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Robustness of Extended Least Squares Based Adaptive Control*

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Abstract

While the standard Least Squares parameter estimation algorithm is applicable for the control of linear stochastic systems in white noise, the Extended Least Squares algorithm is popularly used when the noise is colored. In this paper we examine whether this algorithm, designed to have good stochastic performance, is in fact robust to small unmodeled dynamics and bounded disturbances.

We prove boundedness of a Weighted Extended Least-Squares type *non-interlaced* adaptation law with parameter projection. The nominal plant is assumed to be minimum phase.

1 Introduction

In this paper, we show that the popular Extended Least-Squares (ELS) based one-step ahead adaptive tracking algorithm is robust to both the presence of small unmodeled dynamics and violation of the stochasticity assumption on the noise. Specifically the noise is allowed to be any bounded sequence. The only modification used is a projection of the parameter estimates. The adaptive control algorithm is a weighted extended least-squares type *non-interlaced* adaptation law with projection. Assuming the nominal plant to be minimum-phase (i.e. without the small unmodeled dynamics), we prove that all the signals in the closed-loop system are uniformly bounded.

In [2], Praly *et al.* have proved boundedness of an ELS algorithm which differs in two respects. First, it employs normalization of all the signals entering the update law by a specially constructed signal. The construction of this signal requires a priori knowledge about the nominal plant: namely the stability margin of the B polynomial. Also, it involves additional computation. Second, d *interlaced* algorithms are used, which further adds to the computational burden.

2 Problem Statement

Consider a class of systems of the form (z^{-1} is the unit-delay operator),

$$A(z^{-1})y(t) = z^{-d}B(z^{-1})u(t) + C(z^{-1})\omega(t)\,, \tag{2.1}$$

where $A(z^{-1}) = 1 + a_1z^{-1} + \ldots + a_{p'}z^{-p'}$, $B(z^{-1}) = b_0 + b_1z^{-1} + \ldots + b_{q'}z^{-q'}$, and $C(z^{-1}) = 1 + c_1z^{-1} + \ldots + c_{r'}z^{-r'}$, the coefficients being unknown. $B(z^{-1})$, $C(z^{-1})$ are assumed to have all their zeros within the open unit disk, and $\omega(t)$ is a white noise sequence. The true process under control, however, does not necessarily lie in the above class, but satisfies

$$A(z^{-1})y(t) = z^{-d}B(z^{-1})u(t) + C(z^{-1})w(t) + v'(t)\,, \tag{2.2}$$

where A, B are as earlier, $C(z^{-1}) = 1$, w is any bounded disturbance, and $v'(t)$ represents unmodeled dynamics, which is assumed to satisfy [1]

$$v'^2(t) \le K_v m(t-1) + k_v\,, \tag{2.3}$$

where $m(t)$ is defined by

$$m(t) = \sigma m(t-1) + K_y y^2(t) + K_u u^2(t) + K_3\,, m(0) > 0\,, \tag{2.4}$$

with $0 < \alpha^{1+\delta} < \sigma < 1$, for some $0 \le \delta < 1$ and α is such that all the zeros of $B(z^{-1})$ lie in the open disk $|z| < \alpha$.

It is desired to track a given bounded reference trajectory $y^*(t) \equiv r(t-d)$. Our aim is to study the behavior of an extended-least-squares (ELS) based one-step ahead adaptive tracking law designed under the assumption that the system belongs to the model class (2.1), when the process actually satisfies (2.2).

3 Parametrization and Adaptive Control Law

Define polynomials $F(z^{-1})$ (of degree $d-1$), and $G(z^{-1})$ through,

$$A(z^{-1})F(z^{-1}) + z^{-d}G(z^{-1}) = C(z^{-1})\,. \tag{3.1}$$

Hence, by (2.2) and (3.1), we have $C[y(t) - y^*(t) - Fw(t)] = z^{-d}[BFu(t) + Gy(t) + (1-C)y^*(t+d) - y^*(t+d)] + Fv'(t)$. Define $\phi^T(t) := [u(t), \ldots, u(t-q+1), y(t), \ldots, y(t-p+1), y^*(t+d-1), \ldots, y^*(t+d-r)]^T$, $\theta :=$ [coeffs. of BF, coeffs. of G, coeffs. of $(1-C)]^T$. We will assume that $\|\theta - \theta^0\| \le M_0$ some θ^0, and that $b_0 \ge b_{\min} > 0$, for some positive constants M and $b_{\min}$. This yields

$$C[y(t) - y^*(t) - Fw(t)] = \phi^T(t-d)\theta - y^*(t) + Fv'(t)\,. \tag{3.2}$$

Motivated by this, and assuming $w(\cdot)$ to be a white noise sequence, one designs the certainty-equivalence adaptive control law

$$\phi^T(t)\widehat{\theta}(t) = y^*(t+d)\,, \tag{3.3}$$

where $\widehat{\theta}(t)$ is the estimate of θ at time t.

Assuming that the true process (2.2) is indeed in the class of systems (2.1), one usually uses the well-known ELS algorithm [1] to update the parameter estimate $\widehat{\theta}(\cdot)$:

$$\begin{aligned} \widehat{\theta}(t) &= \widehat{\theta}(t-1) + P(t-1)\phi(t-d)(y(t) - \phi^T(t-d)\widehat{\theta}(t-1) \\ P(t)^{-1} &= P(t-1)^{-1} + \phi(t-d)\phi^T(t-d)\,. \end{aligned}$$

A scheme to bound the condition number of the covariance matrix P is also generally employed.

Usually however, the true plant does not fit into the class of models (2.1). The robustness of ELS-based algorithms in this situation has received very little attention. Here, we establish the boundedness of all closed-loop signals using a *non-interlaced* weighted ELS algorithm where the signals entering the adaptation law need only be normalized by an "extended" regressor.

In [2], Praly et al. show that an *interlaced* ELS-type algorithm with parameter projection, where the signals entering the adaptation law are normalized by a specially constructed signal (which requires knowing the stability margin of the B-polynomial of the nominal plant), ensures bounded closed-loop signals.

We analyze the following parameter update law :

$$\widehat{\theta}''(t) = \widehat{\theta}(t-1) + a(t-1)P(t-1)\overline{\phi}(t-d)\bar{e}_a(t) \tag{3.4}$$

$$P'(t)^{-1} = \lambda(t-1)P(t-1)^{-1} + \overline{\phi}(t-d)\overline{\phi}^T(t-d)\,, \tag{3.5}$$

$$\delta_1 I \ge P(-1) \ge \delta_0 I$$

$$P(t) = \begin{cases} P'(t) & \text{if } trace(P'(t)) \le \delta_1 \\ P(t-1) & \text{otherwise}. \end{cases} \tag{3.6}$$

$$a(t-1) = [1 + \overline{\phi}^T(t-d)P(t-1)\overline{\phi}(t-d)]^{-1} \tag{3.7}$$

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[1] All constants in this paper are positive, unless noted otherwise. K denotes a generic positive constant.

$$\widehat{\theta}'(t) = \widehat{\theta}''(t) + \max\{0, b_{\min} - \widehat{\theta}_1''(t)\}\frac{P_1(t)}{P_{11}(t)} \quad (3.8)$$

where $\widehat{\theta}_1''(t) :=$ first component of the vector $\widehat{\theta}''(t)$ (3.9)

$P_1(t) :=$ first column of the matrix $P(t)$ (3.10)

and $P_{11}(t) := (1,1)$th element of $P(t)$ (3.11)

$$\widehat{\theta}(t) = \theta^0 + [\widehat{\theta}'(t) - \theta^0]\min\{1, \frac{M_0\delta_1}{\delta_0\|\widehat{\theta}'(t) - \theta^0\|}\} \quad (3.12)$$

$$e_a(t) = y(t) - \phi^T(t-d)\widehat{\theta}(t-1) \quad (3.13)$$

$$\overline{\phi}(t) = \frac{\phi(t)}{\sqrt{n(t)}}, \bar{e}_a(t) = \frac{e_a(t)}{\sqrt{n(t)}}, \text{ where} \quad (3.14)$$

$$n(t) = K_1\|\psi(t)\|^2 + K_2 \quad (3.15)$$

$$\psi(t) = [u(t), \ldots, u(t-q-d+2), y(t), \ldots, y(t-p-d+2), y^*(t+d-1), \ldots, y^*(t+1-r)]^T, \quad (3.16)$$

together with the control law (3.3). Note that $\psi(t) \equiv \phi(t)$ in the unit delay case, i.e., when $d = 1$.

Furthermore, the sequence $\lambda(\cdot)$ is assumed to satisfy $(a) 0 < \frac{1}{2} + \epsilon < \lambda' \le \lambda(t) \le 1 - \frac{\delta_0}{K_1} < 1$ $(b) \sum_{t=0}^n [\prod_{j=t+1}^n \lambda(j)] \le K, \forall n \ge 0$ $(c) \prod_{t=0}^n \lambda(t) \le K, \forall n \ge 0$ for some $0 < \epsilon < 1/2$. For example, any $\lambda(\cdot)$ sequence which satisfies $1/2 + \epsilon < \lambda' \le \lambda(t) \le \lambda'' \le 1 - \delta_0/K_1$, $\forall t \ge 0$, is a valid sequence. A common instance of this is $\lambda(t) \equiv \lambda$, $1/2 + \epsilon < \lambda \le 1 - \delta_0/K_1$, which results in an exponentially weighted least-squares based algorithm.

Theorem 1. *The adaptive control law (3.3-3.16), when applied to the true process (2.2), ensures that all closed-loop signals are uniformly bounded.*

The proof of this theorem will proceed through Sections 4-8. The essential idea is to construct a signal W which dominates all other signals, and which has a bounded growth-rate. This signal W is then shown to be bounded. The boundedness of all other signals then follows. The proof begins with some preliminary observations in Section 4. Next, in Section 5, a "switched system" is introduced to construct an intermediate dominating signal z (Such a "switched system" idea was originally used in [3] to study the robustness of a gradient-update based adaptive control algorithm, when applied to a unit-delay, minimum phase plant, to a restricted class of unmodeled dynamics.). In Section 6 we show that over certain finite intervals, z is comparable to the normalization n. Then, a "state error", e_c is defined and the model reference structure of the control law is exploited to show that e_c is generated by a stable system driven by the unmodeled dynamics, and a quantity depending on the parameter estimation error. Finally, the signal W is constructed from the signals z, and the state error e_c. A novel large-signal analysis over finite time-intervals is then shown to yield a contraction property for W. This combined with the bounded growth-rate of W then shows the boundedness of W, and hence of all closed-loop signals.

4 Preliminary Observations

Note that: (1) $\delta_0 I \le P(t) \le \delta_1 I$, $\forall t \ge 0$. (2) $\|\overline{\phi}(t)\|^2 \le 1/K_1$, and $\|P(t)\| \le \delta_1$ together imply $1 \ge a(t) \ge K_1/(\delta_1 + K_1)$, $\forall t \ge 0$. (3) The above scheme ensures $\|\widehat{\theta}(t)\| \le M$ where $M := \delta_1 M_0/\delta_0 + \|\theta^0\|$, and $\widehat{b}_0(t) \ge b_{\min} > 0$, $\forall t \ge 0$.

Next, define $e(t) := y(t) - y^*(t)$, the *output tracking error*. Applying the above-mentioned adaptive control law to the true process (2.2), using (3.2), and (3.3), and letting $\widetilde{\theta}(t) := \widehat{\theta}(t) - \theta$, we get

$$e(t) - Fw(t) = -\phi^T(t-d)\widetilde{\theta}(t-d) + Fv'(t), \quad (4.1)$$

Also, defining $v(t) = F(w(t) + v'(t))$, we have $y(t) = \phi^T(t-d)\theta + v(t)$, which gives, using (3.13),

$$e_a(t) = -\phi^T(t-d)\widetilde{\theta}(t-1) + v(t). \quad (4.2)$$

It is also worth noting that since $e(t) = -\phi^T(t-d)\widetilde{\theta}(t-d) + v(t)$, we have

$$e_a(t) = s(t) + e(t), \text{where } s(t) := \phi^T(t-d)(\widehat{\theta}(t-d) - \widehat{\theta}(t-1)). \quad (4.3)$$

Assume $|w(t)| \le K_w$. Using the definition of v, by (2.3), and (2.4) we get,

$$v^2(t) \le KK_v m(t-1) + k_{vm}. \quad (4.4)$$

The following lemma will be useful. This result essentially states that the "difference" term $s(t)$ can be bounded in terms of the "augmented error" $e_a(t)$.

Lemma 1.

$$s^2(t) \le s_m^2(t) \le K\sum_{j=0}^{d-2} e_a^2(t-1-j), \quad (4.5)$$

$$\text{where } s_m(t) := \xi\sum_{j=0}^{d-2}|e_a(t-1-j)|/K_1. \quad (4.6)$$

$$\text{with } \xi := \left[\sqrt{2\max\left((\delta_1/\delta_0)^2 - 1, 1\right)}\right]^{-1} \quad (4.7)$$

Proof.

$$\begin{aligned}
|s(t)| &\le \sum_{j=0}^{d-2}\|\phi(t-d)\|\|\widehat{\theta}(t-1-j) - \widehat{\theta}(t-2-j)\| \\
&\le \xi\sum_{j=0}^{d-2}\|\phi(t-d)\|\|\widehat{\theta}'(t-1-j) - \widehat{\theta}(t-2-j)\| \\
&= \xi\sum_{j=0}^{d-2}\|\phi(t-d)\|\|a(t-2-j)P(t-2-j) \\
&\quad \cdot\overline{\phi}(t-d-1-j)\frac{e_a(t-1-j)}{\sqrt{n(t-1-j)}}\| \\
&\le \xi\sum_{j=0}^{d-2}|e_a(t-1-j)|/K_1.
\end{aligned}$$

□

5 Bounding signals

Define $z(\cdot)$ through the following "switched system":

$$\begin{aligned}
z(t) &= I(t-1)(\sigma z(t-1) + K_y e_a^2(t) + K_u u^2(t-d) + K_3) \\
&\quad + (1 - I(t-1))(gz(t-1) + 2K_3), \\
&\quad \text{where } 0 < \sigma < g < 1, z(0) > 0,
\end{aligned} \quad (5.1)$$

$$I(t-1) = \begin{cases} 1 & \text{if } \sigma z(t-1) + K_y e_a^2(t) + K_u u^2(t-d) \\ & \quad + K_3 \ge gz(t-1) + 2K_3, \\ 0 & \text{otherwise}. \end{cases} \quad (5.2)$$

Lemma 2.

(a) $m(t) \le Kz(t) + K$.

(b) $z(t) \le K_z z(t-1) + k_z$.

Proof. (a) From the definition of $m(\cdot)$, we have

$$m(t) = \sigma^t m(0) + \sum_{j=0}^{t}\sigma^{t-j}[K_y y^2(j) + K_u u^2(j) + K_3].$$

Note that $y^2(j) \le 2e^2(j) + 2y^{*2}(j)$, and from the control law (3.3),

$$u(t) = \frac{[r(t) - \widehat{\theta}_u^T(t)\phi_u(t) - \widehat{\theta}_y^T(t)\phi_y(t) - \widehat{\theta}_{y1}(t)y(t) - \widehat{\theta}_{y^*}^T(t)\phi_{y^*}(t)]}{\widehat{b}_0(t)}, \quad (5.3)$$

where

$$\phi_u(t)^T = (u(t-1), \ldots, u(t-q+1)),$$

$$\phi_y(t)^T = (y(t-1), \ldots, y(t-p+1)),$$

and

$$\phi_{y^*}(t)^T = (y^*(t+d-1), \ldots, y^*(t+d-r)), \forall t \ge 0.$$

Also note that $\hat{b}_0(t) \geq b_{min} > 0$, $\|\hat{\theta}(t)\| \leq M$, and $y^{*2}(t) \leq k_{y^*}$ $\forall t \geq 0$. The desired result then follows from Lemma 1, by using (5.3) repeatedly $(d-1)$ times, after noting that

$$\begin{aligned} z(t) &\geq \sigma z(t-1) + K_y e_a^2(t) + K_u u^2(t-d) + K_3, \text{ which implies} \\ z(t) &\geq \sigma^t z(0) + \sum_{j=0}^{t} \sigma^{t-j}[K_y e_a^2(t) + K_u u^2(j-d) + K_3]. \end{aligned}$$

(b) The proof is straightforward and omitted. □

Lemma 2(a) when combined with (4.4) gives

$$v^2(t) \leq K K_v z(t-1) + K, \forall t \geq 1. \quad (5.4)$$

Lemma 3.

$$u^2(t-d) \leq K n(t-1) + K \quad (5.5)$$

Proof. The control law (3.3) gives

$$u(t) = (r(t) - \hat{\theta}^T(t)\phi_I(t))/\hat{b}_0(t),$$

where $\phi_I(t)$ equals $\phi(t)$ except for the first component which is set to zero. The result then follows since $\|\phi_I(t)\|^2 \leq \|\psi(t+d-1)\|^2$. □

6 Comparing $n(t)$ and $z(t)$

First, note that we clearly have $n(t) \leq K_n z(t)$, $\forall t \geq 0$.

Lemma 4. *(a) $I(t) = 1 \Rightarrow n(t) \geq K_{nz} z(t)$. (b) Consider a positive integer N'. Let t_1 be such that $I(t_1) = 1$. If $z(t) \geq L, \forall t \in [t_1 - N', t_1]$ where L is a large enough positive constant, then $\exists K_{vmax} > 0$ such that $\forall K_v \in [0, K_{vmax}]$, $n(t) > \delta(N')z(t), \forall t \in [t_1 - N', t_1]$, where $\delta(N') > 0$ is a positive constant which depends only on N'.*

Proof. (a) $I(t-1) = 1 \Rightarrow K_y e_a^2(t) + K_u u^2(t-d) \geq (g-\sigma)z(t-1) + K_3$. Hence, by Lemma 3, $K_y e_a^2(t) + K n(t-1) \geq (g-\sigma)z(t-1) + (K_3 - K) =: \bar{g}z(t-1) + \overline{K} =: RHS$. If $n(t-1) \geq RHS/(2K)$, the result follows. So, suppose $n(t-1) \leq RHS/(2K)$. Then, $e_a^2(t) \geq RHS/(2K_y)$. Using (4.2) and (5.4), we get

$$\begin{aligned} 2M^2\|\psi(t-1)\|^2 &\geq 2M^2\|\phi(t-d)\|^2 \geq e_a^2(t) - v^2(t) \\ &\geq [\bar{g}z(t-1) + \overline{K}]/(2K_y) - K K_v z(t-1) - K \\ &\geq (\bar{g}/(2K_y) - K K_v) z(t-1), \\ &\qquad \text{assuming } \overline{K} \geq 2 K K_y. \end{aligned}$$

(b) First we will bound the growth-rate of $n(t)/z(t)$, and then use this in a reversed time argument. Clearly, $z(t) \geq \sigma z(t-1), \forall t$. First note that

$$\|\psi(t)\|^2 = \sum_{j=0}^{q+d-2} u^2(t-j) + \sum_{j=0}^{p+d-2} y^2(t-j) + \sum_{j=1}^{r+d-1} y^{*2}(t+d-j).$$

Using the control law (3.3) gives,

$$\begin{aligned} u^2(t) &\leq K y^2(t) + K\|\psi(t-1)\|^2 + K \\ &\leq K\|\psi(t-1)\|^2 + K v^2(t) + K \\ \Rightarrow \|\psi(t)\|^2 &\leq K\|\psi(t-1)\|^2 + K v^2(t) + K \\ \Rightarrow n(t) &\leq K n(t-1) + K v^2(t) + K. \end{aligned}$$

Using (5.4) gives $n(t) \leq K n(t-1) + K K_v z(t-1) + K$. This gives

$$\begin{aligned} \frac{n(t)}{z(t)} &\leq K\frac{n(t-1)}{z(t-1)} + K K_v + K/z(t) \\ &\leq K\frac{n(t-1)}{z(t-1)} + (K K_v + K/L) =: K_a \frac{n(t-1)}{z(t-1)} + K_d \\ \Rightarrow \frac{n(t_1)}{z(t_1)} &\leq K_a^{t_1 - t'}[\frac{n(t')}{z(t')} + \sum_{j=t'+1}^{t_1} K_a^{t'-j} K_d], \forall t' \in [t_1 - N', t_1] \\ &\leq K_a^{N'}[\frac{n(t')}{z(t')} + \frac{K_a K_d}{K_a - 1}], \forall t' \in [t_1 - N', t_1] \\ \Rightarrow \frac{n(t')}{z(t')} &\geq K_a^{-N'}\frac{n(t_1)}{z(t_1)} - \frac{K_a K_d}{K_a - 1} \\ &\geq K_a^{-N'} K_{nz} - \frac{K_a K_d}{K_a - 1} =: \delta(N'). \end{aligned}$$

It should be noted that $\delta(N')$ can be made positive for any finite N' by restricting the size of the unmodeled dynamics to be small enough (i.e., making K_v small enough), and making L large enough. This concludes the proof. □

For later use, define $K_{nz}(N') := \min\{K_{nz}, \delta(N')\}$.

7 A Nonminimal System Representation

Recall from (2.2) that

$$y(t) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}u(t) + \frac{1}{A(z^{-1})}v''(t), \quad v''(t) := w(t) + v'(t). \quad (7.1)$$

Consider the following minimal representation of (7.1) :

$$\begin{aligned} x(t) &= A_p x(t-1) + b_{pu} u(t-1) + b_{pv} v''(t), \\ y(t) &= h_p^T x(t). \end{aligned} \quad (7.2)$$

From the control law (3.3), we get

$$\begin{aligned} u(t) &= -\tfrac{1}{b_0}\theta_u^T \phi_u(t) - \tfrac{1}{b_0}\theta_y^T \phi_y(t) - \tfrac{\theta_{y1}}{b_0} h_p^T x(t) \\ &\quad + \tfrac{1}{b_0}(r(t) - \phi(t)^T \tilde{\theta}(t)). \end{aligned} \quad (7.3)$$

Here, ϕ_u and ϕ_y are generated by stably filtering $u(\cdot)$ and $y(\cdot)$ as follows:

$$\phi_u(t) = A_u \phi_u(t-1) + b_u u(t-1) \quad (7.4)$$

$$\phi_y(t) = A_y d_y(t-1) + b_y y(t-1) \quad (7.5)$$

where (A_u, b_u) are such that $(I - z^{-1}A_u)^{-1}b_u = [1, z^{-1}, \cdots, z^{-q+2}]^T$ and (A_y, b_y) are such that $(I - z^{-1}A_y)^{-1}b_y = [1, z^{-1}, \cdots, z^{-p+2}]^T$.

From (7.1)-(7.5), defining $x_c(t) = [x^T(t), \phi_u^T(t), \phi_y^T(t)]$, we get $x_c(t) = A_c x_c(t-1) + b_c[r(t-1) - \phi(t-1)^T\tilde{\theta}(t-1)] + b_{cv}v''(t)$, and $y(t) = h_c^T x_c(t)$.

If $\tilde{\theta} = 0$ and $v'' \equiv 0$, then $x_c(t) = A_c x_c(t-1) + b_c r(t-1)$, and $y(t) = h_c^T x_c(t)$, which implies that $y(t) = h_c^T(I - z^{-1}A_c)^{-1}b_c r(t-1)$. However, we know that $y(t) = y^*(t) = z^{-(d-1)}r(t-1)$ in such a case. This means that (after cancellations) $h_c^T(I - z^{-1}A_c)^{-1}b_c = z^{-(d-1)}$. Since $B(z^{-1})$ has all its zeros within the open disk $|z| < \alpha < 1$, it follows that the cancellations are stable, which in turn implies that A_c is a stable matrix. Finally, defining the closed-loop non-minimal state error $e_c(t) = x_c(t) - x_m(t)$, where x_m is the "corresponding" nonminimal state of $z^{-(d-1)}r(t-1)$, we get,

$$\begin{aligned} e_c(t) &= A_c e_c(t-1) - b_c[\phi(t-1)^T\tilde{\theta}(t-1)] + b_{cv}v''(t), \\ e(t) &= h_c^T e_c(t). \end{aligned}$$

8 Ultimate Boundedness Analysis

Define $W(t) = k_e e_c^T(t-d+1)Pe_c(t-d+1) + z(t)$ where $P = P^T > 0$ satisfies $A_c^T P A_c - P = -I$. Such a P exists since A_c is stable.

Now, $z(t) \leq gz(t-1) + K_y e_a^2(t) + K_u u^2(t-d) + 2K_3$, and $e_c(t-d+1) = A_c e_c(t-d) - b_c[\phi^T(t-d)\tilde{\theta}(t-d)] + b_{cv}v''(t-d+1)$. Recalling that $-\phi(t-d)^T\tilde{\theta}(t-d) = e(t) - v(t)$, we have, $e_c(t-d+1) = A_c e_c(t-d) + b_c e(t) + (-b_c v(t) + b_{cv}v''(t-d+1))$, which gives $W(t) \leq k_e[e_c^T(t-d)A_c^T + b_c^T e(t) + \ell^T(t)]P[A_c e_c(t-d) + b_c e(t) + \ell(t)] + gz(t-1) + K_y e_a^2(t) + K_u u^2(t-d) + 2K_3$, where $\ell(t) := -b_c v(t) + b_{cv}v''(t-d+1)$. This implies

$$\begin{aligned} W(t) &\leq k_e e_c^T(t-d)(P-I)e_c(t-d) \\ &+ 2k_e e_c^T(t-d)A_c^T P b_c e(t) + 2k_e e_c^T(t-d)A_c^T P\ell(t) \\ &+ 2k_e b_c^T P\ell(t)e(t) \\ &+ k_e b_c^T P b_c e^2(t) + k_e \ell^T(t)P\ell(t) \\ &+ gz(t-1) + K_y e_a^2(t) + K_u u^2(t-d) + 2K_3. \end{aligned}$$

Now,

$$\begin{aligned} |u(t-d)| &\leq \tfrac{1}{b_{\min}}(|\hat{\theta}_u^T(t-d)\phi_u(t-d)| + |\hat{\theta}_y^T(t-d)\phi_y(t-d)| \\ &\quad + |\theta_{y1}y(t-d)| + |\hat{\theta}_{y^*}^T(t-d)\phi_{y^*}(t-d)| + |r(t-d)|) \\ &\leq K\|e_c(t-d)\| + K|y(t-d)| + K \\ &\leq K\|e_c(t-d)\| + K. \end{aligned}$$

Also, since $|e(t)| \leq |e_a(t)| + |s(t)|$ and by Lemma 1, $|s(t)| \leq s_m(t)$, we have, after letting $\gamma_1 := \|A_c^T P b_c\|$,

$$\begin{aligned} |2k_e e_c^T(t-d) A_c^T P b_c e(t)| &\leq 2\gamma_1 k_e \|e_c(t-d)\| |e(t)| \\ &\leq 2\gamma_1 k_e \|e_c(t-d)\| (|e_a(t)| + s_m(t)) \\ &\leq \gamma_1 k_e [(\epsilon_1 + \epsilon_2) \|e_c(t-d)\|^2 \\ &\quad + \tfrac{1}{\epsilon_1} e_a^2(t) + \tfrac{1}{\epsilon_2} s_m^2(t)] . \end{aligned}$$

Similarly, after defining $\gamma_2 := \|A_c^T P\|$, we get,

$$\begin{aligned} |2k_e e_c^T(t-d) A_c^T P \ell(t)| &\leq 2\gamma_2 k_e \|e_c(t-d)\| \|\ell(t)\| \\ &\leq \gamma_2 k_e (\epsilon_3 \|e_c(t-d)\|^2 + \tfrac{1}{\epsilon_3} \|\ell(t)\|^2) , \end{aligned}$$

and

$$\begin{aligned} |2k_e b_c^T P \ell(t) e(t)| &\leq 2k_e \gamma_3 \|\ell(t)\| (|e_a(t)| + s_m(t)); \gamma_3 := \|P b_c\| \\ &\leq k_e \gamma_3 ((\epsilon_4 + \epsilon_5) \|\ell(t)\|^2 + \tfrac{1}{\epsilon_4} e_a^2(t) + \tfrac{1}{\epsilon_5} s_m^2(t)) . \end{aligned}$$

Finally, pick $0 < \gamma < 1$, such that $\gamma \geq \max\left\{1 - \frac{1}{\lambda_{\max}(P)}, g\right\}$, and ϵ_1, ϵ_2, ϵ_3 small enough, and k_e large enough so that $-1 + (1-\gamma)\lambda_{\max}(P) + (\epsilon_1 + \epsilon_2)\gamma_1 + \epsilon_3 \gamma_2 + \frac{K}{k_e} \leq 0$. This gives

$$W(t) \leq \gamma W(t-1) + K_{ea} e_a^2(t) + K_s s_m^2(t) + K_\ell \|\ell(t)\|^2 + 2K_3 , \quad (8.1)$$

where $K_{ea} = [k_e(\frac{\gamma_1}{\epsilon_1} + \frac{\gamma_3}{\epsilon_4} + b_c^T P b_c) + K_y]$, $K_s = [k_e(\frac{\gamma_1}{\epsilon_2} + \frac{\gamma_3}{\epsilon_5})]$, and $K_\ell = \left(\frac{\gamma_2}{\epsilon_3} + \gamma_3(\epsilon_4 + \epsilon_5) + \lambda_{\max}(P)\right) k_e$.

From (8.1), since $z(t) \leq W(t)$, we have,

$$\begin{aligned} W(t) &\leq (\gamma + K_{ea} \tfrac{e_a^2(t)}{z(t-1)} + K_s \tfrac{s_m^2(t)}{z(t-1)} + K_\ell \tfrac{\|\ell(t)\|^2}{z(t-1)}) W(t-1) + 2K_3 \\ &=: g(t) W(t-1) + 2K_3 . \end{aligned} \quad (8.2)$$

Using $z(t) \geq \sigma z(t-1)$, we have

$$\frac{s_m^2(t)}{z(t-1)} \leq K \sum_{j=0}^{d-2} \frac{e_a^2(t-1-j)}{z(t-1)} \leq K \sum_{j=0}^{d-2} \sigma^{-j} \frac{e_a^2(t-1-j)}{z(t-1-j)} . \quad (8.3)$$

Lemma 5.

$$W(t) \leq K_{wz} z(t) + k_{wz} .$$

Proof. This follows from Lemma A.2. □

Corollary 6.

$$W(t) \leq K W(t-1) + K .$$

Proof. This follows from Lemma 2(b) and Lemma 5. □

Lemma 7.

Consider a time interval $[a, b]$ such that $W(t) \geq 2K_{wz} L$, $\forall t \in [a-d+1, b]$. Then for L large enough, (i) $z(t) \leq W(t) \leq 2K_{wz} z(t)$. (ii) If $I(t-1) = 0$, $\forall t \in [a, b]$, then $W(b) \leq 2K_{wz}(g^{b-a} + \frac{2K_3}{(1-g)L}) W(a)$. (iii) If for each $t \in [a, b]$ such that $I(t) = 0$, $\exists$ $n' \in [0, N]$ such that $I(t + n') = 1$, then for some $0 < \beta < 1$, $W(b) \leq K \exp[-\beta(b-a)] \left[1 + \frac{K}{L \gamma^{b-a}}\right] W(a)$.

Proof.

(i) This follows by choosing L large enough.

(ii) By (5.1), for $I(t) = 0$, $z(t) = g z(t-1) + 2K_3$, which implies that

$$\begin{aligned} &\tfrac{1}{2K_{wz}} W(b) \leq z(b) \leq g^{b-a} z(a) + 2K_3 \textstyle\sum_{j=a+1}^{b} g^{b-j} \\ &\leq [g^{b-a} + \tfrac{2K_3}{(1-g)L}] W(a) . \end{aligned}$$

(iii) From (8.2), we have

$$\begin{aligned} \frac{W(b)}{W(a)} &\leq \left(\prod_{j=a+1}^{b} g(j)\right) \left[1 + \frac{2K_3}{W(a)} \sum_{t=a+1}^{b} \left(\prod_{j=a+1}^{t} g(j)\right)^{-1}\right] \\ &\leq \exp\left[\sum_{j=a+1}^{b} \ln g(j)\right] \left[1 + \frac{K_3}{K_{wz} L} \sum_{t=a+1}^{b} (\frac{1}{\gamma})^{t-a}\right] \end{aligned} \quad (8.4)$$

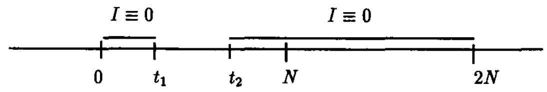

Figure 1: Illustration of Case 2

Now,

$$\sum_{t=a+1}^{b} \ln g(t) \leq \sum_{t=a+1}^{b} g(t) - (b-a).$$

By (8.2), this gives

$$\begin{aligned} \sum_{t=a+1}^{b} \ln g(t) \leq -(1-\gamma)(b-a) + K_{ea} \sum_{t=a+1}^{b} \frac{e_a^2(t)}{z(t-1)} \\ + K_s \sum_{t=a+1}^{b} \frac{s_m^2(t)}{z(t-1)} + K_\ell \sum_{t=a+1}^{b} \frac{\|\ell(t)\|^2}{z(t-1)} . \end{aligned} \quad (8.5)$$

Let $\gamma_6 := 2\|b_c\|^2$, and $\gamma_7 := 2\|b_{cv}\|^2$. Since $z(t) \geq \sigma z(t-1)$, and since $z(t) \leq 2K_z z(t-1)$ for $z(t-1) \geq L$, we have using (8.3),

$$\begin{aligned} \sum_{t=a+1}^{b} \ln g(t) &\leq -(1-\gamma)(b-a) + 2K_{ea} K_z K_n \sum_{t=a+1}^{b} \frac{e_a^2(t)}{n(t)} \\ &\quad + K K_s K_n \sum_{j=0}^{d-2} \sigma^{-j} \cdot (t-1-j) \\ &\quad + \left\{ K_\ell \gamma_6 \left[K K_v + \frac{K}{L}\right] + K_\ell \gamma_7 \sigma^{-(d-1)} \right. \\ &\qquad \left. \left[K K_v + \frac{K}{L}\right] \right\} (b-a) \end{aligned} \quad (8.6)$$

Now, by Lemma A.3 and Lemma 4,

$$\sum_{t=a+1}^{b} \frac{e_a^2(t)}{n(t)} \leq K M^2 + \frac{K(b-a)}{K_{nz}(N)} \left[K K_v + \frac{K}{L}\right] . \quad (8.7)$$

Using (8.6) and (8.7), we get (upon choosing K_v small enough and L large enough),

$$\sum_{t=a+1}^{b} \ln g(t) \leq K M^2 - \beta(b-a), \quad \text{some } 0 < \beta < 1,$$

so that from (8.4),

$$\frac{W(b)}{W(a)} \leq \exp[-\beta(b-a)] \left[1 + \frac{K}{K_{wz} L \gamma^{b-a}} \frac{1}{(1-\gamma)}\right] \exp(K M^2).$$

□

Contraction Lemma.

Pick $0 < \gamma^* < 1$. Then $\exists$ N, L large enough and K_{vmax} small enough, so that if $W(t) \geq 2K_{wz} L$, $\forall t \in [\ell - d, \ell + 2N]$, then $\forall K_v \in [0, K_{v\max}]$,

$$W(\ell + 2N) \leq \gamma^* W(\ell - d) .$$

Proof. There are four cases:

Case 1: $(I(t) = 0, \forall t \in [\ell, \ell + 2N])$

From Lemma 7 (ii),

$$W(\ell + 2N) \leq 2K_{wz} \left(g^{2N} + \frac{2K_3}{(1-g)L}\right) W(\ell) ,$$

and by Corollary 6, $W(\ell) \leq K W(\ell - d)$. Hence, for N, L large enough and $K_{v\max}$ appropriately small, $W(\ell + 2N) \leq \gamma^* W(\ell)$. Define $t_1 = \min\{t \in [0, 2N] : I(t + \ell) = 1\}$, $t_2 = \max\{t \in [0, 2N] : I(t + \ell) = 1\}$.

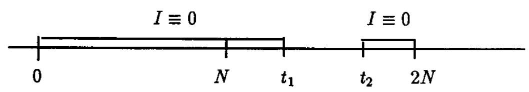

Figure 2: Illustration of Case 3

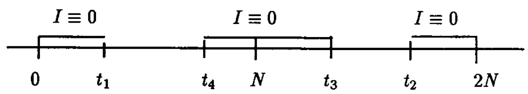

Figure 3: Illustration of Case 4

Case 2: $(0 \leq t_1 \leq t_2 \leq N)$

Using Lemma 7 (i) and (ii), we have

$$\begin{aligned} W(\ell+2N) &\leq 2K_{wz}\left(g^{2N-t_2}+\frac{2K_3}{(1-g)L}\right)W(\ell+t_2) \\ &\leq 2K_{wz}\left(g^{N}+\frac{2K_3}{(1-g)L}\right)Ke^{-\beta t_2} \\ &\quad\cdot\left(1+\frac{K}{L\gamma^{t_2}}\right)W(\ell) \\ &\leq 2K_{wz}K\left(g^{N}+\frac{2K_3}{(1-g)L}\right)\left(1+\frac{K}{L\gamma^{N}}\right)W(\ell). \end{aligned}$$

Since $W(\ell) \leq KW(\ell-d)$ by Corollary 6, we have

$$W(\ell+2N) \leq \gamma^* W(\ell-d),$$

where the last inequality holds for N, L large enough and $K_{v\max}$ appropriately small.

Case 3: $(N < t_1 < t_2 \leq 2N)$

Using Lemma 7 (i) and (ii), we have

$$\begin{aligned} W(\ell+2N) &\leq 2K_{wz}\left(g^{2N-t_2}+\frac{2K_3}{(1-g)L}\right)W(\ell+t_2), \\ W(\ell+t_2) &\leq K\exp\left[-\beta(t_2-N)\right]\left[1+\frac{K}{L\gamma^{t_2-N}}\right]W(\ell+N), \end{aligned}$$

and

$$W(\ell+N) \leq 2K_{wz}\left(g^{N}+\frac{2K_3}{(1-g)L}\right)W(\ell),$$

which implies

$$\begin{aligned} W(\ell+2N) &\leq 4KK_{wz}^2\left(g^{N}+\frac{2K_3}{(1-g)L}\right) \\ &\quad\cdot\left(1+\frac{2K_3}{(1-g)L}\right)\left(1+\frac{K}{L\gamma^{N}}\right)W(\ell). \end{aligned}$$

Since $W(\ell) \leq KW(\ell-d)$ by Corollary 6, we have

$$W(\ell+2N) \leq \gamma^* W(\ell-d),$$

where the last inequality holds for N, L large enough and $K_{v\max}$ appropriately small.

Case 4: $(0 \leq t_1 \leq N < t_2 \leq 2N)$

Define $t_3 := \min\{t \in [N,2N] : I(t+\ell)=1\}$, and $t_4 := \max\{t \in [0,N] : I(t+\ell)=1\}$.

Case 4(a) $(t_3 - t_4 \leq N)$

Using Lemma 7 (i) and (ii), we have

$$W(\ell+2N) \leq 2K_{wz}\left(g^{2N-t_2}+\frac{2K_3}{(1-g)L}\right)W(\ell+t_2),$$

and

$$W(\ell+t_2) \leq K\exp\left[-\beta t_2\right]\left(1+\frac{K}{L\gamma^{t_2}}\right)W(\ell),$$

which gives

$$\begin{aligned} W(\ell+2N) &\leq 2KK_{wz}\left(1+\frac{2K_3}{(1-g)L}\right)\exp(-\beta N) \\ &\quad\cdot\left(1+\frac{K}{L\gamma^{2N}}\right)W(\ell). \end{aligned}$$

Since $W(\ell) \leq KW(\ell-d)$ by Corollary 6, we have

$$W(\ell+2N) \leq \gamma^* W(\ell-d),$$

where the last inequality holds for N, L large enough and $K_{v\max}$ appropriately small.

Case 4(b) $(t_3 - t_4 > N)$.

Using Lemma 7 (i) and (ii), we have

$$\begin{aligned} W(\ell+2N) &\leq 2K_{wz}\left(g^{2N-t_2}+\frac{2K_3}{(1-g)L}\right)W(\ell+t_2), \\ W(\ell+t_2) &\leq K\exp(-\beta(t_2-t_3))\left(1+\frac{K}{L\gamma^{t_2-t_3}}\right)W(\ell+t_3), \\ W(\ell+t_3) &\leq 2K_{wz}\left(g^{t_3-t_4}+\frac{2K_3}{(1-g)L}\right)W(\ell+t_4), \end{aligned}$$

and

$$W(\ell+t_4) \leq K\exp(-\beta t_4)\left(1+\frac{K}{L\gamma^{t_4}}\right)W(\ell),$$

which gives

$$\begin{aligned} W(\ell+2N) &\leq 4K^2K_{wz}^2\left(g^{N}+\frac{2K_3}{(1-g)L}\right) \\ &\quad\cdot\left(1+\frac{K}{L\gamma^{N}}\right)^2\left(1+\frac{2K_3}{(1-g)L}\right)W(\ell). \end{aligned}$$

Since $W(\ell) \leq KW(\ell-d)$ by Corollary 6, we have

$$W(\ell+2N) \leq \gamma^* W(\ell-d),$$

where the last inequality holds for N, L large enough and $K_{v\max}$ appropriately small.

□

Proof of Theorem 1. By Corollary 6, W has a bounded growth rate. The Contraction Lemma then proves that W is uniformly bounded. Since W bounds all other signals (through z and e_c), we conclude that all closed-loop signals are bounded. □

9 Concluding Remarks

In this paper, we have proved uniform boundedness of an ELS-based adaptive tracking scheme subject to unmodeled dynamics, when the stochasticity assumption on the noise is violated. The nominal plant, without unmodeled dynamics, is assumed to be of minimum phase. We believe that to obtain good performance, a primary goal of adaptive control (as in tracking), it is necessary to design control laws on the basis of stochastic considerations. However, it is important to show that such algorithms are robust in a deterministic sense, as we have done in this paper.

Certain modifications to the adaptive law have been employed to achieve this; a weighted ELS update law is used where the signals entering the update were normalized by the norm of an extended regressor (and not by any specially constructed signal), and the parameter estimates are projected onto a compact set. We believe that it would be difficult to ensure boundedness without either of these two modifications.

The results presented in this paper extend the work reported in [4], where the robust boundedness of adaptively controlled continuous time plants in the presence of unmodeled dynamics and bounded disturbances is proved.

References

[1] G. C. Goodwin and K. S. Sin. Adaptive Filtering, Prediction and Control. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1984.

[2] L. Praly, S. -F. Lin and P. R. Kumar. A robust adaptive minimum variance controller. *SIAM J. Control and Optimizn.*, Vol.27, No.2, pp. 235-266, 1989.

[3] B. E. Ydstie. Stability of discrete MRAC - revisited. *Systems & Control Letters*, vol. 13, pp. 429-438, 1989.

[4] S. M. Naik, P. R. Kumar, and B. E. Ydstie. "Robust continuous time adaptive control by parameter projection." *Foundations of Adaptive Control*, Lecture Notes in Control and Information Sciences, Vol. 160, P. V. Kokotovic, Editor, Springer-Verlag, New York, 1991.

10 Appendix A

Lemma A.1 (Filtering).

Consider $w_{out}(t) = H(z^{-1})w_{in}(t)$, where $H(z^{-1})$ has all its poles within $|z| < \alpha$, where $\alpha^{1+\delta} < \sigma$, for some $0 \le \delta < 1$. If $w_{in}^2(t) \le Km(t) + K$, then $w_{out}^2(t) \le Km(t) + K$.

Proof. Let $h(t)$ denote the impulse response of $H(z^{-1})$. Then, letting $\alpha' := \alpha^{1-\delta}$, and $\alpha'' := \alpha^{1+\delta}$,

$$\begin{aligned} w_{out}^2(t) &= [\sum_{j=0}^{t} h(j)w_{in}(t-j)]^2 \le K(\sum_{j=0}^{t}\alpha^j|w_{in}(t-j)|)^2 \\ &= K(\sum_{j=0}^{t}\alpha'^{j/2}\alpha''^{j/2}|w_{in}(t-j)|)^2 \le K\sum_{k=0}^{t}\alpha''^k\sum_{j=0}^{t}\alpha'^j w_{in}^2(t-j) \\ &\le K\sum_{j=0}^{t}\alpha'^j m(t-j) + K \end{aligned}$$

Finally using $m(t) \ge \sigma^j m(t-j)$, we get,

$$w_{out}^2(t) \le K\sum_{j=0}^{t}(\frac{\alpha'}{\sigma})^j m(t) + K,$$

thereby concluding the proof. □

Lemma A.2.

Let the system: $x(t+1) = A_w x(t) + b_w w_{in}(t)$, $w_{out}(t) = h_w^T x(t)$ with zero initial conditions, be a minimal state representation of $w_{out}(t) = \frac{B(z^{-1})}{A(z^{-1})}w_{in}(t) =: H(z^{-1})w_{in}(t)$, where $B(z^{-1})$ has all its zeros within the open disk $|z| < \alpha$, and $\alpha^{1+\delta} < \sigma$, for some $0 \le \delta < 1$. If $w_{out}^2(t) \le Km(t) + K$, then $\|x(t)\| \le Km(t) + K$.

Proof. Without loss of generality, suppose

$$H(z^{-1}) = \frac{B(z^{-1})}{\prod_{i=1}^{k}(1 + a_{i1}z^{-1} + a_{i2}z^{-2})\prod_{j=1}^{n-2k}(1 + a_j z^{-1})}.$$

Since $H(z^{-1})$ is minimal, the corresponding states are the states corresponding to $\frac{1}{1+a_{i1}z^{-1}+a_{i2}z^{-2}}w_{in}(t), i = 1,\ldots,k$ and $\frac{1}{1+a_j z^{-1}}w_{in}(t), j = 1,\ldots,n-2k$. Now,

$$\frac{1}{1+a_l z^{-1}}w_{in}(t) = \frac{\prod_{i=1}^{k}(1 + a_{i1}z^{-1} + a_{i2}z^{-2})\prod_{j=1, j\neq l}^{n-2k}(1 + a_j z^{-1})}{B(z^{-1})} \cdot w_{out}(t)$$

Using Lemma A.1, we get $(\frac{1}{1+a_l z^{-1}}w_{in}(t))^2 \le Km(t) + K, l = 1,\ldots,n-2k$. Define $w_l(t) = \frac{1}{1+a_{l1}z^{-1}+a_{l2}z^{-2}}w_{in}(t) =: H_l(z^{-1})w_{in}(t)$. Then, since

$$\begin{aligned} w_l(t) &= \frac{\prod_{i=1, i\neq l}^{k}(1 + a_{i1}z^{-1} + a_{i2}z^{-2})\prod_{j=1}^{n-2k}(1 + a_j z^{-1})}{B(z^{-1})}w_{out}(t) \\ &=: H_{lo}(z^{-1})w_{out}(t), \end{aligned}$$

using Lemma A.1, we get $w_l^2(t) \le Km(t) + K, l = 1,\ldots,k$. Furthermore, using $m(t) \ge \sigma m(t-1)$, we get $w_l^2(t-1) \le Km(t) + K$. Since $w_l(t)$ and $w_l(t-1)$ are the states corresponding to $H_l(z^{-1})w_{in}(t)$, we are done. □

Lemma A.3.

Let $V(t) = \tilde{\theta}(t)^T P(t)^{-1}\tilde{\theta}(t)$. Then, $V(t) - V(t-1) \le -\beta_1\bar{e}_a(t)^2 + \beta_2\bar{v}(t)^2$ where

$$\begin{aligned} \beta_1 &= \frac{K_1}{(\delta_1 + K_1)}\min\left\{\frac{(\delta_1 + 2K_1)}{(1 + K_1)}\left(\lambda' - \frac{1}{2} - \epsilon\right), \frac{(\delta_1 + 2K_1)}{(1 + K_1)} - \epsilon\right\}, \\ \beta_2 &= \max\left\{1 + \frac{(\delta_1 + K_1)}{K_1}\frac{(1-\lambda')^2}{2\epsilon}, \frac{1}{\epsilon}\right\}. \end{aligned}$$

Proof. Define $\tilde{\theta}''(t) := \hat{\theta}''(t) - \theta$, and $\tilde{\theta}'(t) := \hat{\theta}'(t) - \theta$. Then, from (3.4) we have,

$P(t)^{-1}\tilde{\theta}''(t) = P(t)^{-1}\tilde{\theta}(t-1) + a(t-1)P(t)^{-1}P(t-1)\bar{\phi}(t-d)\bar{e}_a(t)$.

There are two cases to consider.

Case 1. When $P(t) = P'(t)$, from (3.5) we have

$$\begin{aligned} &P(t)^{-1}\tilde{\theta}''(t) = \lambda(t-1)P(t-1)^{-1}\tilde{\theta}(t-1) + (\bar{v}(t) - \bar{e}_a(t))\bar{\phi}(t-d) \\ &+\lambda(t-1)a(t-1)\bar{e}_a(t)\bar{\phi}(t-d) + (1 - a(t-1))\bar{e}_a(t)\bar{\phi}(t-d) \end{aligned}$$

So, $P(t)^{-1}\tilde{\theta}''(t) = \lambda(t-1)P(t-1)^{-1}\tilde{\theta}(t-1) + \{(\lambda(t-1) - 1)a(t-1)\bar{e}_a(t) + \bar{v}(t)\}\bar{\phi}(t-d)$, which gives

$$\begin{aligned} &\tilde{\theta}''^T(t)P(t)^{-1}\tilde{\theta}''(t) - V(t-1) \\ \le\ &\tilde{\theta}''(t)^T P(t)^{-1}\tilde{\theta}''(t) - \lambda(t-1)\tilde{\theta}(t-1)^T P(t-1)^{-1}\tilde{\theta}(t-1) \\ =\ &\lambda(t-1)(\tilde{\theta}''(t) - \tilde{\theta}(t-1))^T P(t-1)^{-1}\tilde{\theta}(t-1) \\ +\ &\{(\lambda(t-1) - 1)a(t-1)\bar{e}_a(t) + \bar{v}(t)\}\left[(\tilde{\theta}(t-1)^T\bar{\phi}(t-d))\right. \\ &\left.+a(t-1)(\bar{\phi}(t-d)^T P(t-1)\bar{\phi}(t-d))\bar{e}_a(t)\right]. \end{aligned}$$

This implies

$$\begin{aligned} &\tilde{\theta}''^T(t)P(t)^{-1}\tilde{\theta}''(t) - V(t-1) \\ \le\ &\lambda(t-1)a(t-1)(\bar{v}(t) - \bar{e}_a(t))\bar{e}_a(t) \\ &+\{[\lambda(t-1) - 1]a(t-1)\bar{e}_a(t) + \bar{v}(t)\}[\bar{v}(t) - a(t-1)\bar{e}_a(t)] \\ \le\ &-[a(t-1)((1 + a(t-1))\lambda(t-1) - a(t-1)) - \delta^2]\bar{e}_a(t)^2 \\ &+\left(1 + \frac{a(t-1)^2(1 - \lambda(t-1))^2}{\delta^2}\right)\bar{v}(t)^2, \end{aligned}$$

the last inequality following after some algebraic manipulations, where $\delta^2 = \alpha_{\min}(1 + \alpha_{\min})\epsilon$, some $0 < \epsilon < 1/2$, where $\alpha_{\min} := K_1/(\delta_1 + K_1)$. Then, since

$$\begin{aligned} &a(t-1)((1 + a(t-1))\lambda(t-1) - a(t-1)) - \delta^2 \\ \ge\ &\frac{K_1(\delta_1 + 2K_1)}{(\delta_1 + K_1)^2}(\lambda' - 1/2 - \epsilon), \text{ and} \\ &\left(1 + \frac{a(t-1)^2[1 - \lambda(t-1)]^2}{\delta^2}\right) \le 1 + \frac{(1-\lambda')^2}{\epsilon}\frac{\delta_1 + K_1}{K_1}, \end{aligned}$$

we get $\tilde{\theta}''^T(t)P(t)^{-1}\tilde{\theta}''(t) - V(t-1) \le -\beta_1\bar{e}_a(t)^2 + \beta_2\bar{v}(t)^2$.

Case 2. When $P(t) = P(t-1)$, we have,

$$P(t)^{-1}\tilde{\theta}''(t) = P(t-1)^{-1}\tilde{\theta}(t-1) + a(t-1)\bar{\phi}(t-d)\bar{e}_a(t).$$

This implies,

$$\begin{aligned} &\tilde{\theta}''^T(t)P(t)^{-1}\tilde{\theta}''(t) - \tilde{\theta}^T(t-1)P(t-1)^{-1}\tilde{\theta}(t-1) \\ =\ &a(t-1)[2\bar{v}(t)\bar{e}_a(t) - [1 + a(t-1)]\bar{e}_a^2(t)] \\ \le\ &-\frac{K_1}{\delta_1 + K_1}[\frac{\delta_1 + 2K_1}{\delta_1 + K_1} - \epsilon]\bar{e}_a^2(t) + \frac{1}{\epsilon}\bar{v}^2(t). \end{aligned}$$

So, in either case, we have $\tilde{\theta}''^T(t)P(t)^{-1}\tilde{\theta}''(t) - V(t-1) \le -\beta_1\bar{e}_a(t)^2 + \beta_2\bar{v}(t)^2$.

The remainder of the proof is similar to that of Lemma 3.2 of [2], and hence will be omitted. □