# Probabilistic Graphical Models

Chao Liu

Ding Xiang Yuan

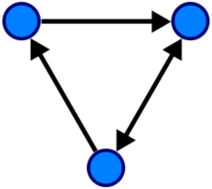
February 26, 2018

# Outline

1. Background
2. Bayesian network(directed graph))
3. Conditional independence
4. Markov random fields(Undirected graph)
5. Inference
6. Reference and further reading

# What is graph

- vertices or nodes or points
- edag, arcs or lines
- direction



# Graph is everywhere in the field of computer science

- all the trees(avl tree, binary search tree, red-black tree...)
- dijkstra's algorithm, maximum flow algorithm...
- deep learning computing framework: Mxnet, Tensorflow

# What is probability graph

- a graph comprises nodes (also called vertices) connected by links (also known as edges or arcs).
- each node represents a random variable (or group of random variables), and the links express probabilistic relation- ships between these variables
- the graph then captures the way in which the joint distribution over all of the random variables can be decomposed into a product of factors each depending only on a subset of the variables.

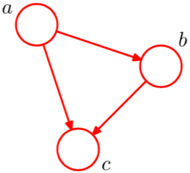
# Why we need to study probability graph

- It provide a simple way to visualize the structure of a probabilistic model and can be used to design and motivate new models
- insights into the properties of the model, including conditional independence properties, can be obtained by inspection of the graph
- complex computations, can be expressed in terms of graphical manipulations, in which underlying mathematical expressions are carried along implicitly.

# Bayesian network(directed graph)

- the links of the graph have a particular directionality indicated arrows
- useful to expressing the causal relationships between random variables
- apply the product rule to the joint distribution over three variables $p(a, b, c) = p(c|a, b)p(b|a)p(a)$

Figure: the directed graph model of joint distribution $p(a, b, c)$



# A example

ploynoimal regression:

$$p(\mathbf{t}, \mathbf{w}) = p(\mathbf{w}) \prod_{n=1}^{N} p(t_n|w) \tag{1}$$

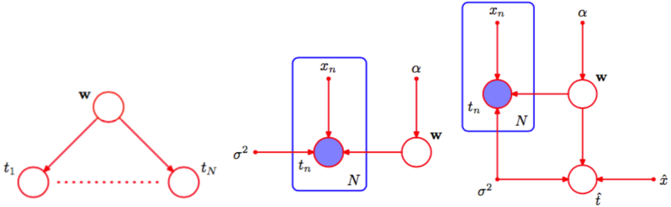
make the parameters of a model, as well as its stochastic variables, explicit

$$p(\mathbf{t}, w|x, \alpha, \sigma^2) = p(w|\alpha) \prod_{n=1}^{N} p(t_n|w, x_n, \sigma^2) \tag{2}$$

where $x$ is input the data, $t$ is the observed data, $\alpha$ is the gaussian prior and $\sigma^2$ is the noise variance.
the form with prediction value:

$$p(\hat{t}, \mathbf{t}, w|\hat{x}, \mathbf{x}, \alpha, \sigma^2) = p(w|\alpha)p(\hat{t}|\hat{x}, w, \sigma^2)[\prod_{n=1}^{N} p(t_n|w, x_n, \sigma^2)] \tag{3}$$

# corresponding graph model



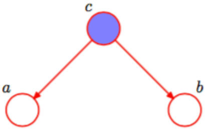
# Conditional independence

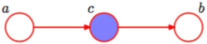
defination: $p(a|b,c) = p(a|c)$, $a \perp\!\!\!\perp b|c$

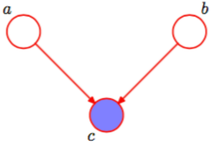
- it is very important for simplifying computation
- using D-sparation in directed graph, undirected graph however, it's easier to judage than the directed graph
- **KEY IDEA**: to see wether a variable is "blocked"

# three example graphs

- tail-to-tail $a \perp\!\!\!\perp b|c$
- head-to-tail $a \perp\!\!\!\perp b|c$
- head-to-head $a \perp\!\!\!\perp b|c$ not hold







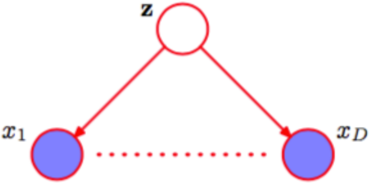
# Example

Naive Bayes:

$$p(D|\mu) = \prod_{n=1}^{N} p(x_n|\mu) \tag{4}$$

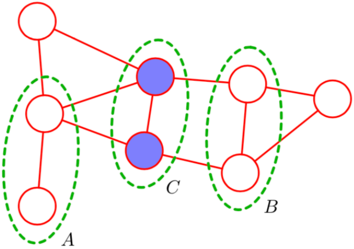
where $\mu$ is the prior

Figure: probalisic graph expression of naive bayes



# Markov random fields(Undirected graph)

- conditional independence properties: $A \perp\!\!\!\perp B | C$, whether all paths that connect nodes in set A to nodes in the set B.



# Factorization properties

- clique: which is defined as a subset of the nodes in a graph such that there exists link between all pairs of nodes in the subset. The set of nodes in a clique is fully connected.
- maximal clique: is a clique such that it is not possible to include any other nodes from the graph
- the joint distribution is written as a product of *potential functions* $\psi(C)$ over the maximal cliques of the graph

$$p(x) = \frac{1}{Z} \prod_{C} \psi(C)(x_C)$$

where $x_C$ denote the node in the clique, and we do not restrict the choice of it. Z is the partition function, give by

$$Z = \sum_{X} \prod_{C} \psi(C)(X_C)$$

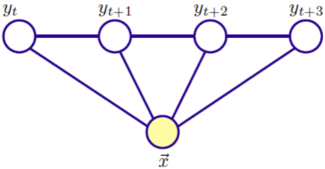
# Example: CRF

- CRF(Conditional Random Field) is just a version of an MRF where all the clique poentials are conditioned on input features
- CRF can be any kind of structure, but usually we use a linear chain structure.

$$p(y|x) = \frac{1}{Z(x)} \prod_{j=1} \psi_j(y, x)$$

where we define the poential function to be:

$$\psi_j(x, y) = \exp(\sum_{i=1}^{m} \lambda_i f_i(y_{j-1}, y_j, x, j))$$



# Inference

- exact inference and approximation method.
- exact inference: sum-product, max-sum algorithm
- approximation: variational methods and sampling methds etc.

# Reference and further reading I

Bishop 2006
*Pattern Recognition and Machine Learning*.
Springer, 2009.

more about CRF and HMM see this material:
`http://www.eng.utah.edu/~cs6961/papers/jerryzhu-crfs.pdf`