AQA A-Level Further Maths: Practice Paper 1

Focus: Pure

Difficulty: Hard

Time: 3 hours

Marks: (Recommended Timings)

Section A (multiple choice): 10 marks (15 minutes)

Section B (standard questions): 70 marks (1 hour 45 minutes)

Section C (extended questions): 45 marks (1 hour)

(Total 125 marks)

Grade Boundaries: (approximate)

A*: 100 (80%)

A: 88 (70%)

B: 75 (60%)

C: 63 (50%)

D: 50 (40%)

Main Topics Examined:

Proof by Induction, Further Algebra, Complex Numbers, Maclaurin Series, Integration, Further Vectors, Matrices, Differential Equations

Advice:

1. The questions in each section are in no particular order.
2. Questions are at A-level standard difficulty, with many above.
3. You may wish to take a short break between Sections B and C.
4. Plan your answers to Section C carefully before beginning.
5. Check the fully worked solutions for any questions you missed.
6. The grade boundaries are based on IAL qualifications.

Section A: Multiple choice. You are advised to spend no more than **15 minutes** in Section A.

1. Four matrices **A**, **B**, **C** and **D** are defined as

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 0 & 7 \\ 4 & 4 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ -9 & 8 \\ 1 & 8 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 4 \\ 3 & -1 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 5 \\ 2 \\ -10 \end{bmatrix}$$

Which of the following products is **not** defined?

O **ABC**
O **CB**
O **CD**
O **BD** [1 mark]

2. The function $f(x)$ satisfies

$$f(x) = \int_a^x \frac{1}{[f(t)]^2}\, \mathrm{d}t$$

for some real constant a and defined for all $x > a$. If $f(1) = 0$, then

O $f(x) = (3x - 3)^{1/3}$
O $f(x) = (3x + 3)^{1/3}$
O $f(x) = (3x - 3)^{-2/3}$
O $f(x) = (3x + 3)^{-2/3}$ [1 mark]

3. The vectors **a** and **b** are such that |**a**| = √10, |**b**| = 10, **a . b** = 30.

The value of |**a** x **b**| is

O 10
O √10
O 30
O √30 [1 mark]

4. The implicit Cartesian form of the polar equation $r^2 = \cot\theta$ for $0 < \theta < \pi$ is

O $x^2y + y^3 - x = 0$, for $x > 0$
O $x^2y - y^3 + x = 0$, for $x < 0$
O $x^2y + y^3 - x = 0$, for all real x
O $x^2y - y^3 + x = 0$, for $|x| < 1$ [1 mark]

5. Consider the differential equation, where $y(x)$ is defined for all real $x > 0$:

$$x^2y' + (1-x)^2y = 1 + x^2$$

If $A(x)$ is the correct integrating factor to solve this differential equation, then

O $\ln A = \dfrac{x^3}{3} + x^2 - x$
O $\ln A = \dfrac{x^3}{3} - x^2 + x$
O $\ln A = x + \dfrac{1}{x} - 2\ln|x|$
O $\ln A = x - \dfrac{1}{x} - 2\ln|x|$ [1 mark]

6. **M** is the matrix transformation in 3D space representing a clockwise rotation by 90° about the z-axis. Which of the following is true?

1 **M** has no real eigenvectors
2 The z-axis is the only invariant line under **M**
3 $\mathbf{M}^2 = \mathbf{I}$

O **1** and **2** only
O **2** and **3** only
O **1** and **3** only
O **1**, **2** and **3** [1 mark]

7. The polynomial $P(x)$ is defined as $P(x) = x^3 - 8x^2 + 7x - 1$. The three distinct roots of $P(x)$ are α, β and γ.

Which of these functions has roots $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$?

O $P(7/x)$
O $P(-7/x)$
O $P(1/x)$
O $P(-1/x)$ [1 mark]

8. Part of the Maclaurin series expansion for a function $f(x)$ is shown below.

$$f(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \ldots \quad \text{for all } x \in \mathbb{R}$$

Which of these could be $f(x)$?

O $f(x) = \sinh x$
O $f(x) = e^x \sin x$
O $f(x) = \cos 2x$
O $f(x) = \cos^{-1} x$ [1 mark]

9. The completed-square form of the function $f(z) = z^2 - (4-2i)z + (3-4i)$, where z is complex, is

O $f(z) = (z + 2 + i)^2$
O $f(z) = (z + 2 - i)^2$
O $f(z) = (z - 2 + i)^2$
O $f(z) = (z - 2 - i)^2$ [1 mark]

10. The domain of $f(x)$ is all real numbers and satisfies $f(x+1) = 2f(x)$ and also,

$$f(x) = x(x-1) \quad \text{for } x \in (0,1]$$

The greatest value of *m* such that $f(x) \geq -8/9$ for all $x \leq m$ is

O 9/4
O 8/3
O 7/3
O 5/2 [1 mark]

Section B: Standard questions. Ensure to leave sufficient time for Section C.

11. The complex number z is defined by the real constants a, b and c as

$$z = \frac{a}{1+i} + \frac{b}{1+2i} + \frac{c}{1+3i}\ , \qquad a, b, c \in \mathbb{R}$$

a. Show that if z is real, then $5a + 4b + 3c = 0$. [3 marks]

b. Find a similar condition on a, b and c if $\arg z = \frac{\pi}{4}$. [3 marks]

[Total for Q11: 6 marks]

12.

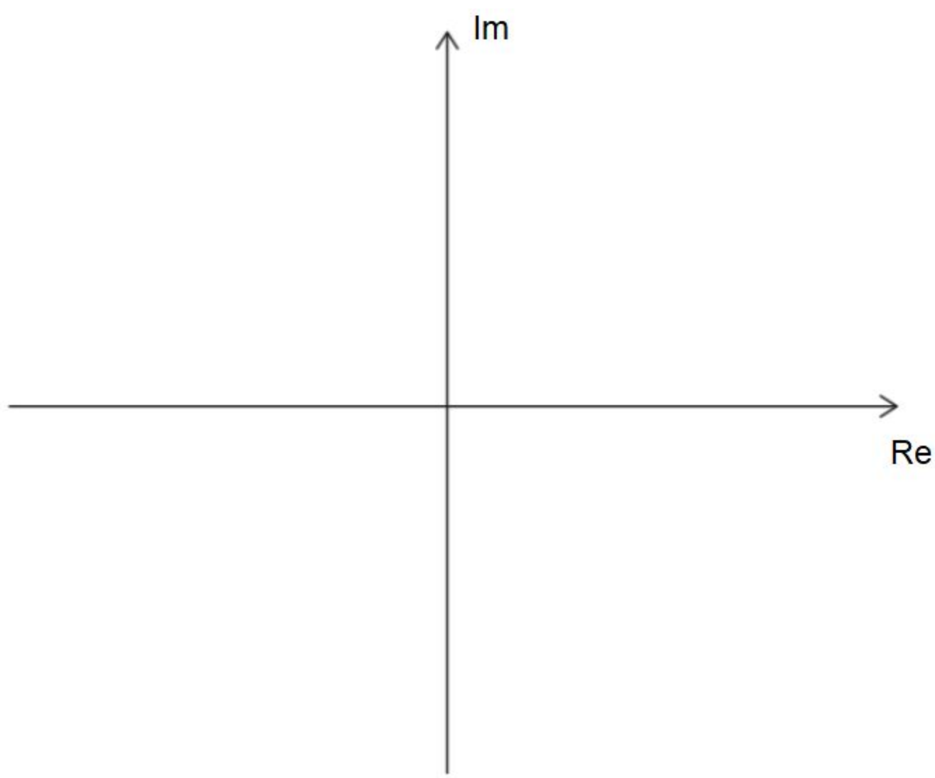
a. Consider the following sets:

$$P = \left\{ z \in \mathbb{C} : 4 < |(z - 2 + i)^2| \leq 9 \right\}$$

$$Q = \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \arg((z - 2 + i)^2) \leq \frac{2\pi}{3} \right\}$$

Sketch, on the Argand diagram below, the locus of $z \in P \cap Q$.

[4 marks]



b. Find the minimum value of Re(z) - Im(z) for all $z \in P \cap Q$.

Give your answer in exact form. [2 marks]

[Total for Q12: 6 marks]

13.

a. Use a suitable Maclaurin series to prove that, for suitable values of *x*,

$$\ln\left(\frac{1}{2-x^2}\right) = \sum_{n=1}^{\infty} \frac{(x^2-1)^n}{n}.$$

[5 marks]

b. State the interval of validity of the series in part (a).

Give your answer in the form $|x| < k$, where k is a constant. [1 mark]

[Total for Q13: 6 marks]

14. Prove by induction that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\mathrm{e}^x \sin(\sqrt{3}x)\right] = 2^n \mathrm{e}^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right) \ , \qquad n \in \mathbb{N}^+$$

where $\mathrm{d}^n y/\mathrm{d}x^n$ denotes the nth derivative of y with respect to x. [10 marks]

[Total for Q14: 10 marks]

15. Using integration by parts, or otherwise, show that

$$\int \sqrt{\ln x}\ \mathrm{d}x = x\sqrt{\ln x} - F(\sqrt{\ln x}) + C$$

for all real $x > 1$, where $F(x)$ is a function satisfying the differential equation $F'(x) = e^{x^2}$ and C is an arbitrary constant.

(You should **not** attempt to find an explicit expression for $F(x)$.) [7 marks]

[Total for Q15: 7 marks]

16. At time t, the rabbit and wolf populations, r and w respectively, on a certain island are described by the coupled system of differential equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 5r - 3w \qquad \text{and} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = r + w$$

Throughout this question, the above system will be referred to as S, while vector $\mathbf{p}$ and matrix $\mathbf{M}$ are defined as

$$\mathbf{p} = \begin{bmatrix} r \\ w \end{bmatrix} \qquad \text{and} \qquad \mathbf{M} = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}.$$

a. Write S in matrix form in terms of $\mathbf{p}$, $\mathbf{M}$ and the differentiation operator d/dt.

[1 mark]

b. Show that if $\mathbf{p} = \mathbf{p}_1(t)$ and $\mathbf{p} = \mathbf{p}_2(t)$ satisfy S then $\mathbf{p} = a\mathbf{p}_1(t) + b\mathbf{p}_2(t)$ also satisfies S, where a and b are constants. [4 marks]

c. Show that if $\mathbf{p} = \mathrm{e}^{\lambda t}\,\mathbf{k}$ satisfies S, where $\mathbf{k}$ is a vector of constants, then $\mathbf{Mk} = \lambda\mathbf{k}$.

[3 marks]

d. Find the eigenvalues and eigenvectors of **M** and hence solve S given that there are 1000 rabbits and 50 wolves at $t = 0$. [8 marks]

[Total for Q16: 16 marks]

17. The straight line L has vector equation

$$\mathbf{r} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

where λ is a scalar parameter.

The point A has coordinates (3, 3, -3) relative to a fixed origin O. The distinct points P and Q lie on L such that $|AP| = |AQ|$.

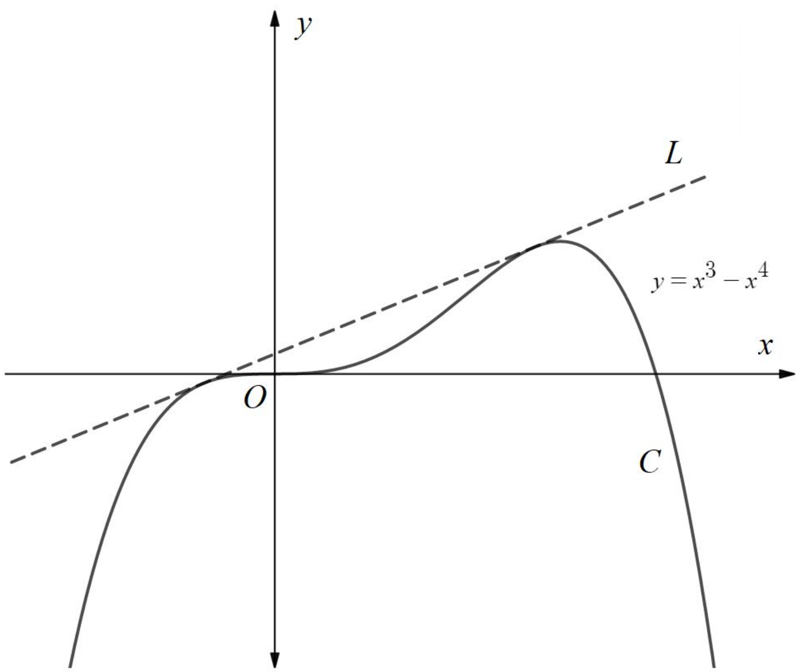
a. Given that angle PAQ is a right-angle and $|OQ| > |OP|$, find the coordinates of P and Q. [10 marks]

[More space for Q17a on the next page]

b. Find the area of triangle *APQ*. [2 marks]

[Total for Q17: 12 marks]

18. The graph below shows the curve C with equation $y = x^3 - x^4$. The straight line L is tangent to C at **two** distinct points as shown.



Find a Cartesian equation for L. Give your answer in the form $ay = bx + c$, where a, b and c are integers.

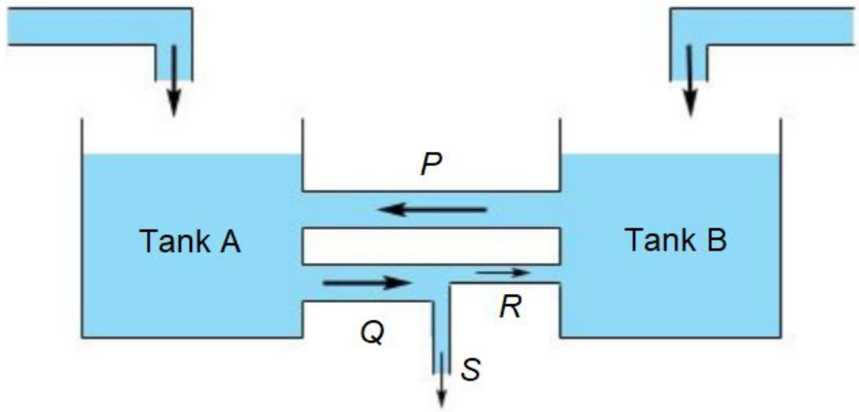
[7 marks]

[More space for Q18 on the next page]

[Total for Q18: 7 marks]

Section C: Extended questions. Attempt **both** questions in this section.

19. The diagram below shows two interconnected tanks of water which are being mixed. The water contains a dissolved chemical contaminant.



Initially:

- Tank A contains 50 litres of water and 25 grams of contaminant
- Tank B contains 100 litres of water and 75 grams of contaminant

Contaminated water with a concentration of 1.5 grams of contaminant per litre enters Tank A at a constant rate of 7 litres per hour, while freshwater enters Tank B at a constant rate of 3 litres per hour.

Water flows from Tank B to Tank A through pipe P at a rate of 5 litres per hour. Pipe Q removes 12 litres from Tank A per hour, pumping 2 litres of it into Tank B through pipe R, while the remaining 10 litres is removed from the system through pipe S.

By forming and solving a suitable model for this system, obtain functions to predict the mass of dissolved contaminant in each tank in grams at any time t.

Fully justify your answer, including any modelling assumptions used.
You are advised to plan your answer first and continue on the pages that follow.

Write your final answer in the space provided on the last question page.

[20 marks]

More space to answer Q19.

More space to answer Q19.

More space to answer Q19.

More space to answer Q19.

[Total for Q19: 20 marks]

20. The function $f(x)$ is defined for all real x by

$$f(x) = \begin{cases} 1 - |x - t| \ , & \text{if } |x - t| \leq 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

where t is real. Furthermore, the function $g(t)$ is defined in terms of $f(x)$ as

$$g(t) = \int_{k}^{k+8} f(x) \cos(\pi x) \ \mathrm{d}x$$

where k is an **odd** integer.

Consider all numbers α such that $g(t)$ is a local **minimum** at $t = \alpha$ and $g(\alpha) < 0$. When these m distinct values of α are listed in ascending order as $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$, it is given that

$$\sum_{i=1}^{m} \alpha_i = 45.$$

Find the value of

$$k - \pi^2 \sum_{i=1}^{m} g(\alpha_i).$$

Fully justify your answer. In your working, you should include simplified sketches of $f(x)$ and $g(t)$ to assist you.

You are advised to plan your answer first and continue on the pages that follow.

Write your final answer in the space provided on the last question page.

[25 marks]

More space to answer Q20.

More space to answer Q20.

More space to answer Q20.

More space to answer Q20.

More space to answer Q20.

[Total for Q20: 25 marks]

Your answers to Q19:

Your answer to Q20:

End of Questions

Question Sources

Q10: Gaokao (Math)

Q11: PAT (Maths)
Q14: MadasMaths Undergraduate Papers
Q16: MEI Practice Problems
Q17: STEP I (Maths)
Q18: MadasMaths Undergraduate Papers

Q19: University of Lamar, Mixing Problems
Q20: CSAT 2019 (Math B)

ENGAA is an entrance exam for studying Engineering at Cambridge.
PAT is an exam to study Physics at Oxford.
STEP I in an entrance exam for studying Maths or Physics at Oxbridge or Durham.
CSAT is an exam in South Korea.
Gaokao is an exam in China.