Lecture 4: Continuos Time Random Walks and Spectral Gap

Instructor: Lorenzo Orecchia Scribe: Erasmo Tani

Continuous-Time Random Walk

The continuous random walk is the process described by the differential equation:

$$\frac{dp(t)}{dt} = -(I - W)p(t)$$

Consider a d-regular graph, i.e. a graph in which every vertex $v \in V$ has degree $d_v = d$. Here $W = \frac{A}{d}$. Using the decomposition $(I - W) = U\Lambda U^T$ we get:

$$\frac{dp(t)}{dt} = -(U\Lambda U^T)p(t)$$

(since in this case $(I - W)$ is symmetric).
By applying a change of basis we can write:

$$\frac{dU^T p(t)}{dt} = -(\Lambda U^T)p(t)$$

Which is equivalent to having, for $i = 1, \cdots, n$:

$$\left(\frac{dU^T p(t)}{dt}\right)_i = -\lambda_i \left(U^T p(t)\right)_i$$

by solving each first order ODE we get:

$$\left(U^T p(t)\right)_i = \left((e^{-\lambda_i} U^T p(0)\right)_i$$

edit:

$$\left(U^T p(t)\right)_i = \left((e^{-\lambda_i t} U^T p(0)\right)_i$$

so that:

$$U^T p(t) = \text{diag}\left(e^{-\lambda_1}, ..., e^{-\lambda_n}\right) U^T p(0)$$

edit:

$$U^T p(t) = \text{diag}\left(e^{-\lambda_1 t}, ..., e^{-\lambda_n t}\right) U^T p(0)$$

and hence:

$$p(t) = U^T \text{diag}\left(e^{-\lambda_1}, ..., e^{-\lambda_n}\right) U^T p(0) \tag{1}$$

edit:

$$p(t) = U \text{diag}\left(e^{-\lambda_1 t}, ..., e^{-\lambda_n t}\right) U^T p(0) \tag{2}$$

Intermezzo: Matrix Exponential

For the sake of convenience, we introduce here the notion of matrix exponential.

Definition 1 (Matrix Exponential)**.** Given a symmetric matrix $M = U\Lambda U^T$, where $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$ the *matrix exponential* of M is the matrix:

$$e^M = U\text{diag}(e^{\lambda_1,...,\lambda_n})U^T$$

edit:

$$e^M = U\text{diag}(e^{\lambda_1}, ..., e^{\lambda_n})U^T$$

so that e^M has the same eigenvector as M but eigenvalues related by: $\lambda_i(e^M) = e^{\lambda_i(M)}$.

Note that:

$$\begin{aligned} e^Y &= U^T \text{diag}_i \left(\sum_{j=0}^{\infty} \frac{\lambda_i^j}{j!} \right) U \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} U^T \text{diag}_i(\lambda_i^j) U \\ &= \sum_{j=0}^{\infty} \frac{Y^j}{j!} \end{aligned}$$

so that we recover an matrix analogue of the familiar Taylor expansion:

$$\boxed{e^Y = \sum_{j=0}^{\infty} \frac{Y^j}{j!}}$$

We also have the following property, which we state without a proof:

Proposition 1. *Given any two symmetric matrices A and B we have:*

$$AB = BA \qquad \Rightarrow \qquad e^A e^B = e^{A+B}$$

Note that in general $e^{A+B} \neq e^A e^B$.

Back to the Markov Chain

In light of Definition 1 we can rewrite Equation (2) as:

$$p(t) = e^{-\left(I - \frac{A}{d}\right)} p(0)$$

$$\begin{aligned} \frac{dD^{-1/2}p(t)}{dt} &= D^{-1/2}(I - AD^{-1})p(t) \\ &= D^{-1/2}(I - AD^{-1})p(t) \\ &= (I - D^{-1/2}AD^{-1/2})D^{-1/2}p(t) \end{aligned}$$

edit: (maybe note that starting here the graph is not necessarily regular?)

$$\begin{aligned}\frac{dD^{-1/2}p(t)}{dt} &= -D^{-1/2}(I - AD^{-1})p(t) \\ &= -(I - D^{-1/2}AD^{-1/2})D^{-1/2}p(t)\end{aligned}$$

Definition 2 (Normalized Laplacian)**.** Given a graph G its *normalized Laplacian* is the matrix:

$$\mathcal{L} = (I - D^{-1/2}AD^{-1/2}) = D^{-1/2}LD^{-1/2}$$

we then have:

$$D^{-1/2}p(t) = e^{-t\mathcal{L}}D^{-1/2}p(0)$$

so that

$$p(t) = D^{1/2}e^{-t\mathcal{L}}D^{-1/2}p(0)$$

so that the derivative of the distance to the stationary distribution is given by:

$$\begin{aligned}\frac{d}{dt}||p - \pi||^2_{D^{-1}} &= \frac{d}{dt}||D^{1/2}e^{-t\mathcal{L}}D^{1/2}(p^{(0)} - \pi)||^2_{D^{-1}} \\ &= -(p(0) - \pi)^T D^{-1/2}e^{-t\mathcal{L}}\mathcal{L}e^{-t\mathcal{L}}D^{-1/2}(p(0) - \pi) \\ &= -(D^{-1}(p(t) - \pi))^{-1}L(D^{-1}(p(t) - \pi))\end{aligned}$$

In order to evaluate our progress we consider the ratio:

$$\frac{\frac{d}{dt}||p(t) - \pi||^2_{D^{-1}}}{||p(t) - \pi||^2_{D^{-1}}}$$

we have:

$$\begin{aligned}\frac{\frac{d}{dt}||p(t) - \pi||^2_{D^{-1}}}{||p(t) - \pi||^2_{D^{-1}}} &= -\frac{(p(t) - \pi)^T D^{-1/2}\mathcal{L}D^{-1/2}(p(t) - \pi)}{(p(t) - \pi)^T D^{-1/2}ID^{-1/2}(p(t) - \pi)} \\ &\leq \min_{D^{1/2}y^T 1 = 0} \frac{y^T\mathcal{L}y}{y^T y}\end{aligned}$$

edit:

$$\begin{aligned}\frac{\frac{d}{dt}||p(t) - \pi||^2_{D^{-1}}}{||p(t) - \pi||^2_{D^{-1}}} &= -\frac{(p(t) - \pi)^T D^{-1/2}\mathcal{L}D^{-1/2}(p(t) - \pi)}{(p(t) - \pi)^T D^{-1/2}ID^{-1/2}(p(t) - \pi)} \\ &\leq -\min_{D^{1/2}y^T 1 = 0} \frac{y^T\mathcal{L}y}{y^T y}\end{aligned}$$

Theorem 1.

$$||p(t) - \pi||^2_{D^{-1}} \leq e^{-t\lambda_2}(p(0) - \pi)^2_{D^{-1}}$$

Definition 3 (Spectral Gap)**.** The spectral Gap is the quantity:

$$\lambda_2(\mathcal{L}) = \min_{D^{1/2}y^T 1=0} \frac{y^T \mathcal{L} y}{y^T y}$$

this equals:

$$\min_{x^T D 1} \frac{x^T L x}{x^T D x}$$

edit:

$$\min_{x^T D 1=0} \frac{x^T L x}{x^T D x}$$

Examples of Spectral Gaps

Spectral Gap of Complete Graphs

The complete graph K_n has Laplacian matrix:

$$L(K_n) = nI - 11^T$$

$$D(K_n) = (n-1) \cdot I$$

And hence the spectral gap is given by:

$$\frac{n}{n-1} = 1 + \frac{1}{n-1}$$

which is the largest possible.

Spectral Gap of n-Cycle

The spectral gap of an n-cycle is $O\left(\frac{1}{n^2}\right)$