☒ THEORETICAL PHYSICS (A)
☐ THEORETICAL PHYSICS OF FUNDAMENTAL INTERACTIONS (A)
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Exercise 1.

Use the (4×4)-generators J_i and K_i of the Lorentz group, to evaluate the following commutators:

1. $[J_2, K_3]$
2. $[K_1, K_3]$
3. $[J_3, J_2]$

Use the *antisymmetric* tensor $M_{\mu\nu}$, whose elements are defined as $M_{0i} \equiv K_i$ and $M_{ij} \equiv \epsilon_{ijk} J_k$ (with $i, j = 1, 2, 3$), to evaluate:

4. $(1/2)\, M^{\mu\nu} M_{\mu\nu}$
5. $(1/2)\, \epsilon^{\mu\nu\rho\sigma} M^{\mu\nu} M_{\rho\sigma}$

Exercise 2.
In Classical Field Theory, consider a generic Lagrangean density $\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$, invariant under Lorentz transformations. Consider the infinitesimal transformation $\phi(x) \to \phi'(x') = S(\Lambda)\phi(x)$, with $\Lambda_{\mu\nu} = g_{\mu\nu} + \delta\omega_{\mu\nu}$ and $S(\Lambda) = \mathbb{I} + (1/2)\delta\omega_{\mu\nu}(I^{\mu\nu})$.

1. Write the Nöther current f_μ in terms of the generalised conjugated momentum $\pi_\mu \equiv \partial\mathcal{L}/\partial(\partial^\mu\phi)$ and of the energy-momentum tensor $\theta_{\mu\nu}$ (the latter can be also written in terms of π_μ and $g_{\mu\nu}$).
2. Show that f_μ can be written as $f_\mu = (1/2)\, \delta\omega^{\nu\rho}\, M_{\mu\nu\rho}$, where $M_{\mu\nu\rho}$ is expressed in terms of π_μ and $\theta_{\mu\nu}$.
3. Analyze the conserved quantities $M_{ij} = \int d^3\vec{x}\, M_{0ij}$ and relate them to physical quantities.
4. Consider the generic charge $Q = \int d^3\vec{x}\, f_0$, and compute the Poisson bracket $\{\phi, Q\}_{P.B.}$. What does it represent?

Exercise 3.
Consider the Dirac γ-matrices, and use their fundamental relation to compute:

1. $\gamma_\mu \gamma^\mu$;
2. $\gamma_\mu \gamma^\nu \gamma^\mu$;
3. $\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu$;
4. $\sigma_{\mu\nu}\, \sigma^{\mu\nu}$.

Exercise 4.
(a). Consider the Langrangean density

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 - ig\,\bar{\psi}\gamma_5\psi\phi$$

where the mass m, and the coupling constants λ and g are real constants.

1. Derive the equation of motion of the fileds $\psi, \bar{\psi}$ and ϕ.
2. Derive the propagators of ψ and ϕ , and intoduce a diagrammatic rule for them
3. Identify the interaction terms and define a graphic rule for them.

(b). Consider the Lagrangean density for a charged Klein-Gordon field,

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

1. Identify the global symmetry of this theory, and, in analogy with QED, build the Lagrangean density of the corresponding gauge theory.

Exercise 5.

The Bhabha scattering is the process $e^-(p_1, s_1) + e^+(p_2, s_2) \to e^-(p_3, s_3) + e^+(p_4, s_4)$, where the external fermions are on the mass shell, $p_i^2 = m^2$ $(i = 1, \ldots, 4)$. The fermion and photon fields are defined as,

$$\begin{aligned}
\psi(x) &= \int d^3\vec{p}\, N_p \sum_s \left(b_{ps}\, u(p,s)\, e^{-ip\cdot x} + d^\dagger_{ps}\, v(p,s)\, e^{+ip\cdot x} \right) = \psi^{(+)}(x) + \psi^{(-)}(x) \\
\bar{\psi}(x) &= \int d^3\vec{p}\, N_p \sum_s \left(d_{ps}\, \overline{v}(p,s)\, e^{-ip\cdot x} + b^\dagger_{ps}\, \overline{u}(p,s)\, e^{+ip\cdot x} \right) = \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x) \\
\hat{A}_\mu(x) &= \int d^3\vec{k}\, N_k \sum_\lambda \left(a_{k\lambda}\, \varepsilon^\mu\, e^{-ik\cdot x} + a^\dagger_{k\lambda}\, \varepsilon^{*\mu}\, e^{+ik\cdot x} \right) = A^{(+)}_\mu(x) + A^{(-)}_\mu(x)
\end{aligned}$$

with

$$N_p \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E_p}}\,, \qquad N_k \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{1}{2\omega_k}}$$

1. Consider the S-matrix expansion at the second order in perturbation theory, and identify the terms which contribute to the Bhabha scattering

   $$S_{fi} = \langle f | S^{(2)} | i \rangle = \langle f | S^{(2)}_{e^+e^- \to e^+e^-} | i \rangle$$

   expressing its integrand in terms of propagators, and positive/negative-frequency fields.

2. Using the Feynman rules in *coordinate space*, Compute S_{fi}, expressing the results in terms of spinors and on the 4-momenta of the particles.
   Hints:
   - keep the normalization factors symbolically till the end;
   - express S_{fi} as an integral of a vacuum expectation value (v.e.v.) of the creation and annihilation operators;
   - use the Fourier decomposition of the propagators.

3. Re-compute the S_{fi} by using the Feynman rules in *momentum-space*, and compare its expression with the earlier result.

Exercise 6.
Use the solutions of the Dirac equation in momentum space,

$$u_r(\vec{p}) = c\,(\not{p} + m) u_r(\vec{0})\,, \qquad v_r = c\,(-\not{p} + m) v_r(\vec{0})\,, \quad (r = 1, 2)\,,$$

where $p^\mu = (p_0, \vec{p})$, and $c = 1/\sqrt{2m(p_0 + m)}$, to verify that:

1. $\sum_{r=1,2} u_r(\vec{p}) \bar{u}_r(\vec{p}) = (\not{p} + m)/(2m)$, corresponding to the positive-energy projector, Λ_+.
2. $\sum_{r=1,2} v_r(\vec{p}) \bar{v}_r(\vec{p}) = (\not{p} - m)/(2m)$, corresponding to opposite of the negative-energy projector, Λ_-.

LORENZO DOMENICHETTI

Ex 1

4x4 Generators J_i, K_i of the Lorentz Group to evaluate the following commutators:

$$J_1 \equiv \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&+i&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&+i&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0&-i&0&0\\ -i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0&0&-i&0\\ 0&0&0&0\\ -i&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ -i&0&0&0 \end{pmatrix}$$

$$\hookrightarrow [J_2, K_3] \equiv J_2K_3 - K_3J_2 \equiv \begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ -i&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ -i&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0\\ +1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&-1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&1&0&0\\ +1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

Expected: $[J_i, K_j] \equiv i\epsilon_{ijk}K_k \Rightarrow [J_2, K_3] = i\underbrace{\epsilon_{231}}_{=+1} K_1 = i \begin{pmatrix} 0&-i&0&0\\ -i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$ ok ✓.

$$[K_1, K_3] \equiv \begin{pmatrix} 0&-i&0&0\\ -i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ -i&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ -i&0&0&0 \end{pmatrix}\begin{pmatrix} 0&-i&0&0\\ -i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}$$

Expected: $[K_i, K_j] = -i\epsilon_{ijk}J_k \Rightarrow [K_1, K_3] = -i\underbrace{\epsilon_{132}}_{=-1} J_2 = +i \begin{pmatrix} 0&0&0&0\\ 0&0&0&+i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}$ ok ✓

OBS: Wrong sign on lecture notes

$$[J_3, J_2] \equiv \begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&+i&0&0\\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&+i&0&0\\ 0&0&0&0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&-1&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&+1&0 \end{pmatrix}$$

Expected: $[J_i, J_j] \equiv i\epsilon_{ijk}J_k \Rightarrow [J_3, J_2] = i\cdot\underbrace{\epsilon_{321}}_{=-1} J_1 = -i \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&+i&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&+1&0 \end{pmatrix}$ ok ✓.

LD

$C_2 = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma}$. $M_{0i} = K_i$, $M_{ij} = \epsilon_{ijk} J_k$

Idea: just ONE index at a time can be 0! Others are "spatial" indices (ijk) (can be 1,2,3)

$$\Rightarrow C_2 = \frac{1}{2}\left[\epsilon^{0ijk} M_{0i}M_{jk} + \epsilon^{i0jk} M_{i0}M_{jk} + \epsilon^{ij0k} M_{ij}M_{0k} + \epsilon^{ijk0} M_{ij}M_{k0}\right]$$

$$= \frac{1}{2}\left[\epsilon^{0ijk} M_{0i}M_{jk} + \epsilon^{0ijk} \underbrace{M_{i0}}_{=-M_{0i}} M_{jk} + \epsilon^{ij0k} M_{ij}M_{0k} + \epsilon^{ij0k} M_{ij}\underbrace{M_{k0}}_{=-M_{0k}}\right]$$

$$= \frac{1}{2}\epsilon^{0ijk}\left(M_{0i}M_{jk} + M_{0i}M_{jk}\right) + \frac{1}{2}\underbrace{\epsilon^{ij0k}}_{=\epsilon^{0ijk}}\left(M_{ij}M_{0k} + M_{ij}M_{0k}\right)$$

2 exchanges

$$= \epsilon^{ijk}\left(K_i \epsilon_{jkl} J_l\right) + \epsilon^{ijk}\left(\epsilon_{ijm} J_m K_k\right)$$

$= \epsilon^{jki}$

If we now remember $\epsilon^{ijk}\epsilon_{ijl} = -2\delta^k_l$

upper vs lower indices!

$$= -2\delta^i_l K_i J_l - 2\delta^k_m J_m K_k$$

$$= -2\cdot 2\cdot \vec{J}\cdot\vec{K} = -4\vec{J}\cdot\vec{K}$$

$C_1 = \frac{1}{2}M^{\mu\nu}M_{\mu\nu}$ ($\Rightarrow$ $M^{i0} = -M_{i0}$, $M^{ij} = M_{ij}$

one spatial component changed

2 spatial component changed

$$= \frac{1}{2}\left[M^{0i}M_{0i} + M^{i0}M_{i0} + \underbrace{M^{ij}}_{=M_{ij}} M_{ij}\right]$$

$$= \frac{1}{2}\left[2M^{i0}M_{i0} + \epsilon_{ijk}\epsilon_{ijl} J_k J_l\right]$$

$$= \frac{1}{2}\left[-2\vec{K}^2 + 2\delta_{kl} J_k J_l\right] = \vec{J}^2 - \vec{K}^2 \quad \text{ok} \checkmark$$

$\mathcal{L}\mathcal{D}$

Esercizio 2:

Consider a Lagrangian Density $\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$, invariant under Lorentz transformation

Consider the inf. tfm $\phi(x) \to \phi'(x') = S(\Lambda)\phi(x)$, $\Lambda_{\mu\nu} \equiv g_{\mu\nu} + \delta\omega_{\mu\nu}$

$S(\Lambda) \equiv \mathbb{1} + \frac{1}{2} \cdot \delta\omega_{\mu\nu} I^{\mu\nu}$

1) $f^\mu := \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}(\delta\phi - \partial_\nu \phi \, \delta x^\nu) + \mathcal{L} \cdot \delta x^\mu$

$\Rightarrow f_\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \cdot \delta\phi - \left(\frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \partial_\nu \phi - \mathcal{L} g_{\mu\nu} \right) \cdot \delta x^\nu$

$\left(\pi_\mu := \text{generalised momentum} \left(= \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \right) \right)$

$\equiv \pi_\mu \delta\phi - \theta_{\mu\nu} \delta x^\nu$, and $\theta_{\mu\nu} \equiv \pi_\mu \cdot \partial_\nu \phi - \mathcal{L} g_{\mu\nu}$

2) Given the transformation (Lorentz) (infinitesimal), $\phi'(x') = \phi(x) + \delta\phi$, with $\delta\phi \equiv \frac{1}{2} \delta\omega_{\mu\nu} I^{\mu\nu} \phi(x)$ (↳ Generators.)

↳ The transformation for coordinates $x'^\mu = x^\mu + \delta\omega^{\mu\nu} x_\nu \iff \delta x^\mu \equiv \delta\omega^{\mu\nu} x_\nu$

$\Rightarrow f_\mu \equiv \pi_\mu \delta\phi - \theta_{\mu\nu} \delta x^\nu = \pi_\mu \cdot \frac{1}{2} \delta\omega_{\nu\lambda} I^{\nu\lambda} \phi - \theta_{\mu\nu} \delta\omega^{\nu\lambda} x_\lambda$

$\equiv \frac{1}{2} \pi_\mu \delta\omega^{\nu\lambda} I_{\nu\lambda} \phi - \left(\frac{1}{2} \delta\omega^{\nu\lambda} (\theta_{\mu\nu} x_\lambda - \theta_{\mu\lambda} x_\nu) \right)$

(Metric is invariant under tfm)

OK ✓

$\underline{\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}}$: Antisymmetric

$\stackrel{!}{=} \frac{1}{2} \delta\omega^{\nu\lambda} (\pi_\mu I_{\nu\lambda} \phi + \theta_{\mu\lambda} x_\nu - \theta_{\mu\nu} x_\lambda) \equiv \frac{1}{2} \delta\omega^{\nu\lambda} M_{\mu\nu\lambda}(x)$

3) Nöther thm $\Rightarrow$ We have 6 conserved quantities

↳ $M_{\nu\lambda} := \int d^3\vec{x} \, M_{0\nu\lambda}$

↳ $M_{0\nu\lambda} \equiv \theta_{0\lambda} x_\nu - \theta_{0\nu} x_\lambda + \pi I_{\nu\lambda} \phi$

($M_{nl} \simeq$ space comp. of ang. mom. Tensor.
↳ can be mapped onto a 3-dim vector)

$\equiv -\mathcal{L} g_{0\lambda} x_\nu + \pi \partial_\lambda \phi x_\nu + \mathcal{L} g_{0\nu} x_\lambda - \pi \partial_\nu \phi x_\lambda + \pi I_{\nu\lambda} \phi$

$\Rightarrow$ Space-Components : $\nu, \lambda \equiv (n, l) = (1,2,3)$.

↳ $M_{nl} = L_{nl} + S_{nl}$, $L_{nl} \equiv \int d^3\vec{x} \cdot (x_n \theta_{0l} - x_l \theta_{0n}) \stackrel{\text{no } g}{\equiv} \int d^3x \cdot \pi \underbrace{(x_n \partial_l - x_l \partial_n)}_{\sim (\vec{x} \times \vec{\nabla})} \phi$ ORBITAL

↳ ANGULAR MOMENTUM

SPIN : = $S_{nl} \equiv \int d^3\vec{x} \, \pi \cdot I_{nl} \phi$

↳ spin depends on the considered fields.
↳ and their intrinsic properties

TOTAL ANGULAR MOMENTUM $J_m := \frac{1}{2} \epsilon_{mnl} M_{nl}$

↳ 3 conserved components.

Not $\vec{L}$ or $\vec{S}$, but only their SUM is conserved under rotations

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Mixed Components $\nu = i,\ \lambda = \phi$

↳ $M_{i0} \equiv L_{i0} + S_{i0}$

$$L_{i0} \equiv \int d^3x \cdot (x^i \theta_{00} - \underbrace{t}_{x^0} \theta_{0i}) =$$

$$\equiv \int d^3x \left[(\pi \cdot \partial_0\phi \cdot x_i - x_i \mathcal{L} \cdot g_{00}) - t \cdot \pi \cdot \partial_i \phi \right]$$

$$\equiv \int d^3x\ x_i (\pi \dot{\phi} - \mathcal{L}) - t P_i \equiv \int d^3\vec{x} \cdot x_i \cdot \underbrace{\mathcal{H}}_{\text{Ham. density}} - t P_i$$

↳ Related to center of mass motion. (and its invariance properties)

$S_{i0} \equiv \int d^3x \cdot \pi\, I_{i0}\, \phi$ ⇝ Boost generators, not interesting.

Generic Charge: $Q \equiv \int_V d^3x\ \underbrace{J_0}_{\text{nöther current } (f_0)} \equiv \int d^3x \cdot (\pi \delta\phi - (\pi \partial_\nu \phi - g_{0\nu} \mathcal{L}) \delta x^\nu)$ $\boxed{\text{ok}\checkmark}$

$$\frac{\delta}{\delta\phi(x)} \int d^3x\, F(x) \equiv \frac{\partial F}{\partial \phi}(x)$$

Calculating the P.B: $\{\phi, Q\}_B =$

$$= \int d^3\vec{x}' \Bigg(\underbrace{\frac{\delta\phi(\vec{x})}{\delta\phi(\vec{x}')}}_{\delta^{(3)}(\vec{x}-\vec{x}')} \frac{\delta Q}{\delta\pi(\vec{x}')} - \frac{\delta\phi(\vec{x})}{\delta\pi(\vec{x}')} \cdot \underbrace{\frac{\delta Q}{\delta\phi(\vec{x}')}}_{=\phi} \Bigg) \equiv \frac{\delta Q}{\delta\pi(x)} \equiv [redacted] = \frac{\partial Q}{\partial \pi} \rightsquigarrow (\text{integrando})$$

Ⓐ At first order $\partial_\nu\phi' \simeq \partial_\nu\phi + \cancel{\partial_\nu\delta\phi}$

From the definition of G_i

$$= \delta\phi - \partial_\nu\phi\, \delta x^\nu \overset{Ⓐ}{\simeq} \delta\phi(\vec{x}) - \partial_\nu\phi' \delta x^\nu = \delta\phi(x) - (\phi(x') - \phi'(x))$$

$$= \phi'(x') - \phi(x) - \phi'(x') + \phi'(x) = \phi'(x) - \phi(x)$$ → "TOTAL VARIATION"

↳ Difference between field and its transformed at the SAME point $\vec{x}$

LD

Exercício 3

Consider Dirac γ-matrices, and use their fundamental relation to compute:

Properties: $\gamma_0\gamma^0 = \mathbb{1}$, $\gamma_i\gamma^i = -\mathbb{1}$, $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$

1) $\gamma_\mu\gamma^\mu \equiv \gamma_0\gamma^0 - \gamma_i\gamma^i \equiv \mathbb{1} - 3\cdot(-\mathbb{1}) = 4\mathbb{1}$

2) $\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(2g^{\mu\nu} - \gamma^\mu\gamma^\nu) \equiv 2\cdot\gamma^\nu - \underbrace{\gamma_\mu\gamma^\mu}_{4\mathbb{1}}\gamma^\nu \equiv -2\gamma^\nu$

3) $\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu \equiv \gamma_\mu\cdot\gamma^\nu[2g^{\rho\mu} - \gamma^\mu\gamma^\rho] = 2\cdot\gamma^\rho\gamma^\nu - \underbrace{(\gamma_\mu\gamma^\nu\gamma^\mu)}_{=-2\gamma^\nu}\gamma^\rho = 2\cdot\gamma^\rho\gamma^\nu + 2\gamma^\nu\gamma^\rho = 4\cdot g^{\rho\nu}$

4) $\sigma_{\mu\nu}\sigma^{\mu\nu} := \left(\frac{i}{2}\right)^2\cdot[\gamma_\mu,\gamma_\nu][\gamma^\mu,\gamma^\nu] = \left(\frac{i}{2}\right)^2\Big[\gamma_\mu\underbrace{\gamma_\nu\gamma^\mu\gamma^\nu}_{-2\gamma^\mu} - \gamma_\nu\underbrace{\gamma_\mu\gamma^\mu}_{4\mathbb{1}}\gamma^\nu - \gamma_\mu\gamma_\nu\gamma^\nu\gamma^\mu + \gamma_\nu\underbrace{\gamma_\mu\gamma^\nu\gamma^\mu}_{-2\gamma^\nu}\Big]$

$4\mathbb{1}_n \Rightarrow -16\mathbb{1}_n$ Analogously

$= -\frac{1}{4}\left[-2\gamma_\mu\gamma^\mu - 16\mathbb{1} - 16\mathbb{1}_n - 2\gamma_\nu\gamma^\nu\right]$

$= -\frac{1}{4}\cdot\left[-8\mathbb{1}_n - 16\mathbb{1} - 16\mathbb{1} - 8\mathbb{1}\right] \equiv -\frac{1}{4}\cdot(-48\mathbb{1}) = +12\mathbb{1}$

Properties: $\gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $\quad (\gamma_0)^2 \equiv \beta^2 \equiv \mathbb{1}_{4\times n}$

$\gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$ $\quad \sigma_i :=$ Pauli Matrices

$(\gamma_i)^2 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i^2 & 0 \\ 0 & -\sigma_i^2 \end{pmatrix} = -\mathbb{1}_{4\times n}$ $\quad (\sigma_i^2 \equiv \mathbb{1}_n)$

+ Anticommuting rule: $\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

LD

Esercizio 4

Consider the Lagrangean Density: $\mathcal{L} \equiv \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 - ig\bar{\psi}\gamma_5\psi\phi$

$m, \lambda, g \in \mathbb{R}$.

1) Equation of motion -> Euler Lagrange Equation: $\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\nu \frac{\partial\mathcal{L}}{\partial\partial_\nu\phi_r} = 0$ $\quad \phi_r = \bar{\psi}, \psi, \phi$

$\bar{\psi}$: $\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma_\mu\partial^\mu - m)\psi - ig\gamma_5\psi\phi$

$\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}} = \phi$ => E.L. eq: $(i\not{\partial} - m)\psi \equiv ig\gamma_5\psi\phi$

ψ: $\frac{\partial\mathcal{L}}{\partial\psi} = -\bar{\psi}m - ig\bar{\psi}\gamma_5\phi$

$\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} \equiv \bar{\psi}(i\gamma^\mu)$

$=> -\bar{\psi}m - ig\bar{\psi}\gamma_5\phi - \partial_\mu(\bar{\psi}i\gamma^\mu) = \phi$

$= \bar{\psi}(i\overleftarrow{\not{\partial}} + m)\cdot(-1) - ig\bar{\psi}\gamma_5\phi$

$=> \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = -ig\bar{\psi}\gamma_5\phi$

ϕ: $\frac{\partial\mathcal{L}}{\partial\phi} \equiv -m^2\phi + \lambda\phi^3 - ig\bar{\psi}\gamma_5\psi$

$\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \equiv 2\cdot\frac{1}{2}\partial^\mu\phi = \partial^\mu\phi$

$-> -m^2\phi + \lambda\phi^3 - ig\bar{\psi}\gamma_5\psi - \partial_\mu\partial^\mu\phi = \phi$

$-> (\Box + m^2)\phi = \lambda\phi^3 - ig\bar{\psi}\gamma_5\psi$

2) Propagators, simple way => $\mathcal{L} = T - V$ -> potential term -> Interaction

-> Kinetic term ~ propagators

$\mathcal{L}_{free}$ can be written as a bi-linear form in fields, $\mathcal{L}_{free} = T = c\cdot\phi^* D\phi$

(c constant, (ϕ, ϕ^*) field + its conjugated, D = polynomial/diff. operator)

In order to find propagators: 1) Identify $\mathcal{L}_{free}$

2) Identify D

3) D to momentum space

4) D^{-1} -> propagator (in mom. space)

ψ propagator: Bilinear term in ψ: $\bar{\psi}(i\not{\partial} - m)\psi$ => $D = (i\not{\partial} - m)$

In mom. space: $D \xrightarrow[\partial_\mu = -ip_\mu]{} (\not{p} - m) \rightarrow D^{-1} := S_F = \frac{1}{\not{p} - m}$ ✓.

ϕ propagator: Bilinear terms (Scalar Real Field) $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \rightarrow D \equiv \overleftarrow{\partial_\mu}\overrightarrow{\partial^\mu} - m^2$, $c = 1/2$.

Acting on the "left"

4D

=> In mom. space: $\mathcal{D} \to$ $\overrightarrow{\partial_\mu} = -ik_\mu$, $\overleftarrow{\partial_\mu} = +ik_\mu$ $\quad k^2 - m^2 \to \mathcal{D}^{-1} = \frac{1}{k^2-m^2} \equiv \Delta_F$

We can introduce diagrammatic rules as we have done for fermion and scalar propagators

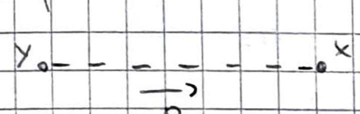
↳ To introduce these rules we need to write propagators in coord. space, i.e. integrating over d^4p

↳ As known (and shown during lectures): $\Delta_F \Rightarrow \Delta_F(x-y) = \int_{C_F} \frac{d^4k}{(2\pi)^4} \cdot e^{-ik\cdot(x-y)} \cdot \underbrace{\frac{1}{k^2-m^2}}_{=\Delta_F(k)}$

(Real path with $+i\epsilon$ prescription)

In the case of a real scalar field

↳ $i\Delta_F(x-y)$ represents a real scalar particle/antiparticle travelling from $\vec{y}$ to $\vec{x}$



(with 4-momentum p)

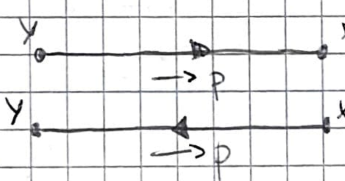
Analogously, Feynman Propagator for Dirac fields: $= S_F(x-y) = \int_{C_F} \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \cdot \underbrace{\frac{1}{\not{p}-m}}_{S_F(p)}$

(we have left here 2 free spinor indices)

In particular, $iS_F(x-y)$ represents:

1) A particle travelling from y to x

2) An antiparticle travelling from y to x



(In both cases, 4-mom. p)

(Real path integration with $+i\epsilon$ prescription in propagators' denominator)

↳ As for the scalar propagator.

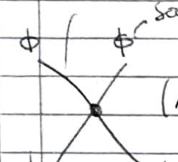
3) Interaction Terms: the remaining ones

(Dashed lines accordingly to propagator)

("Quartic Interaction" $\mathcal{L} + \frac{1}{4}\lambda\phi^4$ (sometimes defined as $-\frac{1}{4!}\lambda\phi^4$)

↳ Self interaction of the scalar field. Diagrams := Quadrilinear interaction

In this case, the coupling constant := $\lambda/4$.



Same field ϕ

(An example)

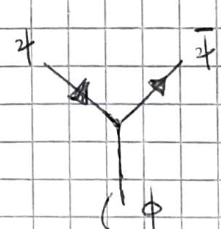
For determining the constant associated to each vertex, one needs to compute explicitly S-Matrix elements.

"Trilinear Interaction" := $-ig\bar{\psi}\gamma_5\psi\phi$

Constant of the interaction: $(-ig\gamma_5)$.

(↳ ~ Analogous to trilinear term in QED.

($\phi \leftrightarrow A^\mu$, $\gamma_5 \leftrightarrow \gamma_\mu$))



+ opposite fermion flow

(~ dashed line accordingly to the propagator)

$\mathcal{L}$ for charged Klein-Gordon fields:

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi$$

Global symmetry = $U(1)$. In fact: if we consider global $U(1)$ tfm

$\phi \to \phi' = U(\Lambda)\phi$, $U(\Lambda) = e^{-ie\Lambda}$ $\quad (U^\dagger(\Lambda)U(\Lambda) = \mathbb{1})$

$\phi^* \to \phi^{*\prime} = \phi^* \cdot U^\dagger(\Lambda)$ $\quad U^\dagger(\Lambda) = e^{+ie\Lambda}$ $\quad \Lambda :=$ Real Constant

$\Rightarrow$ Under $U(1)$ global tfm: $\mathcal{L} \to \mathcal{L}' = (\partial_\mu \phi^* U^\dagger(\Lambda))(\partial^\mu U(\Lambda)\phi) - m^2 \phi^* \underbrace{U^\dagger(\Lambda)U(\Lambda)}_{=\mathbb{1}}\phi$

$= (\partial_\mu \phi^*)\underbrace{U^\dagger(\Lambda)U(\Lambda)}_{=\mathbb{1}}(\partial^\mu \phi) - m^2 \phi^* \phi$ $\quad$ (U does not depend on x)

$\equiv \mathcal{L}$!

BUT under LOCAL $U(1)$ tfm, $\Lambda \equiv \Lambda(x)$. (NOT constant anymore)

$\hookrightarrow \mathcal{L}' = (\partial_\mu \phi^* U^\dagger(\Lambda))(\partial^\mu U(\Lambda)\phi) - m^2 \phi^* \underbrace{U^\dagger(\Lambda)U(\Lambda)}_{=\mathbb{1}}\phi$

$= \overbrace{(\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi}^{\mathcal{L}} + \phi^* \partial_\mu U^\dagger(\Lambda)(\partial^\mu U)\cdot\phi + \phi^* \partial_\mu U^\dagger(\Lambda)\cdot U(\Lambda)\cdot \partial^\mu \phi + (\partial_\mu \phi^*) U^\dagger(\Lambda)\partial^\mu U(\Lambda)\phi$

$\partial_\mu U^\dagger(\Lambda) = +ie(\partial_\mu \Lambda)U^\dagger(\Lambda)$ $\quad \partial^\mu U = -ie(\partial^\mu \Lambda(x))U$

$= \mathcal{L} + e^2 \cdot \phi^* \partial_\mu \Lambda(x) \partial^\mu \Lambda(x)\phi + ie\,\phi^* \partial_\mu \Lambda(x)\cdot \partial^\mu \phi - ie\,\partial_\mu \phi^* \cdot \partial^\mu \Lambda(x)\cdot \phi$

$\to \mathcal{L}$ IS NOT invariant under $U(1)$ tfm (we can observe, ~~at the hidden~~ ... our Nöther charge)

$(J_\mu \Rightarrow i(\partial_\mu \phi^* \phi - \phi^* \cdot \partial_\mu \phi)$ etc.)

We can now introduce, as done for the QED Lagrangian, a COVARIANT DERIVATIVE $D_\mu := \partial_\mu + ieA_\mu(x)$,

where $A_\mu(x)$ is a gauge ~~vector~~ field. (4-~~...~~vector)

"Scalar-Electrodynamics"

$\Rightarrow D_\mu \phi \equiv (\partial_\mu + ieA_\mu(x))\phi$ $\quad ; \quad (D_\mu \phi)^* = (\partial_\mu - ieA_\mu(x))\phi^*$

$$\mathcal{L} = (D_\mu \phi)^*(D^\mu \phi) - m^2 \phi^* \phi = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - ieA_\mu \phi^* \partial^\mu \phi + ie\,\partial_\mu \phi^* A^\mu \phi - e^2 \phi^* A_\mu A^\mu \phi$$

And now we'll ask this $\mathcal{L}$ to be invariant under simultaneous

1) local $U(1)$ tfm $\quad \phi \to \phi' = U(\Lambda)\phi$; $\phi^* \to \phi^{*\prime} = \phi^* \cdot U^\dagger(\Lambda)$ $\quad \Lambda = \Lambda(x)$

2) Gauge tfm $\quad A_\mu \to A_\mu' = A_\mu + \partial_\mu \Lambda$

WITH THE SAME Λ !

$$\mathcal{L}' \equiv (\partial_\mu \phi^* U^+)(\partial^\mu U\phi) - m^2 \phi^* U^+ U \phi - ie(A_\mu + \partial_\mu \Lambda)(\phi^* U^+ \partial^\mu U \phi) + ie(A^\mu + \partial^\mu \Lambda)(\partial_\mu(\phi^* U^+) U\phi) + e^2 \phi^* U^+ (A_\mu + \partial_\mu \Lambda)(A^\mu + \partial^\mu \Lambda) U\phi$$

LD

$(\partial_\mu \Lambda \equiv \partial_\mu \Lambda(x))$

Explicitly:

$\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$

$$\mathcal{L}' \equiv \ldots + ie\,\phi^* \cdot \partial_\mu \Lambda \partial^\mu \phi - ie\,\partial_\mu \phi^* \partial^\mu \Lambda \phi + e^2 \phi^* \partial_\mu \Lambda \partial^\mu \Lambda \phi + V \quad \leftarrow U(\Lambda) \text{ terms (first two terms)}$$

$$- ie A_\mu (\phi^* U^+ (-ie\partial^\mu \Lambda) U\phi + \phi^* \partial^\mu \phi) - ie\,\partial_\mu \Lambda (\phi^* U^+ (-ie\partial^\mu \Lambda) U\phi + \phi^* \partial^\mu \phi)$$

$$+ ie A^\mu (\partial_\mu \phi^* \phi + \phi^* (ie \partial_\mu \Lambda) U^+ U \phi) + ie\,\partial^\mu \Lambda ((\partial_\mu \phi^*) \phi + \phi^* (ie \partial_\mu \Lambda) U^+ U \phi)$$

$$+ e^2 \phi^* (A_\mu A^\mu + A_\mu \partial^\mu \Lambda + \partial_\mu \Lambda A^\mu + \partial_\mu \Lambda \partial^\mu \Lambda) \phi$$

Rearranging the terms:

$$= \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - ie A_\mu \phi^* \partial^\mu \phi + ie A^\mu (\partial_\mu \phi^*) \phi + e^2 \phi^* A_\mu A^\mu \phi + \longrightarrow \text{Original Lagrangian}$$

$$+ ie\,\partial_\mu \Lambda (\phi^* \partial^\mu \phi - \phi^* \partial^\mu \phi) - ie\,\partial^\mu \Lambda ((\partial_\mu \phi^*) \phi - (\partial_\mu \phi^*) \phi) +$$

$$+ e^2 \phi^* \partial_\mu \Lambda \partial^\mu \Lambda \phi + e^2 \phi^* \partial_\mu \Lambda \partial^\mu \Lambda \phi - e^2 \partial^\mu \Lambda \partial_\mu \Lambda \phi^* \phi - e^2 \partial_\mu \Lambda \partial^\mu \Lambda \phi^* \phi$$

$$- e^2 A_\mu (\phi^* \partial^\mu \Lambda \phi - \phi^* \partial^\mu \Lambda \phi) - e^2 A^\mu (\phi^* \partial_\mu \Lambda \phi - \phi^* \partial_\mu \Lambda \phi)$$

$\equiv \mathcal{L}$! $\Rightarrow \mathcal{L}_{SED}$ is effectively invariant under local $U(1)$ & gauge tfm.

For the "final" Lagrangian.. we also need to add a (Gauge Invariant) kinetic term for the Gauge Field just introduced

$$\left(\mathcal{L}_{SED} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - ie A_\mu (\phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \cdot \phi) + e^2 \phi^* A_\mu A^\mu \phi \right.$$

kinetic term for the Gauge Field (As in Maxwell theory).

$= -ie A_\mu J^\mu$ interaction between gauge field and the (scalar) current

$$\left(\mathcal{L}_{SED} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi)' - m^2 \phi^* \phi \right)$$

LD

Exercise 5 : Bhabha Scattering $\rightarrow e^-(p_1,s_1) + e^+(p_2,s_2) \rightarrow e^-(p_3,s_3) + e^+(p_4,s_4)$.

External fermions are ON-shell $p_i^2 \equiv m^2$.

Photon and Fermion fields. $\psi(x) \equiv \int d^3\vec{p}\cdot N_p \sum_s \left(b_{ps} u(p,s) e^{-ip\cdot x} + d^\dagger_{ps} v(p,s) e^{+ip\cdot x} \right) = \psi^{(+)}(x) + \psi^{(-)}(x)$

$N_p \equiv \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}}$ $\qquad \bar{\psi}(x) = \int d^3\vec{p}\cdot N_p \sum_s \left(b^\dagger_{ps} \bar{u}(p,s) e^{+ip\cdot x} + d_{ps} \bar{v}(p,s) e^{-ip\cdot x} \right) = \bar{\psi}^{(+)} + \bar{\psi}^{(-)}(x)$

$N_k \equiv \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}}$ $\qquad \hat{A}_\mu(x) = \int d^3\vec{k}\, N_k \sum_\lambda \left(a_{k\lambda} \epsilon_\mu e^{-ik\cdot x} + a^\dagger_{k\lambda} \epsilon^*_\mu \cdot e^{+ik\cdot x} \right) = A_\mu^{(+)} + A_\mu^{(-)}$

$\mu \sim$ lower index.

1) We can observe that in the initial state we have only 2 fermions (as in the final state). More explicitly, we do NOT have any external photon line.

So, Given the definition of $\hat{S}^{(2)}$ as

$$\hat{S}^{(2)} := \frac{(-ie)^2}{2!} \int d^4x_1 d^4x_2\, T\left[:\hat{\bar{\psi}}(x_1) \gamma_\mu \hat{\psi}(x_1) \hat{A}^\mu(x_1): :\hat{\bar{\psi}}(x_2) \gamma_\nu \hat{\psi}(x_2) \hat{A}^\nu(x_2): \right],$$

We are going to keep, in the Wick expansion, the only term with 4 external fermion lines and with a contraction between gauge fields.

$\hookrightarrow$ The only relevant term for the Bhabha scattering process:

$$\text{"d-term"} := \hat{S}^{(2)}_d = \int d^4x_1 d^4x_2 \underbrace{:\hat{\bar{\psi}}_1 \gamma_\mu \hat{\psi}_1 \hat{\bar{\psi}}_2 \gamma_\nu \psi_2:}_{\text{4 fermion lines.}} \underbrace{\hat{A}_1^\mu \hat{A}_2^\nu}_{\to\text{ photon propagator}}$$

Inside the d-term, various processes are described (Møller Scattering $e^-e^- \rightarrow e^-e^-$ and Bhabha $e^+e^+ \rightarrow e^+e^+$ mainly).

We now have to identify, ~~thro~~ ~~through~~ through the expansion in positive/negative frequencies, the terms related to the $e^-e^+ \rightarrow e^-e^+$ process.

OBS We could consider all 16 terms, but most of them will vanish once considered the expectation value with $\langle f|$ and $|i\rangle$. In a certain sense, we need to "connect" the external fermion lines with the processes inside the S-Matrix.

LD

The full expression: $= \int d^4x_1 d^4x_2 \underbrace{\hat{A}_1^\mu \hat{A}_2^\nu}: \left[\left(\hat{\bar{\psi}}_1^{(+)} + \hat{\bar{\psi}}_1^{(-)}\right) \gamma_\mu \left(\hat{\psi}_1^{(+)} + \hat{\psi}_1^{(-)}\right) \left(\hat{\bar{\psi}}_2^{(+)} + \hat{\bar{\psi}}_2^{(-)}\right) \gamma_\nu \left(\hat{\psi}_2^{(+)} + \hat{\psi}_2^{(-)}\right)\right]:$

$\hookrightarrow$ We are going to consider only terms with:
1. e^- creator operator $b^\dagger$
1 e^+ creator operator $d^\dagger$
1 e^- annihilation operator b
1 e^+ annihilation operator d

The relevant terms:

$$\underset{b^\dagger}{\hat{\bar{\psi}}_1^{(-)}} \gamma_\mu \underset{b}{\hat{\psi}_1^{(+)}} \underset{d}{\hat{\bar{\psi}}_2^{(+)}} \gamma_\nu \underset{d^\dagger}{\hat{\psi}_2^{(-)}}$$
$$\underset{d}{\hat{\bar{\psi}}_2^{(+)}} \gamma_\mu \underset{d^\dagger}{\hat{\psi}_1^{(-)}} \underset{b^\dagger}{\hat{\bar{\psi}}_2^{(-)}} \gamma_\nu \underset{b}{\hat{\psi}_2^{(+)}}$$
) differing only for an $x_1 \leftrightarrow x_2$ exchange

$$\underset{b^\dagger}{\hat{\bar{\psi}}_1^{(-)}} \gamma_\mu \underset{d^\dagger}{\hat{\psi}_1^{(-)}} \underset{d}{\hat{\bar{\psi}}_2^{(+)}} \gamma_\nu \underset{b}{\hat{\psi}_2^{(+)}}$$
$$\underset{d}{\hat{\bar{\psi}}_1^{(+)}} \gamma_\mu \underset{b}{\hat{\psi}_1^{(+)}} \underset{b^\dagger}{\hat{\bar{\psi}}_2^{(-)}} \gamma_\nu \underset{d^\dagger}{\hat{\psi}_2^{(-)}}$$
) differing only for an $x_1 \leftrightarrow x_2$ exchange.

And since we integrate over $d^4x_1 d^4x_2$, we can consider only one term from each couple, multiplying also for a factor 2. $\Rightarrow$ the initial factor $\frac{(-ie)^2}{2!} \cdot 2 \Rightarrow (-ie)^2$.

the S-matrix element $(-ie)^2 \int d^4x_1 d^4x_2 \, \underbrace{\hat{A}_1^\mu \hat{A}_2^\nu} \left[:\hat{\bar{\psi}}_1^{(-)} \gamma_\mu \hat{\psi}_1^{(+)} \hat{\bar{\psi}}_2^{(+)} \gamma_\nu \hat{\psi}_2^{(-)}: + :\hat{\bar{\psi}}_1^{(-)} \gamma_\mu \hat{\psi}_1^{(-)} \hat{\bar{\psi}}_2^{(+)} \gamma_\nu \hat{\psi}_2^{(+)}: \right]$

2) let's now compute S_{fi}: $\quad (\langle f| \equiv \langle 0| d_{p_4} b_{p_3} \qquad |i\rangle \equiv b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle$

$\hookrightarrow \langle f| \hat{S}^{(2)}_d |i\rangle = (-ie)^2 \cdot \int d^4x_1 d^4x_2 \langle 0| d_{p_4} b_{p_3} :\hat{\bar{\psi}}_1^{(-)}(x_1) \gamma_\mu \hat{\psi}_1^{(+)}(x_1) \hat{\bar{\psi}}_2^{(+)}(x_2) \gamma_\nu \hat{\psi}_2^{(+)}(x_2): \underbrace{A_1^\mu A_2^\nu} + :\hat{\bar{\psi}}_1^{(-)}(x_1) \gamma_\mu \hat{\psi}_1^{(-)}(x_1) \hat{\bar{\psi}}_2^{(+)}(x_2) \gamma_\nu \hat{\psi}_2^{(+)}: \underbrace{A_1^\mu A_2^\nu}$

$\times \; b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle$

We can use the Fourier decomposition of positive and negative frequencies:

For the first term (and analogously for the second): $\quad (q_i \; , i = 5 \dots 8)$

~~[illegible]~~ $= \hat{\bar{\psi}}_1^{(-)}(x_1) \equiv \int d^3q_1 \, N_{q_1} \sum_{\sigma_1} e^{+iq_1 x_1} \, b^\dagger_{\sigma_1, q_1} \bar{u}(q_1, \sigma_1)$

$\hat{\psi}_1^{(-)}(x_1) = \int d^3q_2 \, N_{q_2} \cdot \sum_{\sigma_2} e^{+iq_2 x_1} \, d^\dagger_{q_2, \sigma_2} \, v(q_2, \sigma_2)$

$\hat{\psi}_2^{(+)}(x_2) \equiv \int d^3q_4 \, N_{q_4} \sum_{\sigma_3} e^{-iq_4 x_2} \, b_{q_4, \sigma_4} \, u(q_4, \sigma_4)$

$\hat{\bar{\psi}}_2^{(+)}(x_2) \equiv \int d^3q_3 \, N_{q_3} \sum_{\sigma_3} e^{-iq_3 x_2} \, d_{q_3, \sigma_3} \, \bar{v}(q_3, \sigma_3)$

$d_4 = d_{p_4, s_4}$
$b_3 = b_{p_3, \sigma_3} \dots$
Notation: $u_{q_4} = u(q_4, \sigma_4)$
$v_{q_3} = v(q_3, \sigma_3)$
...
And so on

$S_{fi} = (-ie)^2 \cdot \int d^4x_1 d^4x_2 \cdot \sum_{\sigma = 1 \dots 8} \prod_{i=1}^{8} \int d^3q_i \, N_{q_i} \cdot iD^{\mu\nu}(x_1 - x_2) \times$ ✪

$\times \Big\{ \Big[\bar{u}_{q_1} e^{+iq_1 x_1} \gamma_\mu \cdot v_{q_2} e^{+iq_2 x_1} \cdot \bar{v}_{q_3} \cdot e^{-iq_3 x_2} \gamma_\nu u_{q_4} e^{-iq_4 x_2} \langle 0| d_4 b_3 : b^\dagger_{q_1} d^\dagger_{q_2} d_{q_3} b_{q_4} : b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle \Big] +$

$+ \Big[\bar{u}_{q_5} e^{+iq_5 x_1} \gamma_\mu \cdot u_{q_6} \cdot e^{-iq_6 \cdot x_1} \bar{v}_{q_7} e^{-iq_7 x_2} \gamma_\nu v_{q_8} e^{+iq_8 x_2} \langle 0| d_4 b_3 : b^\dagger_{q_5} b_{q_6} d_{q_7} d^\dagger_{q_8} : b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle \Big] \Big\}$ #

$b^\dagger, b$ and $d^\dagger, d$ ANTI-COMMUTE! When mixing $\hat{\ }$ bs and ds

$\{b,d\} = \phi = \{b^\dagger, d\} = \{b, d^\dagger\} = \{b^\dagger, d\}$

$b_{p_1}|0\rangle = \phi$

let's consider the v.e.v: $\textcircled{A} = \langle 0|\, d_{p_4} b_{p_3} \underbrace{b^\dagger_{q_1} d^\dagger_{q_2}}_{} d_{q_3} b_{q_4} \overbrace{b^\dagger_{p_1} d^\dagger_{p_2}}^{\text{2 jumps: + sign}} |0\rangle$ (2 jumps: + sign)

Last four terms: $d_{q_3} b_{q_4} b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle$. If we remember $b_{p_1} b^\dagger_{p_1} |0\rangle = \{b_{p_1}, b^\dagger_{p_1}\}|0\rangle$

$\Rightarrow = + b_{q_4} b^\dagger_{p_1} d_{q_3} d^\dagger_{q_2} |0\rangle = b_{q_4} b^\dagger_{p_1} \{d_{q_3}, d^\dagger_{q_2}\} |0\rangle = \{d_{q_3}, d^\dagger_{p_2}\}\{b_{q_4}, b^\dagger_{p_1}\}|0\rangle$

2 vanishing anticomm. — Annihilation operator $d_{q_3}|0\rangle = \phi$ — Bringing anticommutator in front. It is a $\mathbb{C}$ number! (FINE)

$\equiv \delta^{(3)}(\vec{q}_3 - \vec{p}_2)\, \delta_{\sigma_3, s_2}\, \delta^{(3)}(\vec{q}_4 - \vec{p}_1)\, \delta_{\sigma_4, s_1} |0\rangle$

(I am not considering the term $d^\dagger d|0\rangle$. It vanishes! Annihilation on $|0\rangle$.)

Analogously: $\langle 0|\, d_{p_4} b_{p_3} b^\dagger_{q_1} d^\dagger_{q_2} \underset{\text{2 vanishing } \{,\}}{=} \langle 0|\, \{d_{p_4}, d^\dagger_{q_2}\}\, b_{p_3} b^\dagger_{q_1} = \langle 0|\, \{d_{p_4}, d^\dagger_{q_2}\}\{b_{p_3}, b^\dagger_{q_1}\}$

$= \langle 0|\, \delta^{(3)}(\vec{p}_4 - \vec{q}_2)\, \delta_{s_4, \sigma_2}\, \delta^{(3)}(\vec{p}_3 - \vec{q}_1)\, \delta_{s_3, \sigma_1}$

$\Rightarrow \textcircled{A} = \delta^{(3)}(\vec{p}_4 - \vec{q}_2)\, \delta_{s_4\sigma_2}\, \delta^{(3)}(\vec{p}_3 - \vec{q}_1)\, \delta_{s_3,\sigma_1}\, \delta^{(3)}(\vec{q}_3 - \vec{p}_2)\, \delta_{\sigma_3, s_2}\, \delta^{(3)}(\vec{q}_4 - \vec{p}_1)\, \delta_{\sigma_4 s_1}\, \underbrace{\langle 0|0\rangle}_{=1}$

the second term: $\langle 0|\, d_{p_4} b_{p_3} : b^\dagger_{q_5} b_{q_6} d^\dagger_{q_7} d_{q_8} : b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle = + \langle 0|\, d_{p_4} b_{p_3} b^\dagger_{q_5} d_{q_8} b_{q_6} d^\dagger_{q_7} b^\dagger_{p_1} d^\dagger_{p_2} |0\rangle$

(+1 sign, 2 jumps) (2 jumps: + sign) (1 jump: −1 sign)

1 vanishing ~~commutator~~ ANTI commutator $\Rightarrow$ − sign

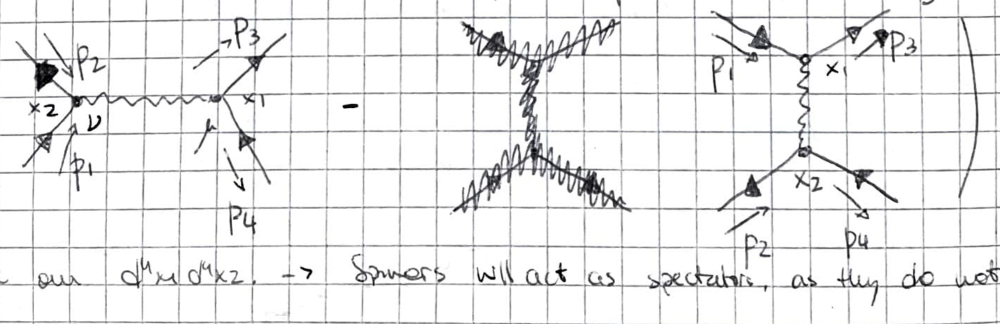
$\Rightarrow$ As before: $(-)\, \delta^{(3)}(\vec{p}_4 - \vec{q}_8)\, \delta_{s_4 \sigma_8}\, \delta^{(3)}(\vec{p}_3 - \vec{q}_5)\, \delta_{s_3 \sigma_5}\, \delta^{(3)}(\vec{q}_6 - \vec{p}_1)\, \delta_{s_1 \sigma_6}\, \delta^{(3)}(\vec{q}_7 - \vec{p}_2)\, \delta_{s_2 \sigma_7}\, \underbrace{\langle 0|0\rangle}_{=1}$

We can now apply the δs on d^3q_i integration and on σ sum.

$\Rightarrow S_{fi} = (-ie)^2 \int d^4x_1\, d^4x_2\, N\, iD^{\mu\nu}(x_1 - x_2) \times \Big\{ \Big[\bar{u}_{p_3} e^{+ip_3x_1} \gamma_\mu v_{p_4} e^{+ip_4x_1} \cdot \bar{v}_{p_2} e^{-ip_2x_2} \gamma_\nu u_{p_1} e^{-ip_1 \cdot x_2} \Big] +$

$- \Big[\bar{u}_{p_3} e^{+ip_3x_1} \gamma_\mu u_{p_1} e^{-ip_1x_1} \bar{v}_{p_2} e^{+ip_2x_2} \gamma_\nu v_{p_4} e^{+ip_4x_2} \Big] \Big\}$

$= N \cdot \int d^4x_1\, d^4x_2$



We now can integrate our $d^4x_1 d^4x_2$ $\to$ Spinors will act as spectators, as they do not have any x_1 or x_2 contribution (we'll forget them for a second)

$\Rightarrow \int d^4x_1 d^4x_2\, e^{-i(p_3 - p_4)x_1} e^{-i(p_2 + p_1)x_2} \cdot iD^{\mu\nu}_F(x_1 - x_2) \overset{\text{Prop. Fourier expansion}}{=} \int d^4x_1 d^4x_2 \cdot e^{-i(-p_3 - p_4)x_1} e^{-i(p_2 + p_1)x_2} \cdot \int \frac{d^4q}{(2\pi)^4} \cdot e^{-iq(x_1 - x_2)}\, iD^{\mu\nu}_F(q)$

$= \int \frac{d^4x_1 d^4x_2 d^4q}{(2\pi)^4}\, e^{-i(q - p_3 - p_4)x_1} e^{-i(p_2 + p_1 - q)} \cdot iD^{\mu\nu}_F(q) \overset{\text{int. over } d^4x_1 d^4x_2}{=} \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \cdot \delta^{(4)}(q - p_3 - p_4)(2\pi)^4 \cdot \delta^{(4)}(p_1 + p_2 - q)\, iD^{\mu\nu}_F(q)$

$\underset{\substack{\text{(int. over } d^4q) \\ \text{over a } \delta^4: q = p_3 + p_4}}{=} \delta^{(4)}(p_1 + p_2 - (p_3 + p_4)) \cdot 2\pi^4 \cdot iD^{\mu\nu}_F(p_3 + p_4)$

the δ insures mom. conservation!

40

~~[illegible]~~

~~[illegible]~~

Exploiting the propagator fermi Decomposition

The second term := $\int d^4x_1 d^4x_2 \frac{d^4q}{(2\pi)^4} iD_F^{\mu\nu}(q) \cdot e^{-i(p_1-p_3+q)x_1} e^{-i(p_2-p_4-q)x_2}$ = integrating on d^4x_i:

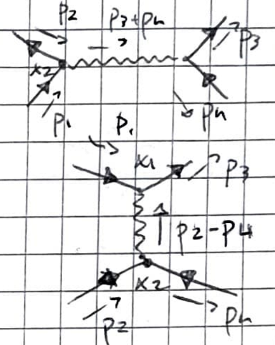
$$= \int \frac{d^4q}{(2\pi)^4} iD^{\mu\nu}(q)\, \delta^{(4)}(p_1-p_3+q)(2\pi)^4 \cdot (2\pi)^4 \cdot \delta^{(4)}(p_2-p_4-q) \Rightarrow q \equiv p_2-p_4$$

$$= (2\pi)^4 \delta^{(4)}(p_1-p_3+p_2-p_4)\, iD_F^{\mu\nu}(p_2-p_4) = (2\pi)^4 \cdot \delta^{(4)}(p_1+p_2-(p_3+p_4))\, iD_F^{\mu\nu}(p_2-p_4)$$

Finally: $S_{fi} = (-ie)^2 N_{p_1}N_{p_2}N_{p_3}N_{p_4} (2\pi)^4 \cdot \delta^{(4)}(p_1+p_2-(p_3+p_4)) \times$

$$\times \Big\{ \left[\bar{u}_{p_3}\gamma_\mu v_{p_4}\, iD^{\mu\nu}(p_3+p_4)\, \bar{v}_{p_2}\gamma_\nu u_{p_1} \right] + $$

$$- \left[\bar{u}_{p_3}\gamma_\mu u_{p_1}\, iD_F^{\mu\nu}(p_2-p_4) \cdot \bar{v}_{p_2}\gamma_\nu v_{p_4} \right] \Big\}$$



ok ✓.

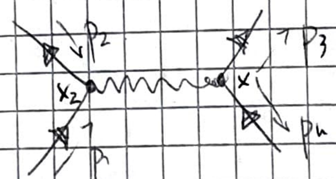
OBS If we did not consider the observations regarding the only four terms, we would have found the same results. The other 12 would have in fact had a ϕ v.e.v. when applying the normal ordered operator to the initial and final states.

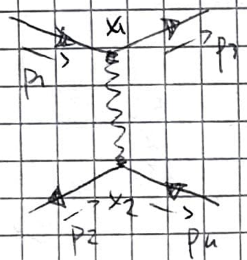
LD

Let's consider now Feynman rules in mom repr'n.

$$\langle f | S^{2} | i \rangle \equiv \delta_{if} + (2\pi)^4 \cdot \delta(p_1+p_2-(p_3+p_4)) \times \text{f. norm} \times \mathcal{M}$$

$\mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}^{(n)}$ -> for $n=2$, we only have 2 diagrams (for Bhabha Scattering of course)

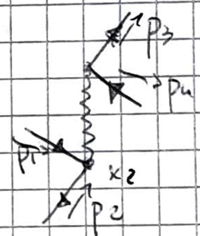
1) 

2) 

Q: What's their relative sign?

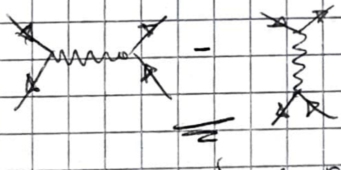
↳ we remember the sign is linked to the number of fermions lines exchanges needed to pass from one diagram to the other.

In the second diagram
We can exchange the incoming electron line with the outgoing positron line!

↳ 

→ We have found back the first diagram!

=> RELATIVE SIGN $\equiv (-1)^N = -1$

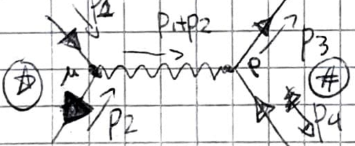
=> Finally: 

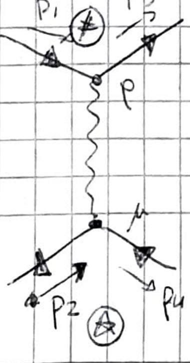
ok

As found before.

We can now compute $\mathcal{M}$, keeping in mind the possibility of opposite fermion flows ($x_1 \leftrightarrow x_2$)

↳ We have to take into account a $\times 2$ factor as done before

$\mathcal{M}_s \equiv$  $\Rightarrow \cancel{2} \cdot \frac{(-ie)^2}{\cancel{2!}} \cdot \bar{v}(p_2)\gamma_\mu \cdot u(p_1) \; -i \frac{g^{\mu\rho}}{(p_1+p_2)^2+i\epsilon} \; \bar{u}(p_3)\gamma_\rho v(p_4)$

$\mathcal{M}_t =$  $\cancel{2}\frac{(-ie)^2}{\cancel{2!}} \Rightarrow \bar{v}(p_2)\gamma_\mu v(p_4) \; \frac{-ig^{\mu\rho}}{(p_2-p_4)^2+i\epsilon} \cdot \bar{u}(p_3)\gamma_\rho u(p_1)$

(Reading from right to left, we need to follow the fermion arrow -> That's the "GOLDEN RULE" I have used to write down the $\mathcal{M}$ terms.)

LD

Putting all results from the $\mathcal{M}_s$ and $\mathcal{M}_t$ calculations, we get exactly the same result as before. (apart from some spinor exchange)

$$S_{fi} = N \cdot (2\pi)^4 \cdot \delta^{(4)}(p_1+p_2-(p_3+p_4)) \times (\mathcal{M}_s - \mathcal{M}_t)$$

Exercise 6

Use the solution of the Dirac equation in mom. space: $u_r(\vec{p}) \equiv c(\not{p}+m)u_r(0)$ $\quad r=1,2$

$v_r(\vec{p}) = c(-\not{p}+m)v_r(0)$

$p^\mu = (p_0, \vec{p})$; $c = \frac{1}{\sqrt{2m(p_0+m)}}$

$\chi_r = \begin{cases} \begin{pmatrix}1\\0\end{pmatrix} \\ \begin{pmatrix}0\\1\end{pmatrix}\end{cases}$

Verify: $\Sigma_{r=1,2}\, u_r(\vec{p})\,\bar{u}_r(\vec{p}) = \frac{\not{p}+m}{2m} := \Lambda_+$ pos. energy projector

Recall $u_r = \begin{pmatrix}\chi_r\\0_2\end{pmatrix}$, $v_r = \begin{pmatrix}0_2\\\chi_r\end{pmatrix}$ ($u_r(0)$ and $v_r(0)$)

$\Sigma_{r=1,2}\, v_r(\vec{p})\,\bar{v}_r(\vec{p}) = \frac{\not{p}-m}{2m} := \Lambda_-$ neg. energy projector

$$\to u_r(\vec{p}) = c\cdot(\underbrace{\not{p}}_{\gamma_\mu p^\mu}+m)\cdot u(0) = c\left[\begin{pmatrix}1\!\!1 & 0\\ 0 & -1\!\!1\end{pmatrix}E - \begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\cdot\vec{p} + \begin{pmatrix}1\!\!1 & 0\\ 0 & 1\!\!1\end{pmatrix}m\right]\begin{pmatrix}\chi_r\\0\end{pmatrix}$$

$$= c\begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p}\\ +\vec{\sigma}\cdot\vec{p} & -E+m\end{pmatrix}\begin{pmatrix}\chi_r\\0\end{pmatrix} = c\cdot(E+m)\begin{pmatrix}\chi_r\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r\end{pmatrix}$$

$$\bar{u}_r(\vec{p}) \equiv u^\dagger(\vec{p})\cdot\gamma_0 = c^*(E+m)\left(\chi_r^\dagger, \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r^\dagger\right)\cdot\underbrace{\begin{pmatrix}1\!\!1 & 0\\ 0 & -1\!\!1\end{pmatrix}}_{(\gamma_0)} = c^*(E+m)\left(\chi_r^\dagger, -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r^\dagger\right)$$

$$u_r(\vec{p})\cdot\bar{u}_r(\vec{p}) \equiv \underbrace{c\cdot c^*}_{|c|^2}(E+m)^2\cdot\begin{pmatrix}\chi_r\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r\end{pmatrix}\left(\chi_r^\dagger, -\chi_r^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) = \begin{pmatrix}\chi_r\chi_r^\dagger & -\chi_r\chi_r^\dagger\,\vec{\sigma}\cdot\vec{p}/(E+m)\\ +\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r\chi_r^\dagger & -\frac{(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{(E+m)^2}\cdot\chi_r\chi_r^\dagger\end{pmatrix}\cdot|c|^2(E+m)^2$$

$\Rightarrow$ We can now observe: $\chi_1\cdot\chi_1^\dagger = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\Rightarrow \Sigma_r\,\chi_r\chi_r^\dagger = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1\!\!1_2$

$\chi_2\chi_2^\dagger = \begin{pmatrix}0\\1\end{pmatrix}(0\ 1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

We can also remember: $(\vec{\sigma}\cdot\vec{p})^2 \equiv (\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) \equiv (\vec{p}\cdot\vec{p})1\!\!1_2 + i\vec{\sigma}\cdot\underbrace{(\vec{p}\times\vec{p})}_{=\phi} = \vec{p}^{\,2}$

And ... $m^2 \equiv p^2 = E^2 - \vec{p}^{\,2} \Leftrightarrow \vec{p}^{\,2} \equiv E^2 - m^2 = (E+m)(E-m)$

$$\Sigma_r\, u_r(\vec{p})\bar{u}_r(\vec{p}) \equiv \begin{pmatrix}1 & 0 & -\vec{\sigma}\cdot\vec{p}/E+m & 0\\ 0 & 1 & 0 & -\vec{\sigma}\cdot\vec{p}/E+m\\ \frac{+\vec{\sigma}\cdot\vec{p}}{E+m} & 0 & -\frac{(E-m)}{(E+m)} & 0\\ 0 & \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & 0 & -\frac{(E-m)}{(E+m)}\end{pmatrix}\cdot\underbrace{|c|^2\cdot(E+m)^2}_{\left(=\frac{1}{2m(E+m)}\right)} = \frac{1}{2m}\begin{pmatrix}E+m & 0 & -\vec{\sigma}\cdot\vec{p} & 0\\ 0 & E+m & 0 & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & 0 & -E+m & 0\\ 0 & \vec{\sigma}\cdot\vec{p} & 0 & -E+m\end{pmatrix}$$

~~And ... $\frac{\not{p}+m}{2m} = \frac{\gamma_0 p_0 + \gamma_i p_i + m}{2m} = \frac{1}{2m}\left(\begin{pmatrix}1\!\!1 & 0\\ 0 & -1\!\!1_2\end{pmatrix}E + m1\!\!1_4 - \gamma_i p_i\right) = \frac{1}{2m}\begin{pmatrix}E+m & 0 & -\vec{\sigma}\cdot\vec{p} & \\ 0 & E+m & & \\ +\vec{\sigma}\cdot\vec{p} & & -E+m & 0\\ & & 0 & -E+m\end{pmatrix}$~~

~~($\Rightarrow$ the two results coincide ..~~ Maybe better to use only "$\begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -E+m\end{pmatrix}$" Represent.

LD

So... $\sum u_r(\vec{p})\bar{u}_r(\vec{p}) = \begin{pmatrix} E+m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix}\cdot\frac{1}{2m}$

And: $\frac{\not{p}+m}{2m} = \frac{\gamma_0 p^0 - \gamma_i p^i + m}{2m} = \left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0\end{pmatrix}\cdot\vec{p} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1}\end{pmatrix}m\right]\frac{1}{2m}$

$= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix}\cdot\frac{1}{2m}$ ok! the two results coincide (we have neglected the $\mathbb{1}_2$ matrices!)

Let's compute now $v_r(\vec{p})\bar{v}_r(\vec{p})$.

$v_r(\vec{p}) \equiv c\cdot(-\not{p}+m)\,v_r(0) = c\left(-\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}E + \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0\end{pmatrix}\vec{p} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1}\end{pmatrix}m\right)\begin{pmatrix} 0 \\ \chi_r\end{pmatrix} =$

$= c\begin{pmatrix} -E+m & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & +E+m\end{pmatrix}\begin{pmatrix} 0 \\ \chi_r\end{pmatrix} = \begin{pmatrix} \vec{\sigma}\cdot\vec{p}\,\chi_r \\ (E+m)\chi_r\end{pmatrix}\cdot c = c\,(E+m)\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \\ \chi_r\end{pmatrix}$

$\bar{v}_r(\vec{p}) \equiv v_r^\dagger(p)\,\gamma_0 = c^*(E+m)\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r^\dagger,\ \chi_r^\dagger\right)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix} = c^*(E+m)\left(\frac{\vec{\sigma}\cdot\vec{p}}{(E+m)}\chi_r^\dagger,\ -\chi_r^\dagger\right)$

$\hookrightarrow \sum_r v_r(\vec{p})\bar{v}_r(p) \equiv \sum_r |c|^2(E+m)^2\begin{pmatrix} \frac{\sigma\cdot p}{E+m}\chi_r \\ \chi_r\end{pmatrix}\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r^\dagger,\ -\chi_r^\dagger\right)$

$E+m \propto (-E+m)$

$(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^2$ $\quad E^2 + \vec{p}^2 = m^2 \to \underline{p^2 = m^2 - E^2}$

$= |c|^2\cdot(E+m)^2\begin{pmatrix} (\vec{\sigma}\cdot\vec{p})^2/(E+m)^2\,\mathbb{1}_2 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\mathbb{1}_2 \\ \frac{(\vec{\sigma}\cdot\vec{p})}{E+m}\mathbb{1}_2 & -\mathbb{1}_2\end{pmatrix}$

$= \frac{1}{2m(\cancel{E+m})}(\cancel{E+m})\cdot\begin{pmatrix} \frac{(E+m)(-E+m)}{E+m} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m)\end{pmatrix} = \frac{1}{2m}\begin{pmatrix} -E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m\end{pmatrix}$

And... $\frac{(+\not{p}-m)}{2m} = \frac{\gamma^0 E - m\mathbb{1}_4 - \gamma_i p^i}{2m} = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m\end{pmatrix}\cdot\frac{1}{2m}$ V.

Ok, again we get the SAME results from the two calculations

(Matrices are always intended as 2x2 Blocks ... I should have multiplied every coefficient for a "factor" $\mathbb{1}_2$. Identity 2x2 Matrix)