Decision-Optimal Probabilistic Forecast Reconciliation for Expected Shortfall in Hierarchical Time Series

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1 Background

Risk managers and regulators oftenly use quantitative risk measures to quantify and manage potential losses in uncertain environments. Two familiar choices are Value-at-Risk (VaR) and Expected Shortfall (ES; also known as Conditional VaR). These tail risks are, however, not managed in isolation. Consider a bank's trading book, which consists of multiple portfolios across different desks, where each desk manages various asset classes, and then individual securities. This is an example of a hierarchical structure. Risk managers need to assess and forecast the risk quantities at each aggregation level, from the overall trading book down to individual securities. Ignoring this naturally-formed hierarchical structure can lead to poor forecasts of uncertainty, resulting in misallocation of capital and suboptimal risk management strategies.

Forecasting such hierarchical time series also arises in many decision-making contexts. From supply chains (Angam et al., 2025; Seaman & Bowman, 2022) and energy planning (Di Modica et al., 2021), to macroeconomic (El Gemayel et al., 2022; Li et al., 2019) and tourism analysis (Athanasopoulos et al., 2009), stakeholders need forecasts at several aggregation levels to allocate resources and manage risk. These hierarchical structures naturally form aggregation constraints.

In practice, when forecasts are produced for all series, they typically violate these aggregation constraints observed in the data. Such forecasts are *incoherent*, which often happens when forecasts at lower-level series (e.g. individual securities) do not sum up to the aggregated series (e.g. overall trading book). This can undermine downstream decisions that require internal consistency. Forecast reconciliation, a post-processing step, addresses this by adjusting the so-called base forecasts so the final adjusted set is *coherent* with the aggregation structure. Approaches such as OLS (Hyndman et al., 2011), WLS (Hyndman et al., 2016), and MinT (Wickramasuriya et al., 2019) are commonly used in practice; see Athanasopoulos et al. (2024) for a comprehensive review.

An additional incentive for the rapid adoption of reconciliation methods is that they improve forecast accuracy. For example, the International Monetary Fund reports improvements in liquidity forecasts of $3 million a day in a medium-sized economy, generating economic benefits via more efficient monetary policy operations (El Gemayel et al., 2022). This context motivates methods that deliver coherent, multi-level forecasts whose uncertainty aligns with the tail-risk metrics managers actually use.

2 Motivation

In the current literature, most reconciliation methodology has been developed for point forecasts, and its benefits for accuracy and coordination across organisation's departments or "silos" are well documented. However, in many applications the quantity of interest is a probabilistic forecast, not just a typical mean or median value. Modern risk management and operations need more than a single number–they need quantities from the full predictive distribution that capture uncertainty and tail behaviour, because many decisions are tail-sensitive (e.g., extreme demand in newsvendor settings or energy generation shortfalls).

Existing reconciliation methods for point forecasts has recently been generalised to reconcile probabilistic forecasts with downstream decisions based on these. However, these approaches originally focus on point forecasts and are optimised for scoring functions such as mean squared error (MSE). There is a gap in the literature on how *optimal* this approach when considering alternative quantities of interest, such as Value-at-Risk or Expected Shortfall. This leaves the question: *"How can we construct probabilistic forecast reconciliation that are explicitly optimised for tail-risk measures?"*

Although Panagiotelis et al. (2023) introduced an score optimisation for reconciliation weights using proper scoring rules (energy score and variogram score) for full multivariate predictive distributions, they do not directly address the tail functionals. Other than that, there is a lack of research on optimising reconciliation for specific risk measures such as VaR and ES.

Both VaR and ES are popular risk measures, but they have different properties. While VaR is conceptually straightforward, it ignores the severity of losses beyond the VaR threshold and is not a coherent risk measure, since it does not satisfy the subadditivity property (Yamai & Yoshiba, 2005). VaR can mislead risk assessments across different levels of the hierarchy, and may underestimate the true risk exposure under market stress. ES, on the other hand, is a coherent metric that captures the average loss in the tail beyond the VaR threshold. This makes ES a more reliable metric for assessing and managing "tail risks", and it has been adopted as the preferred regulatory metric since the Basel III framework (Basel Committee on Banking Supervision, 2019).

Even with these advantages, ES has its own challenges. Notably, ES is not elicitable on its own, meaning there is no scoring rule or loss function that can uniquely identify the ES as the optimal optimiser. However, works from Fissler et al. (2015) and Rockafellar & Uryasev (2000) show that this is not impossible when ES is considered jointly with VaR. This joint elicitability opens the door to optimising reconciliation methods specifically for ES. The biggest objective of this project is to bring these recent theoretical advances into the reconciliation framework, to develop methods that produce coherent probabilistic forecasts optimised for ES.

3 Methodology

Preliminaries

Let $\boldsymbol{y}_t \in \mathbb{R}^K$ denote a vector of K time series variables from the hierarchy, at time t. For simplicity, we suppress the subscript t and denote it as $\boldsymbol{y}$. These variables are subject to linear aggregation constraints represented by a matrix $\boldsymbol{C}$, such that $\boldsymbol{C}\boldsymbol{y} = 0$. For example, in a retail setting, the total sales at a warehouse (y_1) and two stores (y_2 and y_3) might satisfy the constraint $y_1 = y_2 + y_3$, leading to $\boldsymbol{C} = (1, -1, -1)$.

Base forecasts $\hat{\boldsymbol{y}}$, which we generate independently for each series, are typically *incoherent* (i.e. $\boldsymbol{C}\hat{\boldsymbol{y}} \neq 0$), while reconciled forecasts $\tilde{\boldsymbol{y}}$ are coherent by construction (i.e. $\boldsymbol{C}\hat{\boldsymbol{y}} = 0$). We obtain coherence via a linear mapping $\tilde{\boldsymbol{y}} = \boldsymbol{P}\hat{\boldsymbol{y}}$, where $\boldsymbol{P}$ maps the full *incoherent* space $\mathbb{R}^K$ to the linear *coherent* subspace $\mathbb{S} \subset \mathbb{R}^K$, where aggregation constraints hold. This mapping $\boldsymbol{P}$ can be a projection operator $\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$ (Hyndman et al., 2011), where $\boldsymbol{S}$ is a matrix designed such that $\boldsymbol{S}\boldsymbol{C} = 0$. The mapping $\boldsymbol{P}$ is not unique, and selecting an optimal $\boldsymbol{P}$ is the core design problem. In this project, we will focus solely on linear reconciliation.

Talk about the probabilistic reconciliation and how it is extended from point reconciliation.

Optimisation Framework

For a random variable Y, the Expected Shortfall (ES) at level $\alpha \in (0, 1)$ is defined as $\mathrm{ES}_\alpha(Y) = E[Y \mid Y < \mathrm{VaR}_\alpha(Y)]$, where $\mathrm{VaR}_\alpha(Y)$ is the quantile q at level α. The ES alone is not elicitable, meaning there is no scoring rule or loss function that can uniquely identify the ES as the optimal optimiser. However, Fissler et al. (2015) showed that the quantile function and ES are jointly elicitable, minimising the Fissler-Ziegel class of loss functions in expectation. One of which takes the form:

$$L_\alpha^{\mathrm{FZ}}(y, q, e) = (\mathbb{I}(y \leq q) - \alpha)(q - y) + \frac{1}{\alpha}\exp(e)\mathbb{I}(y \leq q)(q - y) + \exp(e)(e - q - 1),$$

where y is the realisation of Y, q is the quantile, and e is the ES. This loss function is non-convex, thus greatly complicating the optimisation problem. Alternatively, we propose to use the Rockafellar-Uryasev joint characterisation of the quantile and ES (Rockafellar & Uryasev, 2000): $\widetilde{\mathrm{ES}}_k(\boldsymbol{P}) = \max_q E_{\tilde{\boldsymbol{y}}}[H_\alpha(\tilde{y}_k(\boldsymbol{P}), q)]$ and $\tilde{q}_k(\boldsymbol{P}) = \arg\max_q E_{\tilde{\boldsymbol{y}}}[H_\alpha(\tilde{y}_k(\boldsymbol{P}), q)]$, where $H_\alpha(y, q) = q - \frac{1}{\alpha}\max\{0, y - q\}$. Thus, we jointly reconcile the quantile and ES using the following bilevel optimisation problem:

$$
\begin{aligned}
\min_{\boldsymbol{P}} \ & \sum_{k=1}^{K} E_{\boldsymbol{y}}\Big[L_{\alpha}^{\mathrm{FZ}}\big(y_k,\, \tilde{q}_k(\boldsymbol{P}),\, \widetilde{\mathrm{ES}}_k(\boldsymbol{P})\big)\Big] \qquad (1)\\
\text{s.t. } & \tilde{q}_k(\boldsymbol{P}) = \arg\max_{q} \ E_{\tilde{\boldsymbol{y}}}\left[H_{\alpha}(\tilde{y}_k(\boldsymbol{P}), q)\right], \\
& \widetilde{\mathrm{ES}}_k(\boldsymbol{P}) = \max_{q} \ E_{\tilde{\boldsymbol{y}}}\left[H_{\alpha}(\tilde{y}_k(\boldsymbol{P}), q)\right], \qquad \forall\, k = 1, \ldots, K.
\end{aligned}
$$

This ensures that the lower-level problem is still convex, however the upper-level objective function with the Fissler-Ziegler loss is non-convex. Irrespective of this non-convexity, smoothing techniques combined with the implicit function and envelope theorems can be used to devise gradient-based methods for locally minimising.

4 Applications

5 Contributions

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