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On Stability Characterization of Discrete-time Piecewise Linear Systems

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***Abstract*— A complete stability characterization is given for a special class of discrete-time piecewise linear systems. The characterization is based on solving and then combining two separate subproblems: one is to obtain stabilizing switching sequences, and the other is to generate switching structure–preserving state-space partitions. Checking the stability of a finite number of periodic switching sequences is sufficient for this characterization.**

I. INTRODUCTION

In the last two decades there has been a rapid increase in interest and study of hybrid systems which exhibit both continuous and discrete dynamics. These systems arise from different areas such as intelligent traffic systems and industrial process control. Modeling and control approaches for hybrid systems are surveyed in [1], [2]. Hybrid systems are widely used to reduce the complexity of modeling nonlinear systems. Many nonlinear systems can be modeled as a set of linear systems with switches occurring between them. This modeling approach introduces a class of hybrid systems which are known as piecewise linear systems.

Although there has been a lot of important progress in hybrid systems theory, stability analysis of piecewise linear systems remains a major challenge. Most of existing results are based on using common Lyapunov function and multiple Lyapunov functions, which require quadratic stability and piecewise quadratic stability, respectively [2]–[5]. All these results are conservative because they give only sufficient conditions for asymptotic stability. The stability of a piecewise system can not be inferred from the stability of its component subsystems because, even in the case where all linear subsystems are stable, the underlying switching structure can make the overall system unstable. In [6] it is shown that the problem of determining the stability of a piecewise linear system is NP-hard even in the simplest case where the system has two subsystems.

A new approach for the stability analysis of discrete-time piecewise linear systems has been presented in [7], [8]. This approach involves solving two subproblems separately: one is to identify all stabilizing sets of switching sequences independently of the underlying switching structure, and the other is to generate switching structure–preserving partitions of the state space without considering stability. In particular, the solution to the first subproblem is based on the results in [9], which give a convex characterization of stabilizing switching sequences. The solutions to the two subproblems are combined in order to obtain a convex programming–based stability analysis of piecewise linear systems.

In this paper, a special class of discrete-time piecewise linear systems is identified, for which using the "separation" approach discussed above yields a complete stability characterization without incurring conservatism. This class consists of all systems such that, after a finite number of steps, one is not able to refine the state-space partition further while preserving the underlying switching structure. It is shown that, for each system within this class, the state space can be divided into two parts such that the system is stable on one part and such that the system is not stable on any subset, with nonempty interior, of the other part. Also, it is inferred that checking the stability of a finite number of eventually periodic switching sequences is sufficient for obtaining this characterization.

The rest of the paper is organized as follows. Section II states the stability analysis problem and introduces the two subproblems which must be solved. An algorithm that combines the subproblems to yield a stability analysis for piecewise linear systems is also given in this section. Section III identifies a class of piecewise linear systems and presents a complete stability characterization for the class. The section also provides some numerical examples to demonstrate the results. Section IV concludes the paper and discusses future work.

II. PROBLEM STATEMENT

Given $A_1, \ldots, A_N \in \mathbb{R}^{n\times n}$, let

$$\mathcal{A} = \{A_1, \ldots A_N\}.$$

Let

$$\mathcal{D} = \{D_1, \ldots, D_N\}$$

be a partition of the Euclidean space $\mathbb{R}^n$ into a number of polyhedral cells; that is, $D_1, \ldots, D_N$ are convex, but not necessarily closed or bounded, polyhedra such that $\bigcup_{i=1}^N D_i = \mathbb{R}^n$ and $D_i \cap D_j = \varnothing$ whenever $i \neq j$. Then the pair $(\mathcal{A}, \mathcal{D})$ defines the *discrete-time piecewise linear system* represented by

$$x(t+1) = A_{\theta(t)}x(t) \tag{1}$$

for $x(0) \in \mathbb{R}^n$ and $t = 0, 1, \ldots,$ where $\theta(t) = i$ whenever $x(t) \in D_i$. Given an initial state $x(0)$, the state-space partition $\mathcal{D}$ completely determines the *switching sequence* $\theta = (\theta(0), \theta(1), \ldots)$.

This paper concerns the stability analysis of discrete-time piecewise linear systems. The stability notion which is used in the paper is that of uniform exponential stability.

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Definition 1: Let $C \subset \mathbb{R}^n$. The discrete-time piecewise linear system $(\mathcal{A}, \mathcal{D})$ is said to be *C-uniformly exponentially stable* or *uniformly exponentially stable on* C if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that

$$\|x(t)\| \leq c\lambda^{t-t_0} \|x(t_0)\| \tag{2}$$

for all $x(t_0) \in C$ and for all t, $t_0 \in \{0, 1, \ldots\}$ with $t \geq t_0$.

Our stability notion is similar to, but different from the notion defined in [3] because we require that the maximum overshoot c and exponential decay rate λ for the state be uniform in time. It is well-known that this added uniformity requirement ensures the state decays robustly against unforeseen perturbation of the state; moreover, it is essential in achieving nonconservatism in our stability analysis.

Our stability analysis problem for discrete-time piecewise linear systems is to find the biggest subset C of the state space $\mathbb{R}^n$ such that $(\mathcal{A}, \mathcal{D})$ is C-uniformly exponentially stable. It has been proposed in [7], [8] to address this problem by solving the following two subproblems separately and then combining their solutions:

- To characterize stabilizing sets of switching sequences for $\mathcal{A}$.
- To generate switching structure–preserving state-space partitions for $(\mathcal{A}, \mathcal{D})$.

We will identify a class of piecewise linear systems where this separation approach gives a complete stability characterization without incurring conservatism. This characterization includes both the boundary points and the interior points of the polyhedral cells that define the switching structure.

A. Stabilizing Sets of Switching Sequences

In this subsection, the first subproblem of characterizing all stabilizing sets of switching sequences for $\mathcal{A}$ is described. In [9] it is shown that obtaining so-called maximal sets of switching paths over different path lengths amounts to identifying all the stabilizing sets of switching sequences.

Definition 2: A set Θ of switching sequences is said to be *uniformly exponentially stabilizing* for $\mathcal{A}$ if there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that the state equation (1) satisfies (2) over all $t_0 \geq 0$, $t \geq t_0$, $x(t_0) \in \mathbb{R}^n$, and all switching sequences $\theta \in \Theta$.

For a given nonnegative integer L, each $(i_0, \ldots, i_L) \in \{1, \ldots, N\}^{L+1}$ shall be called a *switching path of length* L (or an *L-path*).

Definition 3: Let L be a nonnegative integer. A nonempty subset $\mathcal{N}$ of $\{1, \ldots, N\}^{L+1}$ is said to be an *admissible set of L-paths* if for each $(i_0, \ldots, i_L) \in \mathcal{N}$ there exists an integer $M > L$ and a switching path $(i_{L+1}, \ldots, i_M)$ such that $(i_{M-L}, \ldots, i_M) = (i_0, \ldots, i_L)$ and $(i_t, \ldots, i_{t+L}) \in \mathcal{N}$ for $0 \leq t \leq M - L$. Moreover, if there exist an indexed family of matrices $X_{j_1 \cdots j_L} \succ 0$ (i.e., symmetric and positive definite) for $(j_0, \ldots, j_L) \in \mathcal{N}$, such that

$$A_{i_L}^T X_{i_1 \cdots i_L} A_{i_L} - X_{i_0 \cdots i_{L-1}} \prec 0$$

for all $(i_0, \ldots, i_L) \in \mathcal{N}$, then $\mathcal{N}$ is said to be an $\mathcal{A}$-*admissible* set of L-paths.

If $\mathcal{N}$ is an $\mathcal{A}$-admissible set of L-paths, then $\mathcal{N}$ yields a set Θ of switching sequences

$$\Theta = \{\theta : (\theta(t), \ldots, \theta(t + L)) \in \mathcal{N}, t = 0, 1, \ldots\},$$

which is uniformly exponentially stabilizing for $\mathcal{A}$ [9]. Following [9], we define the smallest $\mathcal{A}$-admissible sets and largest $\mathcal{A}$-admissible sets of switching paths.

Definition 4: Let L be a nonnegative integer. Let $\mathcal{N}$ be an $\mathcal{A}$-admissible set of L-paths. If the only $\mathcal{A}$-admissible set $\tilde{\mathcal{N}}$ of L-paths satisfying $\tilde{\mathcal{N}} \subset \mathcal{N}$ is $\mathcal{N}$ itself, then $\mathcal{N}$ is said to be $\mathcal{A}$-*minimal*. Similarly, If the only $\mathcal{A}$-admissible set $\tilde{\mathcal{N}}$ of L-paths satisfying $\mathcal{N} \subset \tilde{\mathcal{N}}$ is $\mathcal{N}$ itself, then $\mathcal{N}$ is called $\mathcal{A}$-*maximal*.

Associated with each $\mathcal{A}$-minimal set is a periodic switching sequence which is uniformly exponentially stabilizing for $\mathcal{A}$; e.g., if $\mathcal{N} = \{(i_0, \ldots, i_L), (i_1, \ldots, i_{L+1}), \ldots, (i_M, i_0, \ldots, i_{L-1})\}$ is an $\mathcal{A}$-minimal set of L-paths, then the periodic switching sequence $\theta = (i_0, \ldots, i_M, i_0, \ldots, i_M, \ldots)$ is uniformly exponentially stabilizing for $\mathcal{A}$. It is readily seen that $\mathcal{A}$-maximal sets are finite unions of $\mathcal{A}$-minimal sets. Finding $\mathcal{A}$-maximal sets of L-paths over all path lengths L leads to identifying all uniformly stabilizing sets of switching sequences [9]. An algorithm for finding the $\mathcal{A}$-maximal sets is presented in [8].

B. Switching Structure–Preserving State-Space Partitions

In this subsection, how to address the second subproblem of recursively refining the state-space partition is presented. The solution to this problem is an increasing sequence of state-space partitions that respect the switching structure of the piecewise linear system which is dictated by the dynamics of the system.

Given $D_1, \ldots, D_N \subset \mathbb{R}^n$, which partition the state space, define $D_{(i_0,\ldots,i_L)} \subset \mathbb{R}^n$ by

$$D_{(i_0,\ldots,i_L)} = \left\{x \in D_{(i_0,\ldots,i_{L-1})} : A_{i_0} x \in D_{(i_1,\ldots,i_L)}\right\} \tag{3}$$

recursively for $L = 1, 2, \ldots$, and for $(i_0, \ldots, i_L) \in \{1, \ldots, N\}^{L+1}$. From (3) it is inferred that $D_{(i_0,\ldots,i_{L+1})}$ is the set of all states in $D_{(i_0,\ldots,i_L)}$ which will evolve to a state in $D_{(i_1,\ldots,i_{L+1})}$ in one step. Then the *L-path partition* of $\mathbb{R}^n$ is defined by the indexed family

$$\mathcal{D}_L = \left\{D_{(i_0,\ldots,i_L)} : (i_0, \ldots, i_L) \in \{1, \ldots N\}^{L+1}\right\}$$

for each path length L, where $\mathcal{D}_0 = \mathcal{D}$.

Each cell $D_{(i_0,\ldots,i_L)}$ in $\mathcal{D}_L$ is assumed polyhedral, so its interior can be represented by a componentwise inequality of the form

$$E_{(i_0,\ldots,i_L)} x + e_{(i_0,\ldots,i_L)} < 0. \tag{4}$$

Using this representation and also the definition of $D_{(i_0,\ldots,i_{L+1})}$ it is obtained that the interior of $D_{(i_0,\ldots,i_{L+1})}$ is the set of all $x \in \mathbb{R}^n$ such that

$$\begin{aligned} E_{(i_0,\ldots,i_L)} x + e_{(i_0,\ldots,i_L)} &< 0, \\ E_{(i_1,\ldots,i_{L+1})} A_{i_0} x + e_{(i_1,\ldots,i_{L+1})} &< 0. \end{aligned} \tag{5}$$

Solving these inequalities for all $(i_0, \ldots, i_{L+1})$, we obtain a state-space partition $\mathcal{D}_{L+1}$ which is finer than $\mathcal{D}_L$ [8].

C. Combination of Two Subproblems

The two subproblems in Subsections II-A and II-B are combined to give an algorithm to obtain a $C \subset \mathbb{R}^n$ such that the system $(\mathcal{A}, \mathcal{D})$ is C-uniformly exponentially stable.

Consider a cell $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ with nonempty interior. Let $\theta(0) = i_0, \ldots, \theta(L) = i_L$. Choose a $D_{(i_0,\ldots,i_{L+1})} \in \mathcal{D}_{L+1}$ with nonempty interior, and put $\theta(L+1) = i_{L+1}$. Choose a $D_{(i_1,\ldots,i_{L+2})} \in \mathcal{D}_{L+1}$ with nonempty interior, and put $\theta(L+2) = i_{L+2}$. Proceeding in this manner, we obtain a switching sequence $\theta = (\theta(0), \theta(1), \ldots)$, which we call a *switching sequence generated by* $D_{(i_0,\ldots,i_L)}$ *and* $\mathcal{D}_{L+1}$. Each switching sequence generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ produces an infinite sequence of L-paths

$$\theta_L = ((\theta(0), \ldots, \theta(L)), (\theta(1), \ldots, \theta(L+1)), \ldots), \qquad (6)$$

which we call a *chain of* L*-paths* (or an L*-path chain*) *generated by* $D_{(i_0,\ldots,i_L)}$ *and* $\mathcal{D}_{L+1}$. The *limit set* of an L-path chain θ_L is the set of all subsequential limits of θ_L; that is, it is the set of L-paths in $\{1, \ldots, N\}^{L+1}$ that occur infinitely many times in the switching sequence θ. If $\theta = (\theta(0), \theta(1), \ldots)$, then the L-path chain θ_L shall be said to be the L*-path chain of* θ.

Each initial state yields a switching sequence θ. Thus the stability of the piecewise linear system is determined by the limit sets of the L-path chains of such θ over all initial states.

Lemma 1: Let L be a nonempty integer. Let $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ have nonempty interior. If each chain of L-paths generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ has a limit set that is contained in an $\mathcal{A}$-maximal set of L-paths, then the discrete-time piecewise linear system $(\mathcal{A}, \mathcal{D})$ is $D_{(i_0,\ldots,i_L)}$-uniformly exponentially stable.

Proof: See [8]. ■

An algorithm to obtain an increasing sequence of subsets C_0, $C_1, \ldots$ of $\mathbb{R}^n$ such that $C_L \subset C_{L+1}$ and the system $(\mathcal{A}, \mathcal{D})$ is C_L-uniformly exponentially stable for all path lengths L is presented in [7], [8], and it is reproduced here:

Algorithm 1: Put $C_{-1} = \emptyset$ and $L = 0$.

- Step 1. Construct $\mathcal{D}_{L+1}$.
- Step 2. Generate all $\mathcal{A}$-maximal sets of L-paths.
- Step 3. Let C_L be the union of C_{L-1} and all $D_{(i_0,\ldots,i_L)}$ such that the limit set of each chain of L-paths generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is contained in an $\mathcal{A}$-maximal set of L-paths.
- Step 4. If $\mathcal{D}_L = \mathcal{D}_{L+1}$ (except possibly for the cell boundaries), then stop. Otherwise, increment L to $L+1$ and go to step 1.

III. Stability Characterization

In this section, we focus on the discrete-time piecewise linear systems for which Algorithm 1 terminates at some path length L. Even though there are cases where the algorithm never terminates and yet we are able to completely determine the stability of the piecewise linear system $(\mathcal{A}, \mathcal{D})$ [8], we will not consider such cases in this paper.

Definition 5: Given a nonnegative integer L, a cell $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ is said to be $(\mathcal{A}, \mathcal{D})$*-invariant* if the interior of $D_{(i_0,\ldots,i_L)}$ is equal to the interior of $D_{(i_0,\ldots,i_{L+1})} \in \mathcal{D}_{L+1}$ for some $i_{L+1} \in \{1, \ldots, N\}$. If every $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ is $(\mathcal{A}, \mathcal{D})$-invariant, then the L-path state-space partition $\mathcal{D}_L$ is said to be $(\mathcal{A}, \mathcal{D})$*-invariant*.

It will be shown that, if $\mathcal{D}_L$ is $(\mathcal{A}, \mathcal{D})$-invariant for some L, a complete stability characterization of the discrete-time piecewise linear system $(\mathcal{A}, \mathcal{D})$ is possible.

A. Stability Analysis of Invariant Cells

In this subsection, we obtain the stability analysis results for the invariant cells. The following definition is adapted from that in [10].

Definition 6: Let q be a nonnegative integer; let p be a positive integer. A switching sequence $\theta = (\theta(0), \theta(1), \ldots)$ is said to be (q,p)*-eventually periodic* (or *eventually* p*-periodic*) if

$$\begin{aligned}&\big(\theta(q+kp), \ldots, \theta(q+(k+1)p-1)\big)\\ &\qquad = \big(\theta(q+(k+1)p), \ldots, \theta(q+(k+2)p-1)\big)\end{aligned}$$

for all $k = 0, 1, \ldots$.

The following lemma establishes that, if the state-space partitions $\mathcal{D}_L$ and $\mathcal{D}_{L+1}$ coincide with each other except possibly for the cell boundaries, then the switching sequence generated by each $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ and $\mathcal{D}_{L+1}$ is unique and eventually periodic.

Lemma 2: If $\mathcal{D}_L$ is $(\mathcal{A}, \mathcal{D})$-invariant for some nonnegative integer L, then the switching sequence generated by each $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ and $\mathcal{D}_{L+1}$ is unique and (q,p)-eventually periodic with $q + p \le N^{L+1}$.

Proof: Suppose there are m cells with nonempty interior in $\mathcal{D}_L$. Consider a cell $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$. Since $D_{(i_0,\ldots,i_L)}$ is $(\mathcal{A}, \mathcal{D})$-invariant, there exists a unique $i_{L+1} \in \{1, \ldots, N\}$ such that $D_{(i_0,\ldots,i_L,i_{L+1})}$ has nonempty interior and such that each state in $D_{(i_0,\ldots,i_L)}$ at some time t will evolve to a point in $D_{(i_1,\ldots,i_{L+1})}$ at time $t+1$. If $(i_0, \ldots, i_L) = (i_1, \ldots, i_{L+1})$, then the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is given by $\theta = (i_0, i_0, \ldots)$ and thus $(0,1)$-periodic. Otherwise, since $D_{(i_1,\ldots,i_{L+1})}$ is $(\mathcal{A}, \mathcal{D})$-invariant, there exists a unique $i_{L+2} \in \{1, \ldots, N\}$ such that $D_{(i_1,\ldots,i_{L+1},i_{L+2})}$ has nonempty interior and such that each state in $D_{(i_1,\ldots,i_{L+1})}$ at time $t+1$ will evolve to a point in $D_{(i_2,\ldots,i_{L+2})}$ at time $t+2$. If either $(i_0, \ldots, i_L) = (i_2, \ldots, i_{L+2})$ or $(i_1, \ldots, i_{L+1}) = (i_2, \ldots, i_{L+2})$ holds, then the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is given by either $\theta = (i_0, i_1, i_0, i_1, \ldots)$, which is $(0,2)$-periodic, or $\theta = (i_0, i_1, i_1, \ldots)$, which is $(1,1)$-periodic.

Suppose we have proceeded in this manner and obtained k disjoint cells $D_{(i_0,\ldots,i_L)}, \ldots, D_{(i_k,\ldots,i_{L+k})} \in \mathcal{D}_L$ with nonempty interior such that, whenever $x(t) \in D_{(i_0,\ldots,i_L)}$, we have $x(t+j) \in D_{(i_j,\ldots,i_{L+j})}$ for all $j \in \{0, \ldots, k\}$. Here, we necessarily have $k+1 \le m$. Since $D_{(i_k,\ldots,i_{L+k})}$ is $(\mathcal{A}, \mathcal{D})$-invariant, there exists a unique $i_{L+k+1} \in \{1, \ldots, N\}$ such that $D_{(i_k,\ldots,i_{L+k},i_{L+k+1})}$ has nonempty interior and such that each state in $D_{(i_k,\ldots,i_{L+k})}$ at some time $t+k$ will evolve to a point in $D_{(i_{k+1},\ldots,i_{L+k+1})}$ at time $t+k+1$. There are two possible cases: One is the case of $k+1 < m$, and the other is that of $k+1 = m$.

- *Case of* $k+1 < m$. If $(i_q, \ldots, i_{L+q}) = (i_{k+1}, \ldots, i_{L+k+1})$ for some $q \in \{0, \ldots, k\}$, then the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is given by
$$\theta = (i_0, \ldots, i_q, \ldots, i_k, i_q, \ldots, i_k, \ldots)$$
and thus $(q, k-q+1)$-periodic; putting $p = k-q+1$ yields that $q+p = k+1 \leq m$. Otherwise, we increment k to $k' = k+1$ and again obtain two possible cases: $k'+1 < m$ and $k'+1 = m$.
- *Case of* $k+1 = m$. In this case, we must have $(i_q, \ldots, i_{L+q}) = (i_{k+1}, \ldots, i_{L+k+1})$ for some $q \in \{0, \ldots, k\}$, and thus the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is (q,p)-periodic with $q+p = k+1 = m$.

Finally, since the initial state-space partition $\mathcal{D}_0$ has N cells, the L-path partition D_L will have at most N^{L+1} cells, so the desired result follows from $m \leq N^{L+1}$. ■

Suppose $\mathcal{D}_L$ is $(\mathcal{A},\mathcal{D})$-invariant for some path length L. Let $\theta = (\theta(0), \theta(0), \ldots)$ be the unique eventually periodic switching sequence generated by some $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ and $\mathcal{D}_{L+1}$. By Lemmas 1 and 2, the discrete-time piecewise linear system $(\mathcal{A},\mathcal{D})$ is $D_{(i_0,\ldots,i_L)}$-uniformly exponentially stable if this switching sequence θ is uniformly exponentially stabilizing for $\mathcal{A}$. Moreover, whether a (q,p)-periodic switching sequence θ is uniformly exponentially stabilizing for $\mathcal{A}$ or not can be determined by simply computing the spectral radius of the matrix product $A_{\theta(q+p-1)} \cdots A_{\theta(q)}$. However, if θ is not uniformly exponentially stabilizing for $\mathcal{A}$, one cannot conclude that none of the states in the cell $D_{(i_0,\ldots,i_L)}$ converges to the origin. Nevertheless, the following lemmas indicate that, even in this case, the states in $D_{(i_0,\ldots,i_L)}$ that converge to the origin reside within a set of measure zero.

Lemma 3: Suppose $\mathcal{D}_L$ is $(\mathcal{A},\mathcal{D})$-invariant for some nonnegative integer L. Let θ be the switching sequence generated by a cell $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ and $\mathcal{D}_{L+1}$. If θ is not uniformly exponentially stabilizing for $\mathcal{A}$, then there is no $T \subset D_{(i_0,\ldots,i_L)}$ with nonempty interior such that the discrete-time piecewise linear system $(\mathcal{A},\mathcal{D})$ is uniformly exponentially stable on the interior of T.

Proof: To prove the desired result by contradiction, suppose there exists a $T \subset D_{(i_0,\ldots,i_L)}$ with nonempty interior such that $(\mathcal{A},\mathcal{D})$ is uniformly exponentially stable on the interior of T. Then there exist $c \geq 1$ and $\lambda \in (0,1)$ such that
$$\left\|A_{\theta(t-1)} \cdots A_{\theta(t_0)} x\right\| < c\lambda^{t-t_0} \|x\|$$
for all $x \in T$ and for all $t,\ t_0 \in \{0, 1, \ldots\}$ with $t \geq t_0$. Since the interior of T is not empty, there exists an open ball $B \subset T$ such that one can choose n linearly independent vectors $x_1, \ldots, x_n$ from B.

Choose a $t_0 \in \{0, 1, \ldots\}$ and an $x(t_0) \in T$. Since $x(t_0)$ is a linear combination of $x_1, \ldots, x_n$, we have
$$x(t_0) = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \end{bmatrix}^T \|x(t_0)\|$$
for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, which satisfy
$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_n \end{bmatrix}^T \|x(t_0)\| = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{-1} x(t_0).$$
As $x_1, \ldots, x_n \in B$, where B is bounded, there exists a $\hat{c} \geq 1$ such that
$$c \sum_{i=1}^{n} |\alpha_i| \|x_i\| \leq \hat{c}$$
holds regardless of $x(t_0)$. Then
$$\begin{aligned} \left\|A_{\theta(t-1)} \cdots A_{\theta(t_0)} x(t_0)\right\| &= \left\| A_{\theta(t-1)} \cdots A_{\theta(t_0)} \left(\sum_{i=1}^{n} \alpha_i x_i \|x(t_0)\| \right) \right\| \\ &\leq \sum_{i=1}^{n} |\alpha_i| \left\|A_{\theta(t-1)} \cdots A_{\theta(t_0)} x_i\right\| \|x(t_0)\| \\ &\leq \sum_{i=1}^{n} |\alpha_i|\, c\lambda^{t-t_0} \|x_i\| \|x(t_0)\| \\ &\leq \hat{c}\lambda^{t-t_0} \|x(t_0)\|. \end{aligned}$$
This implies that θ is uniformly exponentially stabilizing for $\mathcal{A}$, which contradicts our assumption. ■

Lemma 4: Suppose $\mathcal{D}_L$ is $(\mathcal{A},\mathcal{D})$-invariant for some nonnegative integer L. Let θ be the switching sequence generated by a cell $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ and $\mathcal{D}_{L+1}$. If θ is not uniformly exponentially stabilizing for $\mathcal{A}$, and if the discrete-time piecewise linear system $(\mathcal{A},\mathcal{D})$ is T-uniformly exponentially stable for some $T \subset D_{(i_0,\ldots,i_L)}$, then T is the intersection of $D_{(i_0,\ldots,i_L)}$ and a proper subspace of $\mathbb{R}^n$ (i.e., a linear space whose dimension is strictly less than n).

Proof: By lemma 2, the switching sequence θ is (q,p)-periodic and takes the form
$$\theta = (j_1, \ldots, j_q, i_0, \ldots, i_{p-1}, i_0, \ldots, i_{p-1}, \ldots)$$
for some nonnegative integer q and positive integer p, where $(j_1, \ldots, j_q)$ is an empty tuple when $q = 0$. Define
$$\widehat{A} = A_{j_q} \cdots A_{j_1} \quad \text{and} \quad A = A_{i_{p-1}} \cdots A_{i_0}.$$
Let
$$S = \{x \in \mathbb{R}^n \colon A^k \widehat{A} x \to 0 \text{ as } k \to \infty\}.$$
If θ is not uniformly exponentially stabilizing for $\mathcal{A}$, and if $(\mathcal{A},\mathcal{D})$ is T-uniformly exponentially stable for some $T \subset D_{(i_0,\ldots,i_L)}$, then we must have that all $x \in T$ belong to S, which is clearly a subspace of $\mathbb{R}^n$. By Lemma 3, the subspace S does not have interior points, and thus its dimension must be strictly less than n. ■

Since the dimension of the set T in Lemma 4 is strictly less than n, a boundary of a cell can potentially represent such a T. Indeed, if the state-space partition $\mathcal{D}_L$ is $(\mathcal{A},\mathcal{D})$-invariant for some nonnegative integer L and if the switching sequence generated by $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_L$ and $\mathcal{D}_{L+1}$ is not uniformly exponentially stabilizing for $\mathcal{A}$, then the states belonging to the boundary of $D_{(i_0,\ldots,i_L)}$ may or may not converge to the origin. Example 3 will illustrate this point clearly. However, as the example below shows, the set T does not necessarily belong to the boundary of a cell $D_{(i_0,\ldots,i_L)}$.

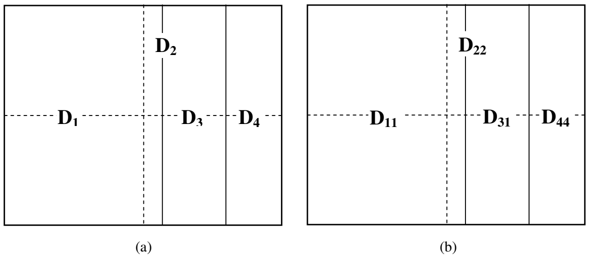

Fig. 1. State-space partitions in *Example 2*, (a) $\mathcal{D}_0$, (b) $\mathcal{D}_1$.

Example 1: Consider the piecewise linear system $(\mathcal{A}, \mathcal{D})$ where $\mathcal{A}$ and $\mathcal{D}$ have

$$A_1 = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$D_1 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 \leq -1 \right\},$$
$$D_2 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon -1 < x_1 < 1 \right\},$$
$$D_3 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 \geq 1 \right\}.$$

It is readily seen that $D_i = D_{ii}$ for all $i = 1, 2, 3$. Thus the state-space partition $\mathcal{D}_0$ is $(\mathcal{A}, \mathcal{D})$-invariant. The cell $D_1 \in \mathcal{D}_0$ and the partition $\mathcal{D}_1$ generate the switching sequence (1, 1, . . .), the cell D_2 and $\mathcal{D}_1$ generate (2, 2, . . .), and D_3 and $\mathcal{D}_1$ generate (3, 3, . . .). Since none of the matrices in $\mathcal{A}$ has spectral radius strictly less than one, these switching sequences are not uniformly exponentially stabilizing for $\mathcal{A}$. Thus, the system is not uniformly exponentially stable on any subset of the state space $\mathbb{R}^2$ with nonempty interior. In fact, letting

$$T = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \end{bmatrix}^T \right\} = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 = 0 \right\}$$

yields that the piecewise linear system is uniformly exponentially stable on T and not uniformly exponentially stable on any subset of $\mathbb{R}^2 \setminus T$. The set T belongs to the interior of the cell D_2, and is a one-dimensional subspace of $\mathbb{R}^2$.

In what has been discussed so far, we have considered the cells with nonempty interior. However, it is possible that the initial state-space partition contains a cell with empty interior, which has its own dynamics that governs the evolution of its states. Then this cell plays the role of a boundary between other cells, although it does not belong to any of them. Since such a cell has measure zero, its stability property is of little importance in many real-world situations where the state is perturbed by noise. Nevertheless, it should be noted that, even in the case where such a cell is a boundary between two cells on which $(\mathcal{A}, \mathcal{D})$ is uniformly exponentially stable, the states belonging to the boundary cell may not converge to the origin as the example below shows.

Example 2: Consider the piecewise linear system $(\mathcal{A}, \mathcal{D})$ where $\mathcal{A}$ and $\mathcal{D}$ have

$$A_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$D_1 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 < 2 \right\},$$
$$D_2 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 = 2 \right\},$$
$$D_3 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon 2 < x_1 < 6 \right\},$$
$$D_4 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 \geq 6 \right\}.$$

The first two state-space partitions are shown in Fig. 1. It is seen that $\mathcal{D}_0$ is an invariant partition. The spectral radii of A_1 and A_3 are less than one, and those of A_2 and A_4 are not less than one. Thus it is deduced that $(\mathcal{A}, \mathcal{D})$ is uniformly exponentially stable on the interior of $D_1 \cup D_3$. However, the states in D_2 do not converge to the origin even though they are boundary points of D_1 and D_3.

B. Stability Characterization

The following theorem is the main result, which gives a stability characterization for discrete-time piecewise linear systems for which finite-path state-space partitions are invariant.

Theorem 5: Suppose $\mathcal{D}_L$ is $(\mathcal{A}, \mathcal{D})$-invariant for some nonnegative integer L. Then, for each $(i_0, \ldots, i_L) \in \{1, \ldots, N\}^{L+1}$ such that $D_{(i_0,\ldots,i_L)}$ has nonempty interior, the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is unique and eventually periodic. Moreover, the following hold true:

(a) If the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is uniformly stabilizing for $\mathcal{A}$, then the discrete-time piecewise linear system $(\mathcal{A}, \mathcal{D})$ is $D_{(i_0,\ldots,i_L)}$-uniformly exponentially stable.

(b) If the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is not uniformly stabilizing for $\mathcal{A}$, then the discrete-time piecewise linear system $(\mathcal{A}, \mathcal{D})$ is not S-uniformly exponentially stable for any subset S of $D_{(i_0,\ldots,i_L)}$ with nonempty interior.

(c) If the switching sequence θ generated by $D_{(i_0,\ldots,i_L)}$ and $\mathcal{D}_{L+1}$ is not uniformly stabilizing for $\mathcal{A}$, and if the discrete-time piecewise linear system $(\mathcal{A}, \mathcal{D})$ is T-uniformly exponentially stable for some $T \subset D_{(i_0,\ldots,i_L)}$, then T is the intersection of $D_{(i_0,\ldots,i_L)}$ and a proper subspace of $\mathbb{R}^n$.

Proof: The result is immediate from Lemmas 1–4. ■

Remark: The key drawback of Algorithm 1 is the high computational complexity of obtaining $\mathcal{A}$-maximal sets. However, in our special case, all the generated switching sequences are (q, p)-eventually periodic with a guaranteed bound on $p + q$, and so computing the spectral radii of associated matrix products is sufficient for checking the

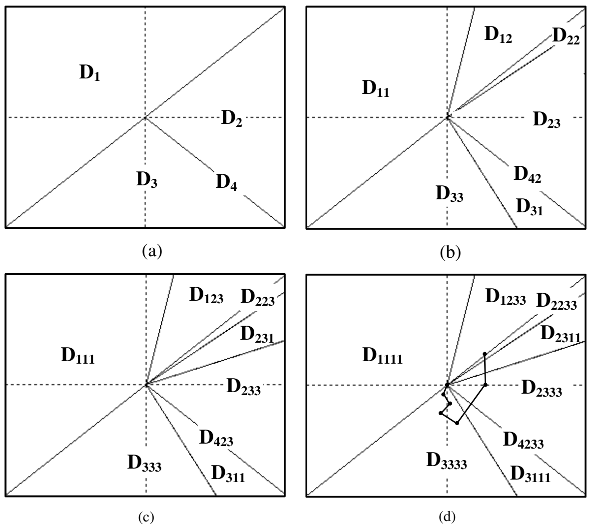

Fig. 2. State-space partitions in *Example 3*, (a) $\mathcal{D}_0$, (b) $\mathcal{D}_1$, (c) $\mathcal{D}_2$, (d) $\mathcal{D}_3$.

stability of the switching sequences and there is no need to obtain the $\mathcal{A}$-maximal sets. Hence the computational burden of the stability analysis is reduced significantly in the identified special case.

Application of Theorem 5 is illustrated by the following example.

Example 3: Consider the piecewise linear system $(\mathcal{A}, \mathcal{D})$ where $\mathcal{A}$ and $\mathcal{D}$ have

$$A_1 = \begin{bmatrix} 5/4 & -1/4 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1/4 & 3 \\ -3 & 1/4 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$D_1 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 \le x_2 \right\},$$
$$D_2 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 > x_2, x_1 > -x_2 \right\},$$
$$D_3 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 > x_2, x_1 < -x_2 \right\},$$
$$D_4 = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2 \colon x_1 = -x_2, x_1 > 0 \right\}.$$

The switching structure–preserving state-space partitions are shown in Fig. 2. It is seen that $\mathcal{D}_2$ is an $(\mathcal{A}, \mathcal{D})$-invariant partition. All the switching sequences generated by each $D_{(i_0,\ldots,i_L)} \in \mathcal{D}_2$ and $\mathcal{D}_3$ are eventually periodic and convergent to either $(1, 1, \ldots)$ or $(3, 3, \ldots)$. It is obtained that $(3, 3, \ldots)$ is a stabilizing switching sequence so $(\mathcal{A}, \mathcal{D})$ is C-uniformly exponentially stable where $C = D_{333} \cup D_{233} \cup D_{123} \cup D_{223} \cup D_{423}$, where D_{423} is a cell with empty interior. Now consider the cell D_{111}, which is not included in C. The boundary between this cell and D_{123} belongs to this cell and forms a subset T with the properties discussed in Lemma 4. Thus all the states on this boundary will converge to the origin. However, the states of the other boundary of this cell will not converge to the origin. Also, it should be mentioned that the states belonging to the boundary of cells on which $(\mathcal{A}, \mathcal{D})$ is uniformly exponentially stable will converge to the origin; a typical state trajectory that starts from a state on the boundary of D_{1233} is shown in Fig. 2. In conclusion, we have divided the state space into two separate parts C and U, where $U = D_{111} \cup D_{311} \cup D_{231}$, such that the system $(\mathcal{A}, \mathcal{D})$ is C-uniformly exponentially stable and not S-uniformly exponentially stable for any $S \subset U$ with nonempty interior.

IV. CONCLUSIONS

A stability characterization for a special class of discrete-time piecewise linear systems was presented. To examine the stability properties of a system in this class, it suffices to identify, and determine the stability of, a finite number of eventually periodic switching sequences. A few numerical examples illustrated various aspects of this result. Despite the inherent difficulty (i.e., NP hardness) of the stability analysis problem for piecewise linear systems, we expect that the proposed approach will be useful in dealing with "small" systems.

This paper was focused on the class of piecewise linear systems for which there exists a partition of the state space which is invariant under switching structure–preserving successive refinement of the initial state-space partition. It has been observed that the proposed stability analysis is potentially applicable to certain piecewise linear systems that do not belong to this class. Thus, future work includes investigation of the situations where the state-space partition can be refined indefinitely and yet a stability characterization can be achieved after a finite amount of computation.

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