Lecture 11: Flow-Based Partitioning and Rounding

Instructor: Lorenzo Orecchia Scribe: Erasmo Tani

Rounding of the Leighton-Rao Relaxation

distance over G by lenght over the edges.

$$W = \sum_{e \in E} \ell_e$$

$$\sum_{v_1, v_2 \in V} d_{i,j} \geq 1$$

Te above says that the average distance is about $\frac{1}{\binom{n}{2}}$. We want to round this to an ℓ_1-metric. We will conside two cases: an easy case and a hard case.

(Easy case) There exists a component $T \subseteq V$ with radius $\frac{1}{2n^2}$ such that:

$$|T| \geq \frac{2}{3}$$

The intuition is that there is a subset of points that has very low variance and hence the remaining points must lie far away from them. Formally we have the following claim:
Claim: in this case we have:

$$\sum_{u \in V \setminus T} d(u, T) \geq \frac{1}{2n}$$

Proof. We have: $\forall u, v$:

$$d(u, v) \leq d(u, T) + d(v, T) + \frac{1}{n^2}$$

since $1/n^2$ is the diameter of T. But then:

$$1 \leq \sum_{\{u,v\}} d_{u,v} = \frac{1}{2} \sum_{(u,v) \in V \times V} \left(d(T, u) + d(T, v) + \frac{1}{n^2} \right) < n \cdot \sum_{w \in V \setminus T} d(T, w) + \frac{1}{2}$$

giving us the desired result. □

Goal: come up with an ℓ_1-embedding $\vec{y}_i$ with $i \in V$ such that:

1.
$$\frac{\sum ||y_i - y_j||}{\sum_{\{u,v\}} ||y_u - y_v||} \leq 3W$$

We will use a Frechet-type embedding:

$$y_i = d(T, i)$$

One nice property of these embeddings is that, for all i, j:

$$|y_i - y_j| \leq d_{ij}$$

so that the embedding is always a contraction. And we then have:

$$\sum_{\{i,j\} \in E} |y_i - y_j| \leq \sum_{\{i,j\} \in E} d_{i,j} = W$$

$$\sum_{\{u,v\}} |y_u - y_v| \geq \sum_{u \in V \setminus T} \sum_{v \in T} |y_u - y_v| = |T| \sum_{u \in V \setminus T} y_u = |T| \cdot \frac{1}{2n} \geq \frac{1}{3}$$

(Hard Case) There exists no component $T \subseteq V$ with radius $\frac{1}{2n^2}$ such that:

$$|T| \geq \frac{2}{3}$$

Here, the following structure lemma comes in handy :

Lemma 1 (Structure Lemma)**.** *Assuming the above setup, for any $\Delta > 0$ we can partition G into components of radius $r \leq \Delta$ such that the number of edges connecting the different components is less than or equal:*

$$\frac{4W \log n}{\Delta}$$

Now, for our proof to work we will also assign a volume of W/n to each vertex, so that the total volume (over edges and vertices together) is $2W$. We will give a constructive proof. Consider the following algorithm:

1. Pick a vertex v and grow a ball in this pipe system. We define the following two quantities:

$$B(v, r) := \{u \in V \mid d_{uv} \leq r\}$$

$$V(s, r) := \frac{W}{n} + \sum_{\{u,v\} \subseteq B(s,r)} c_{uv} \ell_{u,v} + \sum_{\{u,v\} \in \delta(B(s,r))} c(u, v)(r - d_{us})$$

Note that you only count the volume of a vertex if it is at the root. We can also talk about the *capacity* of the ball:

$$C(s, r) = \sum_{\{u,v\} \in \delta(B(s,r)) \cap E} c_{uv}$$

We then have:

$$\frac{dV(s, r)}{dr} = C(s, r)$$

whenever the derivative is defined. (In some points we will instead have a left derivative and a right derivative.) We will now track:

$$\ln V(s,\Delta) - \ln V(s,0) = \ln \frac{V(s,\Delta)}{V(s,0)} = \int_0^1 \frac{1}{V(s,r)} \cdot \frac{\partial V(s,r)}{\partial r} dr = \int_0^1 \frac{C(s,r)}{V(s,r)} dr$$

$$\ln n = \ln \frac{W}{W/n} \geq \ln \frac{V(s,\delta)}{V(s,0)} \geq \Delta \cdot \min_{r\in[0,\Delta]} \frac{C(s,r)}{V(s,r)}$$

You can find $B(s, r_S)$ such that:

$$\frac{C(s,r_s)}{V(s,r_s)} \leq \frac{\log n}{\Delta}$$

$$C(s,r_s) \leq \frac{\log n}{\Delta} \cdot V(s,r(s))$$

We may now remove this ball and reiterate.

$$\sum C(s,r_S) \leq \frac{\log n}{\Delta}$$

. So we can find a cut $(\overline{S}, S)$ where $1/3n \leq |S| \leq 2/3n$

$$\alpha_1(S,\overline{S}) = \frac{E(S,\overline{S})}{|S||\overline{S}} \leq \frac{4Wn^2 \log n}{n^2/9} = O(W \log n)$$

An alternative proof of this can be done by constructing an ℓ_1 embedding and showing that under the assumption of case 2 we can find two balanced sets that are at least δ-separated.