Seoul National University
School of Electrical and Computer Engineering

430.523: Random Signal Theory

Spring Semester, 2018
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Midterm Exam 1
April 17, 2018
75 minutes

This is closed book test. However, one A4 page cheating sheet is allowed.
Make sure to clearly show your work and full justification to get the full credit for the problem.
You have 75 minutes to finish the exam.

Please do not turn this page until requested to do so

Problem 1)[20pt] Let X_i ($i = 1, 2, \cdots, 10$) be the i.i.d. exponential random variables with the parameters λ_i ($i = 1, 2, \cdots, 10$), respectively. Also, let $Y = \min_i X_i$.

(a) What is the PDF of Y? Can you tell me what kind of random variable is Y?

$\Rightarrow$ We have

$$\begin{aligned} F_Y(\alpha) &= P(Y \le \alpha) \\ &= 1 - P(\min_i X_i \ge \alpha) \\ &= 1 - \prod_{i=1}^{10} P(X_i \ge \alpha) \\ &= 1 - \prod_{i=1}^{10} (1 - P(X_i \le \alpha)) \\ &= 1 - \prod_{i=1}^{10} (1 - F_{X_i}(\alpha)). \end{aligned}$$

Also, we note that $F_{X_i}(\alpha) = 1 - \exp(-\lambda_i \alpha)$. Thus, we have $F_Y(\alpha) = 1 - \exp(-(\sum_{i=1}^{10} \lambda_i)\alpha)$, which shows that Y is exponential distributed with the parameter $\sum_{i=1}^{10} \lambda_i$.

(b) Find the probability that $Y = X_1$.

$\Rightarrow$ We have

$$\begin{aligned} P(Y = X_1) &= P(X_1 \le X_2, X_1 \le X_3, \cdots, X_1 \le X_{10}) \\ &= \int_0^\infty \left(\int_{x_1}^\infty \lambda_2 \exp(-\lambda_2 x_2) dx_2 \right) \cdots \left(\int_{x_1}^\infty \lambda_{10} \exp(-\lambda_{10} x_{10}) dx_{10} \right) \lambda_1 \exp(-\lambda_1 x_1) dx_1 \\ &= \int_0^\infty \exp(-\lambda_2 x_1) \cdots \exp(-\lambda_{10} x_1) \lambda_1 \exp(-\lambda_1 x_1) dx_1 \\ &= \int_0^\infty \lambda_1 \exp(-(\lambda_1 + \cdots + \lambda_{10}) x_i) dx_1 \\ &= \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_{10}} \int_0^\infty (\lambda_1 + \cdots + \lambda_{10}) \exp(-(\lambda_1 + \cdots + \lambda_{10}) x_1) dx_1 \\ &= \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_{10}}. \end{aligned}$$

Problem 2)[20pt] Let X and Y be two jointly continuous random variables with the joint PDF as

$$f_{X,Y}(x,y) = \begin{cases} e^{-x} + \lambda y^2 & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{else} \end{cases}$$

(a) Find λ.

$\Rightarrow$ We have

$$\begin{aligned}
1 &= \int_0^1 \int_x^1 (e^{-x} + \lambda y^2) dy dx \\
&= \int_0^1 \left((e^{-x} y + \frac{\lambda}{3} y^3)\Big|_x^1 \right) dx \\
&= \int_0^1 \left((e^{-x} y + \frac{\lambda}{3} y^3)\Big|_x^1 \right) dx \\
&= \int_0^1 (e^{-x} - x e^{-x} + \frac{\lambda}{3} - \frac{\lambda}{3} x^3) dx \\
&= (x e^{-x} + \frac{\lambda}{3} x - \frac{\lambda}{12} x^4)\Big|_0^1 \\
&= e^{-1} + \frac{\lambda}{4}.
\end{aligned}$$

Thus, we have $\lambda = 4 - 4e^{-1}$.

(b) What is $P(X \le \frac{Y}{2})$?

$\Rightarrow$ We have

$$\begin{aligned}
P(X \le \frac{Y}{2}) &= \int_0^1 \int_0^{y/2} (e^{-x} + \lambda y^2) dx dy \\
&= \int_0^1 \left((-e^{-x} + \lambda y^2 x)\Big|_0^{y/2} \right) dy \\
&= \int_0^1 (1 - e^{-y/2} + \frac{\lambda}{2} y^3) dy \\
&= (y + 2e^{-y/2} + \frac{\lambda}{8} y^4)\Big|_0^1 \\
&= 2e^{-1/2} - 1 + \frac{\lambda}{8} \\
&= 2e^{-1/2} - \frac{1 + e^{-1}}{2} \approx 0.5291.
\end{aligned}$$

(c) Are X and Y independent? Justify your answer in details.

$\Rightarrow$ We have

$$
\begin{aligned}
f_X(x) &= \int_x^1 e^{-x} + \lambda y^2 dy \\
&= (e^{-x}y + \frac{\lambda}{3}y^3)\Big|_x^1 \\
&= e^{-x} - xe^{-x} + \frac{\lambda}{3} - \frac{\lambda}{3}x^3.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
f_Y(y) &= \int_0^y e^{-x} + \lambda y^2 dx \\
&= -e^{-x} + \lambda y^2 x\Big|_0^y \\
&= 1 - e^{-y} + \lambda y^3.
\end{aligned}
$$

Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

Problem 3)[20pt] Let X be a random variable. Suppose that the moment generating function $M_X(t)$ of X is $M_X(t) = \exp(e^t - 1)$. Show that

$$P(X \geq \alpha) \leq \exp(-\alpha \ln(\alpha) + \alpha - 1),\ \alpha \geq 1.$$

$\Rightarrow$ For $t \geq 0$, we have

$$
\begin{aligned}
P(X \geq \alpha) &= P(e^{tX} \geq e^{t\alpha}) \\
&\leq \frac{E[e^{tX}]}{e^{t\alpha}} \\
&= \frac{M_X(t)}{e^{t\alpha}} \\
&= \exp(e^t - 1 - t\alpha).
\end{aligned}
$$

Now let $g(t) = \exp(e^t - 1 - t\alpha)$ and $t^* = \arg\min_t g(t)$, then we have

$$P(X \geq \alpha) \quad \leq \quad g(t^*).$$

Also, we have $g^{'}(t) = \exp(e^t - 1 - t\alpha)(e^t - \alpha)$ and $g^{''}(t) = e^t \geq 0$. Thus, $t^* = \ln(\alpha)$ and $g(t^*) = \exp(\alpha - 1 - \alpha\ln(\alpha))$, which is the desired result.

Problem 4)[20pt] Let X and Y be discrete random variables. Show that

$$E[YE[X \mid Y]] = E[E[XY \mid Y]].$$

$\Rightarrow$ First, note that

$$\begin{aligned} E_Y[YE_X[X|Y]] &= \sum_y yE[X|Y=y]P(Y=y) \\ &= \sum_y y(\sum_x xP(X=x|Y=y)P(Y=y) \\ &= \sum_x \sum_y xyP(X=x, Y=y) \\ &= E[XY]. \end{aligned}$$

Next, also note that

$$\begin{aligned} E_y[E_x[XY|Y]] &= \sum_y E[Xy|Y=y]P(Y=y) \\ &= \sum_y \sum_x xyP(X=x|Y=y)P(Y=y) \\ &= \sum_x \sum_y xyP(X=x, Y=y) \\ &= E[XY]. \end{aligned}$$

Thus, we have $E_Y[YE_X[X|Y]] = E_y[E_x[XY|Y]]$, which is the desired result.

Problem 5)[20pt] Let $f_{X,Y}(x,y)$ be given by

	Y=0	Y=1
X = 0	$\frac{1}{4}$	$\frac{1}{4}$
X = 1	0	$\frac{1}{2}$

Find:

(a) $H(X), H(Y)$.

$\Rightarrow$ We have $P(X=0) = P(X=1) = \frac{1}{2}$ and $P(Y=0) = \frac{1}{4}$ $P(Y=1) = \frac{3}{4}$, and

$$\begin{aligned}
H(X) &= -\sum_x P(X=x)\log(P(X=x)) \\
&= -\frac{1}{2}\log(\frac{1}{2}) - \frac{1}{2}\log(\frac{1}{2}) \\
&= 1, \\
H(Y) &= -\sum_y P(Y=y)\log(P(Y=y)) \\
&= -\frac{1}{4}\log(\frac{1}{4}) - \frac{3}{4}\log(\frac{3}{4}) \\
&= \frac{1}{2} + \frac{3}{4}\log(\frac{4}{3}).
\end{aligned}$$

(b) $H(X,Y)$.

$\Rightarrow$ We have

$$\begin{aligned}
H(X,Y) &= -\sum_x\sum_y P(X=x,Y=y)\log(P(X=x,Y=y)) \\
&= -\frac{1}{4}\log(\frac{1}{4}) - \frac{1}{4}\log(\frac{1}{4}) - \frac{1}{2}\log(\frac{1}{2}) \\
&= \frac{3}{2}.
\end{aligned}$$

(c) $H(X \mid Y), H(Y \mid X)$.

$\Rightarrow$ We have

$$\begin{aligned}
H(X|Y) &= H(X,Y) - H(Y) = 1 - \frac{3}{4}\log(\frac{4}{3}), \\
H(Y|X) &= H(X,Y) - H(X) = \frac{1}{2}.
\end{aligned}$$

(d) $I(X;Y)$.

$\Rightarrow$ We have

$$I(X;Y) = H(Y) - H(Y \mid X) = \frac{3}{4}\log(\frac{4}{3}).$$

Problem 6)[20pt] Let a_i and b_i $(i = 1, 2, \cdots, n)$ be positive numbers. Show that

$$\sum_{i=1}^{n} a_i \ln(\frac{a_i}{b_i}) \geq (\sum_{i=1}^{n} a_i) \ln \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i},$$

$\Rightarrow$ First, we note that the function $f(t) = t\ln(t)$ is convex since $f^{''}(t) = \frac{1}{t}>0$ when $t>0$. Then, we apply Jensen's inequality as

$$\sum_i \lambda_i f(t_i) \geq f(\sum_i \lambda_i t_i),$$

where λ_i are some real values satisfying $\lambda_i \geq 0$ and $\sum\limits_i \lambda_i = 1$. Finally, setting $\lambda_i = \frac{b_i}{\sum_i b_i}$ and $t_i = \frac{a_i}{b_i}$, we have

$$\begin{aligned} \sum_i \frac{a_i}{\sum_i b_i} \ln(\frac{a_i}{b_i}) &\geq \sum_i \frac{a_i}{\sum_i b_i} \ln(\sum_i \frac{a_i}{\sum_i b_i}), \\ \sum_i a_i \ln(\frac{a_i}{b_i}) &\geq \sum_i a_i \ln(\frac{\sum_i a_i}{\sum_i b_i}), \end{aligned}$$

which is the desired result.