Seoul National University
School of Electrical and Computer Engineering

430.523: Random Signal Theory

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Midterm Exam 2
May 24, 2018
75 minutes

This is closed book test. However, one A4 page cheating sheet is allowed.
Make sure to clearly show your work and full justification to get the full credit for the problem.
You have 75 minutes to finish the exam.

Please do not turn this page until requested to do so

Problem 1)[20pt] Let X be the exponential random variable with the parameters λ. Also, let $Z = \exp(-X)$. Find the PDF of Z.

$\Rightarrow$ For $\alpha>0$, we have

$$
\begin{aligned}
F_Z(\alpha) &= P(Z \le \alpha) \\
&= P(\exp(-X) \le \alpha) \\
&= P(-X \le \ln(\alpha)) \\
&= P(X \ge -\ln(\alpha)) \\
&= \begin{cases} 1 & \text{if } \alpha>1 \\ 1 - P(X \le -\ln(\alpha)) & \text{if } 0<\alpha \le 1 \end{cases} \\
&= \begin{cases} 1 & \text{if } \alpha>1 \\ 1 - F_X(-\ln(\alpha)) & \text{if } 0<\alpha \le 1 \end{cases} \\
&= \begin{cases} 1 & \text{if } \alpha>1 \\ 1 - (1 - \exp(\lambda \ln(\alpha))) & \text{if } 0<\alpha \le 1 \end{cases} \\
&= \begin{cases} 1 & \text{if } \alpha>1 \\ \alpha^\lambda & \text{if } 0<\alpha \le 1 \end{cases}.
\end{aligned}
$$

Thus, we have $f_Z(\alpha) = \lambda \alpha^{\lambda-1}$ if $0 \le \alpha \le 1$, otherwise zero.

Problem 2)[20pt] Let $X_1, X_2, \cdots, X_n$ be discrete random variables. Also, let $\Phi = \{x : p_i(x)>0 \text{ for all } i\}$ where $p_i(x)$ be the PMF of X_i. Show the following inequalities:

(a) $D(p_1(x)||p_2(x)) \ge 0$, $x \in \Phi$

(b) $I(X_1; X_2) \ge 0$

(c) $H(X_1, X_2, \cdots, X_n) \le \sum_{i=1}^{n} H(X_i)$

⇒ (a) We have

$$\begin{aligned}
-D(p_1(x)||p_2(x)) &= \sum_{x\in\Phi} p_1(x)\log(\frac{p_2(x)}{p_1(x)}) \\
&\overset{(i)}{\leq} \log(\sum_{x\in\Phi} p_1(x)\frac{p_2(x)}{p_1(x)}) \\
&= \log(\sum_{x\in\Phi} p_2(x)) \\
&\overset{(ii)}{\leq} \log(\sum_{x\in\Omega} p_2(x)) \\
&= \log(1) \\
&= 0,
\end{aligned}$$

where (i) is because the application of Jensen's inequality on the concave function $\log(z)$ and (ii) is because $\Phi \subseteq \Omega$

(b) Let $p(x_1, x_2)$ be the joint PMF of X_1 and X_2. Then, we have

$$\begin{aligned}
I(X_1; X_2) &= \sum_{x_1}\sum_{x_2} p(x_1, x_2)\log(\frac{p(x_1, x_2)}{p_1(x)p_2(x)}) \\
&= D(p(x_1, x_2)||p_1(x_1)p_2(x_2)) \\
&\overset{(i)}{\geq} 0,
\end{aligned}$$

where (i) is due to (a).

(c) From (b), we have $I(X_1; X_2) = H(X_2) - H(X_2|X_1) \geq 0$. Hence, $H(X_2) \geq H(X_2|X_1)$. By the chain rule, we have

$$H(X_i) \geq H(X_i|X_{i-1}, \cdots, X_1).$$

Thus, $H(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \cdots, X_1) \leq \sum_{i=1}^{n} H(X_i)$, which is the desired result.

Problem 3)[20pt] Let X_i $(1 \leq i \leq n)$ be independent random variables satisfying $|X_i| \leq M$, $E[X_i] = 0$, and $E[X_i^2] = \sigma_i^2$. Show the following inequalities:

(a) $P(\sum_{i=1}^{n} X_i > t) \leq e^{-\lambda t}\prod_{i=1}^{n} E[e^{\lambda X_i}]$, for any $\lambda > 0$

(b) $E[e^{\lambda X_i}] \leq \exp\left(\frac{\sigma_i^2}{M^2}(e^{\lambda M} - 1 - \lambda M)\right)$

Hint: Note that $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ and $e^y \geq 1 + y$ $(y \geq 0)$.

$\Rightarrow$ (a) We have

$$
\begin{aligned}
P(\sum_{i=1}^{n} X_i > t) &= P\left(\exp(\lambda \sum_{i=1}^{n} X_i) > \exp(\lambda t)\right) \\
&\leq \frac{E[\exp(\lambda \sum\limits_{i=1}^{n} X_i]}{\exp(\lambda t)} \\
&= e^{-\lambda t} \prod_{i=1}^{n} E[e^{\lambda X_i}].
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
E[e^{\lambda X_i}] &= E[\sum_{k=0}^{\infty} \frac{(\lambda X_i)^k}{k!}] \\
&\leq E[1 + \sum_{k=2}^{\infty} \frac{\lambda^k X_i^2 M^{k-2}}{k!}] \\
&= 1 + \frac{\sigma_i^2}{M^2} \sum_{k=2}^{\infty} \frac{\lambda^k M^k}{k!} \\
&= 1 + \frac{\sigma_i^2}{M^2}(e^{\lambda M} - 1 - \lambda M) \\
&\leq \exp\left(\frac{\sigma_i^2}{M^2}(e^{\lambda M} - 1 - \lambda M)\right).
\end{aligned}
$$

Problem 4)[20pt] Let $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}^T$ be the normal random vector where $X_i \sim \mathcal{N}(0, \sigma_i^2)$.

(a) Show that $tr(\mathbf{C}) = \sum\limits_i \lambda_i$ where λ_i is the eigenvalues of the covariance matrix $\mathbf{C}$ of $\mathbf{X}$

Hint: $tr(\mathbf{UVW}) = tr(\mathbf{WUV})$

(b) Show that $\lambda_{\max} \geq \sigma_{\min}^2$ where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{C}$ and $\sigma_{\min}^2 = \min\limits_i \sigma_i^2$.

⇒ (a) Using the eigendecomposition of $\mathbf{C}$, we have $\mathbf{C} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$. Thus, we compute

$$\begin{aligned} tr(\mathbf{C}) &= tr(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}) \\ &\overset{(i)}{=} tr(\mathbf{Q}^{-1}\mathbf{Q}\mathbf{\Lambda}) \\ &= tr(\mathbf{\Lambda}) \\ &= \sum_i \lambda_i, \end{aligned}$$

where (i) is because $tr(\mathbf{AB}) = tr(\mathbf{BA})$.

(b) Since $\mathbf{X}$ has zero mean, we have

$$\begin{aligned} tr(\mathbf{C}) &= tr(E[\mathbf{X}\mathbf{X}^T]) \\ &= tr\left(\begin{bmatrix} E[X_1^2] & E[X_1X_2] & \cdots & E[X_1X_n] \\ E[X_2X_1] & E[X_2^2] & \cdots & E[X_2X_n] \\ \cdots & \cdots & \cdots & \cdots \\ E[X_nX_1] & E[X_nx_2] & \cdots & E[X_n^2] \end{bmatrix}\right) \\ &= \sum_{i=1}^{n} E[X_i^2] \\ &= \sum_{i=1}^{n} \sigma_i^2. \end{aligned}$$

From (a), we have $n\sigma_{\min}^2 \le \sum_{i=1}^{n} \sigma_i^2 = \sum_i \lambda_i \le n\lambda_{\max}$, which is the desired result.

Problem 5)[20pt] Suppose that there are n pairs of shoes in distinct styles and sizes. The shoes are mixed up. Peter randomly selects k shoes (k might not be even). What is the expected number of matched pairs of shoes that Peter selects?

⇒ Let X_i be the random variable defined as

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th pair of shoes is selected by Peter} \\ 0 & \text{else} \end{cases}.$$

The number of matched pairs of shoes that Peter selects is

$$X = X_1 + X_2 + ... + X_n,$$

and also,

$$E[X] = E[X_1] + E[X_2] + ... + E[X_n].$$

Now we compute $E[X_i] = 0P(X_i = 0) + 1P(X_i = 1) = P(X_i = 1)$. Also, we have

$$\begin{aligned} P(X_i = 1) &= P(\text{One shoe in the i-th pair is chosen}) \\ &\quad \times P(\text{The remaining shoe in the i-th pair is also chosen}) \\ &= \frac{k}{2n}\frac{k-1}{2n-1}. \end{aligned}$$

Thus, the desired result is $E[X] = \frac{k(k-1)}{4n-2}$.

Problem 6)[20pt] Let $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^T \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ be the normal random vector with

$$\mathbf{C} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

where $0 < b < a$.

(a) Find the PDF of $X_2 + X_3$.

(b) Find the joint PDF of $X_1 + X_2$ and $X_1 - X_2$.

(c) Find the conditional PDF of $\mathbf{Z} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T$ given $X_3 = x$.

(d) Find the linear transformation $\mathbf{A}$ such that $\mathbf{Y} = \mathbf{A}\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix})$.

$\Rightarrow$ (a) We have $X_2, X_3 \sim \mathcal{N}(0, a)$. Since X_2 and X_3 are independent ($Cov(X_2, X_3) = 0$), we have

$$X_2 + X_3 \sim \mathcal{N}(0, 2a).$$

(b) Let $\mathbf{Z} = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, then we have

$$\mathbf{Z} = \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \mathbf{DX}.$$

Thus, $\mathbf{Z}$ is also a normal random vector with $E[\mathbf{Z}] = E[\mathbf{DX}] = \mathbf{0}$ and the covariance matrix

$$\mathbf{C}_Z = E[\mathbf{ZZ}^T] = E[\mathbf{DXX}^T\mathbf{D}^T] = \mathbf{DCD}^T = \begin{bmatrix} 2a + 2b & 0 \\ 0 & 2a - 2b \end{bmatrix}.$$

The joint PDF of $Z_1 = X_1 + X_2$ and $Z_2 = X_1 - X_2$ is

$$\begin{aligned} f_{Z_1,Z_2}(z_1, z_2) &= \frac{1}{2\pi\sqrt{4a^2 - 4b^2}} \exp(-\frac{1}{2} \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2a+2b} & 0 \\ 0 & \frac{1}{2a-2b} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}) \\ &= \frac{1}{2\pi\sqrt{4a^2 - 4b^2}} \exp(-\frac{z_1^2}{4a + 4b} - \frac{z_2^2}{4a - 4b}). \end{aligned}$$

(c) Since $Cov(X_1, X_3) = Cov(X_2, X_3) = 0$, X_1 and X_2 are uncorrelated with X_3 and thus they are independent from X_3. The conditional PDF of $\mathbf{Z}$ given X_3 is just the same as the PDF of $\mathbf{Z}$, which is

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) &= \frac{1}{2\pi} \det(\begin{bmatrix} a & b \\ b & a \end{bmatrix})^{-1/2} \exp(-\frac{1}{2}\mathbf{z}^T \begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} \mathbf{z}) \\ &= \frac{1}{2\pi\sqrt{a^2 - b^2}} \exp(-\frac{1}{2(a^2 - b^2)}\mathbf{z}^T \begin{bmatrix} a & -b \\ -b & a \end{bmatrix} \mathbf{z}) \\ &= \frac{1}{2\pi\sqrt{a^2 - b^2}} \exp(-\frac{az_1^2 + az_2^2 - 2bz_1z_2}{2(a^2 - b^2)}), \end{aligned}$$

where $\mathbf{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$.

(d) The covariance matrix of $\mathbf{Y}$ is

$$\mathbf{C}_Y = E[\mathbf{YY}^T] = \mathbf{A}E[\mathbf{XX}^T]\mathbf{A}^T = \mathbf{ACA}^T = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}.$$

Since the covariance matrix is psd, we have $\mathbf{C} = \mathbf{C}^{1/2}\mathbf{C}^{1/2}$ and then

$$(\mathbf{AC}^{1/2})(\mathbf{AC}^{1/2})^T = \mathbf{ACA}^T = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}^T.$$

Letting $\mathbf{A}\mathbf{C}^{1/2} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, we have

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{C}^{-1/2}.$$

Note that the fractorization $\begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}^T$ is not unique, resulting in that there are multiple solutions of $\mathbf{A}$.

What remains is to compute $\mathbf{C}^{-1/2}$ using the eigendecomposition of $\mathbf{C}$. We have

$$\begin{aligned}
\det(\mathbf{C} - \lambda \mathbf{I}) &= \det(\begin{bmatrix} a-\lambda & b & 0 \\ b & a-\lambda & 0 \\ 0 & 0 & a-\lambda \end{bmatrix}) \\
&= (a-\lambda)^3 - (a-\lambda)b^2 \\
&= (a-\lambda)((a-\lambda)^2 - b^2)) \\
&= (a-\lambda)(a+b-\lambda)(a-b-\lambda).
\end{aligned}$$

By letting $\det(\mathbf{C} - \lambda \mathbf{I}) = 0$, the eigenvalues are $\lambda_1 = a$, $\lambda_2 = a+b$, and $\lambda_3 = a-b$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

Note that $\mathbf{x}_i$ are orthonormal and the eigendecomposition of $\mathbf{C}$ is $\mathbf{C} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^T$. Thus, $\mathbf{C}^{-1/2} = \boldsymbol{\Lambda}^{-1/2}\mathbf{X}^T$ and

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{C}^{-1/2} \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \boldsymbol{\Lambda}^{-1/2}\mathbf{X}^T \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{a+b}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{a-b}} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3/\sqrt{a} & 0 & 0 \\ 1/\sqrt{a} & 1/\sqrt{a+b} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\frac{3}{\sqrt{a}} \\ \frac{1}{\sqrt{2a+2b}} & \frac{1}{\sqrt{2a+2b}} & -\frac{1}{\sqrt{a}} \end{bmatrix},
\end{aligned}$$

which is the desired result.