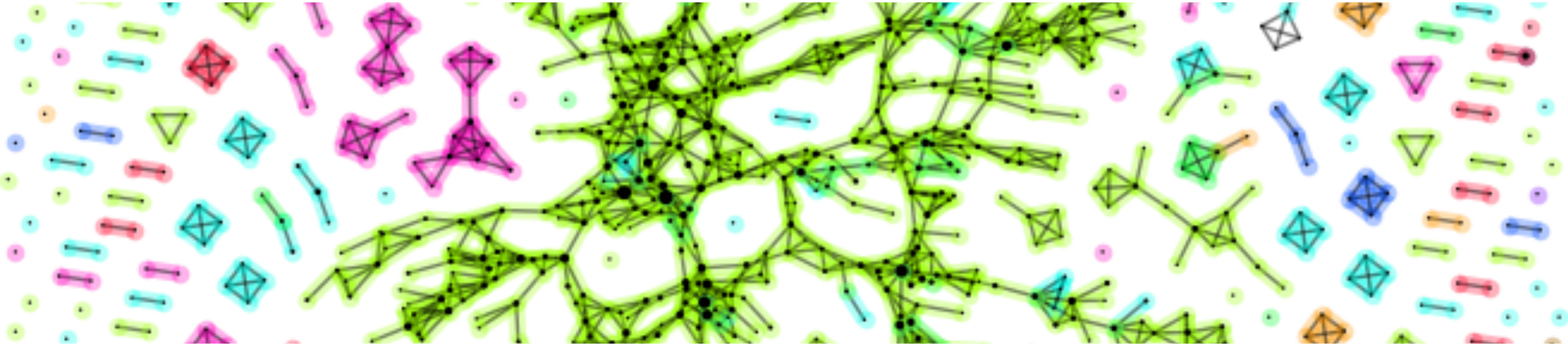
TDA Lecture #3

Mapper and persistent homology

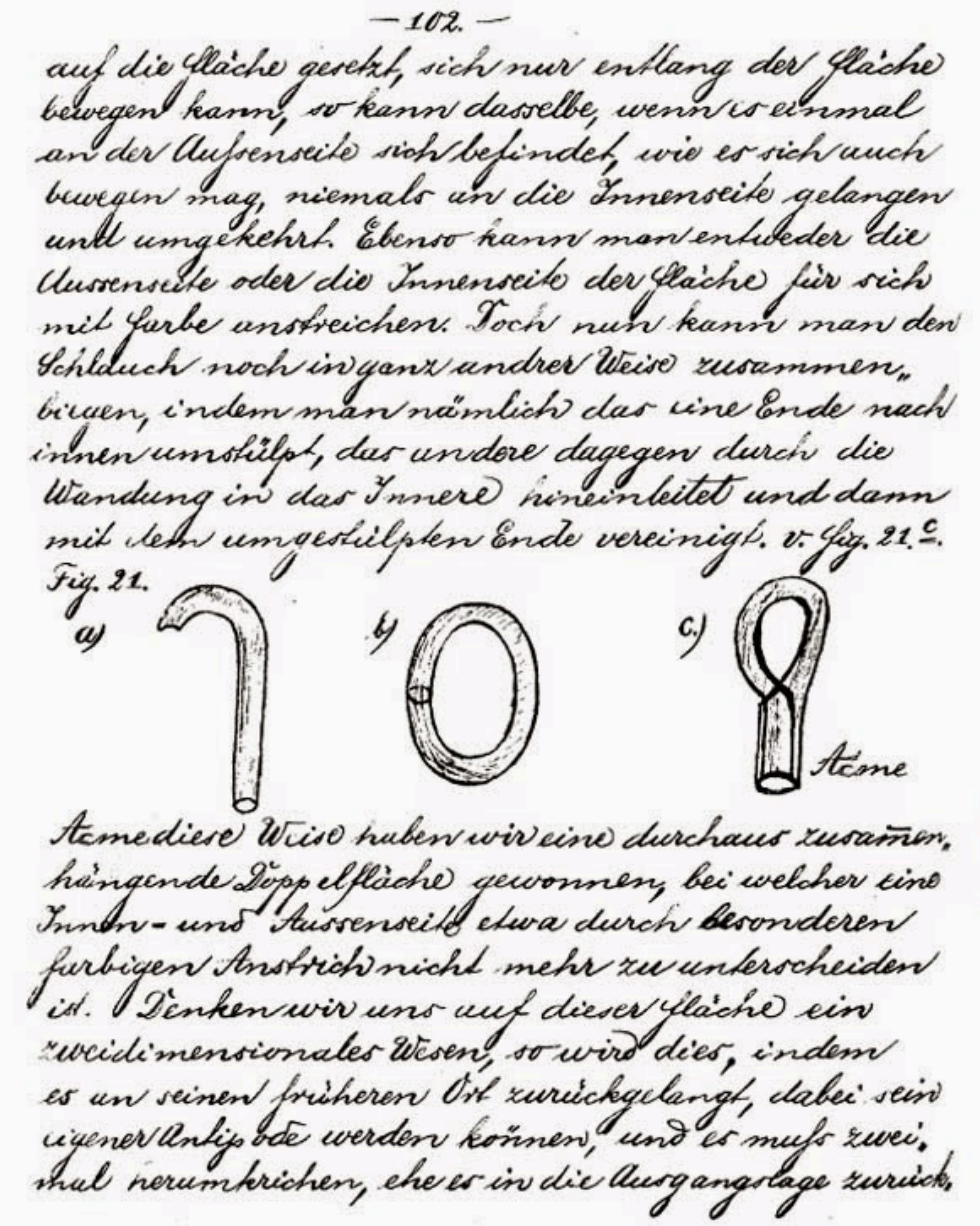


G. Petri/P. Expert

ISI Foundation
& ISI Global Science Foundation

CNTDA2019 , UKM, Kuala Lumpur
Sept 2019

What do we mean by topology?



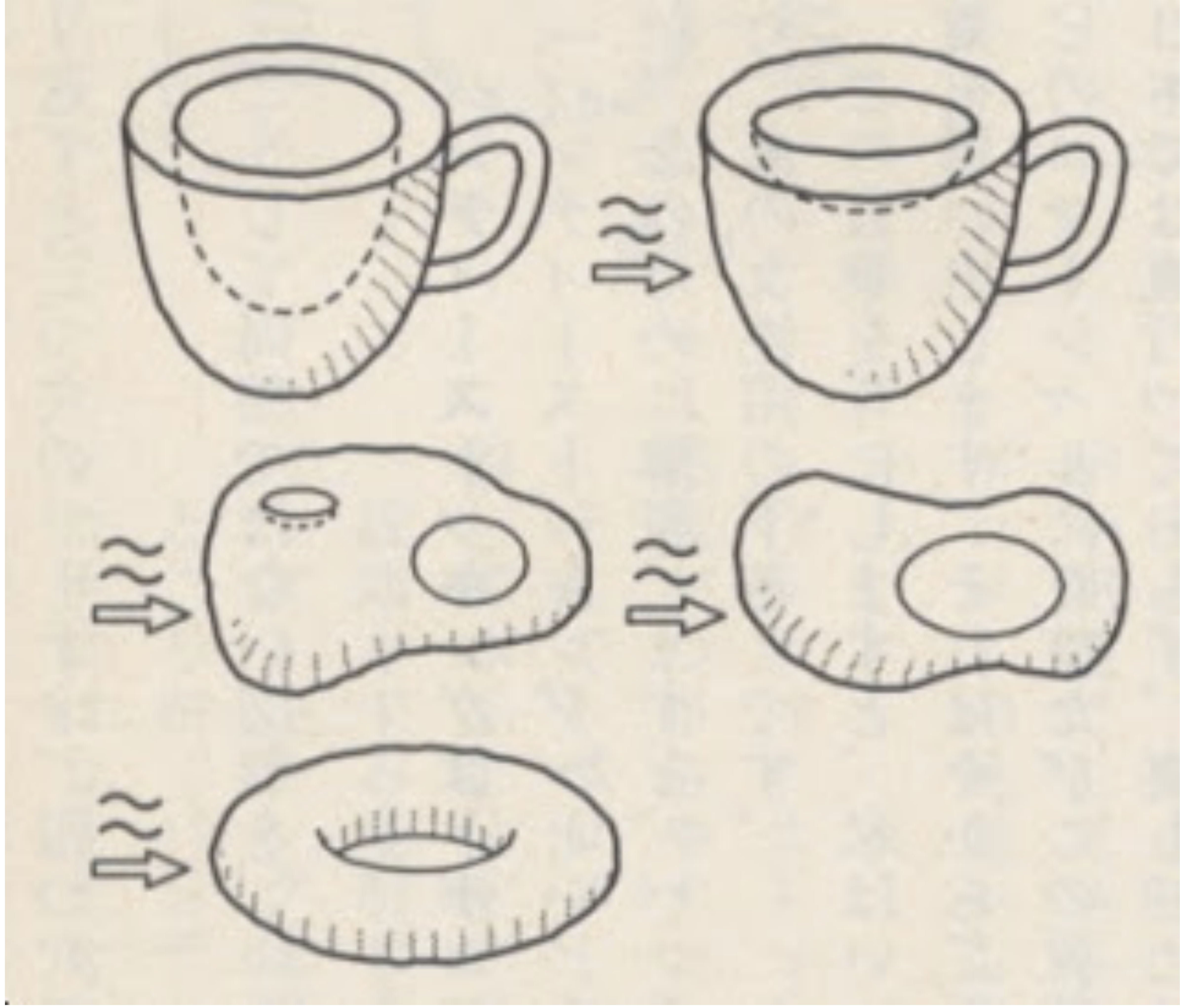
— 102. —

auf die Fläche gesetzt, sich nur entlang der Fläche bewegen kann, so kann dasselbe, wenn es einmal an der Aussenseite sich befindet, wie es sich auch bewegen mag, niemals an die Innenseite gelangen und umgekehrt. Ebenso kann man entweder die Aussenseite oder die Innenseite der Fläche für sich mit Farbe anstreichen. Doch nun kann man den Schlauch noch in ganz andrer Weise zusammenbiegen, indem man nämlich das eine Ende nach innen umstülpt, das andere dagegen durch die Wandung in das Innere hineinleitet und dann mit dem umgestülpten Ende vereinigt. v. Fig. 21 c.

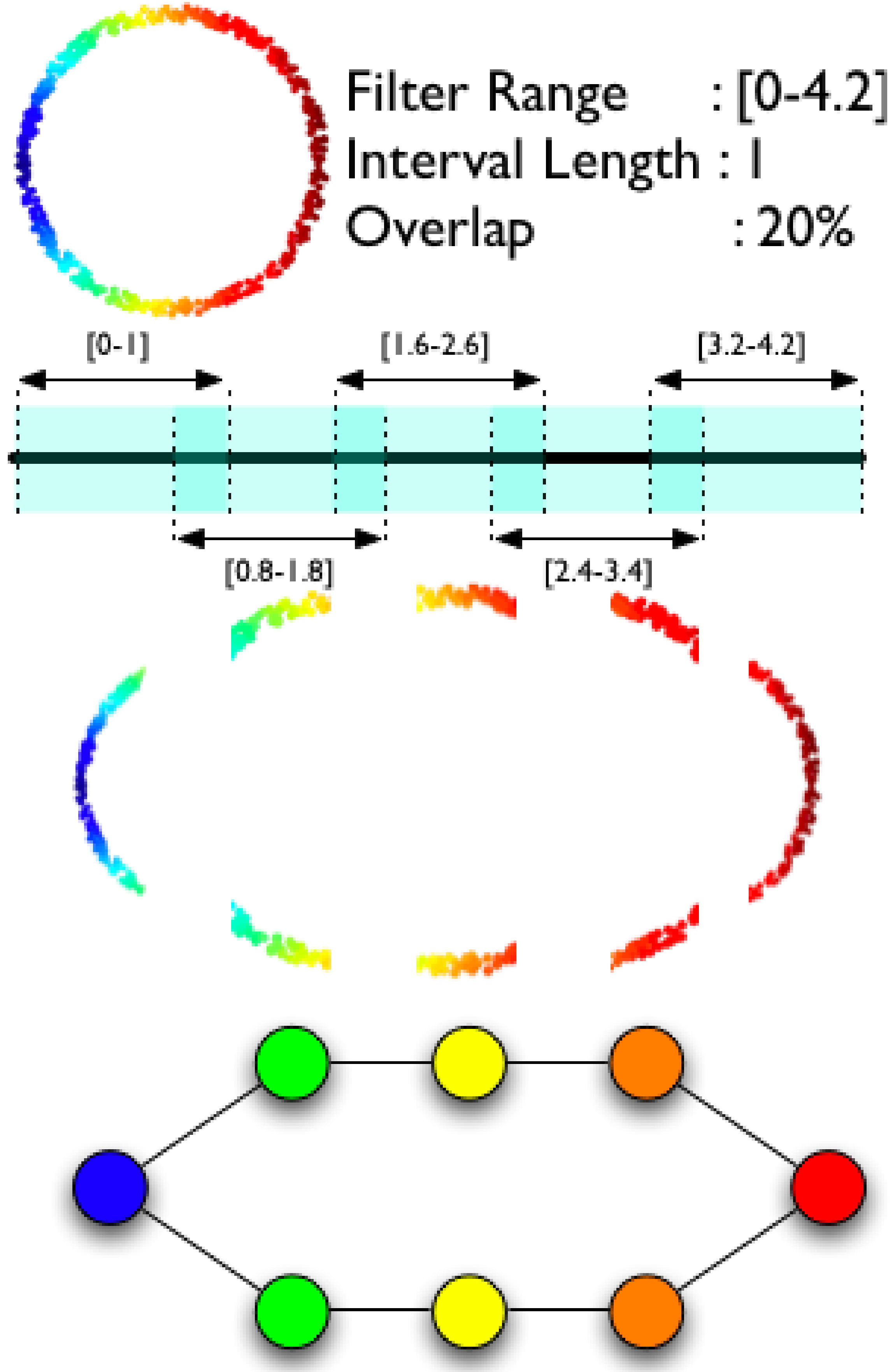
Fig. 21.

a) b) c) Acme

Auf diese Weise haben wir eine durchaus zusammenhängende Doppelfläche gewonnen, bei welcher eine Innen- und Aussenseite etwa durch besonderen farbigen Anstrich nicht mehr zu unterscheiden ist. Denken wir uns auf dieser Fläche ein zweidimensionales Wesen, so wird dies, indem es an seinen früheren Ort zurückgelangt, dabei sein eigener Antipode werden können, und es muss zweimal herumkriechen, ehe es in die Ausgangslage zurück.

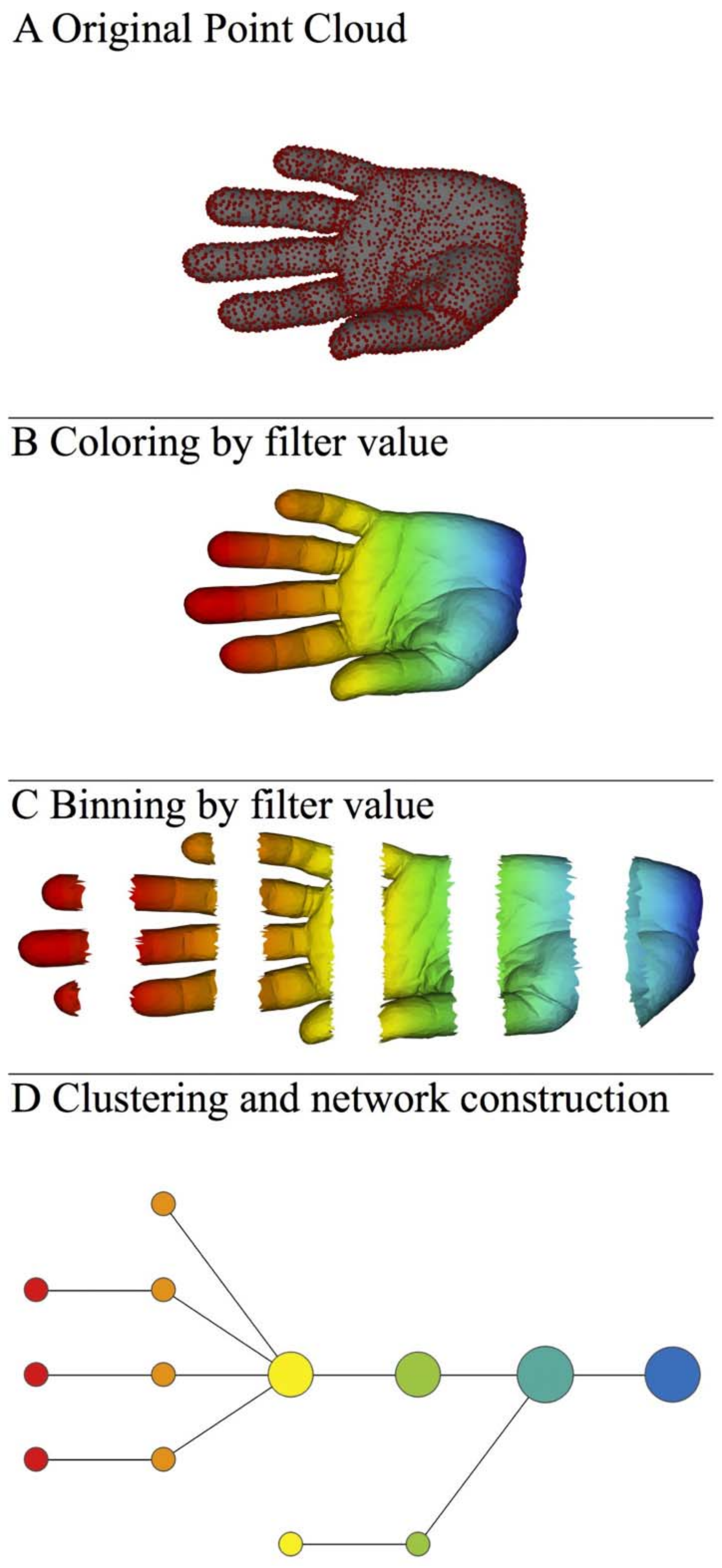


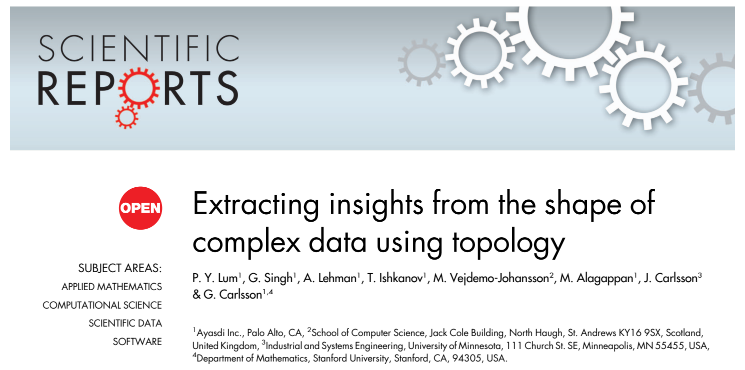
TDA as topological simplification



Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition

Gurjeet Singh[1] , Facundo Mémoli[2] and Gunnar Carlsson[†2]





SCIENTIFIC REPORTS

OPEN

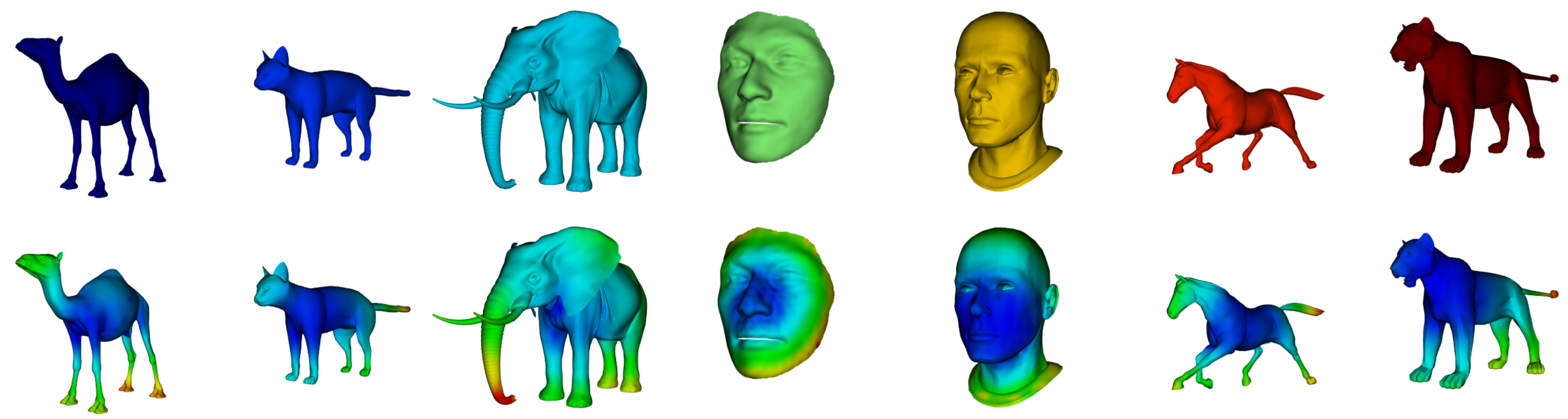
Extracting insights from the shape of complex data using topology

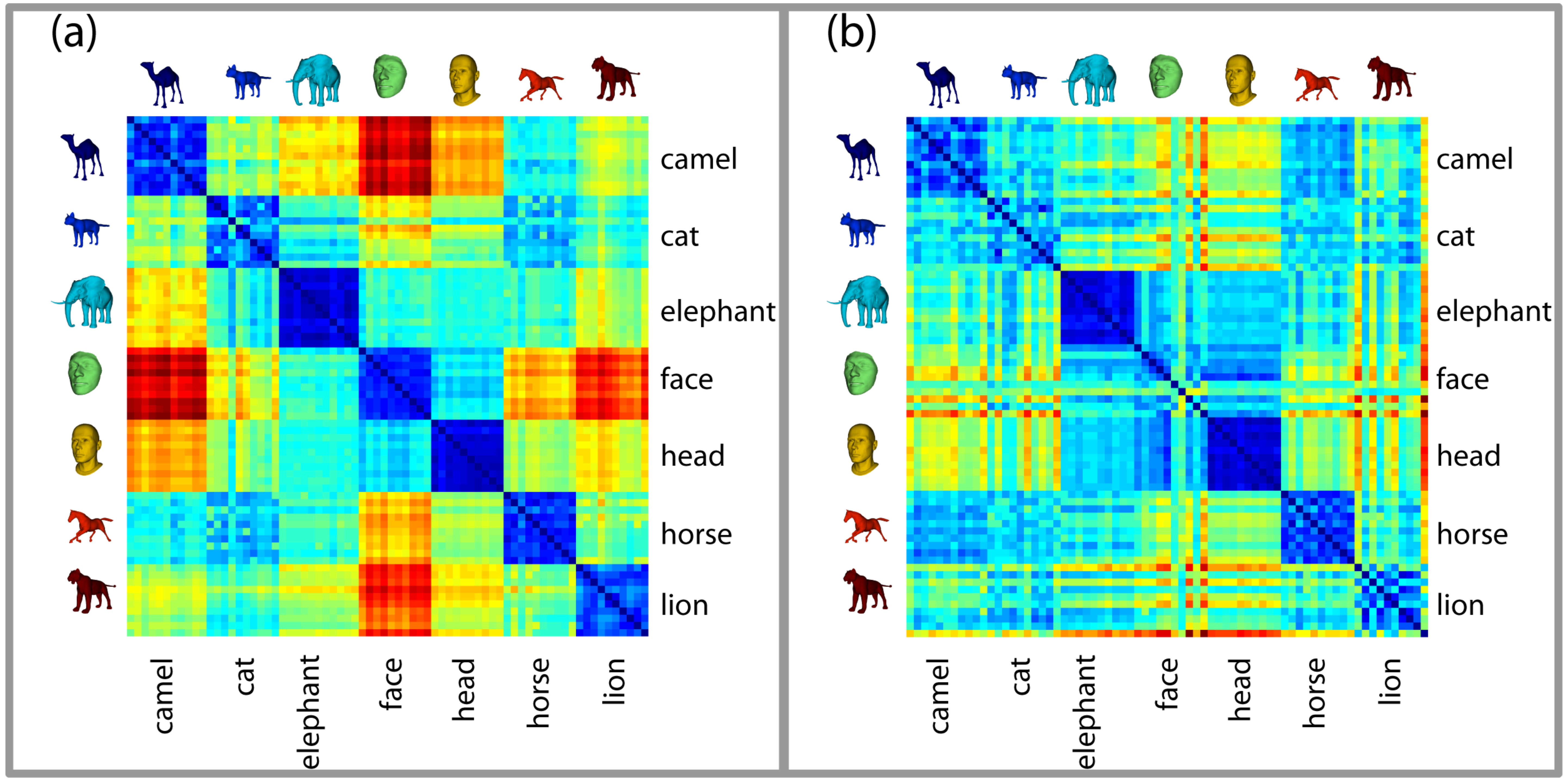
SUBJECT AREAS:
APPLIED MATHEMATICS
COMPUTATIONAL SCIENCE
SCIENTIFIC DATA
SOFTWARE

P. Y. Lum[1], G. Singh[1], A. Lehman[1], T. Ishkanov[1], M. Vejdemo-Johansson[2], M. Alagappan[1], J. Carlsson[3] & G. Carlsson[1,4]

[1]Ayasdi Inc., Palo Alto, CA, [2]School of Computer Science, Jack Cole Building, North Haugh, St. Andrews KY16 9SX, Scotland, United Kingdom, [3]Industrial and Systems Engineering, University of Minnesota, 111 Church St. SE, Minneapolis, MN 55455, USA, [4]Department of Mathematics, Stanford University, Stanford, CA, 94305, USA.

TDA as topological simplification



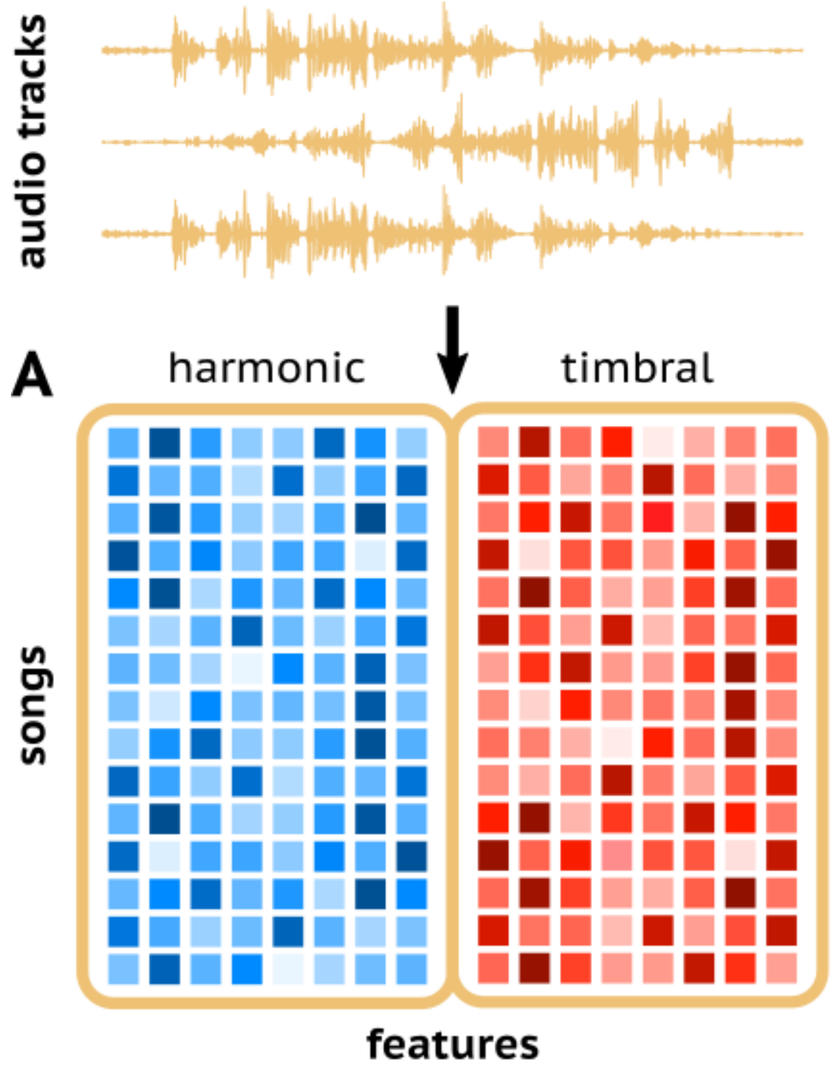


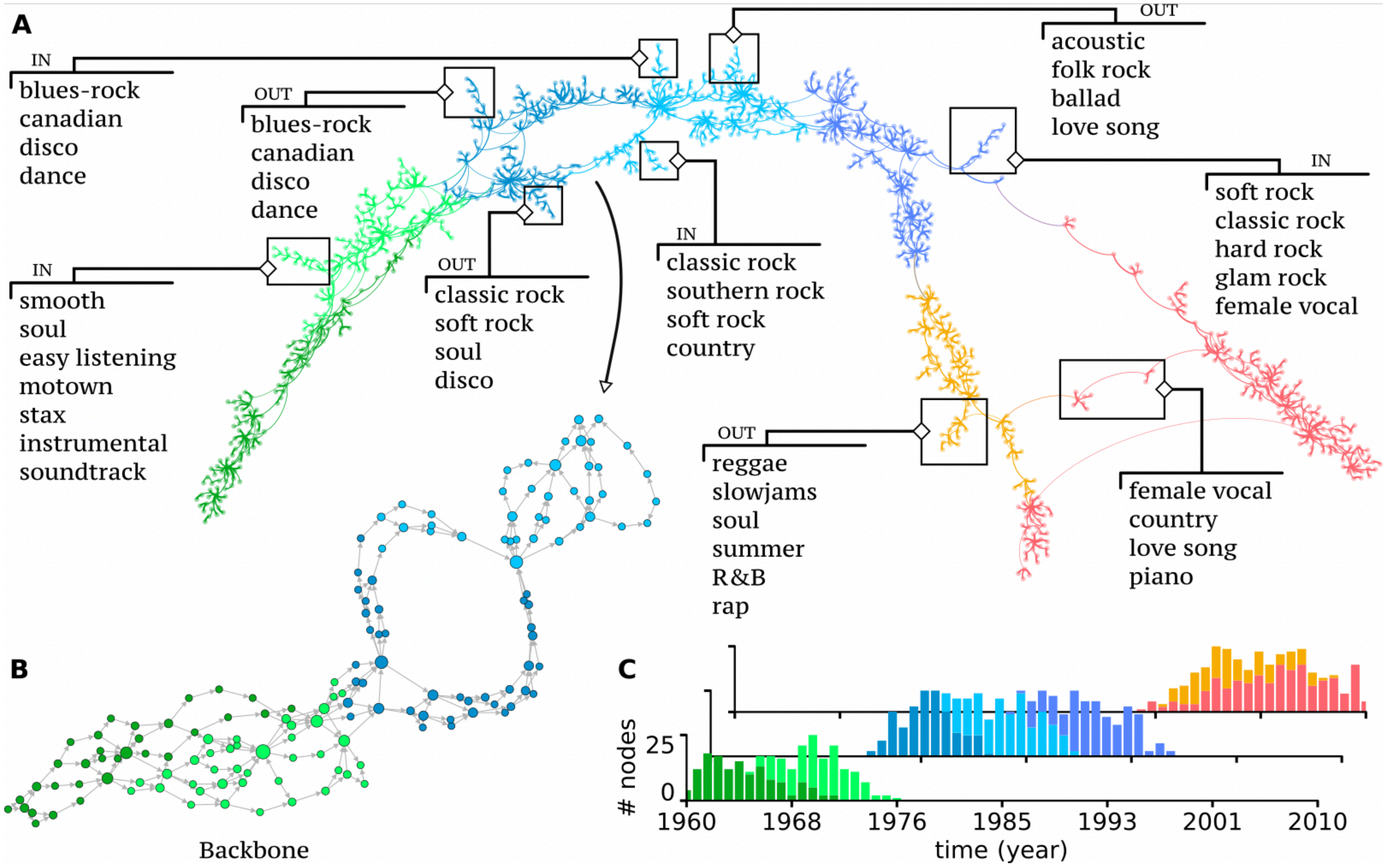
Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition

Gurjeet Singh[1] , Facundo Mémoli[2] and Gunnar Carlsson[†2]

Notebook 01

What does it mean in practice?

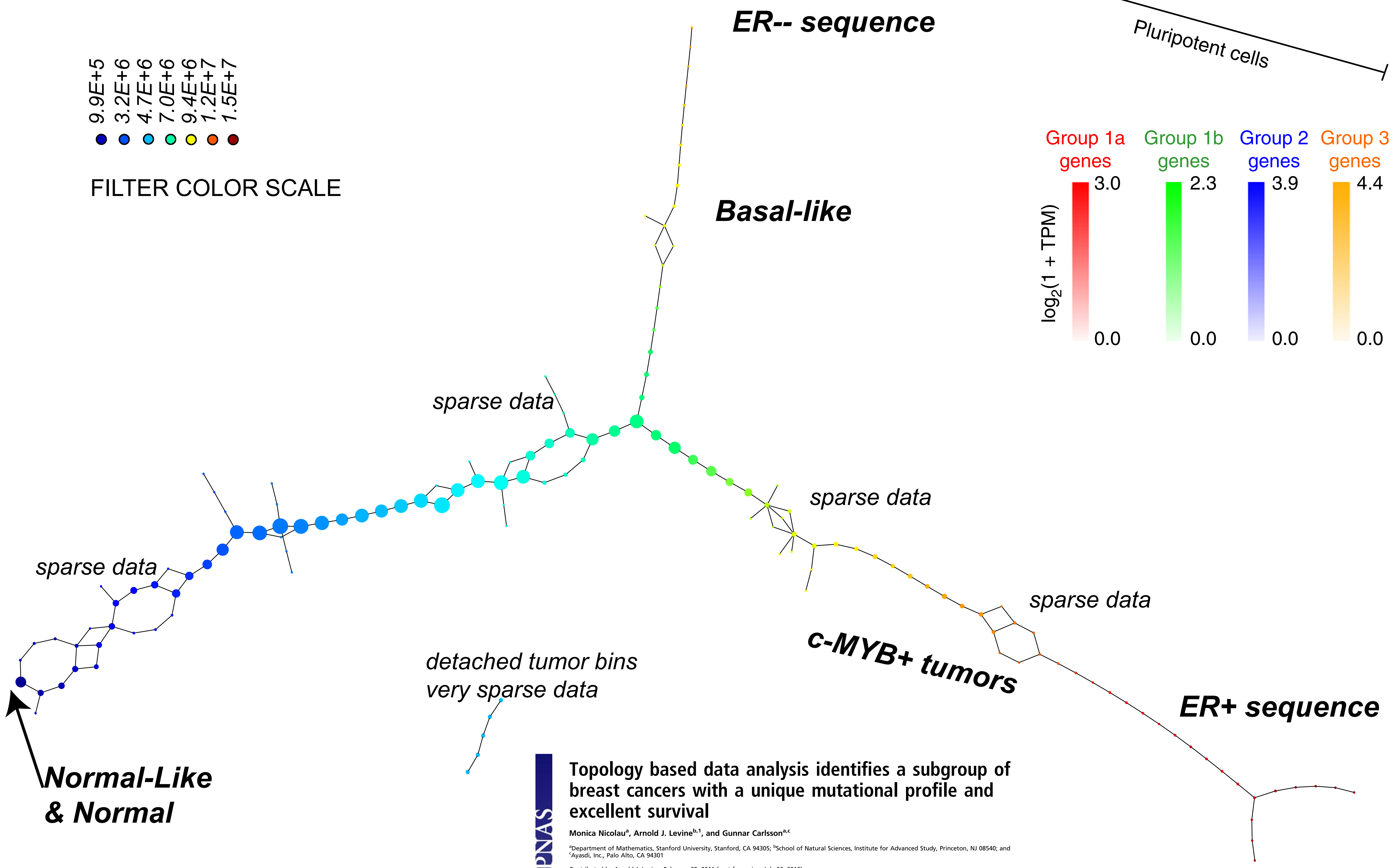




Iacopini, Petri, Latora, (2018) in preparation

Do topological gene-backbones carry information?

SPACE



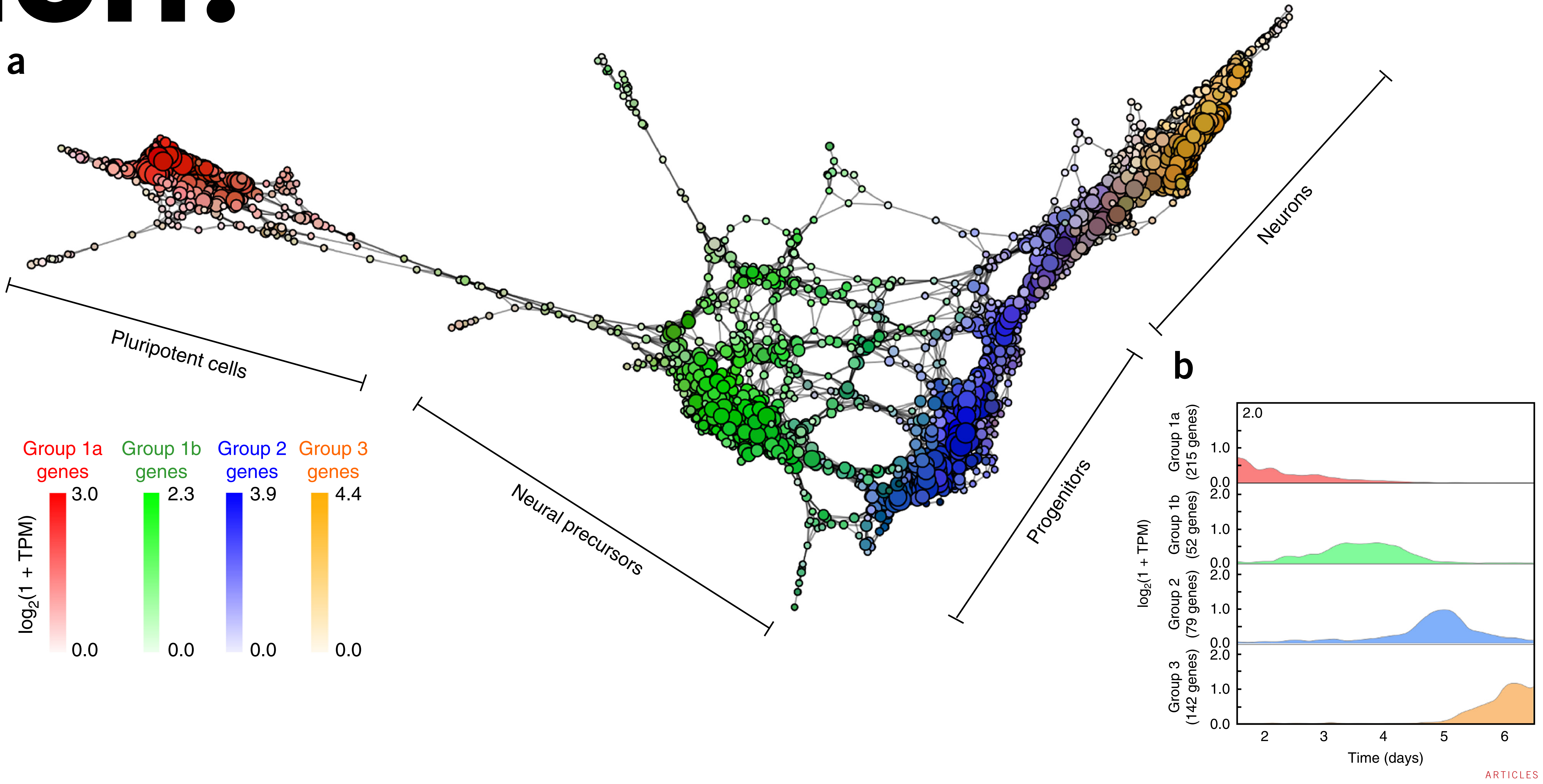
PNAS

Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival

Monica Nicolau[a], Arnold J. Levine[b,1], and Gunnar Carlsson[a,c]

[a]Department of Mathematics, Stanford University, Stanford, CA 94305; [b]School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540; and [c]Ayasdi, Inc., Palo Alto, CA 94301

Contributed by Arnold J. Levine, February 25, 2011 (sent for review July 23, 2010)



ARTICLES

nature biotechnology

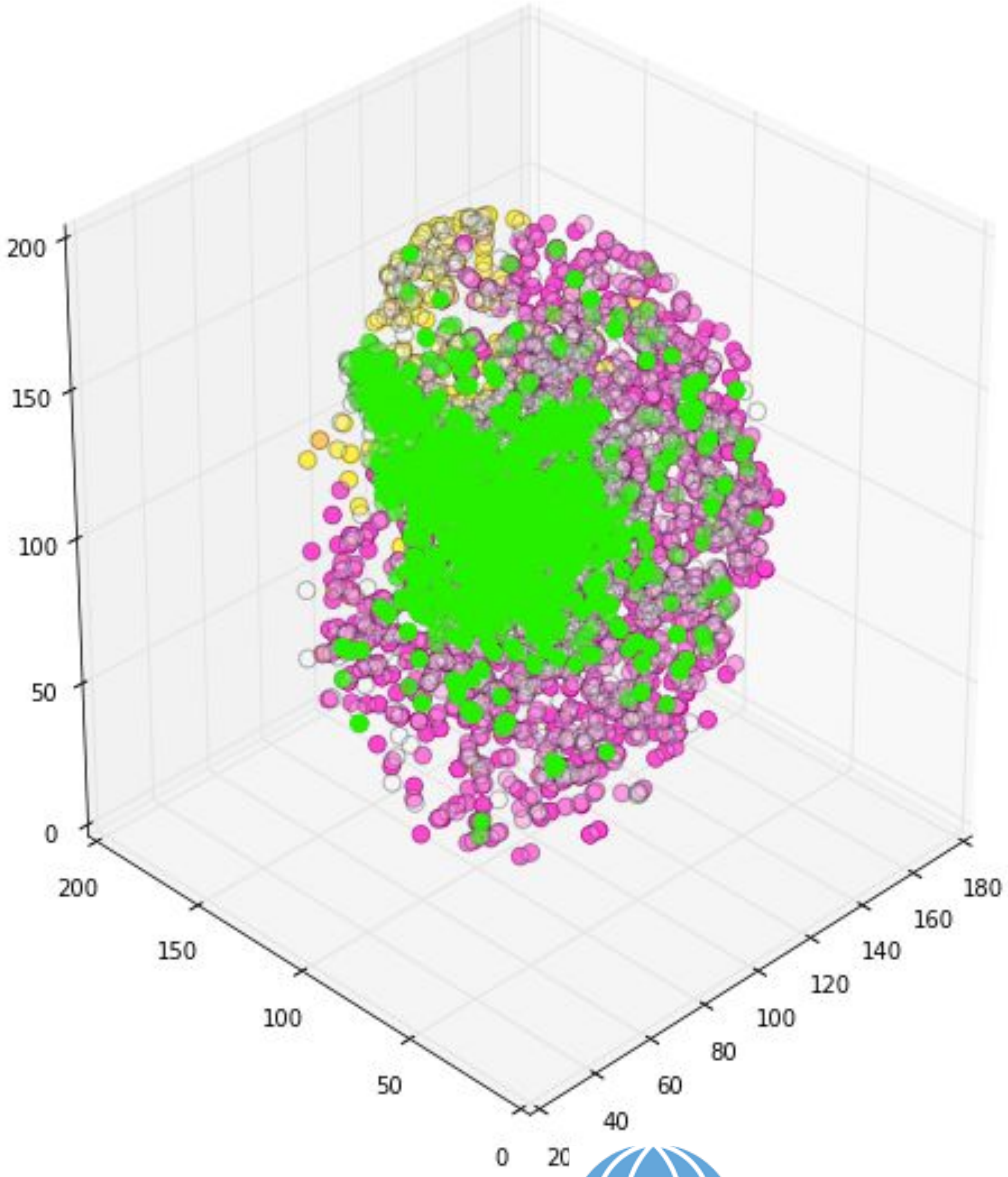
Single-cell topological RNA-seq analysis reveals insights into cellular differentiation and development

Abbas H Rizvi[1,2,6], Pablo G Camara[3,4,6], Elena K Kandror[1,2], Thomas J Roberts[1,2,4], Ira Schieren[2,5], Tom Maniatis[1,2] & Raul Rabadan[3,4]

MAPPER

Notebook 02

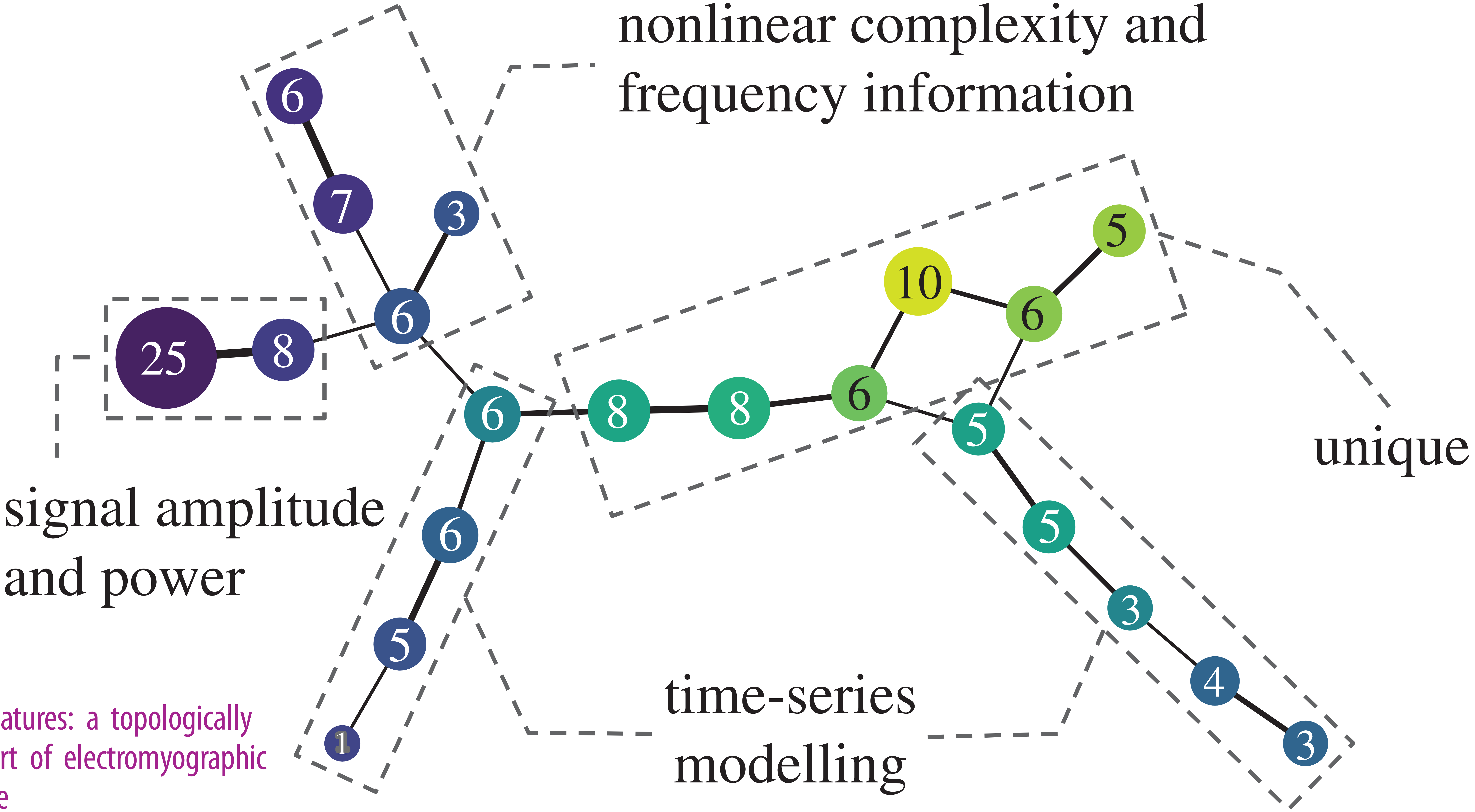
Do topological gene-backbones carry information?



ALLEN INSTITUTE *for* BRAIN SCIENCE

Patania, Selvaggi, di Pasquale, Veronese, Expert, Petri (2019) “Topological gene-expression networks recapitulate brain anatomy and function”, in review Net. Neuroscience **(ISI+Imperial+KCL+IUNI)** https://www.biorxiv.org/content/10.1101/476382v1

Can we build topological feature charts?



INTERFACE

rsif.royalsocietypublishing.org

Research

Navigating features: a topologically informed chart of electromyographic features space

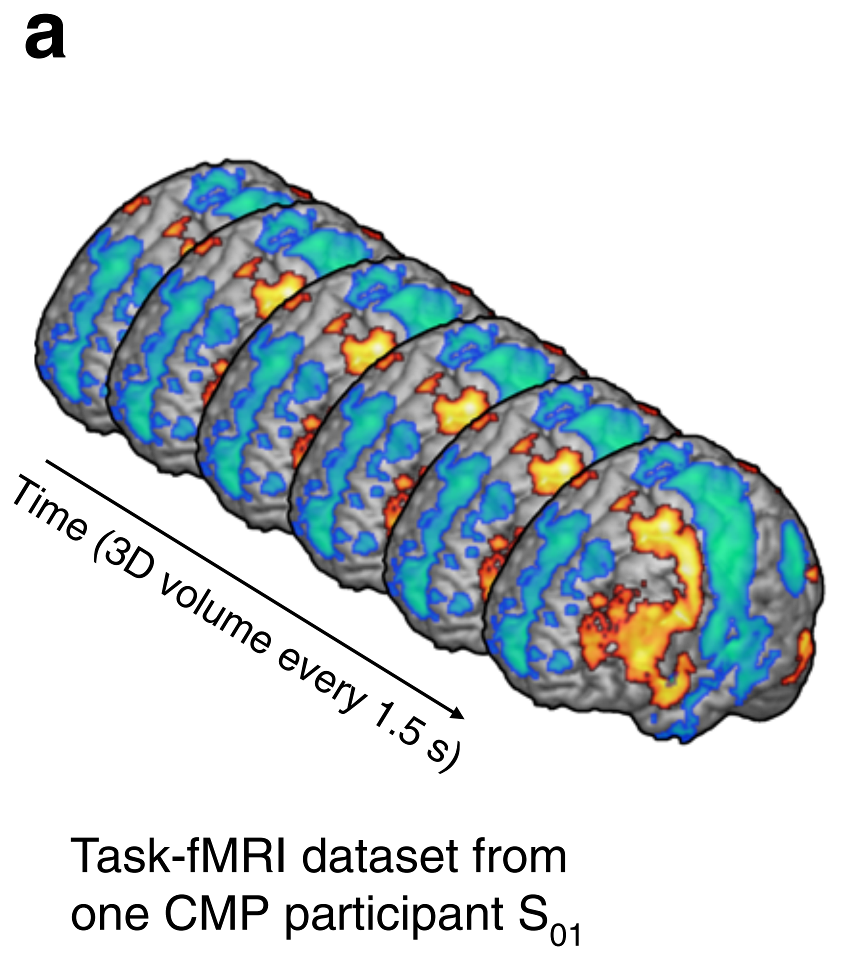
Angkoon Phinyomark[1,2], Rami N. Khushaba[3], Esther Ibáñez-Marcelo[1], Alice Patania[4], Erik Scheme[2] and Giovanni Petri[1]

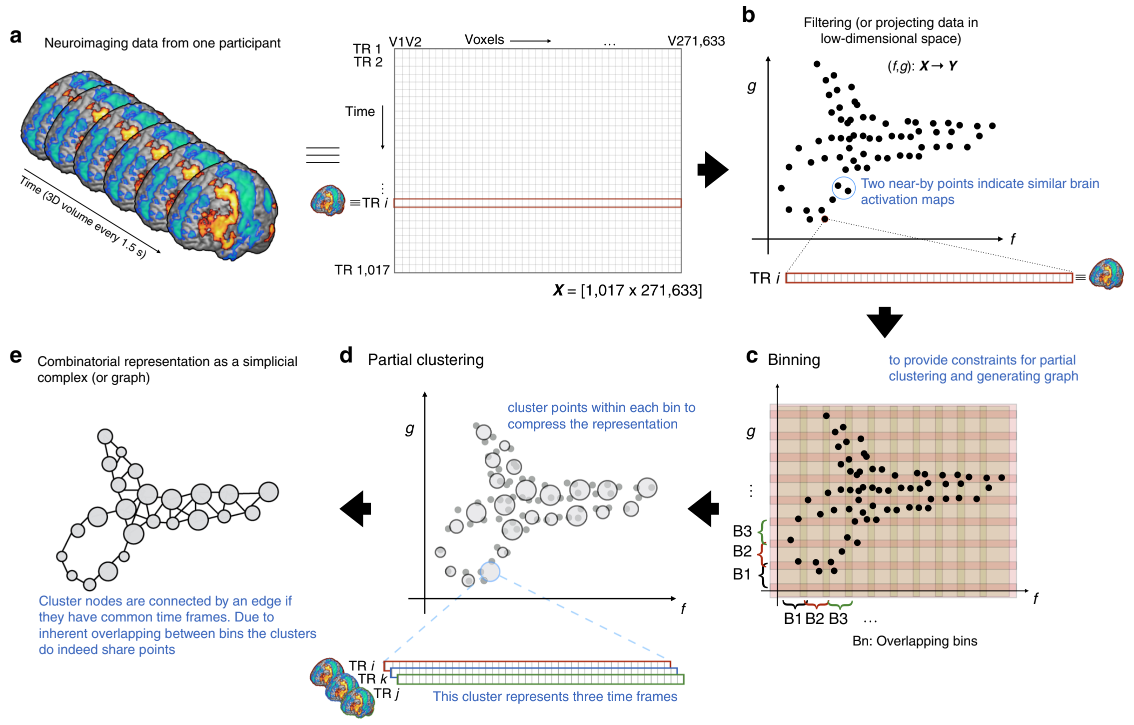
(ISI+Fredericton University+Sydney)

Notebook 03

Approximate activity landscapes using topology

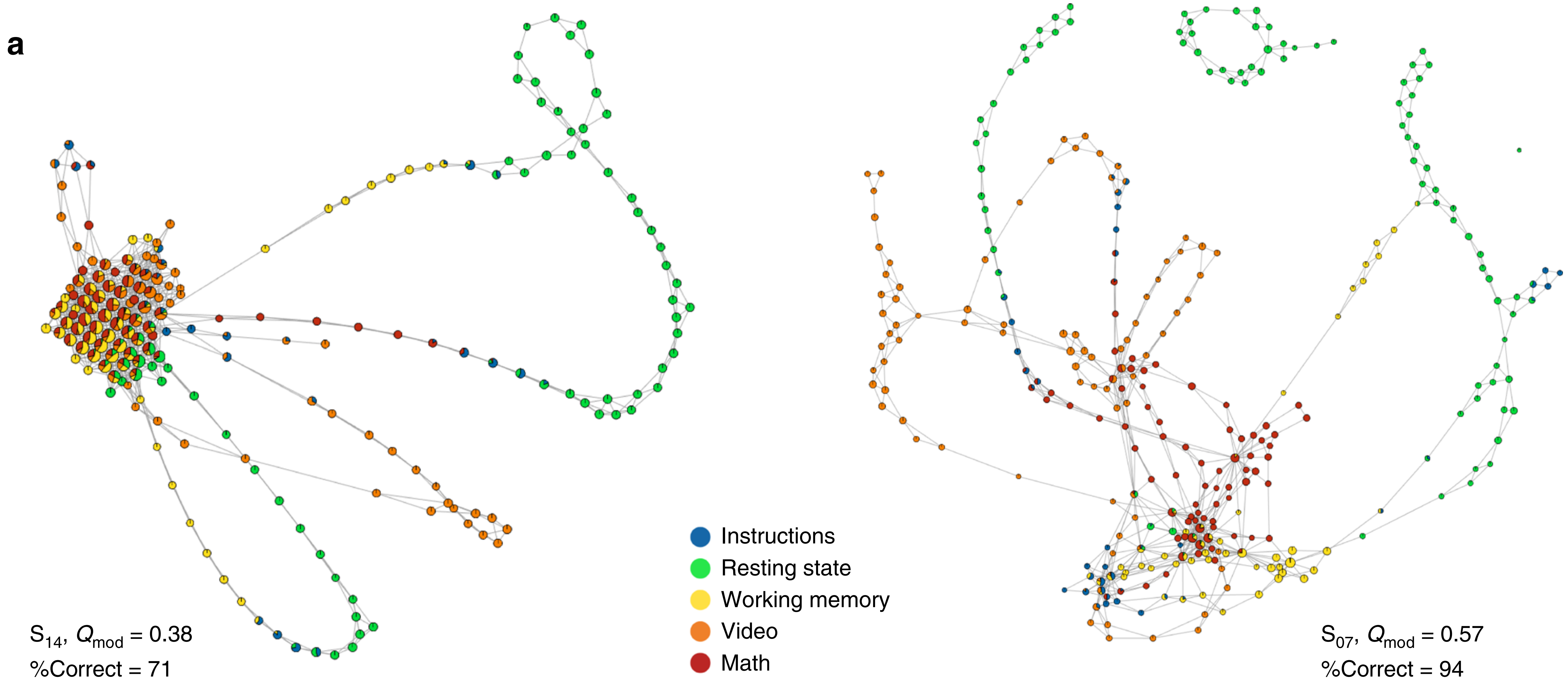
Task brain activation data



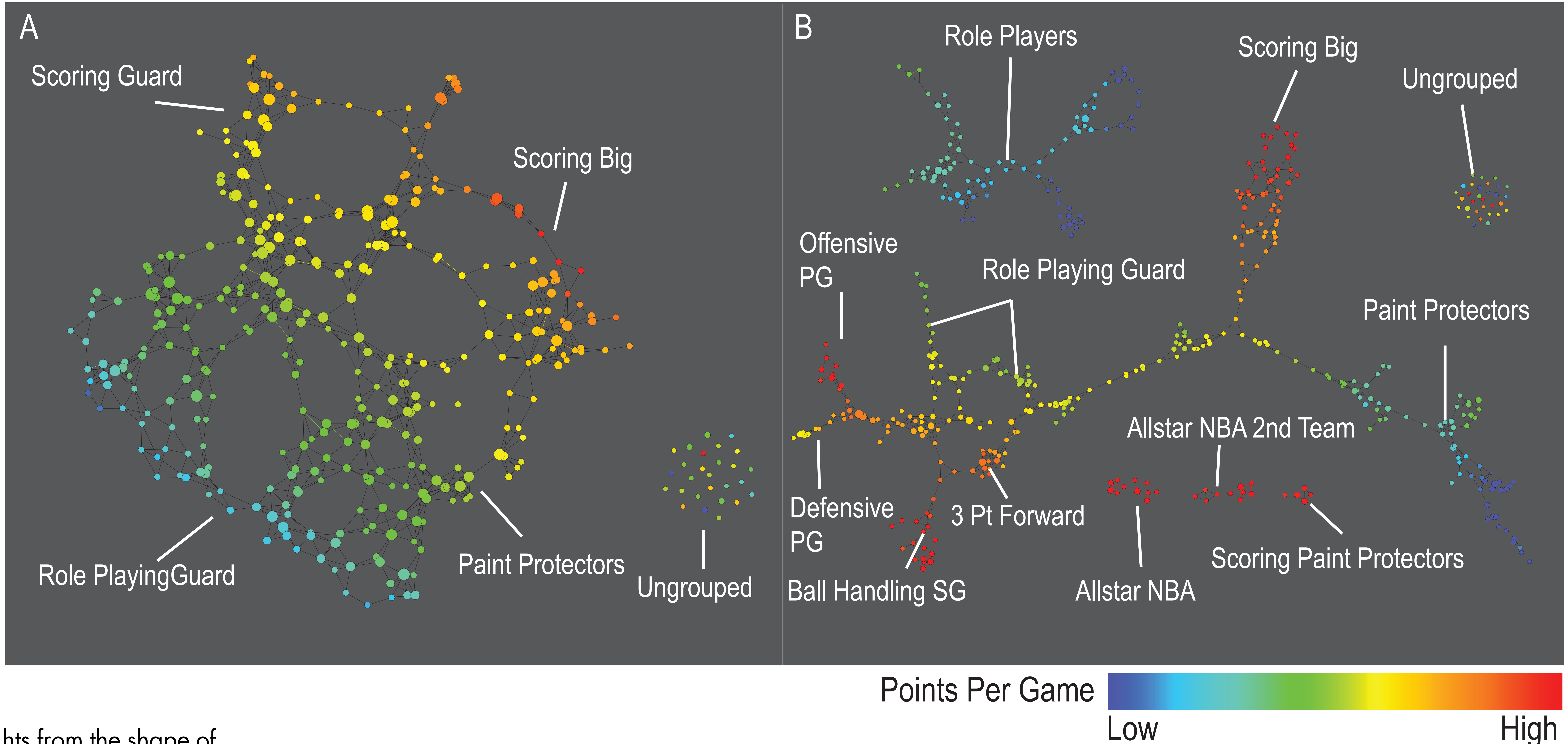


Saggar, Manish, et al. "Towards a new approach to reveal dynamical organization of the brain using topological data analysis." *Nature communications* 9.1 (2018): 1399.

Approximate activity landscapes using topology



Multi-resolution analysis?

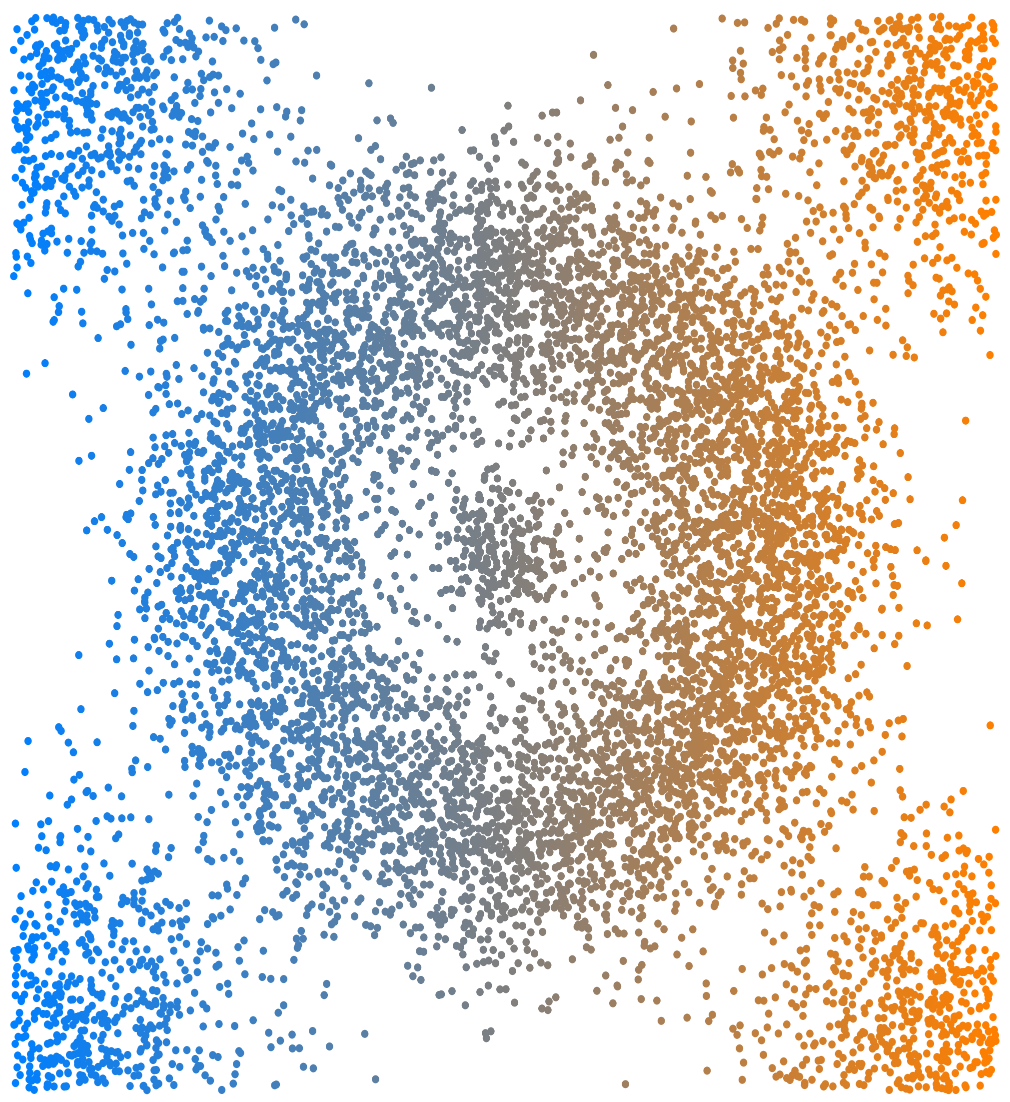


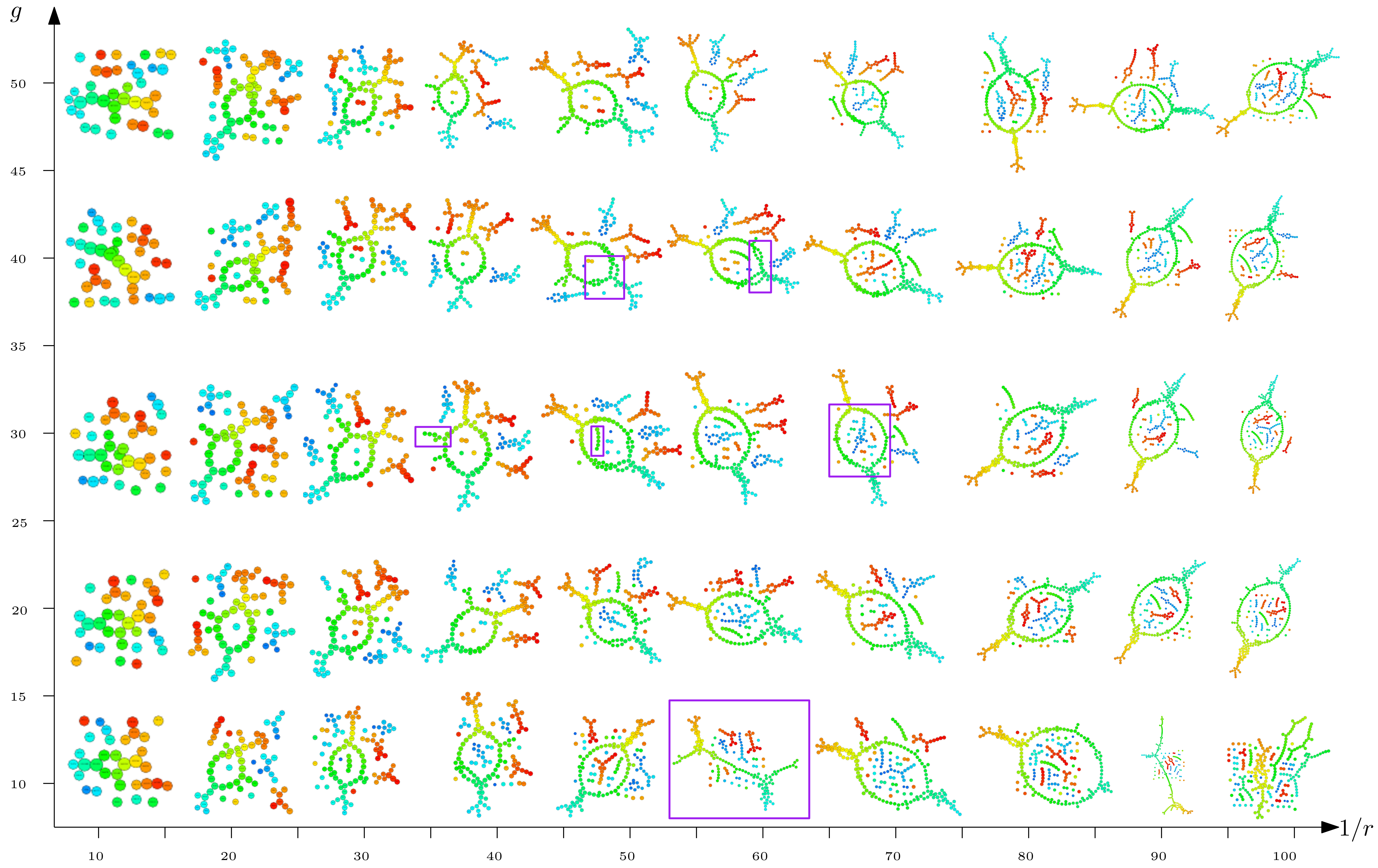
Extracting insights from the shape of complex data using topology

P. Y. Lum[1], G. Singh[1], A. Lehman[1], T. Ishkanov[1], M. Vejdemo-Johansson[2], M. Alagappan[1], J. Carlsson[3] & G. Carlsson[1,4]

https://www.wired.com/2012/04/analytics-basketball/

Automatic Mapper parameter choice





Statistical Analysis and Parameter Selection for Mapper

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Bertrand Michel BERTRAND.MICHEL@EC-NANTES.FR
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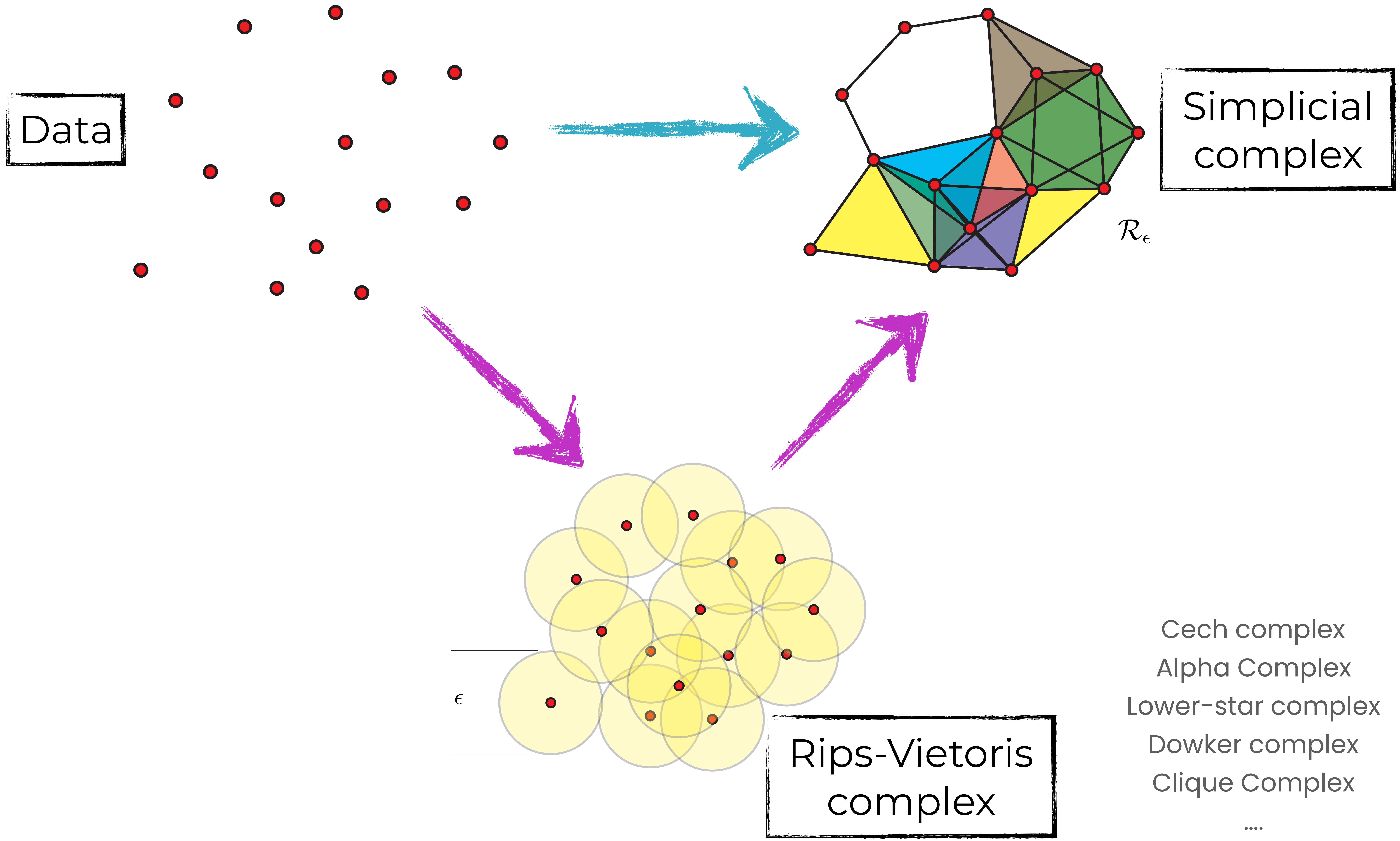
Inria Saclay
91120 Palaiseau, France

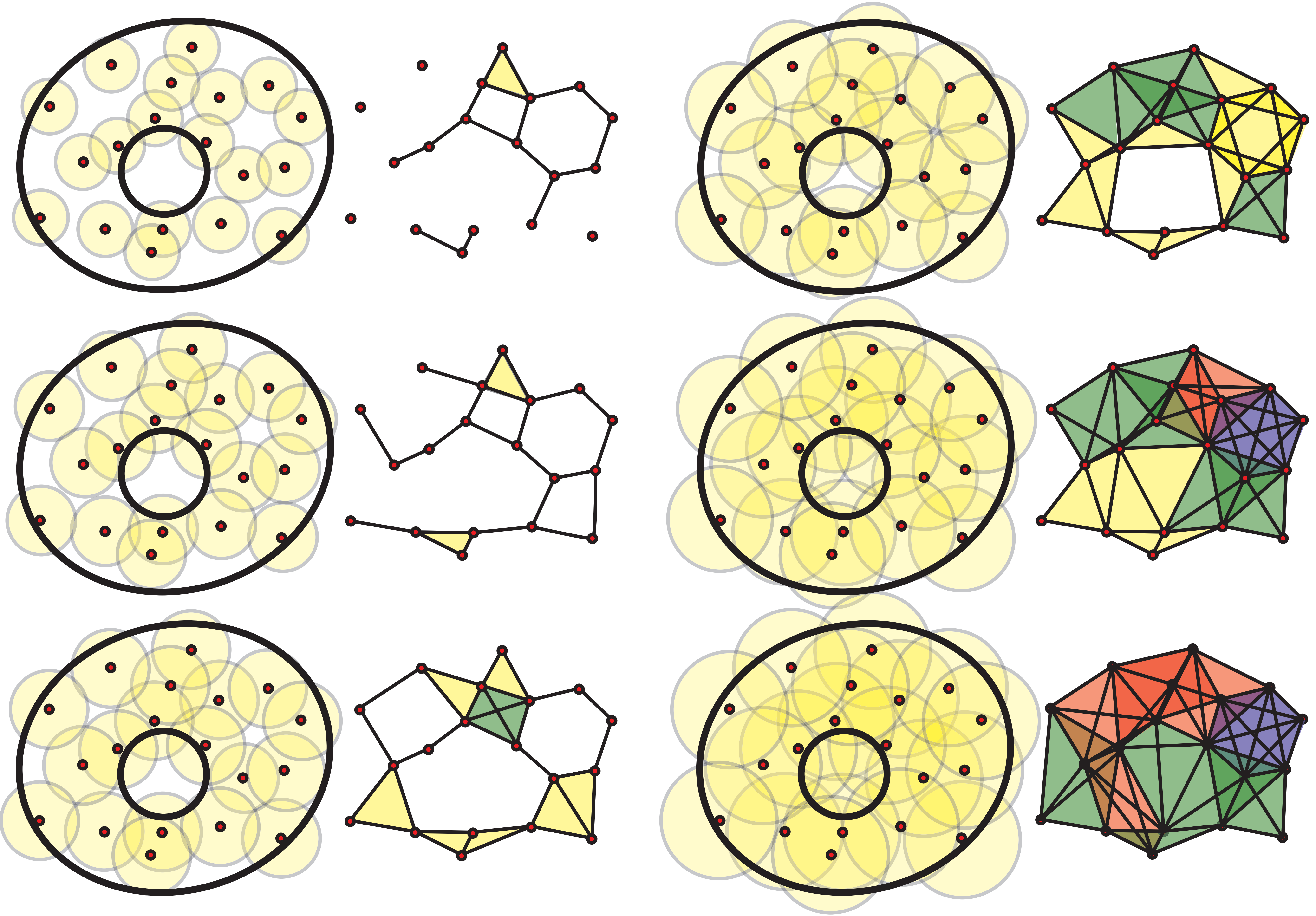
Selection based on “extended” persistent homology

The shape of data: what simplicial complex?



Cech complex
Alpha Complex
Lower-star complex
Dowker complex
Clique Complex
....

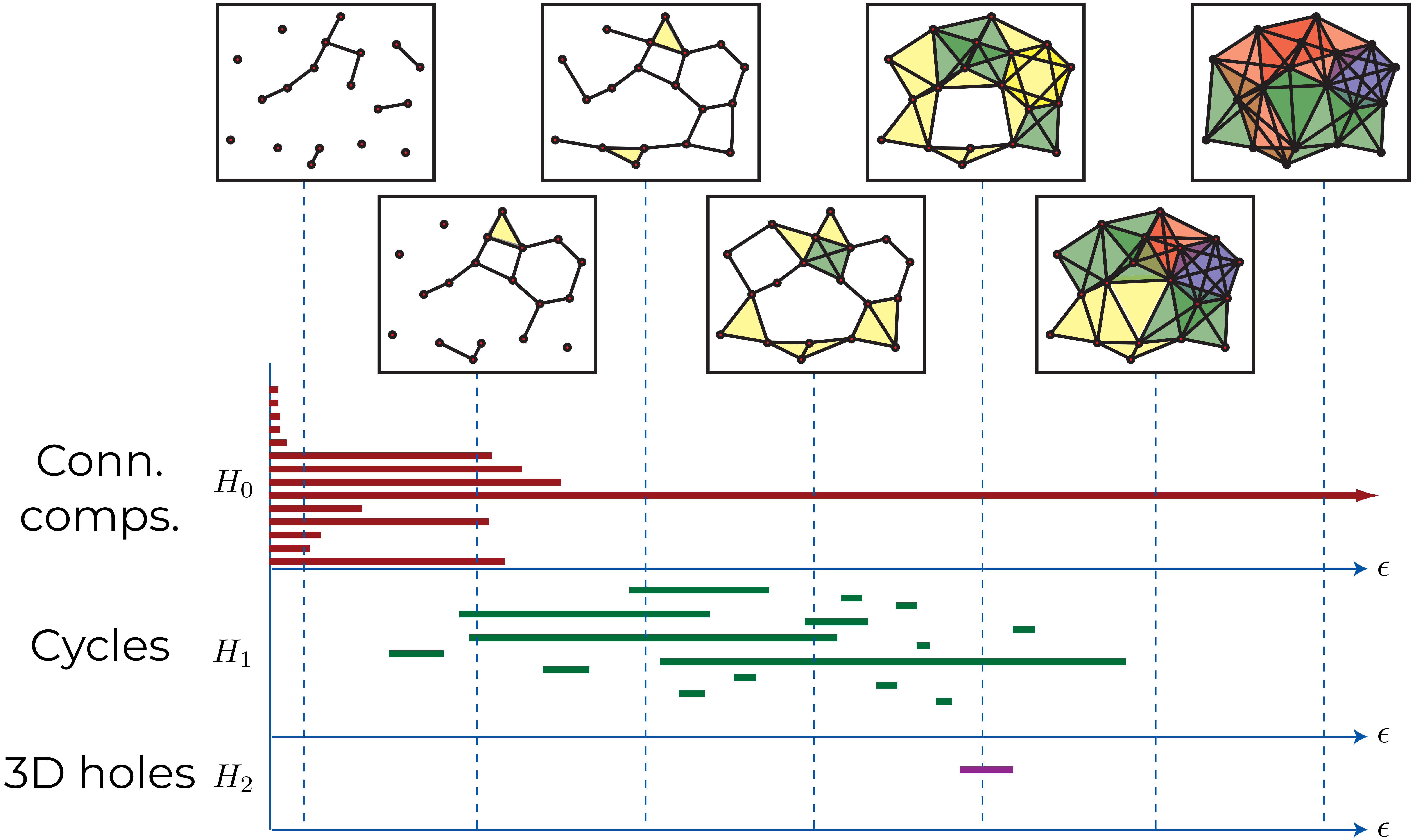
Persistent homology in two slides.. and a donut!



What is the appropriate **radius** for the balls?

Ghrist, R. (2008). Barcodes: The persistent topology of data. *Bulletin-American Mathematical Society*, *45*(1), 61.

Persistent homology: Barcode



Ghrist, R. (2008). Barcodes: The persistent topology of data. *Bulletin-American Mathematical Society*, *45*(1), 61.

Persistent homology in two slides.. and a donut!

The set of vector spaces and linear maps $\{H_n(X_t), H_n(i_t)\}_{t\in\mathbb{N}}$
can be represented as a finitely generated k[x]-module:

$$H_n = \bigoplus_{t\in\mathbb{N}} H_n(X_t) \qquad \text{with} \qquad \cdot x = H_n(X_t) \to H_n(X_{t+1})$$

$$m \to H_n(i_t)(m)$$

Structure theorem

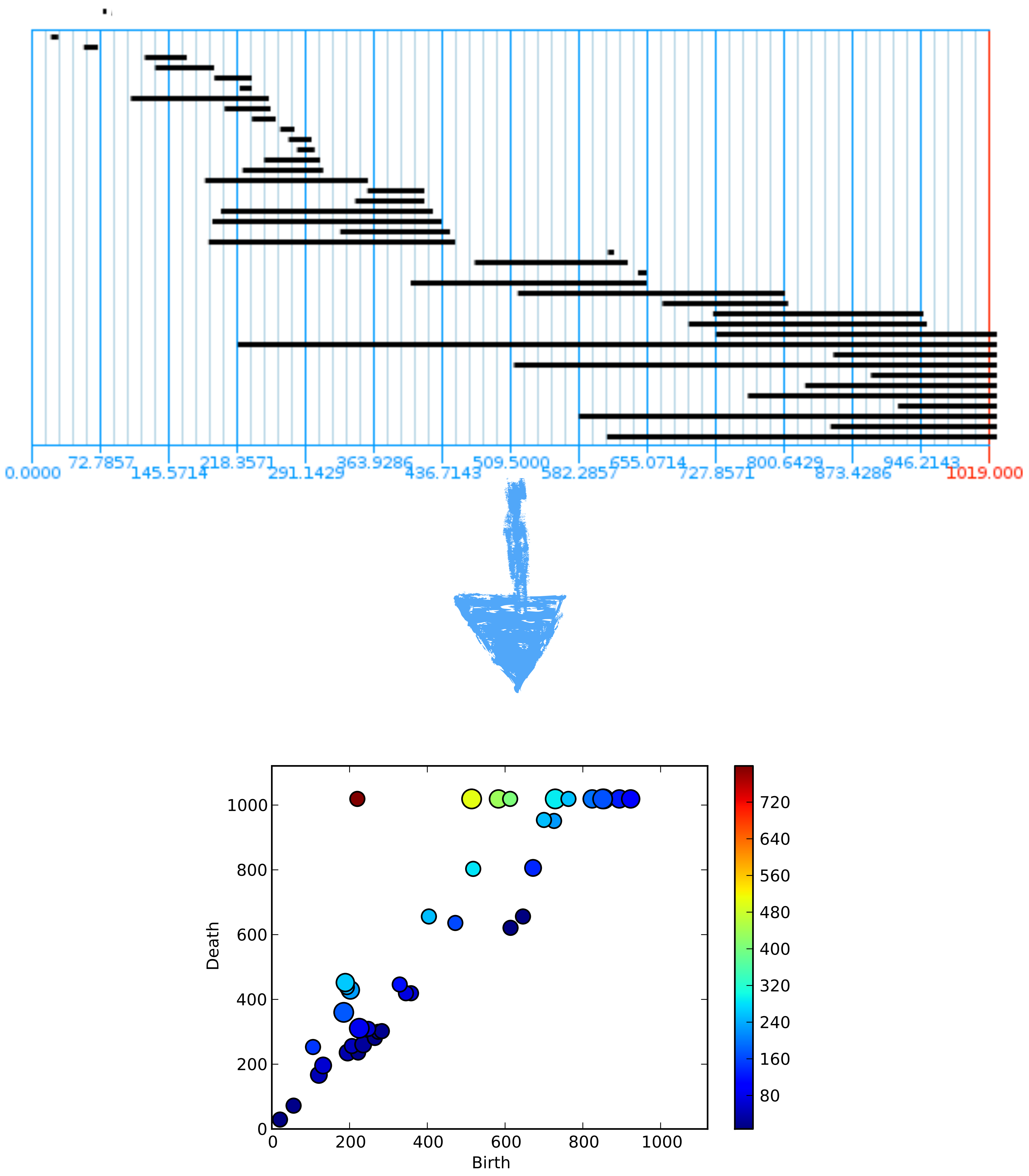
$$M \simeq \bigoplus_{j=1}^{n} k[x](-a_j) \bigoplus_{i=1}^{m} k[x](-c_i)/x_i^{d_i}$$

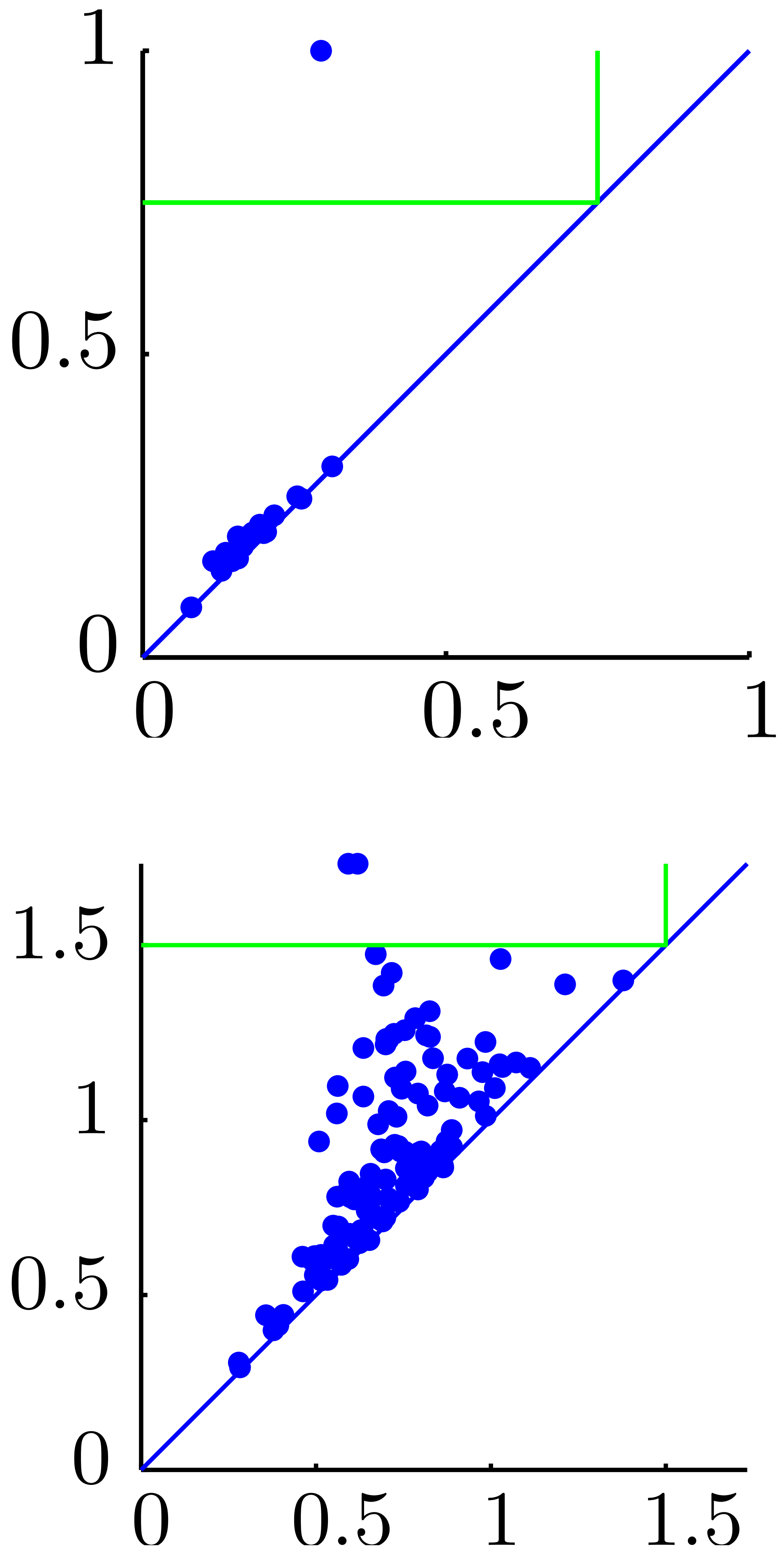
Remark

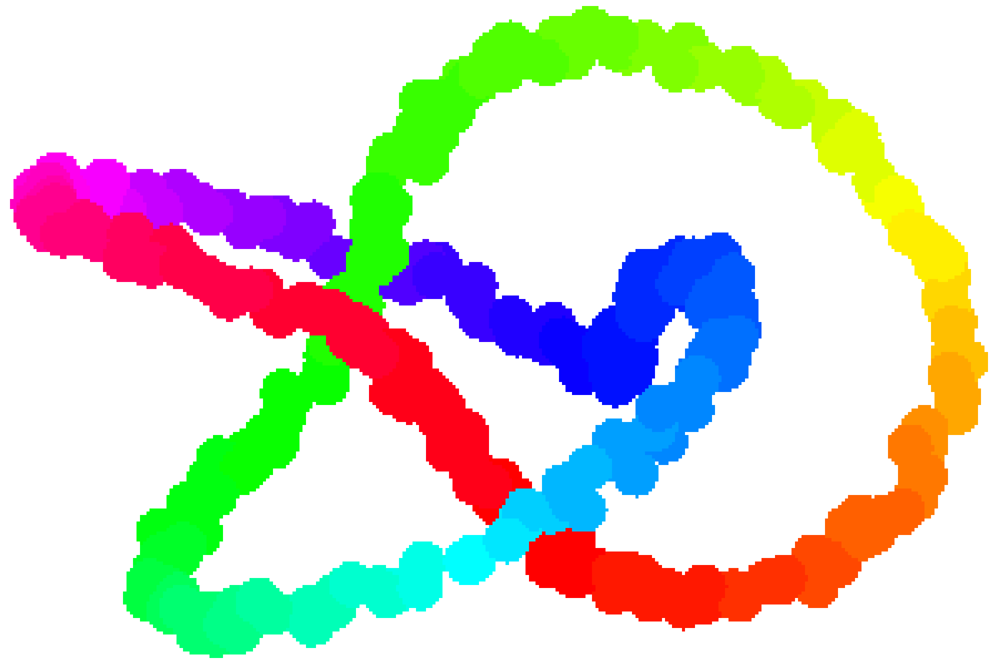
The degrees of the generators and the torsion degrees $\{d_i\}$ completely determine the module

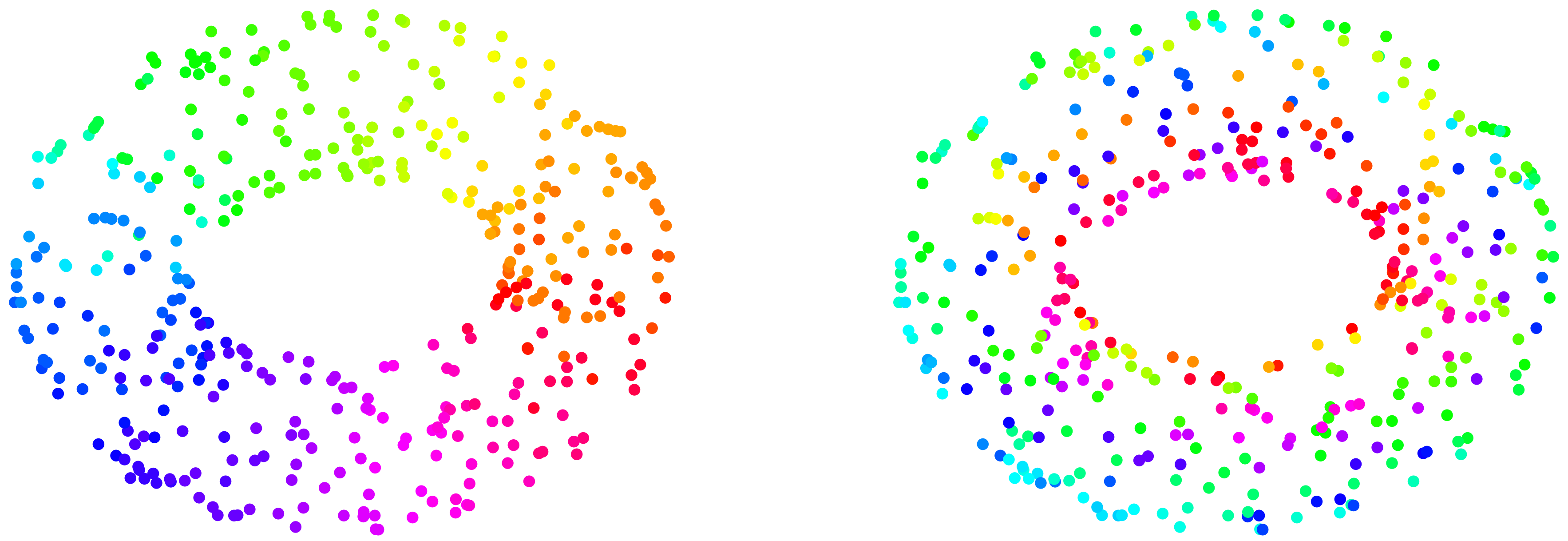
Barcode m finite intervals $[c_i, d_i)$ and n half intervals $[a_j, \infty)$

Barcode to Persistence Diagram



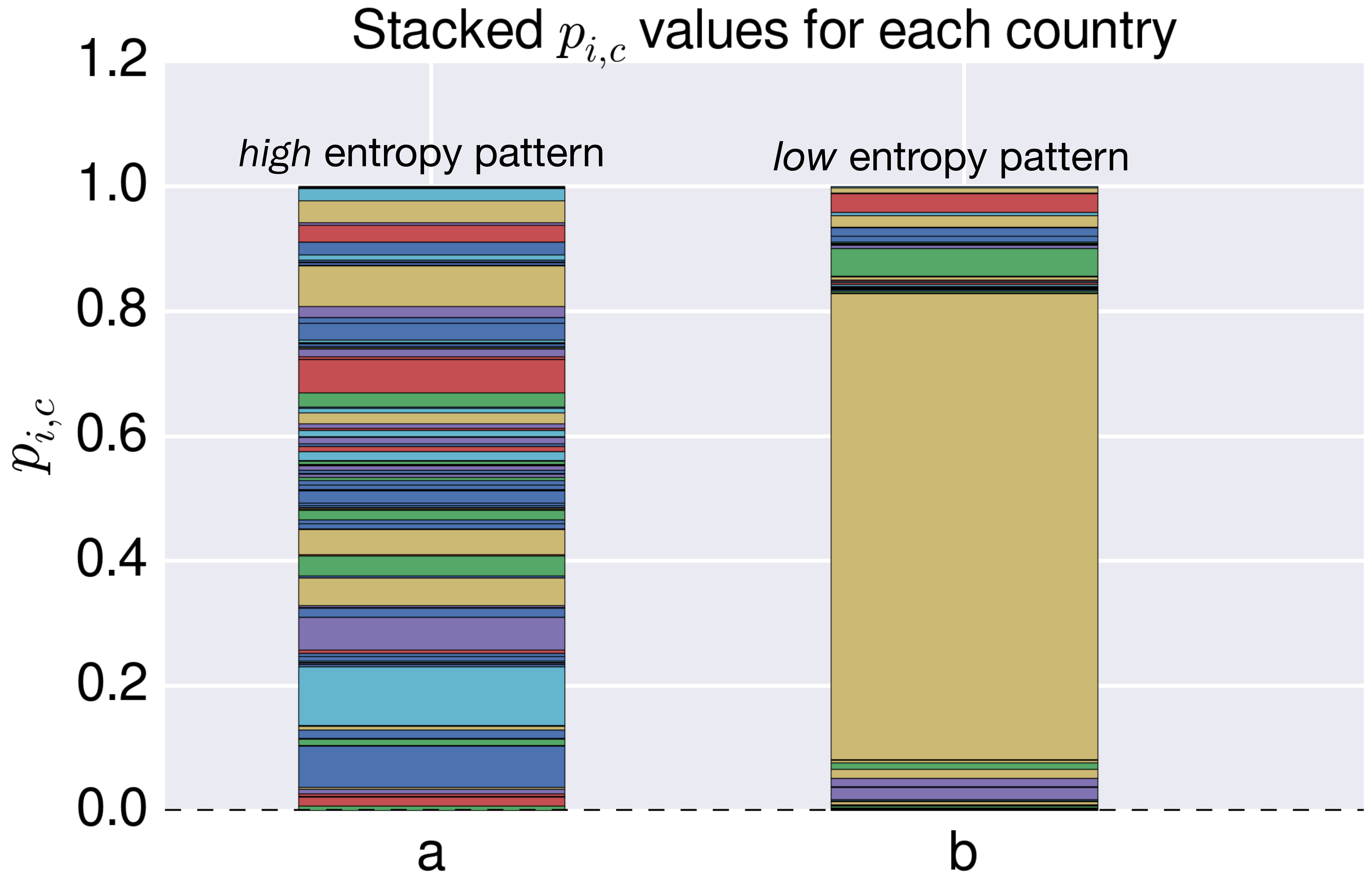






Notebook 04

Phom app: ENTROPY



$A_{i,c}(t)$ i = cell, c = country

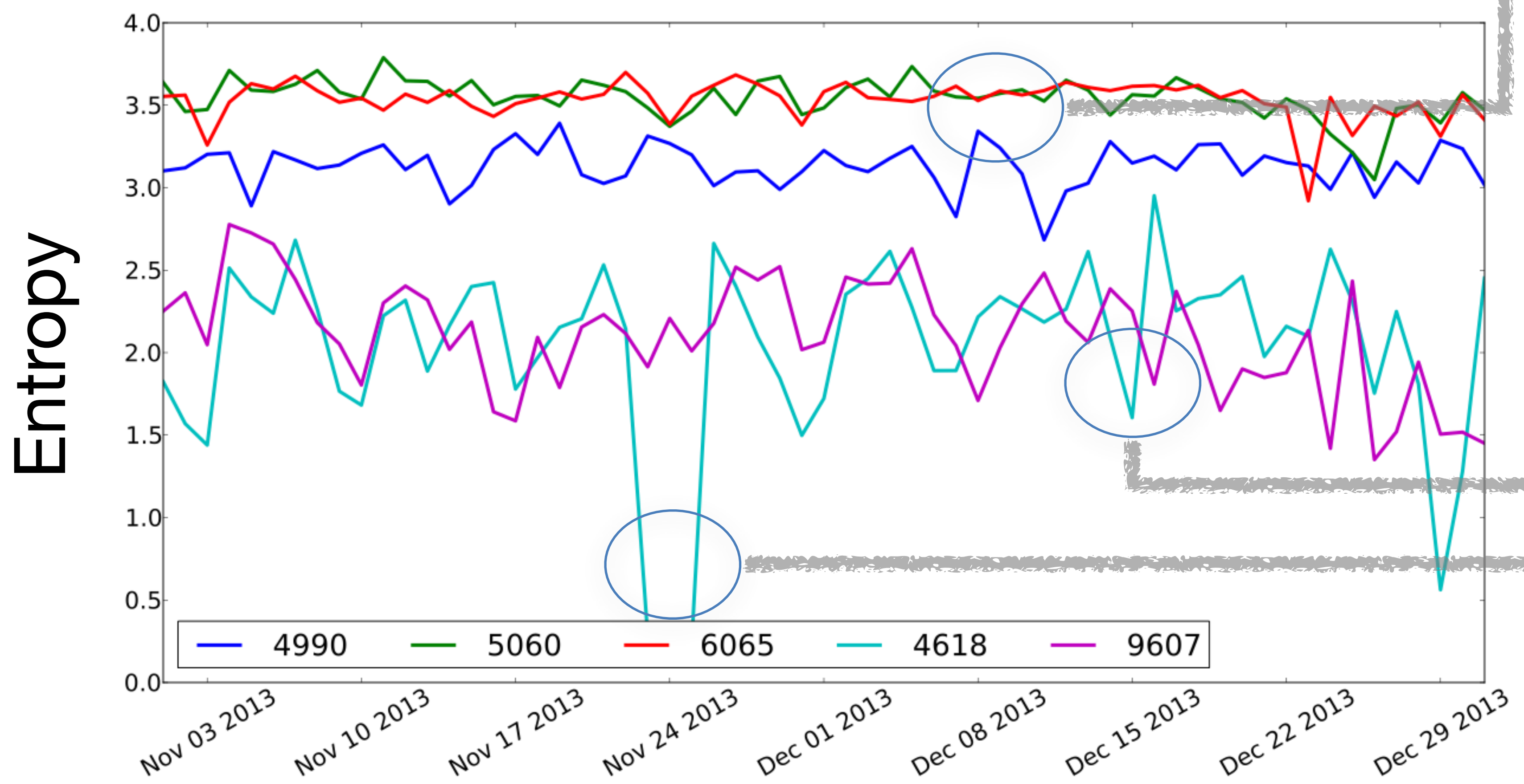
cell level $A_{i,c} = \sum_t A_{i,c}(t)$

T-slices $A_{i,c}(T) = \sum_{t \in T} A_{i,c}(t)$

district level $A_{D,c} = \sum_t \sum_{i \in D} A_{i,c}(t)$

$$\begin{cases} p_{i,c} = A_{i,c} / \sum\limits_c A_{i,c} \\ e_i = \sum\limits_c -p_{i,c}\, log(p_{i,c}) \end{cases}$$

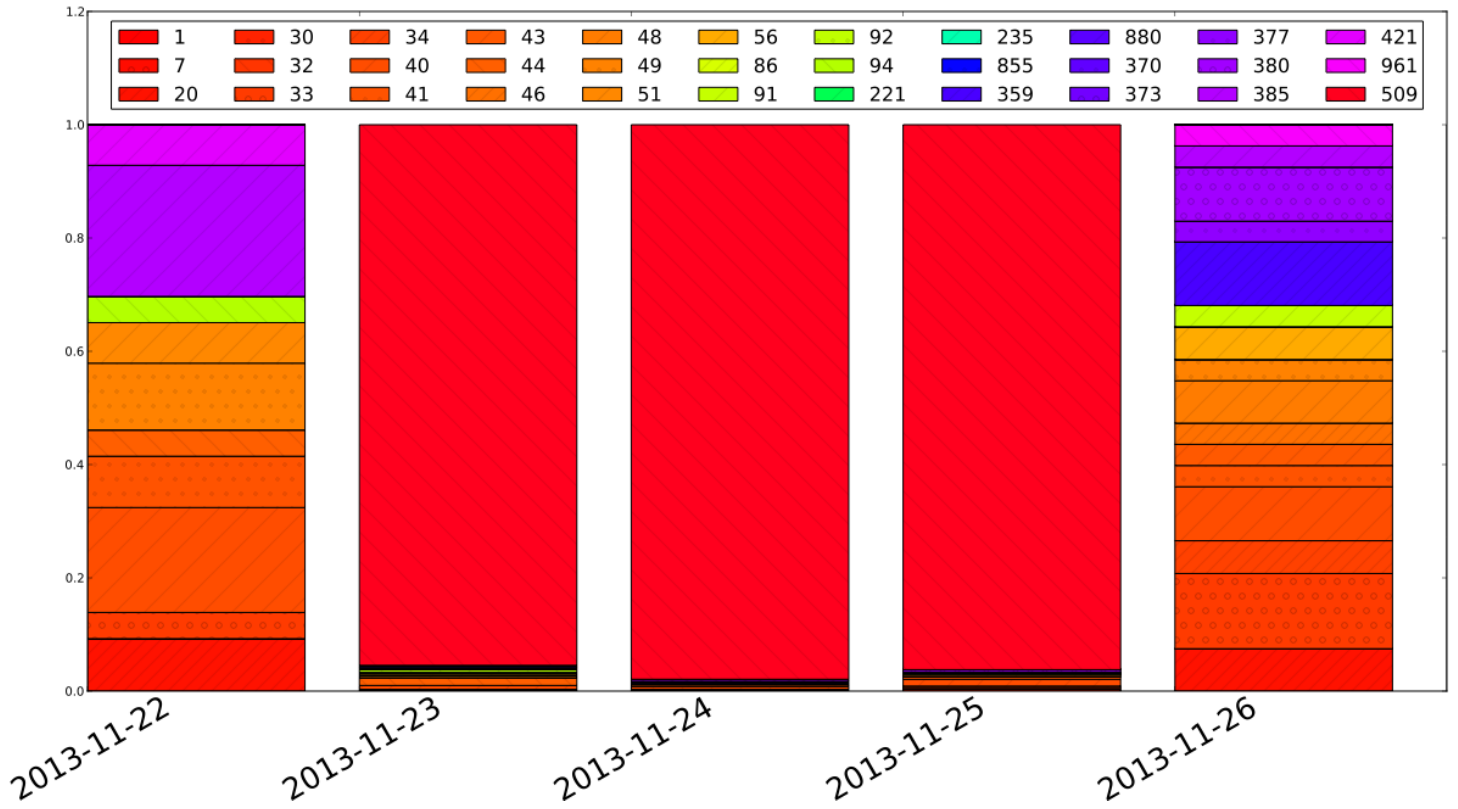
ENTROPY: single cells



High steady entropy areas (Milano's Dome, Central railway station, Linate airport, Milano's stock exchange, etc..)

Low entropy areas with weekly patterns (University, peripheral areas, etc..)

Activity stack chart for cell 4618

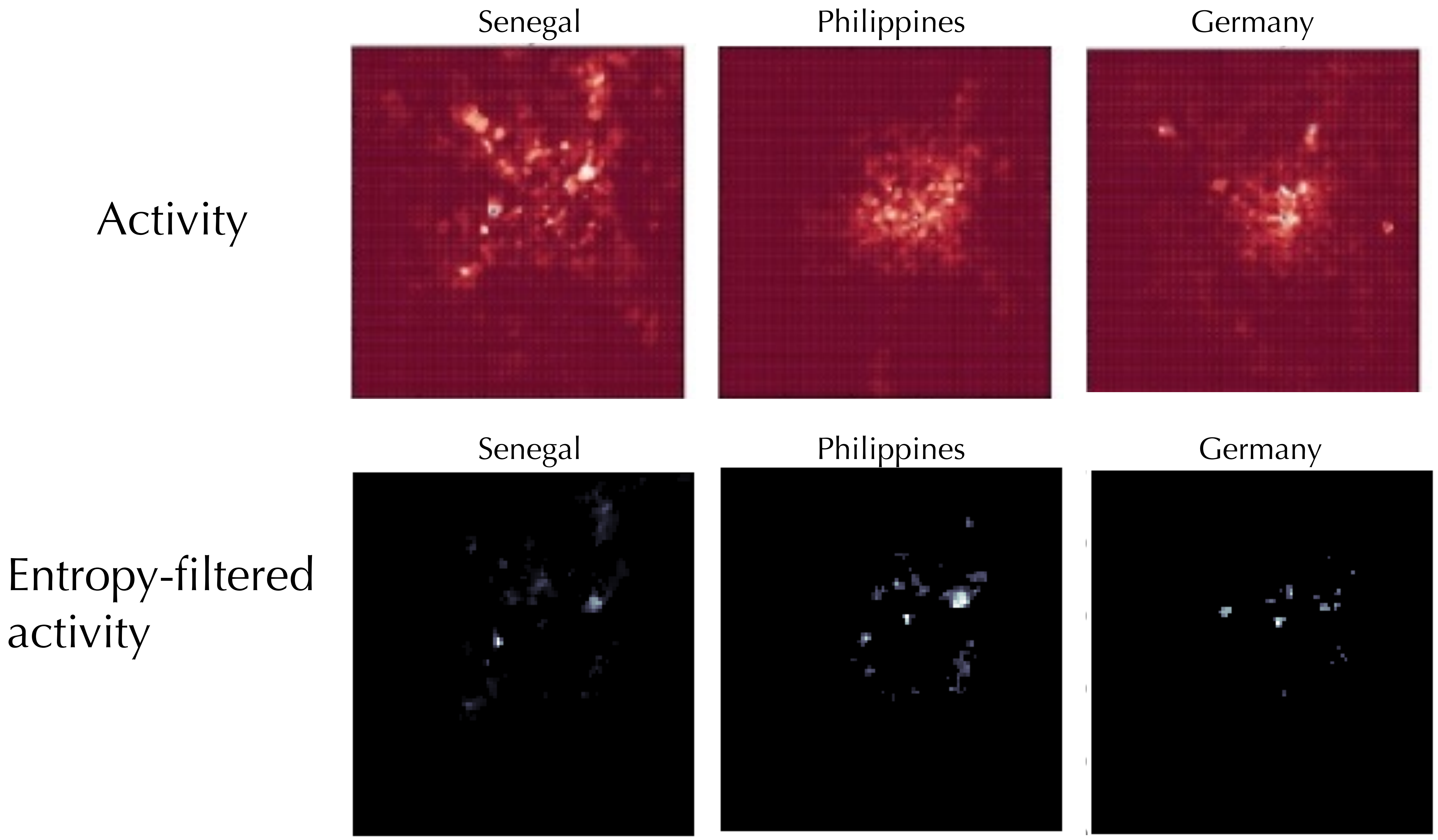


The cell displays very low entropy and a large Haitian activity.

- Haitian Movie Festival Award?
- Big wedding?

ENTROPY: country maps

Country-specific patterns should correspond with low entropy and large country-activity.

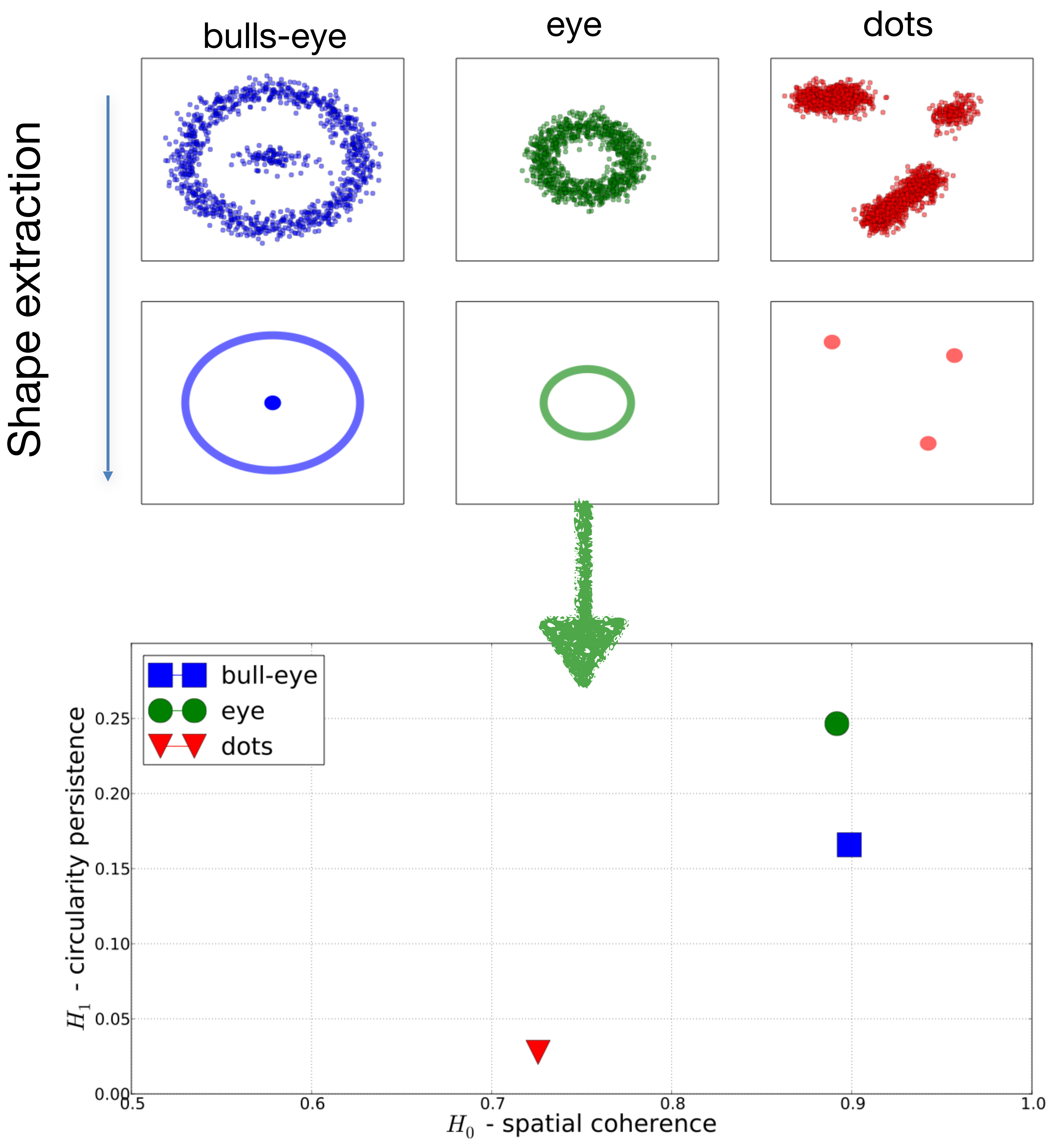


ENTROPY: country maps

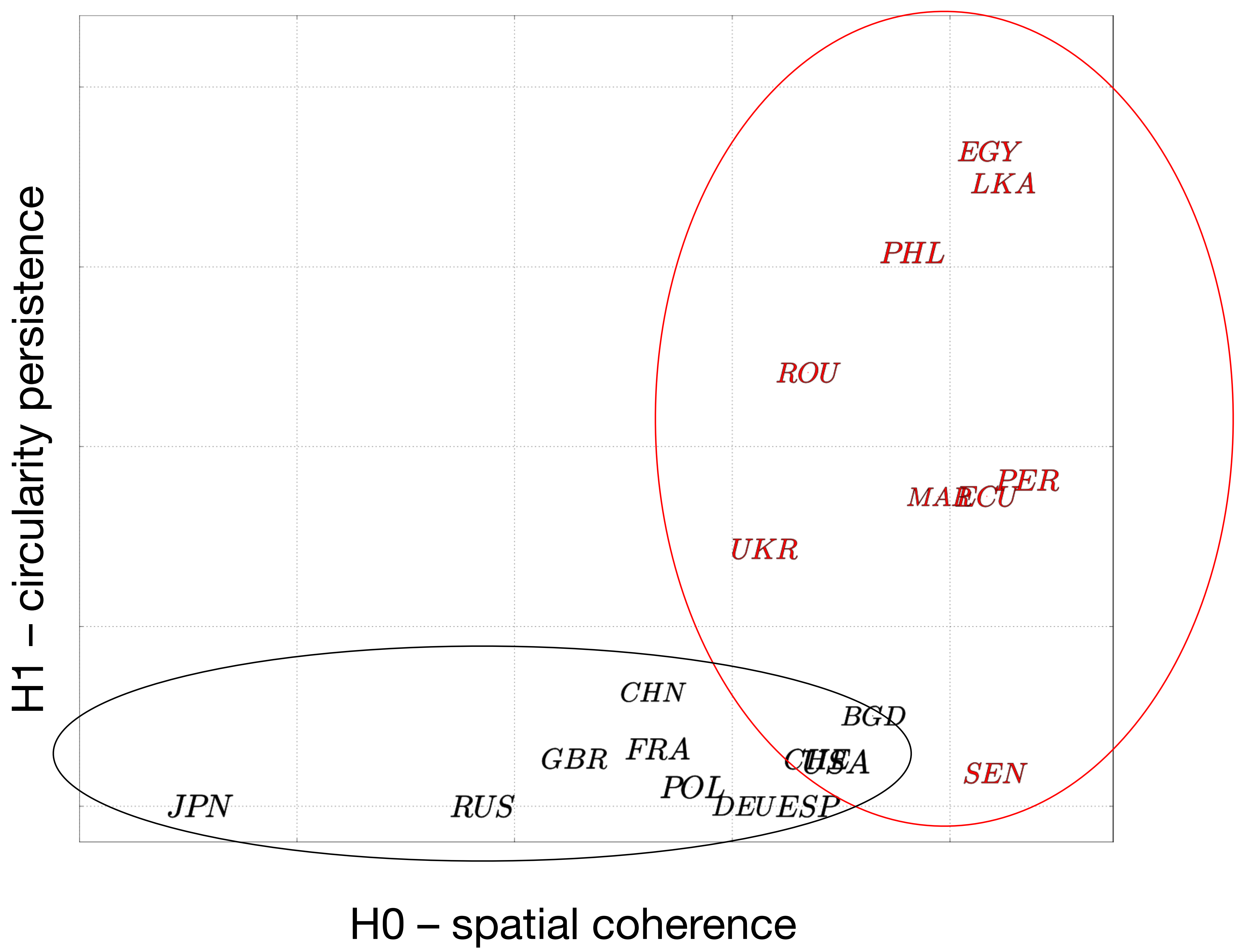
Persistent homology: classifying spatial patterns of international communities

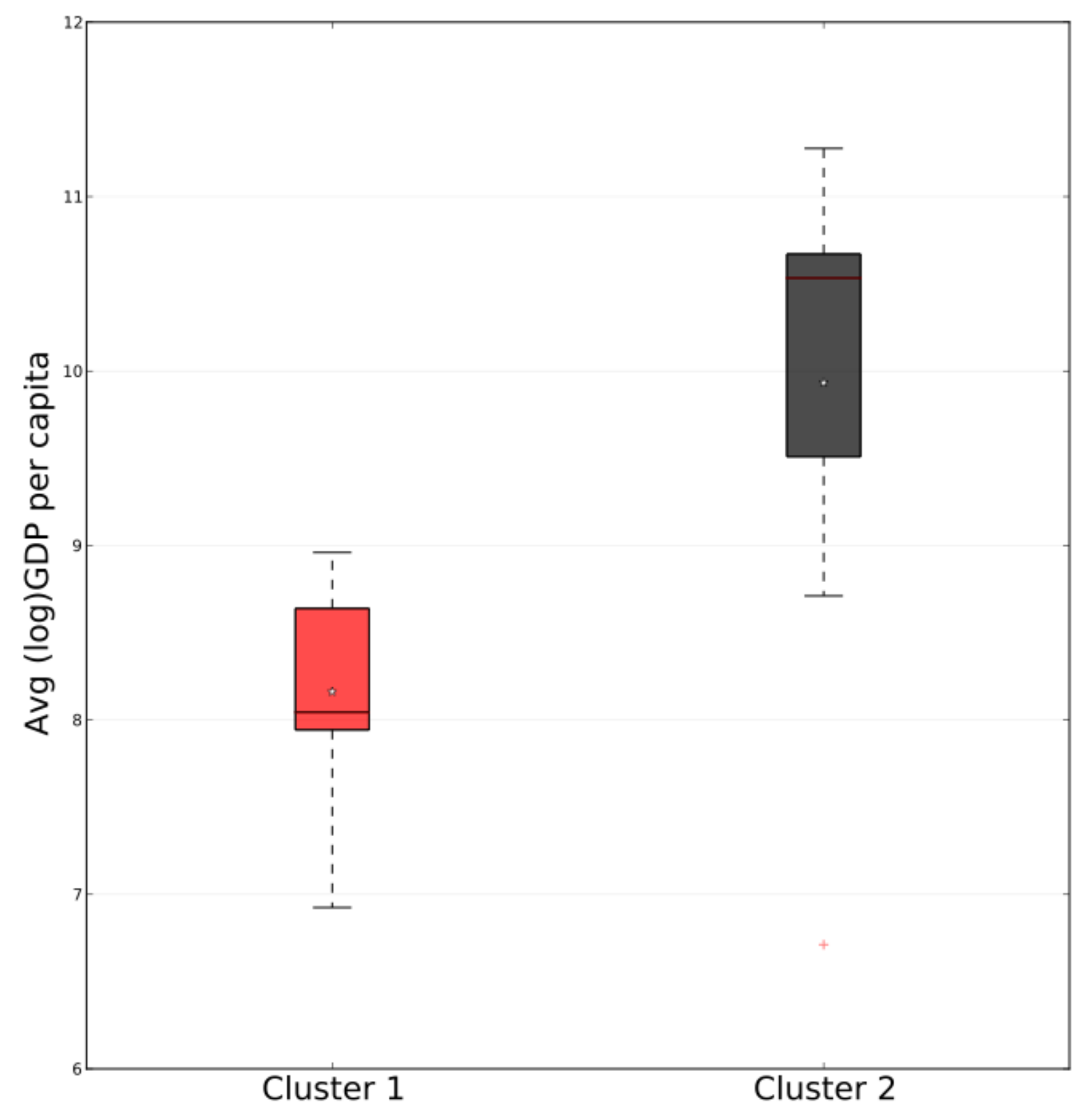
1. Recurring noisy patterns:
    1. disconnected clusters
    2. rings surrounding the city center

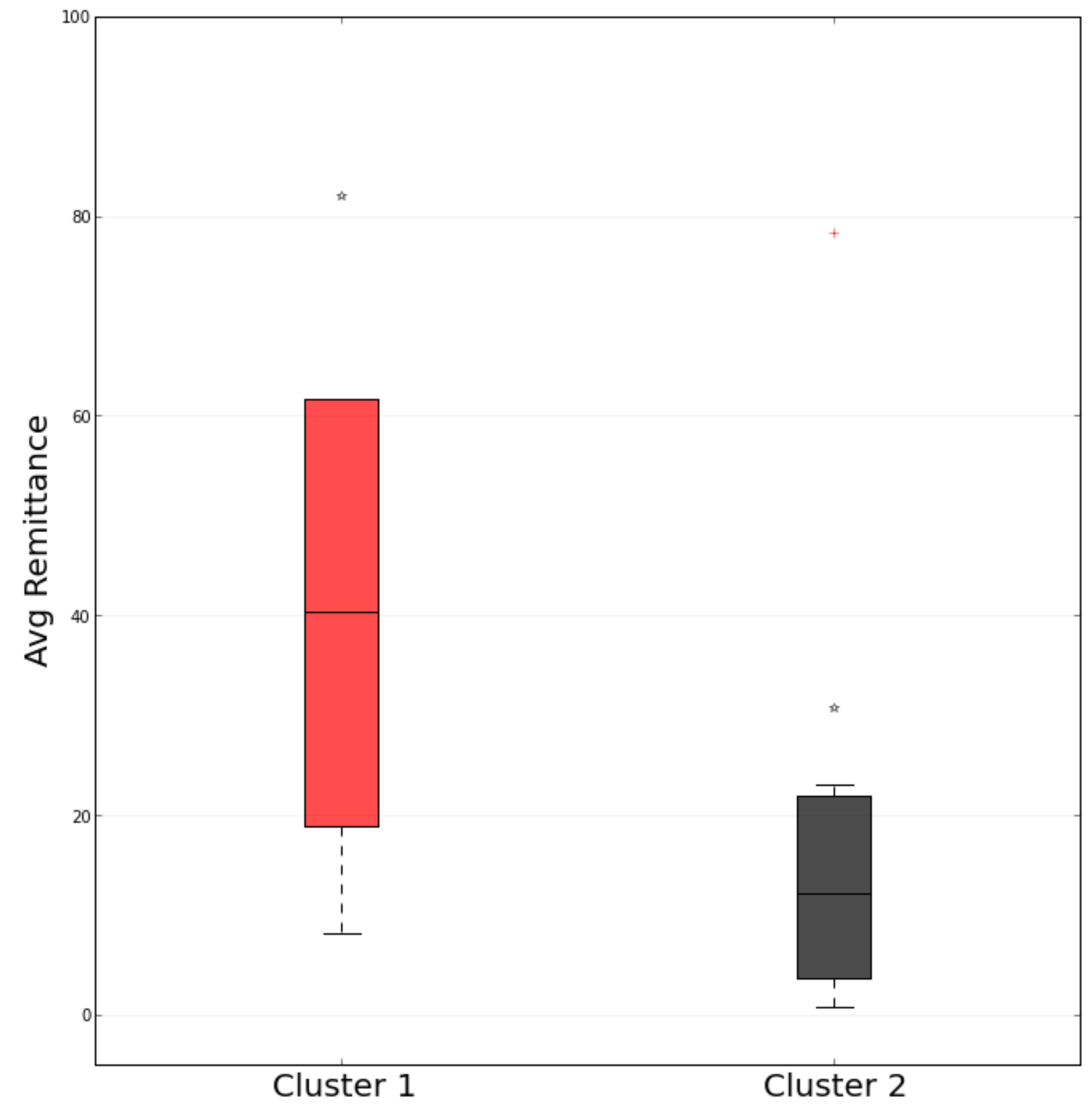
2. indicator based on their topological features
    1. robust to stretching and noise
    2. multiscale shape recognition method (Ghrist, Bull.AMS 45 (2008)).
    3. focus on the spatial coherence of country clusters:
        1. **connected components** (H0),
        2. **circularity features** (H1)



Persistent homology: entropy-activity maps







1. Clusters in 2D-scatter plot are simple k-means.
2. GDP per capita and remittance of countries in the two clusters clearly differentiate between richer migrant communities with respect people from developing countries.

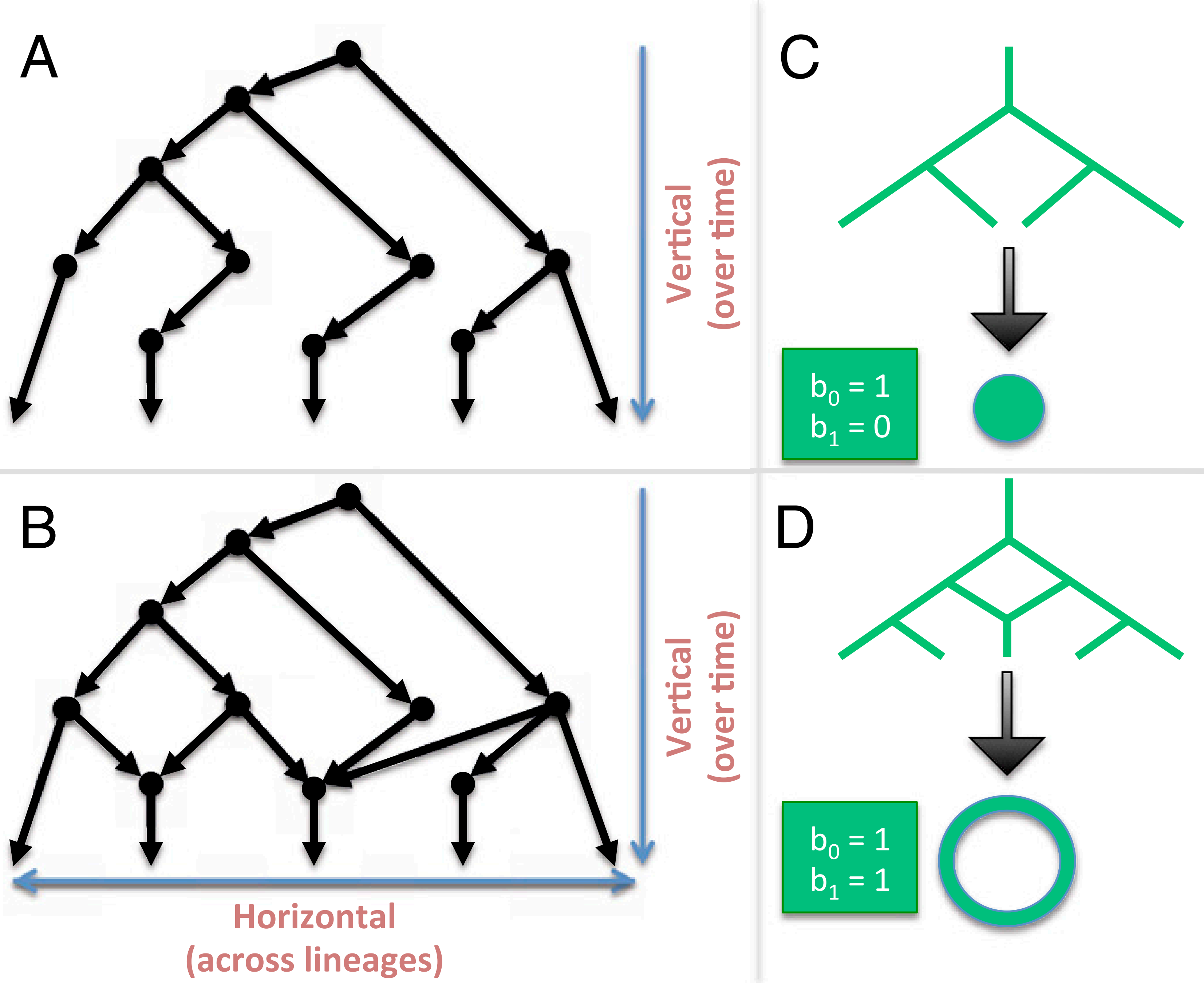
Previous work: phylogenetic trees

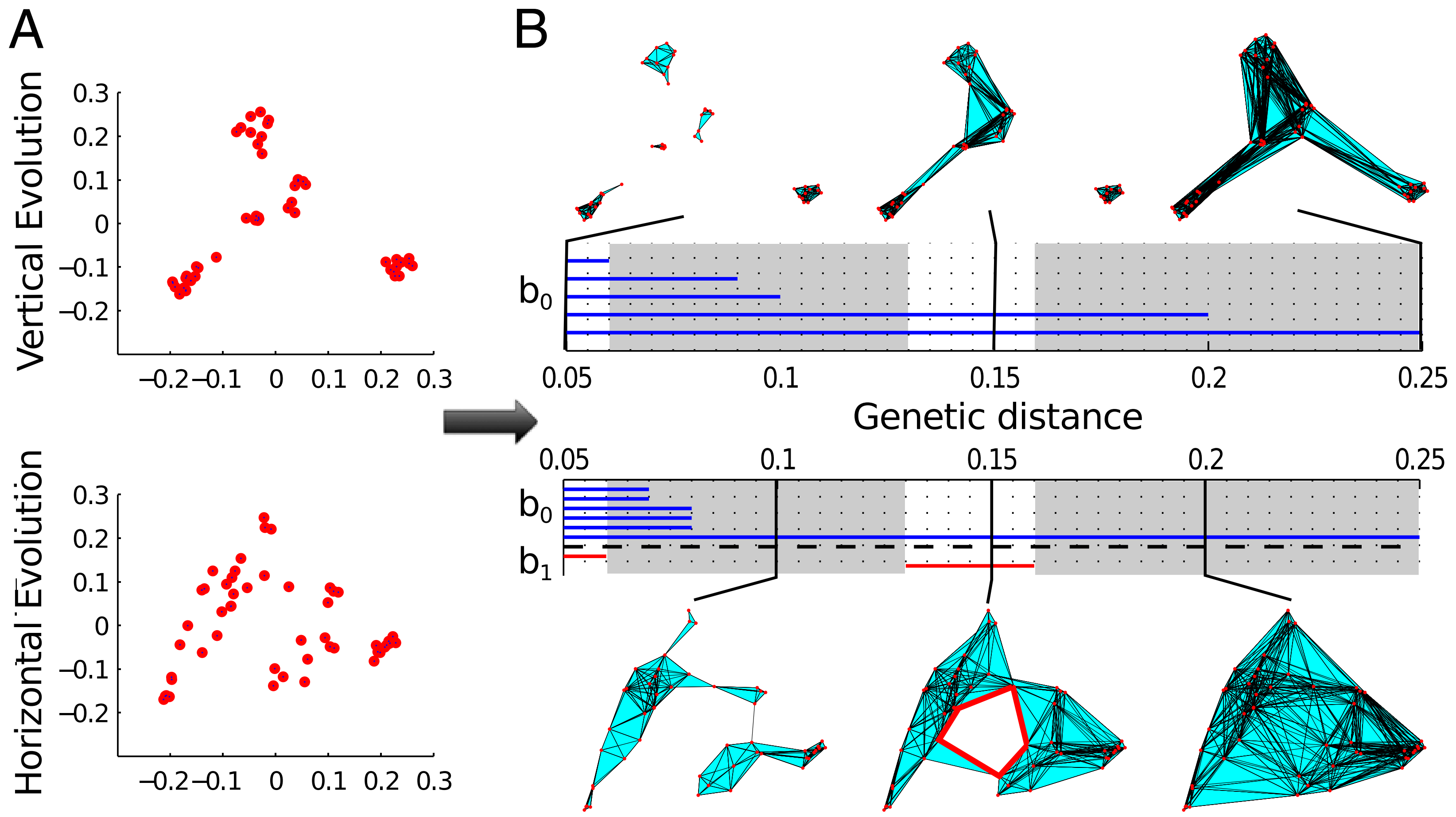
Topology of viral evolution

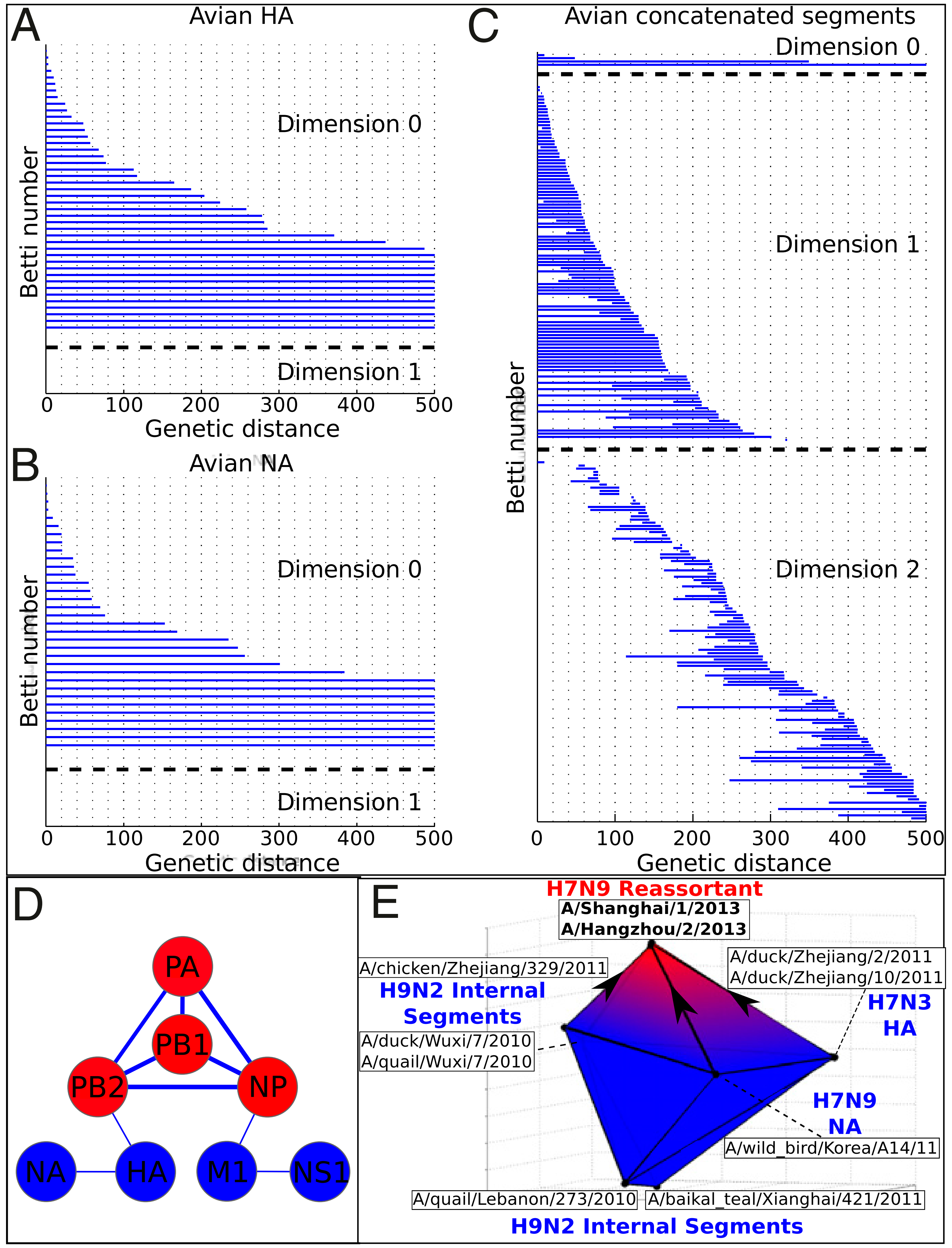
Joseph Minhow Chan[a,b], Gunnar Carlsson[c], and Raul Rabadan[a,b,d,1]

[a]Center for Computational Biology and Bioinformatics and Departments of [b]Biomedical Informatics and [d]Systems Biology, Columbia University Colleg
Physicians and Surgeons, New York, NY 10032; and [c]Department of Mathematics, Stanford University, Stanford, CA 94305

Edited* by Arnold J. Levine, Institute for Advanced Study, Princeton, NJ, and approved October 11, 2013 (received for review July 18, 2013)







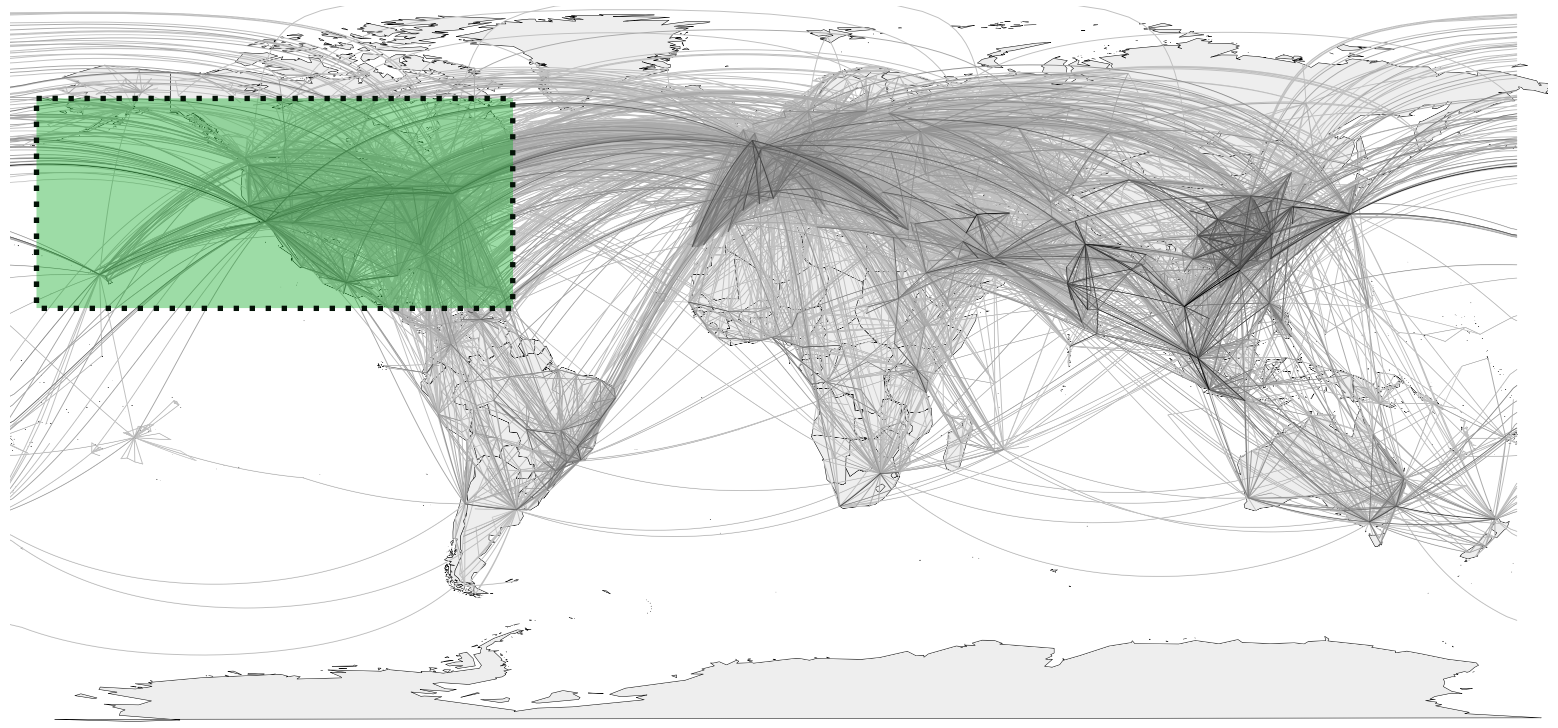
What about networks?

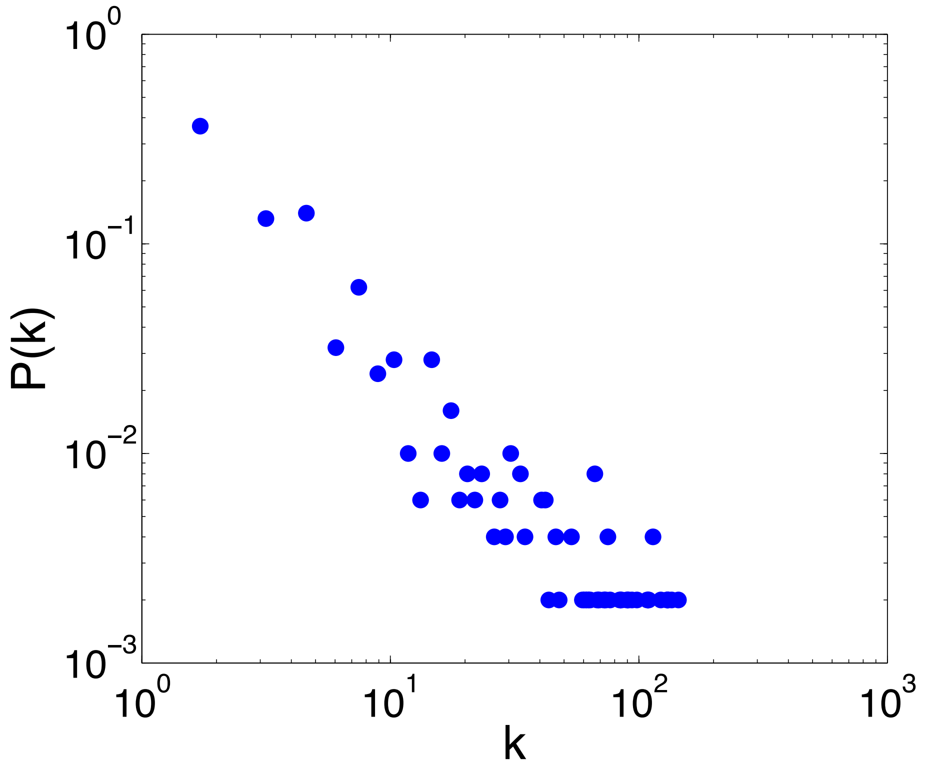
Metrical persistent homology

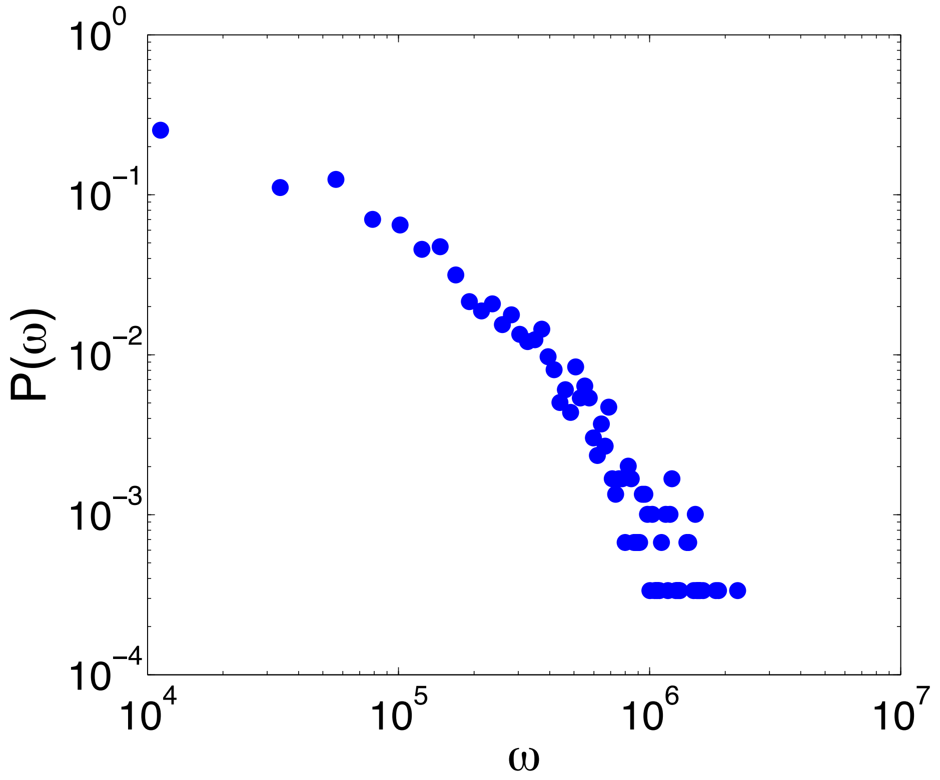
No metric space, but one can define a few:

shortest path,
commute time
other Kernel distances..

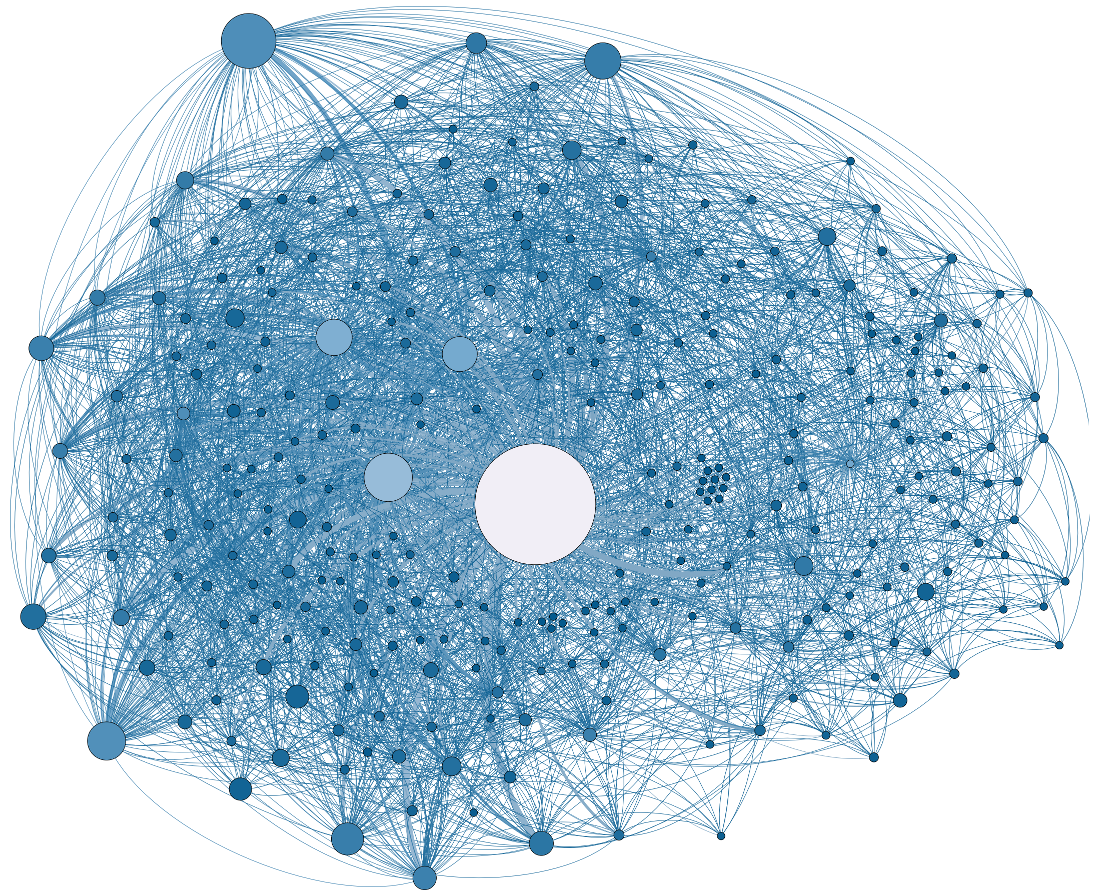
US Top-500 airports network

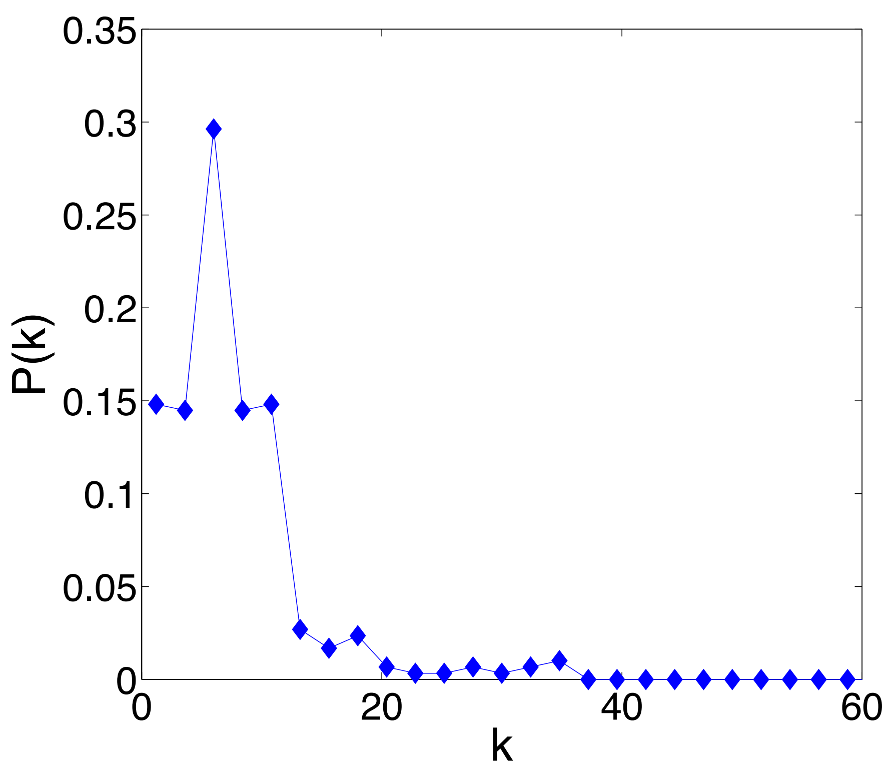


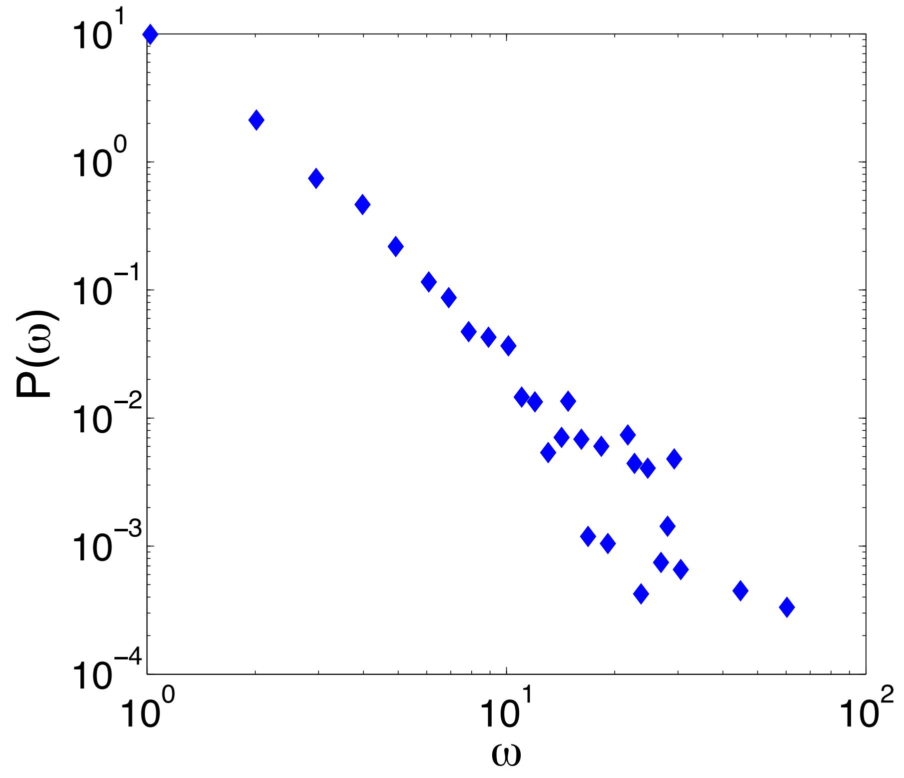




C. Elegans neural network

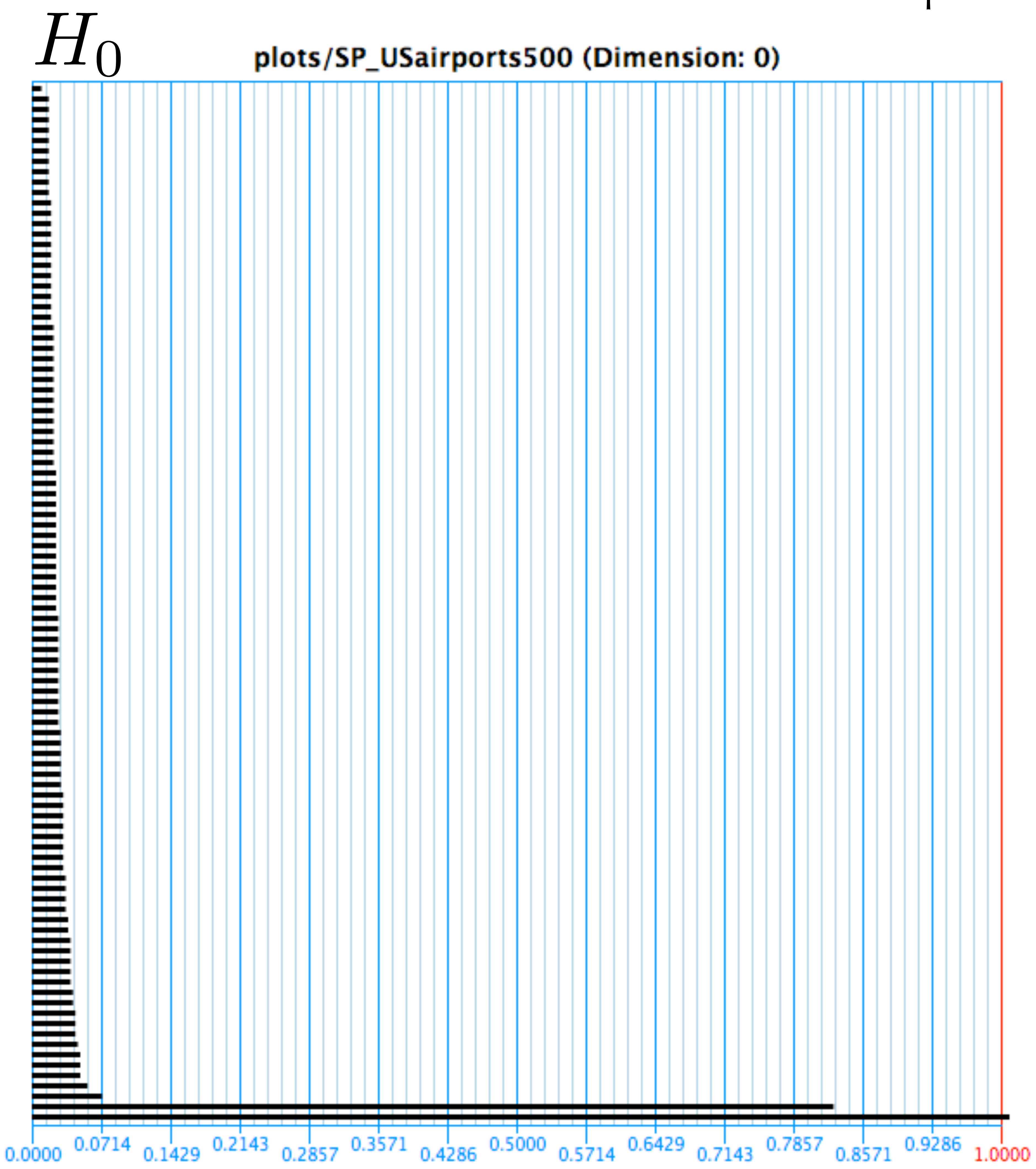




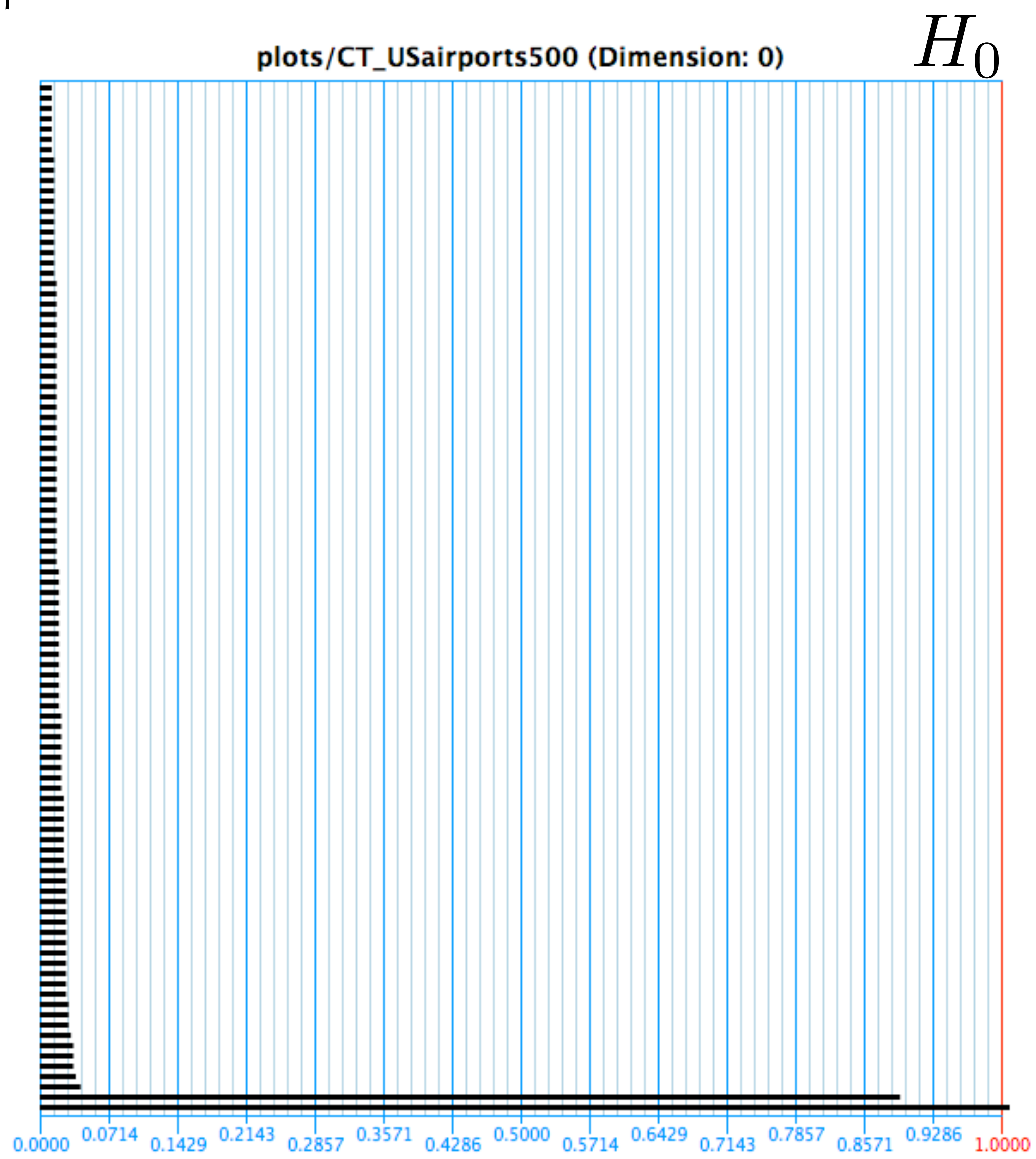


Metrical persistent homology II

US Top500 airports network



Shortest path



Commute time

Metrical persistent homology II bis

C. Elegans

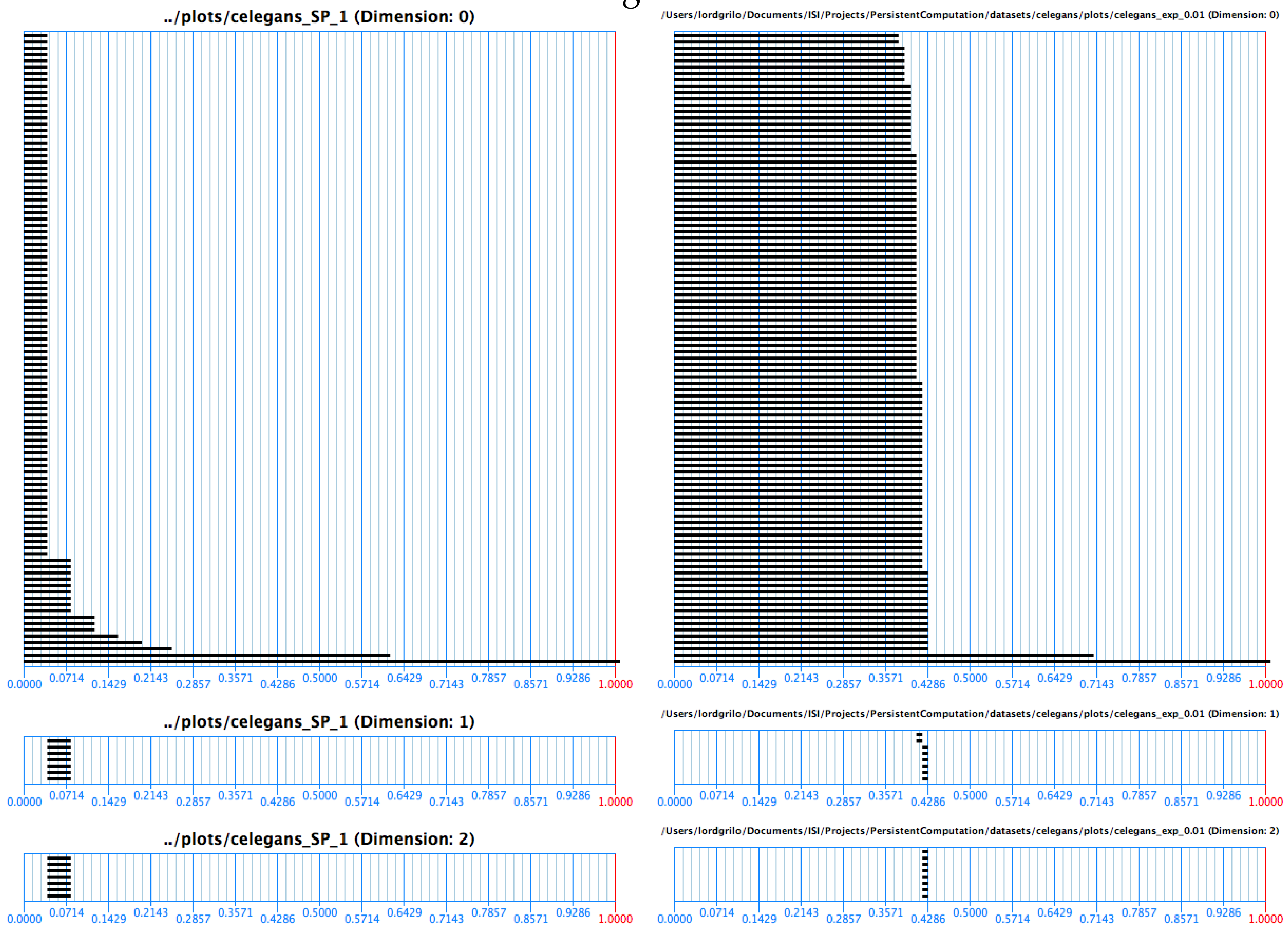
Shortest path

Von Neuman

H_0 H_0

H_1 H_1

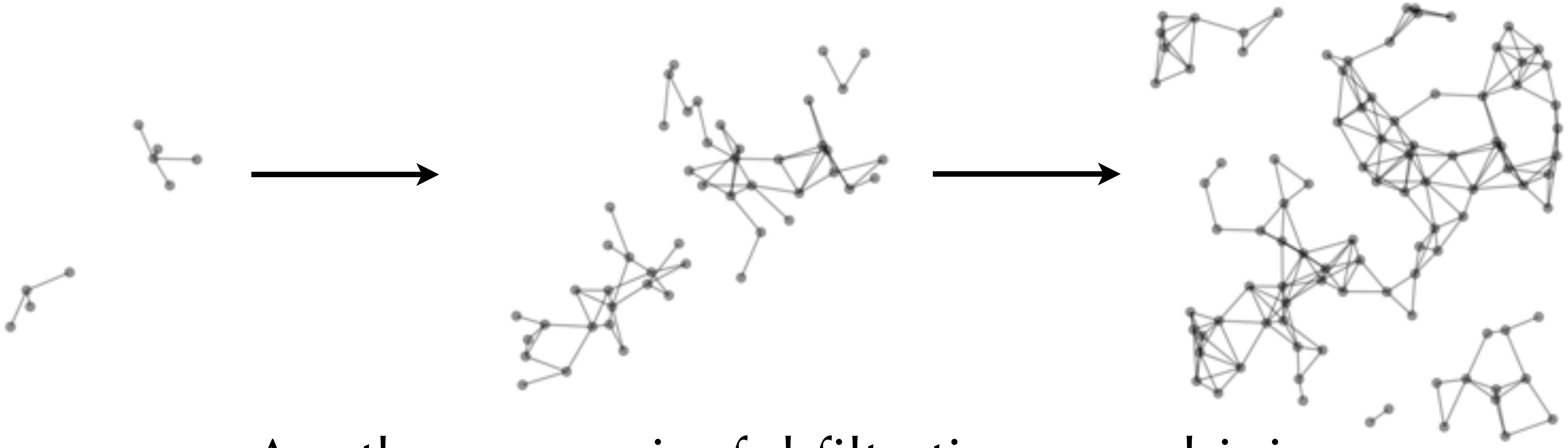
H_2 H_2



1. Different metrics ~ same information
2. Cycles appear at emergence of GCC
3. No need of homology to do this!!

Metrical persistent homology III

Metrics do not convey much information.
What now? The important ingredient is the **filtration**.



Are there meaningful filtrations combining:

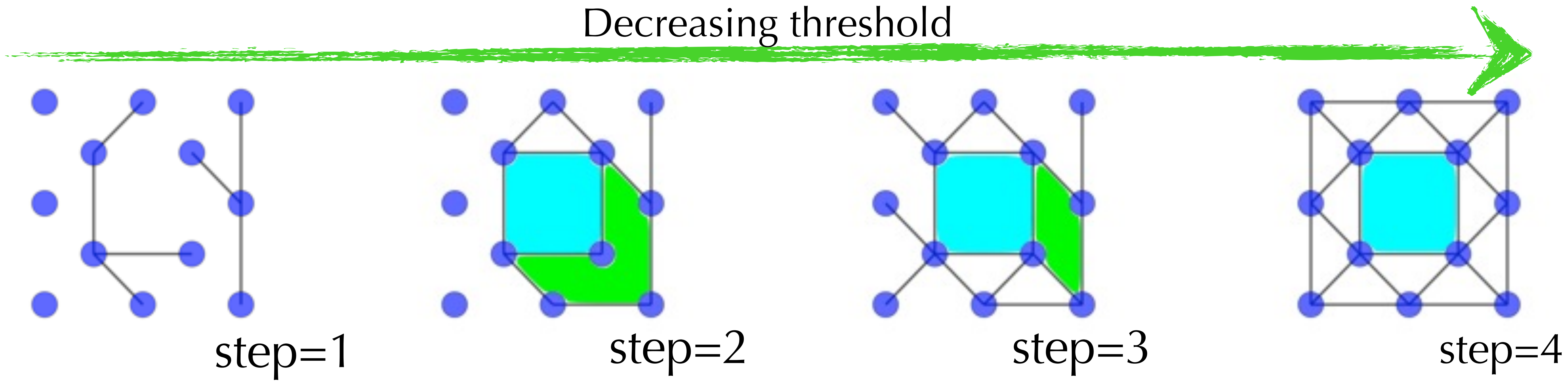
Network "linking patterns"
&
Weight structure
?

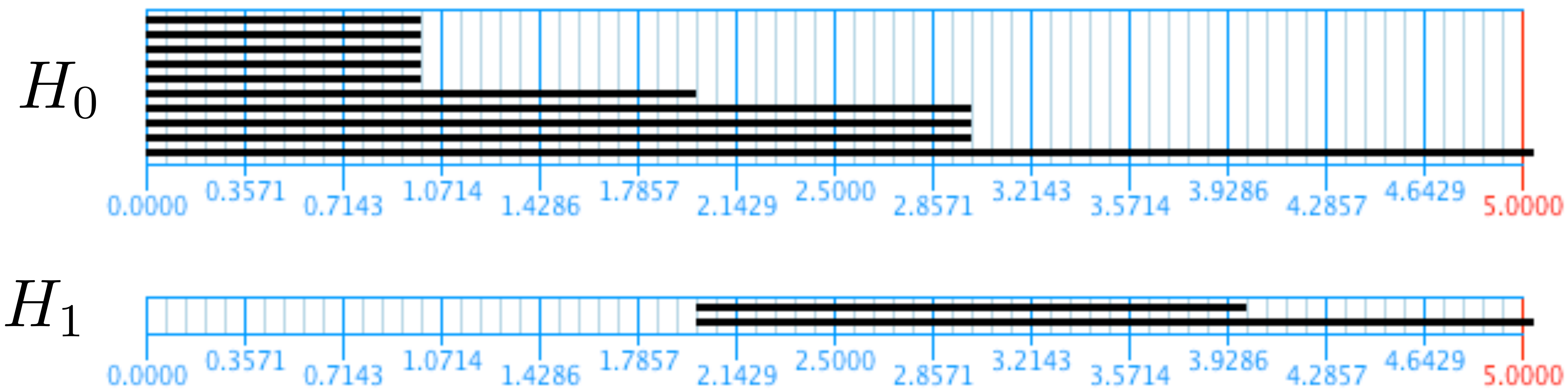
Remark

The Rips complex of points in a metric space is the clique complex of a metric graph.

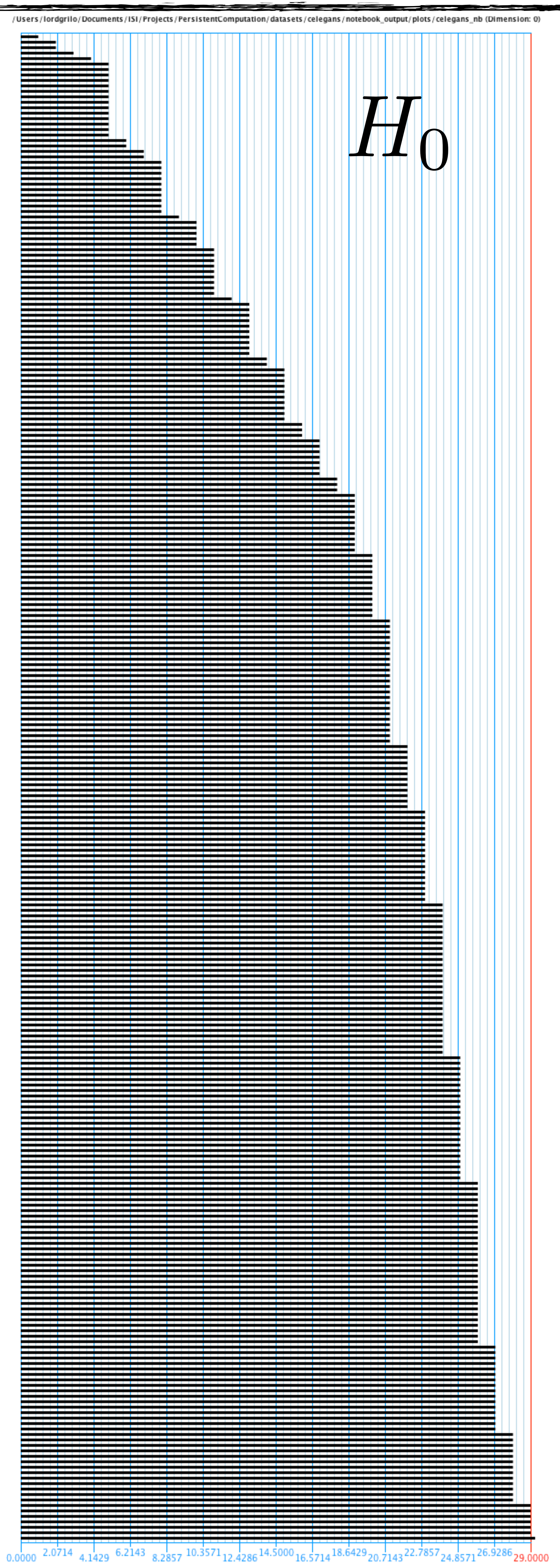
Weighted clique filtration I

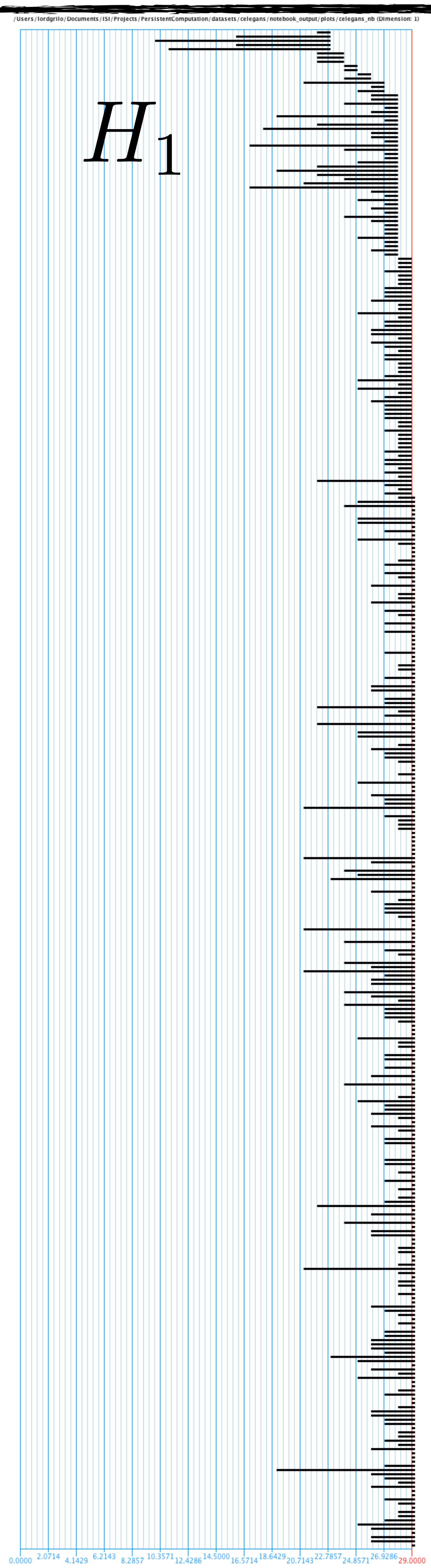
1) Given a real network G=(V,E), consider another one with the same number of node N=|V| and no edges.
2) Start adding edges from G, in decreasing order of weight.
3) At each step, calculate the associated clique complex.

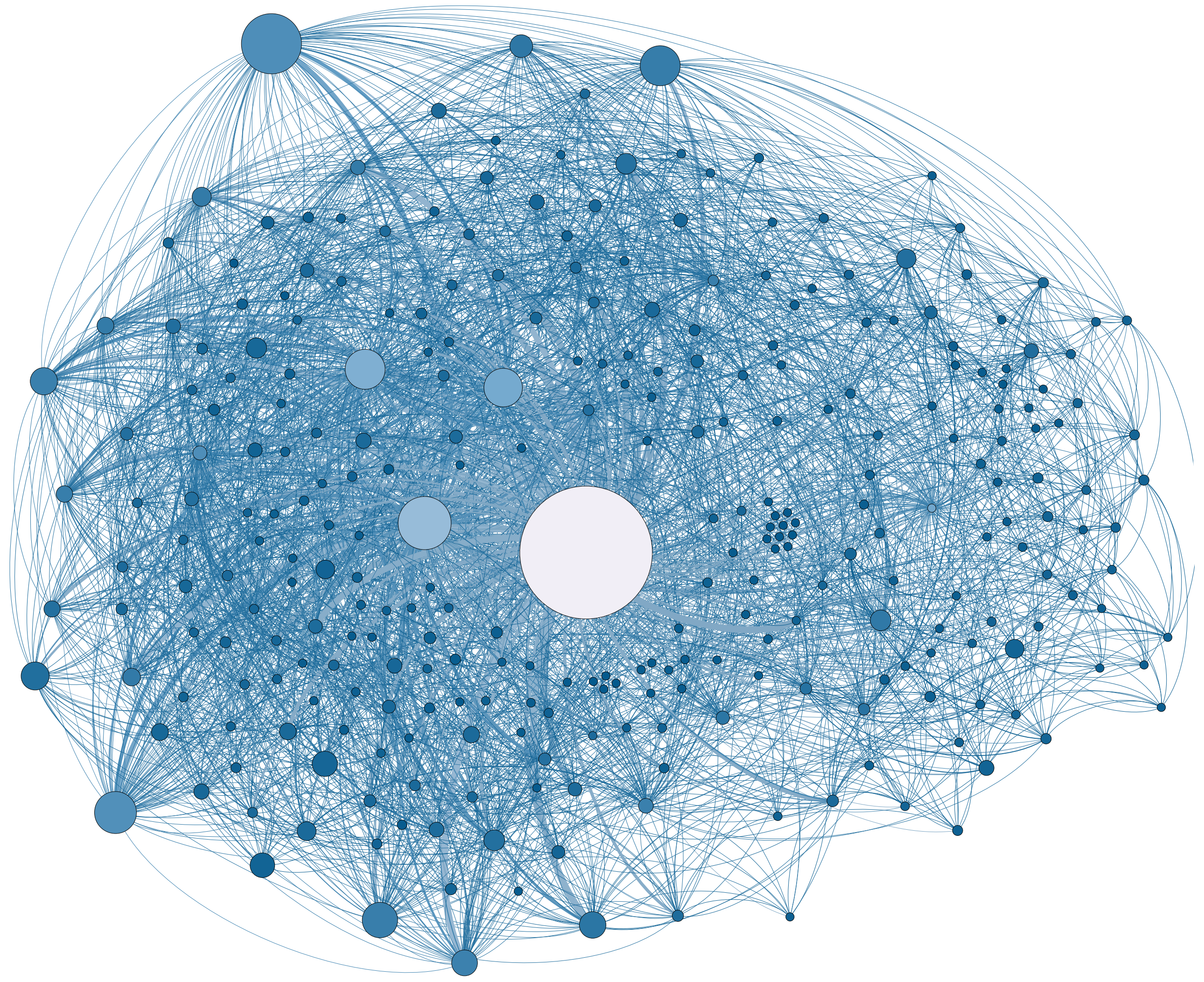




Weighted clique filtration II bis

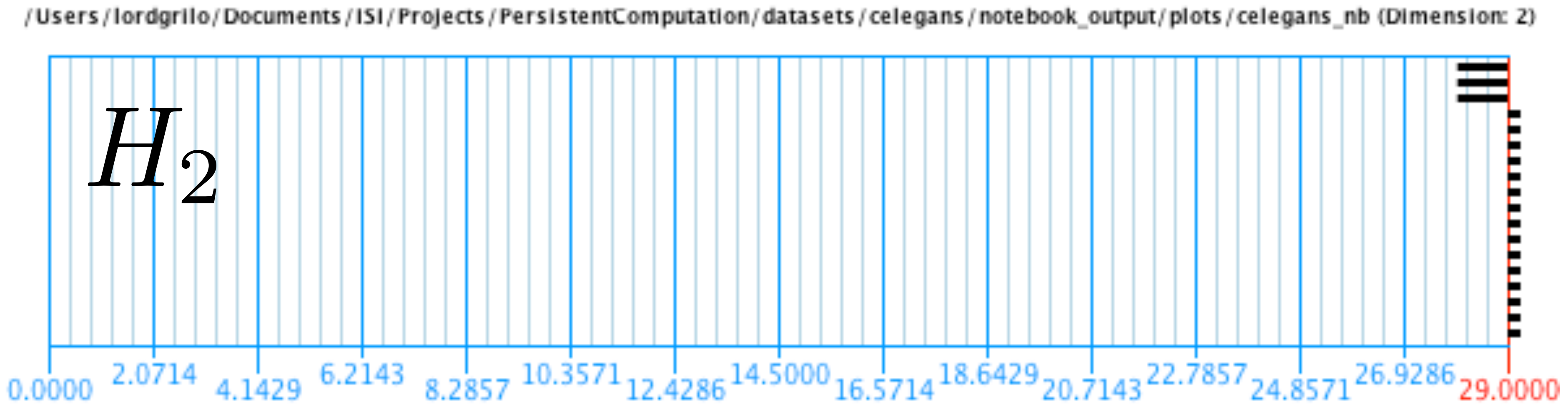




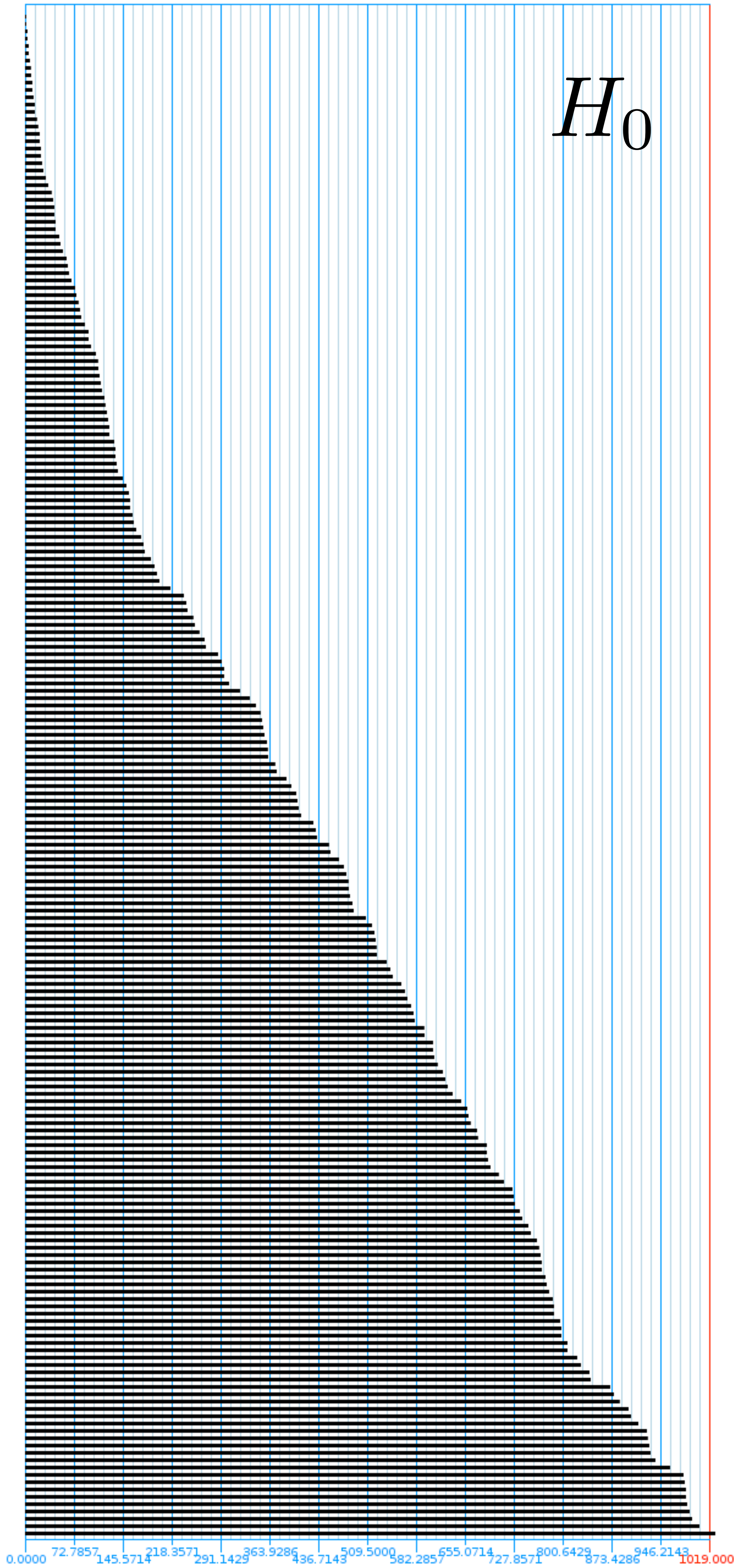


Much more structure appears!

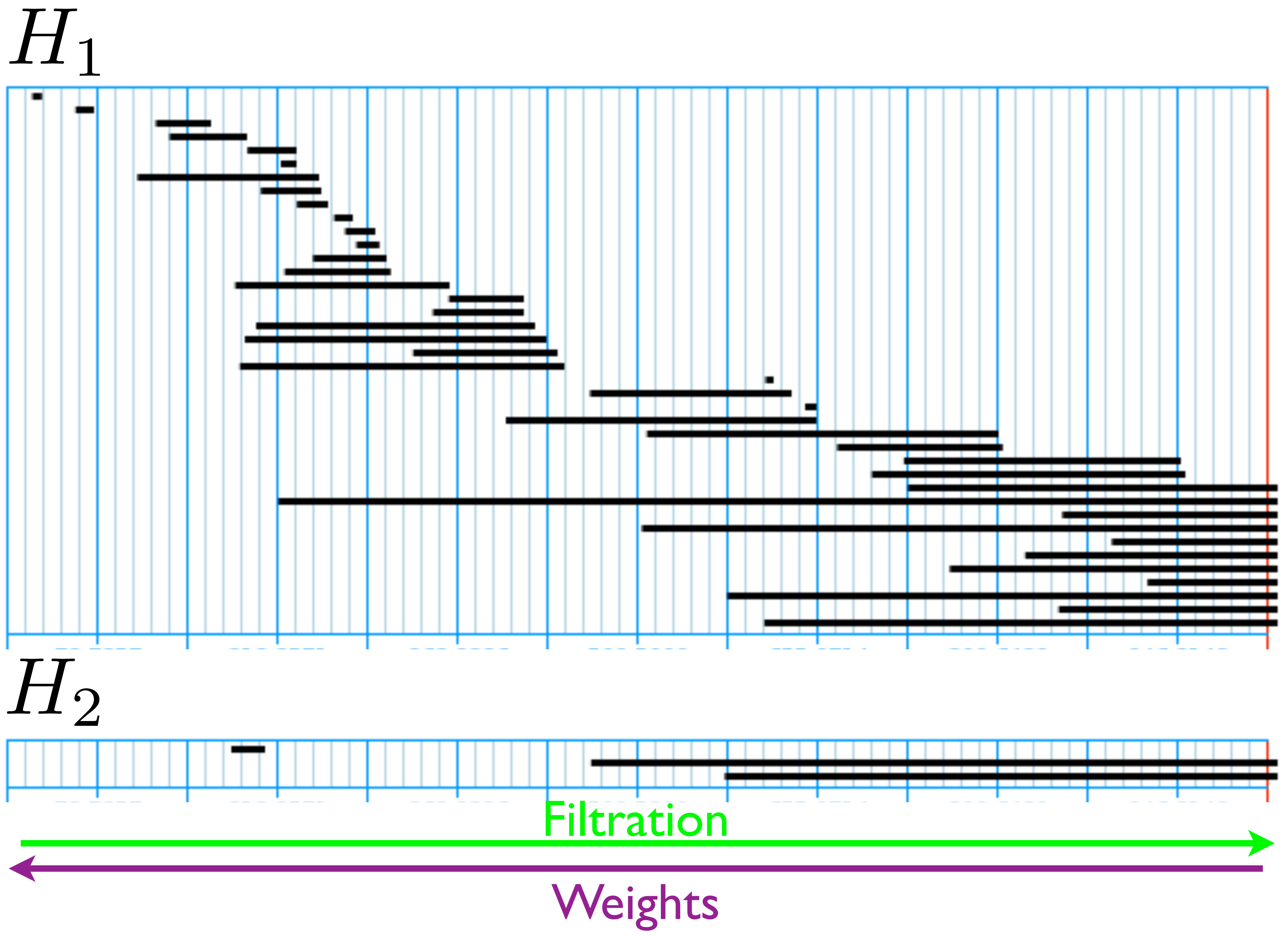
- Conn. comps. smoothly coalesce.
- A large number of loops.
- Even some “3D”-holes for low weights.



Weighted clique filtration II



H_0	Connected components
H_1	1D Cycles (2-simplexes)
H_2	3D holes (3-simplexes)

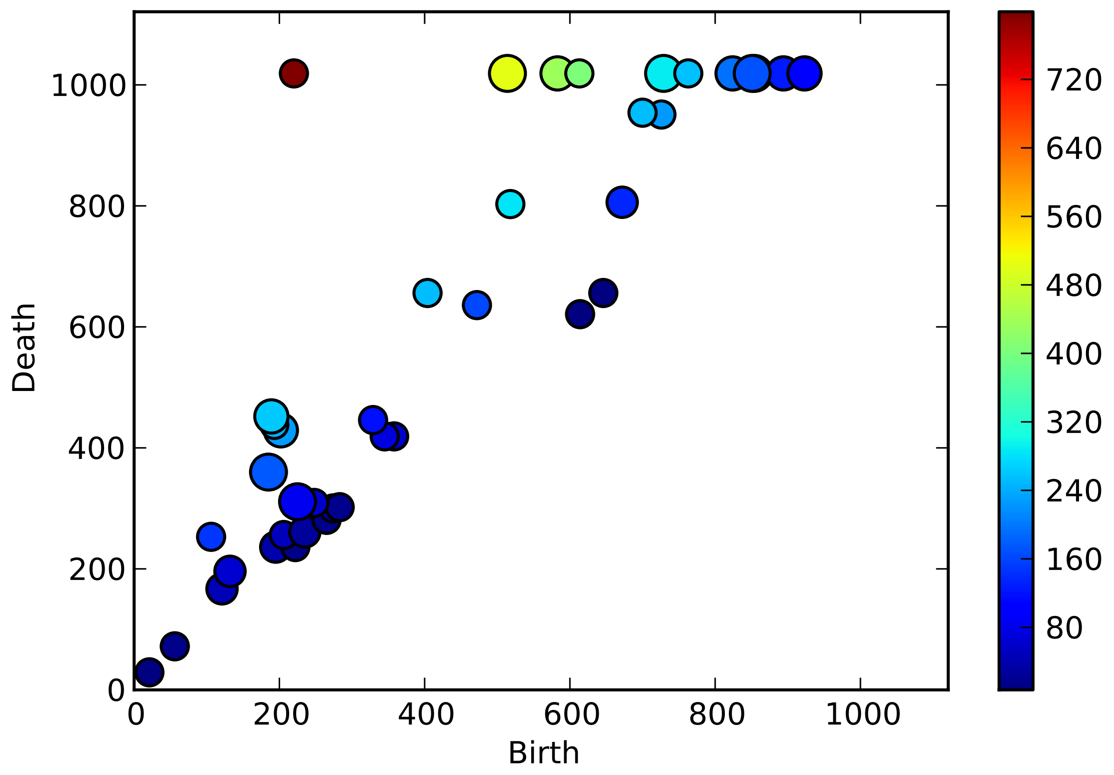


Datasets I

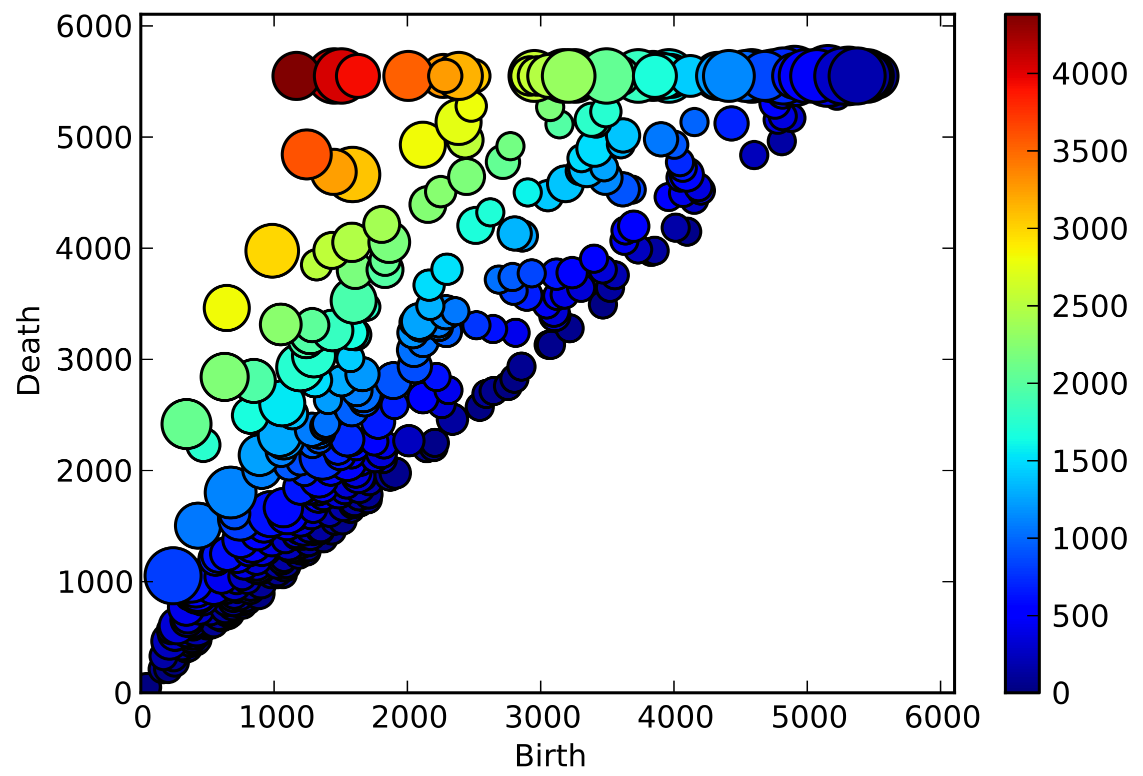
Focus on H_1, map each generator to a point in the plane:

Cycle $\longrightarrow$ (Birth,Death)

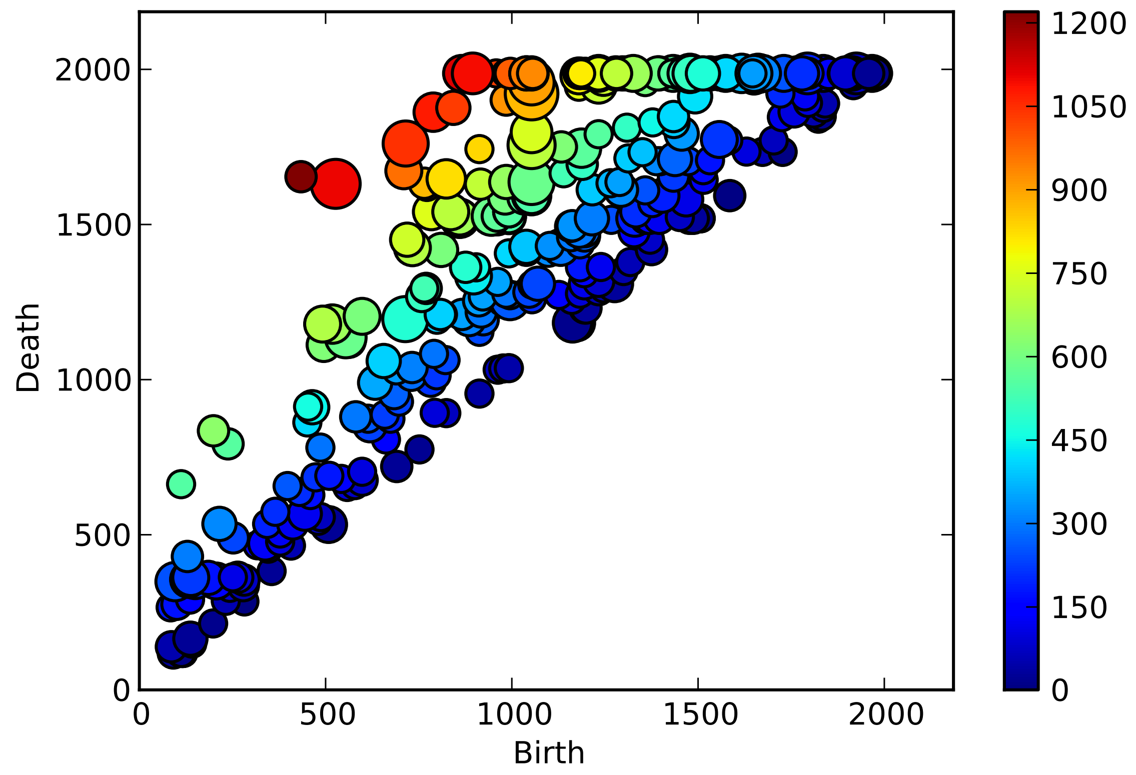
US2000 air passenger network



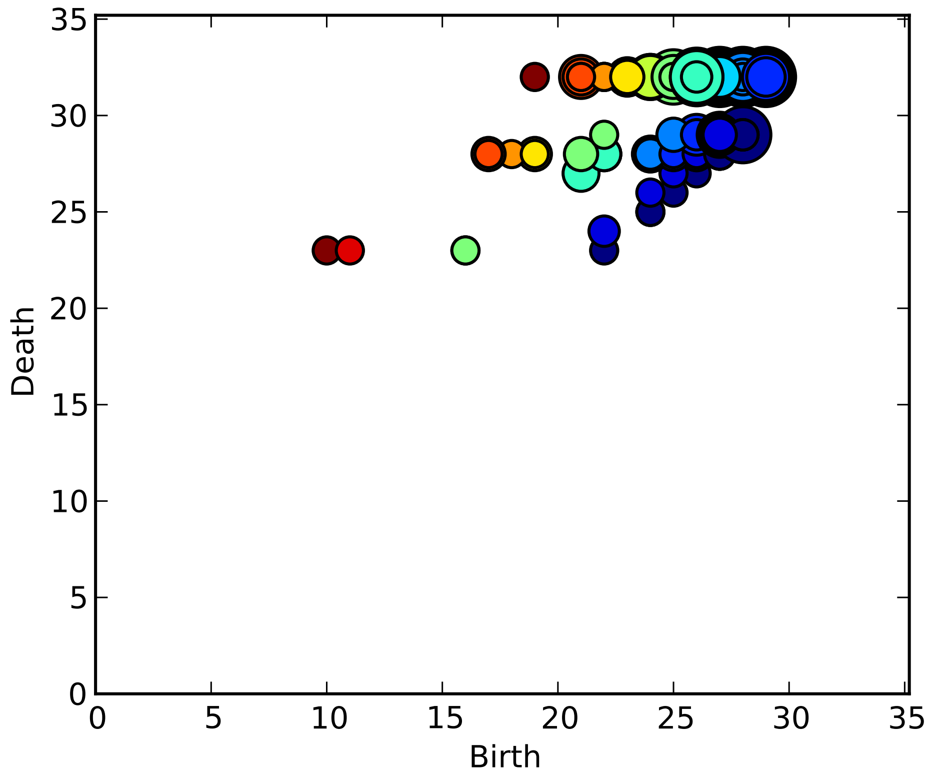
Human gene (sample)



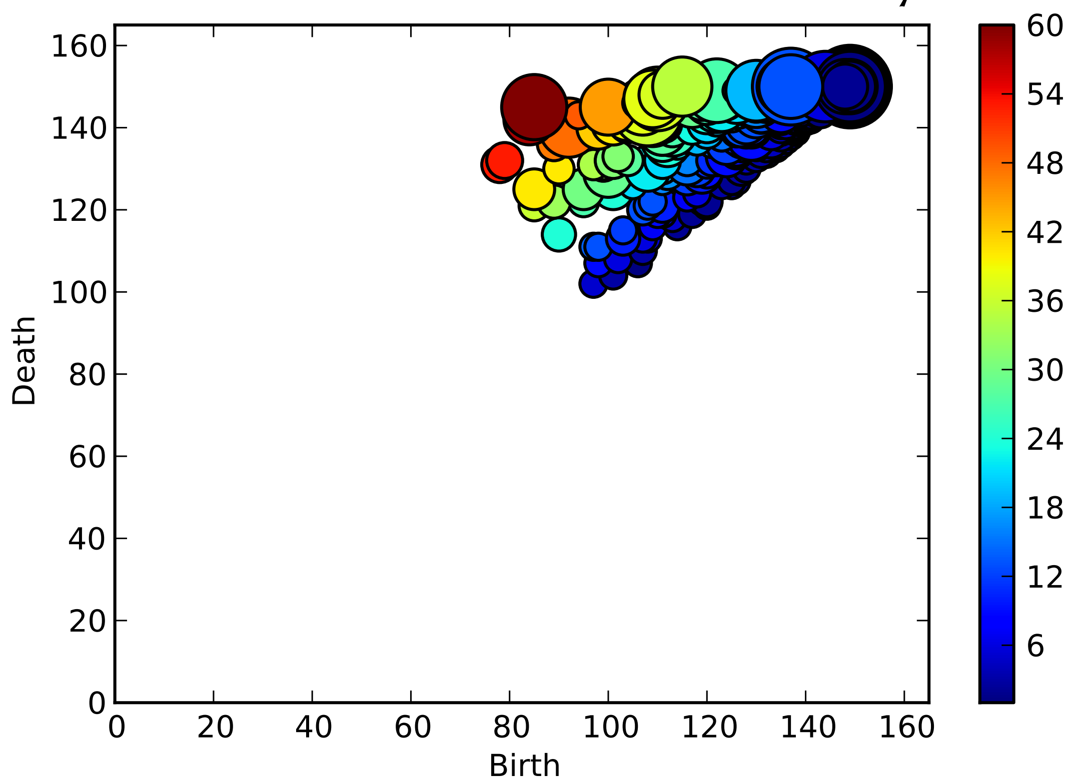
FB-like social network



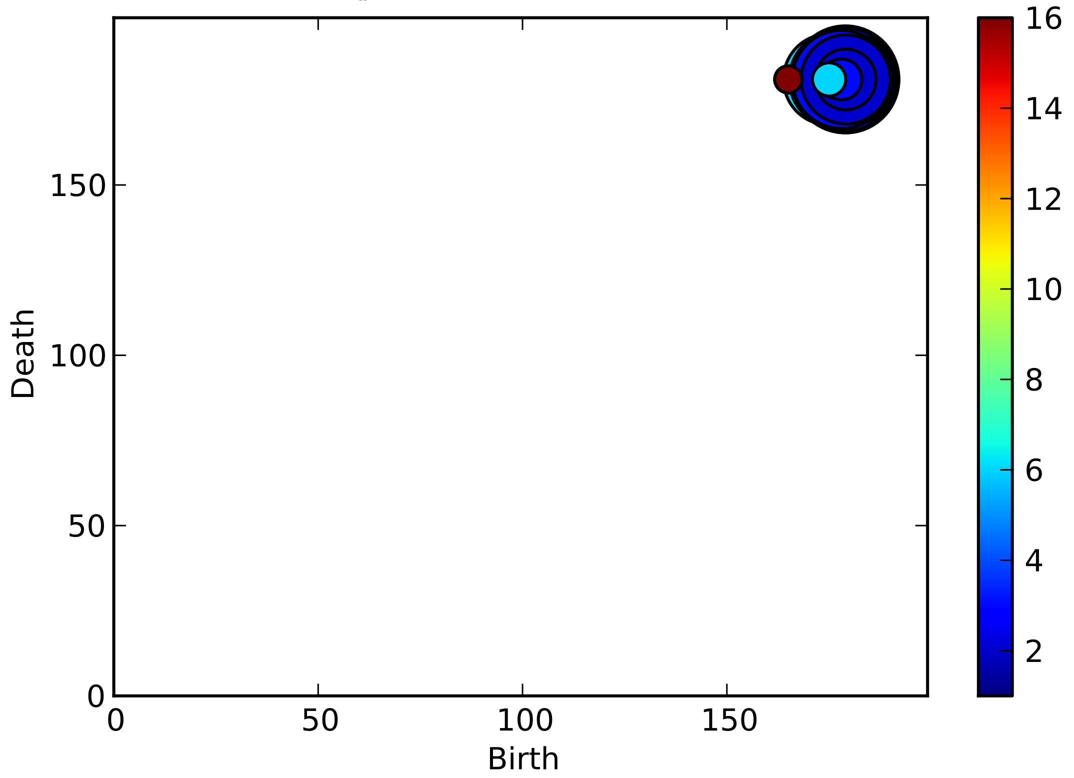
C. elegans



Kids contact duration (day II)



Kumpula et al. model



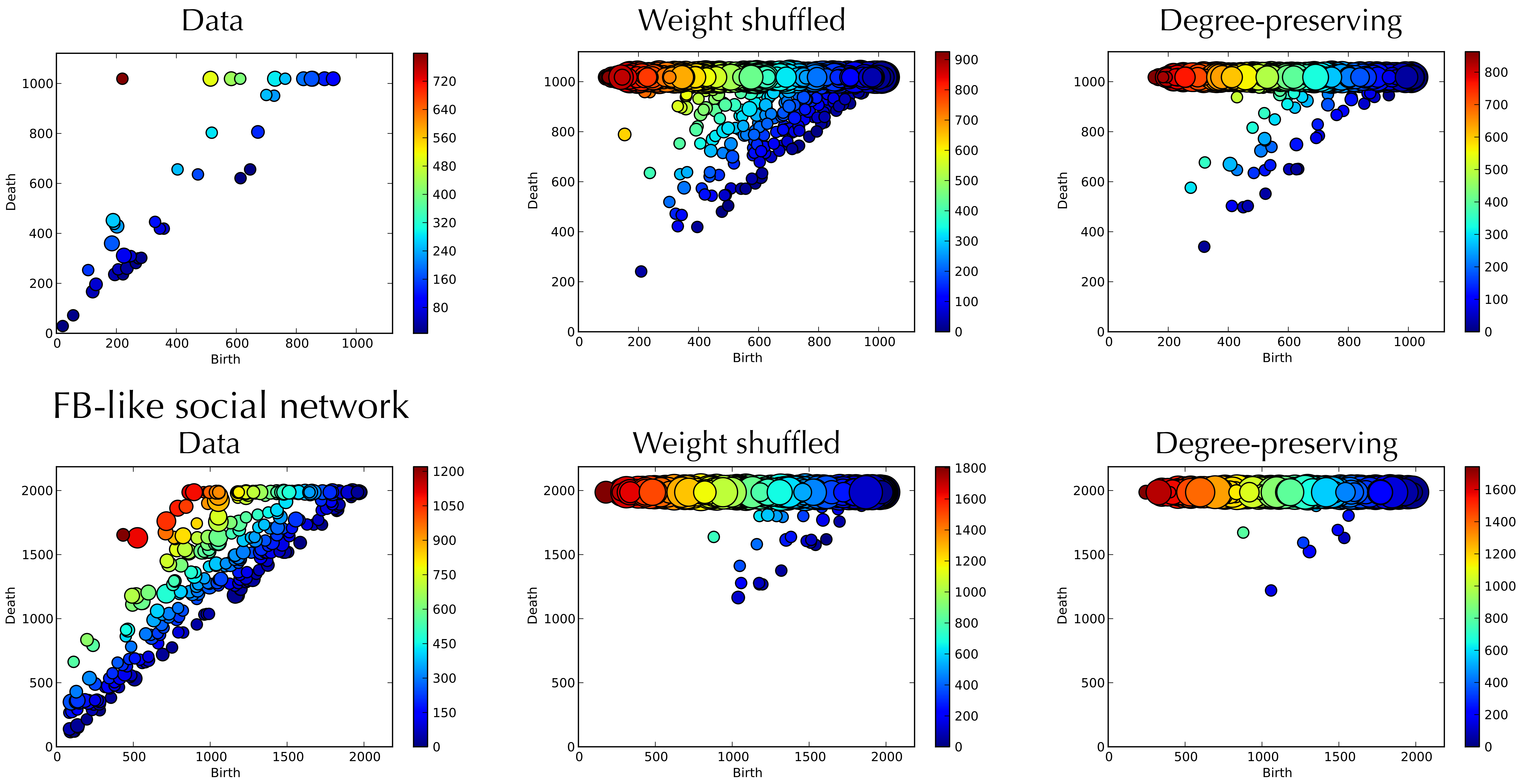
$$H_1$$

What is significant?

Datasets II: null models?

US 2000 air passenger network

FB-like social network

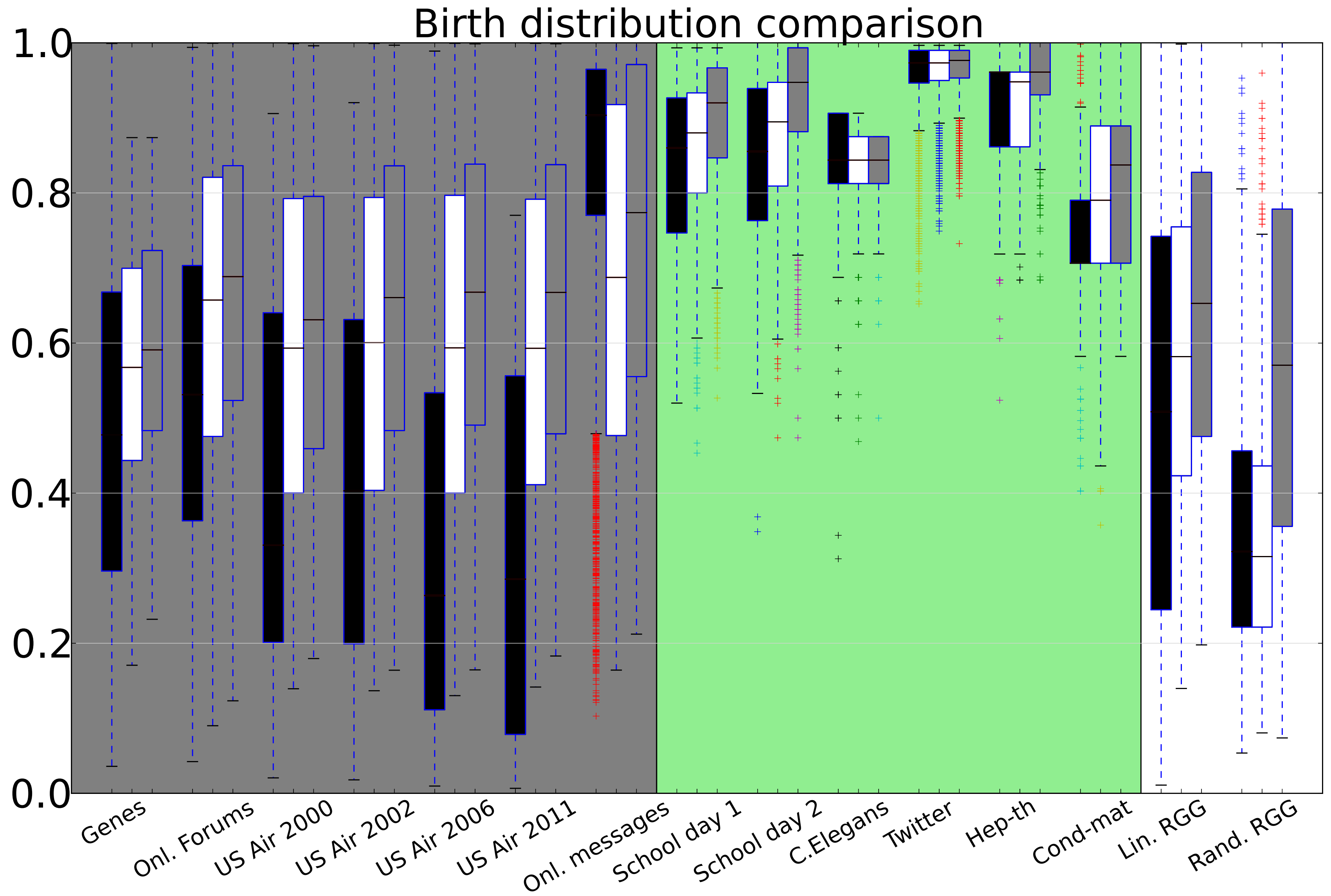


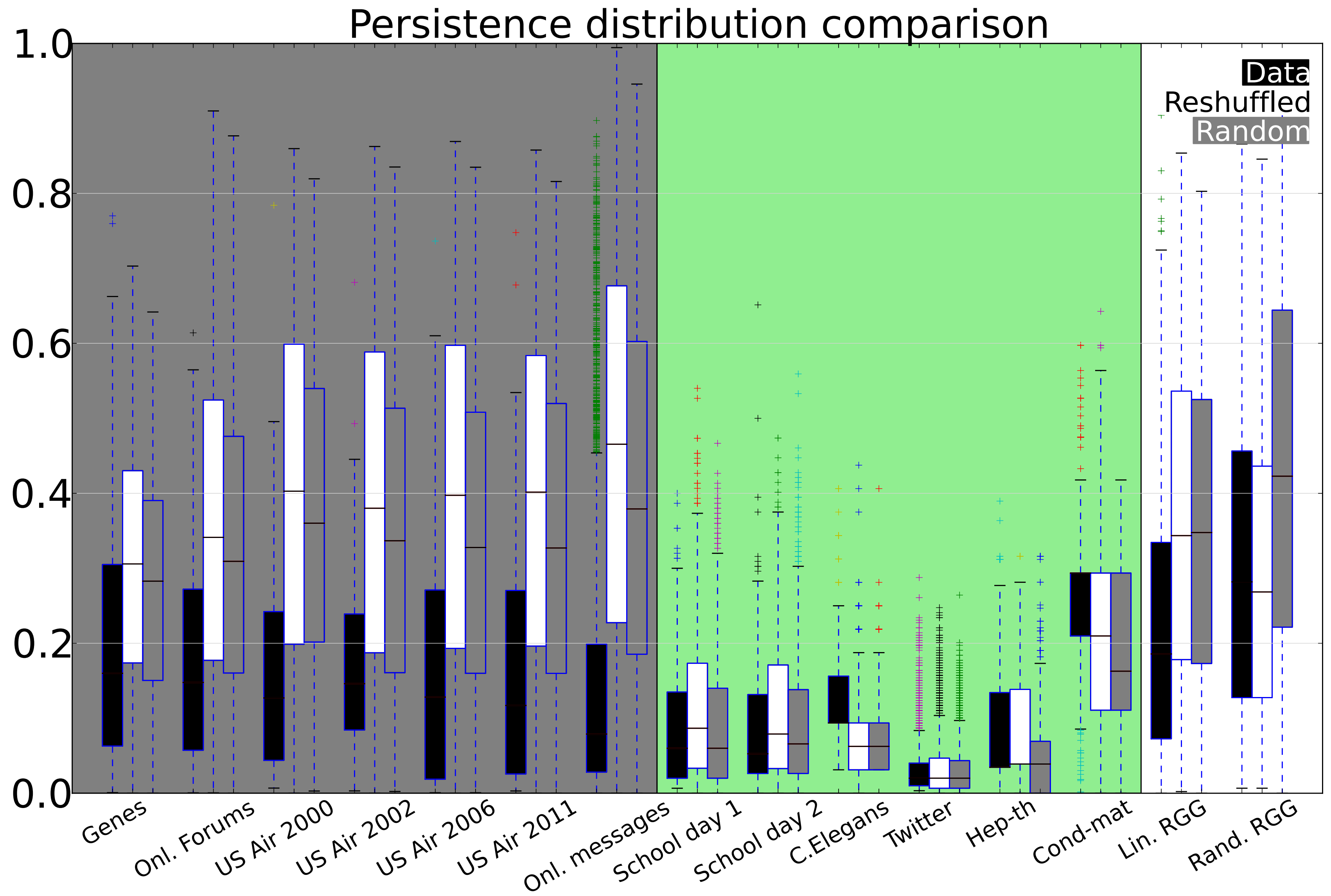
Topological cycles are not simply reproduced by statistical network properties

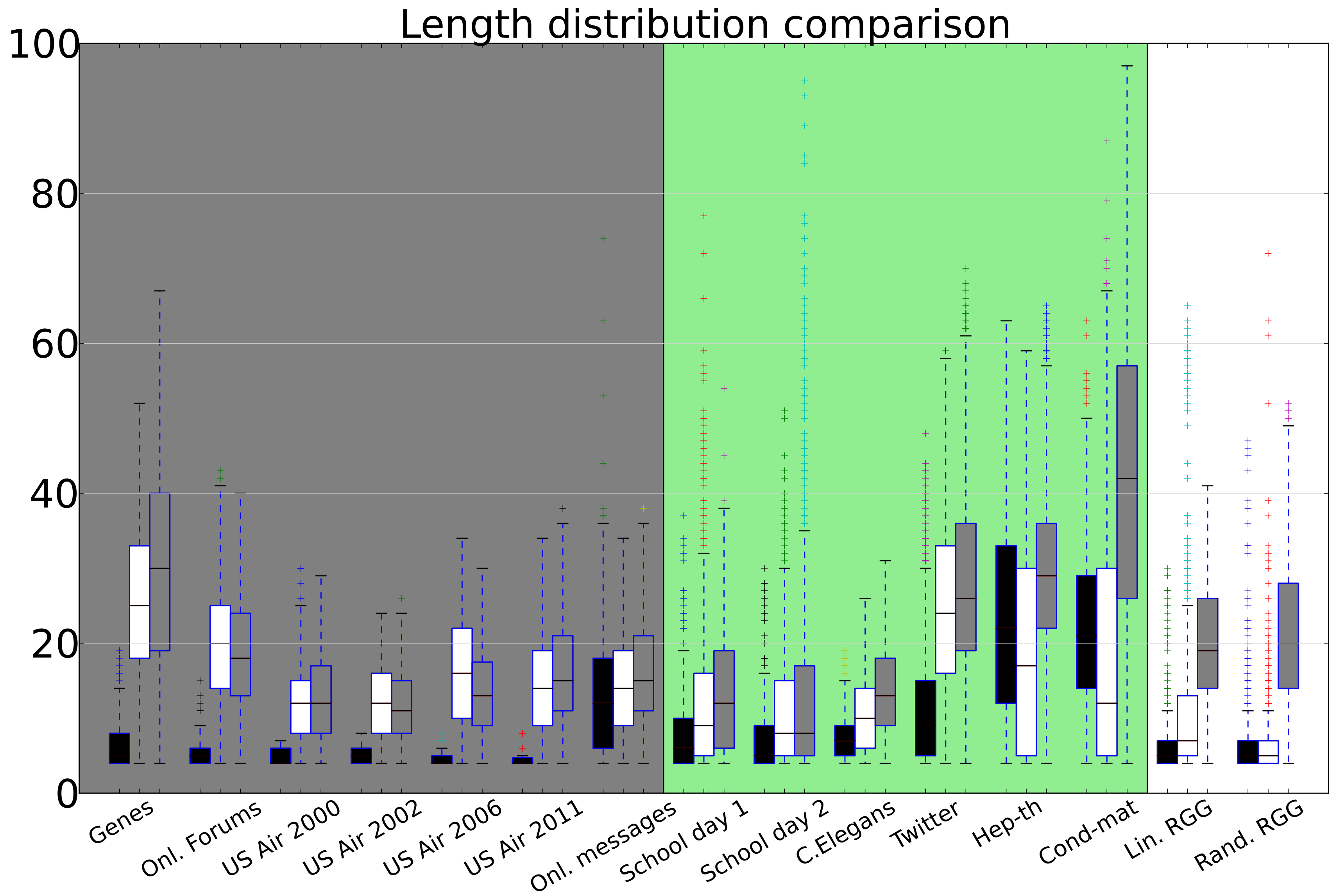
Datasets III: statistics of generators

Generators are characterized by:

1. Birth
2. Persistence
3. Length (of the cycle)







GP, M. Scolamiero, I. Donato, F.Vaccarino,
Topological strata of weighted complex networks, , PLoS ONE 8(6): e66506.

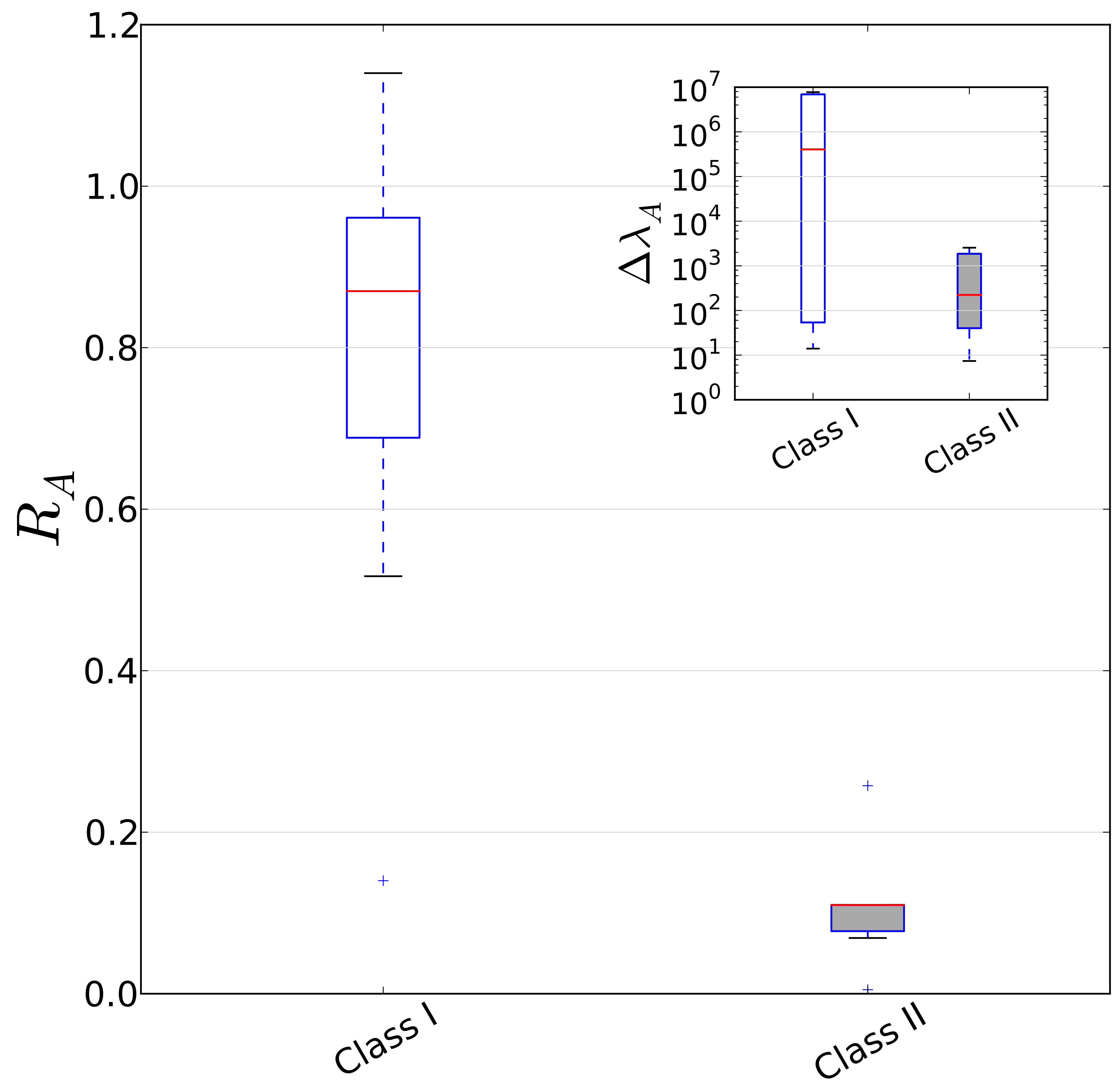
Datasets IV: spectral correlates?

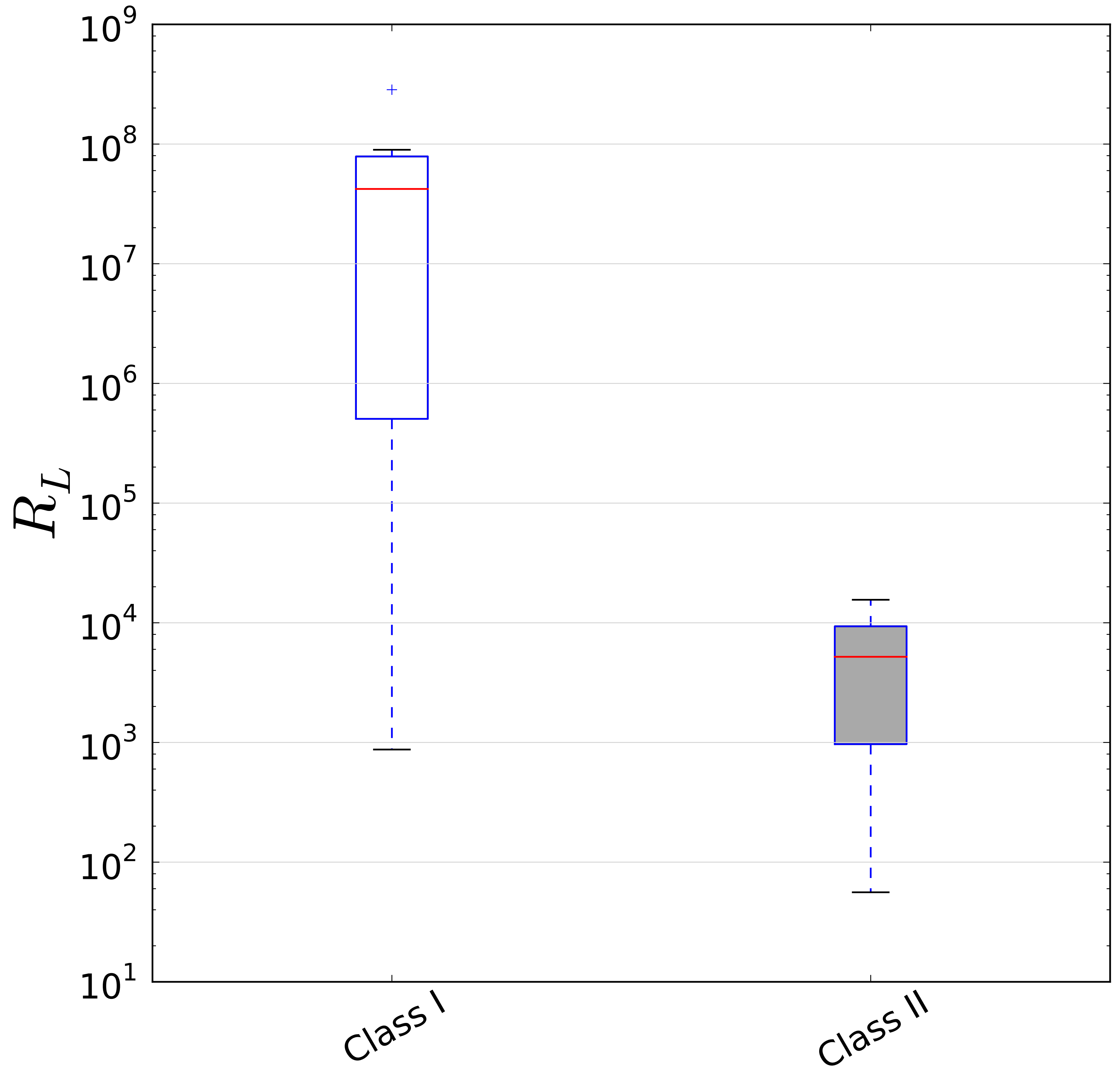
$$R_A = \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_N}$$

Related to graph expansion properties

$$R_L = \frac{\lambda_N^L}{\lambda_2^L}$$

Related to synchronisation and dyn. processes



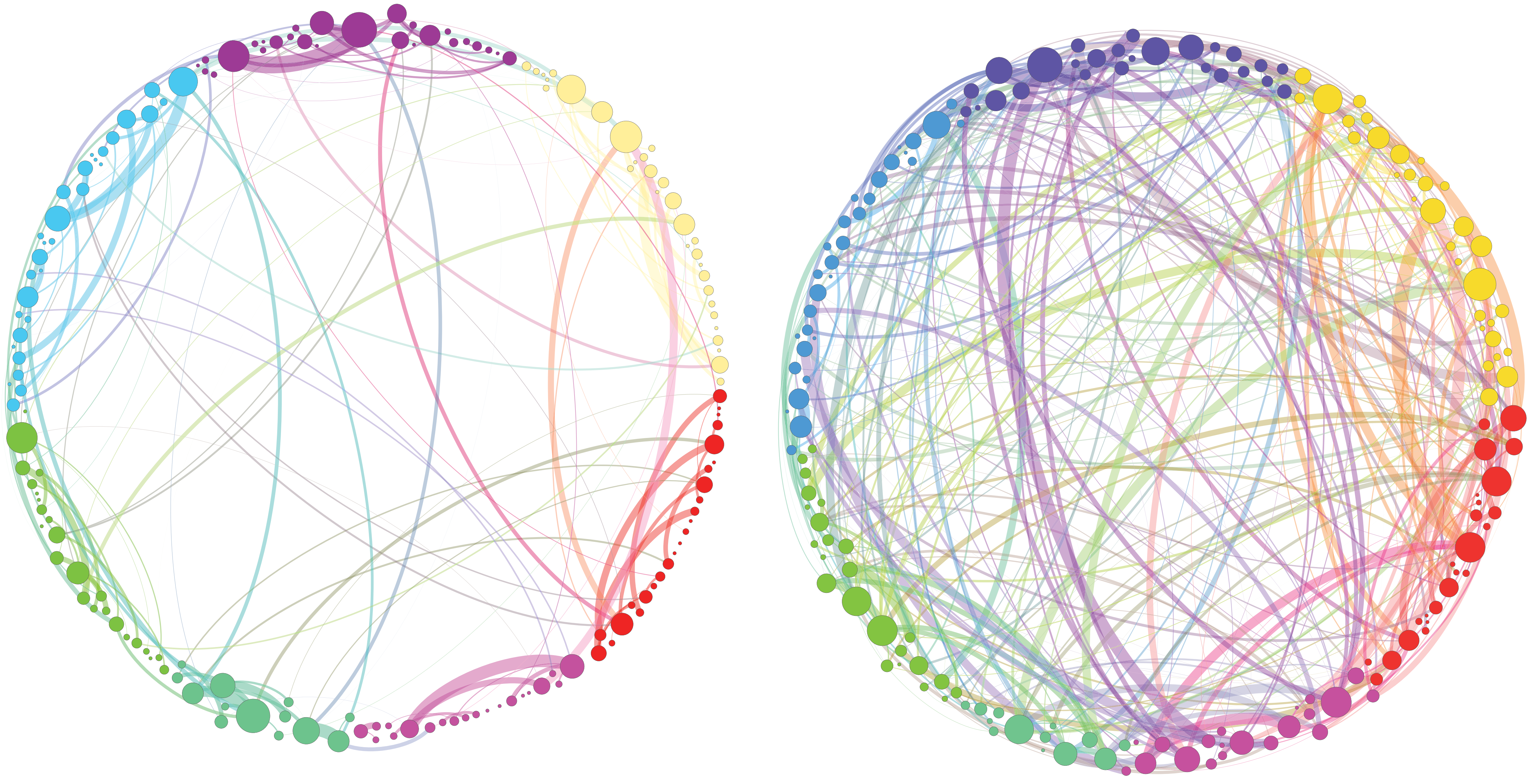


GP, M. Scolamiero, I. Donato, F.Vaccarino,
Topological strata of weighted complex networks, , PLoS ONE 8(6): e66506.

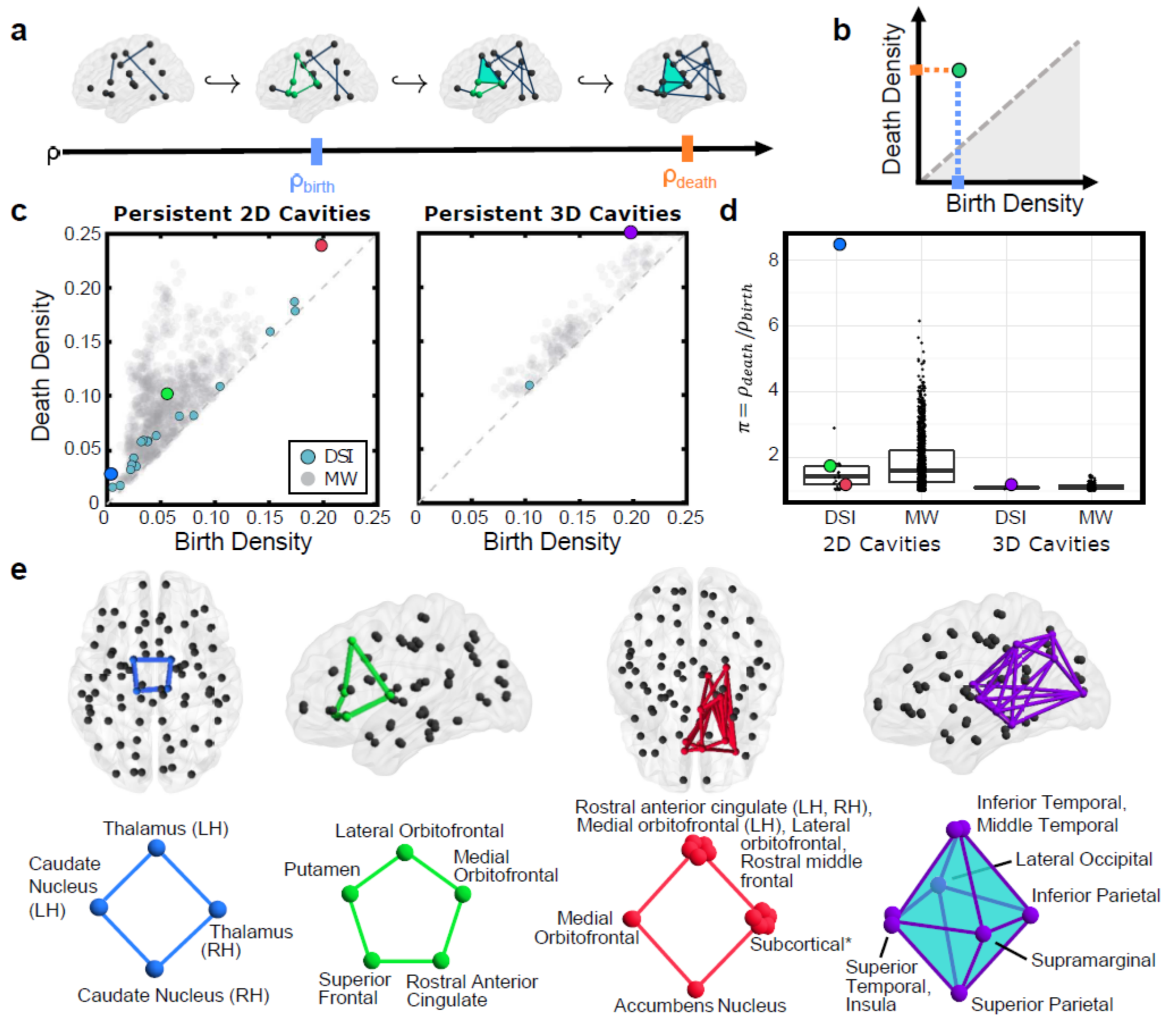
Do homological mesoscales characterise brain networks?

LANGUAGE

Graph-representations of 1-dimensional hole structure

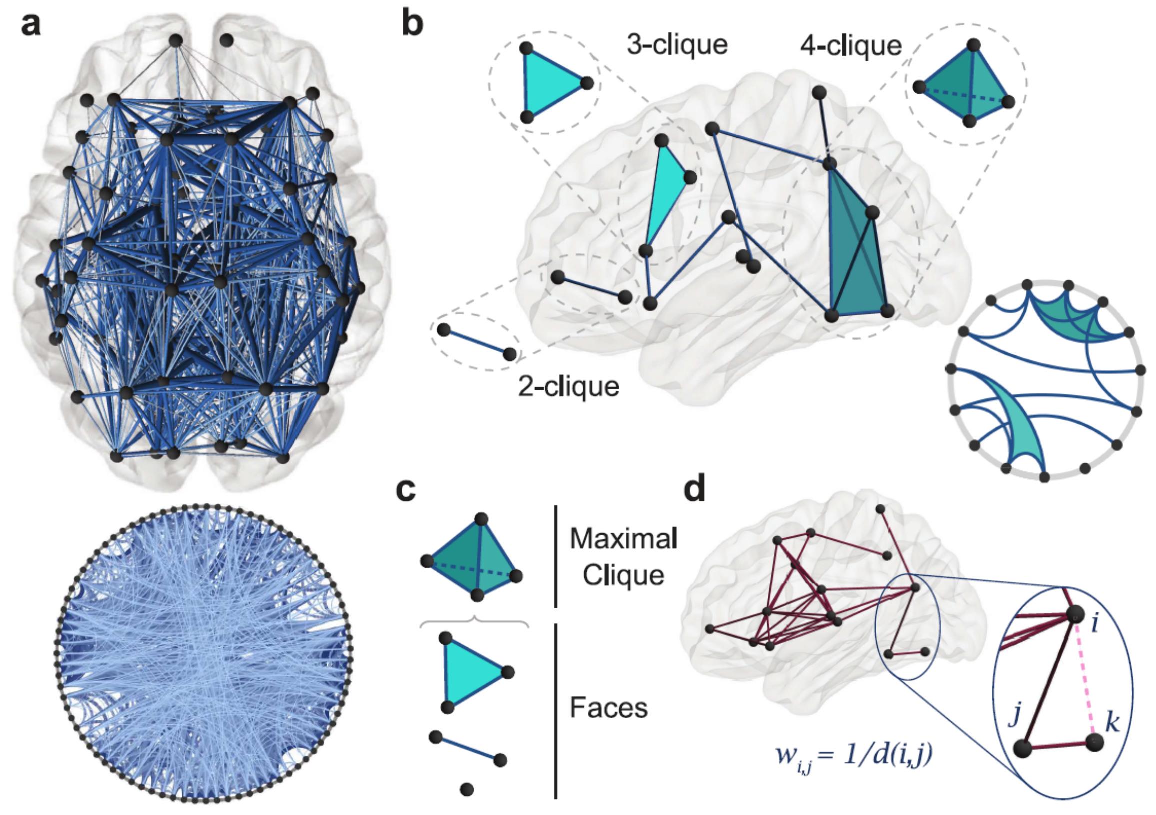


Petri, Giovanni, et al. "Homological scaffolds of brain functional networks." *Journal of The Royal Society Interface* 11.101 (2014): 20140873.

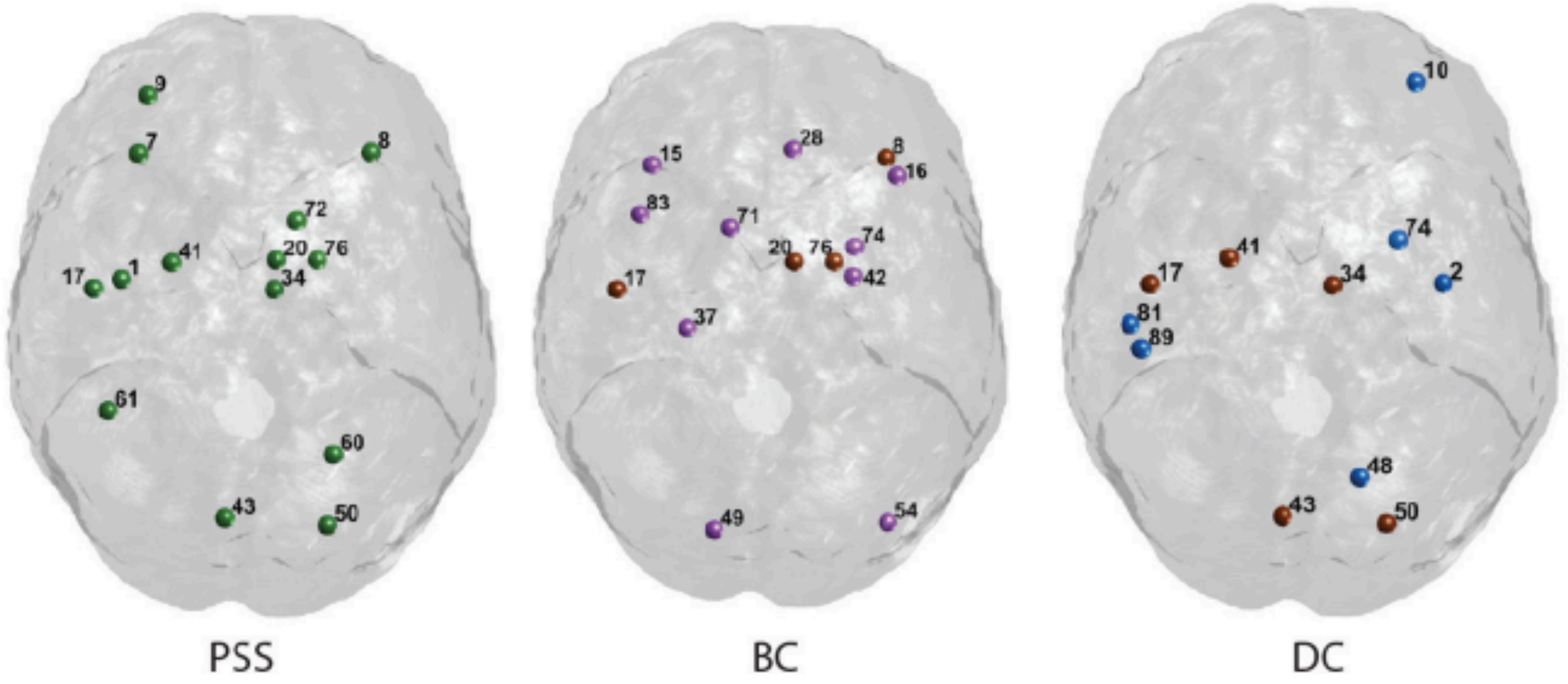


Structural brain cavities

Sizemore, Ann, et al. arXiv:1608.03520 (2016).

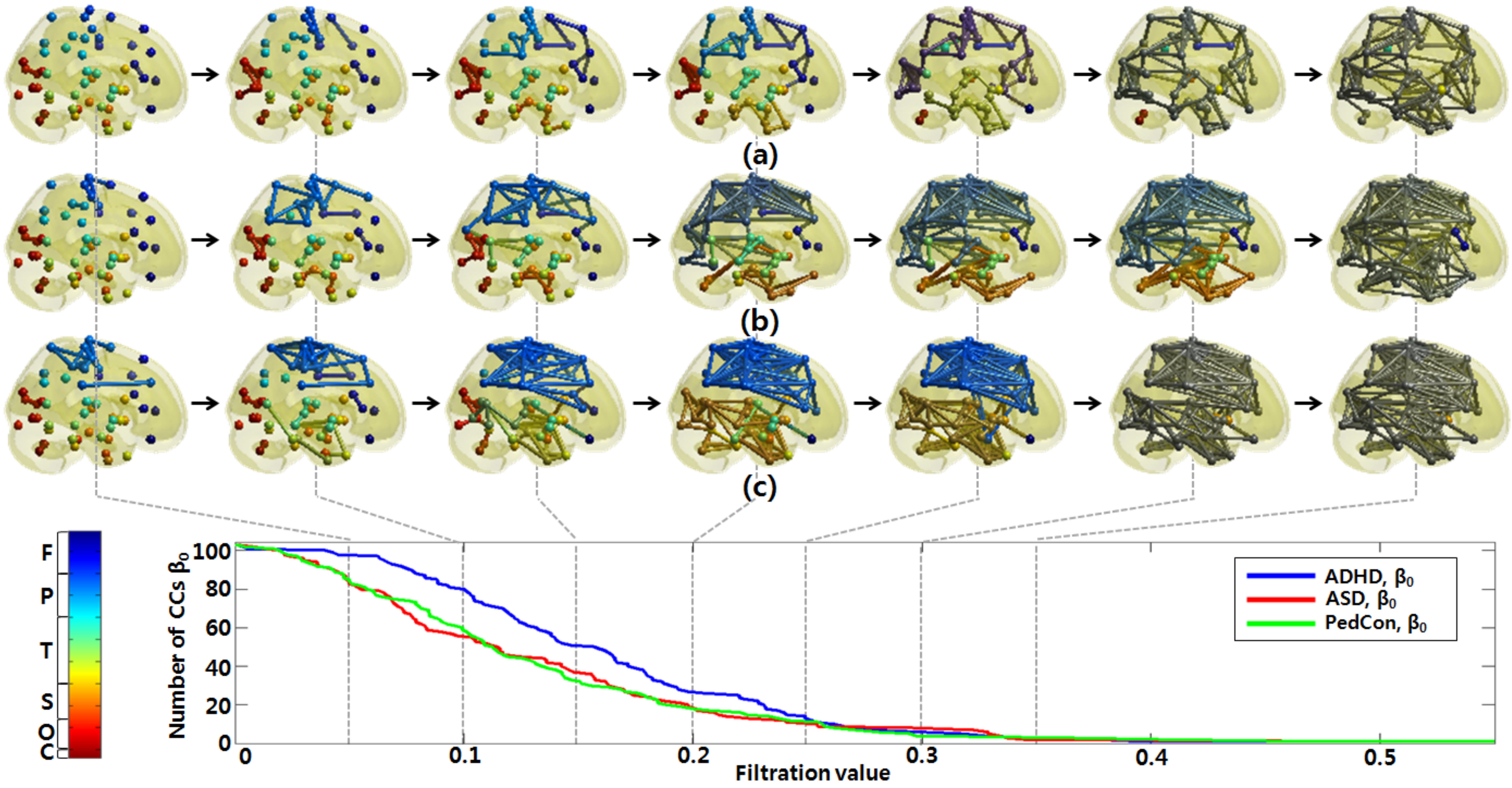


Homological persistence centrality



Lord, Louis-David, et al. "Insights into brain architectures from the homological scaffolds of functional connectivity networks." *Frontiers in systems neuroscience* 10 (2016).

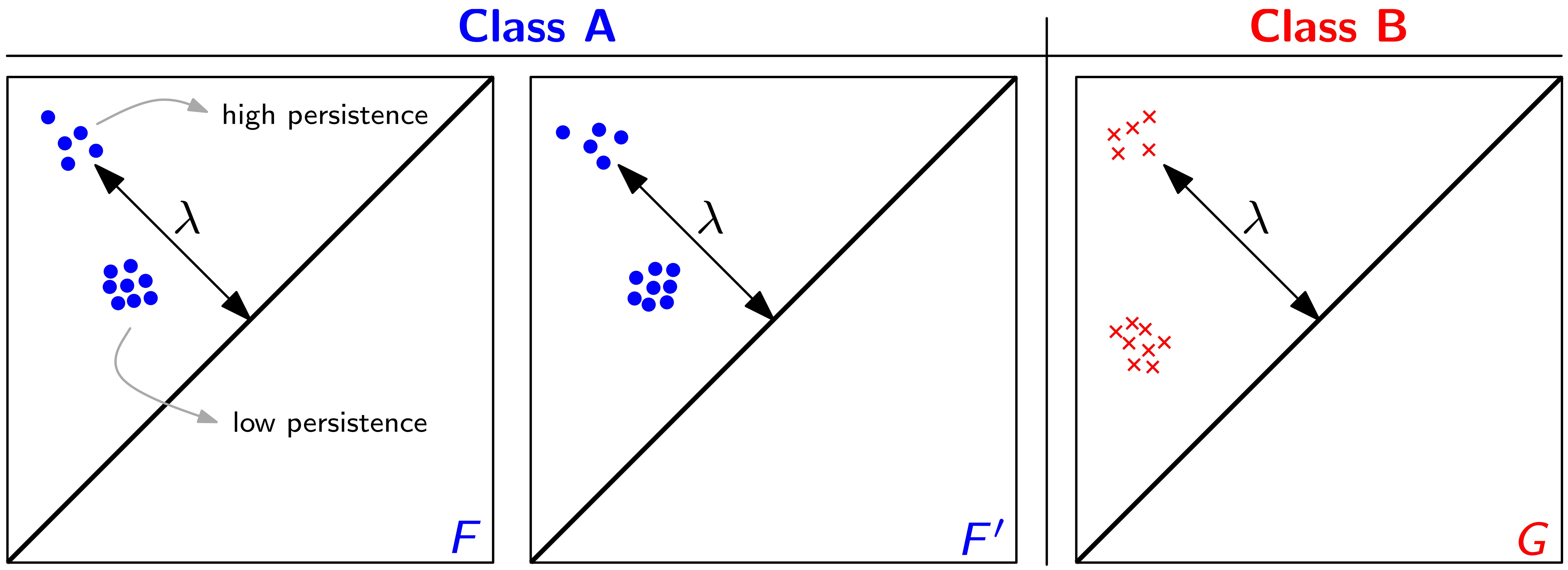
How do we compare this things?



Lee et al. (2011) MICCAI20
Lee et al. (2012) IEEE Trans. on Medical Imaging

Notebook 05

Distances?



Distances and Kernels:

- Bottleneck Distance
- (sliced) Wasserstein distance
- Persistence Scale Kernel (Reininghaus et al 2015)
- Weighted Persistence Kernel (Kusano et al, 2016)

Distances?

Bottleneck Distance

$$\delta_\infty(D_1, D_2) = \inf_{\gamma} \sup_{z \in D_1} ||z - \gamma(z)||_\infty$$

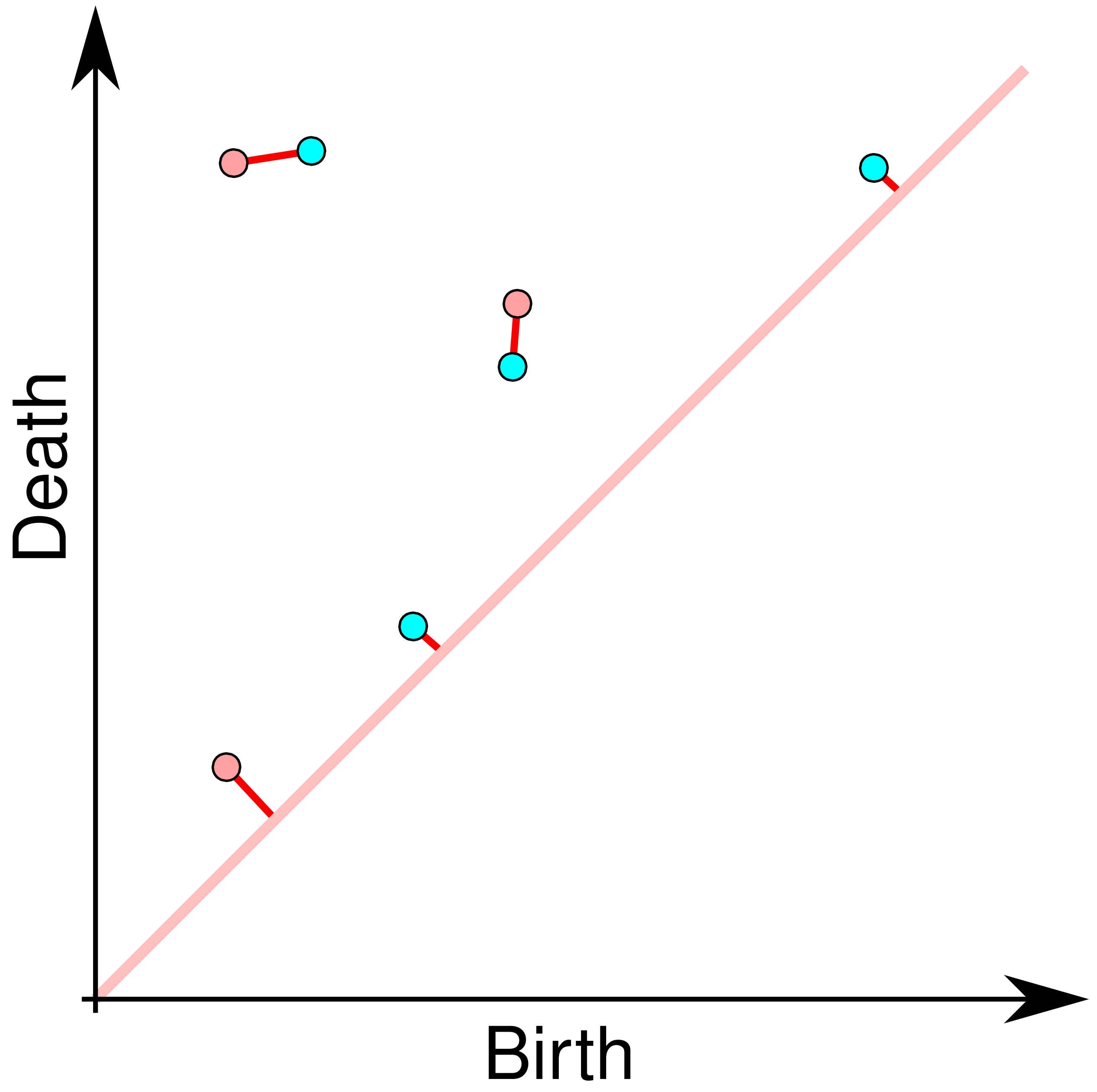
Wasserstein Distance

$$W_p\,(D_1, D_2) \;=\; \inf_{\gamma} \Big(\sum_{u \in D_1} \|u - \gamma(u)\|_\infty^p \Big)^{1/p}$$

Sliced Wasserstein Distance

$$\mathrm{SW}(\mathrm{Dg}_1, \mathrm{Dg}_2) \stackrel{\text{def.}}{=} \frac{1}{2\pi} \int_{\mathbb{S}_1} \mathcal{W}(\mu_1^\theta + \mu_{2\Delta}^\theta, \mu_2^\theta + \mu_{1\Delta}^\theta) \mathrm{d}\theta.$$

$$k_{\mathrm{SW}}(\mathrm{Dg}_1, \mathrm{Dg}_2) \stackrel{\text{def.}}{=} \exp\left(-\frac{\mathrm{SW}(\mathrm{Dg}_1, \mathrm{Dg}_2)}{2\sigma^2} \right).$$



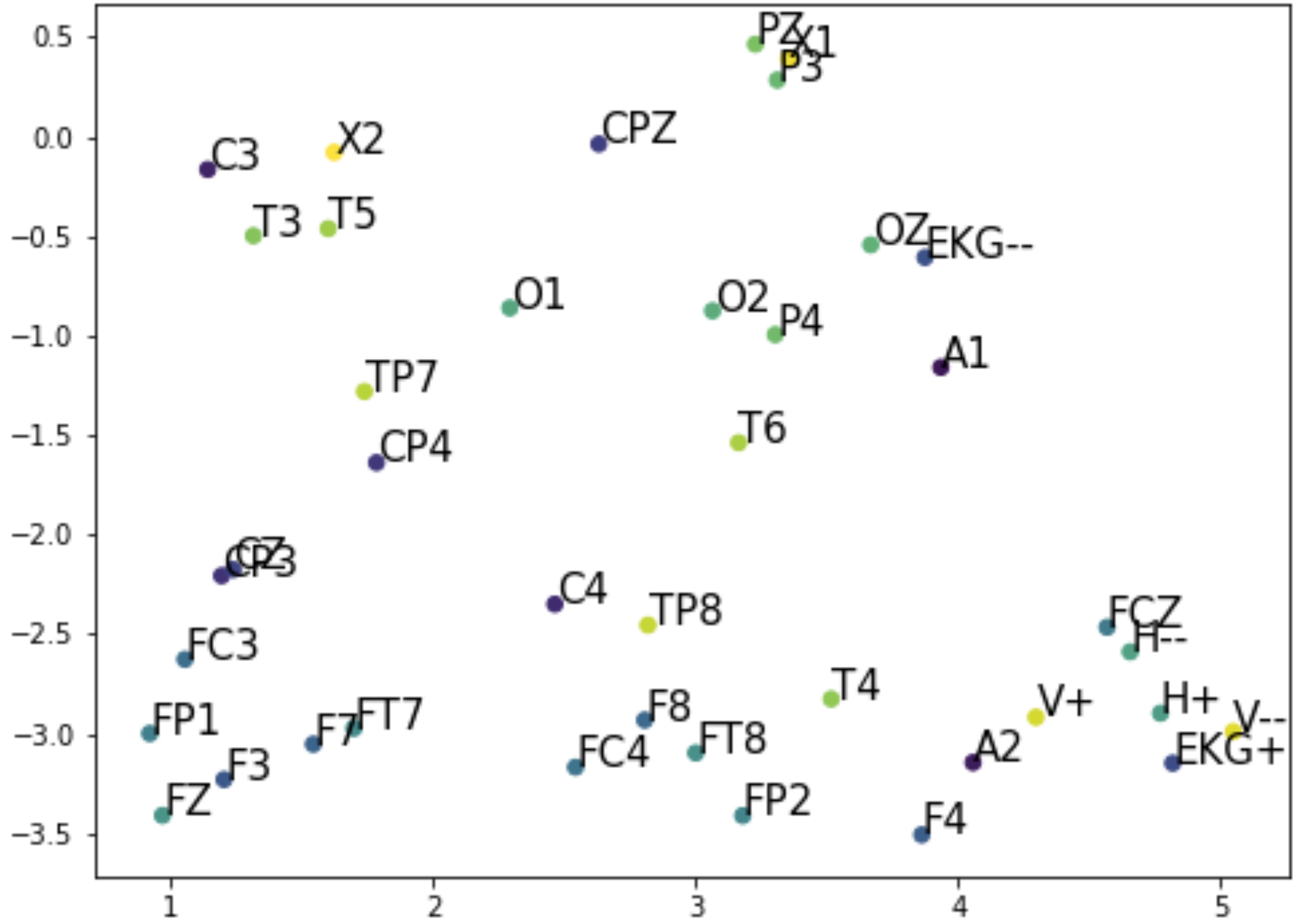
Persistence Scale Kernel

$$k_\sigma(F, G) = \frac{1}{8\pi\sigma} \sum_{\substack{p \in F \\ q \in G}} e^{-\frac{\|p-q\|^2}{8\sigma}} - e^{-\frac{\|p-\bar{q}\|^2}{8\sigma}}.$$

Notebook 06

Do homological shapes capture transitions?

Correlation space of EEG signals



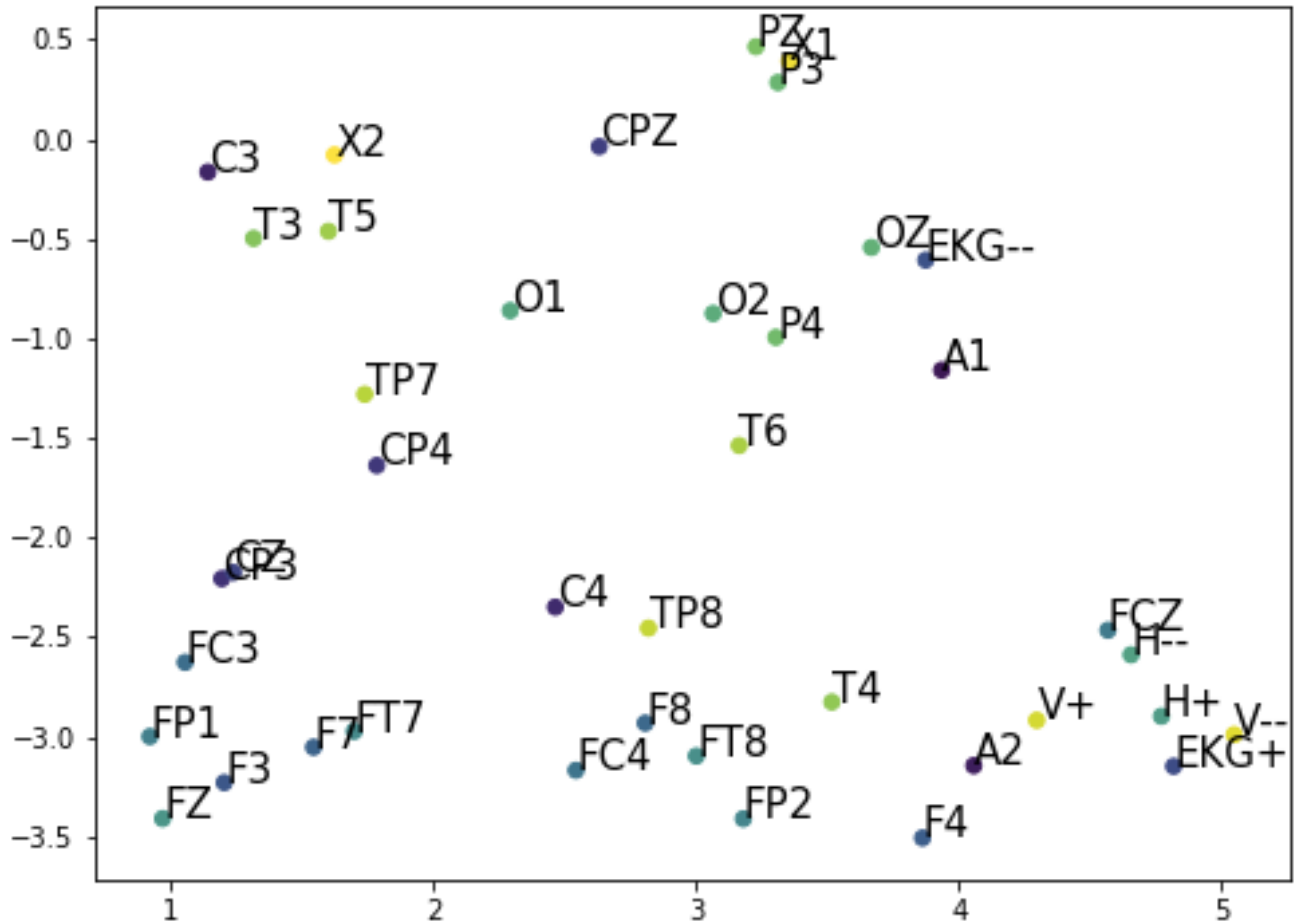
UMAP 2d projection

29 EEG channels

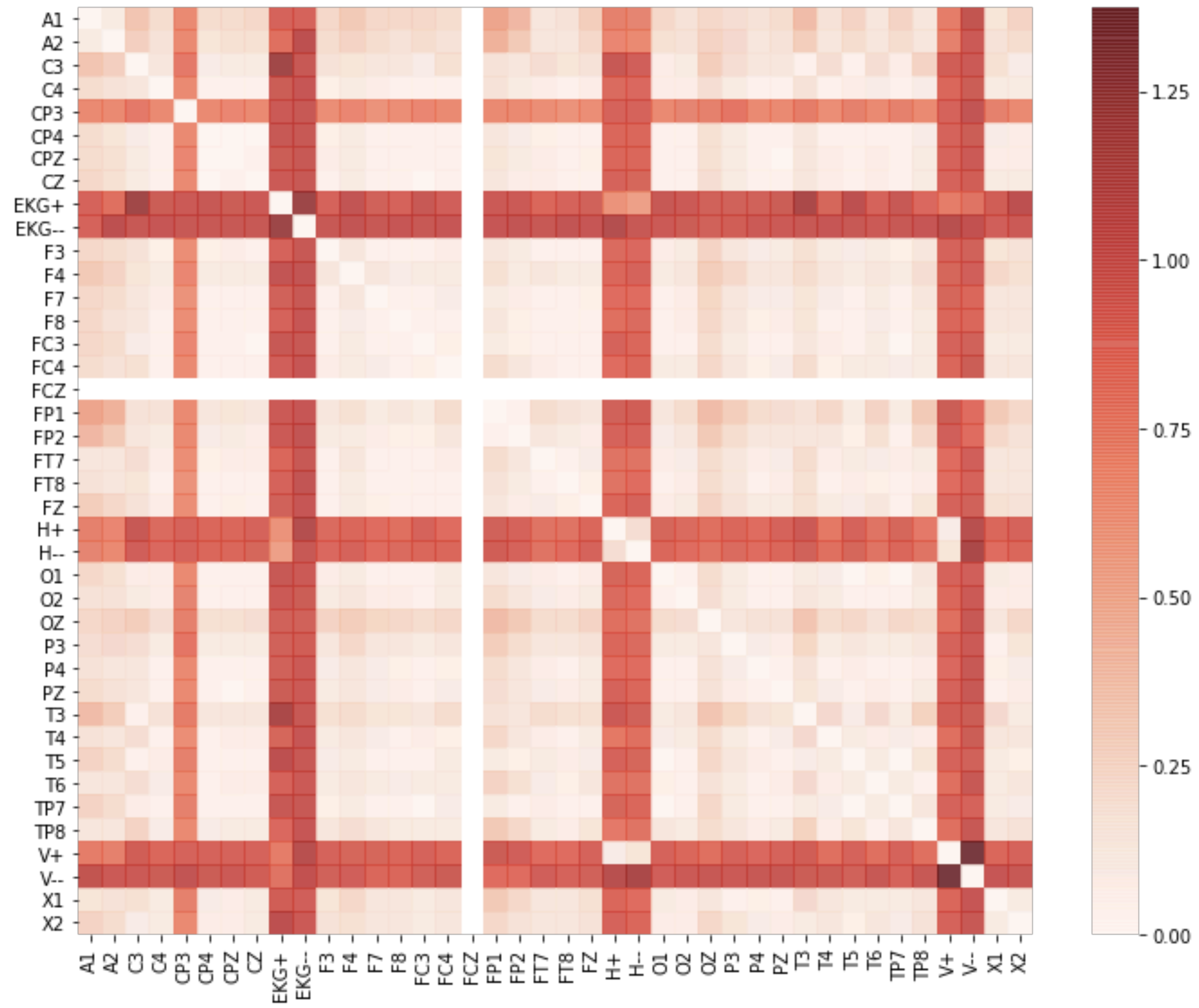
>10^5 timepoints

Do homological shapes capture transitions?

Correlation space of EEG signals

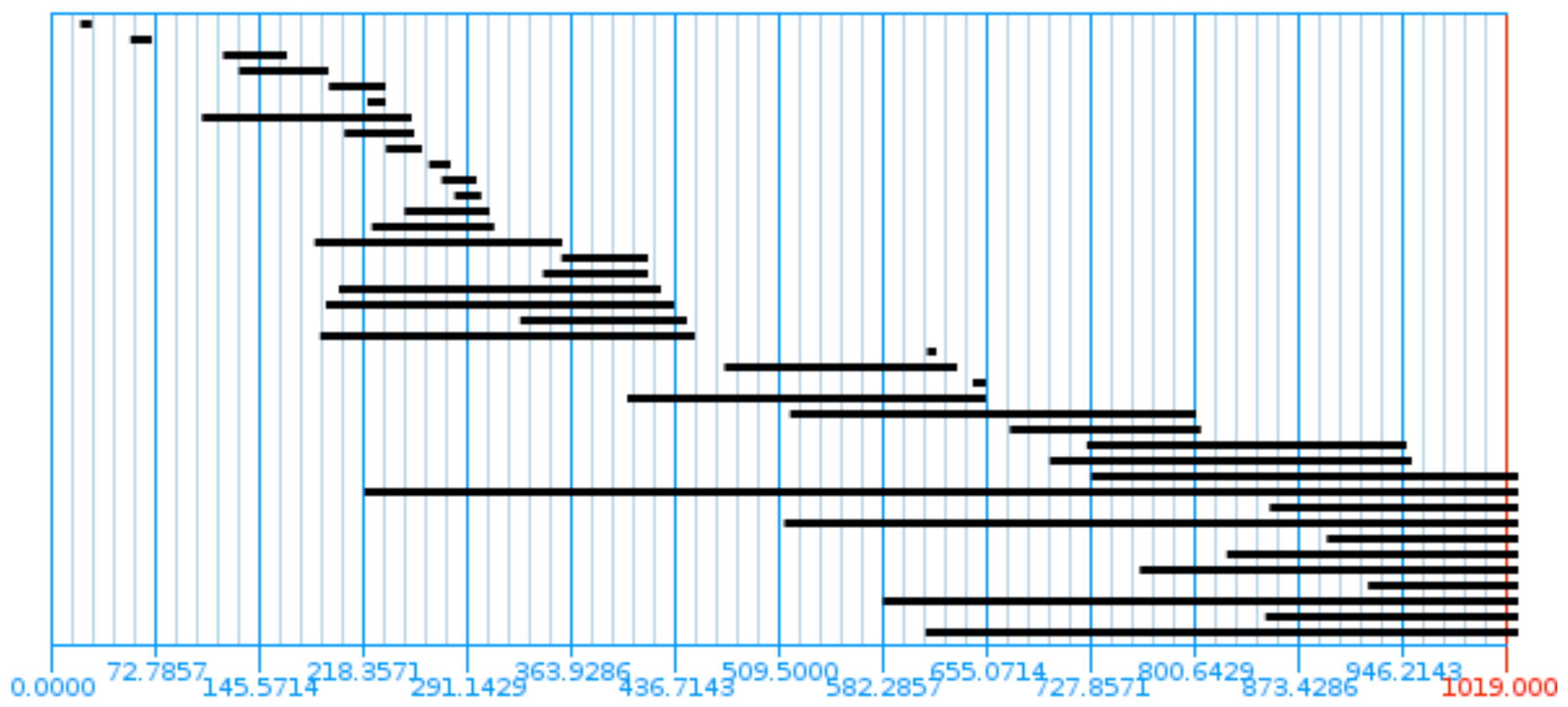


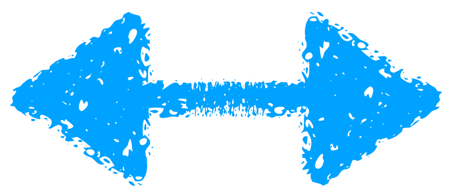
UMAP 2d projection

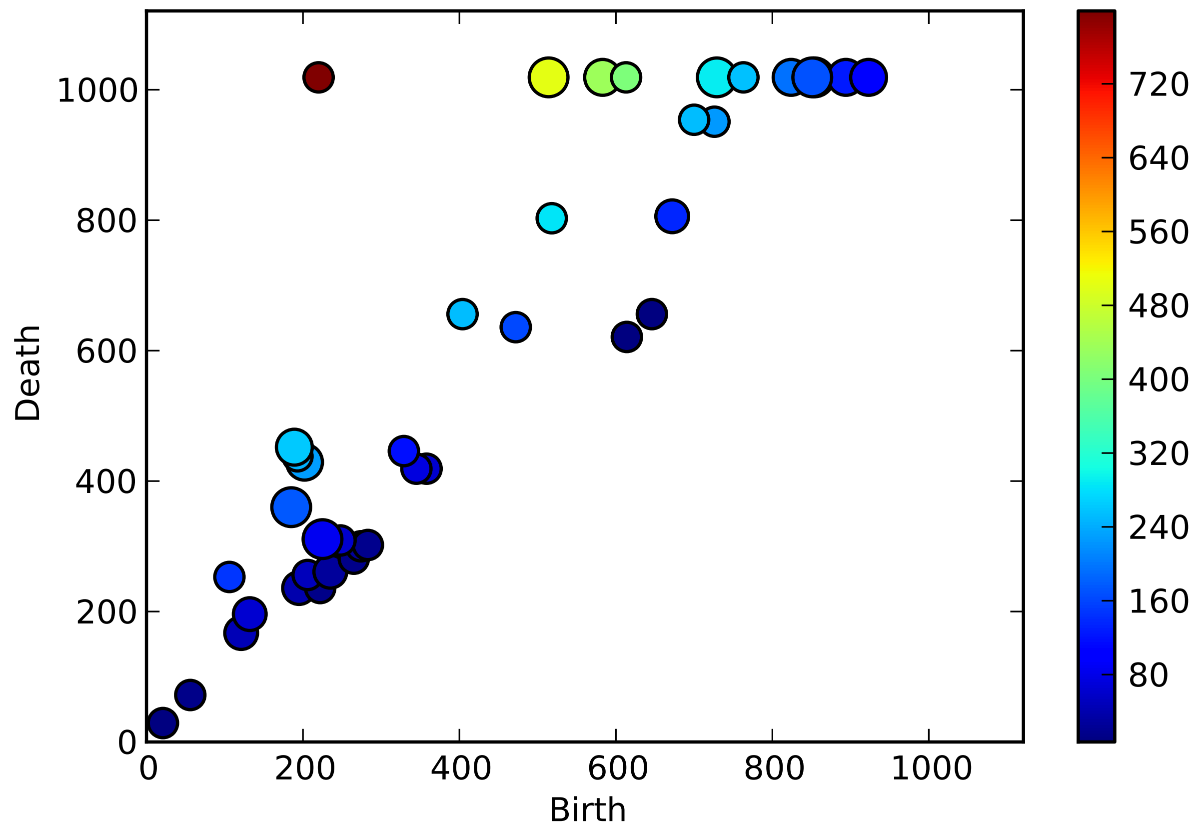


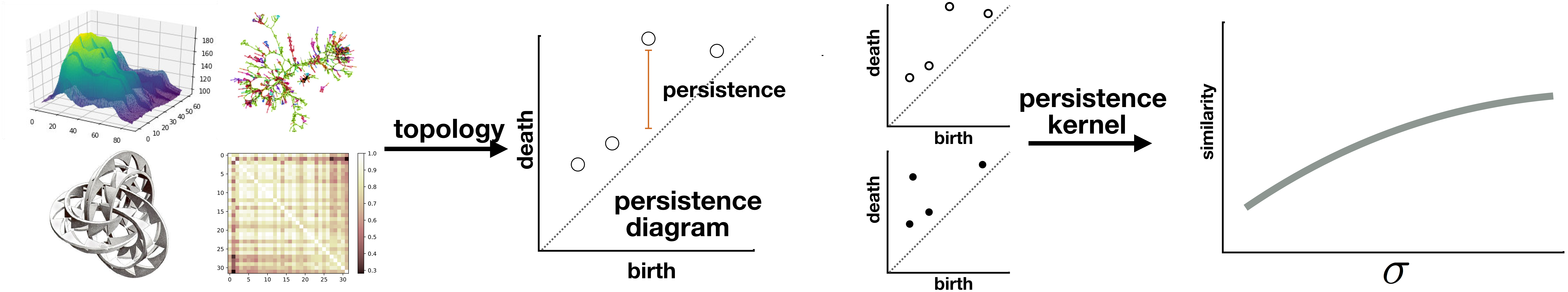
Pearson-correlation distance

Do homological shapes capture transitions?





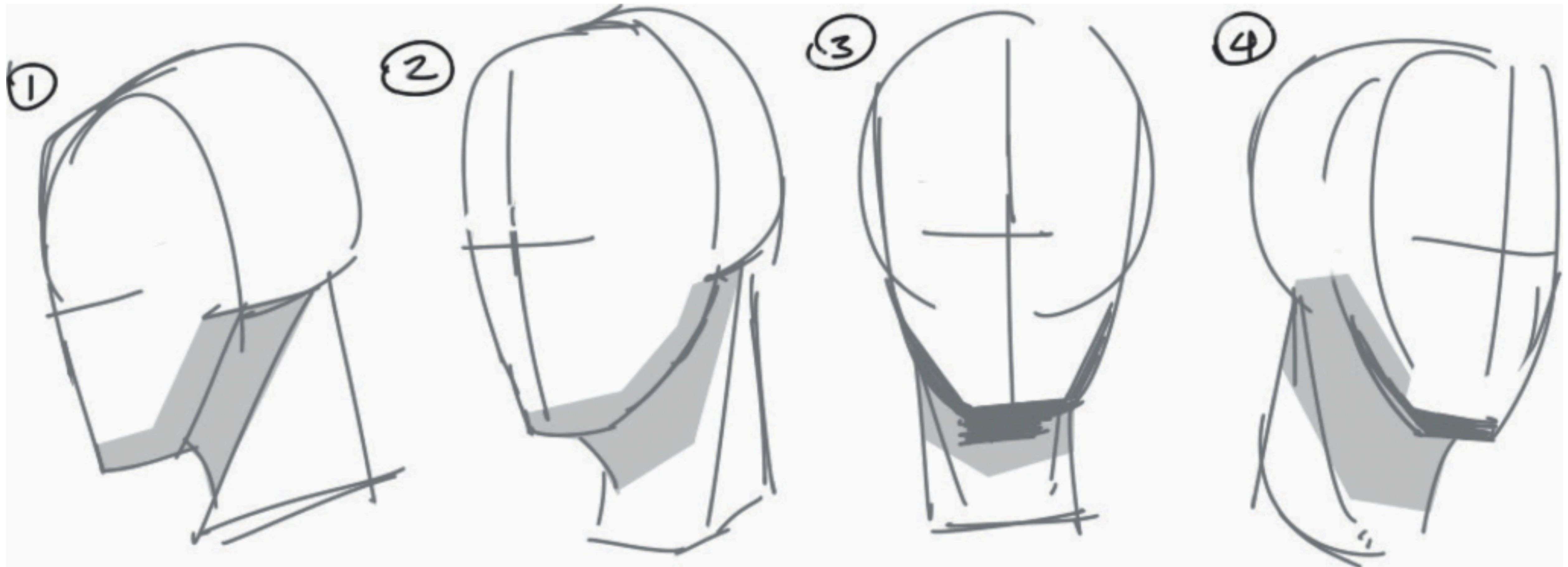




Do homological shapes capture transitions?

Dataset

- 18 highly hypnotisable subjects, 19 lowly hypnotisable subjects
- Hypno-score proxy for increased functional equivalence (FE)
- Scalp EEG recording (29 EEG channels)
- Visual + kinetic mental image / Rotation / visual + kinetic mental image
- **Hypothesis:**
  - high FE -> accurate real-task state reproduction -> vivid images
  - high FE -> small changes from basal state -> effortless imagination

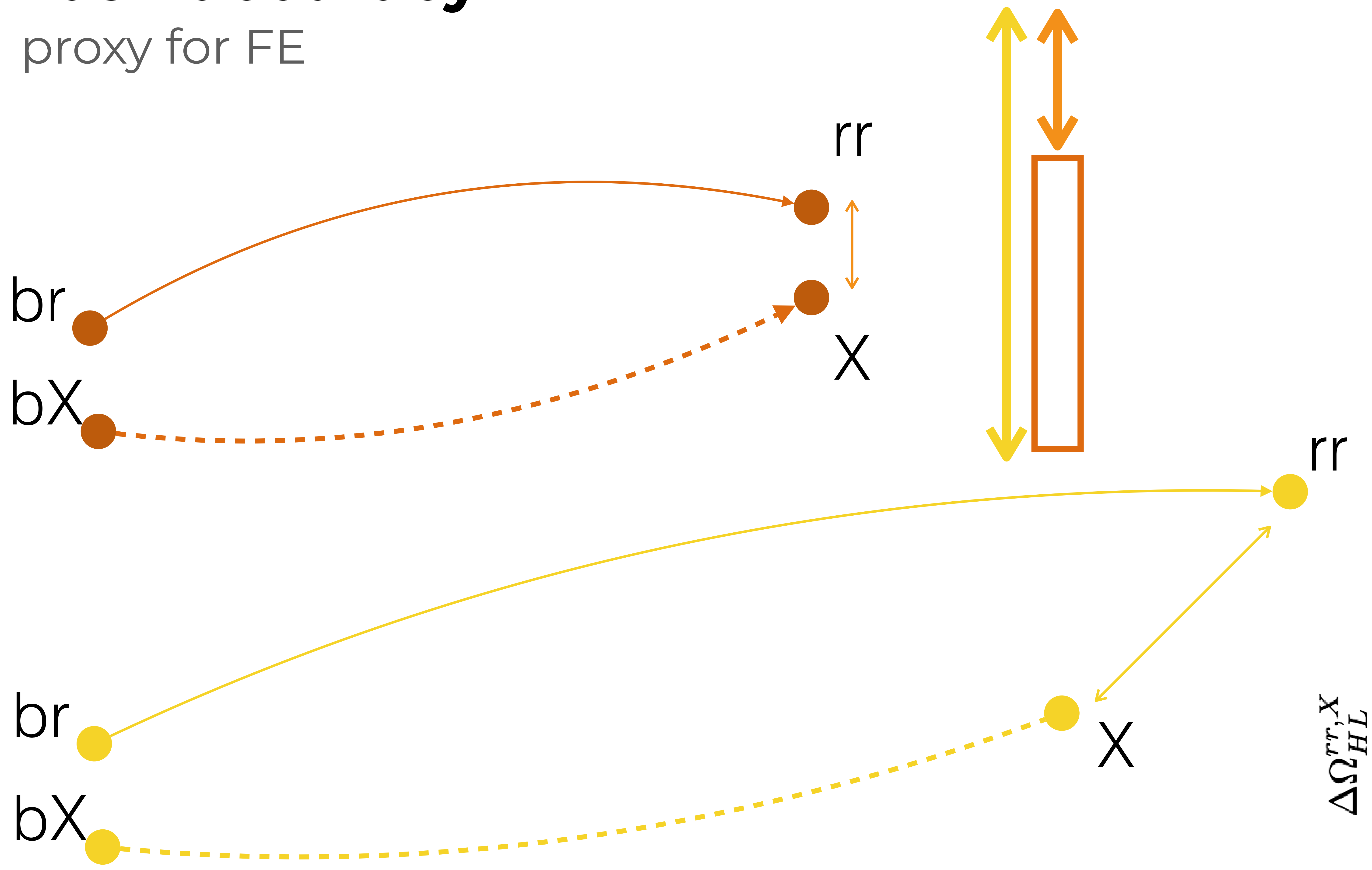


Ibanez-Marcelo, Esther, et al. "Topology highlights mesoscopic functional equivalence between imagery and perception." *bioRxiv* (2018): 268383.

Do homological shapes capture transitions?

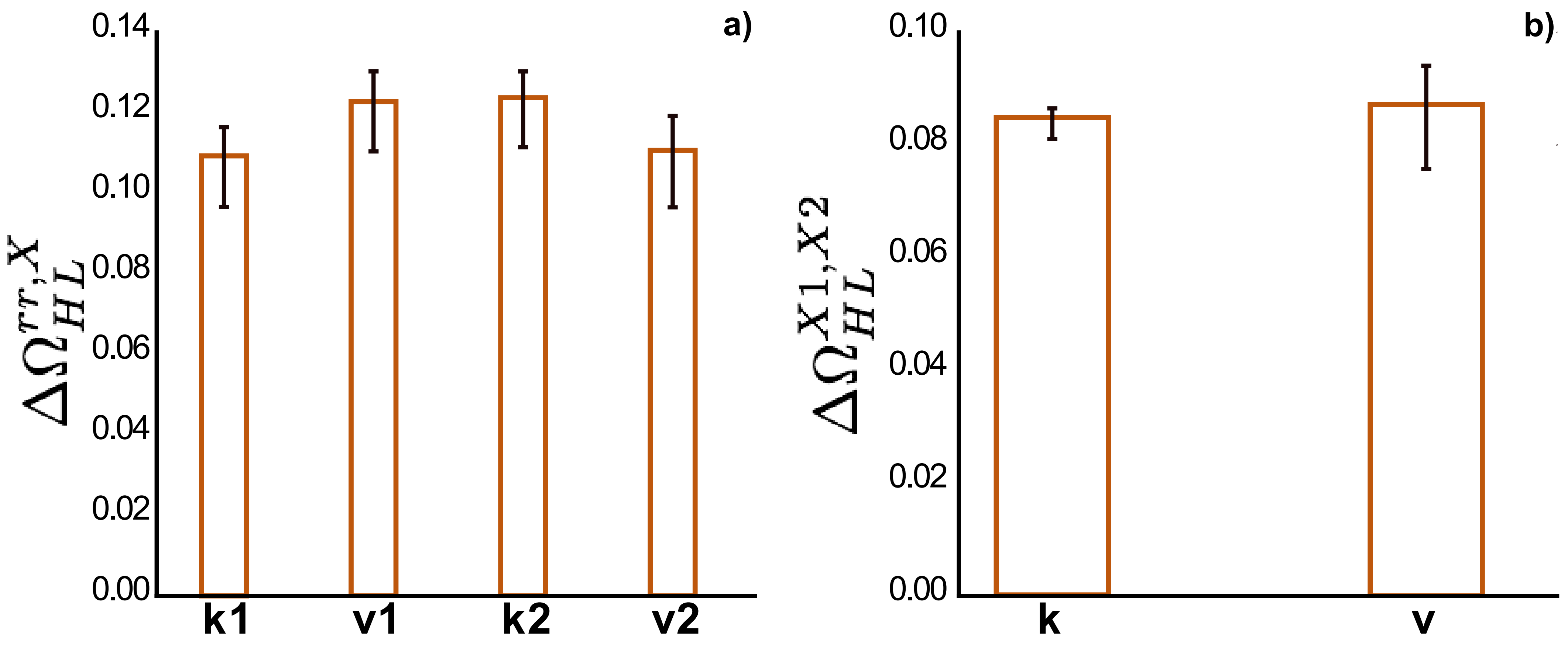
Task accuracy

proxy for FE



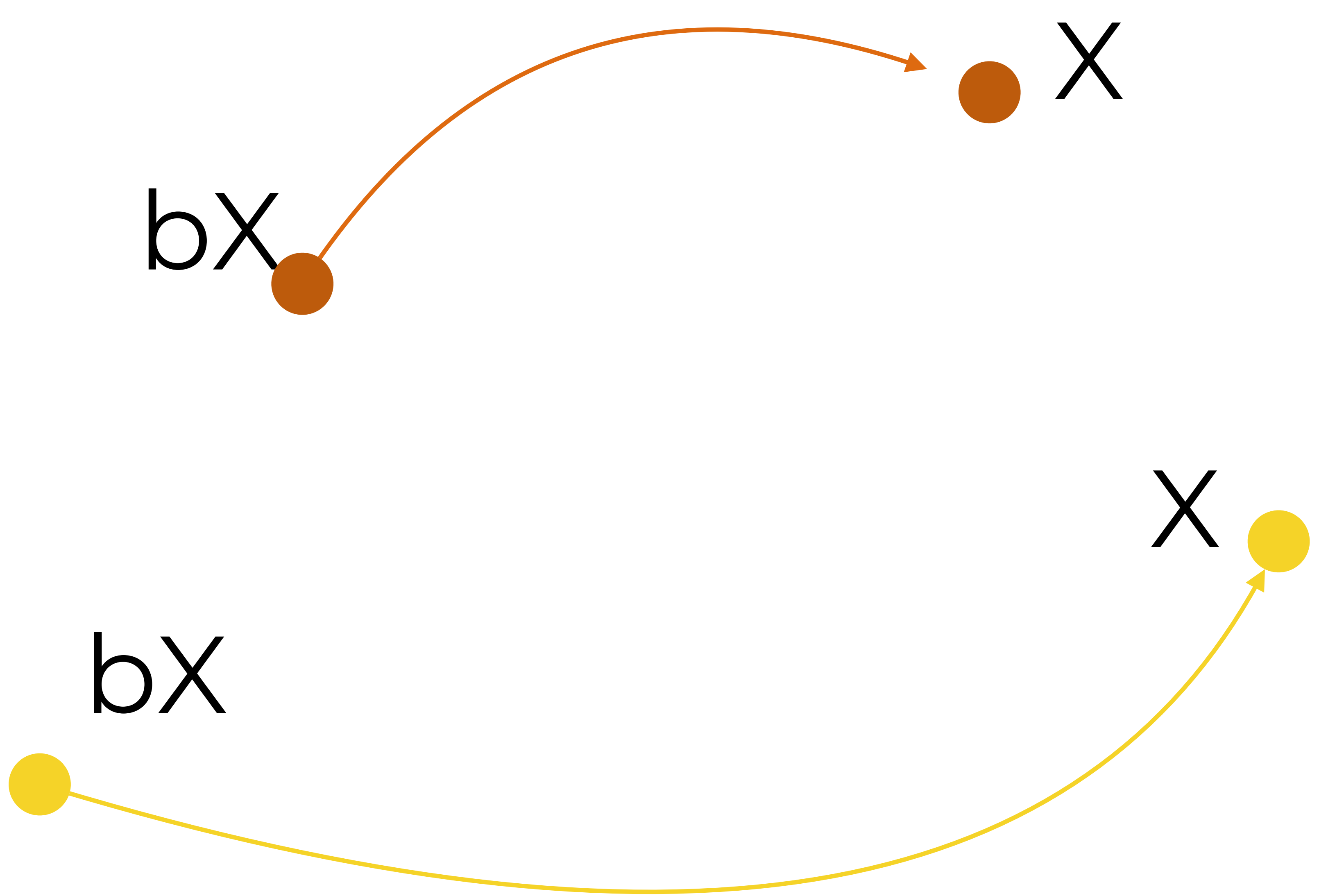
Are they more accurate?
YES!

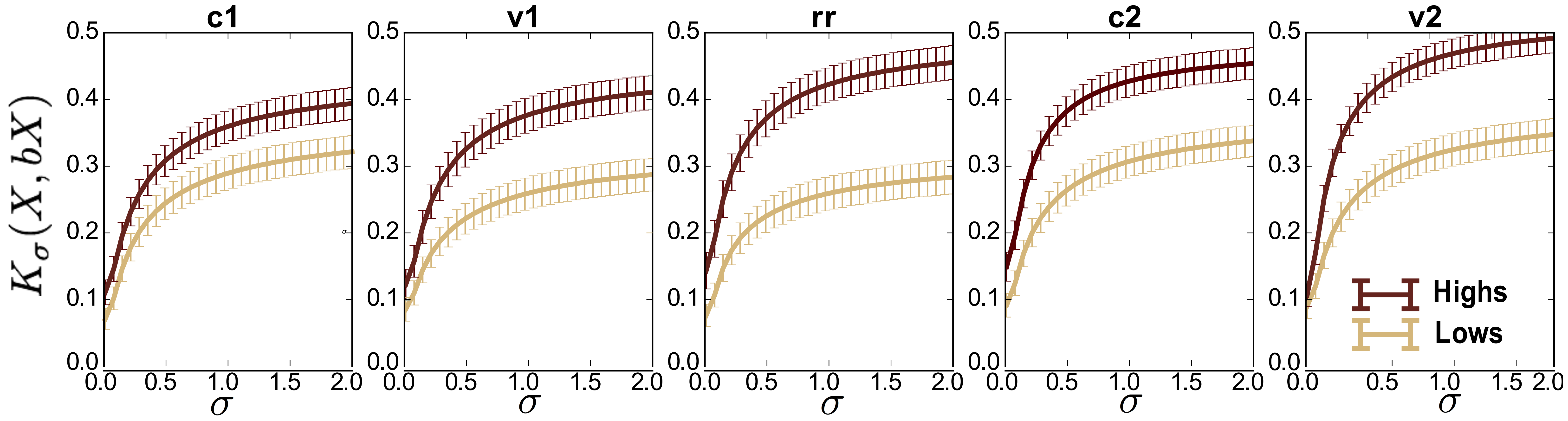
higher vividness in Highs
(group level)



Do homological shapes capture transitions?

Basal deviation

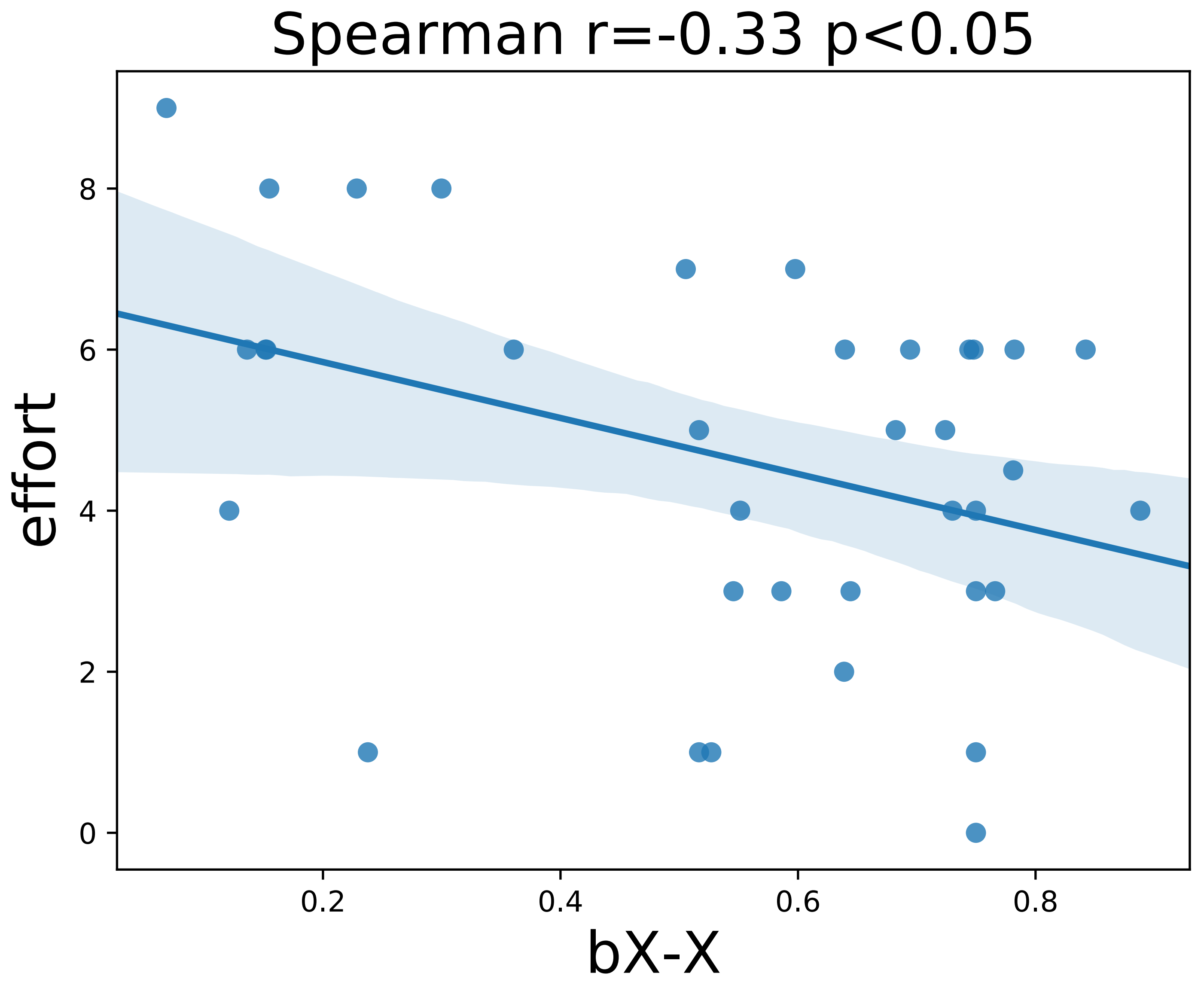




- Do *highs* reconfigure less?
  YES!

- In all sensorimotor and imaginative tasks *highs* show higher topological similarity with basal conditions than *lows*

- Lower reported **effort** in *highs*

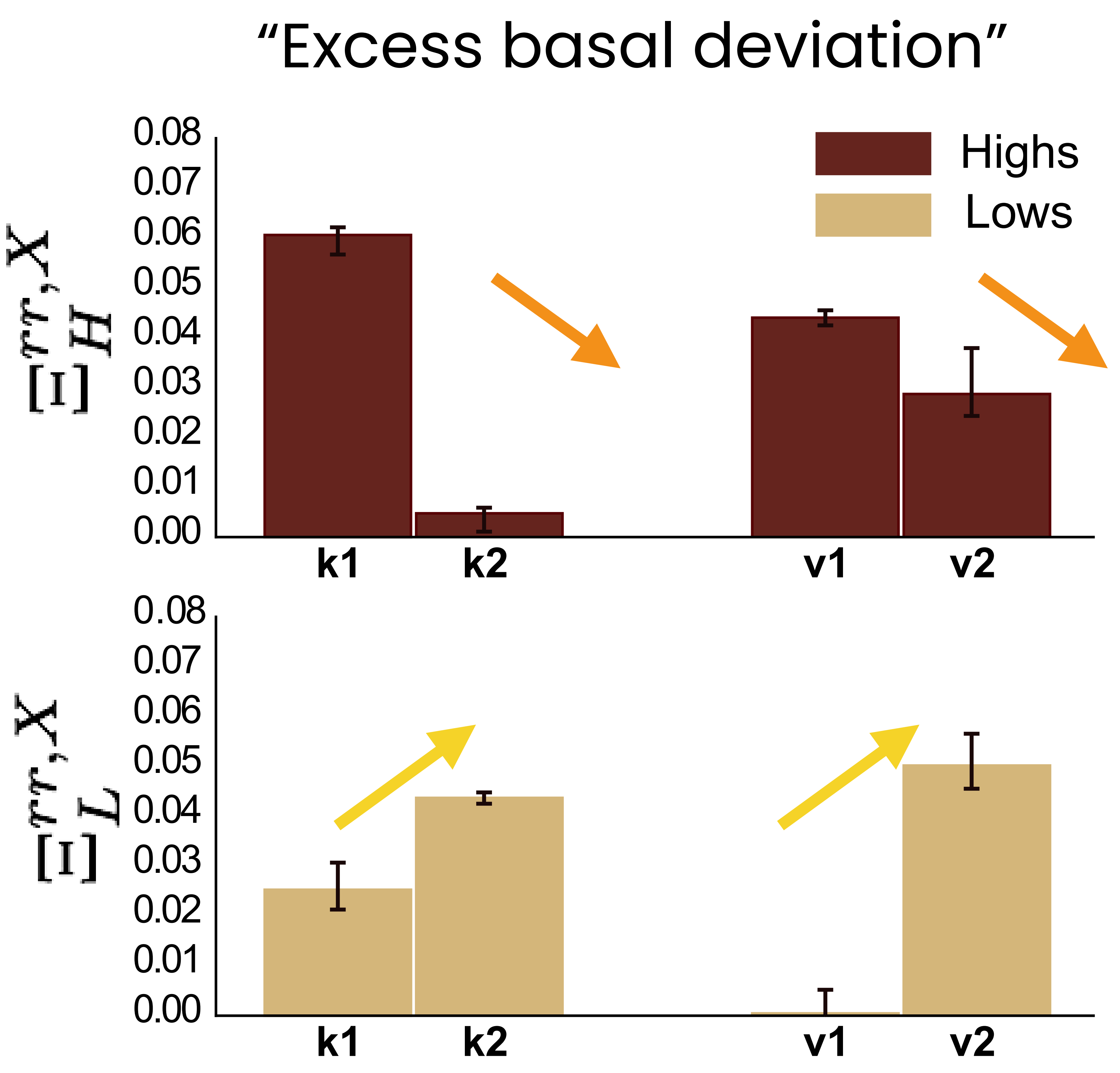


Do homological shapes capture transitions?

Learning

Do they learn?

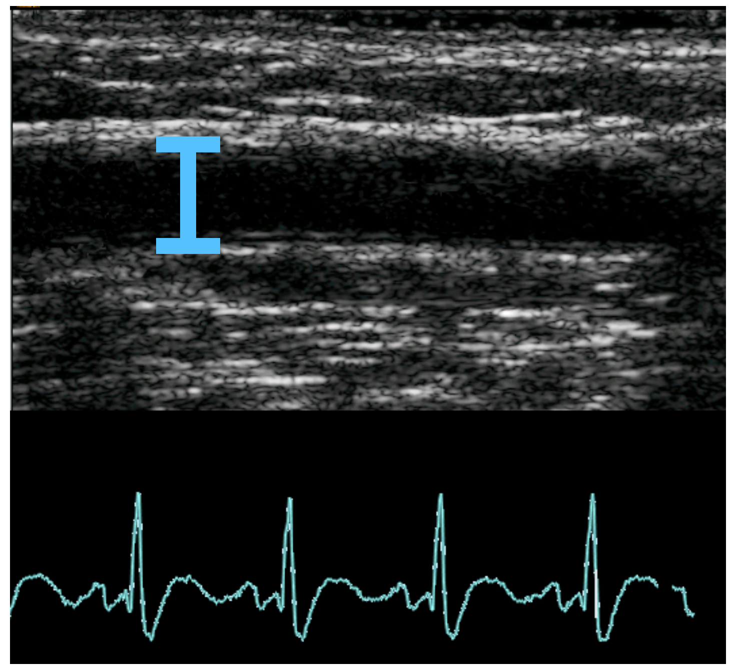
- Highs
  - Reduced effort in Highs
  - same vividness and accuracy
- Lows
  - reduced vividness, same accuracy
  - increased effort
- No measurable improvement in accuracy but “in easiness”

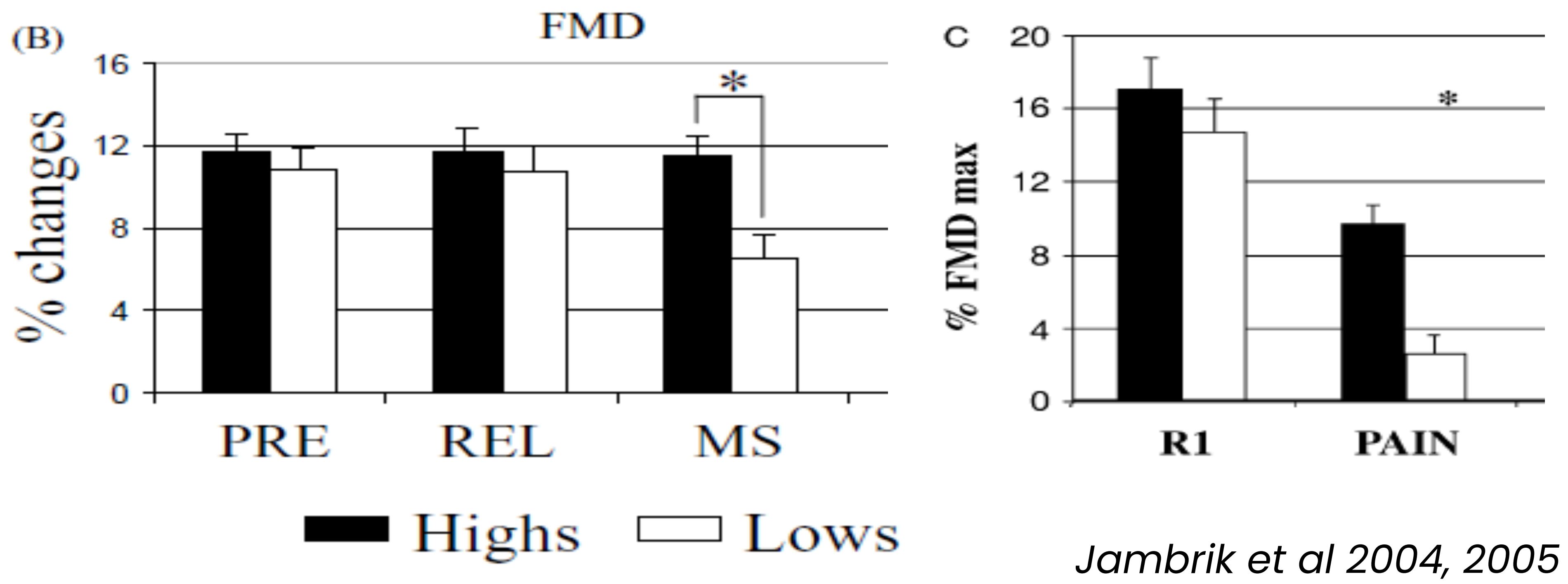


Do homological shapes capture transitions?

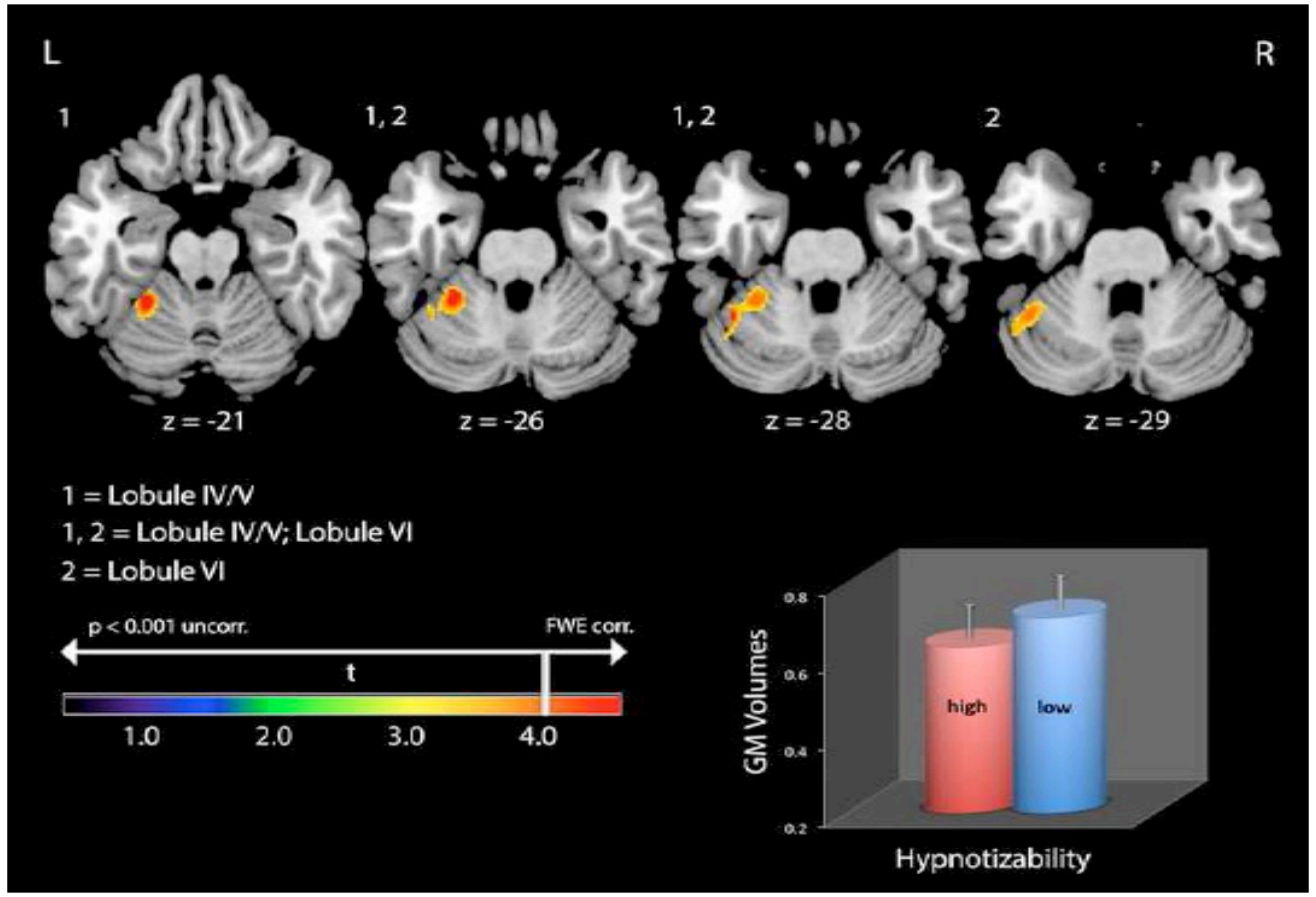
Potential mechanism

- larger FMD in Highs depends on larger release of endothelial nitric oxide
- reduced cerebellar volume
- hence, reduced inhibition of the cerebral cortex by cerebellum causing:
  - stronger FE + highly distributed information processing
  - Paradoxical control of pain after cerebellar anodal stimulation *(Bocci et al, 2017)*





Jambrik et al 2004, 2005

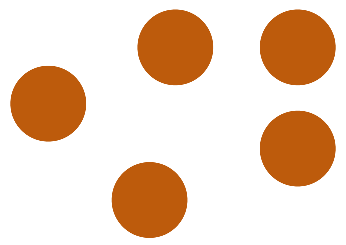


Picerni et al., Cerebellum, 2018

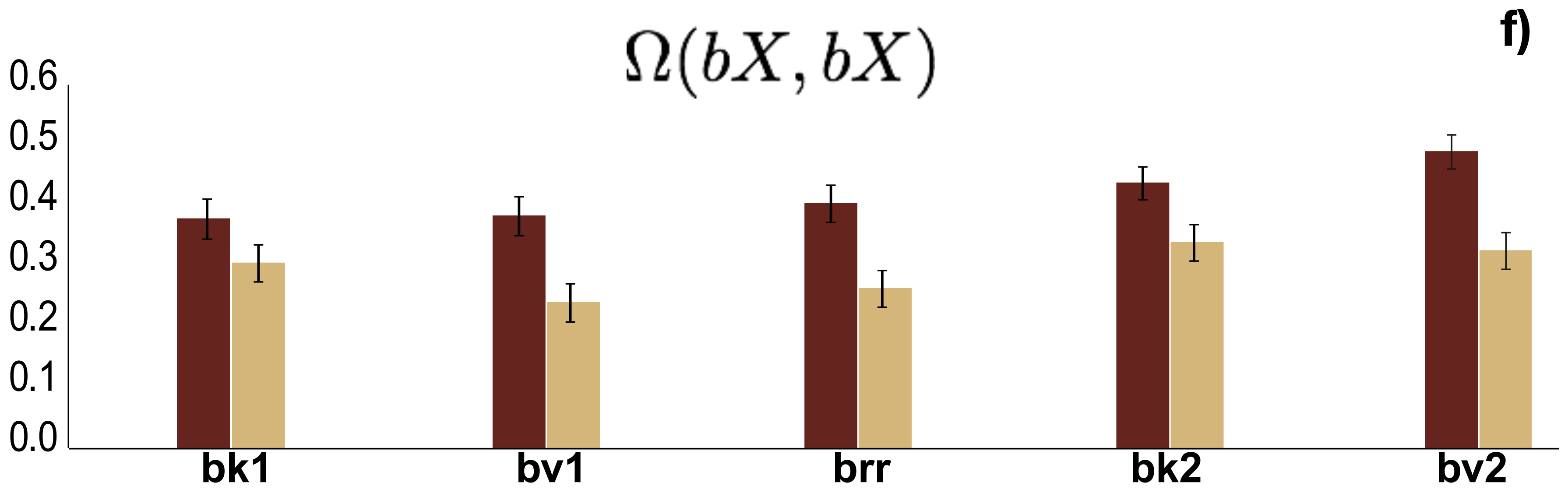
Do homological shapes capture transitions?

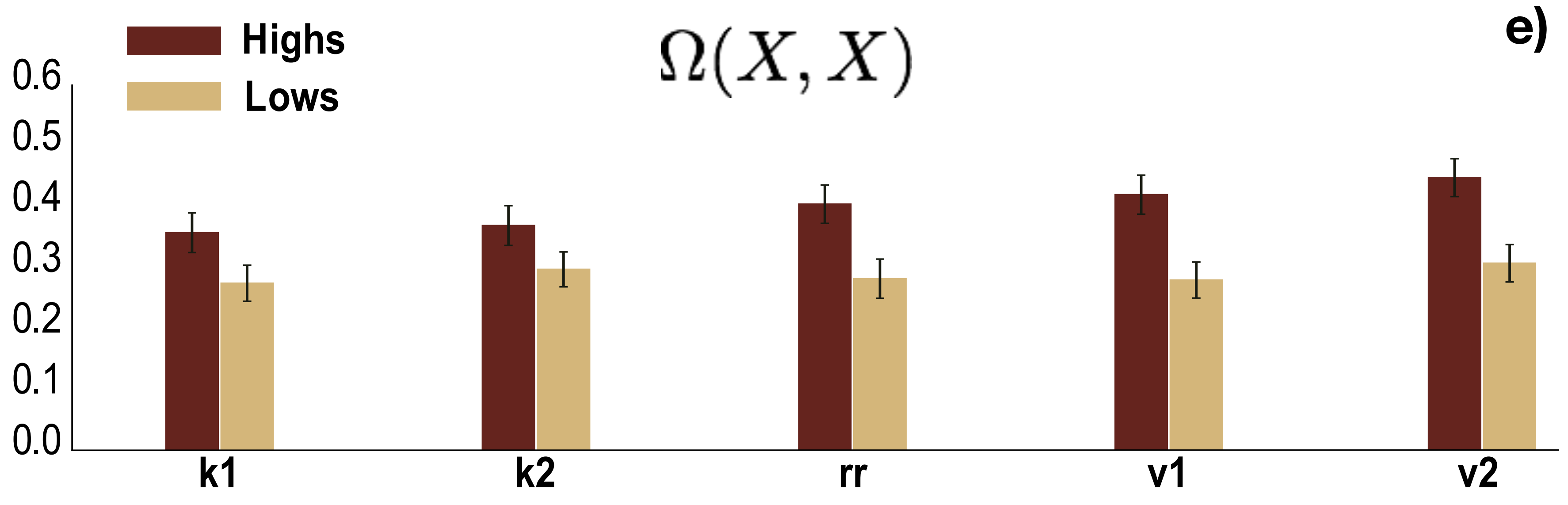
One more thing…

Group homogeneity



bX/X



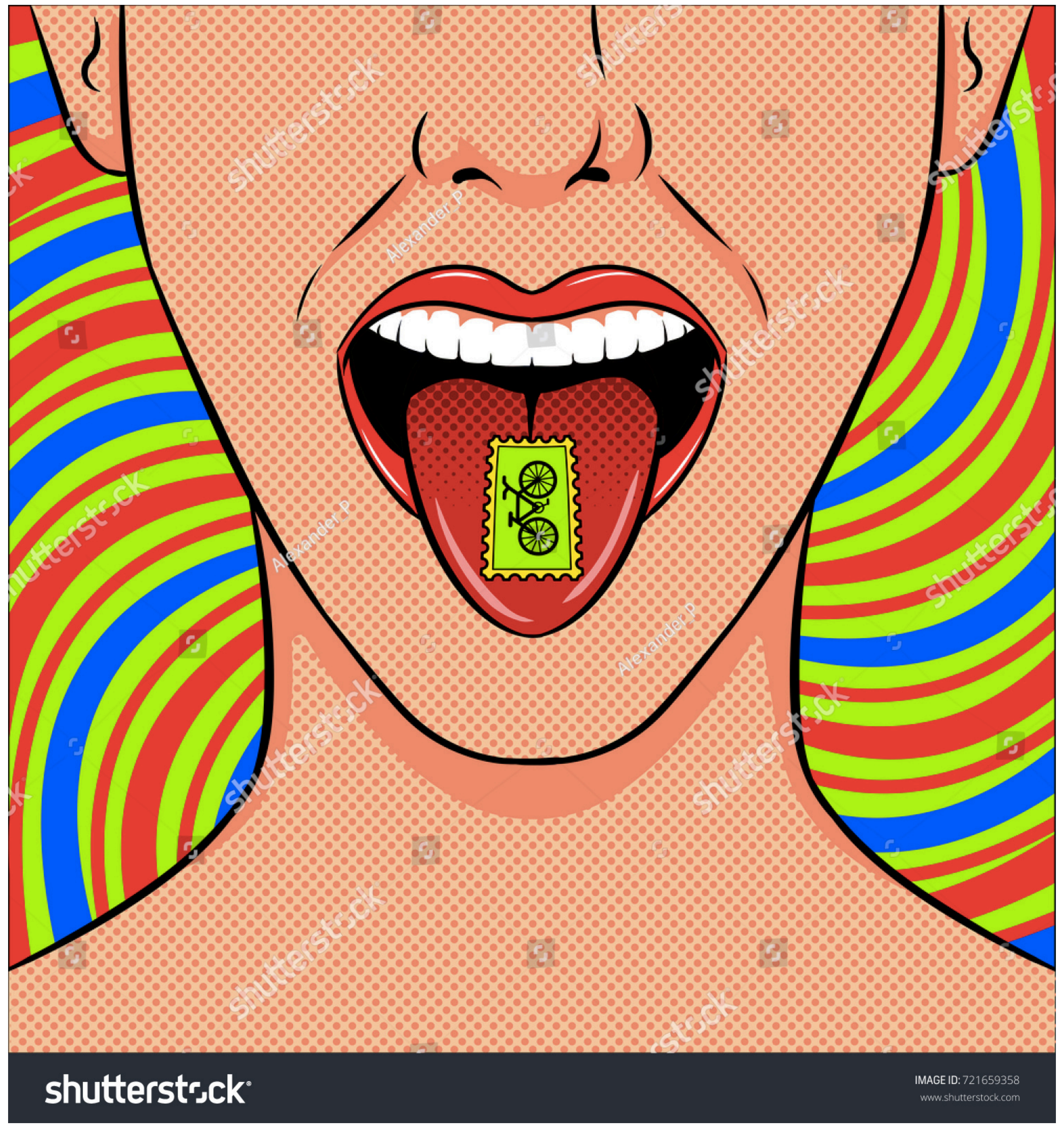


bX/X

- scarce topological changes could be the effect of largely distributed information processing *(Sporns, 2013)*
- highly distributed information could be induced by ascending neuro-modulatory systems *(Shine et al. 2016; Bell and Shine 2016)*
  - NO increases the brain noradrenaline and dopamine availability (Ghasemi et al., 2018)

Can we detect altered states?

Datasets



rs-fMRI
15 subjects, 2 sessions
3 recording conditions

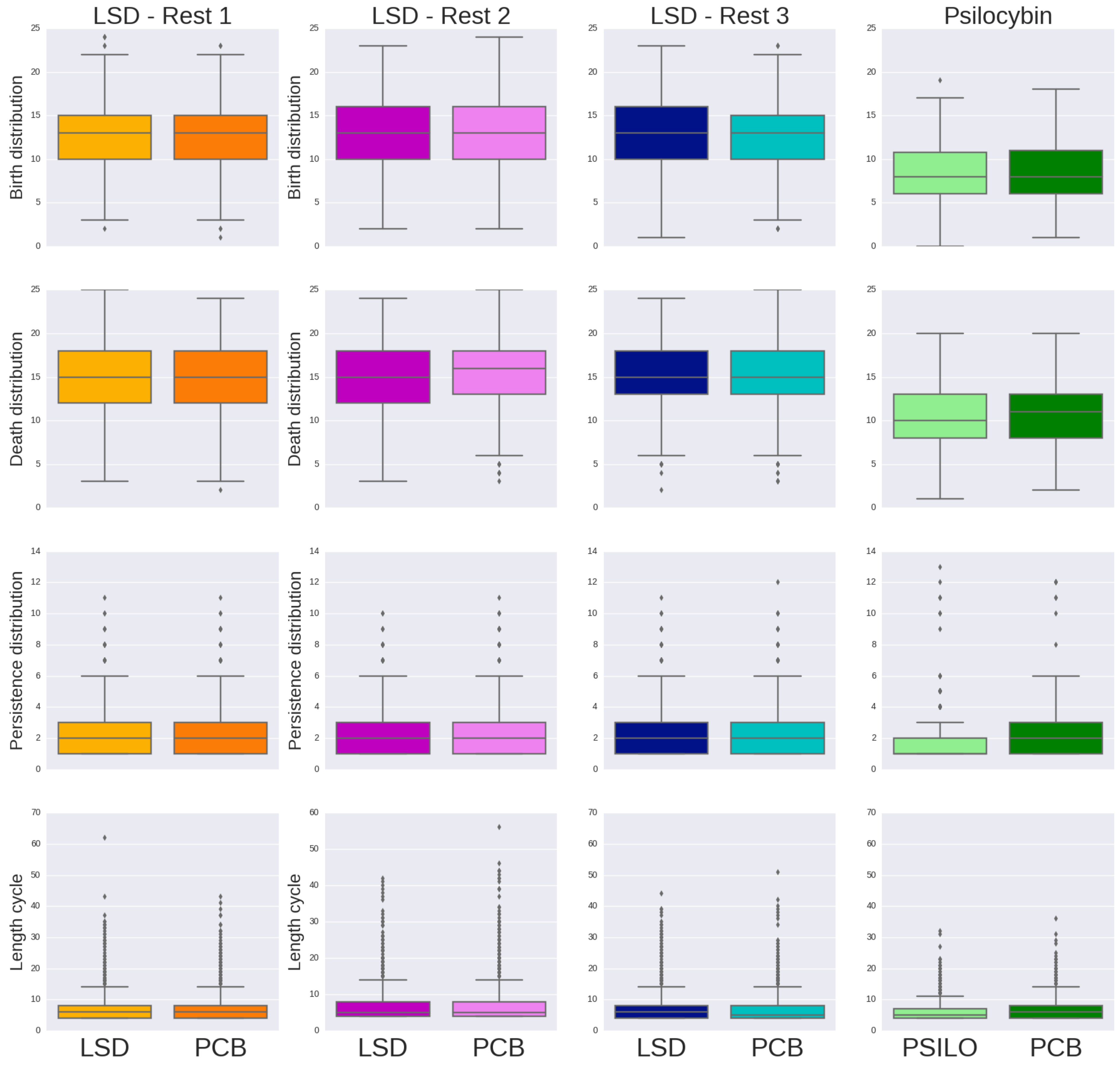
Carhart-Harris, Robin L., et al. "Neural correlates of the LSD experience revealed by multimodal neuroimaging." *Proceedings of the National Academy of Sciences* 113.17 (2016): 4853-4858.

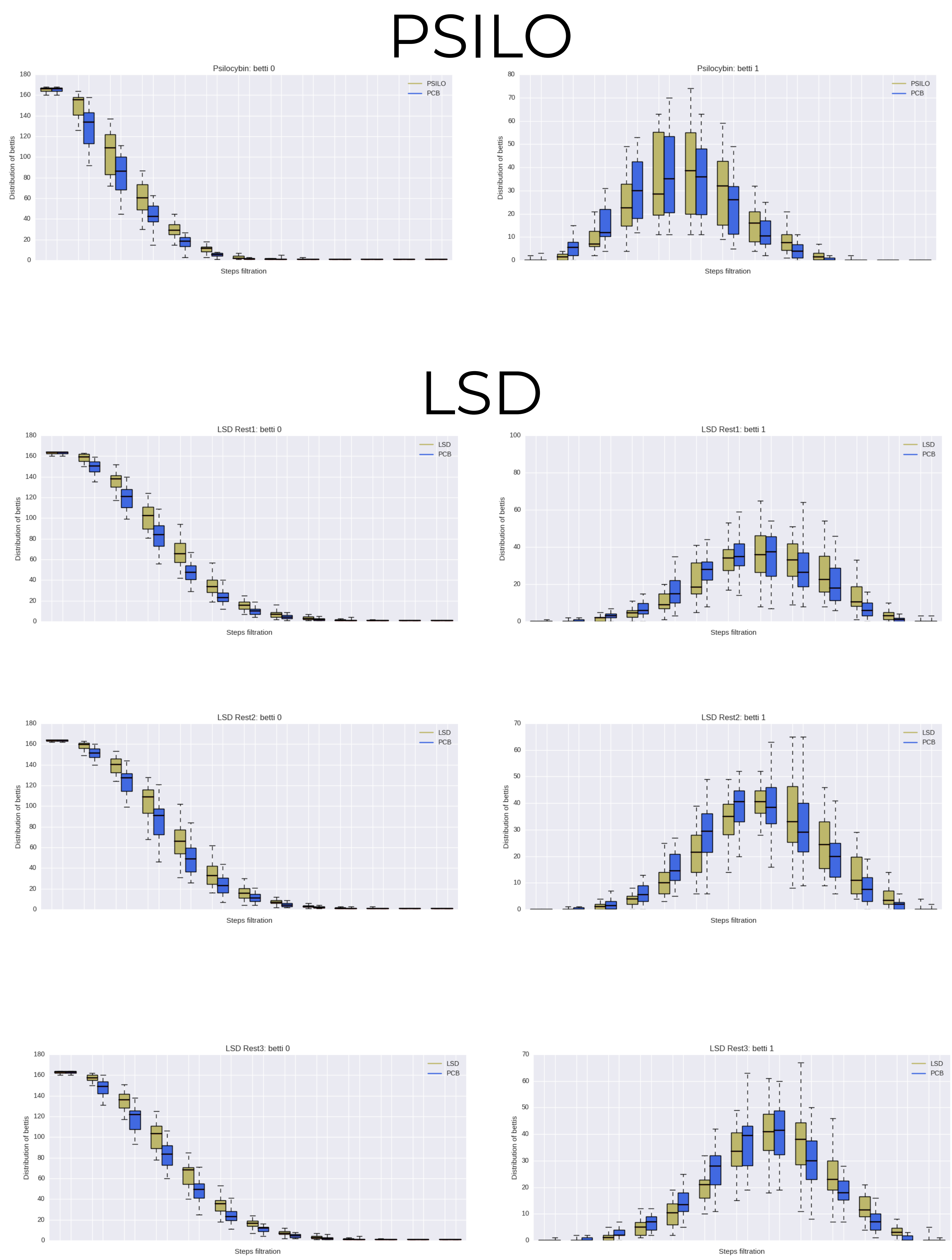
rs-fMRI
15 subjects,2 sessions
1 recording condition

Carhart-Harris, Robin L., et al. "Neural correlates of the psychedelic state as determined by fMRI studies with psilocybin." *Proceedings of the National Academy of Sciences* 109.6 (2012): 2138-2143.

Can we detect altered states?

No changes in the:
- persistences
- length of cycles
- birth/death distribution
- abundance of cycles



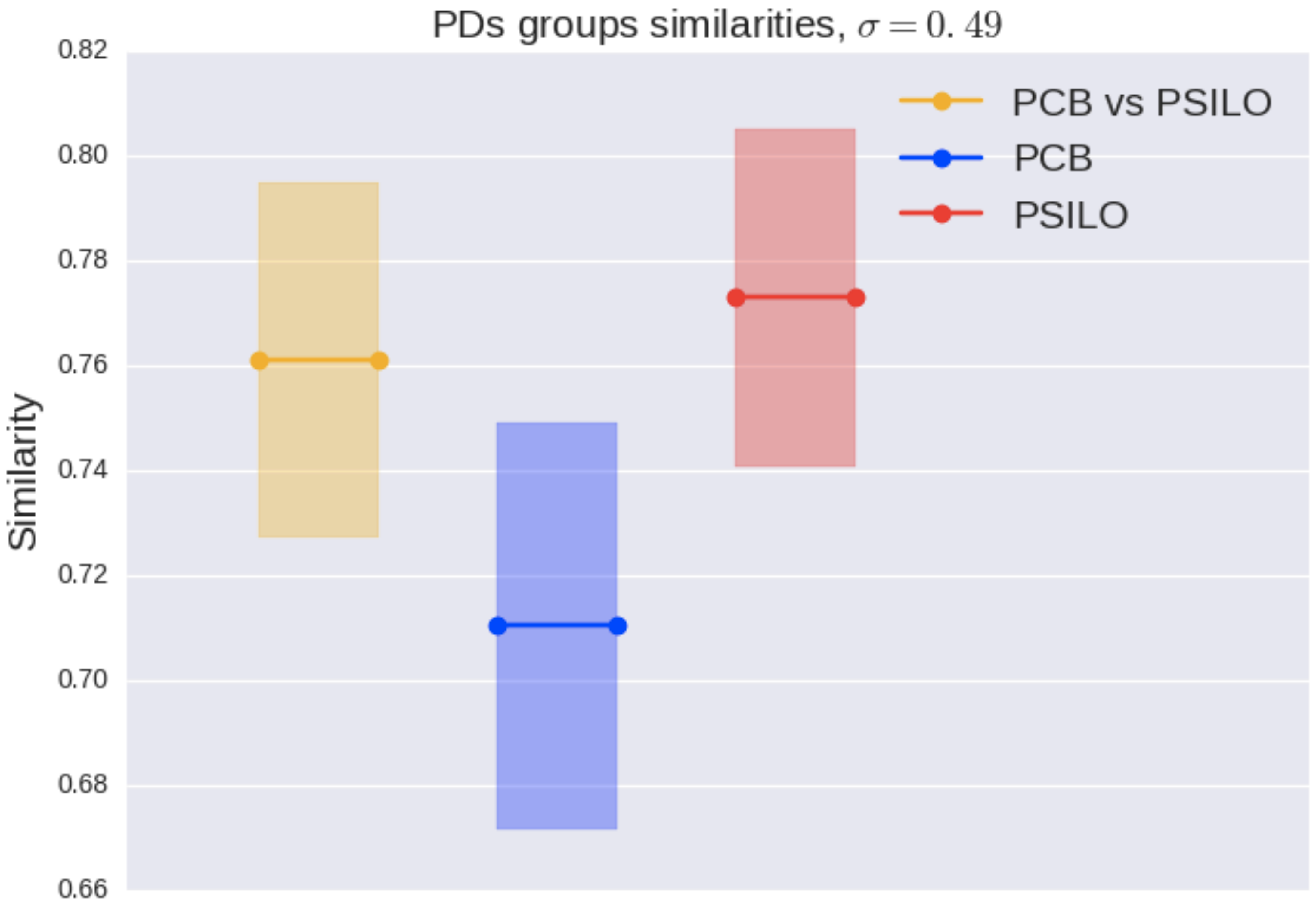


Can we detect altered states?

Opposite results!

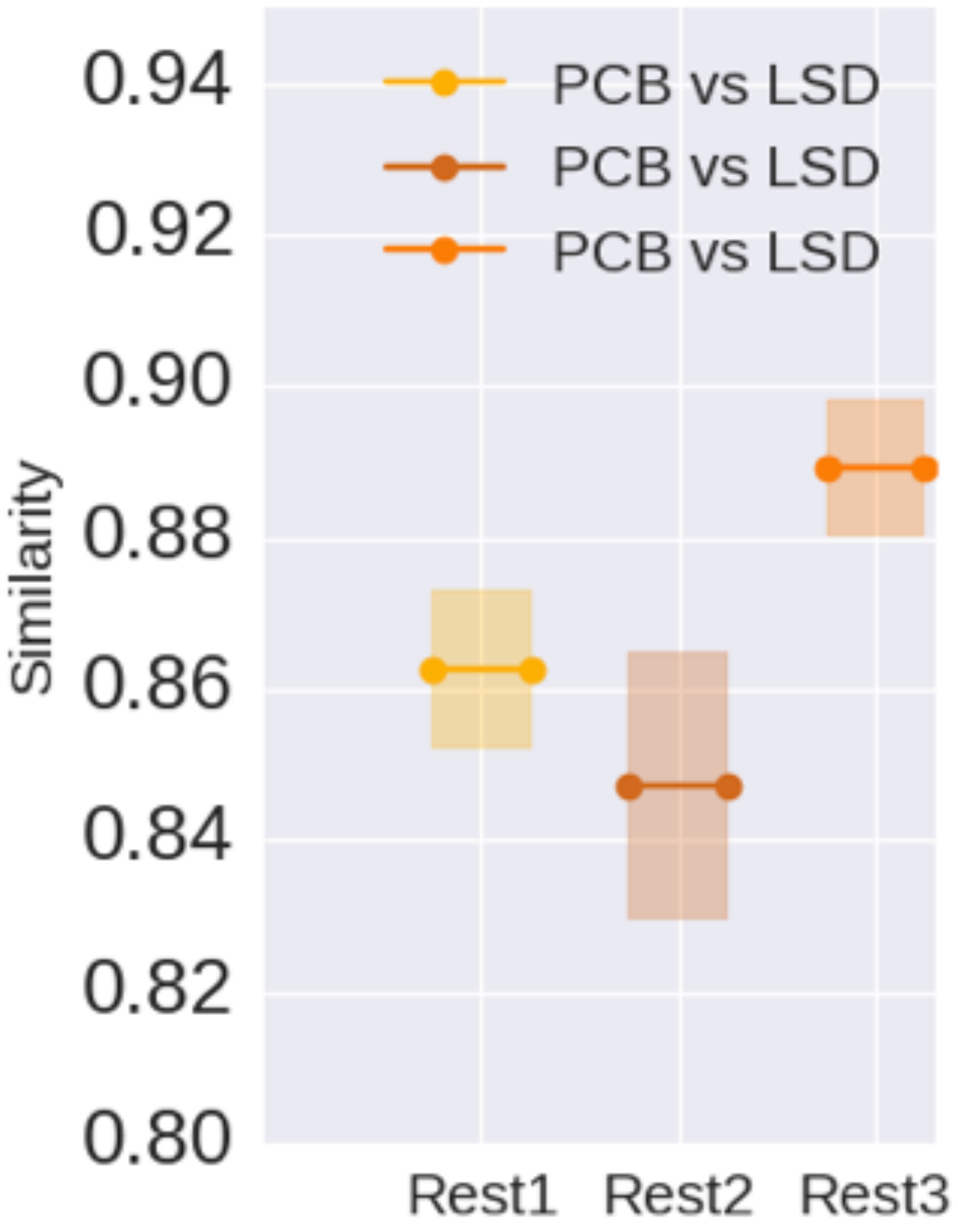
PSILO

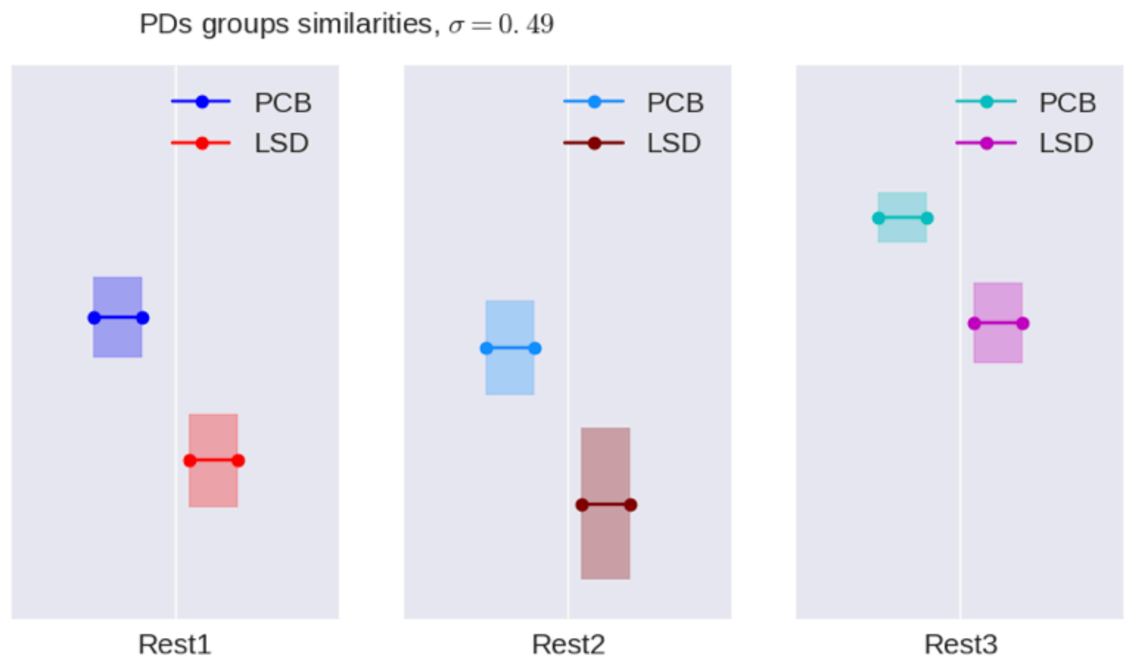
- Topological changes are large
- PSILO group more uniform than PCB



LSD

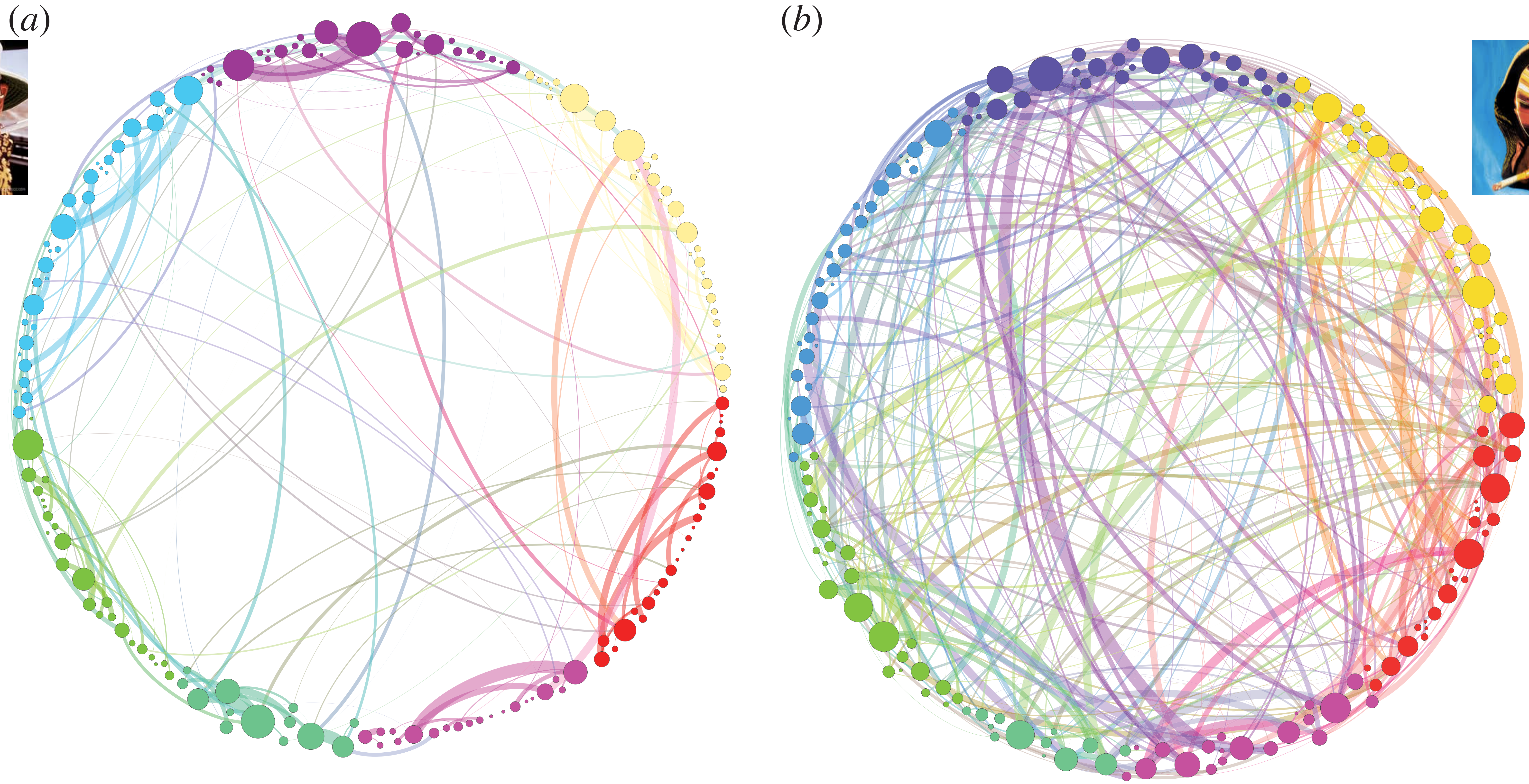
- Topological changes are small overall
- LSD group less uniform than PCB





Is it really topology?

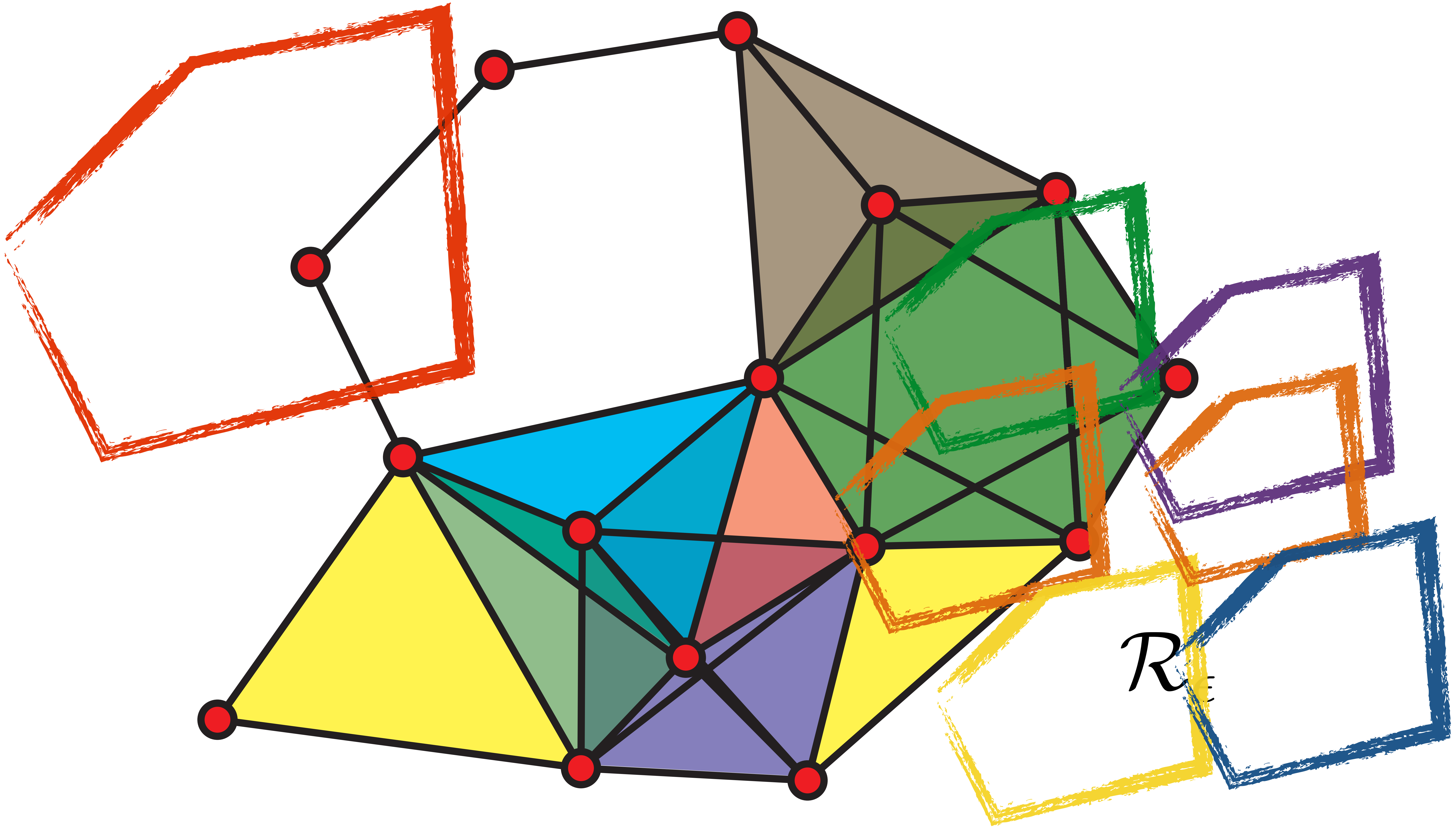
Psilocybin scaffolds



Petri, G. et al. Homological scaffolds of brain functional networks. Journal of The Royal Society Interface 11, 20140873–20140873 (2014).

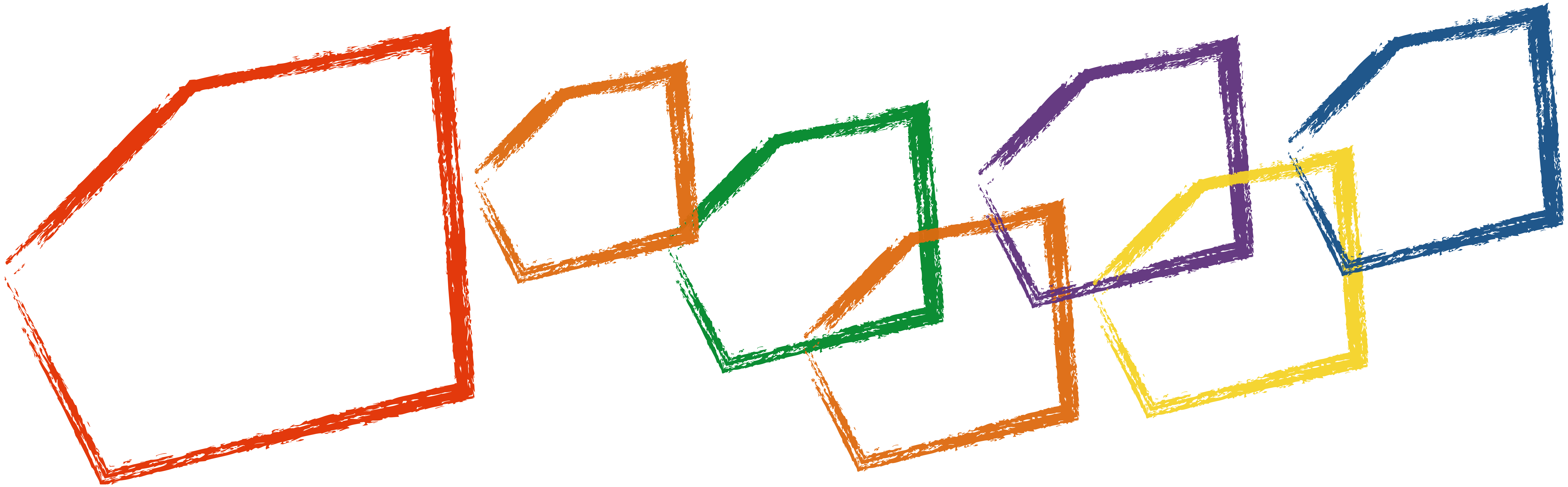
Is it really topology?

Holes as shape indicators?



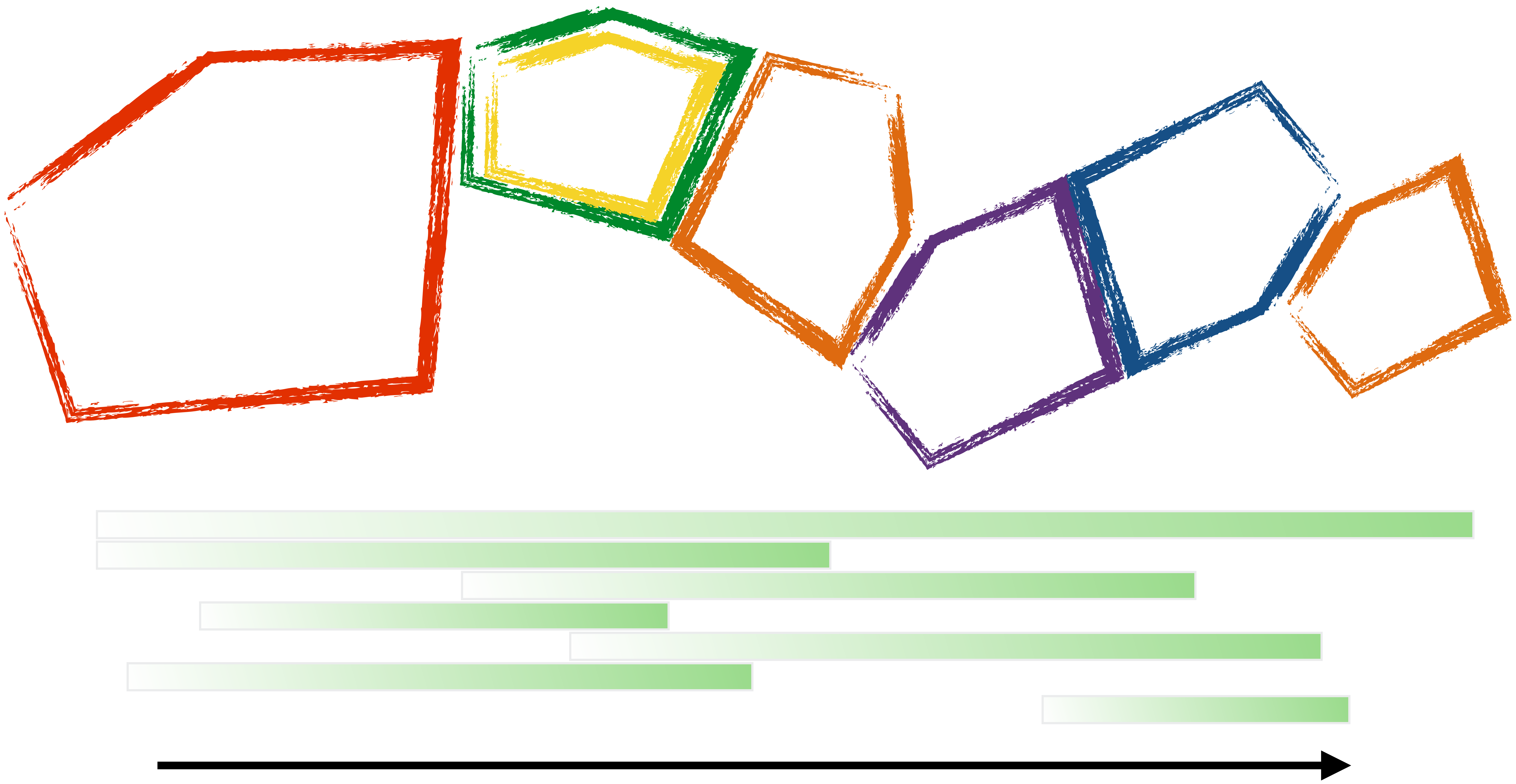
Is it really topology?

How does a *man build a scaffold?



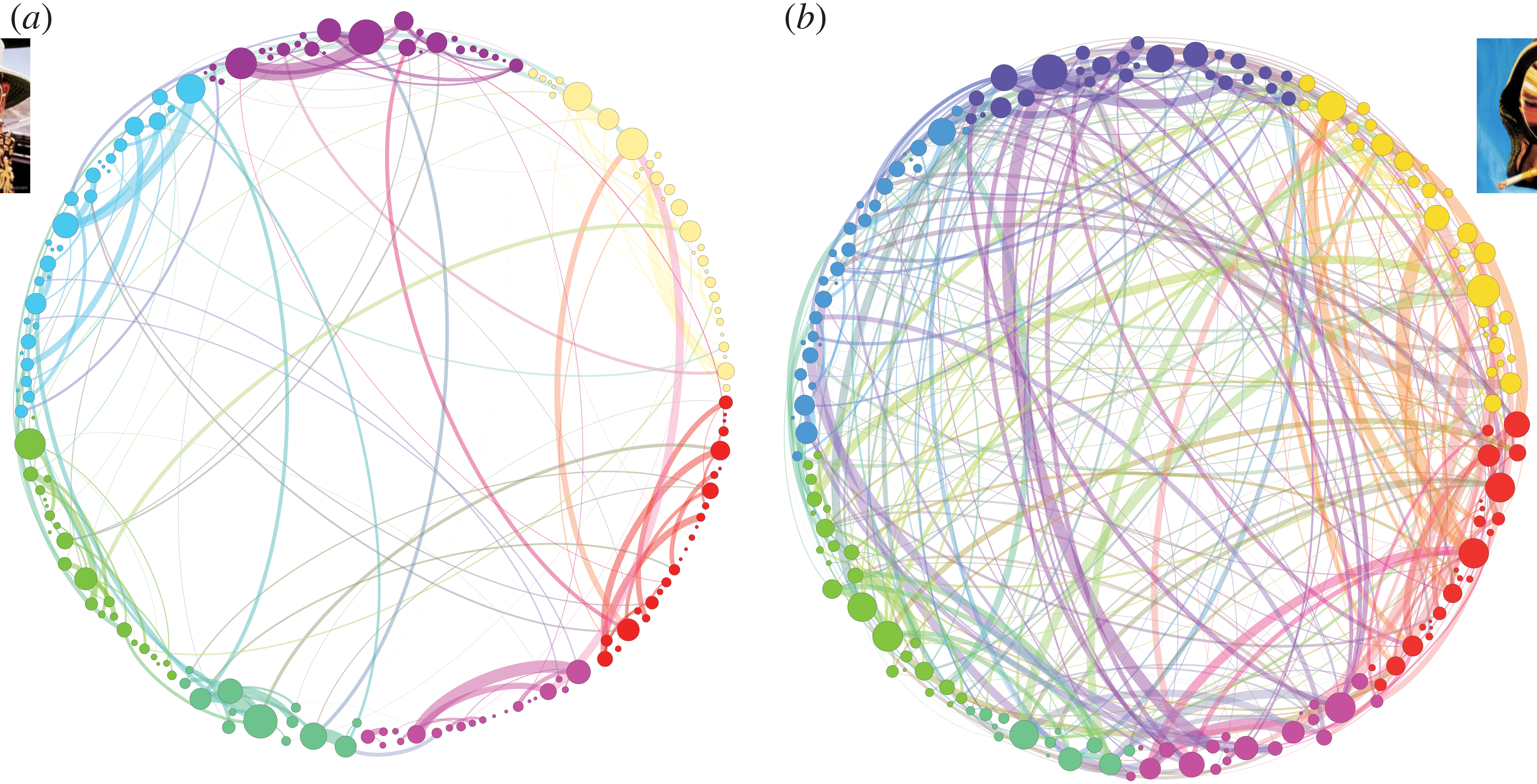
Is it really topology?

How does a *man build a scaffold?



Is it really topology?

Psilocybin scaffolds

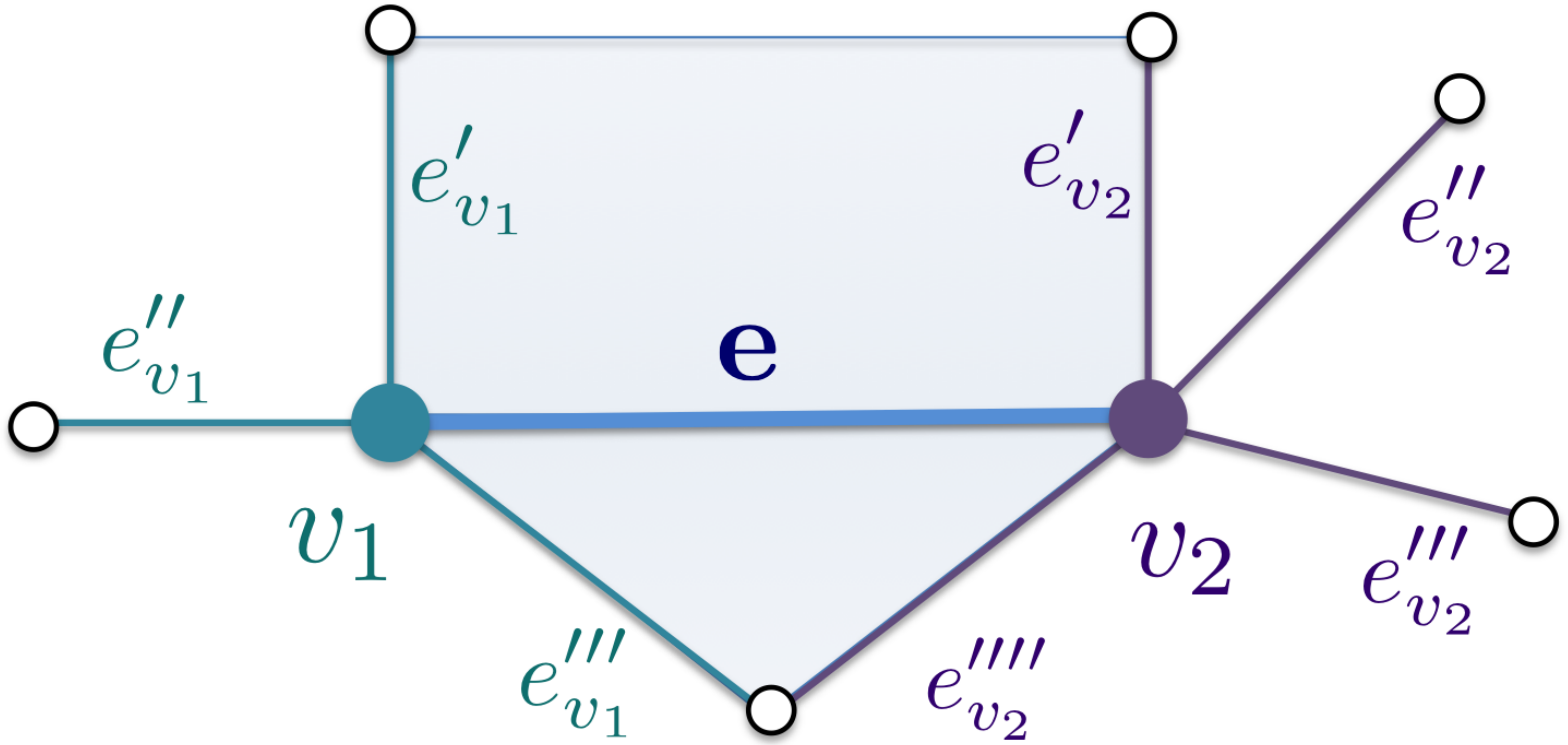


Petri, G. et al. Homological scaffolds of brain functional networks.
Journal of The Royal Society Interface 11, 20140873–20140873 (2014).

Is it really topology?

Ricci-Foreman Curvature

After all these are correlations!



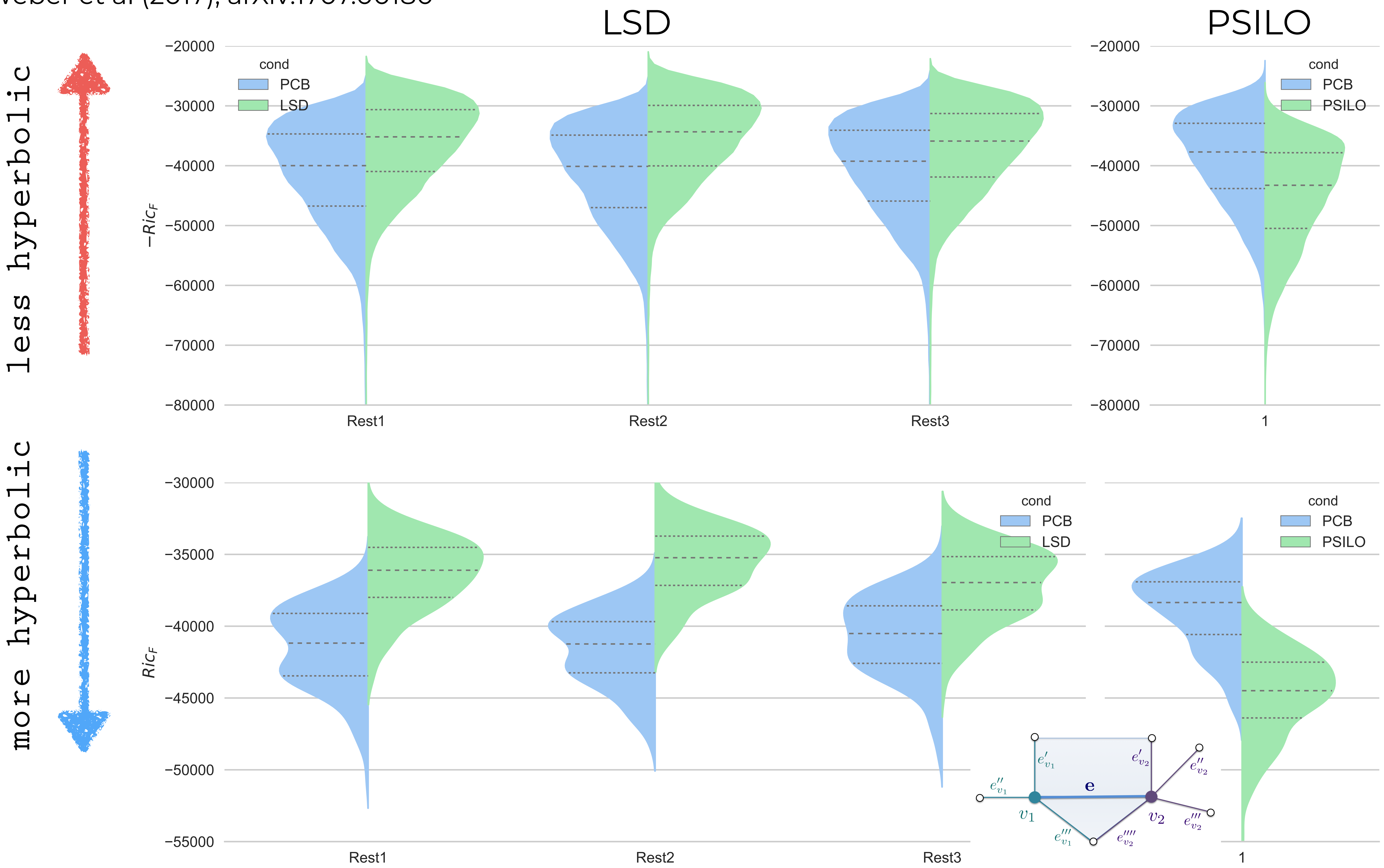
$$Ric_F(e) = w(e)\left(\frac{\omega(v_1)}{\omega(e)} + \frac{\omega(v_2)}{\omega(e)} - \sum_{e_{v_1}\sim e, e_{v_2}\sim e}\left[\frac{\omega(v_1)}{\sqrt{\omega(e)\omega(e_{v_1})}} + \frac{\omega(v_2)}{\sqrt{\omega(e)\omega(e_{v_2})}}\right]\right)$$

Weber et al (2017), arXiv:1707.00180

Is it really topology?

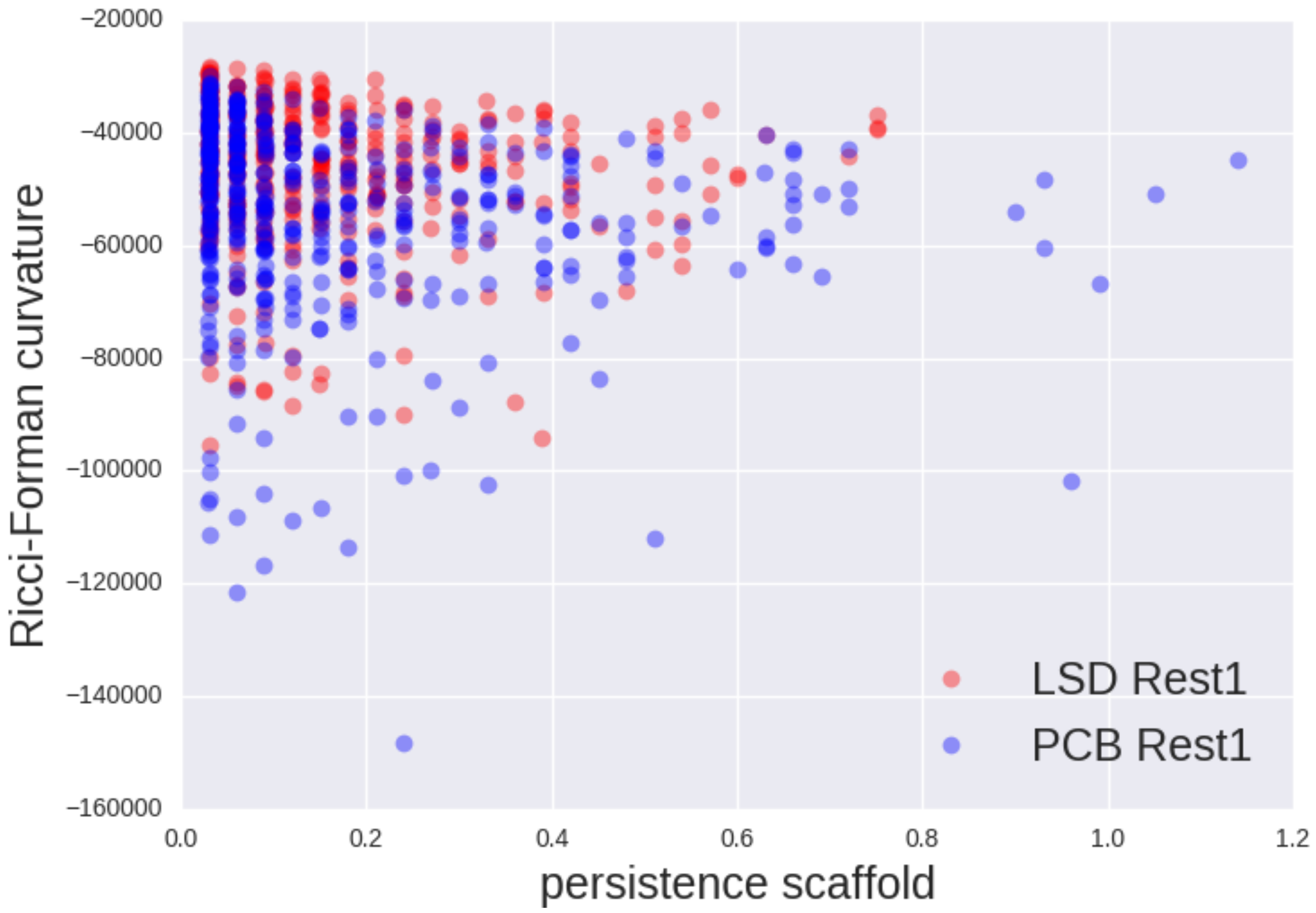
After all these are correlations!
look at Foreman-Ricci curvature

Weber et al (2017), arXiv:1707.00180



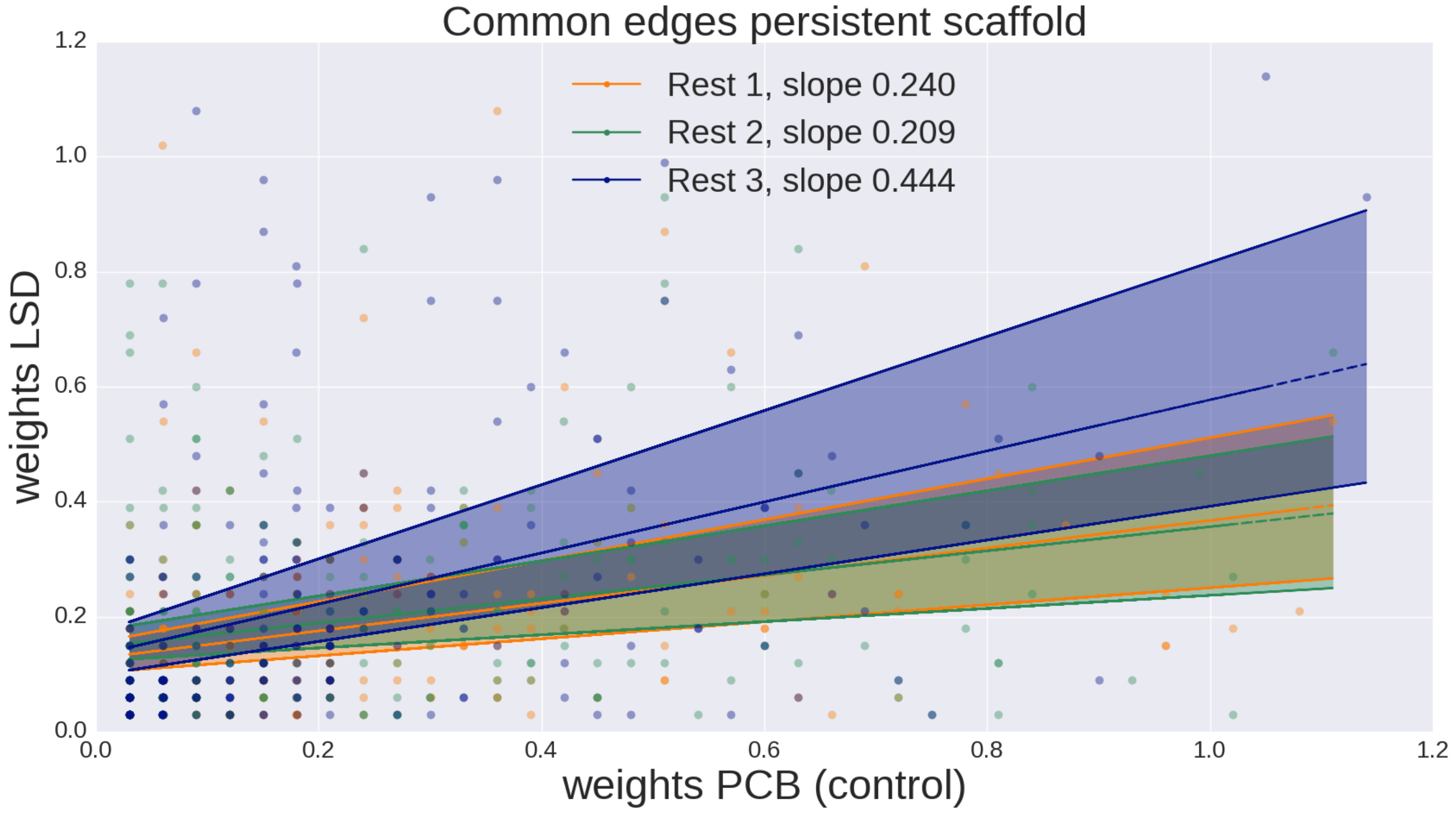
Is it really topology?

Genuine topology!



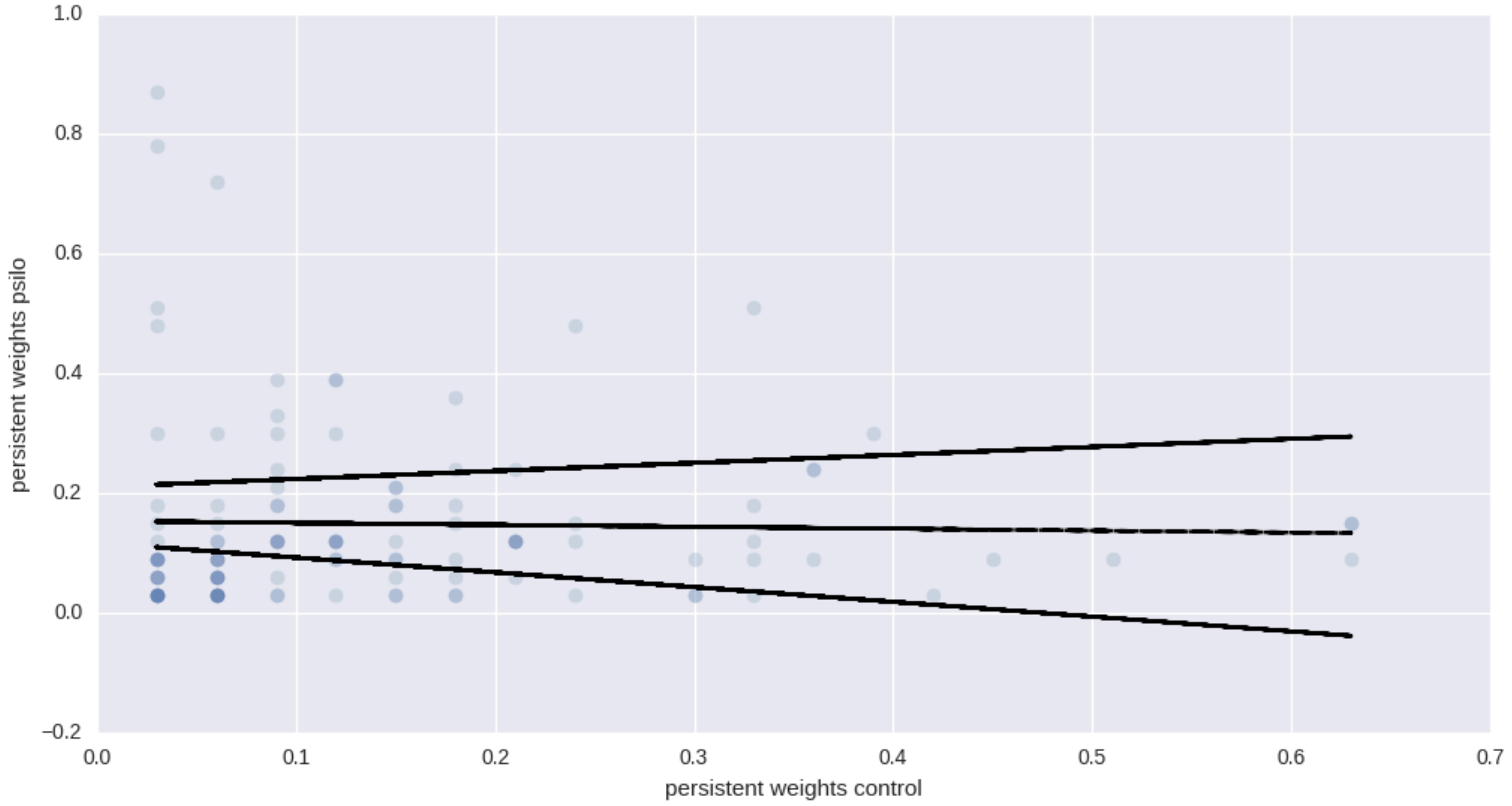
Is it really topology?

Modulation
vs disruption of topology



LSD

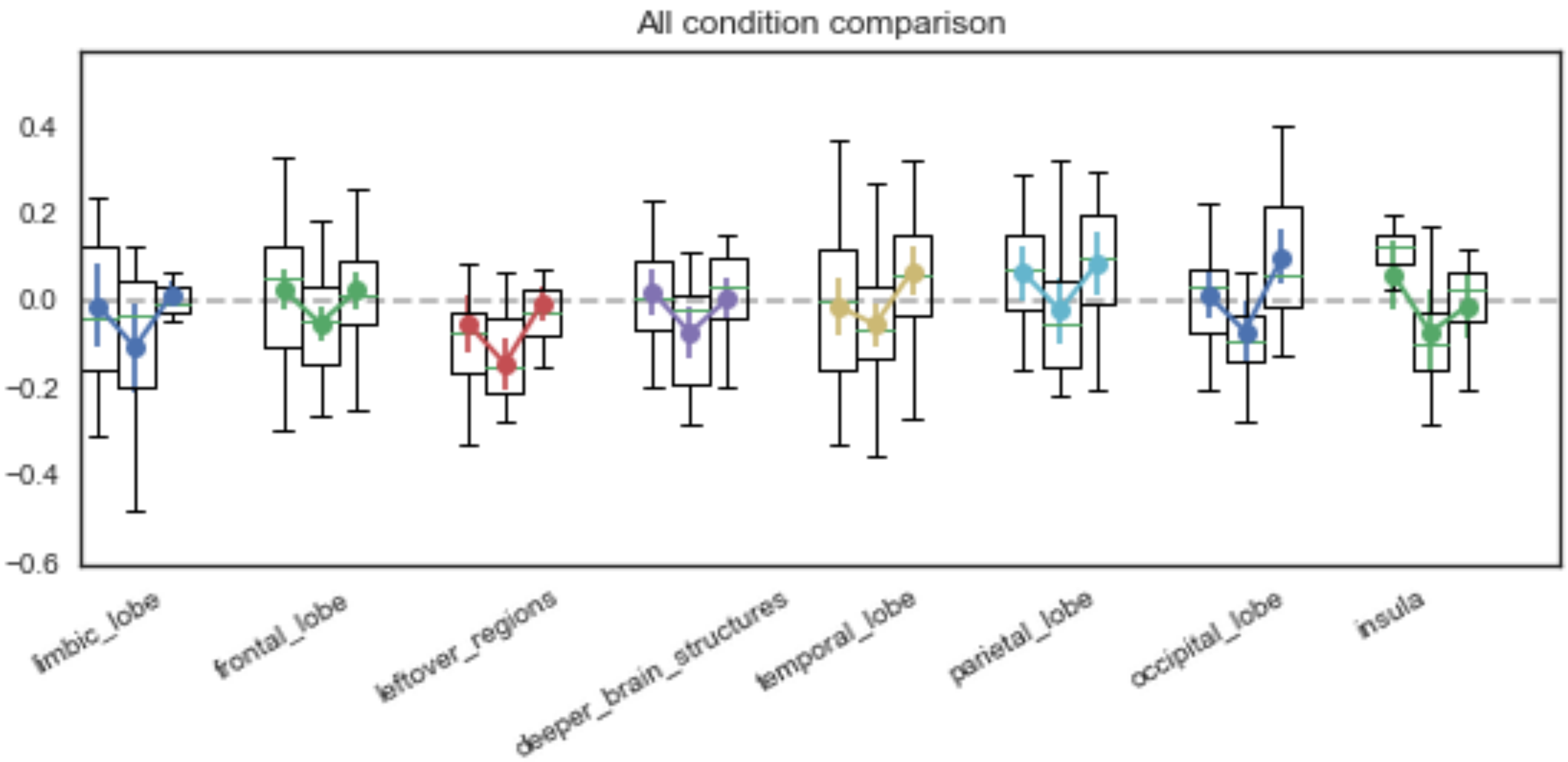
- positive correlation
- modulation of the slope
- core of shared edges between drug and condition



PSILO

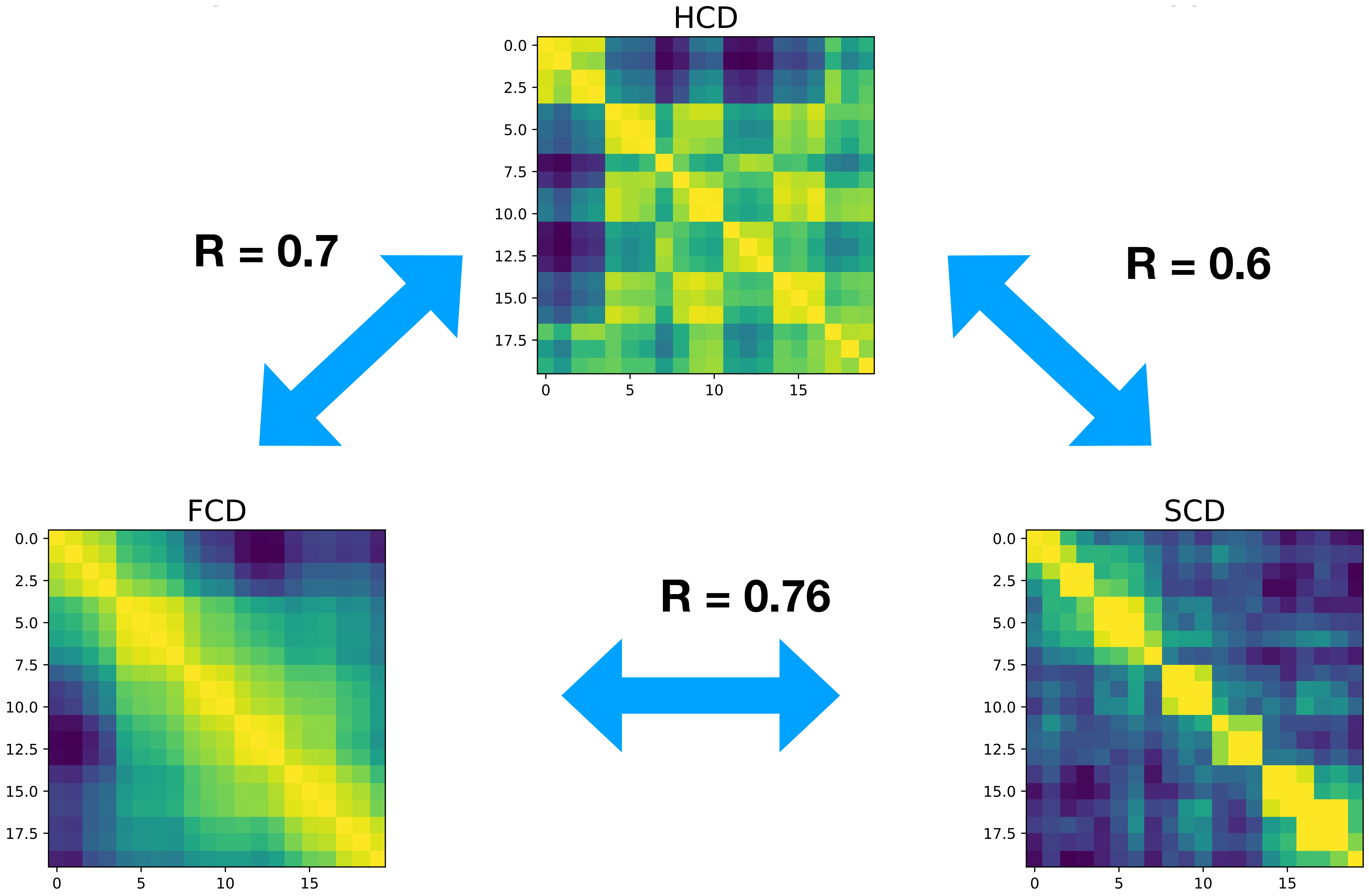
- zero/small correlation
- no overlap in edges between drug and condition
- LARGE disruption

Is it really topology?



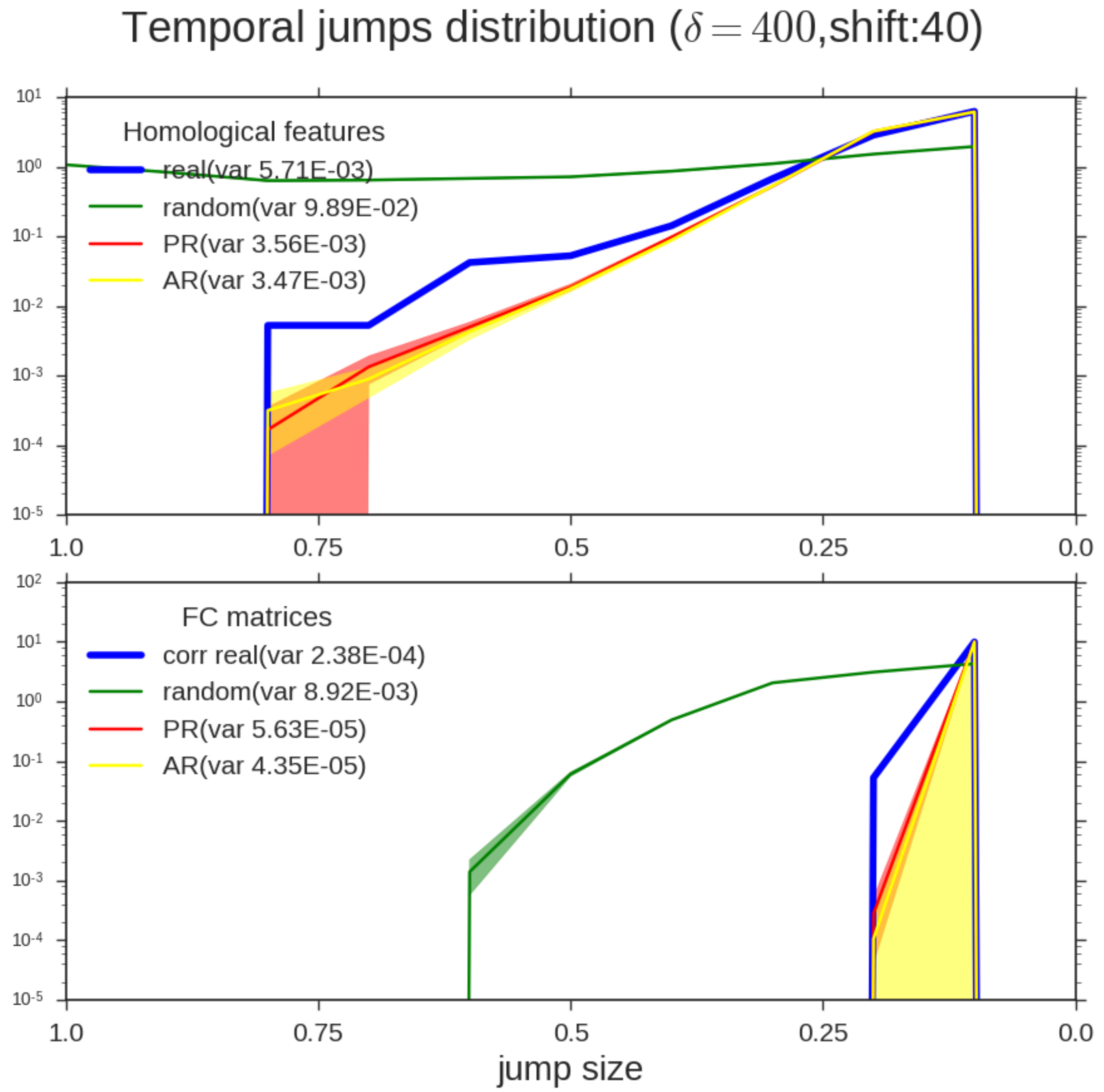
What about temporal evolution?

Homological distances



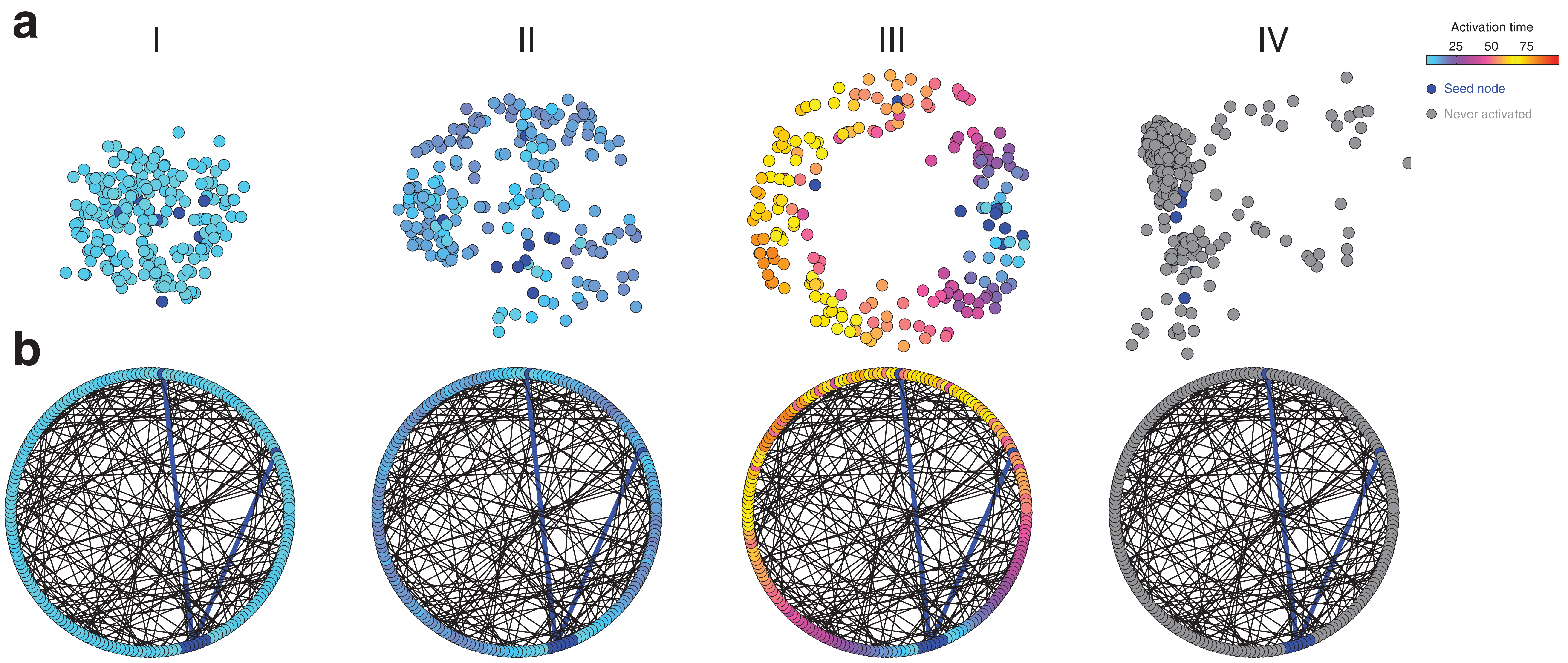
Functional network distances

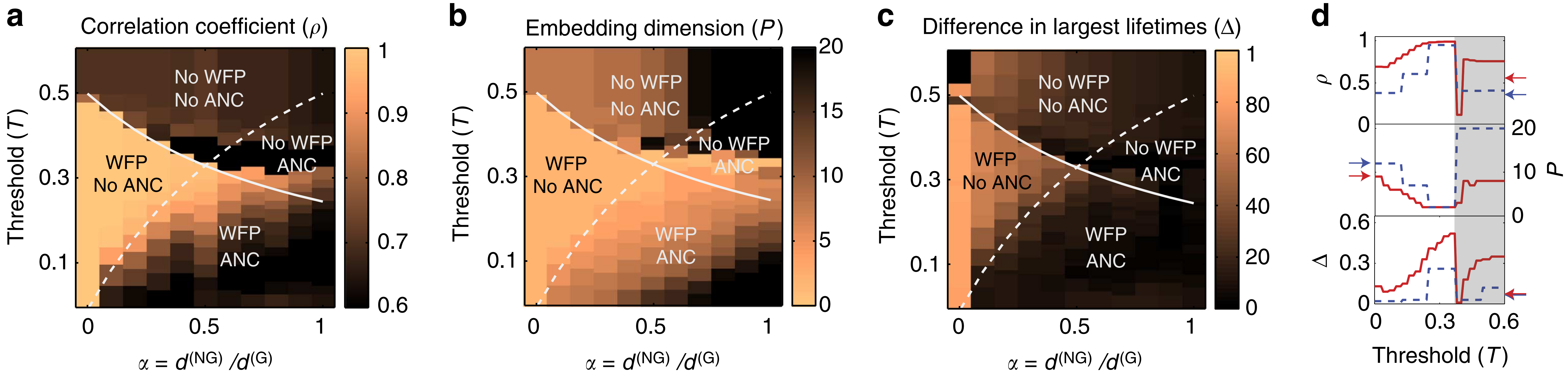
Scaffold distances



Does the shape of activity match the structure?

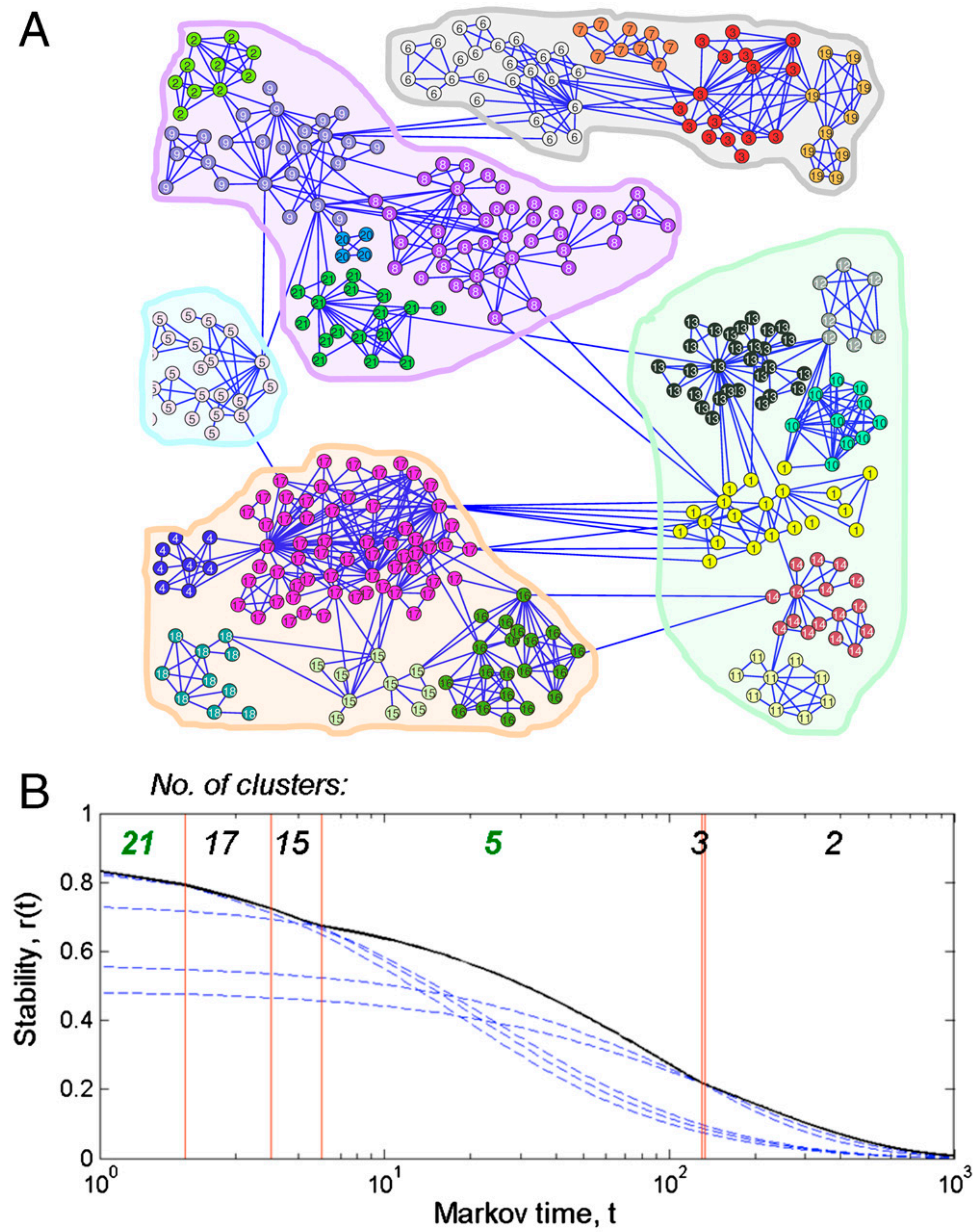
Taylor, Dane, et al. "Topological data analysis of contagion maps for examining spreading processes on networks." *Nature communications* 6 (2015): 7723.

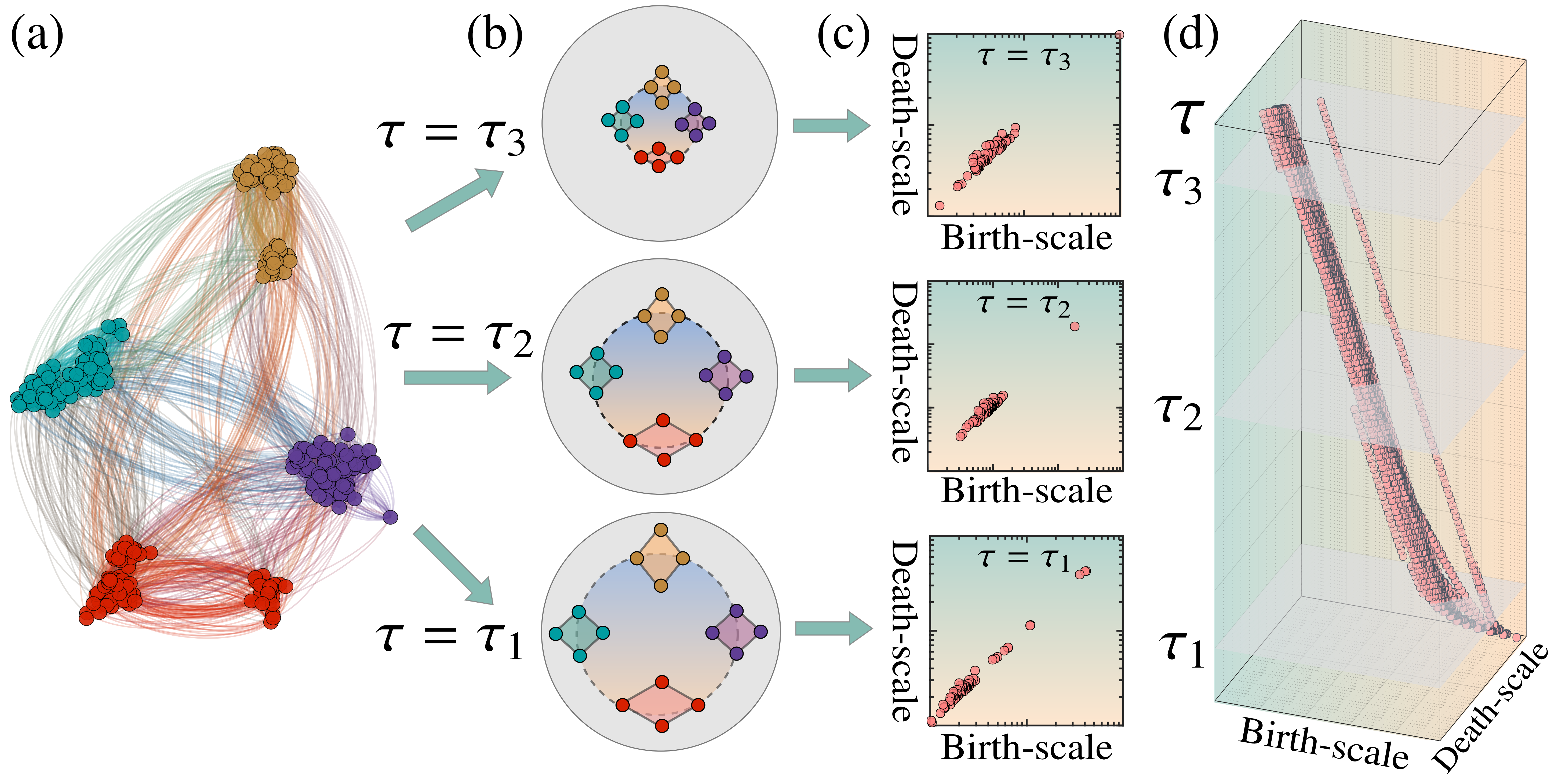




Does the shape of activity match the structure?

Tran, Quoc Hoan, Van Tuan Vo, and Yoshihiko Hasegawa. "Scale-variant topological information for characterizing complex networks." *arXiv preprint arXiv:1811.03573* (2018).

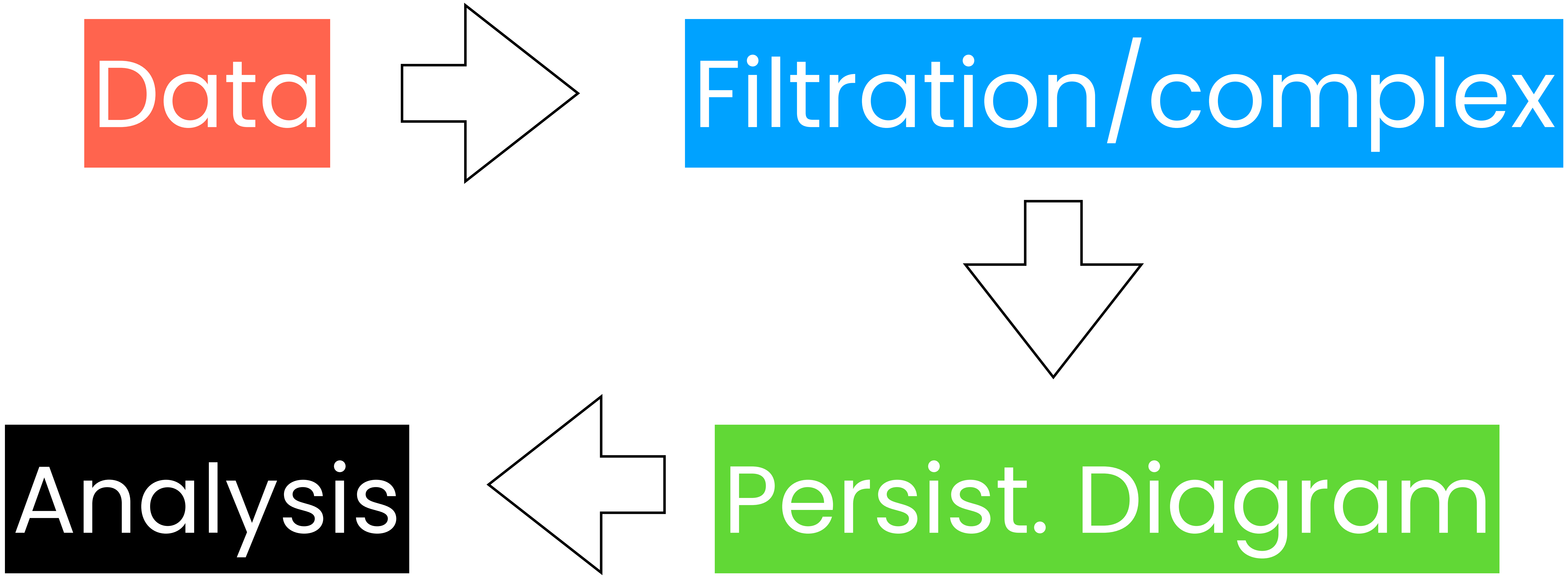




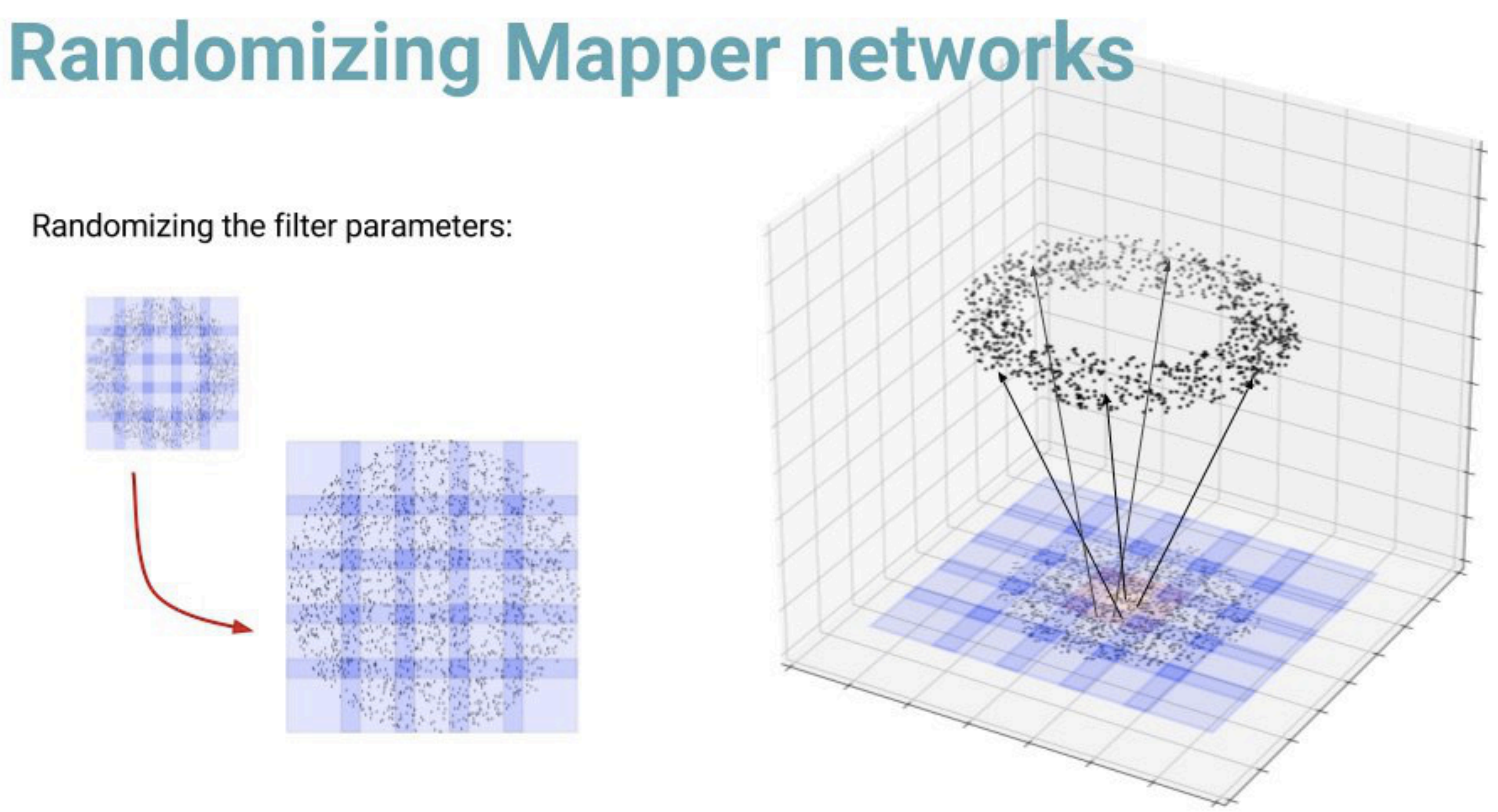
Stability of graph communities across time scales

J.-C. Delvenne[a], S. N. Yaliraki[a,b], and M. Barahona[a,c,1]

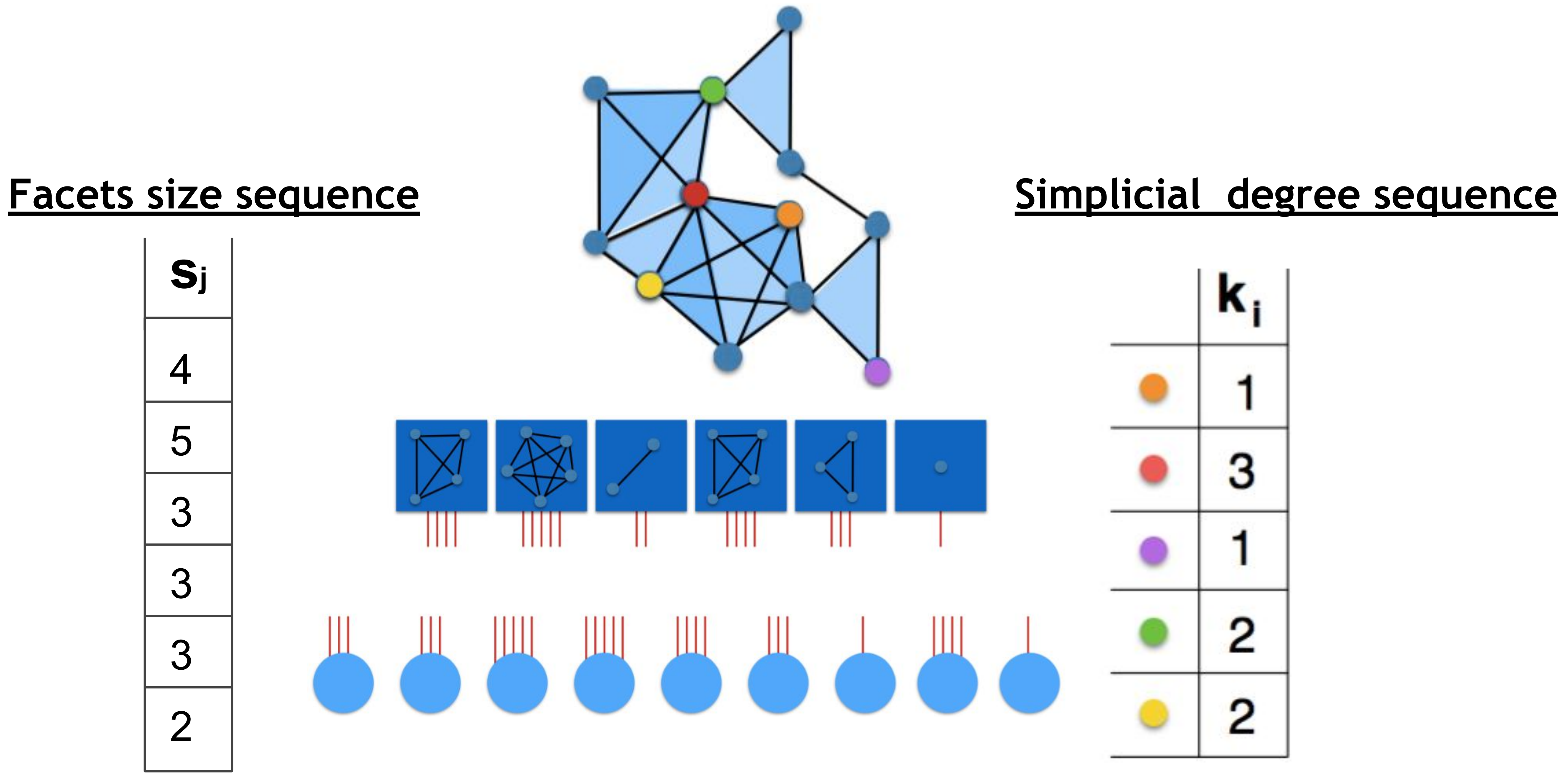
Randomization
+
validation



Randomization?

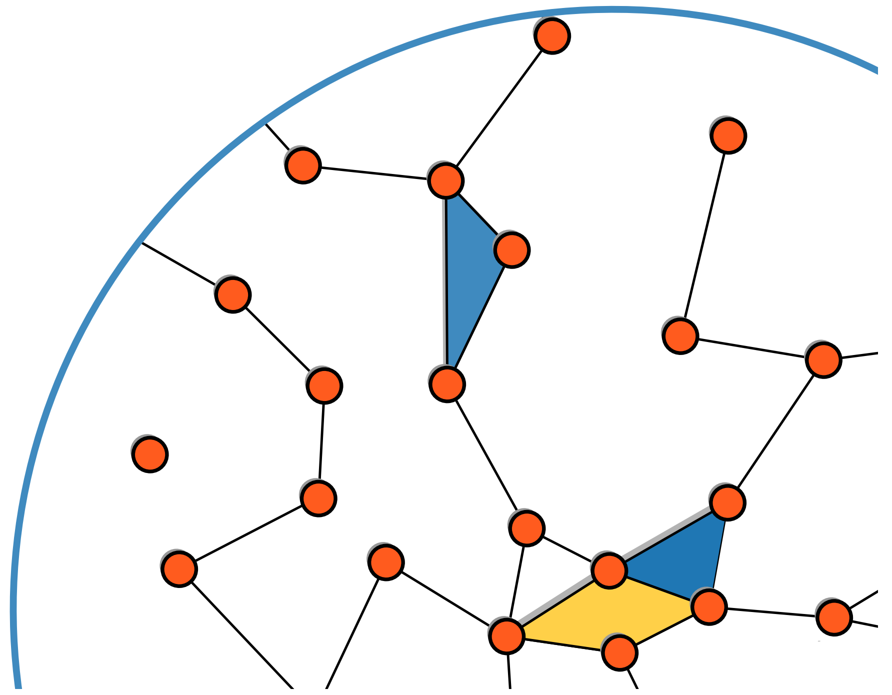


Filter randomization



Simplicial randomization

Joint work with JG Young and A. Patania
Reference : arxiv.org/1705.10298
Software : github.com/jg-you/scm



Confidence sets

Fasy et al. (2014b); Chazal et al. (2014a) suggest the following method. Let

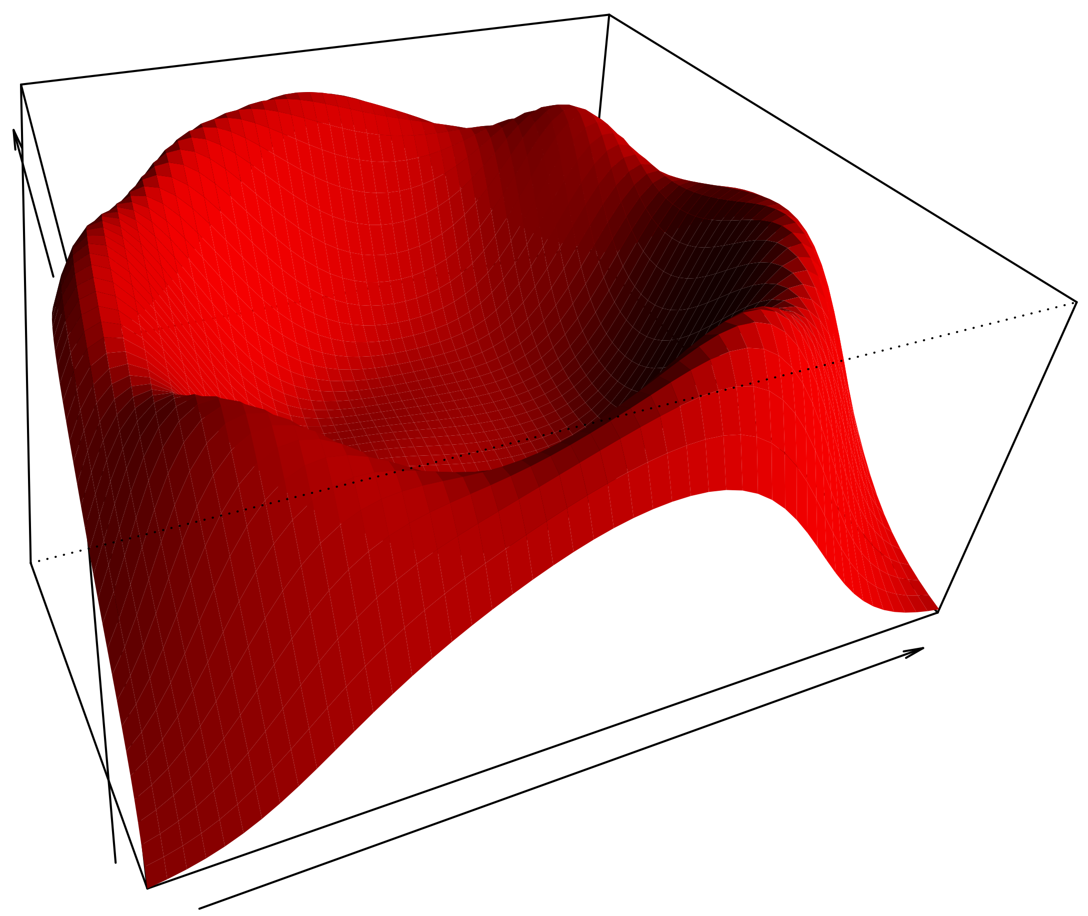
$$F(t) = P(\sqrt{n}\,\delta_\infty(\widehat{D}, D) \le t)$$

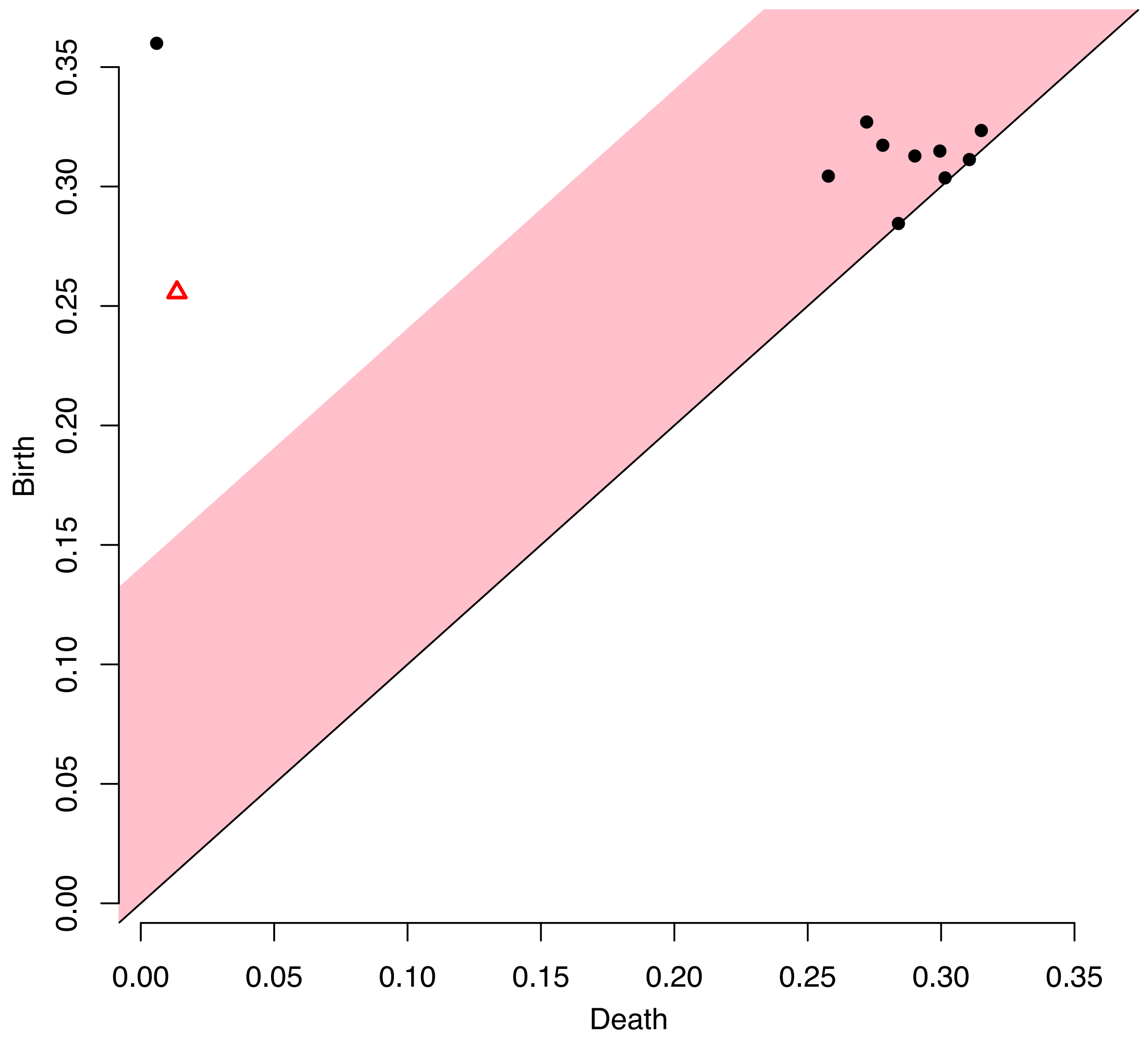
Estimated by bootstrap

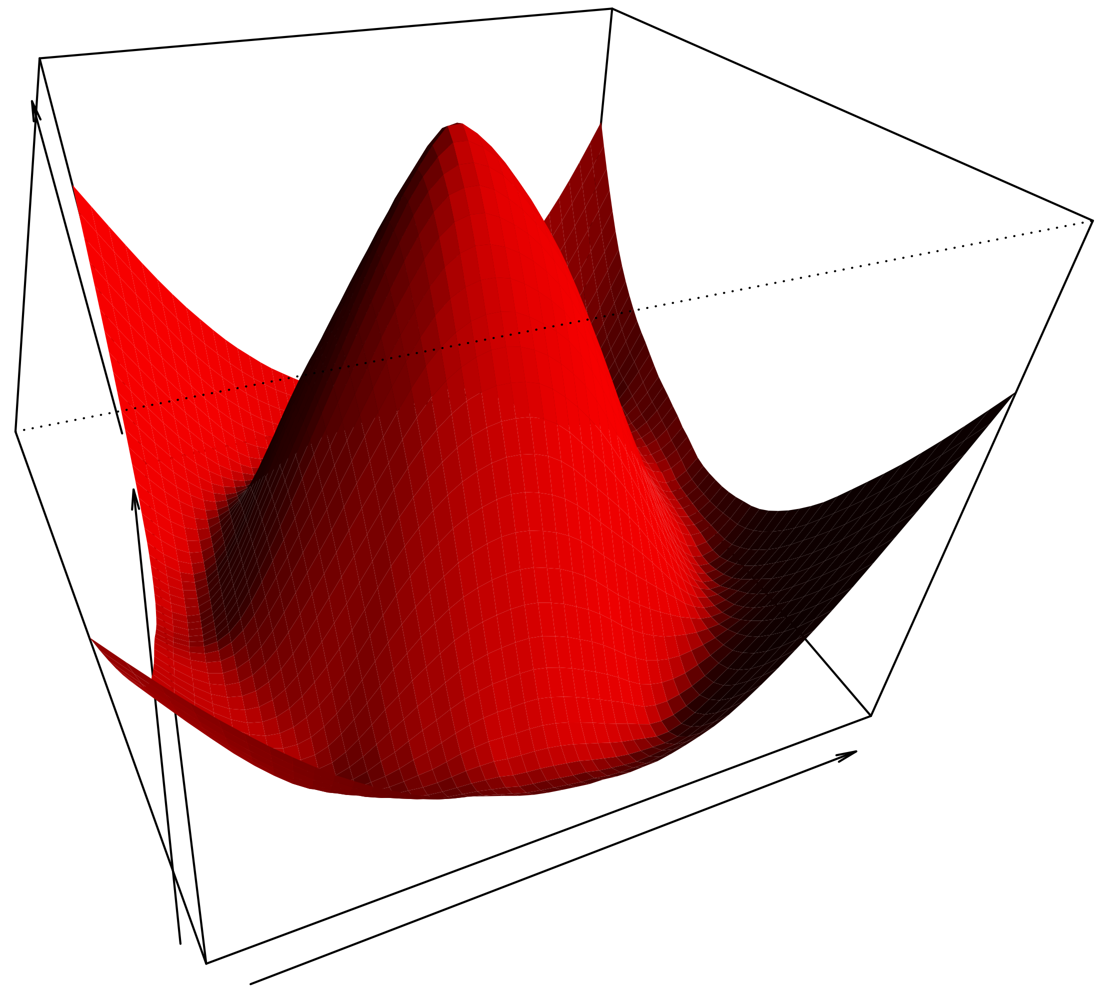
$$\widehat{F}(t) = \frac{1}{B}\sum_{j=1}^{B} I(\sqrt{n} d_\infty(\widehat{D}_j^*, \widehat{D}) \le t)$$

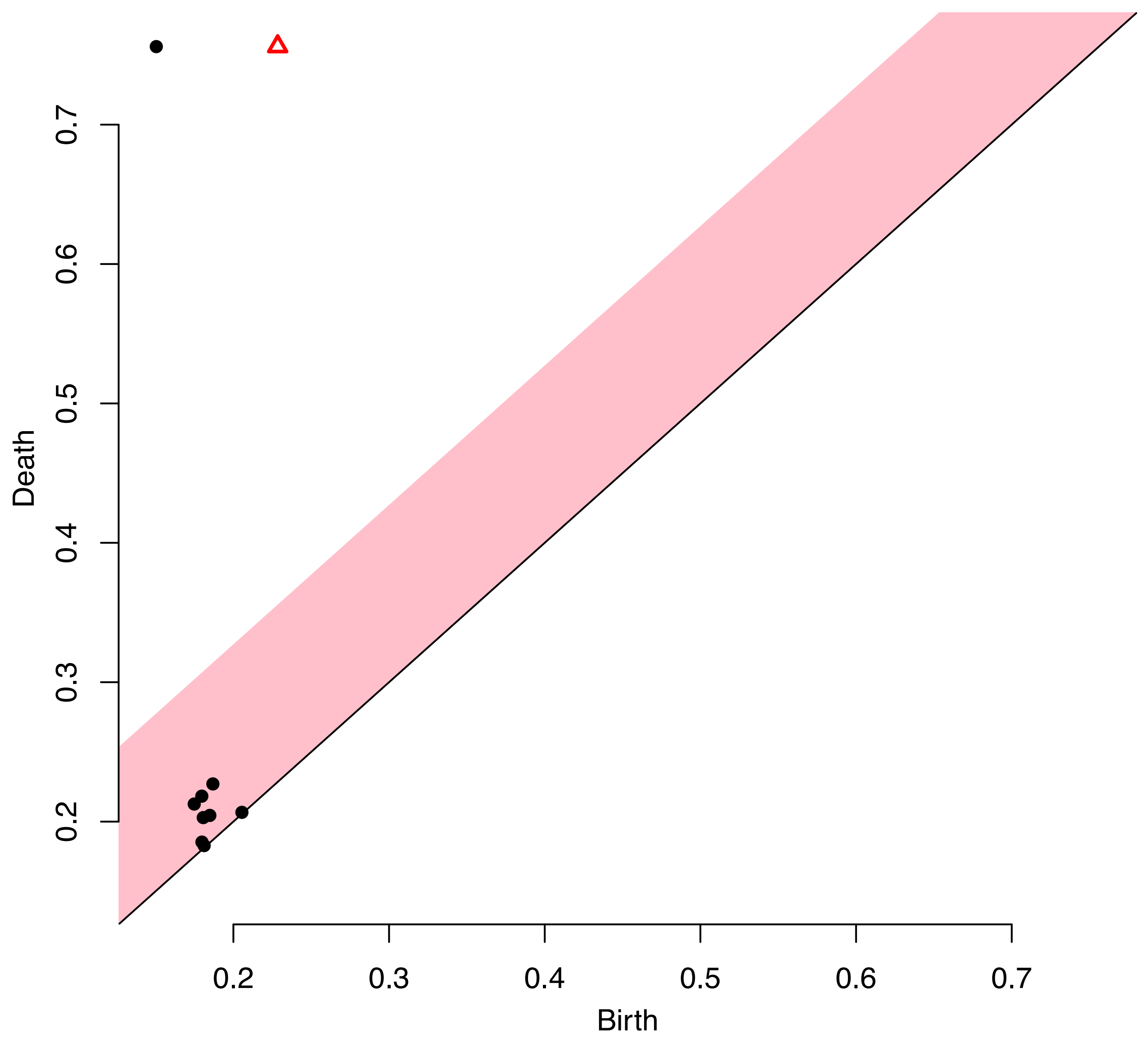
Significance threshold

$$t_\alpha = F^{-1}(1-\alpha)$$









Note: Works for metrical cases.

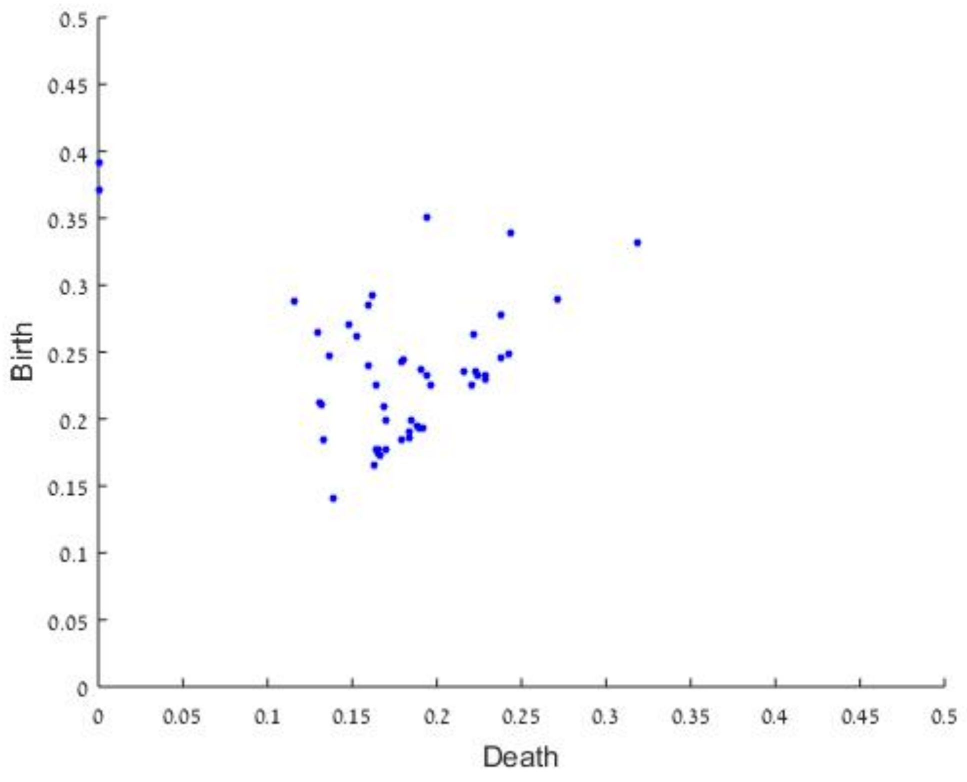
Tip: try this in the TDA r-library

PD “multiplication”

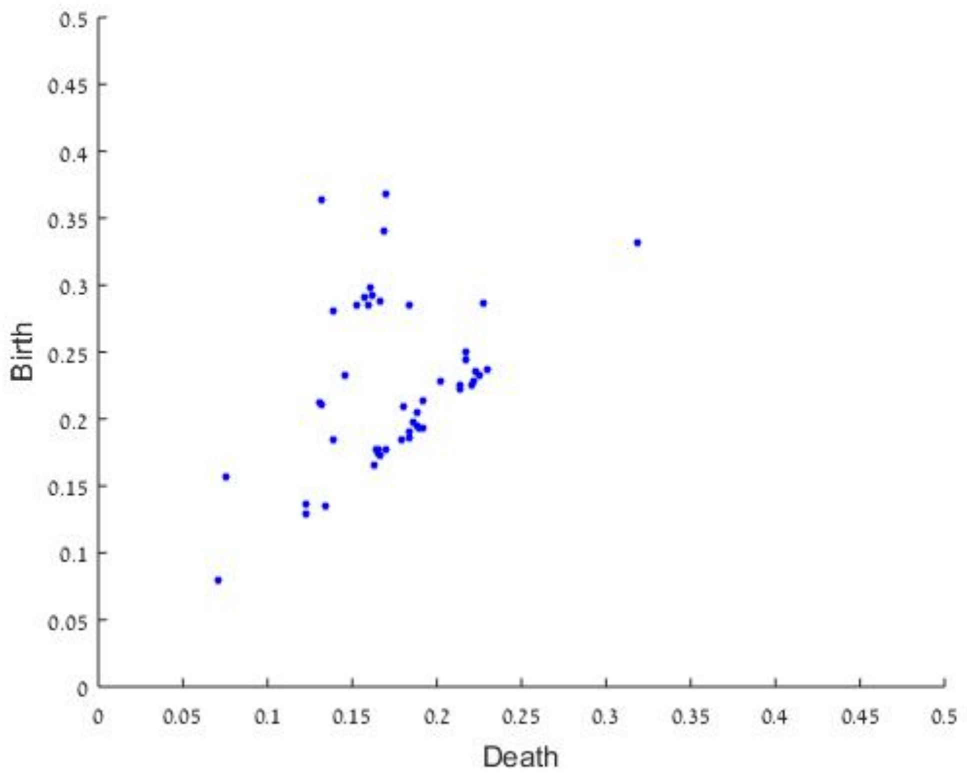
- Compute your PD
- Fit a Gibbs Model with Hamiltonian
- Use it to produce synthetic PDs

$$\varphi_\Theta(\tilde{x}_N) = \frac{1}{Z_\Theta} \exp(-H_\Theta(\tilde{x}_N)),$$

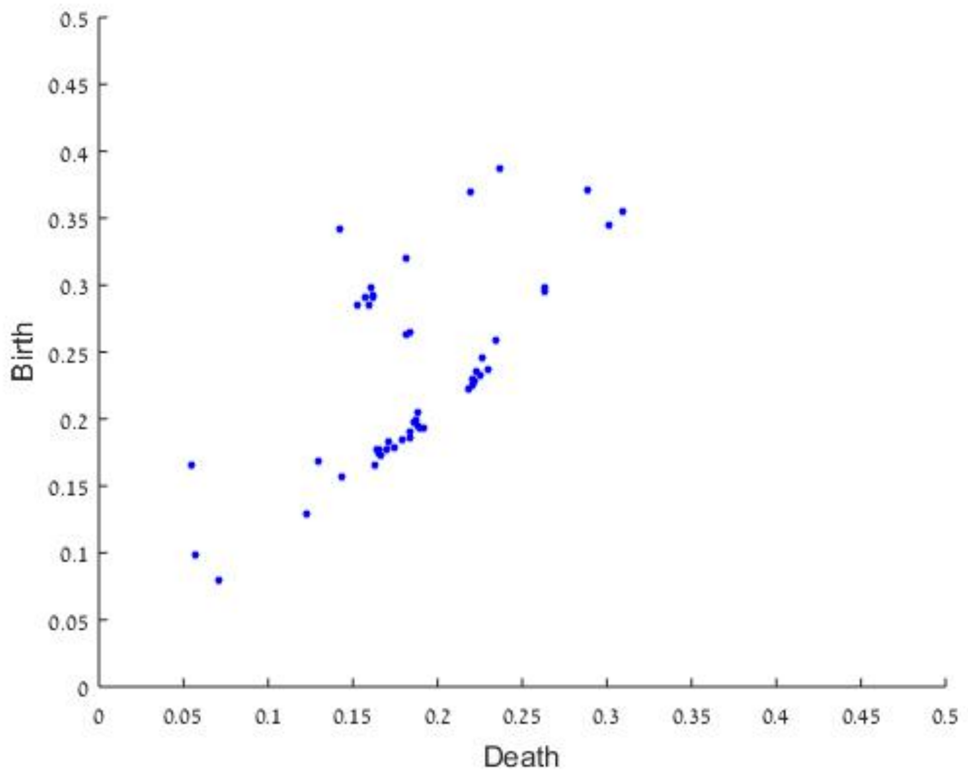
$$H^2_{\delta,\Theta}(\tilde{x}_N) = \theta_H \sigma_H^2 + \theta_V \sigma_V^2 + \sum_{k=1}^{3} \delta^{-2} \theta_k \mathcal{L}_{\delta,k}(\tilde{x}_N),$$



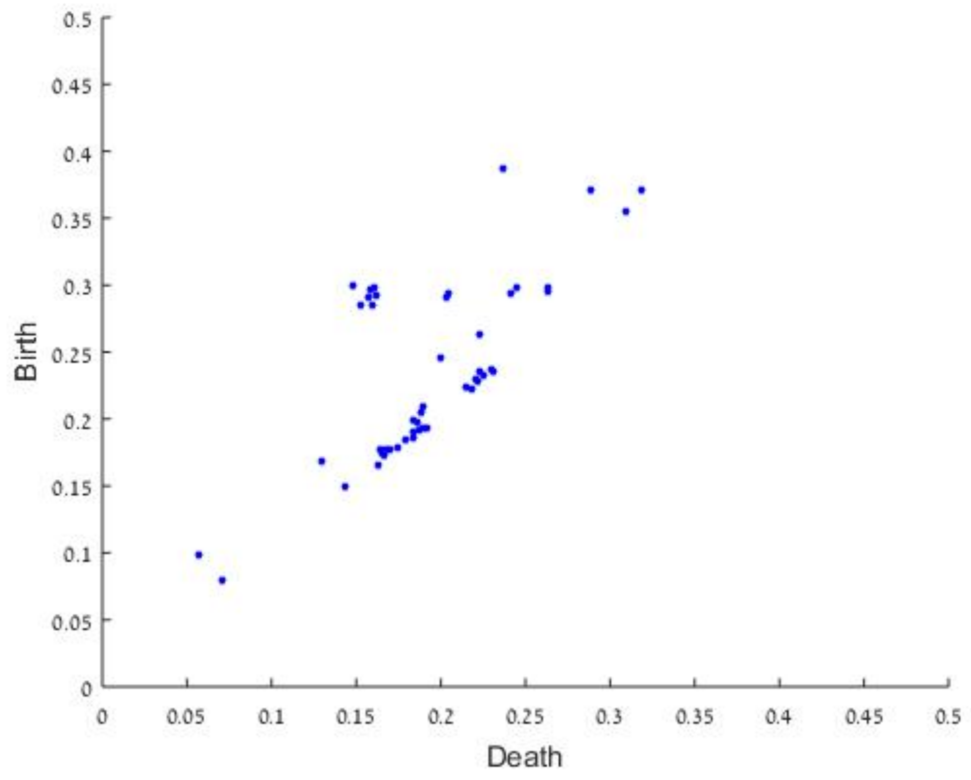
(a)



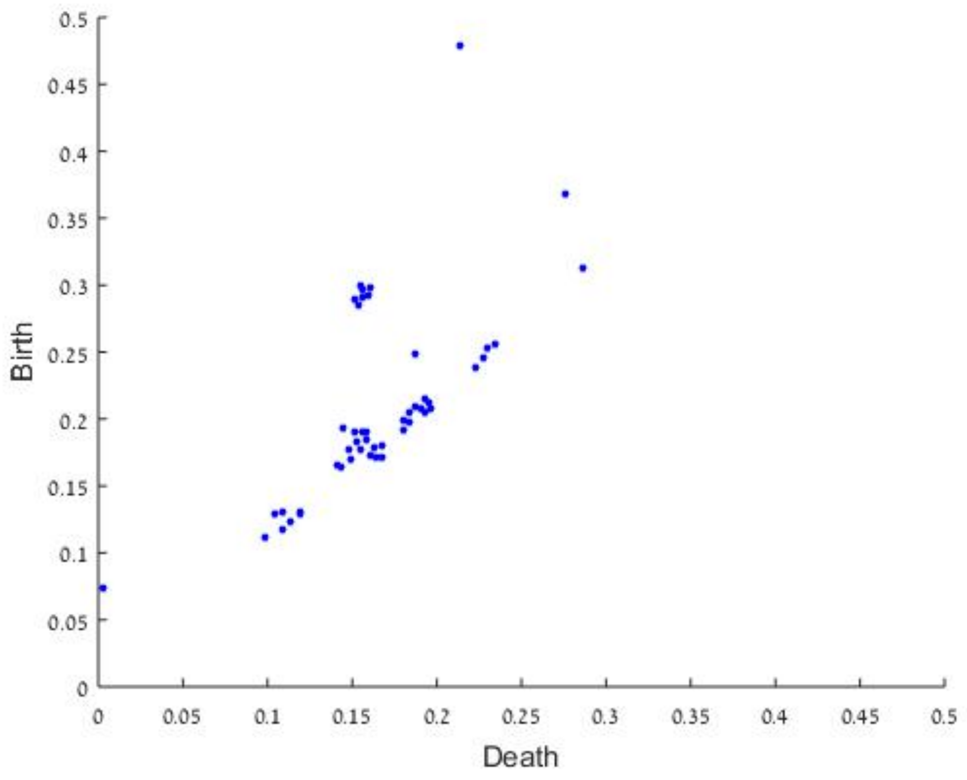
(b)



(c)



(d)



(e)

Modeling and replicating statistical topology and evidence for CMB nonhomogeneity

Robert J. Adler[a,1], Sarit Agami[a], and Pratyush Pranav[a]

Notebook
07-08-09

Thanks!