Nonlinear Time-Varying Stability Analysis of Particle Swarm Optimization

Weibing Fan
Complex System and Computational Intelligence Laboratory
Taiyuan University of Science and Technology, Shanxi, P.R.China, 030024
Email: abing606@126.com

Zhihua Cui
Complex System and Computational Intelligence Laboratory
Taiyuan University of Science and Technology, Shanxi, P.R.China, 030024
Email: cuizhihua@gmail.com

Yongjing Chen
Complex System and Computational Intelligence Laboratory
Taiyuan University of Science and Technology, Shanxi, P.R.China, 030024
Email: yongjing178@163.com

Ying Tan
Complex System and Computational Intelligence Laboratory
Taiyuan University of Science and Technology, Shanxi, P.R.China, 030024
Email: tanying1965@gmail.com

***Abstract*—As a nonlinear time-varying system, it is hard to make the stability analysis for particle swarm optimization (PSO). In this paper, geometric speed stability analysis is introduced to analyze the stability condition of PSO. According to the condition, a new inertia weight selection strategy is proposed. Simulation results show it is effective.**

***Keywords*-particle swarm optimization; nonlinear time-varying system; geometric speed stability analysis;**

I. Introduction

Particle swarm optimization is a novel computational intelligent algorithm which was proposed by Kennedy and Eberhart [1,2] in 1995. Different from other evolutionary algorithms, PSO algorithm has the advantage of being fast convergent speed and easy to implementation. In addition, parameters of PSO is also relatively small. Therefore, after PSO was proposed, many scholars pay their attention to this new algorithm[3].

The standard PSO algorithm maintains a population of m particles in the D-dimensional search space, suppose $x_i = (x_{i1}, x_{i2}, x_{i3}, ..., x_{iD})(i = 1, 2, ..., m)$ is the position of i^{th} particle in the PSO algorithm, each individual represents a potential solution to a problem, optimal position experienced by i^{th} particle known as the optimal location of the personal best solution of of particle i, recorded as $P_i = (p_{i1}, p_{i2}, p_{i3}, ..., p_{iD},)$,optimal position experienced by all particles known as the optimal location of the global best solution, recorded as $P_g = (p_{g1}, p_{g2}, p_{g3}, ..., p_{gD},)$,at each iteration, its d-dimensional ($1 \leq d \leq D$) based on the following iterative equation [4,5]:

$$\begin{cases} v_i(t+1) = \omega v_i(t) + c_1 r_1[p_i - x_i(t)] + c_2 r_2[p_g - x_i(t)] \\ x_i(t+1) = x_i(t) + v_i(t). \end{cases}$$

where ω is the inertia weight;c_1 and c_2 is the normal number, known as the constant accelerators; r_1 and r_2 are two random numbers sampled within the scope of [0,1].

In the update equations, the first part $\omega v_i(t)$ used to guarantee the global convergence performance, while the second and third parts make the capacity of local convergence, the introduction of inertia weight ω may balance the ratio of the global convergence and local convergence, which can improve the performance of algorithm. At the same time, it eliminates the needs of v_{max} for basic PSO algorithm, when v_{max} increased, it may balance search by reducing ω, while reducing of ω can be made to reduce the number of iterations, thereout ,v_{max} can be fixed for the changes in the scope of variables.

Although many work have been done, there still many work need to do. Since PSO is a nonlinear time-varying system, therefore, how to measure the stability is an important problem. In this paper, a new theory, geometric speed stability concept is introduced to analyze the stability of PSO.

II. GEOMETRIC SPEED STABILITY ANALYSIS FOR PSO

Consider the following discrete time-varying dynamic system

$$x(n+1) = f(n, x(n)) n = 0, 1, 2, ... of x(0) \in R^n \tag{1}$$

Definition[6]: Let $L \subset R^m$ be a bounded set, the dynamic system is (1). For the initial value of iterative sequence $\{f_n(n, x(0)), n \geq 1\}$, $r_n(x(0), L)$ indicates to the collection of ordinary Euclidean space distance, namely:

$$r_n(x(o), L) = \inf_{y \in L} \parallel x(n) = y \parallel \tag{2}$$

If $\lim_{n \to \infty} r_n(x(0), L) = 0, \forall x(0) \in R^m$, the dynamic system (1) is called the overall stability.

If convergence rate of the (2) is exponential, in other words, there is a positive number $\theta < 1$ and K_1,K_2, so that $r_n(x(0), L) \leq \theta^n[K_1 r(x(0), L) + K_2], \forall x(0) \in R^m$, while $r(x(0), L)$ represents the distance between $x(0)$ and L, the dynamic system (1) is called geometric speed stable.

III. INERTIA WEIGHT SELECTION STRATEGY BASED ON GEOMETRIC SPEED STABILITY ANALYSIS

The update equations of standard PSO is re-written with

$$\begin{cases} v_i(t+1) = \omega v_i - \varphi x_i(t) + \varphi_1 p_i(t) + \varphi_2 p_g(t) \\ x_i(t+1) = \omega v_i + (1-\varphi) x_i(t) + \varphi_1 p_i(t) + \varphi_2 p_g(t). \end{cases}$$

where $\varphi_1 = c_1 r_1$, $\varphi_2 = c_2 r_2$, and $\varphi = \varphi_1 + \varphi_2$.

Re-format it with matrix manner:

$$\begin{bmatrix} v_i(t+1) \\ x_i(t+1) \end{bmatrix} = \begin{bmatrix} w & -\varphi \\ w & 1-\varphi \end{bmatrix} \begin{bmatrix} v_i(t) \\ x_i(t) \end{bmatrix} + \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} p_i(t) \\ p_g(t) \end{bmatrix}$$

Suppose $y(t+1) = \begin{bmatrix} v_i(t+1) \\ x_i(t+1) \end{bmatrix}$, it can be written

$$y(t+1) = f(t, y(t)), t = 1, 2, ... \quad (3)$$

Therefore,

$$\begin{aligned} &\left\| \begin{bmatrix} v_i(t+1) \\ x_i(t+1) \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} w & -\varphi \\ w & 1-\varphi \end{bmatrix} \begin{bmatrix} v_i(t) \\ x_i(t) \end{bmatrix} + \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} p_i(t) \\ p_g(t) \end{bmatrix} \right\|_\infty \\ &\le \left\| \begin{bmatrix} w & -\varphi \\ w & 1-\varphi \end{bmatrix} \begin{bmatrix} v_i(t) \\ x_i(t) \end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} p_i(t) \\ p_g(t) \end{bmatrix} \right\|_\infty \end{aligned}$$

Because $\| Ay(t) \|_\infty \le \| A \|_\infty \cdot \| y(t) \|_\infty$, we have

$$\le \left\| \begin{bmatrix} w & -\varphi \\ w & 1-\varphi \end{bmatrix} \right\|_\infty \cdot \left\| \begin{bmatrix} v_i(t) \\ x_i(t) \end{bmatrix} \right\|_\infty + \left\| \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} p_i(t) \\ p_g(t) \end{bmatrix} \right\|_\infty$$

But

$$\begin{aligned} \left\| \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1 & \varphi_2 \end{bmatrix} \begin{bmatrix} p_i(t) \\ p_g(t) \end{bmatrix} \right\| &= \left\| \begin{bmatrix} \varphi_1 p_i(t) + \varphi_2 p_g(t) \\ \varphi_1 p_i(t) + \varphi_2 p_g(t) \end{bmatrix} \right\|_\infty \\ = |\varphi_1 p_i(t) + \varphi_2 p_g(t)| &= |c_1 r_1 p_i(t) + c_2 r_2 p_g(t)| \\ \le c_1 d + c_2 d &= (c_1 + c_2) d = c \end{aligned}$$

Clearly, geometric speed stability exists and.

$$\left\| \begin{bmatrix} w & -\varphi \\ w & 1-\varphi \end{bmatrix} \right\| = max \omega + \varphi, \omega + |1-\varphi| < \theta < 1$$

Suppose $\theta = 0.9$, then following discussion can be obtained:

If $0 < \varphi < 0.5$, $\omega + \varphi < \omega + 1 - \varphi < 0.9 < 1$, $0 < \omega < \varphi - 0.1$;

If $0.5 < \varphi < 1$, $\omega + 1 - \varphi < \omega + \varphi < 0.9 < 1$, $0 < \omega < 0.9 - \varphi$;

If $\varphi > 1$, there is no stability parameter settings;

Overall, the parameter choices of ω and φ is

$$\begin{cases} 0 < \omega < \varphi - 0.1 & if\ 0 < \varphi < 0.5 \\ 0 < \omega < 0.9 - \varphi & if\ 0.5 \le \varphi < 1 \end{cases}$$

Under the above conditions,we can guarantee the standard PSO algorithm for the geometric speed stability in

$$L = \{x : \|x\|_v \le \frac{c}{1-\theta} = \frac{(c_1 + c_2)d}{1 - 0.9} = 10c\}$$

In briefly, this new algorithm is called particle swarm optimization with geometric speed stability $PSO - GSS$.

Table I
ALGORITHM'S TESTING ENVIRONMENT

dimension	30	50	100
popsize	100	100	100
max_length	1500	2500	5000
run_times	10	10	10

IV. MUTATION STRATEGY

In order to improve ability jumping out of local extreme points, the paper will introduce the following mutation operator into $PSO - GSS$, at each iteration , one dimension of a component of particle was selected with the following operation :

$$v_{ij} = \begin{cases} 0.5 * v_{max} * rand(1) & r < 0.5 \\ -0.5 v_{max} * rand(1) & \text{otherwise.} \end{cases} \quad (4)$$

Including, r is a random number a range of (0,1).

V. SIMULATION RESULTS

In order to analyze the performance of the particle swarm optimization with geometric speed stability, the standard particle swarm optimization (SPSO) and modified time-varying accelerator coefficients particle swarm optimization ($MPSO - TVAC$) are used to compare, and one test suits consist 7 benchmarks is selected to test. They are Rosenbrock, Schwefel 2.26, Rastrigin, Ackley, Griewank and two penalized functions.

Acceleration constants of SPSO is 2.0, inertia weight ω of SPSO and $MPSO - TVAC$ linearly decreases from 0.9 to 0.4, while the inertia weight of $PSO - GSS$ changes with the above mentioned manner. The testing environment is shown in table 1: The following is test functions based on standard particle swarm optimization (SPSO), Modified time-varying accelerator coefficients particle swarm optimization ($MPSO - TVAC$) and Particle swarm optimization with geometric speed stability ($PSO - GSS$) in the 30 ,50,100,150,200,250,300 dimensions respectively, simulation results are as following:

Simulation results show $PSO - GSS$ is suit for high-dimensional multi-modal problems.

VI. CONCLUSION

In this paper, the geometric speed stability is used to analyze the stability of particle swarm optimization, furthermore, the selection strategy of inertia weight is also derived. Simulation results show PSO-GSS is superior to other two variants of PSO.

ACKNOWLEDGMENT

This paper was supported by Natural Science Foundation of Shanxi under Grant 2008011027-2.

Table II
SIMULATION RESULTS OF ROSENBROCK FUNCTION

Dimension	Algorithm	Mean	Variance
30	SPSO	5.6170e+001	4.3584e+001
	MPSO-TVAC	3.3589e+001	4.1940e+001
	PSO-GSS	2.9738e+001	1.8240e+001
50	SPSO	1.1034e+002	3.7488e+001
	MPSO-TVAC	7.8125e+001	3.2496e+001
	PSO-GSS	8.3396e+001	3.7867e+001
100	SPSO	4.1064e+002	1.0584e+002
	MPSO-TVAC	2.8517e+002	9.8129e+001
	PSO-GSS	1.5130e+002	3.7258e+001
150	SPSO	8.9131e+002	1.6561e+002
	MPSO-TVAC	5.4671e+002	6.4228e+001
	PSO-GSS	3.1032e+002	5.9251e+001
200	SPSO	2.9071e+003	5.4258e+002
	MPSO-TVAC	8.0076e+002	2.0604e+002
	PSO-GSS	4.3420e+002	6.0381e+001
250	SPSO	7.4767e+003	3.2586e+003
	MPSO-TVAC	1.3062e+003	3.7554e+002
	PSO-GSS	4.8766e+002	9.5214e+001
300	SPSO	2.3307e+004	1.9726e+004
	MPSO-TVAC	1.4921e+003	3.4571e+002
	PSO-GSS	5.7904e+002	1.7351e+002

Table III
SIMULATION RESULTS OF SCHWEFEL PROBLEM 2.26 FUNCTION

Dimension	Algorithm	Mean	Variance
30	SPSO	-6.2762e+003	1.1354e+003
	MPSO-TVAC	-6.7672e+003	5.7050e+002
	PSO-GSS	-1.0165e+004	3.2316e+002
50	SPSO	-1.0091e+004	1.3208e+003
	MPSO-TVAC	-9.7577e+003	9.6392e+002
	PSO-GSS	-1.6615e+004	4.1565e+002
100	SPSO	-1.8147e+004	2.2012e+003
	MPSO-TVAC	-1.7943e+004	1.5061e+003
	PSO-GSS	-3.3165e+004	7.8362e+002
150	SPSO	-2.5036e+004	4.7553e+003
	MPSO-TVAC	-2.7863e+004	1.6350e+003
	PSO-GSS	-4.9628e+004	1.3802e+003
200	SPSO	-3.3756e+004	3.4616e+003
	MPSO-TVAC	-4.0171e+004	4.3596e+003
	PSO-GSS	-6.5766e+004	1.1445e+003
250	SPSO	-3.9984e+004	4.7099e+003
	MPSO-TVAC	-4.7337e+004	3.7545e+003
	PSO-GSS	-7.1141e+003	1.2914e+003
300	SPSO	-4.6205e+004	6.0073e+003
	MPSO-TVAC	-5.6873e+004	3.5129e+003
	PSO-GSS	-9.8838e+004	1.1964e+003

Table IV
SIMULATION RESULTS OF RASTRIGIN FUNCTION

Dimension	Algorithm	Mean	Variance
30	SPSO	1.7961e+001	4.2276e+000
	MPSO-TVAC	1.5471e+001	4.2023e+000
	PSO-GSS	1.0292e+000	1.1524e+000
50	SPSO	3.9958e+001	7.9258e+000
	MPSO-TVAC	3.8007e+001	7.0472e+000
	PSO-GSS	1.5588e+000	9.3061e-001
100	SPSO	9.3679e+001	9.9635e+000
	MPSO-TVAC	8.4478e+001	9.4568e+000
	PSO-GSS	2.9849e+000	1.4071e+000
150	SPSO	1.5353e+002	1.2171e+001
	MPSO-TVAC	1.3692e+002	2.0096e+001
	PSO-GSS	4.3786e+000	1.4971e+000
200	SPSO	2.2827e+002	1.1196e+001
	MPSO-TVAC	1.9920e+002	2.8291e+001
	PSO-GSS	7.3627e+000	3.0106e+000
250	SPSO	2.8965e+002	2.8707e+001
	MPSO-TVAC	2.4616e+002	2.4220e+001
	PSO-GSS	6.7658e+000	1.8644e+000
300	SPSO	3.5449e+002	1.9825e+001
	MPSO-TVAC	2.7093e+002	3.7639e+001
	PSO-GSS	8.5618e+000	3.3242e+000

Table V
SIMULATION RESULTS OF ACKLEY FUNCTION

Dimension	Algorithm	Mean	Variance
30	SPSO	5.8161e-006	4.6415e-006
	MPSO-TVAC	7.5381e-007	3.3711e-006
	PSO-GSS	1.3431e-010	5.7852e-010
50	SPSO	1.7008e-004	1.2781e-004
	MPSO-TVAC	4.4132e-002	1.9651e-001
	PSO-GSS	1.6348e-010	2.1430e-010
100	SPSO	3.3139e-001	5.0105e-001
	MPSO-TVAC	4.6924e-001	1.9178e-001
	PSO-GSS	4.2800e-009	7.0165e-009
150	SPSO	1.4839e+000	6.6448e-001
	MPSO-TVAC	4.8264e-001	2.8287e-001
	PSO-GSS	1.9394e-008	3.7933e-008
200	SPSO	2.1693e+000	2.61267e-001
	MPSO-TVAC	6.9455e-001	4.0884e-001
	PSO-GSS	1.4595e-008	8.6428e-009
250	SPSO	2.6212e+000	1.7879e-001
	MPSO-TVAC	4.9552e-001	2.4789e-001
	PSO-GSS	2.2801e-008	1.1337e-008
300	SPSO	2.8959e+000	3.1471e-001
	MPSO-TVAC	7.6695e-001	3.1660e-001
	PSO-GSS	1.9751e-008	2.2749e-008

REFERENCES

[1] J.Kennedy ,C.EberhartR,"Particle swarm optimization",in*Proceeding of 1995 IEEE International Conference on Neural Net-works*,New York, NY, USA: IEEE, 1995. 1942 1948.

[2] C.EberhartR , J.Kennedy ,"A new optimizer using particle swarm theory",in *Proceedings of the Sixth International Symposiumon MicroMachine and Human Science*,New York, NY,USA: IEEE, 1995,39 43.

[3] J.CH.Zeng, J.Jie,ZH.H.Cui,"Particle Swarm Optimization",Beijing: Science Press, 2004 (in Chinese)

[4] Sh,I R C Eberhart. A Modified particle swarm optimizer [C]. Proc. IEEE Int. Con.f Evolutionary, 1998. 69-73.

[5] J.CH.Zeng, ZH.H.Cui,"A guaranteed global convergence particle swarm optimizer",in*Journal of Computer Research and Development*, 2004, 41(8): 1333-1338 (in Chinese)

[6] W.Zhu, "The Asymptotic Stability and Geometry-Velocity Stability of the Time-Varying Discrete Dynamic System", in *Journal of Applied Sciences*,2004,22.

Table VI
SIMULATION RESULTS OF PENALIZED FUNCTION2

Dimension	Algorithm	Mean	Variance
30	SPSO	9.7240e-003	1.1187e-002
	MPSO-TVAC	9.4583e-003	1.6344e-002
	PSO-GSS	1.1396e-002	1.4603e-002
50	SPSO	3.6945e-003	6.3651e-003
	MPSO-TVAC	1.5983e-002	2.7852e-002
	PSO-GSS	9.1489e-003	1.6660e-002
100	SPSO	2.1204e-003	4.3056e-003
	MPSO-TVAC	2.2598e-002	3.1112e-002
	PSO-GSS	4.3391e-002	6.5800e-002
150	SPSO	7.0552e-003	5.6254e-003
	MPSO-TVAC	3.0625e-002	5.0783e-002
	PSO-GSS	5.8220e-002	1.2595e-001
200	SPSO	8.8379e-002	1.2785e-001
	MPSO-TVAC	8.0535e-002	7.5014e-002
	PSO-GSS	4.5156e-002	7.6727e-002
250	SPSO	1.9341e-001	8.8577e-002
	MPSO-TVAC	1.6142e-001	1.5397e-001
	PSO-GSS	2.5390e-002	3.2658e-002
300	SPSO	5.5838e-001	2.1158e-001
	MPSO-TVAC	3.9988e-001	2.4034e-001
	PSO-GSS	4.0535e-002	8.2475e-002

Table VII
SIMULATION RESULTS OF PENALIZED1 FUNCTION

Dimension	Algorithm	Mean	Variance
30	SPSO	6.7461e-002	2.3159e-001
	MPSO-TVAC	1.8891e-017	6.9756e-017
	PSO-GSS	2.8452e-021	6.8157e-021
50	SPSO	5.4175e-002	6.7157e-002
	MPSO-TVAC	3.4248e-002	8.1985e-002
	PSO-GSS	1.1413e-018	2.8138e-018
100	SPSO	2.4899e+000	1.2686e+000
	MPSO-TVAC	2.3591e-001	1.9998e-001
	PSO-GSS	1.0195e-015	2.3660e-015
150	SPSO	9.4218e+000	4.2934e+000
	MPSO-TVAC	4.0495e-001	2.9981e-001
	PSO-GSS	1.2449e-015	2.9291e-015
200	SPSO	2.8059e+001	1.3881e+001
	MPSO-TVAC	5.7757e-001	2.4177e-001
	PSO-GSS	3.3272e-016	3.5664e-016
250	SPSO	1.1076e+002	1.9090e+002
	MPSO-TVAC	7.8355e-001	3.2578e-001
	PSO-GSS	5.3646e-014	1.6649e-013
300	SPSO	5.3088e+002	9.0264e+002
	MPSO-TVAC	4.2045e+000	3.0387e+000
	PSO-GSS	9.9212e-016	1.3695e-015

Table VIII
SIMULATION RESULTS OF PENALIZED 2 FUNCTION

Dimension	Algorithm	Mean	Variance
30	SPSO	5.4943e-004	2.4568e-003
	MPSO-TVAC	9.3610e-027	4.17530e-026
	PSO-GSS	2.9704e-020	1.3854e-019
50	SPSO	6.4279e-003	1.0769e-002
	MPSO-TVAC	4.9270e-002	2.0248e-001
	PSO-GSS	8.9890e-016	4.6121e-015
100	SPSO	3.8087e+001	1.8223e+001
	MPSO-TVAC	3.7771e-001	6.1357e-001
	PSO-GSS	1.1639e-015	1.7114e-015
150	SPSO	1.6544e+002	5.5689e+001
	MPSO-TVAC	1.2655e+000	1.4557e+000
	PSO-GSS	7.1447e-012	2.2412e-011
200	SPSO	1.8029e+003	2.8233e+003
	MPSO-TVAC	2.3221e+000	1.5383e+000
	PSO-GSS	2.9751e-014	6.9682e-014
250	SPSO	6.7455e+003	9.5733e+003
	MPSO-TVAC	2.8990e+000	1.3026e+000
	PSO-GSS	1.1437e-012	2.4563e-012
300	SPSO	3.2779e+004	4.4431e+004
	MPSO-TVAC	3.7343e+000	2.6830e+000
	PSO-GSS	8.7726e-012	1.8712e-011