isn't part of the solution. Here is the solution for this problem.

$$-1 < x < 3 \qquad \text{and} \qquad 3 < x < \infty$$

$$(-1, 3) \qquad \text{and} \qquad (3, \infty)$$

Now, all of the examples that we've worked to this point involved factorable polynomials. However, that doesn't have to be the case. We can work these inequalities even if the polynomial doesn't factor. We should work one of these just to show you how they work.

Example 5 Solve $3x^2 - 2x - 11 > 0$.
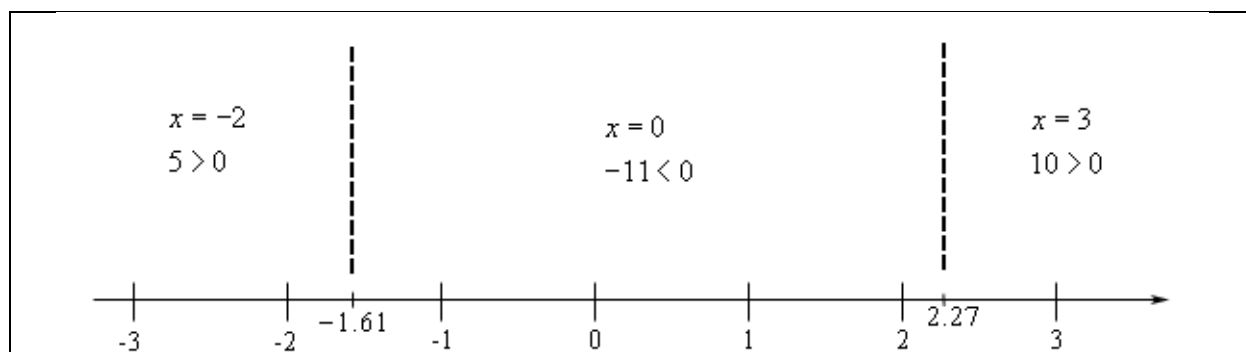
Solution
In this case the polynomial doesn't factor so we can't do that step. However, we do still need to know where the polynomial is zero. We will have to use the quadratic formula for that. Here is what the quadratic formula gives us.

$$x = \frac{1 \pm \sqrt{34}}{3}$$

In order to work the problem we'll need to reduce this to decimals.

$$x = \frac{1 + \sqrt{34}}{3} = 2.27698 \qquad\qquad x = \frac{1 - \sqrt{34}}{3} = -1.61032$$

From this point on the process is identical to the previous examples. In the number line below the dashed lines are at the approximate values of the two decimals above and the inequalities show the value of the quadratic evaluated at the test points shown.

$x = -2$
$5 > 0$

$x = 0$
$-11 < 0$

$x = 3$
$10 > 0$

-1.61   2.27

-3   -2   -1   0   1   2   3

So, it looks like we need the two outer regions for the solution. Here is the inequality and interval notation for the solution.

$$-\infty < x < \frac{1-\sqrt{34}}{3} \qquad \text{and} \qquad \frac{1+\sqrt{34}}{3} < x < \infty$$

$$\left(-\infty, \frac{1-\sqrt{34}}{3}\right) \qquad \text{and} \qquad \left(\frac{1+\sqrt{34}}{3}, \infty\right)$$

Section 2-13 : Rational Inequalities

In this section we will solve inequalities that involve rational expressions. The process for solving rational inequalities is nearly identical to the process for solving polynomial inequalities with a few minor differences.

Let's just jump straight into some examples.

Example 1 Solve $\dfrac{x+1}{x-5} \leq 0$.

Solution

Before we get into solving these we need to point out that these DON'T solve in the same way that we've solve equations that contained rational expressions. With equations the first thing that we always did was clear out the denominators by multiplying by the least common denominator. That won't work with these however.

Since we don't know the value of *x* we can't multiply both sides by anything that contains an *x*. Recall that if we multiply both sides of an inequality by a negative number we will need to switch the direction of the inequality. However, since we don't know the value of *x* we don't know if the denominator is positive or negative and so we won't know if we need to switch the direction of the inequality or not. In fact, to make matters worse, the denominator will be both positive and negative for values of *x* in the solution and so that will create real problems.

So, we need to leave the rational expression in the inequality.

Now, the basic process here is the same as with polynomial inequalities. The first step is to get a zero on one side and write the other side as a single rational inequality. This has already been done for us here.

The next step is to factor the numerator and denominator as much as possible. Again, this has already been done for us in this case.

The next step is to determine where both the numerator and the denominator are zero. In this case these values are.

$$\text{numerator} : x = -1 \qquad\qquad \text{denominator} : x = 5$$

Now, we need these numbers for a couple of reasons. First, just like with polynomial inequalities these are the only numbers where the rational expression *may* change sign. So, we'll build a number line using these points to define ranges out of which to pick test points just like we did with polynomial inequalities.

There is another reason for needing the value of *x* that make the denominator zero however. No matter what else is going on here we do have a rational expression and that means we need to avoid division by zero and so knowing where the denominator is zero will give us the values of *x* to avoid for this.

Here is the number line for this inequality.

$x = -2$

$\dfrac{-1}{-7} > 0$

$x = 0$

$\dfrac{1}{-5} < 0$

$x = 6$

$\dfrac{7}{1} > 0$
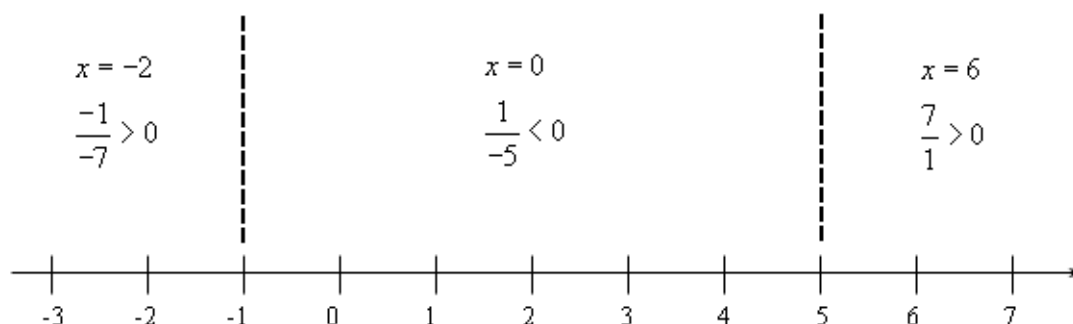
So, we need regions that make the rational expression negative. That means the middle region. Also, since we've got an "or equal to" part in the inequality we also need to include where the inequality is zero, so this means we include $x = -1$. Notice that we will also need to avoid $x = 5$ since that gives division by zero.

The solution for this inequality is,

$$-1 \le x < 5 \qquad\qquad [-1, 5)$$

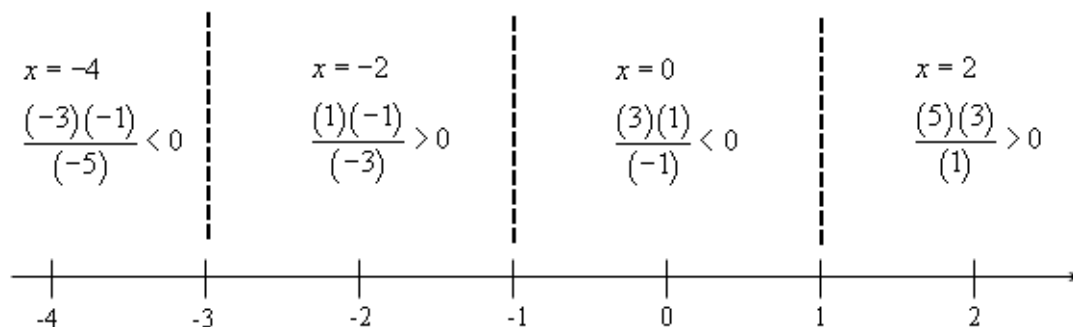***Example 2*** Solve $\dfrac{x^2 + 4x + 3}{x - 1} > 0$.

Solution
We've got zero on one side so let's first factor the numerator and determine where the numerator and denominator are both zero.

$$\frac{(x+1)(x+3)}{x-1} > 0$$

numerator : $x = -1$, $x = -3$ \qquad\qquad denominator : $x = 1$

Here is the number line for this one.

$x = -4$

$\dfrac{(-3)(-1)}{(-5)} < 0$

$x = -2$

$\dfrac{(1)(-1)}{(-3)} > 0$

$x = 0$

$\dfrac{(3)(1)}{(-1)} < 0$

$x = 2$

$\dfrac{(5)(3)}{(1)} > 0$

In the problem we are after values of *x* that make the inequality strictly positive and so that looks like the second and fourth region and we won't include any of the endpoints here. The solution is then,

$$-3 < x < -1 \qquad \text{and} \qquad 1 < x < \infty$$

$$(-3, -1) \qquad \text{and} \qquad (1, \infty)$$

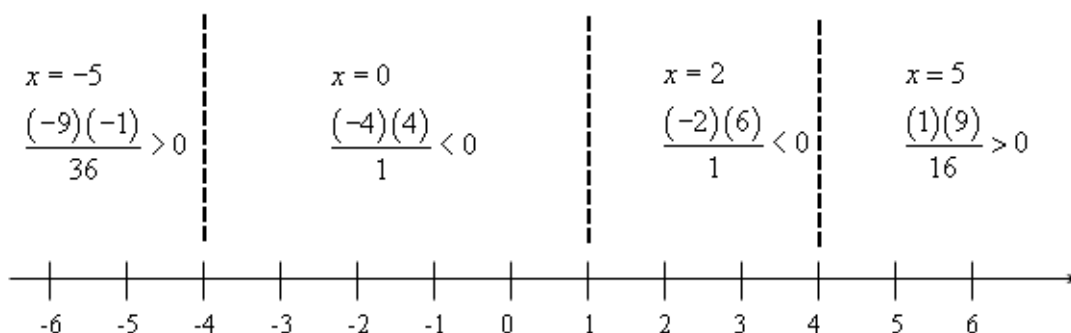**Example 3** Solve $\dfrac{x^2 - 16}{(x-1)^2} < 0$.

Solution

There really isn't too much to this example. We'll first need to factor the numerator and then determine where the numerator and denominator are zero.

$$\frac{(x-4)(x+4)}{(x-1)^2} < 0$$

$$\text{numerator}: x = -4, \quad x = 4 \qquad\qquad \text{denominator}: x = 1$$

The number line for this problem is,

$$x = -5$$
$$\frac{(-9)(-1)}{36} > 0$$

$$x = 0$$
$$\frac{(-4)(4)}{1} < 0$$

$$x = 2$$
$$\frac{(-2)(6)}{1} < 0$$

$$x = 5$$
$$\frac{(1)(9)}{16} > 0$$

So, as with the polynomial inequalities we can not just assume that the regions will always alternate in sign. Also, note that while the middle two regions do give negative values in the rational expression we need to avoid $x = 1$ to make sure we don't get division by zero. This means that we will have to write the answer as two inequalities and/or intervals.

$$-4 < x < 1 \qquad \text{and} \qquad 1 < x < 4$$
$$(-4, 1) \qquad \text{and} \qquad (1, 4)$$

Once again, it's important to note that we really do need to test each region and not just assume that the regions will alternate in sign.

Next, we need to take a look at some examples that don't already have a zero on one side of the inequality.

Example 4 Solve $\dfrac{3x+1}{x+4} \geq 1$.

Solution
The first thing that we need to do here is subtract 1 from both sides and then get everything into a single rational expression.
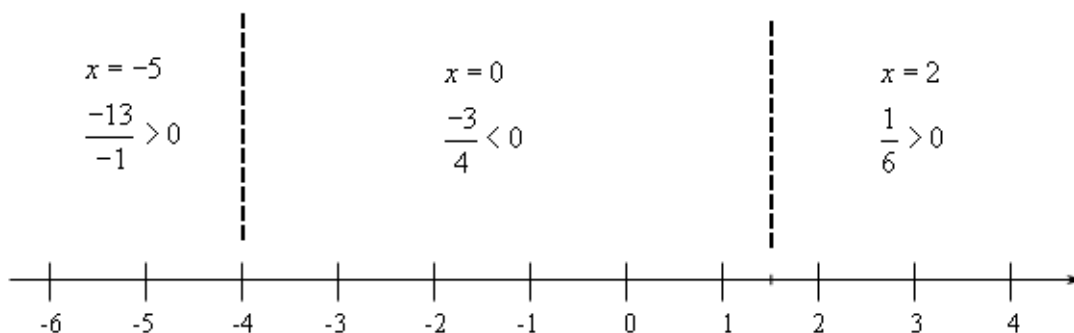
$$\frac{3x+1}{x+4} - 1 \geq 0$$

$$\frac{3x+1}{x+4} - \frac{x+4}{x+4} \geq 0$$

$$\frac{3x+1-(x+4)}{x+4} \geq 0$$

$$\frac{2x-3}{x+4} \geq 0$$

In this case there is no factoring to do so we can go straight to identifying where the numerator and denominator are zero.

$$\text{numerator}: x = \frac{3}{2} \qquad\qquad \text{denominator}: x = -4$$

Here is the number line for this problem.

Okay, we want values of x that give positive and/or zero in the rational expression. This looks like the outer two regions as well as $x = \dfrac{3}{2}$. As with the first example we will need to avoid $x = -4$ since that will give a division by zero error.

The solution for this problem is then,

$$-\infty < x < -4 \qquad \text{and} \qquad \frac{3}{2} \leq x < \infty$$

$$(-\infty, -4) \qquad \text{and} \qquad \left[\frac{3}{2}, \infty\right)$$

Example 5 Solve $\dfrac{x-8}{x} \le 3-x$.

Solution

So, again, the first thing to do is to get a zero on one side and then get everything into a single rational expression.

$$\frac{x-8}{x} + x - 3 \le 0$$

$$\frac{x-8}{x} + \frac{x(x-3)}{x} \le 0$$

$$\frac{x-8+x^2-3x}{x} \le 0$$

$$\frac{x^2-2x-8}{x} \le 0$$

$$\frac{(x-4)(x+2)}{x} \le 0$$

We also factored the numerator above so we can now determine where the numerator and denominator are zero.

$$\text{numerator} : x = -2, \quad x = 4 \qquad\qquad \text{denominator} : x = 0$$

Here is the number line for this problem.
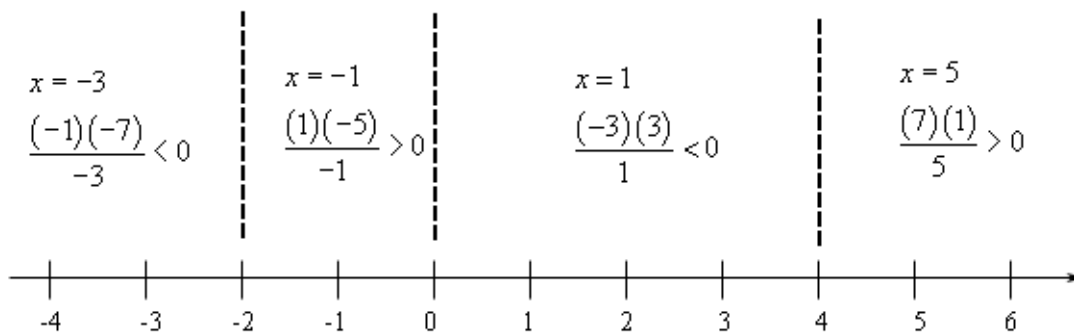
The solution for this inequality is then,

$$-\infty < x \le -2 \qquad\quad \text{and} \qquad\quad 0 < x \le 4$$

$$(-\infty, -2] \qquad\quad \text{and} \qquad\quad (0, 4]$$

# Section 2-14 : Absolute Value Equations

In the final two sections of this chapter we want to discuss solving equations and inequalities that contain absolute values. We will look at equations with absolute value in them in this section and we'll look at inequalities in the next section.

Before solving however, we should first have a brief discussion of just what absolute value is. The notation for the absolute value of p is $|p|$. Note as well that the absolute value bars are NOT parentheses and, in many cases, don't behave as parentheses so be careful with them.

There are two ways to define absolute value. There is a geometric definition and a mathematical definition. We will look at both.

Geometric Definition

In this definition we are going to think of $|p|$ as the distance of p from the origin on a number line. Also, we will always use a positive value for distance. Consider the following number line.
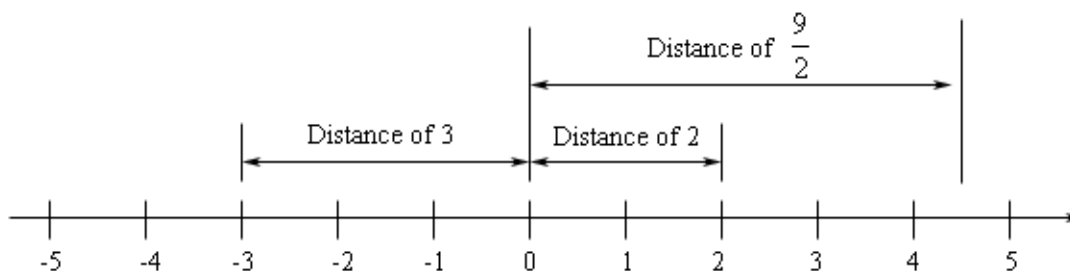
From this we can get the following values of absolute value.

$$|2| = 2 \qquad |-3| = 3 \qquad \left|\frac{9}{2}\right| = \frac{9}{2}$$

All that we need to do is identify the point on the number line and determine its distance from the origin. Note as well that we also have $|0| = 0$.

Mathematical Definition

We can also give a strict mathematical/formula definition for absolute value. It is,

$$|p| = \begin{cases} p & \text{if } p \geq 0 \\ -p & \text{if } p < 0 \end{cases}$$

This tells us to look at the sign of p and if it's positive we just drop the absolute value bar. If p is negative we drop the absolute value bars and then put in a negative in front of it.

So, let's see a couple of quick examples.

$$|4| = 4 \qquad\qquad \text{because } 4 \geq 0$$
$$|-8| = -(-8) = 8 \qquad\qquad \text{because } -8 < 0$$
$$|0| = 0 \qquad\qquad \text{because } 0 \geq 0$$

Note that these give exactly the same value as if we'd used the geometric interpretation.

One way to think of absolute value is that it takes a number and makes it positive. In fact, we can guarantee that,

$$|p| \geq 0$$

regardless of the value of *p*.

We do need to be careful however to not misuse either of these definitions. For example, we can't use the definition on

$$|-x|$$

because we don't know the value of *x*.

Also, don't make the mistake of assuming that absolute value just makes all minus signs into plus signs. In other words, don't make the following mistake,

$$|4x - 3| \neq 4x + 3$$

This just isn't true! If you aren't sure that you believe that then plug in a number for *x*. For example, if $x = -1$ we would get,

$$7 = |-7| = |4(-1) - 3| \neq 4(-1) + 3 = -1$$

There are a couple of problems with this. First, the numbers are clearly not the same and so that's all we really need to prove that the two expressions aren't the same. There is also the fact however that the right number is negative and we will never get a negative value out of an absolute value! That also will guarantee that these two expressions aren't the same.

The definitions above are easy to apply if all we've got are numbers inside the absolute value bars. However, once we put variables inside them we've got to start being very careful.

It's now time to start thinking about how to solve equations that contain absolute values. Let's start off fairly simple and look at the following equation.
$$|p| = 4$$

Now, if we think of this from a geometric point of view this means that whatever *p* is it must have a distance of 4 from the origin. Well there are only two numbers that have a distance of 4 from the origin, namely 4 and -4. So, there are two solutions to this equation,

$$p = -4 \qquad \text{or} \qquad p = 4$$

Now, if you think about it we can do this for any positive number, not just 4. So, this leads to the following general formula for equations involving absolute value.

$$\boxed{\text{If} \quad |p| = b, \; b > 0 \qquad \text{then} \qquad p = -b \; \text{ or } \; p = b}$$

Notice that this does **require** the b be a positive number. We will deal with what happens if b is zero or negative in a bit.

Let's take a look at some examples.

Example 1 Solve each of the following.

    (a) $|2x - 5| = 9$

    (b) $|1 - 3t| = 20$

    (c) $|5y - 8| = 1$

Solution

Now, remember that absolute value does not just make all minus signs into plus signs! To solve these, we've got to use the formula above since in all cases the number on the right side of the equal sign is positive.

(a) $|2x - 5| = 9$

There really isn't much to do here other than using the formula from above as noted above. All we need to note is that in the formula above p represents whatever is on the inside of the absolute value bars and so in this case we have,

$$2x - 5 = -9 \qquad \text{or} \qquad 2x - 5 = 9$$

At this point we've got two linear equations that are easy to solve.

$$
\begin{aligned}
2x &= -4 & \text{or} & & 2x &= 14 \\
x &= -2 & \text{or} & & x &= 7
\end{aligned}
$$

So, we've got two solutions to the equation $x = -2$ and $x = 7$.

(b) $|1 - 3t| = 20$

This one is pretty much the same as the previous part so we won't put as much detail into this one.

$$
\begin{aligned}
1 - 3t &= -20 & \text{or} & & 1 - 3t &= 20 \\
-3t &= -21 & \text{or} & & -3t &= 19 \\
t &= 7 & \text{or} & & t &= -\frac{19}{3}
\end{aligned}
$$

The two solutions to this equation are $t = -\dfrac{19}{3}$ and $t = 7$.

(c) $|5y - 8| = 1$

Again, not much more to this one.

$$5y - 8 = -1 \qquad \text{or} \qquad 5y - 8 = 1$$
$$5y = 7 \qquad \text{or} \qquad 5y = 9$$
$$y = \frac{7}{5} \qquad \text{or} \qquad y = \frac{9}{5}$$

In this case the two solutions are $y = \dfrac{7}{5}$ and $y = \dfrac{9}{5}$.

Now, let's take a look at how to deal with equations for which b is zero or negative. We'll do this with an example.

Example 2 Solve each of the following.
    (a) $|10x - 3| = 0$
    (b) $|5x + 9| = -3$

Solution
(a) Let's approach this one from a geometric standpoint. This is saying that the quantity in the absolute value bars has a distance of zero from the origin. There is only one number that has the property and that is zero itself. So, we must have,

$$10x - 3 = 0 \qquad \Rightarrow \qquad x = \frac{3}{10}$$

In this case we get a single solution.

(b) Now, in this case let's recall that we noted at the start of this section that $|p| \geq 0$. In other words, we can't get a negative value out of the absolute value. That is exactly what this equation is saying however. Since this isn't possible that means there is **no solution** to this equation.

So, summarizing we can see that if b is zero then we can just drop the absolute value bars and solve the equation. Likewise, if b is negative then there will be no solution to the equation.

To this point we've only looked at equations that involve an absolute value being equal to a number, but there is no reason to think that there has to only be a number on the other side of the equal sign. Likewise, there is no reason to think that we can only have one absolute value in the problem. So, we need to take a look at a couple of these kinds of equations.

Example 3 Solve each of the following.

(a) $|x-2| = 3x+1$

(b) $|4x+3| = 3-x$

(c) $|2x-1| = |4x+9|$

Solution

At first glance the formula we used <u>above</u> will do us no good here. It requires the right side of the equation to be a positive number. It turns out that we can still use it here, but we're going to have to be careful with the answers as using this formula will, on occasion introduce an incorrect answer. So, while we can use the formula we'll need to make sure we check our solutions to see if they really work.

(a) $|x-2| = 3x+1$

So, we'll start off using the formula above as we have in the previous problems and solving the two linear equations.

$$x-2 = -(3x+1) = -3x-1 \qquad \text{or} \qquad x-2 = 3x+1$$
$$4x = 1 \qquad\qquad \text{or} \qquad -2x = 3$$
$$x = \frac{1}{4} \qquad\qquad \text{or} \qquad x = -\frac{3}{2}$$

Okay, we've got two potential answers here. There is a problem with the second one however. If we plug this one into the equation we get,

$$\left| -\frac{3}{2} - 2 \right| \overset{?}{=} 3\left(-\frac{3}{2} \right) + 1$$
$$\left| -\frac{7}{2} \right| \overset{?}{=} -\frac{7}{2}$$
$$\frac{7}{2} \neq -\frac{7}{2} \qquad\qquad \text{NOT OK}$$

We get the same number on each side but with opposite signs. This will happen on occasion when we solve this kind of equation with absolute values. Note that we really didn't need to plug the solution into the whole equation here. All we needed to do was check the portion without the absolute value and if it was negative then the potential solution will NOT in fact be a solution and if it's positive or zero it will be solution.

We'll leave it to you to verify that the first potential solution does in fact work and so there is a single solution to this equation : $x = \frac{1}{4}$ and notice that this is less than 2 (as our assumption required) and so is a solution to the equation with the absolute value in it.

So, all together there is a single solution to this equation : $x = \frac{1}{4}$.

(b) $|4x+3| = 3-x$

This one will work in pretty much the same way so we won't put in quite as much explanation.

$$4x+3 = -(3-x) = -3+x \qquad \text{or} \qquad 4x+3 = 3-x$$
$$3x = -6 \qquad\qquad \text{or} \qquad\qquad 5x = 0$$
$$x = -2 \qquad\qquad \text{or} \qquad\qquad x = 0$$

Now, before we check each of these we should give a quick warning. Do not make the assumption that because the first potential solution is negative it won't be a solution. We only exclude a potential solution if it makes the portion without absolute value bars negative. In this case both potential solutions will make the portion without absolute value bars positive and so both are in fact solutions.

So in this case, unlike the first example, we get two solutions : $x = -2$ and $x = 0$.

(c) $|2x-1| = |4x+9|$

This case looks very different from any of the previous problems we've worked to this point and in this case the formula we've been using doesn't really work at all. However, if we think about this a little we can see that we'll still do something similar here to get a solution.

Both sides of the equation contain absolute values and so the only way the two sides are equal will be if the two quantities inside the absolute value bars are equal or equal but with opposite signs. Or in other words, we must have,

$$2x-1 = -(4x+9) = -4x-9 \quad \text{or} \qquad 2x-1 = 4x+9$$
$$6x = -8 \qquad\qquad \text{or} \qquad\qquad -2x = 10$$
$$x = -\frac{8}{6} = -\frac{4}{3} \qquad \text{or} \qquad\qquad x = -5$$

Now, we won't need to verify our solutions here as we did in the previous two parts of this problem. Both with be solutions provided we solved the two equations correctly. However, it will probably be a good idea to verify them anyway just to show that the solution technique we used here really did work properly.

Let's first check $x = -\frac{4}{3}$.

$$\left|2\left(-\frac{4}{3}\right)-1\right| \overset{?}{=} \left|4\left(-\frac{4}{3}\right)+9\right|$$
$$\left|-\frac{11}{3}\right| \overset{?}{=} \left|\frac{11}{3}\right|$$
$$\frac{11}{3} = \frac{11}{3} \qquad\qquad \text{OK}$$

In the case the quantities inside the absolute value were the same number but opposite signs.

However, upon taking the absolute value we got the same number and so $x = -\dfrac{4}{3}$ is a solution.

Now, let's check $x = -5$.

$$\left|2(-5)-1\right| \overset{?}{=} \left|4(-5)+9\right|$$

$$\left|-11\right| \overset{?}{=} \left|-11\right|$$

$$11 = 11 \qquad \text{OK}$$

In the case we got the same value inside the absolute value bars.

So, as suggested above both answers did in fact work and both are solutions to the equation.

So, as we've seen in the previous set of examples we need to be a little careful if there are variables on both sides of the equal sign. If one side does not contain an absolute value then we need to look at the two potential answers and make sure that each is in fact a solution.

Section 2-15 : Absolute Value Inequalities

In the previous section we solved equations that contained absolute values. In this section we want to look at inequalities that contain absolute values. We will need to examine two separate cases.

Inequalities Involving < and $\leq$

As we did with equations let's start off by looking at a fairly simple case.

$$|p| \leq 4$$

This says that no matter what p is it must have a distance of no more than 4 from the origin. This means that p must be somewhere in the range,

$$-4 \leq p \leq 4$$

We could have a similar inequality with the < and get a similar result.

In general, we have the following formulas to use here,

If	$|p| \leq b, \ b > 0$	then	$-b \leq p \leq b$
If	$|p| < b, \ b > 0$	then	$-b < p < b$

Notice that this does **require** b to be positive just as we did with equations.

Let's take a look at a couple of examples.

Example 1 Solve each of the following.

      (a) $|2x - 4| < 10$

      (b) $|9m + 2| \leq 1$

      (c) $|3 - 2z| \leq 5$

Solution

(a) $|2x - 4| < 10$

There really isn't much to do other than plug into the formula. As with equations p simply represents whatever is inside the absolute value bars. So, with this first one we have,

$$-10 < 2x - 4 < 10$$

Now, this is nothing more than a fairly simple double inequality to solve so let's do that.

$$-6 < 2x < 14$$

$$-3 < x < 7$$

The interval notation for this solution is $(-3, 7)$.

(b) $|9m+2| \le 1$

Not much to do here.

$$-1 \le 9m+2 \le 1$$
$$-3 \le 9m \le -1$$
$$-\frac{1}{3} \le m \le -\frac{1}{9}$$

The interval notation is $\left[-\frac{1}{3}, -\frac{1}{9} \right]$.

(c) $|3-2z| \le 5$

We'll need to be a little careful with solving the double inequality with this one, but other than that it is pretty much identical to the previous two parts.

$$-5 \le 3-2z \le 5$$
$$-8 \le -2z \le 2$$
$$4 \ge z \ge -1$$

In the final step don't forget to switch the direction of the inequalities since we divided everything by a negative number. The interval notation for this solution is $[-1, 4]$.

Inequalities Involving > and $\ge$

Once again let's start off with a simple number example.

$$|p| \ge 4$$

This says that whatever p is it must be at least a distance of 4 from the origin and so p must be in one of the following two ranges,

$$p \le -4 \qquad \text{or} \qquad p \ge 4$$

Before giving the general solution we need to address a common mistake that students make with these types of problems. Many students try to combine these into a single double inequality as follows,

$$-4 \ge p \ge 4$$

While this may seem to make sense we can't stress enough that THIS IS NOT CORRECT!! Recall what a double inequality says. In a double inequality we require that both of the inequalities be satisfied simultaneously. The double inequality above would then mean that p is a number that is simultaneously smaller than -4 and larger than 4. This just doesn't make sense. There is no number that satisfies this.

These solutions must be written as two inequalities.

Here is the general formula for these.

$$\text{If} \quad |p| \geq b, \ b > 0 \qquad \text{then} \qquad p \leq -b \quad \text{or} \quad p \geq b$$
$$\text{If} \quad |p| > b, \ b > 0 \qquad \text{then} \qquad p < -b \quad \text{or} \quad p > b$$

Again, we will **require** that *b* be a positive number here. Let's work a couple of examples.

Example 2 Solve each of the following.

      (a) $|2x - 3| > 7$

      (b) $|6t + 10| \geq 3$

      (c) $|2 - 6y| > 10$

Solution

(a) $|2x - 3| > 7$

Again, *p* represents the quantity inside the absolute value bars so all we need to do here is plug into the formula and then solve the two linear inequalities.

$$2x - 3 < -7 \qquad \text{or} \qquad 2x - 3 > 7$$
$$2x < -4 \qquad \text{or} \qquad 2x > 10$$
$$x < -2 \qquad \text{or} \qquad x > 5$$

The interval notation for these are $(-\infty, -2)$ or $(5, \infty)$.

(b) $|6t + 10| \geq 3$

Let's just plug into the formulas and go here,

$$6t + 10 \leq -3 \qquad \text{or} \qquad 6t + 10 \geq 3$$
$$6t \leq -13 \qquad \text{or} \qquad 6t \geq -7$$
$$t \leq -\frac{13}{6} \qquad \text{or} \qquad t \geq -\frac{7}{6}$$

The interval notation for these are $\left(-\infty, -\frac{13}{6} \right]$ or $\left[-\frac{7}{6}, \infty \right)$.

(c) $|2 - 6y| > 10$

Again, not much to do here.

$$2 - 6y < -10 \qquad \text{or} \qquad 2 - 6y > 10$$
$$-6y < -12 \qquad \text{or} \qquad -6y > 8$$
$$y > 2 \qquad \text{or} \qquad y < -\frac{4}{3}$$

Notice that we had to switch the direction of the inequalities when we divided by the negative

number! The interval notation for these solutions is $(2, \infty)$ or $\left(-\infty, -\frac{4}{3} \right)$.

Okay, we next need to take a quick look at what happens if *b* is zero or negative. We'll do these with a set of examples and let's start with zero.

Example 3 Solve each of the following.

      (a) $|3x + 2| < 0$

      (b) $|x - 9| \leq 0$

      (c) $|2x - 4| \geq 0$

      (d) $|3x - 9| > 0$

Solution
These four examples seem to cover all our bases.

(a) Now we know that $|p| \geq 0$ and so can't ever be less than zero. Therefore, in this case there is no solution since it is impossible for an absolute value to be strictly less than zero (*i.e.* negative).

(b) This is almost the same as the previous part. We still can't have absolute value be less than zero, however it can be equal to zero. So, this will have a solution only if
$$|x - 9| = 0$$
and we know how to solve this from the previous section.
$$x - 9 = 0 \qquad \Rightarrow \qquad x = 9$$

(c) In this case let's again recall that no matter what *p* is we are guaranteed to have $|p| \geq 0$. This means that no matter what *x* is we can be assured that $|2x - 4| \geq 0$ will be true since absolute values will always be positive or zero.

The solution in this case is all real numbers, or all possible values of *x*. In inequality notation this would be $-\infty < x < \infty$.

(d) This one is nearly identical to the previous part except this time note that we don't want the absolute value to ever be zero. So, we don't care what value the absolute value takes as long as it isn't zero. This means that we just need to avoid value(s) of *x* for which we get,
$$|3x - 9| = 0 \qquad \Rightarrow \qquad 3x - 9 = 0 \qquad \Rightarrow \qquad x = 3$$

The solution in this case is all real numbers except $x = 3$.

Now, let's do a quick set of examples with negative numbers.

Example 4 Solve each of the following.

      (a) $|4x + 15| < -2$ and $|4x + 15| \leq -2$

      (b) $|2x - 9| \geq -8$ and $|2x - 9| > -8$

Solution
Notice that we're working these in pairs, because this time, unlike the previous set of examples the solutions will be the same for each.

Both (all four?) of these will make use of the fact that no matter what p is we are guaranteed to have $|p| \geq 0$. In other words, absolute values are always positive or zero.

(a) Okay, if absolute values are always positive or zero there is no way they can be less than or equal to a negative number.

Therefore, there is no solution for either of these.

(b) In this case if the absolute value is positive or zero then it will always be greater than or equal to a negative number.

The solution for each of these is then all real numbers.