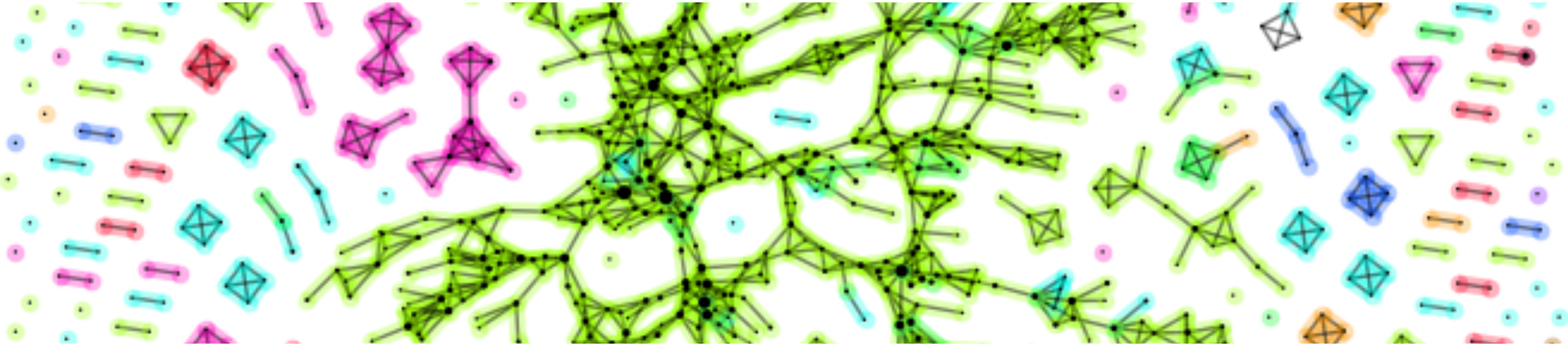
TDA Lecture #2

Dynamics

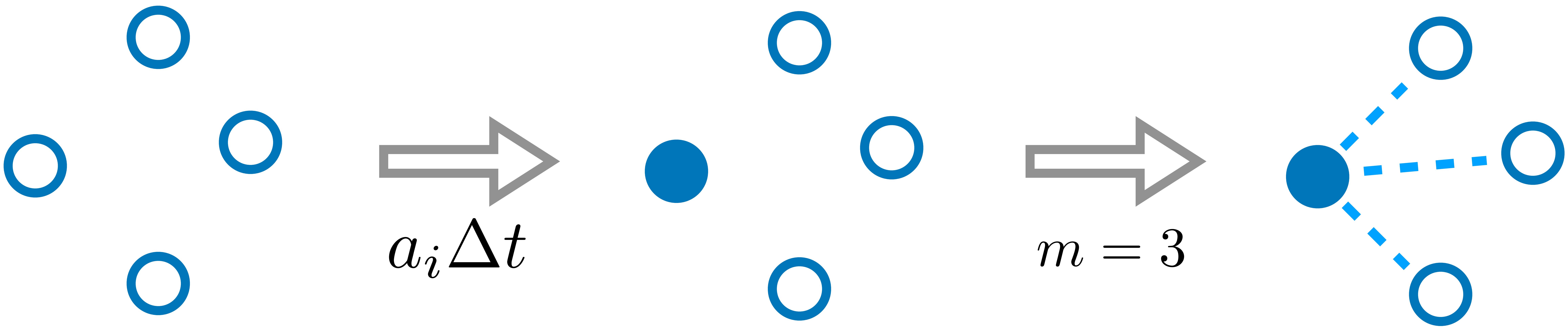


G. Petri/P. Expert

ISI Foundation
& ISI Global Science
Foundation

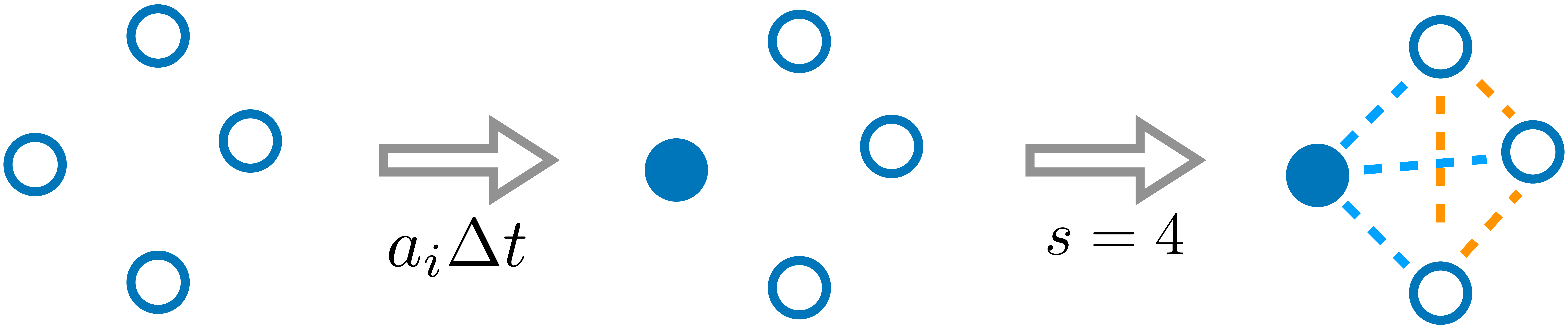
CNTDA2019 , UKM, Kuala Lumpur
Sept 2019

Activity Driven model



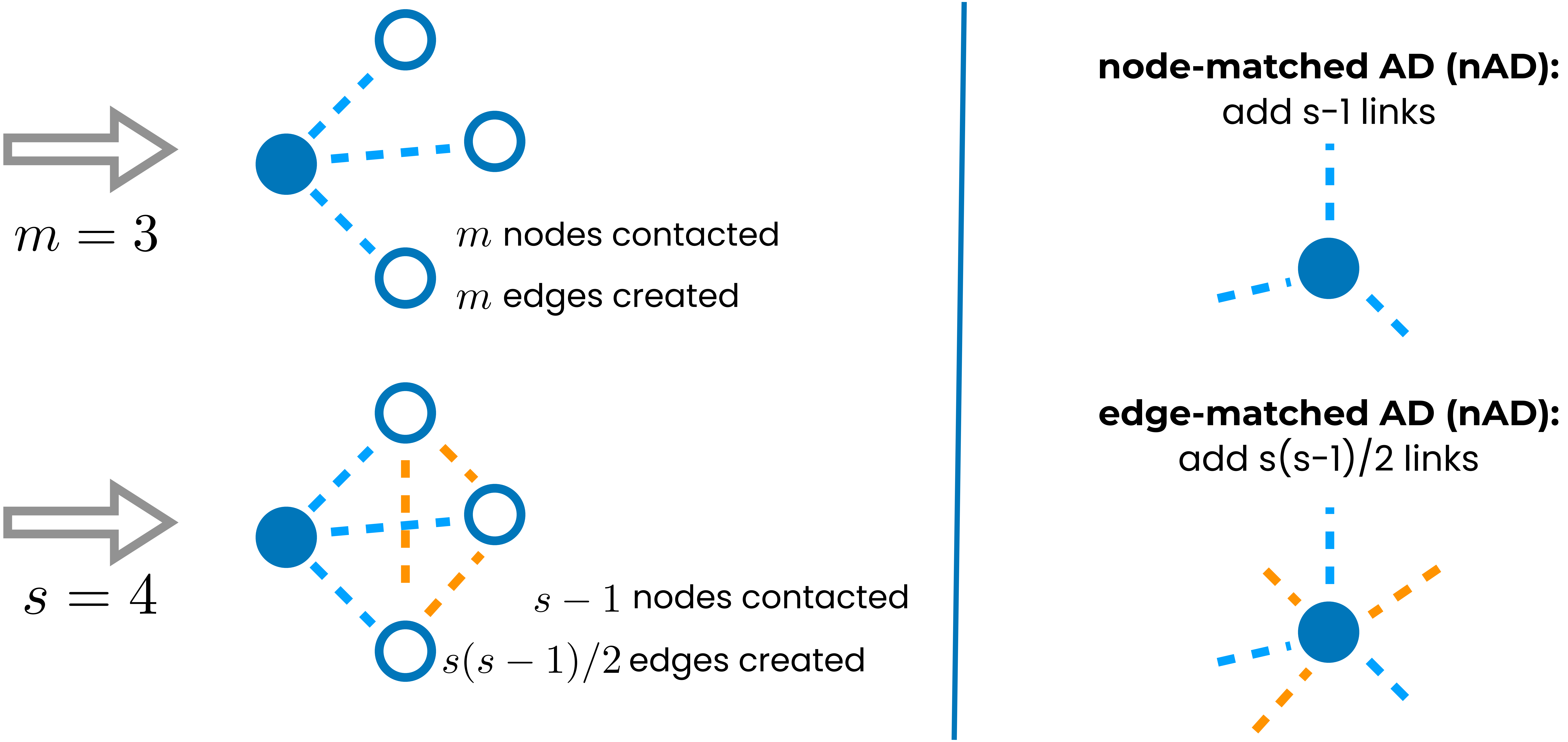
- Random walks: Perra et al (PRL 2012)
- memory: Vestergaard, Génois, Barrat (PRE 2014)
- bursts+aging: Moinet, Starnini, Pastor-Satorras (PRL 2015)
- control: Liu et al (PRL 2014), Pan and Li (PLoS One 2014)
- communities: Salathe and Jones (PLoS Comput Biol, 2010), Huang and Li (JSTAT, 2007) , Stagehuish et al (Sci Rep, 2016)

Simplicial **A**ctivity **D**riven model



NB: different from communities!

AD models for comparison



Structural properties

aggregated degree

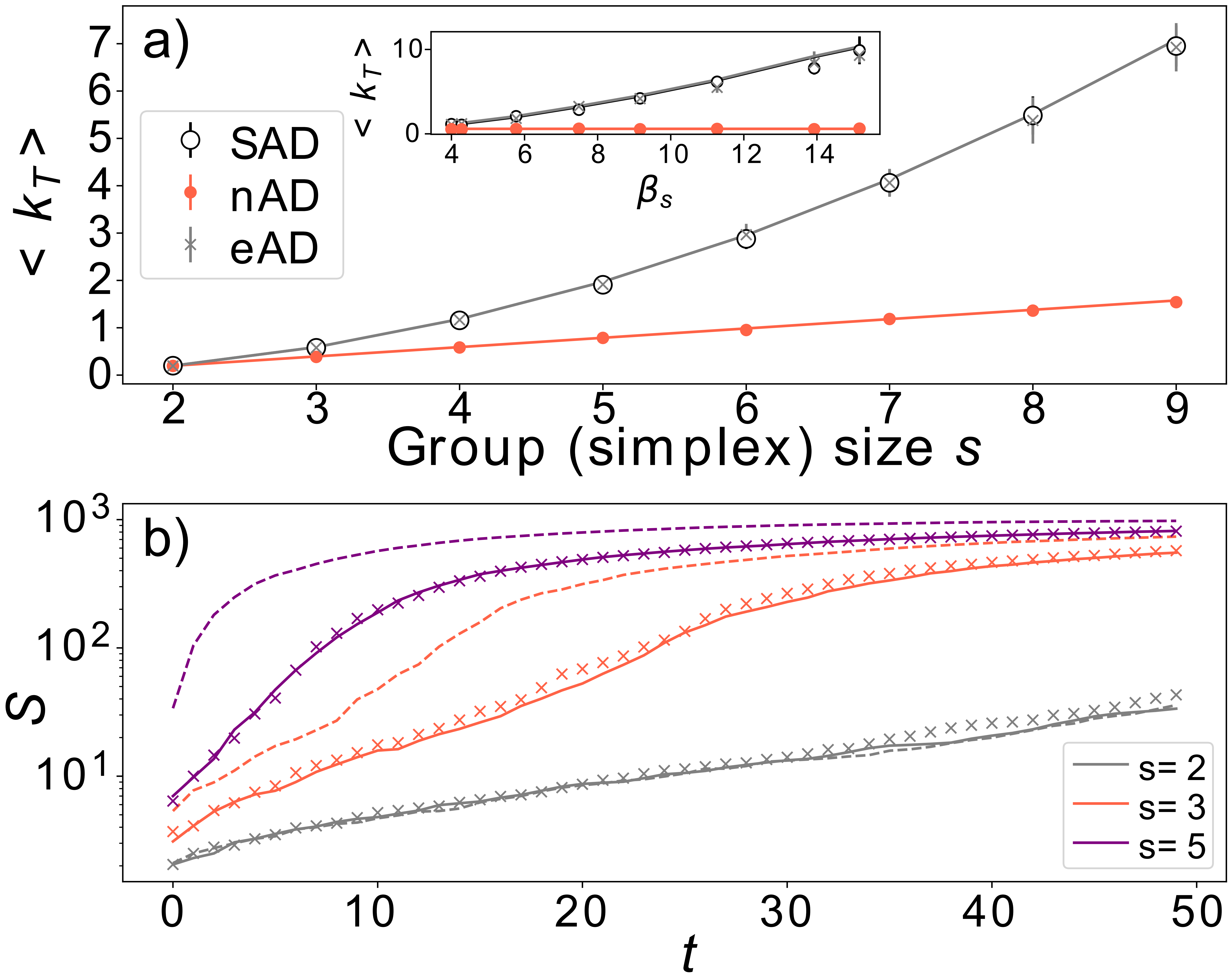
$$k_T^{AD}(i) = N(1 - e^{-Tm(a_i + \langle a \rangle)/N})$$

$$k_T^{SAD}(i) \simeq N\left[1 - e^{-\frac{T\bar{m}(a_i + <a>\bar{m})}{N}}\right]$$

$$\bar{m} = s - 1$$

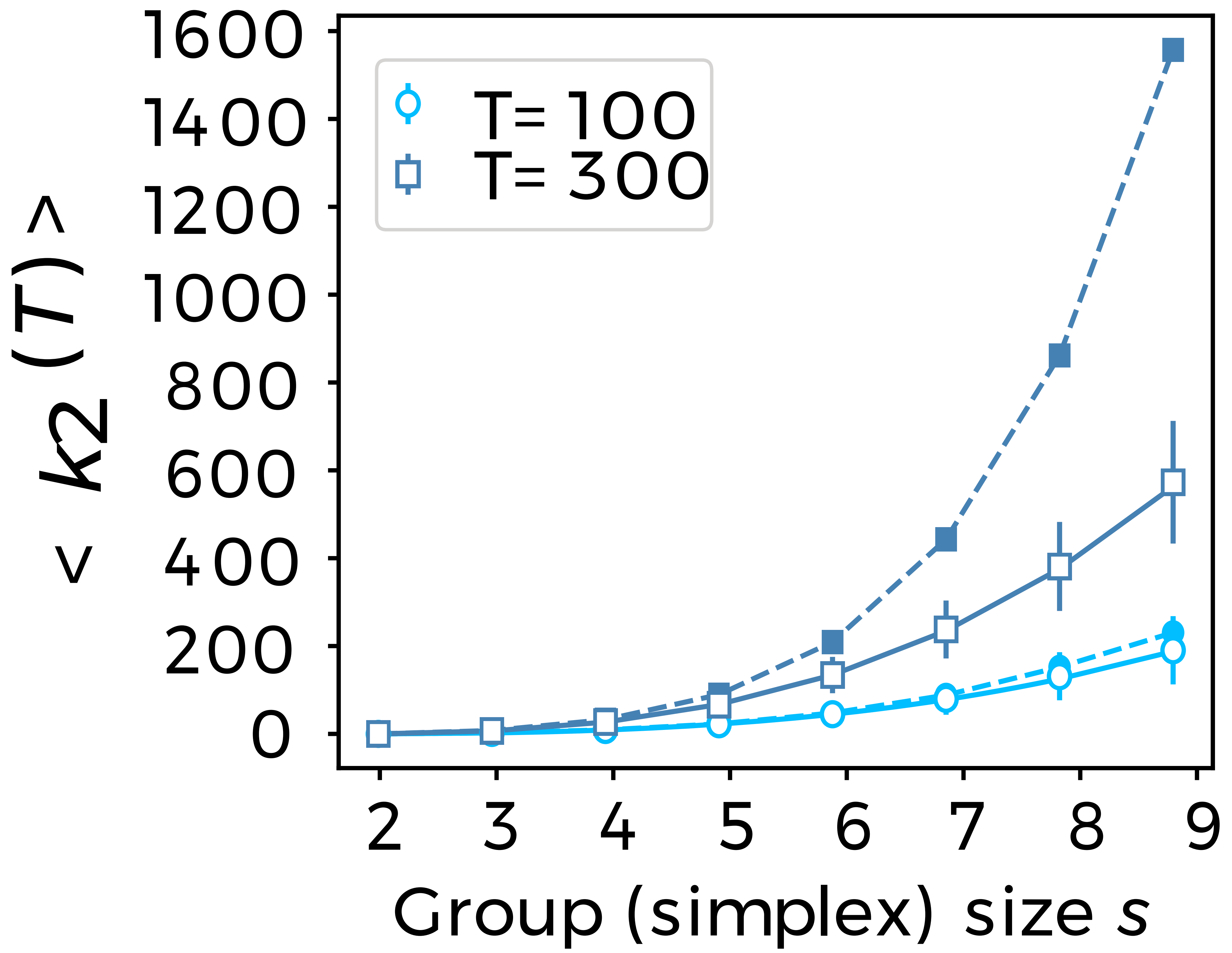
aggregated degree distribution

$$P_T(k) \sim \frac{1}{\bar{m}T\eta} \frac{1}{1 - \frac{k}{N}} F\left[\frac{k}{\bar{m}T\eta} - \frac{\bar{m}\langle a \rangle}{\eta}\right]$$



Structural properties

$$k_2(i,T) = \binom{N-1}{2}\left(1 - e^{-\{[(s-1)(s-2)]/[(N-1)(N-2)]\}T(a_i + \bar{m}\langle a\rangle)}\right).$$



Structural properties

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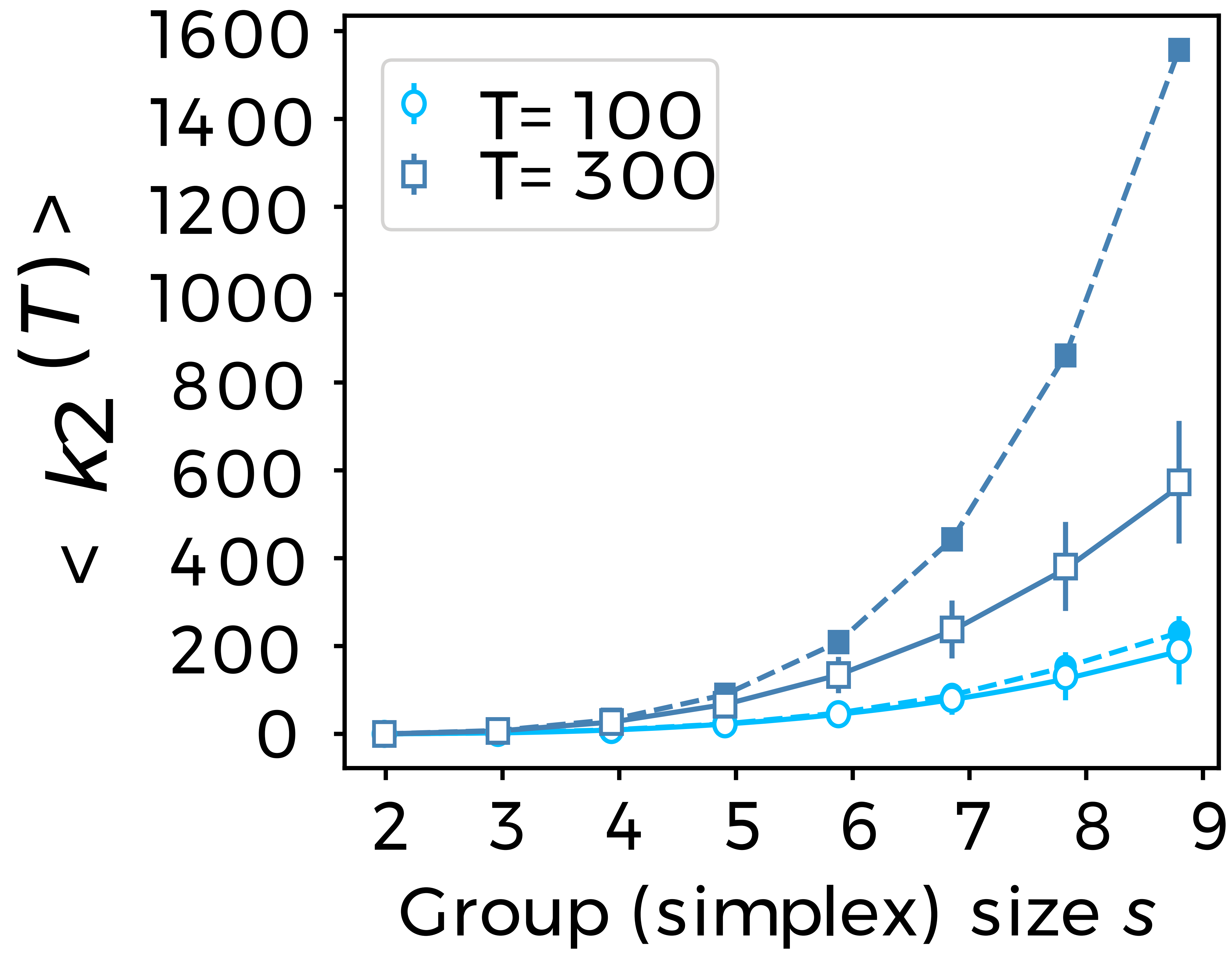
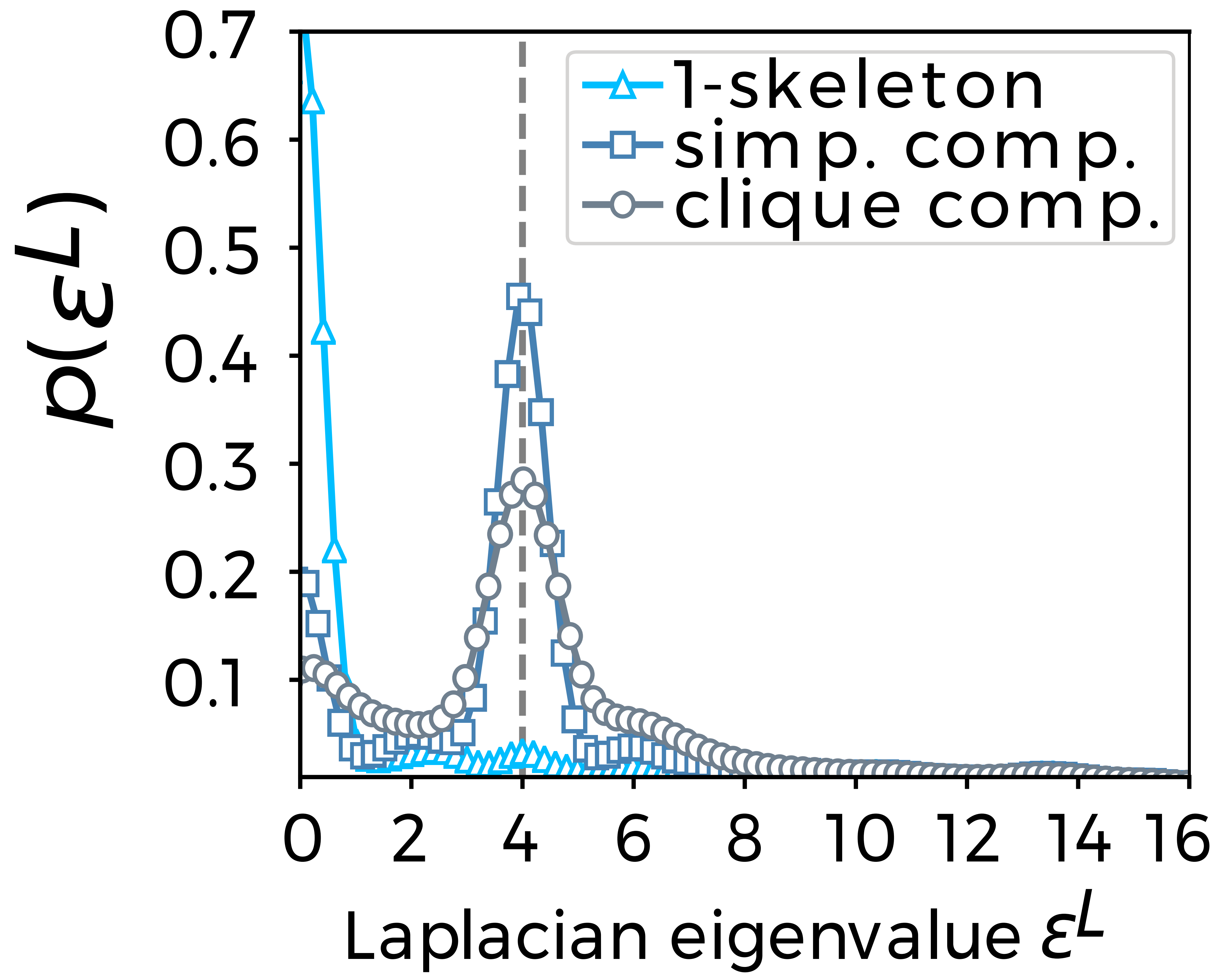
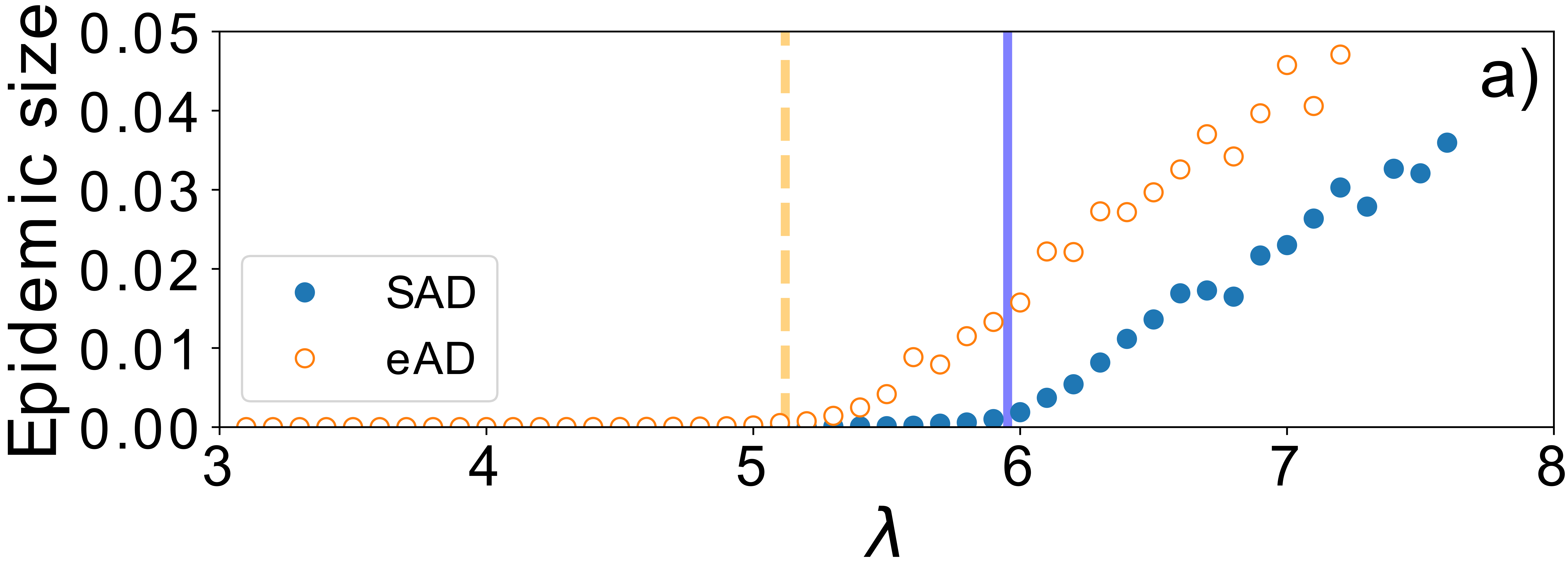




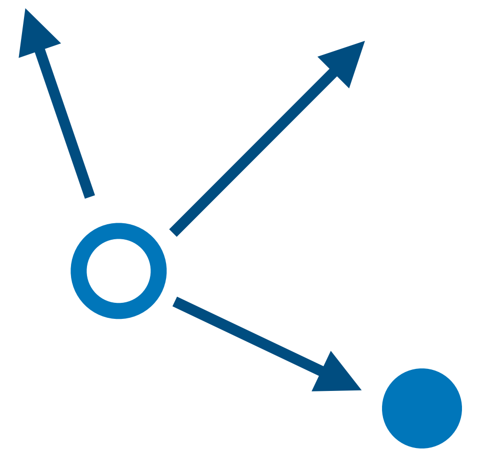
SIS epidemic threshold



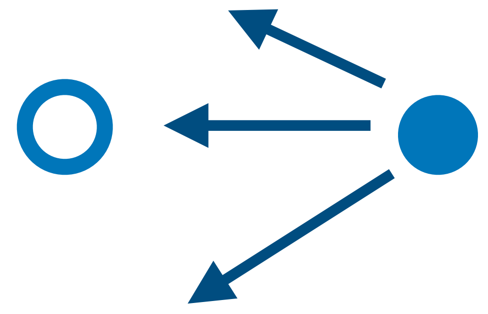
SIS epidemic threshold

$$I_a^{t+\Delta t} - I_a^t = -\mu \Delta t I_a^t + \beta \Delta t S_a a (s-1) \int da' \frac{I_{a'}^t}{N}$$

$$+\beta \Delta t S_a \int da' a' \frac{I_{a'}^t}{N} (s-1)$$

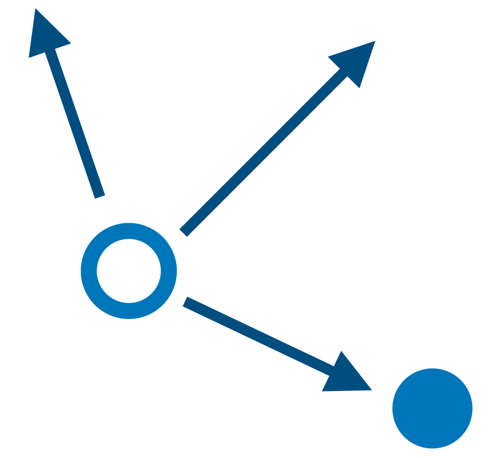


Usual terms

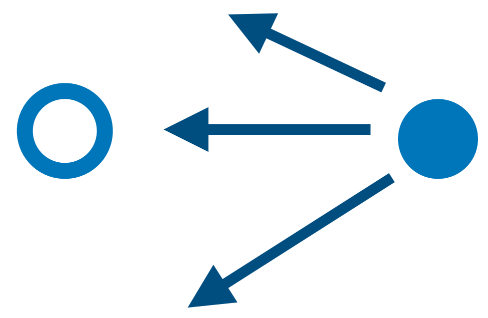


SIS epidemic threshold

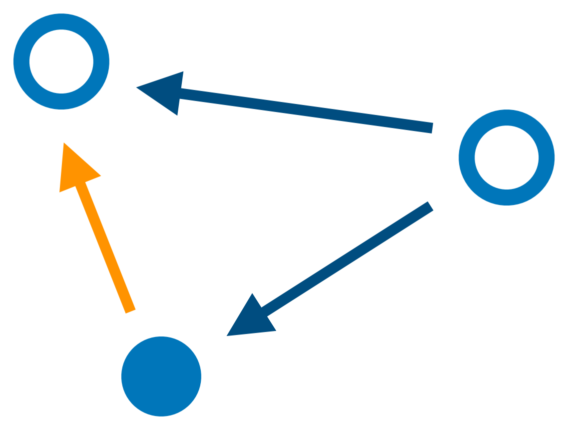
$$
\begin{aligned}
I_a^{t+\Delta t} - I_a^t = -\mu \Delta t I_a^t &+ \beta \Delta t S_a a(s-1) \int da' \frac{I_{a'}^t}{N} \\
&+ \beta \Delta t S_a \int da' a' \frac{I_{a'}^t}{N} (s-1) \\
&+ \beta \Delta t S_a \int da' a' \frac{S_{a'}}{N} (s-1) \int da'' \frac{I_{a''}}{N} (s-2)
\end{aligned}
$$



Usual terms



New term!



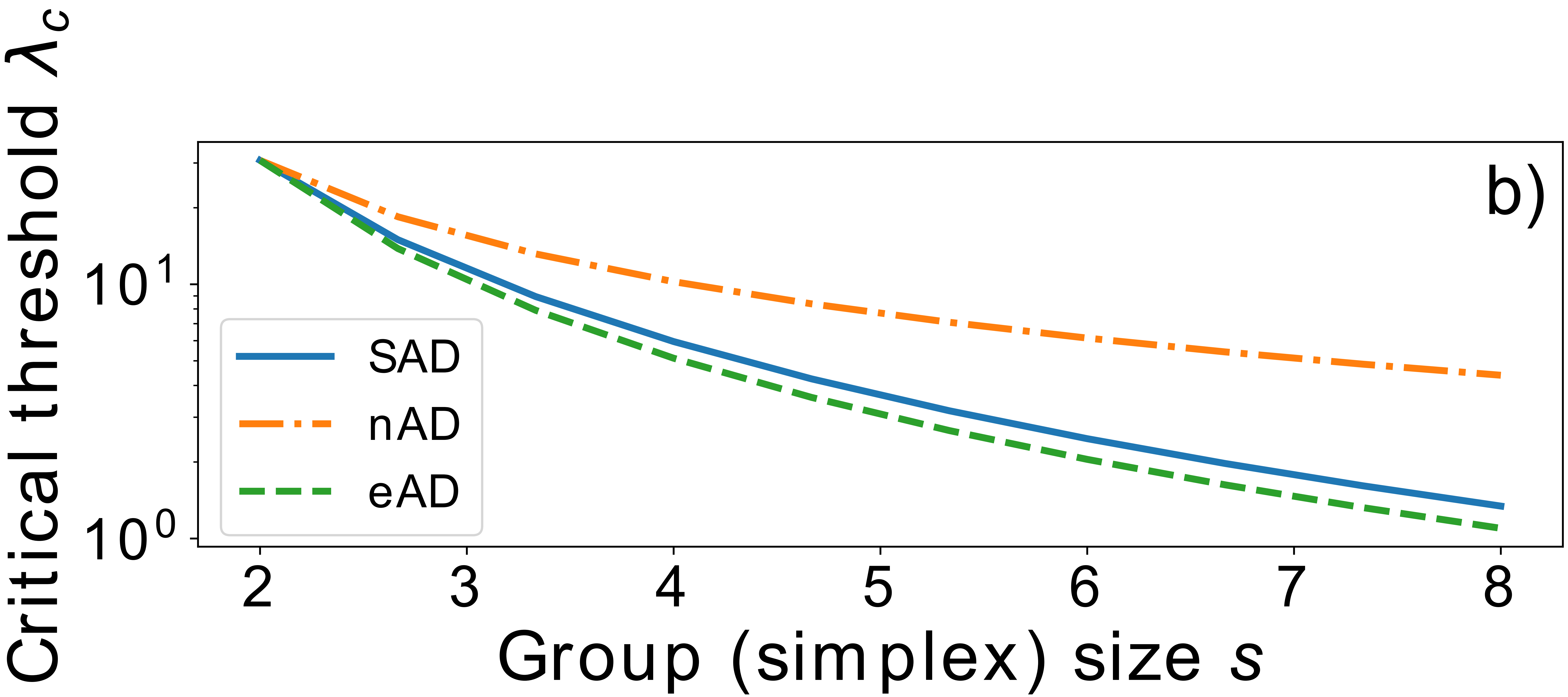
SIS epidemic threshold

regular-s SAD threshold

$$\frac{\beta}{\mu} > \frac{2}{s(s-1)\langle a\rangle + (s-1)\sqrt{s^2\langle a\rangle^2 + 4(\langle a^2\rangle - \langle a\rangle^2)}}$$

AD threshold

$$\frac{\beta}{\mu} > \frac{1}{m\langle a\rangle + m\sqrt{\langle a^2\rangle}}$$

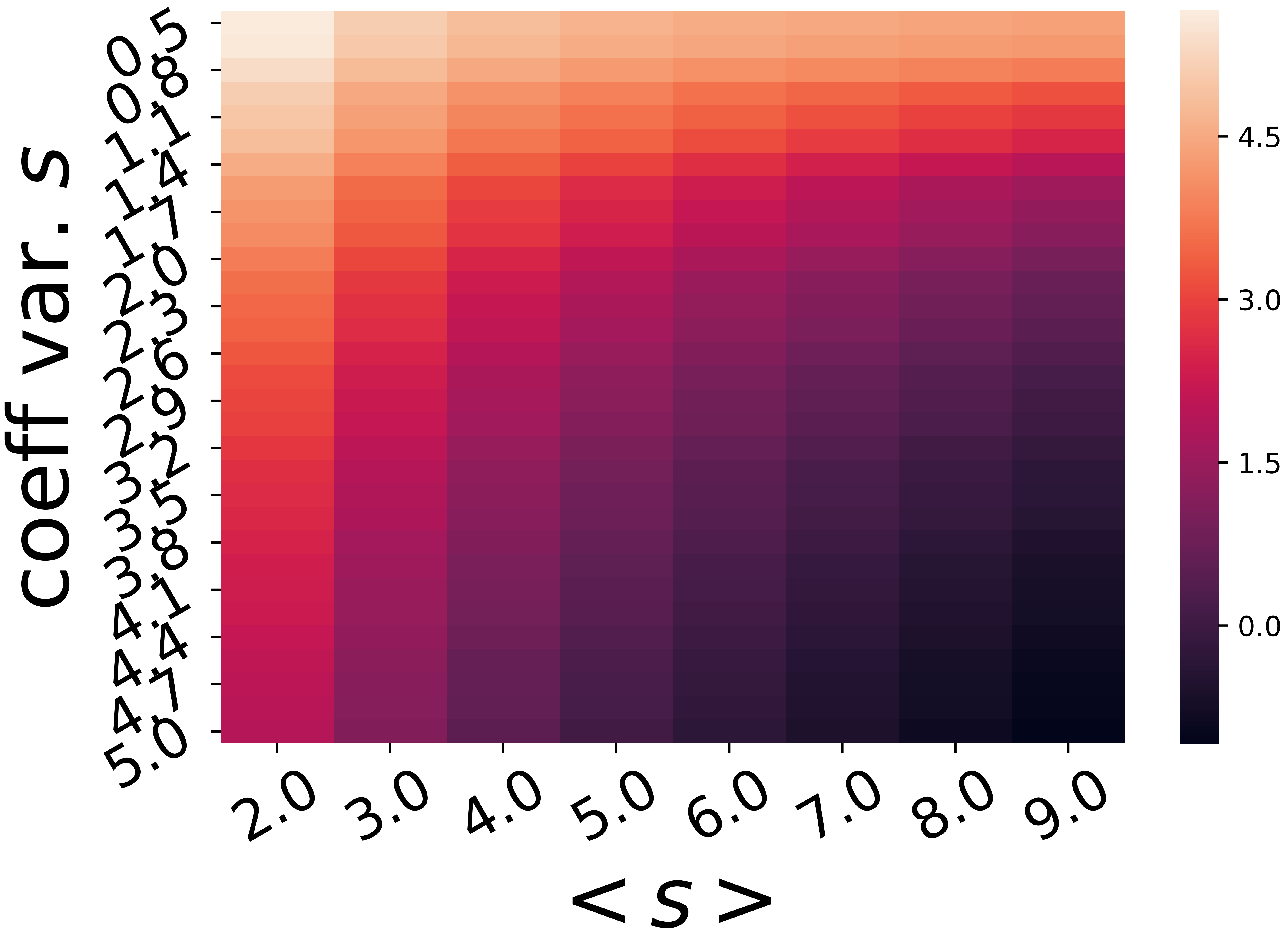


SIS epidemic threshold

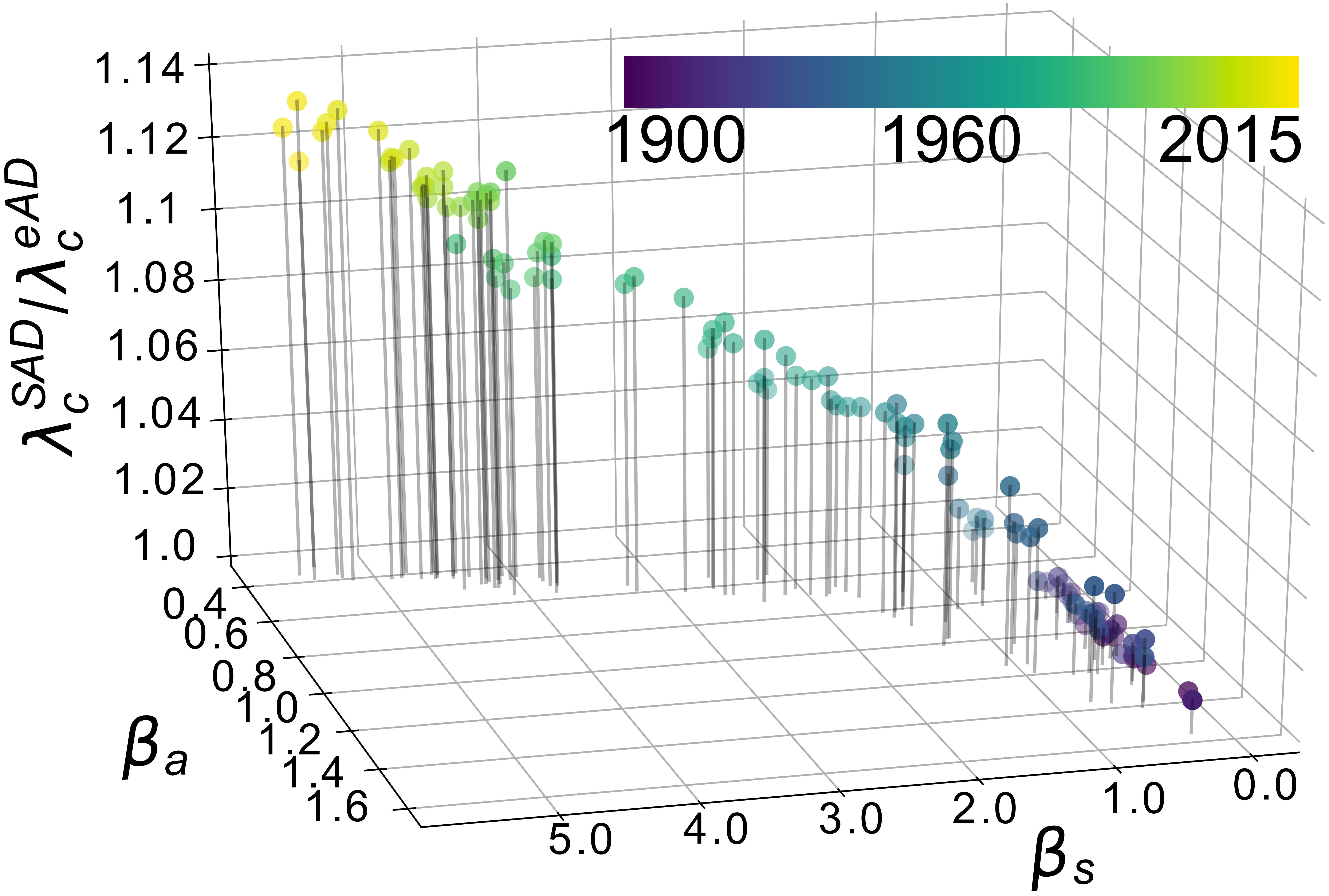
variable-s SAD threshold

$$\frac{\beta}{\mu} > \frac{2}{\langle s(s-1)\rangle\langle a\rangle + \sqrt{\Delta}}$$

$$\begin{aligned}\Delta = \langle (s-1)(s-2)\rangle\langle (s-1)(s+2)\rangle\langle a\rangle^2 \\ +4\langle s-1\rangle^2\langle a^2\rangle\end{aligned}$$



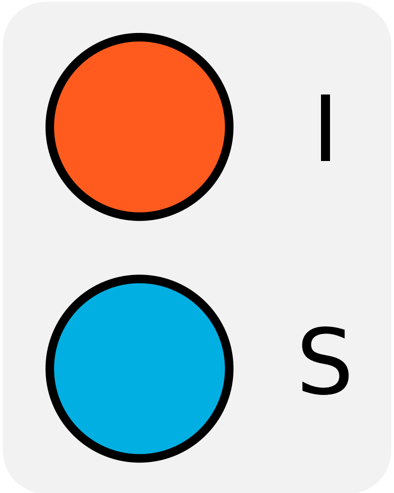
does it matter?

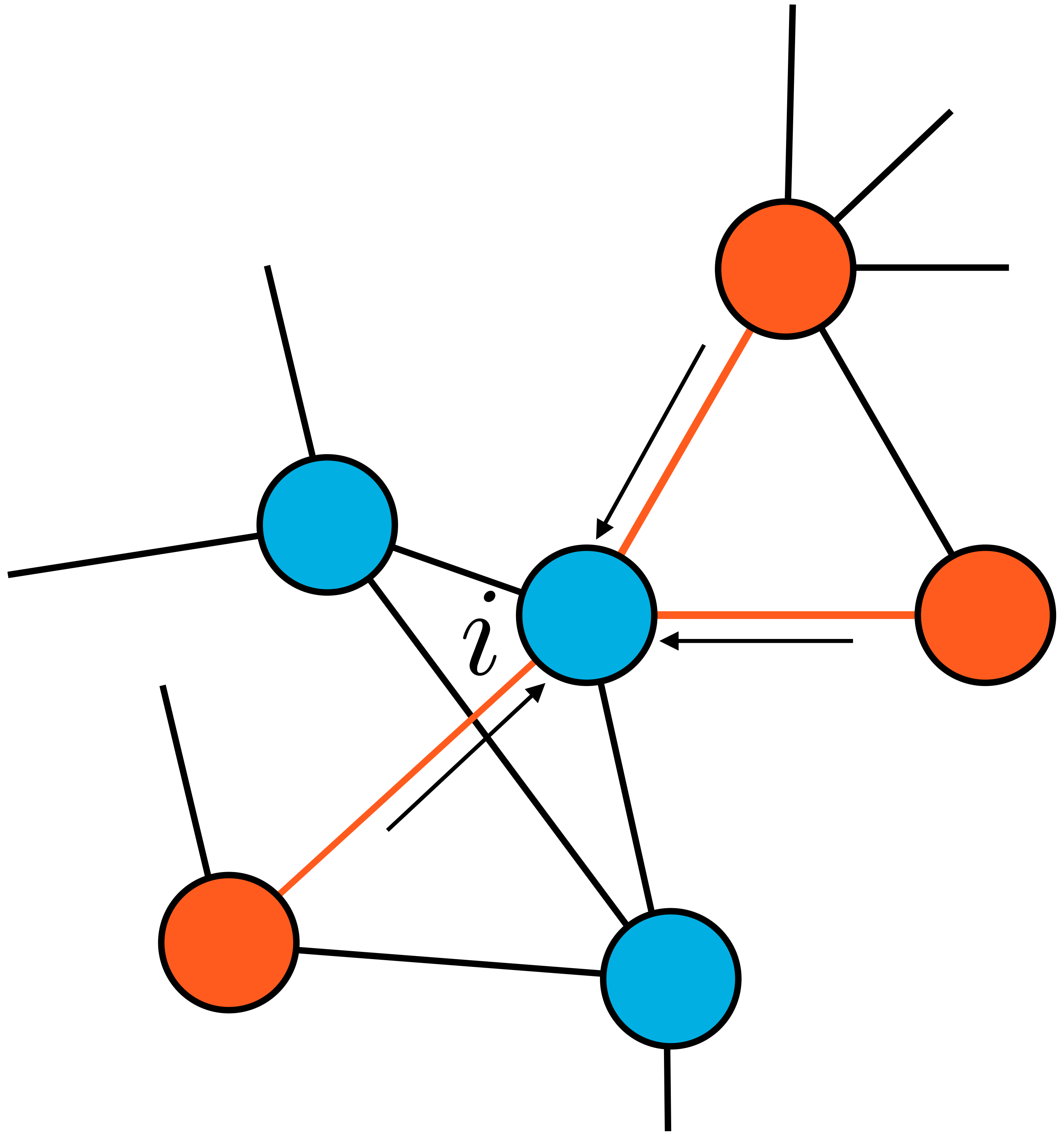


Notebook

Complex Contagion

Social contagion

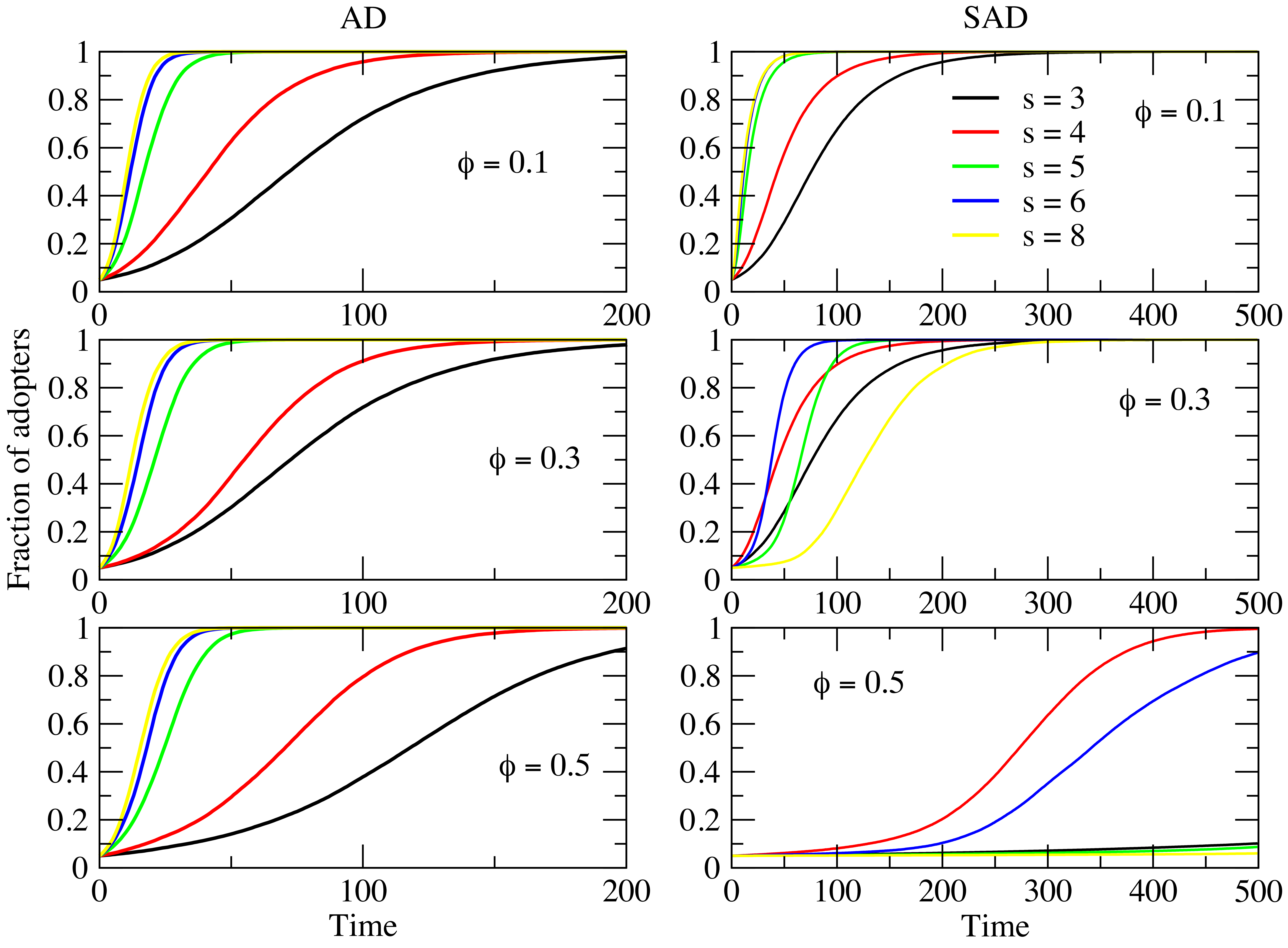




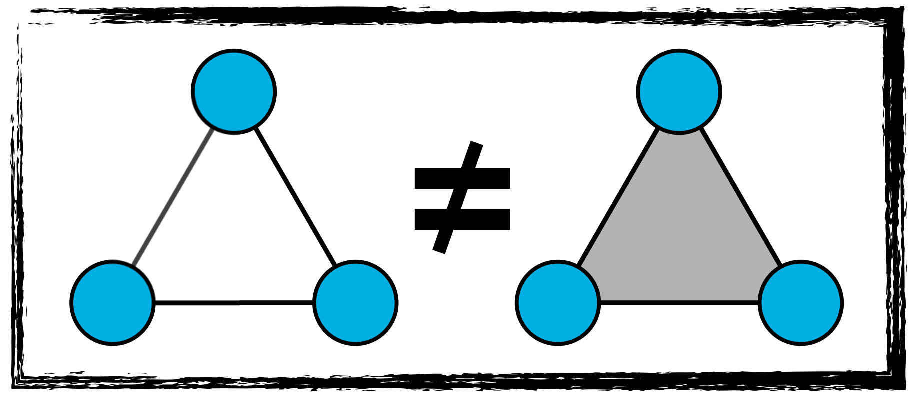
Multiple sources of activation are required for a transmission

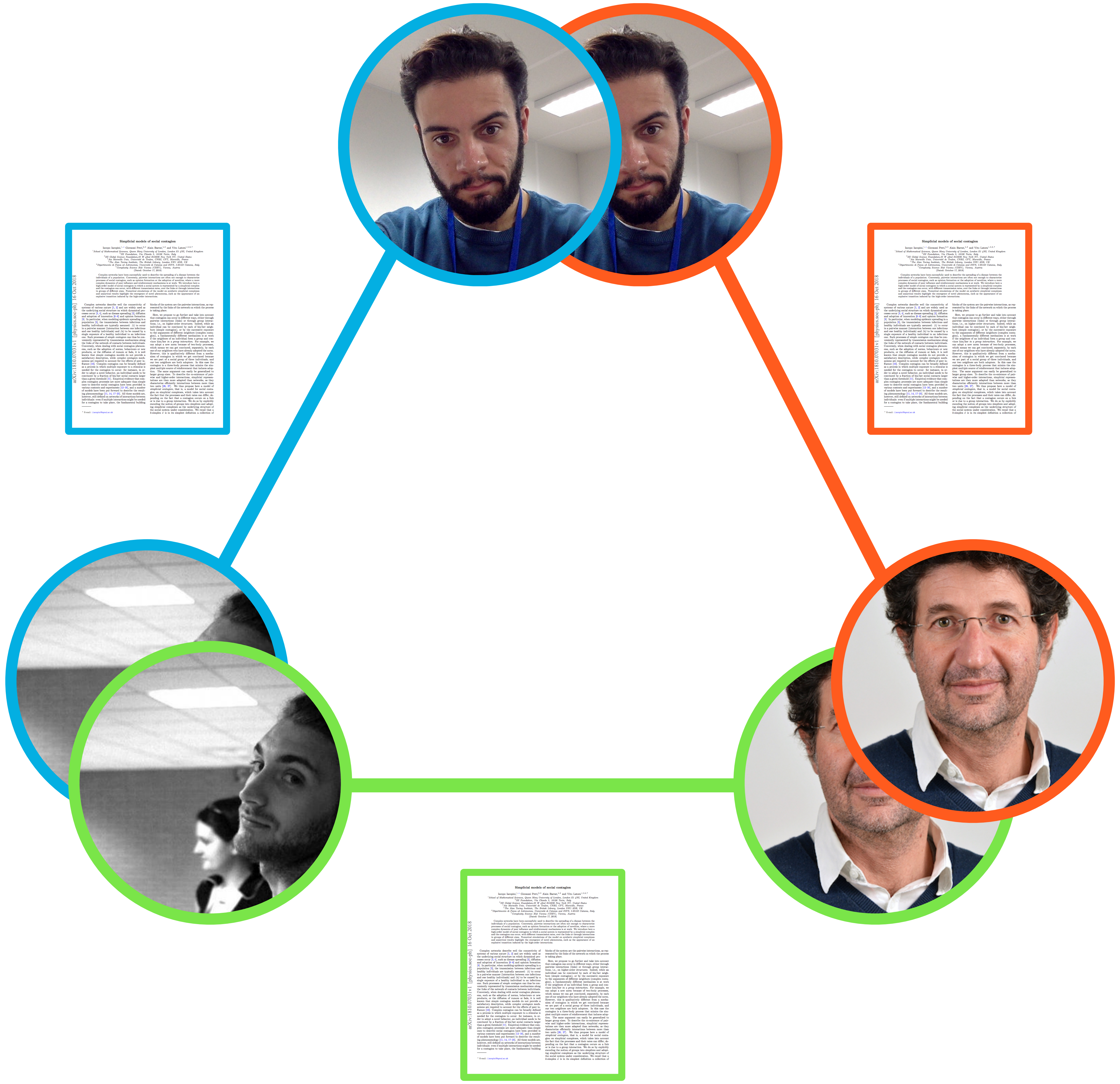
D. Centola & M. Macy, Americ. J. Sociol. (2007); D. Guilbeault et al., Springer (2018)

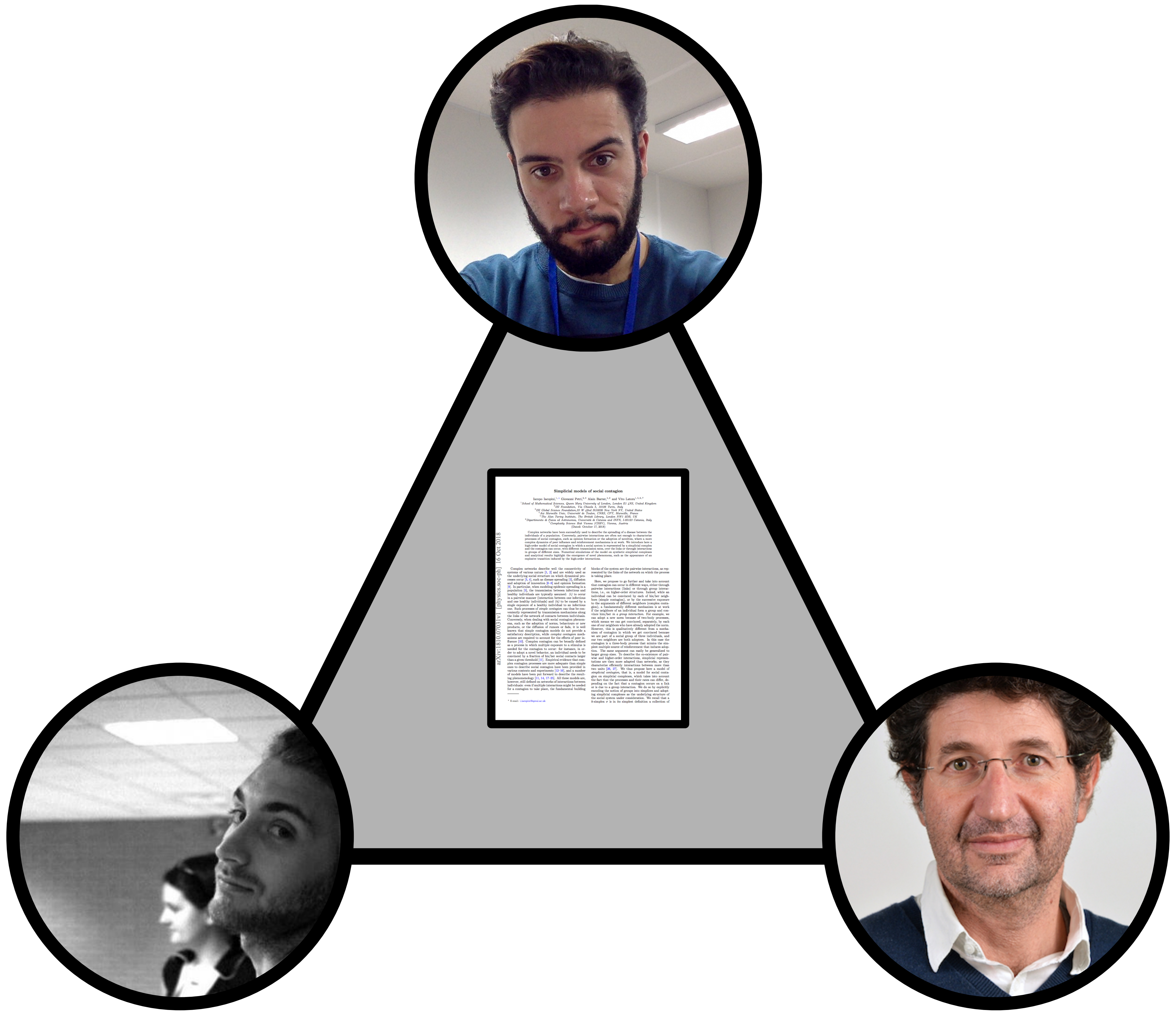
Complex contagion



SIMPLicial ContAGION



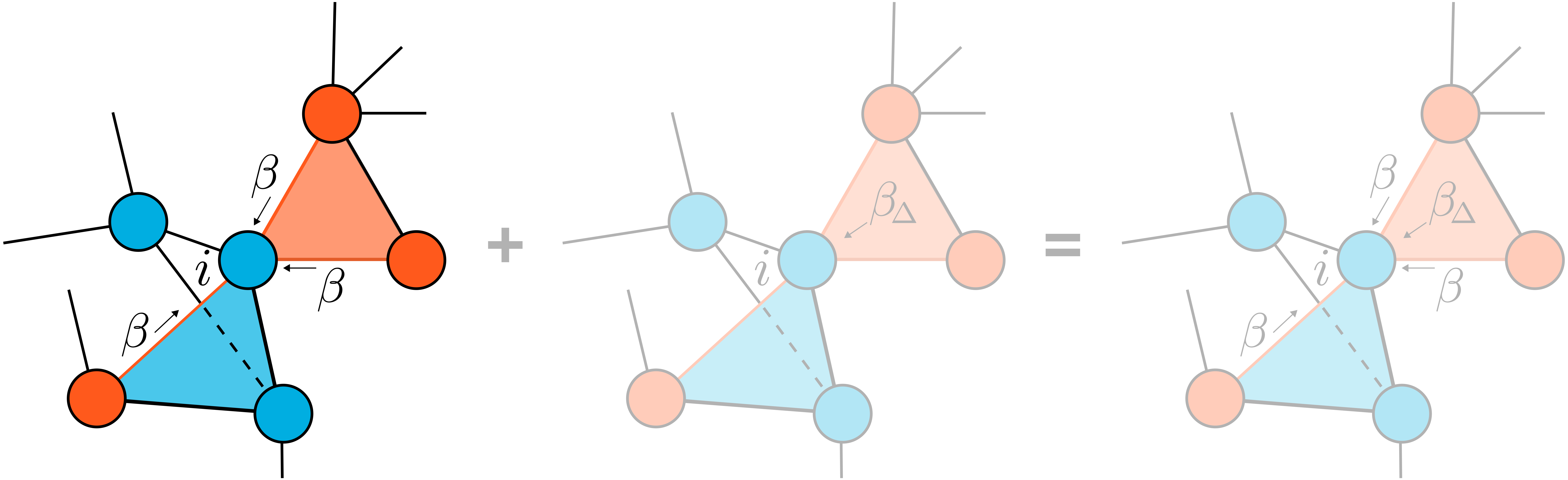




Sorry Alain!

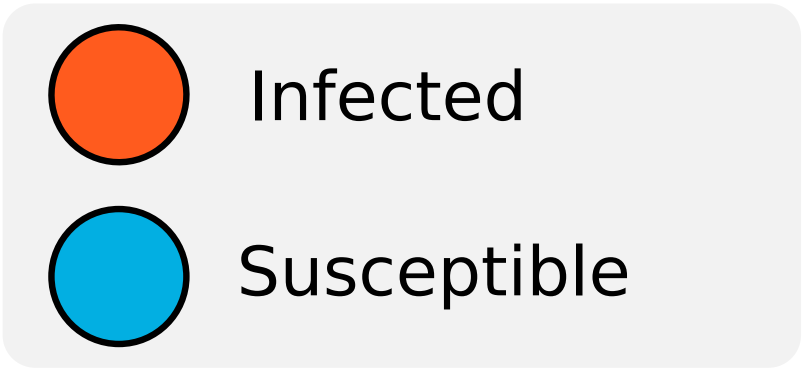
SIMPLicial contAGION

The Model



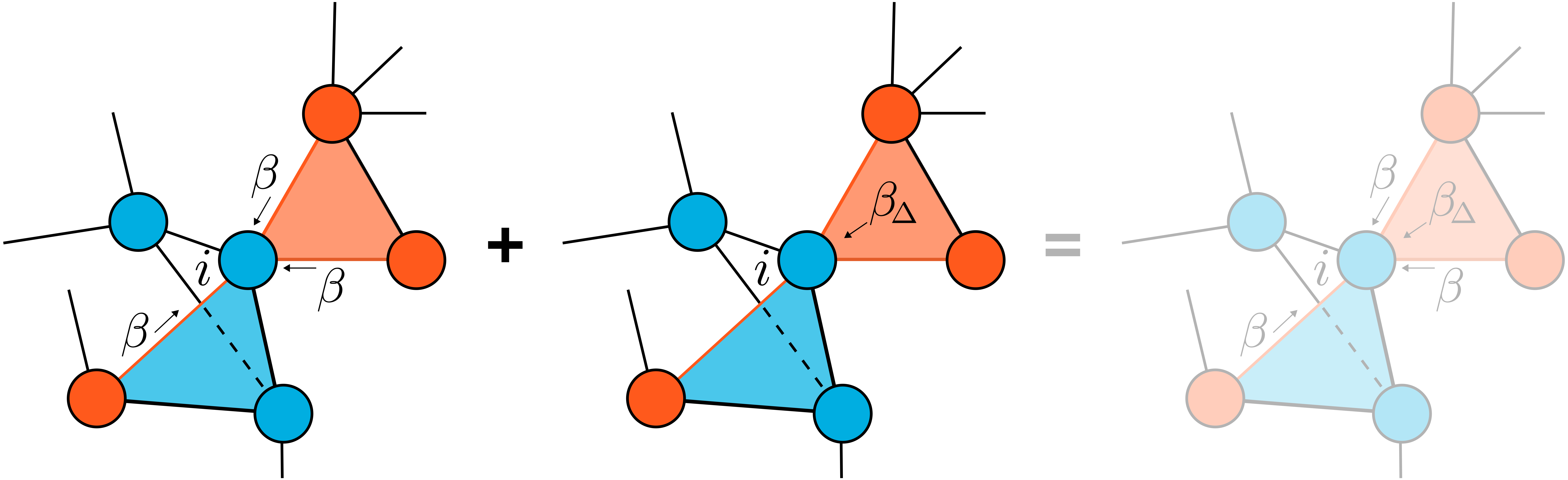
1-simplices (links)

2-simplices (triangles)



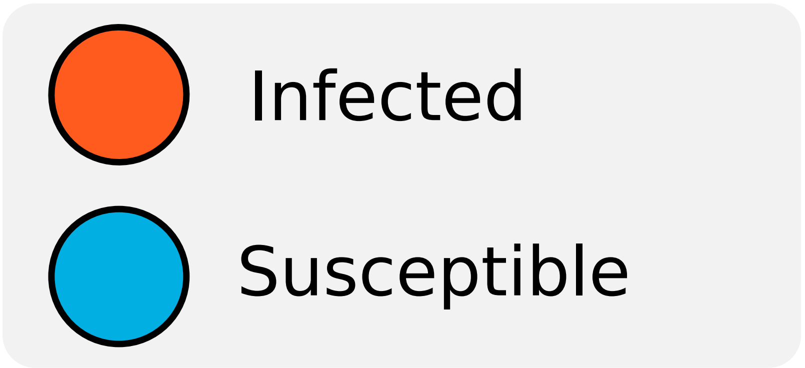
SIMPLicial contAGION

The Model



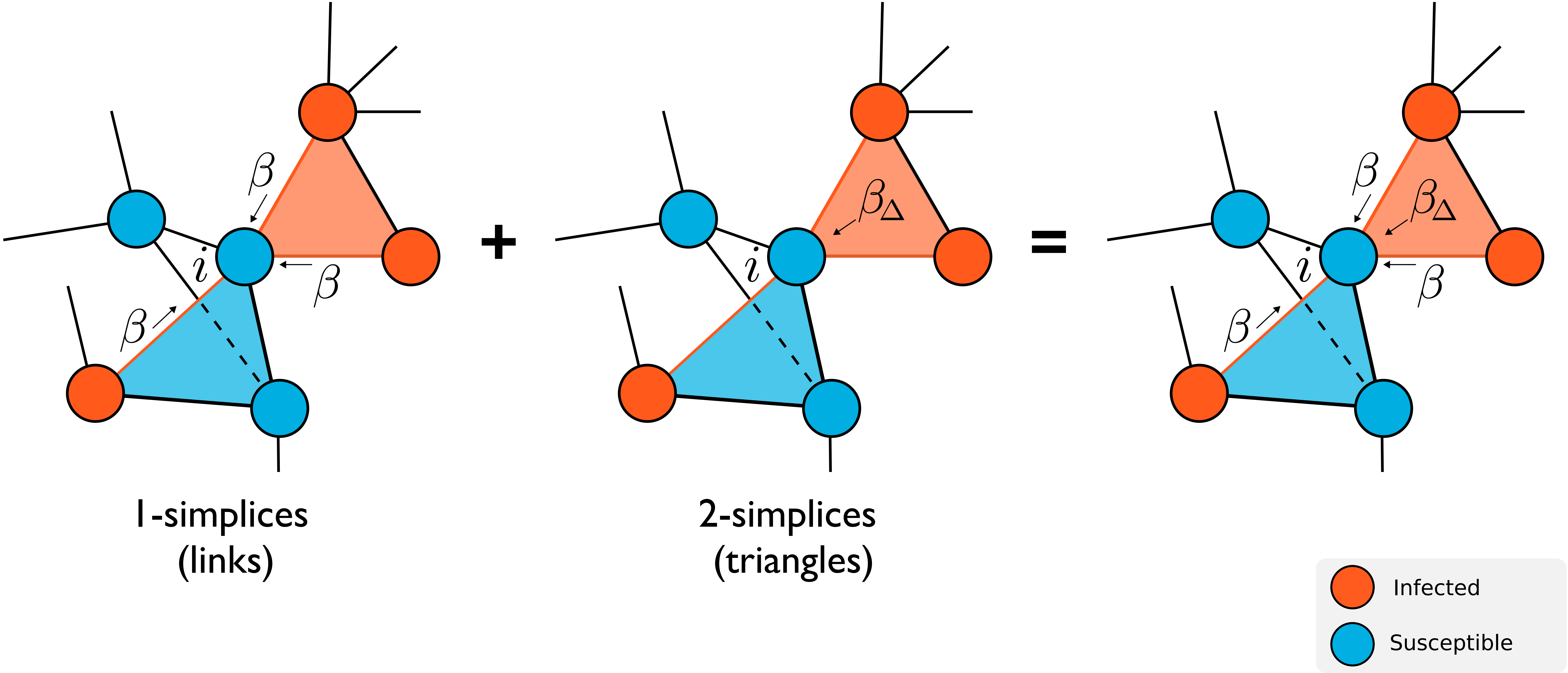
1-simplices
(links)

2-simplices
(triangles)



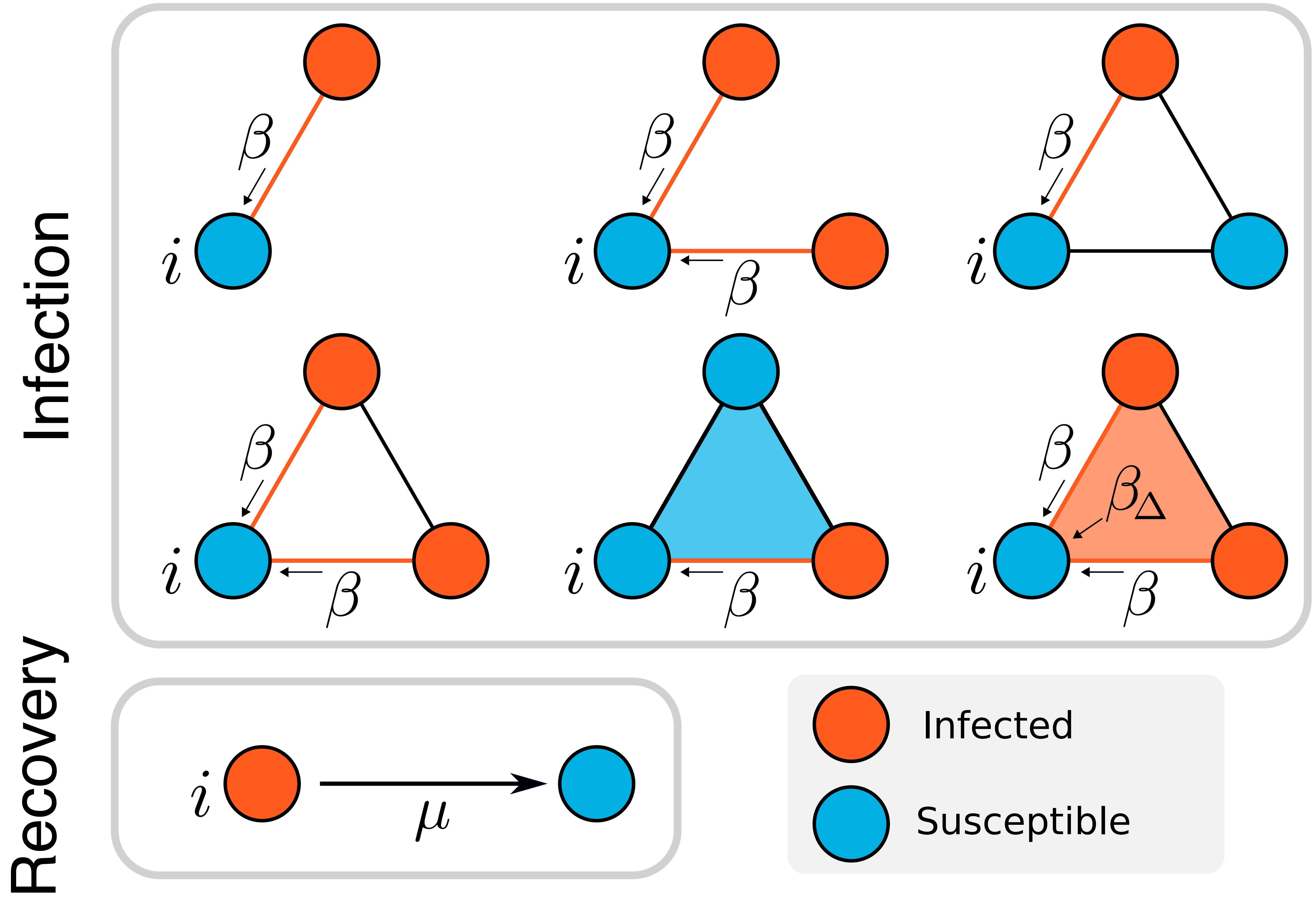
SIMPLicial contAGION

The Model



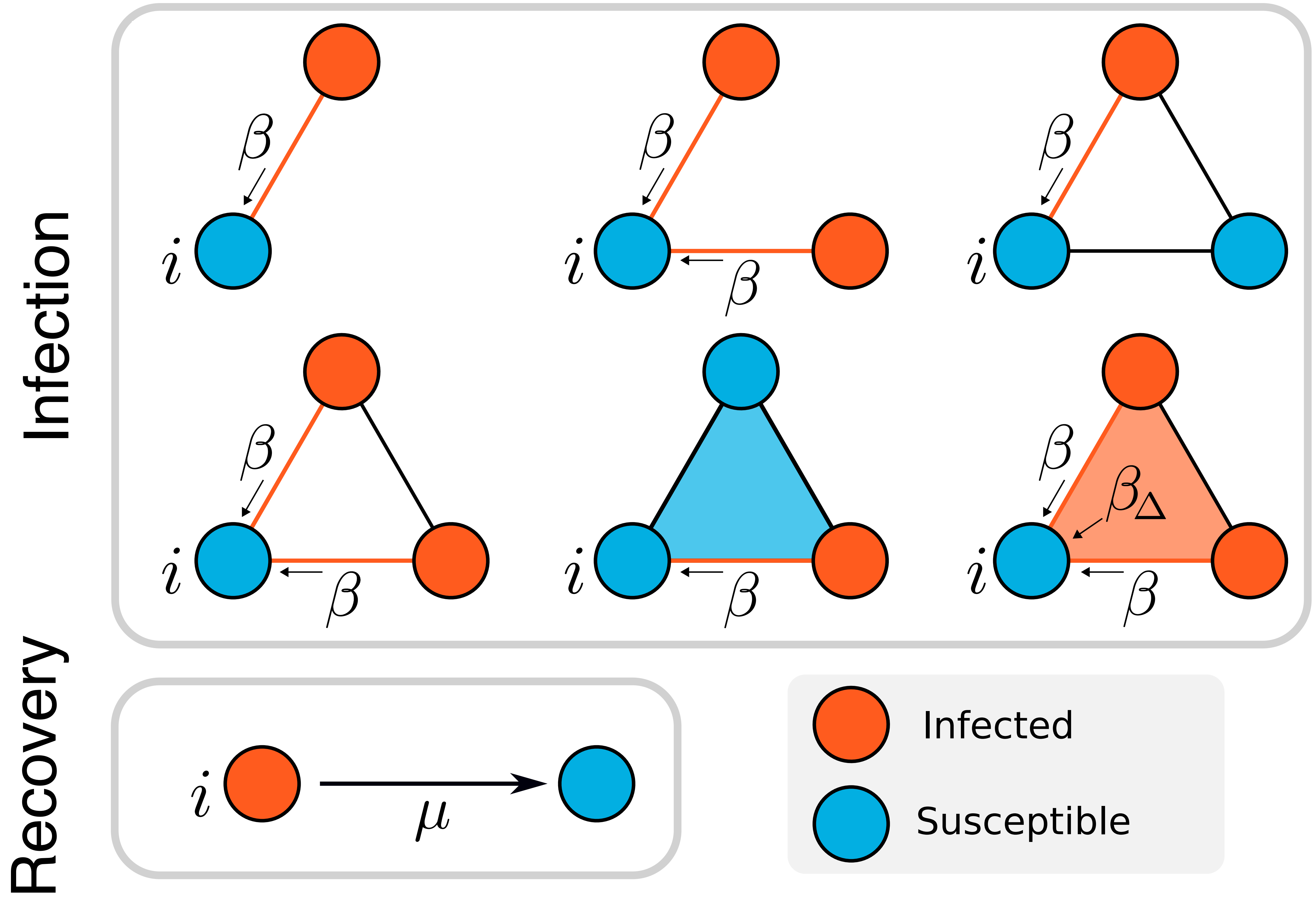
SIMPLicial contAGION

The Model (D=2)



SIMPLicial contAGION

The Model (D=2)



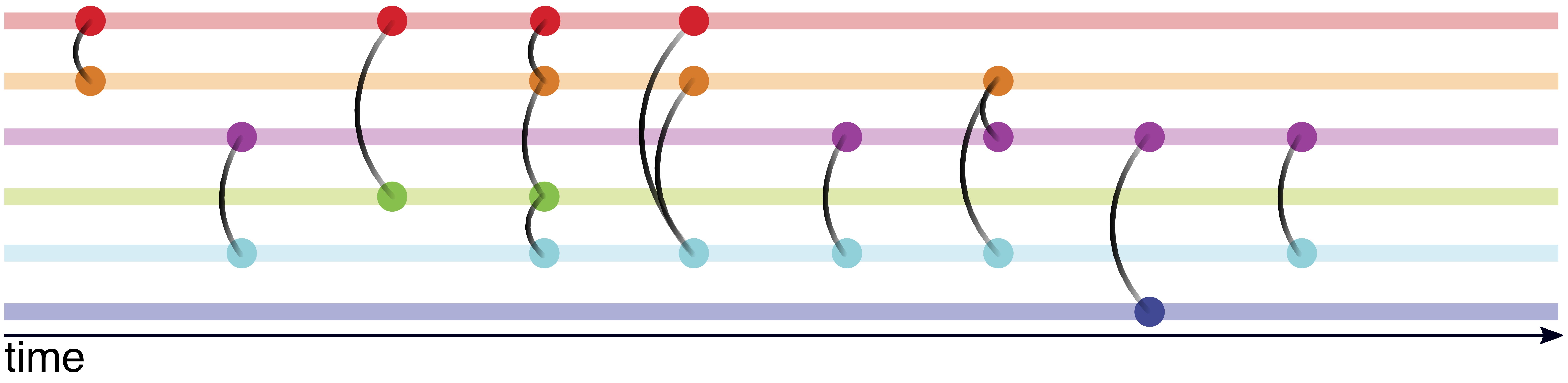
$\lambda = \beta\langle k\rangle/\mu$

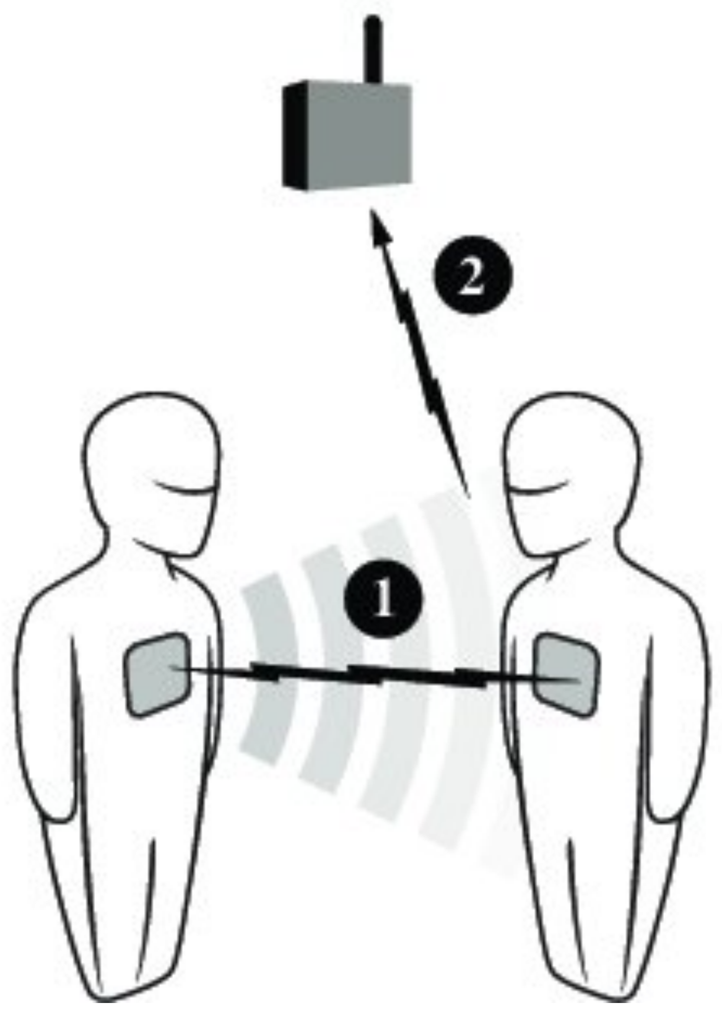
$\lambda_\Delta = \beta_\Delta\langle k_\Delta\rangle/\mu$

O.T. Courtney & G. Bianconi, PRE (2016)

Real world simplicial complexes

High-resolution proximity data

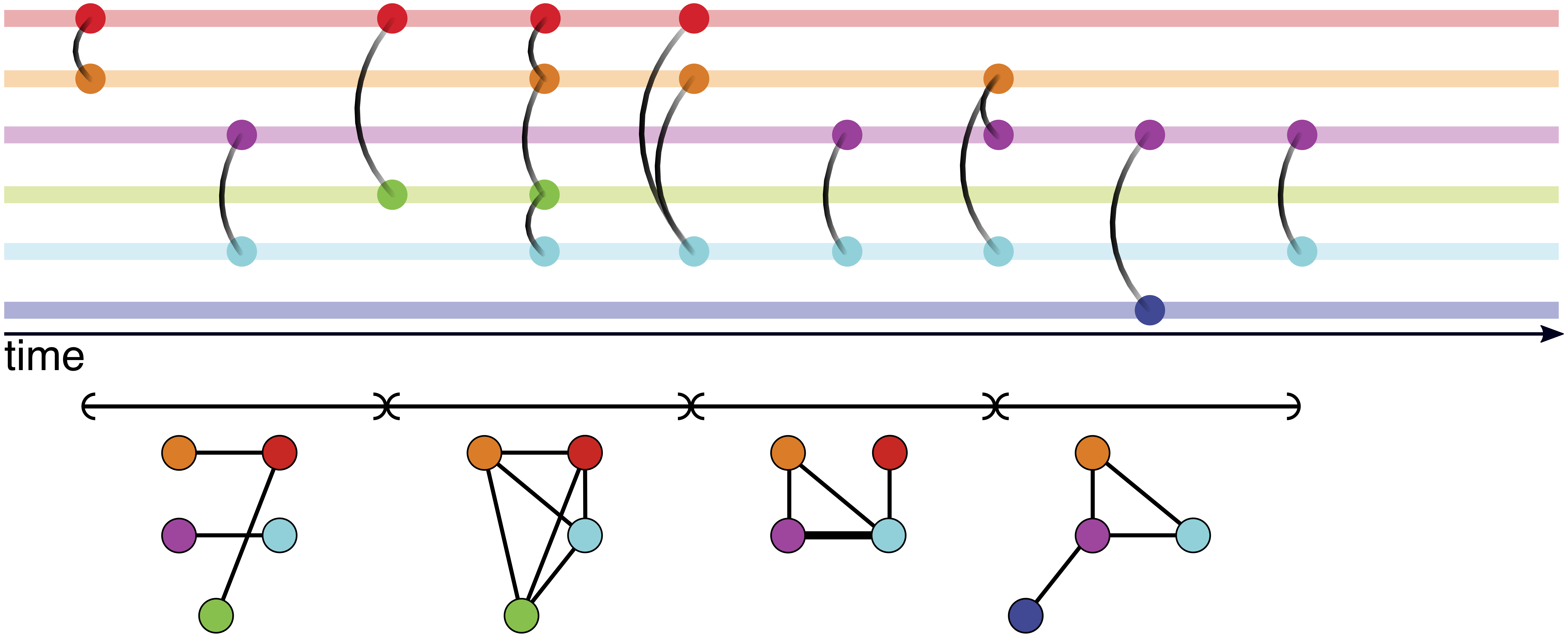


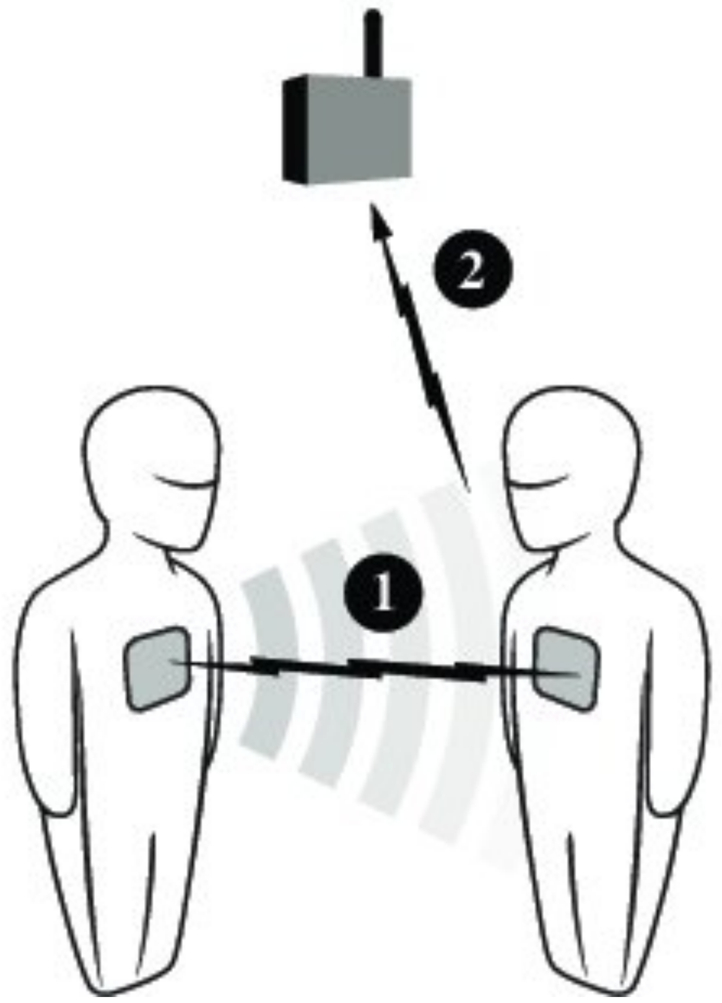


SocioPatterns M. Génois and A. Barrat, EPJ Data Science (2018)

Real world simplicial complexes

High-resolution proximity data





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Real world simplicial complexes

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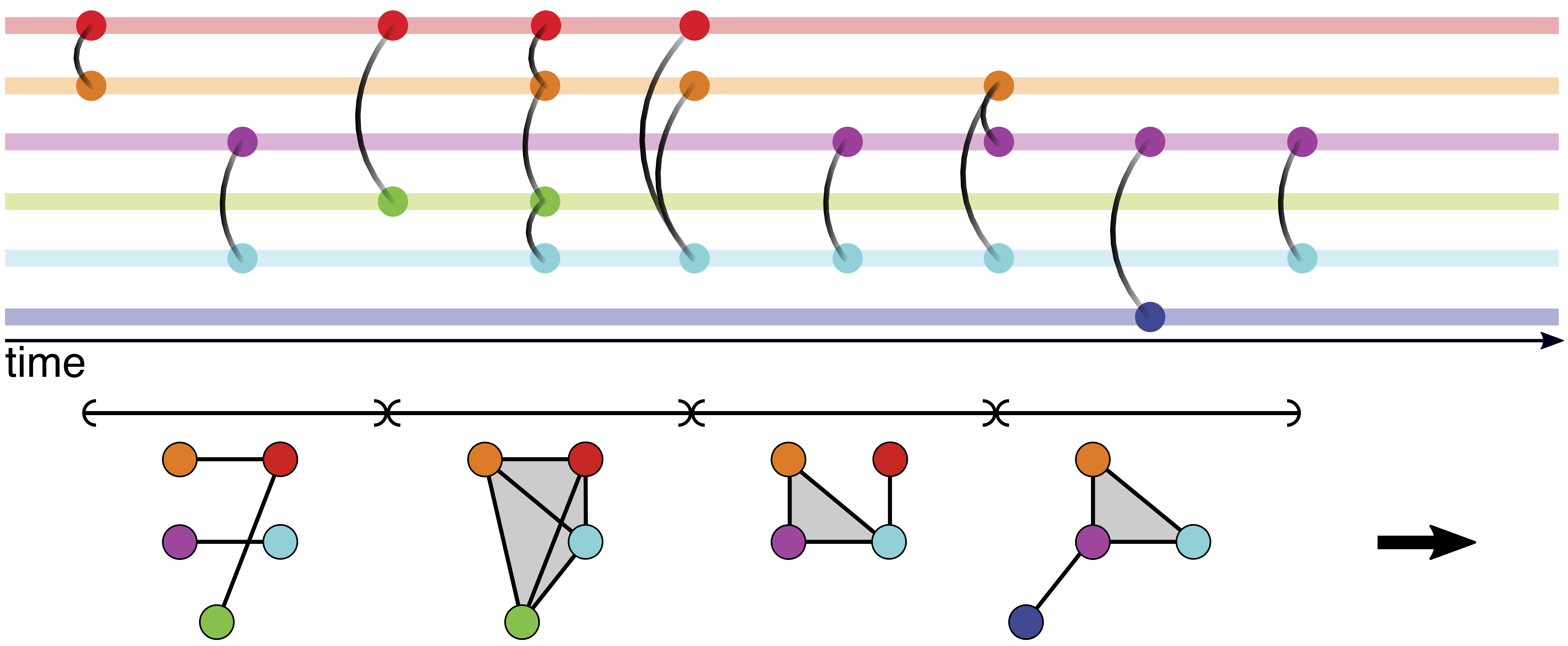

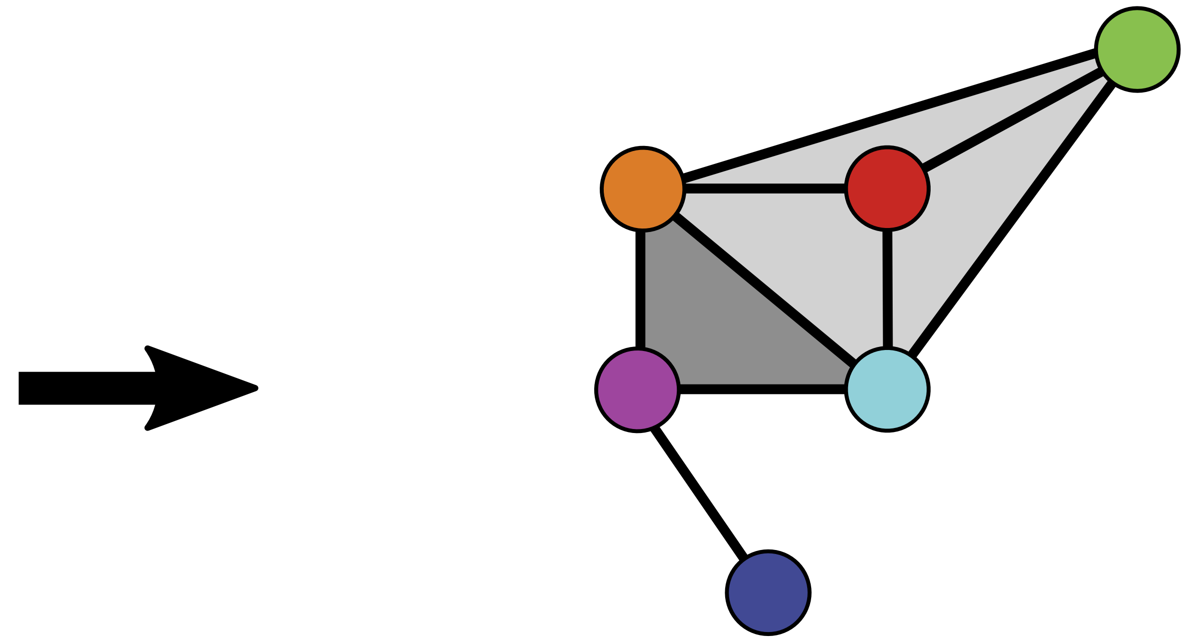

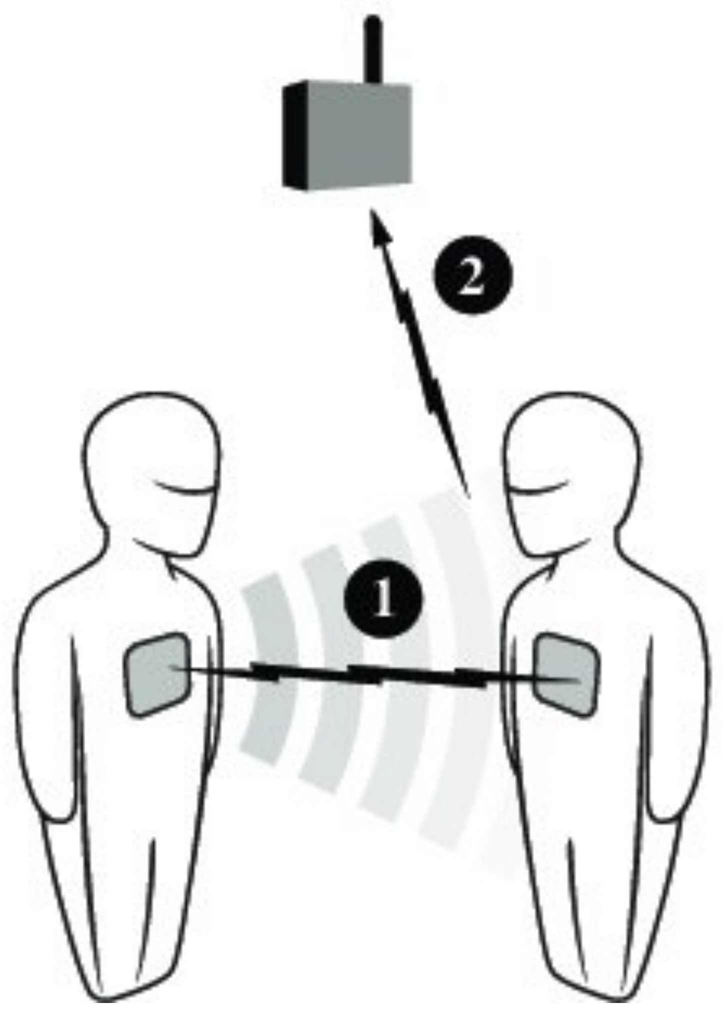

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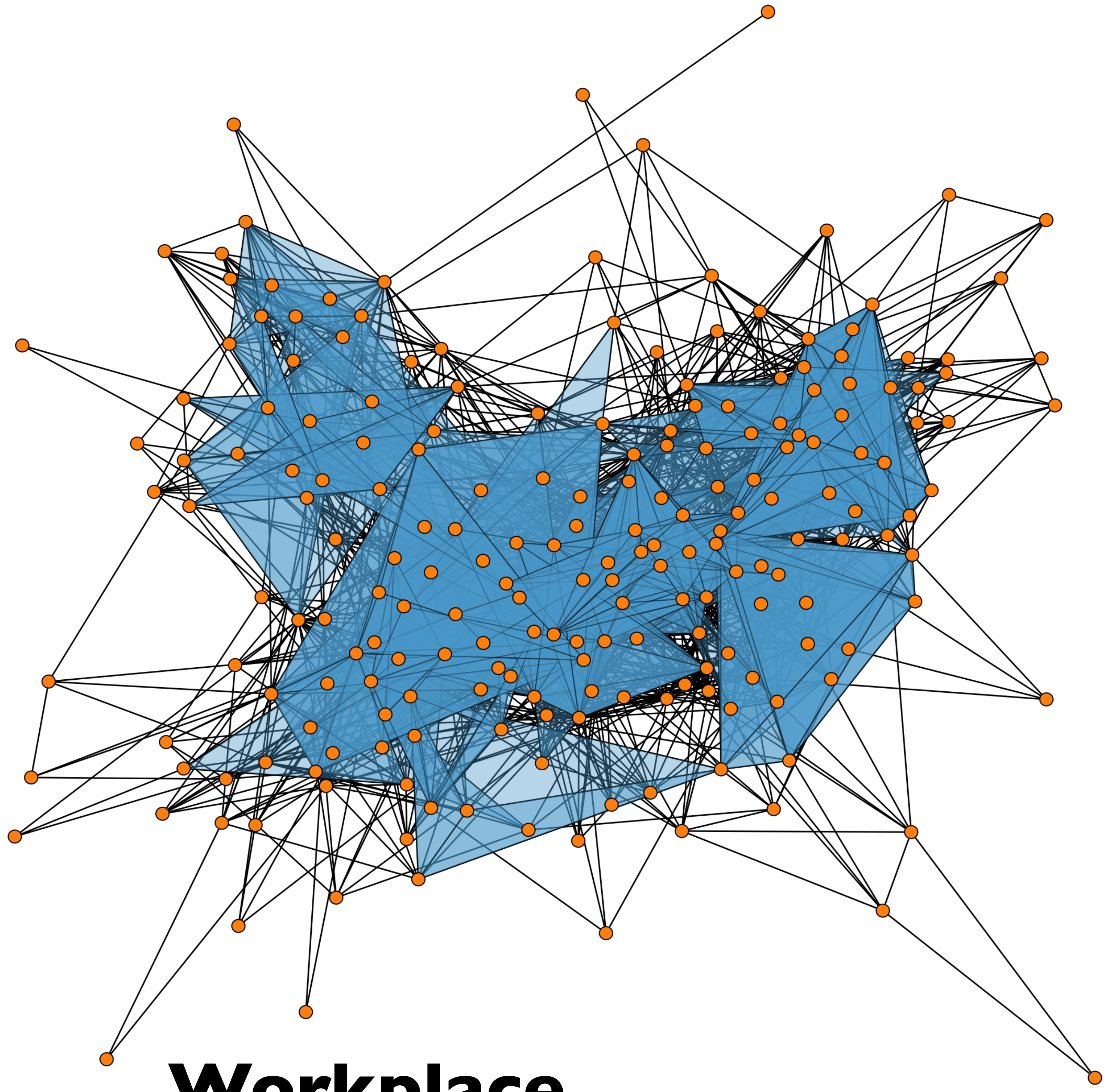
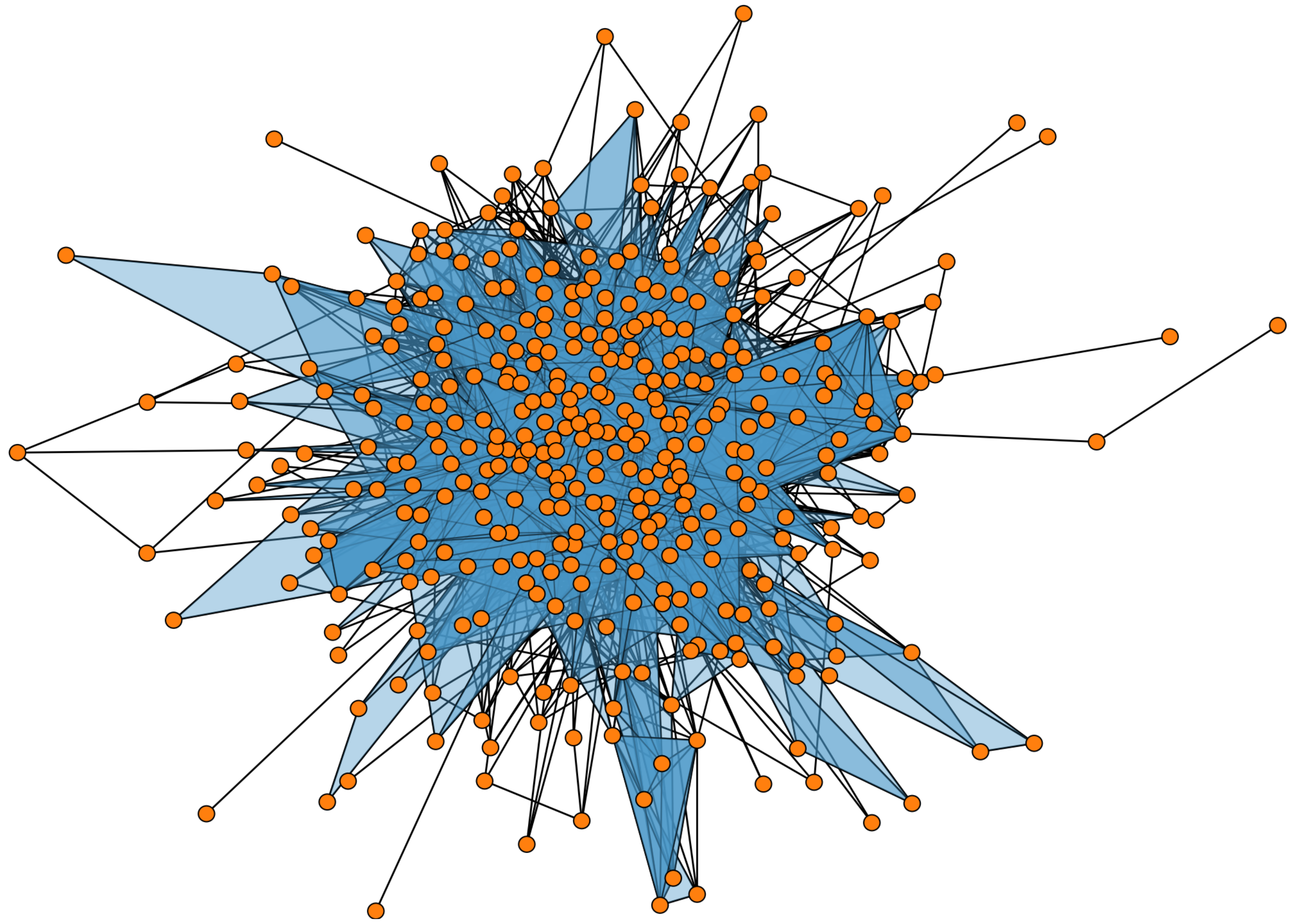
Real world simplicial complexes

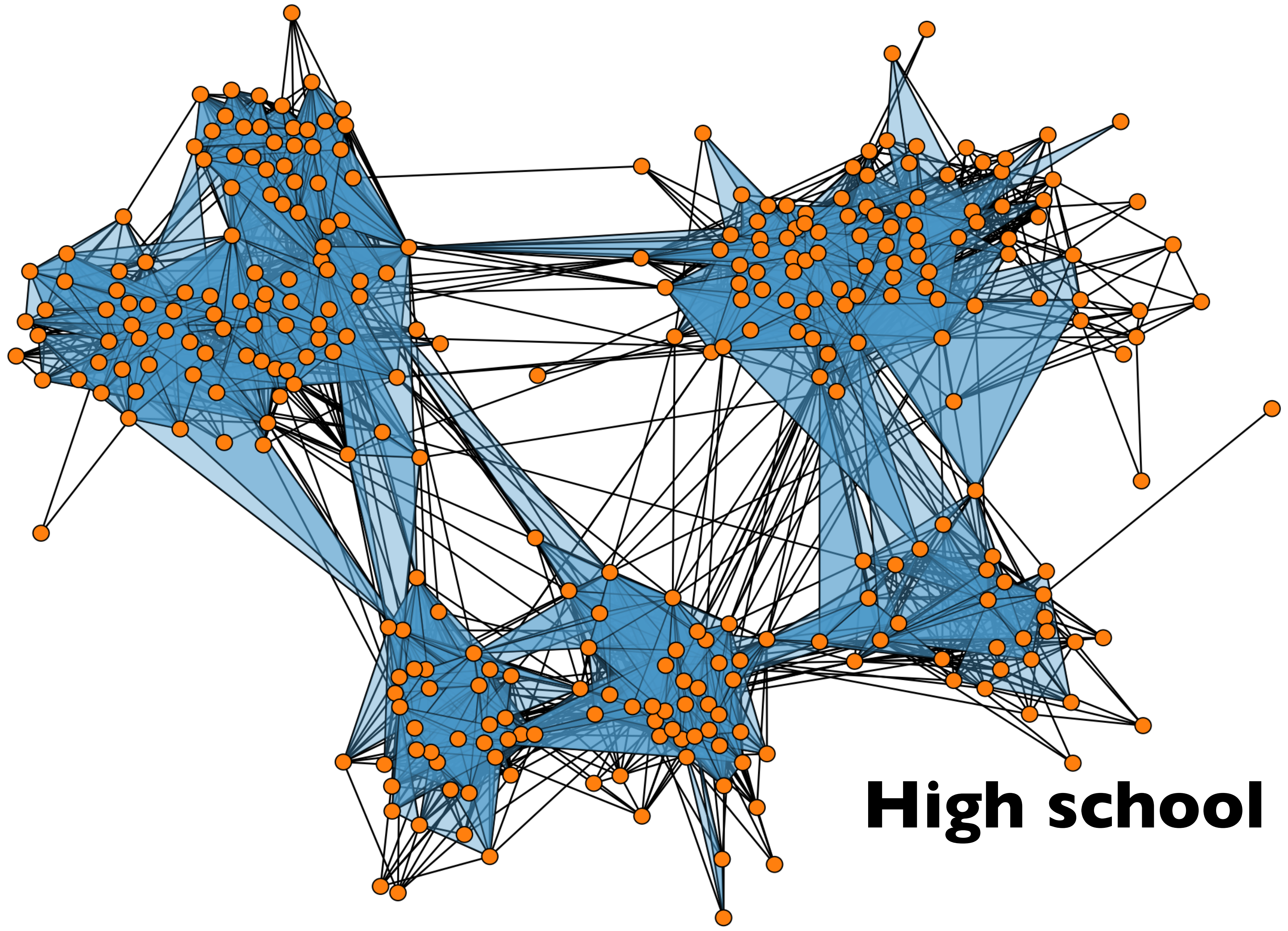
High-resolution proximity data



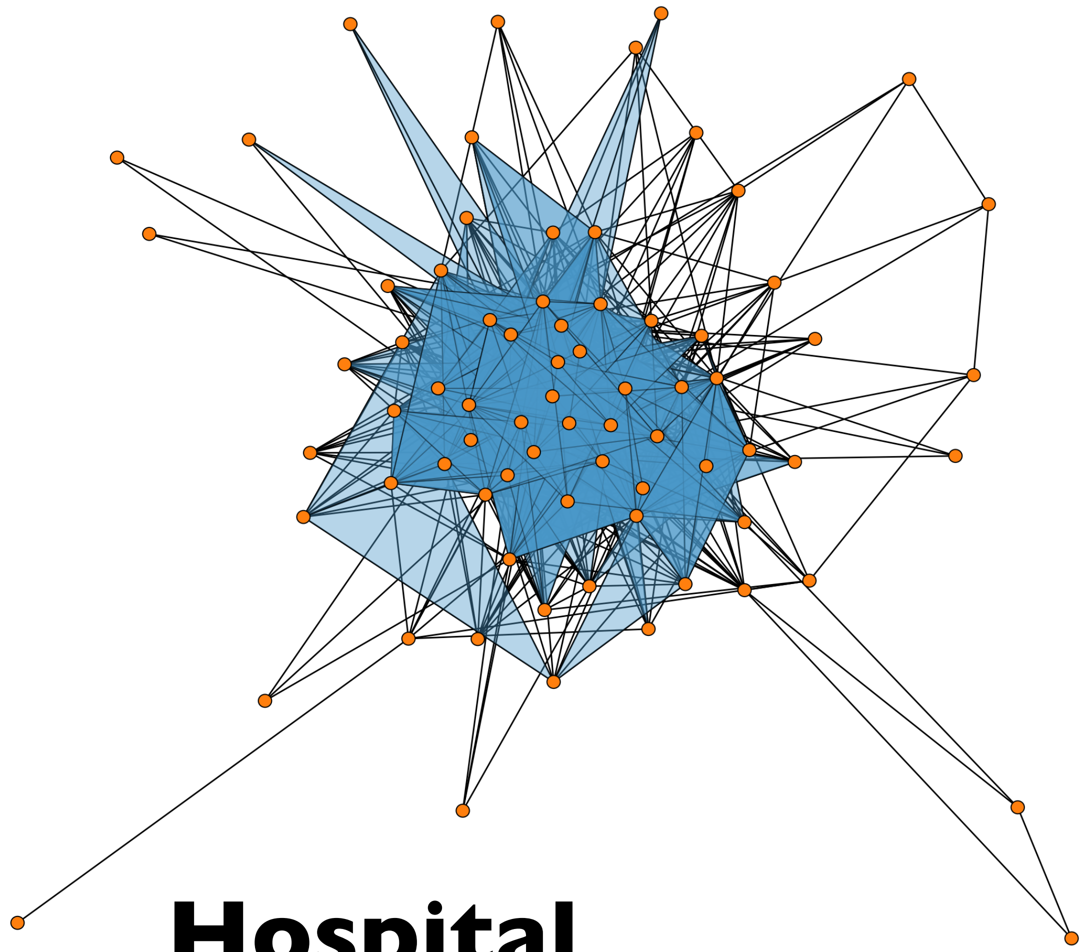
Workplace



Conference



High school



Hospital

SocioPatterns

M. Génois and A. Barrat, EPJ Data Science (2018)

Results

High-resolution proximity data

Results

High-resolution proximity data

Results

High-resolution proximity data

Results

High-resolution proximity data

RSC Model

ER-like Random Simplicial Complexes

- Set of N vertices (0-simplices)
- Set of probabilities $\{p_1, \ldots, p_k, \ldots, p_D\}, \; p_k \in [0, 1]$

RSC Model

ER-like Random Simplicial Complexes

- Set of N vertices (0-simplices)
- Set of probabilities $\{p_1, \dots, p_k, \dots, p_D\}, \; p_k \in [0, 1]$

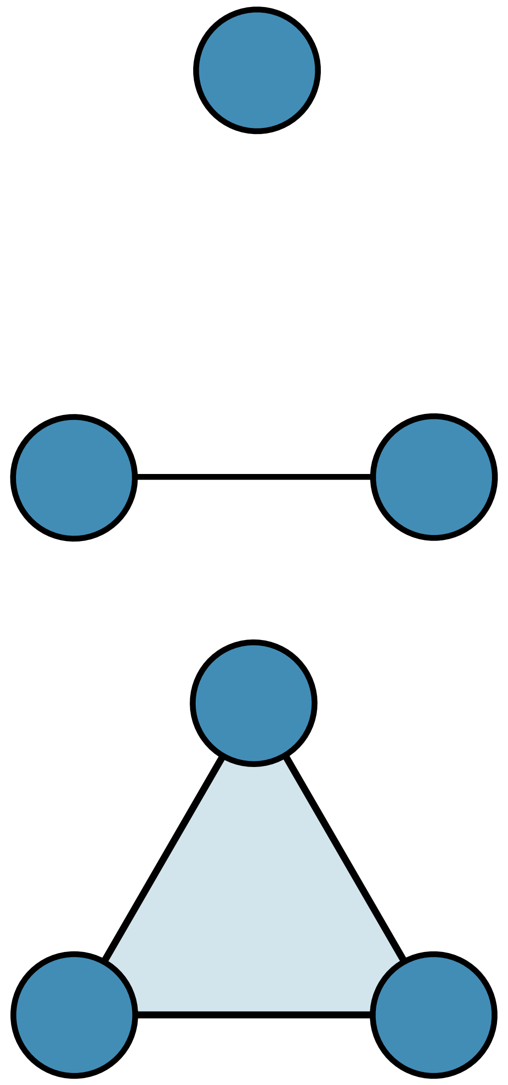
Case D=2

(N, p_1, p_2)

N
Number of vertices

p_1
probability of creating a simplex (i, j)

$p_2 \equiv p_\Delta$
probability of creating a simplex (i, j, k)



RSC Model

ER-like Random Simplicial Complexes

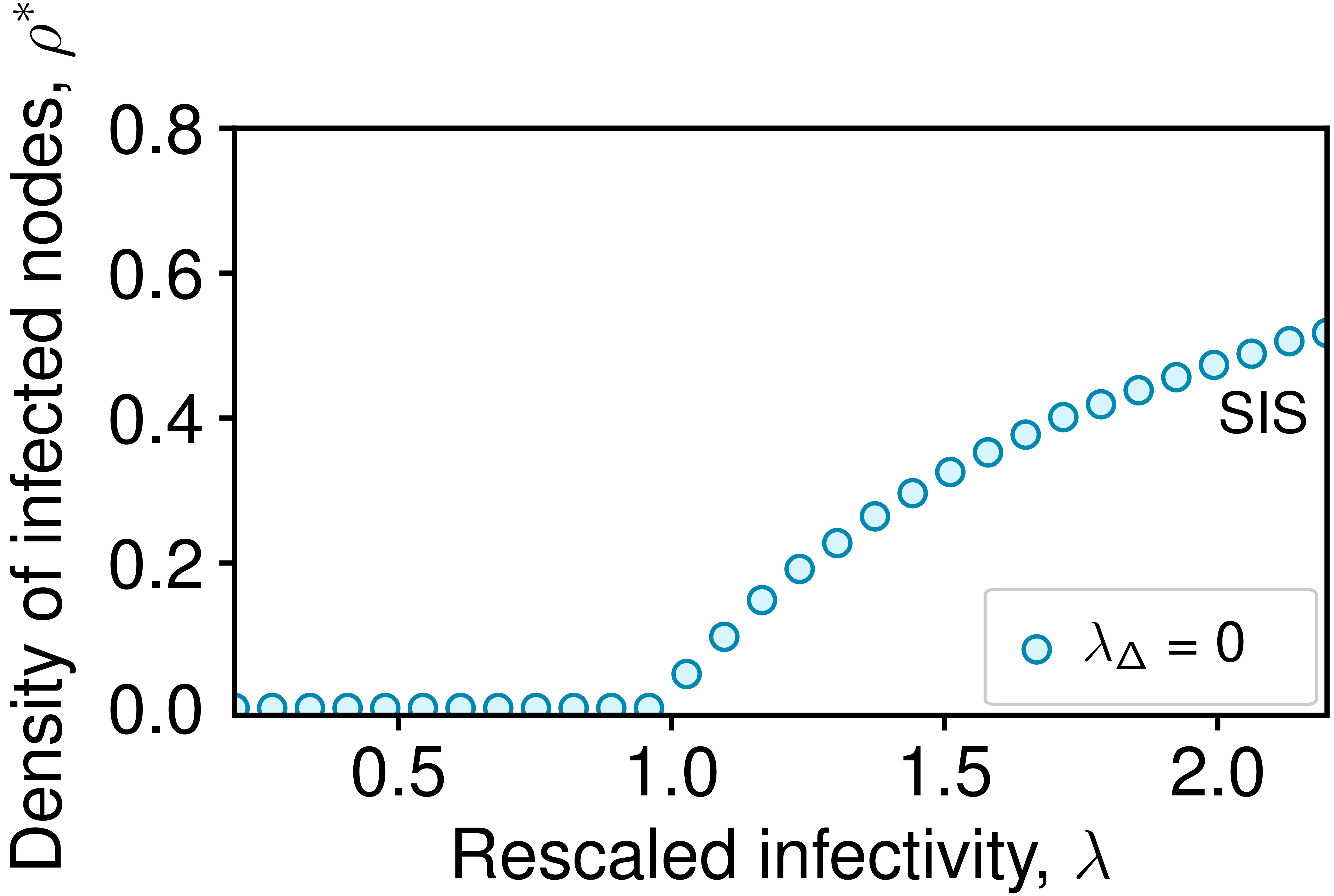
- Set of N vertices (0-simplices)
- Set of probabilities $\{p_1, \ldots, p_k, \ldots, p_D\}, \; p_k \in [0, 1]$

Case D=2

$$(N, p_1, p_2) \longrightarrow \boxed{\begin{aligned} p_1 &= \frac{\langle k \rangle - 2\langle k_\Delta \rangle}{(N-1) - 2\langle k_\Delta \rangle} \\ p_\Delta &= \frac{2\langle k_\Delta \rangle}{(N-1)(N-2)} \end{aligned}} \longrightarrow (N, \langle k \rangle, \langle k_\Delta \rangle)$$

Results

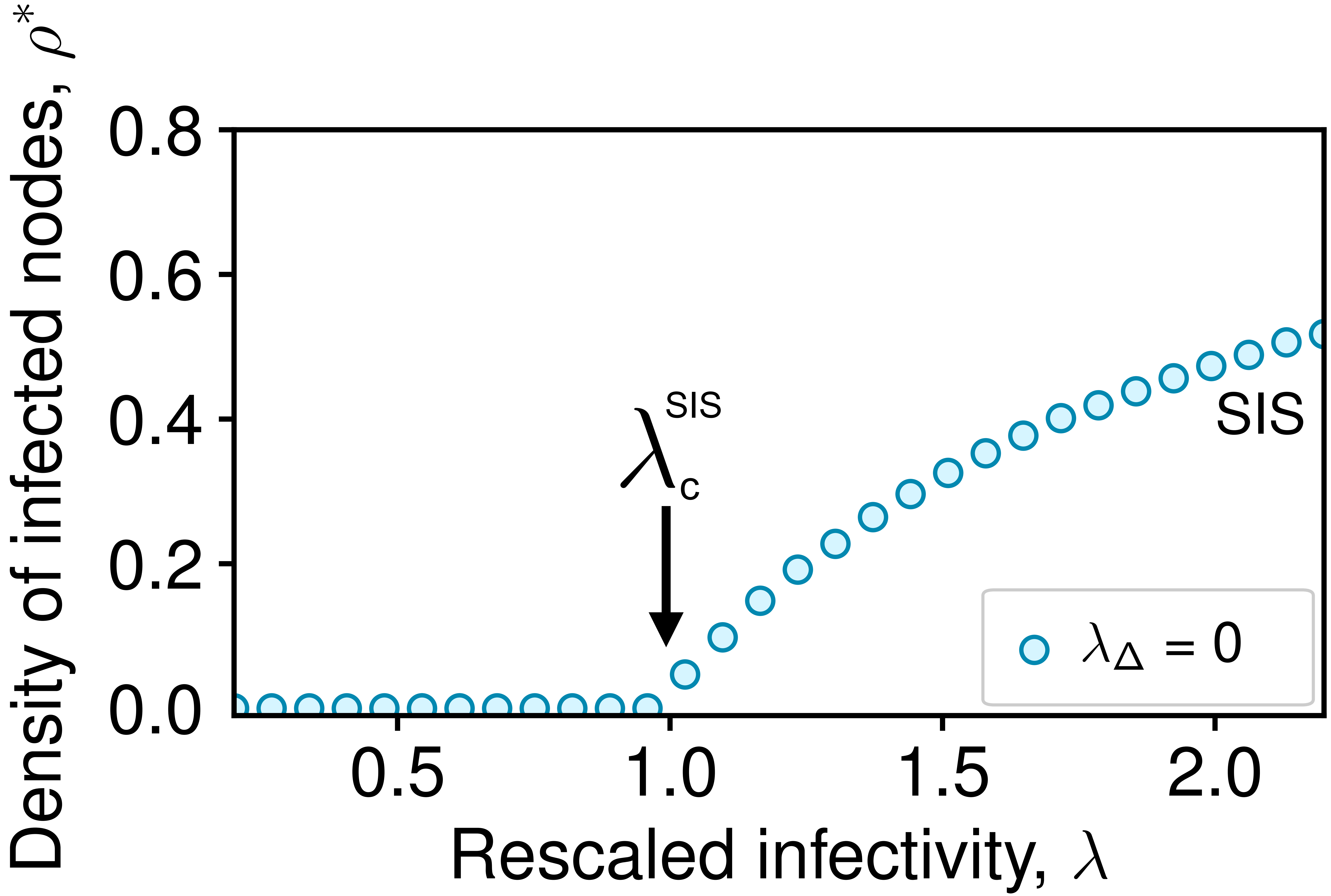
Random Simplicial Complexes



2-simplex infectivity

1-simplex infectivity

Results
Random Simplicial Complexes



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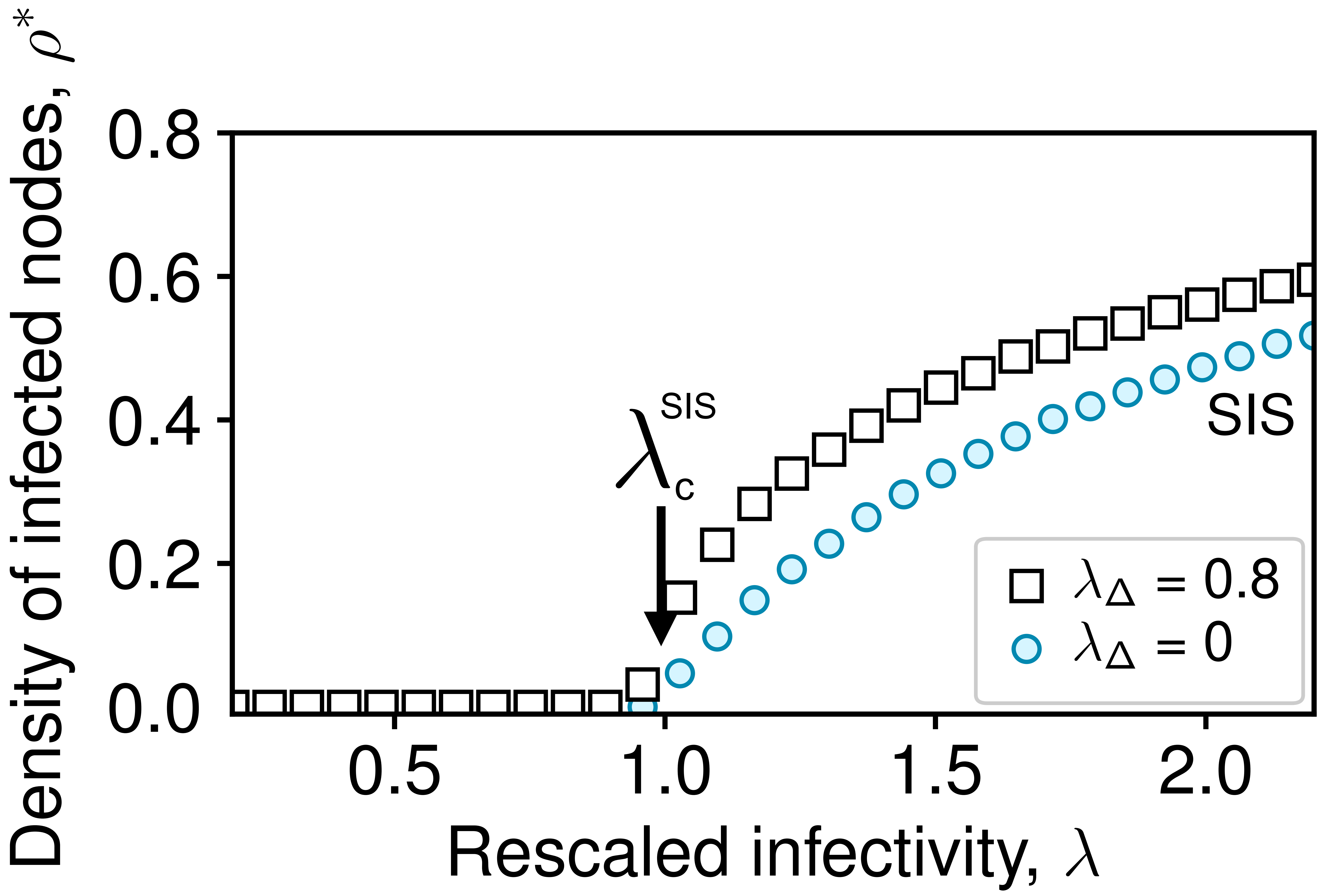
Results

Random Simplicial Complexes



2-simplex infectivity

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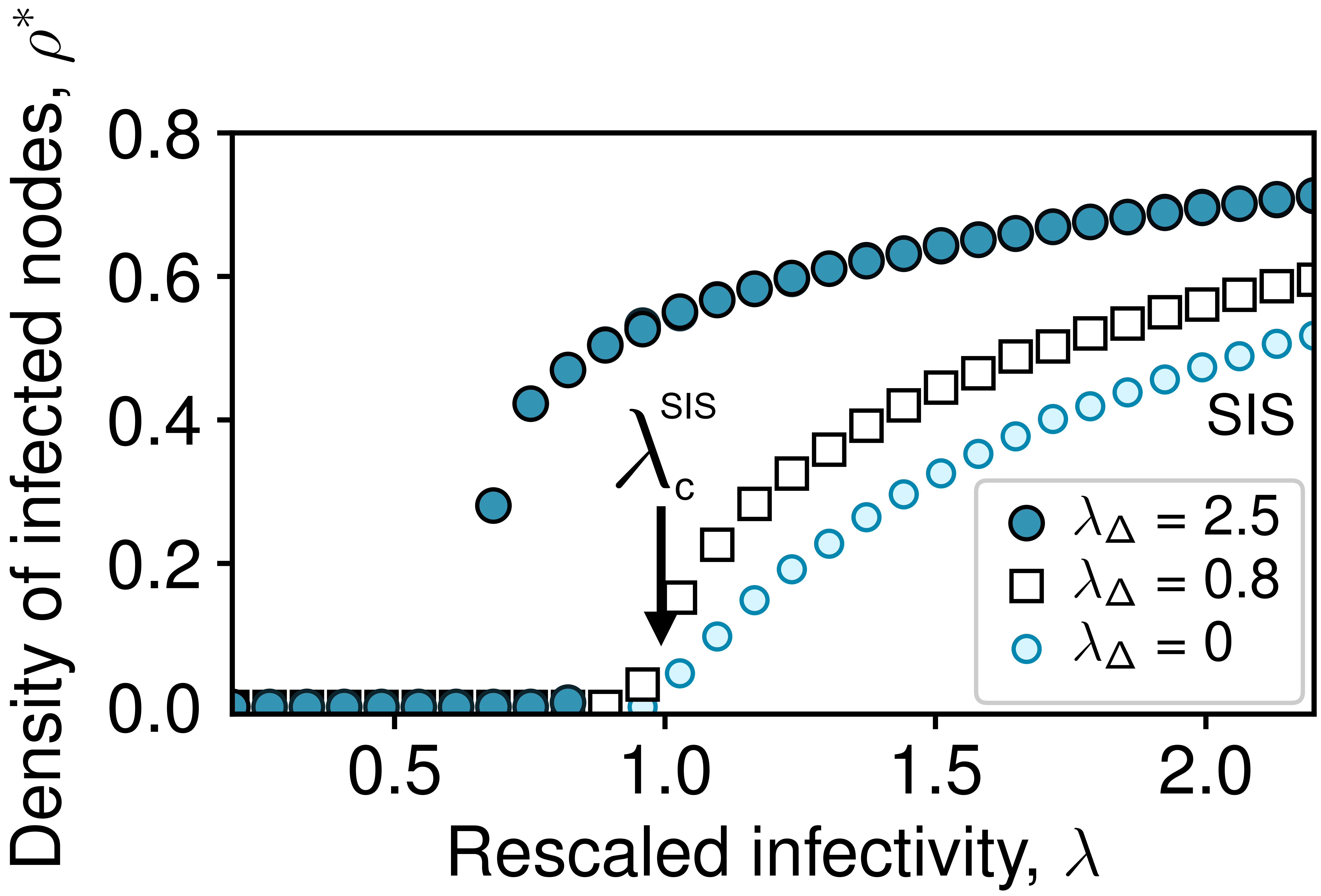
Results

Random Simplicial Complexes



2-simplex infectivity

1-simplex infectivity

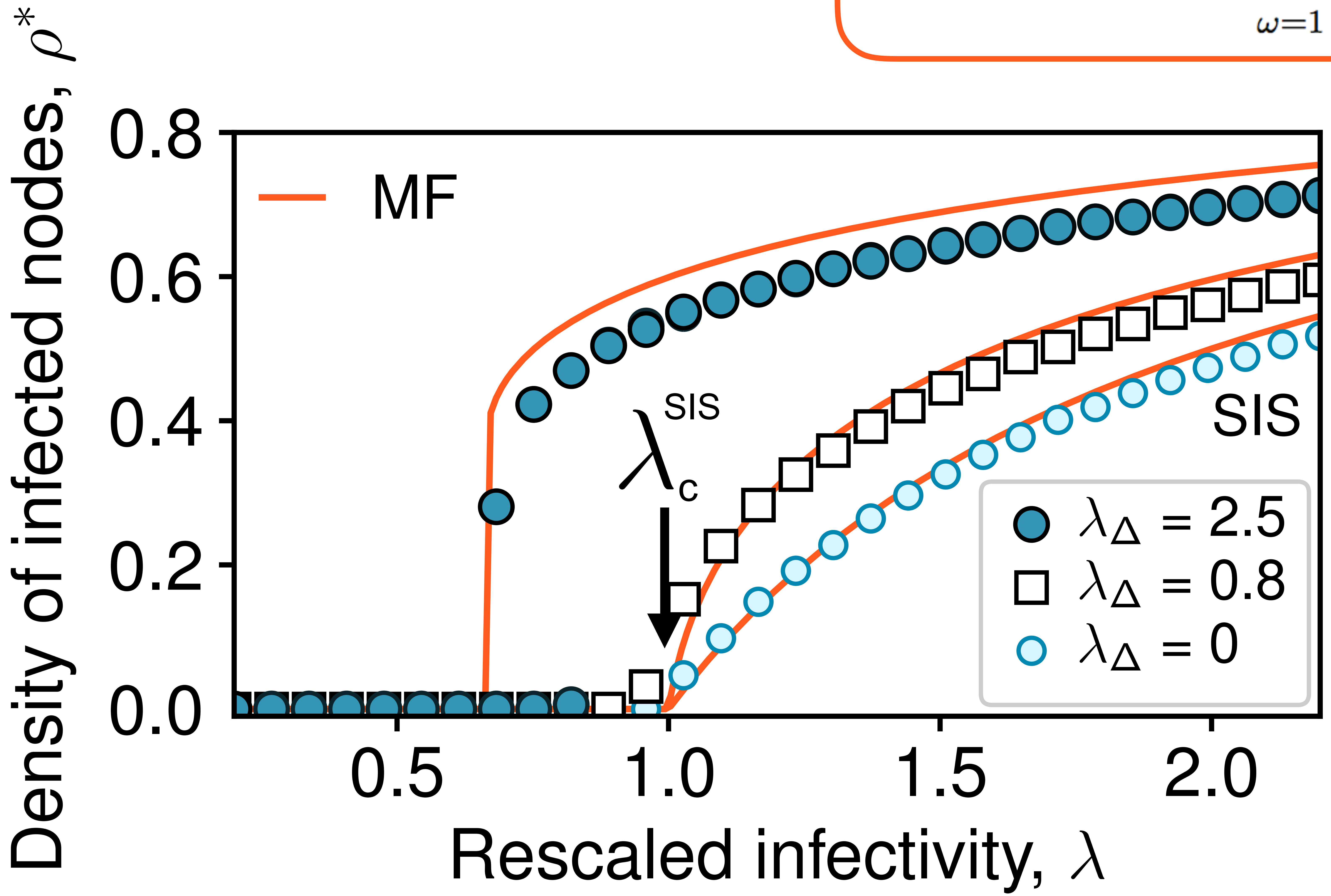
Results

Random Simplicial Complexes

Mean Field

$$d_t\rho(t) = -\mu\rho(t) + \sum_{\omega=1}^{D} \beta_\omega \langle k_\omega \rangle \rho^\omega(t) \left[1 - \rho(t)\right]$$

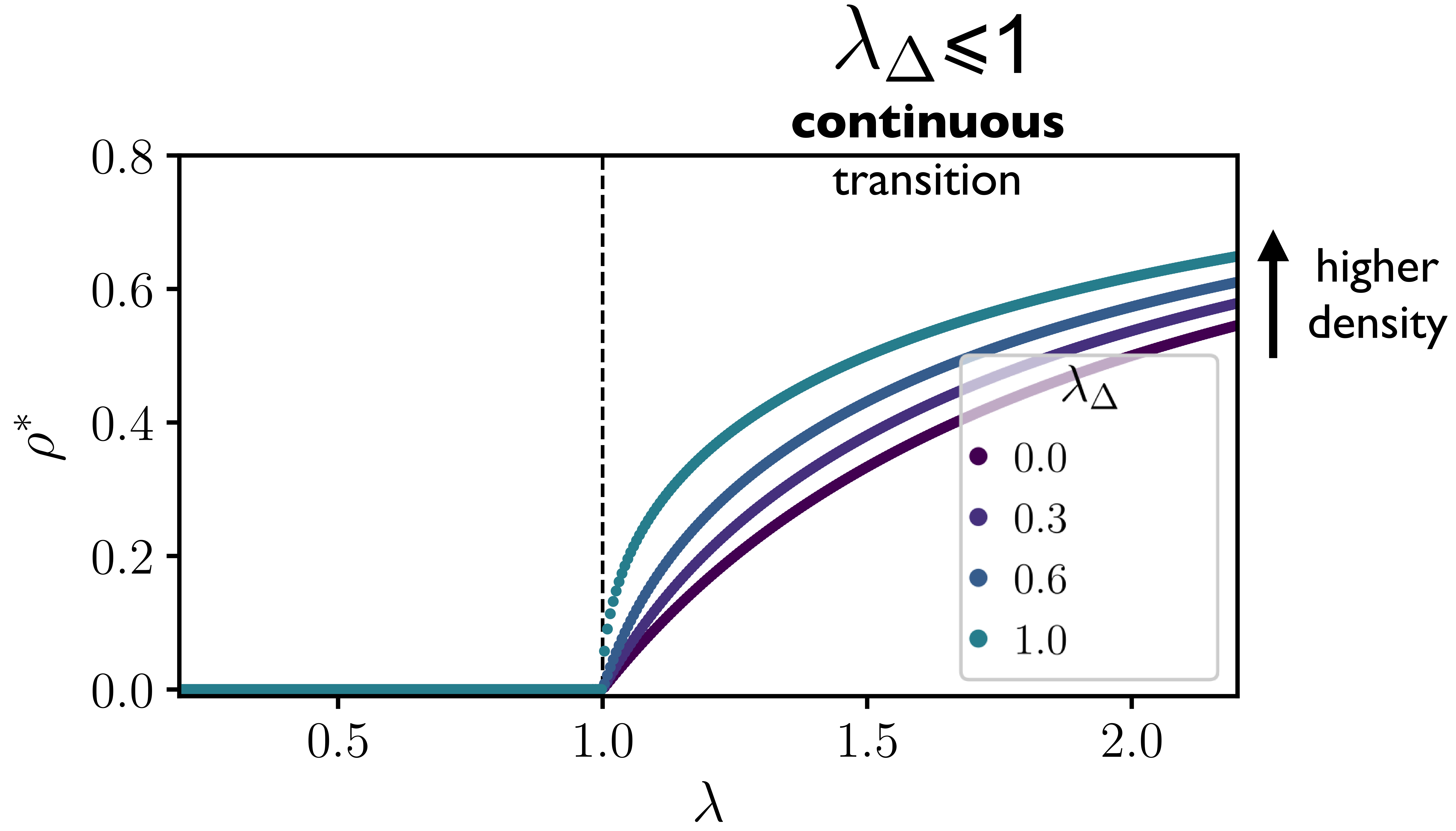
D=2



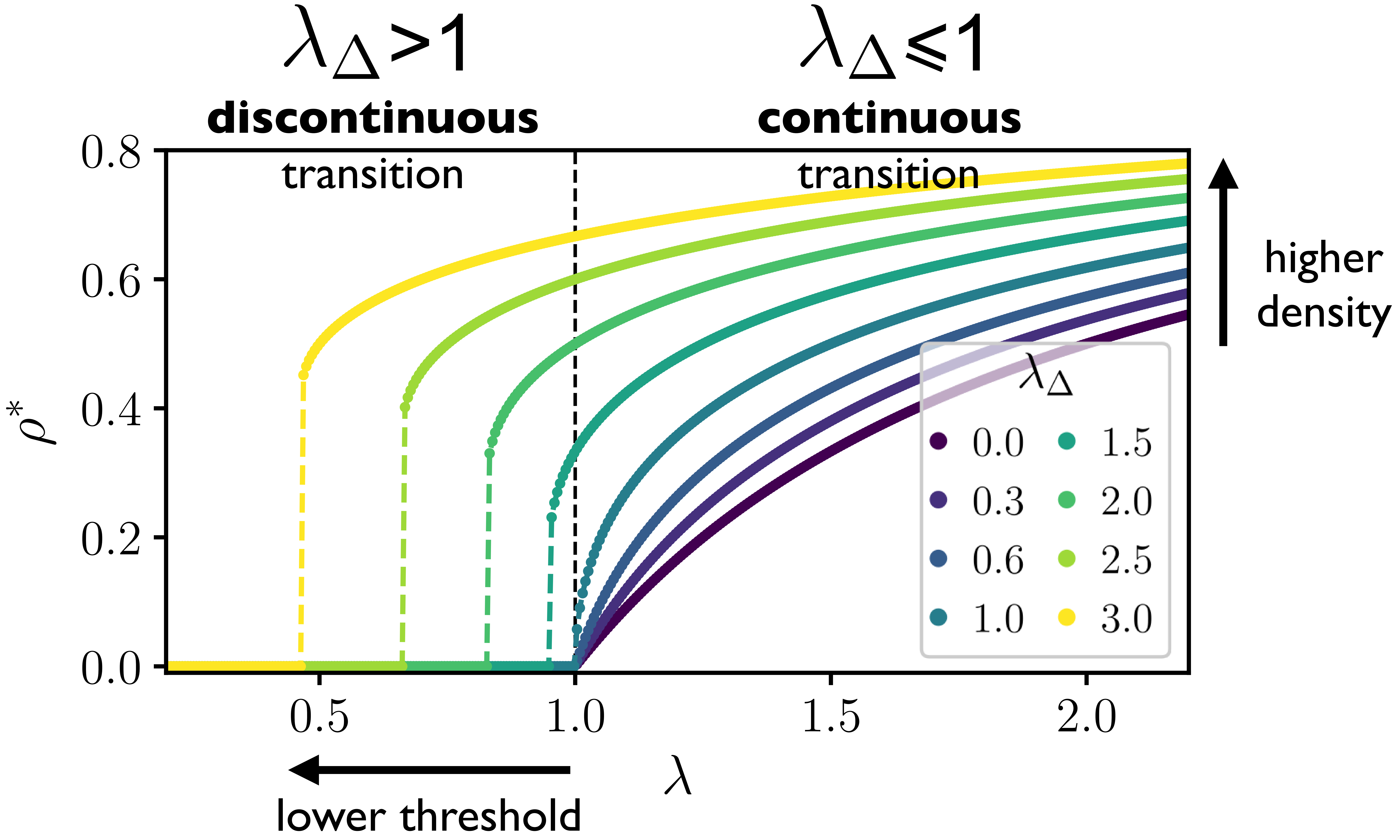
2-simplex infectivity

1-simplex infectivity

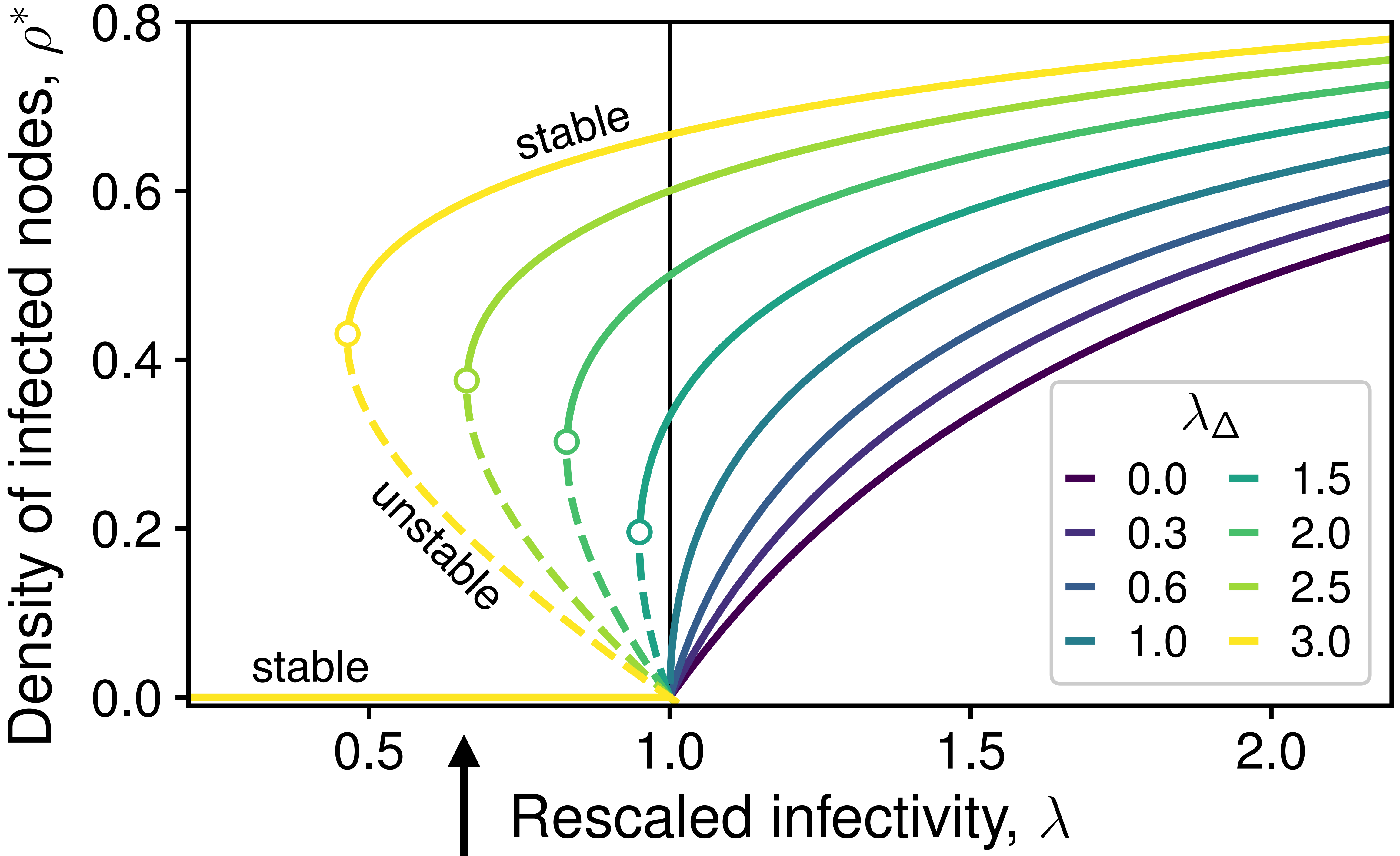
Mean Field Approach



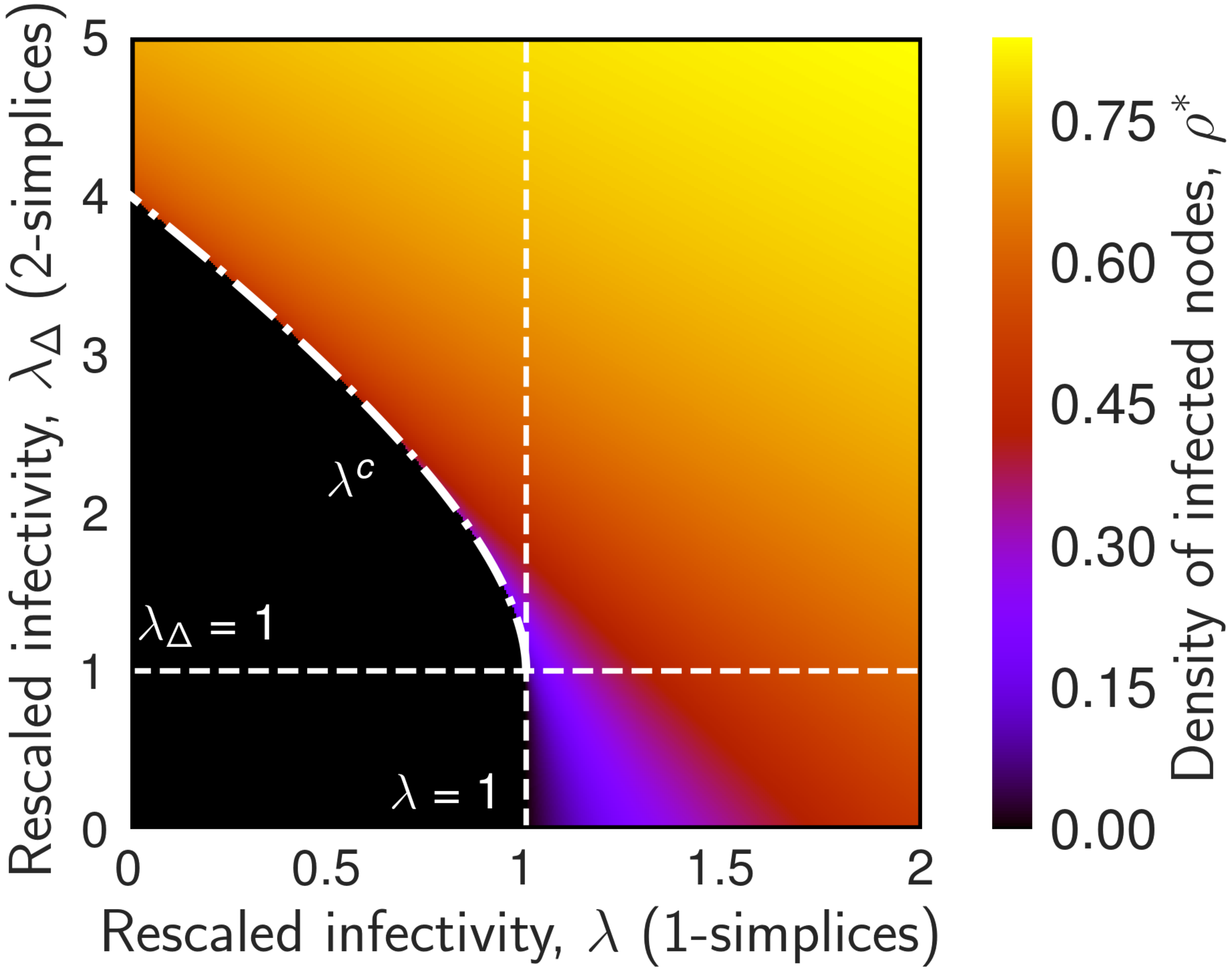
Mean Field Approach



Mean Field Approach

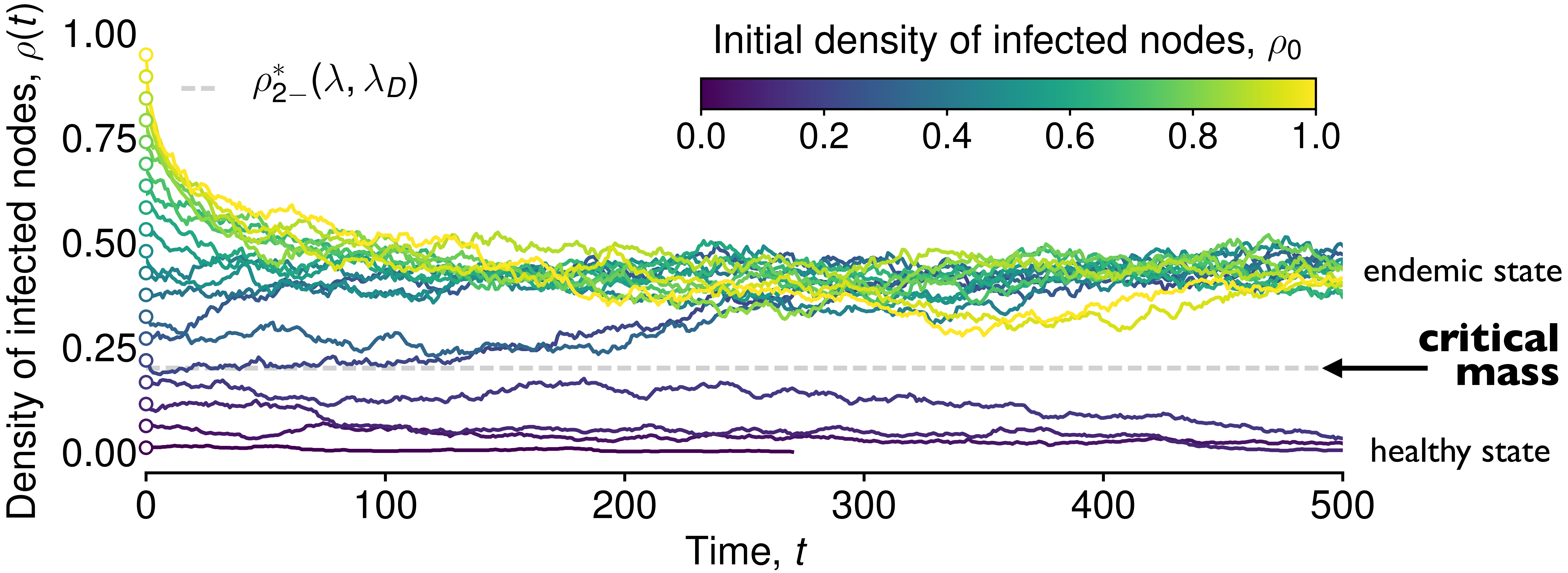


Bistable region



Mean Field Approach

Dependency on initial conditions

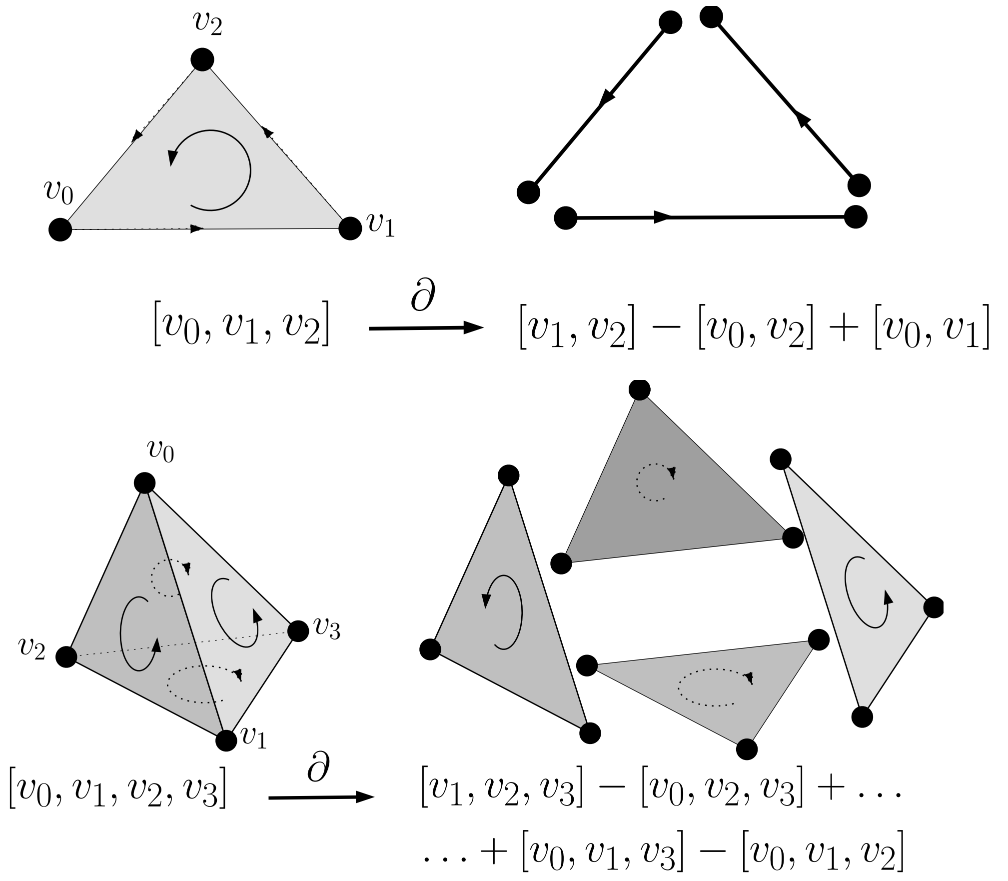


Notebook

What happens to dynamics on complexes?

Combinatorial (Hodge) Laplacian

$$\mathcal{L}_k = \partial_{k+1}\partial_{k+1}^* + \partial_k^*\partial_k.$$



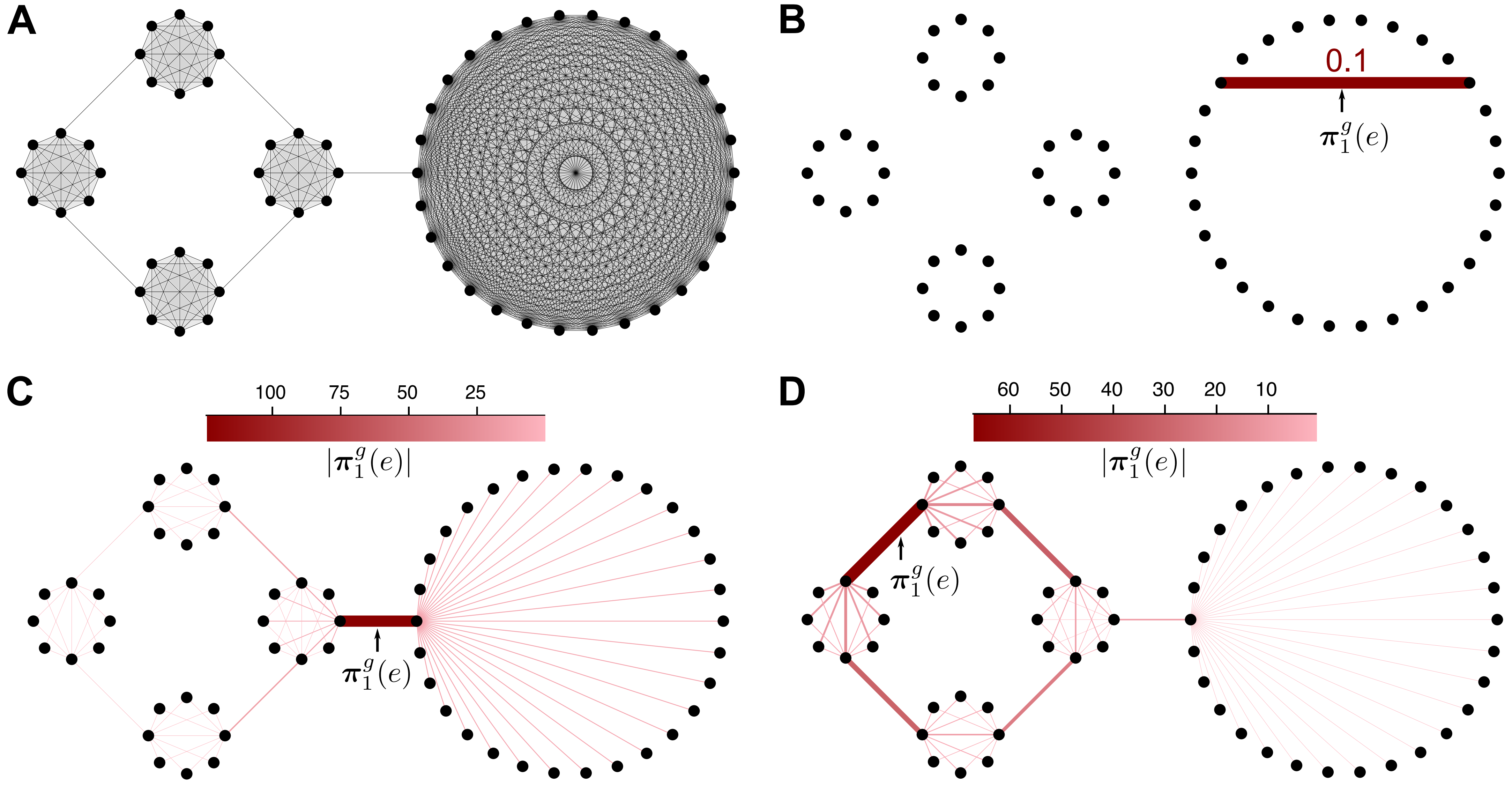
$$C_k(X;\mathbb{R}) = \mathcal{H}_k(X) \oplus \operatorname{im}(\partial_{k+1}) \oplus \operatorname{im}(\partial_k^*),$$

$$\boxed{\mathcal{H}_k(X)} = \{c \in C_k(X) : \mathcal{L}_k c = 0\} = \boxed{\ker \mathcal{L}_k}.$$

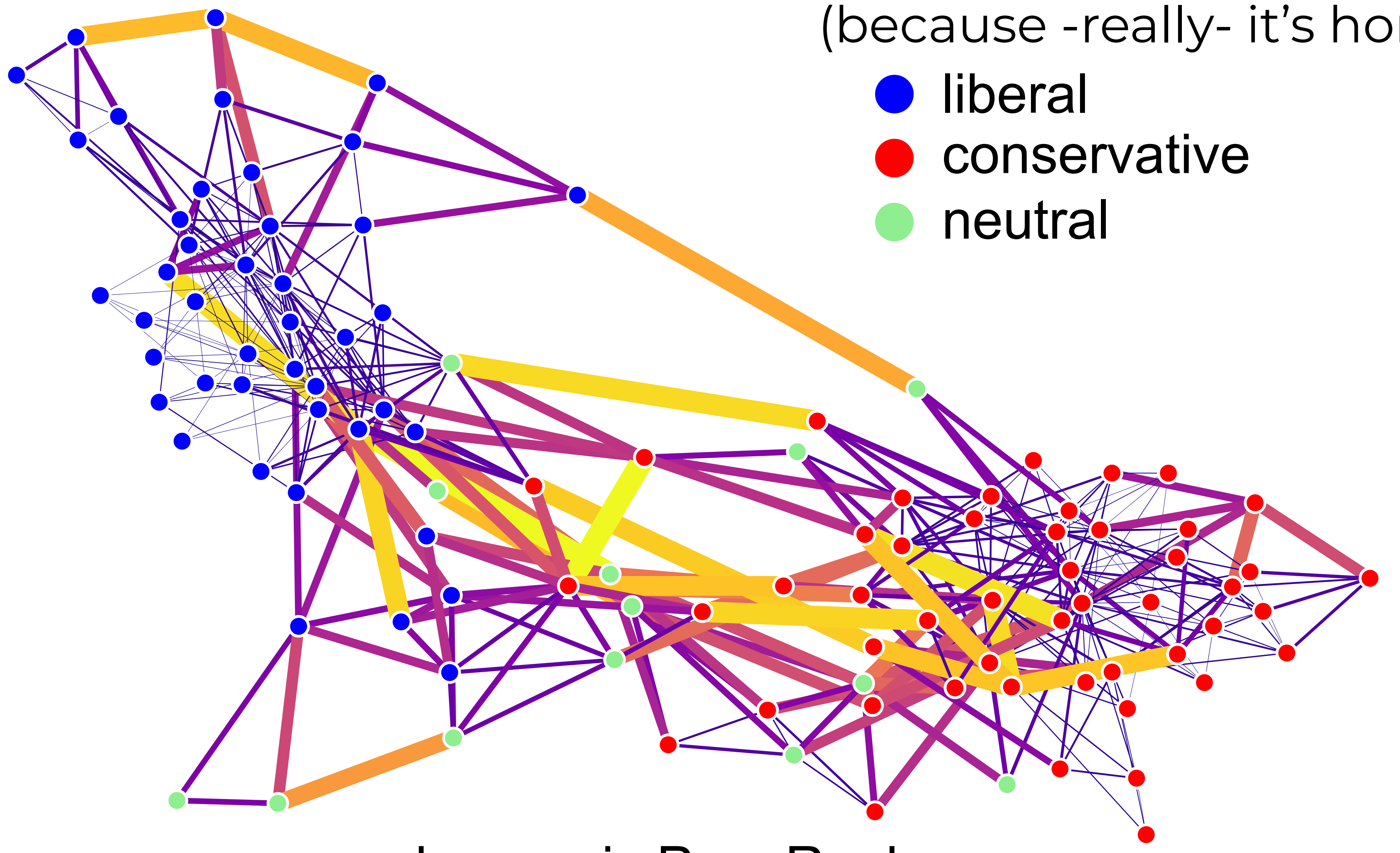
Simplicial PageRank

Based on the k-Laplacian properties

$$\boldsymbol{\pi}_1^g = \boldsymbol{\Theta}\left(\kappa\mathbf{I} + \boldsymbol{\mathcal{L}}_1\right)^{-1}\boldsymbol{\Theta}\mathbf{x}.$$



Similar to Patania et al. EPJ Data Science (2017)
(because -really- it's homology!)

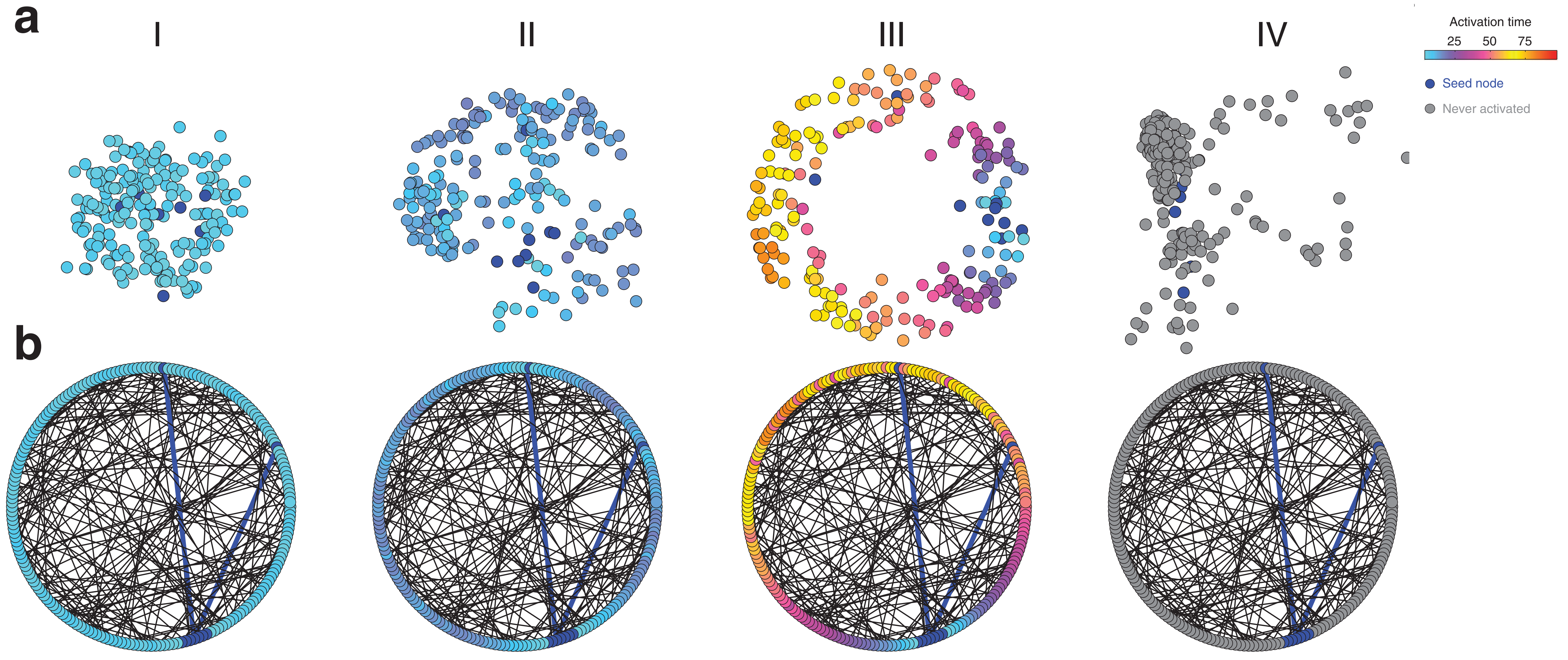


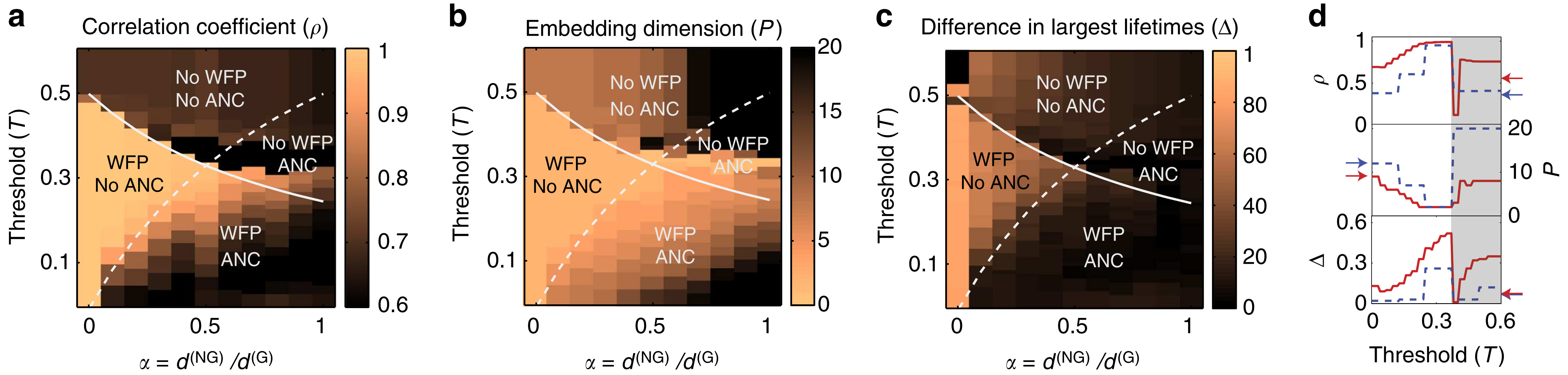
harmonic PageRank

Schaub, Michael T., et al. "Random Walks on Simplicial Complexes and the normalized Hodge Laplacian." *arXiv preprint arXiv:1807.05044* (2018).

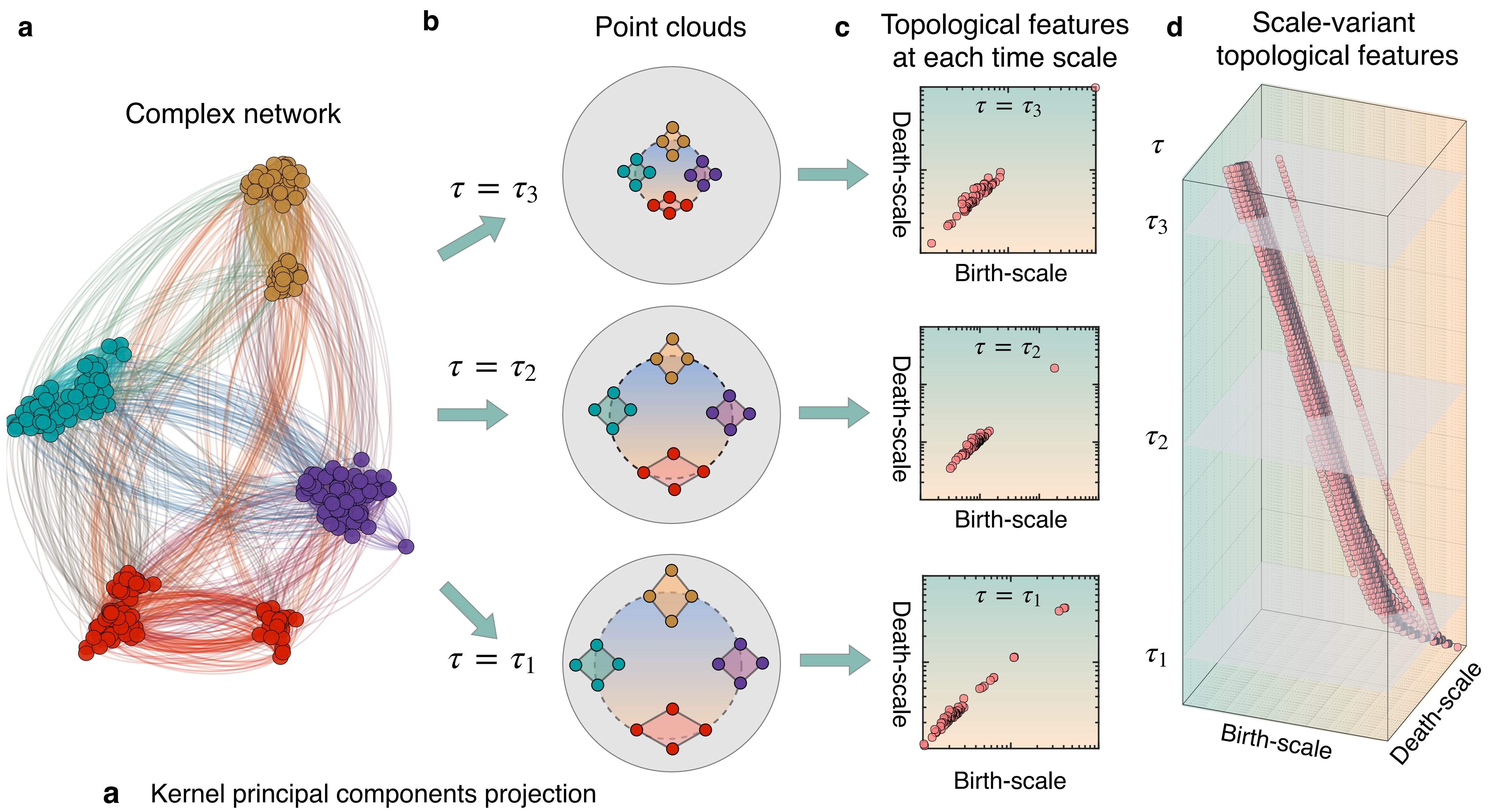
Does the shape of activity match the structure?

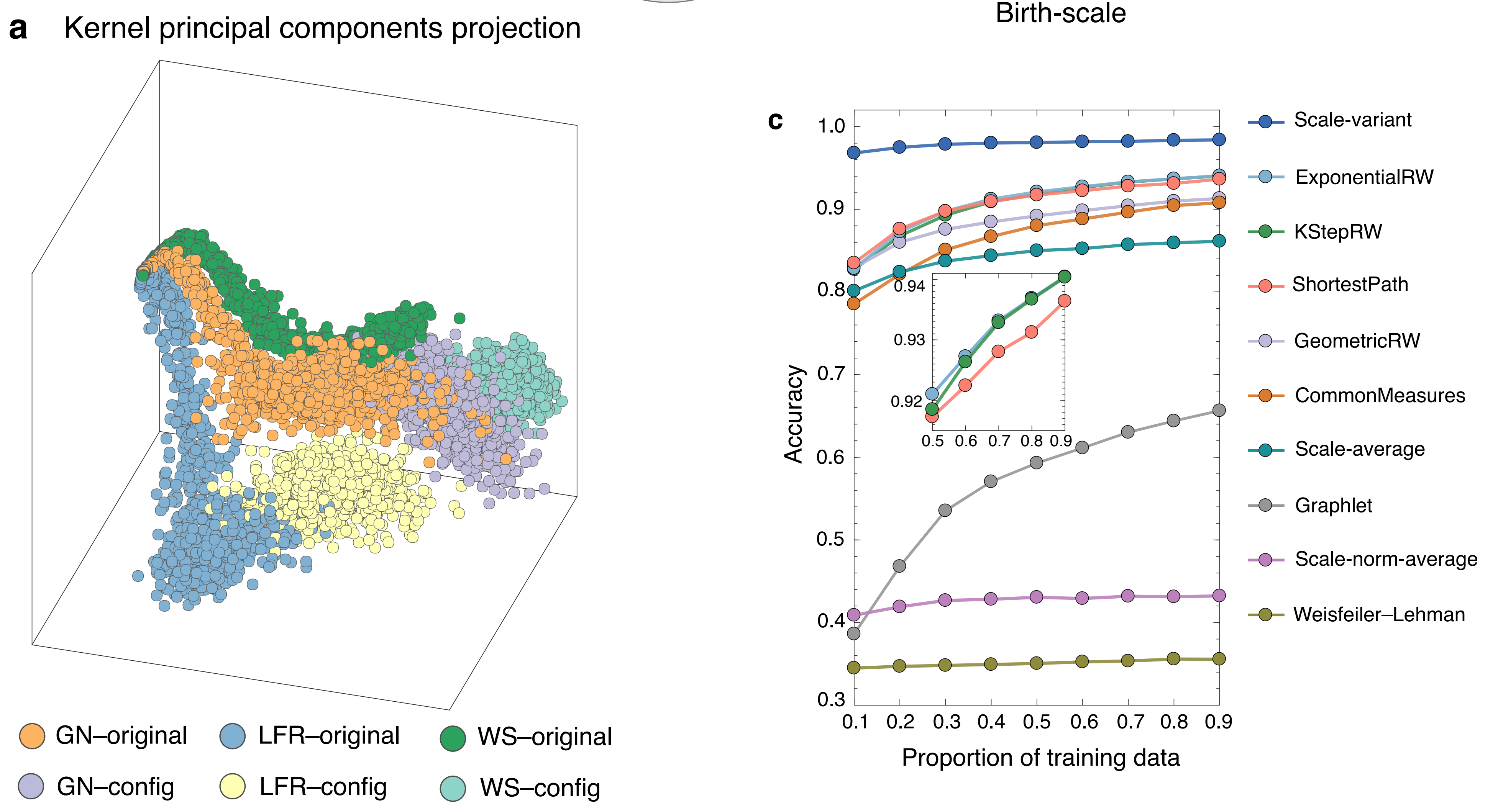
Tran, Quoc Hoan, Van Tuan Vo, and Yoshihiko Hasegawa. "Scale-variant topological information for characterizing complex networks." *arXiv preprint arXiv:1811.03573* (2018).





Taylor, Dane, et al. "Topological data analysis of contagion maps for examining spreading processes on networks." *Nature communications* 6 (2015): 7723.

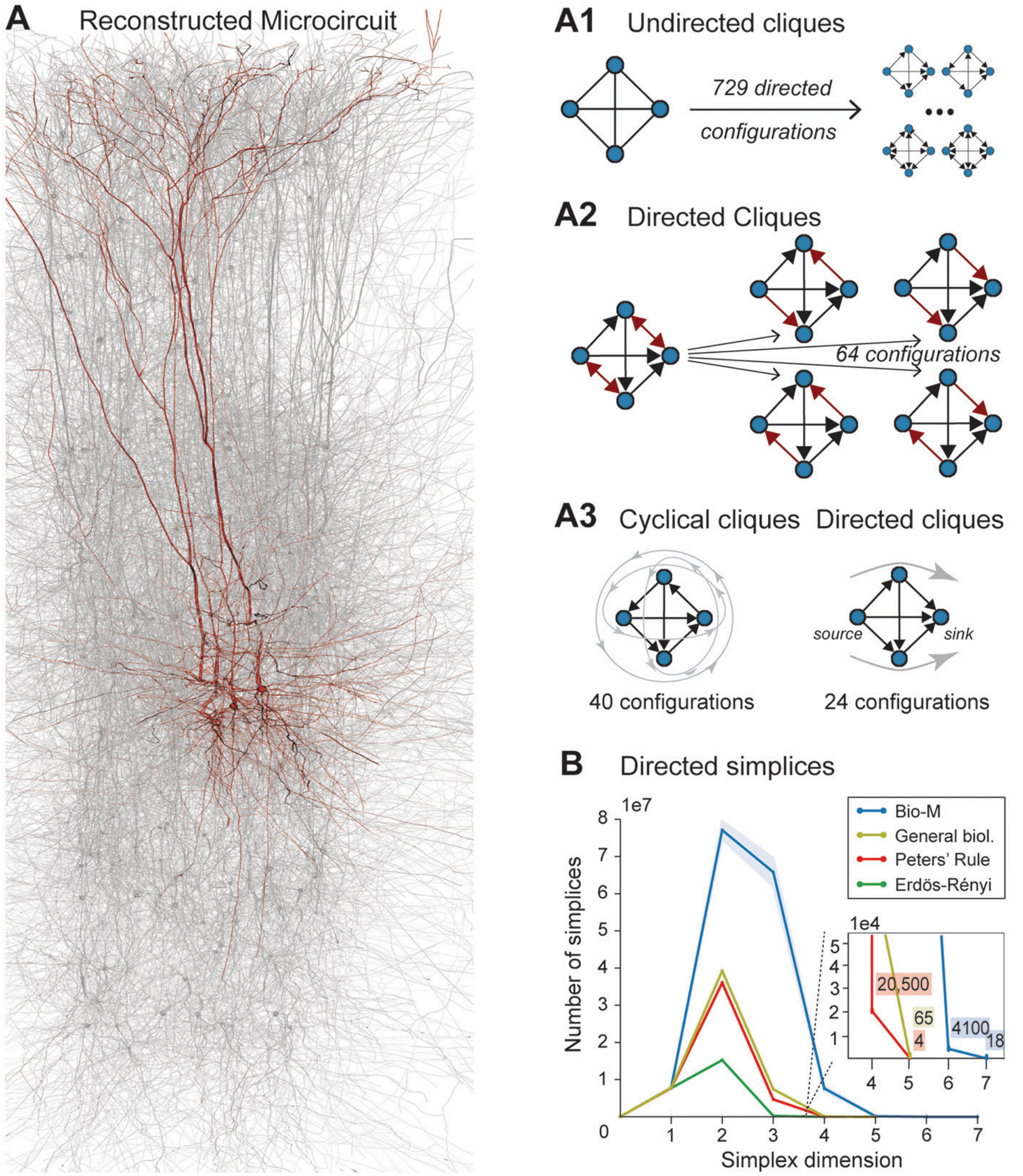




Thank you.

Emotional support?

At great request… directed homology!



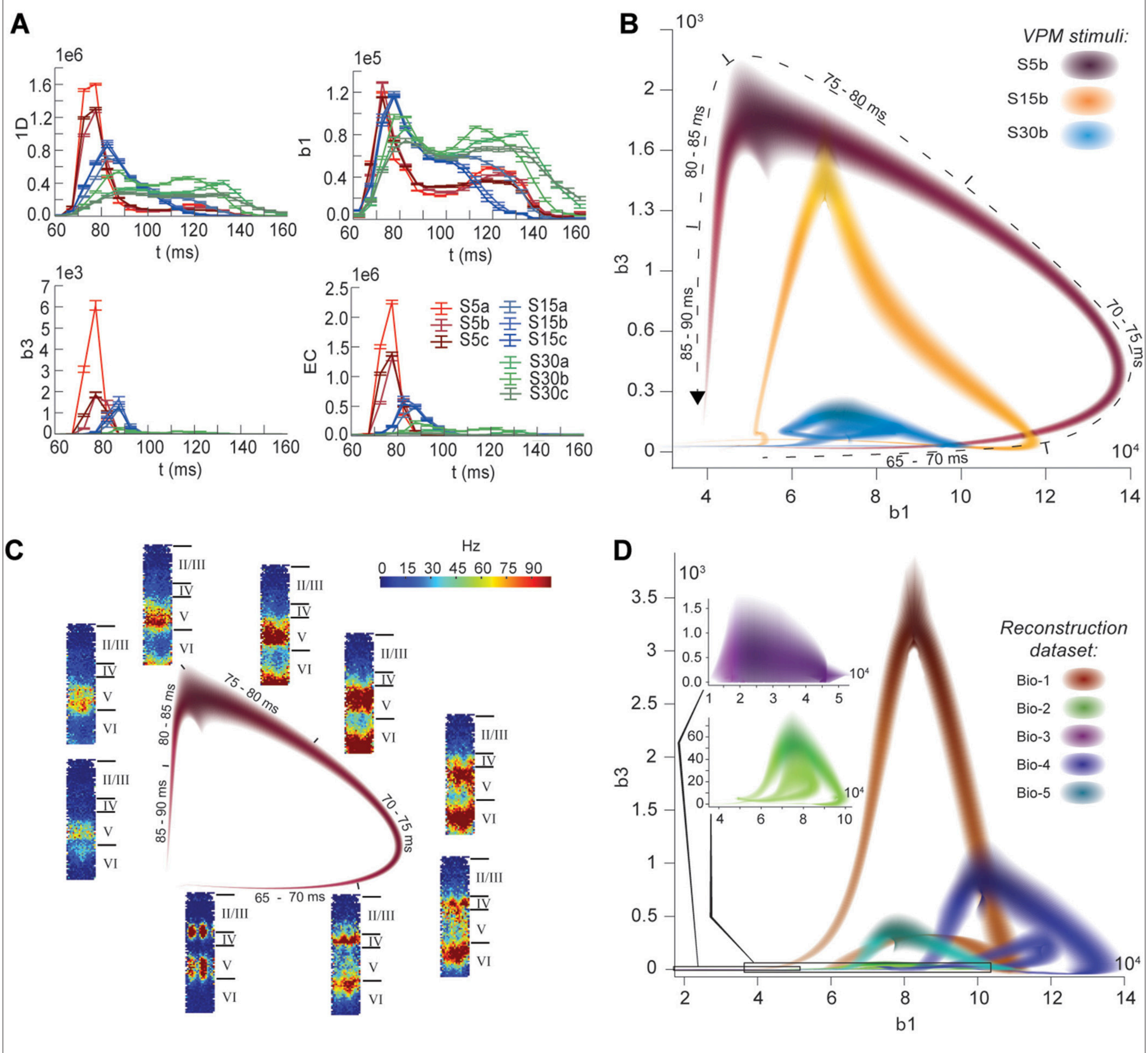


FIGURE 6 | (A) Number of edges, β_1, β_3, and Euler characteristic of the time series of TR graphs in response to the stimulus patterns shown in **Figure 4** (mean and SEM of 30 repetitions of each stimulus). **(B)** Trace of the time series of β_1 against β_3 for three of the stimuli. Shading of colors indicates Gaussian profiles at each time step with means and standard deviations interpolated from 30 repetitions of each stimulus. **(C)** Trace for one of the stimuli in B, along with the mean firing activity at different locations of the microcircuit during time steps of 2 ms. **(D)** Like **(B)**, but for TR graphs of Bio 1-5, in response to stimulus S15b.