CALCULUS II

Vectors

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Chapter 5 : Vectors

This is a fairly short chapter. We will be taking a brief look at vectors and some of their properties. We will need some of this material in the next chapter and those of you heading on towards Calculus III will use a fair amount of this there as well.

Here is a list of topics in this chapter.

Basic Concepts – In this section we will introduce some common notation for vectors as well as some of the basic concepts about vectors such as the magnitude of a vector and unit vectors. We also illustrate how to find a vector from its starting and end points.

Vector Arithmetic – In this section we will discuss the mathematical and geometric interpretation of the sum and difference of two vectors. We also define and give a geometric interpretation for scalar multiplication. We also give some of the basic properties of vector arithmetic and introduce the common *i*, *j*, *k* notation for vectors.

Dot Product – In this section we will define the dot product of two vectors. We give some of the basic properties of dot products and define orthogonal vectors and show how to use the dot product to determine if two vectors are orthogonal. We also discuss finding vector projections and direction cosines in this section.

Cross Product – In this section we define the cross product of two vectors and give some of the basic facts and properties of cross products.

Section 5-1 : Vectors - The Basics

Let's start this section off with a quick discussion on what vectors are used for. Vectors are used to represent quantities that have both a magnitude and a direction. Good examples of quantities that can be represented by vectors are force and velocity. Both of these have a direction and a magnitude.

Let's consider force for a second. A force of say 5 Newtons that is applied in a particular direction can be applied at any point in space. In other words, the point where we apply the force does not change the force itself. Forces are independent of the point of application. To define a force all we need to know is the magnitude of the force and the direction that the force is applied in.

The same idea holds more generally with vectors. Vectors only impart magnitude and direction. They don't impart any information about where the quantity is applied. This is an important idea to always remember in the study of vectors.

In a graphical sense vectors are represented by directed line segments. The length of the line segment is the magnitude of the vector and the direction of the line segment is the direction of the vector. However, because vectors don't impart any information about where the quantity is applied any directed line segment with the same length and direction will represent the same vector.

Consider the sketch below.
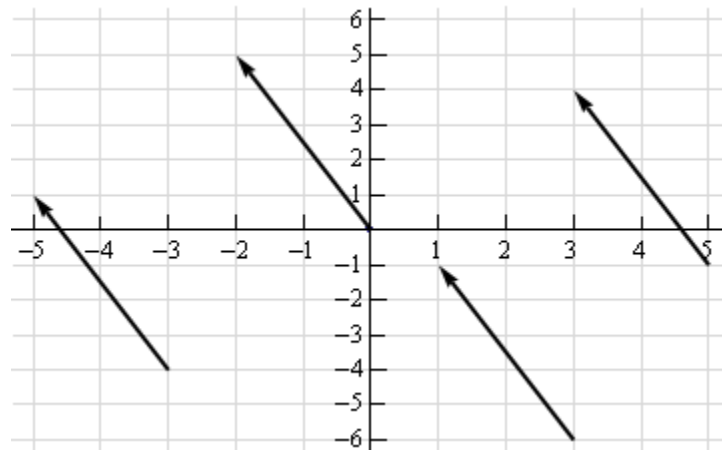
Each of the directed line segments in the sketch represents the same vector. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up. The notation that we'll use for this vector is,

$$\vec{v} = \langle -2, 5 \rangle$$

and each of the directed line segments in the sketch are called **representations** of the vector.

Be careful to distinguish vector notation, $\langle -2, 5 \rangle$, from the notation we use to represent coordinates of points, $(-2, 5)$. The vector denotes a magnitude and a direction of a quantity while the point denotes a location in space. So don't mix the notations up!

A representation of the vector $\vec{v} = \langle a_1, a_2 \rangle$ in two dimensional space is any directed line segment, $\overrightarrow{AB}$, from the point $A = (x, y)$ to the point $B = (x + a_1, y + a_2)$. Likewise a representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ in three dimensional space is any directed line segment, $\overrightarrow{AB}$, from the point $A = (x, y, z)$ to the point $B = (x + a_1, y + a_2, z + a_3)$.

Note that there is very little difference between the two dimensional and three dimensional formulas above. To get from the three dimensional formula to the two dimensional formula all we did is take out the third component/coordinate. Because of this most of the formulas here are given only in their three dimensional version. If we need them in their two dimensional form we can easily modify the three dimensional form.

There is one representation of a vector that is special in some way. The representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ that starts at the point $A = (0, 0, 0)$ and ends at the point $B = (a_1, a_2, a_3)$ is called the **position vector** of the point (a_1, a_2, a_3). So, when we talk about position vectors we are specifying the initial and final point of the vector.

Position vectors are useful if we ever need to represent a point as a vector. As we'll see there are times in which we definitely are going to want to represent points as vectors. In fact, we're going to run into topics that can only be done if we represent points as vectors.

Next, we need to discuss briefly how to generate a vector given the initial and final points of the representation. Given the two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ the vector with the representation $\overrightarrow{AB}$ is,

$$\vec{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

Note that we have to be very careful with direction here. The vector above is the vector that starts at A and ends at B. The vector that starts at B and ends at A, *i.e.* with representation $\overrightarrow{BA}$ is,

$$\vec{w} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

These two vectors are different and so we do need to always pay attention to what point is the starting point and what point is the ending point. When determining the vector between two points we always subtract the initial point from the terminal point.

Example 1 Give the vector for each of the following.

    (a) The vector from $(2, -7, 0)$ to $(1, -3, -5)$.

    (b) The vector from $(1, -3, -5)$ to $(2, -7, 0)$.

    (c) The position vector for $(-90, 4)$

Solution
(a) Remember that to construct this vector we subtract coordinates of the starting point from the ending point.

$$\langle 1-2,-3-(-7),-5-0 \rangle = \langle -1,4,-5 \rangle$$

(b) Same thing here.

$$\langle 2-1,-7-(-3),0-(-5) \rangle = \langle 1,-4,5 \rangle$$

Notice that the only difference between the first two is the signs are all opposite. This difference is important as it is this difference that tells us that the two vectors point in opposite directions.

(c) Not much to this one other than acknowledging that the position vector of a point is nothing more than a vector with the point's coordinates as its components.

$$\langle -90,4 \rangle$$

We now need to start discussing some of the basic concepts that we will run into on occasion.

Magnitude

The **magnitude**, or **length**, of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ is given by,

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 2 Determine the magnitude of each of the following vectors.

(a) $\vec{a} = \langle 3,-5,10 \rangle$

(b) $\vec{u} = \left\langle \dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}} \right\rangle$

(c) $\vec{w} = \langle 0,0 \rangle$

(d) $\vec{i} = \langle 1,0,0 \rangle$

Solution
There isn't too much to these other than plug into the formula.

(a) $\|\vec{a}\| = \sqrt{9+25+100} = \sqrt{134}$

(b) $\|\vec{u}\| = \sqrt{\dfrac{1}{5} + \dfrac{4}{5}} = \sqrt{1} = 1$

(c) $\|\vec{w}\| = \sqrt{0+0} = 0$

(d) $\|\vec{i}\| = \sqrt{1+0+0} = 1$

We also have the following fact about the magnitude.

$$\text{If } \|\vec{a}\| = 0 \text{ then } \vec{a} = \vec{0}$$

This should make sense. Because we square all the components the only way we can get zero out of the formula was for the components to be zero in the first place.

Unit Vector

Any vector with magnitude of 1, *i.e.* $\|\vec{u}\| = 1$, is called a **unit vector**.

Example 3 Which of the vectors from Example 2 are unit vectors?

Solution
Both the second and fourth vectors had a length of 1 and so they are the only unit vectors from the first example.

Zero Vector

The vector $\vec{w} = \langle 0, 0 \rangle$ that we saw in the first example is called a zero vector since its components are all zero. Zero vectors are often denoted by $\vec{0}$. Be careful to distinguish 0 (the number) from $\vec{0}$ (the vector). The number 0 denotes the origin in space, while the vector $\vec{0}$ denotes a vector that has no magnitude or direction.

Standard Basis Vectors

The fourth vector from the second example, $\vec{i} = \langle 1, 0, 0 \rangle$, is called a **standard basis vector**. In three dimensional space there are three standard basis vectors,

$$\vec{i} = \langle 1, 0, 0 \rangle \qquad \vec{j} = \langle 0, 1, 0 \rangle \qquad \vec{k} = \langle 0, 0, 1 \rangle$$

In two dimensional space there are two standard basis vectors,

$$\vec{i} = \langle 1, 0 \rangle \qquad \vec{j} = \langle 0, 1 \rangle$$

Note that standard basis vectors are also unit vectors.

Warning

We are pretty much done with this section however, before proceeding to the next section we should point out that vectors are not restricted to two dimensional or three dimensional space. Vectors can exist in general n-dimensional space. The general notation for a n-dimensional vector is,

$$\vec{v} = \langle a_1, a_2, a_3, \ldots, a_n \rangle$$

and each of the a_i's are called **components** of the vector.

Because we will be working almost exclusively with two and three dimensional vectors in this course most of the formulas will be given for the two and/or three dimensional cases. However, most of the concepts/formulas will work with general vectors and the formulas are easily (and naturally) modified for general n-dimensional vectors. Also, because it is easier to visualize things in two dimensions most of the figures related to vectors will be two dimensional figures.

So, we need to be careful to not get too locked into the two or three dimensional cases from our discussions in this chapter. We will be working in these dimensions either because it's easier to visualize the situation or because physical restrictions of the problems will enforce a dimension upon us.
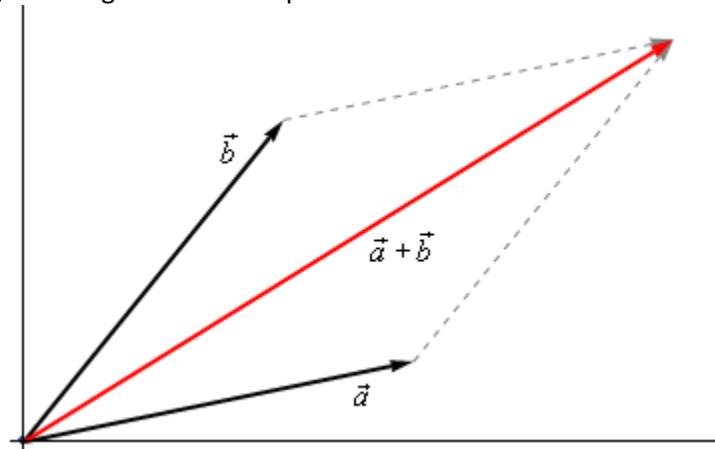
Section 5-2 : Vector Arithmetic

In this section we need to have a brief discussion of vector arithmetic.

We'll start with **addition** of two vectors. So, given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the addition of the two vectors is given by the following formula.

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

The following figure gives the geometric interpretation of the addition of two vectors.
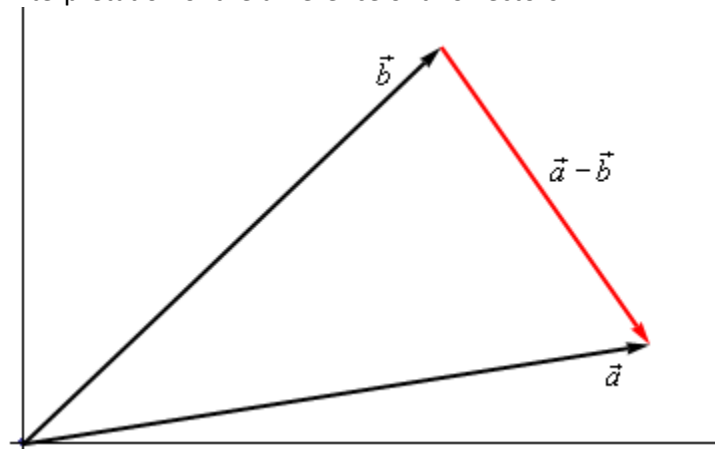

This is sometimes called the **parallelogram law** or **triangle law**.

Computationally, **subtraction** is very similar. Given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the difference of the two vectors is given by,

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Here is the geometric interpretation of the difference of two vectors.

It is a little harder to see this geometric interpretation. To help see this let's instead think of subtraction as the addition of $\vec{a}$ and $-\vec{b}$. First, as we'll see in a bit $-\vec{b}$ is the same vector as $\vec{b}$ with opposite signs on all the components. In other words, $-\vec{b}$ goes in the opposite direction as $\vec{b}$. Here is the vector set up for $\vec{a} + \left(-\vec{b}\right)$.
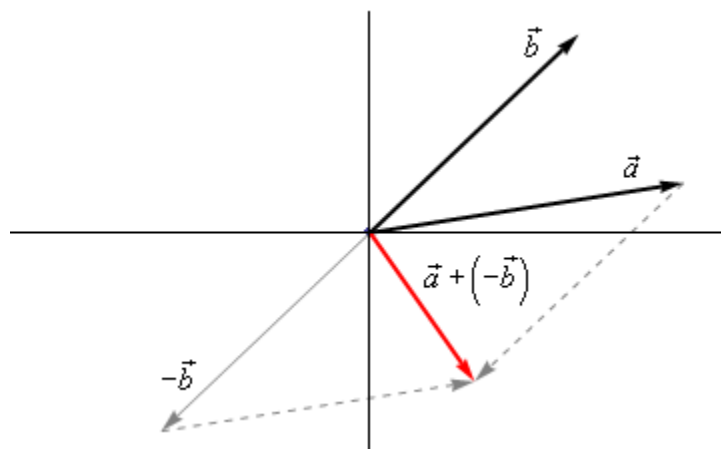
As we can see from this figure we can move the vector representing $\vec{a} + \left(-\vec{b}\right)$ to the position we've got in the first figure showing the difference of the two vectors.

Note that we can't add or subtract two vectors unless they have the same number of components. If they don't have the same number of components then addition and subtraction can't be done.

The next arithmetic operation that we want to look at is **scalar multiplication**. Given the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and any number c the scalar multiplication is,

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

So, we multiply all the components by the constant c. To see the geometric interpretation of scalar multiplication let's take a look at an example.
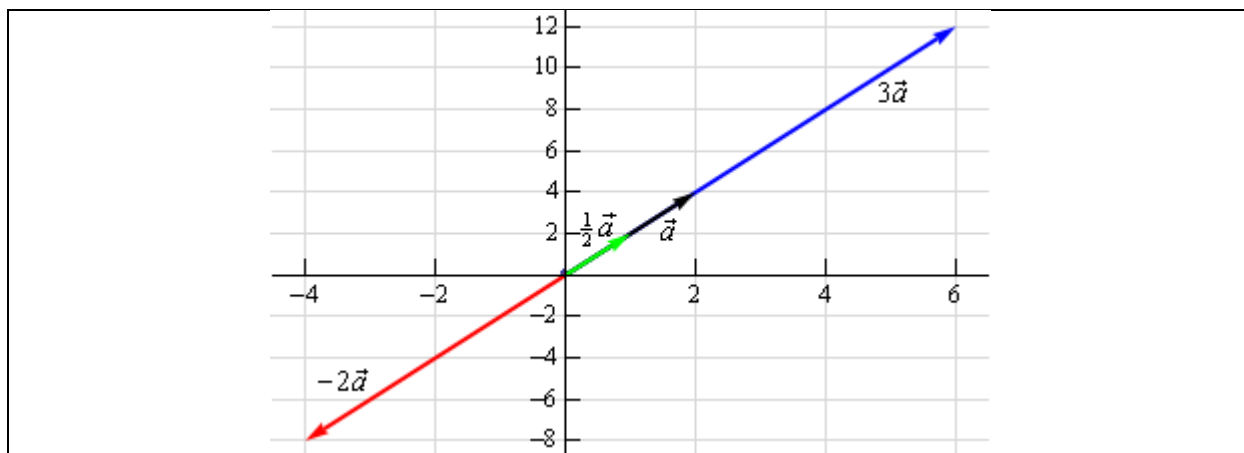

Example 1 For the vector $\vec{a} = \langle 2, 4 \rangle$ compute $3\vec{a}$, $\frac{1}{2}\vec{a}$ and $-2\vec{a}$. Graph all four vectors on the same axis system.

Solution
Here are the three scalar multiplications.

$$3\vec{a} = \langle 6, 12 \rangle \qquad \frac{1}{2}\vec{a} = \langle 1, 2 \rangle \qquad -2\vec{a} = \langle -4, -8 \rangle$$

Here is the graph for each of these vectors.

In the previous example we can see that if *c* is positive all scalar multiplication will do is stretch (if $c > 1$) or shrink (if $c < 1$) the original vector, but it won't change the direction. Likewise, if *c* is negative scalar multiplication will switch the direction so that the vector will point in exactly the opposite direction and it will again stretch or shrink the magnitude of the vector depending upon the size of *c*.

There are several nice applications of scalar multiplication that we should now take a look at.

The first is parallel vectors. This is a concept that we will see quite a bit over the next couple of sections. Two vectors are parallel if they have the same direction or are in exactly opposite directions. Now, recall again the geometric interpretation of scalar multiplication. When we performed scalar multiplication we generated new vectors that were parallel to the original vectors (and each other for that matter).

So, let's suppose that $\vec{a}$ and $\vec{b}$ are parallel vectors. If they are parallel then there must be a number *c* so that,

$$\vec{a} = c\vec{b}$$

So, two vectors are parallel if one is a scalar multiple of the other.

Example 2 Determine if the sets of vectors are parallel or not.

     (a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

     (b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

Solution

(a) These two vectors are parallel since $\vec{b} = -3\vec{a}$

(b) These two vectors aren't parallel. This can be seen by noticing that $4\left(\frac{1}{2}\right) = 2$ and yet $10\left(\frac{1}{2}\right) = 5 \neq -9$. In other words, we can't make $\vec{a}$ be a scalar multiple of $\vec{b}$.

The next application is best seen in an example.

Example 3 Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2, 1 \rangle$.

Solution
Okay, what we're asking for is a new parallel vector (points in the same direction) that happens to be a unit vector. We can do this with a scalar multiplication since all scalar multiplication does is change the length of the original vector (along with possibly flipping the direction to the opposite direction).

Here's what we'll do. First let's determine the magnitude of $\vec{w}$.
$$\|\vec{w}\| = \sqrt{25 + 4 + 1} = \sqrt{30}$$

Now, let's form the following new vector,
$$\vec{u} = \frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{\sqrt{30}}\langle -5, 2, 1 \rangle = \left\langle -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right\rangle$$

The claim is that this is a unit vector. That's easy enough to check
$$\|\vec{u}\| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = \sqrt{\frac{30}{30}} = 1$$

This vector also points in the same direction as $\vec{w}$ since it is only a scalar multiple of $\vec{w}$ and we used a positive multiple.

So, in general, given a vector $\vec{w}$, $\vec{u} = \dfrac{\vec{w}}{\|\vec{w}\|}$ will be a unit vector that points in the same direction as $\vec{w}$.

Standard Basis Vectors Revisited
In the previous section we introduced the idea of standard basis vectors without really discussing why they were important. We can now do that. Let's start with the vector
$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

We can use the addition of vectors to break this up as follows,
$$\begin{aligned}\vec{a} &= \langle a_1, a_2, a_3 \rangle \\ &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \end{aligned}$$

Using scalar multiplication we can further rewrite the vector as,
$$\begin{aligned}\vec{a} &= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \end{aligned}$$

Finally, notice that these three new vectors are simply the three standard basis vectors for three dimensional space.
$$\langle a_1, a_2, a_3 \rangle = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

So, we can take any vector and write it in terms of the standard basis vectors. From this point on we will use the two notations interchangeably so make sure that you can deal with both notations.

Example 4 If $\vec{a} = \langle 3, -9, 1 \rangle$ and $\vec{w} = -\vec{i} + 8\vec{k}$ compute $2\vec{a} - 3\vec{w}$.

Solution
In order to do the problem we'll convert to one notation and then perform the indicated operations.
$$2\vec{a} - 3\vec{w} = 2\langle 3, -9, 1 \rangle - 3\langle -1, 0, 8 \rangle$$
$$= \langle 6, -18, 2 \rangle - \langle -3, 0, 24 \rangle$$
$$= \langle 9, -18, -22 \rangle$$

We will leave this section with some basic properties of vector arithmetic.

Properties

If $\vec{v}$, $\vec{w}$ and $\vec{u}$ are vectors (each with the same number of components) and a and b are two numbers then we have the following properties.

$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \qquad\qquad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{v} + \vec{0} = \vec{v} \qquad\qquad 1\vec{v} = \vec{v}$$

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \qquad\qquad (a + b)\vec{v} = a\vec{v} + b\vec{v}$$

The proofs of these are pretty much just "computation" proofs so we'll prove one of them and leave the others to you to prove.

Proof of $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$

We'll start with the two vectors, $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n \rangle$ and yes we did mean for these to each have *n* components. The theorem works for general vectors so we may as well do the proof for general vectors.

Now, as noted above this is pretty much just a "computational" proof. What that means is that we'll compute the left side and then do some basic arithmetic on the result to show that we can make the left side look like the right side. Here is the work.

$$a(\vec{v} + \vec{w}) = a\left(\langle v_1, v_2, \ldots, v_n \rangle + \langle w_1, w_2, \ldots, w_n \rangle \right)$$
$$= a\langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$$
$$= \langle a(v_1 + w_1), a(v_2 + w_2), \ldots, a(v_n + w_n) \rangle$$
$$= \langle av_1 + aw_1, av_2 + aw_2, \ldots, av_n + aw_n \rangle$$
$$= \langle av_1, av_2, \ldots, av_n \rangle + \langle aw_1, aw_2, \ldots, aw_n \rangle$$
$$= a\langle v_1, v_2, \ldots, v_n \rangle + a\langle w_1, w_2, \ldots, w_n \rangle = a\vec{v} + a\vec{w}$$

Section 5-3 : Dot Product

The next topic for discussion is that of the dot product. Let's jump right into the definition of the dot product. Given the two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the dot product is,

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1}$$

Sometimes the dot product is called the **scalar product**. The dot product is also an example of an inner product and so on occasion you may hear it called an inner product.

Example 1 Compute the dot product for each of the following.

       (a) $\vec{v} = 5\vec{i} - 8\vec{j}$, $\vec{w} = \vec{i} + 2\vec{j}$

       (b) $\vec{a} = \langle 0, 3, -7 \rangle$, $\vec{b} = \langle 2, 3, 1 \rangle$

Solution

Not much to do with these other than use the formula.

(a) $\vec{v} \cdot \vec{w} = 5 - 16 = -11$

(b) $\vec{a} \cdot \vec{b} = 0 + 9 - 7 = 2$

Here are some properties of the dot product.

Properties

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \qquad\qquad (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$$

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \qquad\qquad\qquad\qquad \vec{v} \cdot \vec{0} = 0$$

$$\vec{v} \cdot \vec{v} = \left\| \vec{v} \right\|^2 \qquad\qquad\qquad\qquad \text{If } \vec{v} \cdot \vec{v} = 0 \text{ then } \vec{v} = \vec{0}$$

The proofs of these properties are mostly "computational" proofs and so we're only going to do a couple of them and leave the rest to you to prove.

Proof of $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

We'll start with the three vectors, $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$, $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n \rangle$ and yes we did mean for these to each have *n* components. The theorem works for general vectors so we may as well do the proof for general vectors.

Now, as noted above this is pretty much just a "computational" proof. What that means is that we'll compute the left side and then do some basic arithmetic on the result to show that we can make the left side look like the right side. Here is the work.

$$\vec{u} \bullet (\vec{v} + \vec{w}) = \langle u_1, u_2, \ldots, u_n \rangle \bullet (\langle v_1, v_2, \ldots, v_n \rangle + \langle w_1, w_2, \ldots, w_n \rangle)$$
$$= \langle u_1, u_2, \ldots, u_n \rangle \bullet \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$$
$$= u_1 (v_1 + w_1), u_2 (v_2 + w_2), \ldots, u_n (v_n + w_n)$$
$$= u_1 v_1 + u_1 w_1, u_2 v_2 + u_2 w_2, \ldots, u_n v_n + u_n w_n$$
$$= (u_1 v_1, u_2 v_2, \ldots, u_n v_n) + (u_1 w_1, u_2 w_2, \ldots, u_n w_n)$$
$$= \langle u_1, u_2, \ldots, u_n \rangle \bullet \langle v_1, v_2, \ldots, v_n \rangle + \langle u_1, u_2, \ldots, u_n \rangle \bullet \langle w_1, w_2, \ldots, w_n \rangle$$
$$= \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$$

Proof of : If $\vec{v} \bullet \vec{v} = 0$ then $\vec{v} = \vec{0}$

This is a pretty simple proof. Let's start with $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and compute the dot product.
$$\vec{v} \bullet \vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \bullet \langle v_1, v_2, \ldots, v_n \rangle$$
$$= v_1^2 + v_2^2 + \cdots + v_n^2$$
$$= 0$$

Now, since we know $v_i^2 \geq 0$ for all *i* then the only way for this sum to be zero is to in fact have $v_i^2 = 0$. This in turn however means that we must have $v_i = 0$ and so we must have had $\vec{v} = \vec{0}$.

There is also a nice geometric interpretation to the dot product. First suppose that θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \leq \theta \leq \pi$ as shown in the image below.
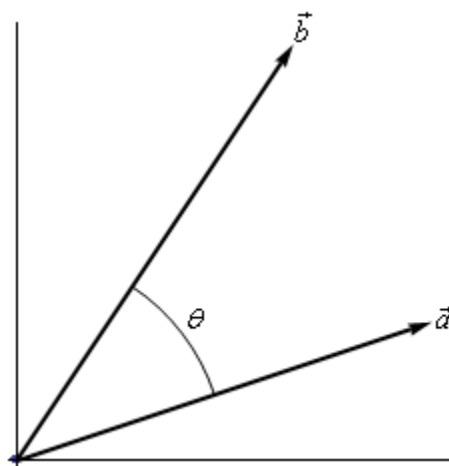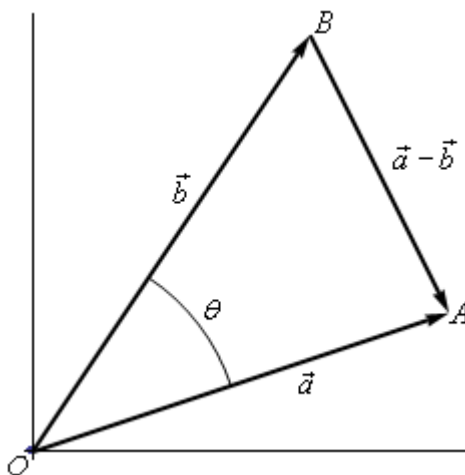

We can then have the following theorem.

Theorem

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos \theta \qquad\qquad (2)$$

Proof

Let's give a modified version of the sketch above.

The three vectors above form the triangle *AOB* and note that the length of each side is nothing more than the magnitude of the vector forming that side.

The Law of Cosines tells us that,

$$\left\|\vec{a} - \vec{b}\right\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \, \|\vec{b}\| \cos \theta$$

Also using the properties of dot products we can write the left side as,

$$\left\|\vec{a} - \vec{b}\right\|^2 = \left(\vec{a} - \vec{b}\right) \bullet \left(\vec{a} - \vec{b}\right)$$
$$= \vec{a} \bullet \vec{a} - \vec{a} \bullet \vec{b} - \vec{b} \bullet \vec{a} + \vec{b} \bullet \vec{b}$$
$$= \|\vec{a}\|^2 - 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2$$

Our original equation is then,

$$\left\|\vec{a} - \vec{b}\right\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \, \|\vec{b}\| \cos \theta$$
$$\|\vec{a}\|^2 - 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \, \|\vec{b}\| \cos \theta$$
$$-2\vec{a} \bullet \vec{b} = -2\|\vec{a}\| \, \|\vec{b}\| \cos \theta$$
$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos \theta$$

The formula from this theorem is often used not to compute a dot product but instead to find the angle between two vectors. Note as well that while the sketch of the two vectors in the proof is for two dimensional vectors the theorem is valid for vectors of any dimension (as long as they have the same dimension of course).

Let's see an example of this.

Example 2 Determine the angle between $\vec{a} = \langle 3, -4, -1 \rangle$ and $\vec{b} = \langle 0, 5, 2 \rangle$.

Solution
We will need the dot product as well as the magnitudes of each vector.

$$\vec{a} \bullet \vec{b} = -22 \qquad \qquad \|\vec{a}\| = \sqrt{26} \qquad \qquad \|\vec{b}\| = \sqrt{29}$$

The angle is then,

$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} = \frac{-22}{\sqrt{26}\sqrt{29}} = -0.8011927$$

$$\theta = \cos^{-1}(-0.8011927) = 2.5 \text{ radians} = 143.24 \text{ degrees}$$

The dot product gives us a very nice method for determining if two vectors are perpendicular and it will give another method for determining when two vectors are parallel. Note as well that often we will use the term **orthogonal** in place of perpendicular.

Now, if two vectors are orthogonal then we know that the angle between them is 90 degrees. From (2) this tells us that if two vectors are orthogonal then,

$$\vec{a} \bullet \vec{b} = 0$$

Likewise, if two vectors are parallel then the angle between them is either 0 degrees (pointing in the same direction) or 180 degrees (pointing in the opposite direction). Once again using (2) this would mean that one of the following would have to be true.

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \ (\theta = 0°) \qquad \text{OR} \qquad \vec{a} \bullet \vec{b} = -\|\vec{a}\| \, \|\vec{b}\| \ (\theta = 180°)$$

Example 3 Determine if the following vectors are parallel, orthogonal, or neither.

  (a) $\vec{a} = \langle 6, -2, -1 \rangle$, $\vec{b} = \langle 2, 5, 2 \rangle$

  (b) $\vec{u} = 2\vec{i} - \vec{j}$, $\vec{v} = -\dfrac{1}{2}\vec{i} + \dfrac{1}{4}\vec{j}$

Solution
(a) First get the dot product to see if they are orthogonal.

$$\vec{a} \bullet \vec{b} = 12 - 10 - 2 = 0$$

The two vectors are orthogonal.

(b) Again, let's get the dot product first.

$$\vec{u} \cdot \vec{v} = -1 - \frac{1}{4} = -\frac{5}{4}$$

So, they aren't orthogonal. Let's get the magnitudes and see if they are parallel.

$$\|\vec{u}\| = \sqrt{5} \qquad \|\vec{v}\| = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$$

Now, notice that,

$$\vec{u} \cdot \vec{v} = -\frac{5}{4} = -\sqrt{5}\left(\frac{\sqrt{5}}{4}\right) = -\|\vec{u}\|\,\|\vec{v}\|$$

So, the two vectors are parallel.

There are several nice applications of the dot product as well that we should look at.

Projections

The best way to understand projections is to see a couple of sketches. So, given two vectors $\vec{a}$ and $\vec{b}$ we want to determine the projection of $\vec{b}$ onto $\vec{a}$. The projection is denoted by $\text{proj}_{\vec{a}}\,\vec{b}$. Here are a couple of sketches illustrating the projection.
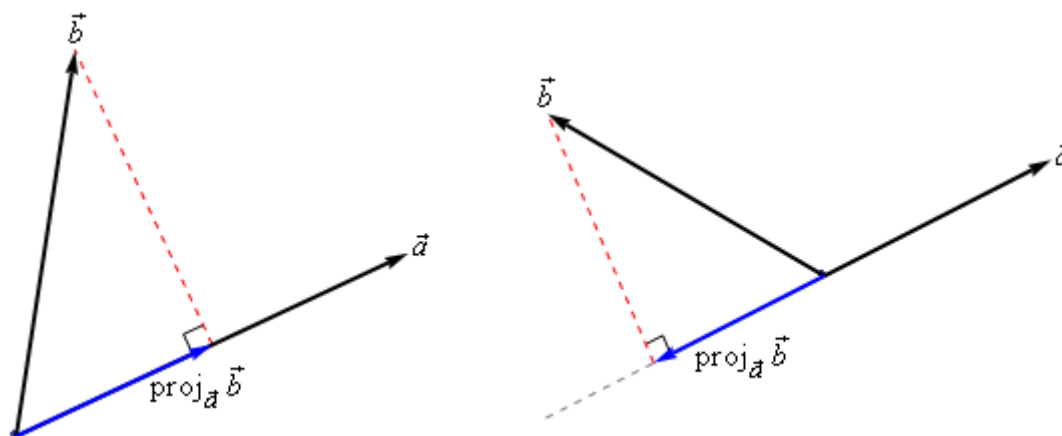
So, to get the projection of $\vec{b}$ onto $\vec{a}$ we drop straight down from the end of $\vec{b}$ until we hit (and form a right angle) with the line that is parallel to $\vec{a}$. The projection is then the vector that is parallel to $\vec{a}$, starts at the same point both of the original vectors started at and ends where the dashed line hits the line parallel to $\vec{a}$.

There is a nice formula for finding the projection of $\vec{b}$ onto $\vec{a}$. Here it is,

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2}\,\vec{a}$$

Note that we also need to be very careful with notation here. The projection of $\vec{a}$ onto $\vec{b}$ is given by

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\bullet\vec{b}}{\left\|\vec{b}\right\|^{2}}\,\vec{b}$$

We can see that this will be a totally different vector. This vector is parallel to $\vec{b}$, while $\text{proj}_{\vec{a}}\,\vec{b}$ is parallel to $\vec{a}$. So, be careful with notation and make sure you are finding the correct projection.

Here's an example.

Example 4 Determine the projection of $\vec{b} = \langle 2,1,-1 \rangle$ onto $\vec{a} = \langle 1,0,-2 \rangle$.

Solution
We need the dot product and the magnitude of $\vec{a}$.

$$\vec{a}\bullet\vec{b} = 4 \qquad \left\|\vec{a}\right\|^{2} = 5$$

The projection is then,

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\bullet\vec{b}}{\left\|\vec{a}\right\|^{2}}\,\vec{a}$$

$$= \frac{4}{5}\langle 1,0,-2 \rangle$$

$$= \left\langle \frac{4}{5},0,-\frac{8}{5} \right\rangle$$

For comparison purposes let's do it the other way around as well.

Example 5 Determine the projection of $\vec{a} = \langle 1,0,-2 \rangle$ onto $\vec{b} = \langle 2,1,-1 \rangle$.

Solution
We need the dot product and the magnitude of $\vec{b}$.

$$\vec{a}\bullet\vec{b} = 4 \qquad \left\|\vec{b}\right\|^{2} = 6$$

The projection is then,

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\bullet\vec{b}}{\left\|\vec{b}\right\|^{2}}\,\vec{b}$$

$$= \frac{4}{6}\langle 2,1,-1 \rangle$$

$$= \left\langle \frac{4}{3},\frac{2}{3},-\frac{2}{3} \right\rangle$$

As we can see from the previous two examples the two projections are different so be careful.

Direction Cosines
This application of the dot product requires that we be in three dimensional space unlike all the other applications we've looked at to this point.

Let's start with a vector, $\vec{a}$, in three dimensional space. This vector will form angles with the *x*-axis (α), the *y*-axis (β), and the *z*-axis (γ). These angles are called **direction angles** and the cosines of these angles are called **direction cosines**.

Here is a sketch of a vector and the direction angles.
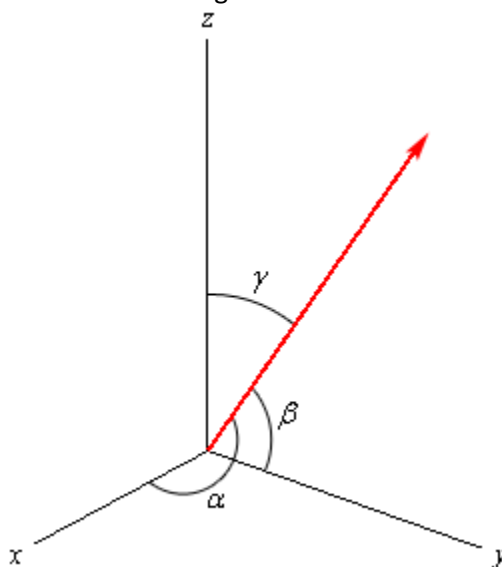
The formulas for the direction cosines are,

$$\cos\alpha = \frac{\vec{a}\bullet\vec{i}}{\|\vec{a}\|} = \frac{a_1}{\|\vec{a}\|} \qquad \cos\beta = \frac{\vec{a}\bullet\vec{j}}{\|\vec{a}\|} = \frac{a_2}{\|\vec{a}\|} \qquad \cos\gamma = \frac{\vec{a}\bullet\vec{k}}{\|\vec{a}\|} = \frac{a_3}{\|\vec{a}\|}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the standard basis vectors.

Let's verify the first dot product above. We'll leave the rest to you to verify.
$$\vec{a}\bullet\vec{i} = \langle a_1, a_2, a_3\rangle\bullet\langle 1,0,0\rangle = a_1$$

Here are a couple of nice facts about the direction cosines.

1. The vector $\vec{u} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$ is a unit vector.
2. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
3. $\vec{a} = \|\vec{a}\|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$

Let's do a quick example involving direction cosines.

Example 6 Determine the direction cosines and direction angles for $\vec{a} = \langle 2, 1, -4 \rangle$.

Solution
We will need the magnitude of the vector.
$$\|\vec{a}\| = \sqrt{4 + 1 + 16} = \sqrt{21}$$
The direction cosines and angles are then,

$$\cos\alpha = \frac{2}{\sqrt{21}} \qquad\qquad \alpha = 1.119 \text{ radians} = 64.123 \text{ degrees}$$

$$\cos\beta = \frac{1}{\sqrt{21}} \qquad\qquad \beta = 1.351 \text{ radians} = 77.396 \text{ degrees}$$

$$\cos\gamma = \frac{-4}{\sqrt{21}} \qquad\qquad \gamma = 2.632 \text{ radians} = 150.794 \text{ degrees}$$

Section 5-4 : Cross Product

In this final section of this chapter we will look at the cross product of two vectors. We should note that the cross product requires both of the vectors to be three dimensional vectors.

Also, before getting into how to compute these we should point out a major difference between dot products and cross products. The result of a dot product is a number and the result of a cross product is a vector! Be careful not to confuse the two.

So, let's start with the two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product is given by the formula,

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

This is not an easy formula to remember. There are two ways to derive this formula. Both of them use the fact that the cross product is really the determinant of a 3x3 matrix. If you don't know what this is that is don't worry about it. You don't need to know anything about matrices or determinants to use either of the methods. The notation for the determinant is as follows,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The first row is the standard basis vectors and must appear in the order given here. The second row is the components of $\vec{a}$ and the third row is the components of $\vec{b}$. Now, let's take a look at the different methods for getting the formula.

The first method uses the Method of Cofactors. If you don't know the method of cofactors that is fine, the result is all that we need. Here is the formula.

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

where,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

This formula is not as difficult to remember as it might at first appear to be. First, the terms alternate in sign and notice that the 2x2 is missing the column below the standard basis vector that multiplies it as well as the row of standard basis vectors.

The second method is slightly easier; however, many textbooks don't cover this method as it will only work on 3x3 determinants. This method says to take the determinant as listed above and then copy the first two columns onto the end as shown below.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix}\begin{matrix}\vec{i} & \vec{j} \\ a_1 & a_2 \\ b_1 & b_2\end{matrix}$$

We now have three diagonals that move from left to right and three diagonals that move from right to left. We multiply along each diagonal and add those that move from left to right and subtract those that move from right to left.

This is best seen in an example. We'll also use this example to illustrate a fact about cross products.

Example 1 If $\vec{a}=\langle 2,1,-1\rangle$ and $\vec{b}=\langle -3,4,1\rangle$ compute each of the following.

      (a) $\vec{a}\times\vec{b}$

      (b) $\vec{b}\times\vec{a}$

Solution

(a) Here is the computation for this one.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1\end{vmatrix}\begin{matrix}\vec{i} & \vec{j} \\ 2 & 1 \\ -3 & 4\end{matrix}$$

$$=\vec{i}\,(1)(1)+\vec{j}\,(-1)(-3)+\vec{k}\,(2)(4)-\vec{j}\,(2)(1)-\vec{i}\,(-1)(4)-\vec{k}\,(1)(-3)$$

$$=5\vec{i}+\vec{j}+11\vec{k}$$

(b) And here is the computation for this one.

$$\vec{b}\times\vec{a}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & 1 \\ 2 & 1 & -1\end{vmatrix}\begin{matrix}\vec{i} & \vec{j} \\ -3 & 4 \\ 2 & 1\end{matrix}$$

$$=\vec{i}\,(4)(-1)+\vec{j}\,(1)(2)+\vec{k}\,(-3)(1)-\vec{j}\,(-3)(-1)-\vec{i}\,(1)(1)-\vec{k}\,(4)(2)$$

$$=-5\vec{i}-\vec{j}-11\vec{k}$$

Notice that switching the order of the vectors in the cross product simply changed all the signs in the result. Note as well that this means that the two cross products will point in exactly opposite directions since they only differ by a sign. We'll formalize up this fact shortly when we list several facts.

There is also a geometric interpretation of the cross product. First we will let θ be the angle between the two vectors $\vec{a}$ and $\vec{b}$ and assume that $0\le\theta\le\pi$, then we have the following fact,

$$\left\|\vec{a}\times\vec{b}\right\|=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta \tag{1}$$

and the following figure.
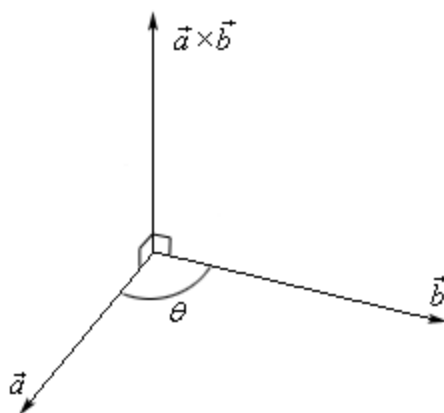
There should be a natural question at this point. How did we know that the cross product pointed in the direction that we've given it here?

First, as this figure implies, the cross product is orthogonal to both of the original vectors. This will always be the case with one exception that we'll get to in a second.

Second, we knew that it pointed in the upward direction (in this case) by the "right hand rule". This says that if we take our right hand, start at $\vec{a}$ and rotate our fingers towards $\vec{b}$ our thumb will point in the direction of the cross product. Therefore, if we'd sketched in $\vec{b} \times \vec{a}$ above we would have gotten a vector in the downward direction.

Example 2 A plane is defined by any three points that are in the plane. If a plane contains the points $P = (1,0,0)$, $Q = (1,1,1)$ and $R = (2,-1,3)$ find a vector that is orthogonal to the plane.

Solution
The one way that we know to get an orthogonal vector is to take a cross product. So, if we could find two vectors that we knew were in the plane and took the cross product of these two vectors we know that the cross product would be orthogonal to both the vectors. However, since both the vectors are in the plane the cross product would then also be orthogonal to the plane.

So, we need two vectors that are in the plane. This is where the points come into the problem. Since all three points lie in the plane any vector between them must also be in the plane. There are many ways to get two vectors between these points. We will use the following two,

$$\overrightarrow{PQ} = \langle 1-1, 1-0, 1-0 \rangle = \langle 0,1,1 \rangle$$
$$\overrightarrow{PR} = \langle 2-1, -1-0, 3-0 \rangle = \langle 1,-1,3 \rangle$$

The cross product of these two vectors will be orthogonal to the plane. So, let's find the cross product.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 0 & 1 \\ 1 & -1 \end{matrix}$$

$$= 4\vec{i} + \vec{j} - \vec{k}$$

So, the vector $4\vec{i} + \vec{j} - \vec{k}$ will be orthogonal to the plane containing the three points.

Now, let's address the one time where the cross product will not be orthogonal to the original vectors. If the two vectors, $\vec{a}$ and $\vec{b}$, are parallel then the angle between them is either 0 or 180 degrees. From (1) this implies that,

$$\left\| \vec{a} \times \vec{b} \right\| = 0$$

From a fact about the magnitude we saw in the first section we know that this implies

$$\vec{a} \times \vec{b} = \vec{0}$$

In other words, it won't be orthogonal to the original vectors since we have the zero vector. This does give us another test for parallel vectors however.

Fact

If $\vec{a} \times \vec{b} = \vec{0}$ then $\vec{a}$ and $\vec{b}$ will be parallel vectors.

Let's also formalize up the fact about the cross product being orthogonal to the original vectors.

Fact

Provided $\vec{a} \times \vec{b} \neq \vec{0}$ then $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Here are some nice properties about the cross product.

Properties

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors and c is a number then,

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \qquad\qquad (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$$

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \qquad\qquad \vec{u} \bullet (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$\vec{u} \bullet (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The determinant in the last fact is computed in the same way that the cross product is computed. We will see an example of this computation shortly.

There are a couple of geometric applications to the cross product as well. Suppose we have three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and we form the three dimensional figure shown below.
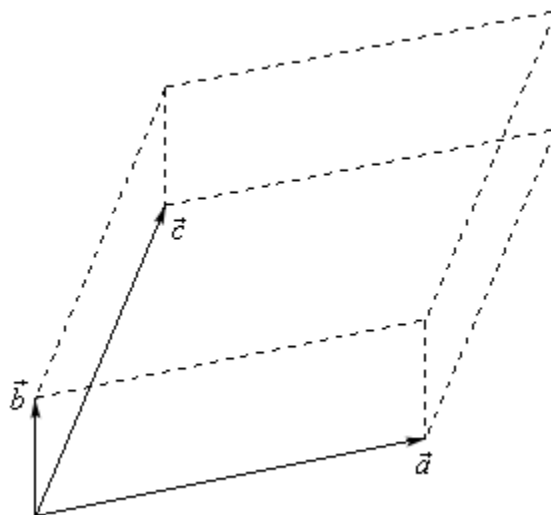
The area of the parallelogram (two dimensional front of this object) is given by,

$$\text{Area} = \left\| \vec{a} \times \vec{b} \right\|$$

and the volume of the parallelepiped (the whole three dimensional object) is given by,

$$\text{Volume} = \left| \vec{a} \bullet \left(\vec{b} \times \vec{c} \right) \right|$$

Note that the absolute value bars are required since the quantity could be negative and volume isn't negative.

We can use this volume fact to determine if three vectors lie in the same plane or not. If three vectors lie in the same plane then the volume of the parallelepiped will be zero.

Example 3 Determine if the three vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$ and $\vec{c} = \langle 0, -9, 18 \rangle$ lie in the same plane or not.

Solution
So, as we noted prior to this example all we need to do is compute the volume of the parallelepiped formed by these three vectors. If the volume is zero they lie in the same plane and if the volume isn't zero they don't lie in the same plane.

$$
\vec{a} \bullet \left(\vec{b} \times \vec{c} \right) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} \begin{matrix} 1 & 4 \\ 2 & -1 \\ 0 & -9 \end{matrix}
$$

$$
= (1)(-1)(18) + (4)(4)(0) + (-7)(2)(-9) -
$$
$$
(4)(2)(18) - (1)(4)(-9) - (-7)(-1)(0)
$$

$$
= -18 + 126 - 144 + 36
$$

$$
= 0
$$

So, the volume is zero and so they lie in the same plane.