Lecture 5: More about λ_2

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Recap

So far we have looked at:

1. Role of the Laplacian,
2. Continuous Random Walk,
3. Heat Kernel: $e^{-t(I-W)}$ this is the solution to the differential equation:

$$\frac{dp(t)}{dt} = -(I-W)p(t)$$

   this is known as the heat equation or diffusion process. We have seen how the convergence of this process depends on the second eigenvalue of the Laplapcian λ_2.:

$$||p(t)-\pi||_{D^{-1}}^2 \leq e^{-t\lambda_2}||p(0)-\pi||_{D^{-1}}^2$$

4. We got some bounds on λ_2 for the cycle graph C_n and we found: $\lambda_2 \leq O(\frac{1}{n^2})$ using a test vector to find x such that: $x^T 1 = 0$:

$$\lambda_2 \leq \frac{x^T L x}{x^T D x} = O\left(\frac{1}{n^2}\right)$$

   this characterization of λ_2 is useful to prove that for some starting conditions the process takes a long time to converge. However it is sometimes useful to try and prove that convergence is fast.

[...]

Lemma 1. *$\lambda_2 = 0$ iff G is disconected.*

Proof. Using the Rayleigh quotient characterization of the eigenvalues:

$$\lambda_2 = \min_{x \perp 1} \frac{x^T L x}{x^T D x}$$

If the graph is disconnected then there exists at least two disconnected components, say S, and $\overline{S}$, we can then take **x** above to be:

$$x_i = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ -\frac{1}{|\overline{S}|} & \text{if } i \in \overline{S} \end{cases}$$

which gives:

$$\lambda_2 \leq 0$$

but we know:

$$L \succeq 0$$

and hence:

$$\lambda_2 = 0.$$

On the other hand, if λ_2 is zero, then there exists some eigenvalue $\mathbf{x} \perp 1$. But then some entries of $\mathbf{x}$ are non-negative and some are negative. By looking at the quadratic form, it should be clear that there cannot be any edge connecting the set of vertices i where the sign of x_i non-negative, and those in which the sign is negative. □

(Spoiler Alert: We will later see a robust version of this fact: if there is a small eigenvalue then there is a small cut. - Cheeger inequality).
We then have the following generalization:

Proposition 1. *G has k connected components if and only if $\lambda_k = 0$.*

Proof. The proof is somewhat similar to the above and makes use of the following characterization of the k^{th} smallest eigenvalue, known as the Courant-Fischer theorem:

$$\lambda_k = \min_{\substack{S \subseteq \mathbb{R}^n, \\ dim(s)=k}} \max_{x \in S} \frac{x^T L x}{x^T D x}$$

□

Lower Bounds on $x^T L x$

Path graph

$$x^T L x = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \geq \frac{(x_1 - x_n)^2}{n-1} \tag{1}$$

where the inequality follows by applying Cauchy-Schwartz with the $\vec{1}$ vector and the $n-1$-dimensional vector with entry i equal to $(x_i - x_{i+1})$. (1) is sometimes known as the *path inequality*. In the above we have essentially compared the Laplacian of the path graph with the Laplacian of another graph. This is sometimes referred to as a *graphic inequality*.

Background: PSD Ordering

We write:

$$\forall x : \qquad x^T A x \geq x^T B x$$

as:

$$A \succeq B$$

this is equivalent to:

$$A - B \succeq 0$$

. When this is true, we have that the k^{th} eigenvalue of A is greater than the k^{th} eigenvalue of B. Some strange things happen when looking at the PSD ordering, for instance:

$$A \succeq B \not\Rightarrow A^2 \succeq B^2$$

this is a consequence of non-commutativity.

Graphic Inequalities

1. $(n-1)L_{path} \succeq L_{1n}$,
2. Question: $\lambda_2(L_{path}) \geq ?$ which graph should I compare this against? We will use the complete graph, the Laplacian of which is $L(K_v) = nI - 11^T$ and hence has eigenvalues $0, n, n, ..., n$.

How does one interpret these graphic inequalities? It helps to think of the consequences that these inequalities have on random walk processes.

Lowerbounds on λ_2

$(n-1)L_p \geq L_{1n}$ for any pairs of vertices i, j: $(j-1)L_P \geq L_{ij}$ We then have:

$$\sum_{i<j}(j-i)L_P \geq \sum_{i<j} L_{i,j} = L(K_v)$$

which gives:

$$O(n^3) \geq L(K_v)$$

$$O(n^3)\lambda_2(L_{path}) \geq n$$

$$O\left(\frac{1}{n^2}\right) \geq \lambda_2(L_{path}) \geq \Omega\left(\frac{1}{n^2}\right)$$

Complete Binary Tree

Just like the path the complete binary tree is acyclic and removing an edge disconnects the graph. However the λ_2 can be very different.