Stability Analysis of the Particle Dynamics in Particle Swarm Optimizer

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***Abstract*—Previous stability analysis of the particle swarm optimizer was restricted to the assumption that all parameters are nonrandom, in effect a deterministic particle swarm optimizer. We analyze the stability of the particle dynamics without this restrictive assumption using Lyapunov stability analysis and the concept of passive systems. Sufficient conditions for stability are derived, and an illustrative example is given. Simulation results confirm the prediction from theory that stability of the particle dynamics requires increasing the maximum value of the random parameter when the inertia factor is reduced.**

***Index Terms*—Absolute stability, circle criterion, Lyapunov stability, particle swarm optimization (PSO), stability analysis.**

I. Introduction

PARTICLE swarm optimization (PSO) is a swarm intelligence technique developed by Eberhart and Kennedy [1], inspired by the social behavior of bird flocking and fish schooling. PSO is a population-based search process where individuals, referred to as particles, are candidate solutions to the optimization problem at hand. Particles change their state by evolving in a multidimensional search space until an equilibrium or optimal state has been reached or until computation limitations are exceeded. PSO has been shown to be a very effective optimizer, especially in large convoluted search spaces [2]. Empirical evidence has been accumulated to show that the algorithm is a useful tool for optimization [3], [4]. PSO has been applied to many optimization problems in engineering [5]–[10]. On the algorithmic front, extensions have been made to deal with dynamical environments and efficient exploration [11], [12]. More recently, multiobjective particle swarm optimizers have also been derived [13]–[16]. Additional operators have been incorporated into the basic particle swarm optimization scheme, such as the selection operator in genetic algorithms [17] and a neighborhood operator [18]. The similarity between a population of particles in swarm optimization and a population of genotypes in genetic algorithms has resulted in a comparison between the two [2].

The first analysis of the simplified particles behavior was carried out by Kennedy [19], who showed the different particle trajectories for a range of design choices for the gain through simulations. In [20], the authors showed that a particle in a simple one-dimensional (1-D) PSO system follows a path defined by a sinusoidal wave, randomly deciding on both its amplitude and frequency. The first formal analysis of the stability properties of the algorithm was carried out in [21]. Essentially, the analysis required the simplification of the standard stochastic PSO to a deterministic dynamical system by treating the random coefficients as constants. The resulting system was a second-order linear dynamical system whose stability depended on the system poles or the eigenvalues of the state matrix. A similar analysis based on the deterministic version of the PSO was also carried out in identifying regions in the parameter space that guarantees stability [22]. The issue of convergence and parameter selection was also addressed in [23] and [24]. However, the authors acknowledged the limitations of their results, which did not take the stochastic nature of the algorithm into account. A similar analysis on a continuous-time version of PSO has also been carried out in [25]. A Lyapunov analysis approach was adapted in [26] for the social foraging swarms, different to the PSO, in a continuous-time setting.

In this paper, we provide a stability analysis of the stochastic particle dynamics. The analysis is made feasible by representing the particle dynamics as a nonlinear feedback controlled system as formulated by Lure [27], [28]. Such systems have a deterministic linear part and a nonlinear and/or time-varying gain in the feedback path. It is well known that the stability of such nonlinear feedback systems cannot be determined by analyzing the stability of all possible linear feedback systems resulting from the nonlinear and/or time varying gain being replaced by constant linear gain values spanning the entire range of the gain [28]. Known as *Aizerman's conjecture*, its implication is that the stability conditions derived by treating the particle dynamics as deterministic is not valid for the stochastic case, in general.

The paper is organized as follows: In Section II, the basic PSO algorithm is given. In Section III, some characteristics of the particle dynamics are elucidated. In Section IV, the main stability analysis result is derived. In Section V, an illustrative example is given, followed by the conclusion of the paper.

II. Particle Swarm Optimization

The PSO formulation defines each particle as a potential solution to a problem in d-dimensional space with a memory of its previous best position and the best position among all particles, in addition to a velocity component. At each iteration, the particles are combined to adjust the velocity along each dimension, which in turn is used to compute the new particle position. Since each dimension is updated independently of others and the only link between the dimensions of the problem space are introduced via the objective functions, an analysis can be carried out on the 1-D case without loss of generality. The original version was found to lack precision in a local search solution.

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This led to the introduction of an inertia factor in the velocity update in [23], giving rise to the commonly used form of the PSO. The particle dynamics in one dimension are given by

$$v_{t+1} = wv_t + \alpha_t^{(l)} \left(p^{(l)} - x_t\right) + \alpha_t^{(g)} \left(p^{(g)} - x_t\right) \quad (1)$$
$$x_{t+1} = x_t + v_{t+1} \quad (2)$$

where v_t is the particle velocity at the tth iteration, x_t is the particle position at the tth iteration, $p^{(l)}$ is the personal best position or the particle's best position thus far, $p^{(g)}$ is the best global position or the best solution among all particles, w is the inertia factor, and $\alpha_t^{(l)} \sim \mathcal{U}[0, c_1]$, and $\alpha_t^{(g)} \sim \mathcal{U}[0, c_2]$ are random parameters with uniform distributions where c_1 and c_2 are constants known as acceleration coefficients.

The following statements can be derived from the particle dynamics of (1).

1) The system dynamics are stochastic and of order two.
2) The system does not have an equilibrium point if $p^{(g)} \neq p^{(l)}$.
3) If $p^{(g)} = p^{(l)} = p$ is time invariant, there is a unique equilibrium point at $v_* = 0$, $x_* = p$.

An equilibrium point thus exists only for the best particle whose local best solution is the same as that of the global best solution. If asymptotic stability of the dynamics for the best particle can be guaranteed, then is it guaranteed that this particle will reach the equilibrium point relating to the best solution. The analysis of the nonbest particle is more challenging and is beyond the scope of this paper. Clearly, the conditions outlined for the existence of an equilibrium point do not hold true for any particle at all times in the particle swarm optimization. There are two points to be made with regard to this. First, convergence to a fixed equilibrium point requires time invariance of the best solution position. Second, particles stop improving their solution after a finite number of iterations so that beyond this point the conditions can be deemed to hold.

We proceed to consider the particle dynamics associated with the best particle

$$v_{t+1} = wv_t + \alpha_t(p - x_t) \quad (3)$$
$$x_{t+1} = x_t + v_{t+1} \quad (4)$$

where $\alpha_t = \alpha_t^{(l)} + \alpha_t^{(g)}$. The combined stochastic parameter is no longer uniformly distributed but satisfies

$$0 < \alpha_t < K \quad (5)$$

where $K = c_1 + c_2$, c_1, and c_2 are constants known as acceleration coefficients. Note that the use of (3) with p as a constant is not valid for nonbest particle dynamics. The following expression used in [21] and [24], for the deterministic PSO, gives that

$$p = \frac{\alpha_t^{(l)} p^{(l)} + \alpha_t^{(g)} p^{(g)}}{\alpha_t} \quad (6)$$

is generally time varying if $p^{(g)} \neq p^{(l)}$ and if $\alpha_t^{(l)}$ and $\alpha_t^{(g)}$ are random.

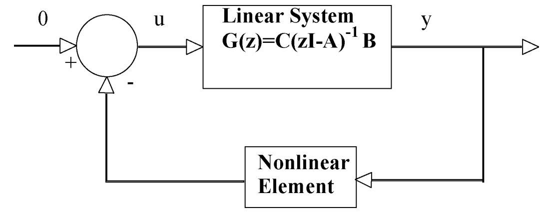

Fig. 1. Feedback control system representation of particle dynamics.

The previous stability analysis [21], [24] represents the system in state-space form

$$\begin{pmatrix} x_{t+1} \\ v_{t+1} \end{pmatrix} = \begin{pmatrix} 1-\alpha_t & w \\ -\alpha_t & w \end{pmatrix} \begin{pmatrix} x_t \\ v_t \end{pmatrix} + \begin{pmatrix} \alpha_t \\ \alpha_t \end{pmatrix} p. \quad (7)$$

By treating the random variable α_t as a constant, essentially deterministic particle dynamics, the system dynamics are reduced to a simple time-invariant linear second-order dynamic model. Stability of such a deterministic particle dynamics can be concluded based on the eigenvalues of the state matrix in (7), as shown in [21], [22], and [24]. The conditions for convergence derived in [22] and [24] in our notation are given by

$$w < 1 \quad (8)$$

and

$$K < 2(w+1). \quad (9)$$

We shall see in Section IV that the sufficient conditions for the stability of the stochastic particle dynamics differ from those given in (8) and (9).

III. System Characteristics

We note that the stability analysis of the particle dynamics can be mapped to the problem of absolute stability of nonlinear feedback systems, known as Lure's stability problem [28], [29]. The stochastic particle dynamics are thus represented as a feedback controlled dynamic system as shown in Fig. 1. The feedback control system representation depicts a time-invariant linear plant in the forward path and an output control with time-varying gain in the feedback path. The equations governing the dynamics in this new representation can be expressed as

$$\begin{pmatrix} x_{t+1} \\ v_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & w \\ 0 & w \end{pmatrix} \begin{pmatrix} x_t \\ v_t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_t \quad (10)$$
$$y_t = (1 \quad 0) \begin{pmatrix} x_t \\ v_t \end{pmatrix} \quad (11)$$
$$u_t = -\alpha_t(y_t - p) \quad (12)$$

where u_t is interpreted as the control input signal.

Under the conditions of p being time invariant, the dynamical system equation can be simplified further by introducing the state vector as follows:

$$\xi_t = \begin{pmatrix} x_t - p \\ v_t \end{pmatrix}.$$

The resulting state-space representation from (10)–(12) is thus

$$\xi_{t+1} = A\xi_t + Bu_t \tag{13}$$
$$y_t = C\xi_t \tag{14}$$
$$u_t = -\alpha_t y_t \tag{15}$$

where the state matrix A, input matrix B, and the output matrix C are given by

$$A = \begin{pmatrix} 1 & w \\ 0 & w \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = (1 \quad 0). \tag{16}$$

Definition (Equilibrium [30]): ξ_* is an equilibrium point of a dynamical system in the state-space form $\xi_{t+1} = f_t(\xi_t)$ if it satisfies $\xi_* = f_t(\xi_*)$ for every $t \geq 0$.

Remark: For the PSO, the dynamical systems with feedback can be rewritten in the following state-space representation:

$$\xi_{t+1} = (A - \alpha_t BC)\xi_t \tag{17}$$
$$(A - \alpha_t BC) = \begin{pmatrix} 1-\alpha_t & w \\ -\alpha_t & w \end{pmatrix}. \tag{18}$$

If $w \neq 0$, then $(A - \alpha_t BC)$ is nonsingular, hence, the only solution that satisfies $\xi_* = (A - \alpha_t BC)\xi_*$ is $\xi_* = 0$. Hence, the particle dynamics specified in (13)–(15) have a unique equilibrium point at the origin in the ξ state space.

Remark: If $p^{(g)} \neq p^{(l)}$, the particle converges to the line that connects its personal best and the global best particle.

The transfer function of the linear plant is then

$$G(z) = C(zI - A)^{-1}B = \frac{z}{(z-1)(z-w)} \tag{19}$$

where z is the complex variable associated with Z transforms [31].

Remark: The linear plant has poles at $z = 1$ and $z = w$ and, hence, is (marginally) stable if $|w| < 1$ and is unstable if $|w| \geq 1$. Poles are also the eigenvalues of A.

For dynamical systems specified in the state-space form, the following properties are of interest and are needed for the analysis in the next section.

Definition (Controllability [32]): A system is completely controllable if the system state $x(t_f)$ at time t_f can be forced to take on any desired value by applying a control input $u(t)$ over a period of time from t_0 until t_f. Suppose n, m, and l are given integers, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{l\times n}$, $D \in \Re^{l\times m}$ and $x_{t+1} = Ax_t + Bu_t$, and $y_t = Cx_t + Du_t$ represents the dynamics of the linear systems. Then, the pair (A, B) is said to be controllable if $\mathrm{Rank}[B\ AB \ldots A^{n-1}B] = n$.

Definiton (Observability [32]): A system is completely observable if any initial state vector $x(t_0)$ can be reconstructed by examining the system output $y(t)$ over some period of time from t_0 until t_f. Suppose n, m, and l are given integers $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{l\times n}$, $D \in \Re^{l\times m}$, and $x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t + Du_t$ represents the dynamics of the linear systems. Then, the pair (C, A) is said to be observable if $\mathrm{Rank}[C\ CA \ldots CA^{n-1}]^T = n$.

State-space representation of the linear part of the PSO system is given by

$$\xi_{t+1} = A\xi_t + Bu_t \tag{20}$$
$$y_t = C\xi_t \tag{21}$$

where the state matrix A, input matrix B, and the output matrix C are given by

$$A = \begin{pmatrix} 1 & w \\ 0 & w \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = (1 \quad 0). \tag{22}$$

According to the controllability definition, the PSO dynamics of (13) give rise to

$$(B\ AB) = \begin{pmatrix} 1 & 1+w \\ 1 & w \end{pmatrix} \tag{23}$$

and $\mathrm{Rank}(B\ AB) = 2$.

Hence, the linear part of the PSO system is controllable.

According to the observability definition, the PSO dynamics of (13) give rise to

$$(C\ CA)^T = \begin{pmatrix} 1 & 0 \\ 1 & w \end{pmatrix}. \tag{24}$$

$\mathrm{Rank}(C\ CA)^T = 2$ if $w \neq 0$.

Hence, the linear part of the PSO systems is observable, provided $w \neq 0$. The linear plant pair $\{A, B\}$ is controllable and pair $\{A, C\}$ is observable.

Remark: The implication of complete controllability and observability of the particle dynamics is that the dynamics are always that of a second-order system (not reduced to first order, for example due to pole-zero cancellation). Such a condition is necessary for us to use the method of positive real lemma in the next section.

The time-varying memoryless feedback gain satisfies the so-called sector condition $\alpha_t \in (0, K)$ and, hence, satisfies

$$u_t^2 + Ku_t y_t \leq 0. \tag{25}$$

IV. Stability Analysis

The stability analysis is carried out using the concept of passive systems and Lyapunov stability [28]. We begin this treatment by explaining some basic concepts and their interpretations.

Definition [28]: The linear plant has a stable matrix A, if its eigenvalues lie strictly inside the unit circle in the Z plane or equivalently $|\lambda_i\{A\}| < 1$ for all i. Here, $\lambda_i\{\cdot\}$ represents the ith eigenvalues of A.

Remark: The linear plant in the feedback representation of the particle dynamics has a semistable A matrix with a simple pole on $|z| = 1$ when $|w| < 1$.

Definition [28]: A dynamical system is said to be *passive* if there is a nonnegative scalar function $V(\xi)$ with $V(0) = 0$ which satisfies

$$V(\xi_{t+1}) - V(\xi_t) \leq y_t u_t. \tag{26}$$

Remark: The equation above can be interpreted as the increase in stored energy is less than or equal to the energy input so that energy is lost in passive systems.

Theorem (Lyapunov Stability [28]): Let $\xi = 0$ be an equilibrium point of the system. The equilibrium point is asymptotically stable if there is a nonnegative scalar function $V(\xi)$ with $V(0) = 0$ which satisfies

$$V(\xi_{t+1}) - V(\xi_t) < 0. \tag{27}$$

Remark: Lyapunov stability analysis is based on the idea that if the total energy in the system continually decreases, then the system will asymptotically reach the zero energy state associated with an equilibrium point of the system.

A system is said to be asymptotically stable if all the states approach zero with time.

The passivity idea and the Lyapunov stability idea are combined to analyze the Lure stability problem [28], whereby if all subsystems in a feedback system are passive, then the total energy can only decrease in an autonomous system (with zero input energy).

For linear systems, the passivity property can be related to a condition in the frequency domain known as positive real transfer functions.

Definition [28]: The transfer function $H(z)$ of a dynamical system is said to be positive real if and only if the system is stable and

$$\Re\left\{H(e^{j\theta})\right\} > 0$$

for every $\theta \in [0, 2\pi)$, where $\Re\{\cdot\}$ indicates the real part of its argument and $j = \sqrt{-1}$ is the imaginary number.

Remark: The transfer function $G(z)$ representing the linear plant in the particle dynamics is not a positive real transfer function. However, a lower limit for $\Re\{G(e^{j\theta})\}$ exists and is given by (see Appendix for details)

$$\Re\left\{G(e^{j\theta})\right\} > -\frac{(1+w)}{2\,(1-2|w|+w^2)} \quad \text{for all} \quad \theta \in [0, 2\pi). \tag{28}$$

An important result that is necessary for the stability analysis is the discrete-time positive real lemma which links the concepts of positive real transfer functions and the existence of a Lyapunov function.

Lemma (Discrete-Time Positive Real Lemma [33], [34]): Let $H(z) = C(zI - A)^{-1}B + D$ be a transfer function, where A is a stable matrix or a semistable matrix with a simple pole on $|z| = 1$, $\{A, B\}$ is controllable, and $\{A, C\}$ is observable. Then, $H(z)$ is strictly positive real if and only if there exist a symmetric positive definite matrix P, matrices W and L, and a positive constant ε such that [33], [34]

$$A^TPA - P = -L^TL \tag{29}$$
$$B^TPA = C - W^TL \tag{30}$$
$$D + D^T - B^TPB = W^TW. \tag{31}$$

Now, we are ready to state the main result of this paper which specifies the conditions that when satisfied by the design parameters w and K guarantee the stability of the particle dynamics.

Theorem (Main Result): Let the particle dynamics be represented by (20)–(22) and satisfy (5) with an equilibrium point at the origin. Then, the origin is asymptotically stable if $|w| < 1$, $w \neq 0$, and

$$K < \left(\frac{2\left(1-2|w|+w^2\right)}{1+w}\right).$$

Proof: Consider the Lyapunov function

$$V(\xi) = \xi_t^T P \xi_t \tag{32}$$

where P is a symmetric positive definite matrix.

The decrease in the system energy as represented by the Lyapunov function between two discrete-time instants is given by

$$\begin{aligned} \Delta V_{t+1} &= V(\xi_{t+1}) - V(\xi_t) && (33)\\ &= \xi_{t+1}^T P \xi_{t+1} - \xi_t^T P \xi_t && (34)\\ &= \xi_t^T (A^TPA - P)\xi_t - 2\alpha_t y_t B^T P A \xi_t \\ &\quad + (\alpha_t y_t)^2 B^T P B. && (35)\end{aligned}$$

Since $-2\alpha_t y_t(\alpha_t y_t - K y_t) \geq 0$, if we add this component to the right-hand side of the equation, we get

$$\begin{aligned} \Delta V_{t+1} &\leq \xi_t^T (A^TPA - P)\xi_t - 2\alpha_t y_t B^T P A \xi_t + \alpha_t y_t^2 B B^T \\ &\quad - 2\alpha_t y_t(\alpha_t y_t - K y_t) && (36)\\ &= \xi_t^T (A^TPA - P)\xi_t - 2\alpha_t y_t (B^TPA - KC)\xi_t \\ &\quad - (\alpha_t y_t)^2 (2 - B^TPB). && (37)\end{aligned}$$

We can show that the right-hand side is negative by completing a square term if the following matrix equations are satisfied:

$$A^TPA - P = -L^TL \tag{38}$$
$$B^TPA = KC - W^TL \tag{39}$$
$$2 - B^TPB = W^TW. \tag{40}$$

Comparing these with the relationship established in the *Positive Real lemma* indicates that if and only if the linear system with the transfer function

$$\tilde{H}(z) = KC(zI - A)^{-1}B + 1$$

satisfies all the conditions stated in the positive real lemma, then (38)–(40) hold.

It is straightforward then to show that $\tilde{H}(z)$ satisfies the conditions in the Positive Real lemma if

$$|w| < 1, \quad w \neq 0 \tag{41}$$

and

$$1 + K\Re\left\{G(e^{j\theta})\right\} > 0 \tag{42}$$

which then leads to

$$K < \left(\frac{2\left(1 - 2|w| + w^2\right)}{1 + w} \right). \tag{43}$$

Then

$$\begin{aligned} \Delta V_{t+1} \leq & - \xi_t^T L^T L \xi_t - 2\alpha_t y_t W^T L \xi_t - (\alpha_t y_t)^2 W^T W \\ = & - (L\xi_t - \alpha_t y_t W)^T (L\xi_t - \alpha_t y_t W) \\ \leq & \, 0. \end{aligned} \tag{44}$$

Since the difference in the Lyapunov function is nonincreasing, the particle dynamics are guaranteed to be stable, according to the Lyapunov stability theorem.

In fact, asymptotic stability can be guaranteed using La Salle's extension [28] to Lyapunov stability, observing that when $\Delta V_{t+1} = 0$, the particle dynamics are such that at the next time point, it will be nonzero except when the particle has reached equilibrium. To see this, consider the following scenarios.

$\Delta V_{t+1} = 0$ implies that $L\xi_t - \alpha_t y_t W = 0$, which can be written as follows with a substitution for $y_t = C\xi_t$:

$$(L - \alpha_t WC)\xi_t = 0. \tag{45}$$

If the rank of $(L-\alpha_t WC) = 0$, then if any solution is to exist, it will be unique $\alpha_t = \alpha_*$. Then, for any $\alpha_{t+1} \neq \alpha_*$, $\Delta V_{t+1} < 0$, given α_t is random, it can be seen that the energy will only continue to decrease, barring time instants when $\Delta V_{t+1} = 0$, at which time it will temporarily stop decreasing.

If the rank of $(L - \alpha_t WC)$ is rank deficient, then this implies $|L - \alpha_t WC| = 0$, which gives at most a quadratic equation in α_t for constants L, W, and C. Hence, at most, α_t can take only two specific values, say α_1^* and α_2^*. Since α_t is random with probability density $P(\alpha_t)$, $P_r(\alpha_t = \alpha_1^*) + P_r(\alpha_t = \alpha_2^*)$ is infinitesimally small. Hence, the probability of the event that $\alpha_t = \alpha_1^*$ or $\alpha_t = \alpha_2^*$ is infinitesimally small. Therefore, the energy will stop decreasing only at infinitesimally small finite time instants, implying that an asymptotically zero energy state will be reached.

If the rank of $(L - \alpha_t WC) = 2$, then the only solution for (45) is $\xi_t = 0$, implying that the energy will stop decreasing only when the system reaches equilibrium.

Hence, $V_t \longrightarrow 0$ as $t \longrightarrow \infty$.

Remark: The equilibrium point at the origin represents the particle position reaching the minimum location p with zero velocity. Lyapunov stability results give only sufficient conditions and, hence, can be very conservative. Violation of the stability conditions do not imply instability, rather that stability cannot be guaranteed.

When $w > 0$, (43) reduces to $K < (2(1 - w)^2/(1 + w))$; and when $w < 0$, (43) reduces to $K < 2(1 + w)$. The sufficient stability conditions derived in the main theorem are illustrated graphically in Fig. 2, which shows the maximum gain for a chosen inertia factor.

Remark: Note that the maximum gain that gives sufficient guarantees for the stability of particle dynamics decreases with the increase in inertia factor when it is positive. This is in contrast to the results derived in [22] and [24] under nonrandom constant gain assumptions where the maximum gain increased with the inertia factor.

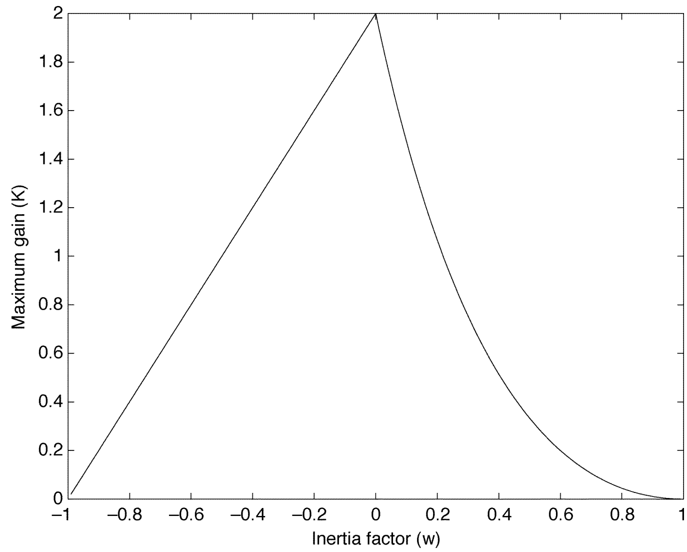

Fig. 2. Maximum gain versus inertia factor for stability.

V. Illustrative Examples

The stability analysis given in this paper can be interpreted in the frequency domain and time domain. Through an illustrative example, we demonstrate their utility and insight.

A. Nyquist Plot and Circle Criterion

The main stability theorem and the proof are based on the discrete-time version of the circle criterion, which can be used as a frequency domain graphical method for stability analysis [28]. The result derived here is a special case when the lower limit for the feedback gain α_t is zero.

The circle criterion when applied to the stability of particle dynamics simply states that the Nyquist plot of the linear plant in the feedback system representation should lie to the right side of the point $-(1/K) + j0$ in the Z plane.

For the general particle dynamics as represented in (7), the discrete-time Nyquist plots of $G(z)$ in (19) with the inertia factor (design parameter) $w = 0.8$ is given in Fig. 3 and with $w = 0.2$ is given in Fig. 4. The Nyquist plots showing the real and imaginary parts of $G(z)$ clearly lie to the right of a limiting vertical line. The required conditions identified to satisfy positive realness in (42) then imply that the real value of this limiting line can be translated into a limiting condition on the gain K. The vertical lines on the figures show the limiting condition for the positive realness

$$\begin{aligned} -\frac{1}{K} &< -22.5, \qquad \text{for} \quad w = 0.8 \\ -\frac{1}{K} &< -0.9375, \qquad \text{for} \quad w = 0.2. \end{aligned}$$

The graphical results match those obtained from the results from the main theorem as expected.

Note, however, the circle criterion can be applied to general sector conditions such as $\alpha_{\min} \leq \alpha_t \leq \alpha_{\max}$ and thus provides us flexibility in designing further parameters.

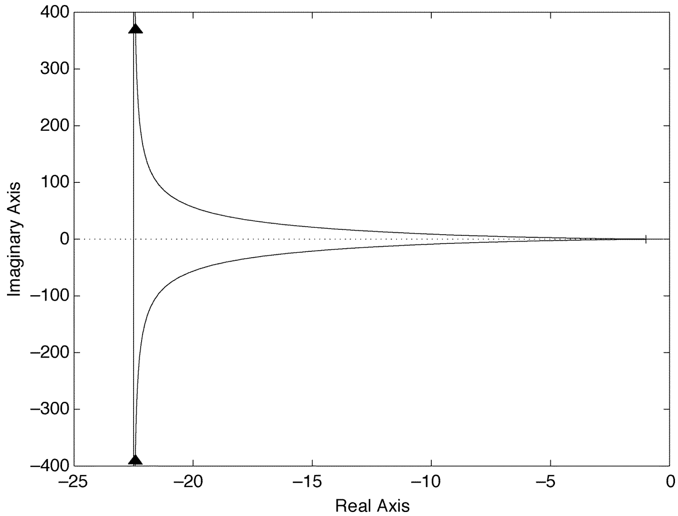

Fig. 3. Discrete-time Nyquist plot for inertia factor $= 0.8$ and limit value for its real part.

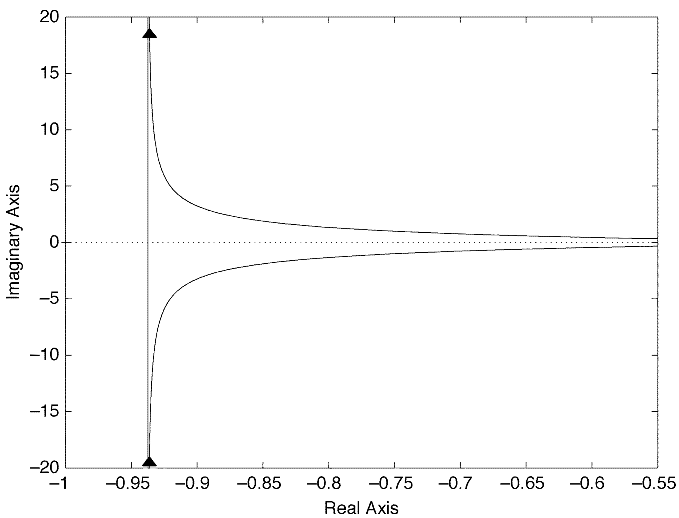

Fig. 4. Discrete-time Nyquist plot for inertia factor $= 0.2$ and limit value for its real part.

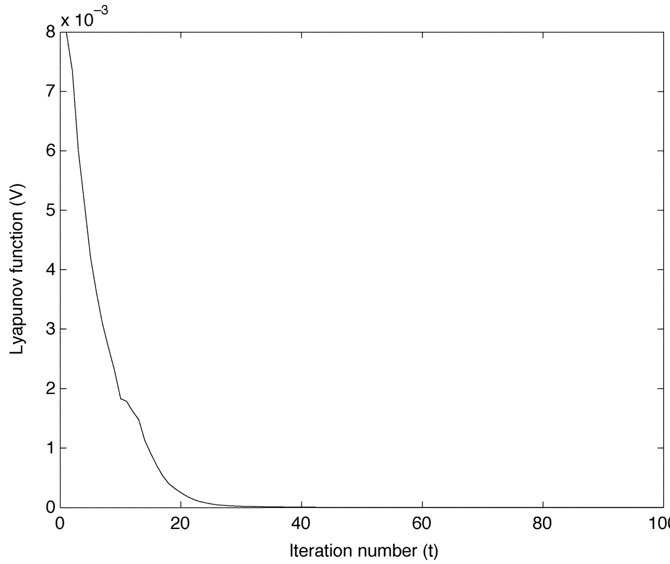

Fig. 5. Lyapunov function with $K = 0.04$ and $w = 0.8$.

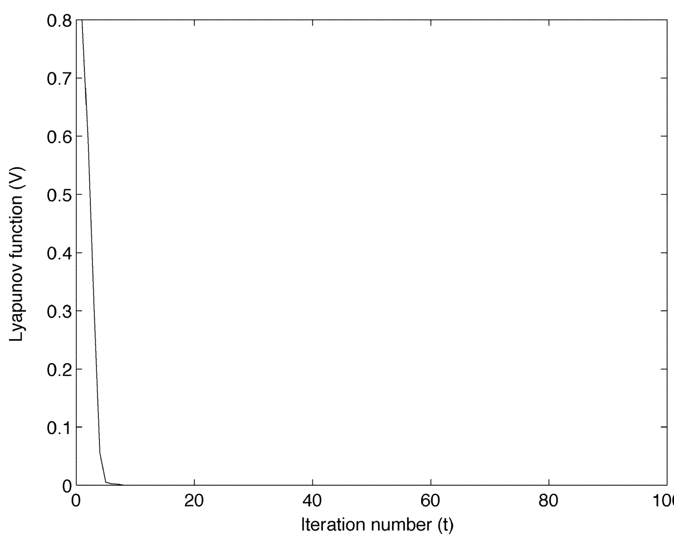

Fig. 6. Lyapunov function with $K = 1$ and $w = 0.2$.

B. Lyapunov Function and Particle Trajectories

The stability conditions derived are based on Lyapunov stability analysis and, hence, are overly conservative. It is therefore important to analyze the impact on the particle dynamics of the choices for the design parameters. In particular, it is of interest to analyze the case when the derived stability conditions are violated.

First, we will determine a candidate positive definite matrix P in the Lyapunov function for the chosen inertia factor w. Consider the system with $w = 0.8$, then the system state matrix is

$$A = \begin{pmatrix} 1 & 0.8 \\ 0 & 0.8 \end{pmatrix}. \tag{46}$$

For this case, stability requires $K < 0.044$. A choice of $K = 0.04$ that satisfies this condition but is close to the limit is made for the analysis of this particle. This is to ensure that while a worse case condition within limits is considered, it gives convenient rounded values for the matrices A and thus $P1$ and $P2$. Note, however, that any value that satisfies this inequality for K will demonstrate the analysis.

By solving for P from (29)–(31), the solutions are given by

$$P_1 = \begin{pmatrix} 0.008 & 0.032 \\ 0.032 & 0.4372 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.008 & 0.032 \\ 0.032 & 0.2108 \end{pmatrix}. \tag{47}$$

Likewise, for the system with $w = 0.2$, the state matrix is

$$A = \begin{pmatrix} 1 & 0.2 \\ 0 & 0.2 \end{pmatrix}. \tag{48}$$

A convenient choice to demonstrate the result is $K = 1$, which satisfies the stability guarantees of the main results. The solutions for the positive definite matrix P are given by

$$P_1' = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.7905 \end{pmatrix}, \quad P_2' = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.1215 \end{pmatrix}. \tag{49}$$

Having computed the Lyapunov function matrix for the two design choices, we can analyze how this function evolves over time. All the simulations are carried out based on (1) and (2)

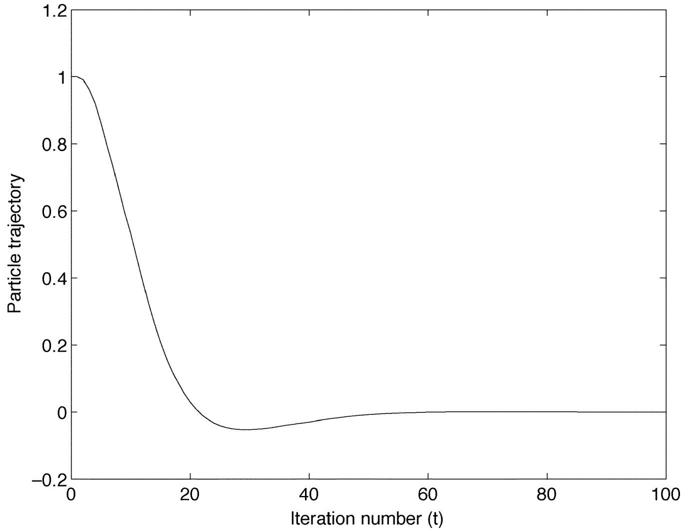

Fig. 7. Particle trajectories with $K = 0.04$ and $w = 0.8$.

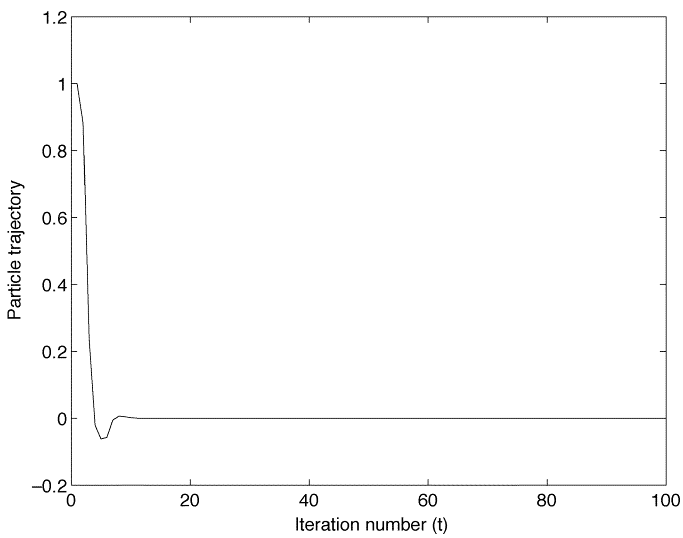

Fig. 8. Particle trajectories with $K = 1$ and $w = 0.2$.

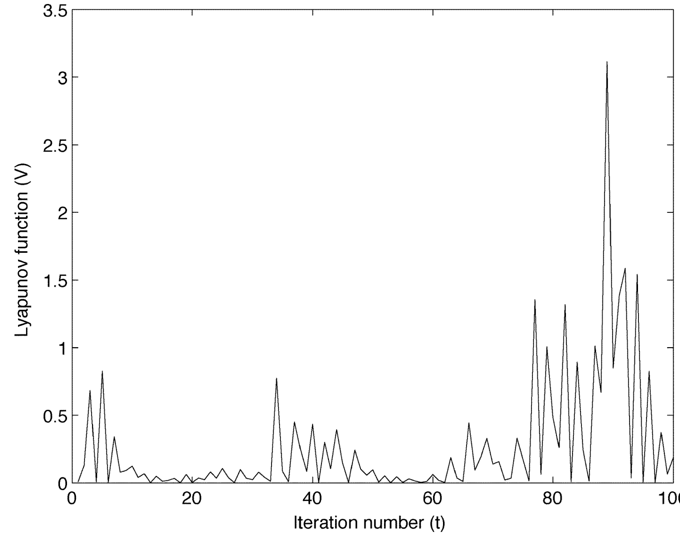

Fig. 9. Lyapunov function with $K = 2.5$ and $w = 0.8$.

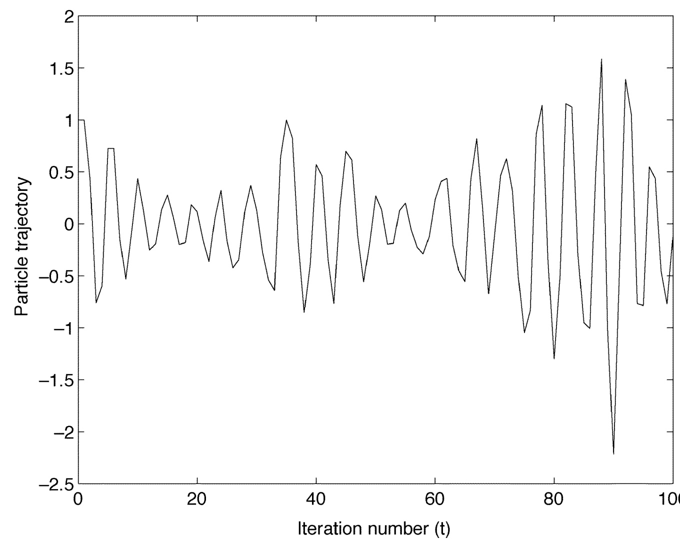

Fig. 10. Particle trajectories with $K = 2.5$ and $w = 0.8$.

and with initial conditions of $x = 1$ and $v = 0$. Figs. 5 and 6 show the Lyapunov energy function based on P_1, and P_1' decreases with time monotonically for the respective values of w. The trajectory of the particles for the two cases above are also given in Figs. 7 and 8, demonstrating the asymptotic stability of the particle dynamics.

In order to analyze the behavior of the particle under conditions that do not guarantee stability, the evolution of the Lyapunov function determined in (47) was observed. As seen in Fig. 9, for a single realization, the energy decreases to zero, but not monotonically, showing an increase at various times. In fact, the results were consistently similar. The associated particle trajectory is given in Fig. 10 and shows asymptotic stability despite the stability conditions not being satisfied. A similar analysis was carried out with the design choices of $w = 0.2$ and $K = 2$, which also violate the required stability conditions. Figs. 11 and 12 show the evolution of the Lyapunov function (49) and the corresponding particle trajectory.

Again, the figures demonstrate the conservativeness of the stability result by showing asymptotic stability for the particle trajectory even when the design parameters do not meet the required conditions.

However, instability does occur even at reasonable design parameter values when the stability conditions are violated, as shown Figs. 13–15.

It may appear at first sight that the conservative stability conditions derived here are not useful for design. However, the utility of such analysis is in providing insights into particular features of the algorithm and thereby guide design choices. In particular, we have shown that for $0 < w < 1$, decreasing w should be associated with increased K if we want to maintain the same level of exploration/convergence. It is also possible to arrive at adaptive designs in which parameters such as w and K are changed over time, while stability is maintained within the analytical framework such as those in control systems literature [28], [30], [32].

When $-1 < w < 0$ shows stability and the particle trajectories have alternating signs which lead to large jumps in the

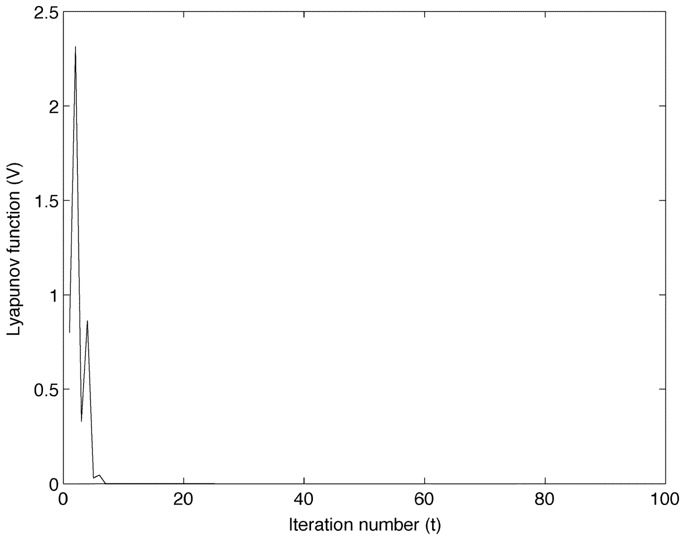

Fig. 11. Lyapunov function with $K = 2$ and $w = 0.2$.

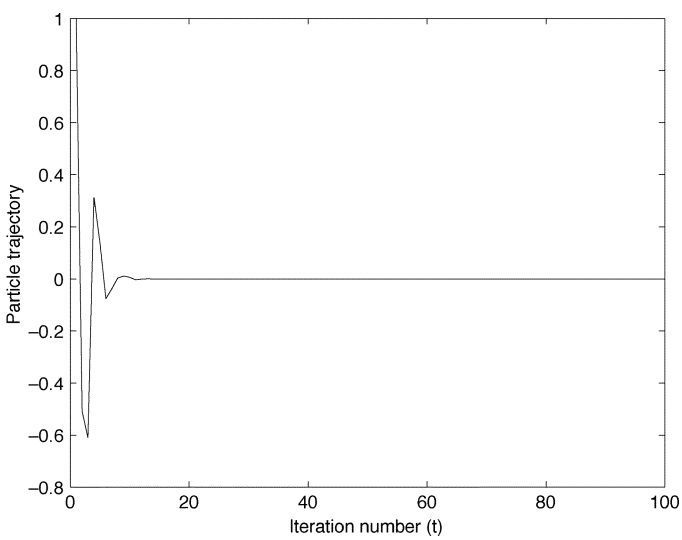

Fig. 12. Particle trajectories with $K = 2$ and $w = 0.2$.

particle motion, this is undesirable for local exploration. Ideally, the choice for w is for it to lie in the region $0 < w < 1$, as identified by [21] and [22].

It is interesting to note that under instability conditions, the particle trajectories reach very high values, suggesting that particles escape from the search region, not monotonically, but at various times. This effect has been observed in the literature and solutions such as imposing a limit on the particle velocity have been proposed, albeit with further problems [23] to mitigate this effect. It is possible that the stability analysis provided here can be used not only to analyze such schemes but also to provide a guide to deriving new stabilizing particle dynamics algorithms.

A characteristic feature of some of the selected particle trajectories with design choices in the region outside stability guarantees is that the particle position magnitudes were very large, albeit temporarily. Such movement of particles outside the relevant search region is undesirable, which is one of the aims of addressing stability. In order to investigate the relationship of the number of times in a simulation, the particles exceed

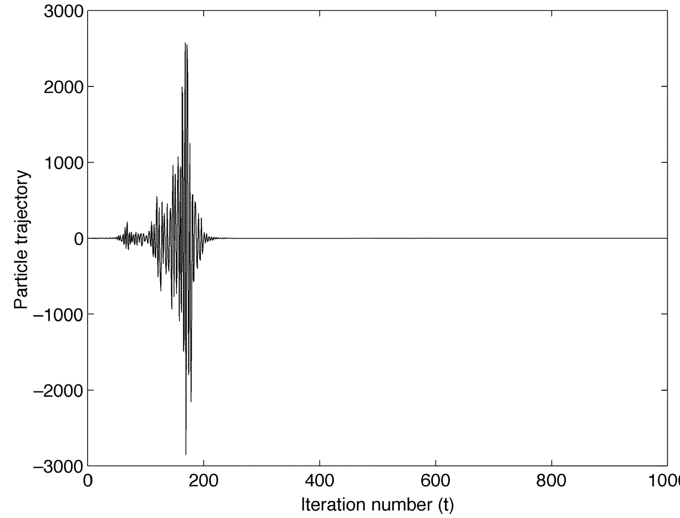

Fig. 13. Particle trajectories with $K = 3.5$ and $w = 0.8$.

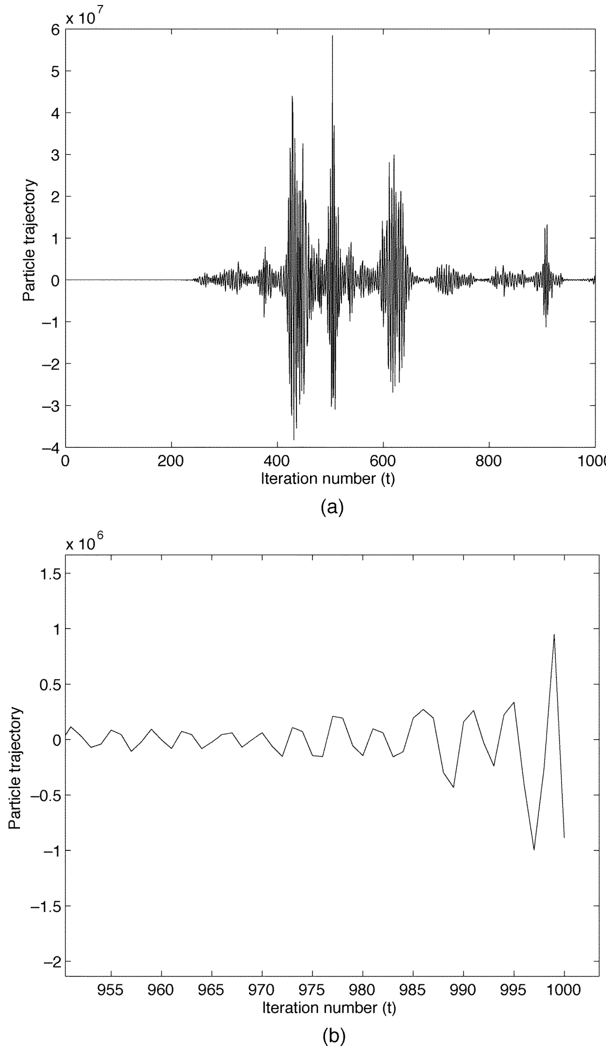

Fig. 14. Particle trajectories with $K = 3.5$ and $w = 0.9$. (a) From initial time to $t = 1000$. (b) Zoomed trajectory in time interval [950, 1000].

some search region defined by a threshold for specific w values and varying K; 1000 Monte Carlo simulations for each design

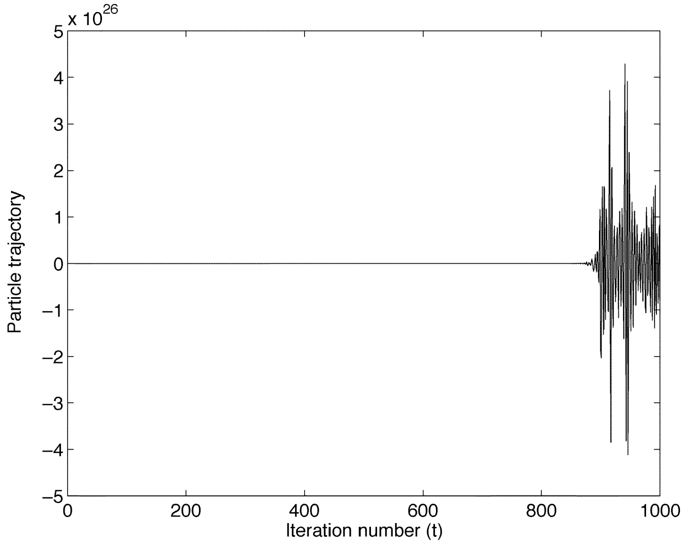

Fig. 15. Particle trajectories with $K = 3.8$ and $w = 0.95$.

TABLE I
THRESHOLD AND INSTABILITY COUNT FOR 1000 MONTE CARLO RUNS

Threshold	w=0.8 and K=3.5	w=0.9 and K=2.5	w=0.95 and K=2
10	93	240	817
100	14	75	609
1000	1	18	374
10000	0	2	215

choice were carried out. The relevant search region was defined as

$$\mathcal{S} = \{x : |x| < \delta\} \tag{50}$$

where δ is a threshold. Simulations were carried out for $\delta = 10, 100, 1000$, and $10\,000$ for three design choices that are outside the stability region identified in this paper but inside the stability region identified in [21], [22], and [24]. The results are given in Table I, where the number of simulations in which the particle escaped the region S at some time during the particle motion is referred to as instability count. A further set of experiments were carried out with $\delta = 100$ and $w = 0.8, 0.9, 0.95$, and 0.99, while varying K in the region $(0, 5)$. Fig. 16 shows the count of the simulations in which the particle escaped region $\mathcal{S}$ for these parameter choices.

The results clearly show the onset of instability, as defined by the count of simulations escaping some search region, and how instability increases with increasing K. The results also show the conservative nature of the theoretical bounds derived here. However, it is also noteworthy that going from $w = 0.8$ to $w = 0.9$, to achieve the same level of stability, the choice for K has to be decreased. This trend is predicted by the theoretical results shown in Fig. 2. The critical values of K for the onset of instability as defined here is also in between the values predicted theoretically in this paper and that advocated in [21], [22], and [24].

VI. CONCLUSION

We have provided a different approach to the stability analysis of PSO with stochastic parameters. The passivity theorem [28] and Lyapunov stability [31] methods were applied to the particle dynamics in determining sufficient conditions for asymptotic stability and, hence, convergence to the equilibrium point. Since the results are based on the Lyapunov function approach, they are conservative, and, hence, violation of these conditions do not imply instability. Nevertheless, the results can be used to infer qualitative design guidelines. Illustrative examples were given to demonstrate the application of the technique.

The analysis provided in this paper has addressed only the issue of absolute stability. The primary aim of PSO, however, is optimization, while maintaining stability. Our future work is aimed at developing adaptation rules on K and/or w design parameters such that exploration is facilitated, while maintaining stability. Another avenue is to investigate the condition for decrease in Lyapunov energy function over a time interval rather than at every time instant, which is likely to be less conservative.

TABLE II
MATLAB CODE FOR MONTE CARLO SIMULATION

```
T=1000;                      % Simulation time interval
w=0.9;                       % Inertia factor
K=3.5;                       % Maximum gain
instabilitycount=0;
threshold=100;
for S=1:1000                 % Number of Monte Carlo trials
  alpha1=K*rand(1,T);
  alpha2=K*rand(1,T);
  alpha=0.5*alpha1+0.5*alpha2;
  x(1)=1; v(1)=0;            % initial parameter
  for t=2:T
  v(t)=w*v(t-1)-alpha(t)*x(t-1);
  x(t)=x(t-1)+v(t);
  end
  if max(abs(x(:))) ≥threshold
  instabilitycount=instabilitycount+1;
  end
end
```

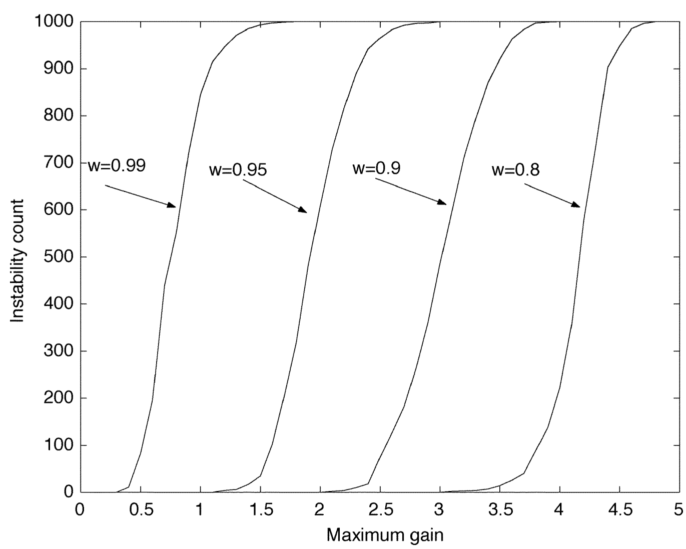

Fig. 16. Monte Carlo trials for different w values with threshold 100.

APPENDIX

Lower Limit for $\Re\{G(e^{j\theta})\}$*:*

Proof: The transfer function of the linear part of (13)–(15) is given by

$$G(z) = \frac{z}{(z-1)(z-w)}. \tag{51}$$

The real part of $G(e^{j\theta})$ is given by

$$\begin{aligned}
&\Re\left\{G(e^{j\theta})\right\} \\
&= \frac{(\cos\theta + j\sin\theta)}{(\cos\theta - 1 + j\sin\theta)\left((\cos\theta - w + j\sin\theta)\right)} && (52)\\
&= -\frac{(w+1)}{2(1 - 2w\cos\theta + w^2)}. && (53)
\end{aligned}$$

This leads to

$$\Re\left\{G(e^{j\theta})\right\} > \frac{-(1+w)}{2\left(1 - 2|w| + w^2\right)} \quad \text{for all} \quad \theta \in [0, 2\pi). \tag{54}$$

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