CS 591 O1: Iterative Methods for Graph Algorithms and Network Analysis Fall 2018

Lecture 9: Metrics, Cuts, Conductance and all That Jazz

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Introduction

Recall:

- (Symmetrized) Conductance: $\overline{\phi}(S) := \frac{|E(S,\overline{S}|}{Vol(S)Vol(\overline{S})} \cdot Vol(V)$,
- $\overline{\phi} = \min_{S \subseteq V} \overline{\phi}(S) = \min_{S \subseteq V} \frac{|E(S,\overline{S})|}{|E_{K_G}(S,\overline{S})|} = \min_{S \subseteq V} \frac{\vec{1}_S^T L_G \vec{1}_S}{|E_{K_G}(S,\overline{S})|} \geq \min_{x \in \mathbb{R}^n} \frac{x^T L_G x}{x^T L_{K_G} x} = \lambda_2$
- if λ_2 is large, all cuts have large conductance.

Metrics

Definition 1 (Metric)**.** A metric can be seen as a matrix $d \in \mathbb{R}^{n \times n}$ satisfying:

1. $\forall i$: $d_{ii} = 0$
2. $\forall i, j$: $d_{ij} = d_{ji}$
3. $\forall i, j, k$: $d_{ij} \leq d_{ik} + d_{kj}$

Examples

1. L_2-distance over $\mathbb{R}^n$,
2. L_2^2-distance is *not* in general a metric.

Definition 2 (Semimetric)**.** A semimetric is a metric which does not necessarily satisfy the triangle inequality.

Given a metric d on a graph G, we can obtain a notion of volume on G:

$$d(G) := \sum_{\{i,j\} \in E} w_{i,j} d_{i,j}$$

Cut Metrics

Definition 3 (Cut Metric)**.** Given a set $S \subseteq V$, the cut metric $\delta^{(S)}$ is:

$$\delta^{(S)}_{i,j} := \begin{cases} 0 & \text{if } i, j \text{ are on the same side of the cut} \\ 1 & \text{otherwise} \end{cases}$$

Relation to Conductance

$$\delta^{(S)}(G) = |E_G(S, \overline{S})|$$
$$\delta^{(S)}(K_G) = \frac{Vol(S)Vol(\overline{S})}{Vol(V)}$$
$$\overline{\phi}(S) = \frac{\delta^{(S)}(G)}{\delta^{(S)}(K_G)}$$
$$\overline{\phi}_G = \min_{\delta \in \text{cut metric}} \frac{\delta(G)}{\delta(K_G)}$$

the above is a more general version of the ratio:

$$\min_{S \subseteq V} \frac{\vec{1}_S^T L \vec{1}_S}{\vec{1}_S^T L(K_G) \vec{1}_S}$$

Spectral Gap

$$\lambda_2 = \min \frac{\sum_{\{i,j\} \in E} w_{i,j}(x_i - x_j)^2}{\sum_{i,j \in V} \frac{d_i d_j}{Vol(G)}(x_i - x_j)^2}$$

this kind of looks like a ratio of volume.

Definition 4. A semimetric g is ℓ_2^2-*embeddable* if there exists an embedding $v_i \in R^\ell$ (for some $\ell \in \mathbb{N}$) s.t.:

$$\forall i, j \in V \, g_{i,j} = ||v_i - v_j||_2^2$$

Lemma 1.

$$\lambda_2 := \min_{g\ \ell_2^2\text{-embeddable}} \frac{g(G)}{g(K_G)}$$

Proof. Note:

$$\boxed{\frac{\sum s_i}{\sum b_i} \geq \min_i \frac{a_i}{b_i}}$$

The above can easily be proved by rewriting the LHS as:

$$\sum_i \frac{b_i}{\sum_i b_i} \cdot \frac{a_i}{b_i}$$

We then have:

$$\begin{aligned}
\min_{g \in \ell_2^2} \frac{g(G)}{g(K_G)} &= \frac{\sum_{\{i,j\} \in E} w_{i,j} ||v)i - v_j||^2}{\sum_{i,j \in V} \frac{d_i d_j}{Vol(V)} ||v_i - v_j||^2} \\
&= \frac{\sum_{u=1}^k \sum_{\{i,j\} \in E} w_{i,j}(v_i - v_j)_u^2}{\sum_{u=1}^k \sum_{i,j \in V} \frac{d_i d_j}{Vol(V)} ||v_i - v_j||_u^2} \\
&\geq \min_{1 \leq u \leq k} \frac{\sum_{\{i,j\} \in E} w_{i,j}(v_i - v_j)_u^2}{\sum_{i,j \in V} \frac{d_i d_j}{Vol(V)} ||v_i - v_j||_u^2} \\
&\geq \lambda_2
\end{aligned}$$

$\square$

Note: Cut metrics are ℓ_2^2-embeddable.

The space of cut metrics is discrete, but we will consider its convex hull.

Cut Cone

Definition 5 (Cone)**.** A set $A \in \mathbb{R}^k$ is a *cone* if for all $x \in A$ and $\alpha > 0$ we have $\alpha x \in A$.

Definition 6 (Convex Cone)**.** The convex cone generated by a set of points $B = \{v_i \in \mathbb{R}^k\}$ is the set:

$$Cone(B) := \left\{ x \in \mathbb{R}^K : x = \sum \alpha_i v_i, \ \ \vec{\alpha} > 0 \right\}$$

Definition 7 (Cut Cone)**.** The *cut cone* CUT_v is:

$$Cone(\{\delta^{(S)}\}_{S \subseteq V})$$

We will show CUT_V =metrics that embeddable in ℓ_1.

Definition 8 (ℓ_1-embeddability)**.** A metric d over V is ℓ_1-embeddable if there exists an embedding $\{v_i \in \mathbb{R}^k\}_{i \in V}$ s.t:

$$d_{i,j} = ||v_i - v_j||_1$$

Theorem 1. *The set of ℓ_1-embeddable metrics over V is CUT_V.*

Proof. We will show in two parts:

1. Any cut metric is ℓ_1-embeddable
2. Any ℓ_1-embeddable metric is a conic combination of cut merics

We make use of the Sweep Cut:

$$S_t = \{1, ..., t\}$$

we will have:

$$d = \sum |v_{t+1} - v_t| \cdot \delta^{(S_t)}$$

$\square$