What's the role of higher-order topology?

Recap and outlook

G. Petri

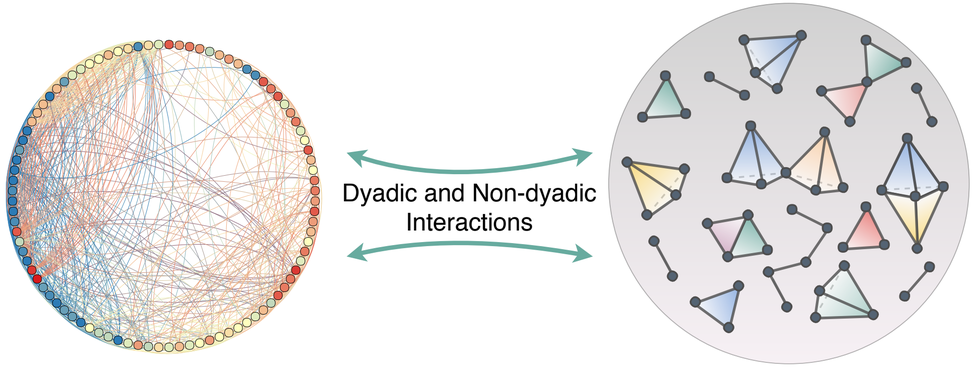
TDA Crash Course, Applied Machine Learning Days

Lausanne, 26th Jan 2019

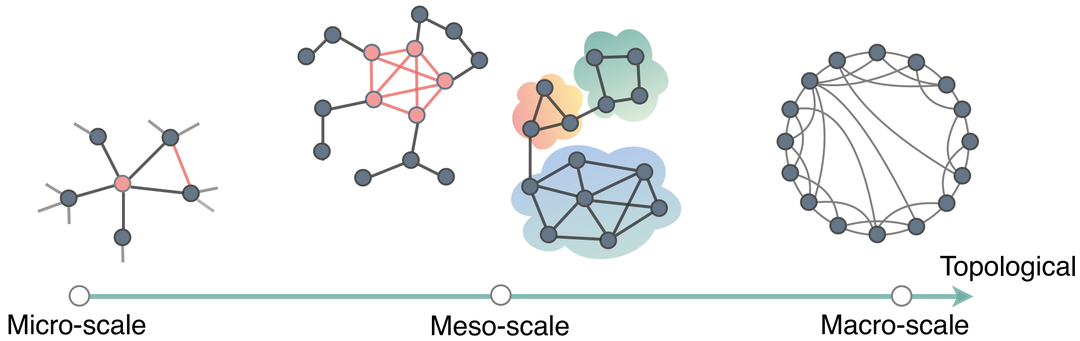
ISI Foundation
& ISI Global Science
Foundation

Why are we talking about it?

1. **Higher-order interactions:**
    1. Paradigmatic example: collaboration
    2. complex contagion / local environment
    3. gene-gene interactions
    4. Some type of neuronal interactions



2. **Shapes / Mesoscale structures**
    1. Organisation on intermediate scales
    2. Non-local features



3. **General comparison of spaces**
    1. Distance manifold to manifold
    2. Compare networks with different structures

What does it mean in practice?

1. **(Persistent) homology**
    1. Multiscale description of a network/space
    2. Complementary to communities/blocks
    3. Categorical foundation

2. **Mapper**
    1. Topological backbone of datasets
    2. Builds networks!
    3. Topological Simplification

TDA

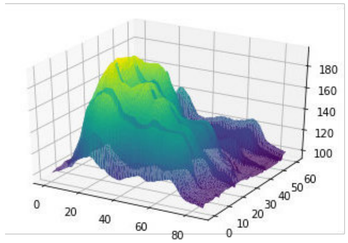
3. **Simplicial models**
    1. Configuration models (SCM, ERSC)
    2. Generative models (Network geo+flavour)
    3. Dynamics on/off models (SAD)

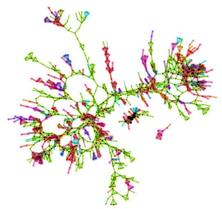
“Netsci”

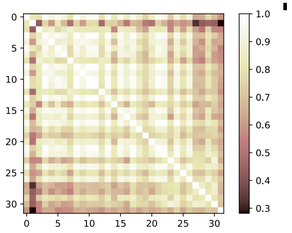
What does it mean in practice?

Persistent homology pipeline (Ghrist 2008)

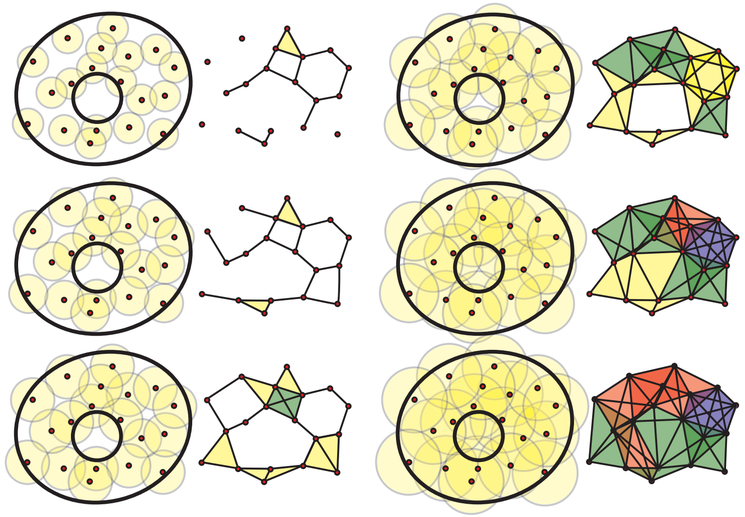
Data of sorts



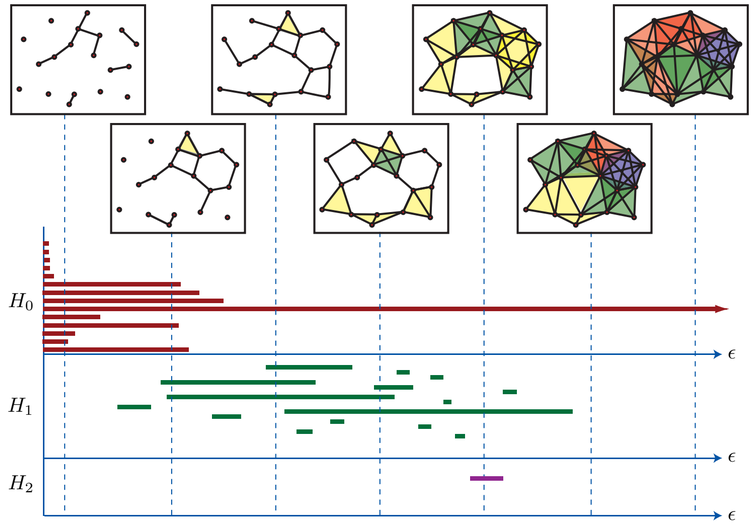




Filtration over distance/density/weights



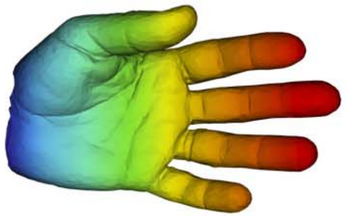
Homological properties



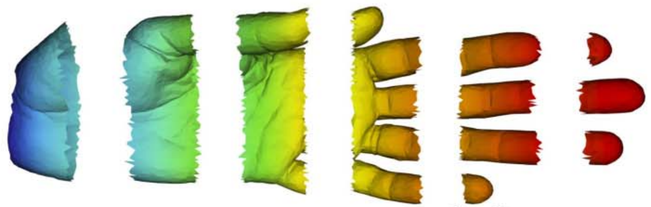
Mapper Pipeline (Singh et al 2007)

Point cloud

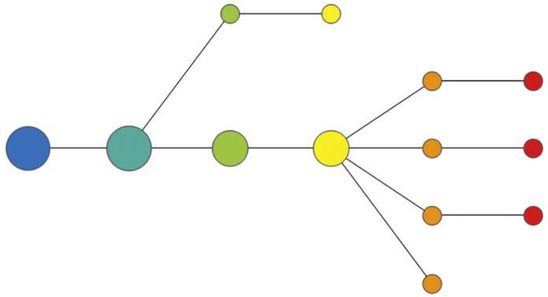
colouring (projecting) using geometric filters



overlapped binning



Clustering and network construction binning



Proposal

Algebraic Topology (AT):

1. as a language to extend and generalise networks
2. as a language to investigate spaces where networks "do" stuff

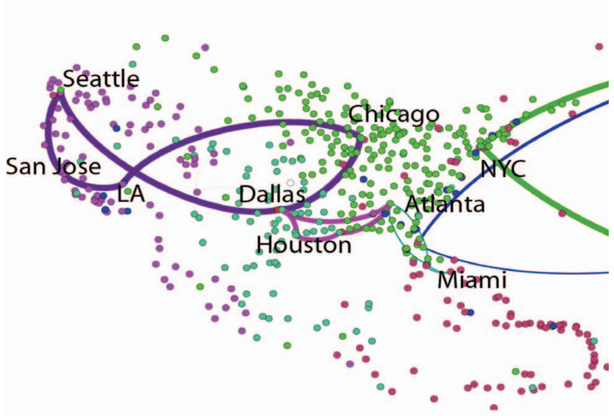
To highlight the difference between:

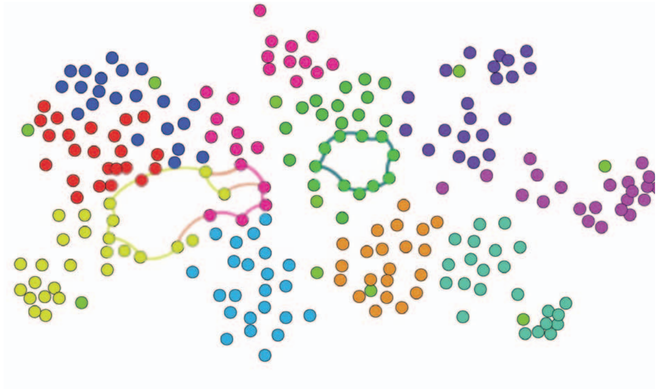
- Perspectives LANGUAGE SPACE
- Tools NETSCI MAPPER PHOM

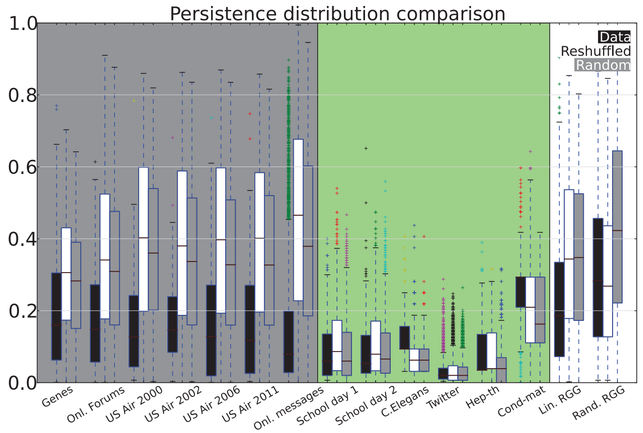
Applications

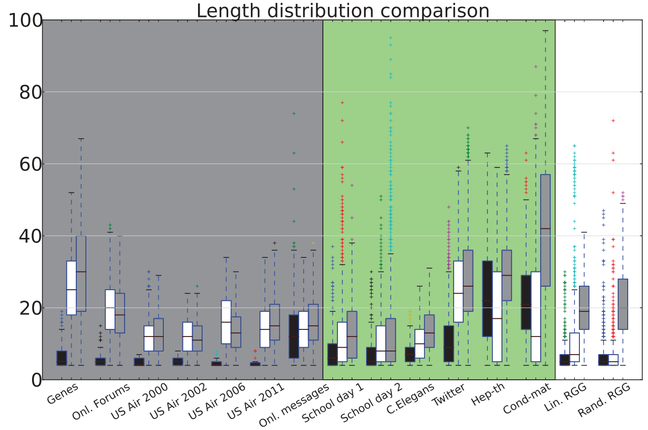
AT as a language to extend networks

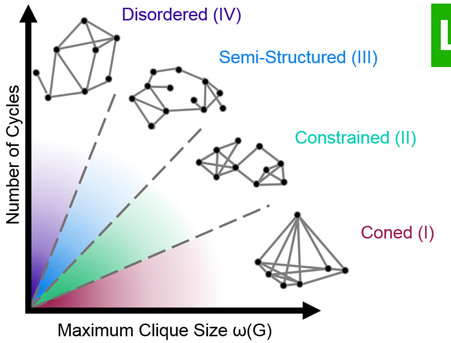
Are there homological features in graphs?



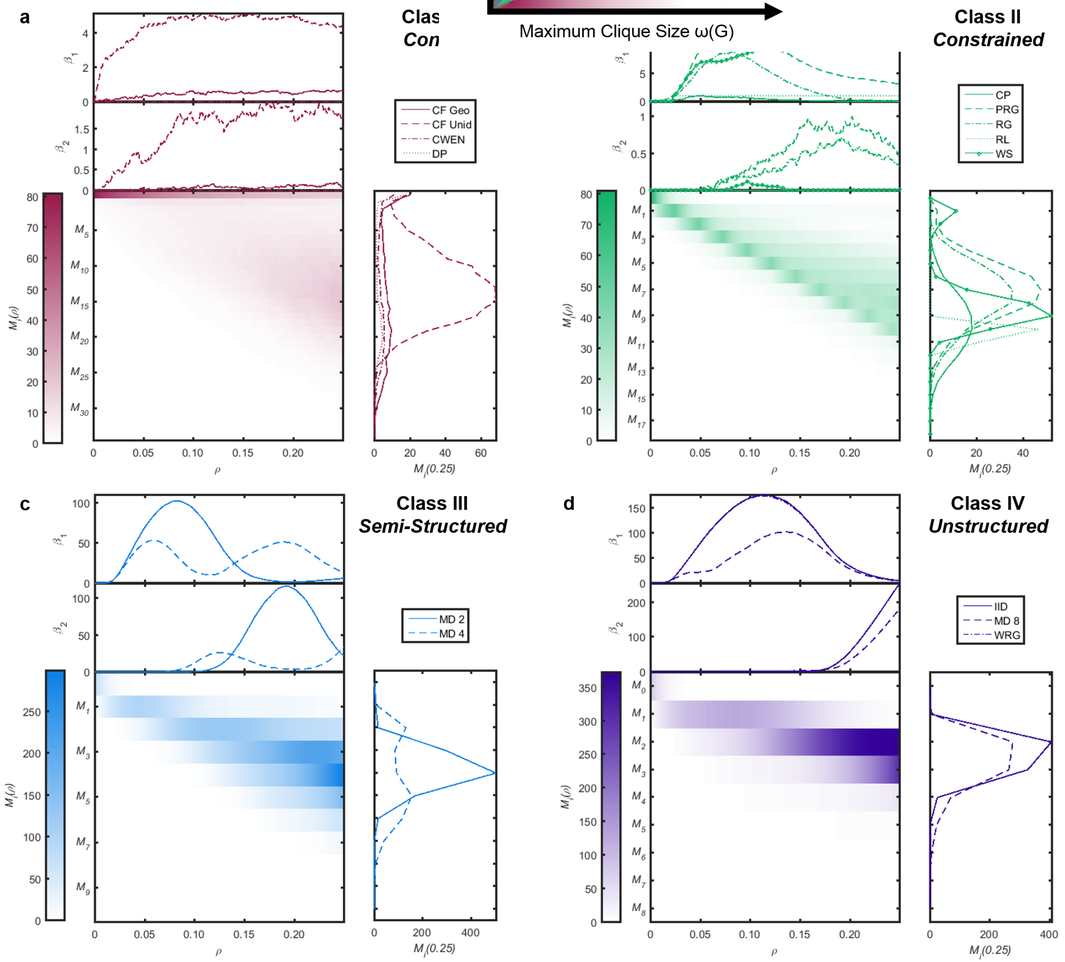








LANGUAGE



Petri, Giovanni, et al. "Topological strata of weighted complex networks." *PloS one* 8.6 (2013): e66506.

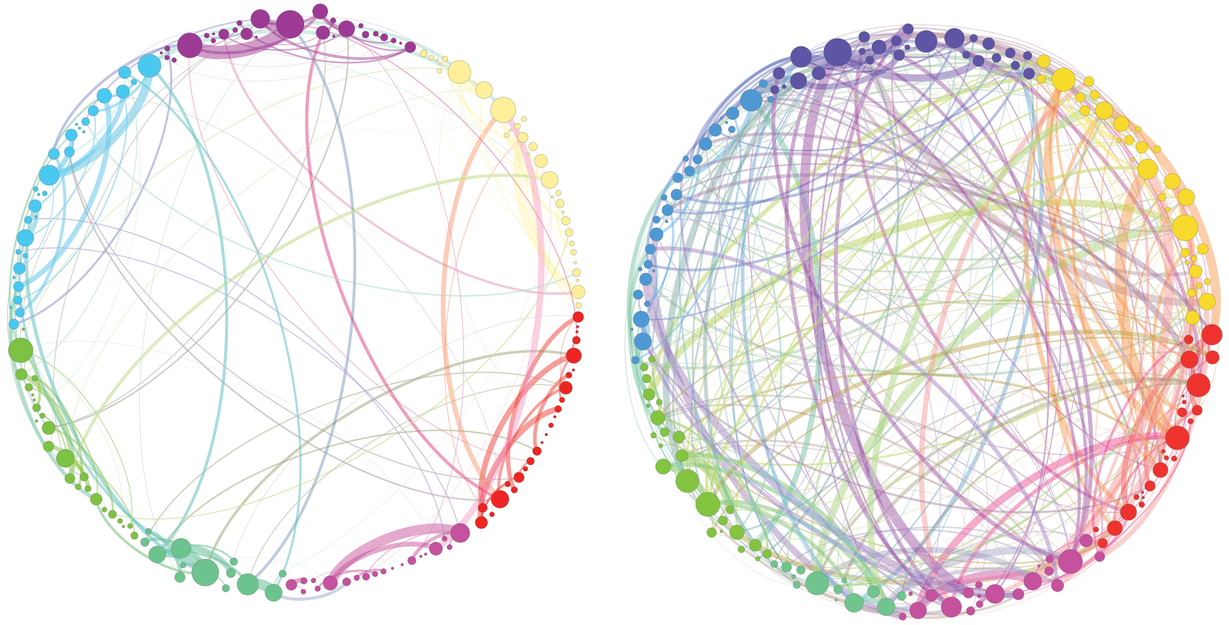
Sizemore, Ann, Chad Giusti, and Danielle S. Bassett. "Classification of weighted networks through mesoscale homological features." *Journal of Complex Networks* 5.2 (2016): 245-273.

PHOM

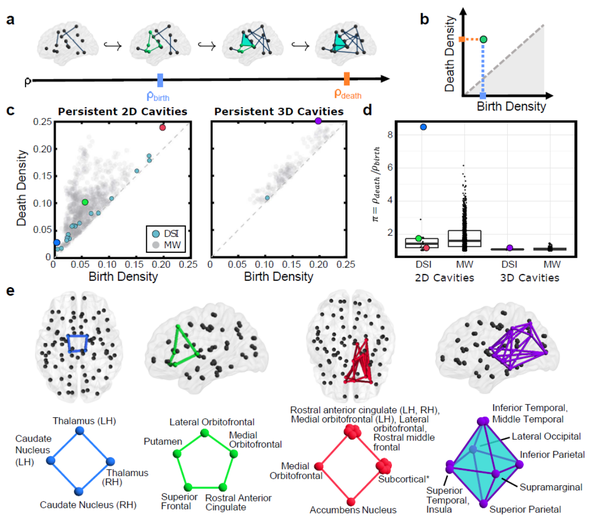
Do homological mesoscales characterise brain networks?

LANGUAGE

Graph-representations of 1-dimensional hole structure

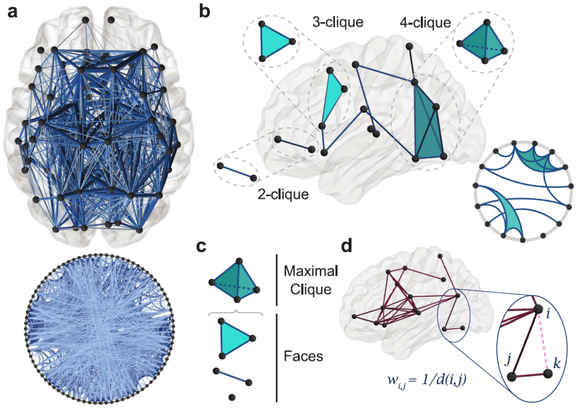


Petri, Giovanni, et al. "Homological scaffolds of brain functional networks." *Journal of The Royal Society Interface* 11.101 (2014): 20140873.

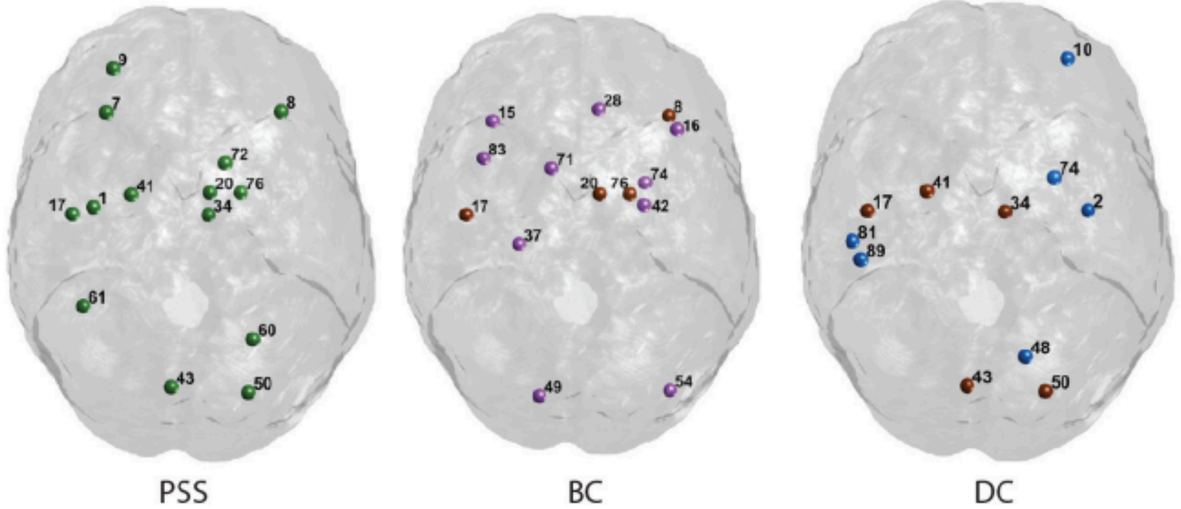


Structural brain cavities

Sizemore, Ann, et al. arXiv:1608.03520 (2016).



Homological persistence centrality

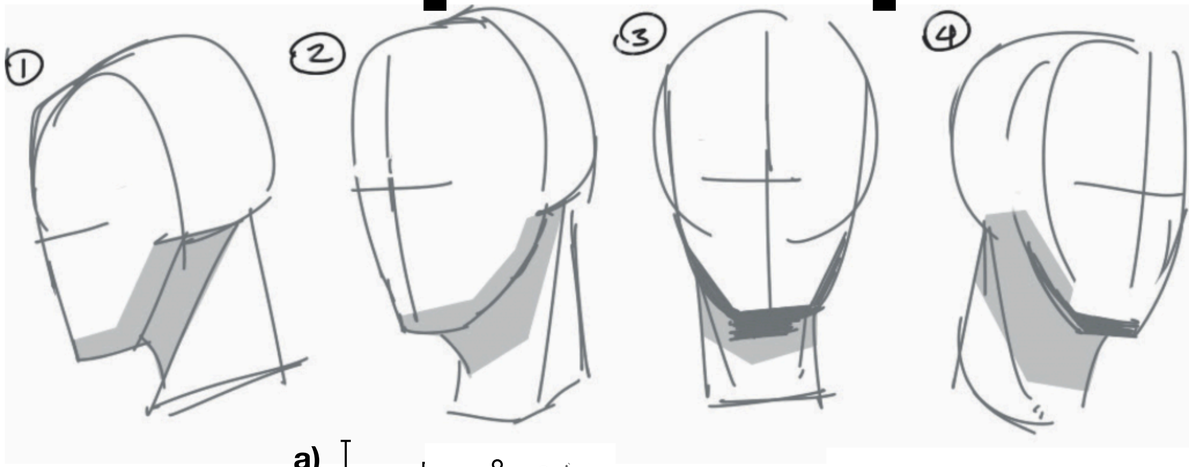


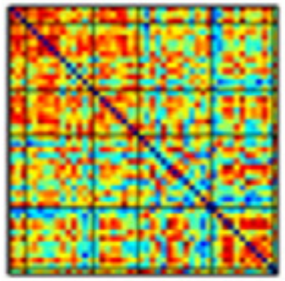
Lord, Louis-David, et al. "Insights into brain architectures from the homological scaffolds of functional connectivity networks." *Frontiers in systems neuroscience* 10 (2016).

PHOM

Do homological shapes capture transitions?

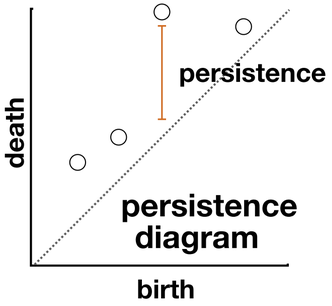
LANGUAGE

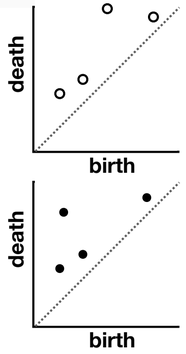




a)

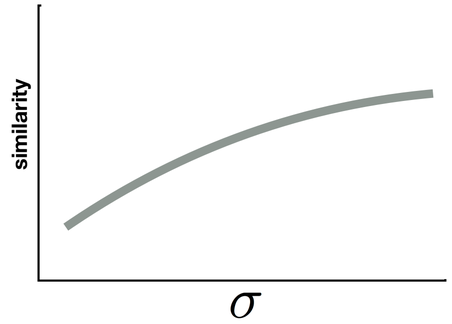
topology



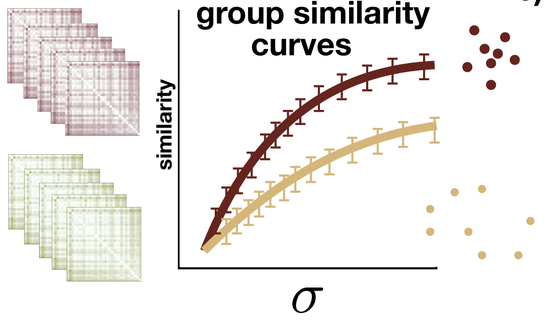


persistence kernel

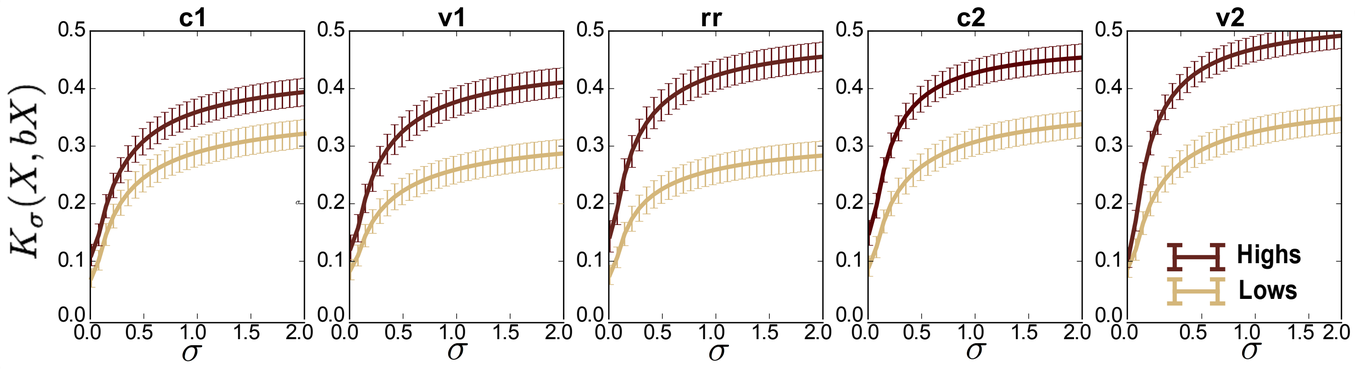
b)



c)



d)

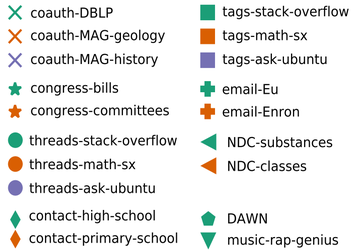


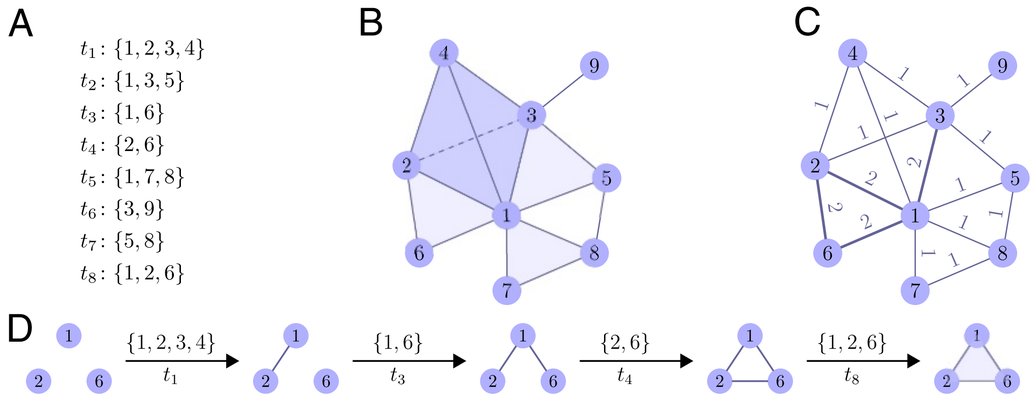
Ibanez-Marcelo, Esther, et al. "Topology highlights mesoscopic functional equivalence between imagery and perception." *bioRxiv* (2018): 268383.

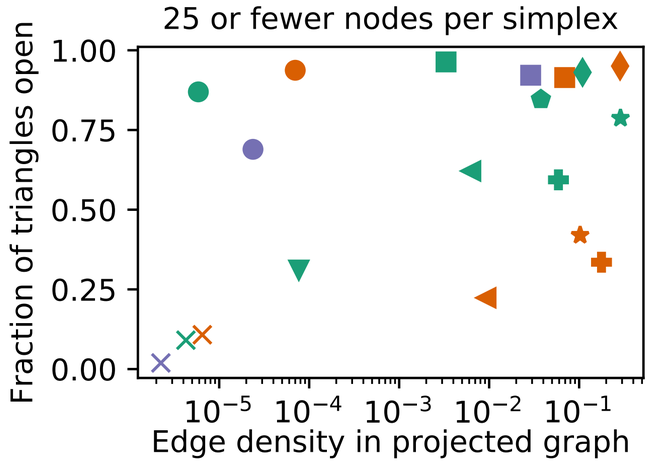
PHOM

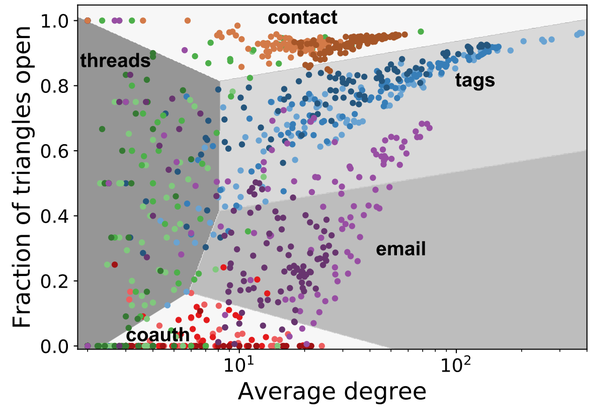
Do high-order interactions predict new interactions?

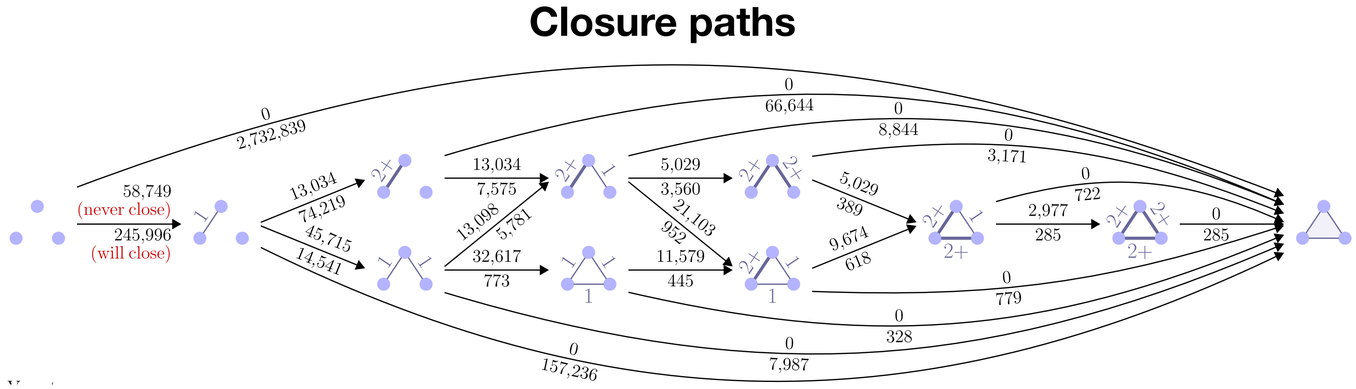
LANGUAGE

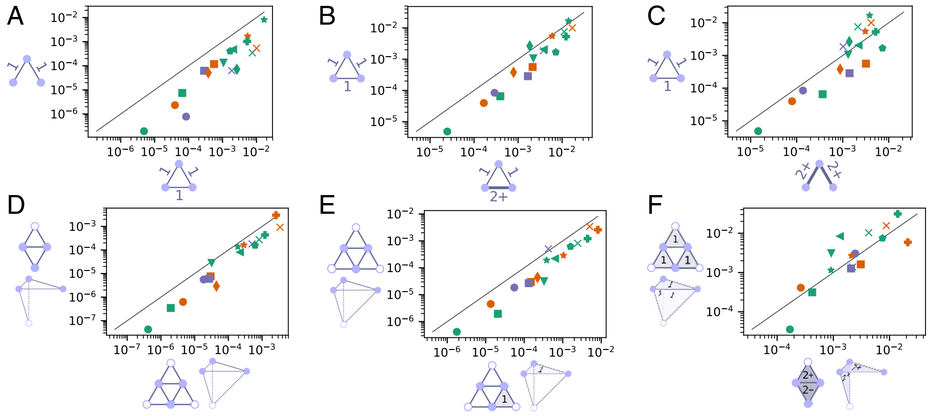












Closure probabilities

Simplicial closure and higher-order link prediction

Austin R. Benson[a], Rediet Abebe[a], Michael T. Schaub[b,c], Ali Jadbabaie[b,d], and Jon Kleinberg[a,1]

[a]Department of Computer Science, Cornell University, Ithaca, NY 14853; [b]Institute for Data, Systems, and Society, Massachusetts Institute of Technology, Cambridge, MA 02139; [c]Department of Engineering Science, University of Oxford, Oxford OX1 3PJ, United Kingdom; and [d]Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139

Edited by Duncan J. Watts, Microsoft Research, New York, NY, and accepted by Editorial Board Member Donald J. Geman October 12, 2018 (received for review January 13, 2018)

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LANGUAGE

What happens to dynamics on complexes?

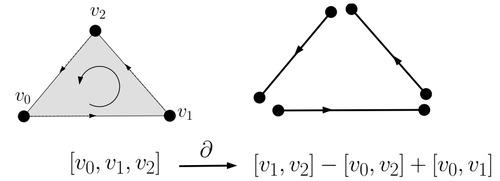
Simplicial PageRank

Based on the k-Laplacian properties

$$\boldsymbol{\pi}_1^g = \boldsymbol{\Theta}\left(\kappa\mathbf{I} + \boldsymbol{\mathcal{L}}_1\right)^{-1}\boldsymbol{\Theta}\mathbf{x}.$$

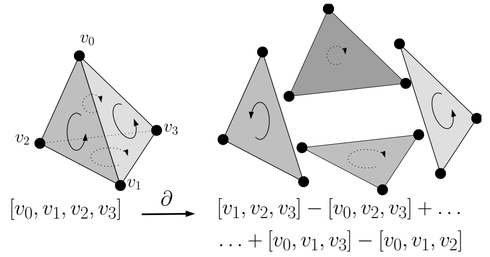
Combinatorial (Hodge) Laplacian

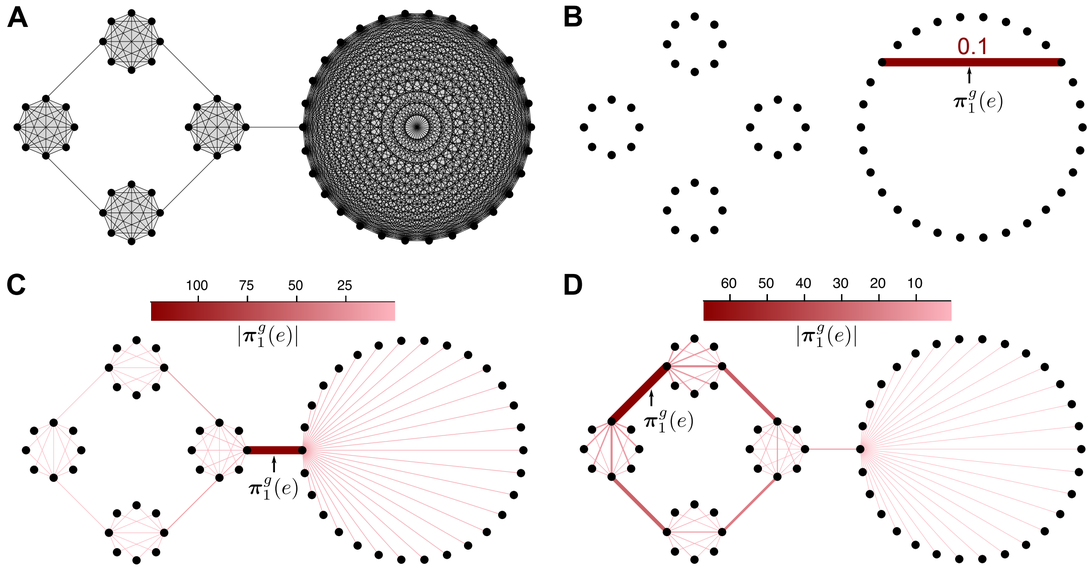
$$\mathcal{L}_k = \partial_{k+1}\partial_{k+1}^* + \partial_k^*\partial_k.$$



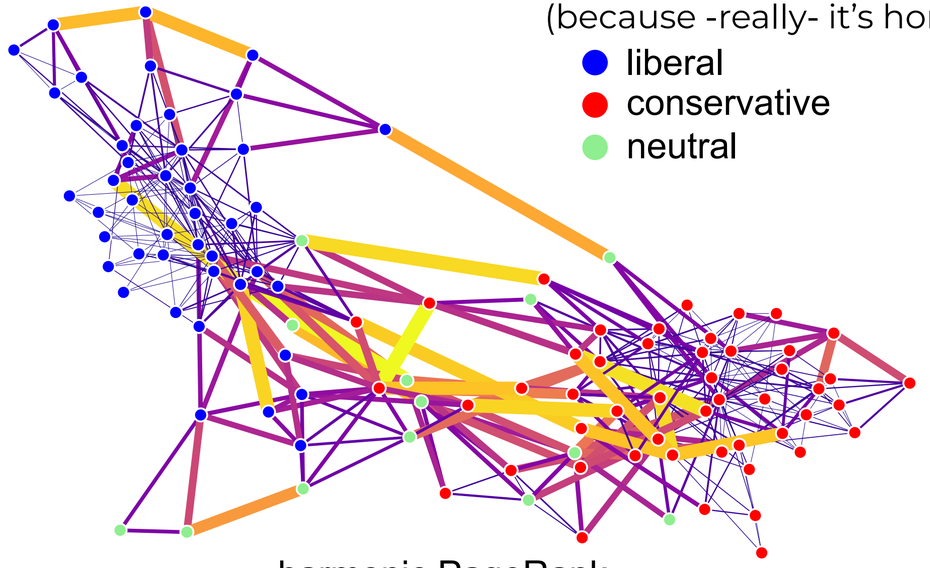
$$C_k(X;\mathbb{R}) = \mathcal{H}_k(X) \oplus \mathrm{im}(\partial_{k+1}) \oplus \mathrm{im}(\partial_k^*),$$

$$\mathcal{H}_k(X) = \{c \in C_k(X) : \mathcal{L}_k c = 0\} = \ker \mathcal{L}_k.$$





Similar to Patania et al. EPJ Data Science (2017)
(because -really- it's homology!)



harmonic PageRank

Schaub, Michael T., et al. "Random Walks on Simplicial Complexes and the normalized Hodge Laplacian." *arXiv preprint arXiv:1807.05044* (2018).

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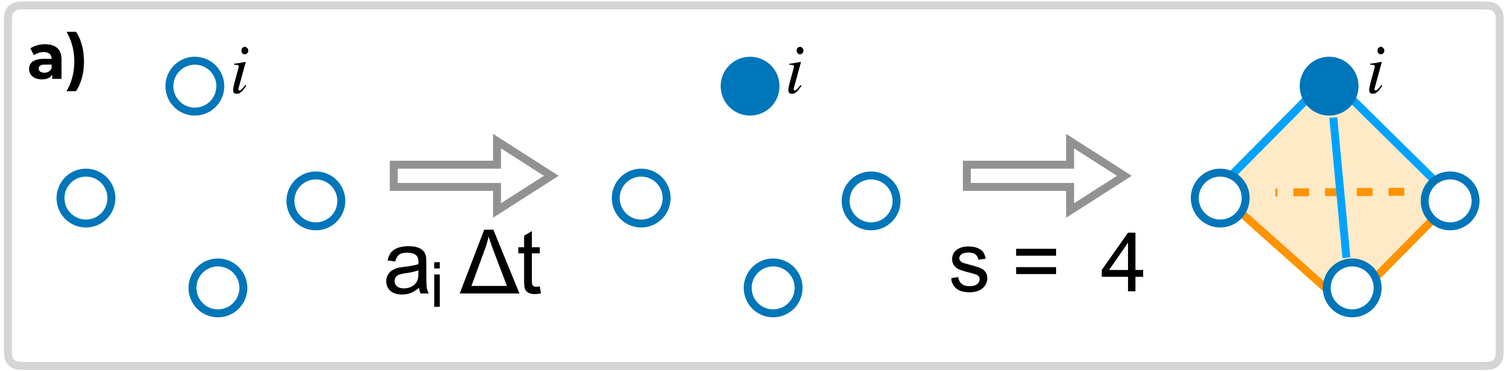
What happens to dynamics on complexes?

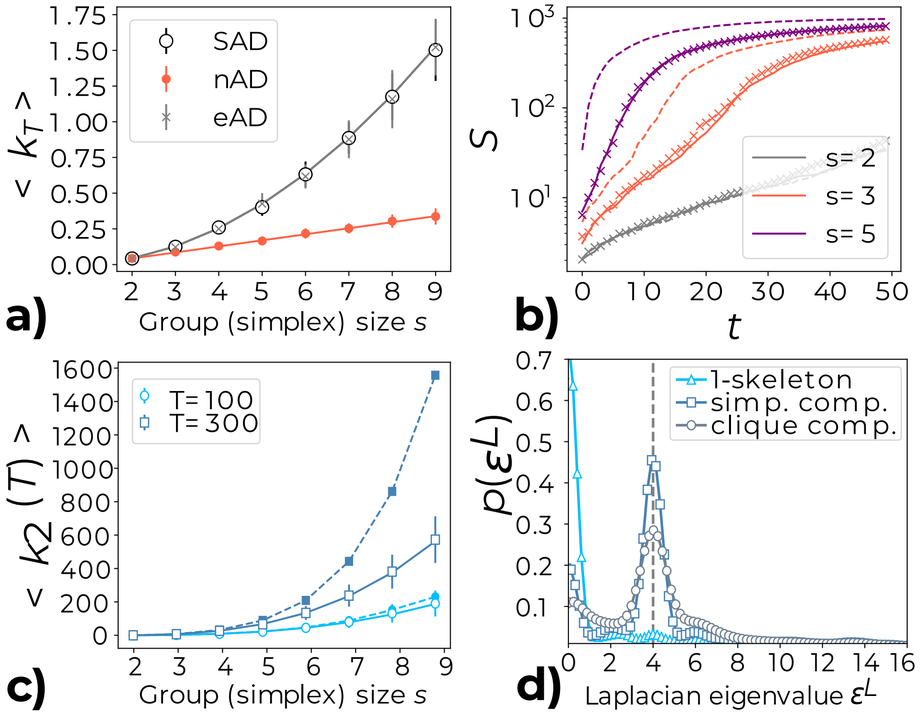
LANGUAGE

Simplicial Activity Driven Model:
Direct extension of the activity driven model to group interactions

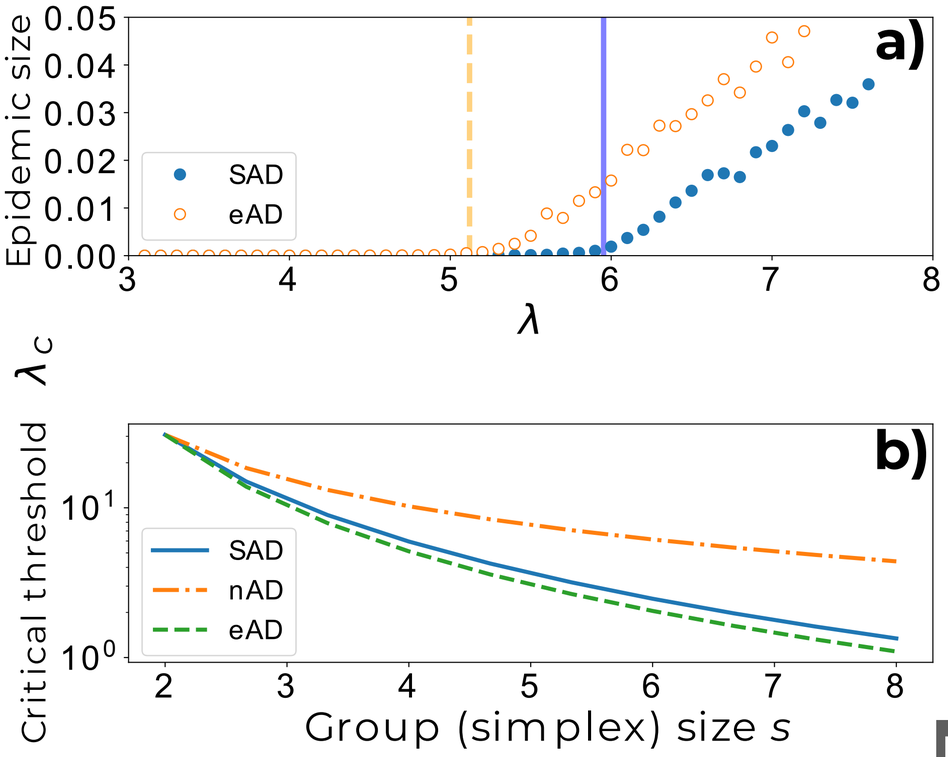
Petri & Barrat, PRL 2018



Structural effects



Dynamical effects



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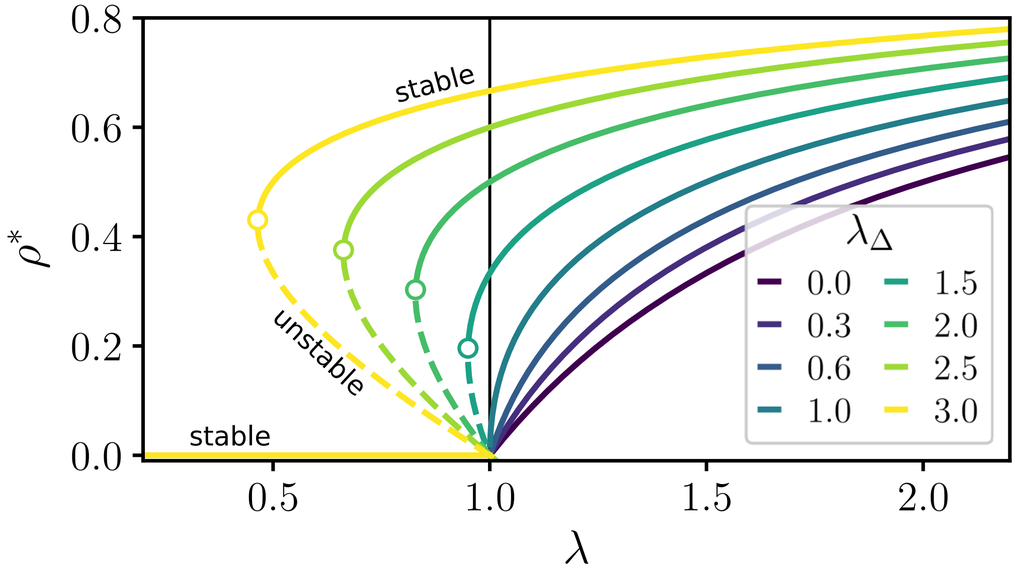
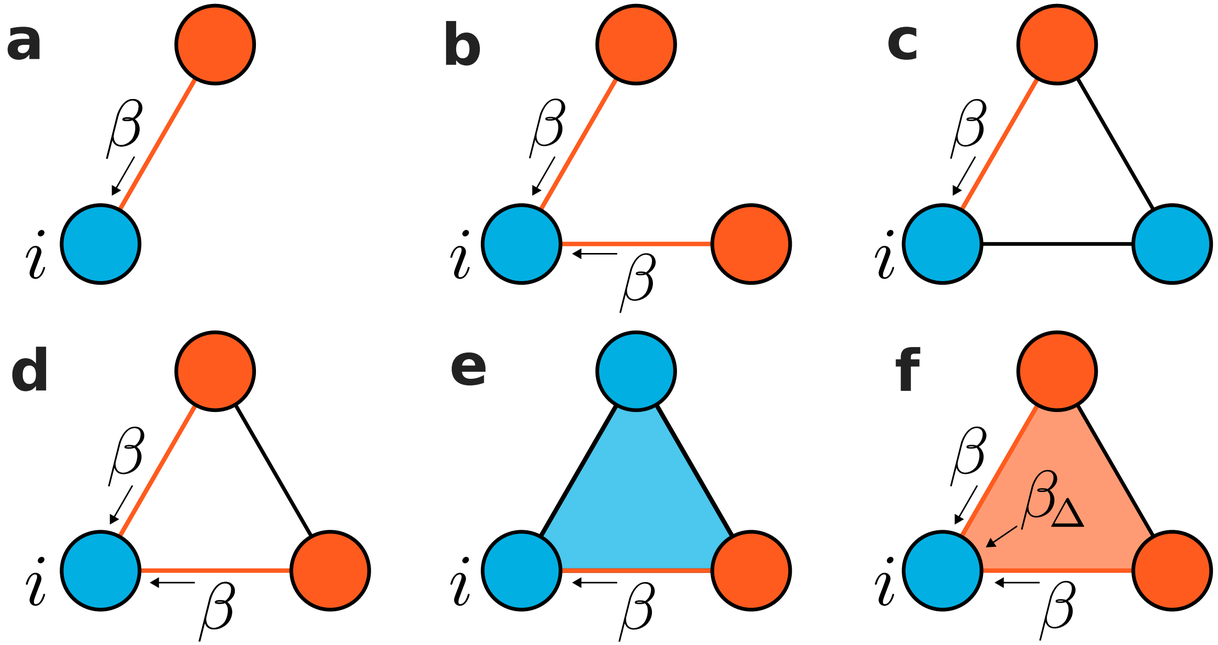
What happens to dynamics on complexes?

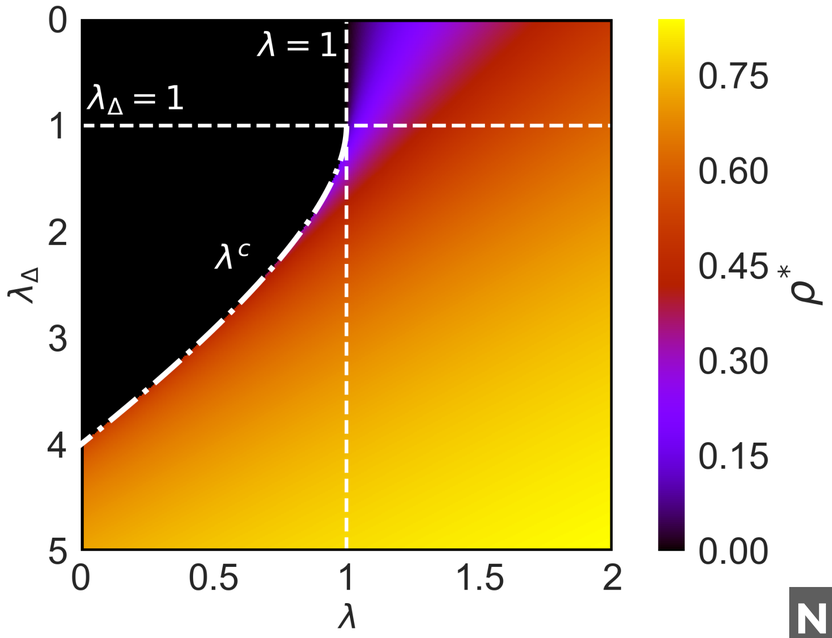
LANGUAGE

Simplagion:
Direct extension of the contagion models to **actual** group interactions

Iacopini, Iacopo, et al. "Simplicial models of social contagion." arXiv preprint arXiv:1810.07031 (2018).







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Applications

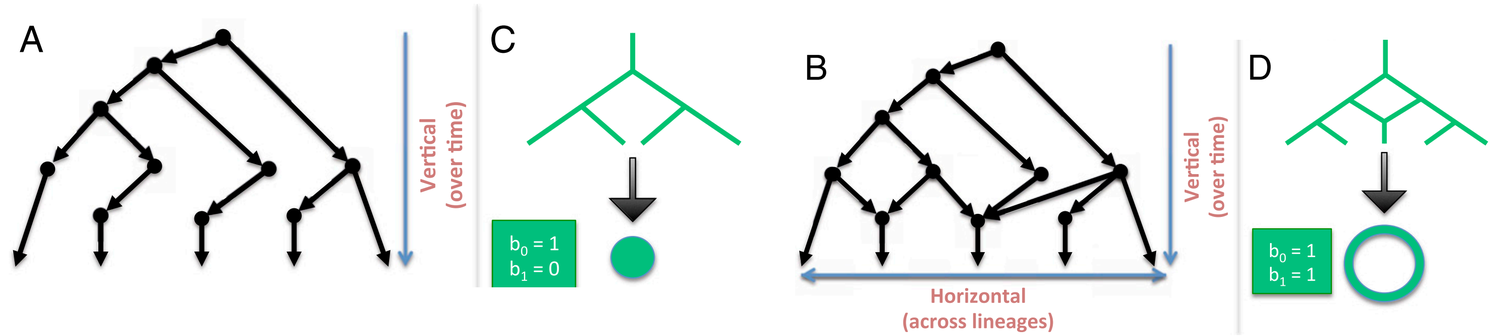
AT as a language to investigate spaces where networks "do" things

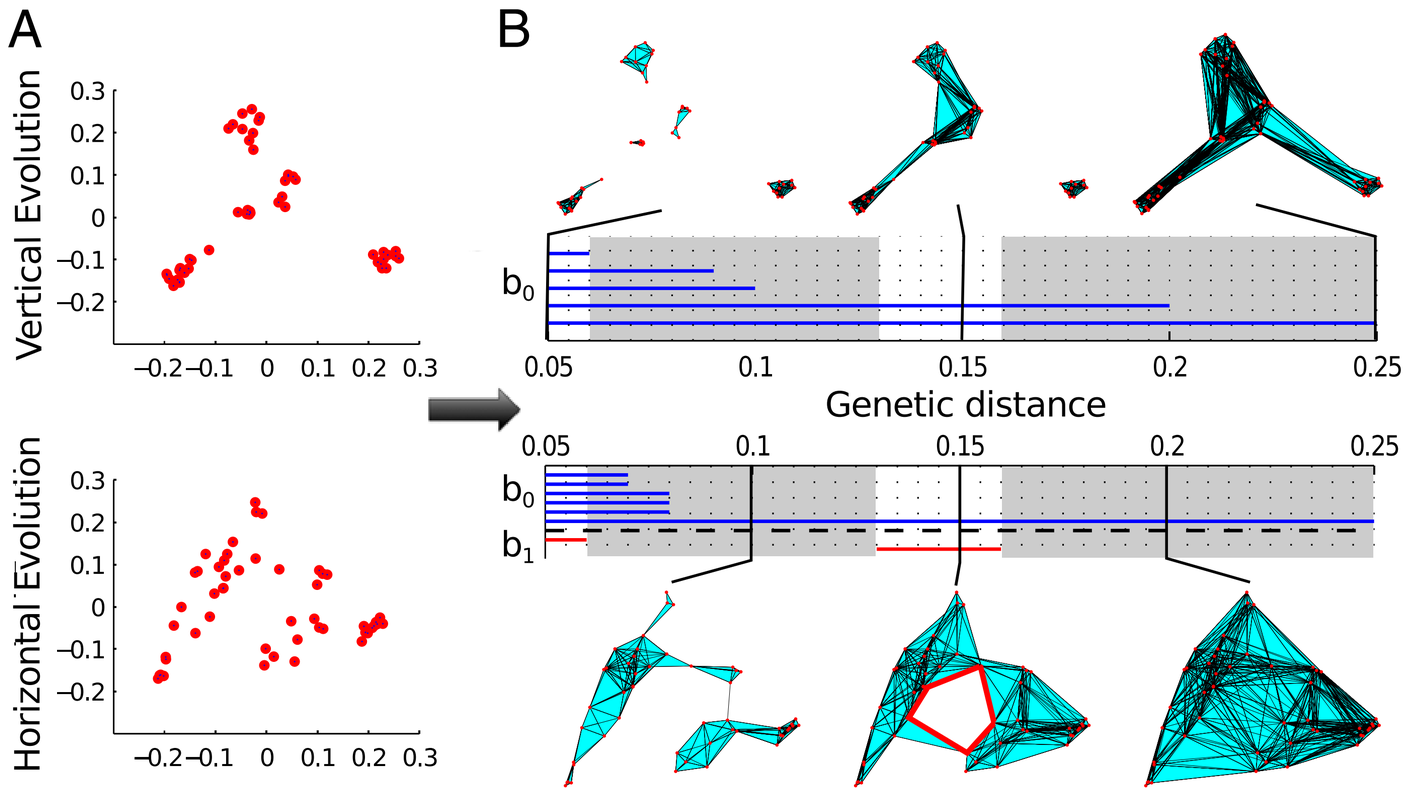
In most cases, these entail

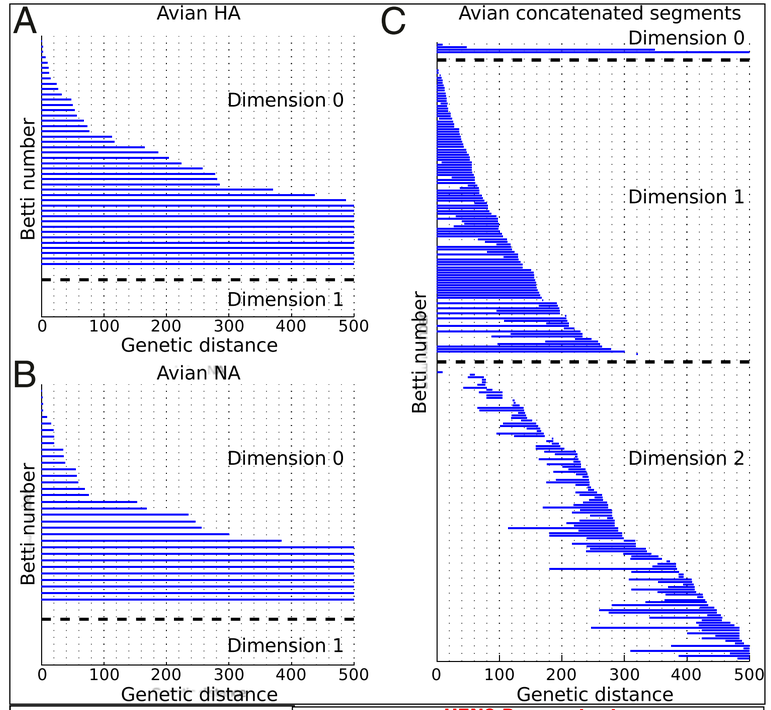
1. **Extracting networks from data spaces**
2. **Network embeddings**
3. **Signal embeddings**

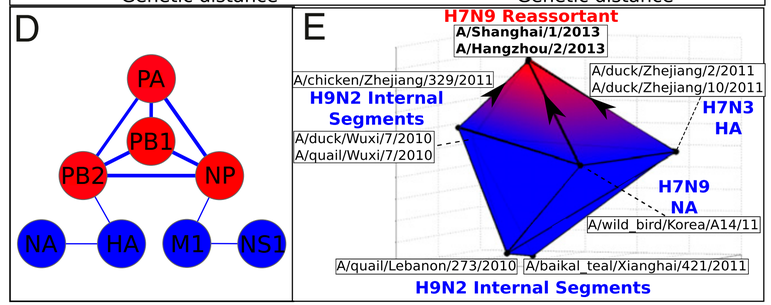
Extracting networks from data spaces

Non-treelike viral evolution









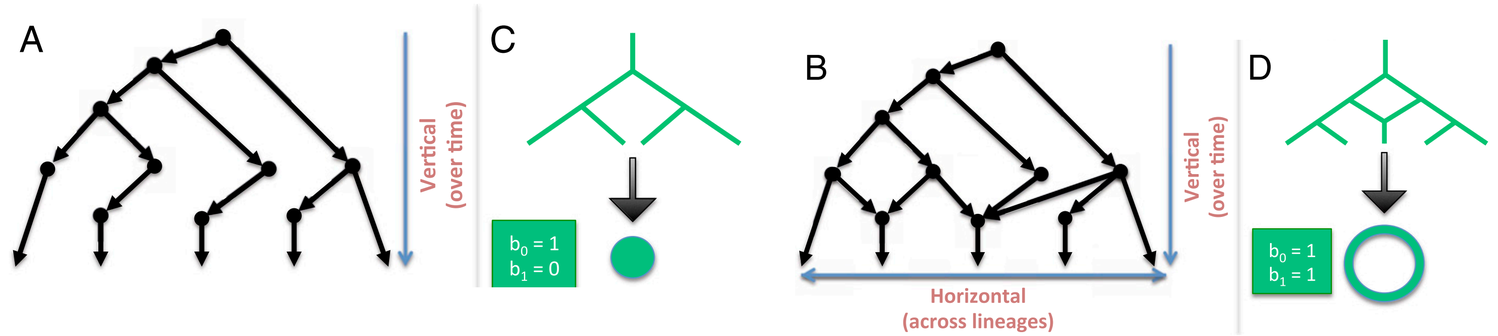
Topology of viral evolution

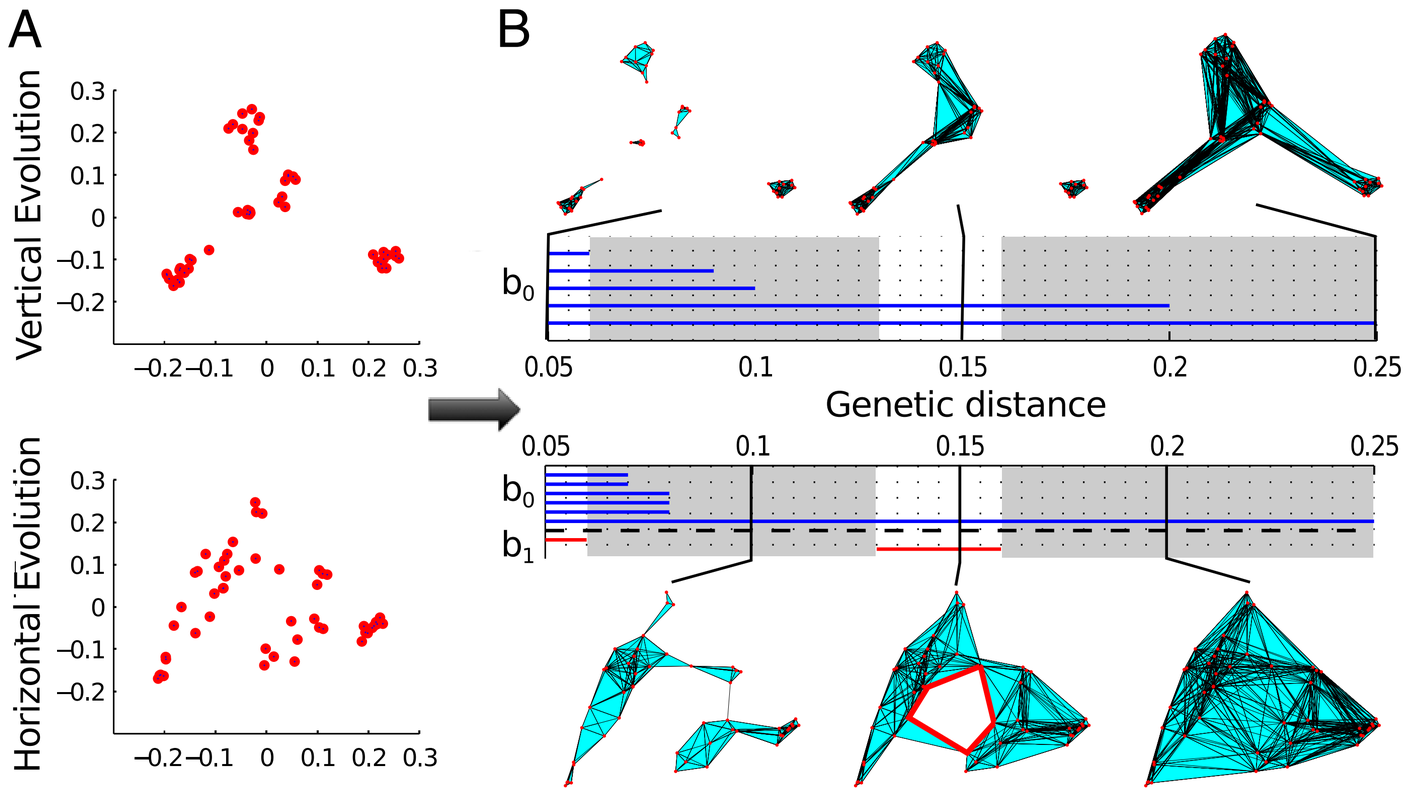
Joseph Minhow Chan[a,b], Gunnar Carlsson[c], and Raul Rabadan[a,b,d,1]

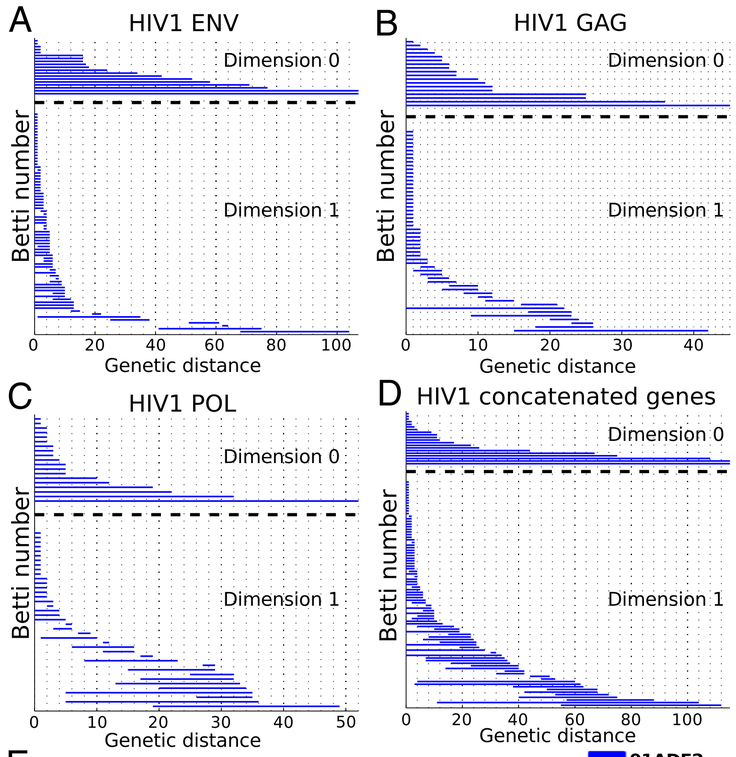
[a]Center for Computational Biology and Bioinformatics and Departments of [b]Biomedical Informatics and [d]Systems Biology, Columbia University College of Physicians and Surgeons, New York, NY 10032; and [c]Department of Mathematics, Stanford University, Stanford, CA 94305

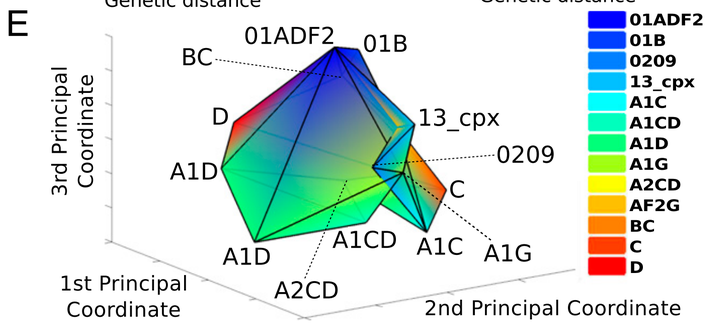
Edited* by Arnold J. Levine, Institute for Advanced Study, Princeton, NJ, and approved October 11, 2013 (received for review July 18, 2013)

Non-treelike viral evolution









Topology of viral evolution

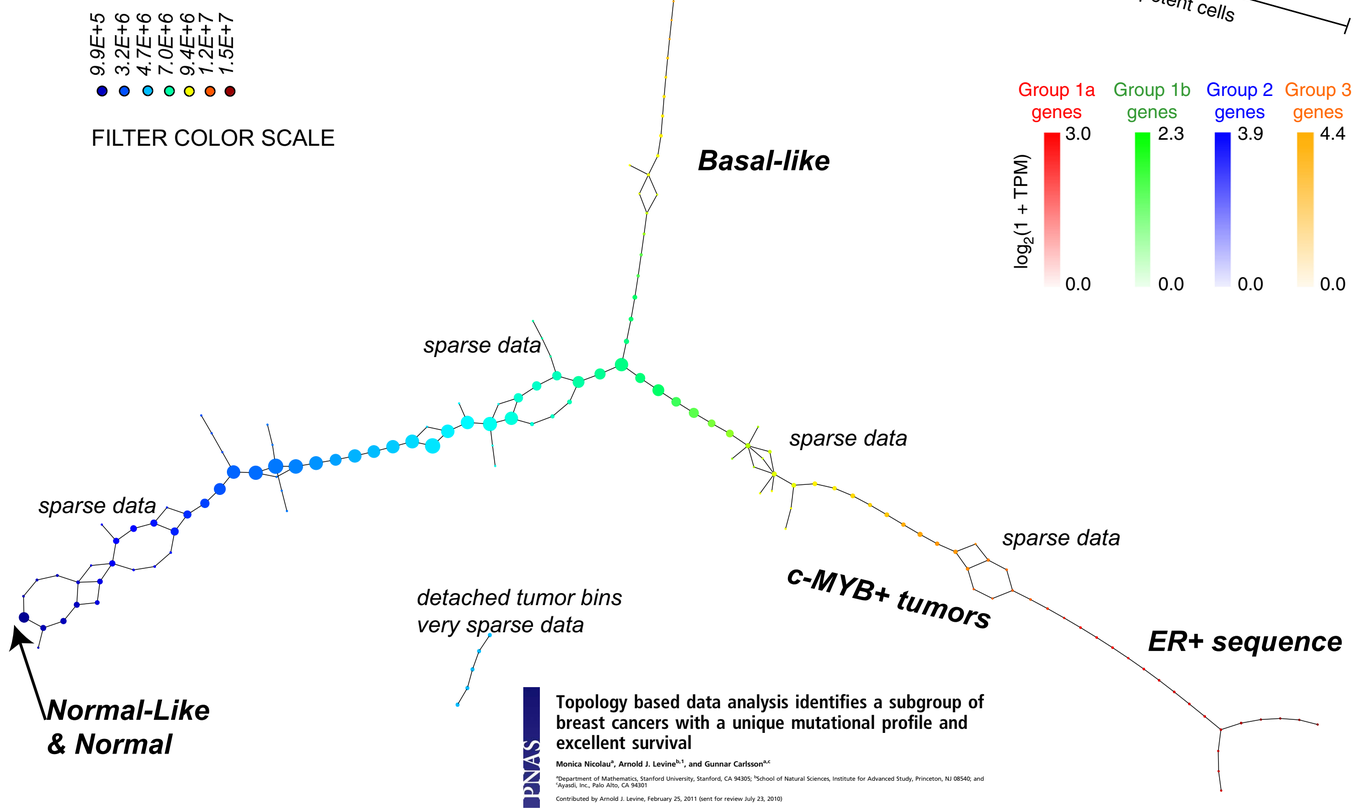
Joseph Minhow Chan[a,b], Gunnar Carlsson[c], and Raul Rabadan[a,b,d,1]

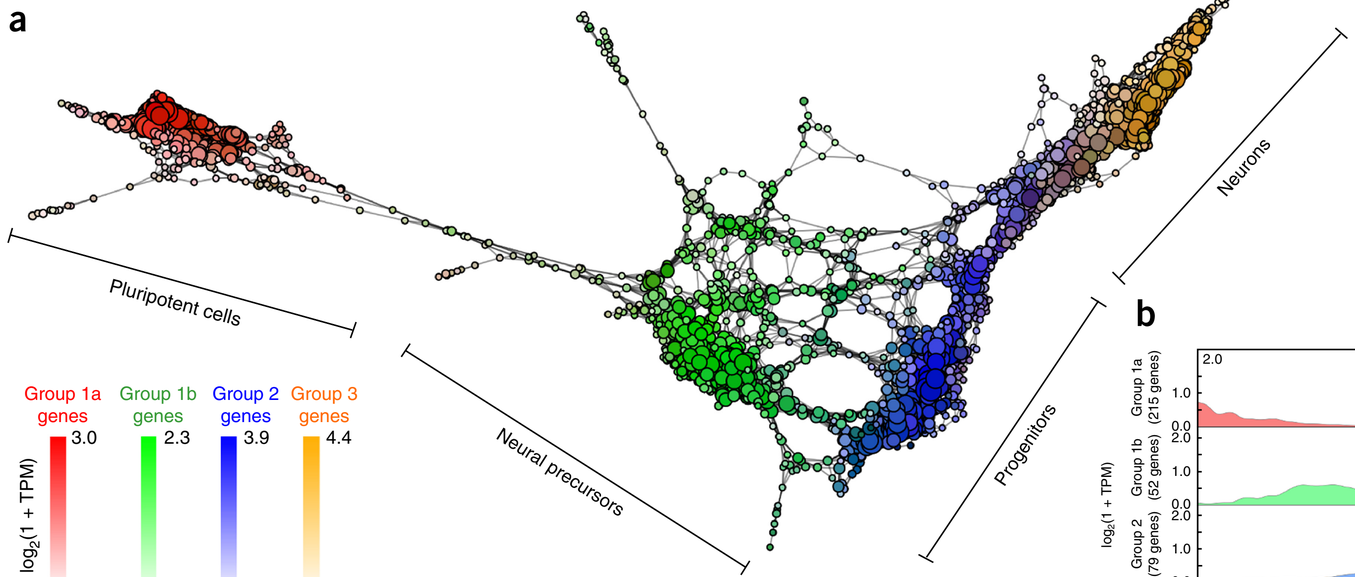
[a]Center for Computational Biology and Bioinformatics and Departments of [b]Biomedical Informatics and [d]Systems Biology, Columbia University College of Physicians and Surgeons, New York, NY 10032; and [c]Department of Mathematics, Stanford University, Stanford, CA 94305

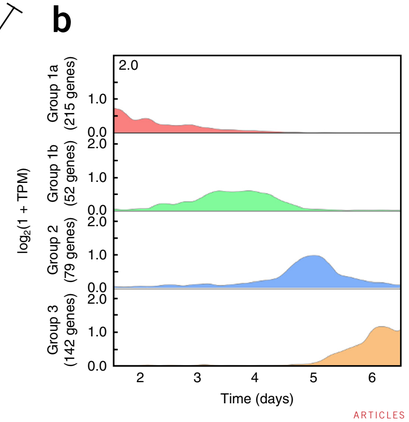
Edited* by Arnold J. Levine, Institute for Advanced Study, Princeton, NJ, and approved October 11, 2013 (received for review July 18, 2013)

Do topological gene-backbones carry information?

SPACE







PNAS

Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival

Monica Nicolau[a], Arnold J. Levine[b,1], and Gunnar Carlsson[a,c]

[a]Department of Mathematics, Stanford University, Stanford, CA 94305; [b]School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540; and [c]Ayasdi, Inc., Palo Alto, CA 94301

Contributed by Arnold J. Levine, February 25, 2011 (sent for review July 23, 2010)

ARTICLES

nature biotechnology

Single-cell topological RNA-seq analysis reveals insights into cellular differentiation and development

Abbas H Rizvi[1,2,6], Pablo G Camara[3,4,6], Elena K Kandror[1,2], Thomas J Roberts[1,2,4], Ira Schieren[2,5], Tom Maniatis[1,2] & Raul Rabadan[3,4]

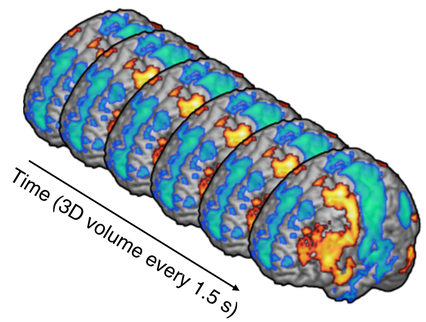
MAPPER

Approximate activity landscapes using topology

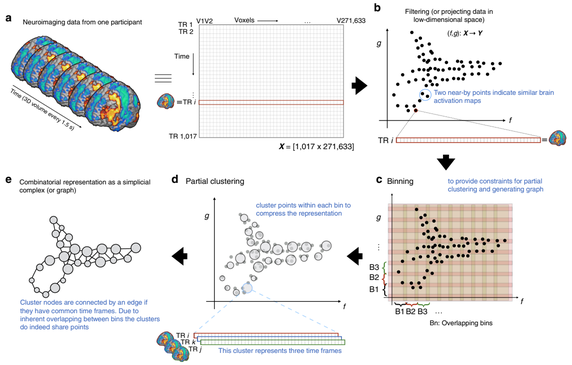
SPACE

a

Task brain activation data



Task-fMRI dataset from one CMP participant S_{01}



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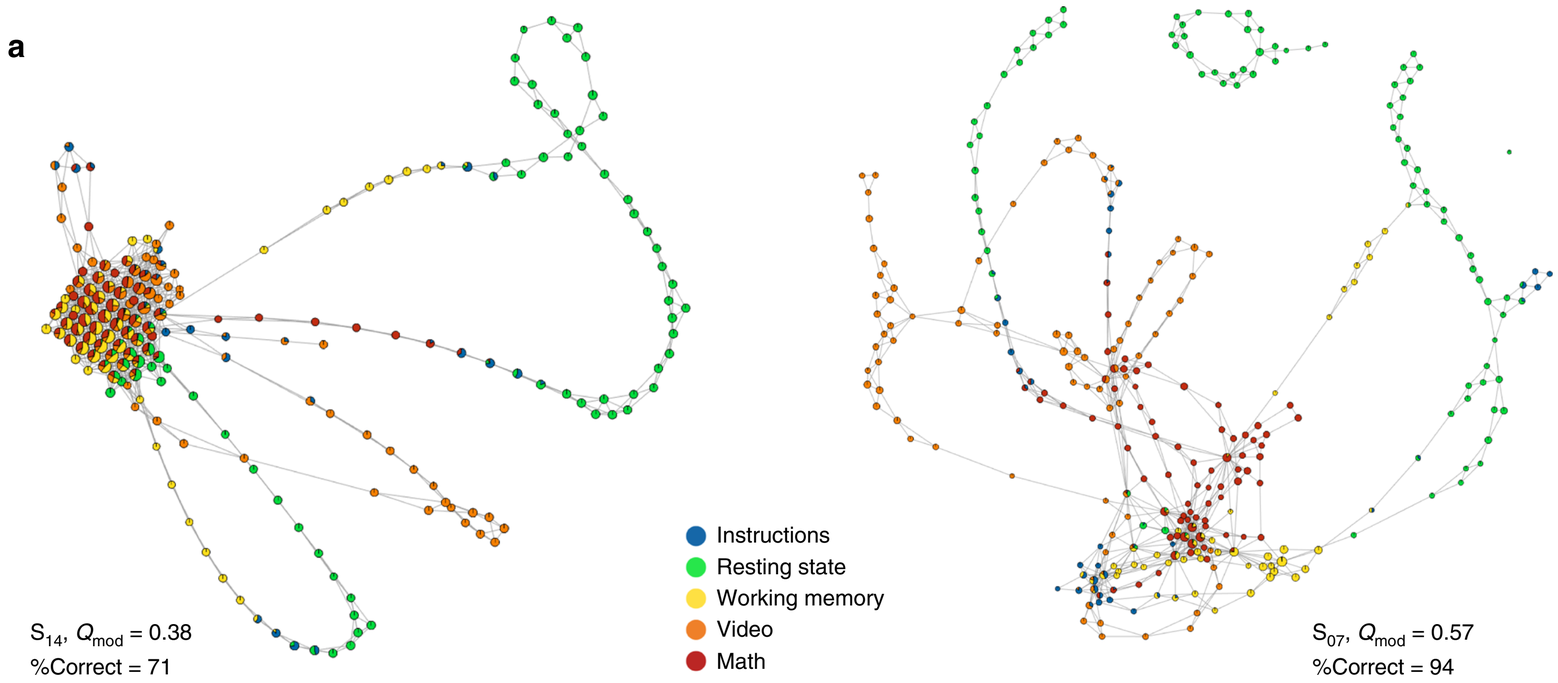
MAPPER

Approximate activity landscapes using topology

LANGUAGE

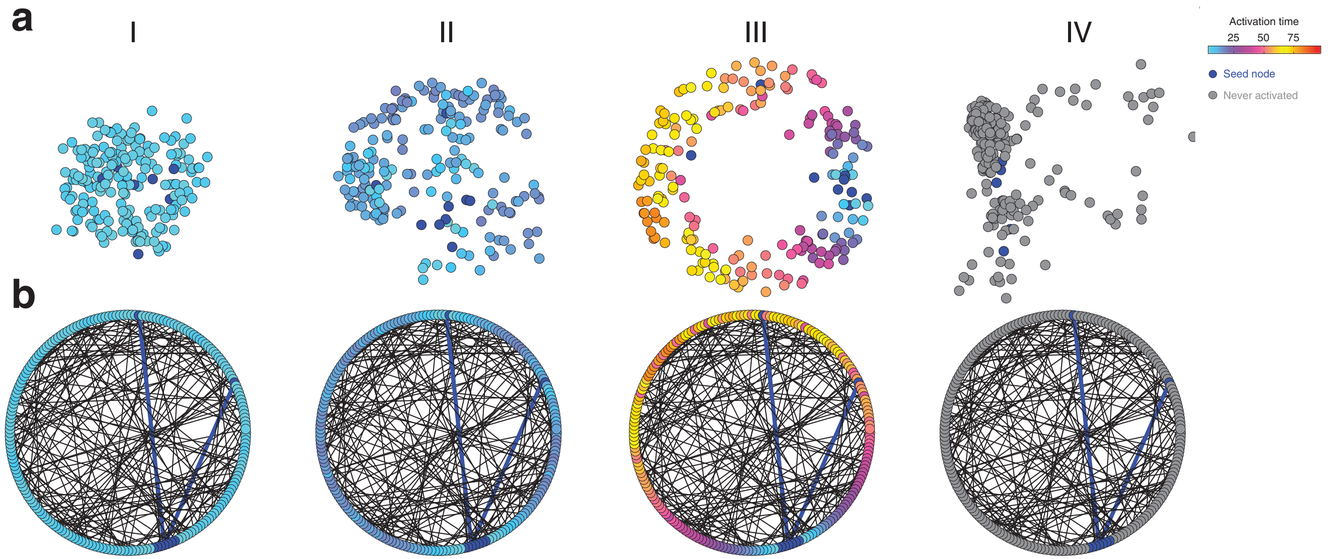


NETSCI MAPPER

Temporal/dynamical Embeddings

Does the shape of activity match the structure?

LANGUAGE



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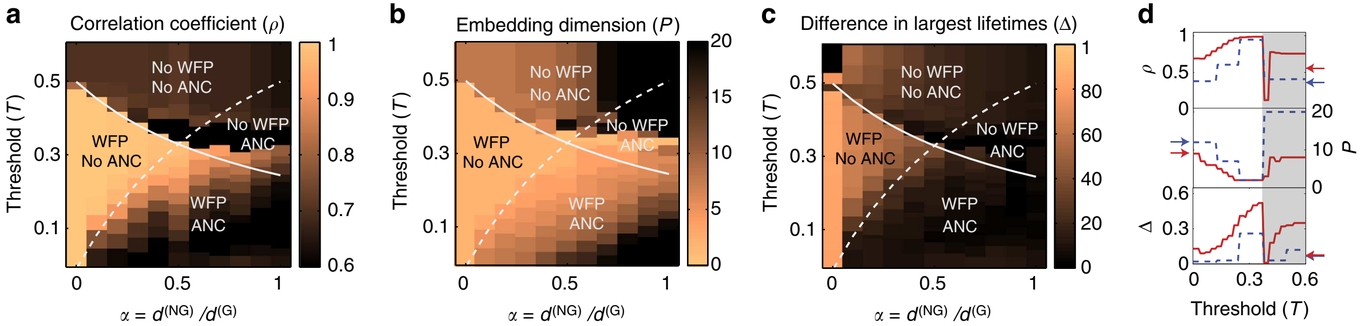

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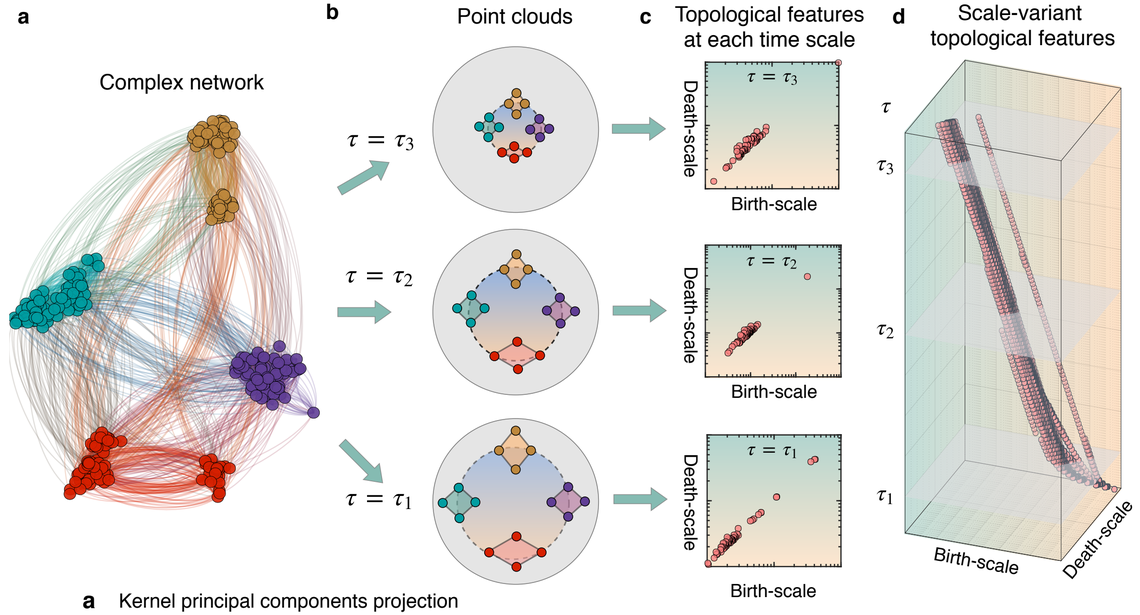
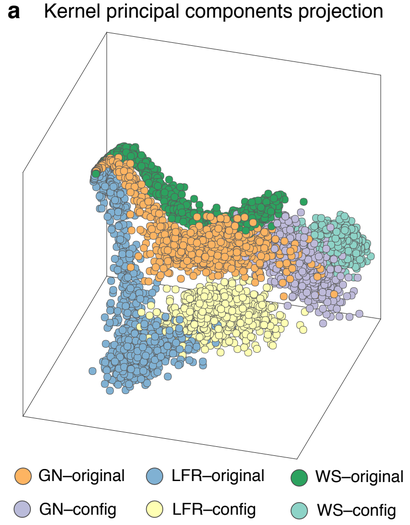
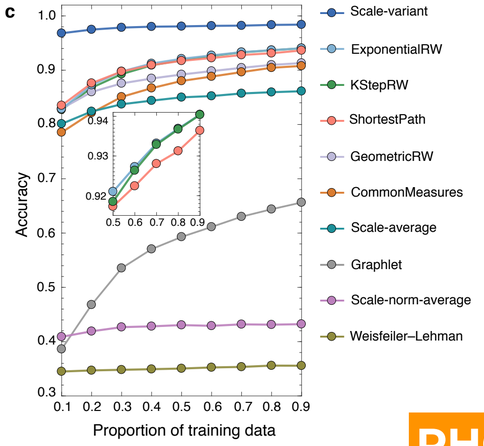


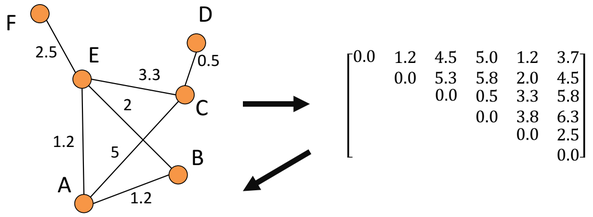




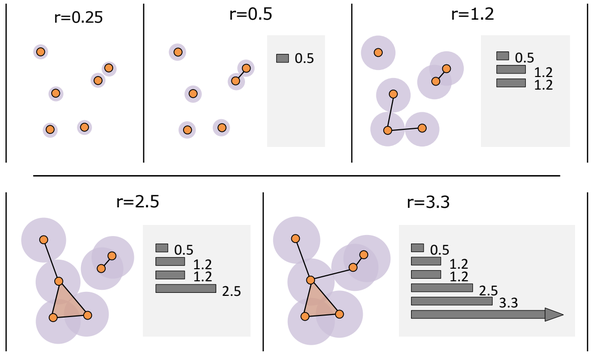
PHOM

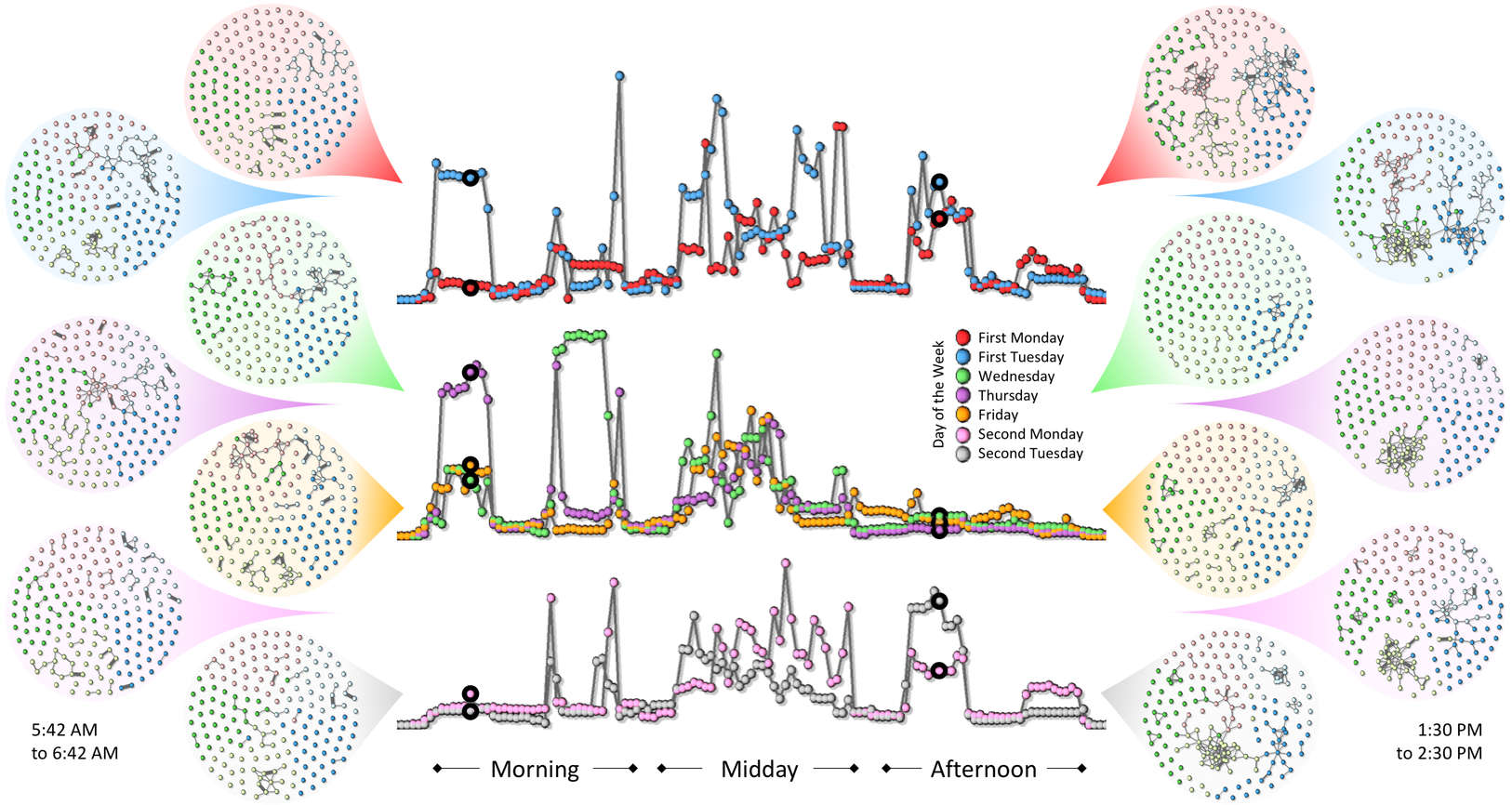
SPACE

Commute time based viz



$$\begin{bmatrix} 0.0 & 1.2 & 4.5 & 5.0 & 1.2 & 3.7 \\ & 0.0 & 5.3 & 5.8 & 2.0 & 4.5 \\ & & 0.0 & 0.5 & 3.3 & 5.8 \\ & & & 0.0 & 3.8 & 6.3 \\ & & & & 0.0 & 2.5 \\ & & & & & 0.0 \end{bmatrix}$$





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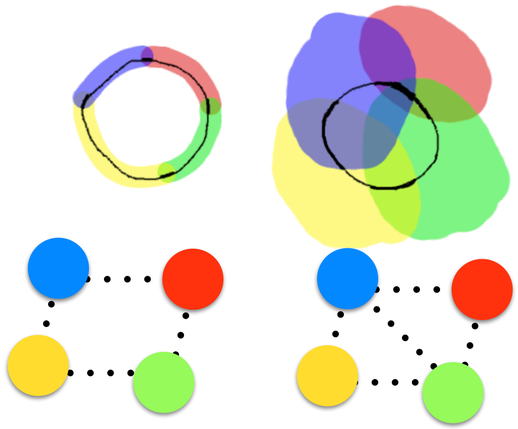
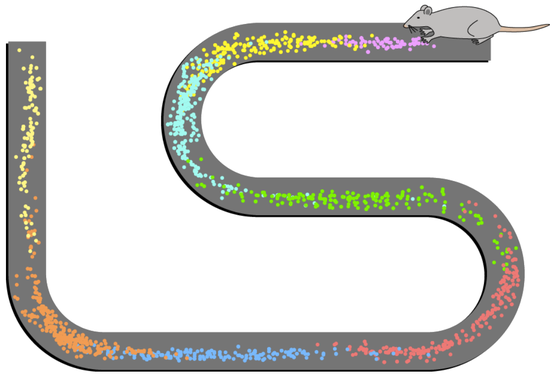
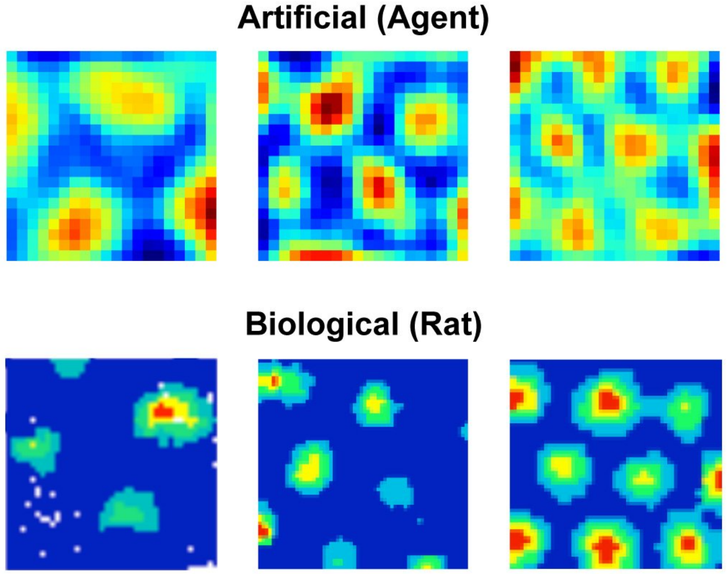
PHOM

SPACE

Is topology really there?







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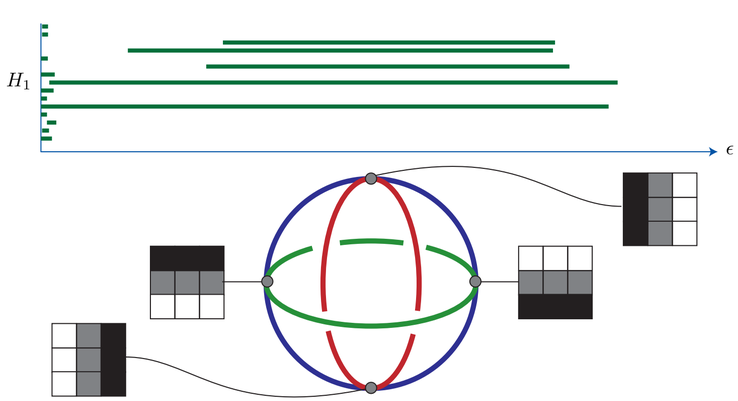

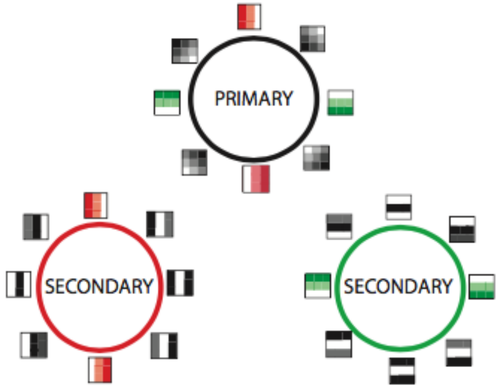

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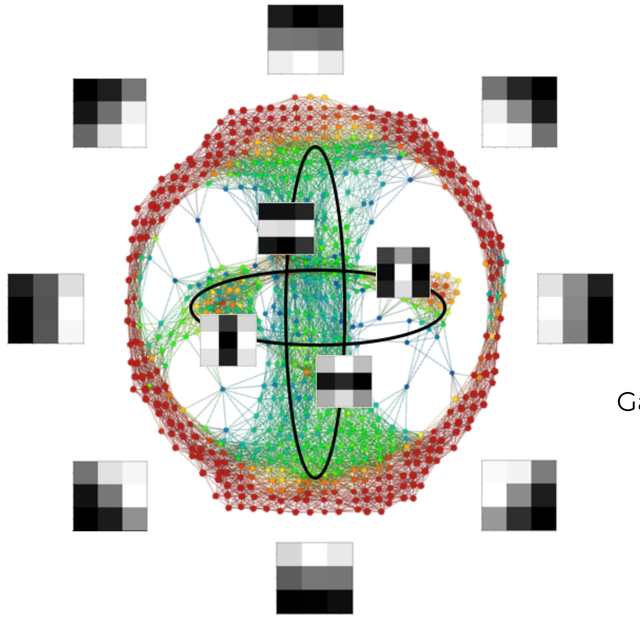

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MAPPER PHOM

What’s next?

A couple of ideas

What’s next? Open problems

Conceptual
Technical

+what higher order interaction would look like?
+ estimation from data of genuine higher-order effects (Multivariate estim, etc..)

+ can we localise shapes or should we give up and think about manifolds?
+ localization of homological/homotopical features in "standard/acceptable" way
+ Homology is a quotient, gauge anyone?

+ what does it mean for a system to be “many-body”?
+ formal extension of dynamical systems to higher-order interactions
+ What is a 3-body Kuramoto model? Environment effects in diseases?

+What are the “TRUE” interactions? How (well) can we reconstruct them?
+ decomposition/reconstruction of simplicial complex from projected graphs

+ How does non-trivial topology emerge?
+ Generative models for target homology

+What is topologically rich structure?
+ Random constrained null models

We need new **datasets, experiments, methods**
to **progress together** in order to give a conclusive answer

Thank you.

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- Song et al. (PlosONE 2011): unsupervised clustering method based on (*g=0)*-filtering;
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