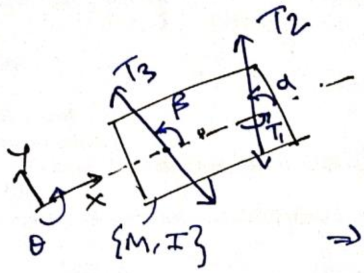
Homework – 1

Problem #1

Under Actuated Robots – HW 1

(1)



Eqns of motion:

$\Sigma F @ Y$

$$\Rightarrow T_2 \cdot \sin\alpha + T_3 \cdot \sin\beta = M\ddot{y}$$

$\Sigma F @ X$

$$\Rightarrow T_3 \cdot \cos\beta + T_2 \cdot \cos\alpha = M\ddot{x}$$

$$\Sigma M @ cm \Rightarrow T_1 + (T_2 \cdot \sin\alpha)(1) - T_3 \cdot \sin\beta\,(1) = I\ddot{\theta}$$

$$\Downarrow$$

$$\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{Bmatrix} = 0 + \begin{bmatrix} 0 & \cos\alpha & \cos\beta \\ 0 & \sin\alpha & \sin\beta \\ 1 & \sin\alpha & -\sin\beta \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}.$$

For fully actuated → this matrix has to be full-rank

Condition for linear independence →

$$\alpha_1 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \alpha_2 \begin{Bmatrix} \cos\alpha \\ \sin\alpha \\ \sin\alpha \end{Bmatrix} + \alpha_3 \begin{Bmatrix} \cos\beta \\ \sin\beta \\ -\sin\beta \end{Bmatrix} = 0$$

$$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$$

$$\Rightarrow \det\left(\begin{bmatrix} 0 & \cos\alpha & \cos\beta \\ 0 & \sin\alpha & \sin\beta \\ 1 & \sin\alpha & -\sin\beta \end{bmatrix}\right) \neq 0$$

$$\therefore \cos\alpha \cdot \sin\beta - \sin\alpha \cdot \cos\beta \neq 0$$

$$\Rightarrow \sin(\alpha - \beta) \neq 0$$

$$\boxed{\alpha - \beta \neq n\pi}, \quad n \in \mathbb{Z}$$

Condition for fully-actuated.

For (α = π/4, β = π/2), the condition for fully actuated is satisfied. Hence, it is a fully actuated system.

Problem #2

(a)

(2) $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

(a) for phase portrait, we need to find $\left(\frac{d\dot{\theta}}{d\theta}\right)$

$$\Rightarrow \frac{d\dot{\theta}}{dt} = \frac{u}{ml^2} - \frac{b\dot{\theta}}{ml^2} - \frac{g}{l}\sin\theta$$

$$\frac{d\dot{\theta}}{d\theta} = \left(\frac{u}{ml^2}\right)\frac{1}{\dot{\theta}} - \frac{b}{ml^2} - \left(\frac{g}{l}\right)\left(\frac{\sin\theta}{\dot{\theta}}\right)$$

for fixed pts we set $\dot{x}^* = \begin{Bmatrix}\dot{\theta}^* \\ \ddot{\theta}^*\end{Bmatrix} = 0$

$\therefore \quad mgl \cdot \sin\theta^* = u$

$$\boxed{\theta^* = (-1)^n \sin^{-1}\left(\frac{u}{mgl}\right) + n(\pi)} \rightarrow \text{fixed points}$$

$n \in \mathbb{Z}$

$\{b, u\} = \{0, 0\}$

$\Rightarrow \boxed{\theta^* = n\pi}$

$\{b, u\} = \{0.25, 0\}$

$\Rightarrow \boxed{\theta^* = n\pi}$

$$\{b, u\} = \left\{\frac{1}{4}, \frac{g}{2l}\right\}$$

$$\theta^* = (-1)^n \sin^{-1}\left(\frac{1}{2ml^2}\right) + n\pi$$

$$\boxed{\theta^* = (-1)^n \cdot \frac{\pi}{6} + n\pi}$$

For phase portrait, we plot ~~f(x), x, given c by~~ $\dot{x}$ over the space given by x

$$\dot{x} = f(x, u)$$

$$\begin{Bmatrix} \dot{\theta} \\ \ddot{\theta} \end{Bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dfrac{u - b\dot{\theta} - mgl \cdot \sin\theta}{ml^2} \end{bmatrix}$$

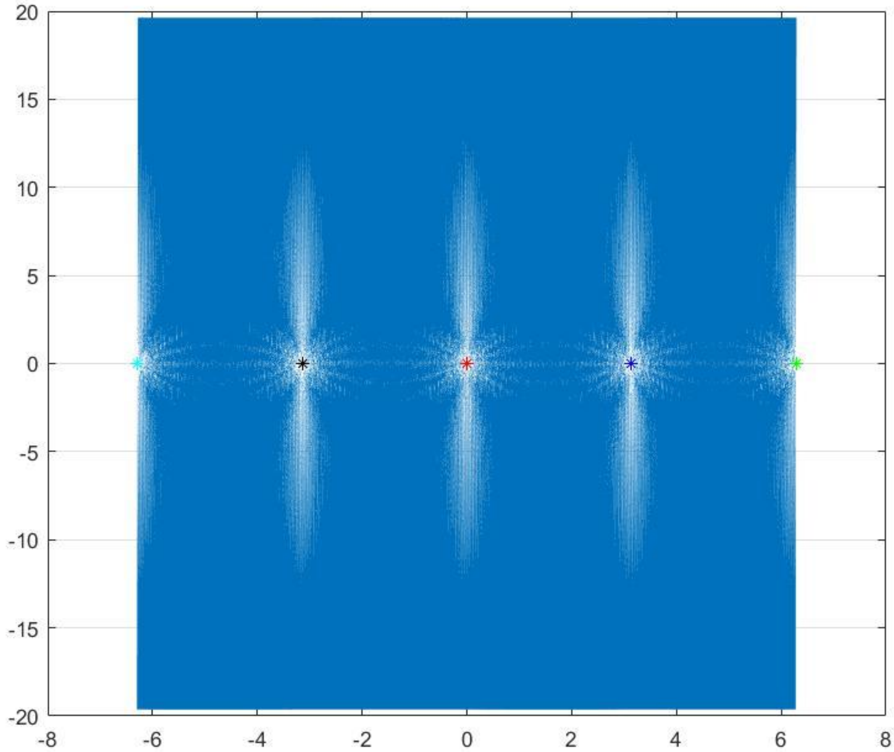


Fig 1: Phase plot over full range for (b, u) = (0, 0)

Basins of attraction:

1) (b, u) = (0, 0) -

In this case, the basin of attraction will be the stable equilibrium point itself. Any other point will not converge and keep circling around the equilibrium point for eternity since damping is absent.

2) (b, u) = (0.25, 0) –

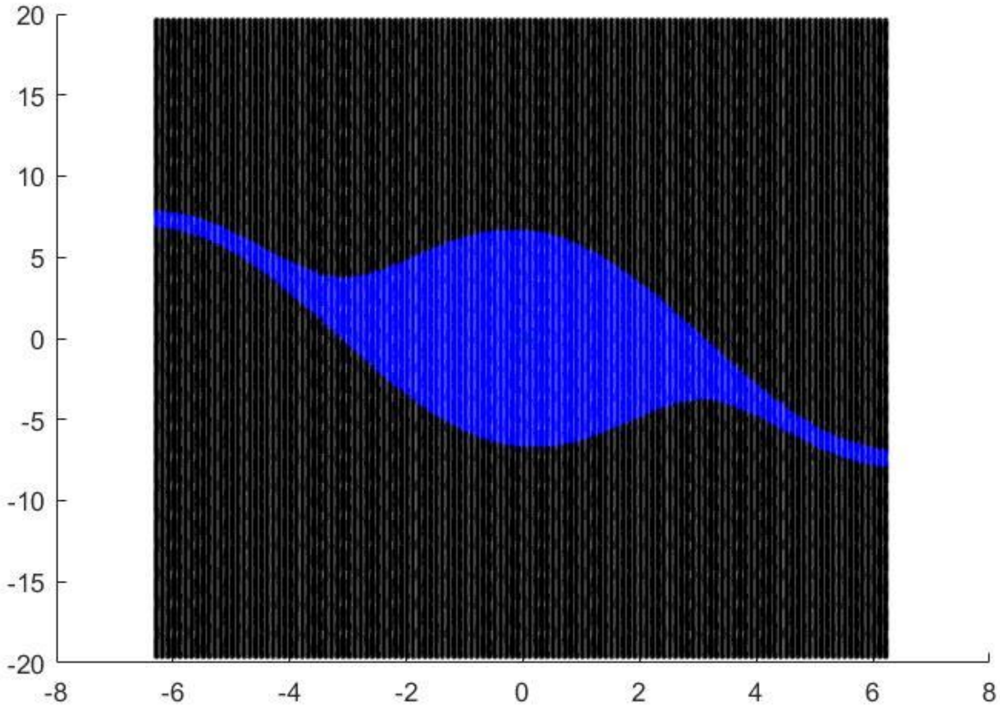


Fig 2: Basin of attraction for (0, 0) with (b, u) = (0.25, 0)

The blue portion indicates the initial conditions that converge to (0, 0) while the black portion indicates points that haven't converged yet. It is expected that all points in the black portion will converge to one of the stable equilibrium points, however due to computational restrictions, it is not shown here.

3) (b, u) = (0.25, 2*g/l) –

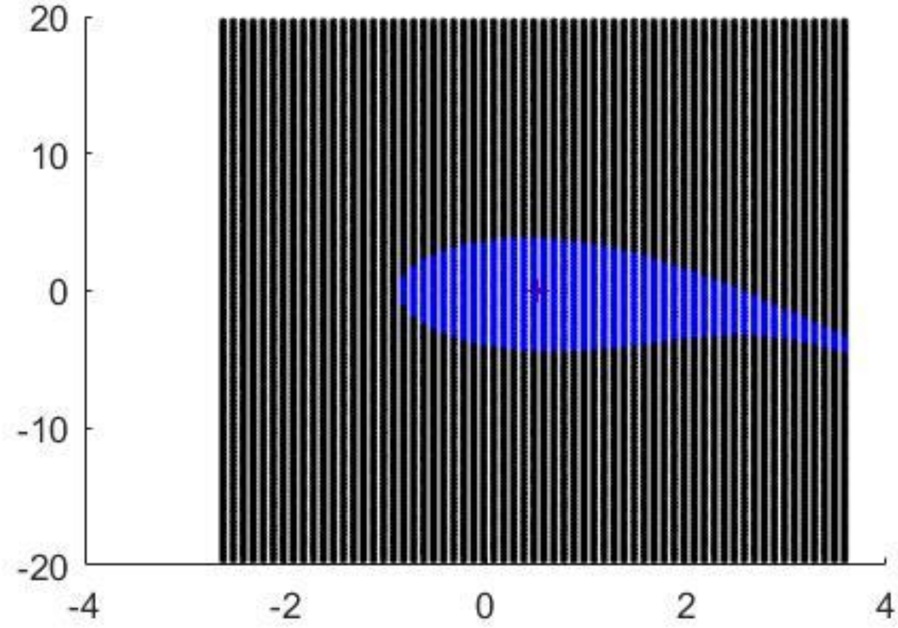


Fig 3: Basin of attraction for (pi/6, 0) with (b, u) = (0.25, 2*g/l)

(b)

(b) $ml^2\ddot{\theta} + b\dot{\theta} + mgl\cdot\sin\theta = u$

Set $u = 2mgl\cdot\sin\theta$

$\Rightarrow ml^2\ddot{\theta} + b\dot{\theta} - mgl\cdot\sin\theta = 0$

$$\begin{Bmatrix}\dot{\theta}\\ \ddot{\theta}\end{Bmatrix} = \begin{bmatrix}\dot{\theta}\\ \dfrac{mgl\cdot\sin\theta - b\dot{\theta}}{ml^2}\end{bmatrix}$$

From observing the phase plots, it is apparent that the nature of these system is the same as with parameters $\{b, u\} = \{0.25, 0\}$; except that there is a phase lag of $\{\theta = \pi\}$ for stable/unstable pts.

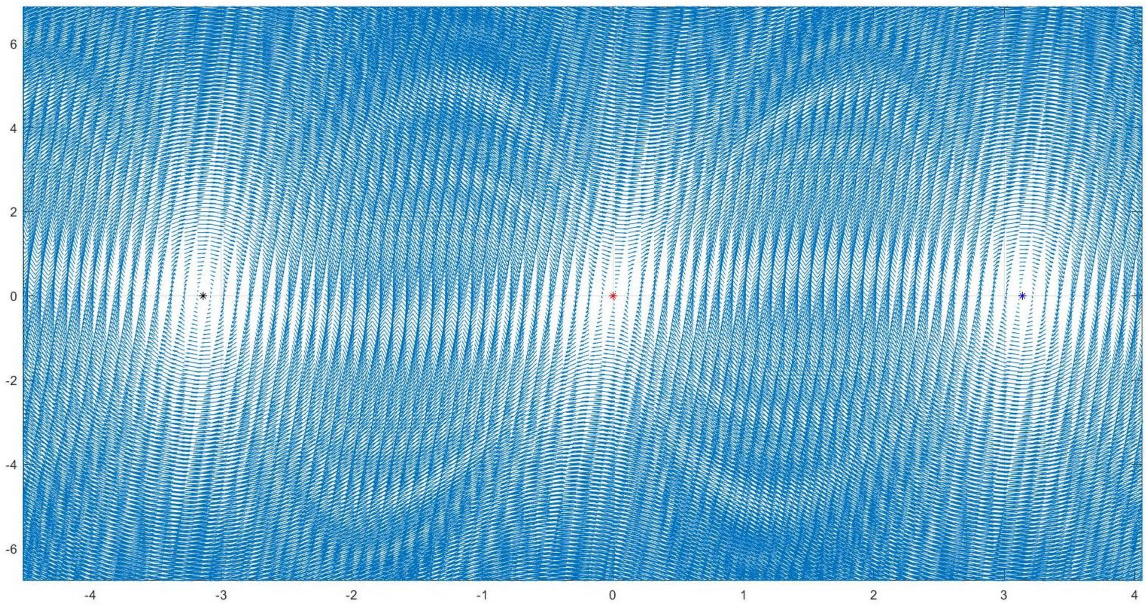


Fig 4: Phase plot for b = 0.25 with gravity inverted.

(c)

(c) $\ddot{\theta} + \frac{b\dot{\theta}}{ml^2} + \frac{g}{l}\cdot\sin\theta = \frac{u}{ml^2}$

Set $u = ml^2\left\{ u' + \frac{b}{ml^2}\dot{\theta} + \frac{g}{l}\sin\theta \right\}$

$\therefore \ddot{\theta} = u'$

Set $u' = -k_p(\theta - \pi) - k_d\dot{\theta}$

$\therefore \ddot{\theta} + k_d\theta + k_p(\theta - \pi) = 0$

$$\begin{Bmatrix} \dot{\theta} \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ -k_d\dot{\theta} - k_p(\theta - \pi) \end{Bmatrix}$$

~~Characteristic polynomial:~~

$\cancel{s^2 + k_d \cdot s + k_p}$

let $\phi = \theta - \pi$

$\Rightarrow \ddot{\phi} + k_d\dot{\phi} + k_p\phi = 0$

$$\boxed{s^2\cancel{\phi} + s.k_d + k_p = 0}$$ — Characteristic polynomial

$$\boxed{s_{1,2} = -k_d \pm \sqrt{k_d^2 - 4k_p}}$$

$\cancel{\text{If } k_d^2 < 4k_p}$

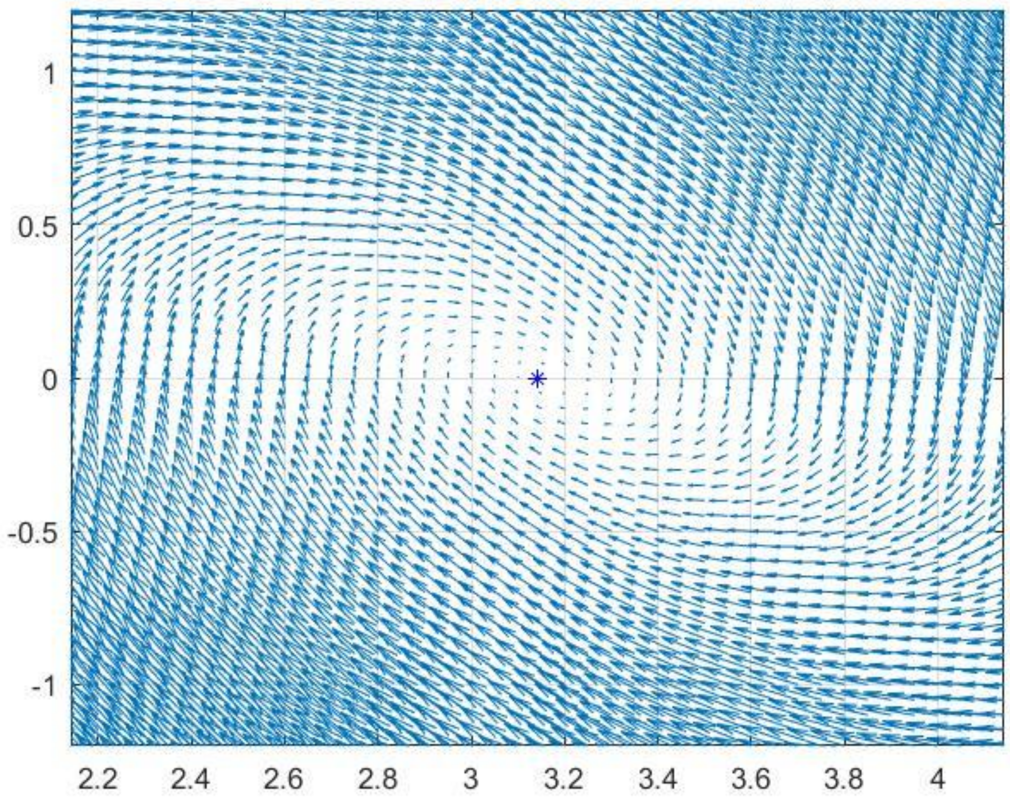


Fig 5: Phase plot for (kp, kd) = (1, 1)

This plot displays an underdamped oscillation about the stable equilibrium point, hence we can observe spiralling tendencies.

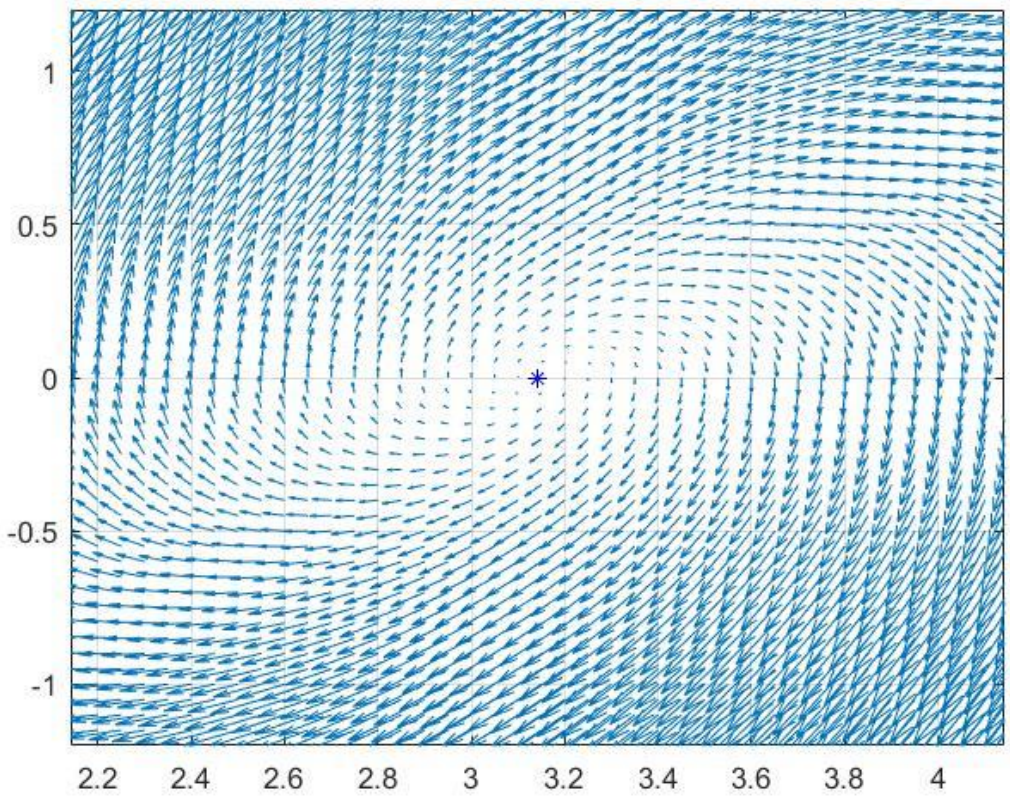


Fig 6: Phase plot for (kp, kd) = (1, -1)

This parameter set shows unstable behaviour about (pi, 0). Thus, our control policy should not allow negative kd values.

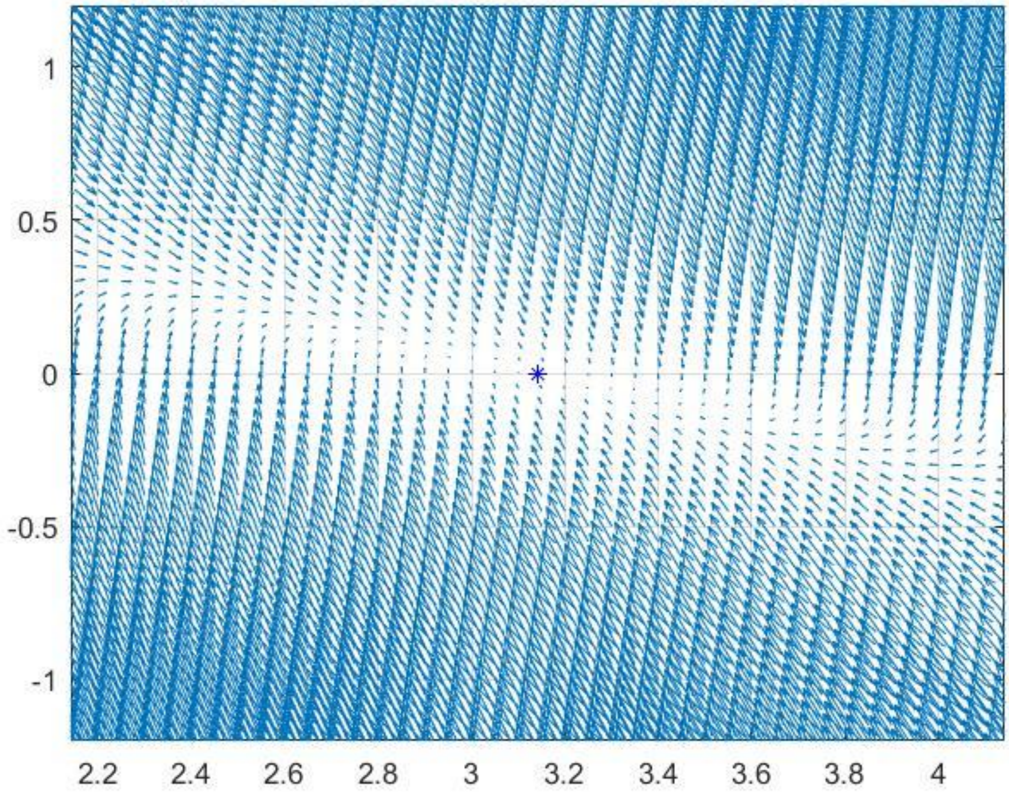


Fig 7: Phase plot for (kp, kd) = (1, 3)

This plot is of an overdamped case. Hence, we can observe that spiralling does not happen. Which means that oscillations about (pi, 0) does not occur.

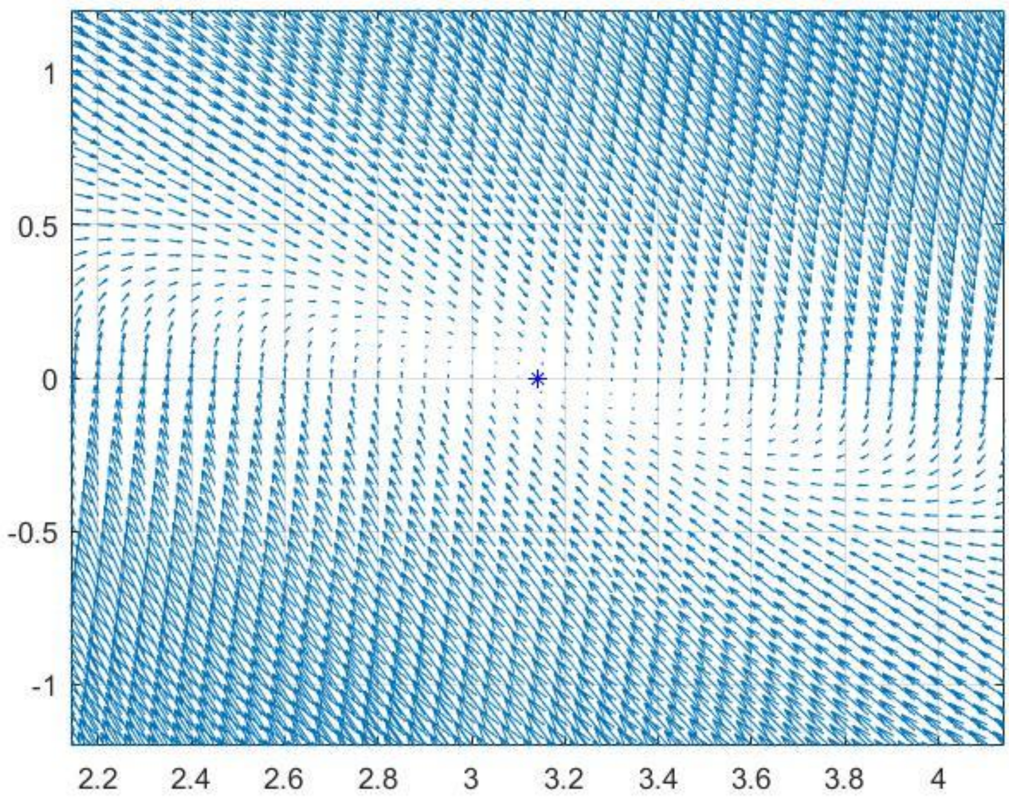


Fig 8: Phase plot for (kp, kd) = (1, 2)

This plot shows a critically damped case, we can observe a balance between the underdamped and overdamped behaviour.

Problem #3

(a)

(3) a) $\ddot{x} = u$ ~~//SX~~

~~U ≠ const.~~
~~work time~~

Objective: $\min T_{final}$, s.t $|u| \leq 1$

For minimum time policy,

let $|u| = 1$ (max allowable input)

we have $\ddot{x} = u$

$$\dot{x} = \dot{x}_0 + ut$$

$$x = x_0 + \dot{x}_0 t + \frac{1}{2} u t^2$$

$$t = \frac{\dot{x} - \dot{x}_0}{u} \Rightarrow x = x_0 + \frac{\dot{x}_0(\dot{x} - \dot{x}_0)}{u} + \frac{1}{2} u \frac{(\dot{x} - \dot{x}_0)^2}{u^2}$$

$$\Rightarrow x = c + \frac{\dot{x}^2}{2u}, \text{ where } c = x_0 - \frac{\dot{x}_0^2}{2u}$$

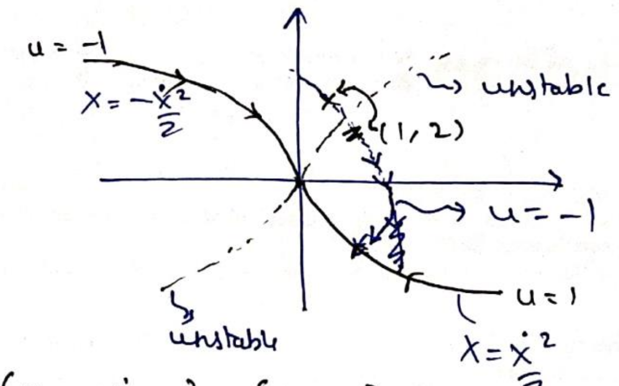
$\Rightarrow u = \pm 1$

$u = 1$	$u = -1$
$x = c + \frac{\dot{x}^2}{2}$	$x = c - \frac{\dot{x}^2}{2}$

To find the ~~line~~ curve of initial conditions that lead directly to origin for const. u, we set $(x(t), \dot{x}(t)) = (0, 0)$ for some 't'

If $u = 1$, $\Rightarrow c = 0 \Rightarrow x_0 = \frac{1}{2}\dot{x}_0^2$

$u = -1$, $\Rightarrow c = 0 \Rightarrow x_0 = -\frac{1}{2}\dot{x}_0^2$



for $\{x(0), \dot{x}(0)\} = \{1, 2\}$

$\underline{u = 1}$	$\underline{u = -1}$
$x = c + \frac{\dot{x}^2}{2}$	$x = c - \frac{\dot{x}^2}{2}$
$c = x_0 - \frac{\dot{x}_0^2}{2} = -1$	$c = x + \frac{\dot{x}_0^2}{2} = 3$
$\therefore x = -1 + \frac{\dot{x}^2}{2}$	$\therefore x = 3 - \frac{\dot{x}^2}{2}$

for min. time policy.

$u = -1$, until $\left(x = 3 - \frac{\dot{x}^2}{2}\right)$ intersects $\left(x = \frac{\dot{x}^2}{2}\right)$

$\Rightarrow \left(x = \frac{3}{2},\ \dot{x} = -\sqrt{3}\right)$

then $u = 1$ till $(0, 0)$

$$\therefore u = \begin{cases} -1, & \text{from } (1,2) \text{ to } (\tfrac{3}{2}, -\sqrt{3}) \\ 1, & \text{from } (\tfrac{3}{2}, -\sqrt{3}) \text{ to } (0,0) \\ -1, & \text{beyond } (0,0) \end{cases}$$

(b)

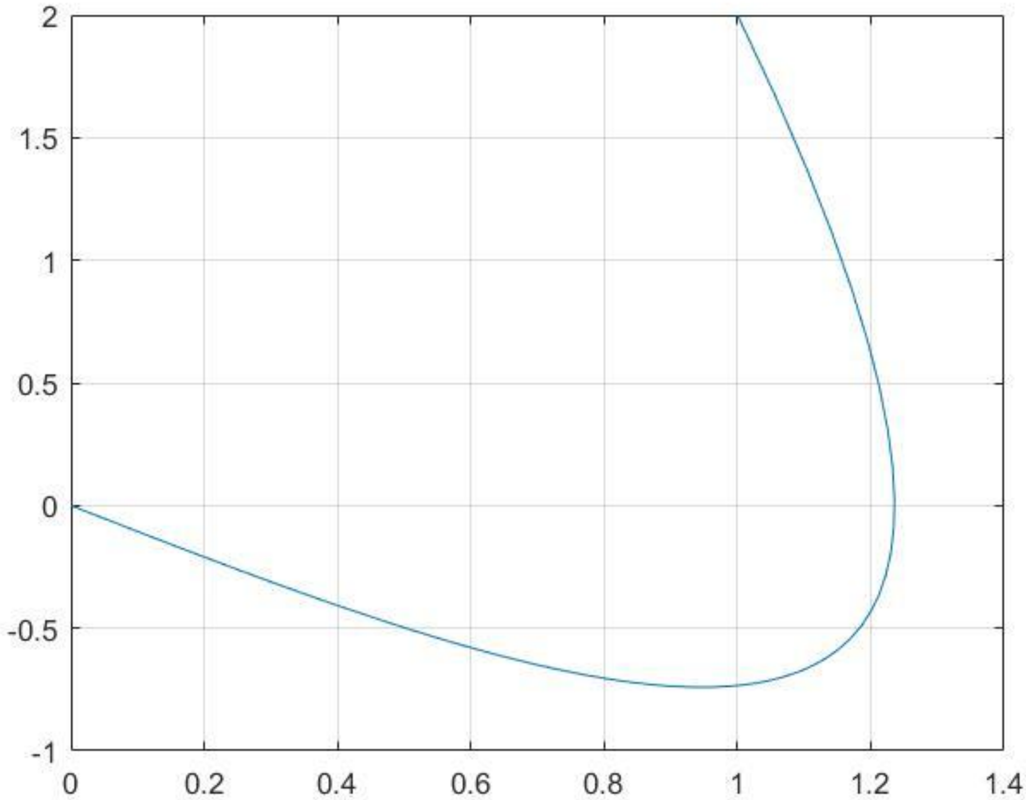


Fig 9: Phase trajectory with initial condition (1,2)

(c)

3(c) for the min. time policy :

$$t_{tot} = t_{(u=-1)} + t_{(u=1)}$$

$$t = \frac{\dot{x} - \dot{x}_0}{u}$$

$$t(u=-1) = \frac{-\sqrt{3}-2}{-1} = 3.732$$ ~~secs~~

$$t(u=1) = \frac{0+\sqrt{3}}{1} = 1.732$$

$$\therefore \; t_{tot} = 5.464$$

From LQR for given parameters,

$$t_{tot} = 14.621 \text{ s}$$

Q	R	Settling time
0.25	5	14.62
0.5	5	11.40
1	5	7.39

5	5	5.32
20	5	4.43
100	5	3.91
1000	5	3.54
0.25	4	13.51
0.25	1	7.08
0.25	0.25	5.32
0.25	0.1	4.31

For (Q, R) = (100, 10) we get settling time as 4.08 as compared to 5.46 in the "time-optimal" policy. This makes sense because the time-optimal policy has an additional actuator constraint that we are not including in the LQR policy. For a proper comparison, actuator constraint will have to be applied equally in both.