Lecture 16: SDPs, Effective Resistance and PageRank.

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SDPs with rank-one solutions

Eigenvalue problem:

$$\begin{aligned} \max \ & A \cdot X \\ & I \cdot X = 1 \\ & X \succeq 0 \end{aligned}$$

Fact: if A and B are symmetric and $\exists X \succeq 0 \in \mathbb{R}^{n \times n}$ and $a, b \in \mathbb{R}$ s.t.:

$$\begin{aligned} A \cdot X = a \\ B \cdot X = b \end{aligned}$$

Then there exists $x \in \mathbb{R}^n$ such that:

$$\begin{aligned} x^T A x = a \\ x^T B x = b \end{aligned}$$

(i.e. X can be taken to be xx^T).
Recall: Balanced Cut SDP:

$$\begin{aligned} \min \ & L \cdot X \\ s.t. \ & L(K_G) \cdot X = Vol(G) \\ & \forall i \in V : \ \ L(S_i) \cdot X \leq \frac{1}{b} \end{aligned}$$

Since we are working with laplacians we may use Johnson-Lindenstrauss to reduce dimensionality. issue: when given a graph with n vertices the SDP solution is some $n \times n$ matrix which is taken to be an inner product matrix. We have $X = V^T V$ since $X \succeq 0$. The columns of V are the embeddings of the vertices in $\mathbb{R}^n$.

Let's now look at another SDP problem:

$$\begin{aligned} \min \ & L \cdot X \\ & L_{st} \cdot X = 1 \end{aligned}$$

L Laplacian of $G = (V, E)$, $\{s, t\} \subseteq V$.

$$\begin{aligned} \min \ & x^T L x \\ & x^T L_{st} x = 1 \\ & (x_s - s_t)^2 = 1 \end{aligned}$$

Reminder: $L_{st} = (e_s - e_t)(e_s - s_t)^T$. Let's look at the dual of this SDP:

$$\begin{aligned} \max \ & \alpha \\ & L \succeq \alpha L_{st} \\ & \alpha \geq 0 \end{aligned}$$

Complementary Slackness: Under mild conditions, which are satisfied here, an optimal solution pair (X^*, α^*) it must be true that $X^* \cdot (L - \alpha^* L_{st}) = 0$.

$$L \succeq \alpha^* L_{st}$$

$$I \succeq \alpha^* L^{-1/2} L_{st} L^{-1/2}$$

$$y = \frac{L^{-1/2}(e_s - e_t)}{(e_s - e_t)^T L^{-1}(e_s - e_t)}$$

$$x^* = \frac{L^{-1}(e_s - e_T)}{(e_s - e_T)^T L^{-1}(e_s - e_T)}$$

$$X^* = x^* {x^*}^T$$

How can we write L^{-1}, given that L is not invertible? We use L^+ where L^+ is the Moore-Penrose pseudoinverse. which satisfies:

$$L^+ \vec{1} = 0$$

and:

$$L^+ L v = v \qquad \text{for} \qquad v \perp \vec{1}$$

Electrical Circuits

Suppose the weighted graph $G = (V, E, w)$ is an electrical circui where the resistance is given by the ivnerse of the of the weight:

$$r_e = \frac{1}{w_e}$$

1. If I have:

$$V_s = 1$$

$$V_t = 0$$

what is the voltage everywhere else? We can cast this as a minimum energy problem:

$$\min \sum_{\{i,j\}\in E} w_{i,j}(V_i - V_j)^2$$
$$V_S - V_t = 1$$

The quantity min $\sum_{\{i,j\}\in E} w_{i,j}(V_i - V_j)^2$ is also known as the inverse of the effective resistance, since it represents the inverse of the resistance of a graph as a whole. (i.e. if the graph was a resistor, what would its resistance be). The inverse effective resistance is also known as the effective conductance.

Personalized PageRank

The personalized PageRank is a kind of random walk over the graph. Normally we define the random walk by the transition matrix:

$$W = AD^{-1}$$

The personalized PageRank random walk is defined by:

$$PR_\alpha = \alpha \sum_{t=0}^{\infty} (1-\alpha)^t W^t$$

This was used by Google to measure the reputation/importance of webpages. Exercise: we can write

$$\alpha \sum_{t=0}^{\infty} (1-\alpha)^t W^t = \alpha(I - (1-\alpha)W)^{-1} = \alpha D(D - (1-\alpha)A)^{-1} = D\left(D + \frac{(1-\alpha)L}{\alpha}\right)^{-1}$$

Remark: PageRank is a mix of the eigenvector and the effective conductance problems.

$$\begin{aligned} \min \quad & \sum_{\{i,j\}\in E} ||v_i - v_j||^2 \\ \text{s.t.} \quad & \sum_{i<j} \frac{d_i d_j}{Vol(G)} ||v_j - v_i||^2 = 1 \\ & (v^T s)^2 \geq 1 \end{aligned}$$

I'm trying to find a vector which has length 1, minimizes $L \cdot X$ and satisfies $X \cdot ss^T \geq c$. the solution to the problem happens to be:

$$X^* = PR_\alpha s.$$