Radio Interferometry with Compressed Sensing

Jonas Schwammberger

Today

Abstract

Solar Flares Abstract

Contents

1 Interferometry and the Inverse Problem **1**
1.1 Inverse Problem for small Field of View Imaging 1
1.2 Deconvolution with CLEAN 2
1.3 From CLEAN to Compressed Sensing 3

2 Inverse Problem for wide Field of View Imaging **5**
2.1 Calibration 6
2.1.1 Self-Calibration 6

3 Compressed Sensing for Radio Astronomy **7**
3.1 Sparseland Prior and Incoherence 7
3.2 Objective Function 8
3.2.1 Implementation In Casa 9
3.3 Compressed Sensing Algorithms in Astronomy 9
3.3.1 SASIR 9
3.3.2 PURIFY 9
3.3.3 Vis-CS 9

4 Results **10**
4.1 Simple Priors 10
4.2 Starlet Transform as Prior 10

5 Conclusion and Future Development **11**

6 Ehrlichkeitserklärung **14**

1 Interferometry and the Inverse Problem

Astronomy requires its instruments to have a high angular resolutions. This is an issue for radio wavelengths: The longer the wavelength, the bigger the diameter of a single dish antenna. Single dish antenna's are expensive to build and harder to steer accurately. Interferometers, where multiple smaller antennas act as a single large instrument, can achieve high angular resolutions while being cheaper to build. In the past, interferometers like VLA, AIMA and LOFAR have made numerous discoveries.

Interferometers do not observe the sky directly. Each antenna pair measure Fourier Components (Visibilities) of the sky brightness. The observed image has to be reconstructed from the measured Visibilites. Since the interferometer can only observe a limited number of Visibilities, the reconstruction is an ill-posed inverse problem. For small Field of View imaging, the CLEAN class of Algorithms[1][2][3][4] have been developed and is the de-facto standard in Radio Astronomy. It is not guaranteed to reconstruct the true image in theory. In practice it has produced remarkable results with expert tuning. New generation Interferometers like ASKAP, Pathfinder and SKA are built with wide Field of View imaging in mind. The CLEAN Algorithms have been extended for Wide Field of Views, but require even more tuning by experts.

The Theory of Compressed Sensing[**?**] has seen success in solving ill-posed inverse problems. It is flexible in its application and has produced remarkable results image de-noising[**?**], in-painting[**?**] and super-resolution[**?**]. Applying Compressed Sensing to wide Field of View imaging is an active field of research. In the last decade numerous approaches have been developed showing the potential of Compressed Sensing: Accurately modelling the effects of wide Field of View imaging, while reducing the tunable parameters and possibly super-resolved images[**?**]. Current research focuses on how the effects of wide Field of View can be accurately modelled while still being computationally efficient.

In this project, a proof of concept Compressed Sensing approach was developed and implemented in the Common Astronomy Software Application(CASA). The approach is focused on small Field of View imaging and the reduction of expert intervention.

1.1 Inverse Problem for small Field of View Imaging

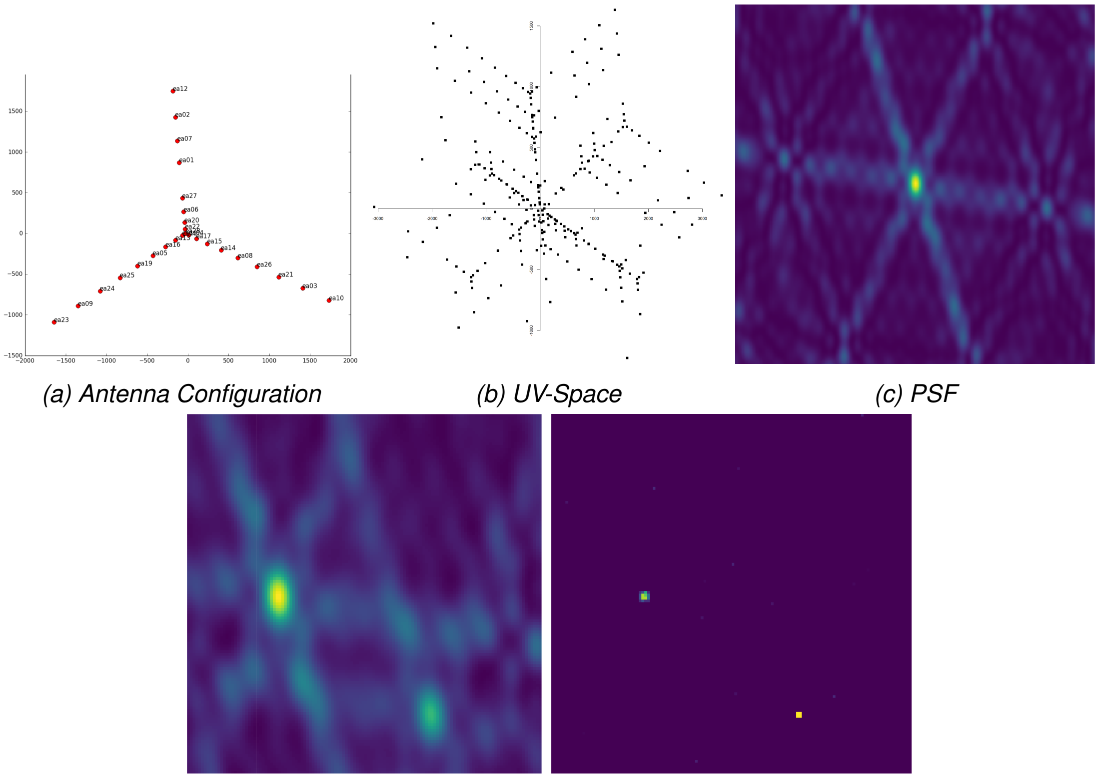
Each antenna pair measures a complex Visibility of the sky brightness. The distance between the antennas, the baseline, dictates the sample point in the Fourier Space (called UV-Space). Longer baselines sample higher frequency components, while shorter baselines sample lower frequency components.

For small Field of View imaging, the measured Visibilities equal two dimensional Fourier components. The observed image can be calculated by the two dimensional Inverse Fourier Transform. However the interferometer cannot sample the whole UV-Space. The image calculated by the Inverse Fourier Transform is 'dirty', it contains artefacts introduced by undersampling.

The Inverse Problem is now to remove the artefacts of the interferometer and retrieve the true image. The effects of the undersampling can be modeled by a Point Spread Function (PSF). The interferometer sees the true image of the sky, but due to undersampling it gets convolved with a PSF, resulting in the dirty image. More formally, we try to find a solution x for equation (1.1), where only the PSF and I_{dirty} are known. This problem is ill-posed: it may have multiple solutions, and a small change in the I_{dirty} or the PSF may result in large changes in x. Furthermore, the whole problem gets corrupted by noise.

$$x \star PSF + N = I_{dirty} \tag{1.1}$$

The PSF is surprisingly easy to calculate. The Fourier Transformed PSF equals the sampling pattern in UV-Space. Remember that a convolution in image space is a multiplication in Fourier. The effects of under-



(a) Antenna Configuration *(b) UV-Space* *(c) PSF*

(d) dirty image *(e) true image*

Figure 1: Inverse Problem example for VLA: Retrieve the true image when only PSF and dirty image are known

sampling in image space are a convolution with the PSF. In the Fourier space it is masking all components other than the measured ones. From the Antenna Configuration we can infer the masking matrix M in UV-Space. Calculating the Inverse Fourier Transform of M results in the PSF.

1.2 Deconvolution with CLEAN

In each iteration of CLEAN, it searches the highest peak of the dirty image and removing a fraction of the PSF at that point. It stops until the next highest peak is below a threshold, or if the maximum number of iterations was reached. The fraction of the PSF, threshold and number of iterations are all tunable by the user. State of the art implementations expose even more parameters. The reconstruction quality depends on the chosen parameters and require extensive user input.

CLEAN does not solve the deconvolution problem (1.1) directly. Instead, it greedily minimizes the objective (1.2). It is easy to see that if CLEAN minimizes the objective to zero, it has found a solution to the original deconvolution problem in a noiseless environment.

$$\underset{x}{minimize} \; \|I_{dirty} - x \star PSF\|_2^2 \tag{1.2}$$

Since the original problem is ill-posed, the objective (1.2) may have several zero points. In practice, CLEAN is stopped before it reaches zero. The addition of noise can add spurious peaks in the dirty image. By stopping early, CLEAN regularizes the objective. It assumes only a limited number of point sources exist in the image. The larger the magnitude of the peak, the more likely it is to be a real point source.

In short, CLEAN does a greedy approximation of the deconvolution problem, and assumes the resulting image consists out of a few point sources. The question remains, how close the CLEAN approximation is to the true image? If the true image consists out of a few point sources, CLEAN produces a good approximation. Extended emissions however are harder for CLEAN to reproduce. The peak of extended sources is lower than that of point sources. It is harder for CLEAN to distinguish extended sources from noise.

The CLEAN regularization scheme is not ideal for extended sources. Ideally another way of regularization would be chosen for extended emissions, but the regularization is a fixed part of the CLEAN algorithm.

1.3 From CLEAN to Compressed Sensing

In the Compressed Sensing Framework, an approach is split into three separate parts:

- An objective with a data and regularization term
- A prior P in which the signal can be sparsely represented.
- An optimization algorithm that is suited for the objective.

The fra To demonstrate the flexibility of the Compressed Sensing Framework, we convert CLEAN into a Compressed Sensing approach. First we add a regularization term to (1.2) and arrive at the new objective (1.3). The objective contains the old CLEAN data term and a new regularization term. The data term forces the reconstruction to be close to the measurement, while the regularization term forces the reconstruction to be plausible. λ models the expected noise in the problem. Note that the zero norm $\left\| Px \right\|_0$ acts as an indicator function.

$$\underset{x}{minimize} \ \left\| D_{dirty} - x \star PSF \right\|_2^2 \ + \ \lambda \left\| Px \right\|_0 \tag{1.3}$$

Now P that models the same regularization scheme. CLEAN assumes the x contains a few point sources. In Compressed Sensing terminology, it assumes x is sparse in image space. Since x is already an image, the Prior P in Compressed Sensing CLEAN is the identity matrix.

CLEAN uses a greedy optimization scheme. In every iteration, CLEAN searches the highest peak in the dirty image. Matching Pursuit is a greedy optimization scheme for $L0$ objectives like our Compressed Sensing CLEAN (1.3).

Both versions of CLEAN produce the same result, but we are getting rid of the tunable parameters in favour of a single λ. There exist schemes that estimate the correct λ for a given problem.

In the noisy environment the true image may not be located at a minimum of the CLEAN objective (1.2). The Compressed Sensing objective is guaranteed to have a unique global minimum and the true image is guaranteed to be there if:

- The Prior models our signal well enough
- Enough samples are available.

How many samples are needed? Interestingly enough, we do not need the Nyquist-Shannon rate. It depends on the prior.

Same as clean, but now we have a global optimization scheme, the regularization is explicit and replaceable.

The important questions are: In an undersampled, noisy environment, does the new objective (1.3) have a global minimum? What is the chances that the minimum is equal to the true image? It turns out that even though we have fewer samples than the Nyquist-Shannon Theorem requires, we can guarantee a global minimum and that the minimum is equal to the true image, if we have enough prior knowledge P about the signal. How we model the signal and by extend, what P we choose is essential for Compressed Sensing

The data term is for small Field of View imaging, it can be modified to solve the wide Field of View problem.

2 Inverse Problem for wide Field of View Imaging

So far the small Field of View inverse problem has been introduced where each antenna pair measures a Visibility of the sky brightness distribution. This leads to the small Field of View measurement equation (2.1). It is identical to the two dimensional Fourier Transform. In practice the Fast Fourier Transform (FFT) is used, since it scales with $n\,log(n)$ instead of n^2 pixels.

$$V(u,v) = \int\int x(l,m)e^{2\pi i(ux+vy)}dldm \tag{2.1}$$

For wide Field of View imaging, two effects break the two dimensional Fourier Transform relationship: Non-coplanar Baselines and the celestial sphere which lead to the measurement equation (2.2). Note that for small Field of View $1 - x^2 - y^2 \ll 1$, and (2.2) reduces to the 2d measurement equation (2.1).

$$V(u,v,w) = \int\int \frac{X(x,y)}{\sqrt{1-x^2-y^2}}e^{2\pi i(ux+vy+w\sqrt{1-x^2-y^2})}dxdy \tag{2.2}$$

Non-coplanar Baselines lead to a third component w for each Visibility. Figure 2 shows the the u v and w coordinate system. w is essentially the pointing direction of the instrument. The UV-Plane is the projection of the antennas on a plane perpendicular to the pointing direction. Which point in the UV-Plane get sampled and what w component it has depends on the pointing direction. If the instrument points straight up, the UV-Plane is a tangent to earth's surface, and the w term compensates for earth's surface curvature. If however the instrument points at the horizon, the projected UV-Plane gets squashed and w compensates for antennas which lie far behind the UV-Plane. In essence, w is a phase delay that corrects antenna positions in three dimensions. The wide Field of View measurement equation (2.2) would account for the w phase delay, but it breaks the the two dimensional Fourier relationship and the FFT cannot be used. The W-Projection [5] algorithm approximates the effect of the w term restores the two dimensional Fourier relationship.

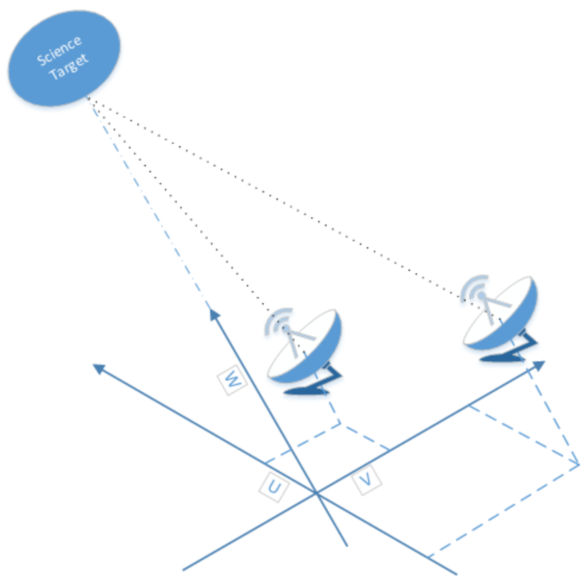


Figure 2: U V and W coordinate space

A-Projection [6]

new instruments like SKA, ASKAP and Pathfinder are constructed with a wide Field of View in mind. The simple two dimensional Fourier Transform does not hold true anymore and we arrive at the wide Field of View measurement equation (2.2).

spread spectrum phenomenon

All here to try to get back to the 2d fourier transform.

Strength of compressed sensing is modelling these effects.

2.1 Calibration

A lot of effects, weather, noise, antenna temperature, drift.

Antenna based calibration, holds true for current interferometers but is not true for SKA. Possible switch to baseline based calibration.

2.1.1 Self-Calibration

?

3 Compressed Sensing for Radio Astronomy

A compressed sensing algorithm consists of three components: A Prior, an Objective and an Optimization Algorithm.

To Include wide field imaging in the objective function directly, which has the potential to improve reconstruction with Model the signal to so the reconstruction is plausible Use an optimization algorithm that is able to handle the expected amount of data. Interferometers produce a large amount of data. In this project, the optimization algorithm was not further investigated. The GuRoBi simplex solver was used.

Compressed Sensing for Radio Astronomy is an active research field. The new instrument's push to wide Field of View imaging has led to increased interest.

The guarantees in Compressed Sensing depend on the signal, in what space the signal is measured and how well our Prior models the signal.

3.1 Sparseland Prior and Incoherence

From compression algorithm, we assume there is a Prior P in which our signal can be sparsely represented. It is not guaranteed that such a space exists, but for natural signals there always seem to be. This has led to the idea of the Sparseland Prior (3.1): We assume for our signal there exists a Dictionary Dic of signal parts. There is potentially a large, but finite number entries of parts. When we measure our signal x, we see a combination of only a few entries non-zero entries α of the Dictionary.

$$\begin{aligned} x = Dic\,\alpha \qquad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m, Dic \in \mathbb{R}^{n*m}, \qquad n \leq m \\ \|\alpha\|_0 = s \qquad s \ll n \leq m \end{aligned} \tag{3.1}$$

In image compression this phenomenon was already observed: image depicting nature scenes tend to be sparse in the wavelet domain. If x in (3.1) are nature scenes, we can create a Dictionary Dic of wavelets. A single image x can be represented with a few wavelets, meaning the number of non-zero entries s in α is far lower than the number of pixels n. All that is left to do for compression is save the non-zero entries of α. An important side-note is the effect of noise: Independent noise tends to affect all elements of the Dictionary: Saving the s largest values of a noisy sparse vector α is the same as de-noising x. The problems of compression, de-noising and indeed sampling are related.

Back to the ill-posed inverse problem of interferometry: We measure the complex Visibilities of a band-limited signal. The Nyquist-Shannon rate states that if our band limited signal has at most frequency f, our sample frequency needs to be higher than $2f$. For n pixels, this is the case when we measure all n complex Visibilites. If we have fewer samples, the Nyquist Shannon theorem says we cannot reconstruct the true image.

But what if we know our image is a Sparseland Signal and we happen to know the dictionary? Let us assume our image consists of $n = 20 * 20$ pixels and our dictionary of $m = 1000$ wavelets. Further assume at most $s = 10$ of the wavelets are non-zero for a given image. Could one just measure the 10 non-zero components of α and reconstruct the image?

If we have prior knowledge about the location of the non-zero components, we would need 10 samples to reconstruct the image. Sadly, this is not the case in general: Every component of α is about equally likely to be non-zero. With the first sample we have a $1/100$ chance of measuring a non-zero component. However note that if we measure a non-zero component, we learn $1/10$ of the information about the image. If we hit a zero component, we learn practically nothing. Is there a way we can maximize our information gain of the non-zero components for each sample?

Surprisingly, the answer is yes: When the measurement space is as incoherent from the dictionary space as possible, we maximize our information gain for the non-zero components.

Interestingly enough, constructing an incoherent measurement space is easy: Random projections are very good at being incoherent from the dictionary.

ndom Projections are not possible, the Interferometer measures complex visibilities. Antenna configuration could help. But luckily what also can help is the Wide Field imaging. McEwen [8] showed the theoretical improvement on synthetic data.

Ra

How many samples are now needed?

CS does not need an over-complete dictionary. It can also work for wavelet domains. overcomplete dictionaries give more freedom in representation.

P in our original objective function. $Dic = P^{-1}$. So if the conversion from image space to Sparseland is only defined when the dictionary is a square matrix. When it has as many signal parts as pixels. The Objective Function can be modified to work with over-complete dictionaries.

3.2 Objective Function

The Compressed Sensing CLEAN objective function (1.3) uses the L0 norm for it's regularization term, which means the Objective Function is not convex. There are specialized solvers for the L0 compressed sensing. The L1 relaxation however is practically guaranteed to have the same minimum as the L0 norm and results in a convex objective function. Since GuRoBi works better on the L1 relaxation it was chosen for this project.

As for the objective function, there are three different spaces in which one might want to reconstruct: The analysis method, where the image x is minimized directly, the synthesis method where the sparse vector α is minimized, or by in-painting the missing Visibilities V_2.

$$
\begin{aligned}
analysis: &\quad \underset{x}{minimize}\ \left\|D_{dirty} - x \star PSF\right\|_2^2 \quad + \lambda \left\|Px\right\|_1 \\
synthesis: &\quad \underset{\alpha}{minimize}\ \left\|D_{dirty} - Dic\,\alpha \star PSF\right\|_2^2 + \lambda \left\|\alpha\right\|_1 \\
in-painting: &\quad \underset{V_2}{minimize}\ \left\|D_{dirty} - F^{-1}MV_2\right\|_2^2 \quad + \lambda \left\|PF^{-1}V_2\right\|_1
\end{aligned}
$$

All three objective functions have the same global minimum. For the second and third objective x can be retrieved by $x = Dic\,\alpha$ and by $x = F^{-1}V_2$ respectively. [Empirical and theoretical studies have shown an advantage of the analysis objective over the other two [**?**]].

However practical considerations play a factor. $Px = \alpha$ is only true P is an orthogonal transformation like the wavelet transform. Over-complete dictionaries would result in $P \in \mathbb{R}^{m*n}, n \ll m$ and may not produce a suitably sparse vector α. Therefore over-complete prior tend to use the synthesis objective, since it only requires a transformation from sparse to image space($x = Dic\,\alpha$).

It is a similar story with in-painting: the transformation $Px2$ may be eaier represented in the fourier space, especially when P contains a convolution.

The strength of Compressed Sensing is that the objective can be modified for the measurement equation, while the Optimization Algorithm and the Prior stay the same. The above objective functions represent the deconvolution problem which is only true for small Field of View imaging.

Either A- and W-Projections are used to transform the wide FoV measurement equation (2.2) back to the deconvolution. Or the measurement equation (2.2) gets included in the data term, resulting in the new objective (3.2).

$$\underset{x}{minimize} \; \left\| V - MF_{wFOV}^{-1}x \right\|_2^2 + \lambda \left\| Px \right\|_1 \tag{3.2}$$

A lot of freedom to choose.

For this project CASA was used, which limits the choices.

3.2.1 Implementation In Casa

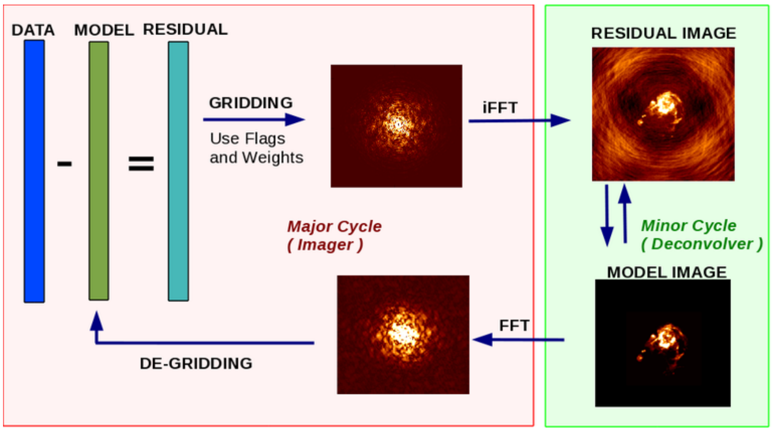


Figure 3: Casa Major Minor Cycle, source [9]

Casa major and minor cycle. Major cycle calculates visibilities in image space. Minor Cycle Deconvolves the Problem, often with a CLEAN class Algorithm. This constrains the algorithm to use the data term in image space.

This forces the objective function to either minimize in the image domain or in the sparsity domain.

3.3 Compressed Sensing Algorithms in Astronomy

3.3.1 SASIR

3.3.2 PURIFY

3.3.3 Vis-CS

4 Results

Physical plausible and shown to produce better results on synthetic data [8]

Clean images:

Gurobi Clean

4.1 Simple Priors

pixels l1 norm

pixels l2 norm

Total Variation

Haar

4.2 Starlet Transform as Prior

starlet decomposition

the cJ map as a smart thresholder

Runtime issues

last comparison, CLean, TV, starlet

5 Conclusion and Future Development

References

[1] JA Högbom. Aperture synthesis with a non-regular distribution of interferometer baselines. Astronomy and Astrophysics Supplement Series, 15:417, 1974.

[2] FR Schwab. Relaxing the isoplanatism assumption in self-calibration; applications to low-frequency radio interferometry. The Astronomical Journal, 89:1076–1081, 1984.

[3] JW Rich, WJG De Blok, TJ Cornwell, Elias Brinks, Fabian Walter, Ioannis Bagetakos, and RC Kennicutt Jr. Multi-scale clean: A comparison of its performance against classical clean on galaxies using things. The Astronomical Journal, 136(6):2897, 2008.

[4] Urvashi Rau and Tim J Cornwell. A multi-scale multi-frequency deconvolution algorithm for synthesis imaging in radio interferometry. Astronomy & Astrophysics, 532:A71, 2011.

[5] Tim J Cornwell, Kumar Golap, and Sanjay Bhatnagar. The noncoplanar baselines effect in radio interferometry: The w-projection algorithm. IEEE Journal of Selected Topics in Signal Processing, 2(5):647–657, 2008.

[6] Cyril Tasse, S van der Tol, J van Zwieten, Ger van Diepen, and S Bhatnagar. Applying full polarization a-projection to very wide field of view instruments: An imager for lofar. Astronomy & Astrophysics, 553:A105, 2013.

[7] Institut de RadioAstronomie Millimétrique. Proceedings from imiss2 “iram millimeter interferometry summer school 2”, 2000.

[8] Jason D McEwen and Yves Wiaux. Compressed sensing for radio interferometric imaging: Review and future direction. In Image Processing (ICIP), 2011 18th IEEE International Conference on, pages 1313–1316. IEEE, 2011.

[9] National Radio Astronomy Observations. tclean overview, 2016.

List of Figures

1 Inverse Problem example for VLA: Retrieve the true image when only PSF and dirty image are known 2
2 U V and W coordinate space 5
3 Casa Major Minor Cycle, source [9] 9

List of Tables

6 Ehrlichkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende schriftliche Arbeit selbstständig und nur unter Zuhilfenahme der in den Verzeichnissen oder in den Anmerkungen genannten Quellen angefertigt habe. Ich versichere zudem, diese Arbeit nicht bereits anderweitig als Leistungsnachweis verwendet zu haben. Eine Überprüfung der Arbeit auf Plagiate unter Einsatz entsprechender Software darf vorgenommen werden.
Windisch, July 4, 2018

Jonas Schwammberger