Lecture 15: More on SDP and MAXCUT

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Last Time...

MAXCUT: Given $G = (V, E)$ we want to find $(S, \overline{S})$ so that $|E(S, \overline{S})|$ is maximized. Recall that there is a simple solution that gives a 1/2-approximation problem. We found that linear programming didn't beat this approximation guarantees. Spectral methods gave a 9/8 integrality gap.

Vector Embedding

Relax: $x_i \in \mathbb{R} \to x_i \in \mathbb{R}^n$ for each vertex v. The SDP:

$$\begin{aligned} \max \ & \frac{1}{4} \sum_{ij \in E} ||v_i - v_j||^2 \\ \text{s.t.} \ & \forall i \ ||v_i||^2 \end{aligned}$$

for any cut S we can consider the solution:

$$v_i^{(S)} = \begin{cases} 1 & if i \in S \\ -1 & if i \notin S \end{cases}$$

we then have:

$$\text{value}(v^{(S)}) = |E(S, \overline{S})|.$$

Note that this is kind of similar to a spectral relaxation except here we have n constraints. We have that the spectral case looks like:

$$\begin{aligned} \max \ & \frac{1}{4} \sum ||v_i - v_j|| \\ s.t. \ & \sum d_i ||v_i||^2 = 2|E|. \end{aligned}$$

Goemans-Williamson Guarantees

The given spectral relaxation is called the Goemans-Williamson relaxation. Today we willl prove that the Geomans-Williamson relaxation can be rounded to a 0.87856-approximation to MAXCUT.

Rounding

Pick a direction s uniformly at random and look at the inner product of the solution vectors with s. We will assign $i \in S$ iff $\langle v_i, s\rangle \geq 0$.

Proof of the Guarantee Take a pair i,j then:

$$\Pr\left(ij \text{ is cut by rounding}\right) = \frac{\theta_{ij}}{\pi}$$

where $\theta_{i,j}$ is the angle between v_i and v_j. Note that one may only consider the projection of s onto the 2-dimensional subspace spanned by v_i and v_j. But recall:

$$\theta_{ij} = \arccos\langle v_i, v_j\rangle$$

but then:

$$\mathbb{E}\left[|E(S,\overline{S})|\right] = \sum_{\{i,j\}\in E} \arccos(\langle v_i, v_j\rangle)$$

We will compare the above expectation with the solution to the SDP that is an upperbound to the value of MAXCUT(G).

$$SDP = \frac{1}{4}\sum_{\{i,j\}\in E} ||v_i - v_j||^2 = \frac{1}{2}\sum_{\{i,j\}\in E} (1 - \langle v_i, v_j\rangle)$$

Consider:

$$\frac{\mathbb{E}\left[E(S,\overline{S})\right]}{SDP} = \frac{2}{\pi}\cdot\frac{\sum_{\{i,j\}\in E} \arccos\,\langle v_i, v_j\rangle}{\sum_{\{i,j\}\in E} 1 - \langle v_i, v_j\rangle}$$

We have:

$$\min_{-1\leq x\leq 1} \frac{2}{\pi}\cdot\frac{\arccos x}{1-x} \geq 0.878...$$

where the minimum is attained at $x = -0.67$. There are also matching integrality gaps between these.

Partitioning: Minimum Conductance

Recall:

$$\min_{S\subseteq V} \frac{|E(S,\overline{S})|}{Vol(S)\cdot Vol(\overline{S})}\cdot Vol(G)$$

Cheeger:

$$\lambda_2 \leq \overline{\phi}_G \leq 2\sqrt{2\lambda_2}$$

Spectral Relaxation:

$$\min \sum_{\{i,j\}} ||v_i - v_j||^2$$

$$s.t. \sum_{i<j} \frac{d_i d_j}{Vol(G)} ||v_i - v_j||^2 = 1$$

Note that this formulation might partition in the graph in two sets of very different volume.

Balanced Minimum Conductance

Divide-and-conquer: we would to get a partition where a relatively large fraction of the volume lies on each side. We define the b-balanced minimum conductance:

$$\overline{\phi}_{G,b} = \min_{S \subseteq V} \overline{\phi}(S)$$

$$Vol(S) \geq b \cdot Vol(G)$$

Side Note: Integral solution to spectral relaxation:

$$(S, \overline{S})$$

we may consider the vector:

$$v_i^{(S)} = \begin{cases} \sqrt{\frac{Vol(G)}{Vol(S)Vol(\overline{S})}} & \text{if } i \in S \\ 0 & \text{if } i \in \overline{S} \end{cases}$$

Suppose now that we want to rule out cuts that are not balanced, i.e. we want to impose the constraint that:

$$\frac{Vol(S)}{Vol(G)} > b$$

or equivalently:

$$1 - \frac{Vol(S)}{Vol(G)} = \frac{Vol(\overline{S})}{Vol(G)} < 1 - b$$

in order to make things balanced we might add a constraint:

$$||v_i||^2 \leq \sqrt{Vol(G)} \cdot \frac{1}{b}$$

Alternatively, we may think of the complete graph K_G as the sum of a bunch of stars:

$$k_G = \frac{1}{2} \sum_{i \in V} d_i \cdot S_i^G$$

where S_i is a star graph centered at i. We may want to change the above blance constraint into a translation-invariant one:

$$d_i \sum_{j \in V} \frac{d_{i,j}}{Vol(G)} ||v_i - v_j||^2$$

where the above is the Laplacian quadratic form of $d_i \cdot S_i$.

SDP Relaxation

$$\begin{aligned} \min \ & L \cdot X \\ \text{s.t.} \ & L(K_G) \cdot X = 1 \\ & \forall i: \ L(S_i) \cdot X \leq \frac{Vol(G)}{b} \end{aligned}$$

or, equivalently:

$$\begin{aligned} \min \ & L \cdot X \\ \text{s.t.} \ & L(K_G) \cdot X = Vol(G) \\ & \forall i: \ d_i \cdot L(S_i) \cdot X \leq d_i \frac{1}{b} \end{aligned}$$

and the dual gives:

$$\begin{aligned} \max \ & \alpha - \sum \frac{d_i \beta_i}{b} \\ \text{s.t.} \ & L + \sum d_i \beta_i L(S_i) \succeq \alpha L(K_G) \end{aligned}$$