

DIFFERENTIAL EQUATIONS

First Order Differential Equations

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Preface

Here are my notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self-contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.
2. In general, I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.
3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.
4. This is somewhat related to the previous three items but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Chapter 2 : First Order Differential Equations

In this chapter we will look at solving first order differential equations. The most general first order differential equation can be written as,

$$\frac{dy}{dt} = f(y, t) \quad (1)$$

As we will see in this chapter there is no general formula for the solution to (1). What we will do instead is look at several special cases and see how to solve those. We will also look at some of the theory behind first order differential equations as well as some applications of first order differential equations. Below is a list of the topics discussed in this chapter.

Linear Equations – In this section we solve linear first order differential equations, *i.e.* differential equations in the form $y' + p(t)y = g(t)$. We give an in depth overview of the process used to solve this type of differential equation as well as a derivation of the formula needed for the integrating factor used in the solution process.

Separable Equations – In this section we solve separable first order differential equations, *i.e.* differential equations in the form $N(y)y' = M(x)$. We will give a derivation of the solution process to this type of differential equation. We'll also start looking at finding the interval of validity for the solution to a differential equation.

Exact Equations – In this section we will discuss identifying and solving exact differential equations. We will develop a test that can be used to identify exact differential equations and give a detailed explanation of the solution process. We will also do a few more interval of validity problems here as well.

Bernoulli Differential Equations – In this section we solve Bernoulli differential equations, *i.e.* differential equations in the form $y' + p(t)y = q(t)y^n$. This section will also introduce the idea of using a substitution to help us solve differential equations.

Substitutions – In this section we'll pick up where the last section left off and take a look at a couple of other substitutions that can be used to solve some differential equations. In particular we will discuss using solutions to solve differential equations of the form $y' = F\left(\frac{y}{x}\right)$ and $y' = G(ax + by)$.

Intervals of Validity – In this section we will give an in depth look at intervals of validity as well as an answer to the existence and uniqueness question for first order differential equations.

Modeling with First Order Differential Equations – In this section we will use first order differential equations to model physical situations. In particular we will look at mixing problems (modeling the amount of a substance dissolved in a liquid and liquid both enters and exits), population problems (modeling a population under a variety of situations in which the population can enter or exit) and falling objects (modeling the velocity of a falling object under the influence of both gravity and air resistance).

Equilibrium Solutions – In this section we will define equilibrium solutions (or equilibrium points) for autonomous differential equations, $y' = f(y)$. We discuss classifying equilibrium solutions as asymptotically stable, unstable or semi-stable equilibrium solutions.

Euler's Method – In this section we'll take a brief look at a fairly simple method for approximating solutions to differential equations. We derive the formulas used by Euler's Method and give a brief discussion of the errors in the approximations of the solutions.

Section 2-1 : Linear Equations

The first special case of first order differential equations that we will look at is the linear first order differential equation. In this case, unlike most of the first order cases that we will look at, we can actually derive a formula for the general solution. The general solution is derived below. However, we would suggest that you do not memorize the formula itself. Instead of memorizing the formula you should memorize and understand the process that I'm going to use to derive the formula. Most problems are actually easier to work by using the process instead of using the formula.

So, let's see how to solve a linear first order differential equation. Remember as we go through this process that the goal is to arrive at a solution that is in the form $y = y(t)$. It's sometimes easy to lose sight of the goal as we go through this process for the first time.

In order to solve a linear first order differential equation we MUST start with the differential equation in the form shown below. If the differential equation is not in this form then the process we're going to use will not work.

$$\frac{dy}{dt} + p(t)y = g(t) \quad (1)$$

Where both $p(t)$ and $g(t)$ are continuous functions. Recall that a quick and dirty definition of a continuous function is that a function will be continuous provided you can draw the graph from left to right without ever picking up your pencil/pen. In other words, a function is continuous if there are no holes or breaks in it.

Now, we are going to assume that there is some magical function somewhere out there in the world, $\mu(t)$, called an **integrating factor**. Do not, at this point, worry about what this function is or where it came from. We will figure out what $\mu(t)$ is once we have the formula for the general solution in hand.

So, now that we have assumed the existence of $\mu(t)$ multiply everything in (1) by $\mu(t)$. This will give.

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \quad (2)$$

Now, this is where the magic of $\mu(t)$ comes into play. We are going to assume that whatever $\mu(t)$ is, it will satisfy the following.

$$\mu(t)p(t) = \mu'(t) \quad (3)$$

Again do not worry about how we can find a $\mu(t)$ that will satisfy (3). As we will see, provided $p(t)$ is continuous we can find it. So substituting (3) into (2) we now arrive at.

$$\mu(t)\frac{dy}{dt} + \mu'(t)y = \mu(t)g(t) \quad (4)$$

At this point we need to recognize that the left side of (4) is nothing more than the following product rule.

$$\mu(t) \frac{dy}{dt} + \mu'(t)y = (\mu(t)y(t))'$$

So we can replace the left side of (4) with this product rule. Upon doing this (4) becomes

$$(\mu(t)y(t))' = \mu(t)g(t) \quad (5)$$

Now, recall that we are after $y(t)$. We can now do something about that. All we need to do is integrate both sides then use a little algebra and we'll have the solution. So, integrate both sides of (5) to get.

$$\begin{aligned} \int (\mu(t)y(t))' dt &= \int \mu(t)g(t) dt \\ \mu(t)y(t) + c &= \int \mu(t)g(t) dt \end{aligned} \quad (6)$$

Note the constant of integration, c , from the left side integration is included here. It is vitally important that this be included. If it is left out you will get the wrong answer every time.

The final step is then some algebra to solve for the solution, $y(t)$.

$$\begin{aligned} \mu(t)y(t) &= \int \mu(t)g(t) dt - c \\ y(t) &= \frac{\int \mu(t)g(t) dt - c}{\mu(t)} \end{aligned}$$

Now, from a notational standpoint we know that the constant of integration, c , is an unknown constant and so to make our life easier we will absorb the minus sign in front of it into the constant and use a plus instead. This will NOT affect the final answer for the solution. So with this change we have.

$$y(t) = \frac{\int \mu(t)g(t) dt + c}{\mu(t)} \quad (7)$$

Again, changing the sign on the constant will not affect our answer. If you choose to keep the minus sign you will get the same value of c as we do except it will have the opposite sign. Upon plugging in c we will get exactly the same answer.

There is a lot of playing fast and loose with constants of integration in this section, so you will need to get used to it. When we do this we will always try to make it very clear what is going on and try to justify why we did what we did.

So, now that we've got a general solution to (1) we need to go back and determine just what this magical function $\mu(t)$ is. This is actually an easier process than you might think. We'll start with (3).

$$\mu(t)p(t) = \mu'(t)$$

Divide both sides by $\mu(t)$,

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

Now, hopefully you will recognize the left side of this from your Calculus I class as nothing more than the following derivative.

$$(\ln \mu(t))' = p(t)$$

As with the process above all we need to do is integrate both sides to get.

$$\begin{aligned}\ln \mu(t) + k &= \int p(t) dt \\ \ln \mu(t) &= \int p(t) dt + k\end{aligned}$$

You will notice that the constant of integration from the left side, k , had been moved to the right side and had the minus sign absorbed into it again as we did earlier. Also note that we're using k here because we've already used c and in a little bit we'll have both of them in the same equation. So, to avoid confusion we used different letters to represent the fact that they will, in all probability, have different values.

Exponentiate both sides to get $\mu(t)$ out of the natural logarithm.

$$\mu(t) = e^{\int p(t) dt + k}$$

Now, it's time to play fast and loose with constants again. It is inconvenient to have the k in the exponent so we're going to get it out of the exponent in the following way.

$$\begin{aligned}\mu(t) &= e^{\int p(t) dt + k} \\ &= e^k e^{\int p(t) dt}\end{aligned}$$

Recall $x^{a+b} = x^a x^b$!

Now, let's make use of the fact that k is an unknown constant. If k is an unknown constant then so is e^k so we might as well just rename it k and make our life easier. This will give us the following.

$$\mu(t) = k e^{\int p(t) dt} \quad (8)$$

So, we now have a formula for the general solution, (7), and a formula for the integrating factor, (8). We do have a problem however. We've got two unknown constants and the more unknown constants we have the more trouble we'll have later on. Therefore, it would be nice if we could find a way to eliminate one of them (we'll not be able to eliminate both....).

This is actually quite easy to do. First, substitute (8) into (7) and rearrange the constants.

$$\begin{aligned}
 y(t) &= \frac{\int k e^{\int p(t) dt} g(t) dt + c}{k e^{\int p(t) dt}} \\
 &= \frac{k \int e^{\int p(t) dt} g(t) dt + c}{k e^{\int p(t) dt}} \\
 &= \frac{\int e^{\int p(t) dt} g(t) dt + \frac{c}{k}}{e^{\int p(t) dt}}
 \end{aligned}$$

So, (7) can be written in such a way that the only place the two unknown constants show up is a ratio of the two. Then since both c and k are unknown constants so is the ratio of the two constants. Therefore we'll just call the ratio c and then drop k out of (8) since it will just get absorbed into c eventually.

The solution to a linear first order differential equation is then

$$y(t) = \frac{\int \mu(t) g(t) dt + c}{\mu(t)} \quad (9)$$

where,

$$\mu(t) = e^{\int p(t) dt} \quad (10)$$

Now, the reality is that (9) is not as useful as it may seem. It is often easier to just run through the process that got us to (9) rather than using the formula. We will not use this formula in any of our examples. We will need to use (10) regularly, as that formula is easier to use than the process to derive it.

Solution Process

The solution process for a first order linear differential equation is as follows.

1. Put the differential equation in the correct initial form, (1).
2. Find the integrating factor, $\mu(t)$, using (10).
3. Multiply everything in the differential equation by $\mu(t)$ and verify that the left side becomes the product rule $(\mu(t)y(t))'$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution $y(t)$.

Let's work a couple of examples. Let's start by solving the differential equation that we derived back in the [Direction Field](#) section.

Example 1 Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Solution

First, we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that $p(t)=0.196$ and so $\mu(t)$ is then.

$$\mu(t) = e^{\int 0.196 dt} = e^{0.196t}$$

Note that officially there should be a constant of integration in the exponent from the integration. However, we can drop that for exactly the same reason that we dropped the k from (8).

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$e^{0.196t} \frac{dv}{dt} + 0.196e^{0.196t}v = 9.8e^{0.196t}$$

$$(e^{0.196t}v)' = 9.8e^{0.196t}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\int (e^{0.196t}v)' dt = \int 9.8e^{0.196t} dt$$

$$e^{0.196t}v + k = 50e^{0.196t} + c$$

Okay. It's time to play with constants again. We can subtract k from both sides to get.

$$e^{0.196t}v = 50e^{0.196t} + c - k$$

Both c and k are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as c . So, we now have

$$e^{0.196t}v = 50e^{0.196t} + c$$

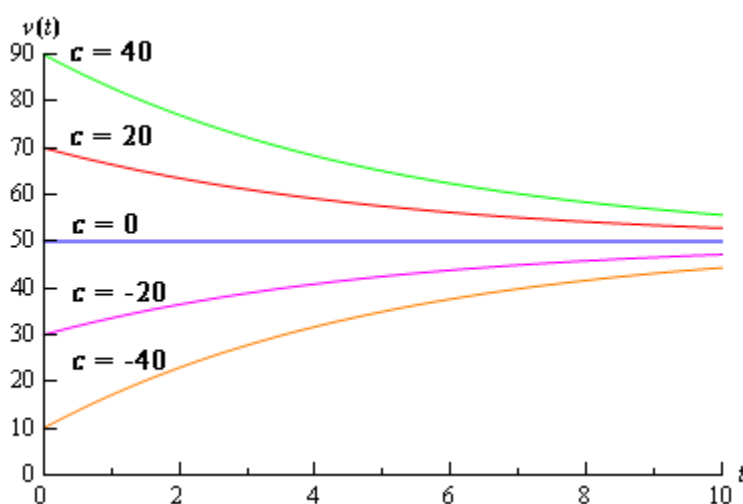
From this point on we will only put one constant of integration down when we integrate both sides knowing that if we had written down one for each integral, as we should, the two would just end up getting absorbed into each other.

The final step in the solution process is then to divide both sides by $e^{0.196t}$ or to multiply both sides by $e^{-0.196t}$. Either will work, but we usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + ce^{-0.196t}$$

From the solution to this example we can now see why the constant of integration is so important in this process. Without it, in this case, we would get a single, constant solution, $v(t)=50$. With the constant of integration we get infinitely many solutions, one for each value of c .

Back in the [direction field](#) section where we first derived the differential equation used in the last example we used the direction field to help us sketch some solutions. Let's see if we got them correct. To sketch some solutions all we need to do is to pick different values of c to get a solution. Several of these are shown in the graph below.



So, it looks like we did pretty good sketching the graphs back in the direction field section.

Now, recall from the Definitions section that the [Initial Condition\(s\)](#) will allow us to zero in on a particular solution. Solutions to first order differential equations (not just linear as we will see) will have a single unknown constant in them and so we will need exactly one initial condition to find the value of that constant and hence find the solution that we were after. The initial condition for first order differential equations will be of the form

$$y(t_0) = y_0$$

Recall as well that a differential equation along with a sufficient number of initial conditions is called an [Initial Value Problem](#) (IVP).

Example 2 Solve the following IVP.

$$\frac{dv}{dt} = 9.8 - 0.196v \quad v(0) = 48$$

Solution

To find the solution to an IVP we must first find the general solution to the differential equation and then use the initial condition to identify the exact solution that we are after. So, since this is the same differential equation as we looked at in Example 1, we already have its general solution.

$$v = 50 + ce^{-0.196t}$$

Now, to find the solution we are after we need to identify the value of c that will give us the solution we are after. To do this we simply plug in the initial condition which will give us an equation we can solve for c . So, let's do this

$$48 = v(0) = 50 + c \quad \Rightarrow \quad c = -2$$

So, the actual solution to the IVP is.

$$v = 50 - 2e^{-0.196t}$$

A graph of this solution can be seen in the figure above.

Let's do a couple of examples that are a little more involved.

Example 3 Solve the following IVP.

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \quad 0 \leq x < \frac{\pi}{2}$$

Solution :

Rewrite the differential equation to get the coefficient of the derivative a one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

Now find the integrating factor.

$$\mu(t) = e^{\int \tan(x) dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)} = \sec(x)$$

Can you do the integral? If not rewrite tangent back into sines and cosines and then use a simple substitution. Note that we could drop the absolute value bars on the secant because of the limits on x . In fact, this is the reason for the limits on x . Note as well that there are two forms of the answer to this integral. They are equivalent as shown below. Which you use is really a matter of preference.

$$\int \tan(x) dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} = \ln|\sec(x)|$$

Also note that we made use of the following fact.

$$e^{\ln f(x)} = f(x) \quad (11)$$

This is an important fact that you should always remember for these problems. We will want to simplify the integrating factor as much as possible in all cases and this fact will help with that simplification.

Now back to the example. Multiply the integrating factor through the differential equation and verify the left side is a product rule. Note as well that we multiply the integrating factor through the rewritten differential equation and NOT the original differential equation. Make sure that you do this. If you multiply the integrating factor through the original differential equation you will get the wrong solution!

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrate both sides.

$$\int (\sec(x)y(x))' dx = \int 2\cos(x)\sin(x) - \sec^2(x) dx$$

$$\sec(x)y(x) = \int \sin(2x) - \sec^2(x) dx$$

$$\sec(x)y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

Note the use of the trig formula $\sin(2\theta) = 2\sin\theta\cos\theta$ that made the integral easier. Next, solve for the solution.

$$\begin{aligned} y(x) &= -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x) \\ &= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x) \end{aligned}$$

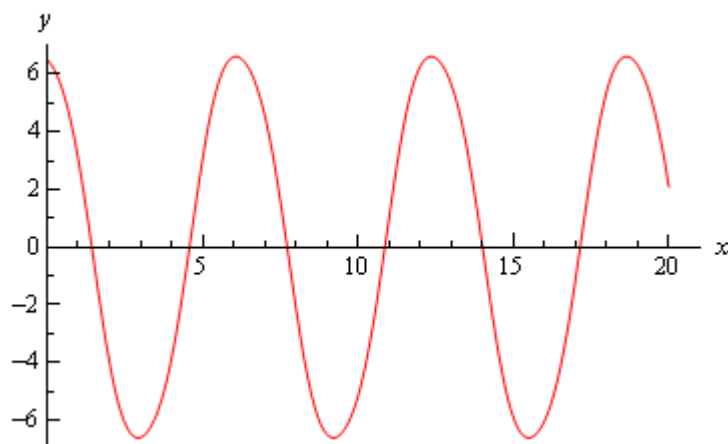
Finally, apply the initial condition to find the value of c .

$$\begin{aligned} 3\sqrt{2} &= y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right) \\ 3\sqrt{2} &= -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2} \\ c &= 7 \end{aligned}$$

The solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Below is a plot of the solution.



Example 4 Find the solution to the following IVP.

$$t y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$$

Solution

First, divide through by the t to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln |t|}$$

Now, we need to simplify $\mu(t)$. However, we can't use (11) yet as that requires a coefficient of one in front of the logarithm. So, recall that

$$\ln x^r = r \ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = e^{2 \ln |t|} = e^{\ln |t|^2} = |t|^2 = t^2$$

We were able to drop the absolute value bars here because we were squaring the t , but often they can't be dropped so be careful with them and don't drop them unless you know that you can. Often the absolute value bars must remain.

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here...) by the integrating factor.

$$(t^2 y)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^2 y &= \int t^3 - t^2 + t \, dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \\ y(t) &= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2} \end{aligned}$$

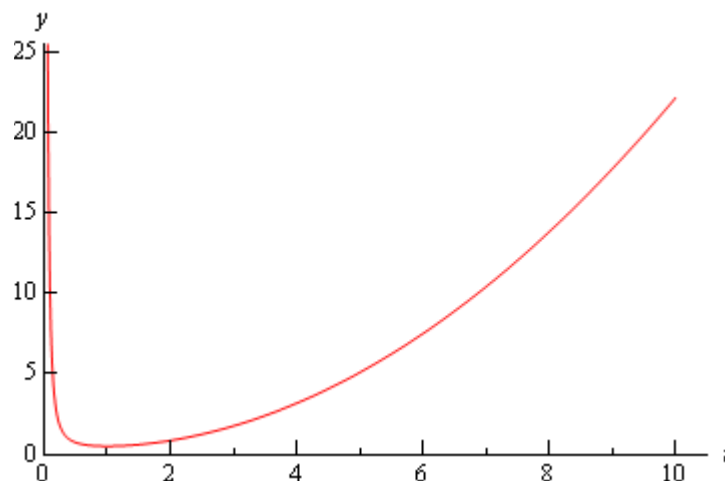
Finally, apply the initial condition to get the value of c .

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \quad \Rightarrow \quad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Here is a plot of the solution.



Example 5 Find the solution to the following IVP.

$$t y' - 2y = t^5 \sin(2t) - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2} \pi^4$$

Solution

First, divide through by t to get the differential equation in the correct form.

$$y' - \frac{2}{t}y = t^4 \sin(2t) - t^2 + 4t^3$$

Now that we have done this we can find the integrating factor, $\mu(t)$.

$$\mu(t) = e^{\int \frac{-2}{t} dt} = e^{-2 \ln|t|}$$

Do not forget that the “-” is part of $p(t)$. Forgetting this minus sign can take a problem that is very easy to do and turn it into a very difficult, if not impossible problem so be careful!

Now, we just need to simplify this as we did in the previous example.

$$\mu(t) = e^{-2 \ln|t|} = e^{\ln|t|^{-2}} = |t|^{-2} = t^{-2}$$

Again, we can drop the absolute value bars since we are squaring the term.

Now multiply the differential equation by the integrating factor (again, make sure it's the rewritten one and not the original differential equation).

$$(t^{-2}y)' = t^2 \sin(2t) - 1 + 4t$$

Integrate both sides and solve for the solution.

$$t^{-2}y(t) = \int t^2 \sin(2t) dt + \int -1 + 4t dt$$

$$t^{-2}y(t) = -\frac{1}{2}t^2 \cos(2t) + \frac{1}{2}t \sin(2t) + \frac{1}{4}\cos(2t) - t + 2t^2 + c$$

$$y(t) = -\frac{1}{2}t^4 \cos(2t) + \frac{1}{2}t^3 \sin(2t) + \frac{1}{4}t^2 \cos(2t) - t^3 + 2t^4 + ct^2$$

Apply the initial condition to find the value of c .

$$\frac{3}{2}\pi^4 = y(\pi) = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2$$

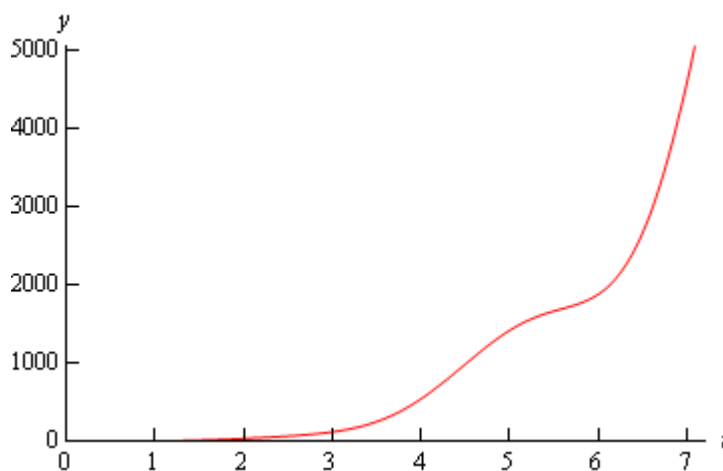
$$\pi^3 - \frac{1}{4}\pi^2 = c\pi^2$$

$$c = \pi - \frac{1}{4}$$

The solution is then

$$y(t) = -\frac{1}{2}t^4 \cos(2t) + \frac{1}{2}t^3 \sin(2t) + \frac{1}{4}t^2 \cos(2t) - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Below is a plot of the solution.



Let's work one final example that looks more at interpreting a solution rather than finding a solution.

Example 6 Find the solution to the following IVP and determine all possible behaviors of the solution as $t \rightarrow \infty$. If this behavior depends on the value of y_0 give this dependence.

$$2y' - y = 4\sin(3t) \quad y(0) = y_0$$

Solution

First, divide through by a 2 to get the differential equation in the correct form.

$$y' - \frac{1}{2}y = 2\sin(3t)$$

Now find $\mu(t)$.

$$\mu(t) = e^{\int -\frac{1}{2} dt} = e^{-\frac{t}{2}}$$

Multiply $\mu(t)$ through the differential equation and rewrite the left side as a product rule.

$$\left(e^{-\frac{t}{2}} y \right)' = 2e^{-\frac{t}{2}} \sin(3t)$$

Integrate both sides (the right side requires integration by parts – you can do that right?) and solve for the solution.

$$e^{-\frac{t}{2}} y = \int 2e^{-\frac{t}{2}} \sin(3t) dt + c$$

$$e^{-\frac{t}{2}} y = -\frac{24}{37} e^{-\frac{t}{2}} \cos(3t) - \frac{4}{37} e^{-\frac{t}{2}} \sin(3t) + c$$

$$y(t) = -\frac{24}{37} \cos(3t) - \frac{4}{37} \sin(3t) + c e^{\frac{t}{2}}$$

Apply the initial condition to find the value of c and note that it will contain y_0 as we don't have a value for that.

$$y_0 = y(0) = -\frac{24}{37} + c \quad \Rightarrow \quad c = y_0 + \frac{24}{37}$$

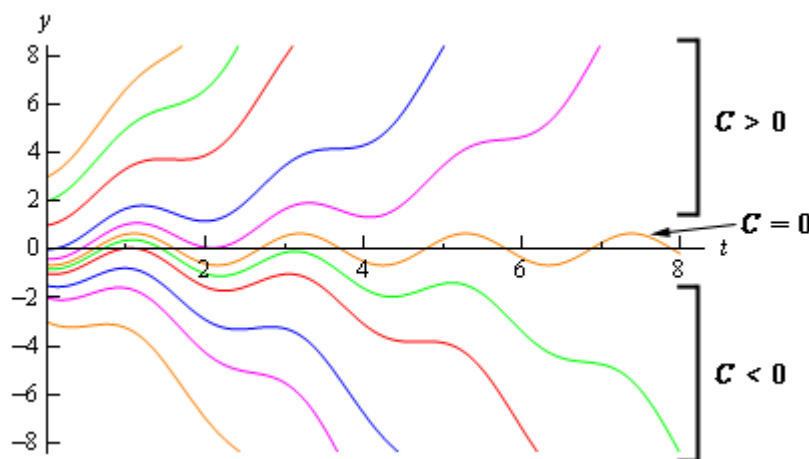
So, the solution is

$$y(t) = -\frac{24}{37} \cos(3t) - \frac{4}{37} \sin(3t) + \left(y_0 + \frac{24}{37} \right) e^{\frac{t}{2}}$$

Now that we have the solution, let's look at the long term behavior (*i.e.* $t \rightarrow \infty$) of the solution. The first two terms of the solution will remain finite for all values of t . It is the last term that will determine the behavior of the solution. The exponential will always go to infinity as $t \rightarrow \infty$, however depending on the sign of the coefficient c (yes we've already found it, but for ease of this discussion we'll continue to call it c). The following table gives the long term behavior of the solution for all values of c .

Range of c	Behavior of solution as $t \rightarrow \infty$
$c < 0$	$y(t) \rightarrow -\infty$
$c = 0$	$y(t)$ remains finite
$c > 0$	$y(t) \rightarrow \infty$

This behavior can also be seen in the following graph of several of the solutions.



Now, because we know how c relates to y_0 we can relate the behavior of the solution to y_0 . The following table give the behavior of the solution in terms of y_0 instead of c .

Range of y_0	Behavior of solution as $t \rightarrow \infty$
$y_0 < -\frac{24}{37}$	$y(t) \rightarrow -\infty$
$y_0 = -\frac{24}{37}$	$y(t)$ remains finite
$y_0 > -\frac{24}{37}$	$y(t) \rightarrow \infty$

Note that for $y_0 = -\frac{24}{37}$ the solution will remain finite. That will not always happen.

Investigating the long term behavior of solutions is sometimes more important than the solution itself. Suppose that the solution above gave the temperature in a bar of metal. In this case we would want the solution(s) that remains finite in the long term. With this investigation we would now have the value of the initial condition that will give us that solution and more importantly values of the initial condition that we would need to avoid so that we didn't melt the bar.

Section 2-2 : Separable Equations

We are now going to start looking at nonlinear first order differential equations. The first type of nonlinear first order differential equations that we will look at is separable differential equations.

A separable differential equation is any differential equation that we can write in the following form.

$$N(y) \frac{dy}{dx} = M(x) \quad (1)$$

Note that in order for a differential equation to be separable all the y 's in the differential equation must be multiplied by the derivative and all the x 's in the differential equation must be on the other side of the equal sign.

To solve this differential equation we first integrate both sides with respect to x to get,

$$\int N(y) \frac{dy}{dx} dx = \int M(x) dx$$

Now, remember that y is really $y(x)$ and so we can use the following substitution,

$$u = y(x) \quad du = y'(x) dx = \frac{dy}{dx} dx$$

Applying this substitution to the integral we get,

$$\int N(u) du = \int M(x) dx \quad (2)$$

At this point we can (hopefully) integrate both sides and then back substitute for the u on the left side. Note, that as implied in the previous sentence, it might not actually be possible to evaluate one or both of the integrals at this point. If that is the case, then there won't be a lot we can do to proceed using this method to solve the differential equation.

Now, the process above is the mathematically correct way of doing solving this differential equation. Note however, that if we "separate" the derivative as well we can write the differential equation as,

$$N(y) dy = M(x) dx$$

We obviously can't separate the derivative like that, but let's pretend we can for a bit and we'll see that we arrive at the answer with less work.

Now we integrate both sides of this to get,

$$\int N(y) dy = \int M(x) dx \quad (3)$$

So, if we compare (2) and (3) we can see that the only difference is on the left side and even then the only real difference is (2) has the integral in terms of u and (3) has the integral in terms of y . Outside of that there is no real difference. The integral on the left is exactly the same integral in each equation. The only difference is the letter used in the integral. If we integrate (2) and then back substitute in for u we would arrive at the same thing as if we'd just integrated (3) from the start.

Therefore, to make the work go a little easier, we'll just use (3) to find the solution to the differential equation. Also, after doing the integrations, we will have an implicit solution that we can hopefully solve for the explicit solution, $y(x)$. Note that it won't always be possible to solve for an explicit solution.

Recall from the Definitions section that an [implicit solution](#) is a solution that is not in the form $y = y(x)$ while an [explicit solution](#) has been written in that form.

We will also have to worry about the [interval of validity](#) for many of these solutions. Recall that the interval of validity was the range of the independent variable, x in this case, on which the solution is valid. In other words, we need to avoid division by zero, complex numbers, logarithms of negative numbers or zero, *etc.* Most of the solutions that we will get from separable differential equations will not be valid for all values of x .

Let's start things off with a fairly simple example so we can see the process without getting lost in details of the other issues that often arise with these problems.

Example 1 Solve the following differential equation and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2x \quad y(1) = \frac{1}{25}$$

Solution

It is clear, hopefully, that this differential equation is separable. So, let's separate the differential equation and integrate both sides. As with the linear first order officially we will pick up a constant of integration on both sides from the integrals on each side of the equal sign. The two can be moved to the same side and absorbed into each other. We will use the convention that puts the single constant on the side with the x 's given that we will eventually be solving for y and so the constant would end up on that side anyway.

$$\begin{aligned} y^{-2} dy &= 6x dx \\ \int y^{-2} dy &= \int 6x dx \\ -\frac{1}{y} &= 3x^2 + c \end{aligned}$$

So, we now have an implicit solution. This solution is easy enough to get an explicit solution, however before getting that it is usually easier to find the value of the constant at this point. So apply the initial condition and find the value of c .

$$-\frac{1}{\cancel{1}/25} = 3(1)^2 + c \quad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$

$$y(x) = \frac{1}{28 - 3x^2}$$

Now, as far as solutions go we've got the solution. We do need to start worrying about intervals of validity however.

Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $x = 1$ in this case.

So, for our case we've got to avoid two values of x . Namely, $x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.05505$ since these will give us division by zero. This gives us three possible intervals of validity.

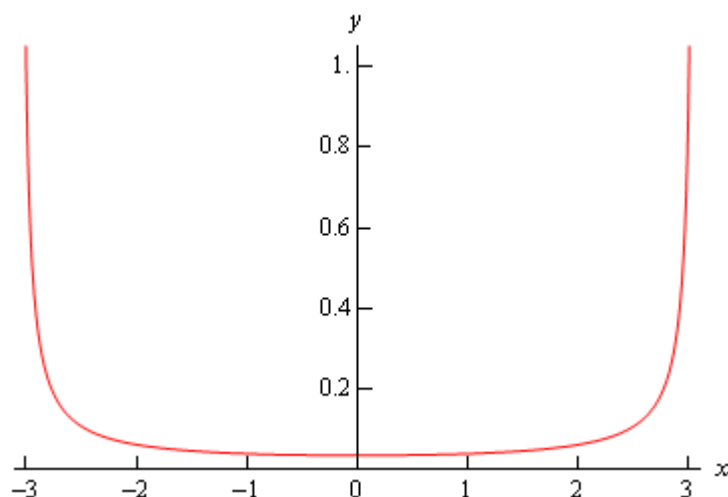
$$-\infty < x < -\sqrt{\frac{28}{3}} \qquad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \qquad \sqrt{\frac{28}{3}} < x < \infty$$

However, only one of these will contain the value of x from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

must be the interval of validity for this solution.

Here is a graph of the solution.



Note that this does not say that either of the other two intervals listed above can't be the interval of validity for any solution to the differential equation. With the proper initial condition either of these could have been the interval of validity.

We'll leave it to you to verify the details of the following claims. If we use an initial condition of

$$y(-4) = -\frac{1}{20}$$

we will get exactly the same solution however in this case the interval of validity would be the first one.

$$-\infty < x < -\sqrt{\frac{28}{3}}$$

Likewise, if we use

$$y(6) = -\frac{1}{80}$$

as the initial condition we again get exactly the same solution and, in this case, the third interval becomes the interval of validity.

$$\sqrt{\frac{28}{3}} < x < \infty$$

So, simply changing the initial condition a little can give any of the possible intervals.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \quad y(1) = 3$$

Solution

This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$\begin{aligned} (2y - 4)dy &= (3x^2 + 4x - 4)dx \\ \int (2y - 4)dy &= \int (3x^2 + 4x - 4)dx \\ y^2 - 4y &= x^3 + 2x^2 - 4x + c \end{aligned}$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of c .

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + c \quad c = -2$$

The implicit solution is then

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First, we need to rewrite the solution a little

$$y^2 - 4y - (x^3 + 2x^2 - 4x - 2) = 0$$

To solve this all we need to recognize is that this is quadratic in y and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant but will in fact involve x 's.

So, upon using the quadratic formula on this we get.

$$y(x) = \frac{4 \pm \sqrt{16 - 4(1)(-(x^3 + 2x^2 - 4x - 2))}}{2}$$

$$= \frac{4 \pm \sqrt{16 + 4(x^3 + 2x^2 - 4x - 2)}}{2}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2...) and then simplify a little.

$$y(x) = \frac{4 \pm 2\sqrt{4 + (x^3 + 2x^2 - 4x - 2)}}{2}$$

$$= 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}$$

We are almost there. Notice that we've actually got two solutions here (the " \pm ") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3 = y(1) = 2 \pm \sqrt{1 + 2 - 4 + 2} = 2 \pm 1 = 3, 1$$

In this case it looks like the "+" is the correct sign for our solution. Note that it is completely possible that the "-" could be the solution (*i.e.* using an initial condition of $y(1) = 1$) so don't always expect it to be one or the other.

The explicit solution for our differential equation is.

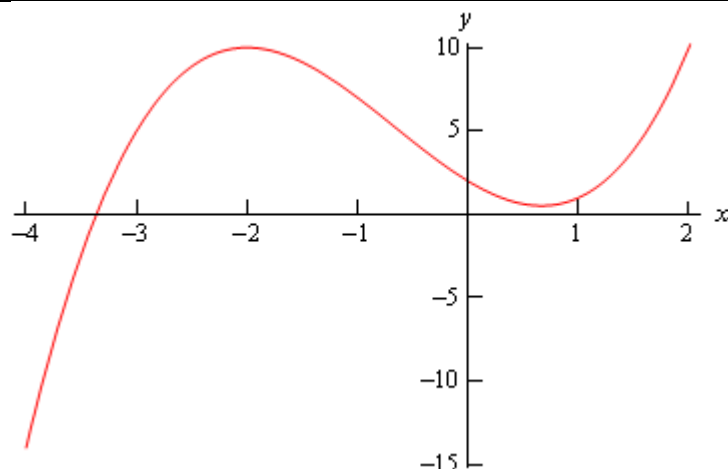
$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

To finish the example out we need to determine the interval of validity for the solution. If we were to put a large negative value of x in the solution we would end up with complex values in our solution and we want to avoid complex numbers in our solutions here. So, we will need to determine which values of x will give real solutions. To do this we will need to solve the following inequality.

$$x^3 + 2x^2 - 4x + 2 \geq 0$$

In other words, we need to make sure that the quantity under the radical stays positive.

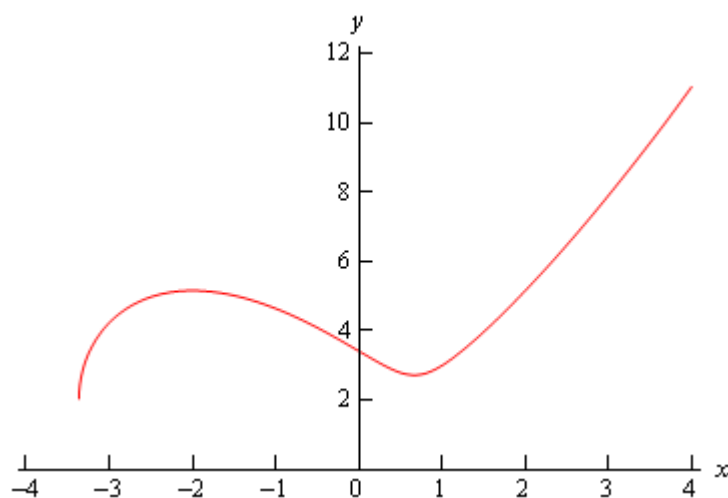
Using a computer algebra system like Maple or Mathematica we see that the left side is zero at $x = -3.36523$ as well as two complex values, but we can ignore complex values for interval of validity computations. Finally, a graph of the quantity under the radical is shown below.



So, in order to get real solutions we will need to require $x \geq -3.36523$ because this is the range of x 's for which the quantity is positive. Notice as well that this interval also contains the value of x that is in the initial condition as it should.

Therefore, the interval of validity of the solution is $x \geq -3.36523$.

Here is graph of the solution.



Example 3 Solve the following IVP and find the interval of validity of the solution.

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$$

Solution

First separate and then integrate both sides.

$$y^{-3} dy = x(1+x^2)^{-\frac{1}{2}} dx$$

$$\int y^{-3} dy = \int x(1+x^2)^{-\frac{1}{2}} dx$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition to get the value of c .

$$-\frac{1}{2} = \sqrt{1} + c \quad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for $y(x)$.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y(x) = \pm \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows us that the “-” is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1+x^2 \geq 0$ the “inner” root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the “outer” root. We can take care of both by requiring

$$3 - 2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2$$

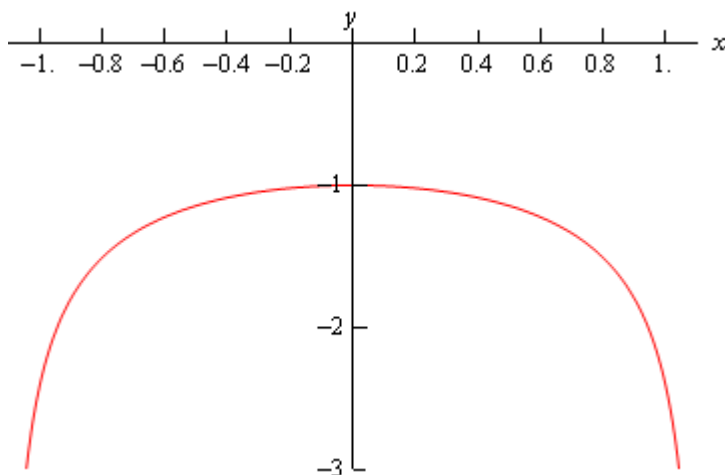
$$\frac{5}{4} > x^2$$

Note that we were able to square both sides of the inequality because both sides of the inequality are guaranteed to be positive in this case. Finally solving for x we see that the only possible range of x 's that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition $x=0$. This interval is therefore our interval of validity.

Here is a graph of the solution.



Example 4 Solve the following IVP and find the interval of validity of the solution.

$$y' = e^{-y}(2x - 4) \quad y(5) = 0$$

Solution

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$e^y dy = (2x - 4) dx$$

$$\int e^y dy = \int (2x - 4) dx$$

$$e^y = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \quad c = -4$$

This then gives an implicit solution of.

$$e^y = x^2 - 4x - 4$$

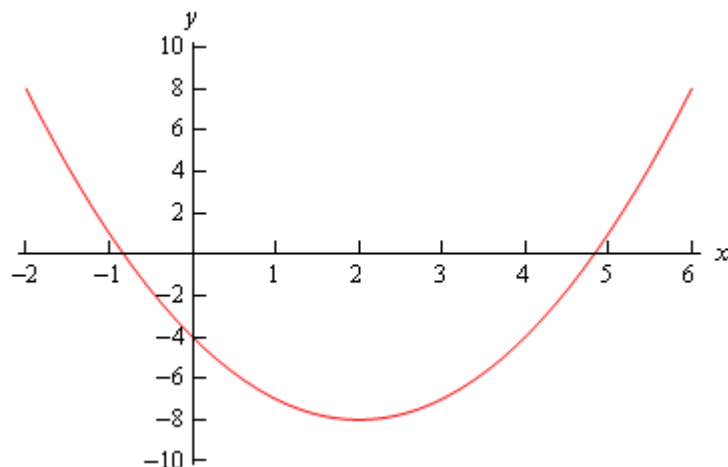
We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y(x) = \ln(x^2 - 4x - 4)$$

Finding the interval of validity is the last step that we need to take. Recall that we can't plug negative values or zero into a logarithm, so we need to solve the following inequality

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at the two points $x = 2 \pm 2\sqrt{2}$. A graph of the quadratic (shown below) shows that there are in fact two intervals in which we will get positive values of the polynomial and hence can be possible intervals of validity.



So, possible intervals of validity are

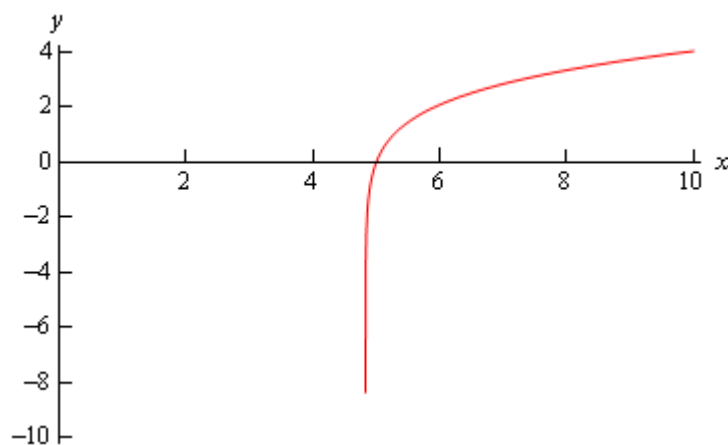
$$-\infty < x < 2 - 2\sqrt{2}$$

$$2 + 2\sqrt{2} < x < \infty$$

From the graph of the quadratic we can see that the second one contains $x = 5$, the value of the independent variable from the initial condition. Therefore, the interval of validity for this solution is.

$$2 + 2\sqrt{2} < x < \infty$$

Here is a graph of the solution.



Example 5 Solve the following IVP and find the interval of validity for the solution.

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \quad r(1) = 2$$

Solution

This is actually a fairly simple differential equation to solve. We're doing this one mostly because of the interval of validity.

So, get things separated out and then integrate.

$$\begin{aligned} \frac{1}{r^2} dr &= \frac{1}{\theta} d\theta \\ \int \frac{1}{r^2} dr &= \int \frac{1}{\theta} d\theta \\ -\frac{1}{r} &= \ln|\theta| + c \end{aligned}$$

Now, apply the initial condition to find c .

$$-\frac{1}{2} = \ln(1) + c \quad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for r gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

Now, there are two problems for our solution here. First, we need to avoid $\theta = 0$ because of the natural log. Notice that because of the absolute value on the θ we don't need to worry about θ being negative. We will also need to avoid division by zero. In other words, we need to avoid the following points.

$$\begin{aligned} \frac{1}{2} - \ln|\theta| &= 0 \\ \ln|\theta| &= \frac{1}{2} && \text{exponentiate both sides} \\ |\theta| &= e^{\frac{1}{2}} \\ \theta &= \pm\sqrt{e} \end{aligned}$$

So, these three points break the number line up into four portions, each of which could be an interval of validity.

$$-\infty < \theta < -\sqrt{e}$$

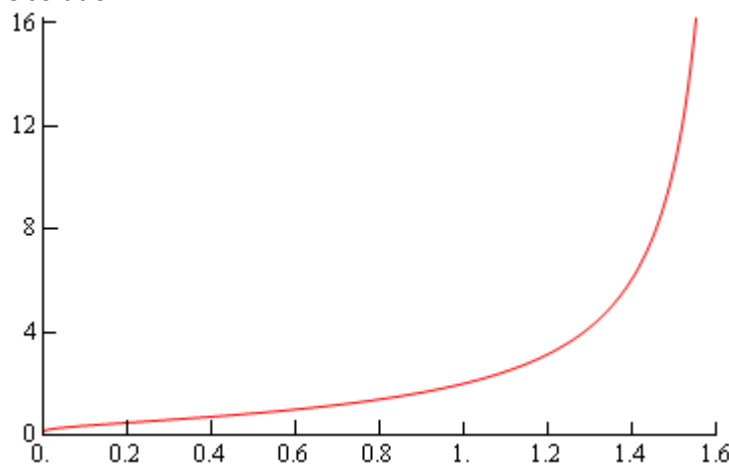
$$-\sqrt{e} < \theta < 0$$

$$0 < \theta < \sqrt{e}$$

$$\sqrt{e} < \theta < \infty$$

The interval that will be the actual interval of validity is the one that contains $\theta = 1$. Therefore, the interval of validity is $0 < \theta < \sqrt{e}$.

Here is a graph of the solution.



Example 6 Solve the following IVP.

$$\frac{dy}{dt} = e^{y-t} \sec(y)(1+t^2) \quad y(0) = 0$$

Solution

This problem will require a little work to get it separated and in a form that we can integrate, so let's do that first.

$$\begin{aligned} \frac{dy}{dt} &= \frac{e^y e^{-t}}{\cos(y)} (1+t^2) \\ e^{-y} \cos(y) dy &= e^{-t} (1+t^2) dt \end{aligned}$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\begin{aligned} \int e^{-y} \cos(y) dy &= \int e^{-t} (1+t^2) dt \\ \frac{e^{-y}}{2} (\sin(y) - \cos(y)) &= -e^{-t} (t^2 + 2t + 3) + c \end{aligned}$$

Applying the initial condition gives.

$$\frac{1}{2}(-1) = -(3) + c \quad c = \frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{e^{-y}}{2}(\sin(y) - \cos(y)) = -e^{-t}(t^2 + 2t + 3) + \frac{5}{2}$$

It is not possible to find an explicit solution for this problem and so we will have to leave the solution in its implicit form. Finding intervals of validity from implicit solutions can often be very difficult so we will also not bother with that for this problem.

As this last example showed it is not always possible to find explicit solutions so be on the lookout for those cases.

Section 2-3 : Exact Equations

The next type of first order differential equations that we'll be looking at is exact differential equations. Before we get into the full details behind solving exact differential equations it's probably best to work an example that will help to show us just what an exact differential equation is. It will also show some of the behind the scenes details that we usually don't bother with in the solution process.

The vast majority of the following example will not be done in any of the remaining examples and the work that we will put into the remaining examples will not be shown in this example. The whole point behind this example is to show you just what an exact differential equation is, how we use this fact to arrive at a solution and why the process works as it does. The majority of the actual solution details will be shown in a later example.

Example 1 Solve the following differential equation.

$$2xy - 9x^2 + (2y + x^2 + 1) \frac{dy}{dx} = 0$$

Solution

Let's start off by supposing that somewhere out there in the world is a function $\Psi(x,y)$ that we can find. For this example the function that we need is

$$\Psi(x,y) = y^2 + (x^2 + 1)y - 3x^3$$

Do not worry at this point about where this function came from and how we found it. Finding the function, $\Psi(x,y)$, that is needed for any particular differential equation is where the vast majority of the work for these problems lies. As stated earlier however, the point of this example is to show you why the solution process works rather than showing you the actual solution process. We will see how to find this function in the next example, so at this point do not worry about how to find it, simply accept that it can be found and that we've done that for this particular differential equation.

Now, take some partial derivatives of the function.

$$\Psi_x = 2xy - 9x^2$$

$$\Psi_y = 2y + x^2 + 1$$

Now, compare these partial derivatives to the differential equation and you'll notice that with these we can now write the differential equation as.

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0 \quad (1)$$

Now, recall from your multi-variable calculus class (probably Calculus III), that (1) is nothing more than the following derivative (you'll need the multi-variable chain rule for this...).

$$\frac{d}{dx}(\Psi(x, y(x)))$$

So, the differential equation can now be written as

$$\frac{d}{dx}(\Psi(x, y(x))) = 0$$

Now, if the ordinary (not partial...) derivative of something is zero, that something must have been a constant to start with. In other words, we've got to have $\Psi(x, y) = c$. Or,

$$y^2 + (x^2 + 1)y - 3x^3 = c$$

This then is an implicit solution for our differential equation! If we had an initial condition we could solve for c . We could also find an explicit solution if we wanted to, but we'll hold off on that until the next example.

Okay, so what did we learn from the last example? Let's look at things a little more generally. Suppose that we have the following differential equation.

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (2)$$

Note that it's important that it must be in this form! There must be an " $= 0$ " on one side and the sign separating the two terms must be a "+". Now, if there is a function somewhere out there in the world, $\Psi(x, y)$, so that,

$$\Psi_x = M(x, y) \quad \text{and} \quad \Psi_y = N(x, y)$$

then we call the differential equation **exact**. In these cases we can write the differential equation as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0 \quad (3)$$

Then using the chain rule from your multi-variable calculus class we can further reduce the differential equation to the following derivative,

$$\frac{d}{dx}(\Psi(x, y(x))) = 0$$

The (implicit) solution to an exact differential equation is then

$$\Psi(x, y) = c \quad (4)$$

Well, it's the solution provided we can find $\Psi(x, y)$ anyway. Therefore, once we have the function we can always just jump straight to (4) to get an implicit solution to our differential equation.

Finding the function $\Psi(x, y)$ is clearly the central task in determining if a differential equation is exact and in finding its solution. As we will see, finding $\Psi(x, y)$ can be a somewhat lengthy process in which there is the chance of mistakes. Therefore, it would be nice if there was some simple test that we could use before even starting to see if a differential equation is exact or not. This will be especially useful if it turns out that the differential equation is not exact, since in this case $\Psi(x, y)$ will not exist. It would be a waste of time to try and find a nonexistent function!

So, let's see if we can find a test for exact differential equations. Let's start with (2) and assume that the differential equation is in fact exact. Since it's exact we know that somewhere out there is a function $\Psi(x,y)$ that satisfies

$$\Psi_x = M$$

$$\Psi_y = N$$

Now, provided $\Psi(x,y)$ is continuous and its first order derivatives are also continuous we know that

$$\Psi_{xy} = \Psi_{yx}$$

However, we also have the following.

$$\Psi_{xy} = (\Psi_x)_y = (M)_y = M_y$$

$$\Psi_{yx} = (\Psi_y)_x = (N)_x = N_x$$

Therefore, if a differential equation is exact and $\Psi(x,y)$ meets all of its continuity conditions we must have.

$$M_y = N_x \quad (5)$$

Likewise, if (5) is not true there is no way for the differential equation to be exact.

Therefore, we will use (5) as a test for exact differential equations. If (5) is true we will assume that the differential equation is exact and that $\Psi(x,y)$ meets all of its continuity conditions and proceed with finding it. Note that for all the examples here the continuity conditions will be met and so this won't be an issue.

Okay, let's go back and rework the first example. This time we will use the example to show how to find $\Psi(x,y)$. We'll also add in an initial condition to the problem.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$2xy - 9x^2 + (2y + x^2 + 1) \frac{dy}{dx} = 0, \quad y(0) = -3$$

Solution

First identify M and N and check that the differential equation is exact.

$$M = 2xy - 9x^2 \quad M_y = 2x$$

$$N = 2y + x^2 + 1 \quad N_x = 2x$$

So, the differential equation is exact according to the test. However, we already knew that as we have given you $\Psi(x,y)$. It's not a bad thing to verify it however and to run through the test at least once however.

Now, how do we actually find $\Psi(x,y)$? Well recall that

$$\Psi_x = M$$

$$\Psi_y = N$$

We can use either of these to get a start on finding $\Psi(x,y)$ by integrating as follows.

$$\Psi = \int M \, dx \qquad \text{OR} \qquad \Psi = \int N \, dy$$

However, we will need to be careful as this won't give us the exact function that we need. Often it doesn't matter which one you choose to work with while in other problems one will be significantly easier than the other. In this case it doesn't matter which one we use as either will be just as easy.

So, we'll use the first one.

$$\Psi(x,y) = \int 2xy - 9x^2 \, dx = x^2y - 3x^3 + h(y)$$

Note that in this case the "constant" of integration is not really a constant at all, but instead it will be a function of the remaining variable(s), y in this case.

Recall that in integration we are asking what function we differentiated to get the function we are integrating. Since we are working with two variables here and talking about partial differentiation with respect to x , this means that any term that contained only constants or y 's would have differentiated away to zero, therefore we need to acknowledge that fact by adding on a function of y instead of the standard c .

Okay, we've got most of $\Psi(x,y)$ we just need to determine $h(y)$ and we'll be done. This is actually easy to do. We used $\Psi_x = M$ to find most of $\Psi(x,y)$ so we'll use $\Psi_y = N$ to find $h(y)$. Differentiate our $\Psi(x,y)$ with respect to y and set this equal to N (since they must be equal after all). Don't forget to "differentiate" $h(y)$! Doing this gives,

$$\Psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$

From this we can see that

$$h'(y) = 2y + 1$$

Note that at this stage $h(y)$ must be only a function of y and so if there are any x 's in the equation at this stage we have made a mistake somewhere and it's time to go look for it.

We can now find $h(y)$ by integrating.

$$h(y) = \int 2y + 1 \, dy = y^2 + y + k$$

You'll note that we included the constant of integration, k , here. It will turn out however that this will end up getting absorbed into another constant so we can drop it in general.

So, we can now write down $\Psi(x,y)$.

$$\Psi(x,y) = x^2y - 3x^3 + y^2 + y + k = y^2 + (x^2 + 1)y - 3x^3 + k$$

With the exception of the k this is identical to the function that we used in the first example. We can now go straight to the implicit solution using (4).

$$y^2 + (x^2 + 1)y - 3x^3 + k = c$$

We'll now take care of the k . Since both k and c are unknown constants all we need to do is subtract one from both sides and combine and we still have an unknown constant.

$$y^2 + (x^2 + 1)y - 3x^3 = c - k$$

$$y^2 + (x^2 + 1)y - 3x^3 = c$$

Therefore, we'll not include the k in anymore problems.

This is where we left off in the first example. Let's now apply the initial condition to find c .

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = c \quad \Rightarrow \quad c = 6$$

The implicit solution is then.

$$y^2 + (x^2 + 1)y - 3x^3 - 6 = 0$$

Now, as we saw in the separable differential equation section, this is quadratic in y and so we can solve for $y(x)$ by using the quadratic formula.

$$\begin{aligned} y(x) &= \frac{-(x^2 + 1) \pm \sqrt{(x^2 + 1)^2 - 4(1)(-3x^3 - 6)}}{2(1)} \\ &= \frac{-(x^2 + 1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2} \end{aligned}$$

Now, reapply the initial condition to figure out which of the two signs in the \pm that we need.

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3, 2$$

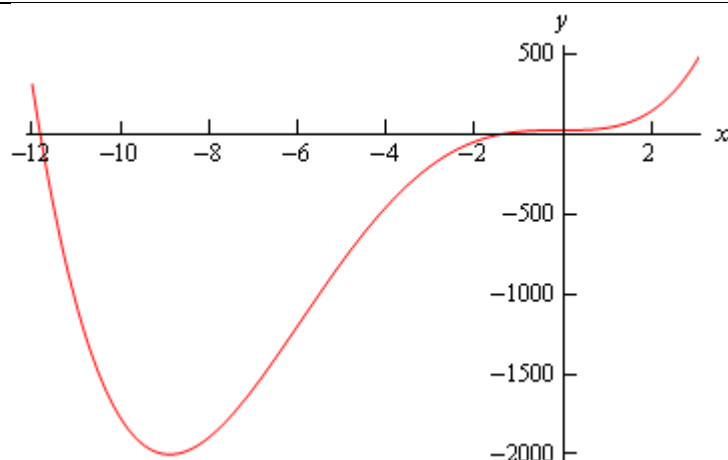
So, it looks like the "-" is the one that we need. The explicit solution is then.

$$y(x) = \frac{-(x^2 + 1) - \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

Now, for the interval of validity. It looks like we might well have problems with square roots of negative numbers. So, we need to solve

$$x^4 + 12x^3 + 2x^2 + 25 = 0$$

Upon solving this equation is zero at $x = -11.81557624$ and $x = -1.396911133$. Note that you'll need to use some form of computational aid in solving this equation. Here is a graph of the polynomial under the radical.



So, it looks like there are two intervals where the polynomial will be positive.

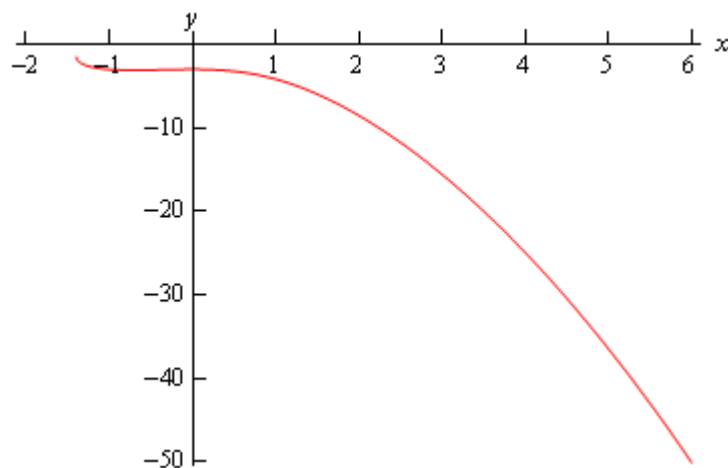
$$-\infty < x \leq -11.81557624$$

$$-1.396911133 \leq x < \infty$$

However, recall that intervals of validity need to be continuous intervals and contain the value of x that is used in the initial condition. Therefore, the interval of validity must be.

$$-1.396911133 \leq x < \infty$$

Here is a quick graph of the solution.



That was a long example, but mostly because of the initial explanation of how to find $\Psi(x, y)$. The remaining examples will not be as long.

Example 3 Find the solution and interval of validity for the following IVP.

$$2xy^2 + 4 = 2(3 - x^2y)y' \quad y(-1) = 8$$

Solution

Here, we first need to put the differential equation into proper form before proceeding. Recall that it needs to be “= 0” and the sign separating the two terms must be a plus!

$$2xy^2 + 4 - 2(3 - x^2y)y' = 0$$

$$2xy^2 + 4 + 2(x^2y - 3)y' = 0$$

So, we have the following

$$M = 2xy^2 + 4 \qquad M_y = 4xy$$

$$N = 2x^2y - 6 \qquad N_x = 4xy$$

and so the differential equation is exact. We can either integrate M with respect to x or integrate N with respect to y . In this case either would be just as easy so we'll integrate N this time so we can say that we've got an example of both down here.

$$\Psi(x, y) = \int 2x^2y - 6 \, dy = x^2y^2 - 6y + h(x)$$

This time, as opposed to the previous example, our "constant" of integration must be a function of x since we integrated with respect to y . Now differentiate with respect to x and compare this to M .

$$\Psi_x = 2xy^2 + h'(x) = 2xy^2 + 4 = M$$

So, it looks like

$$h'(x) = 4 \quad \Rightarrow \quad h(x) = 4x$$

Again, we'll drop the constant of integration that technically should be present in $h(x)$ since it will just get absorbed into the constant we pick up in the next step. Also note that, $h(x)$ should only involve x 's at this point. If there are any y 's left at this point a mistake has been made so go back and look for it.

Writing everything down gives us the following for $\Psi(x, y)$.

$$\Psi(x, y) = x^2y^2 - 6y + 4x$$

So, the implicit solution to the differential equation is

$$x^2y^2 - 6y + 4x = c$$

Applying the initial condition gives,

$$64 - 48 - 4 = c \qquad c = 12$$

The solution is then

$$x^2y^2 - 6y + 4x - 12 = 0$$

Using the quadratic formula gives us

$$\begin{aligned}
 y(x) &= \frac{6 \pm \sqrt{36 - 4x^2(4x - 12)}}{2x^2} \\
 &= \frac{6 \pm \sqrt{36 + 48x^2 - 16x^3}}{2x^2} \\
 &= \frac{6 \pm 2\sqrt{9 + 12x^2 - 4x^3}}{2x^2} \\
 &= \frac{3 \pm \sqrt{9 + 12x^2 - 4x^3}}{x^2}
 \end{aligned}$$

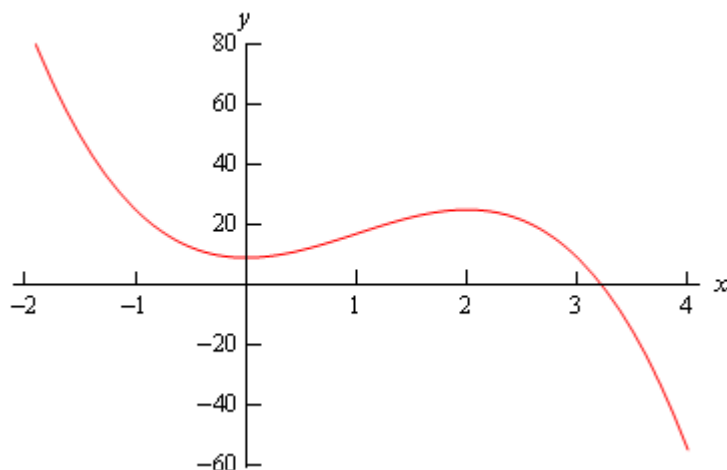
Reapplying the initial condition shows that this time we need the “+” (we’ll leave those details to you to check). Therefore, the explicit solution is

$$y(x) = \frac{3 + \sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Now let’s find the interval of validity. We’ll need to avoid $x = 0$ so we don’t get division by zero. We’ll also have to watch out for square roots of negative numbers so solve the following equation.

$$-4x^3 + 12x^2 + 9 = 0$$

The only real solution here is $x = 3.217361577$. Below is a graph of the polynomial.



So, it looks like the polynomial will be positive, and hence okay under the square root on

$$-\infty < x < 3.217361577$$

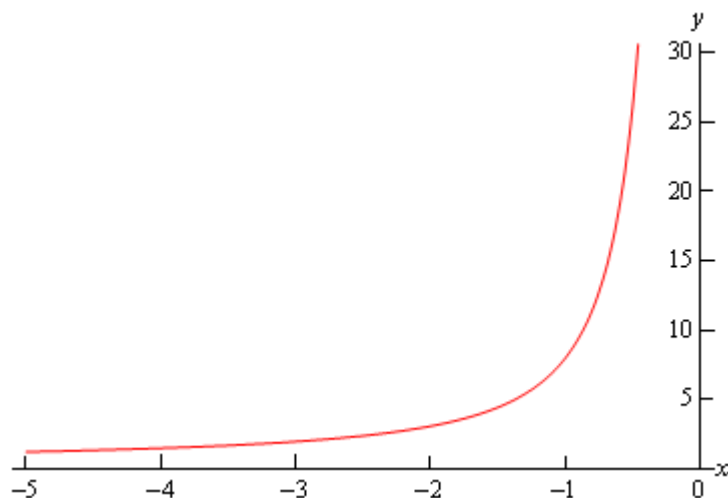
Now, this interval can’t be the interval of validity because it contains $x = 0$ and we need to avoid that point. Therefore, this interval actually breaks up into two different possible intervals of validity.

$$-\infty < x < 0$$

$$0 < x < 3.217361577$$

The first one contains $x = -1$, the x value from the initial condition. Therefore, the interval of validity for this problem is $-\infty < x < 0$.

Here is a graph of the solution.



Example 4 Find the solution and interval of validity to the following IVP.

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 \quad y(5) = 0$$

Solution

So, first deal with that minus sign separating the two terms.

$$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

Now, find M and N and check that it's exact.

$$\begin{aligned} M &= \frac{2ty}{t^2+1} - 2t & M_y &= \frac{2t}{t^2+1} \\ N &= \ln(t^2+1) - 2 & N_t &= \frac{2t}{t^2+1} \end{aligned}$$

So, it's exact. We'll integrate the first one in this case.

$$\Psi(t, y) = \int \frac{2ty}{t^2+1} - 2t \, dt = y \ln(t^2+1) - t^2 + h(y)$$

Differentiate with respect to y and compare to N .

$$\Psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$$

So, it looks like we've got.

$$h'(y) = -2 \quad \Rightarrow \quad h(y) = -2y$$

This gives us

$$\Psi(t, y) = y \ln(t^2+1) - t^2 - 2y$$

The implicit solution is then,

$$y \ln(t^2 + 1) - t^2 - 2y = c$$

Applying the initial condition gives,

$$-25 = c$$

The implicit solution is now,

$$y(\ln(t^2 + 1) - 2) - t^2 = -25$$

This solution is much easier to solve than the previous ones. No quadratic formula is needed this time, all we need to do is solve for y . Here's what we get for an explicit solution.

$$y(t) = \frac{t^2 - 25}{\ln(t^2 + 1) - 2}$$

Alright, let's get the interval of validity. The term in the logarithm is always positive so we don't need to worry about negative numbers in that. We do need to worry about division by zero however. We will need to avoid the following point(s).

$$\ln(t^2 + 1) - 2 = 0$$

$$\ln(t^2 + 1) = 2$$

$$t^2 + 1 = e^2$$

$$t = \pm\sqrt{e^2 - 1}$$

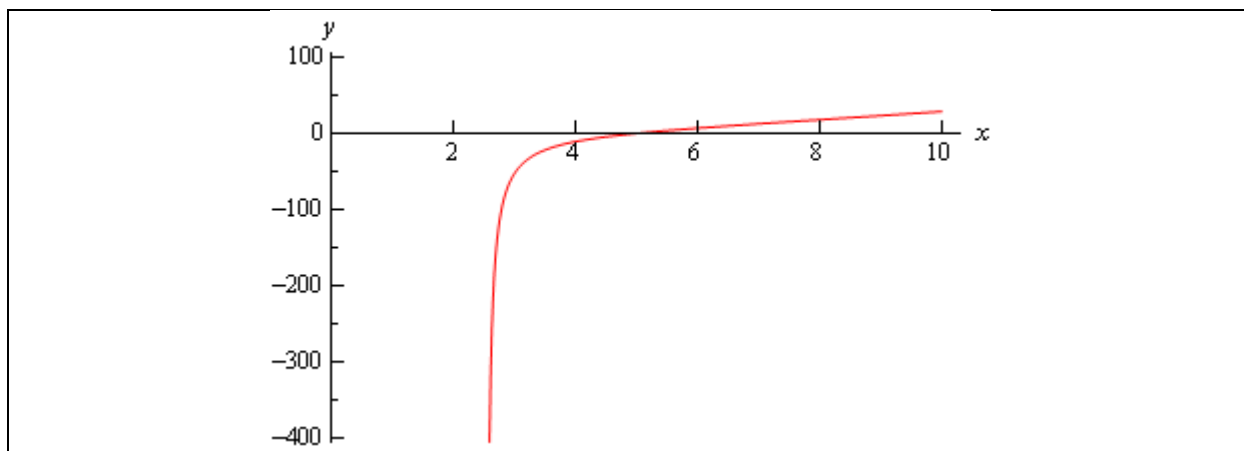
We now have three possible intervals of validity.

$$-\infty < t < -\sqrt{e^2 - 1}$$

$$-\sqrt{e^2 - 1} < t < \sqrt{e^2 - 1}$$

$$\sqrt{e^2 - 1} < t < \infty$$

The last one contains $t = 5$ and so is the interval of validity for this problem is $\sqrt{e^2 - 1} < t < \infty$. Here's a graph of the solution.



Example 5 Find the solution and interval of validity for the following IVP.

$$3y^3 e^{3xy} - 1 + (2ye^{3xy} + 3xy^2 e^{3xy}) y' = 0 \quad y(0) = 1$$

Solution

Let's identify M and N and check that it's exact.

$$M = 3y^3 e^{3xy} - 1 \quad M_y = 9y^2 e^{3xy} + 9xy^3 e^{3xy}$$

$$N = 2ye^{3xy} + 3xy^2 e^{3xy} \quad N_x = 9y^2 e^{3xy} + 9xy^3 e^{3xy}$$

So, it's exact. With the proper simplification integrating the second one isn't too bad. However, the first is already set up for easy integration so let's do that one.

$$\Psi(x, y) = \int 3y^3 e^{3xy} - 1 dx = y^2 e^{3xy} - x + h(y)$$

Differentiate with respect to y and compare to N .

$$\Psi_y = 2ye^{3xy} + 3xy^2 e^{3xy} + h'(y) = 2ye^{3xy} + 3xy^2 e^{3xy} = N$$

So, it looks like we've got

$$h'(y) = 0 \quad \Rightarrow \quad h(y) = 0$$

Recall that actually $h(y) = k$, but we drop the k because it will get absorbed in the next step. That gives us $h(y) = 0$. Therefore, we get.

$$\Psi(x, y) = y^2 e^{3xy} - x$$

The implicit solution is then

$$y^2 e^{3xy} - x = c$$

Apply the initial condition.

$$1 = c$$

The implicit solution is then

$$y^2 e^{3xy} - x = 1$$

This is as far as we can go. There is no way to solve this for y and get an explicit solution.

Section 2-4 : Bernoulli Differential Equations

In this section we are going to take a look at differential equations in the form,

$$y' + p(x)y = q(x)y^n$$

where $p(x)$ and $q(x)$ are continuous functions on the interval we're working on and n is a real number. Differential equations in this form are called **Bernoulli Equations**.

First notice that if $n = 0$ or $n = 1$ then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of n other than these two.

In order to solve these we'll first divide the differential equation by y^n to get,

$$y^{-n} y' + p(x)y^{1-n} = q(x)$$

We are now going to use the substitution $v = y^{1-n}$ to convert this into a differential equation in terms of v . As we'll see this will lead to a differential equation that we can solve.

We are going to have to be careful with this however when it comes to dealing with the derivative, y' . We need to determine just what y' is in terms of our substitution. This is easier to do than it might at first look to be. All that we need to do is differentiate both sides of our substitution with respect to x . Remember that both v and y are functions of x and so we'll need to use the chain rule on the right side. If you remember your Calculus I you'll recall this is just [implicit differentiation](#). So, taking the derivative gives us,

$$v' = (1-n)y^{-n}y'$$

Now, plugging this as well as our substitution into the differential equation gives,

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

This is a [linear differential equation](#) that we can solve for v and once we have this in hand we can also get the solution to the original differential equation by plugging v back into our substitution and solving for y .

Let's take a look at an example.

Example 1 Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{4}{x}y = x^3y^2 \quad y(2) = -1, \quad x > 0$$

Solution

So, the first thing that we need to do is get this into the "proper" form and that means dividing everything by y^2 . Doing this gives,

$$y^{-2} y' + \frac{4}{x} y^{-1} = x^3$$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \qquad v' = -y^{-2} y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x} v = x^3$$

So, as noted above this is a linear differential equation that we know how to solve. We'll do the details on this one and then for the rest of the examples in this section we'll leave the details for you to fill in. If you need a refresher on solving linear differential equations then go back to that [section](#) for a quick review.

Here's the solution to this differential equation.

$$\begin{aligned} v' - \frac{4}{x} v &= -x^3 & \Rightarrow & \mu(x) = e^{\int -\frac{4}{x} dx} = e^{-4 \ln|x|} = x^{-4} \\ \int (x^{-4} v)' dx &= \int -x^{-1} dx \\ x^{-4} v &= -\ln|x| + c & \Rightarrow & v(x) = cx^4 - x^4 \ln x \end{aligned}$$

Note that we dropped the absolute value bars on the x in the logarithm because of the assumption that $x > 0$.

Now we need to determine the constant of integration. This can be done in one of two ways. We can convert the solution above into a solution in terms of y and then use the original initial condition or we can convert the initial condition to an initial condition in terms of v and use that. Because we'll need to convert the solution to y 's eventually anyway and it won't add that much work in we'll do it that way.

So, to get the solution in terms of y all we need to do is plug the substitution back in. Doing this gives,

$$y^{-1} = x^4 (c - \ln x)$$

At this point we can solve for y and then apply the initial condition or apply the initial condition and then solve for y . We'll generally do this with the later approach so let's apply the initial condition to get,

$$(-1)^{-1} = c 2^4 - 2^4 \ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for c and solving for y gives,

$$y(x) = \frac{1}{x^4 \left(\ln 2 - \frac{1}{16} - \ln x \right)} = \frac{-16}{x^4 (1 + 16 \ln x - 16 \ln 2)} = \frac{-16}{x^4 \left(1 + 16 \ln \frac{x}{2} \right)}$$

Note that we did a little simplification in the solution. This will help with finding the interval of validity.

Before finding the interval of validity however, we mentioned above that we could convert the original initial condition into an initial condition for v . Let's briefly talk about how to do that. To do that all we need to do is plug $x = 2$ into the substitution and then use the original initial condition. Doing this gives,

$$v(2) = y^{-1}(2) = (-1)^{-1} = -1$$

So, in this case we got the same value for v that we had for y . Don't expect that to happen in general if you chose to do the problems in this manner.

Okay, let's now find the interval of validity for the solution. First, we already know that $x > 0$ and that means we'll avoid the problems of having logarithms of negative numbers and division by zero at $x = 0$. So, all that we need to worry about then is division by zero in the second term and this will happen where,

$$1 + 16 \ln \frac{x}{2} = 0$$

$$\ln \frac{x}{2} = -\frac{1}{16}$$

$$\frac{x}{2} = e^{-\frac{1}{16}}$$

$$\Rightarrow$$

$$x = 2e^{-\frac{1}{16}} \approx 1.8788$$

The two possible intervals of validity are then,

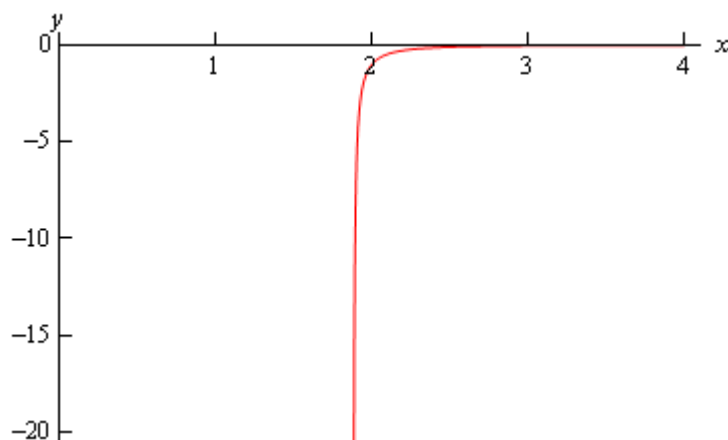
$$0 < x < 2e^{-\frac{1}{16}}$$

$$2e^{-\frac{1}{16}} < x < \infty$$

and since the second one contains the initial condition we know that the interval of validity is then

$$2e^{-\frac{1}{16}} < x < \infty.$$

Here is a graph of the solution.



Let's do a couple more examples and as noted above we're going to leave it to you to solve the linear differential equation when we get to that stage.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$y' = 5y + e^{-2x}y^{-2} \quad y(0) = 2$$

Solution

The first thing we'll need to do here is multiply through by y^2 and we'll also do a little rearranging to get things into the form we'll need for the linear differential equation. This gives,

$$y^2 y' - 5y^3 = e^{-2x}$$

The substitution here and its derivative is,

$$v = y^3 \quad v' = 3y^2 y'$$

Plugging the substitution into the differential equation gives,

$$\frac{1}{3}v' - 5v = e^{-2x} \quad \Rightarrow \quad v' - 15v = 3e^{-2x} \quad \mu(x) = e^{-15x}$$

We rearranged a little and gave the integrating factor for the linear differential equation solution. Upon solving we get,

$$v(x) = ce^{15x} - \frac{3}{17}e^{-2x}$$

Now go back to y 's.

$$y^3 = ce^{15x} - \frac{3}{17}e^{-2x}$$

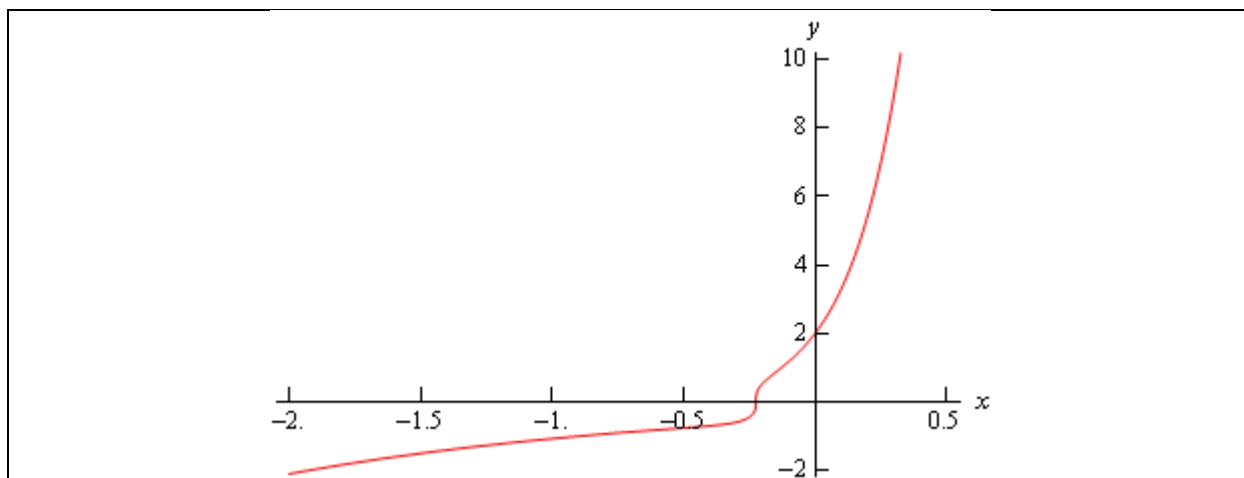
Applying the initial condition and solving for c gives,

$$8 = c - \frac{3}{17} \quad \Rightarrow \quad c = \frac{139}{17}$$

Plugging in c and solving for y gives,

$$y(x) = \left(\frac{139e^{15x} - 3e^{-2x}}{17} \right)^{\frac{1}{3}}$$

There are no problem values of x for this solution and so the interval of validity is all real numbers. Here's a graph of the solution.



Example 3 Solve the following IVP and find the interval of validity for the solution.

$$6y' - 2y = xy^4 \quad y(0) = -2$$

Solution

First get the differential equation in the proper form and then write down the substitution.

$$6y^{-4}y' - 2y^{-3} = x \quad \Rightarrow \quad v = y^{-3} \quad v' = -3y^{-4}y'$$

Plugging the substitution into the differential equation gives,

$$-2v' - 2v = x \quad \Rightarrow \quad v' + v = -\frac{1}{2}x \quad \mu(x) = e^x$$

Again, we've rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

$$v(x) = -\frac{1}{2}(x-1) + ce^{-x}$$

Now back substitute to get back into y 's.

$$y^{-3} = -\frac{1}{2}(x-1) + ce^{-x}$$

Now we need to apply the initial condition and solve for c .

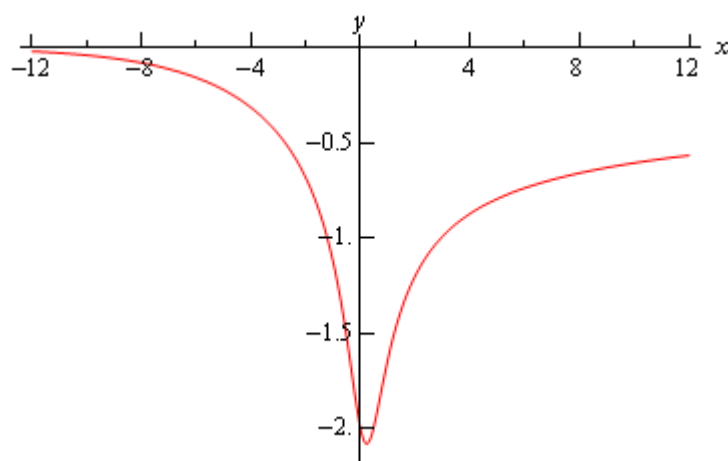
$$-\frac{1}{8} = \frac{1}{2} + c \quad \Rightarrow \quad c = -\frac{5}{8}$$

Plugging in c and solving for y gives,

$$y(x) = -\frac{2}{(4x - 4 + 5e^{-x})^{\frac{1}{3}}}$$

Next, we need to think about the interval of validity. In this case all we need to worry about is division by zero issues and using some form of computational aid (such as Maple or Mathematica) we will see that the denominator of our solution is never zero and so this solution will be valid for all real numbers.

Here is a graph of the solution.



To this point we've only worked examples in which n was an integer (positive and negative) and so we should work a quick example where n is not an integer.

Example 4 Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{y}{x} - \sqrt{y} = 0 \qquad y(1) = 0$$

Solution

Let's first get the differential equation into proper form.

$$y' + \frac{1}{x}y = y^{\frac{1}{2}} \qquad \Rightarrow \qquad y^{-\frac{1}{2}}y' + \frac{1}{x}y^{\frac{1}{2}} = 1$$

The substitution is then,

$$v = y^{\frac{1}{2}} \qquad v' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

Now plug the substitution into the differential equation to get,

$$2v' + \frac{1}{x}v = 1 \qquad \Rightarrow \qquad v' + \frac{1}{2x}v = \frac{1}{2} \qquad \mu(x) = x^{\frac{1}{2}}$$

As we've done with the previous examples we've done some rearranging and given the integrating factor needed for solving the linear differential equation. Solving this gives us,

$$v(x) = \frac{1}{3}x + cx^{-\frac{1}{2}}$$

In terms of y this is,

$$y^{\frac{1}{2}} = \frac{1}{3}x + cx^{-\frac{1}{2}}$$

Applying the initial condition and solving for c gives,

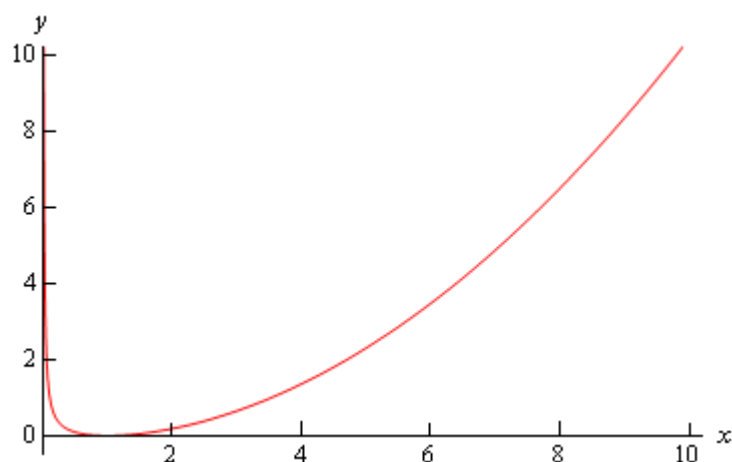
$$0 = \frac{1}{3} + c \quad \Rightarrow \quad c = -\frac{1}{3}$$

Plugging in for c and solving for y gives us the solution.

$$y(x) = \left(\frac{1}{3}x - \frac{1}{3}x^{-\frac{1}{2}} \right)^2 = \frac{x^3 - 2x^{\frac{3}{2}} + 1}{9x}$$

Note that we multiplied everything out and converted all the negative exponents to positive exponents to make the interval of validity clear here. Because of the root (in the second term in the numerator) and the x in the denominator we can see that we need to require $x > 0$ in order for the solution to exist and it will exist for all positive x 's and so this is also the interval of validity.

Here is the graph of the solution.



Section 2-5 : Substitutions

In the previous [section](#) we looked at Bernoulli Equations and saw that in order to solve them we needed to use the substitution $v = y^{1-n}$. Upon using this substitution, we were able to convert the differential equation into a form that we could deal with (linear in this case). In this section we want to take a look at a couple of other substitutions that can be used to reduce some differential equations down to a solvable form.

The first substitution we'll take a look at will require the differential equation to be in the form,

$$y' = F\left(\frac{y}{x}\right)$$

First order differential equations that can be written in this form are called **homogeneous** differential equations. Note that we will usually have to do some rewriting in order to put the differential equation into the proper form.

Once we have verified that the differential equation is a homogeneous differential equation and we've gotten it written in the proper form we will use the following substitution.

$$v(x) = \frac{y}{x}$$

We can then rewrite this as,

$$y = xv$$

and then remembering that both y and v are functions of x we can use the product rule (recall that is [implicit differentiation](#) from Calculus I) to compute,

$$y' = v + xv'$$

Under this substitution the differential equation is then,

$$v + xv' = F(v)$$

$$xv' = F(v) - v \quad \Rightarrow \quad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

As we can see with a small rewrite of the new differential equation we will have a [separable differential equation](#) after the substitution.

Let's take a quick look at a couple of examples of this kind of substitution.

Example 1 Solve the following IVP and find the interval of validity for the solution.

$$x y y' + 4x^2 + y^2 = 0 \quad y(2) = -7, \quad x > 0$$

Solution

Let's first divide both sides by x^2 to rewrite the differential equation as follows,

$$\frac{y}{x} y' = -4 - \frac{y^2}{x^2} = -4 - \left(\frac{y}{x}\right)^2$$

Now, this is not in the officially proper form as we have listed above, but we can see that everywhere the variables are listed they show up as the ratio, y/x and so this is really as far as we need to go.

So, let's plug the substitution into this form of the differential equation to get,

$$v(v + xv') = -4 - v^2$$

Next, rewrite the differential equation to get everything separated out.

$$vxv' = -4 - 2v^2$$

$$xv' = -\frac{4 + 2v^2}{v}$$

$$\frac{v}{4 + 2v^2} dv = -\frac{1}{x} dx$$

Integrating both sides gives,

$$\frac{1}{4} \ln(4 + 2v^2) = -\ln(x) + c$$

We need to do a little rewriting using basic logarithm properties in order to be able to easily solve this for v .

$$\ln(4 + 2v^2)^{\frac{1}{4}} = \ln(x)^{-1} + c$$

Now exponentiate both sides and do a little rewriting

$$(4 + 2v^2)^{\frac{1}{4}} = e^{\ln(x)^{-1} + c} = e^c e^{\ln(x)^{-1}} = \frac{c}{x}$$

Note that because c is an unknown constant then so is e^c and so we may as well just call this c as we did above.

Finally, let's solve for v and then plug the substitution back in and we'll play a little fast and loose with constants again.

$$4 + 2v^2 = \frac{c^4}{x^4} = \frac{c}{x^4}$$

$$v^2 = \frac{1}{2} \left(\frac{c}{x^4} - 4 \right)$$

$$\frac{y^2}{x^2} = \frac{1}{2} \left(\frac{c - 4x^4}{x^4} \right)$$

$$y^2 = \frac{1}{2} x^2 \left(\frac{c - 4x^4}{x^4} \right) = \frac{c - 4x^4}{2x^2}$$

At this point it would probably be best to go ahead and apply the initial condition. Doing that gives,

$$49 = \frac{c - 4(16)}{2(4)} \quad \Rightarrow \quad c = 456$$

Note that we could have also converted the original initial condition into one in terms of v and then applied it upon solving the separable differential equation. In this case however, it was probably a little easier to do it in terms of y given all the logarithms in the solution to the separable differential equation.

Finally, plug in c and solve for y to get,

$$y^2 = \frac{228 - 2x^4}{x^2} \quad \Rightarrow \quad y(x) = \pm \sqrt{\frac{228 - 2x^4}{x^2}}$$

The initial condition tells us that the “ $-$ ” must be the correct sign and so the actual solution is,

$$y(x) = -\sqrt{\frac{228 - 2x^4}{x^2}}$$

For the interval of validity we can see that we need to avoid $x = 0$ and because we can't allow negative numbers under the square root we also need to require that,

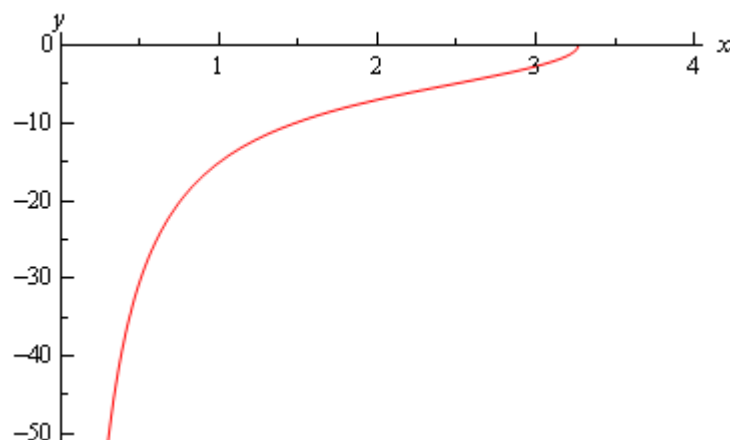
$$\begin{aligned} 228 - 2x^4 &\geq 0 \\ x^4 &\leq 114 \quad \Rightarrow \quad -3.2676 \leq x \leq 3.2676 \end{aligned}$$

So, we have two possible intervals of validity,

$$-3.2676 \leq x < 0 \quad \quad 0 < x \leq 3.2676$$

and the initial condition tells us that it must be $0 < x \leq 3.2676$.

The graph of the solution is,



Example 2 Solve the following IVP and find the interval of validity for the solution.

$$x y' = y(\ln x - \ln y) \quad y(1) = 4, \quad x > 0$$

Solution

On the surface this differential equation looks like it won't be homogeneous. However, with a quick logarithm property we can rewrite this as,

$$y' = \frac{y}{x} \ln \left(\frac{x}{y} \right)$$

In this form the differential equation is clearly homogeneous. Applying the substitution and separating gives,

$$\begin{aligned} v + xv' &= v \ln \left(\frac{1}{v} \right) \\ xv' &= v \left(\ln \left(\frac{1}{v} \right) - 1 \right) \\ \frac{dv}{v \left(\ln \left(\frac{1}{v} \right) - 1 \right)} &= \frac{dx}{x} \end{aligned}$$

Integrate both sides and do a little rewrite to get,

$$\begin{aligned} -\ln \left(\ln \left(\frac{1}{v} \right) - 1 \right) &= \ln x + c \\ \ln \left(\ln \left(\frac{1}{v} \right) - 1 \right) &= c - \ln x \end{aligned}$$

You were able to do the integral on the left right? It used the substitution $u = \ln \left(\frac{1}{v} \right) - 1$.

Now, solve for v and note that we'll need to exponentiate both sides a couple of times and play fast and loose with constants again.

$$\begin{aligned} \ln \left(\frac{1}{v} \right) - 1 &= e^{\ln(x)^{-1} + c} = e^c e^{\ln(x)^{-1}} = \frac{c}{x} \\ \ln \left(\frac{1}{v} \right) &= \frac{c}{x} + 1 \\ \frac{1}{v} &= e^{\frac{c}{x} + 1} \quad \Rightarrow \quad v = e^{-\frac{c}{x} - 1} \end{aligned}$$

Plugging the substitution back in and solving for y gives,

$$\frac{y}{x} = e^{-\frac{c}{x} - 1} \quad \Rightarrow \quad y(x) = x e^{-\frac{c}{x} - 1}$$

Applying the initial condition and solving for c gives,

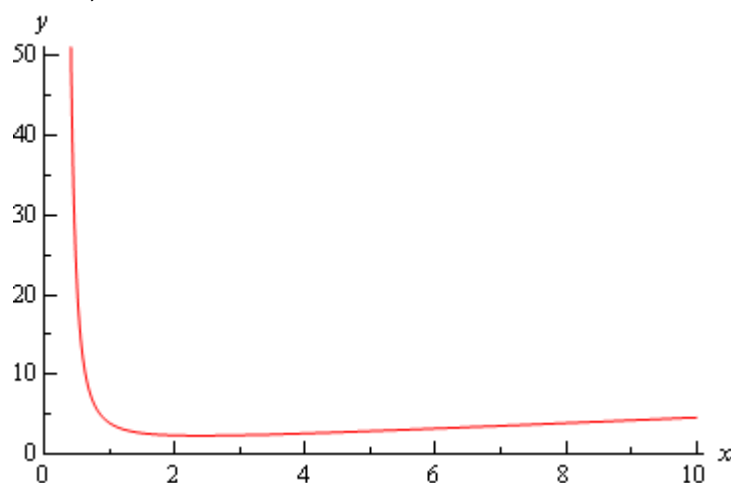
$$4 = e^{-c-1} \quad \Rightarrow \quad c = -(1 + \ln 4)$$

The solution is then,

$$y(x) = xe^{\frac{1+\ln 4}{x}-1}$$

We clearly need to avoid $x = 0$ to avoid division by zero and so with the initial condition we can see that the interval of validity is $x > 0$.

The graph of the solution is,



For the next substitution we'll take a look at we'll need the differential equation in the form,

$$y' = G(ax + by)$$

In these cases, we'll use the substitution,

$$v = ax + by \quad \Rightarrow \quad v' = a + by'$$

Plugging this into the differential equation gives,

$$\frac{1}{b}(v' - a) = G(v)$$

$$v' = a + bG(v) \quad \Rightarrow \quad \frac{dv}{a + bG(v)} = dx$$

So, with this substitution we'll be able to rewrite the original differential equation as a new separable differential equation that we can solve.

Let's take a look at a couple of examples.

Example 3 Solve the following IVP and find the interval of validity for the solution.

$$y' - (4x - y + 1)^2 = 0 \quad y(0) = 2$$

Solution

In this case we'll use the substitution.

$$v = 4x - y \quad v' = 4 - y'$$

Note that we didn't include the "+1" in our substitution. Usually only the $ax + by$ part gets included in the substitution. There are times where including the extra constant may change the difficulty of the solution process, either easier or harder, however in this case it doesn't really make much difference so we won't include it in our substitution.

So, plugging this into the differential equation gives,

$$4 - v' - (v+1)^2 = 0$$

$$v' = 4 - (v+1)^2$$

$$\frac{dv}{(v+1)^2 - 4} = -dx$$

As we've shown above we definitely have a separable differential equation. Also note that to help with the solution process we left a minus sign on the right side. We'll need to integrate both sides and in order to do the integral on the left we'll need to use [partial fractions](#). We'll leave it to you to fill in the missing details and given that we'll be doing quite a bit of partial fraction work in a few chapters you should really make sure that you can do the missing details.

$$\int \frac{dv}{v^2 + 2v - 3} = \int \frac{dv}{(v+3)(v-1)} = \int -dx$$

$$\frac{1}{4} \int \frac{1}{v-1} - \frac{1}{v+3} dv = \int -dx$$

$$\frac{1}{4} (\ln(v-1) - \ln(v+3)) = -x + c$$

$$\ln\left(\frac{v-1}{v+3}\right) = c - 4x$$

Note that we played a little fast and loose with constants above. The next step is fairly messy but needs to be done and that is to solve for v and note that we'll be playing fast and loose with constants again where we can get away with it and we'll be skipping a few steps that you shouldn't have any problem verifying.

$$\frac{v-1}{v+3} = e^{c-4x} = c e^{-4x}$$

$$v-1 = c e^{-4x} (v+3)$$

$$v(1 - c e^{-4x}) = 1 + 3c e^{-4x}$$

At this stage we should back away a bit and note that we can't play fast and loose with constants anymore. We were able to do that in first step because the c appeared only once in the equation. At this point however, the c appears twice and so we've got to keep them around. If we "absorbed" the 3 into the c on the right the "new" c would be different from the c on the left because the c on the left didn't have the 3 as well.

So, let's solve for v and then go ahead and go back into terms of y .

$$v = \frac{1 + 3c e^{-4x}}{1 - c e^{-4x}}$$

$$4x - y = \frac{1 + 3c e^{-4x}}{1 - c e^{-4x}}$$

$$y(x) = 4x - \frac{1 + 3c e^{-4x}}{1 - c e^{-4x}}$$

The last step is to then apply the initial condition and solve for c .

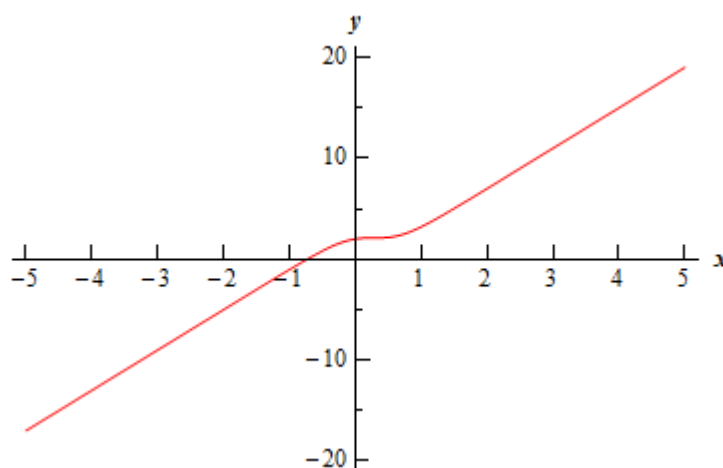
$$2 = y(0) = -\frac{1 + 3c}{1 - c} \quad \Rightarrow \quad c = -3$$

The solution is then,

$$y(x) = 4x - \frac{1 - 9e^{-4x}}{1 + 3e^{-4x}}$$

Note that because exponentials exist everywhere and the denominator of the second term is always positive (because exponentials are always positive and adding a positive one onto that won't change the fact that it's positive) the interval of validity for this solution will be all real numbers.

Here is a graph of the solution.



Example 4 Solve the following IVP and find the interval of validity for the solution.

$$y' = e^{9y-x} \quad y(0) = 0$$

Solution

Here is the substitution that we'll need for this example.

$$v = 9y - x \quad v' = 9y' - 1$$

Plugging this into our differential equation gives,

$$\frac{1}{9}(v' + 1) = e^v$$

$$v' = 9e^v - 1$$

$$\frac{dv}{9e^v - 1} = dx \quad \Rightarrow \quad \frac{e^{-v} dv}{9 - e^{-v}} = dx$$

Note that we did a little rewrite on the separated portion to make the integrals go a little easier. By multiplying the numerator and denominator by e^{-v} we can turn this into a fairly simply substitution integration problem. So, upon integrating both sides we get,

$$\ln(9 - e^{-v}) = x + c$$

Solving for v gives,

$$9 - e^{-v} = e^c e^x = ce^x$$

$$e^{-v} = 9 - ce^x$$

$$v = -\ln(9 - ce^x)$$

Plugging the substitution back in and solving for y gives us,

$$y(x) = \frac{1}{9}(x - \ln(9 - ce^x))$$

Next, apply the initial condition and solve for c .

$$0 = y(0) = -\frac{1}{9}\ln(9 - c) \quad \Rightarrow \quad c = 8$$

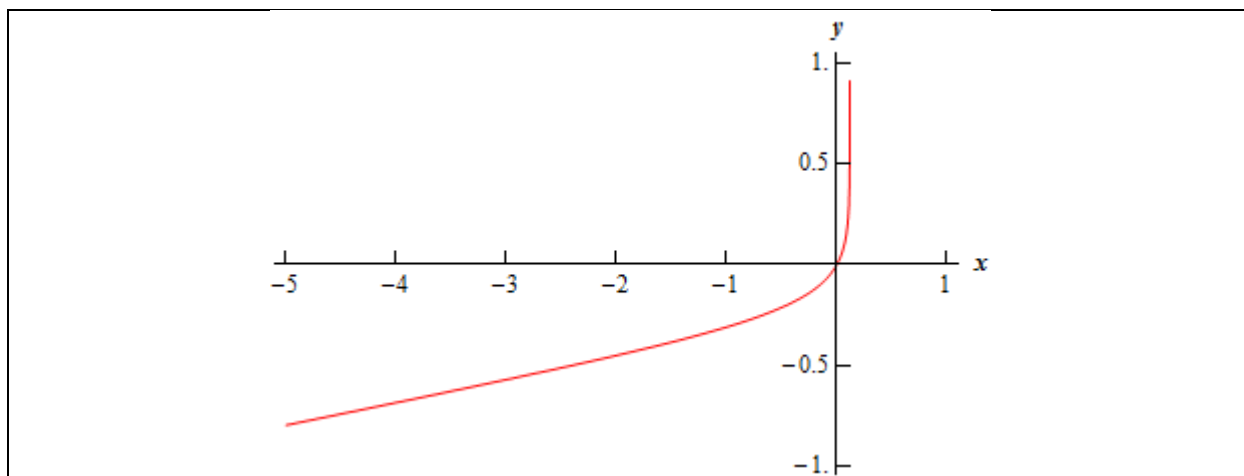
The solution is then,

$$y(x) = \frac{1}{9}(x - \ln(9 - 8e^x))$$

Now, for the interval of validity we need to make sure that we only take logarithms of positive numbers as we'll need to require that,

$$9 - 8e^x > 0 \quad \Rightarrow \quad e^x < \frac{9}{8} \quad \Rightarrow \quad x < \ln \frac{9}{8} = 0.1178$$

Here is a graph of the solution.



In both this section and the previous section we've seen that sometimes a substitution will take a differential equation that we can't solve and turn it into one that we can solve. This idea of substitutions is an important idea and should not be forgotten. Not every differential equation can be made easier with a substitution and there is no way to show every possible substitution but remembering that a substitution may work is a good thing to do. If you get stuck on a differential equation you may try to see if a substitution of some kind will work for you.

Section 2-6 : Intervals of Validity

We've called this section Intervals of Validity because all of the examples will involve them. However, there is a lot more to this section. We will see a couple of theorems that will tell us when we can solve a differential equation. We will also see some of the differences between linear and nonlinear differential equations.

First let's take a look at a theorem about linear first order differential equations. This is a very important theorem although we're not going to really use it for its most important aspect.

Theorem 1

Consider the following IVP.

$$y' + p(t)y = g(t) \qquad y(t_0) = y_0$$

If $p(t)$ and $g(t)$ are continuous functions on an open interval $\alpha < t < \beta$ and the interval contains t_0 , then there is a unique solution to the IVP on that interval.

So, just what does this theorem tell us? First, it tells us that for nice enough linear first order differential equations solutions are guaranteed to exist and more importantly the solution will be unique. We may not be able to find the solution but do know that it exists and that there will only be one of them. This is the very important aspect of this theorem. Knowing that a differential equation has a unique solution is sometimes more important than actually having the solution itself!

Next, if the interval in the theorem is the largest possible interval on which $p(t)$ and $g(t)$ are continuous then the interval is the interval of validity for the solution. This means, that for linear first order differential equations, we won't need to actually solve the differential equation in order to find the interval of validity. Notice as well that the interval of validity will depend only partially on the initial condition. The interval must contain t_0 , but the value of y_0 , has no effect on the interval of validity.

Let's take a look at an example.

Example 1 Without solving, determine the interval of validity for the following initial value problem.

$$(t^2 - 9)y' + 2y = \ln|20 - 4t| \qquad y(4) = -3$$

Solution

First, in order to use the theorem to find the interval of validity we must write the differential equation in the proper form given in the theorem. So we will need to divide out by the coefficient of the derivative.

$$y' + \frac{2}{t^2 - 9}y = \frac{\ln|20 - 4t|}{t^2 - 9}$$

Next, we need to identify where the two functions are not continuous. This will allow us to find all possible intervals of validity for the differential equation. So, $p(t)$ will be discontinuous at $t = \pm 3$ since these points will give a division by zero. Likewise, $g(t)$ will also be discontinuous at $t = \pm 3$ as well as $t = 5$ since at this point we will have the natural logarithm of zero. Note that in this case we won't have to worry about natural log of negative numbers because of the absolute values.

Now, with these points in hand we can break up the real number line into four intervals where both $p(t)$ and $g(t)$ will be continuous. These four intervals are,

$$-\infty < t < -3$$

$$-3 < t < 3$$

$$3 < t < 5$$

$$5 < t < \infty$$

The endpoints of each of the intervals are points where at least one of the two functions is discontinuous. This will guarantee that both functions are continuous everywhere in each interval.

Finally, let's identify the actual interval of validity for the initial value problem. The actual interval of validity is the interval that will contain $t_0 = 4$. So, the interval of validity for the initial value problem is.

$$3 < t < 5$$

In this last example we need to be careful to not jump to the conclusion that the other three intervals cannot be intervals of validity. By changing the initial condition, in particular the value of t_0 , we can make any of the four intervals the interval of validity.

The first theorem required a linear differential equation. There is a similar theorem for non-linear first order differential equations. This theorem is not as useful for finding intervals of validity as the first theorem was so we won't be doing all that much with it.

Here is the theorem.

Theorem 2

Consider the following IVP.

$$y' = f(t, y) \quad y(t_0) = y_0$$

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0) then there is a unique solution to the IVP in some interval $t_0 - h < t < t_0 + h$ that is contained in $\alpha < t < \beta$.

That's it. Unlike the first theorem, this one cannot really be used to find an interval of validity. So, we will know that a unique solution exists if the conditions of the theorem are met, but we will actually need the solution in order to determine its interval of validity. Note as well that for non-linear differential equations it appears that the value of y_0 may affect the interval of validity.

Here is an example of the problems that can arise when the conditions of this theorem aren't met.

Example 2 Determine all possible solutions to the following IVP.

$$y' = y^{\frac{1}{3}} \quad y(0) = 0$$

Solution

First, notice that this differential equation does NOT satisfy the conditions of the theorem.

$$f(y) = y^{\frac{1}{3}} \quad \frac{df}{dy} = \frac{1}{3y^{\frac{2}{3}}}$$

So, the function is continuous on any interval, but the derivative is not continuous at $y = 0$ and so will not be continuous at any interval containing $y = 0$. In order to use the theorem both must be continuous on an interval that contains $y_0 = 0$ and this is problem for us since we do have $y_0 = 0$.

Now, let's actually work the problem. This differential equation is separable and is fairly simple to solve.

$$\int y^{-\frac{1}{3}} dy = \int dt$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

Applying the initial condition gives $c = 0$ and so the solution is.

$$\frac{3}{2} y^{\frac{2}{3}} = t$$

$$y^{\frac{2}{3}} = \frac{2}{3} t$$

$$y^2 = \left(\frac{2}{3} t \right)^3$$

$$y(t) = \pm \left(\frac{2}{3} t \right)^{\frac{3}{2}}$$

So, we've got two possible solutions here, both of which satisfy the differential equation and the initial condition. There is also a third solution to the IVP. $y(t) = 0$ is also a solution to the differential equation and satisfies the initial condition.

In this last example we had a very simple IVP and it only violated one of the conditions of the theorem, yet it had three different solutions. All the examples we've worked in the previous sections satisfied the conditions of this theorem and had a single unique solution to the IVP. This example is a useful reminder of the fact that, in the field of differential equations, things don't always behave nicely. It's easy to forget this as most of the problems that are worked in a differential equations class are nice and behave in a nice, predictable manner.

Let's work one final example that will illustrate one of the differences between linear and non-linear differential equations.

Example 3 Determine the interval of validity for the initial value problem below and give its dependence on the value of y_0

$$y' = y^2 \quad y(0) = y_0$$

Solution

Before proceeding in this problem, we should note that the differential equation is non-linear and meets both conditions of the Theorem 2 and so there will be a unique solution to the IVP for each possible value of y_0 .

Also, note that the problem asks for any dependence of the interval of validity on the value of y_0 . This immediately illustrates a difference between linear and non-linear differential equations. Intervals of validity for linear differential equations do not depend on the value of y_0 . Intervals of validity for non-linear differential can depend on the value of y_0 as we pointed out after the second theorem.

So, let's solve the IVP and get some intervals of validity.

First note that if $y_0 = 0$ then $y(t) = 0$ is the solution and this has an interval of validity of

$$-\infty < t < \infty$$

So, for the rest of the problem let's assume that $y_0 \neq 0$. Now, the differential equation is separable so let's solve it and get a general solution.

$$\int y^{-2} dy = \int dt$$

$$-\frac{1}{y} = t + c$$

Applying the initial condition gives

$$c = -\frac{1}{y_0}$$

The solution is then.

$$-\frac{1}{y} = t - \frac{1}{y_0}$$

$$y(t) = \frac{1}{\frac{1}{y_0} - t}$$

$$y(t) = \frac{y_0}{1 - y_0 t}$$

Now that we have a solution to the initial value problem we can start finding intervals of validity. From the solution we can see that the only problem that we'll have is division by zero at

$$t = \frac{1}{y_0}$$

This leads to two possible intervals of validity.

$$-\infty < t < \frac{1}{y_0}$$

$$\frac{1}{y_0} < t < \infty$$

That actual interval of validity will be the interval that contains $t_0 = 0$. This however, depends on the value of y_0 . If $y_0 < 0$ then $\frac{1}{y_0} < 0$ and so the second interval will contain $t_0 = 0$. Likewise, if $y_0 > 0$ then $\frac{1}{y_0} > 0$ and in this case the first interval will contain $t_0 = 0$.

This leads to the following possible intervals of validity, depending on the value of y_0 .

If $y_0 > 0$ $-\infty < t < \frac{1}{y_0}$ is the interval of validity.

If $y_0 = 0$ $-\infty < t < \infty$ is the interval of validity.

If $y_0 < 0$ $\frac{1}{y_0} < t < \infty$ is the interval of validity.

On a side note, notice that the solution, in its final form, will also work if $y_0 = 0$.

So, what did this example show us about the difference between linear and non-linear differential equations?

First, as pointed out in the solution to the example, intervals of validity for non-linear differential equations can depend on the value of y_0 , whereas intervals of validity for linear differential equations don't.

Second, intervals of validity for linear differential equations can be found from the differential equation with no knowledge of the solution. This is definitely not the case with non-linear differential equations. It would be very difficult to see how any of these intervals in the last example could be found from the differential equation. Knowledge of the solution was required in order for us to find the interval of validity.

Section 2-7 : Modeling with First Order DE's

We now move into one of the main applications of differential equations both in this class and in general. Modeling is the process of writing a differential equation to describe a physical situation. Almost all of the differential equations that you will use in your job (for the engineers out there in the audience) are there because somebody, at some time, modeled a situation to come up with the differential equation that you are using.

This section is not intended to completely teach you how to go about modeling all physical situations. A whole course could be devoted to the subject of modeling and still not cover everything! This section is designed to introduce you to the process of modeling and show you what is involved in modeling. We will look at three different situations in this section : Mixing Problems, Population Problems, and Falling Objects.

In all of these situations we will be forced to make assumptions that do not accurately depict reality in most cases, but without them the problems would be very difficult and beyond the scope of this discussion (and the course in most cases to be honest).

So, let's get started.

Mixing Problems

In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it. If $Q(t)$ gives the amount of the substance dissolved in the liquid in the tank at any time t we want to develop a differential equation that, when solved, will give us an expression for $Q(t)$. Note as well that in many situations we can think of air as a liquid for the purposes of these kinds of discussions and so we don't actually need to have an actual liquid but could instead use air as the "liquid".

The main assumption that we'll be using here is that the concentration of the substance in the liquid is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course.

The main "equation" that we'll be using to model this situation is :

$$\begin{array}{ccccc} \text{Rate of} & & \text{Rate at} & & \text{Rate at} \\ \text{change of} & = & \text{which } Q(t) & - & \text{which } Q(t) \\ Q(t) & & \text{enters the} & & \text{exits the} \\ & & \text{tank} & & \text{tank} \end{array}$$

where,

$$\text{Rate of change of } Q(t) : Q(t) = \frac{dQ}{dt} = Q'(t)$$

Rate at which $Q(t)$ enters the tank : (flow rate of liquid entering) x
(concentration of substance in liquid entering)

Rate at which $Q(t)$ exits the tank : (flow rate of liquid exiting) x
(concentration of substance in liquid exiting)

Let's take a look at the first problem.

Example 1 A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

Solution

First off, let's address the "well mixed solution" bit. This is the assumption that was mentioned earlier. We are going to assume that the instant the water enters the tank it somehow instantly disperses evenly throughout the tank to give a uniform concentration of salt in the tank at every point. Again, this will clearly not be the case in reality, but it will allow us to do the problem.

Now, to set up the IVP that we'll need to solve to get $Q(t)$ we'll need the flow rate of the water entering (we've got that), the concentration of the salt in the water entering (we've got that), the flow rate of the water leaving (we've got that) and the concentration of the salt in the water exiting (we don't have this yet).

So, we first need to determine the concentration of the salt in the water exiting the tank. Since we are assuming a uniform concentration of salt in the tank the concentration at any point in the tank and hence in the water exiting is given by,

$$\text{Concentration} = \frac{\text{Amount of salt in the tank at any time, } t}{\text{Volume of water in the tank at any time, } t}$$

The amount at any time t is easy it's just $Q(t)$. The volume is also pretty easy. We start with 600 gallons and every hour 9 gallons enters and 6 gallons leave. So, if we use t in hours, every hour 3 gallons enters the tank, or at any time t there is $600 + 3t$ gallons of water in the tank.

So, the IVP for this situation is,

$$Q'(t) = (9)\left(\frac{1}{5}(1 + \cos(t))\right) - (6)\left(\frac{Q(t)}{600 + 3t}\right) \quad Q(0) = 5$$

$$Q'(t) = \frac{9}{5}(1 + \cos(t)) - \frac{2Q(t)}{200 + t} \quad Q(0) = 5$$

This is a linear differential equation and it isn't too difficult to solve (hopefully). We will show most of the details but leave the description of the solution process out. If you need a refresher on solving linear first order differential equations go back and take a look at that [section](#).

$$\begin{aligned} Q'(t) + \frac{2Q(t)}{200 + t} &= \frac{9}{5}(1 + \cos(t)) \\ \mu(t) &= e^{\int \frac{2}{200+t} dt} = e^{2\ln(200+t)} = (200 + t)^2 \\ \int \left((200 + t)^2 Q(t) \right)' dt &= \int \frac{9}{5} (200 + t)^2 (1 + \cos(t)) dt \end{aligned}$$

$$(200+t)^2 Q(t) = \frac{9}{5} \left(\frac{1}{3} (200+t)^3 + (200+t)^2 \sin(t) + 2(200+t) \cos(t) - 2 \sin(t) \right) + c$$

$$Q(t) = \frac{9}{5} \left(\frac{1}{3} (200+t) + \sin(t) + \frac{2 \cos(t)}{200+t} - \frac{2 \sin(t)}{(200+t)^2} \right) + \frac{c}{(200+t)^2}$$

So, here's the general solution. Now, apply the initial condition to get the value of the constant, c .

$$5 = Q(0) = \frac{9}{5} \left(\frac{1}{3} (200) + \frac{2}{200} \right) + \frac{c}{(200)^2} \quad c = -4600720$$

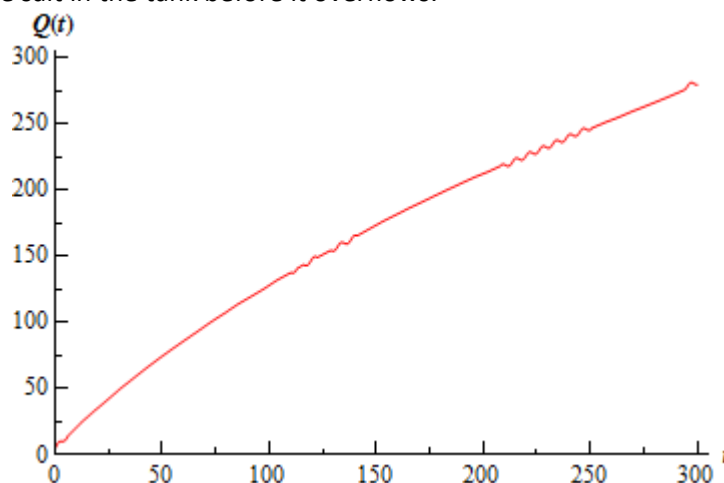
So, the amount of salt in the tank at any time t is.

$$Q(t) = \frac{9}{5} \left(\frac{1}{3} (200+t) + \sin(t) + \frac{2 \cos(t)}{200+t} - \frac{2 \sin(t)}{(200+t)^2} \right) - \frac{4600720}{(200+t)^2}$$

Now, the tank will overflow at $t = 300$ hrs. The amount of salt in the tank at that time is.

$$Q(300) = 279.797 \text{ lbs}$$

Here's a graph of the salt in the tank before it overflows.



Note that the whole graph should have small oscillations in it as you can see in the range from 200 to 250. The scale of the oscillations however was small enough that the program used to generate the image had trouble showing all of them.

The work was a little messy with that one, but they will often be that way so don't get excited about it. This first example also assumed that nothing would change throughout the life of the process. That, of course, will usually not be the case. Let's take a look at an example where something changes in the process.

Example 2 A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gal/hr while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t .

Solution

Okay, so clearly the pollution in the tank will increase as time passes. If the amount of pollution ever reaches the maximum allowed there will be a change in the situation. This will necessitate a change in the differential equation describing the process as well. In other words, we'll need two IVP's for this problem. One will describe the initial situation when polluted runoff is entering the tank and one for after the maximum allowed pollution is reached and fresh water is entering the tank.

Here are the two IVP's for this problem.

$$Q_1'(t) = (3)(5) - (3)\left(\frac{Q_1(t)}{800}\right) \quad Q_1(0) = 2 \quad 0 \leq t \leq t_m$$

$$Q_2'(t) = (2)(0) - (4)\left(\frac{Q_2(t)}{800 - 2(t - t_m)}\right) \quad Q_2(t_m) = 500 \quad t_m \leq t \leq t_e$$

The first one is fairly straight forward and will be valid until the maximum amount of pollution is reached. We'll call that time t_m . Also, the volume in the tank remains constant during this time so we don't need to do anything fancy with that this time in the second term as we did in the previous example.

We'll need a little explanation for the second one. First notice that we don't "start over" at $t = 0$. We start this one at t_m , the time at which the new process starts. Next, fresh water is flowing into the tank and so the concentration of pollution in the incoming water is zero. This will drop out the first term, and that's okay so don't worry about that.

Now, notice that the volume at any time looks a little funny. During this time frame we are losing two gallons of water every hour of the process so we need the "-2" in there to account for that. However, we can't just use t as we did in the previous example. When this new process starts up there needs to be 800 gallons of water in the tank and if we just use t there we won't have the required 800 gallons that we need in the equation. So, to make sure that we have the proper volume we need to put in the difference in times. In this way once we are one hour into the new process (*i.e.* $t - t_m = 1$) we will have 798 gallons in the tank as required.

Finally, the second process can't continue forever as eventually the tank will empty. This is denoted in the time restrictions as t_e . We can also note that $t_e = t_m + 400$ since the tank will empty 400 hours after this new process starts up. Well, it will end provided something doesn't come along and start changing the situation again.

Okay, now that we've got all the explanations taken care of here's the simplified version of the IVP's that we'll be solving.

$$Q_1'(t) = 15 - \frac{3Q_1(t)}{800} \quad Q_1(0) = 2 \quad 0 \leq t \leq t_m$$

$$Q_2'(t) = -\frac{2Q_2(t)}{400 - (t - t_m)} \quad Q_2(t_m) = 500 \quad t_m \leq t \leq t_e$$

The first IVP is a fairly simple linear differential equation so we'll leave the details of the solution to you to check. Upon solving you get.

$$Q_1(t) = 4000 - 3998e^{-\frac{3t}{800}}$$

Now, we need to find t_m . This isn't too bad all we need to do is determine when the amount of pollution reaches 500. So, we need to solve.

$$Q_1(t) = 4000 - 3998e^{-\frac{3t}{800}} = 500 \quad \Rightarrow \quad t_m = 35.4750$$

So, the second process will pick up at 35.475 hours. For completeness sake here is the IVP with this information inserted.

$$Q_2'(t) = -\frac{2Q_2(t)}{435.475 - t} \quad Q_2(35.475) = 500 \quad 35.475 \leq t \leq 435.475$$

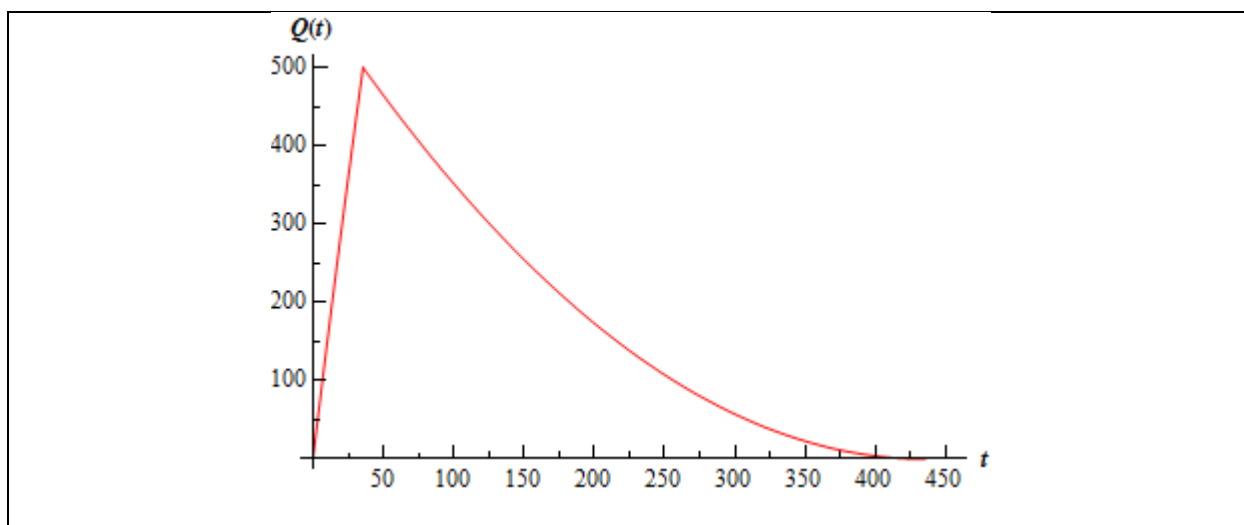
This differential equation is both linear and separable and again isn't terribly difficult to solve so I'll leave the details to you again to check that we should get.

$$Q_2(t) = \frac{(435.476 - t)^2}{320}$$

So, a solution that encompasses the complete running time of the process is

$$Q(t) = \begin{cases} 4000 - 3998e^{-\frac{3t}{800}} & 0 \leq t \leq 35.475 \\ \frac{(435.476 - t)^2}{320} & 35.475 \leq t \leq 435.475 \end{cases}$$

Here is a graph of the amount of pollution in the tank at any time t .



As you can surely see, these problems can get quite complicated if you want them to. Take the last example. A more realistic situation would be that once the pollution dropped below some predetermined point the polluted runoff would, in all likelihood, be allowed to flow back in and then the whole process would repeat itself. So, realistically, there should be at least one more IVP in the process.

Let's move on to another type of problem now.

Population

These are somewhat easier than the mixing problems although, in some ways, they are very similar to mixing problems. So, if $P(t)$ represents a population in a given region at any time t the basic equation that we'll use is identical to the one that we used for mixing. Namely,

$$\begin{array}{ccccc} \text{Rate of} & & \text{Rate at} & & \text{Rate at} \\ \text{change of} & = & \text{which } P(t) & - & \text{which } P(t) \\ P(t) & & \text{enters the} & & \text{exits the} \\ & & \text{region} & & \text{region} \end{array}$$

Here the rate of change of $P(t)$ is still the derivative. What's different this time is the rate at which the population enters and exits the region. For population problems all the ways for a population to enter the region are included in the entering rate. Birth rate and migration into the region are examples of terms that would go into the rate at which the population enters the region. Likewise, all the ways for a population to leave an area will be included in the exiting rate. Therefore, things like death rate, migration out and predation are examples of terms that would go into the rate at which the population exits the area.

Here's an example.

Example 3 A population of insects in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. On any given day there is a net migration into the area of 15 insects and 16 are eaten by the local bird population and 7 die of natural causes. If there are initially 100 insects in the area will the population survive? If not, when do they die out?

Solution

Let's start out by looking at the birth rate. We are told that the insects will be born at a rate that is proportional to the current population. This means that the birth rate can be written as

$$rP$$

where r is a positive constant that will need to be determined. Now, let's take everything into account and get the IVP for this problem.

$$P' = (rP + 15) - (16 + 7) \quad P(0) = 100$$

$$P' = rP - 8 \quad P(0) = 100$$

Note that in the first line we used parenthesis to note which terms went into which part of the differential equation. Also note that we don't make use of the fact that the population will triple in two weeks time in the absence of outside factors here. In the absence of outside factors means that the ONLY thing that we can consider is birth rate. Nothing else can enter into the picture and clearly we have other influences in the differential equation.

So, just how does this tripling come into play? Well, we should also note that without knowing r we will have a difficult time solving the IVP completely. We will use the fact that the population triples in two weeks time to help us find r . In the absence of outside factors the differential equation would become.

$$P' = rP \quad P(0) = 100 \quad P(14) = 300$$

Note that since we used days as the time frame in the actual IVP I needed to convert the two weeks to 14 days. We could have just as easily converted the original IVP to weeks as the time frame, in which case there would have been a net change of -56 per week instead of the -8 per day that we are currently using in the original differential equation.

Okay back to the differential equation that ignores all the outside factors. This differential equation is separable and linear (either can be used) and is a simple differential equation to solve. We'll leave the detail to you to get the general solution.

$$P(t) = ce^{rt}$$

Applying the initial condition gives $c = 100$. Now apply the second condition.

$$300 = P(14) = 100e^{14r} \quad 300 = 100e^{14r}$$

We need to solve this for r . First divide both sides by 100, then take the natural log of both sides.

$$\begin{aligned}
 3 &= e^{14r} \\
 \ln 3 &= \ln e^{14r} \\
 \ln 3 &= 14r \\
 r &= \frac{\ln 3}{14}
 \end{aligned}$$

We made use of the fact that $\ln e^{g(x)} = g(x)$ here to simplify the problem. Now, that we have r we can go back and solve the original differential equation. We'll rewrite it a little for the solution process.

$$P' - \frac{\ln 3}{14}P = -8 \qquad P(0) = 100$$

This is a fairly simple linear differential equation, but that coefficient of P always get people bent out of shape, so we'll go through at least some of the details here.

$$\mu(t) = e^{\int -\frac{\ln 3}{14} dt} = e^{-\frac{\ln 3}{14}t}$$

Now, don't get excited about the integrating factor here. It's just like e^{2t} only this time the constant is a little more complicated than just a 2, but it is a constant! Now, solve the differential equation.

$$\begin{aligned}
 \int \left(P e^{-\frac{\ln 3}{14}t} \right)' dt &= \int -8 e^{-\frac{\ln 3}{14}t} dt \\
 P e^{-\frac{\ln 3}{14}t} &= -8 \left(-\frac{14}{\ln 3} \right) e^{-\frac{\ln 3}{14}t} + c \\
 P(t) &= \frac{112}{\ln 3} + c e^{\frac{\ln 3}{14}t}
 \end{aligned}$$

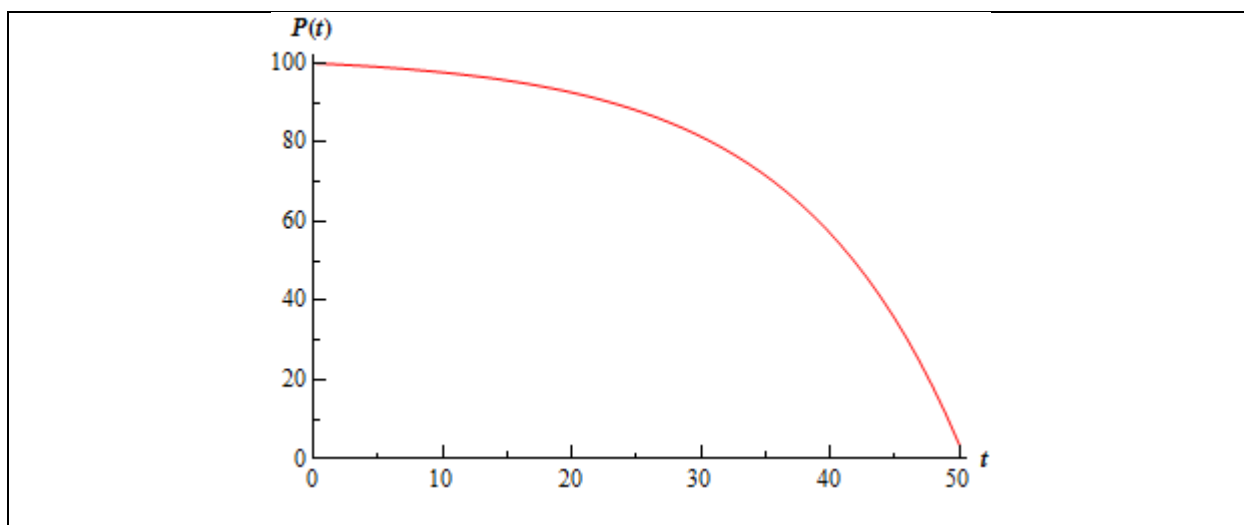
Again, do not get excited about doing the right hand integral, it's just like integrating e^{2t} ! Applying the initial condition gives the following.

$$P(t) = \frac{112}{\ln 3} + \left(100 - \frac{112}{\ln 3} \right) e^{\frac{\ln 3}{14}t} = \frac{112}{\ln 3} - 1.94679 e^{\frac{\ln 3}{14}t}$$

Now, the exponential has a positive exponent and so will go to plus infinity as t increases. Its coefficient, however, is negative and so the whole population will go negative eventually. Clearly, population can't be negative, but in order for the population to go negative it must pass through zero. In other words, eventually all the insects must die. So, they don't survive, and we can solve the following to determine when they die out.

$$0 = \frac{112}{\ln 3} - 1.94679 e^{\frac{\ln 3}{14}t} \quad \Rightarrow \quad t = 50.4415 \text{ days}$$

So, the insects will survive for around 7.2 weeks. Here is a graph of the population during the time in which they survive.



As with the mixing problems, we could make the population problems more complicated by changing the circumstances at some point in time. For instance, if at some point in time the local bird population saw a decrease due to disease they wouldn't eat as much after that point and a second differential equation to govern the time after this point.

Let's now take a look at the final type of problem that we'll be modeling in this section.

Falling Object

This will not be the first time that we've looked into falling bodies. If you recall, we looked at one of these when we were looking at [Direction Fields](#). In that section we saw that the basic equation that we'll use is Newton's Second Law of Motion.

$$mv' = F(t, v)$$

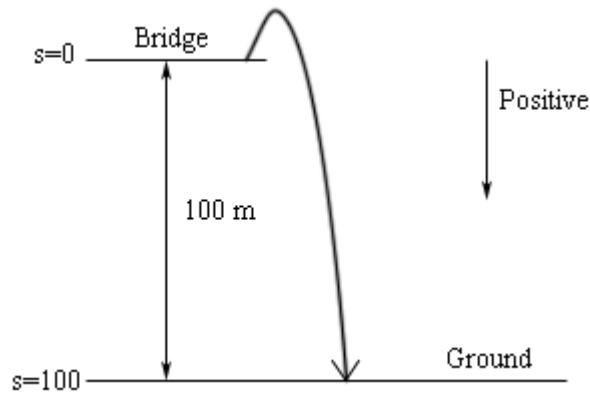
The two forces that we'll be looking at here are gravity and air resistance. The main issue with these problems is to correctly define conventions and then remember to keep those conventions. By this we mean define which direction will be termed the positive direction and then make sure that all your forces match that convention. This is especially important for air resistance as this is usually dependent on the velocity and so the "sign" of the velocity can and does affect the "sign" of the air resistance force.

Let's take a look at an example.

Example 4 A 50 kg object is shot from a cannon straight up with an initial velocity of 10m/s off a bridge that is 100 meters above the ground. If air resistance is given by $5v$ determine the velocity of the mass when it hits the ground.

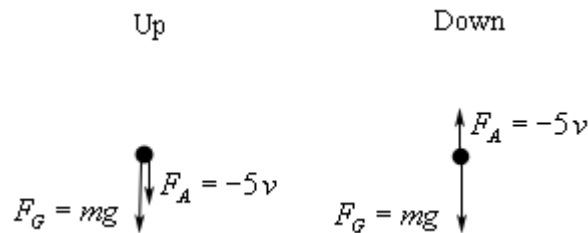
Solution

First, notice that when we say straight up, we really mean straight up, but in such a way that it will miss the bridge on the way back down. Here is a sketch of the situation.



Notice the conventions that we set up for this problem. Since the vast majority of the motion will be in the downward direction we decided to assume that everything acting in the downward direction should be positive. Note that we also defined the “zero position” as the bridge, which makes the ground have a “position” of 100.

Okay, if you think about it we actually have two situations here. The initial phase in which the mass is rising in the air and the second phase when the mass is on its way down. We will need to examine both situations and set up an IVP for each. We will do this simultaneously. Here are the forces that are acting on the object on the way up and on the way down.



Notice that the air resistance force needs a negative in both cases in order to get the correct “sign” or direction on the force. When the mass is moving upwards the velocity (and hence v) is negative, yet the force must be acting in a downward direction. Therefore, the “-” must be part of the force to make sure that, overall, the force is positive and hence acting in the downward direction. Likewise, when the mass is moving downward the velocity (and so v) is positive. Therefore, the air resistance must also have a “-” in order to make sure that it’s negative and hence acting in the upward direction.

So, the IVP for each of these situations are.

$$\begin{aligned} &\text{Up} \\ &mv' = mg - 5v \\ &v(0) = -10 \end{aligned}$$

$$\begin{aligned} &\text{Down} \\ &mv' = mg - 5v \\ &v(t_0) = 0 \end{aligned}$$

In the second IVP, the t_0 is the time when the object is at the highest point and is ready to start on the way down. Note that at this time the velocity would be zero. Also note that the initial condition of the first differential equation will have to be negative since the initial velocity is upward.

In this case, the differential equation for both of the situations is identical. This won't always happen, but in those cases where it does, we can ignore the second IVP and just let the first govern the whole process.

So, let's actually plug in for the mass and gravity (we'll be using $g = 9.8 \text{ m/s}^2$ here). We'll go ahead and divide out the mass while we're at it since we'll need to do that eventually anyway.

$$v' = 9.8 - \frac{5v}{50} = 9.8 - \frac{v}{10} \quad v(0) = -10$$

This is a simple linear differential equation to solve so we'll leave the details to you. Upon solving we arrive at the following equation for the velocity of the object at any time t .

$$v(t) = 98 - 108e^{-\frac{t}{10}}$$

Okay, we want the velocity of the ball when it hits the ground. Of course we need to know *when* it hits the ground before we can ask this. In order to find this we will need to find the position function. This is easy enough to do.

$$s(t) = \int v(t) dt = \int 98 - 108e^{-\frac{t}{10}} dt = 98t + 1080e^{-\frac{t}{10}} + c$$

We can now use the fact that I took the convention that $s(0) = 0$ to find that $c = -1080$. The position at any time is then.

$$s(t) = 98t + 1080e^{-\frac{t}{10}} - 1080$$

To determine when the mass hits the ground we just need to solve.

$$100 = 98t + 1080e^{-\frac{t}{10}} - 1080 \quad t = -3.32203, 5.98147$$

We've got two solutions here, but since we are starting things at $t = 0$, the negative is clearly the incorrect value. Therefore, the mass hits the ground at $t = 5.98147$. The velocity of the object upon hitting the ground is then.

$$v(5.98147) = 38.61841$$

This last example gave us an example of a situation where the two differential equations needed for the problem ended up being identical and so we didn't need the second one after all. Be careful however to not always expect this. We could very easily change this problem so that it required two different differential equations. For instance we could have had a parachute on the mass open at the top of its arc changing its air resistance. This would have completely changed the second differential equation and forced us to use it as well. Or, we could have put a river under the bridge so that before it actually hit the ground it would have first had to go through some water which would have a different "air" resistance for that phase necessitating a new differential equation for that portion.

Or, we could be really crazy and have both the parachute and the river which would then require three IVP's to be solved before we determined the velocity of the mass before it actually hits the solid ground.

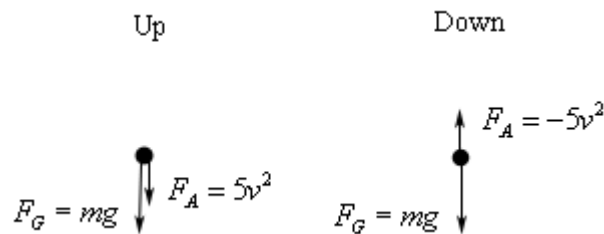
Finally, we could use a completely different type of air resistance that requires us to use a different differential equation for both the upwards and downwards portion of the motion. Let's take a quick look at an example of this.

Example 5 A 50 kg object is shot from a cannon straight up with an initial velocity of 10m/s off a bridge that is 100 meters above the ground. If air resistance is given by $5v^2$ determine the velocity of the mass at any time t .

Solution

So, this is basically the same situation as in the previous example. We just changed the air resistance from $5v$ to $5v^2$. Also, we are just going to find the velocity at any time t for this problem because, we'll the solution is really unpleasant and finding the velocity for when the mass hits the ground is simply more work that we want to put into a problem designed to illustrate the fact that we need a separate differential equation for both the upwards and downwards motion of the mass.

As with the previous example we will use the convention that everything downwards is positive. Here are the forces on the mass when the object is on the way up and on the way down.



Now, in this case, when the object is moving upwards the velocity is negative. However, because of the v^2 in the air resistance we do not need to add in a minus sign this time to make sure the air resistance is positive as it should be given that it is a downwards acting force.

On the downwards phase, however, we still need the minus sign on the air resistance given that it is an upwards force and so should be negative but the v^2 is positive.

This leads to the following IVP's for each case.

Up	Down
$mv' = mg + 5v^2$	$mv' = mg - 5v^2$
$v' = 9.8 + \frac{1}{10}v^2$	$v' = 9.8 - \frac{1}{10}v^2$
$v(0) = -10$	$v(t_0) = 0$

These are clearly different differential equations and so, unlike the previous example, we can't just use the first for the full problem.

Also, the solution process for these will be a little more involved than the previous example as neither of the differential equations are linear. They are both separable differential equations however.

So, let's get the solution process started. We will first solve the upwards motion differential equation. Here is the work for solving this differential equation.

$$\begin{aligned}\int \frac{1}{9.8 + \frac{1}{10}v^2} dv &= \int dt \\ 10 \int \frac{1}{98 + v^2} dv &= \int dt \\ \frac{10}{\sqrt{98}} \tan^{-1}\left(\frac{v}{\sqrt{98}}\right) &= t + c\end{aligned}$$

Note that we did a little rewrite on the integrand to make the process a little easier in the second step.

Now, we have two choices on proceeding from here. Either we can solve for the velocity now, which we will need to do eventually, or we can apply the initial condition at this stage. While, we've always solved for the function before applying the initial condition we could just as easily apply it here if we wanted to and, in this case, will probably be a little easier.

So, to apply the initial condition all we need to do is recall that v is really $v(t)$ and then plug in $t = 0$. Doing this gives,

$$\frac{10}{\sqrt{98}} \tan^{-1}\left(\frac{v(0)}{\sqrt{98}}\right) = 0 + c$$

Now, all we need to do is plug in the fact that we know $v(0) = -10$ to get.

$$c = \frac{10}{\sqrt{98}} \tan^{-1}\left(\frac{-10}{\sqrt{98}}\right)$$

Messy, but there it is. The velocity for the upward motion of the mass is then,

$$\begin{aligned}\frac{10}{\sqrt{98}} \tan^{-1}\left(\frac{v}{\sqrt{98}}\right) &= t + \frac{10}{\sqrt{98}} \tan^{-1}\left(\frac{-10}{\sqrt{98}}\right) \\ \tan^{-1}\left(\frac{v}{\sqrt{98}}\right) &= \frac{\sqrt{98}}{10} t + \tan^{-1}\left(\frac{-10}{\sqrt{98}}\right) \\ v(t) &= \sqrt{98} \tan\left(\frac{\sqrt{98}}{10} t + \tan^{-1}\left(\frac{-10}{\sqrt{98}}\right)\right)\end{aligned}$$

Now, we need to determine when the object will reach the apex of its trajectory. To do this all we need to do is set this equal to zero given that the object at the apex will have zero velocity right

before it starts the downward motion. We will leave it to you to verify that the velocity is zero at the following values of t .

$$t = \frac{10}{\sqrt{98}} \left[\tan^{-1} \left(\frac{10}{\sqrt{98}} \right) + \pi n \right] \quad n = 0, \pm 1, \pm 2, \pm 3, \dots$$

We clearly do not want all of these. We want the first positive t that will give zero velocity. It doesn't make sense to take negative t 's given that we are starting the process at $t = 0$ and once it hits the apex (*i.e.* the first positive t for which the velocity is zero) the solution is no longer valid as the object will start to move downwards and this solution is only for upwards motion.

Plugging in a few values of n will quickly show us that the first positive t will occur for $n = 0$ and will be $t = 0.79847$. We reduced the answer down to a decimal to make the rest of the problem a little easier to deal with.

The IVP for the downward motion of the object is then,

$$v' = 9.8 - \frac{1}{10}v^2 \quad v(0.79847) = 0$$

Now, this is also a separable differential equation, but it is a little more complicated to solve. First, let's separate the differential equation (with a little rewrite) and at least put integrals on it.

$$\int \frac{1}{9.8 - \frac{1}{10}v^2} dv = 10 \int \frac{1}{98 - v^2} dv = \int dt$$

The problem here is the minus sign in the denominator. Because of that this is not an inverse tangent (as was the first integral). To evaluate this integral we could either do a trig substitution ($v = \sqrt{98} \sin \theta$) or use [partial fractions](#) using the fact that $98 - v^2 = (\sqrt{98} - v)(\sqrt{98} + v)$. You're probably not used to factoring things like this but the partial fraction work allows us to avoid the trig substitution and it works exactly like it does when everything is an integer and so we'll do that for this integral.

We'll leave the details of the partial fractioning to you. Once the partial fractioning has been done the integral becomes,

$$\begin{aligned} 10 \left(\frac{1}{2\sqrt{98}} \right) \int \frac{1}{\sqrt{98} + v} + \frac{1}{\sqrt{98} - v} dv &= \int dt \\ \frac{5}{\sqrt{98}} \left[\ln|\sqrt{98} + v| - \ln|\sqrt{98} - v| \right] &= t + c \\ \frac{5}{\sqrt{98}} \ln \left| \frac{\sqrt{98} + v}{\sqrt{98} - v} \right| &= t + c \end{aligned}$$

Again, we will apply the initial condition at this stage to make our life a little easier. Doing this gives,

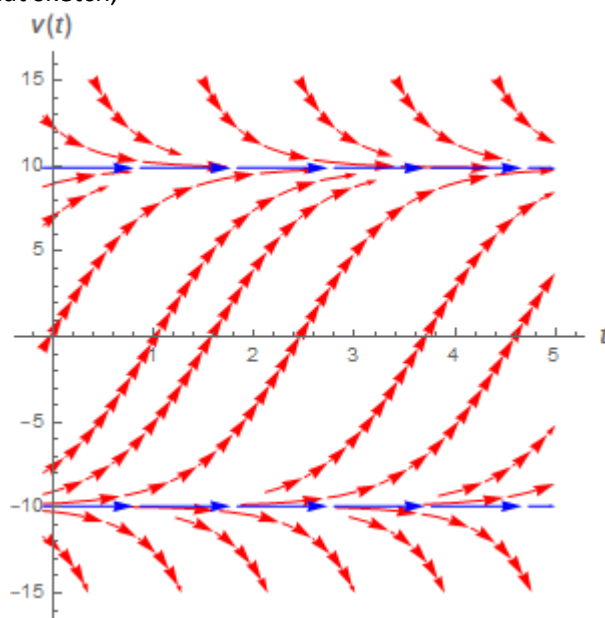
$$\begin{aligned}\frac{5}{\sqrt{98}} \ln \left| \frac{\sqrt{98} + v(0.79847)}{\sqrt{98} - v(0.79847)} \right| &= 0.79847 + c \\ \frac{5}{\sqrt{98}} \ln \left| \frac{\sqrt{98} + 0}{\sqrt{98} - 0} \right| &= 0.79847 + c \\ \frac{5}{\sqrt{98}} \ln |1| &= 0.79847 + c \\ c &= -0.79847\end{aligned}$$

The solutions, as we have it written anyway, is then,

$$\frac{5}{\sqrt{98}} \ln \left| \frac{\sqrt{98} + v}{\sqrt{98} - v} \right| = t - 0.79847$$

Okay, we now need to solve for v and to do that we really need the absolute value bars gone and no we can't just drop them to make our life easier. We need to know that they can be dropped without have any effect on the eventual solution.

To do this let's do a quick direction field, or more appropriately some sketches of solutions from a direction field. Here is that sketch,



Note that $\sqrt{98} = 9.89949$ and so is slightly above/below the lines for -10 and 10 shown in the sketch. The important thing here is to notice the middle region. If the velocity starts out anywhere in this region, as ours does given that $v(0.79847) = 0$, then the velocity must always be less than $\sqrt{98}$.

What this means for us is that both $\sqrt{98} + v$ and $\sqrt{98} - v$ must be positive and so the quantity in the absolute value bars must also be positive. Therefore, in this case, we can drop the absolute value bars to get,

$$\frac{5}{\sqrt{98}} \ln \left[\frac{\sqrt{98} + v}{\sqrt{98} - v} \right] = t - 0.79847$$

At this point we have some very messy algebra to solve for v . We will leave it to you to verify our algebra work. The solution to the downward motion of the object is,

$$v(t) = \sqrt{98} \frac{e^{\frac{1}{5}\sqrt{98}(t-0.79847)} - 1}{e^{\frac{1}{5}\sqrt{98}(t-0.79847)} + 1}$$

Putting everything together here is the full (decidedly unpleasant) solution to this problem.

$$v(t) = \begin{cases} \sqrt{98} \tan \left(\frac{\sqrt{98}}{10} t + \tan^{-1} \left(\frac{-10}{\sqrt{98}} \right) \right) & 0 \leq t \leq 0.79847 \text{ (upward motion)} \\ \sqrt{98} \frac{e^{\frac{1}{5}\sqrt{98}(t-0.79847)} - 1}{e^{\frac{1}{5}\sqrt{98}(t-0.79847)} + 1} & 0.79847 \leq t \leq t_{\text{end}} \text{ (downward motion)} \end{cases}$$

where t_{end} is the time when the object hits the ground. Given the nature of the solution here we will leave it to you to determine that time if you wish to but be forewarned the work is liable to be very unpleasant.

And with this problem you now know why we stick mostly with air resistance in the form $c v$! Note as well, we are not saying the air resistance in the above example is even realistic. It was simply chosen to illustrate two things. First, sometimes we do need different differential equation for the upwards and downwards portion of the motion. Secondly, do not get used to solutions always being as nice as most of the falling object ones are. Sometimes, as this example has illustrated, they can be very unpleasant and involve a lot of work.

Before leaving this section let's work a couple examples illustrating the importance of remembering the conventions that you set up for the positive direction in these problems.

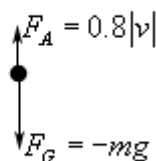
Awhile back I gave my students a problem in which a sky diver jumps out of a plane. Most of my students are engineering majors and following the standard convention from most of their engineering classes they defined the positive direction as upward, despite the fact that all the motion in the problem was downward. There is nothing wrong with this assumption, however, because they forgot the convention that up was positive they did not correctly deal with the air resistance which caused them to get the incorrect answer. So, the moral of this story is : be careful with your convention. It doesn't matter what you set it as but you must always remember what convention to decided to use for any given problem.

So, let's take a look at the problem and set up the IVP that will give the sky diver's velocity at any time t .

Example 6 Set up the IVP that will give the velocity of a 60 kg sky diver that jumps out of a plane with no initial velocity and an air resistance of $0.8|v|$. For this example assume that the positive direction is upward.

Solution

Here are the forces that are acting on the sky diver



Because of the conventions the force due to gravity is negative and the force due to air resistance is positive. As set up, these forces have the correct sign and so the IVP is

$$mv' = -mg + 0.8|v| \quad v(0) = 0$$

The problem arises when you go to remove the absolute value bars. In order to do the problem they do need to be removed. This is where most of the students made their mistake. Because they had forgotten about the convention and the direction of motion they just dropped the absolute value bars to get.

$$mv' = -mg + 0.8v \quad v(0) = 0 \quad (\text{incorrect IVP!!})$$

So, why is this incorrect? Well remember that the convention is that positive is upward. However in this case the object is moving downward and so v is negative! Upon dropping the absolute value bars the air resistance became a negative force and hence was acting in the downward direction!

To get the correct IVP recall that because v is negative then $|v| = -v$. Using this, the air resistance becomes $F_A = -0.8v$ and despite appearances this is a positive force since the “-” cancels out against the velocity (which is negative) to get a positive force.

The correct IVP is then

$$mv' = -mg - 0.8v \quad v(0) = 0$$

Plugging in the mass gives

$$v' = -9.8 - \frac{v}{75} \quad v(0) = 0$$

For the sake of completeness the velocity of the sky diver, at least until the parachute opens, which we didn't include in this problem is.

$$v(t) = -735 + 735e^{-t/75}$$

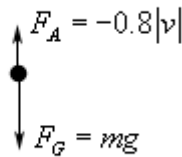
This mistake was made in part because the students were in a hurry and weren't paying attention, but also because they simply forgot about their convention and the direction of motion! Don't fall into this mistake. Always pay attention to your conventions and what is happening in the problems.

Just to show you the difference here is the problem worked by assuming that down is positive.

Example 7 Set up the IVP that will give the velocity of a 60 kg sky diver that jumps out of a plane with no initial velocity and an air resistance of $0.8|v|$. For this example assume that the positive direction is downward.

Solution

Here are the forces that are acting on the sky diver



In this case the force due to gravity is positive since it's a downward force and air resistance is an upward force and so needs to be negative. In this case since the motion is downward the velocity is positive so $|v| = v$. The air resistance is then $F_A = -0.8v$. The IVP for this case is

$$mv' = mg - 0.8v \qquad v(0) = 0$$

Plugging in the mass gives

$$v' = 9.8 - \frac{v}{75} \qquad v(0) = 0$$

Solving this gives

$$v(t) = 735 - 735e^{-\frac{t}{75}}$$

This is the same solution as the previous example, except that it's got the opposite sign. This is to be expected since the conventions have been switched between the two examples.

Section 2-8 : Equilibrium Solutions

In the previous section we modeled a population based on the assumption that the growth rate would be a constant. However, in reality this doesn't make much sense. Clearly a population cannot be allowed to grow forever at the same rate. The growth rate of a population needs to depend on the population itself. Once a population reaches a certain point the growth rate will start reduce, often drastically. A much more realistic model of a population growth is given by the **logistic growth equation**. Here is the logistic growth equation.

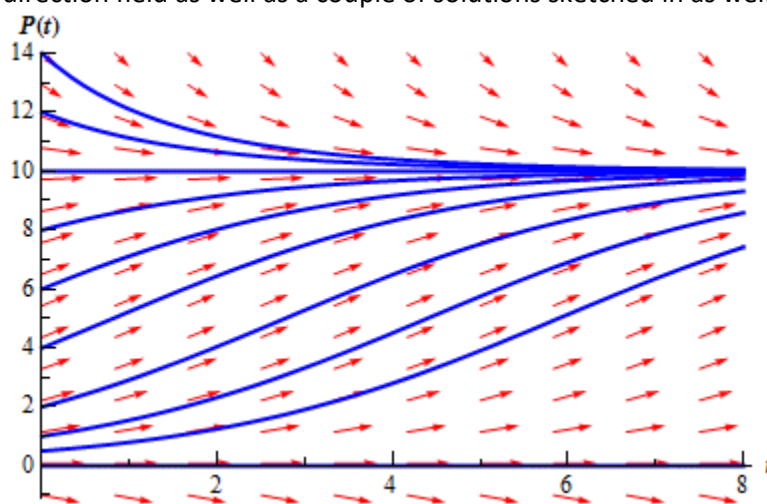
$$P' = r \left(1 - \frac{P}{K} \right) P$$

In the logistic growth equation r is the **intrinsic growth rate** and is the same r as in the last section. In other words, it is the growth rate that will occur in the absence of any limiting factors. K is called either the **saturation level** or the **carrying capacity**.

Now, we claimed that this was a more realistic model for a population. Let's see if that in fact is correct. To allow us to sketch a direction field let's pick a couple of numbers for r and K . We'll use $r = \frac{1}{2}$ and $K = 10$. For these values the logistics equation is.

$$P' = \frac{1}{2} \left(1 - \frac{P}{10} \right) P$$

If you need a refresher on sketching direction fields go back and take a look at that [section](#). First notice that the derivative will be zero at $P = 0$ and $P = 10$. Also notice that these are in fact solutions to the differential equation. These two values are called **equilibrium solutions** since they are constant solutions to the differential equation. We'll leave the rest of the details on sketching the direction field to you. Here is the direction field as well as a couple of solutions sketched in as well.



Note, that we included a small portion of negative P' 's in here even though they really don't make any sense for a population problem. The reason for this will be apparent down the road. Also, notice that a population of say 8 doesn't make all that much sense so let's assume that population is in thousands or millions so that 8 actually represents 8,000 or 8,000,000 individuals in a population.

Notice that if we start with a population of zero, there is no growth and the population stays at zero. So, the logistic equation will correctly figure out that. Next, notice that if we start with a population in the range $0 < P(0) < 10$ then the population will grow, but start to level off once we get close to a population of 10. If we start with a population of 10, the population will stay at 10. Finally, if we start with a population that is greater than 10, then the population will actually die off until we start nearing a population of 10, at which point the population decline will start to slow down.

Now, from a realistic standpoint this should make some sense. Populations can't just grow forever without bound. Eventually the population will reach such a size that the resources of an area are no longer able to sustain the population and the population growth will start to slow as it comes closer to this threshold. Also, if you start off with a population greater than what an area can sustain there will actually be a die off until we get near to this threshold.

In this case that threshold appears to be 10, which is also the value of K for our problem. That should explain the name that we gave K initially. The carrying capacity or saturation level of an area is the maximum sustainable population for that area.

So, the logistics equation, while still quite simplistic, does a much better job of modeling what will happen to a population.

Now, let's move on to the point of this section. The logistics equation is an example of an **autonomous differential equation**. Autonomous differential equations are differential equations that are of the form.

$$\frac{dy}{dt} = f(y)$$

The only place that the independent variable, t in this case, appears is in the derivative.

Notice that if $f(y_0) = 0$ for some value $y = y_0$ then this will also be a solution to the differential equation. These values are called **equilibrium solutions** or **equilibrium points**. What we would like to do is classify these solutions. By classify we mean the following. If solutions start "near" an equilibrium solution will they move away from the equilibrium solution or towards the equilibrium solution? Upon classifying the equilibrium solutions we can then know what all the other solutions to the differential equation will do in the long term simply by looking at which equilibrium solutions they start near.

So, just what do we mean by "near"? Go back to our logistics equation.

$$P' = \frac{1}{2} \left(1 - \frac{P}{10} \right) P$$

As we pointed out there are two equilibrium solutions to this equation $P = 0$ and $P = 10$. If we ignore the fact that we're dealing with population these points break up the P number line into three distinct regions.

$$-\infty < P < 0$$

$$0 < P < 10$$

$$10 < P < \infty$$

We will say that a solution starts "near" an equilibrium solution if it starts in a region that is on either side of that equilibrium solution. So, solutions that start "near" the equilibrium solution $P = 10$ will start in either

$$0 < P < 10 \quad \text{OR} \quad 10 < P < \infty$$

and solutions that start “near” $P = 0$ will start in either

$$-\infty < P < 0 \quad \text{OR} \quad 0 < P < 10$$

For regions that lie between two equilibrium solutions we can think of any solutions starting in that region as starting “near” either of the two equilibrium solutions as we need to.

Now, solutions that start “near” $P = 0$ all move away from the solution as t increases. Note that moving away does not necessarily mean that they grow without bound as they move away. It only means that they move away. Solutions that start out greater than $P = 0$ move away but do stay bounded as t grows. In fact, they move in towards $P = 10$.

Equilibrium solutions in which solutions that start “near” them move away from the equilibrium solution are called **unstable equilibrium points** or **unstable equilibrium solutions**. So, for our logistics equation, $P = 0$ is an unstable equilibrium solution.

Next, solutions that start “near” $P = 10$ all move in toward $P = 10$ as t increases. Equilibrium solutions in which solutions that start “near” them move toward the equilibrium solution are called **asymptotically stable equilibrium points** or **asymptotically stable equilibrium solutions**. So, $P = 10$ is an asymptotically stable equilibrium solution.

There is one more classification, but I’ll wait until we get an example in which this occurs to introduce it. So, let’s take a look at a couple of examples.

Example 1 Find and classify all the equilibrium solutions to the following differential equation.

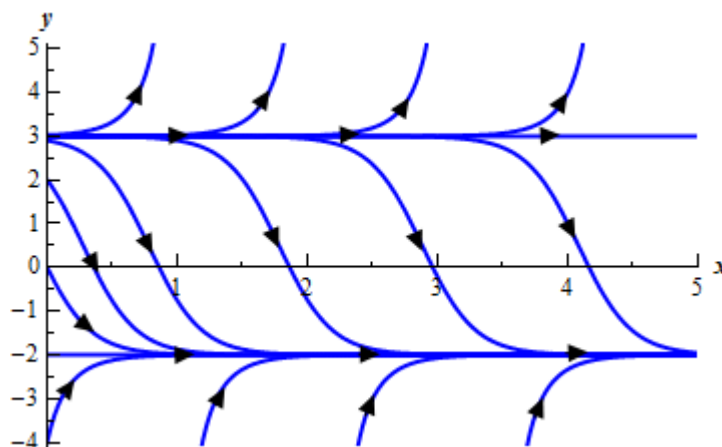
$$y' = y^2 - y - 6$$

Solution

First, find the equilibrium solutions. This is generally easy enough to do.

$$y^2 - y - 6 = (y - 3)(y + 2) = 0$$

So, it looks like we’ve got two equilibrium solutions. Both $y = -2$ and $y = 3$ are equilibrium solutions. Below is the sketch of some integral curves for this differential equation. A sketch of the integral curves or direction fields can simplify the process of classifying the equilibrium solutions.



From this sketch it appears that solutions that start “near” $y = -2$ all move towards it as t increases and so $y = -2$ is an asymptotically stable equilibrium solution and solutions that start “near” $y = 3$ all move away from it as t increases and so $y = 3$ is an unstable equilibrium solution.

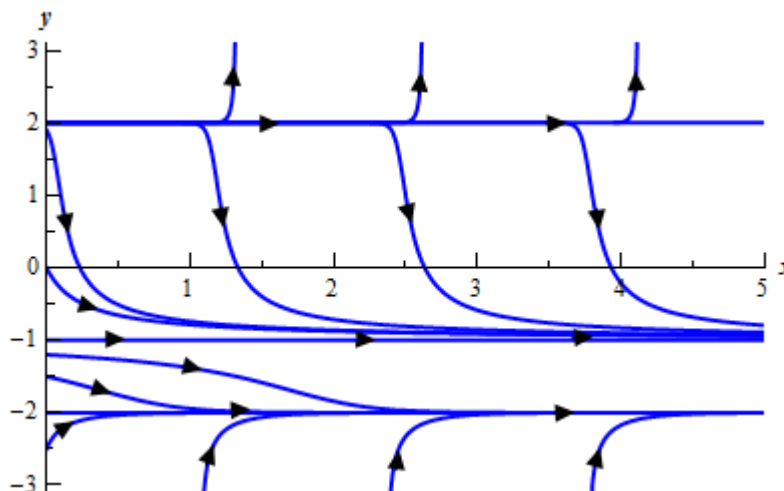
This next example will introduce the third classification that we can give to equilibrium solutions.

Example 2 Find and classify the equilibrium solutions of the following differential equation.

$$y' = (y^2 - 4)(y + 1)^2$$

Solution

The equilibrium solutions to this differential equation are $y = -2$, $y = 2$, and $y = -1$. Below is the sketch of the integral curves.



From this it is clear (hopefully) that $y = 2$ is an unstable equilibrium solution and $y = -2$ is an asymptotically stable equilibrium solution. However, $y = -1$ behaves differently from either of these two. Solutions that start above it move towards $y = -1$ while solutions that start below $y = -1$ move away as t increases.

In cases where solutions on one side of an equilibrium solution move towards the equilibrium solution and on the other side of the equilibrium solution move away from it we call the equilibrium solution **semi-stable**.

So, $y = -1$ is a semi-stable equilibrium solution.

Section 2-9 : Euler's Method

Up to this point practically every differential equation that we've been presented with could be solved. The problem with this is that these are the exceptions rather than the rule. The vast majority of first order differential equations can't be solved.

In order to teach you something about solving first order differential equations we've had to restrict ourselves down to the fairly restrictive cases of linear, separable, or exact differential equations or differential equations that could be solved with a set of very specific substitutions. Most first order differential equations however fall into none of these categories. In fact, even those that are separable or exact cannot always be solved for an explicit solution. Without explicit solutions to these it would be hard to get any information about the solution.

So, what do we do when faced with a differential equation that we can't solve? The answer depends on what you are looking for. If you are only looking for long term behavior of a solution you can always sketch a direction field. This can be done without too much difficulty for some fairly complex differential equations that we can't solve to get exact solutions.

The problem with this approach is that it's only really good for getting general trends in solutions and for long term behavior of solutions. There are times when we will need something more. For instance, maybe we need to determine how a specific solution behaves, including some values that the solution will take. There are also a fairly large set of differential equations that are not easy to sketch good direction fields for.

In these cases, we resort to numerical methods that will allow us to approximate solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation and in fact whole classes can be taught just dealing with the various methods. We are going to look at one of the oldest and easiest to use here. This method was originally devised by Euler and is called, oddly enough, Euler's Method.

Let's start with a general first order IVP

$$\frac{dy}{dt} = f(t, y) \quad y(t_0) = y_0 \quad (1)$$

where $f(t, y)$ is a known function and the values in the initial condition are also known numbers. From the second theorem in the [Intervals of Validity](#) section we know that if f and f_y are continuous functions then there is a unique solution to the IVP in some interval surrounding $t = t_0$. So, let's assume that everything is nice and continuous so that we know that a solution will in fact exist.

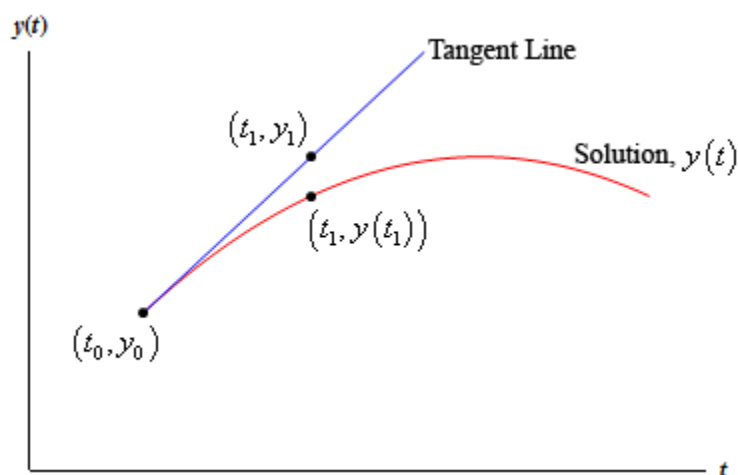
We want to approximate the solution to (1) near $t = t_0$. We'll start with the two pieces of information that we do know about the solution. First, we know the value of the solution at $t = t_0$ from the initial condition. Second, we also know the value of the derivative at $t = t_0$. We can get this by plugging the initial condition into $f(t, y)$ into the differential equation itself. So, the derivative at this point is.

$$\left. \frac{dy}{dt} \right|_{t=t_0} = f(t_0, y_0)$$

Now, recall from your Calculus I class that these two pieces of information are enough for us to write down the equation of the tangent line to the solution at $t = t_0$. The tangent line is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

Take a look at the figure below



If t_1 is close enough to t_0 then the point y_1 on the tangent line should be fairly close to the actual value of the solution at t_1 , or $y(t_1)$. Finding y_1 is easy enough. All we need to do is plug t_1 in the equation for the tangent line.

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

Now, we would like to proceed in a similar manner, but we don't have the value of the solution at t_1 and so we won't know the slope of the tangent line to the solution at this point. This is a problem. We can partially solve it however, by recalling that y_1 is an approximation to the solution at t_1 . If y_1 is a very good approximation to the actual value of the solution then we can use that to estimate the slope of the tangent line at t_1 .

So, let's hope that y_1 is a good approximation to the solution and construct a line through the point (t_1, y_1) that has slope $f(t_1, y_1)$. This gives

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

Now, to get an approximation to the solution at $t = t_2$ we will hope that this new line will be fairly close to the actual solution at t_2 and use the value of the line at t_2 as an approximation to the actual solution. This gives.

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

We can continue in this fashion. Use the previously computed approximation to get the next approximation. So,

$$\begin{aligned}
 y_3 &= y_2 + f(t_2, y_2)(t_3 - t_2) \\
 y_4 &= y_3 + f(t_3, y_3)(t_4 - t_3) \\
 &\text{etc.}
 \end{aligned}$$

In general, if we have t_n and the approximation to the solution at this point, y_n , and we want to find the approximation at t_{n+1} all we need to do is use the following.

$$y_{n+1} = y_n + f(t_n, y_n) \cdot (t_{n+1} - t_n)$$

If we define $f_n = f(t_n, y_n)$ we can simplify the formula to

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n) \quad (2)$$

Often, we will assume that the step sizes between the points t_0, t_1, t_2, \dots are of a uniform size of h . In other words, we will often assume that

$$t_{n+1} - t_n = h$$

This doesn't have to be done and there are times when it's best that we not do this. However, if we do the formula for the next approximation becomes.

$$y_{n+1} = y_n + h f_n \quad (3)$$

So, how do we use Euler's Method? It's fairly simple. We start with (1) and decide if we want to use a uniform step size or not. Then starting with (t_0, y_0) we repeatedly evaluate (2) or (3) depending on whether we chose to use a uniform step size or not. We continue until we've gone the desired number of steps or reached the desired time. This will give us a sequence of numbers $y_1, y_2, y_3, \dots, y_n$ that will approximate the value of the actual solution at $t_1, t_2, t_3, \dots, t_n$.

What do we do if we want a value of the solution at some other point than those used here? One possibility is to go back and redefine our set of points to a new set that will include the points we are after and redo Euler's Method using this new set of points. However, this is cumbersome and could take a lot of time especially if we had to make changes to the set of points more than once.

Another possibility is to remember how we arrived at the approximations in the first place. Recall that we used the tangent line

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

to get the value of y_1 . We could use this tangent line as an approximation for the solution on the interval $[t_0, t_1]$. Likewise, we used the tangent line

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

to get the value of y_2 . We could use this tangent line as an approximation for the solution on the interval $[t_1, t_2]$. Continuing in this manner we would get a set of lines that, when strung together, should be an approximation to the solution as a whole.

In practice you would need to write a computer program to do these computations for you. In most cases the function $f(t, y)$ would be too large and/or complicated to use by hand and in most serious uses

of Euler's Method you would want to use hundreds of steps which would make doing this by hand prohibitive. So, here is a bit of *pseudo-code* that you can use to write a program for Euler's Method that uses a uniform step size, h .

1. **define** $f(t, y)$.
2. **input** t_0 and y_0 .
3. **input** step size, h and the number of steps, n .
4. **for** j from 1 to n **do**
 - a. $m = f(t_0, y_0)$
 - b. $y_1 = y_0 + h * m$
 - c. $t_1 = t_0 + h$
 - d. Print t_1 and y_1
 - e. $t_0 = t_1$
 - f. $y_0 = y_1$
5. **end**

The *pseudo-code* for a non-uniform step size would be a little more complicated, but it would essentially be the same.

So, let's take a look at a couple of examples. We'll use Euler's Method to approximate solutions to a couple of first order differential equations. The differential equations that we'll be using are linear first order differential equations that can be easily solved for an exact solution. Of course, in practice we wouldn't use Euler's Method on these kinds of differential equations, but by using easily solvable differential equations we will be able to check the accuracy of the method. Knowing the accuracy of any approximation method is a good thing. It is important to know if the method is liable to give a good approximation or not.

Example 1 For the IVP

$$y' + 2y = 2 - e^{-4t} \quad y(0) = 1$$

Use Euler's Method with a step size of $h = 0.1$ to find approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4$, and 0.5 . Compare them to the exact values of the solution at these points.

Solution

This is a fairly simple linear differential equation so we'll leave it to you to check that the solution is

$$y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

In order to use Euler's Method we first need to rewrite the differential equation into the form given in (1).

$$y' = 2 - e^{-4t} - 2y$$

From this we can see that $f(t, y) = 2 - e^{-4t} - 2y$. Also note that $t_0 = 0$ and $y_0 = 1$. We can now start doing some computations.

$$f_0 = f(0,1) = 2 - e^{-4(0)} - 2(1) = -1$$

$$y_1 = y_0 + h f_0 = 1 + (0.1)(-1) = 0.9$$

So, the approximation to the solution at $t_1 = 0.1$ is $y_1 = 0.9$.

At the next step we have

$$f_1 = f(0.1, 0.9) = 2 - e^{-4(0.1)} - 2(0.9) = -0.470320046$$

$$y_2 = y_1 + h f_1 = 0.9 + (0.1)(-0.470320046) = 0.852967995$$

Therefore, the approximation to the solution at $t_2 = 0.2$ is $y_2 = 0.852967995$.

I'll leave it to you to check the remainder of these computations.

$$f_2 = -0.155264954 \qquad y_3 = 0.837441500$$

$$f_3 = 0.023922788 \qquad y_4 = 0.839833779$$

$$f_4 = 0.1184359245 \qquad y_5 = 0.851677371$$

Here's a quick table that gives the approximations as well as the exact value of the solutions at the given points.

Time, t_n	Approximation	Exact	Error
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

We've also included the error as a percentage. It's often easier to see how well an approximation does if you look at percentages. The formula for this is,

$$\text{percent error} = \frac{| \text{exact} - \text{approximate} |}{\text{exact}} \times 100$$

We used absolute value in the numerator because we really don't care at this point if the approximation is larger or smaller than the exact. We're only interested in how close the two are.

The maximum error in the approximations from the last example was 4.42%, which isn't too bad, but also isn't all that great of an approximation. So, provided we aren't after very accurate approximations this didn't do too badly. This kind of error is generally unacceptable in almost all real applications however. So, how can we get better approximations?

Recall that we are getting the approximations by using a tangent line to approximate the value of the solution and that we are moving forward in time by steps of h . So, if we want a more accurate

approximation, then it seems like one way to get a better approximation is to not move forward as much with each step. In other words, take smaller h 's.

Example 2 Repeat the previous example only this time give the approximations at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation. We'll leave the computational details to you to check.

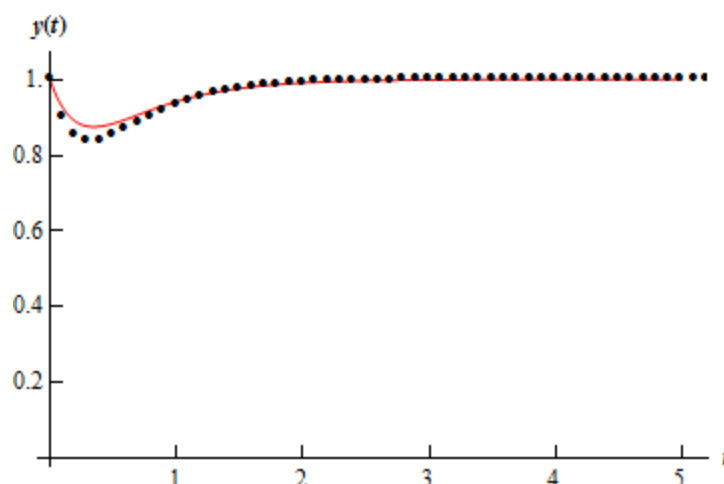
Time	Approximations					
	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$t = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$t = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$t = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$t = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

Time	Percentage Errors				
	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

We can see from these tables that decreasing h does in fact improve the accuracy of the approximation as we expected.

There are a couple of other interesting things to note from the data. First, notice that in general, decreasing the step size, h , by a factor of 10 also decreased the error by about a factor of 10 as well.

Also, notice that as t increases the approximation actually tends to get better. This isn't the case completely as we can see that in all but the first case the $t = 3$ error is worse than the error at $t = 2$, but after that point, it only gets better. This should not be expected in general. In this case this is more a function of the shape of the solution. Below is a graph of the solution (the line) as well as the approximations (the dots) for $h = 0.1$.



Notice that the approximation is worst where the function is changing rapidly. This should not be too surprising. Recall that we're using tangent lines to get the approximations and so the value of the tangent line at a given t will often be significantly different than the function due to the rapidly changing function at that point.

Also, in this case, because the function ends up fairly flat as t increases, the tangents start looking like the function itself and so the approximations are very accurate. This won't always be the case of course.

Let's take a look at one more example.

Example 3 For the IVP

$$y' - y = -\frac{1}{2}e^{\frac{t}{2}}\sin(5t) + 5e^{\frac{t}{2}}\cos(5t) \quad y(0) = 0$$

Use Euler's Method to find the approximation to the solution at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

We'll leave it to you to check the details of the solution process. The solution to this linear first order differential equation is.

$$y(t) = e^{\frac{t}{2}}\sin(5t)$$

Here are two tables giving the approximations and the percentage error for each approximation.

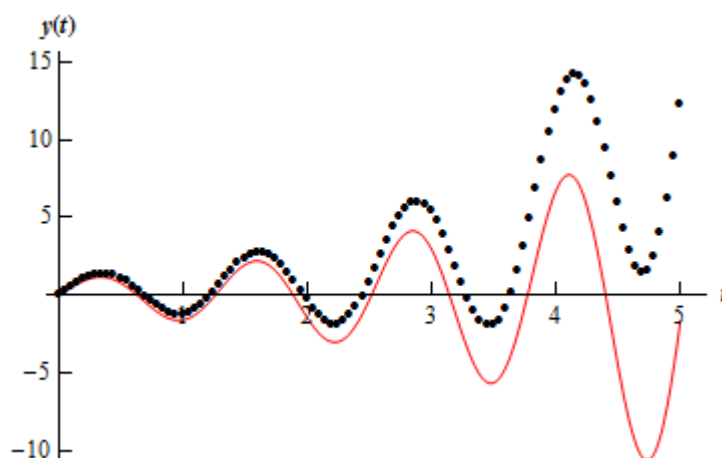
Time	Exact	Approximations				
		$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-1.23907	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

Time	Percentage Errors				
	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	38.54 %	19.98 %	4.12 %	2.07 %	0.42 %
$t = 2$	144.14 %	76.79 %	16.21 %	8.16 %	1.64 %
$t = 3$	150.55 %	83.46 %	18.20 %	9.20 %	1.86 %
$t = 4$	130.68 %	75.64 %	17.08 %	8.67 %	1.76 %
$t = 5$	1461.63 %	859.14 %	196.79 %	100.12 %	20.30 %

So, with this example Euler's Method does not do nearly as well as it did on the first IVP. Some of the observations we made in Example 2 are still true however. Decreasing the size of h decreases the error as we saw with the last example and would expect to happen. Also, as we saw in the last example, decreasing h by a factor of 10 also decreases the error by about a factor of 10.

However, unlike the last example increasing t sees an increasing error. This behavior is fairly common in the approximations. We shouldn't expect the error to decrease as t increases as we saw in the last example. Each successive approximation is found using a previous approximation. Therefore, at each step we introduce error and so approximations should, in general, get worse as t increases.

Below is a graph of the solution (the line) as well as the approximations (the dots) for $h = 0.05$.



As we can see the approximations do follow the general shape of the solution, however, the error is clearly getting much worse as t increases.

So, Euler's method is a nice method for approximating fairly nice solutions that don't change rapidly. However, not all solutions will be this nicely behaved. There are other approximation methods that do a much better job of approximating solutions. These are not the focus of this course however, so I'll leave it to you to look further into this field if you are interested.

Also notice that we don't generally have the actual solution around to check the accuracy of the approximation. We generally try to find bounds on the error for each method that will tell us how well an approximation should do. These error bounds are again not really the focus of this course, so I'll leave these to you as well if you're interested in looking into them.

