Lecture 10: Finishing Off Cheeger and Further Partitioning

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Finishing Cheeger

Recap from last time: We were giving a proof based on a metric point of view:

- There is a characterization of conductance:

$$\overline{\phi} := \min_{S \subseteq V} \frac{|E(S,\overline{S})|}{|E(S,\overline{S})|} = \min_{d \in CUT_V} \frac{d(G)}{d(K_G)}$$

- CUT_V := metrics that are ℓ_1-embeddable

We will follow the following plan:

- Start with an eigenvector $v \perp \vec{1}$ s.t.: $\frac{x^T L x}{v^T L(K_G) v} = \lambda_{2,1}$ use v to constrcut an ℓ_1-metric d:

$$d_{i,j} = ||x_i - x_j||_1$$

  s.t.:

$$\frac{d(G)}{d(K_G)} \leq 2\sqrt{2\lambda_2}$$

We will find a small conductance cut by looking at sweep cuts corresponding to ℓ_1-embeddings.

1. Assume $\vec{x}$ is translated such that:

$$\sum_{x_i \geq 0} d_i = \frac{1}{2} Vol(G)$$

2. We let $y_i = \text{sign}(x_i) \cdot x_i^2$ (tensoring trick)

We have:

$$\begin{aligned}
d(G) = \sum_{\{i,j\} \in E} |y_i - y_j| &= \sum_E |\text{sign}(x_i) \cdot x_i^2 - \text{sign}(x_j) \cdot x_j^2| \\
&\leq \sum_E |x_i - x_j|(|x_i| + |x_j|) \\
&\leq \sqrt{\sum_E (|x_i| + |x_j|)^2 \cdot \sum_E (x_i - x_j)^2} \\
&\leq \sqrt{2 \sum_E (x_i^2 + x_j^2) x^T L x} \\
&= \sqrt{2 x^T D x \cdot x^T L x}
\end{aligned}$$

We can now look at the denominator:

$$d(K_G) = \sum_{i<j} \frac{d_i d_j}{Vol(G)} |y_i - y_j|$$

We can now use the assumption in (1.) and we will only look at contributions that cross the “zero line”:

$$\begin{aligned}
d(K_G) &= \sum_{i<j} \frac{d_i d_j}{Vol(G)} |y_i - y_j| \\
&\geq \sum_{x_i > 0 > x_j} \frac{d_i d_j}{Vol(G)} \left(|y_i| + |y_j|\right) \\
&\geq \sum_{x_i > 0} \frac{d_i \frac{Vol(G)}{2}}{Vol(G)} |y_i| + \sum_{x_j < 0} \frac{d_j \frac{Vol(G)}{2}}{Vol(G)} |y_j| \\
&= \frac{1}{2} \left(\sum_{i \in V} d_i |y_i| \right) = \frac{1}{2} \left(\sum d_i x_i^2 \right) = \frac{1}{2} x^T D x
\end{aligned}$$

Giving:

$$\frac{d(G)}{d(K_G)} \leq 2\sqrt{2\frac{x^T L x}{x^T D x}} \leq 2\sqrt{2\lambda_2}$$

More Approaches to Graph Partitioning

Graph Partitioning by Metric Relaxation

Cheeger:

$$\min d(G)$$

$$\text{s.t. } \; d(K_G) = 1 \quad\; d \in CUT_V$$

We will now use a linear program relaxation instead. In particular we will use a network flow linear program, a type of linear program that can be solved efficiently.

$$\min d(G)$$

$$\text{s.t. } \; d(K_G) = 1 \quad\; d \in \text{metric}_V$$

this is known as the Leighton-Rao relaxation. In this class we will talk about the relaxation and we will postpone the discuss of the rounding to next time.

$$\begin{aligned}
\min & \sum_{\{i,j\} \in E} \delta_{i,j} \\
\text{s.t.} & \sum_{i<j} \frac{d_i d_j}{Vol(G)} \delta_{i,j} = 1 \\
& \forall i, j \forall p \in P_{i,j} \left[\delta_{i,j} \leq \sum_{e \in P} \delta_e \right]
\end{aligned}$$

In the optimal solution, every $\delta_{i,j}$ is given by a path along G

$$\begin{aligned}
\min \sum_{\{i,j\}\in E} \ell_{i,j} \\
\text{s.t.} \sum_{i<j} \frac{d_i d_j}{Vol(G)} \delta_{i,j} = 1 \\
\forall i, j \forall p \in P_{i,j} \left[\delta_{i,j} \leq \sum_{e \in P} \ell_e \right]
\end{aligned}$$

While this primal problem might seem a little daunting due to the exponential number of constraints, it turns out its dual is a relatively simple flow problem:

Dual of the Leighton-Rao problem: Maxima Multicommodity Concurrent Flow

Note that whenever your primal has some kind of triangle inequality constraint, its dual will be a flow problem of some kind.

$$\begin{aligned}
\max \quad & \alpha \\
\text{s.t.} \quad & \forall e \in E : \sum_{e \in P} f_P \leq 1 \\
& \forall i, j \sum_{p \in P_{i,j}} f_P \geq \alpha \cdot \frac{d_i d_j}{Vol(G)}
\end{aligned}$$

in some sense you're trying to route K_G into G as a flow, but that is not always possible so you might have to scale things by a factor of α which you want to maximize. Finding a feasible α immediately certifies the conductance is at least α. We will see next time that LR does well on a path graph.