Radio Interferometry with Compressed Sensing

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Today

Abstract

Solar Flares Abstract

Contents

1 Interferometry and the Inverse Problem

Astronomy requires its instruments to have a high angular resolutions. This is an issue for radio wavelengths: The longer the wavelength, the bigger the diameter of a single dish antenna. Single dish antenna's are expensive to build and harder to steer accurately. Interferometers, where multiple smaller antennas act as a single large instrument, have had great success in Radio Astronomy with instruments like VLA, ALMA and LOFAR.

Interferometers do not observe the sky directly. Each antenna pair measure Fourier Components (Visibilities) of the sky brightness. The observed image has to be reconstructed from the measured Visibilites. Since the interferometer can only observe a limited number of Visibilities, the reconstruction is an ill-posed inverse problem. For small Field of View imaging, the CLEAN class of Algorithms[1][2][3][4] have been developed and is the de-facto standard in Radio Astronomy. It is not guaranteed to reconstruct the true image in theory. In practice it has produced remarkable results with expert tuning. New generation Interferometers like ASKAP, Pathfinder and SKA are built with wide Field of View imaging in mind. The CLEAN Algorithms have been extended for Wide Field of Views, but require even more tuning by experts.

The Theory of Compressed Sensing[**?**] has seen success in solving ill-posed inverse problems. It is flexible in its application and has produced remarkable results image de-noising[**?**], in-painting[**?**] and super-resolution[**?**]. Applying Compressed Sensing to wide Field of View imaging is an active field of research. In the last decade numerous approaches have been developed showing the potential of Compressed Sensing: Accurately modelling the effects of wide Field of View imaging, while reducing the tunable parameters and possibly super-resolved images[**?**]. Current research focuses on how the effects of wide Field of View can be accurately modelled while still being computationally efficient.

In this project, a proof of concept Compressed Sensing approach was developed and implemented in the Common Astronomy Software Application(CASA). The approach is focused on small Field of View imaging and the reduction of expert intervention.

1.1 Inverse Problem for small Field of View Imaging

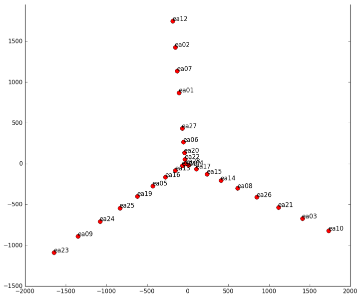
Each antenna pair measures a complex Visibility of the sky brightness. The distance between the antennas, the baseline, dictates the sample point in the Fourier Space (called UV-Space). Longer baselines sample higher frequency components, while shorter baselines sample lower frequency components.

For small Field of View imaging, the measured Visibilities equal two dimensional Fourier components. The observed image can be calculated by the two dimensional Inverse Fourier Transform. However the interferometer cannot sample the whole UV-Space. The image calculated by the Inverse Fourier Transform is 'dirty', it contains artefacts introduced by undersampling.

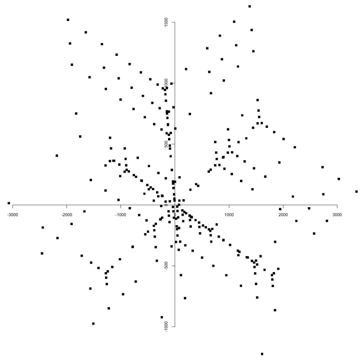
The Inverse Problem is now to remove the artefacts of the interferometer and retrieve the true image. The effects of the undersampling can be modeled by a Point Spread Function (PSF). The interferometer sees the true image of the sky, but due to undersampling it gets convolved with a PSF, resulting in the dirty image. More formally, we try to find a solution x for equation (1.1), where only the PSF and I_{dirty} are known. This problem is ill-posed: it may have multiple solutions, and a small change in the I_{dirty} or the PSF may result in large changes in x. Furthermore, the whole problem gets corrupted by noise.

$$x \star PSF + N = I_{dirty} \tag{1.1}$$

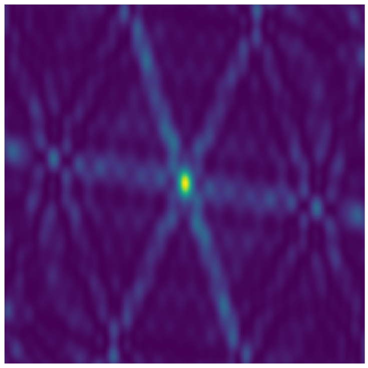
The PSF is surprisingly easy to calculate. The Fourier Transformed PSF equals the sampling pattern in UV-Space. Remember that a convolution in image space is a multiplication in Fourier. The effects of under-



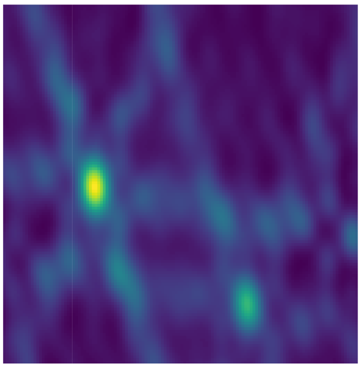
(a) Antenna Configuration



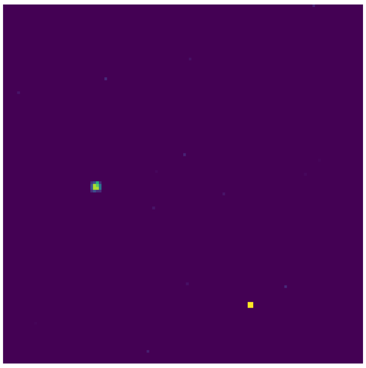
(b) UV-Space



(c) PSF



(d) dirty image



(e) true image

Figure 1: Inverse Problem example for VLA: Retrieve the true image when only PSF and dirty image are known

sampling in image space are a convolution with the PSF. In the Fourier space it is masking all components other than the measured ones. From the Antenna Configuration we can infer the masking matrix M in UV-Space. Calculating the Inverse Fourier Transform of M results in the PSF.

1.2 Deconvolution with CLEAN

In each iteration of CLEAN, it searches the highest peak of the dirty image and removing a fraction of the PSF at that point. It stops until the next highest peak is below a threshold, or if the maximum number of iterations was reached. The fraction of the PSF, threshold and number of iterations are all tunable by the user. State of the art implementations expose even more parameters. The reconstruction quality depends on the chosen parameters and require extensive user input.

CLEAN does not solve the deconvolution problem (1.1) directly. Instead, it greedily minimizes the objective function (1.2). It is easy to see that if CLEAN minimizes the objective to zero, it has found a solution to the original deconvolution problem in a noiseless environment.

$$\underset{x}{minimize}\ \|I_{dirty} - x \star PSF\|_2^2 \tag{1.2}$$

Since the original problem is ill-posed, the objective (1.2) may have several zero points. In practice, CLEAN is stopped before it reaches zero. The addition of noise can add spurious peaks in the dirty image. By stopping early, CLEAN regularizes the objective. It assumes only a limited number of point sources exist in the image. The larger the magnitude of the peak, the more likely it is to be a real point source.

In short, CLEAN does a greedy approximation of the deconvolution problem, and assumes the resulting image consists out of a few point sources. The question remains, how close the CLEAN approximation is to the true image? If the true image consists out of a few point sources, CLEAN produces a good approximation. Extended emissions however are harder for CLEAN to reproduce. The peak of extended sources is lower than that of point sources. It is harder for CLEAN to distinguish extended sources from noise.

CLEAN's regularization scheme is not ideal for extended sources. Ideally another way of regularization would be chosen for extended emissions, but the regularization is a fixed part of the CLEAN algorithm.

1.3 From CLEAN to Compressed Sensing

An Algorithm in the Compressed Sensing Framework has three components:

- An objective function with a data and regularization term
- A Matrix P in which the signal can be sparsely represented.
- An optimization algorithm that is able to handle the objective function

CLEAN can be converted into the Compressed Sensing Framework. First, the regularization term has to be explicit, it gets added to the objective function. The resulting objective (1.3) has two terms: A data term and a regularization term. The data term forces the reconstruction to be close to the measurement, while the regularization term forces the reconstruction to be plausible. λ models the trade off between the terms. Note that the zero norm $\|PX\|_0$ acts as an indicator function and is not technically a norm.

$$\underset{X}{minimize} \; \|D_{dirty} - X \star PSF\|_2^2 \; + \; \lambda \, \|PX\|_0 \tag{1.3}$$

the new objective is the Compressed Sensing version of CLEAN. It assumes that the reconstruction is sparse in the image domain. When P is the identity matrix, This is true when there are only a few point sources located in the image. Extended sources are are not well represented and are harder to detect.

The objective function is optimized by a greedy solver.

in the compressed Sensing Framework CLEAN is a specific objective function with an identity matrix as prior and a specific optimization algorithm

In this part, mostly the prior is

The important questions are: In an undersampled, noisy environment, does the new objective (1.3) have a global minimum? What is the chances that the minimum is equal to the true image? It turns out that even though we have fewer samples than the Nyquist-Shannon Theorem requires, we can guarantee a global minimum and that the minimum is equal to the true image, if we have enough prior knowledge P about the signal. How we model the signal and by extend, what P we choose is essential for Compressed Sensing

2 Inverse Problem for wide Field of View Imaging

So far the simplified inverse problem has been introduced. Each antenna pair measures a Fourier Component of the sky brightness distribution. The distance between antenna pairs dictates what point is sampled in the UV plane. This leads to the measurement equation (2.1). Calculating the true image X is simply inverting the two dimensional Fourier Transform.

$$V(u,v) = \int\int X(x,y)e^{2\pi i(ux+vy)}dxdx \tag{2.1}$$

In reality, each visibility has a third component w. It comes from the fact that the antennas are not on a flat plane but on the curved surface of the earth. Image 2 shows the three dimensional space. w is the vector points from the phase center to the science target. For small Field of Views, the effect is negligible and the measurement equation (2.1) is a good approximation. The field of view is limited by the primary beam of the antennas. Primary beam widens with wavelength. So far, wide Field of View problems were encountered in low frequencies like LOFAR.

new instruments like SKA, ASKAP and Pathfinder are constructed with a wide Field of View in mind. The simple two dimensional Fourier Transform does not hold true anymore and we arrive at the wide Field of View measurement equation (2.2).

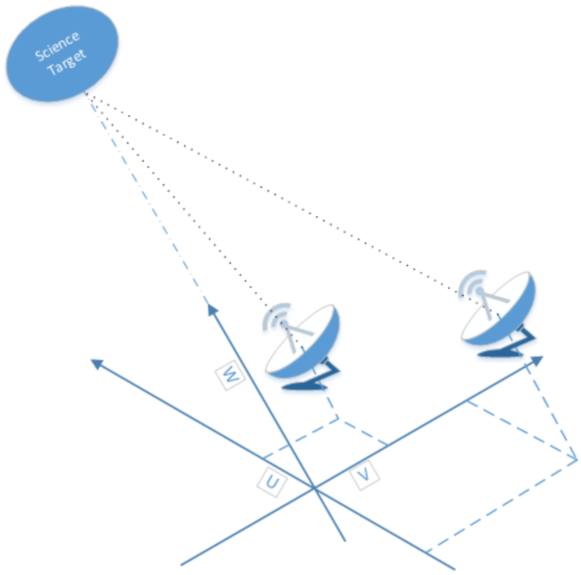


Figure 2: U V and W coordinate space

$$V(u,v,w) = \int\int \frac{X(x,y)}{\sqrt{1-x^2-y^2}}e^{2\pi i(ux+vy+w\sqrt{1-x^2-y^2})}dxdy \tag{2.2}$$

Note that for small Field of View $1 - x^2 - y^2 \ll 1$, and equation (2.2) can be approximated by the 2d Fourier Transform (2.1). Two separate effects: w Component for non coplanar baselines.

W-Projection [5]

A-Projection [6]

All here to try to get back to the 2d fourier transform.

"The field of view of a telescope is limited by the primary beams of the antennas. To map a region of sky where the emission is at a scale larger than the angular width of the primary beams, mosaicing needs to be done. This is discussed in the second part of this lecture." Phase

source [7]

Strength of compressed sensing is modelling these effects.

2.1 Calibration

A lot of effects, weather, noise, antenna temperature, drift.

Antenna based calibration, holds true for current interferometers but is not true for SKA. Possible switch to baseline based calibration.

2.1.1 Self-Calibration

?

3 Compressed Sensing for Radio Astronomy

A compressed sensing algorithm consists of three components: A Prior, an Objective and an Optimization Algorithm.

To Include wide field imaging in the objective function directly, which has the potential to improve reconstruction with Model the signal to so the reconstruction is plausible Use an optimization algorithm that is able to handle the expected amount of data. Interferometers produce a large amount of data. In this project, the optimization algorithm was not further investigated. The GuRoBi simplex solver was used.

Compressed Sensing for Radio Astronomy is an active research field. The new instrument's push to wide Field of View imaging has led to increased interest.

The guarantees in Compressed Sensing depend on the signal, in what space the signal is measured and how well our Prior models the signal.

3.1 Sparseland Prior and Incoherence

From compression algorithm, we assume there is a Prior P in which our signal can be sparsely represented. It is not guaranteed that such a space exists, but for natural signals there always seem to be. This has led to the idea of the Sparseland Prior (3.1): We assume for our signal there exists a Dictionary Dic of signal parts. There is potentially a large, but finite number entries of parts. When we measure our signal x, we see a combination of only a few entries non-zero entries α of the Dictionary.

$$\begin{aligned} x = Dic\,\alpha \qquad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m, Dic \in \mathbb{R}^{n*m}, \qquad n \leq m \\ \|\alpha\|_0 = s \qquad s \ll n \leq m \end{aligned} \tag{3.1}$$

Pictures of nature scenes for example tend to be sparse in the wavelet domain. If x in (3.1) are nature scenes, we can create a Dictionary Dic of wavelets. A single image x then consists a few wavelets, meaning the number of non-zero entries in α is far lower than the number of pixels n.

Noise tends to affect all entries of the Dictionary. Sparseland Prior has had success in image denoising.

The signal parts is not restricted to be in one domain. It can consists of wavelets, cosine functions, a combination of both, or any other function.

Work with over-complete dictionaries, where the number of pixels n is far smaller than the number of signal parts m. There are redundant entries which is OK, as long is it is not too redundant. There is a tradeoff in practice in how sparse a signal can be represented and how redundant the dictionary is.

Back to the ill-posed inverse problem of interferometry. Nyquist Shannon Sampling Theorem requires that if we want an image of n pixels, we need at least $2n$ measurements. The small Field of View Interferometer measures complex visibilities, so for n fully sampled pixels we need n complex Visibility measurements.

But when we know it is a Sparseland signal and we know the Dictionary, we could try to measure in the Dictionary space, so we could try to retrieve the non-zero components and only measure s samples. Sadly, this is only possible if we know which entry of the dictionary are non-zero beforehand, which is in general not possible. So we would measure different α and are more likely to hit a zero component. Note however that if we measure a non-zero component of α, we learn a lot more about our signal than when we hit a zero component. So the next question is, can we measure in a space where we learn more about the non-zero components of α?

Surprisingly, the answer is yes, there is a way. The space in which we measure our signal x should be incoherent from our Dictionary. With that we maximize the amount we learn about the non-zero components of $alpha$ with each measurement. Interestingly enough, constructing an incoherent measurement space is easy: Random projections are very good at being incoherent from the dictionary.

Random Projections are not possible, the Interferometer measures complex visibilities. Antenna configuration could help. But luckily what also can help is the Wide Field imaging. McEwen [8] showed the theoretical improvement on synthetic data.

CS does not need an over-complete dictionary. It can also work for wavelet domains. overcomplete dictionaries give more freedom in representation.

P in our original objective function. $Dic = P^{-1}$. So if the conversion from image space to Sparseland is only defined when the dictionary is a square matrix. When it has as many signal parts as pixels. The Objective Function can be modified to work with over-complete dictionaries.

3.2 Objective Function

The Compressed Sensing CLEAN objective function (1.3) uses the L0 norm for it's regularization term, which means the Objective Function is not convex. There are specialized solvers for the L0 compressed sensing. The L1 relaxation however is practically guaranteed to have the same minimum as the L0 norm and results in a convex objective function. Since GuRoBi works better on the L1 relaxation it was chosen for this project.

As for the objective function, there are three different spaces in which one might want to reconstruct: The analysis method, where the image x is minimized directly, the synthesis method where the sparse vector α is minimized, or by in-painting the missing Visibilities V_2.

$$\begin{aligned}
analysis: \quad & \underset{x}{minimize}\ \left\| D_{dirty} - x \star PSF \right\|_2^2 \quad && + \lambda \left\| Px \right\|_1 \\
synthesis: \quad & \underset{\alpha}{minimize}\ \left\| D_{dirty} - Dic\, \alpha \star PSF \right\|_2^2 && + \lambda \left\| \alpha \right\|_1 \\
in-painting: \quad & \underset{V_2}{minimize}\ \left\| D_{dirty} - F^{-1} M V_2 \right\|_2^2 \quad && + \lambda \left\| P F^{-1} V_2 \right\|_1
\end{aligned}$$

All three objective functions have the same global minimum. For the second and third objective x can be retrieved by $x = Dic\ \alpha$ and by $x = F^{-1}V_2$ respectively. [Empirical and theoretical studies have shown an advantage of the analysis objective over the other two [**?**]]. However practical considerations play a factor. $Px = \alpha$ is only true P is an orthogonal transformation like the wavelet transform. Over-complete dictionaries would result in $P \in \mathbb{R}^{m*n}, n \ll m$ and may not produce a suitably sparse vector α. Therefore over-complete prior tend to use the synthesis objective, since it only requires a transformation from sparse to image space($x = Dic\ \alpha$).

It is a similar story with in-painting: the transformation $Px2$ may be eaier represented in the fourier space, especially when P contains a convolution.

The strength of Compressed Sensing is that the objective can be modified for the measurement equation, while the Optimization Algorithm and the Prior stay the same. The above objective functions represent the deconvolution problem which is only true for small Field of View imaging.

Either A- and W-Projections are used to transform the wide FoV measurement equation (2.2) back to the deconvolution. Or the measurement equation (2.2) gets included in the data term, resulting in the new objective

(3.2).

$$\underset{x}{minimize} \left\| V - MF^{-1}_{wFOV}x \right\|_2^2 + \lambda \left\| Px \right\|_1 \tag{3.2}$$

A lot of freedom to choose.

For this project CASA was used, which limits the choices.

3.2.1 Implementation In Casa

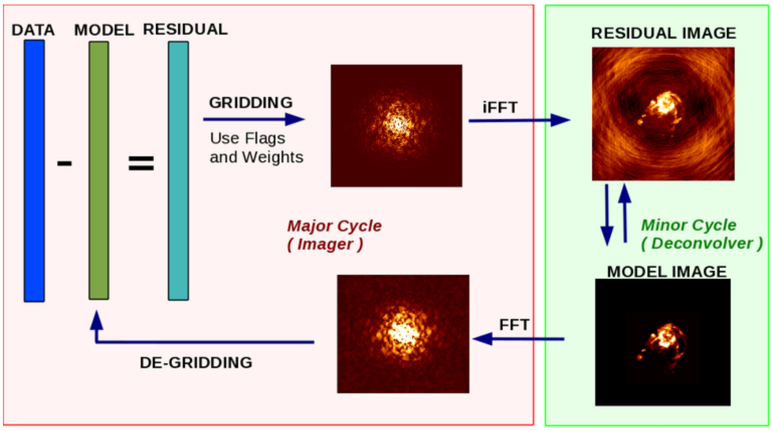


Figure 3: Casa Major Minor Cycle, source [9]

Casa major and minor cycle. Major cycle calculates visibilities in image space. Minor Cycle Deconvolves the Problem, often with a CLEAN class Algorithm. This constrains the algorithm to use the data term in image space.

This forces the objective function to either minimize in the image domain or in the sparsity domain.

3.3 Compressed Sensing Algorithms in Astronomy

3.3.1 SASIR

3.3.2 PURIFY

3.3.3 Vis-CS

4 Results

Physical plausible and shown to produce better results on synthetic data [8]

Clean images:

Gurobi Clean

4.1 Simple Priors

pixels l1 norm

pixels l2 norm

Total Variation

Haar

4.2 Starlet Transform as Prior

starlet decomposition

the cJ map as a smart thresholder

Runtime issues

last comparison, CLean, TV, starlet

5 Conclusion and Future Development

References

[1] JA Högbom. Aperture synthesis with a non-regular distribution of interferometer baselines. Astronomy and Astrophysics Supplement Series, 15:417, 1974.

[2] FR Schwab. Relaxing the isoplanatism assumption in self-calibration; applications to low-frequency radio interferometry. The Astronomical Journal, 89:1076–1081, 1984.

[3] JW Rich, WJG De Blok, TJ Cornwell, Elias Brinks, Fabian Walter, Ioannis Bagetakos, and RC Kennicutt Jr. Multi-scale clean: A comparison of its performance against classical clean on galaxies using things. The Astronomical Journal, 136(6):2897, 2008.

[4] Urvashi Rau and Tim J Cornwell. A multi-scale multi-frequency deconvolution algorithm for synthesis imaging in radio interferometry. Astronomy & Astrophysics, 532:A71, 2011.

[5] Tim J Cornwell, Kumar Golap, and Sanjay Bhatnagar. The noncoplanar baselines effect in radio interferometry: The w-projection algorithm. IEEE Journal of Selected Topics in Signal Processing, 2(5):647–657, 2008.

[6] Cyril Tasse, S van der Tol, J van Zwieten, Ger van Diepen, and S Bhatnagar. Applying full polarization a-projection to very wide field of view instruments: An imager for lofar. Astronomy & Astrophysics, 553:A105, 2013.

[7] Institut de RadioAstronomie Millimétrique. Proceedings from imiss2 “iram millimeter interferometry summer school 2”, 2000.

[8] Jason D McEwen and Yves Wiaux. Compressed sensing for radio interferometric imaging: Review and future direction. In Image Processing (ICIP), 2011 18th IEEE International Conference on, pages 1313–1316. IEEE, 2011.

[9] National Radio Astronomy Observations. tclean overview, 2016.

List of Figures

List of Tables

6 Ehrlichkeitserklärung

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Windisch, July 1, 2018

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