Homework – 2

Problem #1

HW 2

(1) (a) $ml^2 \ddot{\theta} + b\dot{\theta} + mgl \cdot \sin\theta = u$

set $m = l = g = 1$

$\Downarrow$

$\ddot{\theta} + b\dot{\theta} + \sin\theta = u$

linearising @ $\theta = 0$,

$$\boxed{\ddot{\theta} + b\dot{\theta} + \theta = u}$$

Take $b = 1$, $u = 0$

$\Downarrow$

$\ddot{\theta} + \dot{\theta} + \theta = 0$

$$s_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore$ underdamped, Stable

(b) let $\hat{\theta} = \theta - \pi$

$\Downarrow$

$\ddot{\hat{\theta}} + \dot{\hat{\theta}} + \sin(\hat{\theta} + \pi) = 0$

$\ddot{\hat{\theta}} + \dot{\hat{\theta}} - \sin\hat{\theta} = 0$

let $\sin\hat{\theta} \approx \hat{\theta}$

$$\Rightarrow \ \ddot{\hat{\theta}} + \dot{\hat{\theta}} - \hat{\theta} = 0$$

(c) $$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{bmatrix}$$

$$\therefore \ \ddot{\hat{\theta}} + \dot{\hat{\theta}} - \hat{\theta} = -k_1 \hat{\theta} - k_2 \dot{\hat{\theta}}$$

$$\ddot{\hat{\theta}} + \dot{\hat{\theta}}(1 + k_2) + \hat{\theta}(k_1 - 1) = 0$$

(d) $$l(x, u) = x^T Q x + u^T R u$$

$$\phi(x_{(T)}) = x_{(T)}^T Q_f \, x_{(T)} \quad , \quad Q_f = I$$

optimal policy: $$u^* = \underbrace{-\frac{1}{2} R^{-1} B^T S_{(t)}}_{K} \, X$$

we can get $S_{(t)}$ by solving Ricatti eq^n.

$$\begin{cases} -\dot{S}_{(t)} = Q - S_{(t)} B R^{-1} B^T S_{(t)} + S_{(t)} A + A^T S_{(t)} \\ S_{(T)} = Q_f = I \end{cases}$$

For solving backwards in time, ODE45 from T_f to 0.

a) Stable equilibrium point with zero-control

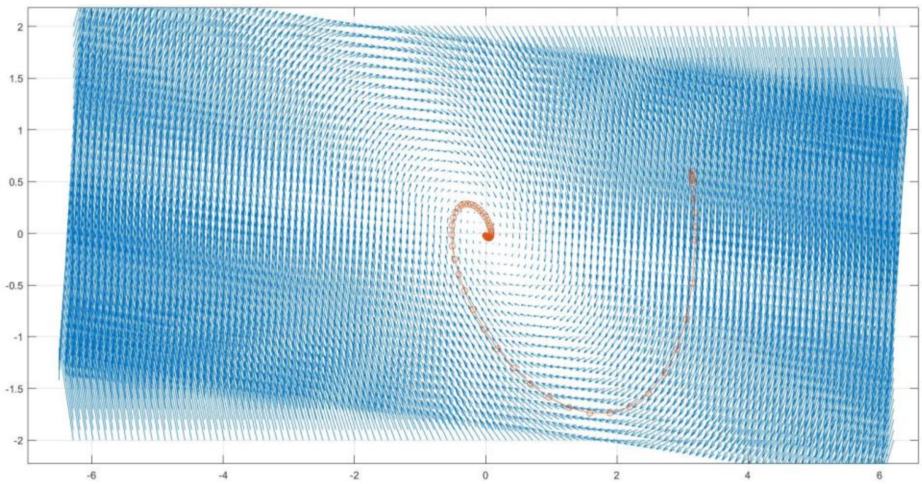

Fig 1: Trajectory overlaid on the phase plot. Initial condition = (pi, 0.6)

- From the characteristic polynomial, we can infer that the system is underdamped.

b) Unstable equilibrium point with zero-control

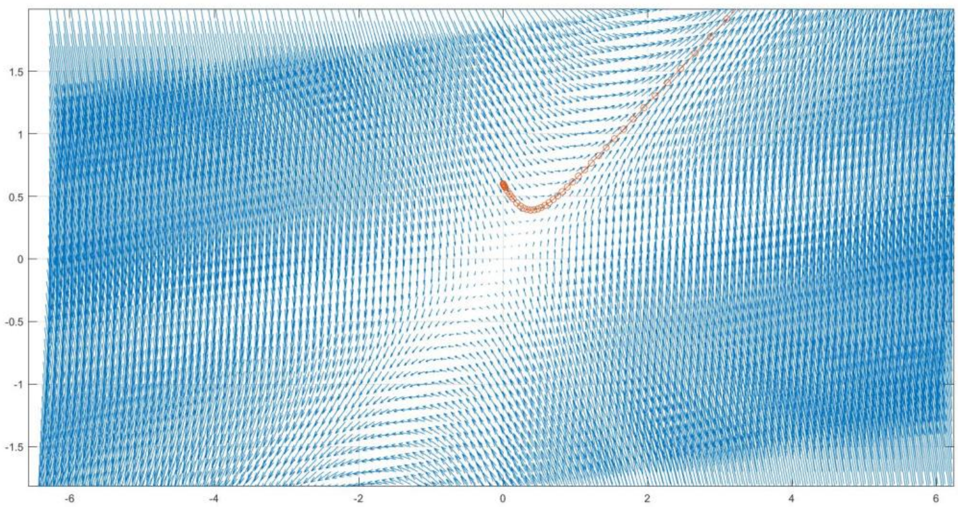

Fig 2: Trajectory overlaid on the phase plot. Initial condition = (0, 0.6)

- From the trajectory and phase plot, we can observe that the system is unstable at (0, 0.6).

c) Unstable equilibrium point with PD control

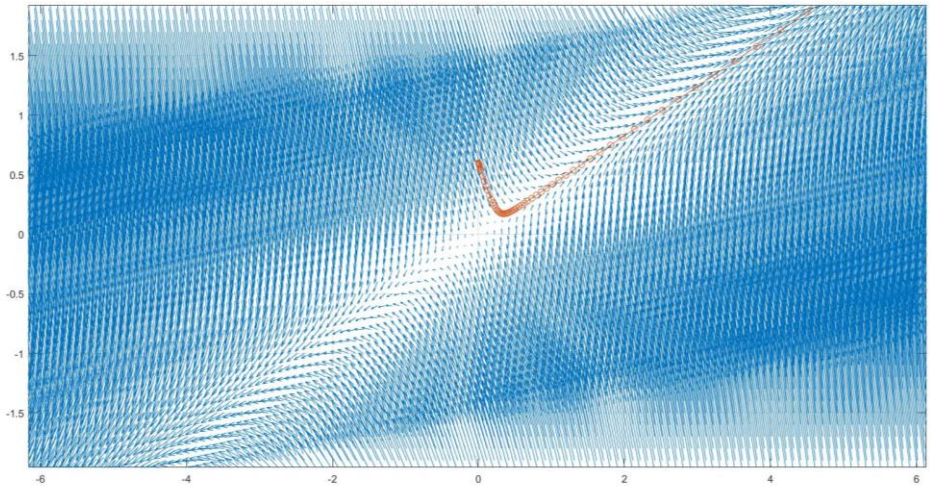

Fig 3: Trajectory overlaid on the phase plot. (k1, k2) = (0, 1)

- Equilibrium point (0, 0) is unstable.

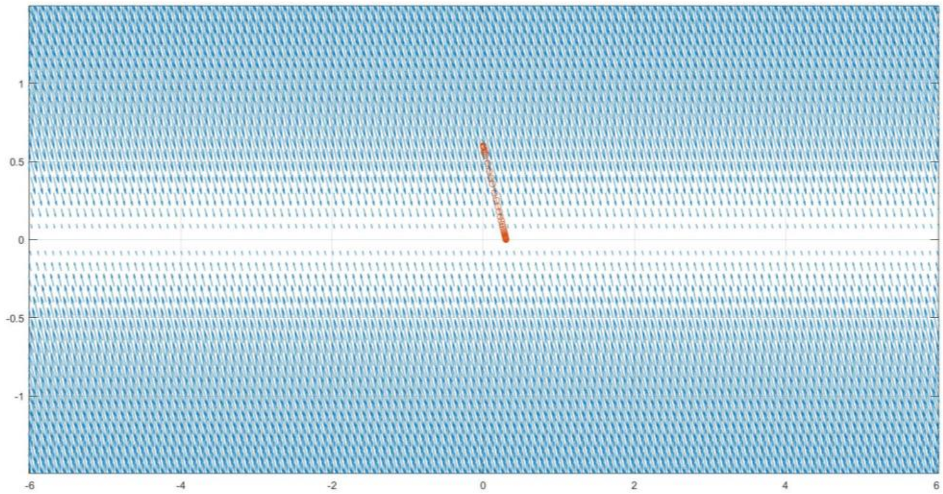

Fig 4: Trajectory overlaid on the phase plot. (k1, k2) = (1, 1)

- Equilibrium point is unstable in general. This becomes like a sliding block with damping. It can only converge to equilibrium if the initial conditions lie on a particular line in the phase plot that meets on the origin.

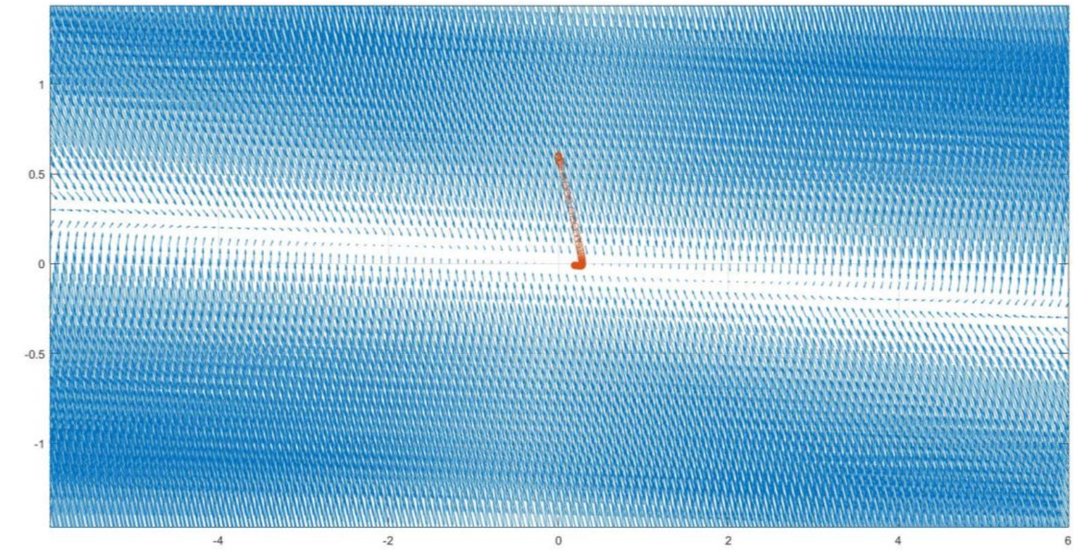

Fig 5: Trajectory overlaid on the phase plot. (k1, k2) = (1.1, 1)

- This becomes similar to an overdamped case. Damping force is high compared to restoring force. Origin is a stable equilibrium point.

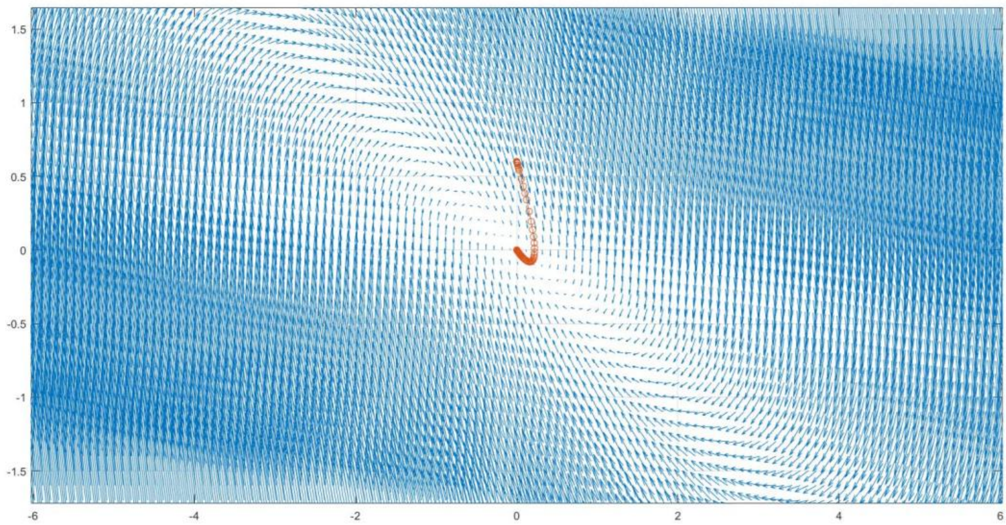

Fig 6: Trajectory overlaid on the phase plot. (k1, k2) = (2, 1)

- Equilibrium point is stable and system is critically damped.

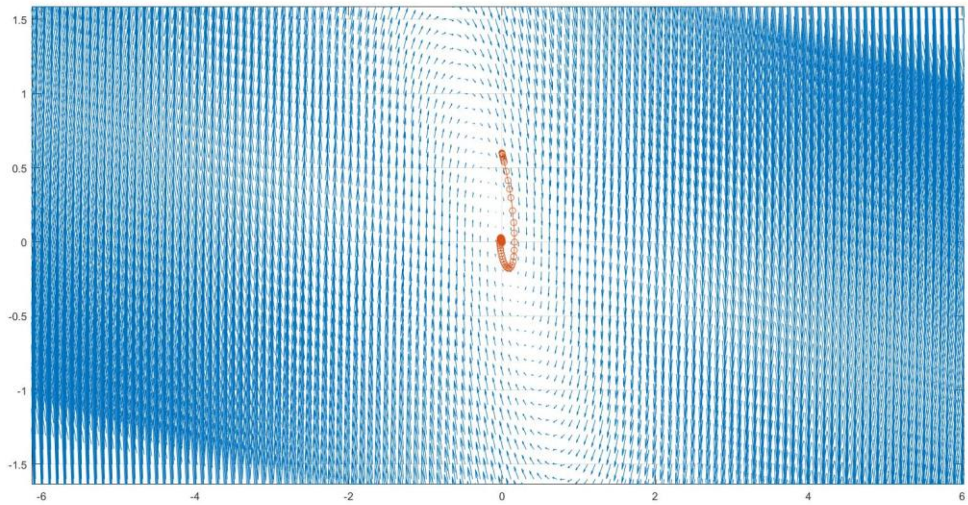

Fig 7: Trajectory overlaid on the phase plot. (k1, k2) = (5, 1)

- Equilibrium point is stable and system is underdamped.

d) Unstable equilibrium point with optimal control

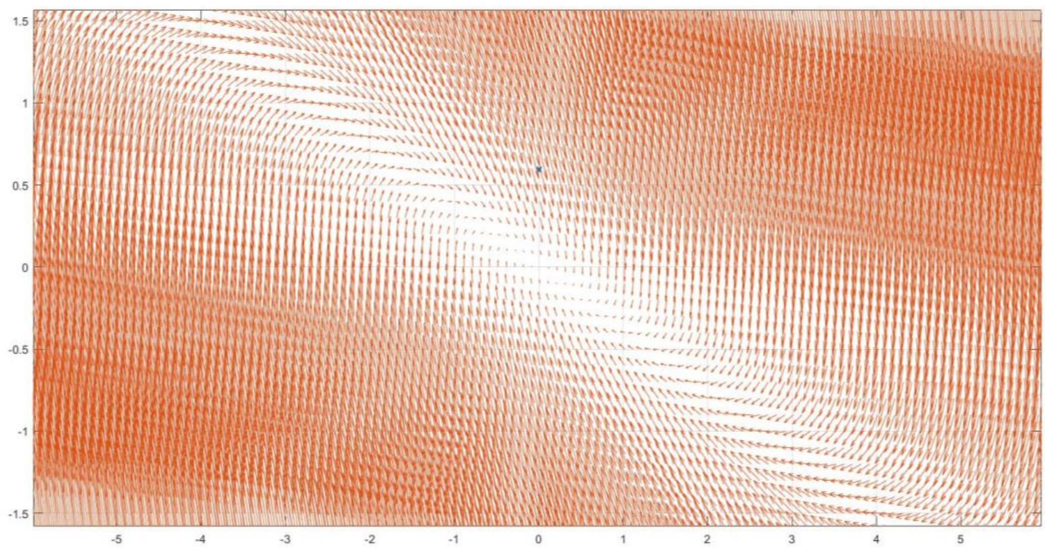

Fig 8: Phase plot for t = 0s. Initial condition (0, 0.6) shown by blue cross-mark.

- The equilibrium point is stable. Appears to be underdamped according to the phase plot.
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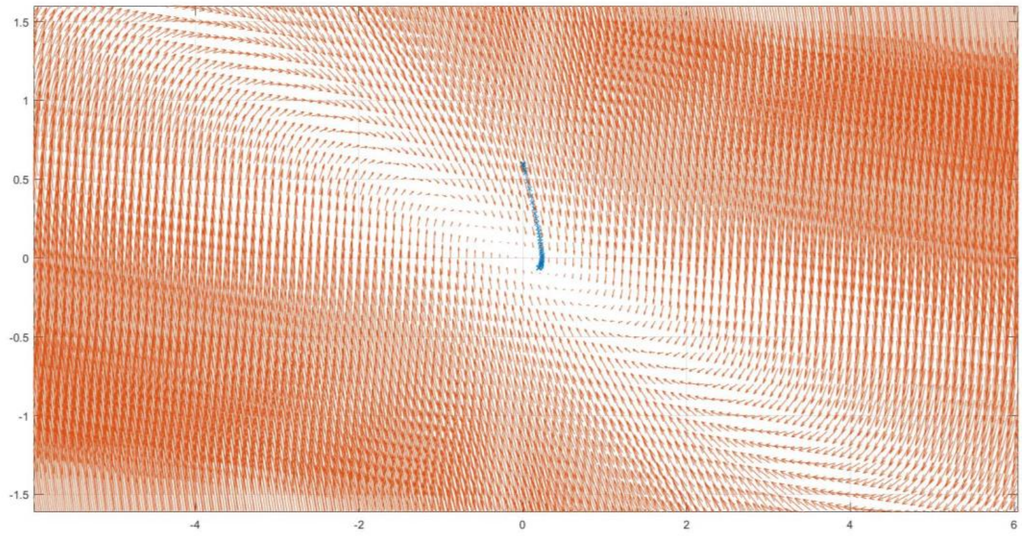


Fig 9: Phase plot for t = 2 s. Trajectory shown in blue

- The system is stable and appears to be underdamped.

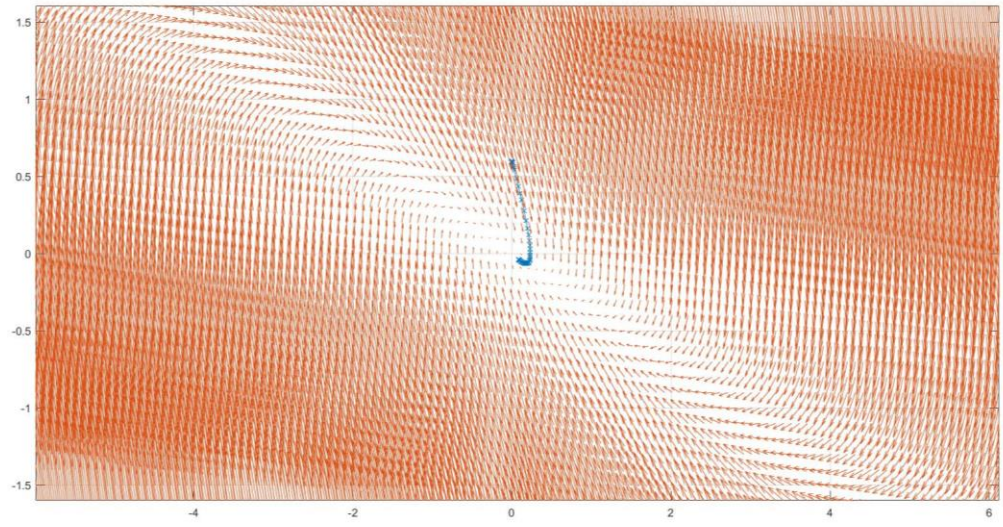


Fig 10: Phase plot for t = 4 s. Trajectory shown in blue

- The system is stable and appears to be under/critically damped.

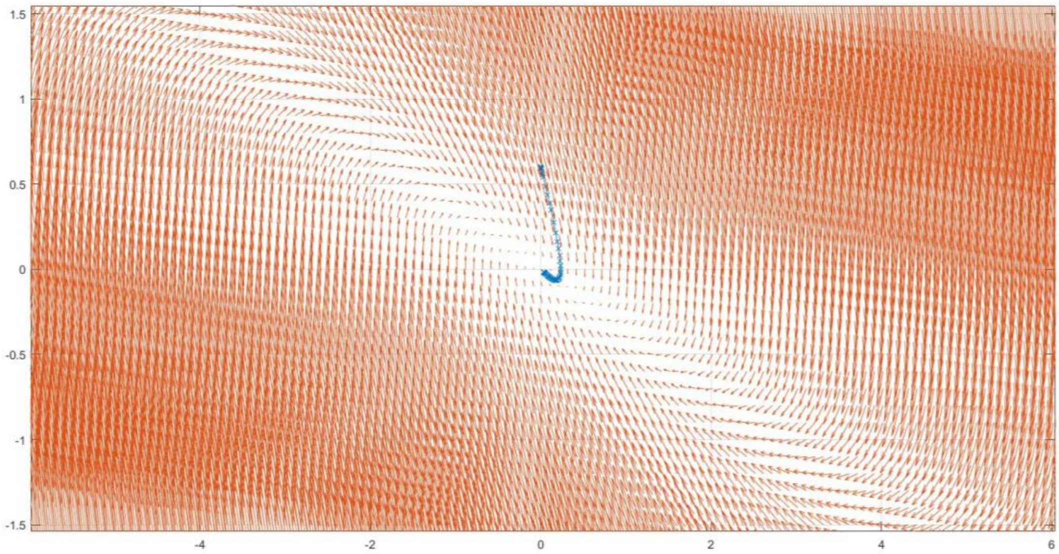


Fig 11: Phase plot for t = 6 s. Trajectory shown in blue

- The system is stable and appears to be under/critically damped.

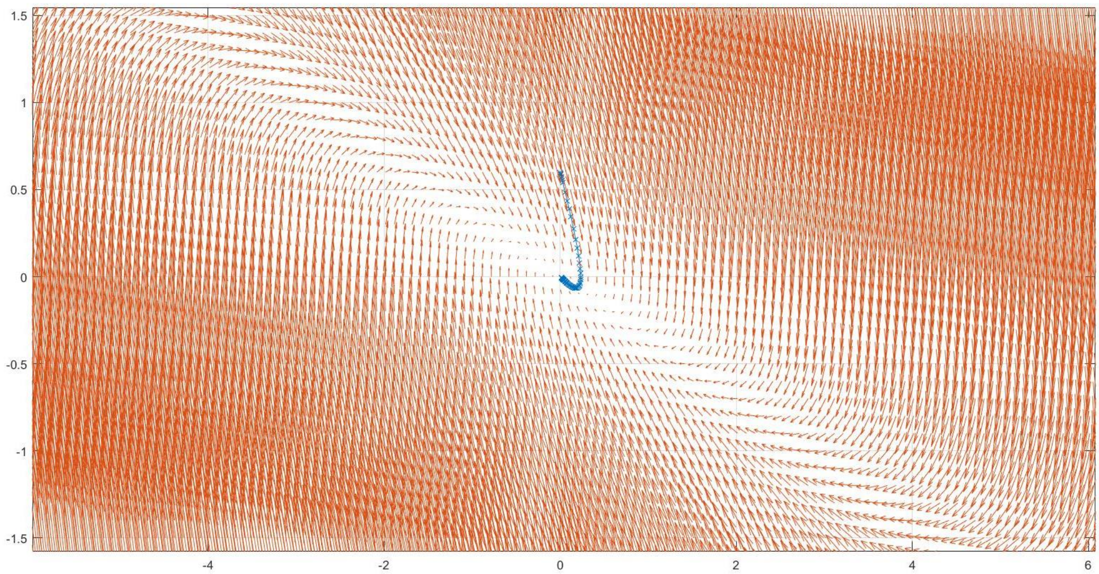


Fig 12: Phase plot for t = 8 s. Trajectory shown in blue

- The system is stable and appears to be underdamped.

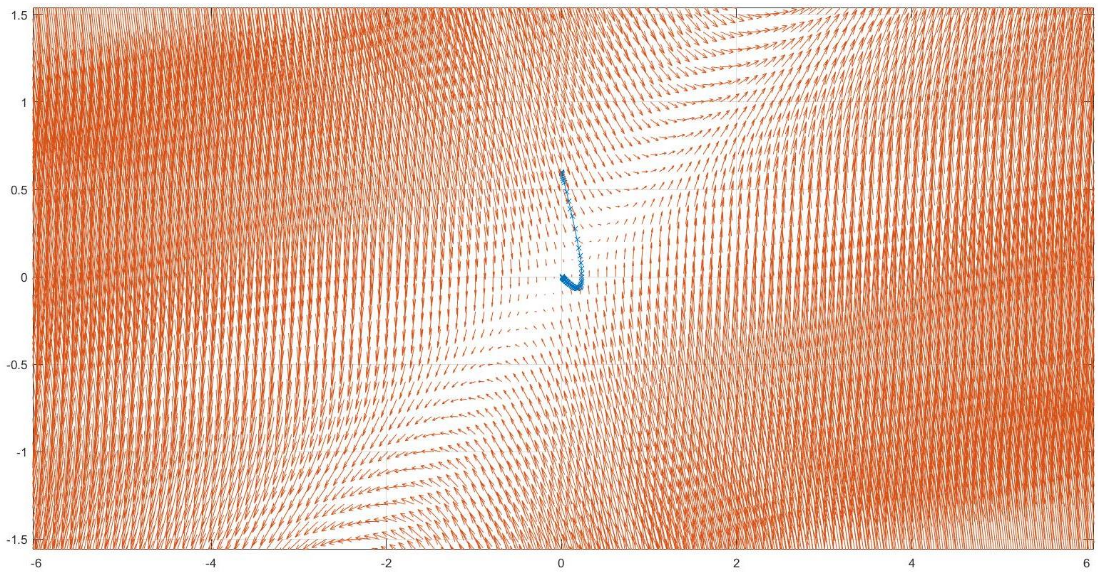


Fig 13: Phase plot for t = 10 s. Trajectory shown in blue

- The phase plot shows the linearised system to be unstable.

Problem #2

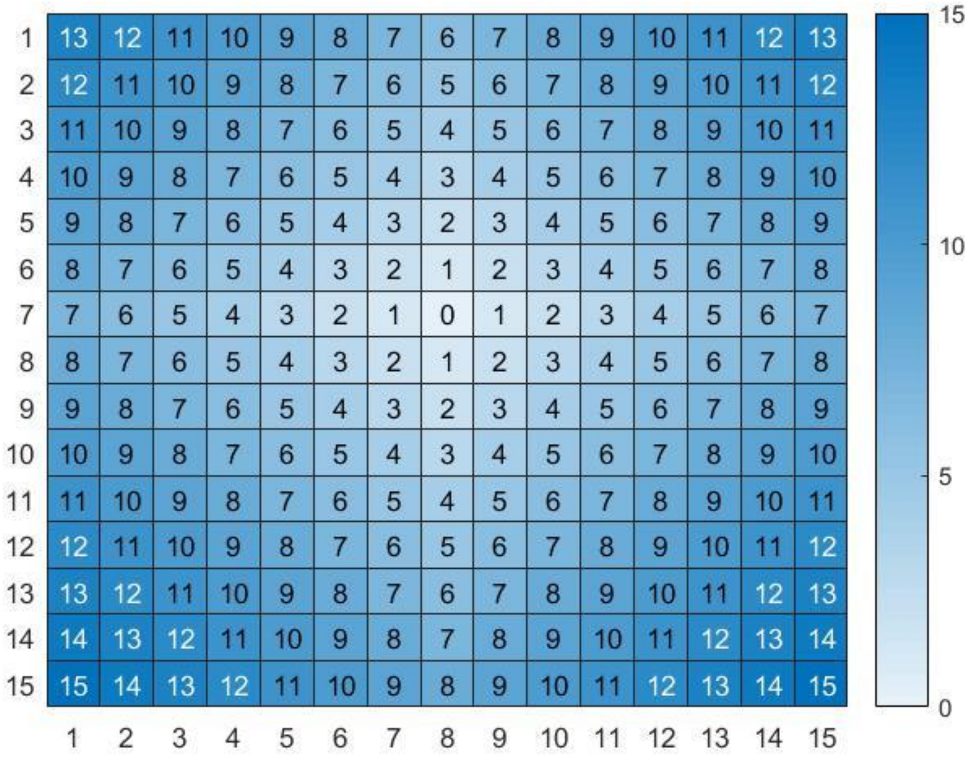


Fig 14: Value function grid plot after convergence

Problem #3

(Q3) $\ddot{\theta} + \dot{\theta} + \sin\theta = u.$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\theta} \\ -\dot{\theta} - \sin\theta + u \end{bmatrix}}_{f(x,u)}$$

$$\dot{x} = f(x, u)$$

Euler integration, $x_{n+1} = x_n + \Delta t \cdot f(x_n, u_n)$

$$u \in [-2 : 0.5 : 0.2]$$

For value iteration,

we know $Q_f = 0$

$\therefore \quad J(s, N) = (x - x_d)^T Q_f (x - x_d) = 0, \ \forall s$

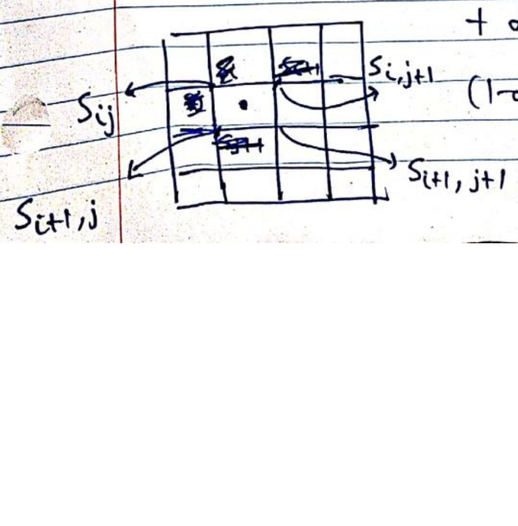
$$J^*_{(s,N-1)} = \min_u \Big[(x-x_d)^T Q (x-x_d) + Ru^2 + \underbrace{J^*(s', N)}_{0} \Big]$$

Thus, we can find $J^*_{(N-1)}$ at every grid point.

$$J^*_{(s,N-2)} = \min_u \Big[(x-x_d)^T Q (x-x_d) + Ru^2 + J^*(x_{n+1}, n+1) \Big]$$

where, $x_{n+1} = x_n + \Delta t \cdot f(x_n, u_n)$

$$J^*(x_{n+1}) = \alpha_1 \cdot \alpha_2 \cdot J(s_{ij})_{n+1} + \alpha_1(1-\alpha_2) J(s_{i+1,j})_{n+1} + \alpha_2(1-\alpha_1) J(s_{i,j+1})_{(n+1)} + (1-\alpha_1)(1-\alpha_2) J(s_{i+1}, s_{j+1})_{n+1}$$



$$\alpha_1 = 1 - \frac{(x - x_{ij})}{\Delta x}$$

$$\alpha_2 = 1 - \frac{(y - y_{ij})}{\Delta y}$$

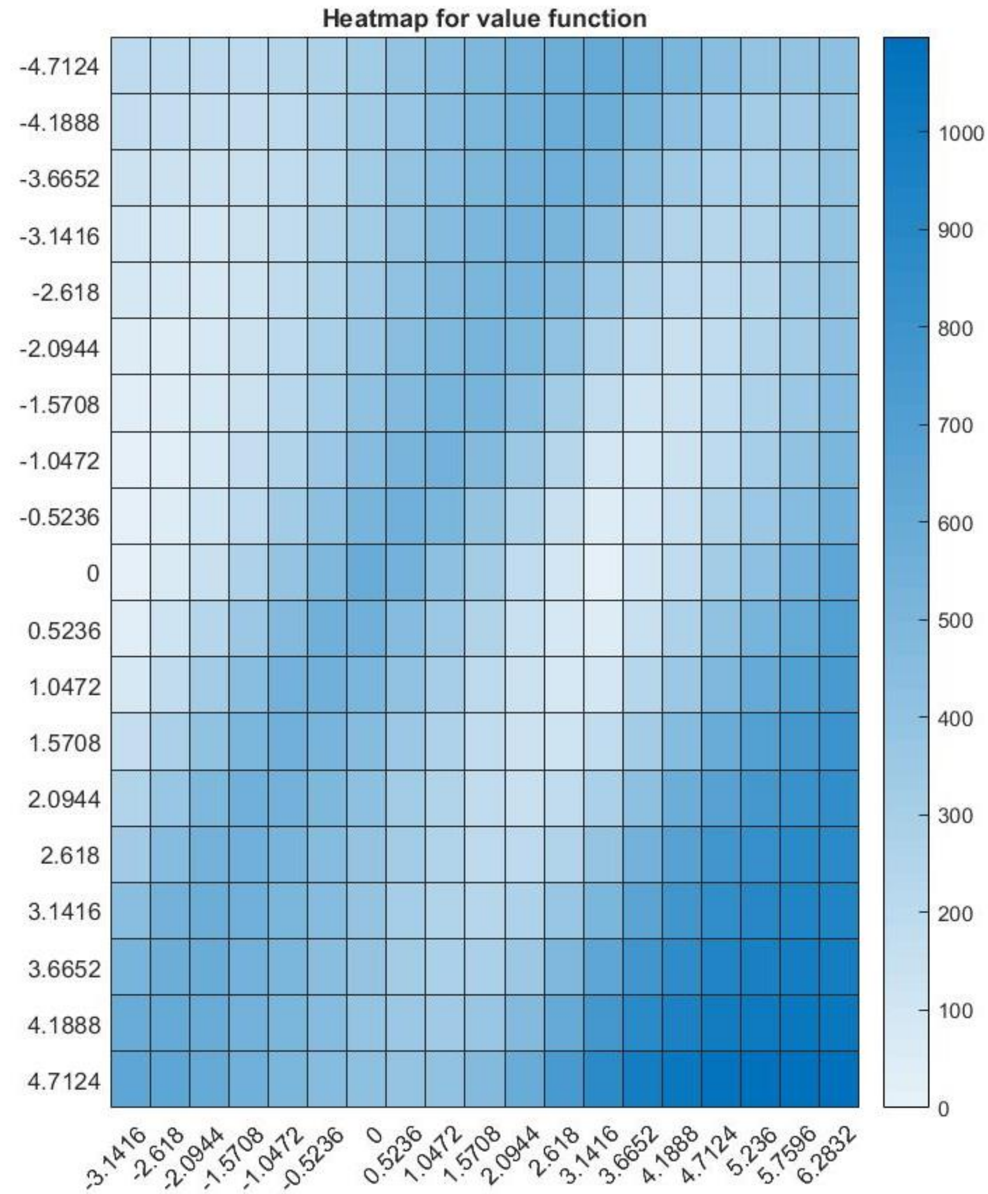


Fig 14: Value function heatmap (converged plot)

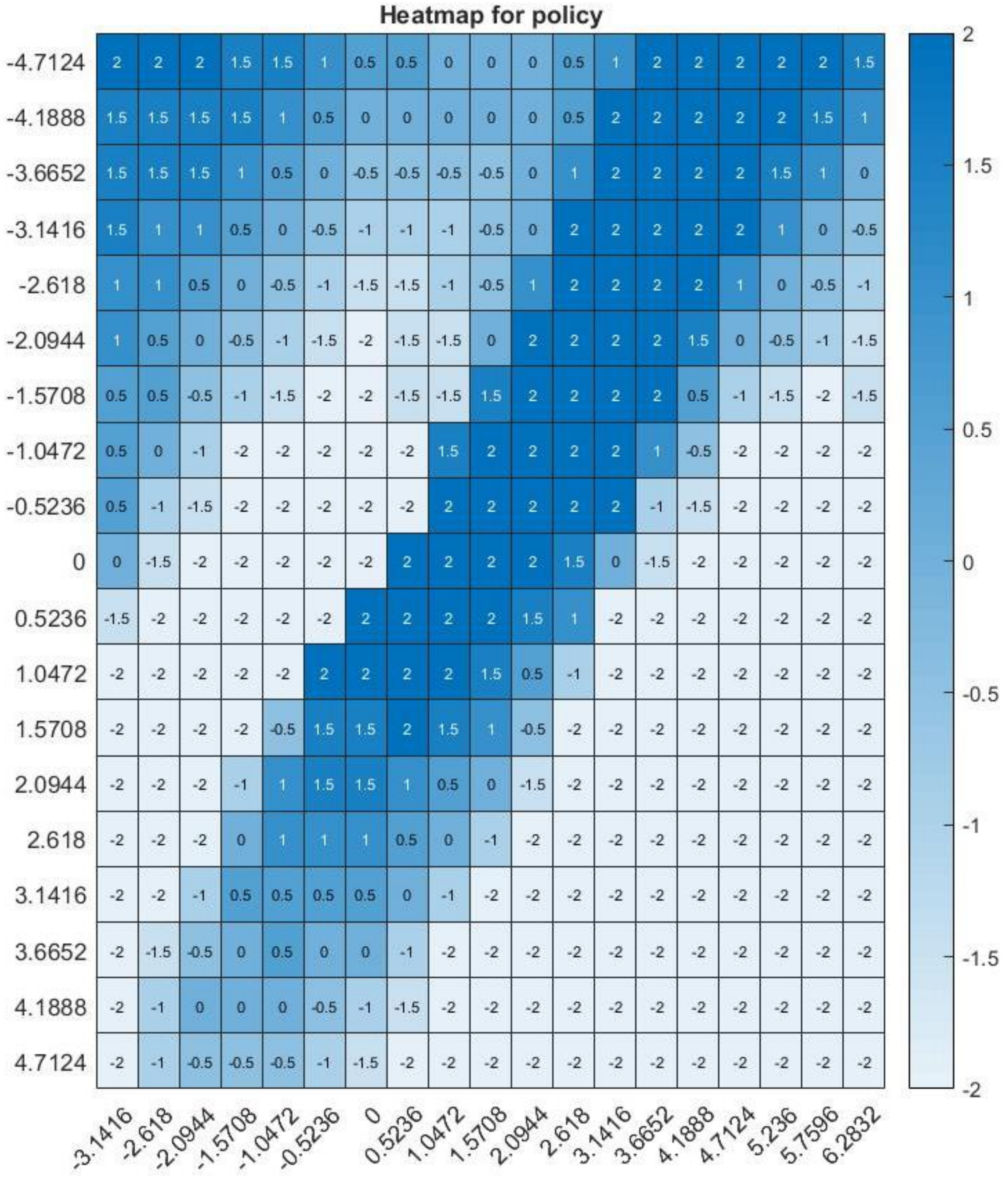


Fig 15: Policy heatmap (converged plot)

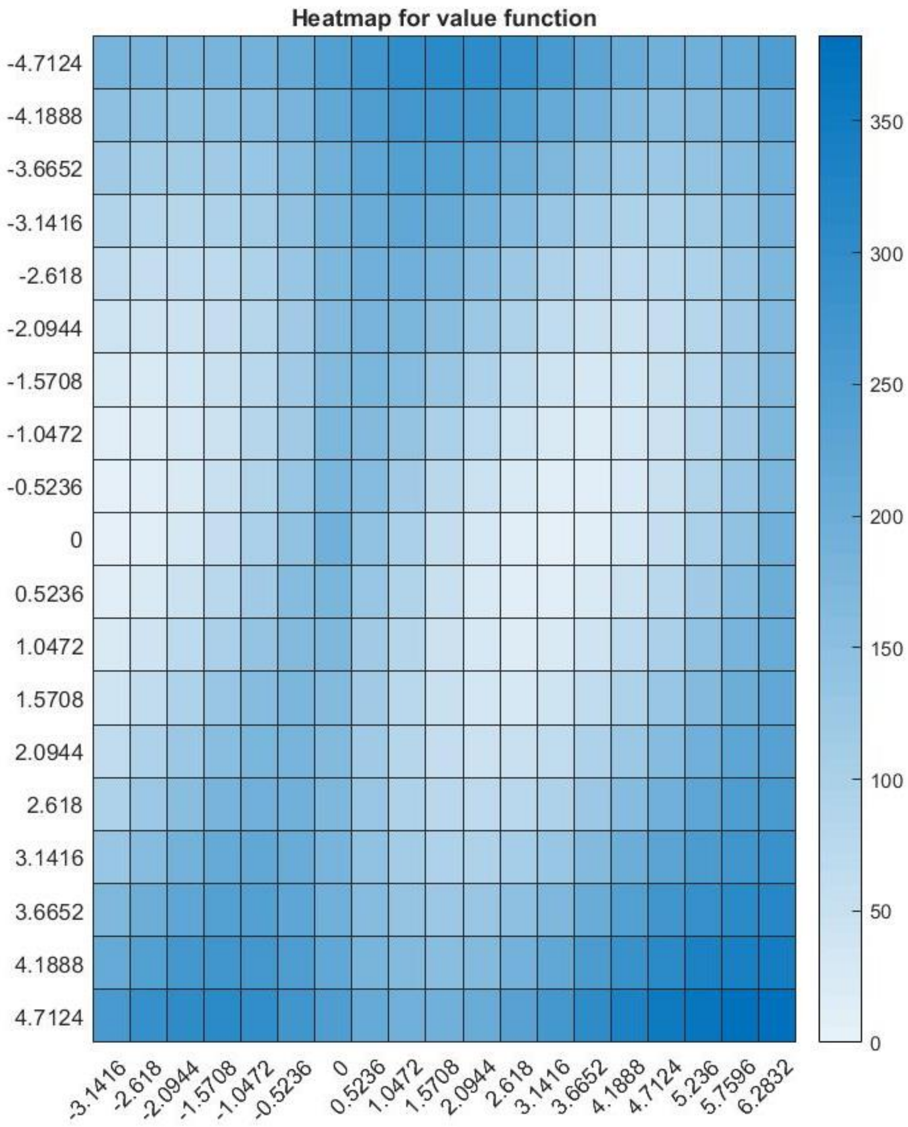


Fig 16: Policy heatmap (20 iterations)

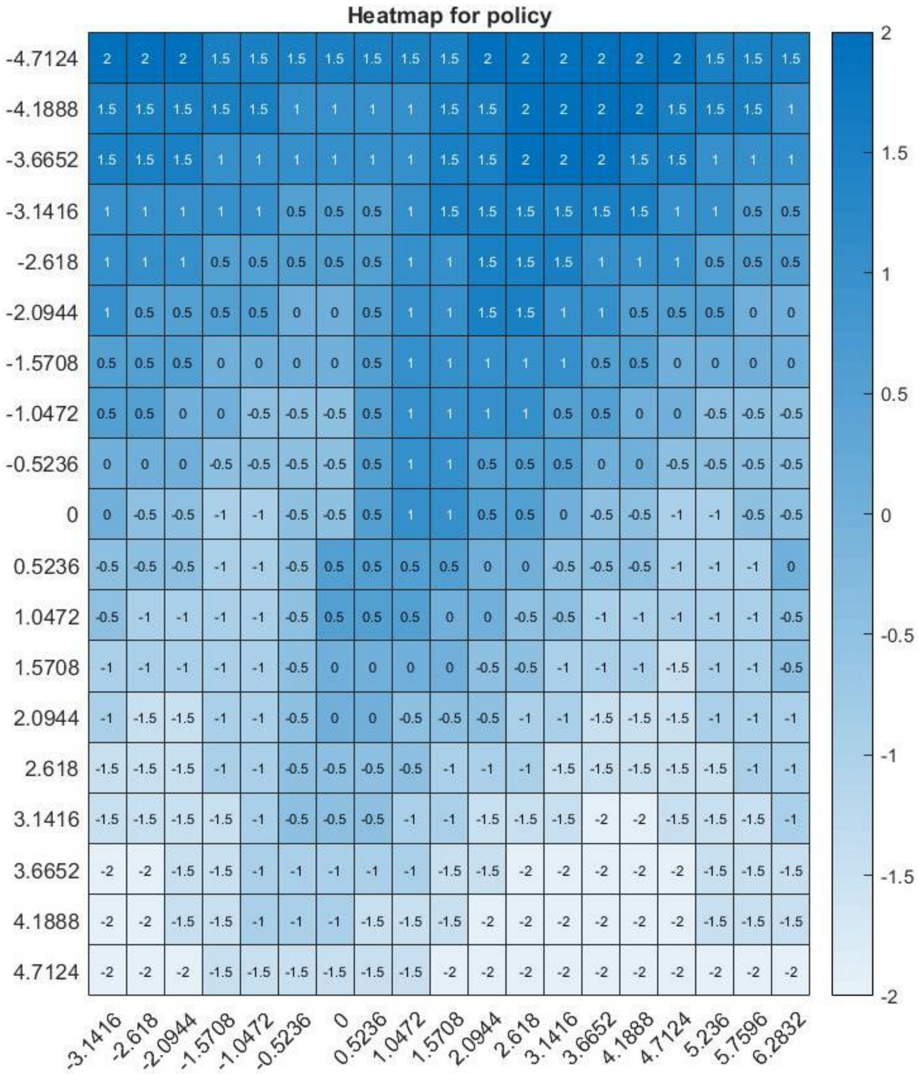


Fig 17: Value function heatmap (20 iterations)