What is Logic?

- Logic is the basis of all mathematical reasoning.
- The rules of logic specify the meaning of mathematical statements.
- These rules help us understand and reason with statements such as –
- $\exists x$ such that $x \neq a^2+b^2$, where $x,a,b \in \mathbb{Z}$

What is a proposition?

- A proposition is the basic building block of logic.
- It is defined as a declarative sentence that is either True or False, but not both.
- The **Truth Value** of a proposition is True(denoted as T) if it is a true statement, and False(denoted as F) if it is a false statement.
- For Example,

1. The sun rises in the East and sets in the West.
2. 1+1=2.
3. ‘a' is a consonant.

1. What time is it?
2. Go out and play.
3. $x + 1 = 2$.

Simple proposition

- Examples :
- It is raining.
- My car is painted silver.
- Snow is white.
- People live on the moon.

Compound proposition

- Formed from atomic formulas using the logical connectives not, and, or, if…then,if and only if.
- Examples:
- It is raining **and** the wind is blowing.
- If you study hard **then** you will be rewarded.
- The sum of 10 **and** 20 is 50.
- The moon is made of green cheese **or** it is not.

Representation of propositions

- propositional variables
- these variables are represented by alphabets such as p,q,r,s.
- The area of logic which deals with propositions is called propositional calculus or propositional logic

Logical connectives

- The propositions are combined together using **Logical Connectives** or **Logical Operators**.
- ▲ : for **not** or **negation**.
- ◀ : for **and** or **conjunction.**
- ☰: for **if … then** or **implication.**
- ▶ : for **or** or **disjunction.**
- ⬇: for **if and only if** or **double implication.**

Syntax

- Sentences in PL are determined according to the rules of propositional syntax.
- This syntax governs the combination of basic building blocks such as propositions and logical connectives
- The syntax of PL is defined recursively as follows:
- If p and q are formulas, the following are formulas:
- (▲p)
- (p ◀q)
- (p ▶ q)
- (p ☰q)
- (p ⬇q)

- An example of compound formula is:
- ((p◀(▲q▶r) [illegible]q[illegible]))
- When no chance for ambiguity , omit parentheses for brevity
- ex: (▲(p ◀(▲ q))) can be written as ▲(P ◀ ▲ Q)

Precedence

- Precedence given to the connectives from highest to lowest is:

1. ▲
2. ◀
3. ▶
4. ☰
5. ⬇

- Ex:
- p◀▲q▶r☰s⬇uvw
- is written as:
- ((((p◀▲(q)) ▶r) ☰s⬇(uvw))

Semantics

- Semantics or meaning of a sentence is just the value true or false; that is ,
- it is an assignment of truth value to the sentence.

Semantics rules for statements

Rule no.	True statements	False statements
1	▲**f**	▲**t**
2	t◀**t’**	f◀**a**
3	t▶**a**	a◀**f**
4	a▶**t**	f▶**f’**
5	a[illegible]	t[illegible]
6	f[illegible]**a**	t⬇**f**
7	t[illegible]**’**	f⬇**t**
8	f⬇**f’**	

- Example:
- The meaning of any statement given an interpretation I for the statement, let I assign true to p, false to q and false to r in the statement
- ((p ◀ ▲q) ▤) ▶q

Properties of semantics

- **Contradiction** : if there are no interpretation for which it is true.
- Ex: p◂~p is contradiction.
- **Tautology** : if it is true for every interpretation .
- Ex: p▸~p is tautology .
- **Equivalence** : two sentences are equivalent if same truth value under every interpretation.
- Ex: p and~(~p) are equivalent .
- **Contingency**: if it is neither necessarily true nor necessarily false
- Ex: p ▸p
- **Satisfiable**: if there is some interpretation for which it is true.

Equivalence laws

- **Idempotency** : $p \lor p = p$
- $p \land p = p$
- **Associativity** : $(p \lor q) \lor r = p \lor (q \lor r)$
- $(p \land q) \land r = p \land (q \land r)$
- **Commutativity**: $p \lor q = q \lor p$
- $p \land q = q \land p$
- $p \leftrightarrow q = q \leftrightarrow p$
- **Distributivity**: $p \land (q \lor r) = (p \land q) \lor (p \land r)$
- $p \lor (q \land r) = (p \lor q) \land (p \lor r)$
- **De Morgan’s Laws**: $\neg(p \lor q) = \neg p \land \neg q$
- $\neg(p \land q) = \neg p \lor \neg q$
- **Conditional Elimination**: $p \rightarrow q = \neg p \lor q$
- **Bi-conditional Elimination**: $p \leftrightarrow q = (p \rightarrow q) \land (q \rightarrow p)$

Truth table for equivalent sentences

p	q	$\sim p$	$(\sim p \vee q)$	$(p \rightarrow q)$	$(q \rightarrow p)$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	F	T	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	T	T	F	F	F
F	F	T	T	T	F	T	T

Truth Table

- The compilation of all possible scenarios in a tabular format is called a **truth table**.
- **1. Negation** – If p is a proposition, then the negation of p is denoted by ¬p, which when translated to simple English means-
  “It is not the case that ” or simply “not “.
  The truth value of ¬p is the opposite of the truth value of p .
  The truth table of ¬p is-

p	**¬p**
F	T
T	F

Conjunction

- **2. Conjunction** – For any two propositions p and q, their conjunction is denoted by p ◂q, which means “ p and q “. The conjunction is True when both p and q are True, otherwise False.
  The truth table of p ◂ q is-

p	**q**	**p◂q**
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLE

- The conjunction of the propositions p – “Today is Friday” and q– “It is raining today”, p ◂ q is “Today is Friday and it is raining today”.
- This proposition is true only on rainy Fridays and is false on any other rainy day or on Fridays when it does not rain.

Disjunction

- **3. Disjunction** – For any two propositions p and q , their disjunction is denoted by p ▸q, which means “ p or q “.
- The disjunction p ▸q is True when either p or q is True, otherwise False.
  The truth table of is-

p	**q**	**p▸q**
T	T	T
T	F	T
F	T	T
F	F	F

Implication

- **4. Implication :** For any two propositions p and q, the statement "if p then q " is called an implication and it is denoted by $p \rightarrow q$.
- In the implication $p \rightarrow q$, p is called the **hypothesis** or **antecedent** or **premise** and q is called the **conclusion** or **consequence**.
- The implication $p \rightarrow q$ is also called a **conditional statement**.
- The implication is false when p is true and q is false otherwise it is true. The truth table of is-

Truth table

- The truth table of $p \rightarrow q$ is-

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- variety of terminology is used to express :
- “if p, then q "
- " p is sufficient for q"
- " q when p"
- "a necessary condition for is p is q"
- " p only if q“
- “q unless ▲ p"
- “q follows from p"

Biconditional or Double Implication

- **5. Biconditional or Double Implication:**
- For any two propositions and , the statement “ if and only if (iff) ” is called a bi-conditional and it is denoted by $p \leftrightarrow q$.
- The statement is also called a **bi-implication**.
- The implication is true when pand q have same truth values, and is false otherwise. The truth table of is-

- The truth table of p⬇q is-

p	q	p⬇q
T	T	T
T	F	F
F	T	F
F	F	T

- Some other common ways of expressing are-
- “p is necessary and sufficient for q"
- "if p then q, and conversely"
- “p iff q "

Example

“It is raining today if and only if it is Friday today.” is a proposition which is of the form $p \leftrightarrow q$

. The above proposition is true if it is not Friday and it is not raining or if it is Friday and it is raining, and it is false when it is not Friday or it is not raining.

Inference Rules

- Inference rules of PL provide the means to perform logical proofs or deductions.
- Proofs are nothing but a set of arguments that are conclusive evidence of the validity of the theory.
- An argument is a sequence of statements.
- The arguments are chained together using Rules of Inferences to deduce new statements and ultimately prove that the theorem is valid.

- The last statement is the conclusion
- all its preceding statements are called premises (or hypothesis).

Rules

- **Modus Ponens**: from p and $p \rightarrow q$ infer q.

  p

  $p \rightarrow q$

$\therefore$ q

For example:

Given: (Joe is a father)

And: (Joe is a father) $\rightarrow$ (Joe has a child)

Conclude: (Joe has a child)

Chain rule

- Chain rule: from $p \rightarrow q$, and $q \rightarrow r$, infer $p \rightarrow r$

  $p \rightarrow q$

  $q \rightarrow r$

  ______

  $\therefore\ p \rightarrow r$

Ex:

given: (programmer likes LISP) $\rightarrow$ (programmer hates COBOL)

and: (programmer hates COBOL) $\rightarrow$ (programmer likes recursion)

conclude: (programmer likes LISP) $\rightarrow$ (programmer likes recursion)

Simplification

- If p◂q is a premise ,we can use Simplification rule to derive p.
- $p \wedge q$

$$\frac{p \wedge q}{\therefore p}$$

ex.:

He studies very hard and he is the best boy in the class , $p \wedge q$

Therefore – "He studies very hard"

Conjunction

- If p and q are two premises, we can use conjunction rule to derive p∧q

  pq

  ______
- ∴p∧q
- Example
- Let p : “He studies very hard”
- Let q : “He is the best boy in the class”
- Therefore − "He studies very hard and he is the best boy in the class"

Addition

- If p is a premise, we can use Addition rule to derive $p \lor q$.
- p

  ______
- $\therefore p \lor q$

  Example
- Let p be the proposition, “He studies very hard” is true
- Therefore $p \lor q$ – "Either he studies very hard Or he is a very bad student."
- Here q is the proposition “he is a very bad student”.

Modus Tollens

- if $p \rightarrow q$ and $\neg q$ are two premises, we can use modus tollens to derive $\neg p$

  $p \rightarrow q$

  $\neg q$

- $\therefore \neg p$

  example
- "if you have a password, then you can log on to facebook", $p \rightarrow q$
- "you cannot log on to facebook", $\neg q$
- therefore – "you do not have a password "

Formal system

- A formal system is a set of statements S and a set of inference rules L from which new statements can be logically derived.
- Sometimes denotes as <S,L> or simply knowledge base

Soundness

- Let <S,L> be a formal system.
- Inference procedures L are sound iff any statement s that can be derived from <S,L> is a logical sequence of <S,L>

Completeness

- Let <S,L> be a formal system.
- Inference procedure L is complete iff any sentence s logically implied by <S,L> can be derived using that procedure