430.523: Random Signal Theory

Electrical and Computer Engineering, Seoul National Univ.
Spring Semester, 2018
Homework #1, Due: In class @ March 29

Note: No late homework will be accepted.

Problem 1) Show that $P(\cup_i E_i) \leq \sum_i P(E_i)$

$\Rightarrow$ Since $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ for any E_1 and E_2 (proved in the class), we have

$$P(E_1 \cup E_2) \leq P(E_1) + P(E_2). \tag{1}$$

Applying (1) recursively, we yield

$$\begin{aligned} P(\bigcup_{i=1}^{n} E_i) &\leq P(\bigcup_{i=2}^{n} E_i) + P(E_1) \\ &\leq P(\bigcup_{i=3}^{n} E_i) + P(E_2) + P(E_1) \\ &\leq P(\bigcup_{i=k}^{n} E_i) + \sum_{j=1}^{k-1} P(E_j) \quad (k<n) \\ &\leq \sum_{j=1}^{n} P(E_j). \end{aligned}$$

Problem 2) Show that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

$\Rightarrow$ By definition, we have

$$\begin{aligned} \binom{n}{r} &= \frac{n!}{(n-r)!r!} \\ &= \frac{(n-1)!(r+n-r)}{(n-r)!r!} \\ &= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} \\ &= \binom{n-1}{r-1} + \binom{n-1}{r}. \end{aligned}$$

Problem 3) A random variable X is called to have gamma distribution with parameters (α, λ),

$\alpha > 0, \lambda > 0$, if its density function is given by

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \quad x \geq 0 \tag{2}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$.

Show that $Var[X] = \frac{\alpha}{\lambda^2}$

$\Rightarrow$ First, we note that $\Gamma(n) = (n-1)!$. In fact, from $\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$, we obtain $\Gamma(1) = \int_0^\infty e^{-y} dy = -e^{-y}\Big|_0^\infty = 1$. Also, we have

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty e^{-y} y^{n-1} dy \\
&= -e^{-y} y^{n-1}\Big|_0^\infty + (n-1)\int_0^\infty e^{-y} y^{n-2} dy \\
&= (n-1)\Gamma(n-1) \\
&= (n-1)(n-2)...\Gamma(1) \\
&= (n-1)!.
\end{aligned}$$

Second, we compute $E[X]$ as

$$\begin{aligned}
E[X] &= \int_0^\infty x f_X(x) dx \\
&= \int_0^\infty x \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx \\
&= -(\lambda x)^{\alpha-1} x \frac{e^{-\lambda x}}{\Gamma(\alpha)}\Big|_0^\infty + \alpha \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx \\
&= \frac{\alpha}{\lambda \Gamma(\alpha)} \int_0^\infty e^{-\lambda x}(\lambda x)^{\alpha-1} d(\lambda x) \\
&= \frac{\alpha}{\lambda \Gamma(\alpha)} \Gamma(\alpha) = \frac{\alpha}{\lambda}.
\end{aligned}$$

Finally, we compute $E[X^2]$ as

$$\begin{aligned}
E[X^2] &= \int_0^\infty x^2 f_X(x) dx \\
&= \int_0^\infty x^2 \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx \\
&= x^2 \frac{e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\Big|_0^\infty + \frac{\alpha+1}{\lambda} \int_0^\infty x \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx \\
&= \frac{\alpha+1}{\lambda} E[X].
\end{aligned}$$

Therefore, we obtain $Var[X] = \frac{\alpha+1}{\lambda} E[X] - E^2[X] = \frac{\alpha}{\lambda^2}$.

Problem 4) Show that the pdf of Gaussian RV X is valid pdf. You need to show that integration of $f_X(x)$ for all real line (i.e., $x \in (-\infty, \infty)$) should be 1.

$\Rightarrow$ For simplicity's sake (without loss of generality), we consider the standard Gaussian distributed RV X with the pdf

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

To show that $f_X(x)$ is valid pdf, we need to show the integration of $f_X(x)$ for all real line should be 1. Denote $I = \int_{-\infty}^{\infty} f_X(x)dx$, we have

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy. \end{aligned}$$

Transform the integral (3) into polar coordinate with $x = rCos(\phi)$, $y = rSin(\phi)$, and the Jacobian matrix $J = \begin{bmatrix} Cos(\phi) & -rSin(\phi) \\ Sin(\phi) & rCos(\phi) \end{bmatrix}$, we obtain

$$\begin{aligned} I^2 &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}|J|drd\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}rdrd\phi \\ &= \int_0^{\infty} e^{-r^2/2}d(r^2/2) \\ &= -e^{-r^2/2}\Big|_0^{\infty} = 1 \end{aligned}$$

Problem 5) Show that the variance of the Binomial random variable Z with parameter n, p (i.e., $B(n,p)$) is $Var(Z) = np(1-p)$

$\Rightarrow$ It is known that $Var(X) = E[X^2] - (E[X])^2$ and $E[X] = np$. What remains is to compute

$$
\begin{aligned}
E[X^2] &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} (x + x^2 - x) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} + \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\
&= E[X] + \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\
&= E[X] + \sum_{x=2}^{n} x(x-1) \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \\
&= E[X] + \sum_{x=2}^{n} \frac{n!}{(n-x)!(x-2)!} p^x (1-p)^{n-x} \\
&= E[X] + n(n-1) \sum_{x=2}^{n} \binom{n-2}{x-2} p^x (1-p)^{n-x} \\
&= E[X] + n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} (1-p)^{n-x} \\
&\overset{(a)}{=} E[X] + n(n-1)p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^y (1-p)^{n-2-y} \\
&\overset{(b)}{=} E[X] + n(n-1)p^2,
\end{aligned}
$$

where (a) is because we denote $y = x - 2$ and (b) is because $\sum_{y=0}^{n-2} \binom{n-2}{y} p^y (1-p)^{n-2-y} = (p + 1 - p)^{n-2} = 1$. Thus, we have

$$Var(X) = np + n(n-1)p^2 - (np)^2 = np(1-p).$$

Problem 6) Let Y follows $B(n, p)$. Show that $E\left(\frac{1}{Y+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$

$\Rightarrow$ By definition, we have

$$\begin{aligned}
E\left(\frac{1}{Y+1}\right) &= \sum_{y=0}^{n}\binom{n}{y}\frac{1}{y+1}p^y(1-p)^{n-y} \\
&= \frac{1}{p}\sum_{y=0}^{n}\binom{n}{y}(1-p)^{n-y}\int_0^p x^y dx \\
&= \frac{1}{p}\int_0^p\left(\sum_{y=0}^{n}\binom{n}{y}(1-p)^{n-y}x^y\right)dx \\
&= \frac{1}{p}\int_0^p(1-p+x)^n\,dx \\
&= \left.\frac{(1-p+x)^{n+1}}{(n+1)p}\right|_0^p \\
&= \frac{1-(1-p)^{n+1}}{(n+1)p}.
\end{aligned}$$

Problem 7) Let T be the random variable that takes on all positive real t. Show that if $P(t_0 \le T \le t_0+t_1|T \ge t_0) = P(T \le t_1)$ for all t_0 and t_1, then $P(T \le t_1) = 1-e^{-ct_1}$.

$\Rightarrow$ We have

$$\begin{aligned}
P(T \le t_1) &= P(t_0 \le T \le t_0+t_1|T \ge t_0) \\
&= \frac{P(t_0 \le T \le t_0+t_1, T \ge t_0)}{P(T \ge t_0)} \\
&= \frac{P(t_0 \le T \le t_0+t_1)}{P(T \ge t_0)} \\
&= \frac{P(T \ge t_0)-P(T \ge t_0+t_1)}{P(T \ge t_0)}.
\end{aligned}$$

Define $G(t) = P(T \ge t)$, then we have

$$1-G(t_1) = \frac{G(t_0)-G(t_0+t_1)}{G(t_0)}.$$

Thus, we have $G(t_0)G(t_1) = G(t_0+t_1)$. Let $g(t) = ln(G(t))$, then we obtain the Cauchy functional equation

$$g(t_0)+g(t_1) = g(t_0+t_1),$$

whose solution is $g(t) = -ct$ for some constant c. Since $g(t) \le 1$ for any t, the constant c is positive. Therefore, we have $P(T \le t_1) = 1-P(T \ge t_1) = 1-e^{-ct_1}$.

Problem 8) Suppose a jar contains $2N$ cards, two of them marked 1, two marked 2, and so on. Draw out m cards at random. What is the expected number of pairs that still remain in the jar?

Hint: this problem is posed and solved by D. Bernoulli, the great mathematician in 18th century. You may define a Bernoulli random variable X_i that takes on value 1 when i-th pair remains in the jar and 0 otherwise.

$\Rightarrow$ Define the Bernoulli RV X_i $(i = 1, 2, ..., N)$ that takes on value 1 when i-th pair remains in the jar and 0 otherwise. The expected number of pairs that still remain in the jar is

$$\begin{aligned} E\left[\sum_{i=1}^{N} X_i\right] &= \sum_{i=1}^{N} E[X_i] \\ &= \sum_{i=1}^{N} Pr(X_i = 1). \end{aligned}$$

Since X_i takes on value 1 when i-th pair remains in the jar, we have

$$Pr(X_i = 1) = \frac{\binom{2N-2}{m}}{\binom{2N}{m}} = \frac{(2N-m)(2N-m-1)}{2N(2N-1)}.$$

Therefore, the expected number is $\frac{(2N-m)(2N-m-1)}{2(2N-1)}$.

Problem 9) Find out the expected value of the Rayleigh random variable R whose density function is given by

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

$\Rightarrow$ We have

$$\begin{aligned} E[R] &= \int_0^{\infty} \frac{r^2}{\sigma^2} e^{-r^2/(2\sigma^2)} dr \\ &= -re^{-r^2/(2\sigma^2)}\Big|_0^{\infty} + \int_0^{\infty} e^{-r^2/(2\sigma^2)} dr \\ &= \int_0^{\infty} e^{-r^2/(2\sigma^2)} dr \end{aligned}$$

From the Problem 2, we prove that $\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} = 1$. Substitute $r = \sigma x$ into (3), then we have

$$\begin{aligned} E[R] &= \sigma \int_0^{\infty} e^{-x^2/2} dx \\ &= \sqrt{\frac{\pi}{2}}\sigma \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \sqrt{\frac{\pi}{2}}\sigma. \end{aligned}$$

Problem 10) Show that

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

$\Rightarrow$ Since for any x, one can write $x^2 = x(x-1) + x = \frac{1}{3}((x+1)x(x-1) - x(x-1)(x-2)) + x$, we have

$$\begin{aligned}
\sum_{x=1}^{n} x^2 &= \sum_{x=1}^{n} x(x-1) + \sum_{x=1}^{n} x \\
&= \frac{1}{3}\left(\sum_{x=1}^{n}(x+1)x(x-1) - \sum_{x=1}^{n} x(x-1)(x-2)\right) + \sum_{x=1}^{n} x \\
&= \frac{1}{3}\left(\sum_{x=1}^{n}(x+1)x(x-1) - \sum_{x=0}^{n-1}(x+1)x(x-1)\right) + \sum_{x=1}^{n} x \\
&= \frac{1}{3}\left((n+1)n(n-1) - (0+1)0(0-1)\right) + \sum_{x=1}^{n} x \\
&= \frac{1}{3}(n+1)n(n-1) + \sum_{x=1}^{n} x.
\end{aligned}$$

The remain is to compute $I = \sum_{x=1}^{n} x$. We have

$$\begin{aligned}
2I &= (1+n) + (2+n-1) + (3+n-2) + ... + (n+1) \\
&= (n+1)n.
\end{aligned}$$

Thus, $\sum_{x=1}^{n} x^2 = \frac{1}{3}(n+1)n(n-1) + \frac{1}{2}(n+1)n = \frac{1}{6}n(n+1)(2n+1)$.