Nonlinear Control Systems Analysis

Andras Boguslaw Arancio

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Chapter 1

Elements of General Systems Theory

1.1 Dynamical Systems and State Space Representation

General systems theory is inspired by the identification of common laws in distinct disciplinary sectors. Its objective is to achieve unification on a modeling basis. System science is inspired by the need to manage complexity and aims to propose a common design approach for problem classes and models. In this context, systems theory sets itself the objective to arrive at a formalization of the concepts and the construction of a framework of methodologies for the systematic study by classes of models.

We identify four levels of abstraction:

1. abstract formulation (model) of the behavior of the studied object;
2. use of the model to infer knowledge about the real object;
3. identification of the generality of the model to represent different phenomena;
4. classification of types of models and development of analysis methodologies for classes of models.

Definition 1 (Abstract System)

An abstract system is a couple

$$\Sigma := \{V, R\}$$

where V represents the set of *variables* and R represents the set of *rules* that define the behaviors of the system.

We now introduce the notion of dynamical system. Let $T \subset \mathbb{R}$ (or $\mathbb{Z}$), be the ordered set of *time instants*. Let

$$T(t_0) = \{t \in T : t \geq t_0\}.$$

Let $W^{T(t_0)}$ be the set of functions defined from $T(t_0) \to W$:

$$W^{T(t_0)} = \{w_0(\cdot) :\ t \to w_0(t) \in W \quad \forall t \geq t_0\}.$$

A subset

$$\Sigma(t_0) \subset W^{T(t_0)}$$

can be used to describe the possible behaviors at t_0. We can in fact define an abstract system as a set of possible behaviors in different time instants.

To formalize the intuitive property of truncation closure, which states that the results of an experiment carried out at time t_0, if viewed at time t_1, must be in the set of possible results starting at time t_1, we introduce the following formal property. For all (t_0, t_1), with $t_1 \geq t_0$, if $w_0 \in \Sigma(t_0)$, then its truncation $w_0|_{T(t_1)}$ belongs to $\Sigma(t_1)$.

Definition 2 (Dynamical System)

A dynamical system is a triplet

$$S := \{T, W, \Sigma\}$$

where:

$$\Sigma := \{\Sigma(t_0), t_0 \in T : \; w_0 \in \Sigma(t_0) \implies w_0|_{T(t_1)} \in \Sigma(t_1), \quad \forall t_1 \in T(t_0)\}.$$

If $T \subset \mathbb{R}$, then the system is a *continuous time system.* Whereas if $T \subset \mathbb{Z}$, the system is a *discrete time system.*

It is useful to note that truncation closure does not imply that $\Sigma(t_1)$ may contain behaviors that are not obtained by truncation of behaviors that belong to $\Sigma(t_0)$. To satisfy the property that the behaviors at any time t_1 can be obtained as a truncation of the behaviors at a previos time instant we introduce the following definition.

Definition 3 (Uniform Dynamical System)

A dynamical system $S := \{T, W, \Sigma\}$ is said to be *uniform* if there exists a unique subset $\Sigma_{un} \subset W^T$ that generates all the possible behaviors $\Sigma(t_0)$, for all t_0:

$$w \in \Sigma_{un} \implies w|_{T(t_0)} \in \Sigma(t_0), \quad \forall t_0$$
$$w_0 \in \Sigma(t_0) \implies \exists w \in \Sigma_{un} : w|_{T(t_0)} = w_0.$$

Definition 4 (Stationary Dynamical System)

A dynamical system is said to be stationary if

$$\Delta_{\bar{t}}\Sigma(t_0) = \Sigma(t_0 + \bar{t})$$

for each t_0 and $\bar{t}$ in T. Where $\Delta_{\bar{t}}$ denotes the traslation operator:

$$(\Delta_{\bar{t}} f)(t') := f(t' - \bar{t}).$$

The stationarity property expresses the fact that by traslating the $\Sigma(t_0)$ function right of $\bar{t}$ (with $\bar{t} > 0$) we get the value $\Sigma(t_0 + \bar{t})$. For discrete time systems, the translation is defined by the unitary translation operator σ and stationarity is expressed as:

$$\sigma\Sigma(t) = \Sigma(t + 1).$$

From the definition of stationary dynamical system follows that any $\Sigma(t_0)$ can be obtained as $\Delta_{t_0}\Sigma(0)$. So we define a stationary dynamical system as a triplet $\{T, W, \Sigma(0)\}$.

Definition 5 (Linear Dynamical System)

A dynamical system $S := \{T, W, \Sigma\}$ is said to be linear if $W \subset \mathbb{R}^n$ is a linear space and if, for all t_0, $\Sigma(t_0)$ is a linear subspace of $W^{T(t_0)}$. That is, for all $w_1, w_2 \in \Sigma(t_0)$ and $\alpha, \beta \in \mathbb{R}$:

$$\alpha w_1 + \beta w_2 \in \Sigma(t_0).$$

A dynamical system is usually described in *implicit form*, by a system of differential equations (continuous time) or a system of difference equations (discrete time) together with an auxiliary set of variables.

Definition 6 (Dynamical System with Auxiliary Variables)

A *dynamical system with auxiliary variables* is a quadruple $S := \{T, W, A, \Sigma_a\}$ where:

- A is the set of values of the auxiliary variables;
- $\Sigma_a = \{\Sigma_a(t_0) \subseteq (W \times A)^{T(t_0)}$ s.t. truncament closure is satisfied$\}$.

S_a is the representation with auxiliary variables of $S := \{T, W, \Sigma\}$ if for all t_0

$$\Sigma(t_0) = \{w_0 : \exists a_0 \in A^{T(t_0)} \text{ s.t. } (w_0, a_0) \in \Sigma_a(t_0)\}.$$

A special class of auxiliary variables is the one of *state variables*.

Definition 7 (State)

A dynamical system with state variables is a dynamic system with auxiliary variables $S_x := \{T, W, X, \Sigma_x\}$, where Σ_x satisfies the *state axiom*:

$$(w_0^1, x_0^1), (w_0^2, x_0^2) \in \Sigma_x(t_0),\ t \geq t_0 \text{ and } x_0^1(t) = x_0^2(t) \implies (w_0, x_0) \in \Sigma_x(t_0)$$

where (w_0, x_0) is defined as:

$$(w_0(t'), x_0(t')) = \begin{cases} (w_0^1(t'), x_0^1(t')) & t' < t \\ (w_0^2(t'), x_0^2(t')) & t' \geq t \end{cases}$$

The state axiom requires that every trajectory that arrives in a fixed state can be concatenated with every trajectory that starts from that same state. In such conditions, once the state is know at a fixed time instant, future behaviors are fixed and no other information is contained in past behaviors. In other words, the state at time t suffices to characterize every possible behavior from t onwards; the state contains every necessary information about the past. Shortly, the state represents the memory of the past.

$S_x := \{T, W, X, \Sigma_x\}$, is the state space representation of a dynamical system $S = \{T, W, \Sigma\}$ such that $\Sigma(t_0) = \{w_0 : \exists x_0 \text{ s.t. } (w_0, x_0) \in \Sigma_x(t_0)\}$.

But, under which conditions does a state space representation exist? The representation problem is exstensively studied in systems theory and will be furhter discussed later on.

1.2 Oriented Dynamical Systems and State Space Representation

The engineering point of view conducts to distinguish variables in causes and effects, inputs and outputs, connected by causal relationships in function of time.

It is worth recalling that in engineering, the modeling of a process or phenomenon represents the first phase of a development process that often has the aim of satisfying some prefix specifications based on a fixed set of variables. The identification of the external variables on which to intervene naturally conducts to a cause-effect, input-output oriented type of modeling. Furthermore, our interest is limited to phenomena and processes where the input-output relationship is *causal*, that is, the output at time t depends upon the past and present input, but not the upon the future input.

Before giving the formal definition of an *Oriented Abstract System*, some further clarification is needed. Suppose that the set of values of variables is a cartesian product $W = U \times Y$ where U indicates the set of input values and Y the set of output values. The orientation of an abstract system corresponds to dividing the variables in causes and effects, and naturally suggest to figure the system as a black box that represents the laws that rule how input variables influence output variables. Behaviors are in this case imagined as experiments conducted in different t_0 instants.

Definition 8 (Oriented Abstract Dynamical System)

An *oriented abstract dynamical system* is a triplet $\{T, U \times Y, \Sigma\}$ where

$$\Sigma = \left\{\Sigma(t_0) \subset U^{T(t_0)} \times Y^{T(t_0)} : t_0 \in T \text{ s.t. truncation closure is satisfied.}\right\}$$

In oriented dynamical systems, truncation closure is defined as: $\forall t_0 \in T,\ \forall t_1 \geq t_0$,

$$(u_0, y_0) \in \Sigma(t_0) \implies (u_0|_{T(t_1)}, y_0|_{T(t_1)}) \in \Sigma(t_1).$$

1.2.1 Causality

In the context of oriented systems, where a behavior at t_0 is thought of as the result of an experiment that corresponds to an external solicitation $u_0(\cdot)$ defined from t_0 onwards, the concept of state variable as defined in definition 7 naturally conducts to identify a state specific property. In fact, the set of states at t_0 constitutes a parametrization of the set $\Sigma(t_0)$ of the possible input-output couples. Fixing u_0 does not suffice for identifying y_0 because $\Sigma(t_0)$ is a relationship (multiple y_0 may correspond to the same u_0); the state x_0 represents the additional information to u_0 to speficy y_0.

From definition 1.2, an abstract oriented system is a set of relationships, subset of $\{U^{T(t_0)} \times Y^{T(t_0)}\}$. The fact that the state, together with u_0, can be used to identify a corresponding output behavior has roots in the following algebra proposition, for which a relationship can be partitioned in equivalence classes.

Let A, B be non empty sets, $R \subset A \times B$, and let $D(R) \subset A$ and $R(R) \subset B$ be the domain and range of the relationship R.

Proposition 9

It is possible to define a set P and a function $\pi : P \times D(R) \to R(R)$ such that

$$\begin{aligned} (a, b) \in R &\implies \exists p : b = \pi(p, a) \\ p \in P, a \in D(R) &\implies (a, \pi(p, a)) \in R \end{aligned}$$

(P, π) is the *parametrization* of R.

It is therefore possible to associate to every $\Sigma(t_0)$ a parametrization, that is a set of parameters X_{t_0} and a function

$$\pi_{t_0} : X_{t_0} \times D(\Sigma(t_0)) \to R(\Sigma(t_0)).$$

Definition 10

A parametric representation of a system S is a set of functions

$$\pi = \left\{\pi_{t_0} : X_{t_0} \times D(\Sigma(t_0)) \to R(\Sigma(t_0)), \quad t_0 \in T\right\}$$

that satisfy the following properties:

$$\begin{aligned} (u_0, y_0) \in \Sigma(t_0) &\implies \exists x_0 : y_0 = \pi_{t_0}(x_0, u_0) \\ x_0 \in X_{t_0}, u_0 \in D(\Sigma(t_0)) &\implies (u_0, \pi_{t_0}(x_0, u_0)) \in \Sigma(t_0) \end{aligned}$$

Note that by the given definition, being u_0 a fixed input at time t_0, the same output y_0 can correspond to different parameter values at time t_0.

We can now formally introduce another fundamental property: *causality.* Given $T \setminus T(\bar{t})$, a function f is *strictly causal* if:

$$\forall \bar{t} \in T, u|_{T \setminus T(\bar{t})} = u'|_{T \setminus T(\bar{t})} \implies [f(u)](\bar{t}) = [f(u')](\bar{t})$$

If it is further needed that $u(\bar{t}) = u'(\bar{t})$, then f is *causal.*

Definition 11

A dynamical system S is causal if there exists at least one causal parametric representation, that is:

$$\forall t_0 \in T, \forall x_0 \in X_{t_0}, \forall \bar{t} \in T(t_0)$$
$$u_{[t_0,\bar{t}]} = u'_{[t_0,\bar{t}]} \implies [\pi_{t_0}(x_0, u)](\bar{t}) = [\pi_{t_0}(x_0, u')](\bar{t}).$$

Note that in the latter formula, choosing $[t_0, \bar{t})$ would be expressing strict causality.

1.2.2 State Space Representation

To understand how it is possible to introduce the concept of state starting from a causal parametrization, consider the following.

The definition of oriented dynamical system requires that the parameters x_0, at different time instants, are connected. If at t_0, $y_0 \in R(\Sigma(t_0))$ corresponds to the couple $(x_0, u_0) \in X_{t_0} \times D(\Sigma(t_0))$, then at the generic time $t_1 \geq t_0$ the couple $(u_0, y_0)|_{T(t_1)}$ belongs to $\Sigma(t_1)$, so it corresponds to one or more parameter values in X_{t_1}. If we say that $\{X_{t_0}, t_0 \in T\}$ are subsets of a unique set X and remember that we want the state to mean "the memory of the system, which contains all necessary information about the past", it seems natural that between the values of the state at time t_1, to which $(u_0, y_0)|_{T(t_1)}$ corresponds, there is one that is related to x_0 and u_0 in a functional way such as:

$$x_1 = x(t_1) = \varphi(t_1, t_0, x_0, u_0)$$

Furthermore, such functional relationship is admitted to be causal, more precisely, strictly casual; so that only the restriction of u_0 to the interval $[t_0, t_1)$ is meaningful.

The previous considerations make light on the opportunity to define an evolution in the parameter space X to connect the parameter values in different time instants.

Definition 12 (State transition function)

Let X be the parameter space, U the set of values of the functions u, $\mathcal{U} \subset U^T$ the space of the input functions,

$$(T \times T)* = \Big\{(t, t_0) : t \geq t_0, \quad t, t_0 \in T\Big\}$$

The state transition function φ is defined as:

$$\varphi : (T \times T)^* \times X \times \mathcal{U} \to X$$
$$x(t) := \varphi(t, t_0, x_0, u)$$

and satisfies the following properties:

P1: consistency

$$\forall t \in T, \ \forall u \in \mathcal{U} \quad \varphi(t, t, x, u) = x$$

P2: causality

$$\forall (t, t_0), \ \forall x_0 \in X \quad u|_{[t_0,t)} = u'|_{[t_0,t)} \implies \varphi(t, t_0, x_0, u) = \varphi(t, t_0, x_0, u')$$

P3: separation

$$\forall (t, t_0), \ \forall x \in X, \ \forall u \in \mathcal{U}$$
$$t > t_1 > t_0 \implies \varphi(t, t_0, x_0, u) = \varphi(t, t_1, \varphi(t_1, t_0, x_0, u), u)$$

P1 and P2 are obvious. P3 states that the state at t can be obtaind from x_0 and $u_{[t_0,t)}$, but even from the state reached at t_1 with x_0 and $u_{[t_0,t_1)}$ and further with $u_{[t_1,t)}$, because $x(t_1)$ contains the history of the state at t_1.

We can now ask ourselves, how does the output depend on x and u? From definition 10 we have that

$$\forall t_0, t \geq t_0, \quad y_0(t) = [\pi_{t_0}(x_0, u_0)](t)$$

where only the restriction of u_0 on $[t_0, t]$ is meaningful, because of causality. Assumed $t_0 = t$, we have

$$y(t) = [\pi_t(x(t), u_0)](t) := \eta(t, x(t), u_0(t))$$

that points out how the output at time t depends on the input values and the state at that same instant.

In conclusion, we have that an output transformation η is defined as:

$$\eta : T \times X \times U \to Y$$
$$y(t) := \eta(t, x(t), u(t))$$

As a last insight, it is useful to note how by assigning $U, Y, \mathcal{U} \subset U^T, X$, functions φ with properties P1-P3 and η make possible to generate a system $\tilde{\Sigma}$. In fact, for any fixed t_0, a relation $\tilde{\Sigma}(t_0)$ is defined:

$$\tilde{\Sigma}(t_0) = \Big\{ (u_0, y_0) \in U^{T(t_0)} \times Y^{T(t_0)}$$
$$u_0 = u|_{T(t_0)},\ y_0 : y_0(t) = \eta(t, \varphi(t, t_0, x_0, u), u(t)) \text{ with } u \in \mathcal{U}, x_0 \in X \Big\}$$

Furthermore, because of property P3, truncament closure is satisfied by the set of relationship $\Sigma(\tilde{t_0})$.

We now understand that φ and η define an oriented causal system in an alternative form. Such definition can be assumed as the starting point in the development of a theory.

We will soon see that it is in fact possible to use a triplet (X, φ, η) with X state space, φ transition function and η output transformation to describe an oriented abstract system.

1.2.3 Existence and uniqueness of state space representations

Definition 13

Given a system S and $\mathcal{U} \subset U^T$ set of input functions, a triplet (X, φ, η), with φ and η as previously defined, is a *state space representation* of S if properties P1, P2 and P3 are satisfied and at any given time t_0, the set of input-output couples $\tilde{\Sigma}(t_0)$ generated by (X, φ, η) coincides with the set of the systems input-output couples $\Sigma(t_0)$:

$$\forall t_0 \quad \tilde{\Sigma}(t_0) = \Sigma(t_0)$$

Given the system S the problem of indentifying a state space representation (X, φ, η) by definition 13 is known as *state association problem.*

The previous considerations are intended to highlight the salient aspects of such a problem. Passing from a parametric causal representation to a state space representation is possible under certain hypotheses about the set of input functions. More precisely:

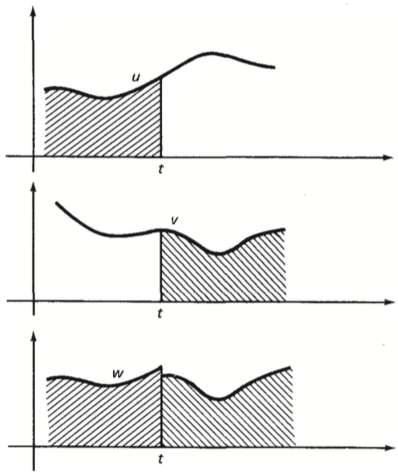

Figure 1.1

Definition 14 (Input functions space)

$\mathcal{U} \subset U^T$ is an *input functions space* for S if:

$$\forall t_0 \quad D(\Sigma(t_0)) = \left\{ u_0 = u|_{T(t_0)} \in U^{T(t_0)}, u \in \mathcal{U} \right\}$$

It is immediate to see that in uniform systems, $D(\Sigma(t_0))$ has such property hence it is an input functions space.

- $\mathcal{U}$ is closed under concatenation if:

$$\forall u, v \in \mathcal{U}, \forall t \in T, \quad \exists w = \begin{cases} w|_{T(t)} = v|_{T(t)} \\ w|_{T \setminus T(t)} = u|_{T \setminus T(t)} \end{cases}$$

  Such property is well described by figure 1.1. If $u, v \in \mathcal{U}$, than $\mathcal{U}$ must also contain w defined as shown.

- $\mathcal{U}$ is complete if it is closed under concatenation and:

$$\forall t \in T \quad U = \left\{ u(t) \in U : u \in \mathcal{U} \right\}$$

Under the aforementioned hypotheses, a fundamental result is verified that resolves the association problem. It is in fact proved that:

Theorem 15

A given oriented abstract system S, defined over a complete input functions space $\mathcal{U}$, admits state space representations if and only if it is causal.

For more details about the proof, the reader may consult Ruberti A., Isidori A., *Teoria dei Sistemi*, 1997, Edizioni Boringhieri.

A system may be associated to multiple state space representation. Two state space representations (X, φ, η) and (X', φ', η') represent the same system if:

$$\begin{gathered} \forall t_0, \forall x_0, \forall u, \exists x_0' : \forall t \in T(t_0) \\ \eta(t, \varphi(t, t_0, x_0, u), u(t)) = \eta'(t, \phi'(t, t_0, x_0', u), u(t)) \end{gathered}$$

and vice versa

$$\begin{gathered} \forall t_0, \forall x_0', \forall u, \exists x_0 : \forall t \in T(t_0) \\ \eta(t, \varphi(t, t_0, x_0, u), u(t)) = \eta'(t, \phi'(t, t_0, x_0', u), u(t)) \end{gathered}$$

Note that, in general, x_0' depends upon x_0 and a particular input u. It is often of interest that two representation have a stronger relationship, where x_0' only depends upon x_0. Such stronger relationship is usually called equivalence between two representations.

Definition 16

Two state space representations (X, φ, η) and (X', φ', η') are said to be equivalent if:

$$\forall t_0, \forall x_0, \exists x_0' : \forall u, \forall t \in T(t_0)$$
$$\eta(t, \varphi(t, t_0, x_0, u), u(t)) = \eta'(t, \phi'(t, t_0, x_0', u), u(t))$$

and vice versa

$$\forall t_0, \forall x_0', \exists x_0 : \forall u, \forall t \in T(t_0)$$
$$\eta(t, \varphi(t, t_0, x_0, u), u(t)) = \eta'(t, \phi'(t, t_0, x_0', u), u(t))$$

The relationship in definition 16 ensures the existence for each state $x_0 \in X$ of at least one state $x_0' \in X'$ (and vice versa) from which the same input-output couples are produced.

Example 17

Given (X, φ, η) and $f : X \to X'$, with f invertible, an equivalent representation (X', φ', η') can be defined in the following way:

$$\varphi'(t, t_0, x_0', u) = f \circ \varphi(t, t_0, f^{-1}(x_0'), u)$$
$$\eta'(t, x'(t), u(t)) = \eta(t, f^{-1}(x'(t)), u(t))$$

It is easy to verify that (X', φ', η') respects the properties of a state space representation. Further, the given definition of equivalence respects the criteria of a mathematical equivalence relation, so the set of all state space representations for a given system S may be partition by using such relation.

Between the possible state space representations, we have a particular interest in those that present a smaller state space. For this sake,

Definition 18

$x_a, x_b \in X$ are equivalent (or indistinguishable) at time t_0 if $\forall u, \forall t \in T(t_0)$

$$\eta(t, \varphi(t, t_0, x_a, u), u(t)) = \eta(t, \varphi(t, t_0, x_b, u), u(t))$$

By applying separation property P3 it is immediate to verify that if x_a and x_b are equivalent at t_0, they're equivalent for each $t_1 > t_0$.

Definition 19

(X, φ, η) associated to system S is said to be *reduced* at time t_0 if no equivalent states exist at time t_0.

1.2.4 From explicit to implicit representations

Discrete Time Systems

A general property of discrete time representations is the possibility to obtain without any additional hypothesis a so called "implicit" representation from (X, φ, η). From

$$x(t) = \varphi(t, t_0, x_0, u)$$

by putting $t = t + 1$, $t_0 = t$, we get:

$$x(t+1) = \varphi(t+1, t, x(t), u|_{[t,t+1)}) = \varphi(t+1, t, x(t), u(t)) \tag{1.1}$$

Such expression makes light on the fact that, in discrete time sistems, the value of the state at time $t + 1$ depends on time t and by the values of the state and input at that time. We can than say, in general, that

$$x(t+1) = f(t, x(t), u(t))$$

where f a function

$$f : \mathbb{Z} \times X \times U \to X \tag{1.2}$$

that is computed, starting from φ, with the rule specified in equation 1.1.

A function f of the form 1.2 is unique for each transition function φ (Ruberti A., Isidori A., *Teoria dei Sistemi*, 1997).

Continuous Time Systems

As pointed out, in the discrete-time case it is always possible to obtain an implicit representation of a system. The existence of implicit representations in the continuous-time case is conditioned by so called *regularity hypotheses*.

A continuous-time systems representation must provide a description of the system "in real time", by the use of causal and differential relationships. The existence hypotheses are thus strictly related to the differentiability of the state transition function. If we put $\varphi(t, t_0, x_0, u)$ to be a solution of

$$\frac{\partial \varphi(t, t_0, x_0, u)}{\partial t} = f(t, \varphi(t, t_0, x_0, u), u(t))$$

with the initial condition $\varphi(t_0, t_0, x_0, u) = x_0$, we can rewrite such last equation in the more compact form:

$$\dot{x}(t) = f(t, x(t), u(t))$$

f is called *generation function*. We can thus use theory about differential equations to study systems theory.

1.3 Linear Finite-Dimensional State Space Representations

We now introduce and discuss the linearity hypotheses of a finite-dimensional state space representation. We also introduce the free and forced response decomposition in the state and in output. We then focus on the relationship between explicit and explicit representations. Further, on the stationarity hypotheses and at last the use of linear representations as approximation of nonlinear models.

1.3.1 Structure and properties of linear representations

Definition 20 (Linear representation)

Let X, U, Y be linear spaces on the same field. (X, φ, η) is a linear representation if: φ is linear $\forall(t, t_0)$ on the set $X \times \mathcal{U}$ and η is linear $\forall t$ on the set $X \times U$.

Proposition 21 (Decomposition of the linear transition function)

An immediate consequence of linearity is: $\forall k_1, k_2, \forall x_{0_1}, x_{0_2} \in X, \forall u_1, u_2$

$$\varphi(t, t_0, k_1 x_{0_1} + k_2 x_{0_2}, k_1 u_1 + k_2 u_2) = k_1 \varphi(t, t_0, x_{0_1}, u_1) + k_2 \varphi(t, t_0, x_{0_2}, u_2) \tag{1.3}$$

If we put $k_1 = k_2, u_1 = 0, u_2 = u, x_{0_1} = x_0, x_{0_2} = 0$ in equation 1.3 we get

$$\varphi(t, t_0, x_0, u) = \varphi(t, t_0, x_0, 0) + \varphi(t, t_0, 0, u) = \varphi_l + \varphi_f$$

where φ_l is called *free response in the state* and φ_f is called *forced response in the state*. The first is linear in the initial state variable x_0 while the second is linear in the input u.

With the additional hypotheses that X, U, Y have finite dimensions n, p, q respectively, free and forced responses assume a particular form, as we'll soon see.

Given a linear dynamical system S as in definition 20, under which conditions do linear finite dimensional state space representations of S exist? It is obvious that a finite dimensional linear representation of the type given in definition 20 "generates" a causal finite dimensional linear system (remember definition 5 and theorem 15), but the vice versa is not always true. Let's think about what we learned about equivalent representations (definition 16 and example 17). If we suppose that (X, φ, η) is a linear representation and $f : X \to X'$ a nonlinear function, we can define a representation (X', φ', η')

$$\varphi'(t, t_0, x'_0, u) = f \circ \varphi(t, t_0, f^{-1}(x'_0), u)$$
$$\eta'(t, x'(t), u(t)) = \eta(t, f^{-1}(x'(t)), u(t))$$

which is a nonlienar representation equivalent to (X, φ, η).

To get a linear representation for a linear system, we need to ensure that the causal parametrisation satisfies the so called *consistency property at state zero*. More precisely, a parametrization π for S is said to be consistent with respect to the state zero if:

$$\forall (t, t_0), \forall u \in U \quad \pi_{t_0}(0, 0_{[t_0, t_1)} * u_[t_1, t)])|T(t_1) = \pi_{t_1}(0, u_{[t_1, t)})$$

A system S, defined on a complete set of input functions $\mathcal{U}$, has at least one linear finite dimensional state space representation if and only if there exists a causal, linear, finite dimensional, zero consistent parametrization. The proof is omitted.

1.3.2 Discrete Time Systems

Proposition 22

Let $T = \mathbb{Z}$ and let $x_0 \in X \cong \mathbb{R}^n$. Because of the linearity of φ_l with respect to x_0, we can express the free response as:

$$\varphi(t, t_0, x_0, 0) = \phi(t, t_0) x_0 \tag{1.4}$$

where $\phi(t, t_0)$ is an $(n \times n)$ matrix of functions defined over $(\mathbb{Z} \times \mathbb{Z})^*$, that is $(t, t_0) : t \geq t_0$.

Proposition 23

The forced response in the state of a finite dimensional linear representation can be expressed in the form:

$$\varphi(t, t_0, 0, u|_{[t_0,t)}) = \sum_{\tau=t_0}^{t} H(t,\tau)u(\tau) \tag{1.5}$$

where $H(t,\tau)$ is an $(n \times p)$ matrix defined on $(\mathbb{Z} \times \mathbb{Z})^*$, that is $(t,\tau) : t \geq \tau$ and is such that:

$$H(t,t) = 0 \tag{1.6}$$

Equations 1.5, 1.6 constitute a synthesis of the two following expressions

$$\varphi(t, t_0, 0, u(\cdot)) = 0 \quad t = t_0 \tag{1.7}$$

$$\varphi(t, t_0, 0, u(\cdot)) = \sum_{\tau=t_0}^{t-1} H(t,\tau)u(\tau) \quad t > t_0 \tag{1.8}$$

1.7 is coherent with consistency property P1, whereas 1.8 is coherent with causality property P2, for which the value of the state at time t depends upon the values of the input in $[t_0, t)$. 1.8 is often used in place of 1.5, with the assumption that $t > t_0$.

We now give the proof of proposition 23.

Proof. Note that the function segment $u|_{[t_0,t)}$ is identified by the following sequence of $t - t_0$ vectors

$$[u(t_0), \ldots, u(\tau), \ldots, u(t-1)]$$

which can be written as

$$[u(t_0), 0, \ldots, 0] + \ldots + [0, \ldots, 0, u(\tau), 0, \ldots, 0] + [0, \ldots, 0, u(t-1)]$$

Because of the linearity of $\varphi_f = \varphi(t, t_0, 0, u)$ with respect to u, we have that

$$\varphi(t, t_0, 0, u|_{[t_0,t)}) = \varphi(t, t_0, 0, [u(t_0), 0, \ldots, 0]) + \ldots + \varphi(t, t_0, 0, [0, \ldots, 0, u(t-1)])$$

The contribution of the generic term $\varphi(t, t_0, 0, [0, \ldots, 0, u(\tau), 0, \ldots, 0])$ is equal to the value assumed by a linear function from U to X and can thus be represented by a product $H(t,\tau)u(\tau)$ where $H(t,\tau)$ is an $(n \times p)$ matrix.

From such considerations follows that equation 1.5 is proved. □

Proposition 24

By putting together the decomposition of the transition function shown in proposition 21, the form of the free response in the state given in proposition 22 and the form of the forced response in the state given in proposition 23, we give the following equation for the value of the state at time t:

$$x(t) = \varphi(t, t_0, x_0, u) = \phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} H(t,\tau)u(\tau) \tag{1.9}$$

Proposition 25

Because of property P1,

$$\phi(t,t) = I$$

H is called *state impulse response matrix.* The columns $h_i(t,\tau)$, $i = 1,\dots,p$ of the matrix $H(t,\tau)$ relative to the generic term $\varphi(t,t_0,0,[u(t_0),0,\dots,0])$ have an interesting interpretation in terms of forced responses to particular inputs. Consider the following function $\delta\colon \mathbb{Z} \to \mathbb{C}$ called *impulse* or *impulse function* centered at 0, defined as:

$$\begin{cases} \delta(0) = 1 \\ \delta(t) = 0 \quad t \neq 0 \end{cases}$$

If we now consider an input of the form

$$u^i(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \delta(t-t')$$

that is, an impulse centered at t' for the i-th component of the input, with $t_0 \le t' < t$, the corresponding forced response in the state is:

$$\varphi(t,t_0,0,u^i|_{[t_0,t)}) = h_i(t,t')$$

Some properties of matrices ϕ and H are an immediate consequence of separation property P3. In fact:

$$\varphi(t,t_0,x_0,u) = \phi(t,t_0)x_0 + \sum_{\tau=t_0}^{t-1} H(t,\tau)u(\tau) = \tag{1.10}$$

$$\varphi(t,t_1,\varphi(t_1,t_0,x_0,u),u) = \phi(t,t_1)\left[\phi(t_1,t_0)x_0 + \sum_{\tau=t_0}^{t_1-1} H(t_1,\tau)u(\tau)\right] + \sum_{\tau=t_1}^{t-1} H(t,\tau)u(\tau) \tag{1.11}$$

Equation 1.10 must be satistied for all u and for all x_0, thus:

Proposition 26

By putting $u(\cdot) = 0$ we note the semi-group property of the transition matrix

$$\phi(t,t_0) = \phi(t,t_1)\phi(t_1,t_0) \quad \forall t \ge t_1 \ge t_0$$

Proposition 27

By putting $x_0 = 0$ we note the separation property of the state impulse response matrix

$$H(t,\tau) = \phi(t,t_1)H(t_1,\tau) \quad \forall t \ge t_1 > \tau \tag{1.12}$$

Proposition 27 makes light on a strong relationship between $H(\cdot,\cdot)$ and $\phi(\cdot,\cdot)$, that is, between the forced response matrix and the free response matrix. It is interesting to write 1.12 with $t_1 = \tau + 1$; in this case, if we consider the i-th column of both members, we get:

$$h_i(t,\tau) = \phi(t,\tau+1)h_i(\tau+1,\tau) \tag{1.13}$$

and by confronting the second member of this equation with 1.4, we can tell that it can be though of as the free response in the state, of the free evolution of the system starting from initial state $h_i(\tau+1,\tau)$, thought of as assumed at time $\tau + 1$. By further remembering the interpretation of the colummns of $H(\cdot,\cdot)$ as impulse response, equation 1.13 make light on the fact that *forced response to an impulse centered in* τ *is equal to free response starting from state* $h_i(\tau+1,\tau)$. In other words, we can think that such equality is implied by the fact that an impulse centered in τ determines, when passing from time τ to $\tau+1$, a transition from state 0 to state $h_i(\tau+1,\tau)$, followed by a free evolution of the system.

Because of the linearity of η over $U \times Y$ for all r and the finite dimensions of $U \cong R^p$ and $Y \cong R^q$, we have that

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1.14}$$

where $C(t)$ and $D(t)$ are matrices in function of time of dimensions $(q \times n)$ and $(q \times p)$ respectively. Note that in 1.14, if $D(t) \neq 0$, then the value of output at time t also depends upon the value of input at time t, that is, the system is causal. If $D(t) = 0$, the value of output at time t depends, through the value of the state, upon the values of that the input u assumed at time instants prior to t, that is, the system is strictly causal.

Proposition 28

By using equation 1.14 for function η and proposition 24 for φ, we can write the equation of the output at time t as:

$$y(t) = C(t)\phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)H(t, \tau)u(\tau) + D(t)u(t)$$

and by putting

$$C(t)\phi(t, t_0) = \psi(t, t_0) \tag{1.15}$$

$$D(t) = W(t, t) \tag{1.16}$$

$$C(t)H(t, \tau) = W(t, \tau) \quad t > \tau \tag{1.17}$$

we get the following equation, which is the one commonly used for describing the output in explicit form:

$$y(t) = \psi(t, t_0)x_0 + \sum_{\tau=t_0}^{t} W(t, \tau)u(\tau) \tag{1.18}$$

1.3.3 Continuous Time Systems

Proposition 29 (Free response in the state)

Let $T = \mathbb{R}$ and $x_0 \in X \cong \mathbb{R}^n$. Because φ_l is linear with respect to x_0, we can write:

$$\varphi(t, t_0, x_0, 0) = \phi(t, t_0)x_0$$

where $\phi(t, t_0)$ is an $(n \times n)$ matrix of functions on $(\mathbb{R} \times \mathbb{R})^*$. Also in this case, P1 implies:

$$\varphi(t, t) = I$$

Proposition 30 (Forced response in the state)

If φ_f is a continuous function for $u|_{[t_0,t)}$, the free response in the state can be written in the following form:

$$\varphi(t, t_0, 0, u_{[t_0,t)}) = \int_{t_0}^{t} H(t, \tau)u(\tau)d\tau \tag{1.19}$$

Proof. By using an analytic result about linear functional operator[1], we can get to an expression of the form:

$$\varphi(t, t_0, 0, u_{[t_0,t)}) = \int_{t_0}^{t} H_{t,t_0}(\tau)u(\tau)d\tau$$

[1] You may consult Schwartz J.T., Dunford N. (1958), *Linear Operators*, Interscience, New York.

where the function $H_{t_0,t}$ is a function of τ that also depends on time instants t_0, t. By using separation property P3 of the function $\varphi(t, t_0, x_0, u|_{[t_0,t)})$, and putting:

$$x_0 = 0$$
$$u|_{[t_0,t_1)} = 0|_{[t_0,t_1)}$$

and by taking into account the fact that, because of linearity

$$\varphi(t, t_0, x_0, 0|_{[t_0,t_1)}) = 0$$

then, the transition from t_0 to t, with a $u|_{[t_0,t_1)} \equiv 0|_{[t_0,t_1)}$ is equal to the transition from t_1 to t with input $u_{[t_1,t)}$. Thus:

$$\int_{t_0}^{t} H_{t,t_0}(\tau)u(\tau)d\tau = \int_{t_0}^{t} H_{t,t_1}(\tau)u(\tau)d\tau$$

Because $u|_{[t_1,t)}$ is arbitrary, we can state that

$$H_{t,t_0}(\tau) = H_{t,t_1}(\tau)$$

for all $t \geq \tau \geq t_1 > t_0$. And because t_0, t_1 are arbitrary too, we can define a matrix $H(\cdot, \cdot)$ with the rule:

$$H(t, \tau) = H_{t,\bar{t}}(\tau)$$

where the only limitation is that $t \geq \tau \geq \bar{t}$. We can thus use a such defined matrix in place of $H_{t,t_0}(\tau)$. □

Matrix $H(t, \tau)$ is called matrix of the impulse responses in the state because, similarly to the discrete time case, it's columns represent the responses in the state to impulse inputs. It is worth underlining the fact that such result is obtained by the use of a general approximation, for which we estimate the responses in the state obtained by the use of a succession that tends to the unitary impulse (Dirac's δ distribution). The limit of the response to a succession that only select the i-th entry of the input converges to the i-th column of H.

Proposition 31

As for discrete time systems, the following properties hold:

$$\phi(t, t_0) = \phi(t, t_1)\phi(t_1, t_0) \quad \forall t \geq t_1 \geq t_0 \tag{1.20}$$
$$H(t, \tau) = \phi(t, t_1)H(t_1, \tau) \quad \forall t \geq t_1 \geq t_0 \tag{1.21}$$

Proof. By applying property P3 we have that:

$$\varphi(t, t_0, x_0, u) = \phi(t, t_0)x_0 + \int_{t_0}^{t} H(t, \tau)u(\tau)d\tau \tag{1.22}$$

$$\varphi(t, t_1, \varphi(t_1, t_0, x_0, u), u) = \phi(t, t_1)\left[\varphi(t_1, t_0)x_0 + \int_{t_0}^{t_1} H(t, \tau)u(\tau)d\tau\right] + \int_{t_1}^{t} H(t, \tau)u(\tau)d\tau \tag{1.23}$$

which is true for all u and x_0. By putting $u(\cdot) \equiv 0$ and $x_0 = 0$ we obtain the thesis. □

As for the discrete time case, we have that

$$y(t) = \eta(t, x(t), u(t)) = C(t)x(t) + D(t)u(t)$$

with $C(t), D(t)$ matrices of dimension $(q \times n)$ and $(q \times p)$ respectively.

Proposition 32

We thus can write the output in the form:

$$y(t) = C(t)\phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)H(t, \tau)u(\tau)d\tau + D(t)u(t) \tag{1.24}$$

Under the hypotheses that the input function is continuous and $\eta(t, \varphi(t, t_0, 0, u|_{[t_0,t)}), u(t))$ is continuous for $u|_{[t_0,t]}$, by setting

$$W(t, \tau) := C(t)H(t, \tau) + D(t)\delta(\tau - t)$$

where $\delta(t)$ is the Dirac's distribution, defined by the following property

$$u(t) = \int_{t-\epsilon}^{t+\epsilon} u(\tau)\delta(\tau - t)d\tau \quad \forall \epsilon$$

W is called matrix of impulse responses in output. The argumentation behind the form of it's columns is analogous to the one made for matrix H. we can give the following explicit representation of the system:

Proposition 33 (Explicit representation, continuous time)

An explicit representation for a continuous time system is given by:

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} H(t, \tau)u(\tau)d\tau \tag{1.25}$$

$$y(t) = \psi(t, t_0)x_0 + \int_{t_0}^{t} W(t, \tau)u(\tau)d\tau \tag{1.26}$$

where $\psi(t, t_0) := C(t)\phi(t, t_0)$ and $W(t, \tau)$ as previously defined.

1.3.4 Implicit representations

We now introduce implicit representations for discrete and continuous systems. An implicit representation is a mathematical description in terms of differential or difference equations, which put light on how the behaviors of a dynamical system are the result of an iterative evolution process in the discrete time case, and a "real time" evolution in the continuous time case.

Discrete time systems

As earlier discussed, the evolution at a state step of a discrete time system is described by an equation of the form:

$$x(t + 1) = f(t, x(t), u(t))$$

f is called generation function. The given equation is explicative about the fact that the evolution is the result of an iterative process that generates the state at time $t + 1$ from the state at time t and the input at time t.

If f is linear over the space $X \times U$ for all t and $X \cong \mathbb{R}^n$ and $U \cong \mathbb{R}^P$ are of finite dimension, we can write

$$x(t + 1) = A(t)x(t) + B(t)u(t) \tag{1.27}$$

where A, B are matrices in function of time of dimensions $(n \times n), (n \times p)$ respectively, defined as

$$A(t) := \phi(t + 1, t), \quad B(t) := H(t + 1, t)$$

Furthermore, we remeber that the linearity of η over $X \times U$ for all r and the fact that $Y \cong \mathbb{R}^q$ imply: +

$$y(t) = C(t)x(t) + D(t)u(t) \tag{1.28}$$

Definition 34

Equations 1.27 and 1.28 define an *implicit state space representation* of a finite dimensional discrete time linear system.

Figure 1.2 shows the realization scheme of a discrete time linear system.

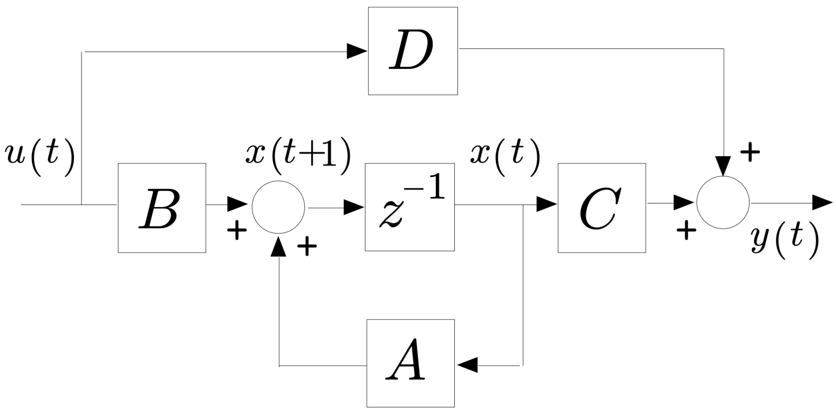

Figure 1.2

Continuous time systems

The existence of an implicit representation in the continuous time case depends on the regularity hypothesis. Such hypothesis consists in assuming that $\varphi(t, t_0, x_0, u)$ is the solution of a differential equation

$$\frac{\partial \varphi}{\partial t} = \dot{x}(t) = f(t, \varphi, u(t))$$

We will now show that if the explicit representation is linear, then the generator function is linear on the product space $X \times U$.

Theorem 35

There exists implicit representation of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.29}$$

with $A(t), B(t)$ matrix of continuous functions of dimensions $(n \times n), (n \times p)$ respectively if and only if the state transition matrix ϕ and the impulse response in the state matrix H are continuous functions over $(\mathbb{R} \times \mathbb{R})^*$ and ϕ is differetiable with respect to its first argument and its derivative is continuous. A system with such hypotheses is called a regular system. Thus, only regular system have an implicit representation.

Proof. Under the given hypotheses,

$$\begin{aligned}\frac{\partial \phi(t,\tau)}{\partial \tau} &= \lim_{\epsilon \to 0} \frac{\phi(t+\epsilon,\tau) - \phi(t,\tau)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \left(\frac{\phi(t+\epsilon,t) - \phi(t,t)}{\epsilon} \right) \phi(t,\tau) \\ &= \left[\frac{\partial \varphi(\xi,t)}{\partial \xi} \right]_{\xi=t} \phi(t,\tau)\end{aligned} \tag{1.30}$$

The derivative in the last line is a single variable function (because the other is assinged to be $\xi = t$), defined over $\mathbb{R}$. We can identify it as:

$$\left[\frac{\partial \varphi(\xi,t)}{\partial \xi} \right]_{\xi=t} = A(t) \tag{1.31}$$

Equation 1.30 becomes

$$\frac{\partial \phi(t,\tau)}{\partial \tau} = A(t)\phi(t,\tau)$$

By hypotheses, the derivative of $\phi(\cdot,\cdot)$ is a continuous functions, thus also $A(\cdot)$ is a continuous function, defined as 1.31.

Furthermore,

$$\begin{aligned}\frac{\partial}{\partial t} H(t,\tau) &= \frac{\partial}{\partial t} \phi(t,t_1) H(t_1,\tau) \\ &= A(t)\phi(t,t_1)H(t_1,\tau) = A(t)H(t,\tau)\end{aligned}$$

and by deriving $x(t)$ in its explicit form we get

$$\begin{aligned}\dot{x}(t) &= \frac{\partial}{\partial t}\left(\phi(t,t_0)x_0 + \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau\right) \\ &= A(t)\phi(t,t_0)x_0 + \int_{t_0}^{t} A(t)H(t,\tau)u(\tau)d\tau + H(t,t,)u(t) \\ &= A(t)x(t) + B(t)u(t)\end{aligned}$$

where

$$B(t) := H(t,t) \tag{1.32}$$

is continuous by hypotheses. □

The necessity of these hypotheses, assumed the existence of a linear generator function over $X \times U$, with $A(t), B(t)$ continuous, comes directly from differential equations theory.

Equations 1.31 and 1.32 are the formulas for passing from explicit to implicit form.

We now describe how to pass from implicit to explicit form, that is, how $A(\cdot), B(\cdot)$ can be computed from $\phi(\cdot, cdot), H(\cdot,\cdot)$. For a good understanding on the theory behind the transition from implicit to explicit, you may read appendix A. By using equations A.7 and A.9 we get to the following expression

$$\phi(t,\tau) = X(t)X^{-1}(\tau) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)d\tau_1 \int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 d\tau_1 + \dots \tag{1.33}$$

Appendices

Appendix A

First order vectorial, linear differential equations

Consider the following differential equation:

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{A.1}$$

where $t \in \mathbb{R}, x(t) \in \mathbb{C}^n$, $A(\cdot)$ is an $(n \times n)$ matrix and $b(\cdot)$ is an (n) vector of continuous functinos in the variable t with codomain $\mathbb{C}$.

In such hypotheses, given $t_0 \in \mathbb{R}$ and $c \in \mathbb{C}^n$, there exists a unique solution $\xi(\cdot)$ with codomain $\mathbb{R}$ to equation A.1 that satisfies the initial condition

$$\xi(t_0) = c \tag{A.2}$$

For computing the solution of A.1, consider the associated homogeneous equation

$$\dot{x}(t) = A(t)x(t) \tag{A.3}$$

For the solutions of such homogeneous equation, the following theorem holds.

Theorem 36

The set of all solutions with codomain $\mathbb{R}$ of A.3 is a vector space of dimension n, with field $\mathbb{C}$.

Proof. Let $\xi_1(\cdot), \xi_2(\cdot)$ be two generic solutions of equation A.3 and let c_1, c_2 be two complex numbers. Then also $c_1\xi_1(\cdot) + c_2\xi_2(\cdot)$ is a solution of A.3. In fact:

$$A(t)(c_1\xi(t) + c_2\xi(t)) = A(t)c_1\xi(t) + A(t)c_2\xi_2(t) = \dot{\xi_1}(t) + \dot{\xi_2}(t) = \big(c_1\xi_1(t) \overset{\bullet}{+} c_2\xi_2(t)\big)$$

The set of all solutions is thus a vector space with field $\mathbb{C}$.

To show that the dimension of the solutions space is n, consider n solutions $\xi_1(\cdot), \ldots, \xi_n(\cdot)$, each satisfying an initial condition $\xi_1(t_0) = e_1, \ldots, \xi_n(t_0) = e_n$, where $e_1, \ldots, e_n$ are the vectors of the standard basis. Such solutions are linearly independent on R; if they weren't there should exist a vector $a \neq 0, a \in \mathbb{C}^n$ such that

$$\big(\xi_1(t) \ldots \xi_n(t)\big)a = 0 \quad \forall t \in \mathbb{R}$$

but then we would have that, for $t = t_0$,

$$\big(\xi_1(t_0) \ldots \xi_n(t_0)\big)a = \big(e_1 \ldots e_n\big)a = 0$$

which is absurd.

We must now show that each solution can be written as a linear combination of $\xi_1(\cdot), \ldots, \xi_n(\cdot)$. For this sake, it is enough to think about the fact that any initial value $c \in \mathbb{C}^n$ can be written in the form

$$c = c_1e_1 + \ldots + c_ne_n$$

where $c_1, \ldots, c_n$ are the entries of c. It is thus immediately verified that function

$$x(\cdot) = c_1\xi_1(\cdot) + \ldots + c_n\xi_n(\cdot)$$

(which is a solution of A.3 because of linearity of the solutions space) satisfies initial condition A.2. □

Because of the given result, to have complete knowledge of the solutions space of equation A.3 it is sufficient to have a set of n linearly independent solutions which form a basis of the solutions space. We now introduce what a fundamental matrix of solutions is.

Definition 37

A *fundamental matrix* of solutions is any matrix $X(\cdot)$ that has as columns n linearly independent solutions of an equation of the form A.3 (that is, the elements of a basis of the solutions space).

With one of such matrices, because of theorem 36, any solution of A.3 can be written as:

$$\xi(\cdot) = X(\cdot)k \tag{A.4}$$

where $k \in \mathbb{C}^n$.

For fundamental matrices, the next theorem holds:

Theorem 38

Every fundamental matrix $X(t)$ is non singular for all $t \in \mathbb{R}$

Proof. We will at first prove that, for at least one value $\bar{t} \in \mathbb{R}$, $X(\bar{t})$ is non singular. For this sake, consider an arbitrary non zero solution $\xi(\cdot)$ of equation A.3 and let $\bar{t}$ be such that $\xi(\bar{t}) \neq 0$. Being true equation A.4 for an appropriate k, we can write

$$0 \neq \xi(\bar{t}) = X(\bar{t})k$$

If $X(\bar{t})$ was singular, such last equation would have infinite solutions in the variable k, but that is absurd because of the uniqueness of the solution of A.3 with initial condition $\xi(\bar{t})$. Thus $X(\bar{t})$ is non singular.

For completing the proof, we show that $X(t)$ cannot be singular for any other value of t. In fact, if for some $\hat{t} \in \mathbb{R}$ we had that $\det\big[X(\hat{(t)})\big] = 0$, there would exists a constant (non zero) vector β such that

$$X(\hat{t})\beta = 0$$

Consider now function

$$\xi(t) = X(t)\beta$$

which, by equation A.4 is the only solution of A.3 that satisfies initial condition $\xi(\hat{t}) = 0$. But, because even the identically null solution satisfies the same initial condition, we conclude that because of the uniqueness of the solution of A.3, $\xi(t) = X(t)\beta = 0$ for all $t \in \mathbb{R}$. This would imply that $\det\big[X(t)\big] = 0$ for all $t \in \mathbb{R}$, contradicting what initially proved. □

We now observe the fact that every fundamental matrix has the following property:

$$\dot{X}(t) = A(t)X(t) \tag{A.5}$$

because every column of X satisfies equation A.3.

For finding a fundamental matrix of solutions we thus consider (in place of the vectoral equation A.3) the associated *matricial* homogeneous equation

$$\dot{S}(t) = A(t)S(t) \tag{A.6}$$

where $S(\cdot)$ is an $(n \times n)$ matrix of functions of t. Between the solutions of A.6 are the fundamental matrices. For identifying these, we use the following theorem.

Theorem 39

A solution $\Sigma(\cdot)$ of equation A.6 is a fundamental matrix if and only if, arbitrarily chosen $t \in \mathbb{R}$, we have $\det\big[\Sigma(t)\big] \neq 0$.

Proof. If $\Sigma(\cdot)$ is a fundamental matrix, for theorem 38 $\det\big[\Sigma(t)\big] \neq 0$ for all $t \in \mathbb{R}$. Vice versa, if $\det\big[\Sigma(\bar{t})\big] \neq 0$ for at least one point $\bar{t} \in \mathbb{R}$, there cannot exist a constant (non zero) vector such that $\Sigma(\bar{t})a = 0$ and thus no constant vector β exists such that $\Sigma(t)\beta = 0$ for all $t \in \mathbb{R}$. As a consequence, the columns of $\Sigma(\cdot)$ are linearly independent solutions with codomain $\mathbb{R}$, in other words $\Sigma(\cdot)$ is a fundamental matrix. □

Starting from A.4 we can explicit the solution of the associated homogeneous equation A.3 that satisfies condition A.2. By imposing such condition, we have

$$\xi(t_0) = X(t_0)k = c$$

and because of the nonsingularity of $X(t_0)$ we have

$$k = X^{-1}(t_0)c$$

from which

$$\xi(t) = X(t)X^{-1}(t_0)c \quad \forall t \in \mathbb{R}$$

The solution of the non homogeneous equation A.1 can be obtained by constant variation method ad has the following form

$$\xi(t) = X(t)X^{-1}(t_0)c + X(t)\int_{t_0}^{t} X^{-1}(\tau)b(\tau)d\tau \tag{A.7}$$

The proof that A.7 is the unique solution of A.1 can be done by direct substitution.

To use A.7 it is necessary to dispose of a fundamental matrix of solutions and thus, for theorem 39, of a solution of equation A.5 which is not singular in at least one point $\bar{t} \in \mathbb{R}$.

Such a matrix is the following one:

$$X(t) = \Big\{ I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)d\tau_1\big[\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2\big]d\tau_1 + \dots \Big\} C \tag{A.8}$$

where C is an $(n \times n)$ non singular matrix with elements in $\mathbb{C}$.

It is proved that equation A.8 converges absolutely for every finite t and uniformly on every closed interval in $\mathbb{R}$. It is further verified that $C = X(t_0)$ and thus

$$X(t)X^{-1}(t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)d\tau_1\big[\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2\big]d\tau_1 + \dots \tag{A.9}$$