430.523: Random Signal Theory

Electrical and Computer Engineering, Seoul National Univ.
Spring Semester, 2018
Homework #4, Due: June 19 (Room 1117 building 301)

Note: No late homework will be accepted.

Problem 1) Let's check the interesting property of LMMSE operator.

(a) Show the following equality

$$\mathbf{H}^T(\mathbf{H}\mathbf{H}^T + \sigma^2\mathbf{I})^{-1} = (\mathbf{H}^T\mathbf{H} + \sigma^2\mathbf{I})^{-1}\mathbf{H}^T$$

(b) Run the simulation for tall and fat matrices (e.g., 500×10 vs. 10×500) for 1000 times (in each run, you need to generate the matrix at random). What can you say for the computational complexity?

Problem 2) Find the MAP estimate when we have multiple measurements $y_i = \theta + v_i$ $(i = 1, \cdots n)$.

Problem 3) Let $y_1, \cdots y_n$ be samples obtained from the exponential density with parameter θ. That is,

$$f_Y(y_i) = \frac{1}{\theta}\exp(-\frac{y_i}{\theta}).$$

Find the ML estimate of θ.

Problem 4) Let $X_t = R\cos(\omega t + \theta)$ where R is a Rayleigh RV and the θ is a uniform RV over $(0, 2\pi)$. R and θ are independent of each other.

(a) Find $\mu_X(t)$ and $R_X(s,t)$.

(b) Is X_t WSS-RP?

Problem 5) Answer to the following question.

- Let $\mathbf{U}_{all} = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n] \in \mathbb{R}^{n\times n}$ be the orthonormal matrix.
- Let $\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_r] \in \mathbb{R}^{n\times r}$ be a submatrix of $\mathbf{U}_{all}$.

- Let $\mathbf{e}_i$ be the standard basis vector (i-th element is 1 and all others are zero). For example, if $n = 4$, then $\mathbf{e}_1 = [1\ 0\ 0\ 0]$ and $\mathbf{e}_2 = [0\ 1\ 0\ 0]$, and so on.
- $\mathbf{P_U}$ is the projection matrix. That is, $\mathbf{P_U x}$ is the projection of $\mathbf{x}$ onto the space spanned by the columns of $\mathbf{U}$.

Show that $\max_i \|\mathbf{P}_U \mathbf{e}_i\|_2^2 \geq \frac{r}{n}$.

Hint: You may use the property $\max \|\mathbf{x}_i\|_2^2 \geq \frac{1}{n}\sum_{i=1}^{n} \|\mathbf{x}_i\|_2^2$.

Problem 6) Let X_i be i.i.d. with mean μ. Show that $Z_n = \sum_{k=1}^{n}(X_k - \mu)$ is Martingale process.

Problem 7) We have process with two state with the following state transition probability $P(X_k = 0|X_{k-1} = 0) = 0.55$, $P(X_k = 1|X_{k-1} = 0) = 0.45$, $P(X_k = 0|X_{k-1} = 1) = 0.3$, $P(X_k = 1|X_{k-1} = 1) = 0.7$.

(a) Find the (state) transition diagram.

(b) Find $P_{01}(3)$ using the direct calculation (in general difficult)

(c) Find $P_{01}(3)$ using the transition matrix computation (in general easy)

Hint: The state transition matrix P models $P_{ij}(1)$.

(d) What is the steady state probability (limiting probability) of this process?

Problem 8) (a) Show that WSS Gaussian random process is SSS.

(b) Show that if the input X_t to the linear system is Gaussian, then the output $Y_t = X_t * h_t$ is also Gaussian.