
INTRODUCING TYPE PROPERTIES

Per-Lorean Graph

August 25, 2021

ABSTRACT

1 Introduction

One of our main goals in type theory is to provide meticulous reasoning about programs. In consequence, a considerable number of works in this field has been dedicated to incorporate broader reasoning, i.e., on the static level, to give more accurate specifications of programs through types and, based on the information from such specifications, to reveal the utmost properties of our programs.

The key thing to enable broader reasoning about, in fact, anything in this world is to provide more information about the subject of reasoning: to *say more*, we need to *know more*. In type theory, the conventional way to do this is to merely, syntactically annotate language constructions we reason about with some additional data and, later, manipulate the data in inference rules to state its relative meaning (that is, to signify what do these data really tell us about the behavior of language constructions they are annotated to). This can be illustrated by some recently-proposed substructural type systems ([3]) that, in their typing contexts, annotate variables with an element of a semiring ([5]) — 0, 1, or σ — which that tells us how many times a variable is used at runtime:

$$x : \text{Int}, y : \text{Int}, z : \text{Int} \vdash y + z : \text{Int}$$

Figure 1: An ordinary typing judgment. Variables are not annotated with any additional information except types.

$$x :^0 \text{Int}, y :^1 \text{Int}, z :^1 \text{Int} \vdash y + z :^\sigma \text{Int}$$

Figure 2: The same typing judgment, but now variables are annotated with their usage information.

As a result, the ability to explicitly impose such restrictions empowers us to reason about variables as resources ([1]) since the only difference between ordinary data and a resource is that the latter is subject to further usage constraints.

In fact, by pushing the boundaries of what we can say about our programs, this pattern can be seen as a unified way to endow type theories (and, consequently, programming languages) with new useful features. In the example above, it is clear that the theory has been augmented with a notion of resource-awareness, which has shown to be tremendously beneficial and found its applications in many areas ([7, 4]). Another specimen of features introduced in this way is the notion of a computational effect; effect systems merely attach some additional information, F , to their typing judgments: $\Gamma \vdash e : T, F$. This additional information represents the effect on the environment when the expression e is computed.

To not fall into a deep examination of useful notions introduced in this way but still give the reader flavor of what they are, we will finally mention some recently-proposed generalized modal type systems ([2, 6]), which subsume a considerable amount of such features at once, including the above-mentioned quantitative typing, differential privacy, session types, modal necessity and possibility, and so forth. These great systems encompass that discernible number of features because they are parametrized by algebraic structures that obey certain laws, which they attach to certain types to incorporate these structures within the language; as they comprise those features with the support of attaching additional information (those algebraic structures) to types, they greatly enlarge the expanse of useful notions introduced by this pattern with the features they encompass.

In the rest of this paper, we will refer to the data annotated using this pattern as to static annotations.

For most theories, the difficulty with using this pattern to add new features is that one becomes a subject to a complete alteration; that is, one needs to thoroughly refine its inference rules to let them collect more information about certain language constructions and signify the information's meaning:

$$\frac{\Gamma \vdash t_1 : \Pi(x : A)B \quad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1 t_2 : B[a/x]}$$

Figure 3: The "App" rule in ordinary ITT

$$\begin{array}{c} \sigma' = 0 \Leftrightarrow (\pi = 0 \vee \sigma = 0) \quad 0\Gamma_1 = 0\Gamma_2 \\ \Gamma_1 \vdash M \overset{\sigma}{:} (x \overset{\pi}{:} S) \rightarrow T \quad \Gamma_2 \vdash N \overset{\sigma'}{:} S \\ \hline \Gamma_1 + \pi\Gamma_2 \vdash \text{App}_{(x \overset{\pi}{:} S)T}(M, N \overset{\sigma}{:} T[N/x]) \end{array}$$

Figure 4: The same "App" rule in QTT

To our regret, this process is shown to be laborious, time-consuming, and error-prone. Even worse, as it incorporates changing the theory itself, it feasibly forces us to reengineer a compiler's algorithms (such as type-checking) and reprove most previously-held theorems. A well-known example of such features is linear (or, most often, quantitative) types, adding which programming language implementors still have difficulties with. As a result, such impracticalities, altogether, may easily postpone the augmentation of possibly demanded features for years, perceptibly slowing down a language's development. With this in mind, it would be superfluous to say that a better approach would be to somehow allow programmers on their own to associate additional information with language constructions and somehow signify its meaning without altering the language and its compiler.

In this paper, we introduce a new programming language concept that allows end programmers to implicitly annotate language expressions with additional data and, to designate the data's meaning, write some code that instructs the compiler what to do when it finds out that a particular expression has some associated data.

As a matter of fact, the idea of attaching additional metadata has been residing in many programming languages with competent metaprogramming systems. In particular, many Lisp-like languages implement some essential functionality through an attachment of metadata to particular kinds of language constructs; these include syntax object properties and transformation bindings from Racket, identifier properties from Scheme, Clojure's metadata, and so forth. In our work, we unify this idea as a separate, distinct programming language concept that is independent of any language ecosystem and that programming language implementors can equip their languages with to provide end-users with the functionality to associate additional data. In fact, the unification of this idea is, most probably, an inescapable consequence of the novelty of our approach; the main novelty of our approach is that we treat this problem primarily from the type-theoretic perspective and bring this notion to type theory. That is, we have attempted to design this concept in such a way that if we annotate expressions with some additional data, then it must be as similar to how we do it in type theory as possible (and, as we will see in Section 2, this imposes some indispensable requirements), presented a simple type theory that has this concept, and, most notably, we used a special case of dependent types to keep, save, and retrieve these annotations.

The main aim of this concept is to provide facilities to annotate expressions with additional data as we do it in type theory and, most importantly, do it using types, and designate its meaning using the existing language constructions. This means that the expanse of features that become expressible with a language's encompassment of our concept is bounded *only* by the existing language constructions. That is, if existing language constructions allow one to "interpret" those 0, 1, or σ as usage constraints (which is, in fact, not that difficult to achieve)¹, then, as our concept provides facilitates to annotate language expressions with those elements of the semiring and interpret them through custom code, it becomes possible to express linear (quantitative) types as just functionality in a separate program module, i.e., within the language itself and without the need to introduce a complex metaprogramming system into the language. Finally, bringing this idea in type theory makes it possible to draw connections with arising beneficial notions and bringing them to programming faster, because in this field, in a field that provides programming languages with their practical foundations, such notions arise much earlier, sometimes even decades, than they enter the industry.

In short, our **contributions** are as follows:

- We provide an informal description of the concept and develop the intuition about our approach (Section 2), thus preparing readers for later sections whose content is merely technical. While describing, we define so-called *propertied types*, the very types we use to keep, retrieve, and propagate static annotations.

¹A possible solution would be to allow one to "overload" a variable's usage behavior (variable as in software engineering, not as in type theory). That is, the ability to write a function invoked when a variable is used, which allows us to keep track of one's usage. Provided that such language is equipped with ordinary destructors from the RAI world and means to stop the compilation, we could compare the user's tracking number with the quantity we initially annotated to the variable (those 0, 1, and σ). If they mismatch in the destructor (or even in the function where we keep track of its usage), we terminate the compilation.

- We present a simple type theory, namely, λ_{\rightarrow_p} , augmented with the notion of type properties, thus formalizing our concept (Section 3).
- We then prove some important properties of λ_{\rightarrow_p} (Section 4). In particular, we prove that the transformation, a phase where we handle all property types, is deterministic, that all property types are handled before runtime, and that the type system is type sound.

2 Type Properties

Presenting our solution as a programming language concept means equipping a language with novel constructions that reflect our solution's ideas. In our case, this means we are obliged to provide new constructions for: annotating another construction with some additional data, retrieving these data by our need, and specifying the data's meaning.

To put it in another way, here is what we want. Firstly, we want our compiler to internally remember that we associate some additional data with an expression. This corresponds to annotating expressions we reason about with some additional information in type theory. The second wish is in the ability to be aware of whether an expression has some associated data with it at any point of the program and, consequently, to retrieve such data. Finally, since, in programming, to designate one's meaning, we tend to express it in terms of existing language constructions, i.e., we write code that interprets it, we wish to be able to write code that will tell a compiler what shall it do when it sees an expression that is annotated with such data.

Moreover, as we already mentioned in the introduction, we address this problem primarily with the type-theoretic perspective. This means that when there are some expressions annotated with additional data by means of our concept, for a user, it should look as if the data's presence would be built-in into the language's type theory. From this, we can infer two things. Firstly, in type theory, to provide more information about construction, we typically associate it with the construction in our contexts, thereby making the information's presence implicit for programmers. A case in point is the same substructural type systems:

$$x : \text{Int}, y : \text{Int} \vdash x + y : T$$

Therefore, provided that a programmer writes the expression $x + y$, they do not have to be aware of the collection of variables' usage unless they are the one explicitly imposing these usage restrictions on their code. The same must hold for our constructions; we must design them that for a user, it shall look that a compiler magically remembers what data we associate with which expressions. Secondly, since type systems usually define the static part of programming languages, the additional data must be given meaning already at the stage of compilation. In other words, we should evaluate all these constructions during compile-time.

This section serves two purposes. The first is to attempt to describe what these constructions might be. The second is to build a proper intuition on our approach; we will smoothly skim through their obligations and informal descriptions to proper type theoretical notations, thereby preparing the reader for Section 3, whose content, consisting only of inference rules and their descriptions, is merely technical.

2.1 Constructions

The best way to create the proper impression about programming language constructions is to illustrate them in examples. Consider the following pseudo-code snippet:

```

1  let x = 1 in
2  let x' = set(x, 1) in
3  x + x' + get(x')
```

Following our intuition, we can think of the expression $\text{set}(x, 1)$ as an appeal to the compiler to treat this expression as just x , but also remember that we associate with it the additional data of the integer 1. Thereafter, the construction $\text{get}(x')$, on the third line, is an appeal to substitute it with the data we previously requested to remember about x' . Thus, we expect this program, right before we run it, to look like the following:

```

1  let x = 1 in
2  let x' = x in
3  x + x' + 1
```

Seem to be effortless and, to be honest, useless: almost any modern programming language provides facilities to define and substitute constants at the stage of compilation. Thus, for demonstrational purposes, let us investigate a more suggestive example. For instance, consider that, in our program, we agree that if there is an integer constant associated with an expression of the integer type, then it means that this expression, at runtime, evaluates to the value of this constant. For this to have any sense, we must be able to be acknowledged that an expression has some associated data at any point of the program, to later instruct the compiler with what to do when it finds out that there is some data associated. For instance, we must be able to retrieve this information even in a function's body, which we intend to execute at runtime:

```

1  func greater_than_five (x : Int) with
2      if-has x n : Int then n > 5
3      else x > 5 in
4
5      let x = 1 in
6      let x' = set(x, 1) in
7
8      let a = greater_than_five x in
9      let b = greater_than_five x' in
10     let c = greater_than_five (x + x')
```

The code on lines 2 and 3 merely reads as: "*if the expression that has been used to construct the value of 'x' has been annotated with additional data of the type Int, then bind this data to the name 'n' and compile itself into the first branch, or to the second, otherwise*", by which we instruct the compiler how to interpret the case when it finds out that the expression has some data has been associated with it. In our case, we tell it to use the associated data and perform computation with the data rather than with the real argument that shall be evaluated at runtime.

But now it does not seems to be so straightforward how to implement this. The thing is, the function's body is intended to be evaluated at runtime, but we promised that we handle all static annotations at the stage of compilation. On the other hand, lifting the construction's evaluation to the compile-time prevents us from compiling functions ahead: the function's body for calls on lines 8 and 9 must be different.

As the investigation of the problem requires some understanding of our approach, it seems that it is the very time to start discussing how we can use types to handle static annotations and then return to the problem later.

2.2 Implementation

Using types to keep, retrieve, and propagate annotations has definitely something to do with the type of $set(e, v)$; at least because one is responsible for setting such annotations. As a suggestion, we can impose its type, for instance, to keep the associated data and e 's type within itself:

$$\frac{\Gamma \vdash e : \text{Int} \quad \Gamma \vdash v : \text{Int}}{\Gamma \vdash \text{set}(e, v) : [\text{Int}] \langle v : \text{Int} \rangle}$$

Storing data in our types might be a reasonable way to go, because what we might end up doing is expressing our solution using dependent or, at least, indexed types, which is an already exausive, well-studied notion. To not confuse readers, let us return to the first example and ascribe types to some expressions we are interested about in comments:

```

1  let x = 1 in # x : Int
2  let x' = set(x, 1) in # x' : [Int] (a : Int)
3  x + x' + get(x') # ???
```

Now, as we can see, the only thing the construction $get(x')$ needs to do is to introspect the type of x' and substitute itself with the data it acquires. Its typing rule may be defined as follows:

$$\frac{\Gamma \vdash e_1 : [T_1] \langle e_2 : T_2 \rangle}{\Gamma \vdash \text{get}(e_1) : T_2}$$

It seems that using types in such a way solves the first two problems: we are now able to save and retrieve these associations. Hopefully, it does; but now we are in troubles: the expression $x + x' + get(x')$ is not well-typed anymore. The reason for that is because now the type of x' is not integer, but rather the type $[\text{Int}] \langle 1 : \text{Int} \rangle$. However, such expression is indeed valid since, before we run a program, x' becomes an ordinary integer variable.

From now on, such types, which are created to handle static annotations, will be called *propertied types*, the expression we annotate with some data (in our example, the variable x) will be called its *underlying expression*, and we will refer to the underlying expression's type as to the *base type* of a propertied one. As an example, in $e : T_e, v : T_v \vdash \text{set}(e, v) : [T_e]\langle v : T_v \rangle$: the type $[T_e]\langle v : T_v \rangle$ is a propertied type based on T_e , whose underlying expression is e . The v , with respect to this propertied type, will be called its property. As one may notice, a propertied types is just a type parametrized by values, except that it is an anonymous type that also stores its underlying expression type and is created to handle static annotations (and, as we will see in Section 3, follows a specific structure to be more general than its this section's version). We can thus speak about propertied types as just of the special case of dependent ones.

Now, when we have established the terminology, we can return to the problem. A possible solution would be to simply write down rules that cover all possible cases, i.e., merely enforce arithmetic operators to work on propertied types too:

$$\frac{\Gamma \vdash e_1 : \text{Int} \quad \Gamma \vdash e_2 : \text{Int}}{\Gamma \vdash e_1 + e_2 : \text{Int}}$$

$$\frac{\Gamma \vdash e_1 : [\text{Int}]\langle \dots \rangle \quad \Gamma \vdash e_2 : \text{Int}}{\Gamma \vdash e_1 + e_2 : \text{Int}}$$

$$\frac{\Gamma \vdash e_1 : \text{Int} \quad \Gamma \vdash e_2 : [\text{Int}]\langle \dots \rangle}{\Gamma \vdash e_1 + e_2 : \text{Int}}$$

$$\frac{\Gamma \vdash e_1 : [\text{Int}]\langle \dots \rangle \quad \Gamma \vdash e_2 : [\text{Int}]\langle \dots \rangle}{\Gamma \vdash e_1 + e_2 : \text{Int}}$$

Although it works, repeatedly writing the same pattern may be annoying. Further, the difficulty of proving meta-theorems about a type theory increases as the number of presented rules does. A better approach would be to introduce one more construction that, when receives an expression of propertied type, evaluates to its underlying value or to the expression itself otherwise:

$$\frac{\Gamma \vdash e : [T]\langle \dots \rangle}{\Gamma \vdash \text{refresh}(e) : T}$$

$$\frac{\Gamma \vdash e : T}{\Gamma \vdash \text{refresh}(e) : T}$$

Having that, we can pack those four rules into just one:

$$\frac{\Gamma \vdash \text{refresh}(e_1) : \text{Int} \quad \Gamma \vdash \text{refresh}(e_2) : \text{Int}}{\Gamma \vdash e_1 + e_2 : \text{Int}}$$

Now we can return to our problem from the last example with a function. Recall its code is:

```

1  func greater_than_five (x : Int) with
2    if-has x n : Int then n > 5
3    else x > 5 in
4
5  let x = 1 in
6  let x' = set(x, 1) in
7
8  let a = greater_than_five x in
9  let b = greater_than_five x' in
10 let c = greater_than_five (x + x')
```

And here we face another typing problem; this time, the ill-typed expression is *greater_than_five x'*. This is because, just as arithmetic operators, our function expects to receive arguments of the integer type, but at line 9, we pass it x' , which is of a propertied one. To solve this, again, we need to force functions that work on the type X to receive arguments of propertied types based on X as well:

$$\frac{\Gamma \vdash \text{refresh}(t_1) : T_1 \rightarrow T_2 \quad \Gamma \vdash \text{refresh}(t_2) : T_1}{\Gamma \vdash t_1 t_2 : T_2}$$

In other words, we make our functions *polymorphic with respect to propertied type*. As it turned out, proper implementation of the polymorphism solves the problem of correct propagation of static annotations from Section 2.1. That is, as long as we store all our annotations in a propertied type, propagating the type of an argument makes it possible for constructions *get* and, essentially, *if-has*, to acquire the data and, consequently, decide how to compile themselves.

As not all implementations of polymorphism satisfy our requirements, we now have to choose the appropriate one. For instance, we cannot follow the classical, i.e., "boxed", approach, in which bodies are compiled ahead, an argument is set to be of an unknown type and internal runtime checks are present; the presence of runtime checks is what makes it incompatible for us. An approach that seems to fit our requirements is what people usually call a "compile-time", "static", "early-binding", or "static-binding" polymorphism, depending on the language they came from. This approach mainly relies on the notion of *function monomorphization*, meaning that instead of boxing types and compiling functions ahead, we generate a specialized version of a function each time one is invoked and substitute that unknown type with the type of calling argument, so the resulting call is made to its specialized version. Indeed, if there is a specialized version that has the same body as we need, a new version shall not be generated; the resulting call shall just use it. Following this intuition, a moment before it goes to the runtime, we expect our program to look like the following:

```

1  func greater_than_five_1 (x : Int) with x > 5 in
2  func greater_than_five_2 (x : Int) with 1 > 5 in
3
4  let x = 1 in
5  let x' = x in
6
7  let a = greater_than_five_1 x in
8  let b = greater_than_five_2 x' in
9  let c = greater_than_five_1 (x + x')
```

As we can see, we expressed a trivial optimization that propagates the value of x' . Before we go further, it is worth noting that our concept should never be used to express such optimizations; most modern compilers, without any notion of type properties, may effectively apply an optimization called *constant propagation*, reducing these whole constant expressions to their equivalent values at the stage of compilation, and obtain even a better result. Propagating values of constant expressions in this way rather consumes a discernible amount of a compiler's resources and bloats the output binary file's size than works as an optimization. We have chosen it just as a simple example and primarily to the reason that expressing something like linear types using this concept in an elegant manner is a whole separate research topic.

Nevertheless, sometimes, such optimizations in the code itself may be beneficial. As an example, we can annotate an array with its size hiddenly, provided the one is, of course, known. After that, in functions working on arrays, just as we did in the example above, we can instruct the compiler that if there is some data associated with an array that represents its size, then use this data to lift some runtime checks to compile-time. Another such specimen is annotating instances of 'std::any', whether from the C++'s or Rust's world, with their inhabited types. Both cases, in fact, would consume a significant amount of the compiler's resources but will make it possible to lift 100% of runtime checks that can be made at the stage of compilation to the compile-time, without bloating the output binary's size when used appropriately, which can be beneficial for performance-critical applications that are compiled rarely.

2.3 Transformation

This is the very time to discuss how exactly we evaluate all these static annotations during compilation. This is the last part of this section, proceeding after which the reader shall be ready for the presentation of our type theory, where we formalize type properties.

As we said many times, we must handle all these annotations before we run a program. This means that, right before we pass it to the runtime, it must proceed through an additional phase where all these annotations are handled. We call this phase *transformation* since it is the standard name for phases introduced for partially evaluating programs before one is run.

Formally, we can express transformation as the judgment $\Gamma \vdash e_1 \rightarrow e_2 : T$, which merely reads as " e_1 , under the typing context Γ , is transformed into e_2 , and the type of e_2 is T ". However, we also need to monomorphize our functions,

which means that we are supposed to somewhere store their both ordinary and specialized versions. For this purpose, we introduce a new context on which we operate — the functional context Δ :

$$\Delta ::= \diamond \mid \Delta, f :: x : T_1 . M : T_2 \mid \Delta, f[n] \triangleright x : T_1 . M : T_2$$

The entry $f :: x : T_1 . M : T_2$ keeps ordinary user functions (such as *greater_than_five*), while $f[n] \triangleright x : T_1 . M : T_2$ designates f 's specialized version (such as *greater_than_five_1* and *greater_than_five_2*). n here is just a natural number to distinguish different specialized versions.

Now our transformation shall look like the following:

$$\Gamma \vdash \langle \Delta_1; e_1 \rangle \Rightarrow \langle \Delta_2; e_2 \rangle : T$$

which now also means that to pass the transformation, it is required to have the functional context of Δ_1 , which, after the transformation, becomes Δ_2 .

Just as the usual typing judgment works under a typing context, the same way the transformation judgments tend to work under assumptions about transformations of different parts of its program. For this purpose, as long as the transformation works under the typing context and is actually typed itself, rather than introducing another context, we can store transformation judgments right in Γ :

$$\Gamma ::= \dots \mid \langle \Delta; x \rangle \Rightarrow \langle \Delta'; e \rangle : T$$

where the judgment $\langle \Delta; x \rangle \Rightarrow \langle \Delta'; e \rangle : T$ means that the variable x is transformed into the expression e , and the context changes from Δ to Δ' .

Now let us attempt to design transformation for simple cases. Constants, for instance, have nothing to do with type properties, so we transform them into themselves:

$$\frac{(\Gamma \text{ is well-formed})}{\Gamma \vdash \langle \Delta; n \rangle \Rightarrow \langle \Delta; n \rangle : \text{Int}}$$

where n is an integer constant.

Now it is the time to give a turn for *set* and *get*. As we said in previous sections, expressions of propertied type are transformed into their underlying expressions; we, however, have not yet defined what is a canonical term of a propertied type itself. As a suggestion, since propertied types serve only to hang on some additional data on an expression, and we store both the additional data and the expression's type in the propertied type itself, we could consider making any underlying expression be its canonical term. But this would break the uniqueness of typing for all other expressions, so we could instead uniformly denote *propertied*[e] to be a canonical term of any propertied type, whose underlying expression is e :

$$\frac{\Gamma \vdash e : T \quad \Gamma \vdash [T]\langle v : P \rangle}{\Gamma \vdash \text{propertied}[e] : [T]\langle v : P \rangle}$$

Following this intuition, *set* has no choice but to be transformed into a proper term of a propertied type:

$$\frac{\Gamma \vdash \langle \Delta; e \rangle \Rightarrow \langle \Delta_1; e' \rangle : T \quad \Gamma \vdash \langle \Delta_1; v \rangle \Rightarrow \langle \Delta'; v' \rangle : P}{\Gamma \vdash \langle \Delta; \text{set}(e, v) \rangle \Rightarrow \langle \Delta'; \text{propertied}[e'] \rangle : [T]\langle v' : P \rangle}$$

The transformation of *get* is simple and we can transform it as follows:

$$\frac{\Gamma \vdash \langle \Delta; M \rangle \Rightarrow \langle \Delta'; \text{propertied}[e'] \rangle : [T]\langle v : P \rangle}{\Gamma \vdash \langle \Delta; \text{get}(M) \rangle \Rightarrow \langle \Delta'; v \rangle : P}$$

Thus, having defined transformation for simple cases, we can now switch on slightly more difficult ones. For example, here is how we can design the transformation for the expression *let*:

$$\frac{(T_1 \neq [T]\langle \dots \rangle) \quad \Gamma \vdash \langle \Delta; M \rangle \Rightarrow \langle \Delta_M; M' \rangle : T_1 \quad \Gamma, \langle \Delta; x \rangle \Rightarrow \langle \Delta_M; x \rangle : T_1 \vdash \langle \Delta_M; N \rangle \Rightarrow \langle \Delta_N; N' \rangle : T_2}{\Gamma \vdash \langle \Delta; \text{let } x = M \text{ in } N \rangle \Rightarrow \langle \Delta_N; \text{let } x = M' \text{ in } N' \rangle : T_2}$$

$$\frac{\begin{array}{c} \Gamma, \Gamma' \vdash \langle \Delta_M; N[x/y] \rangle \rightarrow \langle \Delta_N; N' \rangle : T_2 \\ \Gamma \vdash \langle \Delta; M \rangle \rightarrow \langle \Delta_M; \text{propertied}[e] \rangle : [T_1] \langle a : T_a \rangle \\ (\Gamma' \equiv \langle \Delta; x \rangle \rightarrow \langle \Delta; x \rangle : T_1, \langle \Delta; y \rangle \rightarrow \langle \Delta_M; \text{propertied}[x] \rangle : [T_1] \langle a : T_a \rangle) \end{array}}{\Gamma \vdash \langle \Delta; \text{let } x = M \text{ in } N \rangle \rightarrow \langle \Delta_N; \text{let } x = e \text{ in } N' \rangle : T_2}$$

which is, as opposed to simple cases above, defined by two distinct rules. Consider an expression $\text{let } x = M \text{ in } N$. For now, there are two cases whose transformation we shall treat differently. The first is when M is an ordinary expression and is not of a propertied typed. In this case, there is nothing special to do: under the assumptions that M is transformed into M' , that x transforms into itself, and that N transforms into N' , we can transform the expression into $\text{let } x = M' \text{ in } N'$, taking in account all rules with composing typing and functional contexts. This is what the first rule deals with. The second rule, in turn, deals with the case when M is of a propertied type. In this case, we shall transform M into its underlying expression and properly propagate its information to the continuation. That is, we must propagate the propertied type but must not propagate the underlying expression since the construction *let* is here to abstract from concrete values.

Following these rules, we obtain that the example above is transformed into the following program:

```

1  let x = 1 in
2  let x' = x in
3  x + x' + 1
    
```

just as we expected.

Rules for transforming the expression *if – has* follow the same strategy; they explore the type of their expression and are transformed into one of their branches. Transformation of the function definition is just the expansion of our functional context with one more function definition:

$$\frac{\Gamma \vdash T_e \quad \Gamma, x : T_1 \vdash M : T_2 \quad \Gamma, f : T_1 \rightarrow T_2, \vdash \langle \Delta, f :: x : T_1 . M : T_2; e \rangle \rightarrow \langle \Delta'; e' \rangle : T_e}{\Gamma \vdash \langle \Delta; \text{func } f (x : T_1) \text{ with } M \text{ in } e \rangle \rightarrow \langle \Delta'; e' \rangle : T_e}$$

Rules for the transformation of function application are where things get complicated. In short, we take out the body of the function, perform its transformation, place the acquired instantiation into the functional context, and replace the function in the former expression with its specialized version. The complicated thing is that there are four cases of function applications that we have to treat differently, i.e., for which the transformation shall be different. Moreover, generating specialized versions of a function for exactly each function call may be inefficient since a specialized version that fits our requirements may be already present in the context. So, to build a comprehensive intuition on how to design such rules, too many notions from our type theory must be introduced. Therefore, it seems that it is the very time to start presenting our type theory.

3 The system $\lambda_{\rightarrow p}$: Formalization of Type Properties

In this section, we present a simple type theory ² augmented with the notion of type properties. We call this system $\lambda_{\rightarrow p}$.

3.1 Syntax

The concrete syntax of $\lambda_{\rightarrow p}$ is defined as follows:

²The theory was originally based on *Simply Typed Lambda Calculus* (...), but augmenting it with type properties eliminated almost all observable similarities.

(typing contexts)	$\Gamma ::= \diamond \mid \Gamma, x : T^+$
(types)	$T, P ::= \text{int} \mid \text{unit} \mid T_1 \rightarrow T_2$
(types and propertied types)	$T^+ ::= T \mid [T]\langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n] \rangle$
(expressions)	$F, M, N, L, e, t ::= \text{func } f \ x : T \text{ with } M \text{ in } N \mid \text{let } x = e \text{ in } L$ $\mid \text{if-has } L \ p : T \text{ bind-as } x \text{ in } M \text{ else } N$ $\mid \text{extract}(M) \mid \text{set}(M, p, e) \mid \text{get}(M, p)$ $\mid \text{erase}(M, p) \mid \text{refresh}(M) \mid M \ N \mid e_1 + e_2 \mid e_1 - e_2$ $\mid \text{assert}(e_1, e_2) \text{ before } e \mid f[k] \ e \mid x \mid v$
(expressions and internal terms)	$e^+ ::= e \mid \text{propertied}[e]$
(values)	$v ::= n$
(identifiers and variables)	x, y, f, p
(integer constants)	n, k

Here, $[T]\langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n] \rangle$ represents a propertied type based on the type T and has properties in the form of $p \hookrightarrow e[P]$, where p is an identifier used to refer to the property, and e is the property's expression of the type P .

The expressions *if-has*, *set*, and *get* are those constructions from pseudo-code samples from Section 2 but are generalized versions of them; the crucial distinction is that now each construction takes an identifier as an additional argument, and its justification is as follows. In a real language, a user may want to store different data in propertied types for different purposes. For this reason, we now make it possible to store an arbitrary amount of data in propertied types, each marked with a distinct name.

In contrast with Section 2, two more expressions operate on propertied types. These are *extract* and *erase*, the purposes of which one can infer from their names directly: to extract underlying expressions and erase one's certain properties, respectively.

The expression $f[k] \ e$ represent a call to a monomorphized (specialized) version of the function f . Programmers must not call monomorphized functions directly, as a compiler on its own translates all function calls to their appropriate monomorphized versions.

Finally, there is *assert*, a facility to make sure that two expressions, e_1 and e_2 , are transformed into the same expression, before the evaluation of e .

3.2 Typing Rules

Below we present inference rules for typing contexts and type formations.

$\frac{}{\diamond \text{ ctx}} \text{ Ctx-1}$		$\frac{\Gamma \vdash T^+ \quad (x \notin \Gamma)}{\Gamma, x : T^+ \text{ ctx}} \text{ Ctx-2}$	
$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \text{unit}} \text{ F-Unit}$	$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \text{int}} \text{ F-Int}$	$\frac{\Gamma \vdash T_1 \quad \Gamma \vdash T_2}{\Gamma \vdash T_1 \rightarrow T_2} \text{ F-Func}$	$\frac{\Gamma \vdash T_1}{\Gamma \vdash [T_1]\langle \rangle} \text{ F-Prop-1}$
$\frac{\Gamma \vdash [T]\langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n] \rangle \quad \Gamma \vdash e : P \quad (p_1 \neq p \wedge \dots \wedge p_n \neq p)}{\Gamma \vdash [T]\langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n], p \hookrightarrow e[P] \rangle} \text{ F-Prop-2}$			

The notation $(p_1 \neq p \wedge \dots \wedge p_n \neq p)$ in the rule *F-Prop-2* indicates that p must not be already present in the properties.

As we go through this section, in our inference rules, we use some non-standard notations. We list them, along with their corresponding meaning, below:

- We use $\Gamma \uplus \{\Gamma'\}$ to denote a context obtained by taking union of Γ and Γ' , preferring the latter's type if the types of variables with the same name mismatch. That is, $x : T_1, y : T_2 \uplus \{z : T_3\}$ is $x : T_1, y : T_2, z : T_3$, but $x : T_1, y : T_2 \uplus \{x : T_3\}$ is $x : T_3, y : T_2$.
- We use $[T]\langle \dots \rangle$ to represent any propertied type based on the type T .
- We use $T_1^+ \neq T_2^+$ to indicate that the type T_1^+ must not have the form of the type T_2^+ . For instance, $T \neq P_1 \rightarrow P_2$ states that T must not be a function type.

- We use $T_1^+ = T_2^+$ to indicate that the type T_1^+ must be of the type's T_2^+ form. Following the previous example, $T \neq P_1 \rightarrow P_2$ states that T must be a function type.
- We use $e^+[M/N]$ to designate the substitution of M with N in e^+ .

Next, we give a turn for introduction rules:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash () : \text{unit}} \text{ I-Unit}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash n : \text{int}} \text{ I-Int}$$

$$\frac{\begin{array}{c} \Gamma \uplus \{x : T_1\} \vdash M : T_2 \quad (T_2 \neq P_1 \rightarrow P_2) \quad \Gamma \vdash T_1 \\ \Gamma \uplus \{x : [T_1]\langle \rangle\} \vdash M : T_2 \quad \Gamma \uplus \{f : T_1 \rightarrow T_2\} \vdash e : T^+ \quad \Gamma \vdash T^+ \end{array}}{\Gamma \vdash \text{func } f \ x : T_1 \text{ with } M \text{ in } e : T^+} \text{ I-Func}$$

$$\frac{\Gamma \vdash M : T \quad \Gamma \vdash e : P}{\Gamma \vdash \text{set}(M, p, e) : [T]\langle p \hookrightarrow e[P] \rangle} \text{ I-Prop-1}$$

$$\frac{\Gamma \vdash M : [T]\langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n] \rangle \quad \Gamma \vdash e : P \quad (p_1 \neq p \wedge \dots \wedge p_n \neq p)}{\Gamma \vdash \text{set}(M, p, e) : [T]\langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n], p \hookrightarrow e[P] \rangle} \text{ I-Prop-2}$$

$$\frac{\Gamma \vdash M : [T]\langle \text{props}_1, p \hookrightarrow e[P_1], \text{props}_2 \rangle \quad \Gamma \vdash e' : P_2}{\Gamma \vdash \text{set}(M, p, e') : [T]\langle \text{props}_1, p \hookrightarrow e'[P_2], \text{props}_2 \rangle} \text{ I-Prop-3}$$

Notice that we prohibited functions to accept arguments of propertied types *only*, meaning that they must accept their base types as well. This is because we have attempted to hide for a user the fact that we use them to handle static annotations since it must look like the compiler mysteriously handles static annotations on its own.

Notice that, in this type system, functions can't return expressions of ones either. We must note that the incorporation of our concept subsumes the ability to return propertied types of functions as it is essential to facilitate the expression of beneficial features (such as linear types, for instance). In our type system, we lack this opportunity only because this system stands for demonstrational purposes (i.e., formalization of type properties), and it is already overcomplicated to be called simple. That is, such an opportunity, for us, is pretty effortless to bring but will introduce some further complications to the system and will probably make 8-level rules for function monomorphization even more cumbersome. We thus preferred to prohibit some expressions to return or evaluate to terms of propertied types, such a function application and *let*-expressions. The same applies to functions as we specially treat such values since, as we will see in Section 3.3, they represent names of entries of Δ contexts, the very contexts that keep information on such functions.

The rule *I-Prop-3* states that we can update a specific's property expression in a propertied type. In this rule, we write props_1 for all properties $p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n]$ that come before one with the name p and props_2 for all that come after.

We present rules that cover typing judgment of the remaining expressions below.

$$\frac{\Gamma \vdash e : T}{\Gamma \vdash \text{refresh}(e) : T} \text{ E-Refresh-1}$$

$$\frac{\Gamma \vdash e : [T]\langle \dots \rangle}{\Gamma \vdash \text{refresh}(e) : T} \text{ E-Refresh-2}$$

$$\frac{\Gamma \vdash \text{refresh}(t_1) : T_1 \rightarrow T_2 \quad \Gamma \vdash \text{refresh}(t_2) : T_1 \quad (T_2 \neq P_1 \rightarrow P_2)}{\Gamma \vdash t_1 \ t_2 : T_2} \text{ E-App}$$

$$\frac{\Gamma \vdash N : T^+ \quad \Gamma \uplus \{x : T^+\} \vdash M : T \quad (T \neq P_1 \rightarrow P_2)}{\Gamma \vdash \text{let } x = N \text{ in } M : T} \text{ E-Let}$$

$$\begin{array}{c}
 \frac{\Gamma \vdash \text{refresh}(e_1) : \text{int} \quad \Gamma \vdash \text{refresh}(e_2) : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} \text{ E-Plus} \\
 \frac{\Gamma \vdash \text{refresh}(e_1) : \text{int} \quad \Gamma \vdash \text{refresh}(e_2) : \text{int}}{\Gamma \vdash e_1 - e_2 : \text{int}} \text{ E-Minus} \\
 \frac{\Gamma \vdash M : [T] \langle \dots, p \hookrightarrow e[P], \dots \rangle}{\Gamma \vdash \text{get}(M, p) : P} \text{ E-Get} \\
 \frac{\begin{array}{c} (y \notin \Gamma) \quad \Gamma \vdash L : T_L \\ \Gamma \vdash T_x \quad \Gamma \vdash T^+ \quad \Gamma, y : [T_L] \langle \rangle \vdash N[L/y] : T^+ \\ \Gamma \uplus \{x : T_x\}, y : [T_L] \langle p \hookrightarrow x[T_x] \rangle \vdash M[L/y] : T^+ \end{array}}{\Gamma \vdash \text{if-has } L \text{ } p : T_x \text{ bind-as } x \text{ in } M \text{ else } N : T^+} \text{ E-If-Has-1} \\
 \frac{\begin{array}{c} (y \notin \Gamma) \quad \Gamma \vdash L : [T_L] \langle \text{props} \rangle \\ \Gamma \vdash T_x \quad \Gamma \vdash T^+ \quad \Gamma, y : [T_L] \langle \text{props} \setminus p \rangle \vdash N[L/y] : T^+ \\ \Gamma \uplus \{x : T_x\}, y : [T_L] \langle \text{props} \setminus p, p \hookrightarrow x[T_x] \rangle \vdash M[L/y] : T^+ \end{array}}{\Gamma \vdash \text{if-has } L \text{ } p : T_x \text{ bind-as } x \text{ in } M \text{ else } N : T^+} \text{ E-If-Has-2} \\
 \frac{\Gamma \vdash M : [T] \langle \dots \rangle}{\Gamma \vdash \text{extract}(M) : T} \text{ E-Extract} \\
 \frac{\Gamma \vdash M : [T] \langle \text{props}_1, p \hookrightarrow e[P], \text{props}_2 \rangle}{\Gamma \vdash \text{erase}(M, p) : [T] \langle \text{props}_1, \text{props}_2 \rangle} \text{ E-Erase} \\
 \frac{\Gamma \vdash e_1 : P \quad \Gamma \vdash e_2 : P \quad \Gamma \vdash e : T^+}{\Gamma \vdash \text{assert}(e_1, e_2) \text{ before } e : T^+} \text{ E-Assert}
 \end{array}$$

The rules *E-If-Has-1* and *E-If-Has-2* represent typing rules for the expression *if-has*. The rule *E-If-Has-1* states that if one tries to check a property's presence even on expressions of non-property types, then it is still well-typed. This is crucial because the expression that we investigate may easily be a function's argument, the real type of which is not known until the function is monomorphized. The reason that they require the code for the branch *else* to be well-typed when *L* is of an empty property type will be explored in Section 3.3.

Notice that the expression *get*, as opposed to *if-has*, works only on expressions of property types. This means that we are unable to retrieve a property from a function's argument without checking that it is present, since it is impossible for a function to work only on property types — they must work on their base types too.

The notation $\text{props} \setminus p$ represents the list of all properties of *props* excluding the one with name *p* (provided that one exists, of course).

As already said in Section 2, we write $\text{property}[e]$ to a term of any property type with the underlying expression *e*. We express it formally with this rule:

$$\frac{\Gamma \vdash e : T \quad \Gamma \vdash [T] \langle \text{props} \rangle}{\Gamma \vdash \text{property}[e] : [T] \langle \text{props} \rangle} \text{ E-Prop-Repr}$$

As we will see in Section 3.3, we write $f[k] \ e$ to designate an application of monomorphized version of the function *f* with the argument *e*. To preserve typing for transformation, we present its typing rule below:

$$\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1 \quad (T_2 \neq P_1 \rightarrow P_2)}{\Gamma \vdash t_1[k] \ t_2 : T_2} \text{ E-App-Compiled}$$

For now, to complete typing rules, we are only left with the structural rule that deduces variables from typing contexts — namely, the rule *Var*:

$$\frac{\Gamma \text{ ctx } (x : T^+ \in \Gamma)}{\Gamma \vdash x : T^+} \text{ S-Var}$$

3.3 Transformation

In this section, we formalize the transformation phase, the very phase where we handle all static annotations. Rules presented in this section do not assume that terms they operate on are well-typed explicitly. This is because, as we will see in Section 3.4, only well-typed programs may be passed to the runtime, so it would be meaningless to perform transformation if a program is ill-typed.

Just as in Section 2, for the transformation judgment we introduce an additional context that keeps relevant information about functions, which we call a *functional context*. This context is defined in the same way as it is done in Section 2, except that now it encompasses one more entry — \square — which we call the universal Δ context:

$$\Delta ::= \diamond \mid \Delta, f :: x : T_1 . M : T_2 \mid \Delta, f[n] \triangleright x : T_1 . M : T_2 \mid \square$$

Here, $f :: x : T_1 . M : T_2$ indicates that f is a function that takes x of the type T_1 as an argument, its body is M , and the return type is T_2 . It stands for a function that contains a raw code (not transformed yet) and therefore cannot be propagated to a runtime call. $f[n] \triangleright x : T_1 . M : T_2$, where n is any natural number, stands for a monomorphized version of f , meaning that its body has already gone through transformation and it is ready to be called. n here stands just for distinguishing different monomorphizations of the function f . The meaning of the universal Δ context is signified by the rule R-S-Uni.

Below we present rules for Δ contexts formation:

$$\frac{}{\diamond \text{dctx}} \text{Delta-Ctx-1}$$

$$\frac{\Delta \text{dctx} \quad (f \notin \Delta)}{\Delta, f :: x : T_1 . M : T_2 \text{dctx}} \text{Delta-Ctx-2}$$

$$\frac{\Delta \text{dctx} \quad (f[n] \triangleright x : T_1 . M : T_2 \notin \Delta)}{\Delta, f[k] \triangleright x : T_1 . M : T_2 \text{dctx}} \text{Delta-Ctx-3}$$

As we have seen in Section 2, functional contexts are intended to work under the new transformation judgment, which itself is intended to work under a typing context. Here is the extended version of the context Γ and the corresponding structural rule:

$$\Gamma ::= \dots \mid \Gamma, \langle \Delta_1; x \rangle \twoheadrightarrow_p \langle \Delta_2; e^+ \rangle : T^+$$

$$\frac{\Gamma \text{ctx} \quad \Delta \text{dctx} \quad \Delta' \text{dctx} \quad \Gamma \vdash x : T^+ \quad \Gamma \vdash e^+ : T^+ \quad (x \twoheadrightarrow_p \notin \Gamma)}{\Gamma, \langle \Delta; x \rangle \twoheadrightarrow_p \langle \Delta'; e^+ \rangle : T^+ \text{ctx}} \text{Ctx-3}$$

To illustrate, the judgment $\Gamma \vdash \langle \Delta; e \rangle \twoheadrightarrow_p \langle \Delta'; e^+ \rangle : T^+$ merely means that the program e , under the functional context Δ , is transformed into the program e^+ , the functional context became Δ' , and the type of e^+ is T^+ .

We extend the list of notations we use in this section:

- $f \notin \Delta$ — there is no entry $f :: x : T_1 . M : T_2$, where f matches exactly this f , T_1 and T_2 match *arbitrary* types, and M matches *any* term, in the context Δ .
- $f[n] \triangleright x : T_1 . M : T_2 \notin \Delta$ — there is no entry $f[k] \triangleright x : T_1 . M : T_2$, where f matches exactly this f , T_1 and T_2 match exactly these types, M matches exactly this term, and k matches any natural number, in the context Δ .
- $x \twoheadrightarrow_p \notin \Gamma$ — there is no entry $\langle \Delta_1; x \rangle \twoheadrightarrow_p \langle \Delta_2; e^+ \rangle : T^+$, where x matches exactly this x , Δ_1 and Δ_2 match any contexts, e^+ matches any term, and T^+ matches any type, in the context Γ .

We also extend structural rules that work on typing context to cover the transformation judgment as follows:

$$\frac{\Gamma \text{ctx} \quad (\langle \Delta; x \rangle \twoheadrightarrow_p \langle \Delta'; e^+ \rangle : T^+ \in \Gamma)}{\Gamma \vdash \langle \Delta; x \rangle \twoheadrightarrow_p \langle \Delta'; e^+ \rangle : T^+} \text{R-S-Red}$$

$$\frac{\Delta \text{dctx} \quad \Gamma \vdash \langle \square; x \rangle \rightarrow_p \langle \square; e^+ \rangle : T^+}{\Gamma \vdash \langle \Delta; x \rangle \rightarrow_p \langle \Delta; e^+ \rangle : T^+} \text{R-S-Uni}$$

$$\frac{\Gamma \vdash \langle \Delta_1, \Delta_2; e \rangle \rightarrow_p \langle \Delta'_1, \Delta'_2; e^+ \rangle : T^+}{\Gamma \vdash \langle \Delta_2, \Delta_1; e \rangle \rightarrow_p \langle \Delta'_1, \Delta'_2; e^+ \rangle : T^+} \text{R-S-Exchange-Delta-1}$$

$$\frac{\Gamma \vdash \langle \Delta_1, \Delta_2; e \rangle \rightarrow_p \langle \Delta'_1, \Delta'_2; e^+ \rangle : T^+}{\Gamma \vdash \langle \Delta_1, \Delta_2; e \rangle \rightarrow_p \langle \Delta'_2, \Delta'_1; e^+ \rangle : T^+} \text{R-S-Exchange-Delta-2}$$

The rule *R-S-Uni* states that if a transformation has the universal context at both sides, then we can derive transformation for any valid context. This is useful when, for instance, assuming transformation of variables to themselves — in any valid context, one should be transformed into itself. Now it is the turn for the actual transformation rules. Constants and literals have nothing to do with propertied types, so it would be reasonable to transform them into themselves:

$$\frac{\Gamma \text{ctx} \quad \Delta \text{dctx}}{\Gamma \vdash \langle \Delta; () \rangle \rightarrow_p \langle \Delta; () \rangle : \text{unit}} \text{R-V-Unit}$$

$$\frac{\Gamma \text{ctx} \quad \Delta \text{dctx}}{\Gamma \vdash \langle \Delta; n \rangle \rightarrow_p \langle \Delta; n \rangle : \text{int}} \text{R-V-Int}$$

$$\frac{\Delta \text{dctx} \quad \Gamma \vdash f : T_1 \rightarrow T_2 \quad (f :: x : T_1 . M : T_2 \in \Delta)}{\Gamma \vdash \langle \Delta; f \rangle \rightarrow_p \langle \Delta; f \rangle : T_1 \rightarrow T_2} \text{R-V-Func}$$

Next, we give turn to the expression *set*:

$$\frac{\Gamma \vdash \langle \Delta; M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T \quad \Gamma \vdash \langle \Delta_M; e \rangle \rightarrow_p \langle \Delta'; e' \rangle : P}{\Gamma \vdash \langle \Delta; \text{set}(M, p, e) \rangle \rightarrow_p \langle \Delta'; \text{propertied}[M'] \rangle : [T] \langle p \hookrightarrow e'[P] \rangle} \text{R-Set-1}$$

$$\frac{\Gamma \vdash \langle \Delta_M; e \rangle \rightarrow_p \langle \Delta'; e' \rangle : P \quad (p_1 \neq p \wedge \dots \wedge p_n \neq p)}{\Gamma \vdash \langle \Delta; M \rangle \rightarrow_p \langle \Delta_M; \text{propertied}[M'] \rangle : [T] \langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n] \rangle} \text{R-Set-2}$$

$$\frac{\Gamma \vdash \langle \Delta_M; e_2 \rangle \rightarrow_p \langle \Delta'; e'_2 \rangle : P_2}{\Gamma \vdash \langle \Delta; \text{set}(M, p, e_2) \rangle \rightarrow_p \langle \Delta'; \text{propertied}[M'] \rangle : [T] \langle \text{props}_1, p \hookrightarrow e'_2[P_2], \text{props}_2 \rangle} \text{R-Set-3}$$

One more thing to note before we continue. One may observe that we do not evaluate the expression of the property but rather store its transformed version. A better approach would be to impose some restrictions, i.e., permit occurrence only of constant expressions. Then, given that we store only constant expressions, we can evaluate them already at the stage of compilation and do not store expressions but rather their concrete values. Although it is a great way to go, two reasons made us choose the other. The first thing to note is that all expressions in our type theory are constant expressions, and thus we can evaluate all of them at compile-time. The second is that, again, as with the allowance for functions to have a range of propertied types, this would bring the theory to another level of difficulty and cause more confusion in the reader's eyes, which we cannot afford, as the theory stands only for demonstrational purposes.

Next, we give a turn for rules that work on type properties as well, but, this time, they are responsible for retrieving, erasing, extracting their underlying, and so forth:

$$\frac{\Gamma \vdash \langle \Delta; e \rangle \rightarrow_p \langle \Delta'; e' \rangle : T}{\Gamma \vdash \langle \Delta; \text{refresh}(e) \rangle \rightarrow_p \langle \Delta'; e' \rangle : T} \text{R-Refresh-1}$$

$$\frac{\Gamma \vdash \langle \Delta; e \rangle \rightarrow_p \langle \Delta_e; \text{propertied}[M] \rangle : [T] \langle \dots \rangle \quad \Gamma \vdash \langle \Delta_e; M \rangle \rightarrow_p \langle \Delta'; M' \rangle : T}{\Gamma \vdash \langle \Delta; \text{refresh}(e) \rangle \rightarrow_p \langle \Delta'; M' \rangle : T} \text{R-Refresh-2}$$

$$\begin{array}{c}
 \frac{\Gamma \vdash \langle \Delta; M \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : [T] \langle \dots, p \hookrightarrow e[P], \dots \rangle \quad \Gamma \vdash \langle \Delta_M; e \rangle \twoheadrightarrow_p \langle \Delta_e; e' \rangle : P}{\Gamma \vdash \langle \Delta; \text{get}(M, p) \rangle \twoheadrightarrow_p \langle \Delta_e; e' \rangle : P} \text{R-Get} \\
 \\
 \frac{\Gamma \vdash \langle \Delta; M \rangle \twoheadrightarrow_p \langle \Delta_e; \text{propertied}[e] \rangle : [T] \langle \dots \rangle \quad \Gamma \vdash \langle \Delta_e; e \rangle \twoheadrightarrow_p \langle \Delta'; e' \rangle : T}{\Gamma \vdash \langle \Delta; \text{extract}(M) \rangle \twoheadrightarrow_p \langle \Delta'; e' \rangle : T} \text{R-Extract} \\
 \\
 \frac{\Gamma \vdash \langle \Delta; M \rangle \twoheadrightarrow_p \langle \Delta'; \text{propertied}[M'] \rangle : [T] \langle \text{props}_1, p \hookrightarrow e[P], \text{props}_2 \rangle}{\Gamma \vdash \langle \Delta; \text{erase}(M, p) \rangle \twoheadrightarrow_p \langle \Delta'; \text{propertied}[M'] \rangle : [T] \langle \text{props}_1, \text{props}_2 \rangle} \text{R-Erase} \\
 \\
 \frac{\Gamma \vdash \langle \Delta; e_1 \rangle \twoheadrightarrow_p \langle \Delta_1; e' \rangle : P \quad \Gamma \vdash \langle \Delta_1; e_2 \rangle \twoheadrightarrow_p \langle \Delta_2; e' \rangle : P \quad \Gamma \vdash \langle \Delta_2; e \rangle \twoheadrightarrow_p \langle \Delta'; e^+ \rangle : T^+}{\Gamma \vdash \langle \Delta; \text{assert}(e_1, e_2) \text{ before } e \rangle \twoheadrightarrow_p \langle \Delta'; e^+ \rangle : T^+} \text{R-Assert}
 \end{array}$$

The rules below define the transformation of the expression *if-has*. These rules enforce L to enter the branch *else* being of a *propertied* type, even if the one is originally not. This is crucial because, otherwise, the L 's type with which it enters the branch would be unknown. This is because there are three cases in which we can send L to the *else* branch. The first is when L is of a *propertied* type but does not have the desired property (the rule *R-If-Has-2*). The second is when L is again of a *propertied* type, but the property we are looking for has a different type (the rule *R-If-Has-3*). The third is when L is not of a *propertied* type, meaning that it obviously lacks the property we are searching for (the rule *R-If-Has-1*); because L , in two cases above, enters the branch *else* being of a *propertied* type, we shall send it to the branch L being of a *propertied* type as well.

$$\begin{array}{c}
 \frac{(y \notin \Gamma) \quad \Gamma \vdash \langle \Delta; L \rangle \twoheadrightarrow_p \langle \Delta_L; L' \rangle : T_L \quad \Gamma, y : [T_L] \langle \rangle, \langle \Delta; y \rangle \twoheadrightarrow_p \langle \Delta_L; \text{propertied}[L'] \rangle : [T_L] \langle \rangle \vdash \langle \Delta_L; N[L/y] \rangle \twoheadrightarrow_p \langle \Delta_N; N' \rangle : T^+}{\Gamma \vdash \langle \Delta; \text{if-has } L \text{ } p : T_x \text{ bind-as } x \text{ in } M \text{ else } N \rangle \twoheadrightarrow_p \langle \Delta_N; N' \rangle : T^+} \text{R-If-Has-1} \\
 \\
 \frac{\Gamma \vdash \langle \Delta_L; N \rangle \twoheadrightarrow_p \langle \Delta_N; N' \rangle : T^+ \quad (p_1 \neq p \wedge \dots \wedge p_n \neq p) \quad \Gamma \vdash \langle \Delta; L \rangle \twoheadrightarrow_p \langle \Delta_L; L' \rangle : [T] \langle p_1 \hookrightarrow e_1[P_1], \dots, p_n \hookrightarrow e_n[P_n] \rangle}{\Gamma \vdash \langle \Delta; \text{if-has } L \text{ } p : T_x \text{ bind-as } x \text{ in } M \text{ else } N \rangle \twoheadrightarrow_p \langle \Delta_N; N' \rangle : T^+} \text{R-If-Has-2} \\
 \\
 \frac{\Gamma \vdash \langle \Delta_L; N \rangle \twoheadrightarrow_p \langle \Delta_N; N' \rangle : T^+ \quad (P \neq T_x) \quad \Gamma \vdash \langle \Delta; L \rangle \twoheadrightarrow_p \langle \Delta_L; L' \rangle : [T] \langle \dots, p \hookrightarrow e[P], \dots \rangle}{\Gamma \vdash \langle \Delta; \text{if-has } L \text{ } p : T_x \text{ bind-as } x \text{ in } M \text{ else } N \rangle \twoheadrightarrow_p \langle \Delta_N; N' \rangle : T^+} \text{R-If-Has-3} \\
 \\
 \frac{\Gamma \vdash \langle \Delta_L; e \rangle \twoheadrightarrow_p \langle \Delta_e; e' \rangle : P \quad \Gamma \vdash \langle \Delta; L \rangle \twoheadrightarrow_p \langle \Delta_L; L' \rangle : [T] \langle \dots, p \hookrightarrow e[P], \dots \rangle \quad \Gamma \uplus \{x : P, \langle \Delta_L; x \rangle \twoheadrightarrow_p \langle \Delta_e; e' \rangle : P\} \vdash \langle \Delta_e; M \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T^+}{\Gamma \vdash \langle \Delta; \text{if-has } L \text{ } p : P \text{ bind-as } x \text{ in } M \text{ else } N \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T^+} \text{R-If-Has-4}
 \end{array}$$

The transformation rule for a function definition is as follows:

$$\frac{\Gamma \uplus \{x : T_1\} \vdash M : T_2 \quad \Gamma \uplus \{f : T_1 \rightarrow T_2\} \vdash \langle \Delta, f :: x : T_1 . M : T_2; e \rangle \twoheadrightarrow_p \langle \Delta'; e' \rangle : T^+ \quad (f \notin \Delta)}{\Gamma \vdash \langle \Delta; \text{func } f \text{ } x : T_1 \text{ with } M \text{ in } e \rangle \twoheadrightarrow_p \langle \Delta'; e' \rangle : T^+} \text{R-Func}$$

The requirement on the typing judgment $\Gamma \uplus \{x : T_1\} \vdash M : T_2$ is only needed to infer the type T_2 , not to ensure that the body is well-typed. As we already said, this is because it is meaningless to perform the transformation on an ill-typed program.

$$\begin{array}{c}
 (T_1 \neq P_1 \rightarrow P_2) \\
 (\Delta_f \equiv f :: x : T_1 . M : T_2) \\
 (f[n] \triangleright x : T_1 . M' : T_2 \notin \Delta) \\
 (\Delta_{f[k]} \equiv f[k] \triangleright x : T_1 . M' : T_2) \\
 \Gamma \vdash \langle \Delta_M, \Delta_f, \Delta_{f[k]}; N \rangle \twoheadrightarrow_p \langle \Delta'; N' \rangle : T_1 \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \twoheadrightarrow_p \langle \Delta, \Delta_f; f \rangle : T_1 \rightarrow T_2 \\
 \Gamma \uplus \{x : T_1, \langle \square; x \rangle \twoheadrightarrow_p \langle \square; x \rangle : T_1\} \vdash \langle \Delta; M \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T_2 \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \twoheadrightarrow_p \langle \Delta'; f[k] N' \rangle : T_2
 \end{array}
 \text{R-App-Compile}$$

$$\begin{array}{c}
 (T_1 \neq P_1 \rightarrow P_2) \\
 (\Delta_f \equiv f :: x : T_1 . M : T_2) \\
 (\Delta_{f[k]} \equiv f[k] \triangleright x : T_1 . M' : T_2) \\
 \Gamma \vdash \langle \Delta_M, \Delta_f, \Delta_{f[k]}; N \rangle \twoheadrightarrow_p \langle \Delta'; N' \rangle : T_1 \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \twoheadrightarrow_p \langle \Delta, \Delta_f, \Delta_{f[k]}; f \rangle : T_1 \rightarrow T_2 \\
 \Gamma \uplus \{x : T_1, \langle \square; x \rangle \twoheadrightarrow_p \langle \square; x \rangle : T_1\} \vdash \langle \Delta; M \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T_2 \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \twoheadrightarrow_p \langle \Delta'; f[k] N' \rangle : T_2
 \end{array}
 \text{R-App-Ready}$$

$$\begin{array}{c}
 (\Delta_f \equiv f :: x : P_1 \rightarrow P_2 . M : T_M) \\
 (\Delta_{f[k]} \equiv f[k] \triangleright x : P_1 \rightarrow P_2 . M' : T_M) \\
 (f[n] \triangleright x : P_1 \rightarrow P_2 . M' : T_M \notin \Delta \cup \Delta_{f'}) \\
 \Gamma \vdash \langle \Delta, \Delta_f; N \rangle \twoheadrightarrow_p \langle \Delta_{f'}, \Delta_f; f' \rangle : P_1 \rightarrow P_2 \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \twoheadrightarrow_p \langle \Delta, \Delta_f; f \rangle : P_1 \rightarrow P_2 \rightarrow T_M \\
 \Gamma \uplus \{x : P_1 \rightarrow P_2, \langle \Delta; x \rangle \twoheadrightarrow_p \langle \Delta_{f'}, f' \rangle : P_1 \rightarrow P_2\} \vdash \langle \Delta_{f'}, M \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \twoheadrightarrow_p \langle \Delta_M, \Delta_f, \Delta_{f[k]}; f[k] f' \rangle : T_M
 \end{array}
 \text{R-App-Compile-Func}$$

$$\begin{array}{c}
 (\Delta_f \equiv f :: x : P_1 \rightarrow P_2 . M : T_M) \\
 (\Delta_{f[k]} \equiv f[k] \triangleright x : P_1 \rightarrow P_2 . M' : T_M) \\
 \Gamma \vdash \langle \Delta, \Delta_f; N \rangle \twoheadrightarrow_p \langle \Delta_{f'}, \Delta_f, \Delta_{f[k]}; f' \rangle : P_1 \rightarrow P_2 \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \twoheadrightarrow_p \langle \Delta, \Delta_f; f \rangle : P_1 \rightarrow P_2 \rightarrow T_M \\
 \Gamma \uplus \{x : P_1 \rightarrow P_2, \langle \Delta; x \rangle \twoheadrightarrow_p \langle \Delta_{f'}, f' \rangle : P_1 \rightarrow P_2\} \vdash \langle \Delta_{f'}, M \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \twoheadrightarrow_p \langle \Delta_M, \Delta_f, \Delta_{f[k]}; f[k] f' \rangle : T_M
 \end{array}
 \text{R-App-Ready-Func}$$

$$\begin{array}{c}
 (f[n] \triangleright x : T_1 . M' : T_2 \notin \Delta \cup \Delta_y) \\
 (\Delta_{f[k]} \equiv f[k] \triangleright x : T_1 . M' : T_2) \\
 (T_1 \neq P_1 \rightarrow P_2) \wedge (y \notin \Gamma) \\
 (\Delta_f \equiv f :: x : T_1 . M : T_2) \\
 \Gamma \vdash \langle \Delta_y; e \rangle \twoheadrightarrow_p \langle \Delta_y; e' \rangle : T_1 \\
 \Gamma \uplus \{\Gamma_f\} \vdash \langle \Delta_y; M[x/y] \rangle \twoheadrightarrow_p \langle \Delta_M; M' \rangle : T_2 \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \twoheadrightarrow_p \langle \Delta, \Delta_f; f \rangle : T_1 \rightarrow T_2 \\
 \Gamma \vdash \langle \Delta, \Delta_f; N \rangle \twoheadrightarrow_p \langle \Delta_y, \Delta_f; \text{propertied}[e] \rangle : [T_1] \langle \text{props} \rangle \\
 (\Gamma_f \equiv x : T_1, y : [T_1] \langle \text{props} \rangle, \langle \Delta; y \rangle \twoheadrightarrow_p \langle \Delta_y; \text{propertied}[x] \rangle : [T_1] \langle \text{props} \rangle) \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \twoheadrightarrow_p \langle \Delta_M, \Delta_f, \Delta_{f[k]}; f[k] e' \rangle : T_2
 \end{array}
 \text{R-App-Compile-Prop-1}$$

$$\begin{array}{c}
 (\Delta_{f[k]} \equiv f[k] \triangleright x : T_1 . M' : T_2) \\
 (T_1 \neq P_1 \rightarrow P_2) \wedge (y \notin \Gamma) \\
 (\Delta_f \equiv f :: x : T_1 . M : T_2) \\
 \Gamma \vdash \langle \Delta_y; e \rangle \rightarrow_p \langle \Delta_y; e' \rangle : T_1 \\
 \Gamma \uplus \{\Gamma_f\} \vdash \langle \Delta_y; M[x/y] \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_2 \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \rightarrow_p \langle \Delta, \Delta_f; f \rangle : T_1 \rightarrow T_2 \\
 \Gamma \vdash \langle \Delta, \Delta_f; N \rangle \rightarrow_p \langle \Delta_y, \Delta_f, \Delta_{f[k]}; \text{propertyied}[e] \rangle : [T_1] \langle \text{props} \rangle \\
 (\Gamma_f \equiv x : T_1, y : [T_1] \langle \text{props} \rangle, \langle \Delta; y \rangle \rightarrow_p \langle \Delta_y; \text{propertyied}[x] \rangle : [T_1] \langle \text{props} \rangle) \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \rightarrow_p \langle \Delta_M, \Delta_f, \Delta_{f[k]}; f[k] e' \rangle : T_2
 \end{array}
 \quad \text{R-App-Ready-Prop-1}$$

$$\begin{array}{c}
 (f[n] \triangleright x : P_1 \rightarrow P_2 . M' : T_M \notin \Delta \cup \Delta_{f'}) \\
 (\Delta_{f[k]} \equiv f[k] \triangleright x : P_1 \rightarrow P_2 . M' : T_M) \\
 (\Delta_f \equiv f :: x : P_1 \rightarrow P_2 . M : T_M) \\
 \Gamma \vdash \langle \Delta_{f'}; f' \rangle \rightarrow_p \langle \Delta_{f'}; f' \rangle : P_1 \rightarrow P_2 \\
 \Gamma \uplus \{\Gamma_f\} \vdash \langle \Delta_{f'}; M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \rightarrow_p \langle \Delta, \Delta_f; f \rangle : P_1 \rightarrow P_2 \rightarrow T_M \\
 \Gamma \vdash \langle \Delta, \Delta_f; N \rangle \rightarrow_p \langle \Delta_{f'}, \Delta_f; \text{propertyied}[f'] \rangle : [P_1 \rightarrow P_2] \langle \text{props} \rangle \\
 (\Gamma_f \equiv x : [P_1 \rightarrow P_2] \langle \text{props} \rangle, \langle \Delta; x \rangle \rightarrow_p \langle \Delta_{f'}; \text{propertyied}[f'] \rangle : [P_1 \rightarrow P_2] \langle \text{props} \rangle) \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \rightarrow_p \langle \Delta_M, \Delta_f, \Delta_{f[k]}; f[k] f' \rangle : T_M
 \end{array}
 \quad \text{R-App-Compile-Prop-2}$$

$$\begin{array}{c}
 (\Delta_{f[k]} \equiv f[k] \triangleright x : P_1 \rightarrow P_2 . M' : T_M) \\
 (\Delta_f \equiv f :: x : P_1 \rightarrow P_2 . M : T_M) \\
 \Gamma \vdash \langle \Delta_{f'}; f' \rangle \rightarrow_p \langle \Delta_{f'}; f' \rangle : P_1 \rightarrow P_2 \\
 \Gamma \uplus \{\Gamma_f\} \vdash \langle \Delta_{f'}; M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \Gamma \vdash \langle \Delta_i; \text{refresh}(F) \rangle \rightarrow_p \langle \Delta, \Delta_f; f \rangle : P_1 \rightarrow P_2 \rightarrow T_M \\
 \Gamma \vdash \langle \Delta, \Delta_f; N \rangle \rightarrow_p \langle \Delta_{f'}, \Delta_f, \Delta_{f[k]}; \text{propertyied}[f'] \rangle : [P_1 \rightarrow P_2] \langle \text{props} \rangle \\
 (\Gamma_f \equiv x : [P_1 \rightarrow P_2] \langle \text{props} \rangle, \langle \Delta; x \rangle \rightarrow_p \langle \Delta_{f'}; \text{propertyied}[f'] \rangle : [P_1 \rightarrow P_2] \langle \text{props} \rangle) \\
 \hline
 \Gamma \vdash \langle \Delta_i; FN \rangle \rightarrow_p \langle \Delta_M, \Delta_f, \Delta_{f[k]}; f[k] f' \rangle : T_M
 \end{array}
 \quad \text{R-App-Ready-Prop-2}$$

$$\begin{array}{c}
 \Delta \text{ dctx} \quad \Gamma \vdash f : T_1 \rightarrow T_2 \quad \Gamma \vdash \langle \Delta; N \rangle \rightarrow_p \langle \Delta; N' \rangle : T_1 \quad (f[k] \triangleright x : T_1 . M : T_2 \in \Delta) \\
 \hline
 \Gamma \vdash \langle \Delta; f[k] N \rangle \rightarrow_p \langle \Delta; f[k] N' \rangle : T_2
 \end{array}
 \quad \text{R-App-Compiled}$$

The rules above are responsible for the transformation of a function application. As we discussed in Section 2, in our system, functions, at the stage of transformation are monomorphized. Rules *R-App-Compile**, given that there is no suitable monomorphized version yet, perform this monomorphization for a function application. Rules *R-App-Ready** handle cases when the appropriate function is already in the context, thus they do not generate its new version but rather compile the call to that specialized one.

Now let us investigate these rules. In our system, there are four different types of function applications that we should treat differently:

- The first is when an argument of a non-propertyied and non-function type is applied to a function. This is the simplest case, and its transformation is as follows. Firstly, to perform monomorphization, we need to obtain the name of the target function to retrieve its body from the functional context. For this, we perform the transformation of a function's position expression itself. Then, under the assumption that the argument is transformed into itself, we transform the function's body. Finally, we perform transformation for an argument's position expression and then transform the whole expression into the call to the function's specialized version. This is what the rules *R-App-Compile* and *R-App-Ready* do.
- The second is when a function is applied to another function. The difference with the first case is that this time, we cannot hide the function's value behind an arbitrary variable while transforming the body of the former, so we perform the body's transformation under the assumption that the argument's variable is transformed

not into itself but rather to a concrete function. This is because, as we have seen in the first case, to transform a function application, we need its concrete value to retrieve its body from a Δ context, meaning that the expression in a function's position must be transformed into a proper function name. This is what the rules *R-App-Compile-Func* and *R-App-Ready-Func* do.

- The third is when an argument of a propertied type the underlying expression of which is not a function is applied to a function. In this case, we need to transform the body under the assumption that the argument is of the propertied type, not of its original one. As long as the underlying expression is not a function, we can abstract over them, i.e., say that the underlying expression is an arbitrary variable. This is what the rules *R-App-Compile-Prop-1* and *R-App-Ready-Prop-1* do.
- The last case is when an argument of a propertied type the underlying expression of which is a function applied to another function. This case is different from the previous one in the same way as the first is different from the second. The rules *R-App-Compile-Prop-1* and *R-App-Ready-Prop-1* handle this case.

Further, every *R-App-Compile-** rule places an acquired monomorphized version into the resulting Δ context.

The last rule, *R-App-Compiled*, transforms an already transformed function call. Since the function call is already transformed, this rule transforms one to itself. Below we present rules for transformation of the *let* expression:

$$\begin{array}{c}
 (T \neq P_1 \rightarrow P_2) \wedge (y \notin \Gamma) \\
 \Gamma \vdash \langle \Delta_N; e \rangle \rightarrow_p \langle \Delta_N; e' \rangle : T \\
 \Gamma \uplus \{\Gamma_f\} \vdash \langle \Delta_N; M[x/y] \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \Gamma \vdash \langle \Delta; N \rangle \rightarrow_p \langle \Delta_N; \text{propertied}[e] \rangle : [T]\langle \text{props} \rangle \\
 (\Gamma_f \equiv x : T, y : [T]\langle \text{props} \rangle, \langle \Delta; y \rangle \rightarrow_p \langle \Delta_N; \text{propertied}[x] \rangle : [T]\langle \text{props} \rangle) \\
 \hline
 \Gamma \vdash \langle \Delta; \text{let } x = N \text{ in } M \rangle \rightarrow_p \langle \Delta_M; \text{let } x = e' \text{ in } M' \rangle : T_M \quad \text{R-Let-Prop-1}
 \end{array}$$

$$\begin{array}{c}
 \Gamma \vdash \langle \Delta_N; f \rangle \rightarrow_p \langle \Delta_N; f \rangle : P_1 \rightarrow P_2 \\
 \Gamma \uplus \{\Gamma_M\} \vdash \langle \Delta_N; M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \Gamma \vdash \langle \Delta; N \rangle \rightarrow_p \langle \Delta_N; \text{propertied}[f] \rangle : [P_1 \rightarrow P_2]\langle \text{props} \rangle \\
 (\Gamma_M \equiv x : [P_1 \rightarrow P_2]\langle \text{props} \rangle, \langle \Delta; x \rangle \rightarrow_p \langle \Delta_N; \text{propertied}[f] \rangle : [P_1 \rightarrow P_2]\langle \text{props} \rangle) \\
 \hline
 \Gamma \vdash \langle \Delta; \text{let } x = N \text{ in } M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \quad \text{R-Let-Prop-2}
 \end{array}$$

$$\begin{array}{c}
 \Gamma \vdash \langle \Delta; F \rangle \rightarrow_p \langle \Delta_F; f \rangle : T_1 \rightarrow T_2 \\
 \Gamma \uplus \{x : T_1 \rightarrow T_2, \langle \Delta; x \rangle \rightarrow_p \langle \Delta_F; f \rangle : T_1 \rightarrow T_2\} \vdash \langle \Delta_F; M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \hline
 \Gamma \vdash \langle \Delta; \text{let } x = F \text{ in } M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \quad \text{R-Let-Func}
 \end{array}$$

The rest of the rules are those which don't transform/evaluate their expressions but instead, propagate the transformation of their subexpressions as follows:

$$\begin{array}{c}
 \Gamma \vdash \langle \Delta; \text{refresh}(e_1) \rangle \rightarrow_p \langle \Delta_{e_1}; e'_1 \rangle : \text{int} \quad \Gamma \vdash \langle \Delta_{e_1}; \text{refresh}(e_2) \rangle \rightarrow_p \langle \Delta_{e_2}; e'_2 \rangle : \text{int} \\
 \hline
 \Gamma \vdash \langle \Delta; e_1 + e_2 \rangle \rightarrow_p \langle \Delta_{e_2}; e'_1 + e'_2 \rangle : \text{int} \quad \text{R-Plus}
 \end{array}$$

$$\begin{array}{c}
 \Gamma \vdash \langle \Delta; \text{refresh}(e_1) \rangle \rightarrow_p \langle \Delta_{e_1}; e'_1 \rangle : \text{int} \quad \Gamma \vdash \langle \Delta_{e_1}; \text{refresh}(e_2) \rangle \rightarrow_p \langle \Delta_{e_2}; e'_2 \rangle : \text{int} \\
 \hline
 \Gamma \vdash \langle \Delta; e_1 - e_2 \rangle \rightarrow_p \langle \Delta_{e_2}; e'_1 - e'_2 \rangle : \text{int} \quad \text{R-Minus}
 \end{array}$$

$$\begin{array}{c}
 (T \neq P_1 \rightarrow P_2) \\
 \Gamma \vdash \langle \Delta; N \rangle \rightarrow_p \langle \Delta_N; N' \rangle : T \\
 \Gamma \uplus \{x : T, \langle \square; x \rangle \rightarrow_p \langle \square; x \rangle : T\} \vdash \langle \Delta_N; M \rangle \rightarrow_p \langle \Delta_M; M' \rangle : T_M \\
 \hline
 \Gamma \vdash \langle \Delta; \text{let } x = N \text{ in } M \rangle \rightarrow_p \langle \Delta_M; \text{let } x = N' \text{ in } M' \rangle : T_M \quad \text{R-P-Let}
 \end{array}$$

3.4 Operational Semantics

In this section, we present a small-step operational semantics that describes the runtime behavior of our programs. Passing a program to the runtime starts with obtaining a Φ context, a runtime version of functional context from the transformation stage that keeps relevant information on functions. The difference is that this context doesn't contain

a function's typing information, i.e., an argument's and the return type, because, as long as we ensure that only well-typed programs can enter the runtime, there is no more need for that:

$$\Phi ::= \diamond \mid \Phi, f[n] :: x \otimes M$$

To ensure that the real program evaluation can be started *only* after a program passes a type checker and goes through the transformation phase, we give this Φ context only to programs that satisfy these conditions:

$$\frac{\vdash t_1 : T \quad \vdash \langle t_1 \rangle \twoheadrightarrow_p \langle \Delta; t_2 \rangle : T \quad \Phi \equiv \{f[n] :: x \otimes M \mid (f[n] \triangleright x : T_1 . M : T_2) \in \Delta\}}{\Phi \triangleright \langle t_2 \rangle} \text{ Ready}$$

The rule above states that if under empty typing context, the program t_1 is well-typed, transformed into t_2 , and t_2 is an expression of a propertied type, then the program t_2 is ready to be executed and is given a Φ -context. We call the program t_1 *well-transformed* if it satisfies the conditions above. The restriction with t_2 not being a value of a propertied type is required because we prohibit all that stuff with type properties to occur at runtime. In the next section, we will prove that this is the only source of them.

Our list of notations now will encounter one more — we will write $f[n] \notin \Phi$ to impose a requirement on a Φ context that one must not have a record $f[n] :: x \otimes M$, where f is exactly this f , while n matches any natural number, x matches any variable, and M matches any function's body. Formation rules of Φ contexts are given as follows:

$$\frac{}{\diamond pctx} \text{ Phi-Ctx-1}$$

$$\frac{\Phi pctx \quad (f[n] \notin \Phi)}{\Phi, f[n] :: x \otimes M pctx} \text{ Phi-Ctx-2}$$

A context that will keep track of variables and their corresponding values is defined as follows:

$$\sigma ::= \diamond \mid \sigma, x \hookrightarrow v$$

Below we present rules for σ contexts formation:

$$\frac{}{\diamond sctx} \text{ Sigma-Ctx-1}$$

$$\frac{\sigma sctx \quad \diamond \triangleright v \text{ val} \quad (x \notin \sigma)}{\sigma, x \hookrightarrow v sctx} \text{ Sigma-Ctx-2}$$

Obtaining a Φ context for an expression ensures that the expression is ready to process its computation. Since the resulting expression, to proceed with the computation, must have the one too, we added a rule that does just that:

$$\frac{\Phi \triangleright \langle \sigma; e \rangle \quad \Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle}{\Phi \triangleright \langle \sigma'; e' \rangle} \text{ Phi-Preserve}$$

We also add two more expressions to the language that are not present in the language itself, but which instead help us dropping or retrieving previous values of variables:

$$e ::= \dots \mid \text{drop } x \text{ after } e \mid \text{retrieve } x = v \text{ after } e$$

To preserve the preservation theorem (one of the theorems of type soundness), we also present their typing rules:

$$\frac{\Gamma \vdash e : T \quad (x \in \Gamma)}{\Gamma \vdash \text{drop } x \text{ after } e : T} \text{ E-Drop}$$

$$\frac{\Gamma \vdash e : T \quad \Gamma \vdash v : P \quad (x \in \Gamma)}{\Gamma \vdash \text{retrieve } x = v \text{ after } e : T} \text{ E-Retrieve}$$

Programmers are not allowed to use these expressions as they exist only to facilitate the computation of some expressions below. Closed values are evaluated by the following rules:

$$\begin{array}{c}
 \frac{\Phi \text{ pctx}}{\Phi \triangleright () \text{ val}} \text{ V-Unit} \\
 \\
 \frac{\Phi \text{ pctx}}{\Phi \triangleright n \text{ val}} \text{ V-Int} \\
 \\
 \frac{\Phi, f[n] :: x \otimes M, \Phi' \text{ pctx}}{\Phi, f[n] :: x \otimes M, \Phi' \triangleright f \text{ val}} \text{ V-Func} \\
 \\
 \frac{\Phi, f[n] :: x \otimes M, \Phi' \text{ pctx}}{\Phi, f[n] :: x \otimes M, \Phi' \triangleright f[n] \text{ val}} \text{ V-Func-Compiled}
 \end{array}$$

And, finally, the transition rules are:

$$\begin{array}{c}
 \frac{\Phi \text{ pctx} \quad \sigma, x \hookrightarrow v, \sigma' \text{ sctx}}{\Phi \triangleright \langle \sigma, x \hookrightarrow v, \sigma'; x \rangle \mapsto \langle \sigma, x \hookrightarrow v, \sigma'; v \rangle} \text{ Var} \\
 \\
 \frac{\Phi \triangleright \langle \sigma; e_1 \rangle \mapsto \langle \sigma'; e'_1 \rangle}{\Phi \triangleright \langle \sigma; e_1 e_2 \rangle \mapsto \langle \sigma'; e'_1 e_2 \rangle} \text{ App-P-1} \\
 \\
 \frac{\Phi \triangleright e_1 \text{ val} \quad \Phi \triangleright \langle \sigma; e_2 \rangle \mapsto \langle \sigma'; e'_2 \rangle}{\Phi \triangleright \langle \sigma; e_1 e_2 \rangle \mapsto \langle \sigma'; e_1 e'_2 \rangle} \text{ App-P-2} \\
 \\
 \frac{\Phi \triangleright v \text{ val} \quad \sigma \text{ sctx} \quad (f[n] :: x \otimes M \in \Phi) \wedge (x \notin \sigma)}{\Phi \triangleright \langle \sigma; f[n] v \rangle \mapsto \langle \sigma, x \hookrightarrow v; \text{drop } x \text{ after } M \rangle} \text{ App-1} \\
 \\
 \frac{\Phi \triangleright v \text{ val} \quad \sigma, x \hookrightarrow v_{\text{prev}} \text{ sctx} \quad (f[n] :: x \otimes M \in \Phi)}{\Phi \triangleright \langle \sigma, x \hookrightarrow v_{\text{prev}}; f[n] v \rangle \mapsto \langle \sigma, x \hookrightarrow v; \text{retrieve } x = v_{\text{prev}} \text{ after } M \rangle} \text{ App-2} \\
 \\
 \frac{\Phi \text{ pctx} \quad \sigma \text{ sctx} \quad (g[k] :: y \otimes M_g \in \Phi) \quad (f[n] :: x \otimes M_f \in \Phi)}{\Phi \triangleright \langle \sigma; f[n] g \rangle \mapsto \langle \sigma; M_f \rangle} \text{ App-With-Func} \\
 \\
 \frac{\Phi \triangleright \langle \sigma; n_1 \rangle \mapsto \langle \sigma'; n'_1 \rangle}{\Phi \triangleright \langle \sigma; n_1 + n_2 \rangle \mapsto \langle \sigma'; n'_1 + n_2 \rangle} \text{ Plus-P-1} \\
 \\
 \frac{\Phi \triangleright n_1 \text{ val} \quad \Phi \triangleright \langle \sigma; n_2 \rangle \mapsto \langle \sigma'; n'_2 \rangle}{\Phi \triangleright \langle \sigma; n_1 + n_2 \rangle \mapsto \langle \sigma'; n_1 + n'_2 \rangle} \text{ Plus-P-2} \\
 \\
 \frac{\Phi \triangleright n_1 \text{ val} \quad \Phi \triangleright n_2 \text{ val} \quad \sigma \text{ sctx} \quad (n_1 + n_2 = n)}{\Phi \triangleright \langle \sigma; n_1 + n_2 \rangle \mapsto \langle \sigma; n \rangle} \text{ Plus} \\
 \\
 \frac{\Phi \triangleright \langle \sigma; n_1 \rangle \mapsto \langle \sigma'; n'_1 \rangle}{\Phi \triangleright \langle \sigma; n_1 - n_2 \rangle \mapsto \langle \sigma'; n'_1 - n_2 \rangle} \text{ Minus-P-1} \\
 \\
 \frac{\Phi \triangleright n_1 \text{ val} \quad \Phi \triangleright \langle \sigma; n_2 \rangle \mapsto \langle \sigma'; n'_2 \rangle}{\Phi \triangleright \langle \sigma; n_1 - n_2 \rangle \mapsto \langle \sigma'; n_1 - n'_2 \rangle} \text{ Minus-P-2} \\
 \\
 \frac{\Phi \triangleright n_1 \text{ val} \quad \Phi \triangleright n_2 \text{ val} \quad \sigma \text{ sctx} \quad (n_1 - n_2 = n)}{\Phi \triangleright \langle \sigma; n_1 - n_2 \rangle \mapsto \langle \sigma; n \rangle} \text{ Minus}
 \end{array}$$

$$\begin{array}{c}
 \frac{\Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle}{\Phi \triangleright \langle \sigma; \text{let } x = e \text{ in } M \rangle \mapsto \langle \sigma'; \text{let } x = e' \text{ in } M \rangle} \text{Let-P} \\
 \\
 \frac{\Phi \triangleright v \text{ val} \quad \sigma \text{ sctx} \quad (x \notin \sigma)}{\Phi \triangleright \langle \sigma; \text{let } x = v \text{ in } M \rangle \mapsto \langle \sigma, x \hookrightarrow v; \text{drop } x \text{ after } M \rangle} \text{Let-1} \\
 \\
 \frac{\Phi \triangleright v \text{ val} \quad \sigma, x \hookrightarrow v_{\text{prev}} \text{ sctx}}{\Phi \triangleright \langle \sigma, x \hookrightarrow v_{\text{prev}}; \text{let } x = v \text{ in } M \rangle \mapsto \langle \sigma, x \hookrightarrow v; \text{retrieve } x = v_{\text{prev}} \text{ after } M \rangle} \text{Let-2} \\
 \\
 \frac{\Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle}{\Phi \triangleright \langle \sigma; \text{drop } x \text{ after } e \rangle \mapsto \langle \sigma'; \text{drop } x \text{ after } e' \rangle} \text{Drop-After-1} \\
 \\
 \frac{\Phi \triangleright v \text{ val} \quad \sigma, x \hookrightarrow v_x \text{ sctx}}{\Phi \triangleright \langle \sigma, x \hookrightarrow v_x; \text{drop } x \text{ after } v \rangle \mapsto \langle \sigma; v \rangle} \text{Drop-After-2} \\
 \\
 \frac{\Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle}{\Phi \triangleright \langle \sigma; \text{retrieve } x = v \text{ after } e \rangle \mapsto \langle \sigma'; \text{retrieve } x = v \text{ after } e' \rangle} \text{Retrieve-After-1} \\
 \\
 \frac{\Phi \triangleright v \text{ val} \quad \sigma, x \hookrightarrow v_x \text{ sctx}}{\Phi \triangleright \langle \sigma, x \hookrightarrow v_x; \text{retrieve } x = v_{\text{prev}} \text{ after } v \rangle \mapsto \langle \sigma, x \hookrightarrow v_{\text{prev}}; v \rangle} \text{Retrieve-After-2}
 \end{array}$$

4 Properties of $\lambda_{\rightarrow p}$

In this section, we are going to explore some important properties of $\lambda_{\rightarrow p}$. We start with properties of compile-time program transformation.

4.1 Program transformation properties

One of the important claims we made in previous sections was that all static annotations are handled at the stage of compilation and are, therefore, not present at runtime. The following lemma represents this claim in a formal manner.

Lemma 4.1 *If $\Gamma \vdash e_1^+ : T_1^+$ and $\Gamma \vdash \langle \Delta; e_1^+ \rangle \rightarrow_p \langle \Delta'; e_2^+ \rangle : T_2^+$, then either T_2^+ is of the form $[T']\langle \dots \rangle$, or no subexpression in e_2^+ is of a type $[T']\langle \dots \rangle$. Moreover, for every function $f [n] \triangleright x : T_1 . M : T_2$ in Δ' , no subexpression of M and M itself is of the form $[T']\langle \dots \rangle$.*

We allow e_2^+ to be of a propertied type, since we do not give Φ contexts to programs of propertied types (the rule *Ready*), meaning that such programs are not passed to the runtime.

Proof 4.1 *By induction on the derivation rules of the judgment \rightarrow_p and the typing rules from Section 3.2.*

The proof for functions is almost straightforward: as T_2 is M 's type, and it has syntactic category T , which does not range over propertied types, M a-priori cannot be its inhabitant. Next, one should recall rules which put monomorphized functions into Δ contexts; the rules transform M , a function's body, which, by induction, if is not of a propertied type, must have no subexpressions being of one.

For the rules that act on values — *R-V-Unit* and *R-V-Int* — the proof is immediate since they are transformed into themselves. The rules that act on the expression *set* make it be of type $[T']\langle \dots \rangle$, meaning the proof is immediate too. Next, the rules *R-Refresh-**, *R-Get* and *R-Extract* transform their expressions into their subexpressions, which are, by induction, assumed to have the desired property.

What is interesting are the rules *R-App-*.Prop-1*: when the argument of a function application is a propertied type's inhabitant, it is transformed into its underlying expression. Since the underlying expression cannot be of an inhabitant of propertied type, we eliminated the presence of one in function applications.

Proofs for the remaining rules follow the same structure and are obtained by using the same techniques; we omit them here for space reasons.

The next important property of the \rightarrow_p -transformation is that it must be deterministic, since none of our compile-time constructions is expected to do something that causes non-deterministic behavior.

Lemma 4.2 *If $\Gamma \vdash \langle \Delta; e^+ \rangle \rightarrow_p \langle \Delta_1; e_1^+ \rangle : T_1^+$ and $\Gamma \vdash \langle \Delta; e^+ \rangle \rightarrow_p \langle \Delta_2; e_2^+ \rangle : T_2^+$, then $e_1^+ = e_2^+$ and $T_1^+ = T_2^+$*

We have not defined equality judgment for types and, so what we mean is the ordinary syntactic one.

Proof 4.2 *By induction on the derivation rules of the judgment \rightarrow_p .*

The proof is similar to the proof of *Lemma 4.1*. For some cases, there is only one rule defining its transformation, so assuming that the condition holds for all subexpressions makes the proof for them. Other rules explore their subexpressions' structure so that no expression can be transformed by two rules simultaneously.

To illustrate, let us investigate the rules *R-Let-Prop-1*, *R-Let-Prop-2*, *R-Let-Func*, and *R-P-Let*. They all operate on an expression *let* $x = N$ in M , but:

- *R-Let-Prop-2* works only when N is of a propertied type and the underlying value is of a function one.
- *R-Let-Prop-1* works only when N is of a propertied type too but the underlying value is *not* of a function type, thus covering all cases that *R-Let-Prop-2* excludes.
- *R-Let-Func* works only when N is transformed into a function.
- *R-P-Let* works only when N is neither transformed into a function nor to a propertied type's inhabitant.

thus mutually excluding each other.

4.2 Type soundness

Now, when we showed that the compile-time part has the desired properties, we are able to prove one of the most important properties of a programming language — the property of being *type sound*. The *type soundness* theorem states that if a program passes its programming language's type checker, then it is guaranteed that it has a well-defined behavior when executed. It consists of two theorems — the first, called the *progress* theorem, which states that if an expression e is well-typed (and, in our case, well-transformed under \rightarrow_p) then either it is already a value, or we can proceed with its computation. And the second, called the *preservation* theorem, which states that if an expression e has the type T and $\Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle$, then e' is of type T too.

Lemma 4.3 (Progress) *If $\Phi \triangleright \langle \sigma; e \rangle$, then either $\Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle$ or $\Phi \triangleright e \text{ val}$*

Proof 4.3 *By induction on the rules given in Section 3.4*

The proof for expressions that represent values is immediate. Next, the key thing is to observe that the operational semantics is defined for all expressions, except ones that work with type properties, such as *set*, *get*, and so forth. Since we proved (Lemma 4.1) that one cannot occur at runtime, so the assumption is justified.

Lemma 4.4 (Preservation) *if $\Gamma \vdash e : T$ and $\Phi \triangleright \langle \sigma; e \rangle \mapsto \langle \sigma'; e' \rangle$, then $\Gamma \vdash e' : T$*

Proof 4.4 *By induction on the typing rules from Section 3.2 and the rules given in Section 3.4*

5 Future Work

As this paper stands as the introduction, presentation, or, if one wishes, the "specification" our concept, which language implementors can use it when they wish to implement it in their programming language, surely, for the foreseeable future, our the primary goal is to prepare some works that will expose the concept's applications.

The first obvious application is that, in a language, where those 0, 1, and σ can be "interpreted" as usage constraints (which, as we said in the introduction, is not that difficult to achieve), our concept, by providing appropriate means to annotate language expressions with such elements of the semiring as we do it in type theory, makes linear (quantitative) types expressible in the language itself, without the need to transfer complex metaprogramming systems into an already overcomplicated language to express them.

To present applications at their full glance, at first, we are going to implement a simple, real-world alike programming language that will incorporate our concept. Having such a language can be beneficial for several reasons; first of all, given that it will be capable of treating those elements of the semiring as usage constraints, it will allow us to illustrate the first application on an actual programming language and a real example. Further, such a language will give an opportunity for interested readers, when one wishes to investigate its applicability for its problems, instead of

skimming through its large type-theoretic specification, to install such a language and start investigating its question without any delay.

Finally, it is worth noting that our true exploration only begins; expressing linear types in the way we described above is just a first, an apparent specimen that follows on from what we have right now, and it seems that there are much more connections with recent advances in the field of type theory to be drawn. As we said in the introduction, the expanse of idioms that become expressible with a language's endowment with our concept is limited to the rest of the language's constructions, as this concept primarily targets the annotation of language expressions with additional data, the meaning of which is expressed in terms of those existing constructions. The above-mentioned language, in fact, by encompassing our concept will greatly facilitate this investigation since any future discovery or connection, when drawn, may be instantaneously examined more studiously in the actual programming language and upon real examples.

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