А-08-19
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Типовой расчёт № 27

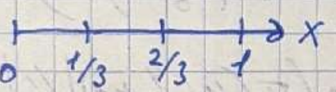
Методом конечных разностей найти решение краевой задачи $\begin{cases} -y''+q(x)y=f(x) \\ y(0)=y_0,\ y(1)=y_1 \end{cases}$ с шагами $h_1=\frac{1}{3}$, $h_2=\frac{1}{6}$

и оценить погрешность по правилу Рунге

$q(x)=\frac{5\pi^2}{9}$; $f(x)=\pi^2\sin(\pi(4x+1)/6)$

$y_0=1/2$; $y_1=1/2$

а) $h=h_1=\frac{1}{3}$



$X_0=0$; $X_1=\frac{1}{3}$; $X_2=\frac{2}{3}$; $X_3=1$

$$-y_{i-1}+(2+h^2q(x_i))y_i-y_{i+1}=h^2f(x_i)$$

$$\begin{cases} -y_0+(2+h^2q(x_1))y_1-y_2=h^2f(x_1) \\ -y_1+(2+h^2q(x_2))y_2-y_3=h^2f(x_2) \end{cases}$$

$$\begin{cases} -0{,}5+\left(2+\frac{1}{9}\cdot\frac{5\pi^2}{9}\right)y_1-y_2=\frac{1}{9}\pi^2\sin\left(\pi\cdot\left(4\left(\frac{1}{3}+1\right)/6\right)\right) \\ -y_1+\left(2+\frac{1}{9}\cdot\frac{5\pi^2}{9}\right)y_2-0{,}5=\frac{1}{9}\pi^2\sin\left(\pi\left(4\left(\frac{2}{3}+1\right)/6\right)\right) \end{cases}$$

$$\begin{cases} 2{,}609235y_1-y_2=0{,}875067 \\ -y_1+2{,}609235y_2=0{,}124933 \end{cases}$$

Методом прогонки найду y_1 и y_2:

$$\alpha_1=-\frac{-1}{2{,}609235}=0{,}383254 \quad \beta_1=\frac{0{,}875067}{2{,}609235}=0{,}335373$$

$$\beta_2=\frac{0{,}124933-(-1)\cdot 0{,}335373}{2{,}609235+(-1)\cdot 0{,}383254}=0{,}206788$$

$$y_2=\beta_2=0{,}206788$$

$$y_1=\alpha_1\cdot\beta_2=0{,}383254\cdot 0{,}206788+\overset{0{,}206788}{\cancel{0{,}079252}}=$$
$$=0{,}286040$$

б) $h = h_2 = \frac{1}{6}$ 0 $\frac{1}{6}$ $\frac{2}{6}$ $\frac{3}{6}$ $\frac{4}{6}$ $\frac{5}{6}$ 1 $x_0 = 0;\ x_1 = \frac{1}{6};\ x_2 = \frac{1}{3};\ x_3 = \frac{1}{2};\ x_4 = \frac{2}{3};\ x_5 = \frac{5}{6};\ x_6 = 1$

$$\begin{cases} -y_0 + (2 + h^2 q(x_1))y_1 - y_2 = h^2 f(x_1) \\ -y_1 + (2 + h^2 q(x_2))y_2 - y_3 = h^2 f(x_2) \\ -y_2 + (2 + h^2 q(x_3))y_3 - y_4 = h^2 f(x_3) \\ -y_3 + (2 + h^2 q(x_4))y_4 - y_5 = h^2 f(x_4) \\ -y_4 + (2 + h^2 q(x_5))y_5 - y_6 = h^2 f(x_5) \end{cases}$$

$$\begin{cases} -0,5 + \left(2 + \frac{1}{36} \cdot \frac{5\pi^2}{9}\right)y_1 - y_2 = \frac{1}{36} \cdot \pi^2 \sin\left(\pi\left(4\left(\frac{1}{6}+1\right)\right)/6\right) \\ -y_1 + \left(2 + \frac{1}{36} \cdot \frac{5\pi^2}{9}\right)y_2 - y_3 = \frac{1}{36} \cdot \pi^2 \sin\left(\pi\left(4\cdot\left(\frac{2}{6}+1\right)\right)/6\right) \\ -y_2 + \left(2 + \frac{1}{36} \cdot \frac{5\pi^2}{9}\right)y_3 - y_4 = \frac{1}{36} \pi^2 \sin\left(\pi\left(4\left(\frac{3}{6}+1\right)\right)/6\right) \\ -y_3 + \left(2 + \frac{1}{36} \cdot \frac{5\pi^2}{9}\right)y_4 - y_5 = \frac{1}{36} \pi^2 \sin\left(\pi\left(4\left(\frac{4}{6}+1\right)\right)/6\right) \\ -y_4 + \left(2 + \frac{1}{36} \cdot \frac{5\pi^2}{9}\right)y_5 - 0,5 = \frac{1}{36} \pi^2 \sin\left(\pi\left(4\left(\frac{5}{6}+1\right)\right)/6\right) \end{cases}$$

$$\begin{cases} 2,152309 y_1 - y_2 = 0,676234 \\ -y_1 + 2,152309 y_2 - y_3 = 0,093767 \\ -y_2 + 2,152309 y_3 - y_4 = 0 \\ -y_3 + 2,152309 y_4 - y_5 = -0,093767 \\ -y_4 + 2,152309 y_5 = -0,176224 = 0,323776 \end{cases}$$

$$\alpha_1 = -\frac{-1}{2,152309} = 0,464617 \qquad \beta_1 = \frac{0,676234}{2,152309} = 0,314190$$

$$\alpha_2 = -\frac{-1}{2,152309 + (-1) \cdot 0,464617} = 0,592525$$

$$\beta_2 = \frac{0,093767 - (-1) \cdot 0,314190}{2,152309 + (-1) \cdot 0,464617} = 0,241725$$

$$\alpha_3 = -\frac{-1}{2,152309 + (-1) \cdot 0,592525} = 0,641114$$

$$\beta_3 = \frac{0 - (-1) \cdot 0,241725}{2,152309 + (-1) \cdot 0,592525} = 0,154973$$

$$\alpha_4 = -\frac{-1}{2{,}152309+(-1)\cdot 0{,}641114} = 0{,}661728$$

$$\beta_4 = \frac{-0{,}093767-(-1)\cdot 0{,}154973}{2{,}152309+(-1)\cdot 0{,}641114} = 0{,}040502$$

$$\beta_5 = \frac{0{,}323776-(-1)\cdot 0{,}040502}{2{,}152309+(-1)\cdot 0{,}661728} = 0{,}244387$$

$$y_5 = \beta_5 = 0{,}244387$$

$$y_4 = \alpha_4 \cdot y_5 + \beta_4 = 0{,}661728\cdot 0{,}244387 + 0{,}040502 = 0{,}202219$$

$$\cancel{y_3 = \alpha_3 \beta_4 + \beta_3 = 0{,}641114\cdot 0{,}040502 + 0{,}154923 = 0{,}180889}$$

$$y_2 = \alpha_2 \beta_3 + \beta_2 = 0{,}592525\cdot 0{,}154973$$

$$y_3 = \alpha_3 y_4 + \beta_3 = 0{,}641114\cdot 0{,}202219 + 0{,}154923 = 0{,}284568$$

$$y_2 = \alpha_2 y_3 + \beta_2 = 0{,}592525\cdot 0{,}284568 + 0{,}241725 =$$
$$= 0{,}410339$$

$$y_1 = \alpha_1 y_2 + \beta_1 = \cancel{0{,}3} = 0{,}464617\cdot 0{,}410339 + 0{,}314190 =$$
$$= 0{,}504840$$

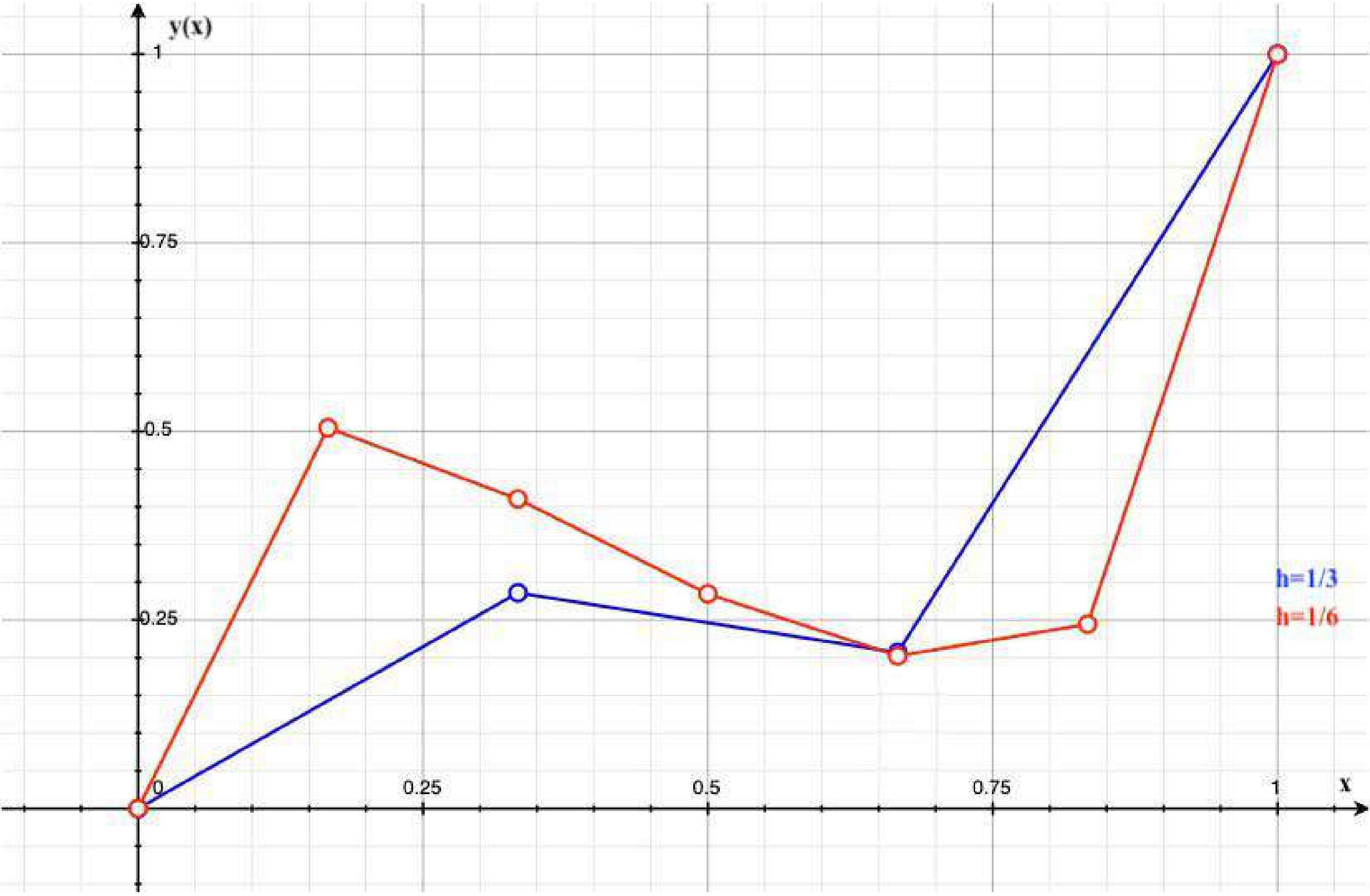
· Оценить погрешность по правилу Рунге!

$x_i = \frac{2}{6}$; $x_i = \frac{4}{6}$

$$r:\ y\left(\frac{1}{3}\right) - y_2^{1/6} = \frac{|y_2^{1/6} - y_1^{1/3}|}{2^2-1} = \frac{|0{,}410339 - 0{,}286040|}{3} = 0{,}041433$$

$$y\left(\frac{2}{3}\right) - y_4^{1/6} = \frac{|y_4^{1/6} - y_2^{1/3}|}{2^2-1} = \frac{|0{,}202219 - 2067881|}{3} = 0{,}001523$$

$$R = \max(|r_i|) = 0{,}041433$$


y(x)
1
0.75
0.5
0.25
0
0.25
0.5
0.75
1
x
h=1/3
h=1/6