L08RulesofInferenceIntrotoProofs

March 4, 2017

1 Lecture 8

2 Midterm Post Mortem

STATISTIC	Q1 (18)	Q2 (22)	Q3 (22)	Q4 (18)	TOTAL (80)
AVERAGE	11.9	16.8	13.3	5	47.0
MAX	18	22	22	18	71
MEDIAN	13	17	14	3	46
STDEV	2.9	3.7	5.3	5.4	11.1

Let Q be the set of questions and S the set of students. Let the predicate $C(q, s)$ mean: "Question q was answered correctly by student s".

Then, the following is true:

$$\forall q \in Q, \exists s \in S, C(q, s)$$

Unfortunately, this was not true:

$$\exists s \in S, \forall q \in Q, C(q, s)$$

3 Midterm Post Mortem

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My take-aways: 1) Students found question 4 very hard. Probably because it was the most different from questions in homework.

2) Students found question 3 harder than I thought. This was mostly due to problems understanding what the question asked. It was also different from homework questions.

3) Many points were lost because of lack of formality: $(P(x) \implies B(X)) \wedge (B(x) \implies F(x))$ is not the same as $P(x) \implies B(x) \implies F(x)$.

Notice however that $P(x) \implies F(x)$ is a valid conclusion given the premise.

3.1 Going forward

Projected Grades:

Range	Letter Grade
0-10	F
10-20	D
20-25	C-
25-30	C
30-35	C+
35-40	B-
40-50	B
50-55	B+
55-60	A-
60-80	A

Class Trajectory: Class material will become harder, but more fun, as we begin to work on proofs.

Homework Trajectory: I will start adding some harder questions in the homework, so that you see a larger variety of problems besides those in the book.

3.2 Back to Work: Logical Arguments and Rules of Inferences

Last formal topic in logic. From next week, we'll talk about how to use our understanding of logic to write correct and clear **mathematical proofs.**

In order to write correct mathematical proof, we need to understand what a **valid logical argument** is.

Definition: A (logical) *argument* is a sequence of statements, ending with a *conclusion*.

Definition: An argument is *valid* if its conclusion must logically follow from the prececeding statements, which are known as *premises*.

Equivalently, an argument is valid if and only if it is impossible for the premises to be true and the conclusion to be false, i.e., it is true that the premises __imply_ the conclusion.

For propositions, we have: Premises: P_1, P_2, P_3,. . . ., P_m Conclusion: Q This is a valid argument if and only if:

$$P_1 \wedge P_2 \wedge \ldots \wedge P_m \implies Q$$

is a **tautology.**

3.3 Rules of Inference

What are they?: Logica shortcuts: Simple rules that allow us to *generate a conclusion from a set of premises* to form a valid argument.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Rules of Inference

3.3.1 Two simple rules of inference:

1) *Modus Ponens*

$$\begin{aligned} & P \\ P \implies & Q \\ \therefore & Q \end{aligned}$$

2) *Modus Tollens*

$$\begin{aligned} & \neg Q \\ P \implies & Q \\ \therefore & \neg P \end{aligned}$$

3.4 Rules of Inference

3.5 Application of Rules of Inference

Premises you may know:
1) No one, who is going to a party, ever fails to brush his or her hair.

$$\forall x, P(x) \implies B(x)$$

2) No one looks fascinating, if he or she is untidy.

$$\forall x, U(x) \implies \neg F(x)$$

3) Everyone who has brushed his or her hair looks fascinating.

$$\forall x, B(x) \implies F(x)$$

4) A person is always untidy if he or she has no self-command.

$$\forall x, N(x) \implies U(x)$$

Additional premise: 5) Everyone is going to a party.

$$\forall x, P(x)$$

3.6 Building an argument:

Premises: Take $\wedge$ of premises to form single premise

$$(\forall x, P(x) \implies B(x)) \wedge (\forall x, U(x) \implies \neg F(x)) \wedge (\forall x, B(x) \implies F(x)) \wedge (\forall x, N(x) \implies U(x)) \wedge (\forall x, P(x))$$

Apply equivalences: Take quantifier out

$$\forall x, [(P(x) \implies B(x)) \ \wedge \ (U(x) \implies \neg F(x)) \ \wedge \ (B(x) \implies F(x)) \ \wedge \ (N(x) \implies U(x)) \ \wedge \ P(x)]$$

Apply Rule of Inference: *Simplification*

$$\begin{array}{r} P \wedge Q \\ \therefore P \end{array}$$

Simplification allows us to keep only two terms in the conjunction:

$$\forall x, [(P(x) \implies B(x)) \ \wedge \ P(x)]$$

Argument Over?: We already finished the argument. We applied simplication to obtain:

$$(\forall x, P(x) \implies B(x)) \wedge (\forall x, U(x) \implies \neg F(x)) \wedge (\forall x, B(x) \implies F(x)) \wedge (\forall x, N(x) \implies U(x)) \wedge (\forall x, P(x)) \therefore \forall$$

Note: This was not a very interesting argument. To get more interesting conclusions, we *keep the argument going by including our conclusion as a new premise*.

3.7 Applying Modus Ponens

Premise:

$$\forall x, [(P(x) \implies B(x)) \land P(x)]$$

Apply Modus Ponens:

$$\begin{array}{r} P \\ P \implies B \\ \therefore B \end{array}$$

Conclusion:

$$\forall x, B(x)$$

3.8 Keep the argument going

We have a growing set of available premises we can use.
The original premises: 1)

$$\forall x, P(x) \implies B(x)$$

2)

$$\forall x, U(x) \implies \neg F(x)$$

3)

$$\forall x, B(x) \implies F(x)$$

4)

$$\forall x, N(x) \implies U(x)$$

5)

$$\forall x, P(x)$$

And the conclusions to our previous arguments, which we can now use as premises:
6)

$$\forall x, [(P(x) \implies B(x)) \land P(x)]$$

7)

$$\forall x, B(x)$$

Let's make another step:
Premises: 3 and 7

$$\forall x, B(x) \land B(x) \implies F(x)$$

Conclusion:

$$\therefore \forall x, F(x)$$

by modus ponens again.

3.9 Question: Apply Modus Tollens

Recall: *Modus Tollens*

$$\begin{aligned} &\neg Q \\ P \implies &Q \\ \therefore &\neg P \end{aligned}$$

Question: To which of the following two premises should we apply modus tollens to conclude that

$$\forall x, \neg U(x)?$$

Premises: Click two

1)
$$\forall x, P(x) \implies B(x)$$

2)
$$\forall x, U(x) \implies \neg F(x)$$

3)
$$\forall x, B(x) \implies F(x)$$

4)
$$\forall x, N(x) \implies U(x)$$

5)
$$\forall x, P(x)$$

6)
$$\forall x, B(x)$$

7)
$$\forall x, F(x)$$

Answer: 2 and 7.

3.10 Other Rules of Inference

3.11 Hypothetical Syllogism

Premises:

$$P \implies Q Q \implies R$$

Conclusion:

$$\therefore P \implies R$$

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Rules of Inference

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Rules of Inference with Quantifiers

3.12 Application of Hypothetical Syllogism

Hypothetical syllogism is useful in cases when we have a number of implications: 1)

$$\forall x, P(x) \implies B(x)$$

2)

$$\forall x, U(x) \implies \neg F(x) \equiv \forall x, F(X) \implies \neg U(x)$$

by contrapositive 3)

$$\forall x, B(x) \implies F(x)$$

4)

$$\forall x, N(x) \implies U(x) \equiv \forall x, \neg U(x) \implies \neg N(x)$$

By repeatedly applying hypothetical conclusions, we can get the following conclusions:

$$\therefore P(x) \implies F(x) \therefore P(x) \implies \neg U(x) \therefore P(x) \implies \neg N(x)$$

3.13 Rules of Inference involving Quantified Statements

3.14 Example

Recall our usual premises:

1)

$$\forall x, P(x) \implies B(x)$$

2)

$$\forall x, F(x) \implies \neg U(x)$$

3)

$$\forall x, B(x) \implies F(x)$$

4)

$$\forall x, \neg U(x) \implies \neg N(x)$$

Suppose we have the additional premise:

$$P(\text{John})$$

stating that John is going to a party.

Using **universal instantiation**, we can turn our premises into the conclusions:

1)

$$P(\text{John}) \implies B(\text{John})$$

2)

$$F(\text{John}) \implies \neg U(\text{John})$$

3)

$$B(\text{John}) \implies F(\text{John})$$

4)

$$\neg U(\text{John}) \implies \neg N(\text{John})$$

Now, by using hypothetical syllogism, we conclude that:

$$\therefore P(\text{John}) \implies \neg N(\text{John})$$

Finally, using $P(\text{John})$ together with the previous statement and applying modus ponens, we have:

$$\therefore \neg N(\text{John})$$

3.15 Introduction to Proofs

Definition: A proof is a valid argument that establishes the truth of a statement.

Informal Proofs: In math, CS, and other disciplines, we generally use **informal proofs**.
1) More than one rule of inference are often used in a step.
2) Steps may be skipped.
3) The rules of inference used are not explicitly stated. 4) Easier for to understand and to explain to people. 5) But it is also easier to introduce errors.

Importance: Proofs have many practical applications: 1) verification that computer programs and algorithms are correct 2) establishing that operating systems are secure 3) enabling programs to make inferences in artificial intelligence 4) showing properties of algorithms

3.16 Definitions

- A **theorem** is a statement that can be proved to be true starting from the premises:

    1) definitions

    2) other theorems

    3) **axioms** (statements which are given as true, aka postulates)

- A **lemma** is a 'helping theorem' or a result which is needed to prove a theorem.
- A **corollary** is a result which follows directly from a theorem.

3.17 Example of an informal proof

Theorem: If $x \geq y$, where x and y are positive real numbers, then $x^2 \geq y^2$.

3.18 What are the axioms?

Peano-Dedekind axioms from Wikipedia: https://en.wikipedia.org/wiki/Peano_axioms
Zermelo-Fraenkel axioms from Wikipedia: https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_se

3.19 Goedel's Incompleteness Theorem

IDEA: It is possible to write a statement P whose meaning is:
> P is not provable.
What happens if P is true? If P is false?

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In [ ]:
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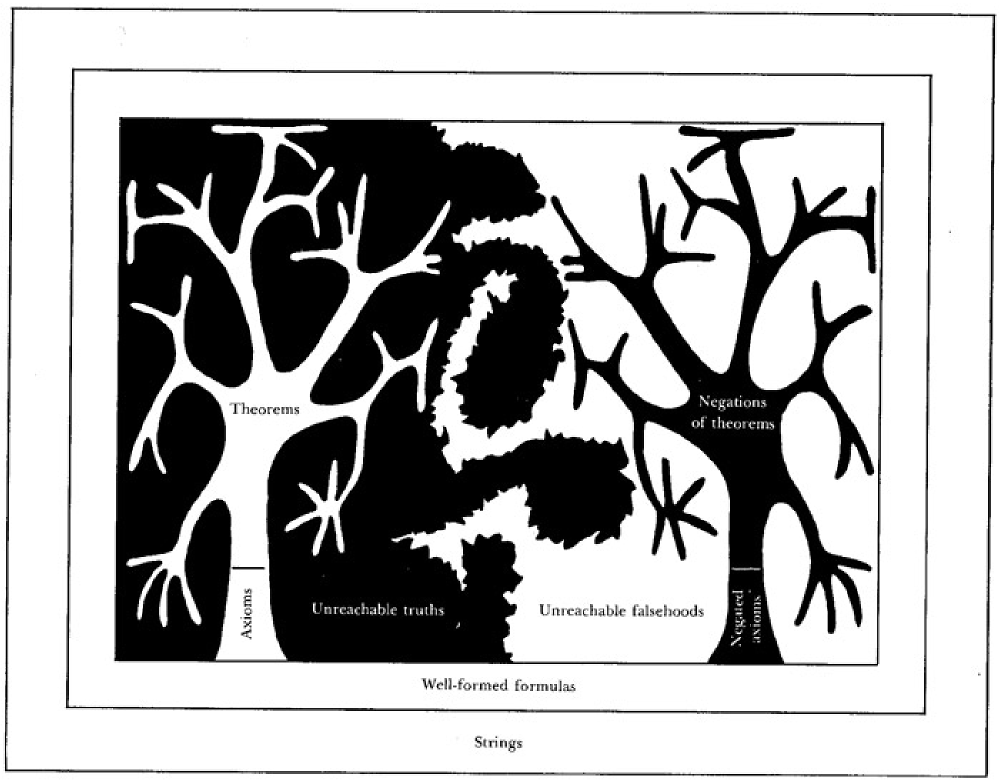

Incompleteness