430.523: Random Signal Theory

Electrical and Computer Engineering, Seoul National Univ.
Spring Semester, 2018
Homework #3, Due: In class @ May 10

Note: No late homework will be accepted.

Problem 1) Find out the moment generating function (MGF) of the RV X when:

(a) X is the Gaussian RV $\mathcal{N}(\mu, \sigma^2)$.

$\Rightarrow$ We have

$$\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{tx} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu-t\sigma^2)^2}{2\sigma^2}} e^{\frac{t^2\sigma^2}{2}+\mu t} dx \\
&= e^{\frac{t^2\sigma^2}{2}+\mu t} \int_{-\infty}^{\infty} \mathcal{N}(x-\mu-t\sigma^2, \sigma) dx \\
&= e^{\frac{t^2\sigma^2}{2}+\mu t}
\end{aligned}$$

(b) X is the uniform RV distributed in (0,1).

$\Rightarrow$ We have

$$M_X(t) = E[e^{tX}] = \int_0^1 e^{tx} dx = \frac{1}{t} e^{tx}\Big|_0^1 = \frac{e^t - 1}{t}$$

(c) X is the exponential RV with the parameter λ.

$\Rightarrow$ For $t<\lambda$, we have

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} \lambda e^{-\lambda x} e^{tx} dx = \frac{\lambda}{\lambda - t} e^{-(\lambda-t)x}\Big|_0^{\infty} = \frac{\lambda}{\lambda - t}$$

Problem 2) Show that the expected value of negative Binomial RV X with parameters l and p (l is the number of success and p is the success probability in a trial) is $\frac{l}{p}$.

(a) Show the answer by direct way.

(b) Show by using MGF.

$\Rightarrow$ (a) The pmf of the negative binomial RV X is $P(X = n) = \binom{n-1}{l-1} p^l (1-p)^{n-l}$. By definition, we have

$$
\begin{aligned}
E[X] &= \sum_{n=l}^{\infty} n \binom{n-1}{l-1} p^l (1-p)^{n-l} \\
&= \sum_{n=k}^{\infty} \frac{l}{p} \binom{n}{k} p^{l+1} (1-p)^{n-l}.
\end{aligned}
$$

Further, for a negative Binomial RV Y with parameter $l+1$ and p, we have its pmf $P(Y = n+1) = \binom{n}{l} p^{k+1} (1-p)^{n-l}$. Thus, $1 = \sum\limits_{n=l}^{\infty} P(Y = n+1) = \sum\limits_{n=l}^{\infty} \binom{n}{l} p^{l+1} (1-p)^{n-l}$ and $E[X] = \frac{l}{p}$.

(b) We have

$$
\begin{aligned}
E[X] &= \frac{d}{dt} M_X(t) \Big|_{t=0} \\
&= \frac{d}{dt} \left(\frac{p^l e^{tl}}{(1 - e^t(1-p))^l} \right) \Big|_{t=0} \\
&= \frac{l p^l e^{tl} (1 - e^t(1-p))^l - p^l e^{tl} l (1 - e^t(1-p))^{l-1} (-e^t(1-p))}{(1 - e^t(1-p))^{2l}} \Bigg|_{t=0} \\
&= \frac{l p^{2l} + p^l l p^{l-1} (1-p)}{p^{2l}} \\
&= \frac{l}{p}.
\end{aligned}
$$

Problem 3) Let X_1, X_2, and X_3 be the i.i.d. exponential RVs with parameter $\lambda = 2$. Suppose $Y = \max_i X_i$ and $Z = \mathit{median}\ X_i$ where *median* is the middle value (e.g., if $X_1 = 3$, $X_2 = 7$, and $X_3 = 1$, then *median* is 3).

(a) Find the CDF of Y by direct calculation.

(b) Find the CDF of Y and Z using order statistics approaches.

⇒ (a)

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(\max_i X_i \le y) \\
&= P(X_1 \le y, X_2 \le y, X_3 \le y) \\
&= P(X_1 \le y)^3 \\
&= (1 - e^{-2y})^3.
\end{aligned}
$$

(b)

$$
\begin{aligned}
F_Y(y) &= \frac{3!}{2!0!}\int_0^y F_X(\alpha)^2 f_X(\alpha)(1 - F_X(\alpha))^0 d\alpha \\
&= 3\int_0^y F_X(\alpha)^2 d(F_X(\alpha)) \\
&= F_X(\alpha)^3 \Big|_0^y \\
&= (1 - e^{-2y})^3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F_Z(z) &= \frac{3!}{1!1!}\int_0^z F_X(\alpha) f_X(\alpha)(1 - F_X(\alpha)) d\alpha \\
&= 6\int_0^z (F_X(\alpha) - F_X(\alpha)^2) d(F_X(\alpha)) \\
&= 6(\frac{1}{2}F_X(\alpha)^2 - \frac{1}{3}F_X(\alpha)^3)\Big|_0^z \\
&= (1 - e^{-2z})^2(1 + 2e^{-2z}) \\
&= 1 - 3e^{-4z} + 2e^{-6z}.
\end{aligned}
$$

Problem 4) Fair dice is rolled. Let X and Y be the number of rolls to obtain 3 and 5. What is $E[X]$? Also, what is $E[X|Y = 1]$?

⇒ Since X is geometric random variable with $p = 1/6$, we have $E[X] = 1/p = 6$. Also, $E[X|Y = 1] = E[X] + 1 = 7$.

Problem 5) Let X and Y be i.i.d. geometric RVs with the parameter p. Also, let $Z = e^X + Y$. Find $E[Z|X]$.

⇒ We have

$$
\begin{aligned}
E[Z|X] &= E[e^X + Y|X] \\
&= e^X + E[Y] \\
&= e^X + 1/p
\end{aligned}
$$

Problem 6) Show the following equality

$$E[E[Z|X,Y]] = E[Z].$$

Assume X, Y, and Z are discrete RVs. The first expectation is w.r.t X, Y and the second expectation is w.r.t Z.

$\Rightarrow$ We have

$$\begin{aligned}
E[E[Z|X,Y]] &= \sum_x \sum_y E[Z|X=x, Y=y] P(X=x, Y=y) \\
&= \sum_x \sum_y \sum_z z P(Z=z|X=x, Y=y) P(X=x, Y=y) \\
&= \sum_z z \sum_x \sum_y P(Z=z, X=x, Y=y) \\
&= \sum_z z P(Z=z) \\
&= E[Z].
\end{aligned}$$

Problem 7) Suppose X and Y be the i.i.d. geometric RVs with parameter $p = 0.2$. Find $P(X < Y)$.
$\Rightarrow$ We have

$$\begin{aligned}
P(X < Y) &\stackrel{(a)}{=} \sum_\alpha P(X < Y | Y = \alpha) P(Y = \alpha) \\
&= \sum_\alpha P(X < \alpha)(1-p)^{\alpha-1} p \\
&= \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\alpha-1} (1-p)^{\beta-1} p (1-p)^{\alpha-1} p \\
&= \sum_{\alpha=1}^{\infty} (1 - (1-p)^{\alpha-1})(1-p)^{\alpha-1} p \\
&= \sum_{\alpha=1}^{\infty} ((1-p)^{\alpha-1} p - (1-p)^{2\alpha-2} p) \\
&= 1 - \frac{p}{1-(1-p)^2} \\
&= \frac{1-p}{2-p},
\end{aligned}$$

where (a) is due to the law of total probability.

Problem 8) Let X_1, X_2, and X_3 be the i.i.d. geometric RVs with parameter p. Suppose

$Y_k = \sum_{j=1}^{k} X_j$.

(a) Find the joint probability mass function of Y_1, Y_2, Y_3.

⇒ We have

$$\begin{aligned}
p(y_1, y_2, y_3) &= P(Y_1 = y_1, Y_2 = y_2, Y_3 = y_3) \\
&= P(X_1 = y_1, X_1 + X_2 = y_2, X_1 + X_2 + X_3 = y_3) \\
&= P(X_1 = y_1, X_2 = y_2 - y_1, X_3 = y_3 - y_2) \\
&= p(1-p)^{y_1 - 1} p(1-p)^{y_2 - y_1 - 1} p(1-p)^{y_3 - y_2 - 1} \\
&= p^3 (1-p)^{y_3 - 3},
\end{aligned}$$

where $0 < y_1 < y_2 < y_3$.

(b) Find the probability mass functions of Y_1, Y_2, and Y_3.

⇒ We have $p(y_1) = P(Y_1 = y_1) = p(1-p)^{y_1 - 1}$ and

$$\begin{aligned}
p(y_2) &= P(X_1 + X_2 = y_2) \\
&\overset{(a)}{=} \sum_{\alpha=1}^{y_2} P(X_1 + X_2 = y_2 | X_1 = \alpha) P(X_1 = \alpha) \\
&\overset{(b)}{=} \sum_{\alpha=1}^{y_2} P(X_2 = y_2 - \alpha) P(X_1 = \alpha) \\
&= \sum_{\alpha=1}^{y_2 - 1} P(X_2 = y_2 - \alpha) P(X_1 = \alpha) \\
&= \sum_{\alpha=1}^{y_2 - 1} p(1-p)^{\alpha - 1} p(1-p)^{y_2 - \alpha - 1} \\
&= p^2 (y_2 - 1)(1-p)^{y_2 - 2},
\end{aligned}$$

where (a) is due to the law of total probability and (b) is due to the independence between X_1 and X_2. Similarly, we have

$$
\begin{aligned}
p(y_3) &= P(X_1 + X_2 + X_3 = y_3) \\
&\overset{(a)}{=} \sum\sum_{2 \le \alpha + \beta \le y_3 - 1} P(X_1 + X_2 + X_3 = y_3 | X_1 = \alpha, X_2 = \beta) P(X_1 = \alpha, X_2 = \beta) \\
&= \sum\sum_{2 \le \alpha + \beta \le y_3 - 1} P(X_3 = y_3 - \alpha - \beta | X_1 = \alpha, X_2 = \beta) P(X_1 = \alpha, X_2 = \beta) \\
&\overset{(b)}{=} \sum\sum_{2 \le \alpha + \beta \le y_3 - 1} P(X_3 = y_3 - \alpha - \beta) P(X_1 = \alpha) P(X_2 = \beta) \\
&= \sum\sum_{2 \le \alpha + \beta \le y_3 - 1} p(1-p)^{\beta - 1} p(1-p)^{\alpha - 1} p(1-p)^{y_3 - \alpha - \beta - 1} \\
&= \sum\sum_{2 \le \alpha + \beta \le y_3 - 1} p^3 (1-p)^{y_3 - 3} \\
&= \sum_{\alpha = 1}^{y_3 - 2} \sum_{\beta = 1}^{y_3 - 1 - \alpha} p^3 (1-p)^{y_3 - 3} \\
&= \sum_{\alpha = 1}^{y_3 - 2} (y_3 - 1 - \alpha) p^3 (1-p)^{y_3 - 3} \\
&= ((y_3 - 1)(y_3 - 2) - \frac{1}{2}(y_3 - 1)(y_3 - 2)) p^3 (1-p)^{y_3 - 3} \\
&= \frac{1}{2}(y_3 - 1)(y_3 - 2) p^3 (1-p)^{y_3 - 3},
\end{aligned}
$$

where (a) is due to the law of total probability and (b) is due to the independence of X_i ($i = 1, 2, 3$).

Problem 9) Suppose X and Y are normal random variables, both with mean 1 and variance 10. Suppose $\rho(X, Y) = 0.4$. Find the variance of $3X + 5Y$.

$\Rightarrow$ We have

$$
\begin{aligned}
Var(3X + 5Y) &= Var(3X) + Var(5Y) + 2Cov(3X, 5Y) \\
&= 9Var(X) + 25Var(Y) + 30Cov(X, Y) \\
&= 9Var(X) + 25Var(Y) + 30\rho(X, Y)\sqrt{Var(X)Var(Y)} \\
&= 90 + 250 + 120 \\
&= 460.
\end{aligned}
$$

Problem 10) Let $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ be jointly normal distributed RVs. Suppose that X and Y are uncorrelated. Show that X and Y are independent.

$\Rightarrow$ Since X and Y are uncorrelated, we should have $Cov(X, Y) = 0$. The joint pdf of X and Y is

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp(-\frac{1}{2} \begin{bmatrix} x-\mu_1 & y-\mu_2 \end{bmatrix} \begin{bmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{bmatrix} \begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}) \\
&= \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp(-\frac{1}{2\sigma_1^2}(x-\mu_1)^2) \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp(-\frac{1}{2\sigma_2^2}(x-\mu_2)^2) \\
&= f_X(x) f_Y(y).
\end{aligned}$$

Thus, X and Y are independent.