Adaptive Particle Swarm Optimization*

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Abstract – *The Particle Swarm Optimization (PSO) method is one of the most powerful methods for solving unconstrained and constrained global optimization problems. Little is, however, known about how the PSO method works or finds a globally optimal solution of a global optimization problem when the method is applied to global optimization problems. This paper deals with the analysis of the dynamics of PSO in order to obtain an understanding about how it searches a globally optimal solution and a strategy about how to tune its parameters. While a generalized reduced model of PSO is proposed in order to analyze the dynamics of PSO, the stability analysis is carried out on the basis of both the eigenvalue analysis and some numerical simulations on a typical global optimization problem.*

Keywords: Global Optimization, Metaheuristics, Particle Swarm Optimization, Stability Analysis.

1 Introduction

In real optimization problems, we encounter many cases in which a problem can be formulated as a global optimization problem of the objective function having nonlinear or multipeaked characteristics. Since it is felt necessary in recent years to derive a global solution for nonlinear and multi-peaked optimization problems, global optimization is one of the most important topics in optimization.

Moreover, with the popularity of high-speed digital computers, tremendous research activity is going on in the area of global optimization in nonlinear optimization problems. Many global optimization methods have been reported in the literatures [4],[5],[2],[3].

The Particle Swarm Optimization (PSO) method is one of the most powerful methods for solving unconstrained and constrained global optimization problems. The method was originally proposed by J. Kennedy as an optimization method in 1995 and has been proved to be an efficient method for many global optimization problems. Little is, however, known about how the PSO method works or finds a globally optimal solution of a global optimization problem when the method is applied to global optimization problems.

This paper deals with the analysis of the dynamics of PSO in order to obtain an understanding about how it searches a globally optimal solution and a strategy about how to tune its parameters. While a generalized reduced model of PSO is proposed in order to analyze the dynamics of PSO, the stability analysis is carried out on the basis of both the eigenvalue analysis and some numerical simulations on a typical global optimization problem.

2 The Particle Swarm Optimization algorithm and its reduced system

2.1 The Particle Swarm Optimization algorithm

While the PSO algorithm was proposed in terms of social and cognitive behavior, two different types of algorithms exist - binary-number and real-number forms. The PSO in a real-number form is only analyzed in this paper.

The projected position of i-th particle of the swarm and the velocity of the particle at $(k+1)$-th iteration are defined as the following two equations:

$$v_{ij}^{k+1} = w \cdot v_{ij}^{k} + c_1 \cdot rand() \cdot (pbest_{ij} - x_{ij}^{k}) + c_2 \cdot rand() \cdot (gbest_j - x_{ij}^{k}) \quad (1)$$

$$x_{ij}^{k+1} = x_{ij}^{k} + v_{ij}^{k+1} \quad (2)$$

where, $i = 1, ..., n$ and n is the size of the swarm; c_1 and c_2 are positive constants; rand() is random number which is uniformly distributed in $[0, 1]$; and k determines the iteration number.

2.2 A reduced system of Particle Swarm Optimization

The above algorithm will be simplified in order to analyze the dynamics of the PSO algorithm. If the particle x is reduced to one dimension, then vector notation

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can be thrown out and the particle's trajectory can be displayed as a simple graph.

Without any information, the two terms of the formula can be transformed into one term:

$$v^{k+1} = wv^k + \phi \cdot (p - x^k) \quad (3)$$

where p is the weighted average of the two best, and ϕ is a random number distributed in $[0, (c_1 + c_2)]$ and its average value is $\frac{c_1+c_2}{2}$.

$$\begin{aligned} \phi &= c_1 \cdot \mathrm{rand}() + c_2 \cdot \mathrm{rand}() \quad (4) \\ p &= \frac{c_1 \cdot \mathrm{rand}() \cdot pbest + c_2 \cdot \mathrm{rand}() \cdot gbest}{c_1 \cdot \mathrm{rand}() + c_2 \cdot \mathrm{rand}()} \quad (5) \end{aligned}$$

Though the exact location of the particle x changes on every iteration because of the random number ϕ, the random number ϕ is made constant for the simplicity in this paper.

The trajectory of the particle can be plotted and studied when the weighted best point p and the random number ϕ are made constant $\phi = \frac{c_1+c_2}{2}$.

The reduced formulas of PSO can be expressed as follows:

$$\begin{aligned} v^{k+1} &= wv^k + \phi y^k \quad (6) \\ y^{k+1} &= -wv^k + (1 - \phi) y^k \end{aligned}$$

where

$$y^k = p - x^k \quad (7)$$

Keep in mind that the simplified model does not represent the original PSO situation. In general population sizes are greater than one and seldom a particle operates on only one dimension.

Furthermore, the weighted best point p is not static but dynamic, often exhibiting complex behavior. Therefore this simplified model is contrived only to provide some information on how each particle behave without the interaction among the particles.

The stability of each particle, however, can be exactly evaluated based on this model when the fluctuation of p is bounded. Since the search space (feasible region) is generally limited in global optimization problems, the fluctuation of p is also bounded. Therefore, it can be applied in the evaluation of the stability of each particle in usual global optimization problems.

3 Stability analysis of the reduced system

3.1 Stability analysis in terms of matrix algebra

The reduced system, which is a deterministic system, can be written in terms of matrix algebra.

$$\begin{aligned} \begin{bmatrix} v^{k+1} \\ y^{k+1} \end{bmatrix} &= \begin{bmatrix} w & \phi \\ -w & 1-\phi \end{bmatrix} \begin{bmatrix} v^k \\ y^k \end{bmatrix} \\ &= M \begin{bmatrix} v^k \\ y^k \end{bmatrix} \quad (8) \end{aligned}$$

$$\begin{bmatrix} v^n \\ y^n \end{bmatrix} = M^n \begin{bmatrix} v^0 \\ y^0 \end{bmatrix} \quad (9)$$

$$P^{-1}MP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (10)$$

$$M^n = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1} \quad (11)$$

where M is the matrix of the system.

The state of the particle at iteration k can be represented as follows:

$$\begin{bmatrix} v^k \\ y^k \end{bmatrix} = P \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} P^{-1} \begin{bmatrix} v^0 \\ y^0 \end{bmatrix} \quad (12)$$

P is a diagonal transformation matrix and

$$\lambda_1, \lambda_2 = \frac{w + 1 - \phi \pm \sqrt{(w+1-\phi)^2 - 4w}}{2} \quad (13)$$

are the eigenvalues of M.

The dynamics of the reduced system is completely defined by M. Based on the stability theory in a linear dynamical system, the stability of each particle is evaluated. According to the stability theory, the behavior of the particle is stable if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

Since the eigenvalues λ_1, λ_2 are a function of parameters w, c_1 and c_2, the eigenvalue analysis will be carried out in the following four conditions.

(a) $w = 0$

From equation (13)

$$\lambda = 1 - \phi \quad (14)$$

The condition in which absolute values of the eigenvalues λ_1, λ_2 become under 1 is the following equation.

$$0 < \phi < 2 \quad (15)$$

Therefore, the reduced system is stable when $0 < \phi < 2$, and each particle x converges to p.

(b) $\phi < w + 1 - 2\sqrt{w}$

Since the above condition can be rewritten as

$$w + 1 - \phi > 2\sqrt{w} > 0 \quad (16)$$

The condition in which absolute values of the eigenvalues λ_1, λ_2 become under 1 is the following equation.

$$|\lambda|_{max} = \frac{w + 1 - \phi + \sqrt{(w+1-\phi)^2 - 4w}}{2} < 1 \quad (17)$$

In addition, if the following relationships are considered

$$\sqrt{(w+1-\phi)^2-4w} \quad < \quad \phi - w + 1 \qquad (18)$$

$$(w+1-\phi)^2 - 4w < (\phi - w + 1)^2 \qquad (19)$$

and

$$\phi > 0 \qquad (20)$$

$$0 \le w < 1 \qquad (21)$$

Under the condition $\phi < w + 1 - 2\sqrt{w}$, the reduced system is stable when $0 \le w < 1$, and each particle x converges to stable equilibrium point p.

(c) $w + 1 - 2\sqrt{w} < \phi < w + 1 + 2\sqrt{w}$

Since $(w+1-\phi)^2 - 4w < 0$ when $w+1-2\sqrt{w} < \phi < w + 1 + 2\sqrt{w}$, λ becomes a complex number. Therefore $|\lambda|$ can be given in the following equation.

$$\begin{aligned} \forall|\lambda| &= \left|\frac{w+1-\phi \bullet \sqrt{(w+1-\phi)^2-4w}}{2}\right| \\ &= \sqrt{\left(\frac{w+1-\phi}{2}\right)^2 + \left(\frac{\sqrt{-(w+1-\phi)^2+4w}}{2}\right)^2} \\ &= \sqrt{w} \end{aligned} \qquad (22)$$

Under the condition $w+1-2\sqrt{w} < \phi < w+1+2\sqrt{w}$, the reduced system is stable when $0 \le w < 1$, and each particle x converges to stable equilibrium point p.

(d) $w + 1 + 2\sqrt{w} < \phi$

$$w + 1 - \phi < -2\sqrt{w} < 0 \qquad (23)$$

$|\lambda|$ has an absolute maximum when the molecule of equation (13) is less than 0. In this condition, λ is also negative and the condition in which the reduced system is stable is given in the following equation.

$$|\lambda|_{max} = -\left(\frac{w+1-\phi-\sqrt{(w+1-\phi)^2-4w}}{2}\right) < 1 \qquad (24)$$

Therefore

$$\sqrt{(w+1-\phi)^2-4w} < w - \phi + 3 \qquad (25)$$

When $0 < w - \phi + 3$, the following relationship holds.

$$(w+1-\phi)^2 - 4w < (w - \phi + 3)^2 \qquad (26)$$

From the above equation, the following relationship can be obtained.

$$\phi < 2w + 2 \qquad (27)$$

Since ϕ which satisfies Equation (27) must satisfy the condition $0 < w - \phi + 3$, and w is a non-negative coefficient, then

$$0 \le w < 1 \qquad (28)$$

Under the condition $w + 1 + 2\sqrt{w} < \phi < 2w + 2$, the reduced system is stable when $0 \le w < 1$, and each particle x converges to stable equilibrium point p.

Based on the above analysis and by using the parameters ϕ and w, the criterion of convergence $|\lambda_1| < 1$ and $|\lambda_2| < 1$ can be written as follows:

$$0 < \quad \phi \quad < 2w + 2 \qquad (29)$$

$$0 \le \quad w \quad < 1 \qquad (30)$$

3.2 More detailed analysis of the reduced system

While only the stability of the reduced system is discussed in the previous section, more detailed analysis such as the analysis on period of vibration will be investigated in this section.

In case of (c) $w + 1 - 2\sqrt{w} < \phi < w + 1 + 2\sqrt{w}$, λ is a complex number. Therefore the following relationship holds.

$$|\lambda| = \sqrt{w} \qquad (31)$$

That is, whether the reduced system converges or diverges is dependent on w. In other words, the adjustment of the dynamics of the reduced system is possible by only adjusting w.

By using a parameter

$$0 < \kappa_1 < 1 \qquad (32)$$

and buy using the following condition

$$w + 1 - 2\sqrt{w} < \phi < w + 1 + 2\sqrt{w} \qquad (33)$$

ϕ can be expressed as follows:

$$\phi = w + 1 - 2\sqrt{w} + 4\kappa_1\sqrt{w} \qquad (34)$$

The polar coordinates expression of λ is the following equation when λ is a complex number.

$$\lambda^k = |\lambda|^k e^{k\theta} \qquad (35)$$

It is clear from Equations (12) and (35) that the trajectory of the reduced system vibrates periodically.

If T is to be the period of the reduced system (12), then

$$\begin{aligned} T &= \frac{2\pi}{\theta} = \frac{2\pi}{\tan^{-1}\frac{Im}{Re}} \\ &= \frac{2\pi}{\tan^{-1}\frac{\sqrt{-(w+1-\phi)^2+4w}}{w+1-\phi}} \quad (\phi < w + 1) \end{aligned} \qquad (36)$$

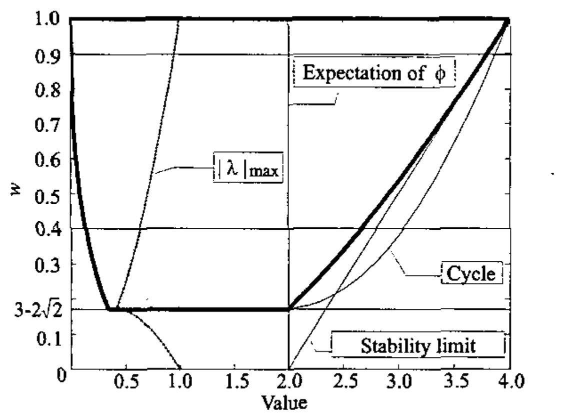

Figure 1: Parameter w and stability limit of the reduced system of PSO.

where

$$\begin{aligned} \mathrm{Re} &= \frac{w+1-\phi}{2} \\ \mathrm{Im} &= \frac{\sqrt{-(w+1-\phi)^2+4w}}{2} \end{aligned} \tag{37}$$

By substituting $w+1-\phi = 2\sqrt{w} - 4\kappa_1\sqrt{w}$ in equation (42) for equation (36), then

$$T = \frac{2\pi}{\tan^{-1}\frac{2\sqrt{\kappa_1-\kappa_1^2}}{1-2\kappa_1}} \quad (\kappa_1 < 0.5), \qquad \frac{2\pi}{\pi + \tan^{-1}\frac{2\sqrt{\kappa_1-\kappa_1^2}}{1-2\kappa_1}} \quad (\kappa_1 > 0.5) \tag{38}$$

By using a parameter $0 < \kappa_2 < 1$, we have

$$\frac{c_2}{c_1} = \frac{\kappa_2}{1-\kappa_2} \tag{39}$$

and

$$\begin{aligned} c_1 &= 2(1-\kappa_2)\phi \\ c_2 &= 2\kappa_2\phi \end{aligned} \tag{40}$$

The above results provide a projected position of i-th particle of the swarm is defined as the following equation:

$$v_{ij}^{k+1} = w \cdot v_{ij}^k + 2(1-\kappa_2)\phi \cdot \mathrm{rand}() \cdot (pbest_{ij} - x_{ij}^k) + 2\kappa_2\phi \cdot \mathrm{rand}() \cdot (gbest_j - x_{ij}^k) \tag{41}$$

where

$$\phi = w + 1 - 2\sqrt{w} + 4\kappa_1\sqrt{w}\kappa_2 = \frac{c_2}{c_1+c_2} \tag{42}$$

The qualitative relation between the search trajectory of the reduced model of PSO and the parameters (w, κ_1 and κ_2) is summarized as follows:

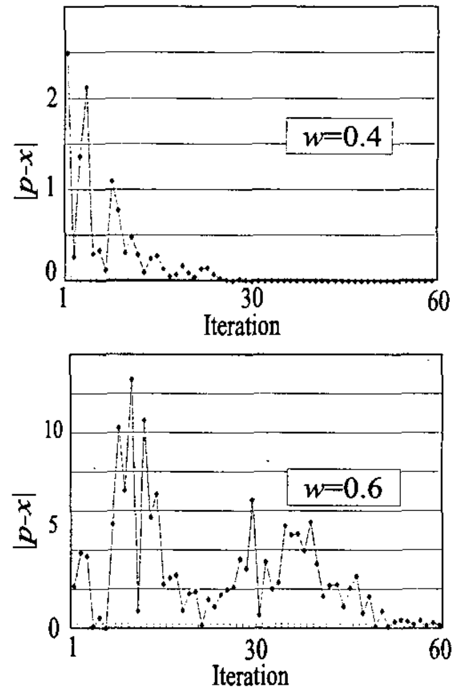

Figure 2: Relationship between search trajectories of the reduced model of PSO and parameter w.

w : If $0 \leq w < 1$, the convergence tendency of the reduced system strengthens as w becomes small. If $1 < w$, the divergence tendency of the reduced system strengthens as w becomes large.

κ_1: $0 < \kappa_1 < 1$, the dynamics of the reduced system becomes vibrational as κ_1 becomes large.

κ_2: $0 < \kappa_2 < 1$, while p is put in the neighborhood of *pbest* if $\kappa_2 \approx 0$, p is put in the neighborhood of *gbest* if $\kappa_2 \approx 1$.

3.3 Analysis of Inertia Weights Approach

As a parameter adjustment method of PSO, the IWA (Inertia Weights Approach) has been proposed. Figure 1 shows the stability region obtained by the above analysis.

While the value of w is made to decrease gradually with the increase in the number of iterations in IWA, parameters $w_{max} = 0.9, w_{min} = 0.4, c_1 = c_2 = 2.0$ are recommended in order to improve the search efficiency of PSO. $w = 1$ corresponds to a stability limit of the reduced system, and the system has the high stability when $w < 0.5$.

Therefore, w is adjusted in order to reach the region $w < 0.5$ with the high stability margin from the stability limit $w = 1$. By this adjustment strategy of parameter w, while the diversification in search is realized in the

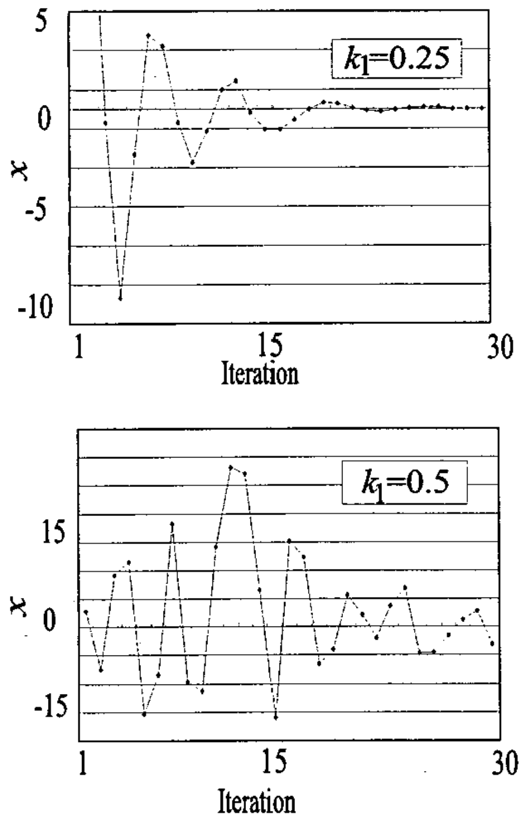

Figure 3: Relationship between search trajectories of the reduced model of PSO and parameter κ_1.

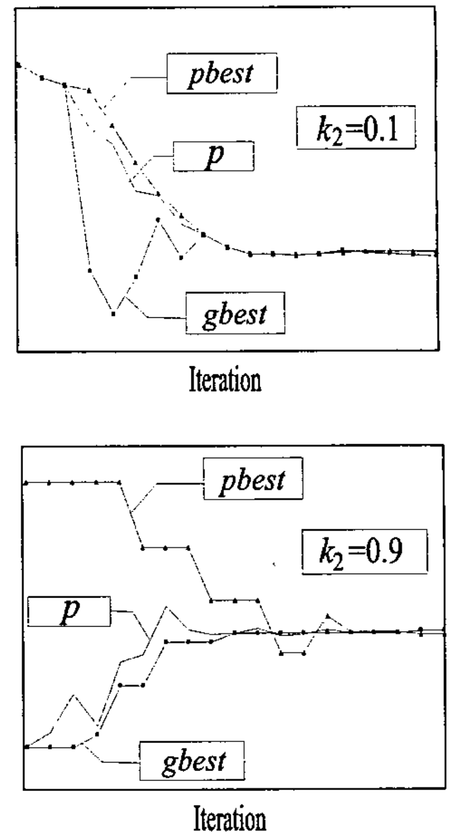

Figure 4: Relationship between parameter κ_2 and parameter p.

initial stage of the search, the intensification in search is also realized in the final stage of the search.

4 Experimental results

Some simulations were carried out in order to examine the relationship between search trajectories of the reduced model of PSO and the parameters w, κ_1 and κ_2.

Figure 2 shows simulation results which were carried out under the condition $\kappa_1, \kappa_2 =$ constant value in order to examine the relationship between a search trajectory and parameter w. It is seen from Figure 2 that the trajectory of the reduced system stably approaches to point p within 30 iterations when w is small, but the trajectory does not approach to point p within 30 iterations when w is large.

Figure 3 shows simulation results which were carried out under the condition $w, \kappa_2 =$ constant value in order to examine the relationship between a search trajectory and parameter κ_1. It is seen from Figure 3 that the trajectory of the reduced system is asymptotic when κ_1 is small, but the trajectory is vibrational when κ_1 is large.

Figure 4 shows simulation results which were carried out under the condition $w, \kappa_1 =$ constant value in order to examine the relationship between a search trajectory and parameter κ_2. It is seen from figure that p is put in the neighborhood of *pbest* if $\kappa_2 \approx 0$ and p is put in the neighborhood of *gbest* if $\kappa_2 \approx 1$.

Figure 5 indicates the relationship between search trajectories of the reduced model of PSO and varying parameter p. It is verified that the system follows changing p.

On the other hand, in order to analyze the behavior of trajectories several types of PSO were applied to the following typical nonconvex optimization problem:

$$\text{min.} \quad f(x_1, x_2) = \sum_{i=1}^{2} (x_i^4 - 16x_i^2 + 0.5x_i) \qquad (43)$$

$$\text{subj. to} \quad -50.0 \le x_1, x_2 \le 50.0$$

Contours for objective function $f(x_1, x_2)$ are shown in Figure 6. The success rates of each simulation using the above optimization problems are shown in Figures 7 and 8; SA indicates simulated annealing method in these figures.

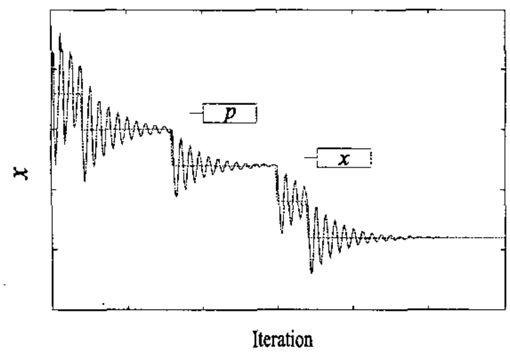

Figure 5: Relationship between search trajectories of the reduced model of PSO and varying parameter p.

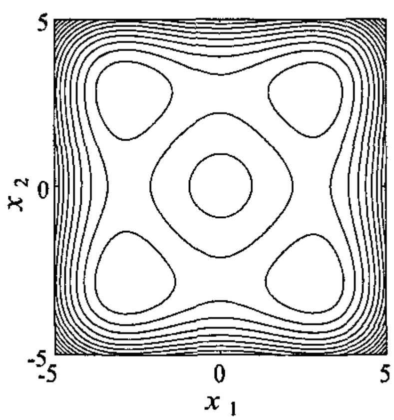

Figure 6: Contours for objective function $f(x_1, x_2)$

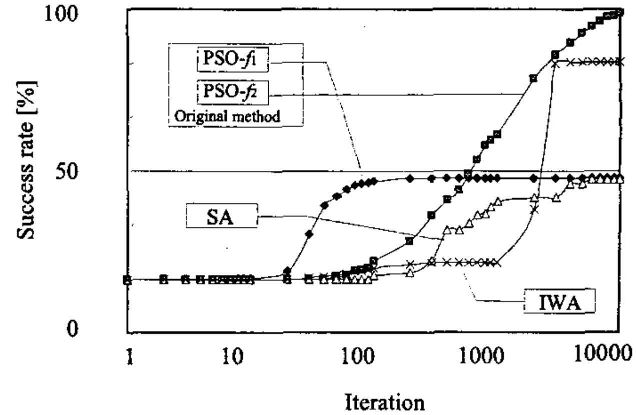

Figure 7: Searching ability of each method for problem (43)

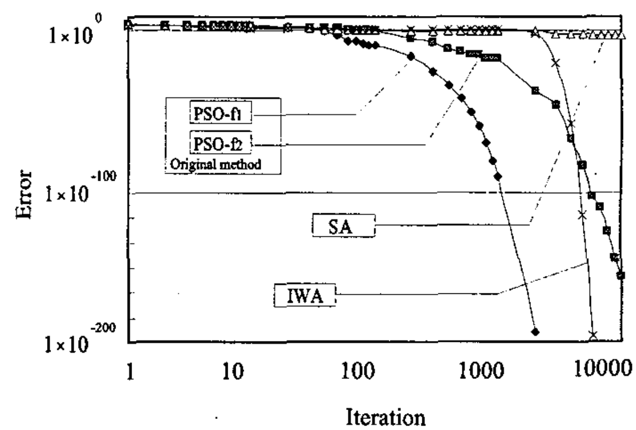

Figure 8: Searching ability of each method for problem (43)

5 Conclusion

In this paper, how the Particle Swarm Optimization algorithm works is explored on the basis of the stability analysis in discrete dynamical systems. Some improvements to the original PSO is proposed and tested based on the analysis and the simulation. The proposed approach for tuning the parameters of PSO allows control over the dynamical characteristics of the particles in PSO.

It should be minded that the simplified model does not represent the original PSO situation because population sizes are generally greater than one and seldom a particle operates on only one dimension.

Furthermore, the weighted best point p is not static but dynamic, often exhibiting complex behavior. Therefore this simplified model is contrived only to provide some information on how each particle behave without the interaction among the particles.

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