Seoul National University
Dept. of Electrical and Computer Engineering

430.523: Random Signal Theory

Spring Semester, 2018
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Quiz1 (March 29, 75 minutes)

You can use ONLY class notes. Other than that no other things (electronic device, computer, book) is allowed. The purpose of this test is to check your basic knowledge on probability.

Following quantities might be helpful in your problem solving: $e^1 = 2.7$, $e^2 = 7.4$, $e^3 = 20$, $e^{-1} = 0.4$, $e^{0.5} = 1.6$, $e^{-0.5} = 0.6, e^{-0.25} = 0.78, e^{-0.125} = 0.88$

Problem 1)[20pt] Let Y follows $B(n,p)$. Show that $E\left(\frac{1}{Y+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$.

$\Rightarrow$ By definition, we have

$$\begin{aligned}
E\left(\frac{1}{Y+1}\right) &= \sum_{y=0}^{n} \binom{n}{y} \frac{1}{y+1} p^y (1-p)^{n-y} \\
&= \frac{1}{p} \sum_{y=0}^{n} \binom{n}{y} (1-p)^{n-y} \int_0^p x^y dx \\
&= \frac{1}{p} \int_0^p \left(\sum_{y=0}^{n} \binom{n}{y} (1-p)^{n-y} x^y \right) dx \\
&= \frac{1}{p} \int_0^p (1-p+x)^n \, dx \\
&= \left. \frac{(1-p+x)^{n+1}}{(n+1)p} \right|_0^p \\
&= \frac{1-(1-p)^{n+1}}{(n+1)p}.
\end{aligned}$$

Problem 2)[20pt] A discrete random variable X and Y are expressed as follows.

	Y=-1	Y=0	Y=1	Y=2
X = 0	0	0.15	0	0.35
X = 2	0.2	0	0.3	0

(a) Are X and Y independent?

$\Rightarrow$ We have $P(X = 0) = P(X = 2) = 0.5$, $P(Y = -1) = 0.2$, $P(Y = 0) = 0.15$, $P(Y = 1) = 0.3$, and $P(Y = 2) = 0.35$. Also, we have

$$P(X = 0|Y = y) = \frac{P(X = 0, Y = y)}{P(Y = y)} = \begin{cases} 0 & \text{if } y = -1 \text{ or } y = 1 \\ 1 & \text{if } y = 0 \text{ or } y = 2 \end{cases} \tag{1}$$

Thus, $P(X = 0) \neq P(X = 0|Y = y)$ for all y. In other words, X and Y are not independent.

(b) Find $E[7X - 3Y]$.

$\Rightarrow$ We have

$$\begin{aligned} E[7X - 3Y] &= 7E[X] - 3E[Y] \\ &= 7(0P(X = 0) + 2P(X = 2)) - \\ &\quad 3(-1P(Y = -1) + 0P(Y = 0) + 1P(Y = 1) + 2P(Y = 2)) \\ &= 4.6 \end{aligned}$$

Problem 3)[20pt] In each week, you buy a lottery ticket from the store. It is known that the chance of winning the lottery is only 0.0015. In every week, you buy a ticket for 5 consecutive years (let's assume that there are 50 weeks in one year). You want to find out the probability that you win at least one time during this period.

(a) Find out the exact probability (you don't need to calculate the exact value. Just get the simplest expression).

$\Rightarrow$ Let $p = 0.0015$ and $n = 5\times 50 = 250$. Then using the Poisson distribution with $\lambda = np = 0.0015\times 250 = 3/8$, we have the probability that we win at least one time as

$$P_1 = 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-3/8}.$$

Alternatively, one might use the binomial distribution with $p = 0.0015$ and $n = 250$. In doing so, the desired probability is computed as

$$P_2 = 1 - \binom{250}{0} p^0 (1-p)^{250} = 1 - (1 - 0.0015)^{250}.$$

(b) Use a proper approximation of the random variable and get the probability.

$\Rightarrow$ We have

$$P_1 = 1 - e^{-3/8} \approx 1 - 0.88^3 \approx 1 - 0.68 = 0.32$$

The approximation error is $|1 - e^{-3/8} - 0.32| \approx 0.0073$. Note that the approximation $P_1 = 1 - e^{-3/8} \approx 1 - 20^{-1/8}$ is also acceptable.

Alternatively, we might have

$$P_2 = 1 - (1 - 0.0015)^{250} \approx 1 - (1 - 250 \times 0.0015) = 0.375$$

Problem 4)[20pt] Let X_i, $i = 1, 2, \cdots, 20$ be the Gaussian random variable with mean $\mu = 0$ and variance $\sigma^2 = 9$. Suppose $Z_1 = \min_i X_i$ and $Z_2 = \max_i X_i$.

(a) Find the $P(Z_1 \leq z)$.

$\Rightarrow$

$$\begin{aligned}
P(Z_1 \leq z) &= P(\min_i X_i \leq z) \\
&= 1 - P(\min_i X_i \geq z) \\
&= 1 - P(X_1 \geq z, X_2 \geq z, ..., X_{20} \geq z) \\
&= 1 - P(X_1 \geq z)P(X_2 \geq z)...P(X_{20} \geq z) \\
&= 1 - Q\left(\frac{z}{3}\right)^{20}.
\end{aligned}$$

(Alternatively, it can be shown that $P(Z_1 \leq z) = 1 - (1 - P(X_1 \leq z))^{20} = 1 - (1 - F_X(z))^{20}$ where $F_X(x)$ is the CDF of X_1.)

(b) Find the $P(Z_2 \leq z)$.

$\Rightarrow$

$$\begin{aligned}
P(Z_2 \leq z) &= P(\max_i X_i \leq z) \\
&= P(X_1 \leq z, X_2 \leq z, ..., X_{20} \leq z) \\
&= P(X_1 \leq z)P(X_2 \leq z)...P(X_{20} \leq z) \\
&= \left(1 - Q\left(\frac{z}{3}\right)\right)^{20}
\end{aligned}$$

(Alternatively, it can be shown that $P(Z_1 \leq z) = P(X_1 \leq z)^{20} = F_X(z)^{20}$)

Problem 5)[20pt] Each computer in a lab has a 20% chance to be infected with a virus. If a computer is infected, an antivirus software finds the virus with probability 0.9. If a computer is not infected, the software will still generate a false alarm and report a virus with probability 0.1. If the antivirus software reports a virus, what is the probability that indeed, the computer is infected?

$\Rightarrow$ Let E be the event that the software reports a virus. Also, let F be the event that this computer is infected. Then, we have

$$
\begin{aligned}
P(F|E) &= \frac{P(F,E)}{P(E)} \\
&= \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)} \\
&= \frac{0.9 \times 0.2}{0.9 \times 0.2 + 0.1 \times 0.8} \\
&= \frac{9}{13}.
\end{aligned}
$$

Problem 6)[20pt] Find out the variance of Binomial random variable X with parameter n and p.

$\Rightarrow$ It is known that $Var(X) = E[X^2] - (E[X])^2$ and $E[X] = np$. What remains is to compute

$$
\begin{aligned}
E[X^2] &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} (x + x^2 - x) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} + \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\
&= E[X] + \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\
&= E[X] + \sum_{x=2}^{n} x(x-1) \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \\
&= E[X] + \sum_{x=2}^{n} \frac{n!}{(n-x)!(x-2)!} p^x (1-p)^{n-x} \\
&= E[X] + n(n-1) \sum_{x=2}^{n} \binom{n-2}{x-2} p^x (1-p)^{n-x} \\
&= E[X] + n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} (1-p)^{n-x} \\
&\overset{(a)}{=} E[X] + n(n-1)p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^y (1-p)^{n-2-y} \\
&\overset{(b)}{=} E[X] + n(n-1)p^2,
\end{aligned}
$$

where (a) is because we denote $y = x - 2$ and (b) is because $\sum_{y=0}^{n-2} \binom{n-2}{y} p^y (1-p)^{n-2-y} = (p+1-p)^{n-2} = 1$. Thus, we have

$$Var(X) = np + n(n-1)p^2 - (np)^2 = np(1-p).$$