Problem Statement:

Consider two players who each choose k items out of n distinct items with replacement. Given that the order of the items does not matter, what is the probability that the two players chose the same items?

Notation:

$P \vdash k$ indicates that P is a partition of k. That is, $k = q_1 + q_2 + ... + q_s$ for positive integers $q_1, q_2, ...q_s$, and $P = \{q_1, q_2, ...q_s\}$ is a multiset of positive integers representing this integer partition.

Given a multiset P, the support of P, $\text{Supp}(P) = \{q \mid q \in P\}$, is the set of all elements in P *without* repetition.

For any element $q \in P$, where P is a multiset, the multiplicity of q is written as $m_P(q)$ and is equal to the number of times q occurs in P. In other words it is a function $m_P : \text{Supp}(P) \to \mathbb{Z}^+$ which maps elements of P to their multiplicities.

Let $\mathbb{M}_P$ be the multiset $\mathbb{M}_P = \{m_p(q) \mid q \in \text{Supp}(P)\}$ (*with* repetition). In other words, for every distinct element of P, $\mathbb{M}_P$ contains the multiplicity of that element, and if k distinct elements of P have the same multiplicity, then $\mathbb{M}_P$ contains that multiplicity k times. (This is equivalent to the "Dedekind multiset determined by the function m_P," which is the image of m_P where every element has multiplicity equal to the size of its preimage.)

The multinomial coefficient $\binom{n}{k_1, k_2, ..., k_r} = \frac{n!}{k_1!k_2!...k_r!} = \frac{n!}{\prod_{i=1}^{r} k_i!}$, where $k_1 + k_2 + ... + k_r = n$.

Let the notation $\binom{n}{S}$, where S is a set, multiset, or tuple with $\sum_{k \in S} k \ = n$, denote the multinomial coefficient $\frac{n!}{\prod_{k \in S} k!}$.

Given an ordered k-tuple $t = (e_1, e_2, ..., e_k)$, let t^* denote the corresponding unordered multiset $t^* = \{e_1, e_2, ..., e_k\}$. Given a set S of tuples, let $S^* = \{t^* \mid t \in S\}$ denote the set (without repetitions) of corresponding unordered multisets.

We say that a tuple t is "equivalent to" a tuple s iff $t^* = s^*$. In other words, t is equivalent to s if they are equal up to reordering.

Solution

Let N be the set of items we are choosing from, $|N| = n$. Let $\Lambda = \{(e_1, e_2, ..., e_k) \mid e_1, e_2, ..., e_k \in N\}$ be the set of *ordered* choosings of k items out of n items with repetition. Let our sample space $\Omega = \Lambda \times \Lambda$.

We say an ordered choosing $t \in \Lambda$ or its corresponding unordered choosing $t^* \in \Lambda^*$ is "of the form P," where $P \vdash k$, if the elements of t^* have multiplicities equal to P, that is if $\mathbb{M}_{t^*} = P$. In other words, if $P = \{q_1, q_2, ..., q_s\}$, a choosing of the form P consists of a group of q_1 identical items, a different group of q_2 identical items, etc., all the way to a different group of q_s identical items (where each group is made up of a different element from all the other groups).

Let $t \in \Lambda$ be an ordered choosing of the form $P \vdash k$. Let $[t] = \{s \mid s \in \Lambda,\ s^* = t^*\}$ be the equivalence class of t, the set of elements in Λ which are equivalent to t. In other words, it is the set of reorderings of t, or the multiset permutations of t^*. The number of permutations of a multiset is given by the multinomial coefficient of its multiplicities, and $\mathbb{M}_{t^*} = P$ so $|[t]| = \binom{|t^*|}{P} = \binom{k}{P}$.

Let $P = \{q_1, q_2, ..., q_s\}$ be a partition $P \vdash k$. The number of *unordered* choosings of the form P is equal to $\binom{n}{|P|}\binom{|P|}{\mathbb{M}_P}$, since we choose $|P|$ distinct elements from N, then we assign each one to a group of size q_i. Since groups of the same size are not distinct, we multiply by the multinomial coefficient to express the number of distinct ways of assigning elements to groups, rather than simply multiplying by the factorial.

Thus the number of *ordered* choosings of the form $P \vdash k$ is equal to $\binom{n}{|P|}\binom{|P|}{\mathbb{M}_P}\binom{k}{P}$, since for each unordered choosing we count the number of ways of reordering it.

There are n^k total ordered choosings, and they are all equally likely. The probability that the first player chooses a choosing of the form $P \vdash k$ is $\frac{1}{n^k}\binom{n}{|P|}\binom{|P|}{\mathbb{M}_P}\binom{k}{P}$.

Given that the first player chose a choosing of the form $P \vdash k$, the second player chooses an equivalent choosing (that is, chooses the same items) with probability $\frac{\binom{k}{P}}{n^k}$.

Thus the probability that both players choose the same k items out of n distinct items with repetition is:

$$\sum_{P \vdash k} \frac{1}{n^k}\binom{n}{|P|}\binom{|P|}{\mathbb{M}_P}\binom{k}{P}\frac{\binom{k}{P}}{n^k} = \boxed{\frac{1}{n^{2k}}\sum_{P \vdash k}\binom{n}{|P|}\binom{|P|}{\mathbb{M}_P}\binom{k}{P}^2}$$

We can obtain a simpler expression by taking a sum over all n-dimensional multi-indices that sum to k. We take the sum over all $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = k$: $\boxed{\frac{1}{n^{2k}}\sum_{|\alpha|=k}\binom{k}{\alpha}^2}$