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Relay Based Gain and Phase Margins PI Controller Design

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Abstract *– In this paper an iterative procedure for achieving gain and phase margin specifications for a PI controller is presented. The iteration scheme is based on the use of two relay tests applied to the closed loop system. The first relay test is standard and it is used here to obtain the gain margin of the closed loop system at each iteration step. The second one is applied to the closed loop system such that a limit cycle is developed at the loop gain crossover frequency. Under this condition it is possible to obtain an estimate of the phase margin of the loop transfer function. The procedure is applied to a PI controller tuning in a heat exchanger in laboratory scale using desired gain and phase margin specifications.*

Keywords *– Relay tests, PI controller tuning, gain and phase margins, iterative procedure.*

I. INTRODUCTION

The relay test introduced in [1] has been used with success for PID controllers tuning. In fact, many commercial PID controllers based on this one-shoot identification-and-tune technique are available. From this relay test, a single point of the frequency response of a process, the critical point (phase angle equals to $-180°$), can be estimated.

In [2] a new relay feedback structure was presented, allowing the user to estimate the frequency at which a desired transfer function achieves a selected gain. The new relay test was applied to a closed loop system during normal operation and its use for estimation of the loop transfer function and sensitivity function was studied. Later, in [3], this identification technique was applied to tune PI controllers using the symmetrical optimum approach.

The tuning of PI controllers based on gain and phase margins specifications has been recently studied in [4], [5], [6], [7], [8], [9] and references therein. In [4], [5], [6], [7] it was assumed that the process is first order plus time delay, which may not be representative for typical industrial processes, as reported in [9]. The controller parameters were calculated by solving a set of non-linear equations, and an analytical solution was obtained using approximations.

In [9], it was shown that with the information of a single point of the process frequency response, it is possible to design a PI controller with gain and phase margins specifications. This is not an exact solution, because the identified point is moved to another location, and it is expected that the resulting loop transfer function is close to the desired gain and phase margins. In [8] an exact solution was derived from a graphical approach, but the knowledge of the process frequency response is required.

In this paper it is presented a new combination of sequential relay tests for achieving gain and phase margins specifications in a closed loop system. The nonlinear problem is solved in an iterative manner, and two relay tests are used in each iteration. The advantage of this procedure is that no assumptions on process model structure is required, neither full knowledge of its frequency response.

II. PROBLEM STATEMENT

Consider the closed loop shown in Fig. (1). The unknown plant transfer function is given by $G(s)$ while the PI controller is given by

$$C(s) = K_p\left(1+\frac{1}{T_i s}\right) = K_p\frac{s+1/T_i}{s}\,. \qquad (1)$$

The closed loop transfer function from reference $y_r(t)$ to output $y(t)$ is given by

$$\frac{Y(s)}{Y_r(s)} = T(s) = \frac{G(s)C(s)}{1+G(s)C(s)}\,. \qquad (2)$$

For this closed loop configuration $T(s)$ is also known as the Complementary Sensitivity Function and $L(s) = G(s)C(s)$ is the Loop Transfer Function.

The Gain Margin (GM) and Phase Margin (PM) of a closed loop are defined as

$$GM = \frac{1}{|L(j\omega_u)|} \text{ and } PM = \pi + \angle L(j\omega_g)\,, \qquad (3)$$

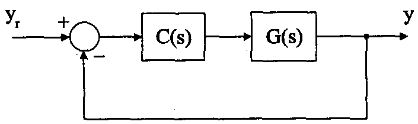

Fig. 1. Closed Loop System.

where ω_u and ω_g are obtained from

$$\angle L(j\omega_u) = -\pi \text{ and } |L(j\omega_g)| = 1 . \qquad (4)$$

The problem posed in this paper is to find the PI controllers parameters using relay tests, such that a closed loop system with specified gain and phase margins is obtained.

III. RELAY BASED IDENTIFICATION

The estimation of the critical point using a relay test is presented in [9] and is a standard in autotune procedures. The idea is to substitute the controller in Fig. 1 by a relay with amplitude d. For most types of processes, this configuration leads to limit cycle operation, with oscillation conditions given by

$$G(j\omega) \cong -\frac{\pi a}{4d} ,$$

which is at the process critical point. It can be shown [10] that if this relay test is applied to a closed loop system, $T(s)$, the limit cycle occurs at the critical frequency of the loop transfer function, i.e.,

$$L(j\omega) = G(j\omega)C(j\omega) \cong \frac{m}{1-m},$$

where

$$m = -\frac{\pi a}{4d} ,$$

and a is the process output amplitude in closed loop. In this paper, this relay test will be referred as the *loop critical gain relay test.*

A basic procedure for the estimation of a general frequency point of a given transfer function using a relay feedback is presented in [2]. The feedback structure applied for loop transfer function estimation is presented in Fig. 2. The conditions of the limit cycle operation are defined by the following proposition.

Proposition 1: Consider the closed loop system shown in Fig (2). Assume that for a stable closed loop $T(s)$ and a real positive number r, the transfer function

$$F(s) = \frac{2}{r}\frac{T(s)}{T(s)\left(\frac{1-r}{r}\right)+1} - 1 \qquad (5)$$

is also stable. Then if a limit cycle is present it oscillates at a frequency ω_o such that

$$|L(j\omega_o)| \approx r .$$

Proof: See [2]. ■

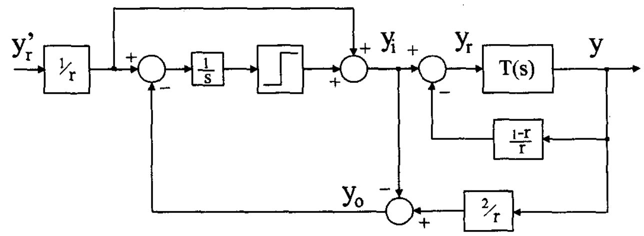

Fig. 2. Relay Closed Loop Experiment for Loop Transfer Function Estimation.

This procedure allows the estimation of the frequency at which the loop transfer function magnitude is close to r. It can be seen as a generalization of the feedback structure presented in [10]. In that work, it was used $r = 1$, and a limit cycle was obtained at the loop gain crossover frequency. Here it will be used to estimate the phase angle at this frequency by observing the phase between the output and input of the closed loop. This relay test will be referred as the *loop gain crossover relay test.*

IV. PI CONTROLLER ITERATIVE TUNING

Consider the closed loop system shown in Fig. 1 and gain and phase margin specifications given by A_m and ϕ_m respectively. For the desired margin specifications the controller must be designed to satisfy the following set of equations:

$$\angle G(j\omega_u)C(j\omega_u) = -\pi , \qquad (6)$$

$$|G(j\omega_u)C(j\omega_u)| = \frac{1}{A_m} , \qquad (7)$$

$$|G(j\omega_g)C(j\omega_g)| = 1 , \qquad (8)$$

$$\angle G(j\omega_g)C(j\omega_g) = -\pi + \phi_m . \qquad (9)$$

If $G(j\omega)$ is known, the PI controller given by Eq. (1) requires the solution of the set of equations for K_p, T_i, ω_u and ω_g, a total of four unknowns and four equations. This is not a trivial task due to the non-linearity of the problem. A procedure was presented in [8] for numerically calculating the controller parameters for both PI and PID controllers, but it requires the knowledge of the process frequency response $G(j\omega)$. It is also not guaranteed that such a solution exists, as pointed out by the authors.

In this paper, instead, an iterative algorithm is used which requires the estimation of the frequencies ω_u and ω_g and the margins at each iteration. Observe that these frequencies are the solutions to Eqs. (6) and (8), and estimates can be obtained using the *loop critical gain relay test* and the *loop gain crossover relay test* respectively.

The following two step design approach can be used to tune the PI controller, where a superscript k denotes the current iteration.

1. **Initial Conditions:** Start from an initial controller $C^0(s)$, with K_p^0 and T_i^0 (in this step it is assumed that the system is operating in closed loop).
2. **Gain Margin Estimation:** use the *loop critical gain relay test* to obtain estimates for ω_u^k and GM^k (as the solution of Eq. (6)), i.e.,
$$\left[\omega_u^k,\, GM^k\right] = \underset{\omega}{\mathrm{sol}}\ \left[\angle G(j\omega)\, C^k(j\omega) = -\pi\right]. \quad (10)$$
3. **Controller Redesign for Gain Margin:** The controller gain can now be calculated to achieve the gain margin A_m using Eq. (7). That is, with the current gain margin, GM^k, and the critical frequency, ω_u^k, compute the controller proportional gain, $\bar{K}_p^{k+1}$, from
$$\bar{K}_p^{k+1} = \frac{K_p^k GM^k}{A_m}. \quad (11)$$
Now a new intermediate controller is
$$\bar{C}^{k+1}(s) = \bar{K}_p^{k+1}\left(\frac{s+1/T_i^k}{s}\right). \quad (12)$$
Note that the closed loop is stable by construction.
4. **Phase Margin Estimation:** Use the *loop gain crossover relay test* to get an estimate of both ω_g^k and PM^k from
$$\left[\omega_g^k,\, PM^k\right] = \underset{\omega}{\mathrm{sol}}\left|G(j\omega)\, C^{k+1/2}(j\omega)\right| = 1. \quad (13)$$
Here PM^k denotes the phase margin estimated at this step, and given by the second part of the expression (3).
5. **Controller Redesign for Phase Margin:** This step is separated into two parts:
(a) Determine T_i^{k+1} such that Eq. (9) is satisfied, i.e.,
$$T_i^{k+1} = \frac{\tan\left[-\pi+\phi_m - PM^k + \tan^{-1}\left(\omega_g^k T_i^k\right)\right]}{\omega_g^k}. \quad (14)$$
The phase contribution from the PI controller ranges from $-90°$ to $0°$, and this information must be used in order to avoid invalid values of T_i^{k+1}. Since $\angle G\left(j\omega_g^k\right) + \angle C^{k+1/2}\left(j\omega_g^k\right) = -180° + \phi_m$, then the following condition must be satisfied in the above steps
$$-180° + \phi_m < \angle G\left(j\omega_g^k\right) < -90° + \phi_m, \quad (15)$$
If Eq. (15) is not satisfied, stop the iteration.
(b) Now, update the controller proportional gain K_p^{k+1} such that the loop gain at the frequency ω_g^k is equal to one,
$$K_p^{k+1} = \bar{K}_p^{k+1}\frac{\sqrt{\left(1/T_i^k\right)^2+\omega_g^2}}{\sqrt{\left(1/T_i^{k+1}\right)^2+\omega_g^2}}. \quad (16)$$
The controller at the end of the iteration is finally given by
$$C^{k+1} = K_p^{k+1}\left(\frac{s+1/T_i^{k+1}}{s}\right). \quad (17)$$
Note that, in this last step, stability of the closed loop is not easily verified as in step 3.
6. If $\left|GM^k - A_m\right| \le \varepsilon_1$ and $\left|PM^k - \phi_m\right| \le \varepsilon_2$, with ε_1 and ε_2 small, then stop. Else, increment k and go to step 2.

Extensive simulations using the exact values of ω_g and ω_u at each iteration instead of the ones obtained with the relay tests were performed. The results point out that the algorithm converges to the same solution obtained using the method in [8], whenever such a solution exists.

V. SIMULATION EXAMPLE

To illustrate the capabilities of the proposed method, a simulation example is presented. The process is third order, given by

$$G(s) = \frac{1}{(s+1)^3},$$

and the initial controller is calculated from Ziegler-Nichols tuning [9], using the critical point information from a relay test. The following controller is obtained

$$K_p = 2.9575 \text{ and } T_i = 3.0161, \quad (18)$$

and the gain and phase margins obtained with this controller are

$$GM_0 = 1.8518 \text{ and } PM_0 = 22.5703°.$$

It is desired a gain margin of $A_m = 3$ and a phase margin of $\phi_m = 40°$. The controller parameters calculated using the procedure presented in [8] are

$$K_{p1} = 1.4139 \text{ and } T_{i1} = 1.9372. \quad (19)$$

It should be clear that this controller can not be obtained without fully knowing the process frequency response.

Now the iteration scheme is used with the relay feedback experiments. Using the same specifications for gain and phase margins, $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.1°$, the following controller is obtained after 7 iterations:

$$K_{p2}^{(7)} = 0.8995 \text{ and } T_{i2}^{(7)} = 1.2984\,.$$

For this controller the gain and phase margins are

$$GM_2^{(7)} = 3.1829 \text{ and } PM_2^{(7)} = 37.7457°$$

Although the obtained controllers are different, the gain and phase margins obtained with the use of the relay test are very close to the desired ones. The difference is explained by the error in the approximation of the describing function of the relay which yields a bias in the estimated gain margin and phase margin. In Fig. 3 the values of the controller parameters are shown for each iteration.

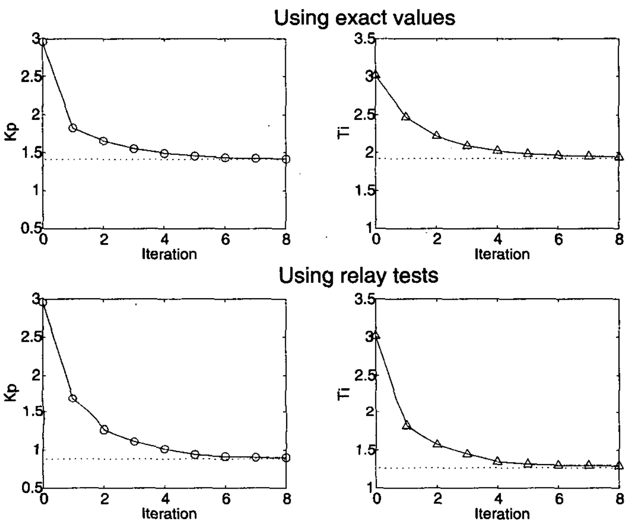

Fig. 3. Controller Parameters at each Iteration.

The gain and phase margins computed using the plant transfer function and the designed controller at each iteration are shown in Table I. Note that a reasonable solution is already obtained at third iteration. The controller parameters at the end of each iteration during simulation are shown in Table II.

Finally, closed loop step responses using the obtained controllers in Eq. (18) and (19) are shown in Fig. 4.

VI. EXPERIMENTAL RESULTS

In this section, experimental results obtained with a laboratory scale heat exchanger are presented. A block diagram of the system is show in Fig. 5.

Iter.	Gain Margin	Phase Margin
0	1.8518	22.5703°
1	2.7585	35.9957°
2	2.9890	37.4756°
3	3.0506	37.2655°
4	3.0889	37.0439°
5	3.1628	37.6582°
6	3.1764	37.7176°
7	3.1829	37.7457°

TABLE I
GAIN AND PHASE MARGIN AT EACH ITERATION.

Iter.	K_p	T_i
0	2.9575	3.0161
1	1.4605	1.8156
2	1.1881	1.5786
3	1.0625	1.4453
4	0.9698	1.3484
5	0.9225	1.3188
6	0.9070	1.3051
7	0.8995	1.2984

TABLE II
CONTROLLER PARAMETERS AT THE END OF EACH ITERATION.

The process consists of two pipes conected to a resevoir. Small fans are used to force the air from the outside environment into the resevoir and then back to the environment. The air flow in the pipe 1 is heated using an electrical resistance. The resistance voltage is controlled by a full wave thyristor drive with fire angle varying according to the control signal, which is an electrical signal ranging from $0 - 2.5V$. The second pipe is used to disturb the system.

The output of the system, $y(t)$, is the temperature in the resevoir, which is measured by a $0 - 2.5V$ electrical signal from the thermal sensor S_1 plus amplifier. The PI controller is implemented in a microcomputer with integrated data acquisition system (A/D and D/A converters), not shown in Figure 5. The objective is to control the temperature at the resevoir at desired setpoints.

The chosen setpoint is $y_r = 1.25V$. The initial controller is obtained from a Ziegler-Nichols frequency response table (see [9]). The critical gain of the process is estimated with a relay feedback,

$$K_u = 6.7977 \text{ and } T_u = 9.4537,$$

and the following controller parameters are obtained,

$$K_p = 2.7191 \text{ and } T_i = 7.5630\,.$$

A step response of the closed loop system with the Ziegler-Nichols controller is shown in Fig. (6). The *loop critical*

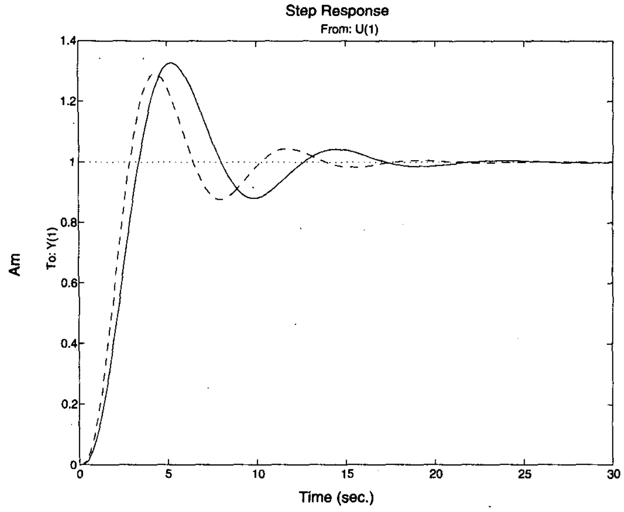

Fig. 4. Closed loop step responses for obtained controllers: exact solution (dashed) and using relay tests (continuous).

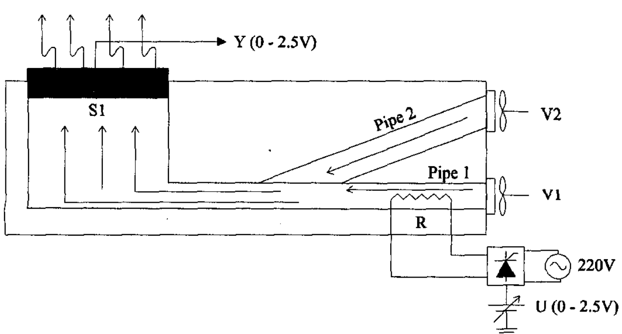

Fig. 5. Block Diagram of the Temperature Control System.

gain and the *loop gain crossover* relay tests applied to the closed loop gives the following estimates,

$$GM^0 = 3.7226 \quad \text{and} \quad PM^0 = 67.987^\circ \, .$$

Now the iterative algorithm starts, with $C^0(s)$ being the Ziegler-Nichols controller. It is desired a closed loop system with gain margin $A_m = 3$ and phase margin $\phi_m = 60^\circ$. The stop criterions are defined as $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 2^\circ$. After 4 iterations, the following controller is obtained,

$$K_p^4 = 1.3952 \quad \text{and} \quad T_i^4 = 3.1873 \, .$$

The stability margins estimated at each iteration step (Eq. (10) and (13)) are shown in Table III. The controller parameters at the end of each iteration step are shown in Table IV.

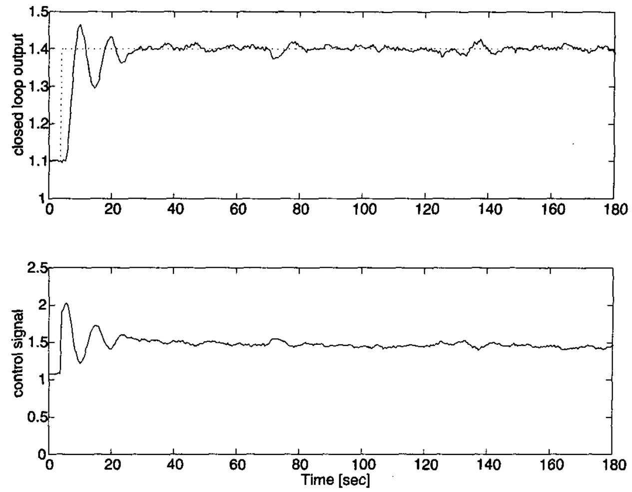

Fig. 6. Closed loop step response for the Ziegler-Nichols controller.

Iter.	Gain Margin	Phase Margin
0	3.7226	62.103°
1	2.4933	67.3798°
2	2.4930	68.0788°
3	2.7811	64.3603°
4	2.9847	(stopped)

TABLE III

GAIN AND PHASE MARGIN AT EACH ITERATION.

Iter.	K_p	T_i
0	2.7191	7.5630
1	3.3259	6.7872
2	2.5765	4.9866
3	1.8923	3.7475
4	1.3952	3.1873

TABLE IV

CONTROLLER PARAMETERS AT THE END OF EACH ITERATION.

To illustrate the estimation procedure, the process output at third iteration step during the relay tests are shown in Fig. 7.

The closed loop step response using the controller obtained at the end of the iteration scheme is shown in Fig. (8). The bandwidth of this system (defined as the frequency at which $\angle L(j\omega) = -180^\circ$) is estimated as

$$\omega_u^4 = 0.6083 \, rad/s \, ,$$

which is smaller than the bandwidth obtained with $C^0(s)$.

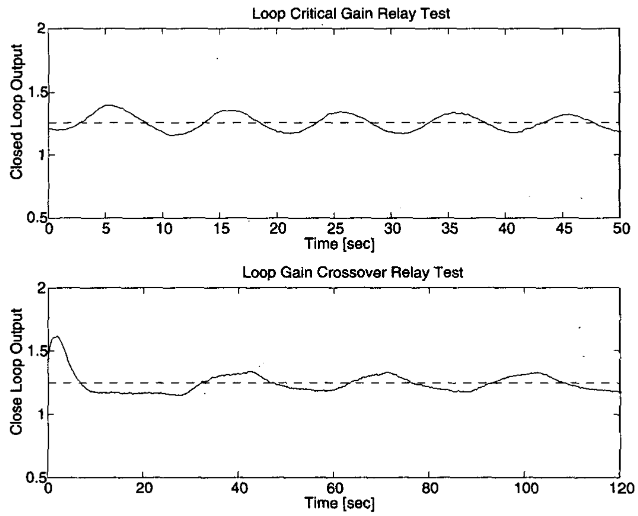

Fig. 7. Closed loop output during relay experiments at third iteration: loop critical gain relay test (above) and loop gain crossover relay test (bellow).

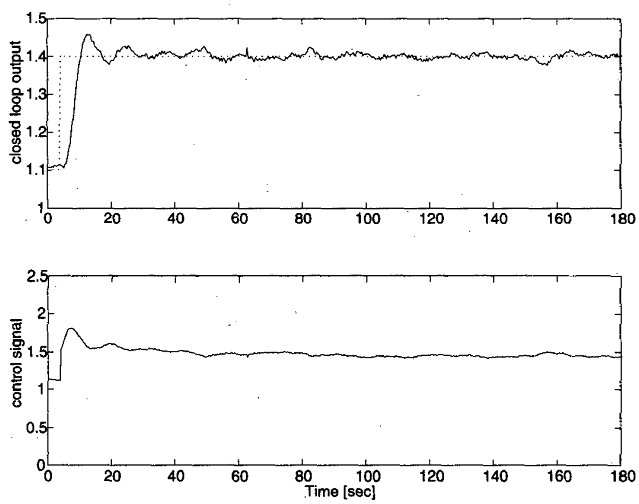

Fig. 8. Closed loop step response for the obtained controller.

VII. CONCLUSIONS

In this paper an iterative procedure combining two relay experiments is applied to PI controller tuning based on gain and phase margins. The relay tests are used to estimate the critical point and the gain crossover point of the loop transfer function of a closed loop system. This information is used to solve a set of nonlinear equations in order to achieve the gain and phase margins specifications. The advantage is that no assumptions on process model structure is required, neither complete knowledge of its frequency response. A simulation example is used to illustrate the procedure, and results with a heat exchanger system in laboratory scale are presented. In both cases, a reasonable controller is obtained with a few iteration steps, which qualifies the method to industrial applications.

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