2010 8th IEEE International Conference on
Control and Automation
Xiamen, China, June 9-11, 2010

Necessary and Sufficient Conditions for Asymptotic Swarm Stability of High Order Swarm Systems

Ning Cai, Jianxiang Xi, and Yisheng Zhong

Abstract—**In this paper, the asymptotic swarm stability problem of high order swarm systems is dealt with. The concept of swarm stability is formally defined based on consensus. Necessary and sufficient conditions for asymptotic swarm stability are respectively presented for two types of swarm systems. The first type is LTI, while the second is nonlinear time-varying. For general LTI systems, the condition depends on the graph topology, the dynamics of agents and the interactions among the neighbors; while for curtain nonlinear time-varying systems, the condition is determined by two independent aspects: agent attraction and graph connection.**

I. Introduction

For systems composed of interacting multi-agents, or "swarm systems", the concept "consensus" means to asymptotically reach an agreement over certain variables of interest determined by the states of all agents [1]-[3].

The consensus problems originated in computer science long ago and in some sense formed the foundation of distributed computing [4]. Since about five years ago, consensus has become a major topic in the study of swarm systems. Olfati-Saber *et al.* first introduced the term "consensus" into control theory, and they have performed much inaugurating research work on it. For instance [1], they studied some first order swarm systems and discovered that strong connection of the digraph is a sufficient condition for consensus achievement. Ren *et al.* [2] relaxed the condition in [1] and proved that the requirement for the digraph to include a spanning tree is necessary and sufficient. Based on a different approach, Moreau [3] presented a condition for the consensus of curtain swarm system models. Until about 2007, the majority of the researchers studying consensus problems had dealt with first order models without agent dynamics since the analysis of high order swarm systems is much more involved and challenging. Recently there have appeared a few papers on the consensus problems for more general high order swarm systems. For instance, Xiao *et al.* [5] proposed a criterion by checking the structure of curtain high dimensional matrices. Ren *et al.* [6] gave a necessary and sufficient condition relying on the spectrum of a high dimensional matrix and Wieland *et al.* [7] gave a criterion by checking whether a given polynomial is Hurwitz. Wang *et al.* [8] endeavored to determine whether an appropriate linear high order consensus algorithm exists under a given undirected graph. Li *et al.* [9] studied the robust stability problem of linear swarm systems with observer type agent interactions.

The current paper will mainly focus on the asymptotic stability problem of high order swarm systems. Based on the concept of consensus, a formal definition of swarm stability will be given, which is clearly different from Lyapunov stability. Such a definition helps to stress that consensus is essentially a stability problem. Two types of swarm systems will be concerned. The systems of first type are general LTI swarm systems and the second are a specific type of nonlinear time-varying swarm systems, namely compartmental systems. Primary contribution of this paper is the presentation of necessary and sufficient conditions for asymptotic swarm stability of both the two types of models.

By far, most of the linear high order swarm system models studied by other scholars on consensus problems are more restricted than the one in the current paper. For some of those models, the matrices of coefficients in the differential equations bear specific structures [6]-[7], while ours demands no structure limitation. The graph topology of some models is undirected [8], while ours is directed. The linear model concerned in the current paper is general LTI homogeneous swarm system and most of the other models in literature are its special cases.

Most of the necessary and sufficient conditions for consensus of linear systems in the literature depend upon some technical assumptions. The models studied by Ren *et al.* [6] and Wieland *et al.* [7] have specific structure analogous to controllable canonical form. The linear model studied by Wang [8], Xiao [5] and Li [9] is similar with ours, while they all assumed that the graph must include a spanning tree. Xiao [5] even assumed that the dynamics of agents must be stable. Actually, these assumptions are not necessary. In the current paper, the necessary and sufficient condition for LTI swarm systems depends upon no accessional assumption.

Few papers in the literature have discussed the consensus problem of high order nonlinear and/or time-varying models. For these models, the conventional technique of Laplacian spectrum analysis is no longer applicable. On the basis of local convex analysis and set-valued Lyapunov function,

Manuscript received October 29, 2009. This work was supported by the National Natural Science Foundation of China under Grants 60674014 and 60736017.

Ning Cai, Jianxiang Xi, and Yisheng Zhong are with the Department of Automation, School of Information Science and Technology, Tsinghua University, Beijing, 100084, P. R. China (phone: 86-010-62790793; e-mail: cain04@mails.tsinghua.edu.cn).

978-1-4244-5196-8/10/$26.00 ©2010 IEEE

Moreau [3] devised an approach to deal with such models; however, his technical assumption that the motion of any agent is toward the convex hull formed by its neighbors is strict, because many swarm systems that can achieve consensus do not satisfy this assumption. Later, there appeared a few papers [10]-[11] toward extending the results in [3], whereas the essential limitation of this approach has not been removed. In this paper, by a different technique based on relative Lyapunov function, it will be shown that for curtain high order nonlinear systems, the attractive interaction and the joint connection of graph can assure the asymptotic swarm stability.

This paper is organized as follows. In Section 2, both the two types of swarm systems that will be dealt with are introduced. Besides, the asymptotic swarm stability problem is described in this section. In Section 3, a necessary and sufficient condition is given for asymptotic swarm stability of the LTI system. A necessary and sufficient condition for global uniform asymptotic swarm stability of the nonlinear time-varying system is presented in Section 4. Section 5 shows some numerical examples. Finally, Section 6 concludes this paper.

II. Description About the Problem and the Models

Some phenomena investigated by the researchers, e.g. consensus achievement, flocking, and formation keeping, require a swarm system to have curtain property of stability. A bird flock, vehicle platoon or robot crew may navigate to anywhere, meanwhile the relative positions among its members should not go unbounded. The states of agents in these systems might still oscillate or even diverge. For swarm systems, due to their structural characteristic, the concept of stability ought to be redefined concerning the relative motions in order to differentiate it from Lyapunov stability of isolated systems. Actually, it has been a common knowledge that the stability of swarm systems means cohesion. Our definitions of swarm stability below are consistent with this idea. In the current paper, only the global uniform asymptotic swarm stability is concerned, which is independent of the initial states and time.

Definition 1: (Swarm Stability) For a swarm system that may be nonlinear and/or time-varying with $x_1, x_2, ..., x_N \in R^d$ the states of N agents, if for $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, *s.t.* $\|x_i(t) - x_j(t)\| < \varepsilon$ ($t > 0$) as $\|x_i(0) - x_j(0)\| < \delta(\varepsilon)$ ($\forall i, j \in \{1, 2, ..., N\}$), then the swarm system is uniformly *swarm stable*. If $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$, the system is globally uniformly swarm stable.

Definition 2: (Asymptotic Swarm Stability) If a swarm system is globally uniformly swarm stable and for $\forall \varepsilon, c > 0$ $\exists T(\varepsilon, c) > 0$ *s.t.* if $\|x_i(0) - x_j(0)\| < c$ ($\forall i, j \in \{1, 2, ..., N\}$), then $\|x_i(t) - x_j(t)\| < \varepsilon$ ($\forall i, j \in \{1, 2, ..., N\}$) as $t > T(\varepsilon, c)$, the system is globally uniformly *asymptotically swarm stable*.

Remark 2.1: For a swarm system that may be nonlinear and/or time-varying, global uniform asymptotic swarm stability is equivalent to full state consensus.

Remark 2.2: If a swarm system is nonlinear, then there should also be different styles of swarm stability such as local swarm stability, regional swarm stability and global swarm stability concerning different initial relative states of agents. A nonlinear system which is locally or regionally swarm stable might not be globally swarm stable.

Such a problem will be dealt with in the remaining part of this paper: how to check the global uniform asymptotic swarm stability of the two types of high order swarm systems? The first type is LTI, while the second is nonlinear and time-varying.

A swarm system considered in this paper has N agents of d th order. The state of agent i is denoted by $x_i \in R^d$. The configuration of information flow among agents is represented by a graph G of order N, with each agent corresponding to a vertex. The edge weight of G between agent i and j is denoted by $w_{ij} \geq 0$, which can be regarded as the strength of information link. If $w_{ij} > 0$, then agent j is a neighbor of agent i. The graph can be denoted by its adjacency matrix W:

$$G : W = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1N} \\ w_{21} & w_{22} & \dots & w_{2N} \\ \dots & \dots & \dots & \dots \\ w_{N1} & w_{N2} & \dots & w_{NN} \end{bmatrix}$$

If G is undirected, W is symmetric; otherwise W may be asymmetric if G is a directed graph or digraph. In the current paper, it is assumed that if G is time-varying, then $W(t)$ is continuous in t.

A. LTI Model

In the LTI model, suppose the dynamics of each agent can be described as:

$$\dot{x}_i = Ax_i + F\sum_{j=1}^{N} w_{ij}(x_j - x_i), \ i \in \{1, 2, ..., N\} \tag{1}$$

with $A \in R^{d \times d}$ and $F \in R^{d \times d}$. If define the stack state vector of the system as $x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T$, then the system dynamics can be described by

$$\dot{x} = (I_N \otimes A - L \otimes F)x \tag{2}$$

In the equation above, $L = L(G)$ is the Laplacian matrix [12] of the graph G.

In this model, A indicates the dynamics of any agent. If without any neighbor, the dynamics of an agent will satisfy $\dot{\xi} = A\xi$. The matrix F indicates the interactive dynamics among the neighboring agents. It intuitively corresponds to the attraction/repulsion relationship [13] in a first order system: e.g. if F is Hurwitz, in an unforced swarm system this relationship can be regarded as being attractive. One can see the complexity of high order swarm systems in comparison with the typical first order systems such as $\dot{x}_i = k\sum_{j=1}^{N} w_{ij}(x_j - x_i)$, e.g. in [1]-[2].

B. Nonlinear Time-varying Model

In the second model, the dynamics of each agent can be described as:

$$\dot{x}_i = \sum_{j=1}^{N} w_{ij}(t) f_{ij}(x_i, x_j, t) \quad (i \in \{1,2,...,N\}) \tag{3}$$

where $w_{ij}(t) = w_{ji}(t) \geq 0$ and the graph is undirected. The function $f_{ij}(x_i, x_j, t) \in R^d$ stands for the nonlinear interaction between agents i and j.

One of the main differences between systems (3) and (1) is that system (3) lacks the agent dynamics, which is represented by Ax_i in system (1). The state of any agent of system (3) will be constant if the agent has no neighbor. A background of model (3) is compartmental system, which has been widely studied in biological, chemical and economic fields. A compartmental system [14] is a kind of swarm system comprised of interconnected agents called compartments, each containing some substance or information of interest, and the substance flows from neighbor to neighbor. A compartmental system being of d th order implies that the agents contain d different types of substance, where any type of substance may be transformed into other types during the flowing process. An undirected graph can be interpreted by that if some substance flows from agent i to agent j, then both the quantities of substance in two agents must be simultaneously affected.

III. Asymptotic Swarm Stability of LTI High Order Swarm Systems

Apparently, the swarm stability of a swarm system is at least jointly determined by two aspects: 1) the interactive algorithm or the relative dynamics among the neighbors; 2) the graph topology. As to the first aspect, there must be some attraction mechanism between any neighboring pair. Without attraction, the agents might depart away from each other. As to the second aspect, the graph topology must be sufficiently connected. Without appropriate connection, two different agents might never have any information exchange, either directly or indirectly. Usually these two aspects are not independent with each other. One will see this from the subsequent analysis. The main purpose of this section is to expound a necessary and sufficient condition for asymptotic swarm stability of LTI swarm systems. Several lemmas should first be introduced to support the main result.

Lemma 1 [2]: The Laplacian matrix L of a directed graph G has exactly a single zero eigenvalue $\lambda_1 = 0$ if and only if G includes a spanning tree, with the corresponding eigenvector $\phi = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. Meanwhile, all the other eigenvalues $\lambda_2, ..., \lambda_N$ locate in the open right half plane.

The readers may refer to [2] and other relational articles by Ren for the detailed discussions of the characteristic of a directed graph that it includes a spanning tree. Intuitively speaking, including a spanning tree can be regarded as the least requirement for a directed graph to allow information to be transferred through it.

Lemma 2: For swarm system (1), if it is asymptotically swarm stable, then as $t \to \infty$, the trajectory of each agent tends to approach that of the system $\dot{\xi} = A\xi$.

Proof: Without loss of generality, consider agent i. The difference between $\dot{x}_i$ and Ax_i is

$$\dot{x}_i - Ax_i = F\sum_{j=1}^{N} w_{ij}(x_j - x_i)$$

Because the system is asymptotically swarm stable, $(x_j - x_i) \to 0$ ($j \in \{1,2,...,N\}$) and $F\sum_{j=1}^{N} w_{ij}(x_j - x_i) \to 0$. Thus, as $t \to \infty$, the difference $(\dot{x}_i - Ax_i) \to 0$. □

Lemma 2 naturally leads to the following two corollaries.

Corollary 1: For swarm system (1), if A is Hurwitz, then the system is asymptotically swarm stable if and only if it is asymptotically Lyapunov stable and the consensus states of all agents must be zero.

Corollary 2: If swarm system (1) is asymptotically swarm stable and the consensus states are not zero, then A is not Hurwitz.

Lemma 2 implies that a swarm system may still be unstable even if it is asymptotically swarm stable. The states of the agents in consensus may keep on oscillating or even diverge to be unbounded.

Remark 3.1: Even for a first order swarm system, the consensus value could also diverge if the agents have unstable dynamics.

Theorem 1: For a swarm system depicted by (1) with $\lambda_1 = 0, \lambda_2, \cdots, \lambda_N \in C$ as the eigenvalues of the Laplacian matrix of G, a necessary and sufficient condition for the system to be asymptotically stable is that if A is not Hurwitz, then both 1) and 2) below are true:

1) The graph topology G includes a spanning tree
2) All the matrices $A - \lambda_i F$ ($i \in \{1,2,...,N\}$ $\lambda_i \neq 0$) are Hurwitz

If A is Hurwitz, then only 2) above is true.

Proof: (Part I) A is not Hurwitz

Assume that G does not include a spanning tree while the system is asymptotically swarm stable. G must contain $k \geq 2$ different subgraphs $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_k$, each receiving no information. According to Lemma 2, as $t \to \infty$, the consentaneous trajectory $\xi_1(t)$ of the agents associated with $\hat{G}_1$ is that of system $\dot{\xi}_1 = A\xi_1$ while the consentaneous trajectory $\xi_2(t)$ of the agents associated with $\hat{G}_2$ is that of system $\dot{\xi}_2 = A\xi_2$. The two trajectories are independent because there is never information exchange between each other, directly or indirectly. Therefore, the difference between the two trajectories meets $\dot{\xi}_1 - \dot{\xi}_2 = A(\xi_1 - \xi_2)$. If A is not Hurwitz, $\xi_1 - \xi_2$ will never approach 0. This contradicts the assumption that the system is asymptotically swarm stable. Thus, 1) is necessary.

Let J denote the Jordan canonical form of L with $\lambda_1 = 0, \lambda_2 ..., \lambda_N \in C$ its eigenvalues, i.e.

$$J = \begin{bmatrix} 0 & \times & 0 & \cdots & 0 \\ 0 & \lambda_2 & \times & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \times \\ 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix} \tag{4}$$

where '$\times$' may either be 1 or 0. Suppose that $TLT^{-1} = J$ and let $\tilde{x} = (T \otimes I_d)x$, then (2) is transformed into

$$\dot{\tilde{x}} = (I_N \otimes A - J \otimes F)\tilde{x} \tag{5}$$

The structure of $I_N \otimes A - J \otimes F$ is of the form:

$$\begin{bmatrix} A & \times & 0 & \cdots & 0 \\ 0 & A - \lambda_2 F & \times & \ddots & \\ & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \times \\ 0 & \cdots & & 0 & A - \lambda_N F \end{bmatrix} \tag{6}$$

where each '$\times$' represents a block in $R^{d \times d}$ that may either be $-F$ or 0. Define the auxiliary variables $\eta_i = \sum_{j=1}^{N} w_{ij}(x_j - x_i)$ ($i \in \{1,2,...,N\}$) and the stack vector $\eta = \begin{bmatrix} \eta_1^T & \cdots & \eta_N^T \end{bmatrix}^T$. It follows that

$$\eta = (L \otimes I_d)x \tag{7}$$

Because of (6) and $x = (T^{-1} \otimes I_d)\tilde{x}$,

$$\begin{aligned} (T \otimes I_d)\eta &= (TL \otimes I_d)x \\ &= (TL \otimes I_d)(T^{-1} \otimes I_d)\tilde{x} = (J \otimes I_d)\tilde{x} \end{aligned} \tag{8}$$

With the condition 1), according to Lemma 1, any vector in the null space of $L \otimes I_d$ is of the form $\phi \otimes \xi$, where $\phi = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in R^N$ and $\xi \in R^d$. Thus, $x_1 = x_2 = \cdots = x_N$ if and only if $\eta = 0$. According to Lemma 1, $\lambda_2, \cdots, \lambda_N$ are all nonzero. From the linear equation (8), we know that $\eta = 0$ if and only if $\tilde{x}_2 = \cdots = \tilde{x}_N = 0$ with considering the triangular structure of J. By (6), one knows that $I_N \otimes A - J \otimes F$ is a strictly upper triangular block matrix and the stability of (5) is determined by the eigenvalues of the diagonal blocks. Meanwhile, one can see that the output stability of the system with respect to $\tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_N$ is independent of $\tilde{x}_1$. Consequently, $\tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_N \to 0$ if and only if $A - \lambda_2 F$, …, $A - \lambda_N F$ are Hurwitz.

(Part II) A is Hurwitz

When A is Hurwitz, according to Corollary 1, the swarm system's asymptotic swarm stability is equivalent to its asymptotic Lyapunov stability. The stability of the system (1) is equivalent to that of (5). By observing the structure of $I_N \otimes A - J \otimes F$ in (6), it is easy to infer that (5) is asymptotically stable if and only if the condition 2) is true.□

Remark 3.2: In the proof of Theorem 1, the variables η_i ($i = 1,..., N$) represent the relative measurements derived by the agents. Vectors in $\{\eta_i \mid i = 1,..., N\}$ are linearly dependent. (7) implies that the rank of this set is $N-1$ at the most.

From the proof of Theorem 1, one can sense that the asymptotic swarm stability of a swarm system corresponds to the asymptotic Lyapunov stability of a dynamic system of lesser order. This can intuitively be interpreted as that the zero eigenvalues of L reflect the unobservability of the absolute motion of the swarm system in the relative measurements η_i ($i = 1,2,..., N$) [15]. The relative motions of a swarm stable system are asymptotically stable, while the asymptotic swarm stability is independent of the absolute motion.

IV. Global Uniform Asymptotic Swarm Stability of LTI Nonlinear Time-varying Swarm Systems

The asymptotic swarm stability of swarm system (3) will be considered in this section. First, several rational assumptions about compartmental systems are listed as follows.

(A1) The interaction between any neighboring pair is skew-symmetric:

$$f_{ij}(x_i, x_j, t) = -f_{ji}(x_j, x_i, t) \quad (i, j \in \{1,2,..., N\})$$

(A2) The interactions are independent of translation:

$$f_{ij}(x_i, x_j, t) = f_{ij}(x_i - x_j, 0, t) \quad (i, j \in \{1,2,..., N\})$$

(A3) Any neighboring pair of agents are always attractive with each other: $\exists V(\alpha, \beta) \ge 0$ ($\alpha, \beta \in R^d$) *s.t.* $V(\alpha, \beta) = V(\beta, \alpha)$, $V(\alpha, \beta) = 0 \Leftrightarrow \alpha = \beta$, and as $\xi_1 \ne \xi_2$,

$$\begin{cases} \dfrac{\partial V(\alpha, \beta)}{\partial \alpha}\Big|_{(\xi_1, \xi_2)} f_{ij}(\xi_1, \xi_2, t) < 0 \\ \dfrac{\partial V(\alpha, \beta)}{\partial \beta}\Big|_{(\xi_1, \xi_2)} f_{ji}(\xi_2, \xi_1, t) < 0 \end{cases} \quad (i, j \in \{1,2,..., N\})$$

Besides, $\dfrac{\partial V(\alpha, \beta)}{\partial \alpha}$ is a linear function of α and

$\frac{\partial V(\alpha,\beta)}{\partial \beta}$ is a linear function of β. $V(\alpha,\beta)$ is called here *relative Lyapunov function*.

Lemma 3: For swarm system (3), with the assumption (A1), the sum of the states of agents keeps constant.

Proof: The sum is $s=\sum_{i=1}^{N} x_i$. Its derivative is

$$\dot{s}=\sum_{i=1}^{N}\dot{x}_i=\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}(t) f_{ij}(x_i,x_j,t) \tag{9}$$

By rearranging the terms in (9) according to the edges in G, one has

$$\dot{s}=\sum_{(i,j)\in G}(w_{ij}(t) f_{ij}(x_i,x_j,t)+w_{ji}(t) f_{ji}(x_j,x_i,t))=0 \ \square$$

For compartmental systems, Lemma 3 verifies the rationality of (A1) because the substance conservation is guaranteed by it. Since there is nothing supplied from external source and nothing lapsed to external sink, the total quantity of substance in the system should be constant.

Definition 3 (Joint Connection): A time-varying undirected N th order graph $G(t)$ continuous in t is *jointly connected* if $\exists T>0$ *s.t.* for $\forall t>0$, $\int_t^{t+T} W(t)dt$ represents a connected graph, where $W(t)\in R^{N\times N}$ is the adjacency matrix of $G(t)$.

Joint connection is the least requirement for a time-varying undirected graph to allow information to be transferred through it.

Theorem 2: For swarm system (3), with the assumptions (A1) ~ (A3), the system is globally uniformly asymptotically swarm stable if and only if G is jointly connected.

Proof: According to Lemma 3, the average of agent states $x_0=(\sum_{i=1}^{N} x_i)/N$ is constant. Let $\Gamma=\sum_{i=1}^{N} V(x_i,x_0)$ and consider its derivative:

$$\begin{aligned}\dot{\Gamma}&=\sum_{i=1}^{N}\dot{V}(x_i,x_0)=\sum_{i=1}^{N}\Big(\frac{\partial V(\alpha,\beta)}{\partial\alpha}\Big|_{(x_i,x_0)}\dot{x}_i\Big)\\&=\sum_{i=1}^{N}\Big(\frac{\partial V(\alpha,\beta)}{\partial\alpha}\Big|_{(x_i,x_0)}\sum_{j=1}^{N} f_{ij}(x_i,x_j,t)\Big)\end{aligned} \tag{10}$$

By rearranging the terms in (10) according to the edges in G with considering the assumptions, one obtains

$$\begin{aligned}\dot{\Gamma}&=\sum_{(i,j)\in G} w_{ij}(t) f_{ij}(x_i,x_j,t)\Big(\frac{\partial V(\alpha,\beta)}{\partial\alpha}\Big|_{(x_i,x_0)}-\frac{\partial V(\alpha,\beta)}{\partial\alpha}\Big|_{(x_j,x_0)}\Big)\\&=\sum_{(i,j)\in G} w_{ij}(t) f_{ij}(x_i-x_j,0,t)\frac{\partial V(\alpha,\beta)}{\partial\alpha}\Big|_{(x_i-x_j,0)}\end{aligned}$$

where $(i,j)\in G$ means that the edge connecting vertexes i and j belongs to graph G. According to (A3), each term $f_{ij}(x_i-x_j,0,t)\frac{\partial V(\alpha,\beta)}{\partial\alpha}\Big|_{(x_i-x_j,0)}$ in the equation above is negative if $x_i\neq x_j$. Suppose G is jointly connected, then $\exists T>0$ *s.t.* for $\forall t_0\geq 0$, $\Gamma(t_0+T)-\Gamma(t_0)<0$ so long as the states of agents are different. The value of $\Gamma(t)$ decreases with a lower bound, according to LaSalle's invariant principle, it must approaches a limit as $t\to\infty$ and $\lim_{t\to\infty}\dot{\Gamma}(t)=0$. It follows that

$$\lim_{t\to\infty} x_1(t)=\lim_{t\to\infty} x_2(t)=...=\lim_{t\to\infty} x_N(t)$$

Because x_0 is constant, $\lim_{t\to\infty} x_i(t)=x_0$ ($i=1,...,N$). Hence, $\lim_{t\to\infty}\Gamma(t)=0$. This conclusion is independent of the initial state and time. Thus, the system is globally uniformly asymptotically swarm stable.

Suppose G is not jointly connected, then there exist two distinct subgraphs $\hat{G}_1$ and $\hat{G}_2$ of G, each of which is jointly connected and receives no substance. Assume that the system is still asymptotically swarm stable. From the analysis above one knows that the agents associated with $\hat{G}_1$ converge to $x_0^{(1)}$, which is the average of them, and the agents associated with $\hat{G}_2$ converge to $x_0^{(2)}$. If $x_0^{(1)}\neq x_0^{(2)}$, these two constant values will never be equal. So joint connection of G is necessary. □

The attraction of agents is prescribed by some relative Lyapunov function. Intuitively, a relative Lyapunov function indicates a measure of distance between two agents. Such a measure is more generic than the ordinary concept of distance, i.e. $\|x_i-x_j\|$. If two neighbors are attractive, their Euclidean distance could still sometimes increase. In this sense, our result is more general than Moreau's about high order nonlinear swarm systems because his technical assumptions implicate that the norm of difference between any two neighbors must decrease at any time.

As a simple example, let $f_{ij}(x_i,x_j,t)=F(x_j-x_i)$, where $F\in R^{d\times d}$. An appropriate relative Lyapunov function is $V(x_i,x_j)=(x_i-x_j)^T P(x_i-x_j)$ if P meets $F^TP+PF=Q$, where $P=P^T>0$ and $Q=Q^T>0$. It is obvious that the interaction between any two neighbors is attractive if and only if $-F$ is Hurwitz. Note that for high order cases, such an example usually does not meet Moreau's assumption in [3] even if the interactions are attractive.

V. Numerical Examples

In this section, two numerical examples will be exhibited to illustrate the theoretic results in the previous sections.

Example 1: Consider an second order LTI swarm system (1) with five agents. Let

$$A=\begin{bmatrix}0 & 1.00\\ -5.00 & 2.00\end{bmatrix},\ F=\begin{bmatrix}5.26 & -9.21\\ 18.42 & 5.26\end{bmatrix}$$

The graph G is described by the following adjacency matrix

$$W=\begin{bmatrix} 0 & 0 & 0 & 0 & 0.38 \\ 0.50 & 0 & 0.02 & 0 & 0.55 \\ 0 & 0.28 & 0 & 0 & 0 \\ 0.84 & 0 & 0 & 0 & 0.40 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A is unstable with eigenvalues $1\pm 2i$, while the eigenvalues of $A+\lambda_i F$ ($\lambda_i \neq 0$) are: $-1.00\pm 7.28i$, $-4.63\pm 16.35i$, $-5.53\pm 18.57i$. Though the eigenvalues of $A+\lambda_i F$ ($\lambda_i \neq 0$) are all with negative real parts, the system is swarm unstable because A is unstable and G includes no spanning tree. □

Example 2: Consider a nonlinear time-varying compartmental system of second order with five agents

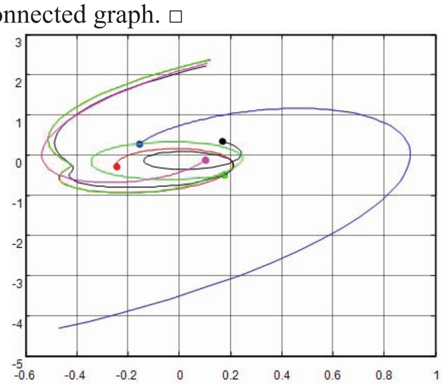
$$\begin{cases} \dot{x}_{i1} = \sum_{j=1}^{5} w_{ij}(t)(x_{j1}-x_{i1}+2(x_{i2}-x_{j2})\cos(x_{j1}-x_{i1})) \\ \dot{x}_{i2} = \sum_{j=1}^{5} w_{ij}(t)(x_{j2}-x_{i2}+(x_{j1}-x_{i1})\cos(x_{j1}-x_{i1})) \end{cases} \quad (11)$$

$$(i=1,2,\ldots,5)$$

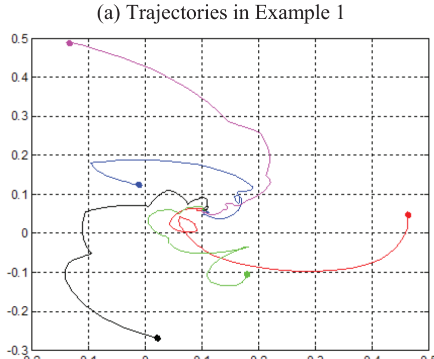
with $x_i=\begin{bmatrix} x_{i1} & x_{i2} \end{bmatrix}^T \in R^2$ as the state of agent i. Obviously this system satisfies (A1)~(A2). If a relative Lyapunov function candidate is selected as

$$V(x_i,x_j)=x_i^T P x_i + x_j^T P x_j - x_i^T P x_j - x_j^T P x_i \quad (12)$$

with $P=diag([1 \quad 2])$, then it is easy to verify that the system with the relative Lyapunov function (12) satisfies (A3). Hence, it is asymptotically swarm stable if with an undirected jointly connected graph. □



(a) Trajectories in Example 1



(b) Trajectories in Example 2

Fig.1. Trajectories of two examples
(Thick dots indicate starting positions)

VI. Conclusion

The asymptotic swarm stability problem was dealt with for two types of high order swarm system models: LTI model and nonlinear time-varying model. The LTI model is general and most of the LTI swarm system models concerned in the literature about consensus problems are special cases of this one. The nonlinear time-varying model concerned in this paper is called compartmental model. Formal definition of swarm stability was given based on consensus. Necessary and sufficient conditions for the global uniform asymptotic swarm stability of the two models were presented respectively. For an LTI system with unstable agent dynamics, a sufficient connection of the graph topology is required. However, for LTI system with asymptotically stable agent dynamics, connection is not necessary. For nonlinear time-varying compartmental models, the condition is determined by two independent factors: attraction and connection. A new concept named relative Lyapunov function was introduced for analyzing the swarm stability problem.

References

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1520-1533, 2004.

[2] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies", *IEEE Trans. Automat. Contr.*, vol. 50, pp. 655-661, 2005.

[3] L. Moreau, "Stability of multiagent systems with time-dependent communication links", *IEEE Trans. Automat. Contr.*, vol. 50, pp. 169-182, 2005.

[4] N. A. Lynch, *Distributed Algorithms*, San Mateo, CA: Morgan Kaufmann, 1997.

[5] F. Xiao and L. Wang, "Consensus problems for high-dimensional multi-agent systems", *J. Contr. Theoy. Applicat.*, vol. 1, pp. 830-837, 2007.

[6] W. Ren, K. L. Moore, and Y. Chen, "High-order and model reference consensus algorithms in cooperative control of multi-vehicle systems", *ASME J. Dynam. Syst. Meas. Contr.*, vol. 129, pp. 678-688, 2007.

[7] P. Wieland, J. Kim, H. Scheu, and F. Allgower, "On consensus in multi-agent systems with linear high-order agents", in *Proc. 17th IFAC World Congress*, Seoul, Korea, Jul. 2008, pp.1541-1546.

[8] J. Wang, D. Cheng, and X. Hu, "Consensus of multi-agent linear dynamic systems", *Asian J. Contr.*, vol. 10, pp. 144-155, 2008.

[9] Z. Li, Z. Duan, and L. Huang, "Leader-follower consensus of multi-agent systems", in *Proc. American Control Conf.*, St. Louis, USA, Jun. 2009, pp. 3256-3261.

[10] Z. Lin, B. Francis, and M. Maggiore, "On the state agreement problem for multiple nonlinear dynamical systems", in *Proc. 16th IFAC World Congress*, Prague, Czech, Jan. 2005.

[11] D. Angeli and P. -A. Bliman, "Extension of a result by Moreau on stability of leaderless multi-agent systems", in *Proc. Conf. Decision Control*, Seville, Spain, Dec. 2005, pp. 759-764.

[12] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, 2000.

[13] V. Gazi and K. M. Passino, "Stability analysis of swarms", *IEEE Trans. Automat. Contr.*, vol. 48, pp. 692-697, 2003.

[14] R. M. Zazworsky and H. K. Knudsen, "Controllability and observability of linear time-invariant compartmental models", *IEEE Trans. Automat. Contr.*, vol. 23, pp. 872-877, 1978.

[15] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1465-1476, 2004.