Northumbria University NEWCASTLE

MOCK - COMPONENT TWO (EXAMINATION PAPER) - MOCK

Module Title:	Fluid Dynamics
Module Number:	KC6027
Examiner Name(s):	Professor James McLaughlin
Month:	April
Academic Year:	2022/2023
Normal Duration of Exam:	3 hours + 10 minutes reading time

INSTRUCTIONS TO CANDIDATES

Please read carefully before you begin your examination.

SIX questions are set.
Candidates may attempt FOUR questions (these are your choice)
Marks will be awarded for these FOUR solutions.
Each question is worth 25 marks.
Marks for parts of questions are shown in parentheses.

You MUST show all your working clearly and carefully to get full credit for each question.

Provided:

- Figures 1, 2, 3 and 4 can be found on the pages 9, 10, 11 and 12.
- Mathematics Formula Sheet 1 is attached.

Allowed Materials

- Candidates may bring a two-sided sheet of A4 paper into the exam, with their own revision notes on both sides, as confirmed by the Departmental Management Group
- Candidates may use a scientific calculator but programmable calculators are not allowed.

PLEASE NOTE:

You must abide by the University's regulations on academic misconduct.

1. For a two-dimensional incompressible inviscid irrotational flow, the stream function $\psi(x, y)$ may be defined by:

$$u = \frac{\partial \psi}{\partial y}\ , \ \ v = -\frac{\partial \psi}{\partial x}$$

where u and v are the fluid velocity components in the x and y directions respectively. Show that the irrotational condition leads to the Laplace governing equation. (It is sufficient to show this in two dimensions).

(3 marks)

Consider the flow region shown in Figure 1. Fluid enters the region *OABCDEFG* through inlets *GO* and *AB,* moving normally across the inlets at speeds $u = 6$ and $u = -6$ respectively, as shown. It leaves via outlet *DE*, moving normally across the outlet with speed v .

Explain why $v = 24$.

(3 marks)

Choosing $\psi = 0$ on the boundary wall *OA,* formulate the boundary conditions that $\psi(x, y)$ must satisfy.

(12 marks)

Carefully sketch the streamlines corresponding to $\psi = -24, -12,\ 0,\ 12$ and $24.$

(7 marks)

[Total 25 marks]

2. An incompressible, inviscid liquid of density ρ is flowing along a pipe of small uniform bore. If q is its speed at time t and p is its pressure, it may be shown that:

$$\frac{dq}{dt} = \boldsymbol{F} \cdot \hat{\boldsymbol{s}} - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

where s is the distance along the pipe from origin O, $\boldsymbol{F}$ is the body force vector and $\hat{\boldsymbol{s}}$ is a unit vector in the direction of s increasing.

Figure 2 shows such a pipe *OAB* emptying under gravity. Section *OA* is vertical and of length a , whilst section *AB* is at an angle of $\pi/3$ to the downward vertical and is of length $14a$. Initially the pipe is full of liquid. Its ends are then opened to the atmosphere and after time t the level of the liquid in section *OA* has fallen by a distance S.

Show that $S(t)$ is governed by:

$$\ddot{S} = \frac{g(8a - S)}{15a - S}.$$

(12 marks)

Show further that at time t the speed at which the level of the liquid is falling is given by:

$$\dot{S} = \sqrt{2g}\left[S + 7a\ln\left(1 - \frac{S}{15a}\right)\right]^{1/2}.$$

(8 marks)

Using a suitable approximation (which you must clearly state), estimate the time taken for section *OA* to empty.

(5 marks)

[Total 25 marks]

3. An incompressible viscous fluid of viscosity μ is flowing steadily in the axial x-direction along a pipe of uniform annular cross-section:

$$b \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

where (r, θ, x) are cylindrical polar coordinates. The axial speed of flow u is governed by:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{P}{\mu}$$

with $P = -dp/dx$ representing the **constant** applied axial pressure gradient.

Establish that:

$$u = \frac{P}{4\mu}(a^2 - r^2) - \frac{P}{4\mu}(a^2 - b^2)\frac{\ln(r/a)}{\ln(b/a)}$$

(12 marks)

Calculate the flow rate when $b = 0$.

(8 marks)

Is this the same physical situation as when $b \to 0$? Give a detailed motivation for your answer.

(5 marks)

[Total 25 marks]

4. Referred to Cartesian coordinates (x, y), the plane wall $y = 0$ forms the lower boundary to the two-dimensional steady flow of a fluid of viscosity μ and density ρ. The mainstream flow speed, outside the boundary layer, is U in the x-direction and is constant. Under these circumstances the Momentum Integral Equation reduces to:

$$\frac{\tau_w}{\rho} = U^2 \frac{d\delta_2}{dx}$$

where τ_w is the skin friction and δ_2 is the momentum thickness, defined by:

$$\delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

where u is the component of fluid velocity in the x-direction.

Suppose that $u(x, y)$ is approximated by:

$$u = UF(\eta) \text{ where } \eta = y/\delta(x)$$

and $\delta(x)$ is a measure of boundary layer thickness.

Establish the cubic expression for $F(\eta)$ such that the conditions:

$$\text{on } y = 0, u = 0$$
$$\text{on } y = \delta, u = U, \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0$$

are satisfied leading to:

$$F(\eta) = 3\eta - 3\eta^2 + \eta^3 \ .$$

(6 marks)

Use the Momentum Integral Equation to show that $\delta(x)$ satisfies the differential equation:

$$\delta \frac{d\delta}{dx} = \frac{28\upsilon}{U}, \text{ where } \upsilon = \frac{\mu}{\rho}$$

(12 marks)

Question 4 continues on the next page.

Question 4 continued:

Assuming $\delta = 0 \text{ when } x = 0$, solve for $\delta(x)$ and hence establish that the drag (per unit width) exerted on the section of wall $0 \leq x \leq L$ is equal to:

$$3\mu\ U\left(\frac{UL}{14\upsilon}\right)^{1/2}.$$

(7 marks)

[Total 25 marks]

Question 5 can be found on the next page.

5. Figure 3 shows a diagram of a specific example of a Rayleigh step bearing of length L (and effectively infinite width). It consists of a lower plane slider, $y = 0$, moving with constant speed U in the x-direction, and an upper stationary stepped surface:

$$\begin{aligned} y &= h_1 \quad \text{for} \quad 0 \le x < L/3 \\ y &= h_2 \quad \text{for} \quad L/3 \le x \le L \end{aligned}$$

where h_1 and h_2 are constant, with $h_2 < h_1$.

The upper surface is supported by the pressure generated in the oil film between the two surfaces. The ends of the bearing are open to the atmosphere. Lubricant pressure $p(x)$ is governed by:

$$\frac{dp}{dx} = \frac{6\mu}{h^3}(Uh - 2Q)$$

where $h(x)$ is the gap width between the bearing surfaces, μ is the oil viscosity and Q is the flow rate (per unit width) in the x-direction.

Show that:

$$p = \begin{cases} \frac{6\mu}{h_1^3}(Uh_1 - 2Q)x & \text{for} \quad 0 \le x < L/3 \,, \\ \frac{6\mu}{h_2^3}(Uh_2 - 2Q)(x - L) & \text{for} \quad L/3 \le x \le L \,. \end{cases}$$

(6 marks)

Find the flow rate Q.

(8 marks)

Sketch the lubricant pressure distribution $p(x)$.

(4 marks)

Show that the maximum pressure generated in the lubricant film is given by:

$$p_{max} = 4\mu LU \frac{(h_1 - h_2)}{(2h_1^3 + h_2^3)}$$

(7 marks)

[Total 25 marks]

6. Figure 4 depicts a highly viscous fluid being removed from the upper surface of a plane, $\theta = 0$, by moving the plane slowly, with constant speed U, past a stationary plane surface, $\theta = \pi/3$, in the direction indicated.

The flow is governed by the biharmonic equation:

$$\nabla^2(\nabla^2\psi) = 0$$

where, referred to (r, θ) coordinates:

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and the stream function ψ is defined by:

$$\frac{\partial\psi}{\partial\theta} = ru \quad \text{and} \quad \partial\psi/\partial r = -v,$$

u and v being the radial and transverse velocity components, respectively.

Assuming a trial separable solution of the form:

$$\psi = Urf$$

where $f = f(\theta)$, show that the general solution for f is given by:

$$f(\theta) = (A\theta + B)\cos\theta + (C\theta + D)\sin\theta,$$

where A, B, C and D are constants.

(10 marks)

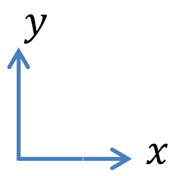
Hence establish an expression for ψ.

(13 marks)

Show further that on any radial line from O the fluid speed is constant.

(2 marks)

[Total 25 marks]



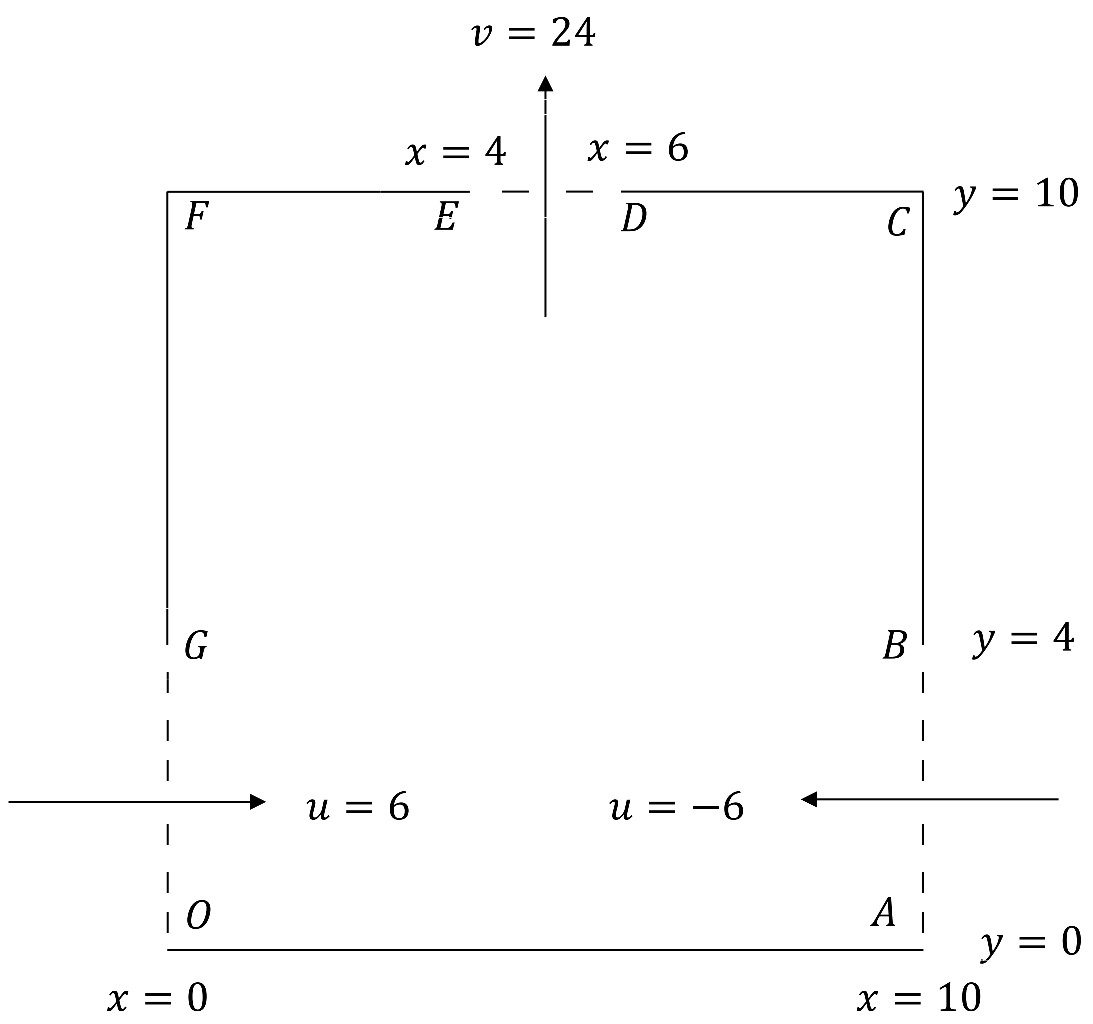


Figure 1

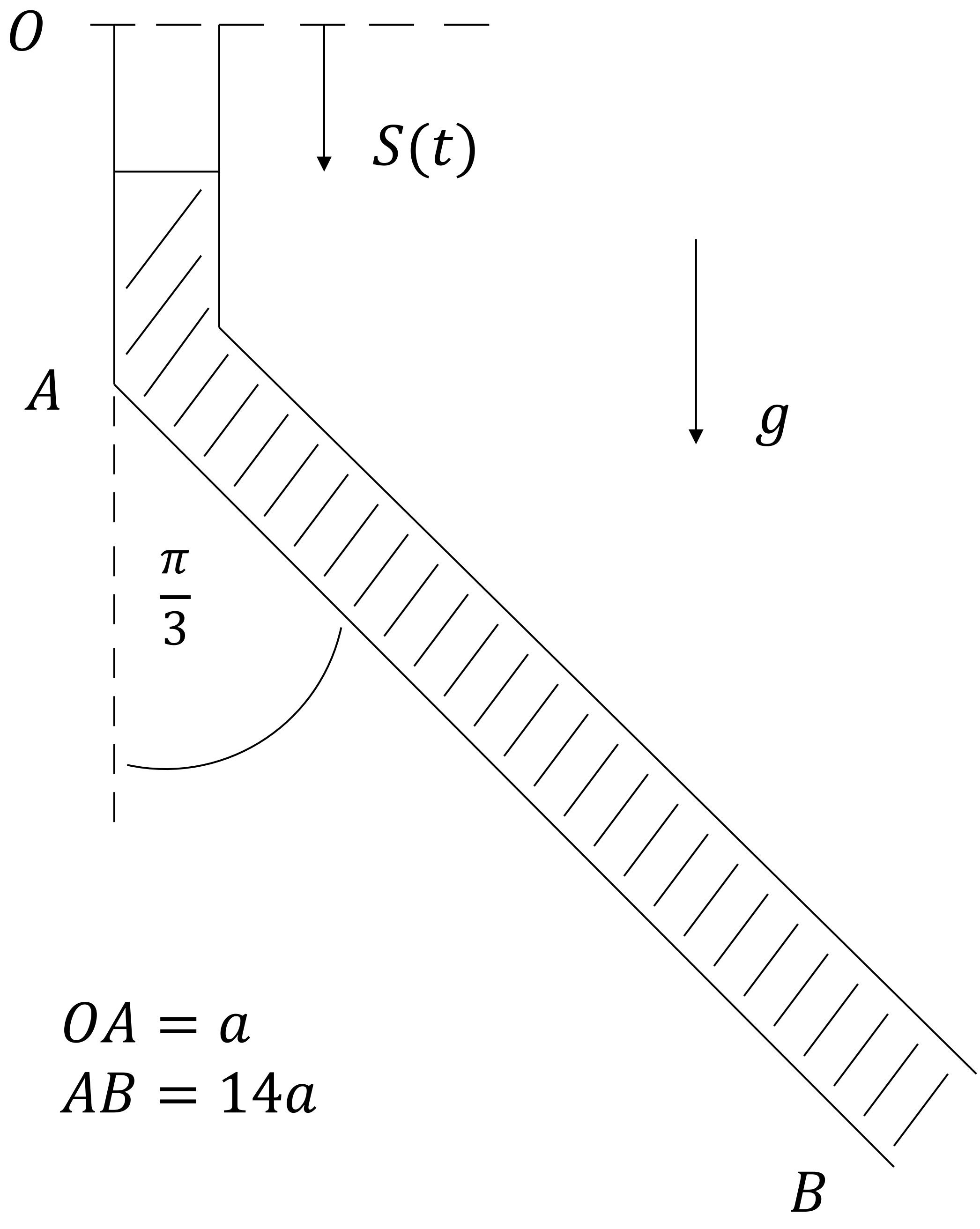


Figure 2 (not to scale)

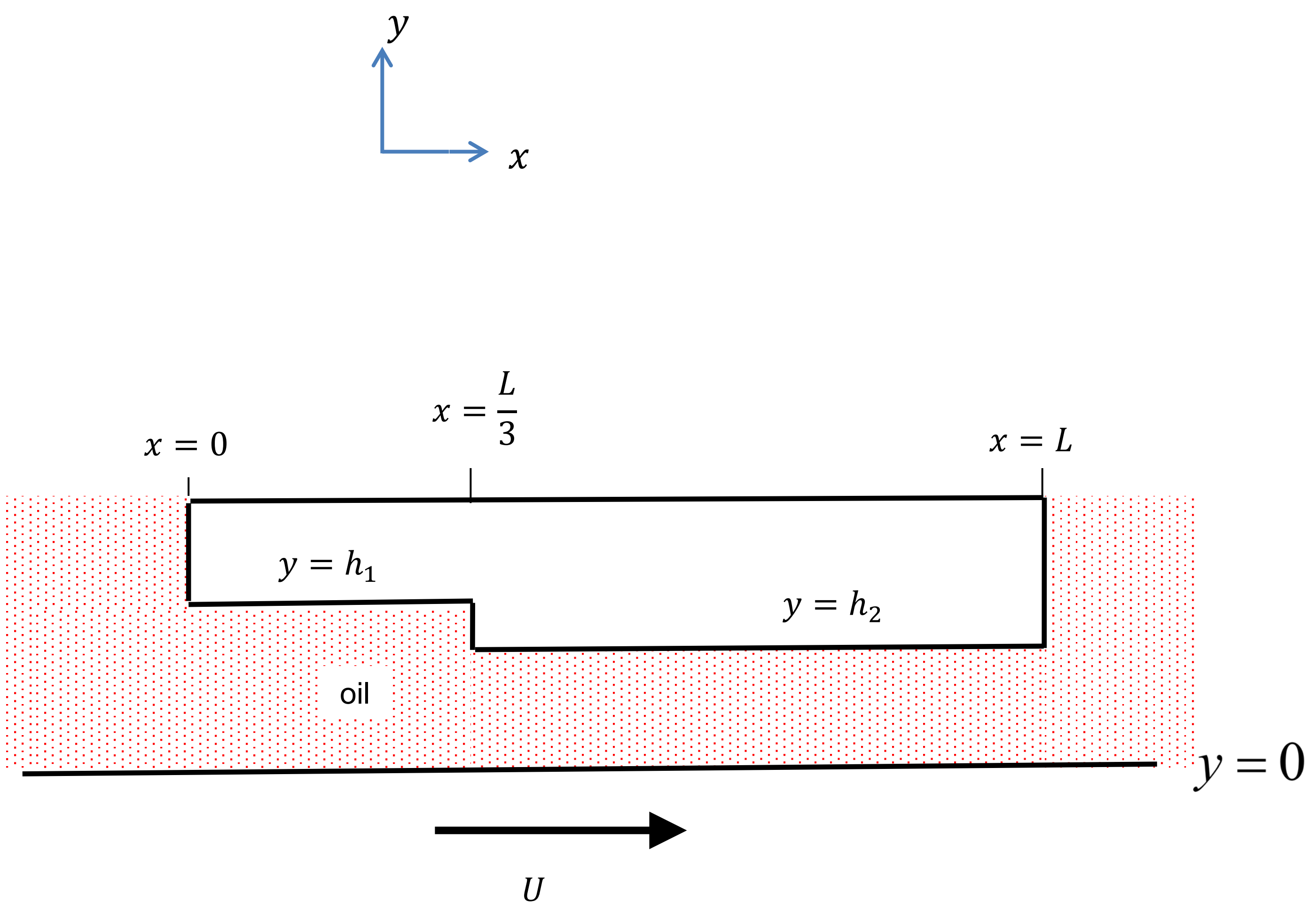


Figure 3 : Rayleigh Step Bearing (not to scale)

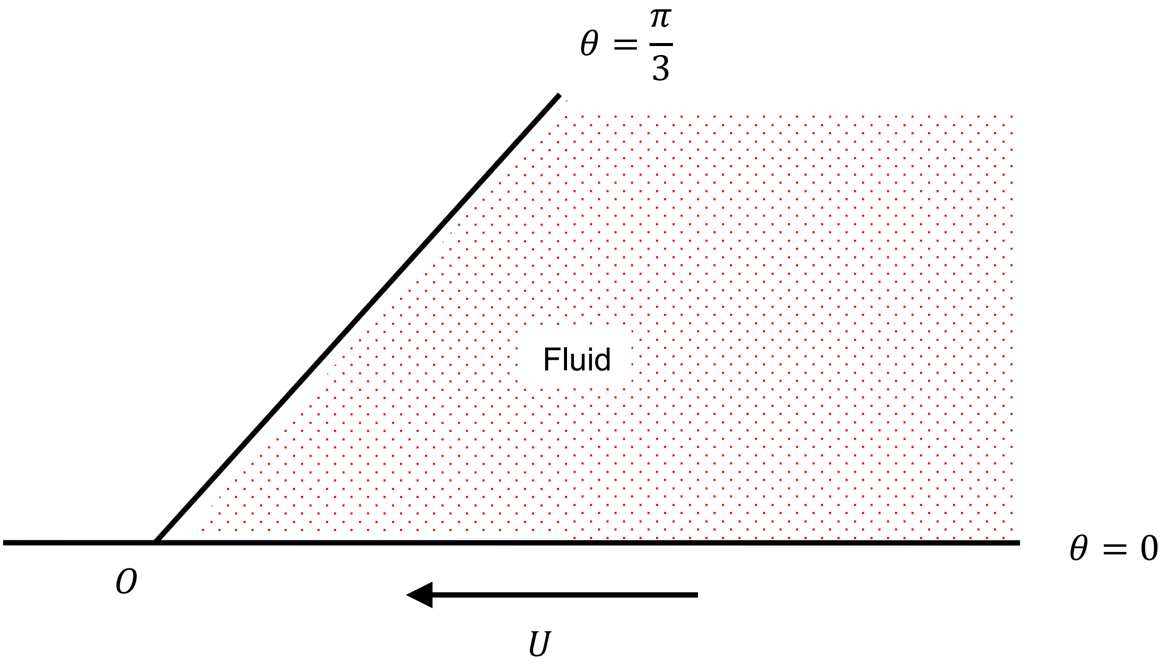


Figure 4 (not to scale)

Mathematics Formula Sheet

Derivative	Function	Integral
$\dfrac{dy}{dx}$	y	$\displaystyle\int y\,dx$
0	k	kx
nx^{n-1}	x^n	$\dfrac{x^{n+1}}{n+1}$
$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln x$
$k\,e^{kx}$	e^{kx}	$\dfrac{1}{k}e^{kx}$
$a^x \ln a$	a^x	$\dfrac{a^x}{\ln a}$
$\dfrac{1}{x}$	$\ln x \quad (\log x)$	$x \ln x - x$
$\dfrac{1}{x \ln a}$	$\log_a x$	$x(\log_a x - \ln a)$
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$	$-\ln(\cos x)$
$\sec x \tan x$	$\sec x$	$\ln(\sec x + \tan x)$
$-\csc^2 x$	$\cot x$	$\ln(\sin x)$
$2 \sin x \cos x$	$\sin^2 x$	$\frac{1}{4}(2x - \sin(2x))$
$-2 \cos x \sin x$	$\cos^2 x$	$\frac{1}{4}(2x + \sin(2x))$
...	$\dfrac{1}{x^2 + a^2}$	$\dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right)$
...	$\dfrac{1}{x^2 - a^2}$	$\dfrac{1}{a} \ln\left(\dfrac{x-a}{x+a}\right)$
...	$\dfrac{1}{\sqrt{x^2 + a^2}}$	$\ln\left(x + \sqrt{x^2 + a^2}\right)$
...	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\ln\left(x + \sqrt{x^2 - a^2}\right)$

Rules for differentiation and integration

Product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}.$

Chain rule: $\dfrac{d}{dx}(u(v(x))) = \dfrac{du}{dv} \times \dfrac{dv}{dx}.$

Quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

Integration by parts: $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$

Series

Maclaurin series: $f(x) \approx f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \ldots$

Taylor series: $f(x+h) \approx f(h) + xf'(h) + \dfrac{x^2}{2!}f''(h) + \dfrac{x^3}{3!}f'''(h) + \ldots$

Numerical integration

Simpsons's rule: $\displaystyle\int_a^b f(x)dx = \frac{h}{3}\left[(f_0 + f_n) + 4(f_1 + f_3 + \ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots + f_{n-2})\right]$

Quadratic formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Euler's equation

If θ is in radians, then $e^{i\theta} = \cos\theta + i\sin\theta$.

Trigonometric identities

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	$\sin(2A) = 2\sin A\cos A$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	$\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$
$\cos^2 A + \sin^2 A = 1$	$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$
$1 + \tan^2 A = \sec^2 A$	$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$
$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$	$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$	$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Hyperbolic functions

Definition: $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Osborn's rule: Change sin to sinh and cos to cosh, then change the sign of any term involving the product, or implied product, of two sines.