Relay-Based Gain and Phase Margins PI Controller Design

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***Abstract*—In this paper, an iterative procedure for achieving gain and phase margin specifications for a PI controller is presented. The iteration scheme is based on the use of two relay tests applied to the closed-loop system. The first relay test is standard, and it is used here to obtain the gain margin of the closed-loop system at each iteration step. The second one is applied to the closed-loop system such that a limit cycle is developed at the loop gain crossover frequency. Under this condition, it is possible to obtain an estimate of the phase margin of the loop transfer function. The procedure is used for PI controller tuning in a vanadium-dioxide (VO_2) thin-film characterization experiment, with desired gain and phase margin specifications.**

***Index Terms*—Gain and phase margins, iterative procedure, PI controller tuning, relay tests.**

I. Introduction

PI CONTROLLERS are widely used in industry and frequently found in instrumentation and measurement systems, where certain operating conditions should be met. For instance, in order to properly characterize the thermodynamical behavior of vanadium-dioxide (VO_2) thin films [1], a PID controller is used to regulate the temperature of the film, using a thermoelectric module or Peltier module as the actuator (see [2]). In such situations, a well tuned controller can reduce experiment duration and increase data reliability.

The relay test introduced in [3] has been used with success for PI controllers tuning. In fact, many commercial PI controllers based on this one-shoot identification-and-tune technique are available. From this relay test, a single point of the frequency response of a process, the critical point (phase angle equals to $-180°$), can be estimated.

The tuning of PI controllers based on gain and phase margins specifications has been recently studied in [4]–[7] and references therein. In [4], [5], it was assumed that the process is first order plus time delay, which may not be representative for typical industrial processes, as reported in [6]. The controller parameters were calculated by solving a set of nonlinear equations, and an analytical solution was obtained using approximations.

In [7], it was shown that with the information of a single point of the process frequency response, it is possible to design a PI controller with gain and phase margins specifications. This is not an exact solution, because the identified point is moved to another location, and it is expected that the resulting loop transfer function is close to the desired gain and phase margins. In [6], an exact solution was derived from a graphical approach, but the knowledge of the process frequency response is required.

In this paper, it is presented a new combination of sequential relay tests for achieving gain and phase margins specifications in a closed-loop system. The nonlinear problem is solved in an iterative manner, and two relay tests are used in each iteration. The advantage of this procedure is that no assumptions on process model structure is required, neither full knowledge of its frequency response.

II. Problem Statement

Consider the closed loop shown in Fig. 1. The unknown process transfer function is given by $G(s)$ while the PI controller is given by

$$C(s) = K_p\left(1 + \frac{1}{T_i s}\right) = K_p \frac{s + \frac{1}{T_i}}{s}. \tag{1}$$

The closed-loop transfer function from reference $y_r(t)$ to output $y(t)$ is given by

$$\frac{Y(s)}{Y_r(s)} = \frac{G(s)C(s)}{1 + G(s)C(s)} = T(s). \tag{2}$$

For this closed-loop configuration, $T(s)$ is also known as the complementary sensitivity function and $L(s) = G(s)C(s)$ is the loop transfer function.

The gain margin (GM) and phase margin (PM) of a closed loop are defined as

$$GM = \frac{1}{|L(j\omega_u)|} \text{ and } \mathrm{PM} = \pi + \angle L(j\omega_g), \tag{3}$$

where ω_u and ω_g are obtained from

$$\angle L(j\omega_u) = -\pi \text{ and } |L(j\omega_g)| = 1. \tag{4}$$

The problem posed in this paper is to find the PI controllers' parameters using relay tests, such that a closed-loop system with specified gain and phase margins is obtained.

III. Relay–Based Identification

The estimation of the critical point using a relay test is presented in [7] and is a standard in autotune procedures. The idea is to substitute the controller in Fig. 1 by a relay with amplitude d. For most types of processes, this configuration leads to limit cycle operation, with oscillation conditions given by

$$G(j\omega) \cong -\frac{\pi a}{4d}$$

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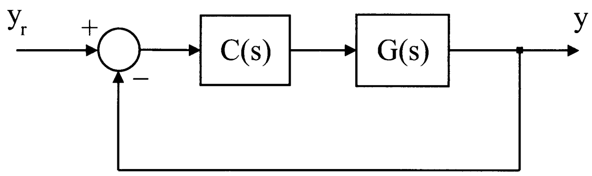

Fig. 1. Closed-loop system.

which is at the process critical point. It can be shown [8] that if this relay test is applied to a closed-loop system $T(s)$, the limit cycle occurs at the critical frequency of the loop transfer function, i.e.

$$L(j\omega) = G(j\omega)C(j\omega) \cong \frac{m}{1-m}$$

where

$$m = -\frac{\pi a}{4d}$$

and a is the process output amplitude in closed loop. In this paper, this relay test will be referred as the *loop critical gain relay test*. An estimate of the gain margin of the feedback system in Fig. 1 is then obtained with the standard relay test on closed loop, and calculated as

$$\hat{G}M = \left|\frac{1-m}{m}\right| = 1 + \frac{4d}{\pi a}, \tag{5}$$

and the critical frequency estimate $\hat{\omega}_u$ is obtained from the frequency of the limit cycle.

A basic procedure for the estimation of a general frequency point of a given transfer function using a relay feedback is presented in [9]. The feedback structure applied for loop transfer function estimation is presented in Fig. 2. The conditions of the limit cycle operation are defined by the following proposition.

Proposition 1: Consider the closed-loop system shown in Fig. 2. Assume that for a stable closed-loop $T(s)$ and a real positive number r, the transfer function

$$F(s) = \frac{Y_o(s)}{Y_i(s)} = \frac{2}{r}\frac{T(s)}{T(s)\left(\frac{1-r}{r}\right)+1} - 1 \tag{6}$$

is also stable. Then, if a limit cycle is present it oscillates at a frequency ω_o such that

$$|L(j\omega_o)| \approx r$$

Proof: Rewrite $F(s)$ as

$$F(s) = \frac{L(s)-r}{L(s)+r}$$

and note that the feedback is constructed by an integrator followed by a relay and $F(s)$ as above. Thus, if a limit cycle is present in the relay closed-loop system, then describing function analysis implies that at frequency ω_o, the phase of $F(j\omega_o)/j\omega_o$ is approximately $-\pi$, which implies that $\angle F(j\omega_o) \approx -\pi/2$. Then

$$F(j\omega_o) = \frac{L(j\omega_o)-r}{L(j\omega_o)+r} \approx -kj$$

for some real $k > 0$, which gives

$$L(j\omega_o) \approx r\frac{1-kj}{1+kj}$$

so that $|L(j\omega_o)| \approx r$. ■

This procedure allows the estimation of the frequency at which the loop transfer function magnitude is close to r. It can be seen as a generalization of the feedback structure presented in [8]. In that work, it was used $r = 1$, and a limit cycle was obtained at the loop gain crossover frequency. Here, it will be used to estimate the phase angle at this frequency by observing the phase between the output and input of the closed loop. This relay test will be referred as the *loop gain crossover relay test*.

To estimate the phase margin of the feedback system in Fig. 1, it is necessary to obtain the phase lag of $L(j\omega)$ at crossover frequency ω_g. That can be obtained using data from closed-loop input and output during one period of the oscillation, and use discrete Fourier techniques to estimate the magnitude and phase at that particular frequency. The procedure to estimate the phase margin is then as follows. Collect N samples of the closed-loop input and output over one period of the oscillation during the frequency response relay test with $r = 1$. Estimate the crossover frequency, $\hat{\omega}_g$, measuring the frequency of the limit cycle, and obtain the estimate of the loop transfer function using

$$\hat{L}\left(e^{j\hat{\omega}_g}\right) = \frac{\hat{T}\left(e^{j\hat{\omega}_g}\right)}{1-\hat{T}\left(e^{j\hat{\omega}_g}\right)} \tag{7}$$

and

$$\hat{T}\left(e^{j\omega}\right) = \frac{Y_N(\omega)}{Yr_N(\omega)} \tag{8}$$

with $Y_N(\omega)$ and $Yr_N(\omega)$ as the N-point the discrete Fourier transform of the N-samples of the signals $y[k]$ and $y_r[k]$ collected over one period, respectively. The phase margin estimate of the feedback system in Fig. 1 is obtained from

$$P\hat{M} = \pi + \angle\hat{L}\left(e^{j\hat{\omega}_g}\right). \tag{9}$$

IV. PI Controller Iterative Tuning

Consider the closed-loop system shown in Fig. 1 and gain and phase margin specifications given by A_m and ϕ_m, respectively. For the desired specifications, the controller must be designed to satisfy the following set of equations:

$$\angle G(j\omega_u)C(j\omega_u) = -\pi \tag{10}$$

$$|G(j\omega_u)C(j\omega_u)| = \frac{1}{A_m} \tag{11}$$

$$|G(j\omega_g)C(j\omega_g)| = 1 \tag{12}$$

$$\angle G(j\omega_g)C(j\omega_g) = -\pi + \phi_m. \tag{13}$$

If $G(j\omega)$ is known, the PI controller given by (1) requires the solution of the set of equations for K_p, T_i, ω_u, and ω_g, a total of four unknowns and four equations. This is not a trivial task due to the nonlinearity of the problem. A procedure was presented in [6] for numerically calculating the controller parameters for both PI and PID controllers, but it requires the knowledge of the

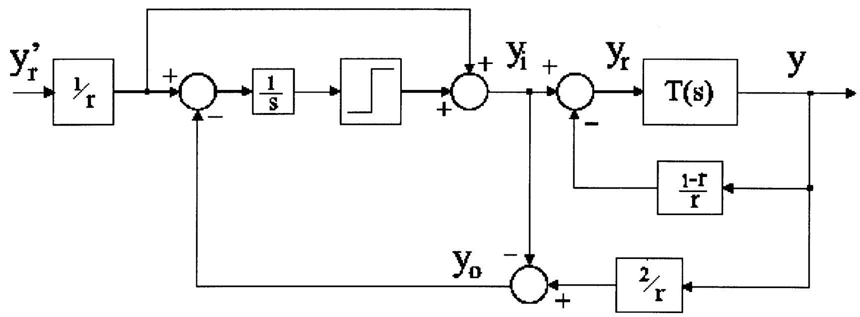

Fig. 2. Relay closed-loop experiment for loop transfer function estimation.

process frequency response $G(j\omega)$. It is also not guaranteed that such a solution exists, as pointed out by the authors.

In this paper, instead, an iterative algorithm is used which requires the estimation of the frequencies ω_u and ω_g and the gain and phase margins at each iteration. Observe that these frequencies are the solutions to (10) and (12), and estimates can be obtained using the *loop critical gain relay test* and the *loop gain crossover relay test*, respectively.

The following two step design approach can be used to tune the PI controller, where a superscript k denotes the current iteration.

1. **Initial Conditions:** Start from an initial controller $C^0(s)$, with K_p^0 and T_i^0 (in this step it is assumed that the closed-loop system is operating and $k=1$).
2. **Gain Margin Estimation:** Use the loop critical gain relay test and obtain the critical frequency estimate, $\hat{\omega}_u^k$, and the gain margin estimate, $\hat{GM}^k$.
3. **Test condition:** If $\left|\hat{GM}^k - A_m\right| \leq \varepsilon_1$ and this is not the first iteration, stop the procedure and make $C(s) = C^k(s)$. Else, continue.
4. **Controller Redesign for Gain Margin:** The controller gain is calculated to achieve the gain margin A_m using **(11)**. With $\hat{GM}^k$, compute the controller proportional gain, $\bar{K}_p^{k+1}$, as

$$\bar{K}_p^{k+1} = \frac{K_p^k \hat{GM}^k}{A_m}. \tag{14}$$

Now a new intermediate controller is

$$\bar{C}^{k+1}(s) = \bar{K}_p^{k+1}\left(\frac{s+\frac{1}{T_i^k}}{s}\right) \tag{15}$$

which can be applied immediately because the closed loop is stable by construction.

5. **Phase Margin Estimation:** Use the loop gain crossover relay test with $r = 1$ and obtain the gain crossover frequency estimate, $\hat{\omega}_g^k$, and the phase margin estimate, $\hat{PM}^k$.
6. **Test condition:** If $\left|\hat{PM}^k - \phi_m\right| \leq \varepsilon_2$, then stop the procedure and make $C(s) = \bar{C}^{k+1}(s)$. Else, continue.
7. **Controller Redesign for Phase Margin:** This step is separated into two parts:

a. Determine T_i^{k+1} such that **(13)** is satisfied, i.e.

$$T_i^{k+1} = \frac{\tan\left[\phi_m - \hat{PM}^k + \tan^{-1}\left(\hat{\omega}_g^k T_i^k\right)\right]}{\hat{\omega}_g^k}. \tag{16}$$

The phase contribution from the PI controller ranges from $-\pi/2$ to 0, and this information is used in order to avoid invalid values of T_i^{k+1}. Since $\angle\hat{G}\left(j\hat{\omega}_g^k\right) + \angle C^{k+1/2}\left(j\hat{\omega}_g^k\right) = -\pi + \phi_m$, then the following condition must be satisfied before calculating T_i^{k+1}

$$-\pi + \phi_m < \angle\hat{G}\left(j\hat{\omega}_g^k\right) < -\frac{\pi}{2} + \phi_m \tag{17}$$

If **(17)** is not satisfied, stop the iteration and return to the initial controller $C^0(s)$. Note that $\angle\hat{G}\left(j\hat{\omega}_g^k\right)$ can be obtained from $\angle\hat{G}\left(j\hat{\omega}_g^k\right) = -\pi + \hat{PM}^k - \angle C\left(j\hat{\omega}_g^k\right)$.

b. Now, update the controller proportional gain K_p^{k+1} such that the loop gain at frequency $\hat{\omega}_g^k$ is equal to one

$$K_p^{k+1} = \bar{K}_p^{k+1}\frac{\sqrt{\left(\frac{1}{T_i^k}\right)^2 + \left(\hat{\omega}_g^k\right)^2}}{\sqrt{\left(\frac{1}{T_i^{k+1}}\right)^2 + \left(\hat{\omega}_g^k\right)^2}}. \tag{18}$$

The controller at the end of the iteration is finally given by

$$C^{k+1} = K_p^{k+1}\left(\frac{s+\frac{1}{T_i^{k+1}}}{s}\right). \tag{19}$$

8. Increment k and go to step 2.

In steps 3 and 6, it is considered that ε_1 and ε_2 are small tolerances. Note that both specifications A_m and ϕ_m are satisfied,

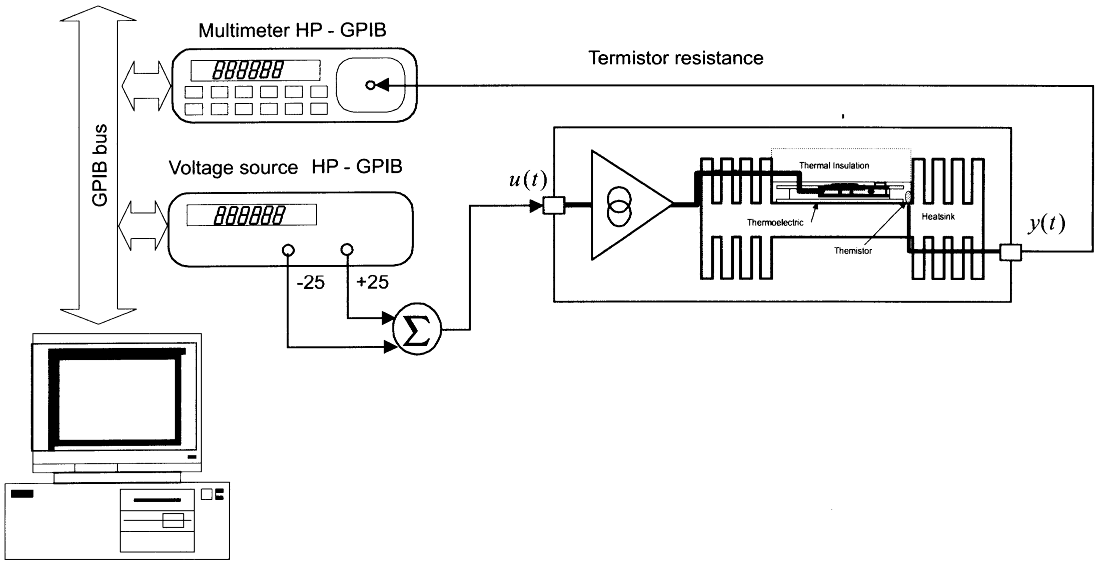

Fig. 3. Experimental setup of the data acquisition system.

TABLE I
GAIN AND PHASE MARGIN AND CONTROLLER PARAMETERS AT EACH ITERATION

Iter.	$\hat{GM}$	$\hat{PM}$	K_p	T_i
$\frac{1}{2}$	1.65	–	260.4542	6.0000
1	–	49.03°	306.1164	8.8747
$1+\frac{1}{2}$	2.06	–	210.3053	8.8747
2	–	61.95°	–	–

except if an invalid value of T_i is calculated in step 7(a). Extensive simulations using the exact values of ω_g and ω_u at each iteration instead of the estimates were performed. The results point out that the algorithm converges to the same solution obtained using the graphical method in [6], whenever such a solution exists. It should be noted that convergence of iterative tuning schemes is not guaranteed in general. This is a common fact in iterative design schemes, except for a few particular cases (see [10]). Instead, the user can detect a possible nonconvergent situation by observing the behavior of the estimated gain and phase margins, as well as the frequencies ω_g and ω_u at each iteration. Also, the scheme can be interrupted whenever a reasonable solution is achieved.

V. EXPERIMENTAL RESULTS

The experimental setup employed for the VO_2 thin-film characterization experiment is shown in Fig. 3. This setup has been used in [1] and [2] to determine the Resistance versus Temperature characteristics of the thin film of VO_2. As indicated in the figure, all the instruments, including the microcomputer are GPIB/IEEE-488 compatible. The platform is composed by a Peltier device, a heatsink and a thermistor. A thermoelectric module or Peltier module is a semiconductor device that operates as heat pump controlled by electric current. Applying a direct current to the Peltier module, heat will be moved through the module from one side to the other. One module face, therefore, will be cooled while the opposite face simultaneously is heated. A change in the polarity of the current through the module will cause heat to be moved into the opposite direction. Peltier modules are used for cooling electronic devices. Usually, the temperature of the cold side should be held at a constant and stable value even if the hot-side as well as the ambient temperatures presents variations. Suitable temperature control techniques are required to achieve this feature. The dynamic behavior of a Peltier module can be accurately studied by using a model consisting of two coupled partial differential equations. However, this kind of model is not suitable for designing PI controllers, which qualifies the method presented here, since knowledge of the process model is not necessary.

The VO_2 thin film should be held at constant temperature for proper characterization of the material. The operating point is chosen as the room temperature (25°), and both performance and stability of the closed loop should be evaluated around this operating condition.

The initial controller is obtained from a Ziegler–Nichols frequency response table (see [7]). Therefore, the critical gain of the process estimated with a standard relay test gives

$$\hat{K}_u = 1183.5\ \mathrm{V/mC^\circ}\ \text{and}\ \hat{T}_u = 7.50\ \mathrm{s}$$

and the following controller parameters are obtained

$$K_p = 473.3923\ \text{and}\ T_i = 6.0000.$$

The *loop critical gain* and the *loop gain crossover* relay tests applied to the closed loop gives the following estimates

$$\hat{GM}^0 = 1.6506\ \text{and}\ \hat{PM}^0 = 36.57^\circ.$$

which indicates a poorly stable closed loop (typical values are in the range 2–3 for the gain margin and 30°–60° for the phase margin, see [7]).

Now the iterative algorithm starts, with $C^0(s)$ being the Ziegler–Nichols controller. It is desired a closed-loop system with gain margin $A_m = 3$ and phase margin $\phi_m = 60^\circ$. The

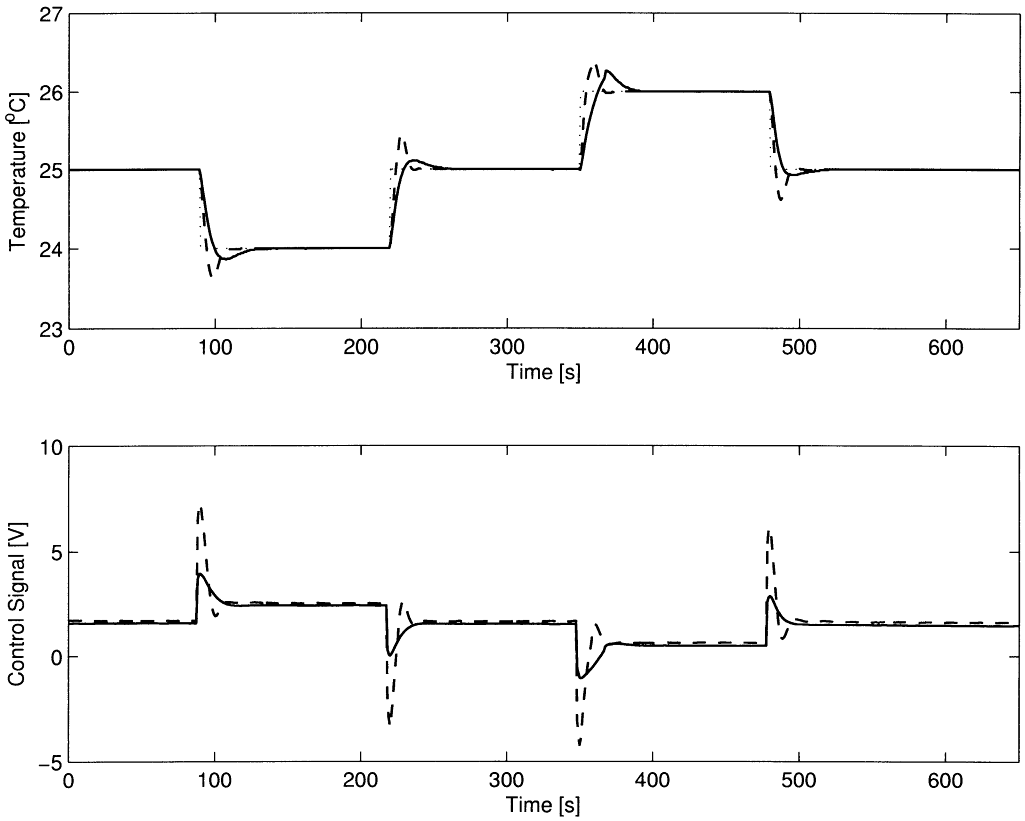

Fig. 4. Upper graphic: closed-loop step responses for both initial (- -) and final (–) controller, for the reference signal $y_r(\cdot)$. Lower graphic: corresponding control signals for initial (- -) and final (–) controller.

tolerances ε_1 and ε_2 are defined as $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 2^\circ$. After two iterations, the following controller is obtained

$$K_p^4 = 210.3053 \text{ and } T_i^4 = 8.8747.$$

The gain and phase margins estimated at each iteration step and the controller parameters are shown in Table I.

Closed-loop step responses using the initial controller and the one obtained at the end of the iteration scheme are shown in Fig. 4. The bandwidth (defined here as the gain crossover frequency, ω_g) for both initial and final controller are estimated as

$$\hat{\omega}_g^{(0)} = 0.20 \text{ rad/s and } \hat{\omega}_g^{(2)} = 0.16 \text{ rad/s}.$$

It can be seen that increasing the stability of the closed loop has decreased its bandwidth, which gave a slower, but much more stable closed-loop system.

VI. Conclusion

In this paper, an iterative procedure combining two relay experiments is applied to PI controller tuning based on gain and phase margins. The relay tests are used to estimate the critical point and the gain crossover point of the loop transfer function of a closed-loop system. This information is used to solve a set of nonlinear equations in order to achieve the gain and phase margins specifications. The advantage is that no assumption on process model structure is required, neither complete knowledge of its frequency response. Results with a vanadium-dioxide (VO_2) thin-film characterization experiment are presented, and a reasonable controller is obtained with a few iteration steps. This qualifies the method for these types of applications, where only a complex model is available.

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