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Sufficient Conditions for Local Asymptotic Stability and Stabilization for Discrete-Time Varying Systems

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***Abstract*—The purpose of this paper is to establish sufficient conditions for local asymptotic stability and feedback stabilization for discrete-time systems with time depended dynamics. Our main results constitute generalizations of those developed by same authors in a recent paper, published in same journal, for the case of continuous-time systems.**

***Index Terms*—Asymptotic stability, averaging, discrete-time systems, stabilization.**

Notations: We adopt the following notations. For $x \in \mathbb{R}^n$, $|x|$ denotes its usual Euclidean norm. Given a matrix $A \in \mathbb{R}^{n\times m}$ we denote by $|A| := \sup_{x\neq 0}(|Ax|/|x|)$ its induced norm. By $S[0,R]$ we denote the closed ball of radius $R>0$ around zero $0\in\mathbb{R}^n$. $\mathcal{N}$ denotes the set of all C^0 functions $\phi:\mathbb{R}^+\to\mathbb{R}^+$ and $\mathcal{K}$ is the set of all functions $\phi\in\mathcal{N}$ which are strictly increasing and vanishing at zero. $\mathcal{K}_\infty$ denotes the subset of $\mathcal{K}$ that constitutes by all $\phi\in\mathcal{K}$ with $\phi(t)\to\infty$ as $t\to\infty$.

I. Introduction

The present work provides sufficient conditions for local asymptotic stability and feedback stabilization for the case of discrete-time systems with time depended dynamics. Our results generalize those in existing works (see for instance [1], [2], [4], [5], [8]). Propositions 1 and 2 in Section II are the main results of the paper establishing Lyapunov-like sufficient conditions for asymptotic stability for systems

$$x(n+1) = f(n, x(n)),\ (n,x)\in\mathbb{N}\times\mathbb{R}^n. \tag{1}$$

These results constitute, in some sense, the discrete analogue to [9, Proposition 1]. It should be emphasized however, that Proposition 1 and 2, as well as the averaging result of Proposition 5 in Section IV, are based on weaker hypotheses than the discrete-analogue conditions imposed in earlier works concerning continuous-time systems (see for instance, [2], [3], [7], [9] and relative references therein). The result of Proposition 1 is applied in Sections III and IV for the establishment of sufficient conditions for the solvability of the feedback stabilization problem for control systems

$$x(n+1) = F(n, x(n), u(n)),\ (n,x,u)\in\mathbb{N}\times\mathbb{R}^n\times\mathbb{R}^m \tag{2}$$

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and to derive an averaging type sufficient condition for local asymptotic stability for the case

$$x(n+1)=x(n)+\varepsilon f(\varepsilon,n,x(n)),\ (\varepsilon,n,x)\in\mathbb{R}\times\mathbb{N}\times\mathbb{R}^n,\ \varepsilon>0. \tag{3}$$

We next provide the concepts of stability, local asymptotic stability and local exponential stability for the case (1). In what follows, we assume that $0\in\mathbb{R}^n$ is an equilibrium, i.e., $f(\cdot,0)=0$. We say that $0\in\mathbb{R}^n$ is stable with respect to (1), if for each $\varepsilon>0$ and given bounded $I\subset\mathbb{N}$ there exists a constant $\delta=\delta(\varepsilon,I)>0$ such that

$$|x(n_0)|\le\delta\Rightarrow|x(n)|\le\varepsilon,\ \forall n\ge n_0, n_0\in I \tag{4}$$

where $x(n)=x(n,n_0,x_0)$, $n=n_0,n_0+1,n_0+2,\ldots$ denotes the solution of (1) initiated from $x_0:=x(n_0)$ at time n_0. We say that $0\in\mathbb{R}^n$ is an attractor for (1), if there exists a constant $\rho>0$ such that for every $\varepsilon>0$ and given bounded $I\subset\mathbb{N}$, a time $\tau=\tau(\varepsilon,I)\in\mathbb{N}$ can be found with

$$|x(n_0)|\le\rho\Rightarrow|x(n)|\le\varepsilon,\ \forall n\ge n_0+\tau,\ n_0\in I. \tag{5}$$

We say that (1) is Asymptotically Stable (AS) (at zero $0\in\mathbb{R}^n$), if zero is stable and an attractor. We say that (1) is Uniformly in time Asymptotically Stable (UAS), if both (4) and (5) hold for every $n_0\in\mathbb{N}$ and for δ and τ depending only on ε. We say that (1) is Exponentially AS (expo-AS), if there exists a constant $\lambda>0$ such that for any given bounded $I\subset\mathbb{N}$, a constant $C=C(I)>0, n_0\in I$ can be found with

$$\begin{aligned}&|x(n)|\le C\,|x(n_0)|\exp(-\lambda(n-n_0)),\\ &\qquad\forall n\ge n_0, n_0\in I, x(n_0)\,near\ zero.\end{aligned} \tag{6}$$

Finally, (1) is Exponentially UAS (expo-UAS), if (6) holds for every $n_0\in\mathbb{N}$ and for certain $C>0$ being independent of the initial values n_0 of time.

II. Main Result

The aim of this section is to establish an extension of the main result in [9] for the discrete-time systems (1). We assume that there exists a constant $R>0$ such that the following properties hold:

A1. There exists a function $L\in\mathcal{N}$ such that

$$|f(n,x)|\le L(n)|x|\ \forall(n,x)\in\mathbb{N}\times S[0,R] \tag{7}$$

moreover, we assume that one of the following conditions is fulfilled:

A2. There exist functions $V:\mathbb{N}\times\mathbb{R}^n\to\mathbb{R}^+, a,b\in\mathcal{K}_\infty, c\in\mathcal{N}, r\in\mathcal{K}$, a sequence $\{\sigma_i\ge 0, i\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\}$, a function $m_0\in\mathcal{N}$ and a constant $m>0$, such that

$$\sum_{i=0}^{\infty}\sigma_i=\infty \tag{8a}$$

$$a(|x|)\le V(n,x)\le b(|x|)\,c(n),\ \forall(n,x)\in\mathbb{N}\times S[0,R] \tag{8b}$$

and further the following hold for the solution $x(\cdot)=x(\cdot,\ell_0,x_0)$, $(\ell_0,x_0)\in\mathbb{N}\times S[0,R]$, $x_0=x(\ell_0)$ of (1)

$$\begin{aligned}&V(n_{i+1},x(n_{i+1}))-V(n_i,x(n_i))\le-\sigma_i r(V(n_i,x(n_i))),\\ &n_i=n_i(\ell_0,x_0), x_0\in S[0,R]\ for\ i\in\mathbb{N}_0\ away\ from\ zero,\\ &provided\ that\ x(\nu)\in S[0,R],\\ &\qquad\nu=n_i,n_i+1,n_i+2,\ldots,n_{i+1}\end{aligned} \tag{8c}$$

for certain strictly increasing sequence $\{n_i = n_i(\ell_0, x_0), i \in \mathbb{N}_0\}$ with $n_0 = n_0(\ell_0, x_0) \geq \ell_0$, in such a way that

$$n_i \to \infty \; as \; i \to \infty \tag{9a}$$

$$\sum_{\nu=n_i(\ell_0,x_0)}^{\nu=n_{i+1}(\ell_0,x_0)} L(\nu) \leq m, \forall i \in \mathbb{N}_0, \ell_0 \in \mathbb{N}, x_0 \in S[0,R] \tag{9b}$$

$$\sum_{\nu=\ell_0}^{\nu=n_0(\ell_0,x_0)} L(\nu) \leq m_0(\ell_0), \; \forall \ell_0 \in \mathbb{N}, x_0 \in S[0,R] \tag{9c}$$

$$c\left(n_0(\ell_0, x_0)\right) \leq m_0(\ell_0). \tag{9d}$$

A'2. There exist functions $V : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^+, a, b \in \mathcal{K}_\infty, c \in \mathcal{N}, r \in \mathcal{K}$, a sequence $\{\sigma_i \in \mathbb{R}, i \in \mathbb{N}_0\}$, a function $m_0 \in \mathcal{N}$ and a constant $m > 0$, such that (8a) and (8b) hold and further

$$\begin{aligned} &V\left(n_{i+1}, x(n_{i+1})\right) - V\left(n_i, x(n_i)\right) \leq -\sigma_i V\left(n_i, x(n_i)\right) \\ &\quad provided\; that\; x(\nu) \in S[0,R], \\ &\quad \nu = n_i, n_i+1, n_i+2, \ldots, n_{i+1} \end{aligned} \tag{10}$$

for certain strictly increasing sequence $\{n_i = n_i(\ell_0, x_0) \in \mathbb{N}, i \in \mathbb{N}_0\}$ with $n_0 \geq \ell_0$ in such a way that (9a), (9b), (9c) and (9d) hold.

Proposition 1: (*i*) Under the assumptions A1 and A2(A'2) the system (1) is AS; (ii) If, in addition to A1 and A'2, we assume

$$\sigma_i \geq \sigma \; for \; some \; constant \; \sigma > 0 \tag{11}$$

$$\begin{aligned} a_0|x|^2 \leq V(n,x) \leq b_0|x|^2 \; \forall (n,x) \in \mathbb{N} \times S[0,R], \\ for \; certain \; constants \; a_0, b_0 > 0 \end{aligned} \tag{12}$$

and there is an integer $N > 0$ such that

$$n_{i+1}(\ell_0, x_0) - n_i(\ell_0, x_0) \leq N, \; \forall i \in \mathbb{N}_0, \ell_0 \in \mathbb{N}, x_0 \in S[0,R] \tag{13}$$

then (1) is expo-AS; (iii) If, in addition to A1 and A2(A'2), we assume that (11) holds and there exist functions $a, b \in \mathcal{K}_\infty$ and a *constant* $m_0 > 0$ such that

$$\begin{aligned} a\left(|x|\right) &\leq V(n,x) \\ &\leq b\left(|x|\right), \; \forall (n,x) \in \mathbb{N} \times S[0,R] \end{aligned} \tag{14}$$

$$\begin{aligned} \sum_{\nu=\ell_0}^{\nu=n_0(\ell_0,x_0)} L(\nu) \leq m_0, \; \forall \ell_0 \in \mathbb{N}, \; x_0 \in S[0,R], \\ x_0 = x(\ell_0) \end{aligned} \tag{15}$$

then (1) is UAS; (iv) If, A1, A'2, (11), (12), (13) and (15) hold, then (1) is expo-UAS.

Remark 1: Obviously, (9d) is not required in statements (ii), (iii) and (iv) of Proposition 1.

Proof: (i) **A1, A2 $\Rightarrow$ AS**. Some parts of proof of this implication, constitute extensions of the approach employed in [9, Proposition 1]. We denote by $x(\ell) = x(\ell, \ell_0, x_0)$, $\ell = \ell_0, \ell_0+1, \ell_0+2, \ldots$ the trajectory of (1) with $x_0 = x(\ell_0)$. By invoking (7) we have:

$$\begin{aligned} |x(\ell)| \leq \left(\prod_{\nu=\ell_0}^{\nu=\ell} L(\nu)\right)|x_0| \leq \exp\left(\sum_{\nu=\ell_0}^{\nu=\ell} L(\nu)\right)|x_0|, \\ provided\; that\; x(\nu) \in S[0,R], \; \nu = \ell_0, \ell_0+1, \ldots, \ell. \end{aligned} \tag{16}$$

Let I be a bounded subset of $\mathbb{N}$ and, for $\ell_0 \in I$ and $x_0 \in S[0,R]$, consider the sequence $\{n_i = n_i(\ell_0, x_0), i \in \mathbb{N}_0\}$ satisfying (8c), (9a)–(9d) and $\ell_0 \leq n_0$. Without any loss of generality we may assume next that $\ell_0 < n_0$ and (8c) holds for all $i \in \mathbb{N}_0$. It follows from (9c) and (16) that

$$\begin{aligned} |x(\ell)| &\leq \exp\left(\sum_{\nu=\ell_0}^{\nu=n_0(\ell_0,x_0)} L(\nu)\right)|x_0| \leq \exp\left(m_0(\ell_0)\right)|x_0|, \\ &for\; all\; \ell = \ell_0, \ell_0+1, \ldots, n_0, \\ &provided\; that\; x(\nu) \in S[0,R], \\ &\nu = \ell_0, \; \ell_0+1, \ldots, n_0. \end{aligned} \tag{17}$$

Likewise, by denoting $M := \exp m$, (9b) and (16) imply that

$$\begin{aligned} &|x(n)| \leq M\,|x(n_i)|\,, \; \forall n \in [n_i, n_{i+1}], \\ &\quad provided\; that\; x(n) \in S[0,R], \; for\; all\; integers \\ &\quad n \in [n_i, n_{i+1}], \; i \in \mathbb{N}_0, \; x_0 = x(\ell_0) \in S[0,R]. \end{aligned} \tag{18}$$

By employing both inequalities in (17) and (18) respectively and taking into account boundedness of I, it follows that for any given $\varepsilon \in (0,R]$, a constant $\varepsilon_1 \in (0,\varepsilon]$, being independent of I, can be found such that the solution $x(\cdot) = x(\cdot, \ell_0, x_0)$ of (1) satisfies:

$$|x(\ell_0)| \leq \varepsilon_1 \Rightarrow |x(n)| \leq \varepsilon, \; \forall n = \ell_0, \ell_0+1, \ldots, n_0; \tag{19a}$$

$$\begin{aligned} &|x(n_i)| \leq \varepsilon_1 \Rightarrow |x(n)| \leq \varepsilon, \\ &\qquad \forall\, n = n_i, n_i+1, \ldots, n_{i+1}, \; i \in \mathbb{N}_0, \ell_0 \in I. \end{aligned} \tag{19b}$$

For simplicity, we denote in the sequel

$$\begin{aligned} x\left(n, \ell, S[0,\varepsilon]\right) = \{x \in \mathbb{R}^n : x = x(n, \ell, w), \\ w = x(\ell) \in S[0,R]\}\,, \; n = \ell, \ell+1, \ell+2, \ldots \end{aligned}$$

We next show that there exists a constant $R' \in (0,R]$, such that the following properties are satisfied:

$$\begin{aligned} &a\left(|x(n_{i+1})|\right) \leq V\left(n_{i+1}, x(n_{i+1})\right) \\ &\quad \leq V\left(n_i, x(n_i)\right) \leq b\left(|x(n_i)|\right) c(n_i), \\ &\quad i \in \mathbb{N}_0, \ell_0 \in I, \; x_0 = x(\ell_0) \in S[0,R'], \; \ell_0 < n_0 \end{aligned} \tag{20a}$$

$$\begin{aligned} &V\left(n_{i+1}, x(n_{i+1})\right) - V\left(n_i, x(n_i)\right) \leq -\sigma_i r\left(V\left(n_i, x(n_i)\right)\right), \\ &n_i = n_i(\ell_0, x_0), \; x_0 \in S[0,R'] for \; i \in \mathbb{N}_0, \; \ell_0 < n_0. \end{aligned} \tag{20b}$$

Indeed, by exploiting (19a) and (19b), a positive constant $R' \leq R$ can be determined with

$$\begin{aligned} &x\left(n, \ell_0, S[0,R']\right) \subset S[0,R], \; n = \ell_0, \ell_0+1, \ldots, n_0; \\ &x\left(n, n_i, S[0,R']\right) \subset S[0,R], \; n = n_i, n_i+1, n_i+2, \ldots, n_{i+1}, \\ &\qquad i \in \mathbb{N}_0. \end{aligned} \tag{21}$$

From (8b), (8c) and (21) we obtain the desired (20a) and (20b). From (20b) we obtain

$$\begin{aligned} V\left(n_i, x(n_i)\right) &\leq V\left(n_0, x(n_0)\right) - \sum_{\nu=0}^{\nu=i-1} \sigma_\nu r\left(V\left(n_\nu, x(n_\nu)\right)\right) \\ &\leq V(n_0, x(n_0)) - \left(\sum_{\nu=0}^{\nu=i-1} \sigma_\nu\right) r(V(n_i, x(n_i)))\,, \\ &\quad \forall\, i \in \mathbb{N}_0, x(n_0) \in S[0,R'], \\ &\quad \ell_0 \in I, \ell_0 < n_0. \end{aligned}$$

The latter in conjunction with (8b), (9d), (17) and (20a) imply

$$\begin{aligned} &a\left(|x(n_i)|\right) + \left(\sum_{\nu=0}^{\nu=i-1} \sigma_\nu\right) r\left(a\left(|x(n_i)|\right)\right) \\ &\quad \leq V\left(n_i, x(n_i)\right) + \left(\sum_{\nu=0}^{\nu=i-1} \sigma_\nu\right) r\left(V\left(n_i, x(n_i)\right)\right) \\ &\quad \leq V\left(n_0, x(n_0)\right) \leq b\left(\exp\left(m_0(\ell_0)\right)|x_0|\right) m_0(\ell_0), \; \forall i \in \mathbb{N}_0, \\ &\qquad x_0 = x(\ell_0) \in S[0,R'], \; \ell_0 \in I, \; \ell_0 < n_0. \end{aligned} \tag{22}$$

From (8a), (9a), (22) and boundedness of I it follows:

$$x(n_i) \to 0 \; as \; n_i \to \infty, \; uniformly \; in \quad x_0 \in S[0, R'] \; and \; \ell_0 \in I. \quad (23)$$

It turns out from (23) that for any $\varepsilon \in (0, R']$ and $\varepsilon_1 \in (0, \varepsilon]$, for which (19a) and (19b) hold, a pair of positive integers $k = k(\varepsilon_1, I)$ and $\tau = \tau(\varepsilon_1, I)$ can be found with

$$n_k \geq \ell_0 + \tau, \quad (24a)$$
$$|x(n_i)| \leq \frac{\varepsilon_1}{2}, \; \forall i = k, k+1, k+2, \ldots, x_0 \in S[0, R'], \; \ell_0 \in I. \quad (24b)$$

We are in a position to establish stability and attractivity of zero with respect to (1):

Stability: Let ε, ε_1, k, τ, R and R' as defined above. Let $I \subset \mathbb{N}$ be a given bounded set. We show that for every $\varepsilon \in (0, R)$ there is a constant $0 < \delta := \delta(\varepsilon, I) < \varepsilon_1$ such that (4) is fulfilled. From (1), (7), (9b), (9c) and (21) we get

$$\begin{aligned} |x(n)| &\leq \exp\left(\sum_{\nu=\ell_0}^{\nu=n} L(\nu)\right)|x_0| \\ &\leq \exp\left(\sum_{\nu=\ell_0}^{\nu=n_k(\ell_0,x_0)} L(\nu)\right)|x_0| \\ &\leq \exp\left(\sum_{\nu=\ell_0}^{\nu=n_0(\ell_0,x_0)} L(\nu) + \sum_{\nu=n_0(\ell_0,x_0)}^{\nu=n_k(\ell_0,x_0)} L(\nu)\right)|x_0| \\ &\leq (\exp(m_0(\ell_0)+m))|x_0|, \\ &\quad \forall n = \ell_0, \ell_0+1, \ell_0+2, \ldots, n_k, \\ &\quad \ell_0 \in I, x_0 \in S[0, R']. \end{aligned} \quad (25)$$

Therefore, by (25) and boundedness of I there is a constant $\delta := \delta(\varepsilon, I) < \varepsilon_1$ such that

$$x(n, \ell_0, S[0, \delta]) \in S[0, \varepsilon_1], \; \forall n = \ell_0, \ell_0+1, \ldots, n_k, \ell_0 \in I. \quad (26)$$

From (24) we also obtain

$$|x(n_i, \ell_0, S[0, \delta])| \leq \frac{\varepsilon_1}{2}, \; \forall i = k, k+1, k+2, \ldots, \ell_0 \in I. \quad (27)$$

Thus, from (26) it follows that $x(n, \ell_0, S[0, \delta]) \subset S[0, \varepsilon_1] \subset S[0, \varepsilon]$ for $n = \ell_0, \ell_0+1, \ldots, n_k$. The latter in conjunction with (27) imply that for all positive integers $n = n_k, n_k+1, n_k+2, \ldots$ the following holds:

$$\begin{aligned} x(n, \ell_0, S[0, \delta]) &= x(n, n_k, x(n_k, \ell_0, S[0, \delta])) \\ &\subset x(n, n_k, S[0, \varepsilon_1]) \subset S[0, \varepsilon] \end{aligned}$$

and therefore

$$x(n, \ell_0, S[0, \delta]) \subset S[0, \varepsilon], \forall n = \ell_0, \ell_0+1, \ell_0+2, \ldots, \ell_0 \in I \subset \mathbb{N} \quad (28)$$

and the latter establishes (4).

Attractivity: We show that for any given bounded subset I of $\mathbb{N}$ there exists a constant $0 < \rho \leq R$ in such a way that for every $\varepsilon > 0$ a positive integer $\tau = \tau(\varepsilon, I)$ can be determined such that (5) holds. Due to stability proven above, for every $0 < \xi \leq R$ there exists a strictly positive constant $\rho = \rho(\xi, I) < \xi$ and an arbitrary constant $\varepsilon \in (0, \rho)$ in such a way that $|x(n, \ell_0, x_0)| \leq \xi, \forall n = \ell_0, \ell_0+1, \ldots, \ell_0 \in I, |x_0| \leq \rho$. Also, for every $\varepsilon > 0$ a constant $0 < \varepsilon_1 < \varepsilon$ can be found such that (19a) and (19b) hold and by recalling (23) there exists an integer $k \geq 1$ such that $|x(n_i, \ell_0, S[0, \rho])| \leq \varepsilon_1/2, i = k, k+1, k+2, \ldots, \ell_0 \in I$. Combining the previous inequality together with (19) and (28) we can establish, as in the case of the proof of stability that $x(\nu, \ell_0, S[0, \rho]) \subset S[0, \varepsilon], \forall \nu \geq \ell_0, \ell_0 \in I$. Hence, attractivity is established and we conclude that under A1 and A2 system (1) is AS.

A1, A'2 $\Rightarrow$ AS. In order to establish AS, we use an analogous procedure, under the presence of A'2. For completeness we note that what differs here, is that, instead of estimation (22), we have by taking into account (8b), (9c), (9d), (10) and (17) that there exists a constant $R' \in (0, R]$ such that

$$\begin{aligned} a(|x(n_i)|) &\leq V(n_i, x(n_i)) \\ &\leq V(n_0, x(n_0)) \exp\left(-\sum_{\nu=0}^{\nu=i-1} \sigma_\nu\right) \\ &\leq b\left(\exp\left(\sum_{\nu=\ell_0}^{\nu=n_0} L(\nu)|\right)|x_0|\right) c(n_0) \\ &\quad \times \exp\left(-\sum_{\nu=0}^{\nu=i-1} \sigma_\nu\right) \\ &\leq b(\exp(m_0(\ell_0))|x_0|)\, m_0(\ell_0) \\ &\quad \times \exp\left(-\sum_{\nu=0}^{\nu=i-1} \sigma_\nu\right), \\ &\quad i \in \mathbb{N}, \; x_0 = x(\ell_0) \in S[0, R'], \\ &\quad \ell_0 \in I, \; \ell_0 < n_0. \end{aligned} \quad (29)$$

(ii) **A1, A'2, (11), (12), (13) $\Rightarrow$ expo-AS**. Let us now assume that, in addition to A1 and A'2 conditions (11), (12) and (13) hold. Then by virtue of (17) and (29) we get

$$\begin{aligned} a_0|x(n_i)|^2 &\leq V(n_i, x(n_i)) \leq V(n_0, x(n_0)) \exp(-(i-1)\sigma) \\ &\leq V(n_0, x(n_0)) \exp(-i\lambda(n_i - n_0 + \lambda N)) \\ &\leq b_0|x(n_0)|^2 \exp(-\lambda(n_i - n_0 + \lambda N)) \\ &\leq b_0 \exp(2m(\ell_0) + \lambda N) \exp(-\lambda(n_i - n_0))|x_0|^2, \\ &\quad i \in \mathbb{N}_0, x_0 = x(\ell_0) \in S[0, R'], \\ &\quad \ell_0 \in I, \ell_0 < n_0 \end{aligned} \quad (30)$$

for certain $R' \in (0, R]$ and a strictly positive constant $\lambda \leq \sigma/N$, where N is defined in (13). The desired (6) is an immediate consequence of (18) and (30). Details are left to the reader. The proofs of Statements (iii) are (iv) are quite analogous to those given in [9] and are omitted.■

The result of Proposition 1 can be extended for the case of systems (1), whose dynamics are in general unbounded in time, as follows:

Proposition 2: (i) The same conclusions of Statements (i) and (iii) of Proposition 1 are valid, under same assumptions, and by replacing (9b) by the weaker hypothesis that there exists a function $\xi \in \mathcal{K}$ such that

$$|x(n)| \leq \xi(|x(n_i)|), \; \forall n \in [n_i(\ell_0, x_0), n_{i+1}(\ell_0, x_0)] \quad (31)$$

provided that $x(n) \in S[0, R]$ for all positive integers $n = n_i, n_i+1, \ldots, n_{i+1}, i \in \mathbb{N}_0, x_0 = x(\ell_0)$, where $x(\cdot)$ denotes the solution of (1) initiated from $x(n_i)$ at time n_i.

(ii) The same conclusions of Statements (ii) and (iv) of Proposition 1 are valid, under same assumptions, and by replacing (9b) by the weaker hypothesis that there exists a constant $\Xi > 0$ such that

$$|x(n)| \leq \Xi|x(n_i)|, \; \forall n \in [n_i(\ell_0, x_0), n_{i+1}(\ell_0, x_0)] \quad (32)$$

provided that $x(n) \in S[0, R]$ for all positive integers $n = n_i, n_i+1, \ldots, n_{i+1}, i \in \mathbb{N}_0$ where $x(\cdot)$ denotes the solution of (1) initiated from $x(n_i)$ at time n_i.

Proof: The proof of Proposition 2 is essentially the same with the part of proof of Proposition 1 after (18) plus some elementary appropriate modifications. ■

Remark 2: It should be emphasized here that under stronger hypotheses, all results of Propositions 1 and 2 take a global nature, namely, in addition to (4), attractivity condition (5) is fulfilled with $\rho = +\infty$. This occurs if for instance we assume that (7), as well as rest assumptions imposed in Propositions 1 and 2 are fulfilled with $R = +\infty$.

The following example illustrates the nature of Proposition 2 and Remark 2. Consider the linear case

$$x(n+1) = A(n)x(n),\ (n,x) \in \mathbb{N} \times \mathbb{R}^n \tag{33}$$

where $A : \mathbb{N} \to \mathbb{R}^{n\times n}$, and assume that there exists a constant $\Xi > 0$ a strictly increasing sequence $\{N_i, i \in \mathbb{N}_0\}$ with $N_i \to \infty$, and a function $\mu \in \mathcal{K}$ such that

$$N_{i+1} - N_i \leq \mu(N_i); \tag{34a}$$

$$|A(n)A(n-1)A(n-2)\dots A(N_i+1)A(N_i)| \leq \Xi,\quad \forall n \in [N_i, N_{i+1}],\ \forall i \in \mathbb{N}_0 \tag{34b}$$

and further there exists a sequence $\{\sigma_i \geq 0, i \in \mathbb{N}_0\}$ such that

$$\sum_{i=0}^{\infty} \sigma_i = \infty \tag{35a}$$

$$|A(N_{i+1})A(N_{i+1}-1)A(N_{i+1}-2)\dots A(N_i)| \leq 1 - \sigma_i,\quad i \in \mathbb{N}\ away\ from\ zero. \tag{35b}$$

From (33), (34a), (35a) and (35b) we may easily conclude that A1, (8a), (9a), (9c), (10) and (12) are fulfilled with $R = \infty$, $V = |x|^2$, appropriate constants m_0, m and constant gain $L := L(\cdot)$ and, due to (34b), instead of (9b), (32) holds as well. Specifically, for any given $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{N}$ each term n_i of the sequence involved in A2 depends in our case on the intial value of time ℓ_0; particularly, is defined as $n_0 := N_{\bar{k}} := \min\{N_k, k \in \mathbb{N}_0\ such\ that\ N_k \geq \ell_0\}$ and $n_i := N_{\bar{k}+i}, i \in \mathbb{N}$ The desired (9c) is a consequence of (34a). We conclude, according to third statement of Proposition 2 that system (33) is UAS. Particularly, we may show, according to Remark 2, that (33) is *globally* UAS, namely, in addition to stability, implication (5) is fulfilled with $\rho = +\infty$ and for certain $\tau \in \mathbb{N}$ being independent of I.

III. Application to Feedback Stabilization

This section is devoted to some applications of Proposition 1 to the feedback stabilization problem for systems (2). For simplicity, we consider the single-input case and assume that (2) takes the form

$$\begin{aligned} x(n+1) &= F(n, x(n), u(n)) \\ &:= f(n, x(n)) + u(n)g(n, x(n)) \\ &\quad + h(n, x(n), u(n)) \end{aligned} \tag{36}$$

where $f, g : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{N} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfy

$$|g(n,x)| \leq C,\ \forall n \in \mathbb{N},\ x\ near\ zero \tag{37a}$$

$$|h(n,x,u)| \leq C|u|^2, \forall n \in \mathbb{N}, x \in \mathbb{R}^n, u \in \mathbb{R}\ near\ zero \tag{37b}$$

for certain constant $C > 0$. Moreover, we make the following hypotheses:

(**H1**) There exists an inner product $\langle \cdot,\cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, if we denote $\|v\| := \langle v, v\rangle^{1/2}$, then

$$\|f(n,x)\| \leq \|x\|,\ \forall n \in \mathbb{N},\ x \in \mathbb{R}^n\ near\ zero. \tag{38}$$

(**H2**) There exists an integer $N > 1$ and a function $\varsigma \in \mathcal{K}$ such that for every integer $n \in \mathbb{N}$ and nonzero $x \in \mathbb{R}^n$ for which $\langle f(n,x), g(n,x)\rangle = 0$ there exists an integer $k : n < k \leq n+N$ in such a way that

$$\begin{aligned} &\langle f(i, f(i-1, \dots, f(n,x), \dots)),\\ &\quad g(i, f(i-1, \dots, f(n,x), \dots))\rangle = 0,\\ &\quad i = n+1, n+2, \dots, k-1 \end{aligned} \tag{39a}$$

$$\begin{aligned} &|\langle f(k, f(k-1, \dots, f(n,x), \dots)),\\ &\quad g(k, f(k-1, \dots, f(n,x), \dots))\rangle|^2 \geq \varsigma(|x|). \end{aligned} \tag{39b}$$

The following proposition generalizes [8, Proposition 2.4]:

Proposition 3: Under previous assumptions there exists a constant $\varepsilon > 0$ such that the map

$$u = u(n,x) := -\varepsilon \langle f(n,x), g(n,x)\rangle \tag{40}$$

exhibits (uniform in time) local asymptotic stabilization of (36), namely, the closed-loop (36) with (40) is UAS.

Proof: Consider the closed-loop dynamics

$$E(n,x) := f(n,x) + ug(n,x) + h(n,x,u)|_{u=-\varepsilon\langle f(n,x), g(n,x)\rangle}. \tag{41}$$

Let $x_0 \neq 0$ and $\ell_0 \in \mathbb{N}$ and consider the increasing sequence $\{n_i = n_i(\ell_0, x_0), i \in \mathbb{N}_0\}$ with $n_0 := \ell_0$ and $n_{i+1} - n_i \leq N, i \in \mathbb{N}_0$ for every $x \in \mathbb{R}^n$ and in such a way that for every $x \neq 0$ near zero, either

$$|\langle f(n_i,x), g(n_i,x)\rangle|^2 \geq \varsigma(|x|)\ and\ n_{i+1} = n_i + 1 \tag{42}$$

or

$$\begin{aligned} &\langle f(n_i,x), g(n_i,x)\rangle = 0;\\ &\langle f(k, f(k-1, \dots, f(n_i,x), \dots)),\\ &\quad g(k, f(k-1, \dots, f(n_i,x), \dots))\rangle = 0,\\ &k = n_i+1, n_i+2, \dots, n_{i+1}-1 \end{aligned} \tag{43a}$$

$$\begin{aligned} &\langle f(n_{i+1}, f(n_{i+1}-1, \dots, f(n_i,x), \dots,)),\\ &\quad g(n_{i+1}, f(n_{i+1}-1, \dots, f(n_i,x), \dots))\rangle \geq \varsigma(|x|) \end{aligned} \tag{43b}$$

Existence of sequence n_i above is guaranteed from hypothesis (H2). Without any loss of generality assume that (43) holds for all $x \in \mathbb{R}^n$. Then by taking into account (37b), (38), (40), (41) and (43a) we find

$$\|E(n_i,x)\|^2 = \|f(n_i,x)\|^2 \leq \|x\|^2$$

and by induction

$$\begin{aligned} &\|E(k, E(k-1, \dots, E(n_i,x), \dots))\|^2\\ &\quad = \|f(k, f(k-1, \dots, f(n_i,x), \dots))\|^2 \leq \|x\|^2\\ &for\ k = n_i+1, \dots, n_{i+1}-1,\ x\ near\ zero. \end{aligned} \tag{44}$$

Also, by taking into account (37a), (37b) and (43b) we get

$$\begin{aligned} &\|E(n_{i+1}, E(n_{i+1}-1, \dots, E(n_i,x), \dots))\|^2\\ &\quad = \|f(n_{i+1}, f(n_{i+1}-1, \dots, f(n_i,x), \dots))\|^2\\ &\quad\ - \varepsilon R\,|\langle f(n_{i+1}, f(n_{i+1}-1, \dots, f(n_i,x), \dots)),\\ &\qquad\quad g(n_{i+1}, f(n_{i+1}-1, \dots, f(n_i,x), \dots))\rangle|^2\\ &\quad \times\left(2 - \varepsilon\,|g(n_{i+1}, f(n_{i+1}-1, \dots, f(n_i,x), \dots))|^2 + \varepsilon\rho\right) \end{aligned} \tag{45}$$

for appropriate $\varepsilon > 0$ near zero, certain $\rho > 0$ and x near zero. It turns out from (37a), (38), (43b) and (45) that there is a constant $\bar{C} > 0$ such that

$$\|E\left(n_{i+1}, E\left(n_{i+1}-1, \ldots, E(n_i, x), \ldots\right)\right)\|^2 \leq \|x\|^2 - \varepsilon \bar{C} \zeta\left(|x|\right), \; x \; near \; zero \quad (46)$$

and for sufficiently small $\varepsilon > 0$. Also, by taking into account (37), (38), (40) and (41) a constant $L > 0$ can be found such that $|E(n,x)| \leq L|x|$ for x and $\varepsilon > 0$ near zero. We now may conclude that all conditions (7), (11)–(13) hold with constant gain L, $V = \|x\|^2$, $r(s) = \varepsilon \bar{C}\zeta(s)$, $\sigma_i = \sigma := 1$ and $m_0 = m = LN$, therefore, according to Proposition 1(iii) system $x(n+1) = E(n, x(n))$ is UAS. ■

Another interesting case arises for affine in the control systems (2), where again, for reasons of simplicity we consider here the single-input case

$$x(n+1) = F\left(n, x(n), u(n)\right) := f\left(n, x(n)\right) + u(n) g\left(n, x(n)\right) \quad (47)$$

where $f, g : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy

$$|f(n,x)| \leq C|x| \; and \; |g(n,x)| \leq C, \; \forall n \in \mathbb{N}, \; x \; near \; zero \quad (48)$$

for certain constant $C > 0$ and we make the following assumption:

(**H**) There exist an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a function $r \in \mathcal{K}$ and an integer $N > 1$ such that, if we denote $\|v\| := \langle v, v \rangle^{1/2}$, then for every $x \neq 0$ near zero and integer $n \in \mathbb{N}$, one of the following properties hold:

- $g(n,x) = 0$ and there exists an integer $k : n < k \leq n + N$ in such a way that

$$g\left(i, f\left(i-1, \ldots, f(n,x), \ldots\right)\right) = 0; \quad i = n+1, n+2, \ldots, k-1 \quad (49a)$$

$$g\left(k, f\left(k-1, \ldots, f(n,x), \ldots\right)\right) \neq 0 \quad (49b)$$

- $g(n,x) \neq 0$ and

$$\|f(n,x)\|^2 - \frac{\langle f(n,x), g(n,x)\rangle^2}{\|g(n,x)\|^2} \leq \|x\|^2 - r\left(|x|\right). \quad (50)$$

Proposition 4: Under previous assumptions the feedback law defined as

$$u = u(n,x) := \begin{cases} = 0, & if \; either \; g(n,x) = 0 \; or \; x = 0 \\ \frac{= \langle f(n,x), g(n,x)\rangle}{\|g(n,x)\|^2}, & if \; (n,x) \neq 0 \end{cases} \quad (51)$$

exhibits (uniform in time) local asymptotic stabilization of (47).

Proof: The proof is similar to that given in Proposition 3. For completeness, we note that the closed-loop system (47) with (51) satisfies (7), (9)–(13) with certain constant gain $L(\cdot) = L$, $V = \|x\|^2$, $m_0 = m = LN$, where N, ρ and $r(\cdot)$ as given in Hypothesis (H). Therefore, according to Proposition 1(iii) the closed-loop (47) with (51) is UAS. Details are left to the reader.

IV. Averaging

In this section we use the result derived in Section II to get an averaging-type sufficient condition for local asymptotic stability for systems (3). We make the following assumptions:

B1. We assume that zero is an equilibrium

$$f(\varepsilon, n, 0) = 0, \; \forall n \in \mathbb{N}, \varepsilon > 0 \quad (52a)$$

and there exists constant $R > 0$ and a function $L \in \mathcal{N}$ such that

$$|f(\varepsilon, n, x_1) - f(\varepsilon, n, x_2)| \leq L(n)|x_1 - x_2|, \quad \forall (n, x_i) \in \mathbb{N} \times S[0,R], \; i = 1, 2, \varepsilon > 0 \; near \; zero. \quad (52b)$$

B2. Moreover, assume that there exist constants $m, c > 0$, a map $f_{av}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ with $f_{av}(0) = 0$, sequences $\{N_i \in \mathbb{N}, i \in \mathbb{N}_0\}$ and $\{c_i \in \mathbb{R}^+, i \in \mathbb{N}_0\}$, the first being strictly increasing, and a sequence of functions $\{T_i = T_i(x) : \mathbb{R}^n \to N, i \in \mathbb{N}_0\}$ such that

$$N_i \to \infty \text{ as } i \to \infty \quad (53a)$$

$$\sum_{\nu=N_i}^{\nu=N_{i+1}} L(\nu) \leq m, \; \forall i \in \mathbb{N}_0 \quad (53b)$$

$$c_i \leq T_i(x) \leq m, \; \forall i \in \mathbb{N}, x \in S[0,R] \quad (53c)$$

$$\underline{\lim}_{i \to \infty} \frac{1}{i} \sum_{j=0}^{j=i} c_j \geq c \quad (53d)$$

and in such a way that for every constant $\xi > 0$ there is an integer $\bar{n} \in \mathbb{N}$ such that

$$\left| f_{av}(x) - \frac{1}{T_i(x)} \sum_{\nu=N_i}^{\nu=N_{i+1}-1} f(\varepsilon, \nu, x) \right| \leq |x|\xi, \quad for \; all \; i \in \mathbb{N}, i \geq \bar{n}, x \in S[0,R] \; and \; \varepsilon > 0 \; near \; zero \quad (53e)$$

B3. For the map f_{av} introduced in B2 we assume that there exist a function $V(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and positive constants C_i, $i = 0, 1, 2, 3, 4$ in such a way that for all $x, y \in S[0,R]$ and $\varepsilon > 0$ near zero the following hold:

$$|f_{av}(x)| \leq C_0|x|, \quad (54a)$$

$$C_1|x|^2 \leq V(x) \leq C_2|x|^2, \quad (54b)$$

$$V(x) - V(y) \leq C_3|x-y|\left(|x| + |y|\right), \quad (54c)$$

$$\overline{\lim}_{\varepsilon \to 0^+} \frac{V\left(x + \varepsilon f_{av}(x)\right) - V(x)}{\varepsilon} \leq -C_4|x|^2. \quad (54d)$$

Remark 3: (i) Condition B3 is fulfilled, if for instance we assume that f_{av} is C^1 and zero is exponentially stable with respect to $\dot{x} = f_{av}(x)$ (see for instance [6]). (ii) Condition (53e) is a weaker version of the familiar averaging assumption used in the existing works in the literature (see [1], [4], [5]).

Proposition 5: (i) For system (3), assume that B1–B3 are fulfilled. Then hypotheses A1 and A'2 are fulfilled for (3) for $\varepsilon > 0$ near zero, hence, by Proposition 1(i), for each $\varepsilon > 0$ near zero, the corresponding system (3) is AS; (ii) If in addition we assume that there exist a constant $c > 0$ and an integer N with

$$c_i \geq c, \; \forall i \in \mathbb{N} \quad (55)$$

$$N_{i+1} - N_i \leq N, \; \forall i \in \mathbb{N}_0 \quad (56)$$

then all assumptions of Proposition 1(iv) are satisfied, therefore, for $\varepsilon > 0$ near zero, (3) is expo-UAS.

Proof: **(i)B1, B2, B3 ⇒ AS**. We establish that for $\varepsilon > 0$ near zero, all hypotheses of first statement of Proposition 1(i) are fulfilled for (3). For any initial $\ell_0 \in \mathbb{N}$, consider the sequence

$$\{n_i = n_i(\ell_0), i \in \mathbb{N}\} \; with \; n_0 = n_0(\ell_0) := N_{\bar{k}} = \min\left\{N_k, k \in \mathbb{N}_0 : N_k \geq \ell_0\right\}, \; n_i := N_{\bar{k}+i}, i \in \mathbb{N}_0. \quad (57)$$

Without loss of generality, assume next that $n_{i+1} > n_i + 1, \forall i \in \mathbb{N}_0$. By (3) and (52) the trajectory $x(n) = x(n, n_i, x(n_i))$ of (3) satisfies

$|x(n_i+1)| \leq |x(n_i)|(1+\varepsilon L(n_i))$, thus by induction and taking into account (52), (53b) and (57) we get:

$$|x(n)| \leq \left(\prod_{\nu=n_i}^{\nu=n-1}(1+\varepsilon L(\nu))\right)|x(n_i)| \leq \exp\left(\varepsilon \sum_{\nu=n_i}^{\nu=n-1} L(\nu)\right)|x(n_i)| \leq \exp(\varepsilon m)\,|x(n_i)|$$

therefore for $n = n_i+1, n_i+2, \ldots, n_{i+1}$ it holds

$$|x(n)| \leq K\,|x(n_i)|\,, K := \exp(\varepsilon m),\ \varepsilon > 0\ near\ zero \tag{58a}$$

provided that

$$x(\nu) \in S[0,R]\ for\ all\ \nu = n_i, n_i+1, \ldots, n_{i+1}, i \in \mathbb{N}_0. \tag{58b}$$

We also may show that

$$|x(n)-x(n_i)| \leq \varepsilon K m\,|x(n_i)|\,, \quad \forall\, n = n_i, n_i+1, \ldots, n_{i+1}, \varepsilon > 0\ near\ zero \tag{59}$$

provided that (58b) holds. Indeed, by (3) and (52), we obtain $|x(n+1)-x(n)| \leq \varepsilon L(n)|x(n)|$, $\forall n : n_i \leq n \leq n_{i+1}$, provided that $x(n) \in S[0,R]$ for $n : n_i \leq n \leq n_{i+1}$. It then follows by taking into account (53b):

$$\begin{aligned}
&|x(n)-x(n_i)| \\
&\quad \leq |x(n)-x(n-1)| + |x(n-1)-x(n-2)| + \ldots \\
&\qquad + |x(n_i+2)-x(n_i+1)| + |x(n_i+1)-x(n_i)| \\
&\quad \leq \varepsilon L(n-1)\,|x(n-1)| + \varepsilon L(n-2)\,|x(n-2)| + \ldots \\
&\qquad + \varepsilon L(n_i+1)\,|x(n_i+1)| + \varepsilon L(n_i)\,|x(n_i)| \\
&\quad \leq \varepsilon\left(L(n-1)\exp\left(\varepsilon\sum_{\nu=n_i}^{\nu=n-1}L(\nu)\right)\right. \\
&\qquad\qquad + L(n-2)\exp\left(\varepsilon\sum_{\nu=n_i}^{\nu=n-2}L(\nu)\right) + \ldots \\
&\qquad\qquad \left. + L(n_i+1)\exp\left(\varepsilon\sum_{\nu=n_i}^{\nu=n_i+1}L(\nu)\right) + L(n_i)\right) \\
&\qquad \times |x(n_i)| \\
&\quad \leq \varepsilon\left(\sum_{\nu=n_i}^{\nu=n-1}L(\nu)\right)\exp\left(\varepsilon\sum_{\nu=n_i}^{\nu=n-1}L(\nu)\right)|x(n_i)| \\
&\qquad for\ n = n_i+1, n_i+2, \ldots, n_{i+1}.
\end{aligned} \tag{60}$$

By (53b), (57) and (60) we get (59), which in conjunction with (52) and (58), imply

$$\sum_{\nu=n_i}^{\nu=n_{i+1}-1}|f(\varepsilon,\nu,x(\nu))| \leq \sum_{\nu=n_i}^{\nu=n_{i+1}-1}L(\nu)\,|x(\nu)| \leq mK\,|x(n_i)| \tag{61a}$$

$$\sum_{\nu=n_i}^{\nu=n_{i+1}-1}|f(\varepsilon,\nu,x(n_i))| \leq \sum_{\nu=n_i}^{\nu=n_{i+1}-1}L(\nu)\,|x(n_i)| \leq m\,|x(n_i)| \tag{61b}$$

$$\left|\sum_{\nu=n_i}^{\nu=n_{i+1}-1}(f(\varepsilon,\nu,x(\nu)) - f(\varepsilon,\nu,x(n_i)))\right| \leq \varepsilon mK\sum_{\nu=n_i}^{\nu=n_{i+1}-1}L(\nu)\,|x(n_i)| \leq \varepsilon m^2 K\,|x(n_i)| \tag{61c}$$

provided that (58b) holds. Now, define

$$\xi := \frac{C_4}{2C_3(1+\varepsilon m)(1+\varepsilon m C_0)} \tag{62}$$

and suppose that (53e) holds with this ξ and $N_i := n_i$. We are in position to show that all assumptions of first claim of Proposition 1(i) are fulfilled for sufficiently small $\varepsilon > 0$, with $V(\cdot)$ as given in B3. We first show that for each $\varepsilon > 0$ near zero the trajectories of system (3) satisfy (10). Indeed, we evaluate

$$V(x(n_{i+1})) - V(x(n_i)) \leq \Xi_1(\varepsilon,i) + \Xi_2(\varepsilon,i) + \Xi_3(\varepsilon,i),\ i \in \mathbb{N}_0 \tag{63a}$$

$$\Xi_1(\varepsilon,i) := V(x(n_{i+1})) - V\left(x(n_i) + \varepsilon\left(\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(n_i))\right)\right), \tag{63b}$$

$$\Xi_2(\varepsilon,i) := V\left(x(n_i) + \varepsilon\left(\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(n_i))\right)\right) - V(x(n_i) + \varepsilon T_i f_{av}(x(n_i)))\,, \tag{63c}$$

$$\Xi_3(\varepsilon,i) := V(x(n_i) + \varepsilon T_i f_{av}(x(n_i))) - V(x(n_i))\,. \tag{63d}$$

We first estimate an upper bound $|\Xi_1(\varepsilon,i)|$. By (54c) we get

$$\begin{aligned}
|\Xi_1(\varepsilon,i)| \leq\ & C_3\left|x(n_{i+1}) - \left(x(n_i) + \varepsilon\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(n_i))\right)\right| \\
& \times\left(|x(n_{i+1})| + \left|x(n_i) + \varepsilon\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(n_i))\right|\right)
\end{aligned} \tag{64}$$

and since $x(n_{i+1}) = x(n_i) + \varepsilon(\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(\nu)))$ it follows from (52b), (59), (61a)–(61c) and (64):

$$\begin{aligned}
|\Xi_1(\varepsilon,i)| &\leq \varepsilon C_3\left|\sum_{\nu=n_i}^{\nu=n_{i+1}-1}(f(\varepsilon,\nu,x(\nu)) - f(\varepsilon,\nu,x(n_i)))\right| \\
&\quad \times\left(\left|x(n_i) + \varepsilon\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(\nu))\right|\right. \\
&\quad \left. + \left|x(n_i) + \varepsilon\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(n_i))\right|\right) \\
&\leq \varepsilon^2 C_3 m^3 K\left[(1+\varepsilon mK) + (1+\varepsilon m)\right]|x(n_i)|^2
\end{aligned}$$

which implies the existence of a constant $\vartheta > 0$ such that

$$|\Xi_1(\varepsilon,i)| \leq \varepsilon^2\vartheta\,|x(n_i)|^2\,,\ for\ all\ i \in \mathbb{N}_0\ away\ from\ zero \tag{65}$$

provided that (58b) holds.

Likewise, we find an upper bound for (63c). By (53b), (53c), (53e), (54a), (54c), (57) and (61b) we obtain

$$\begin{aligned}
|\Xi_2(\varepsilon,i)| &\leq \varepsilon C_3 T_i(x(n_i))\,\xi\,|x(n_i)| \\
&\quad \times\left(\left|x(n_i) + \varepsilon\left(\sum_{\nu=n_i}^{\nu=n_{i+1}-1} f(\varepsilon,\nu,x(n_i))\right)\right|\right. \\
&\quad \left. + |x(n_i) + \varepsilon T_i(x(n_i))\,f_{av}(x(n_i))|\right) \\
&\leq \varepsilon T_i(x(n_i))\,C_3\xi\left[(1+\varepsilon m) + (1+\varepsilon m C_0)\right]|x(n_i)|^2 \\
&\leq \frac{1}{2}\varepsilon T_i(x(n_i))\,C_4\,|x(n_i)|^2
\end{aligned} \tag{66}$$

provided that (58b) holds for all $i \in \mathbb{N}_0$. Finally, to get an upper bound estimation for (63d), we invoke (53c) and (54d), which imply

$$\overline{\lim_{\varepsilon\to 0+}} \frac{V\left(x(n_i)+\varepsilon T_i\left(x(n_i)\right) f_{av}\left(x(n_i)\right)\right)-V\left(x(n_i)\right)}{\varepsilon T_i\left(x(n_i)\right)} \le -C_4\left|x(n_i)\right|^2$$

therefore

$$\Xi_3(\varepsilon, i) \le -\varepsilon T_i\left(x(n_i)\right) C_4 \left|x(n_i)\right|^2 \tag{67}$$

for $\varepsilon > 0$ near zero, provided that (58b) holds. We conclude from (65), (66) and (67) that

$$\begin{aligned} &V\left(x(n_{i+1})\right) - V\left(x(n_i)\right) \\ &\quad \le \varepsilon^2 \vartheta \left|x(n_i)\right|^2 - \frac{1}{2}\varepsilon T_i\left(x(n_i)\right) C_4 \left|x(n_i)\right|^2, \\ &\quad\quad \varepsilon > 0 \; near\ zero,\; i \in \mathbb{N}_0 \; away\ from\ zero \end{aligned} \tag{68}$$

provided that (58b) holds. It follows by taking into account (53c), (54b) and (68):

$$V\left(x(n_{i+1})\right) - V\left(x(n_i)\right) \le -\sigma_i V\left(x(n_i)\right); \sigma_i := -\varepsilon^2 \frac{\vartheta}{C_1} + \varepsilon \frac{C_4}{2C_2} c_i \tag{69}$$

provided that (58b) holds, for all $i \in \mathbb{N}_0$ away from zero, which establishes (10). Also, notice that

$$\sum_{j=0}^{j=i} \sigma_j \ge i\left(-(i+1)\frac{\varepsilon^2 \vartheta}{iC_1} + \frac{\varepsilon C_4}{2C_2}\left(\frac{1}{i}\sum_{j=0}^{j=i} c_j\right)\right) \tag{70}$$

which by virtue of (53d) guarantees existence of a constant $\varepsilon^* > 0$ sufficiently small, such that (8a) holds as well. It can be easily verified that rest assumptions in A1 and A'2 are fulfilled, hence, according to Proposition 1(i), (3) is AS for every $\varepsilon \in (0, \varepsilon^*)$. Proof of statement (ii) is similar to the procedure above, plus some appropriate modifications, and is left to the reader. ■

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Consensus Over Numerosity-Constrained Random Networks

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***Abstract*—We analyze the discrete-time consensus problem for a group of agents that communicate through a stochastic directed network with fixed out-degree. This network construction models the perceptual phenomenon of numerosity observed in animal groups exhibiting collective behavior. We find necessary and sufficient conditions for mean square consentability of the averaging protocol and we derive a closed form expression for the asymptotic convergence factor. Analytical results are illustrated through simulations.**

***Index Terms*—Consensus, network analysis and control, random graphs, stability of linear systems, stochastic systems.**

I. INTRODUCTION

Collective behavior of animal groups, such as schools of fish and flocks of birds, is characterized by coordinated maneuvers of individuals with matched velocities. These large-scale social actions provide security from predators and efficient movement to the entire community, see for example [24]. Since no single member of the group acts as a permanent leader, the group's ability to make decisions on a fast time scale can be interpreted as the result of local compromises among group mates. The salient features of animal grouping have been successfully condensed into consensus problems, see for example [17]. Consequently, the mechanics of coordination in animal groups can be viewed as distributed averaging over networks of coupled agents. The topology of the underlying network describes the information flow among the agents and thus dictates their ability to achieve consensus.

Information flow among group mates is generally affected by the so-called numerosity that quantifies a critical limit to the species perception of natural numbers. Groups with cardinality less than the numerosity are perceived by an individual as a specific collection, while groups of more than this limit are perceived only as "many". This limit varies among species ranging in complexity from fish [2] to humans [23]. The phenomenon of numerosity has been found to contribute to collective responses of fish schools [1], [29] and bird flocks [5] by setting a limit to the individual perception of the animal group size. Numerosity acts as a further constraint to the group geometry that practically dictates possible communication among the mates through physical cues, including vision and flow, see for example [1]. In this technical note, we imbed the biologically-observed phenomenon of perceptual numerosity in a consensus problem towards establishing a first analytical understanding of the impact of psychological factors on collective behavior of animal groups. More specifically, we consider the consensus problem over a stochastic network in which every agent receives information from n agents, defined as its neighbors, that are randomly

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