Seoul National University
School of Electrical and Computer Engineering

430.523: Random Signal Theory

Spring Semester, 2018
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Midterm Exam 2
May 24, 2018
75 minutes

This is closed book test. However, one A4 page cheating sheet is allowed.
Make sure to clearly show your work and full justification to get the full credit for the problem.
You have 75 minutes to finish the exam.

Please do not turn this page until requested to do so

Problem 1)[20pt] Let X be the exponential random variable with the parameters λ. Also, let $Z = \exp(-X)$. Find the PDF of Z.

Problem 2)[20pt] Let $X_1,\ X_2,\ \cdots,\ X_n$ be discrete random variables. Also, let $\Phi = \{x : p_i(x){>}0 \text{ for all } i\}$ where $p_i(x)$ be the PMF of X_i. Show the following inequalities:

(a) $D(p_1(x)||p_2(x)) \geq 0,\ x \in \Phi$

(b) $I(X_1; X_2) \geq 0$

(c) $H(X_1, X_2, \cdots, X_n) \leq \sum\limits_{i=1}^{n} H(X_i)$

Problem 3)[20pt] Let X_i $(1 \leq i \leq n)$ be independent random variables satisfying $|X_i| \leq M$, $E[X_i] = 0$, and $E[X_i^2] = \sigma_i^2$. Show the following inequalities:

(a) $P(\sum\limits_{i=1}^{n} X_i{>}t) \leq e^{-\lambda t} \prod\limits_{i=1}^{n} E[e^{\lambda X_i}]$, for any $\lambda{>}0$

(b) $E[e^{\lambda X_i}] \leq \exp\left(\frac{\sigma_i^2}{M^2}(e^{\lambda M} - 1 - \lambda M)\right)$

Hint: Note that $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ and $e^y \geq 1 + y$ $(y \geq 0)$.

Problem 4)[20pt] Let $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}^T$ be the normal random vector where $X_i \sim \mathcal{N}(0, \sigma_i^2)$.

(a) Show that $tr(\mathbf{C}) = \sum_i \lambda_i$ where λ_i is the eigenvalues of the covariance matrix $\mathbf{C}$ of $\mathbf{X}$

Hint: $tr(\mathbf{UVW}) = tr(\mathbf{WUV})$

(b) Show that $\lambda_{\max} \geq \sigma_{\min}^2$ where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{C}$ and $\sigma_{\min}^2 = \min_i \sigma_i^2$.

Problem 5)[20pt] Suppose that there are n pairs of shoes in distinct styles and sizes. The shoes are mixed up. Peter randomly selects k shoes (k might not be even). What is the expected number of matched pairs of shoes that Peter selects?

Problem 6)[20pt] Let $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^T \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ be the normal random vector with

$$\mathbf{C} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

where $0 < b < a$.

(a) Find the PDF of $X_2 + X_3$.

(b) Find the joint PDF of $X_1 + X_2$ and $X_1 - X_2$.

(c) Find the conditional PDF of $\mathbf{Z} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T$ given $X_3 = x$.

(d) Find the linear transformation $\mathbf{A}$ such that $\mathbf{Y} = \mathbf{AX} \sim \mathcal{N}(\mathbf{0}, \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix})$.