AQA A-Level Further Maths: Practice Paper 2

Focus: Pure

Difficulty: Hard

Time: 3 hours

Marks: (Recommended Timings)

Section A (multiple choice): 10 marks (15 minutes)

Section B (standard questions): 70 marks (1 hour 45 minutes)

Section C (extended questions): 45 marks (1 hour)

(Total 125 marks)

Grade Boundaries: (approximate)

A*: 100 (80%)

A: 88 (70%)

B: 75 (60%)

C: 63 (50%)

D: 50 (40%)

Main Topics Examined:

Proof by Induction, Complex Numbers, Hyperbolic Functions,

Differential Equations, Numerical Methods, Polar Coordinates

Advice:

1. The questions in each section are in no particular order.
2. Questions are at A-level standard difficulty, with many above.
3. You may wish to take a short break between Sections B and C.
4. Plan your answers to Section C carefully before beginning.
5. Check the fully worked solutions for any questions you missed.
6. The grade boundaries are based on IAL qualifications.

Section A: Multiple choice. You are advised to spend no more than **15 minutes** in Section A.

1. Consider the function $f(x) = 2 \sinh x$ defined for all real x.
   If $f(x) \geq ax$ for all $x \geq 0$, then the possible values of a are

   O $a \geq 2$
   O $a \leq e$
   O $a \leq 2$
   O $|a| \leq 2$ [1 mark]

2. The four roots of a quartic polynomial $P(x)$ form a square when plotted in an Argand diagram. Which of these could **not** be $P(x)$?

   O $P(x) = x^4 - 1$
   O $P(x) = x^4 - 2x^2 + 2$
   O $P(x) = x^4 + 4x^3 + 6x^2 + 4x + 2$
   O $P(x) = x^4 - 4ix^3 - 6x^2 + 4ix + 2$ [1 mark]

3. Factorise $(x + y)^4 - (x - y)^4$.

   O $xy(x^2 - y^2)$
   O $2xy(x - y)^2$
   O $4(x - y)^4$
   O $8xy(x^2 + y^2)$ [1 mark]

4. The correct identity for $\cosh(x + y)$, for all real x and y, is

   O $\cosh(x + y) \equiv \sinh x \cosh x + \sinh y \cosh y$
   O $\cosh(x + y) \equiv \sinh x \sinh y + \cosh x \cosh y$
   O $\cosh(x + y) \equiv \sinh x \cosh x - \sinh y \cosh y$
   O $\cosh(x + y) \equiv \sinh x \sinh y - \cosh x \cosh y$ [1 mark]

5. A computer is used to execute a numerical method to approximate the solution to a first-order differential equation in the form dy/dx = $f(x, y)$. The standard formula for this numerical method is applied without further manipulation and the iterative formulae for y_n in this process is

$$y_n = \begin{cases} 0, & \text{if } n = 0 \\ y_{n-1} + 0.1 \times (x_{n-1}^2 + y_{n-1}^2 + 2x_{n-1}y_{n-1}), & \text{if } n = 1 \\ y_{n-2} + 0.2 \times (x_{n-1}^2 + y_{n-1}^2 + 2x_{n-1}y_{n-1}), & \text{otherwise} \end{cases}$$

If $x_0 = 1$, then based on the above information, which of these **cannot** be true?

O The differential equation is dy/dx = $(x + y)^2$

O The iterative formula for x_n is $x_n = x_{n+1} + 0.2$ for all $\{n \in \mathbb{Z} : n \geq 0\}$

O The function $y(x)$ satisfies $y(1) = 0$

O If $u_n = x_n + y_n$, then the new iterative formula approximates the solution to the differential equation du/dx = u^2.

[1 mark]

6. The circle with centre (0, 5) and radius 1 unit is revolved by 2π radians about the x-axis. The area, S, of the resulting closed surface is given by

O $$S = 2\pi \int_{-1}^{1} (5 + \sqrt{1 - x^2})\sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

O $$S = 4\pi \int_{0}^{1} (5 + \sqrt{1 - x^2})\sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

O $$S = 20\pi \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

O $$S = 40\pi \int_{0}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}}\, \mathrm{d}x$$

[1 mark]

7. The plane Π is such that any point on Π is equidistant from the points (2, -1, 3) and (8, 11, -5). The equation of Π in its simplest Cartesian form is

O $3x + 6y - 4z = 49$
O $3x + 6y - 4z = 98$
O $x + 2y - z = 0$
O $x + 2y - z = 98$ [1 mark]

8. Matrix **M** is a square, nonsingular 3 x 3 matrix representing a sequence of two transformations in 3D space.

Given that det **M** = -1, and the second transformation is a reflection in the plane $y = 0$, which of these could describe the first transformation?

O Enlargement with respect to origin, scale factor -1
O Rotation about the y-axis by 180°
O Reflection in the plane $x = 0$
O Projection onto the plane $x = y$ by the mapping $(x, y, z) \Rightarrow (x + y, x + y, z)$
[1 mark]

9. **M** is a 3 x 3 matrix representing a transformation in 3D space. **M** has an eigenvalue of 2 with corresponding eigenvector **i** - 2**j**. **M** also has a repeated eigenvalue of λ, with corresponding distinct eigenvectors **u** and **v**.

Define the plane Π as $\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and the line L as $\mathbf{r} \times (\mathbf{i} - 2\mathbf{j}) = 0$.

Which of these is **false**?

O For all nonzero λ, the plane Π is invariant under **M**.
O If λ = 1, Π is a plane of invariant points under **M**.
O If λ = 2, **M** represents an enlargement with scale factor 2.
O The line L is a line of invariant points under **M**. [1 mark]

10. Consider the function $f(x)$, defined as

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, \mathrm{d}t\ ,\ x > 0$$

Which of these functions has a discontinuity in the interval $0 < x \leq 2\pi$?

- O $f^{-1}(x)$
- O $f'(x)$
- O $f''(x)$
- O $\ln f(x+1)$

[1 mark]

Section B: Standard questions. Ensure to leave sufficient time for Section C.

11.

a. Prove De Moivre's theorem, namely $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, by induction, for all $n \in \mathbb{Z}^+$. [6 marks]

b. Use De Moivre's theorem to show that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

[7 marks]

c. i) Explain why $x = \sin^2(\pi/7)$ is a solution to the equation

$$64x^3 - 112x^2 + 56x - 7 = 0,$$

and write down the two other roots in trigonometric form. [3 marks]

ii) Hence show that the value of

$$\operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7}$$

is an integer. [3 marks]

d. The infinite series C and S are defined as

$$C = 1 + a\cos\theta + a^2\cos 2\theta + \ldots,$$
$$S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots,$$

where a is a real number such that $|a| < 1$.

i) By considering $C + \mathrm{i}S$, show that

$$C = \frac{1 - a\cos\theta}{1 + a^2 - 2a\cos\theta}$$

[7 marks]

ii) Find a similar corresponding expression for S. [1 mark]

[Total for Q11: 27 marks]

12.

a. Use Maclaurin's series for ln(1 + x) and the method of differences to prove that

$$\tanh^{-1}(x^2) = \sum_{n=1}^{\infty} \left(\frac{x^{4n-2}}{2n-1} \right) \ , \ |x| < 1$$

[6 marks]

b. Let p and q be positive real numbers and define

$$A = \ln\frac{p+q}{2} \quad \text{and} \quad B = \frac{\ln p + \ln q}{2}$$

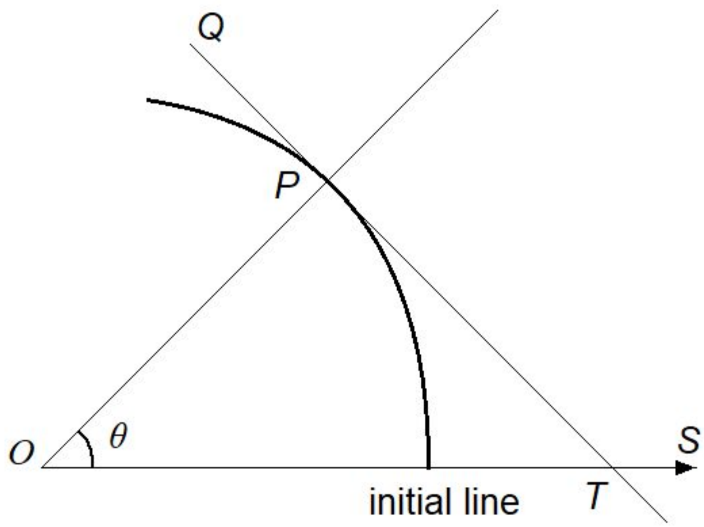
By making a suitable substitution and considering the previously obtained series expansion for $\tanh^{-1}(x^2)$, prove that

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = A - B.$$

[9 marks]

[Total for Q12: 15 marks]

13. The diagram shows part of a spiral curve. The point P has polar coordinates (r, θ) where $0 \le \theta \le \pi/2$. The points T and S lie on the initial line and O is the pole. TPQ is the tangent to the curve at P.



a. Show that the gradient of TPQ is equal to

$$\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

[4 marks]

b. The curve has polar equation $r = e^{(\cot b)\,\theta}$ where b is a constant and $0 < b < \pi/2$.

Use the result of part a) to show that the angle between the line *OP* and the tangent *TPQ* does not depend on θ. [9 marks]

[Total for Q13: 13 marks]

14.

a. y is a function of x such that

$$4x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4x \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2 + 2 \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

for all $x > 0$.

By making a substitution $t = \sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y = A - \sqrt{x} + \ln\left(1 + B\mathrm{e}^{2\sqrt{x}}\right)$$

where A and B are arbitrary constants. Fully justify your answer. [12 marks]

(More space to answer Q14.a)

b. Comment on the validity of the solution in part (a) when $B < 0$.

Fully justify your answer. [3 marks]

[Total for Q14: 15 marks]

Section C: Extended questions. Attempt **both** questions in this section.

15. By integrating with an initial substitution of $u = \tan^{2/3}(x)$, or otherwise, find the exact value of

$$\int_0^{\pi/4} \sqrt[3]{\tan(x)}\ \mathrm{d}x .$$

Give your answer in terms π, $\sqrt{3}$ and $\ln 2$.

Fully justify your answer. You are advised to plan your answer first.

Write your final answer in the space provided on the last question page.

[20 marks]

More space to answer Q15.

More space to answer Q15.

More space to answer Q15.

More space to answer Q15.

[Total for Q15: 20 marks]

16. Consider a coordinate system in which the *xy* Cartesian plane is superimposed on a polar coordinate system, with the *x*-axis parallel to the initial line and the origin at the pole.

A straight line *L*, whose gradient in the *xy* plane is -3/11, is a tangent to the curve with polar equation

$$r = 25\cos(2\theta)\ ,\quad 0 \le \theta \le \frac{\pi}{4}$$

Find, in exact form in terms of $\tan^{-1}(1/3)$, the area of the finite region bounded by the curve, the straight line *L* and the initial line.

Give your answer in the form $A(B + C\tan^{-1} 1/3)$, where *A* is rational and *B* and *C* are integers. Fully justify your answer. You are advised to plan your answer first and continue on the pages that follow.

Write your final answer in the space provided on the last question page.

[25 marks]

More space to answer Q16.

More space to answer Q16.

More space to answer Q16.

More space to answer Q16.

More space to answer Q16.

[Total for Q16: 25 marks]

Your answer to Q15:

Your answer to Q16:

End of Questions

Question Sources

Q1: Gaokao (Math)
Q10: IIT JEE Advanced

Q11: AQA and Edexcel A-Level Further Maths Past Paper
Q12: MadasMaths
Q13: AQA A-Level Further Maths Past Paper
Q14: MadasMaths

Q15: Blackpenredpen YouTube channel
Q16: MadasMaths

ENGAA is an entrance exam for studying Engineering at Cambridge.
PAT is an exam to study Physics at Oxford.
STEP I in an entrance exam for studying Maths or Physics at Oxbridge or Durham.
CSAT is an exam in South Korea.
Gaokao is an exam in China.
IIT JEE is an exam in India.