Stars and Planets I

Homework 2

February 4, 2025

Question 1

(30 points). We will have you make a toy star and derive its basic properties. First, assume a density profile,

$$\rho(r) = \rho_c(1 - \frac{r}{R})$$

where ρ_c is the central density and R is the radius of the star.

a)Find an expression for the central density in terms of R and the total mass, M . Remember that the star is a sphere.

Solution:

From:

$$M = \int_V \rho(r)\, d\tau$$

Using spherical symmetry:

$$M = 4\pi \int_0^R r^2 \rho(r)\, dr$$

Substitute $\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$:

$$M = 4\pi \int_0^R r^2 \rho_c \left(1 - \frac{r}{R}\right)\, dr$$

Expanding:

$$M = 4\pi \rho_c \int_0^R \left(r^2 - \frac{r^3}{R}\right)\, dr$$

Integrating each term:

$$M = 4\pi \rho_c \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]_0^R$$

At the limits:

$$M = 4\pi \rho_c \left(\frac{R^3}{3} - \frac{R^4}{4R}\right)$$

Simplify:

$$M = 4\pi\rho_c\left(\frac{R^3}{3} - \frac{R^3}{4}\right)$$

Combine terms:

$$M = 4\pi\rho_c\frac{4R^3 - 3R^3}{12}$$

$$M = \frac{4\pi\rho_c R^3}{12}$$

$$M = \frac{\pi\rho_c R^3}{3}$$

Solve for ρ_c:

$$\rho_c = \frac{3M}{\pi R^3}$$

The final result:

$$\boxed{\rho_c = \frac{3M}{\pi R^3}}$$

b) Find an equation for the pressure as a function of radius. As part of this, you will need to determine an equation for the central pressure. As we did in class, assume the zero boundary condition, $P(R) = 0$.

Solution:

First we find $m(r)$

From:

$$m(r) = \int \rho(r)\,d\tau$$

Using spherical symmetry:

$$m(r) = 4\pi\int_0^r r'^2\rho(r')\,dr'$$

Substitute $\rho(r') = \rho_c\left(1 - \frac{r'}{R}\right)$:

$$m(r) = 4\pi\rho_c\int_0^r\left(r'^2 - \frac{r'^3}{R}\right)dr'$$

Integrating each term:

$$m(r) = 4\pi\rho_c\left[\frac{r'^3}{3} - \frac{r'^4}{4R}\right]_0^r$$

At the limits:

$$m(r) = 4\pi\rho_c\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

Using $\rho_c = \frac{3M}{\pi R^3}$:

$$m(r) = 4\pi \left(\frac{3M}{\pi R^3}\right) \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]$$

Simplify:

$$m(r) = \frac{4 \cdot 3M}{R^3} \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]$$

Combine terms:

$$m(r) = \frac{M}{R^3} \left[4r^3 - \frac{3r^4}{R}\right]$$

Factorize:

$$m(r) = M \left(\frac{r}{R}\right)^3 \left[4 - 3\left(\frac{r}{R}\right)\right]$$

Starting with:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

Substitute $m(r) = M \left(\frac{r}{R}\right)^3 \left[4 - 3\left(\frac{r}{R}\right)\right]$ and $\rho(r) = \frac{3M}{\pi R^3}\left[1 - \frac{r}{R}\right]$:

$$\frac{dP}{dr} = -\frac{GM}{R^3} \frac{\frac{3M}{\pi R^3}\left[1 - \frac{r}{R}\right] \cdot \left(\frac{r}{R}\right)^3 \left[4 - 3\left(\frac{r}{R}\right)\right]}{r^2}$$

Simplify:

$$\frac{dP}{dr} = -\frac{3GM^2}{\pi R^6} \frac{\left(1 - \frac{r}{R}\right) \cdot r^3 \left(4 - 3\frac{r}{R}\right)}{r^2}$$

$$\frac{dP}{dr} = -\frac{3GM^2}{\pi R^6} \left(4r - \frac{7r^2}{R} + \frac{3r^3}{R^2}\right)$$

Integrating:

$$dP = -\frac{3GM^2}{\pi R^4} \left(\frac{4r}{R^2} - \frac{7r^2}{R^3} + \frac{3r^3}{R^4}\right) dr$$

Assuming central pressure P_c and $P(r) = 0$:

$$\int_{P_c}^{0} dP = -\frac{3GM^2}{\pi R^4} \int_0^R \left(\frac{4r}{R^2} - \frac{7r^2}{R^3} + \frac{3r^3}{R^4}\right) \; dr$$

$$P_c = \frac{3GM^2}{\pi R^4} \left[\frac{2r^2}{R^2} - \frac{7r^3}{3R^3} + \frac{3r^4}{4R^4}\right]_0^R$$

$$P_c = \frac{5}{4} \frac{GM^2}{\pi R^4}$$

Now for $P(r)$

$$\int_{P_c}^{P(r)} dP = -\frac{3GM^2}{\pi R^4} \int_0^r \left(\frac{4r'}{R^2} - \frac{7r'^2}{R^3} + \frac{3r'^3}{R^4}\right) \; dr$$

$$P(r) - P_c = -\frac{3GM^2}{\pi R^4}\left[\frac{2r'^2}{R^2} - \frac{7r'^3}{3R^3} + \frac{3r'^4}{4R^4}\right]_0^r$$

$$P(r) = -\frac{3GM^2}{\pi R^4}\left[\frac{2r^2}{R^2} - \frac{7r^3}{3R^3} + \frac{3r^4}{4R^4}\right] + P_c$$

$$P(r) = -\frac{3GM^2}{\pi R^4}\left[\frac{2r^2}{R^2} - \frac{7r^3}{3R^3} + \frac{3r^4}{4R^4}\right] + \frac{5}{4}\frac{GM^2}{\pi R^4}$$

Final result:

$$\boxed{P(r) = \frac{GM^2}{\pi R^4}\left[7\left(\frac{r}{R}\right)^3 - 6\left(\frac{r}{R}\right)^2 - \frac{9}{4}\left(\frac{r}{R}\right)^4 + \frac{5}{4}\right]}$$

c) What is the central temperature, T_c? Assume an ideal gas equation of state. How does the central temperature scale with the mean molecular weight, μ?

Solution:

Starting with:

$$T = \frac{\mu m_p P}{\rho k}$$

Substitute P_c and ρ_c:

$$T_c = \frac{\mu m_p P_c}{\rho_c k}$$

$$T_c = \frac{\mu m_p}{k} \cdot \frac{\frac{5}{4}\frac{GM^2}{\pi R^4}}{\frac{3M}{\pi R^3}}$$

Final result:

$$\boxed{T_c = \frac{5}{12}\frac{\mu m_p}{k}\frac{GM}{R}}$$

The central Temperature T_c increases with increase in Mean Molecular Weight μ.

d) Write down an equation for the total gravitational potential energy for this toy star. Verify the virial theorem by comparing this to the volume-averaged pressure.

Solution:

Starting with:

$$E_g = -\int_0^M \frac{Gm(r)}{r}\,dm$$

Using spherical symmetry and substituting $dm = 4\pi r^2 \rho(r)\,dr$:

$$E_g = -\int_0^R \frac{Gm(r)}{r} \cdot 4\pi r^2 \rho(r)\,dr$$

Substitute $m(r) = M\left(\frac{r}{R}\right)^3 \left[4 - 3\frac{r}{R}\right]$ and $\rho(r) = \frac{3M}{4\pi R^3}\left(1 - \frac{r}{R}\right)$:

$$E_g = -\int_0^R \frac{GM}{r}\left(\frac{r}{R}\right)^3 \left[4 - 3\frac{r}{R}\right] \cdot 4\pi r^2 \frac{3M}{4\pi R^3}\left(1 - \frac{r}{R}\right)\,dr$$

Simplifying:

$$E_g = -\frac{12GM^2}{R}\int_0^R \left(\frac{4r^4}{R^5} - \frac{7r^5}{R^6} + \frac{3r^6}{R^7}\right)\,dr$$

Integrating term by term:

$$E_g = -\frac{12GM^2}{R}\left[\frac{4r^5}{5R^5} - \frac{7r^6}{6R^6} + \frac{3r^7}{7R^7}\right]_0^R$$

Final result:

$$\boxed{E_g = -\frac{26GM^2}{35R}}$$

The Volume-averaged pressure is given by:

$$\langle P\rangle V = \int_V P(r)\,d\tau$$

Substitute $P(r)$ and $d\tau = 4\pi r^2 dr$:

$$\langle P\rangle V = \frac{GM^2}{\pi R^4}\int_0^R 4\pi r^2 \left[7\left(\frac{r}{R}\right)^3 - 6\left(\frac{r}{R}\right)^2 - \frac{9}{4}\left(\frac{r}{R}\right)^4 + \frac{5}{4}\right] dr$$

Simplify:

$$\langle P\rangle V = \frac{4GM^2}{R}\int_0^R \left[7\frac{r^5}{R^6} - 6\frac{r^4}{R^5} - \frac{9}{4}\frac{r^6}{R^7} + \frac{5}{4}\frac{r^2}{R^3}\right]\,dr$$

Integrate each term:

$$\langle P\rangle V = \frac{4GM^2}{R}\left[\frac{7}{6}\frac{r^6}{R^6} - \frac{6}{5}\frac{r^5}{R^5} - \frac{9}{28}\frac{r^7}{R^7} + \frac{5}{12}\frac{r^3}{R^3}\right]_0^R$$

Simplify the numerical factor:

$$\langle P \rangle V = \frac{26}{105} \frac{GM^2}{R}$$

$$\langle P \rangle V = -\frac{1}{3} \left(-\frac{26}{35} \frac{GM^2}{R} \right)$$

Therefore:

$$\langle P \rangle = -\frac{1}{3} \frac{E_g}{V}$$

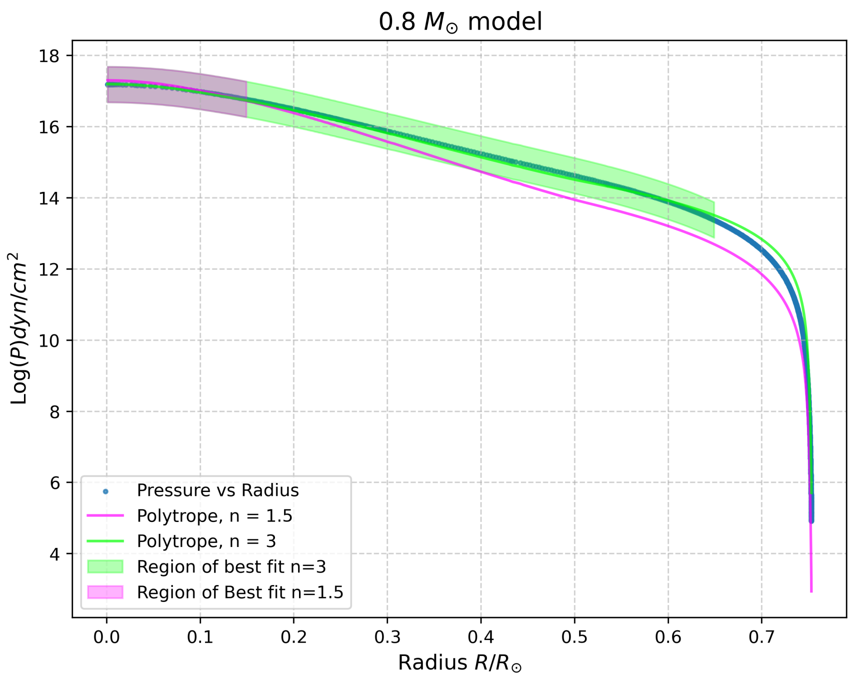
This is the Virial Theorem.

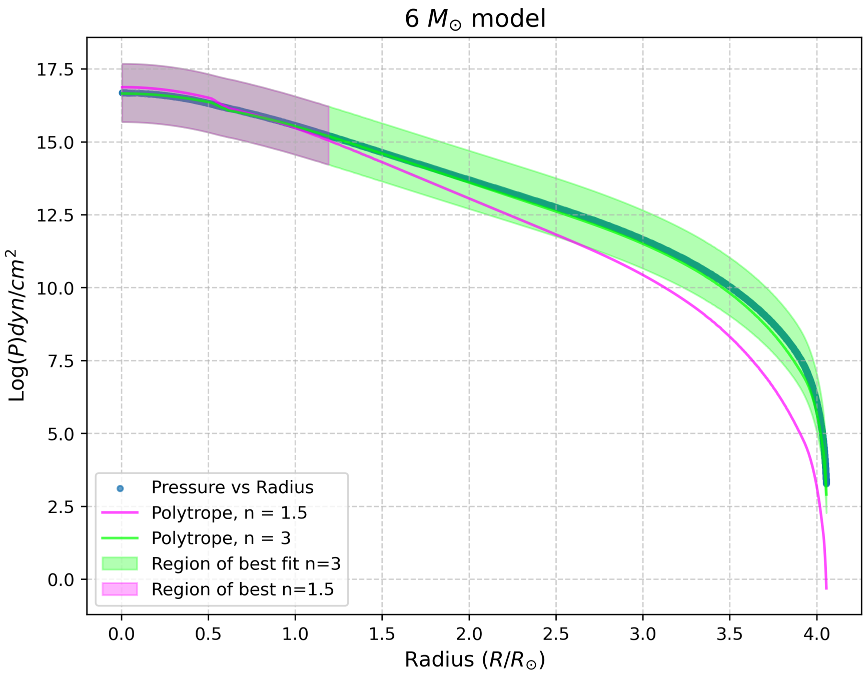
Question 2:

(40 points). Using MESA, create main-sequence models of $0.8M_\odot$ and $6M_\odot$ stars. Choose times roughly half-way between ZAMS and TAMS for examination. You should have created the 6 $M_\odot$ model in Homework 1.

a) Make a plot of the pressure profile (d log Tpressure as a function of stellar radius) for each model (separately). Make sure the plots look good; numerical artifacts can cause some weird things at the ends. You might also want to plot on a log scale to see things better. Overplot pressure profiles given by n = 3 and n = 1.5 polytropes (scaled appropriately to roughly match the MESA model). Mark the regions of each star that is best approximated by each model. If you do not find either model to be good for a large fraction of either star, what is the best polytropic index for that region (overplot it)?

Solution:





What can you infer about the energy transport and structure of each star in different regions?

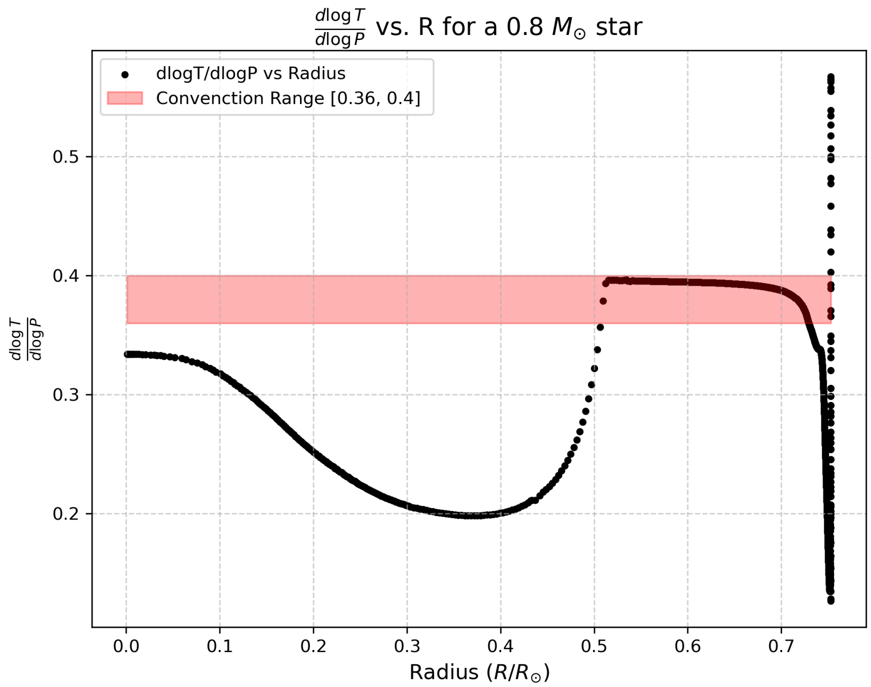
(b) Compute $\frac{d\log T}{d\log P}$ as a function of radius for the MESA model and plot it as a function of stellar radius. Mark the regions with

$$\frac{d\log T}{d\log P} \approx \frac{2}{5}$$

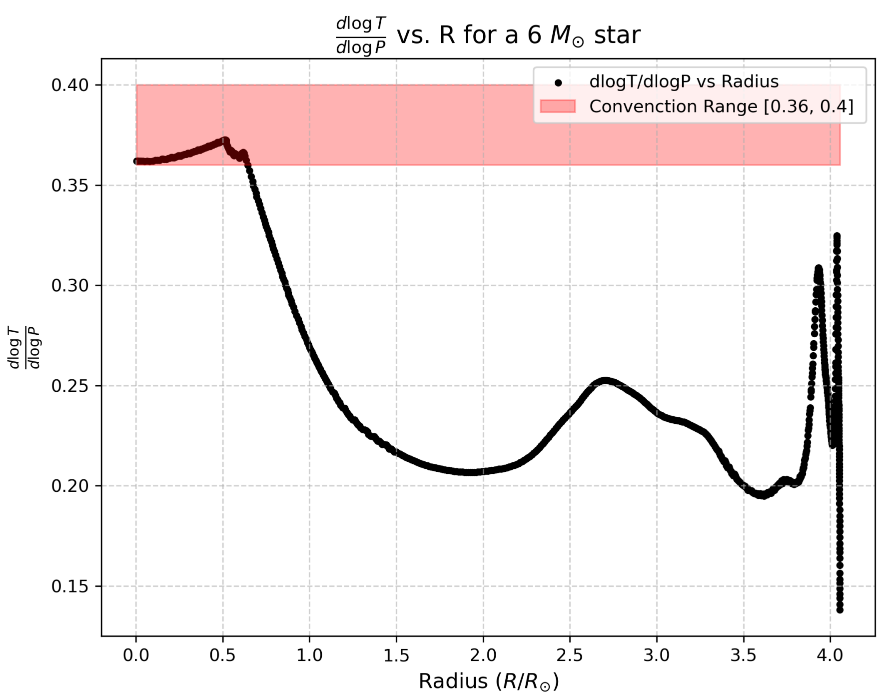
which corresponds to regions of convection. (Strictly, the criterion for convection is when the ratio is $> 2/5$ for ideal gas, but generally adiabatic expansion occurs so that it is right at that limit.) Describe the physical implications and relate this to your results in (a).

Answer: This indicates that at the inner region between $0 \lessapprox R_\odot \lessapprox 0.7$, most of the energy transport is through convention.

Solution:



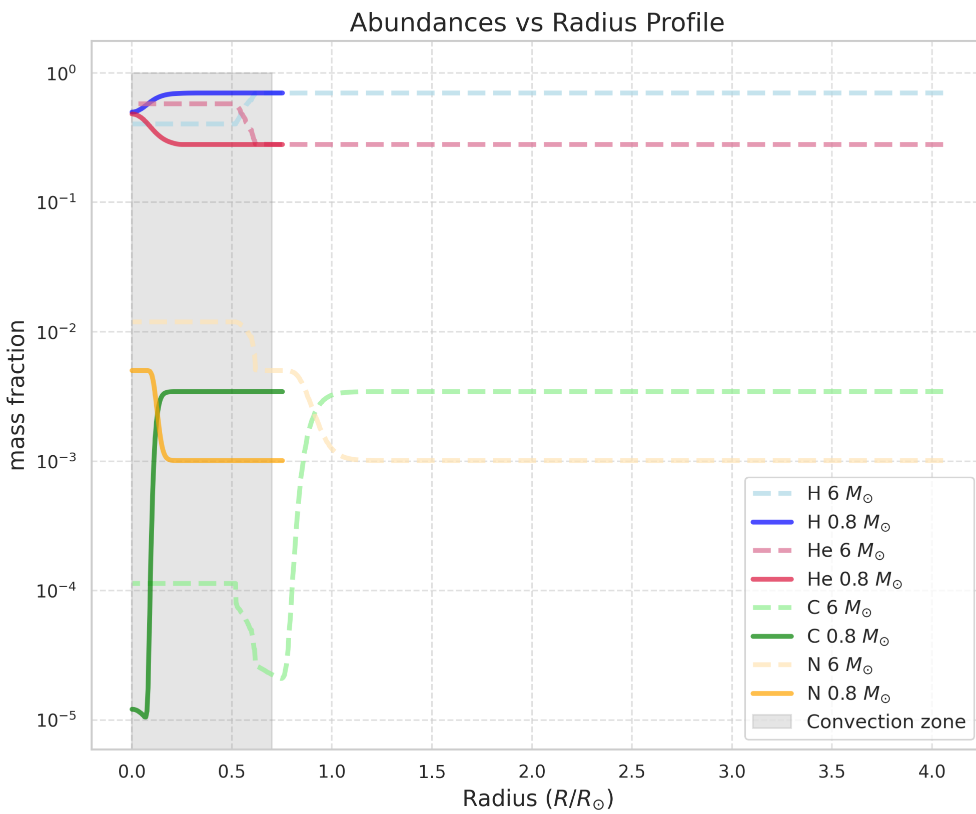
$\frac{d\log T}{d\log P}$ vs. R for a 0.8 $M_\odot$ star
dlogT/dlogP vs Radius
Convection Range [0.36, 0.4]
$\frac{d\log T}{d\log P}$
Radius ($R/R_\odot$)



$\frac{d\log T}{d\log P}$ vs. R for a 6 $M_\odot$ star
dlogT/dlogP vs Radius
Convection Range [0.36, 0.4]
$\frac{d\log T}{d\log P}$
Radius ($R/R_\odot$)

(c)Plot the abundances of H, He, C, and N for both models on the same plot (make sure the lines are easily distinguished. Mark the convective regions.

Solution:

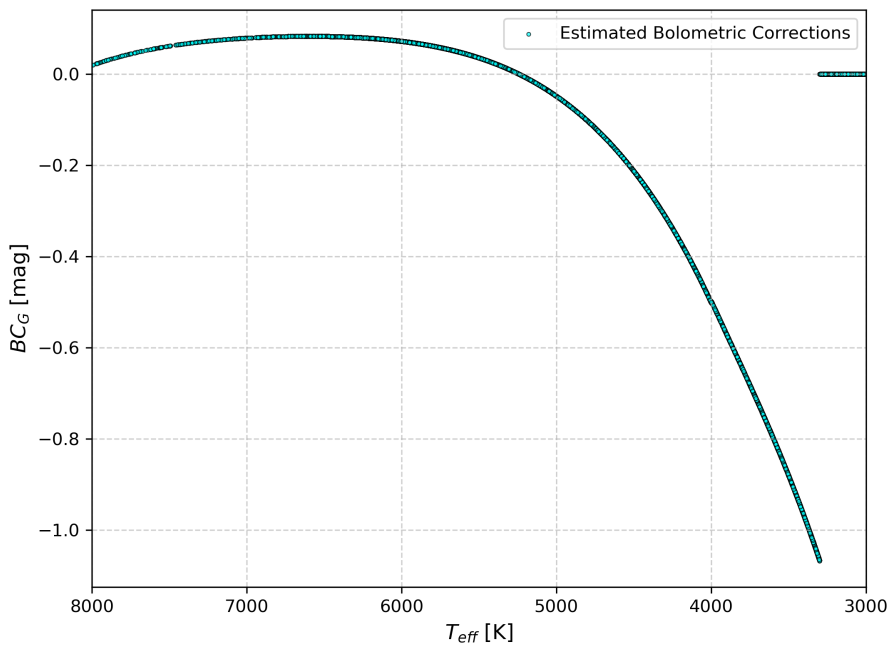


How are the abundances related to convection?

Answer: Convection happens in the region of the star where the composition is mostly heavy elements.

3 (a)). Select the subset of all stars within 70 pc from Gaia with good colors (S/N in the blue and red bands$\geq$ 5) and excellent parallaxes ($S/N > 20$). Make sure to select all of the stars. At this distance, the reddening is minimal and can be ignored. When getting the data, make sure to include columns for the luminosity, effective temperature, metallicity (the mh gspphot column), and radius. To determine the bolometric luminosity of a star from its Gaia magnitude, we need a bolometric correction. This is an adjustment that depends on effective temperature. It essentially says how much light is emitted outside of the observed band compared to a reference (often the Sun). Gaia has its bolometric corrections empirically determined and presented at `https://gea.esac.esa.int/archive/documentation/GDR2/Data_analysis/chap_cu8par/sec_cu8par_process/ssec_cu8par_process_flame.html`

Solution:

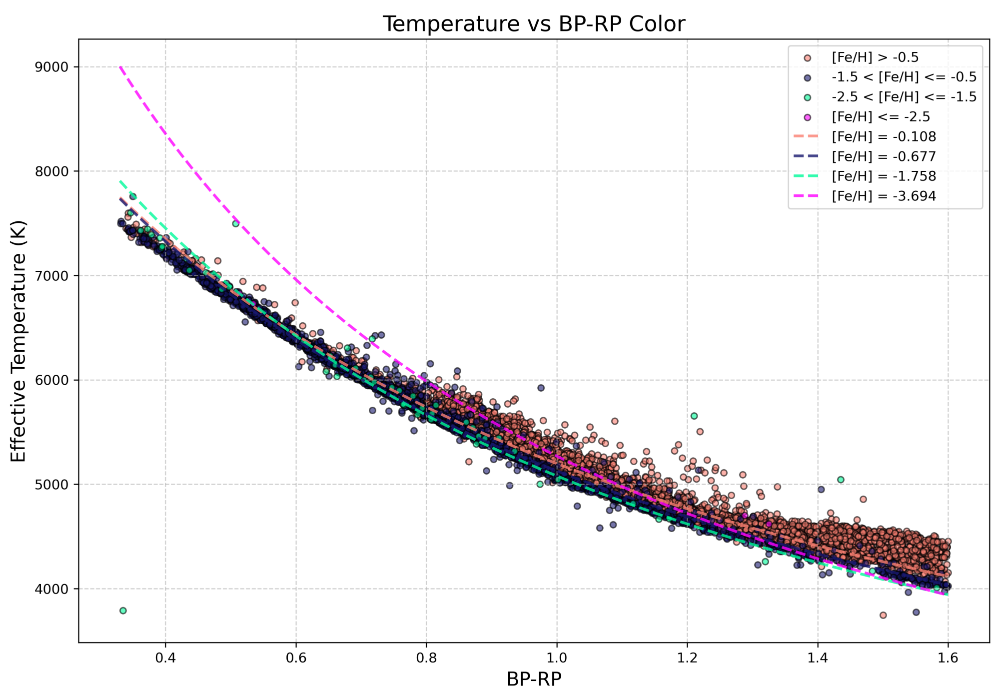


The Gaia temperatures are also determined empirically. They find that the temperature depends on several factors:

$$\frac{5040K}{T} = b_0 + b_1C + b_2C^2 + b_3[Fe/H] + b_4[Fe/H]^2 + b_5[Fe/H]C$$

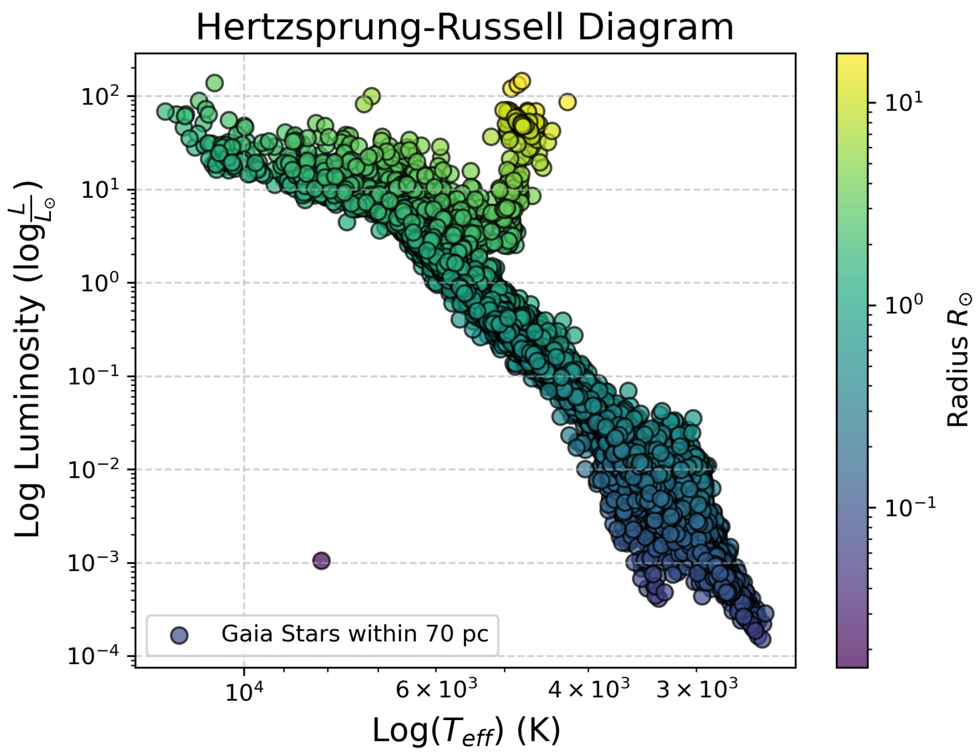
where C is a color, [Fe/H] is the metallicity, and the other factors are fit constants. Gaia uses many colors, but we will just use $BP - RP$. Using your sample of stars, but restricting to just those with $0.33 < BP - RP < 1.6$ mag and those with a metallicity measurement, fit for the constants. Plot the temperature as a function of $BP - RP$. color-code the stars based on their metallicity with bins of $[Fe/H] > -0.5$, $-1.5 < [Fe/H] \geq -0.5$, $-2.5 < [Fe/H] \geq -1.5$ and $[Fe/H] \leq -2.5$. Overplot four lines with metallicity corresponding to the median value for the stars within those bins.

Solution:



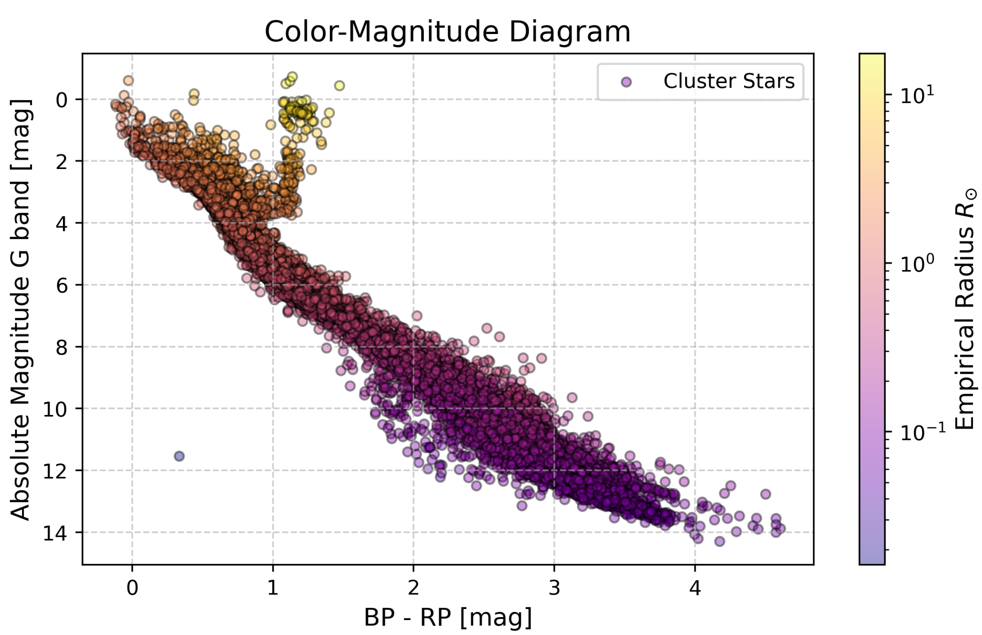
(c) Now use the bolometric correction and effective temperature equations to plot an H-R diagram for all stars. Include stars that do not have a metallicity measurement, and in those cases, assume a reasonable value. Overplot lines corresponding to stars with radii of 0.03, 0.1, 0.3, 1, 3, 10, 30, 100, and 300 solar radii. Remark on why some stars are called “giants” and others “dwarfs.”

Answer: The HR Diagram shows stars of similar size occupy regions in the H-R diagram, the larger stars referred to as giants and the smaller referred to as dwarfs. Dwarfs occupy most of the main sequence while the giants branch out.



(d) Finally, make a MG vs. BP -RP CMD for your sample with each star color-coded by radius. Use a colorbar to indicate the radius.

Solution:



Where does this CMD break down when considering radius?
Answer: There is a red giant branch when $radius > 10R_\odot$