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Relay Based Closed Loop Transfer Function Estimation

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Abstract

This paper describes a new technique for relay based closed loop transfer function estimation. Given a closed loop system, a user selected transfer function and a defined magnitude, then a relay experiment is used to obtain the frequency corresponding to the selected transfer function magnitude. In this paper the loop transfer function and the sensitivity transfer function are considered. This information can be used to evaluate certain transfer function properties or to redesign the controller.

Keywords: Identification; frequency domain identification; transfer function estimation.

1 Introduction

The relay experiment introduced in Åström and Hägglund [1] has gained widespread use as an estimation tool for real time controller tuning. In its original form it was used to estimate the process ultimate gain and ultimate frequency under very well defined operating conditions. The theory for analyzing the plant under relay feedback was based on the well know describing function analysis, which also contributed to make the method widely accepted.

Some attention has been given to use the relay experiment to obtain more than a single point of the plant frequency response. As a recent reference, Wang *et al.* [9] used the transient from the relay experiment to estimate other frequency points. An alternative method, this time based on the frequency sampling filter model, was used in the Wang *et al.* [8]. In Schei [6] the relay was applied to the closed loop such that a limit cycle oscillation occurs in the frequency for which the Complementary Sensitivity Function phase is -90^o. It was shown that this frequency is between crossover frequency and critical frequency of the Loop Transfer Function. In Schei [7] a new relay experiment is included such that the closed loop system oscillates approximately at the crossover frequency.

In this paper, the relay is applied to a closed loop system to obtain the frequency in which a given transfer function has a user defined magnitude. The proposed procedure can be seen as a generalization of the procedure used in Schei [7]. To our best knowledge it was never presented before. Here it is applied in frequency response estimation of the Loop Transfer Function and Sensitivity Function.

In Section 2 the identification problem is stated and in Section 3 a basic procedure for relay based frequency domain identification is presented. This basic procedure is used to derive a Loop Transfer Function estimation method (Section 4) and a Sensitivity Function estimation method (Section 5). A stability analysis for both experiments is developed in Section 6, as well as conditions for bounded closed loop plant signals. Finally, a simulation example is shown in Section 7.

2 Problem Statement

In this paper, and without loss of generality, consider the closed loop shown in Fig. (1). The unknown plant transfer function is given by $G(s)$ while the known controller is given by $C(s)$. The closed loop transfer function from reference $y_r(t)$ to output $y(t)$ is given by

$$\frac{Y(s)}{Y_r(s)} = M(s) = \frac{G(s)\,C(s)}{1+G(s)\,C(s)}. \tag{1}$$

For this closed loop configuration $M(s)$ is also known as the Complementary Sensitivity Function, $L(s) = G(s)\,C(s)$ is the Loop Transfer Function, and

$$\frac{E(s)}{Y_r(s)} = S(s) = \frac{1}{1+G(s)\,C(s)} \tag{2}$$

is the Sensitivity Function.

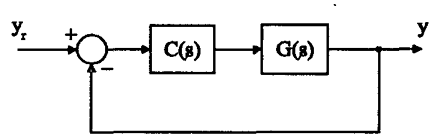


Figure 1: Closed Loop System.

3 The Basic Procedure

Consider the relay closed loop shown in Fig. (2). Assume that transfer function $F(s)$ is stable. Then, describing function analysis (See [2]) points out that a limit cycle oscillation may occur at a frequency near the point where $\angle F(j\omega) = -90^o$. If $F(s) = M(s)$ then one arrives at the structure presented in Schei [6] while if $F(s) = 2M(s) - 1$ then one arrives at the structure presented in Schei [7]. In the latter, it was shown that the oscillation occurs at a frequency ω_{lc} approximately equal to the crossover frequency. For clarity in later developments, that will be restated here:

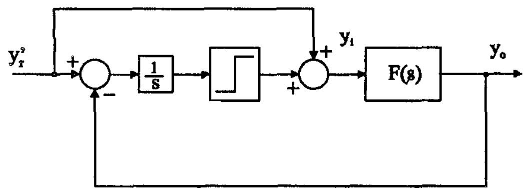


Figure 2: Relay Closed Loop System.

Proposition 1 *Consider the relay closed loop system shown in Fig (2), where $F(s) = 2M(s) - 1$, and assume that it is under a stable limit cycle oscillation. Then the closed loop system oscillates at frequency ω_{lc} for which*

$$|G(j\omega_{lc})\, C(j\omega_{lc})| \approx 1.$$

Proof: See [7]. ∎

The above procedure can be generalized for obtaining a controlled oscillation at a frequency for which a transfer function $H(s)$ has a user defined gain. That defines the basic procedure which is given as the following proposition:

Proposition 2 *Consider the closed loop system shown in Fig (2). Assume that for a stable transfer function $H(s)$ and a real positive number r, transfer function*

$$F(s,r) = \frac{H(s) - r}{H(s) + r} \tag{3}$$

is also stable. Then if a limit cycle is present it oscillates at a frequency ω_o such that

$$|H(j\omega_o)| \approx r.$$

Proof: See [4]. ∎

4 Loop Transfer Function Estimation

The estimation of the magnitude of the Loop Transfer Function $L(s)$ is of practical importance for controller design. Loop shaping controller design techniques use a desired Loop Transfer Function $L(s)$ to obtain disturbance attenuation and robust stability [5]. There exists also several loop shaping techniques which uses a few points of the frequency response in $L(s)$ to design, for instance, PID controllers (see [2]). Now, the basic procedure is applied to estimate other points of the Loop Transfer Function.

Proposition 3 *Consider the closed loop system shown in Fig (2). Assume that for a stable Complementary Sensitivity Function $M(s)$ and a real positive number r, transfer function*

$$F(s,r) = \frac{2}{r}\frac{M(s)}{M(s)\left(\frac{1-r}{r}\right) + 1} - 1 \tag{4}$$

is also stable. Then if a limit cycle is present it oscillates at a frequency ω_o such that

$$|L(j\omega_o)| \approx r.$$

Proof: See [4]. ∎

The closed loop system which implements the procedure is presented on Fig. (3).

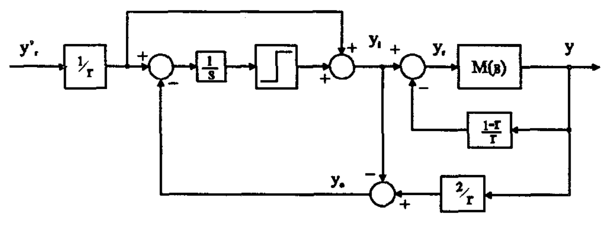


Figure 3: Relay Closed Loop Experiment for Loop Transfer Function Estimation.

5 Sensitivity Function Estimation

The Sensitivity Function brings important information on the closed loop such as disturbance attenuation and stability margins [5]. Sensitivity Function shaping in an alternative approach for controller design is presented in Doyle *et al.* [5] and Barros and Wittenmark [3]. Experimentally evaluating the Sensitivity Function is important for determining if disturbance attenuation and stability margin specifications are being met. It can also be used for controller redesign. Now the basic procedure is applied for estimating the Sensitivity Function.

Proposition 4 *Consider the closed loop system shown in Fig (2). Assume that for a stable transfer function $M(s)$ and a positive real number r, transfer function*

$$F(s,r) = 2\frac{\frac{1}{r+1}}{1 - \frac{r}{r+1}M(s)} - 1 \tag{5}$$

is also stable. Then if and a limit cycle is present it oscillates at a frequency ω_o such that

$$|S(j\omega_o)| \approx \frac{1}{r}.$$

Proof: See [4]. ∎

The closed loop system which estimates the Sensitivity Function is shown in Fig. (4).

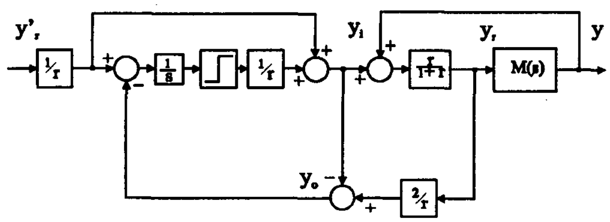


Figure 4: Relay Closed Loop Experiment for Sensitivity Estimation.

6 Experimental Conditions Issues

It is a basic requirement that the closed loop relay experiments presented here are performed under well defined operation conditions. In this Section operating condition issues such as relay amplitude, stability and signal boundedness are considered.

6.1 Relay Amplitude

6.1.1 Loop Transfer Function Case: Consider the closed loop system in Fig. (3). The signal $y(t)$ can be computed as the contribution of two terms: the overall input $y_r'(t)$ and the relay output (with amplitude $\pm d$). Assume a constant reference signal, $M(s)$ stable and $M(0)=1$. If $d=0$ then the transfer function from $y_i(t)$ to $y(t)$ in Fig. (3) is given by

$$\frac{Y(s)}{Y_i(s)} = \frac{rM(s)}{r+(1-r)M(s)} . \tag{6}$$

In steady state

$$y_{ss} = \frac{r}{r+1-r} y_{i_{ss}} = r y_{i_{ss}} = y_r' .$$

Now, if $y_r'(t)=0$ and the closed loop oscillates at frequency ω_o then amplitude Y_1 of the first harmonic of the plant output signal $y(t)$ is given by

$$Y_1 = \frac{rM(j\omega_o)}{r+(1-r)M(j\omega_o)} Y_{i_1} , \tag{7}$$

where Y_{i_1} is the first harmonic amplitude of input signal $y_i(t)$. Since $y_i(t)$ is a periodic symmetrical square wave with amplitude d, then its first harmonic amplitude is given by [2]:

$$Y_{i_1} = \frac{4d}{\pi} .$$

Using the triangle inequality in Eq. (7),

$$\begin{aligned} Y_1 &= \frac{|rM(j\omega_o)|}{|r-(r-1)M(j\omega_o)|}\frac{4d}{\pi} \\ &\le \frac{|rM(j\omega_o)|}{|r|-|(r-1)M(j\omega_o)|}\frac{4d}{\pi} = \frac{r\,|M(j\omega_o)|}{r-|r-1|\,|M(j\omega_o)|}\frac{4d}{\pi} . \end{aligned}$$

Now if $|M(j\omega_o)|=1$ and $r>1$, then

$$Y_1 \le \frac{4dr}{\pi} .$$

This information can be used to keep the output between desired operating levels. Suppose $\Delta y_{\max}$ is the plant output maximum range, such that $y_r' - \Delta y_{\max} \le y(t) \le y_r' + \Delta y_{\max}$. Then,

$$Y_1 \le \frac{4dr}{\pi} = \Delta y_{\max} ,$$

and

$$d = \frac{\pi \Delta y_{\max}}{4} \cdot \frac{1}{r} . \tag{8}$$

That is, the relay amplitude should be weighted by $1/r$ in order to keep the output oscillation amplitude at the same level for different r.

6.1.2 Sensitivity Function Case: Consider now the closed loop system in Fig. (4). Then the transfer function from signal $y_i(t)$ to plant output $y(t)$ is given now by

$$\frac{Y(s)}{Y_i(s)} = \frac{rM(s)}{1+r-rM(s)} . \tag{9}$$

Following the same steps as above,

$$\begin{aligned} Y_1 &= \frac{|rM(j\omega_o)|}{|1+r-rM(j\omega_o)|}\frac{4d}{\pi} \\ &\le \frac{|rM(j\omega)|}{|1+r|-|rM(j\omega_o)|}\frac{4d}{\pi} = \frac{r\,|M(j\omega_o)|}{1+r-r\,|M(j\omega_o)|}\frac{4d}{\pi} . \end{aligned}$$

which for $|M(j\omega_o)|=1$ yields

$$Y_1 \le \frac{4dr}{\pi} .$$

6.2 Stability and Signal Boundedness

Consider the relay closed loop system structure of Fig. (2). If $y_r'(t)$ is bounded, then $y_i(t)$ is a bounded signal by construction. Since $Y_o(s) = F(s,r)\,Y_i(s)$, then $y_o(t)$ is bounded if $F(s,r)$ is stable for some r. Now consider the basic procedure with $F(s)$ implemented as shown in Fig. (5),

$$\frac{Y_o}{Y_i} = F(s,r) = \frac{H(s)-r}{H(s)+r} .$$

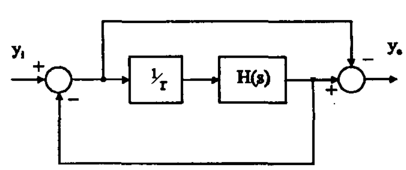
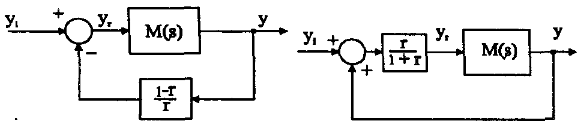


Figure 5: Block Diagram of Transfer Function $F(s,r)$.

The closed loop stability of the structure can be evaluated as follows.

Proposition 5 *The set of real numbers r for which $F(s,r)$ is stable can be evaluated directly from the root locus of $H(s)$.*



(a) Loop Transfer Function. (b) Sensitivity Function.

Figure 6: Stability of Experiment.

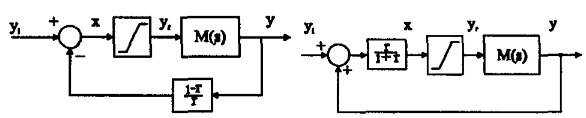
Proof: See [4]. ∎

Remark 1 *In Schei [7] $H(s) = L(s)$ and $r = 1$ so that the resulting $F(s,r)$ is stable by construction.*

From the above result, the closed loop can be unstable depending on $H(s)$ and r. For the Loop Transfer Function example shown in Fig. (3), stability depends only on the feedback loop shown in Fig. (6(a)), and for the Sensitivity Function case shown in Fig. (4), stability is dependent only on the feedback loop shown in Fig. (6(b)). Under the assumptions that $M(s)$ is stable and $M(0) = 1$, then in order to avoid unbounded signals one can use a saturation nonlinearity before the closed loop plant as shown in Fig. (7(a)) and Fig. (7(b)). Assume that $y(t)$ should vary between $y_r'(t) - \Delta y_{\max}$ and $y_r'(t) + \Delta y_{\max}$ where $\Delta y_{\max}$ is the desired variation of the closed loop output. The saturation nonlinearity can be chosen as

$$y_r(t) = \begin{cases} x(t) \text{ if} \\ \quad y_r'(t) - \Delta y_{\max} < x(t) < y_r'(t) + \Delta y_{\max}, \\ y_r'(t) - \Delta y_{\max} \text{ if } x(t) \leq y_r'(t) - \Delta y_{\max}, \\ y_r'(t) + \Delta y_{\max} \text{ if } x(t) \geq y_r'(t) + \Delta y_{\max}. \end{cases} \tag{10}$$

During the experiment one can vary r until the saturation region is achieved. When it occurs, the experiment should be turned off as it is not valid anymore.



(a) Loop Transfer Function. (b) Sensitivity Estimation.

Figure 7: Saturation in Closed Loop Input.

7 Simulation Example

Now simulation results with the estimation methods are presented. The plant is second order with unitary delay and a PI controller is used. The plant delay is approximated by a first order Padé approximation,

$$G(s) = \frac{(1-2s)}{(s^2 + 2(0.8)s + 1)(1+2s)},$$

and the PI controller is given by,

$$C(s) = \frac{0.5(s+0.4)}{s}.$$

7.1 Loop Transfer Function Estimation

The closed loop experiment is implemented as shown in Fig. (3), with saturation in closed loop input as described in Section 6. The amplitude of the oscillation is chosen as 20% of setpoint condition. Thus, with $y_r' = 1$, the relay amplitude should be $d = 0.2 * \pi/4$.

The estimation results for the Loop Transfer Function are shown in Fig. (8), with r chosen as [0.6 1.0 2.0 4.0 8.0 16.0 32.0].

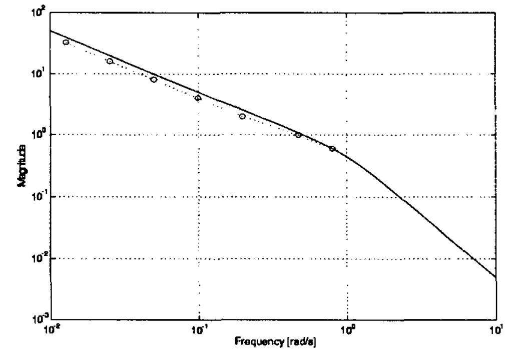
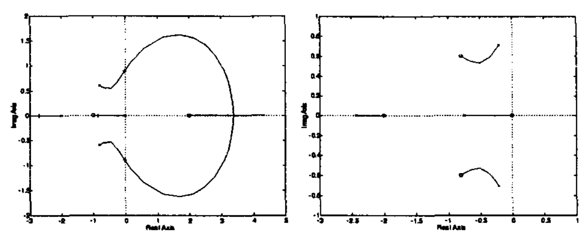


Figure 8: Loop Transfer Function Estimation.

Experiment instability was detected for $r < 0.6$. This situation was expected, as can be seen from the root locus plot of $H(s) = L(s)$, shown in Fig. (9(a)). In fact, it points out that for $r < 0.5076$ transfer function $F(s,r)$ becomes unstable.



(a) $H(s) = L(s)$. (b) $H(s) = 1 + L(s)$.

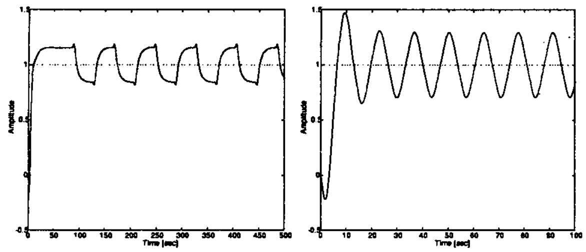
Figure 9: Root locus of $H(s)$.

Plant outputs during relay experiment with $r = 2$ and $r = 0.3$ are shown in Fig. (10(a)) and Fig. (10(b)), respectively. Even for $r = 0.3$, which leads $F(s)$ to instability, plant output boundedness was guaranteed and within ±20% of variation.

7.2 Sensitivity Function Estimation

Now the closed loop experiment for Sensitivity Function estimation is performed as shown in Fig. (4), with saturation in closed loop input. Estimation results for the Sensitivity Function are shown in Fig. (11). The identification experiment was performed for r^{-1} in [0.1 0.25 0.5 1.0 1.5 2.0 2.2].

In this case, experiment instability was not detected for any value of r in the specified set. Indeed, it can be



(a) $r = 2$. (b) $r = 0.3$.

Figure 10: Time Responses for Loop Transfer Function Estimation.

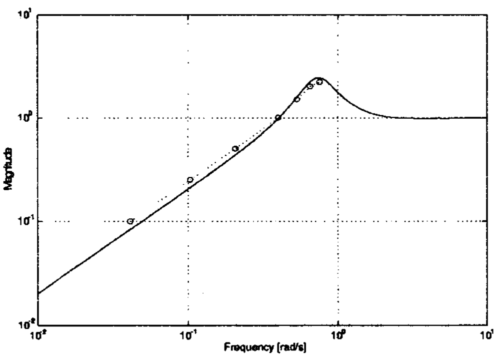


Figure 11: Sensitivity Estimation.

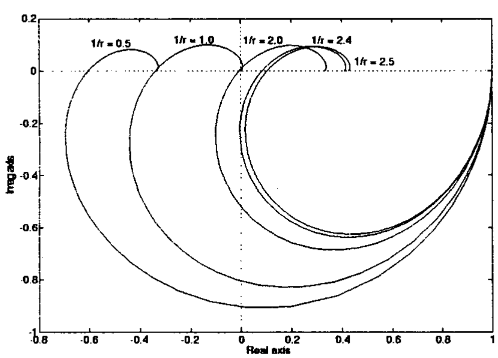
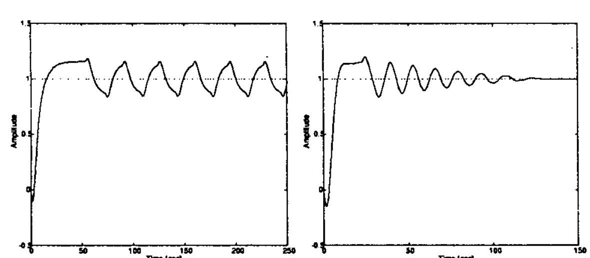


Figure 12: Nyquist Plot for $F(s, r)$ for some values of r.



(a) $r^{-1} = 1$. (b) $r^{-1} = 2.5$.

Figure 13: Time Responses for Sensitivity Estimation.

seen from root locus plot of $H(s) = 1 + L(s)$, shown in Fig. (9(b)), that the experiment is stable for all $r > 0$.

The peak on the Sensitivity Function was obtained by decreasing gain r until oscillation has stopped. Note from the Nyquist plot of $F(s)$, shown in Fig. (12), does not intercept the negative imaginary axis for $r^{-1} > 2.4$, indicating that no limit cycle oscillation occurs.

Time responses for $r^{-1} = 1$ and $r^{-1} = 2.5$ are shown in Fig. (13(a)) and Fig. (13(b)). It can be seen that the limit cycle oscillation vanishes for $r^{-1} = 2.5$, which gives approximately the peak of the Sensitivity Function.

8 Conclusions

In this paper a new technique for transfer function magnitude estimation is presented. A basic procedure is developed and two new relay experiments are derived: Loop Transfer Function and Sensitivity Function Estimation. Stability conditions for using the basic procedure are established and the relay experiments are performed on existing closed loop. Practical issues concerning relay amplitude and signal bounds are discussed. These methods can be used for robust controller analysis or controller tuning based on specifications on Loop Transfer Function magnitude and/or Sensitivity magnitude.

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