430.306: Signals and Systems

Electrical and Computer Engineering, Seoul National Univ.
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Homework #2

Problem 1) Suppose that $x[n]$ is an input signal to an LTI system whose impulse response function is $h[n]$, and let $y[n]$ be the corresponding output signal. Prove that

$$y[n] = x[n] * h[n].$$

State where you use the linearity or time invariance property.

Solution. We have $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$. Note that since the system is time-invariant, the output for $\delta[n-k]$ is $h[n-k]$. Also, since the system is linear,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n].$$

Problem 2) Prob. 2.3 of Oppenheim (p.138)

Solution. Note that

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} u[n-k+2]. \end{aligned} \tag{1}$$

If $n \geq 0$, (1) becomes

$$\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} u[n-k+2] = \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} = 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right].$$

If $n < 0$, (1) becomes 0. Therefore,

$$y[n] = 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] u[n].$$

Problem 4-1) Prob 2.5 of Oppenheim (p.138)

Solution. Note that $h[n] = \delta[n] + \delta[n-1] + \cdots + \delta[n-N]$. Thus, we have

$$y[n] = x[n] + x[n-1] + \cdots + x[n-N].$$

Therefore, in order to satisfy $y[4] = 5$ and $y[14] = 0$, N should be 4.

Problem 5) Prob. 2.8 of Oppenheim (p.139)

Solution. Note that $y(t) = x(t+2) + 2x(t+1)$. Therefore

$$y(t) = \begin{cases} t+3 & \text{if} \quad -2 \le t < -1 \\ 4 & \text{if} \quad t = -1 \\ t+4 & \text{if} \quad -1 < t \le 0 \\ 2-2t & \text{if} \quad 0 < t \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Problem 6) Prob. 2.19 of Oppenheim (p.140)

Solution. (a) Consider the difference equation relating $y[n]$ and $w[n]$ for S_2:

$$y[n] = \alpha y[n-1] + \beta w[n]$$

From this, we have

$$w[n] = \frac{1}{\beta} y[n] - \frac{\alpha}{\beta} y[n-1] \tag{2}$$

and

$$w[n-1] = \frac{1}{\beta} y[n-1] - \frac{\alpha}{\beta} y[n-2]. \tag{3}$$

Plugging (2) and (3) to the difference equation relating $w[n]$ and $x[n]$ for S_1 yields

$$\frac{1}{\beta} y[n] - \frac{\alpha}{\beta} y[n-1] - \frac{1}{2\beta} y[n-1] + \frac{\alpha}{2\beta} y[n-2] = x[n]$$

Equivalently,

$$y[n] = \left(\alpha + \frac{1}{2}\right) y[n-1] - \frac{\alpha}{2} y[n-2] + \beta x[n] \tag{4}$$

Comparing (4) with the given equation relating $y[n]$ and $x[n]$, we obtain

$$\alpha = \frac{1}{4}, \quad \beta = 1$$

(b) From the given difference equation,

$$h[n] - \frac{3}{4} h[n-1] + \frac{1}{8} h[n-2] = \delta[n]. \tag{5}$$

Since this system is causal LTI system, we have $h[n] = 0$ for $n < 0$. Moreover, by putting $n = 0$ to (5), we know that $h[0] = 1$. Now consider the case for $n > 0$. By (5), we have

$$h[n] - \frac{3}{4} h[n-1] + \frac{1}{8} h[n-2] = 0.$$

Then we can write $h[n]$ as $h[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{4}\right)^n$ for $n > 0$. Finally, by putting $n = 1$ and $n = 2$, we obtain $A = 2$ and $B = -1$. Therefore,

$$h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[n].$$

Problem 7) Prob. 2.31 of Oppenheim (p.145)

Solution. Since $x[n] = 0$ for $n < -2$, by initial rest condition, $y[n] = 0$ for $n < -2$. Then by using the given difference equation $y[n] = x[n] + 2x[n-2] - 2y[n-1]$, we can get

$$y[-2] = 1,\ y[-1] = 0,\ y[0] = 5,\ y[1] = -4,\ y[2] = 16,\ y[3] = -27,\ y[4] = 58,\ y[5] = -114$$

Moreover since $x[n] = 0$ for $n \geq 4$, we have

$$y[n] = -2y[n-1]$$

for $n \geq 6$. Therefore we can acquire

$$y[n] = -114\,(-2)^{n-5}$$

for $n \geq 6$.

Problem 8) Prob. 2.32 of Oppenheim (p.145)

Solution. (a) If $y_h[n] = A\left(\frac{1}{2}\right)^n$, then we need to verify

$$A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} = 0,$$

which is trivial.

(b) We now require that for $n \geq 0$

$$B\left(\frac{1}{3}\right)^n - \frac{1}{2}B\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n$$

Therefore, $B = -2$.

(c) We know that $y[0] = x[0] + \frac{1}{2}y[-1] = x[0] = 1$. Now we also have

$$y[0] = A + B \qquad \Rightarrow \qquad A = 1 - B = 3.$$

Problem 9) Suppose that $x[n]$ is an input signal to a causal LTI system such that $x[n] = 0$ if $n < 0$. Prove that the corresponding output signal $y[n]$ satisfies that $y[n] = 0$ for $n < 0$.

Solution. Let $h[n]$ be the impulse response function for this system. Since the system is causal and LTI, we have $h[n] = 0$ for $n < 0$ and thus

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

Noting that $x[n] = 0$ for $n < 0$, we have $y[n] = 0$ for $n < 0$.

Problem 10) In class, we prove that absolute summability of an impulse response function is a sufficient condition for BIBO stability of an LTI system. In this problem, we will show that it is also a necessary condition. Solve the Problem 2.49 of Oppenheim (p.153)

Solution. (a) Since $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, there exists an integer m such that $|h[-m]| \neq 0$. Note that for this m, $|x[m]| = 1$. Moreover it is clear that $|x[n]| \leq 1$ for all $n \in \mathbb{Z}$. Therefore the smallest value of B is 1.
(b) Since the system is LTI,

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k].$$

Define $T = \{n : |h[-n]| \neq 0\}$. Then

$$\begin{aligned} y[0] &= \sum_{k\in T} x[k]h[-k] \\ &= \sum_{k\in T} |h[-k]| \\ &= \sum_{k=-\infty}^{\infty} |h[k]| = \infty. \end{aligned}$$

Therefore we can conclude that absolute summability is a necessary condition for BIBO stability.
(c) Let $h(t)$ be the impulse response function of the LTI system. Suppose that $h(t)$ is not absolutely integrable. If we consider the input signal $x(t)$ defined by $x(t) = \begin{cases} 0 & \text{if} \quad h(-t) = 0 \\ \frac{h(-t)}{|h(-t)|} & \text{if} \quad h(-t) \neq 0 \end{cases}$, by the similar way as above, we can show that $y(0) = \infty$.

Problem 11) Prob. 2.45 of Oppenheim (p.151)

Solution. To solve this problem, we need to understand new concept which we didn't cover in the class. ($u_1(t)$ and $u_{-1}(t)$, for example) I will not grade this problem. Sorry for my mistake.