430.523: Random Signal Theory

Electrical and Computer Engineering, Seoul National Univ.
Spring Semester, 2018
Homework #2, Due: In class @ April 12

Note: No late homework will be accepted.

Problem 1) X, Y, and Z are independent and uniformly distributed RVs defined over $(0, 10)$. Compute $P(X \geq YZ)$.

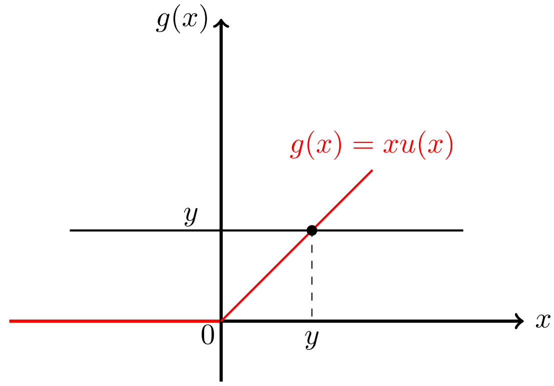
$\Rightarrow$ We have

$$\begin{aligned}
P(X \geq YZ) &= P(X \geq YZ \mid YZ > 10)P(YZ > 10) + P(X \geq YZ \mid YZ \leq 10)P(YZ \leq 10) \\
&= P(X \geq YZ \mid YZ \leq 10)P(YZ \leq 10) \\
&= P(X \geq YZ, YZ \leq 10) \\
&= \int_0^{10} \int_0^{\min(10, \frac{10}{z})} \int_{yz}^{10} f_{X,Y,Z}(x, y, z) dx dy dz \\
&= \int_0^{10} \int_0^{\min(10, \frac{10}{z})} \int_{yz}^{10} f_X(x) f_Y(y) f_Z(z) dx dy dz \\
&= \int_0^{10} \int_0^{\min(10, \frac{10}{z})} \int_{yz}^{10} \frac{1}{10^3} dx dy dz \\
&= \frac{1}{10^3} \int_0^{10} \int_0^{\min(10, \frac{10}{z})} (10 - yz) dy dz \\
&= \frac{1}{10^3} \int_0^{1} \int_0^{10} (10 - yz) dy dz + \frac{1}{10^3} \int_1^{10} \int_0^{\frac{10}{z}} (10 - yz) dy dz \\
&= \frac{1}{10^3} \int_0^{1} \left[(10y - \frac{1}{2} z y^2) \Big|_{y=0}^{y=10} \right] dz + \frac{1}{10^3} \int_1^{10} \left[(10y - \frac{1}{2} z y^2) \Big|_{y=0}^{y=\frac{10}{z}} \right] dz \\
&= \frac{1}{10} \int_0^{1} (1 - \frac{1}{2} z) dz + \frac{1}{10} \int_1^{10} \frac{1}{2z} dz \\
&= \frac{1}{20} (2z - \frac{z^2}{2}) \Big|_{z=0}^{z=1} + \frac{1}{20} \ln(z) \Big|_{z=1}^{z=10} \\
&= \frac{3 + 2\ln(10)}{40}.
\end{aligned}$$

(Alternatively, one can compute $P(X \geq YZ \mid YZ \leq 10)$ and $P(YZ \leq 10)$ and then take the product of them. However, in order to compute $P(X \geq YZ \mid YZ \leq 10)$, it is worth to note that the conditional probability $f_{X,Y,Z|YZ \leq 10}(x, y, z)$ should be used instead of the joint probability $f_{X,Y,Z}(x, y, z)$.)

Problem 2) Prove a rectified linear unit (a.k.a. ReLu) has the transfer function $y = xu(x)$ where $u(x) = 1$ for $x \geq 0$. Suppose X is uniform over $(0, 2)$. Plot the CDF $F_Y(y)$.

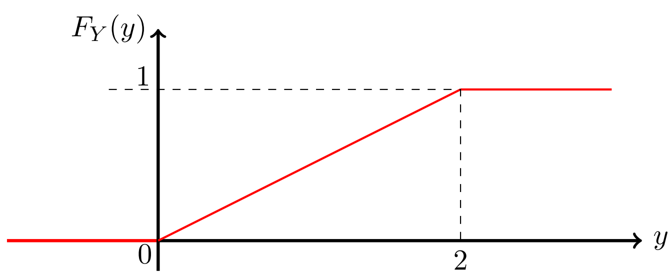
$\Rightarrow$



The cdf $F_Y(y)$ of Y is computed as

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(Xu(X) \leq y) \\ &= \begin{cases} P(X \leq y) & \text{if } y \geq 0 \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} F_X(y) & \text{if } y \geq 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Since X is uniform over (0,2), one can easily find out $F_Y(y) = F_X(y)$



Problem 3) Let X_1, X_2, and X_3 be i.i.d. continuous random variables with the CDF $F(x)$. Also, let $Y_i = F(X_i)$ be the random variables where $F(X_i)$ is the CDF of X_i. Find the joint distribution of Y_1, Y_2, and Y_3.

$\Rightarrow$ Y_i are i.i.d. since they are functions of i.i.d. X_i. The joint distribution of Y_i is

$$f_{Y_1,Y_2,Y_3}(\alpha, \beta, \gamma) = f_{Y_1}(\alpha) f_{Y_2}(\beta) f_{Y_3}(\gamma).$$

Now we compute $f_{Y_1}(\alpha)$. We have

$$\begin{aligned} F_{Y_1}(\alpha) &= P(Y_1 \le \alpha) \\ &= P(F(X_1) \le \alpha) \\ &= P(X_1 \le F^{-1}(\alpha)) \\ &= F(F^{-1}(\alpha)) \\ &= \alpha. \end{aligned}$$

thus, $f_{Y_1}(\alpha) = 1$ $(0 \le \alpha \le 1)$, meaning that $Y_1 \sim \mathcal{U}(0,1)$. Similarly, we have $f_{Y_2}(\beta) = f_{Y_3}(\gamma) = 1$ $(0 \le \beta, \gamma \le 1)$. Thus, $f_{Y_1,Y_2,Y_3}(\alpha, \beta, \gamma) = 1$ $(0 \le \alpha, \beta, \gamma \le 1)$.

Problem 4) Let X_1, X_2, $\cdots$, X_n be i.i.d. random variables. Also, let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the order statistic of X_i. Suppose that $Y_i = g(X_{(i)})$ where g is some monotonically increasing and invertible function.

(a) Find the distribution of Y_1

$\Rightarrow$ Let $F(x)$ be the cdf of X_i. We have

$$\begin{aligned} F_{Y_1}(\alpha) &= P(Y_1 \le \alpha) \\ &= P(g(X_{(1)}) \le \alpha) \\ &= P(X_{(1)} \le g^{-1}(\alpha)) \\ &= 1 - P(X_{(1)} \ge g^{-1}(\alpha)) \\ &= 1 - P(X_1 \ge g^{-1}(\alpha), \cdots, X_n \ge g^{-1}(\alpha)) \\ &= 1 - \prod_{i=1}^{n} P(X_i \ge g^{-1}(\alpha)) \\ &= 1 - (1 - F(g^{-1}(\alpha)))^n. \end{aligned}$$

(b) Find the distribution of Y_n

$\Rightarrow$ Similarly to (a), we have

$$\begin{aligned} F_{Y_n}(\beta) &= P(g(X_{(n)}) \le \beta) \\ &= P(X_{(n)} \le g^{-1}(\beta)) \\ &= \prod_{i=1}^{n} P(X_i \le g^{-1}(\beta)) \\ &= F(g^{-1}(\beta))^n. \end{aligned}$$

(c) Find the joint distribution of Y_1 and Y_n

$\Rightarrow$ We have

$$
\begin{aligned}
F_{Y_1,Y_n}(\alpha,\beta) &= P(Y_1 \le \alpha, Y_n \le \beta) \\
&= P(Y_n \le \beta) - P(Y_1 \ge \alpha, Y_n \le \beta) \\
&= P(Y_n \le \beta) - \prod_{i=1}^{n} P(\alpha \le g(X_i) \le \beta) \\
&= P(Y_n \le \beta) - \prod_{i=1}^{n} P(g^{-1}(\alpha) \le X_i \le g^{-1}(\beta)) \\
&= F_{Y_n}(\beta) - (F(g^{-1}(\beta)) - F(g^{-1}(\alpha)))^n \\
&= F(g^{-1}(\beta))^n - (F(g^{-1}(\beta)) - F(g^{-1}(\alpha)))^n
\end{aligned}
\tag{1}
$$

(d) Are Y_1 and Y_n independent?

$\Rightarrow$ Since $F_{Y_1,Y_n}(\alpha,\beta) \neq F_{Y_1}(\alpha)F_{Y_n}(\beta)$, Y_1 and Y_n are not independent.

Problem 5) Show that if X and Y are independent gamma RVs with parameter (α,λ) and (β,λ), respectively, then the $X+Y$ is also gamma RV with parameters $(\alpha+\beta,\lambda)$.

$\Rightarrow$ Let Z = X + Y, then

$$
\begin{aligned}
f_Z(z) &= f_X(z) * f_Y(z) \\
&= \int_0^\infty f_X(x) f_Y(z-x)dx \\
&= \int_0^z \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda(z-x)}(\lambda(z-x))^{\beta-1}}{\Gamma(\beta)} dx \\
&= \frac{\lambda e^{-\lambda z}(\lambda z)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^z \left(\frac{x}{z}\right)^{\alpha-1} \left(1-\frac{x}{z}\right)^{\beta-1} d\left(\frac{x}{z}\right) \\
&= \frac{\lambda e^{-\lambda z}(\lambda z)^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha,\beta),
\end{aligned}
$$

where $B(\alpha,\beta)$ is Beta function satisfying $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Therefore,

$$
f_Z(z) = \frac{\lambda e^{-\lambda z}(\lambda z)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}
$$

is pdf of gamma distribution with parameter $(\alpha+\beta,\lambda)$.

Problem 6) X and Y are independent gamma RVs with parameter (α,λ) and (β,λ), respectively.

Compute the joint density of $U = X + Y$ and $V = \frac{X}{X+Y}$.

Hint: U will be gamma distributed with parameters $(\alpha+\beta, \lambda)$ (see also previous problem) and V will be beta distributed with parameters (α, β) and they will be independent each other.

$\Rightarrow$ Since X and Y are independent RVs, the joint pdf $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Using the rule of changing variables, one obtains

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

For $u = x + y$ and $v = \frac{x}{x+y}$, we compute

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(u,v)}\right| &= \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} \\
&= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1} \\
&= \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix}^{-1} \\
&= x + y = u,
\end{aligned}$$

also $x = uv$, and $y = u - uv$. Thus,

$$\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(uv, u-uv)u \\
&= f_X(uv)f_Y(u-uv)u \\
&= \frac{\lambda e^{-\lambda uv}(\lambda uv)^{\alpha-1}}{\Gamma(\alpha)} \frac{\lambda e^{-\lambda u(1-v)}(\lambda u(1-v))^{\beta-1}}{\Gamma(\beta)} u \\
&= \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1} v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \\
&= \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} v^{\alpha-1}(1-v)^{\beta-1}.
\end{aligned}$$

Due to the properties of beta function $B(\alpha,\beta)$, we have $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{1}{B(\alpha,\beta)}$. Therefore,

$$f_{U,V}(u,v) = \frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)}.$$

We note that from the Prob. 5, $U = X+Y$ is gamma distributed with the parameters $(\alpha+\beta, \lambda)$. The joint pdf $f_{U,V}(u,v)$ can be written as $f_{U,V}(u,v) = f_U(u)f_V(v)$, where $f_V(v) = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)}$ is the pdf of beta distribution with the parameters (α,β) (U and V are independent).

Problem 7) The joint pdf of RVs X and Y is defined as

$$f_{X,Y}(x,y) = \alpha e^{-5y}$$

where $0 < x < 2$ and $y > 0$.

(a) Find α

$\Rightarrow$ We have

$$
\begin{aligned}
1 &= \int_0^\infty \int_0^2 \alpha e^{-5y} dx dy \\
&= 2\alpha \int_0^\infty e^{-5y} dx dy \\
&= \frac{2\alpha}{-5} e^{-5y}\Big|_0^\infty \\
&= \frac{2\alpha}{5}.
\end{aligned}
$$

Thus, we have $\alpha = 5/2 = 2.5$.

(b) Find the marginal pdfs of X and Y.

$\Rightarrow$ The marginal pdfs of X and Y as follows.

$$
\begin{aligned}
f_X(x) &= \int_0^\infty 2.5 e^{-5y} dy \\
&= -0.5 e^{-5y}\Big|_0^\infty = 0.5.
\end{aligned}
$$

$$
f_Y(y) = \int_0^2 2.5 e^{-5y} dx = 5e^{-5y}
$$

(c) What is the covariance of X and Y?

$\Rightarrow$ From the results of (a), one has

$$
f_{X,Y}(x,y) = f_X(x) f_Y(y).
$$

X and Y are thus independent RVs. Therefore, $Cov(X,Y) = 0$.

Problem 8) Show that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

$\Rightarrow$ We have

$$\begin{aligned}
\binom{n}{r} &= \frac{n!}{(n-r)!r!} \\
&= \frac{(n-1)!(r+n-r)}{(n-r)!r!} \\
&= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} \\
&= \binom{n-1}{r-1} + \binom{n-1}{r}.
\end{aligned}$$