# L05Quantifiers

February 2, 2017

# 1 Lecture 5

## 1.1 Last Lecture: Predicates

*Informal Definition*: Predicates are propositions that contain variables.

**Examples**: - P(x): The integer $x$ is prime. - Q(z): $z \geq 0$.

The truth value of these propositions depends on the setting of the variables: - P(10) is false, but P(11) is true. - Q(1) is true, but Q(-1) is false.

*Formal Definition*: $P(x)$ is a function that maps a universe $U$ to the set of propositions.

$P$ takes as input an element $x$ from a *universe* $U$ and returns a proposition $P(x)$.

**Note**: Be careful that you understand which universe $P(x)$ operates over.

## 1.2 Last Lecture: Truth Sets

Given a predicate $P(x)$ over a universe $U$, we can define a corresponding **truth set**, i.e., the set of elements $x$ in $U$ that make $P(x)$ true.

**Example**:

- For $P(x)$: "The integer $x$ is prime" over the universe of integers, the truth set is exactly the set of prime integers. - For $Q(z)$: "$x^2 \leq 1$" over the universe of reals, the truth set is the interval$[-1, 1]$.

**Formal Definition**: The *truth set* of a predicate $P(x)$ over a universe $U$ is defined as

$$T_P = \{x \in U | P(x)\}.$$

## 1.3 Compound Predicates

Because predicates map to propositions, it is easy to construct new predicates by combining existing predicates using *logical operators*.

**Example**:

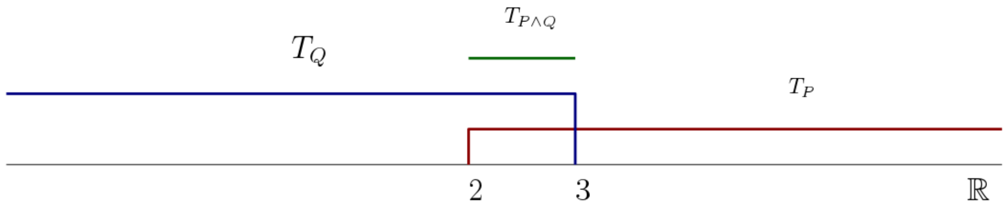
P(x): $x \geq 2$.

Q(x): $x \leq 3$.

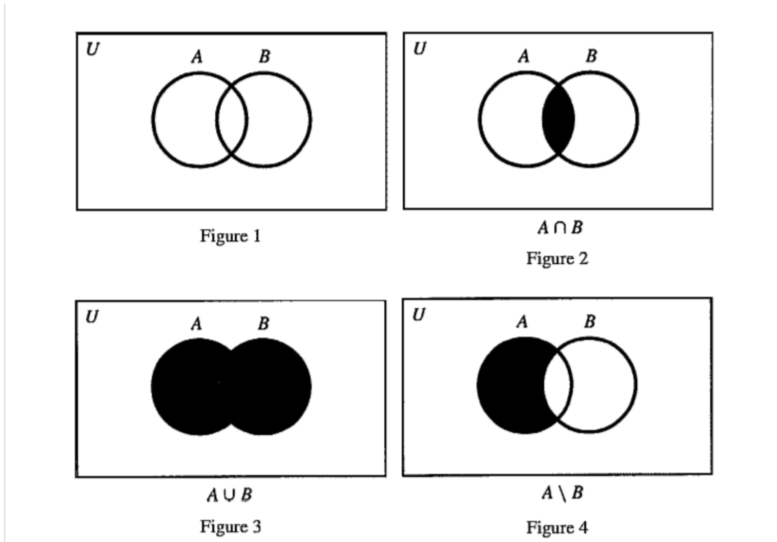
Compound proposition:

$$P(x) \land Q(x) : x \geq 2 \land x \leq 3.$$

**Question:** what is the truth set of the compound proposition? **Answer**: It is the set of reals that satisfy both $P(x)$ **and** $Q(x)$. This set is called the **intersection** of the truth sets of $P(x)$ and \$Q(x). \$




Truth Sets of $P(x), Q(x)$ and $P(x) \wedge Q(x)$




Set Operations

## 1.4 Logical Connectives and Set Operations

- Suppose we have predicates $A(x)$ and $B(x)$ over universe $U$ with truth sets $A$ and $B$ respectively (Figure 1).
- The truth set of $A(x) \wedge B(x)$ is the *intersection* $A \cap B$ (Figure 2),
- The truth set of $A(x) \vee B(x)$ is the *union* $A \cup B$ (Figure 3),
- The truth set of $A(x) \wedge \neg B(x)$ is the *difference* $A \setminus B$ (Figure 4).

**Upshot**: - It also reveals the completely analogous structure of logical connectives and set operations. - This is a formal way of introducing set operations using propositional logic. GOAL: Prove theorems about sets formally using logic.

## 1.5 Review Question

**Question**: Assume that for all $x$ in the universe $P(x) \implies Q(x)$ is a tautology. Which one of the following statement about the truth sets $T_P$ and $T_Q$ is true?

a. $T_P = T_Q$,

b. $T_P \in T_Q$,

c. $T_Q \subseteq T_P$,

d. $T_P \subseteq T_Q$,

e. None of the above

**Answer**: d.
The implication can be written as $x \in T_P \implies x \in T_Q$. When this is true for all $x$, it must be the case that $T_P \subset T_Q$.

### 1.6 Quantifiers

As in the previous example, it is often useful to express the following ideas *quantifying* the truth set of a predicate $P(x)$:

1) (Universal Quantification) Predicate $P(x)$ has truth set equal to the whole universe $U$.

$$T_P = U.$$

This is the same as saying that *for all* $x \in U$, $P(x)$ is true.
**Notation**:

$$\forall x, P(x)$$

$\forall$ is known as the **universal** quantifier.

2) (Existential Quantification) Predicate $P(x)$ has non-empty truth set

$$T_P \neq \emptyset.$$

This is the same as saying that *there exists* $x \in U$ such that $P(x)$ is true.
**Notation**:

$$\exists x, P(x)$$

$\exists$ is known as the **existential** quantifier.

### 1.7 Practice Question

Which of the following formulae are true?

a. $\forall x\ (x^2 \geq 0)$, where the universe is $\mathbb{R}$.

b. $\exists x\ (x^2 - 2x + 3 = 0)$, where the universe is $\mathbb{R}$.

c. $\exists x\ 3 + x = 0$, where the universe is $\mathbb{Z}$.

d. $\exists x\ 3 \cdot x = 1$, where the universe is $\mathbb{Z}$.

e. $\forall y\ (y \text{ is even }) \implies (y + 1 \text{ is odd })$, where the universe is $\mathbb{Z}$.

**Answers**: a. True. b. False. Check discriminant. c. True. $x = -3$. d. False because the universe is $\mathbb{Z}$. e. True.

## 1.8 Exercise

Recall the last example in the question:
$\forall y$ ($y$ is even ) $\implies$ ($y+1$ is odd ), where the universe is $\mathbb{Z}$.

Notice that I have used plain English to define the concepts of even and odd. **Exercise**: Use quantifiers to construct a predicate whose truth sets is the set of even integers.

**Answer**:

$$E(x) : \exists k, x = 2k.$$

Indeed, $x$ is even if and only if it is divisible by 2, i.e., it can be expressed as $2 * k$ for some integer $k$.

**Note**: One can define the set of odd integers similarly:

$$O(x) : \exists k, x = 2k + 1$$

Notice that, once we introduce $E$ and $O$, it is very simple to prove the truth of our starting example. Here is a proof. We will discuss some of the details of this proof once we start discussing proofs in the coming lectures.

**Proof**: As we want to show a universal statement over the integers, we start by considering a *generic* integer $x$. If $x$ is even, we know that

$$\exists k, x = 2k.$$

But this is equivalent to:

$$\exists k, x + 1 = 2k + 1,$$

which is equivalent to saying that $x+1$ is odd. This completes the proof.

**Note**: Notice that, as we were trying to show an implication, we only had to check the case in which the premise of the implication was true, i.e., $x$ is even.

## 1.9 Quantification over a Set

Sometimes we wish to quantify over a set $S \subset U$, rather than over all the universe $U$. In the previous question, the meaning of the last answer was:

For all even integers $y$, we have that $y+1$ is odd.

In general, we may want to write:

$$\forall x \in S, P(x).$$

Fortunately, we can write this as a universal quantification. **Question**: Can you see how? **Answer**: Just like in the previous example regarding even and odd numbers.

$$\forall x, x \in S \implies P(x)$$

## 1.10 Expressing Set Concepts

Using quantifiers, we can write logical formulae to express two important set concepts:

1. Containment: $A \subseteq B$.
2. Disjointness: $A \cap B = \emptyset$.

### 1.11 Question 1

Let $A$ and $B$ be the truth sets for predicates $A(x)$ and $B(x)$. Identify which of the following formulae is equivalent to $A \subseteq B$.

a. $\exists x, (x \notin A) \wedge x \in B$,

b. $\forall x, (x \notin B) \implies (x \in A)$,

c. $\forall x, (x \in A) \implies (x \in B)$,

d. $\exists x, (x \in A) \vee (x \in B)$.

**Answer**: c.

### 1.12 Question 2

Let $A$ and $B$ be the truth sets for predicates $A(x)$ and $B(x)$. Identify which of the following formulae is equivalent to $A \cap B = 0$.

a. $\exists x, (x \notin A) \wedge (x \notin B)$,

b. $\forall x, (x \notin A) \vee (x \notin B)$,

c. $\forall x, (x \in A) \implies (x \in A \wedge x \in B)$,

d. $\forall x, \neg(x \in A) \wedge \neg(x \in B)$.

**Answer**: b.