L02PropositionalLogic

January 25, 2017

0.1 Propositional Logic

Inspired from the Introduction of "How To Prove It" by Daniel J. Velleman. Question 1: modification of Exercise 7a) in chapter 1.1 of HTPI

In our first meeting, we talked briefly about puzzles. Let's start with one today.

Question 1: Jane and Pete won't both win the math prize. Pete will win the math prize or the chemistry prize. If Jane does not win the math prize, Pete will win the chemistry prize. Based on these *premises*, which one of the following *conclusions* is true?

a. Neither Jane nor Pete will win the chemistry prize.

b. Pete will win the chemistry prize.

c. Jane and Pete will both win the chemistry prize.

d. Jane will win the math prize.

Answer: b.

- PREMISE 1: Jane and Pete won't both win the math prize.

- PREMISE 2: Pete will win the math prize or the chemistry prize.
- PREMISE 3: If Jane does not win the math prize, Pete will win the chemistry prize.

ARGUMENT: - If Jane does win the math prize, Peter won't by Premise 1. -Then, By Premise 2, Pete will win the chemistry prize. - Else, if Jane does not win the math prize, by Premise 3, Pete will win the chemistry prize. - Hence, Pete will win the chemistry prize.

PROPOSITIONAL LOGIC: How do we formally reason about problems like this?

0.2 Propositional Logic

Notice that the context of our problem (Jane, Pete, prizes, etc.) did not really matter to the solution. What mattered was the logical connections between the different statements. Indeed, we could replace each statement with a **variable**, while preserving the struture of the problem:

- variable P: Pete will win the math prize.
- variable Q: Pete will win the chemistry prize.
- variable R: Jane will win the math prize.

NOTE: A *proposition* is a declarative sentence that is either *true*(T) or *false*(F), but not both. New propositions, called *compound propositions*, are formed from existing propositions using *logical operators*.

In our case, P, Q and R are *propositional variables*, each representing the corresponding proposition on the right.

The premises of our puzzle are propositions formed by combining the propositions P, Q and R using *logical operators* (aka logical connectives). Next, we will introduce some of the basic such operators

0.3 Common Logical Operators

The most common logical operators are *negation*(NOT), *disjunction*(OR), *conjunction*(AND) and *implication*(IMPLIES). They are denoted as follows:

Symbol	Meaning	Arity
$\neg$	NOT	unary
$\vee$	OR	binary
$\wedge$	AND	binary

0.4 Negation (NOT)

Notation: Given a proposition P, the negation of P is denoted $\neg P$.

To properly define the negation operator, we need to define the truth value of $\neg P$ as a function of the truth value of P. For this purpose, we use a **TRUTH TABLE**:

p	$\neg p$
T	F
F	T

Example: - P: Pete will win the math prize. - $\neg P$: Pete will not win the math prize.

0.5 Disjunction (OR)

Notation: Given propositions P and Q, their disjunction is denoted $P \vee Q$.

Truth Table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

- P: Pete will win the math prize.
- Q: Pete will win the chemistry prize.
- Pete will win the math prize or the chemistry prize: $P \vee Q$.

0.6 Conjunction (AND)

Notation: Given propositions P and Q, their conjunction is denoted $P \wedge Q$.
Truth Table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:

- P: Pete will win the math prize.
- R: Jane will win the math prize.
- Jane and Pete will both win the math prize: $R \wedge P$.

0.7 Back To Our Puzzle

VARIABLES: P: Pete will win the math prize.
Q: Pete will win the chemistry prize.
R: Jane will win the math prize.
PREMISE 1: Jane and Pete won't both win the math prize.

$$\neg(R \wedge P)$$

PREMISE 2: Pete will win the math prize or the chemistry prize.

$$P \vee Q$$

PREMISE 3: If Jane does not win the math prize, Pete will win the chemistry prize.
How do we express this?

0.8 What do you think?

Question: Let S and T be given propositions. We want to build a truth table for the (English) proposition

$$\text{if } S\text{, then } T$$

.

Which of the following truth tables should we use?

Propositon S	Proposition T	Answer a)	Answer b)	Answer c)	Answer d)
F	F	T	F	F	T
F	T	T	T	F	T
T	F	F	F	F	T
T	T	T	T	T	F

The correct answer is a).

Proposition S	Proposition T	If S then T
F	F	T
F	T	T
T	F	F
T	T	T

Think: What's the reason behind the first two rows?
Example: *Proposition A*:If $x^2 = 9$ and $x \geq 0$, then $x = 3$ or $x = 1$. Let us write this as: $A : B \implies C$.
We should all agree that A is true. If we do, then we can use it to justify the answer to the question above as follows:

- consider $x = 3$: then B and C are both true.
- consider $x = -3$: then B and C are both false.
- consider $x = 1$: then B is false and C are true.

Yet, in all these cases the original proposition A is true.

0.9 Another Logical Connective: Implication

Notation: Given propositions P and Q, the implication P implies Q is denoted by $P \implies Q$.
Truth Table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: R: Jane will win the math prize.
Q: Pete will win the chemistry prize.
If Jane does not win the math prize, Pete will win the chemistry prize.

$$(\neg R) \implies Q$$

0.9.1 Looking ahead ...

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that there is a different way fo getting the same truth table:

Proposition P	Proposition Q	(not P) or Q
T	T	T
T	F	F
F	T	T
F	F	T

We say that

$$P \implies Q$$

and

$$(\neg P) \vee Q$$

are *logically equivalent*.

0.10 A Formal Solution to Our Puzzle

Recall our **notation**: - variable P: Pete will win the math prize. - variable Q: Pete will win the chemistry prize. - variable R: Jane will win the math prize.
PUZZLE: - PREMISE 1: Jane and Pete won't both win the math prize.

$$\neg(R \wedge P)$$

- PREMISE 2: Pete will win the math prize or the chemistry prize.

$$P \vee Q$$

- PREMISE 3: If Jane does not win the math prize, Pete will win the chemistry prize.

$$(\neg R) \implies Q$$

- CONCLUSION: Pete will win the chemistry prize.

$$Q$$

Solution on the blackboard.

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In [19]: a = Truths(['P', 'Q', 'R'], ['not(R and P)','P or Q', 'R or Q']).__str__()
         print(a.replace('+','|'));
```

```
|---|---|---|--------------|--------|--------|
| P | Q | R | not(R and P) | P or Q | R or Q |
|---|---|---|--------------|--------|--------|
| 0 | 0 | 0 |      1       |   0    |   0    |
| 0 | 0 | 1 |      1       |   0    |   1    |
| 0 | 1 | 0 |      1       |   1    |   1    |
| 0 | 1 | 1 |      1       |   1    |   1    |
| 1 | 0 | 0 |      1       |   1    |   0    |
| 1 | 0 | 1 |      0       |   1    |   1    |
| 1 | 1 | 0 |      1       |   1    |   1    |
| 1 | 1 | 1 |      0       |   1    |   1    |
|---|---|---|--------------|--------|--------|
```

0.11 Another important connective: Equivalence

Notation: Given propositions P and Q, the equivalence of P and Q is denoted by $P \iff Q$.
Truth Table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example:
R: Jane will win the math prize.
Q: Pete will win the chemistry prize.
Jane will win the math prize if and only if Pete will win the chemistry prize.

$$R \iff Q$$

0.12 Trick Question

How many T's are there in the truth table for the compound proposition:

$$[(\neg P) \vee Q] \iff [P \implies Q]$$

a. 0

b. 1

c. 2

d. 3

e. 4