University of Trento
Department of Industrial Engineering

Mechanical Vibrations

Laboratory experience

Analysis of the vibrations of a beam

Professor:
Prof. Daniele Bortoluzzi

Assistant:
Dott. Edoardo Dalla Ricca

Student:
Lorenzo Colturato, 233301

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Chapter 1

Introduction

The aim of the project is to analyze the vibrations of a beam pinned by two rods which are connected to the ground, like the one shown in Figure 1.1.

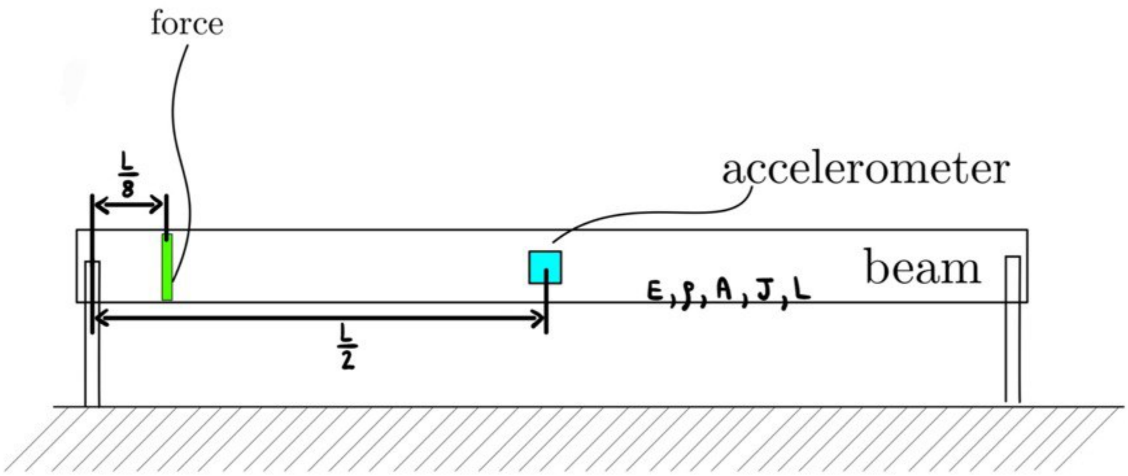


Figure 1.1: System sketch with accelerometers and force sensor

In the middle of the length of the beam there are two accelerometers to record the accelerations of the beam and the cart caused by the impact of a hammer at a distance $\frac{L}{8}$ from the left end of the beam. The generated force $f(t, x)$ is measured by a force sensor which is included in the hammer. The data were acquired with a sampling frequency of 51200 Hz. The beam data are:

- Length: $L = 0.7\text{m}$
- Young modulus: $E = 206\text{GPa}$
- Mass density: $\rho = 7850\text{kg/m}^3$
- Beam cross-section area: $A = 111\text{mm}^2$
- Beam cross-section moment of inertia: $J = 6370\text{mm}^4$

Chapter 2

Analytical model

Figure 2.1 shows a hand-made sketch of the system with the chosen reference frame $x - z$ placed in the left end of the beam.

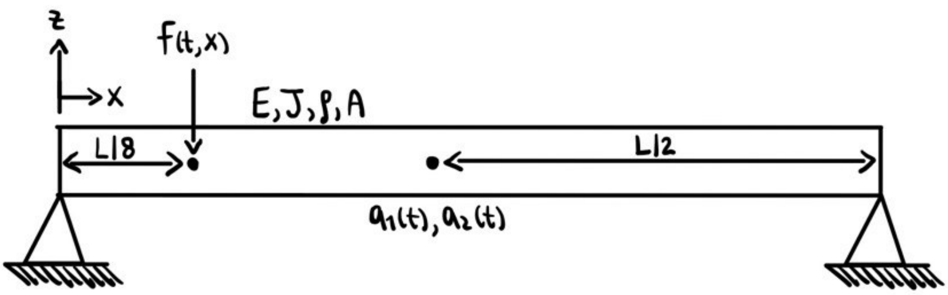


Figure 2.1: System sketch with reference frame

where a_1(t) and a_2(t) are the accelerations of recorded by the two accelerometers. The general equation of forced lateral vibrations of a beam is the following:

$$EJ\frac{d^4}{dx^4}[w(t,x)] + A\rho\frac{d^2}{dt^2}[w(t,x)] = f(t,x) \tag{2.1}$$

where the displacement $w(t,x)$ can be written as $w(t,x) = W(x) \cdot q(t)$ for the separation of variables principle. Considering only the first three modes to describe the system, equation 2.1 can be rewritten in its modal expansion up to the third mode:

$$EJ\sum_{n=1}^{3}(\frac{d^4}{dx^4}[W_{\mathrm{n}}(x)] \cdot q_{\mathrm{n}}(t)) + A\rho\sum_{n=1}^{3}(W_{\mathrm{n}}(x) \cdot \frac{d^2}{dt^2}[q_{\mathrm{n}}(t)]) = f(t,x) \tag{2.2}$$

First the system is considered as undamped in order to compute the natural frequencies and the mode shapes, then it is considered as damped to compute and plot the analytical transfer function between the impulse force and the beam acceleration.

2.1 Undamped system

To compute the natural frequencies and then the mode shapes of the beam it is necessary to consider the free vibrations of the beam, hence the homogeneous differential equation:

$$c^2 \frac{d^4}{dx^4}[w(t,x)] + \frac{d^2}{dt^2}[w(t,x)] = 0 \tag{2.3}$$

where $c = \sqrt{\frac{EJ}{A\rho}} = 38.8067\text{m/s}$ is the speed of sound inside the beam. What we are really interested in to compute the natural frequencies and consequently the mode shapes is the space component of $w(t,x)$, which is $W(x)$, which is equal to:

$$W(x) = \text{c1}\cos(\beta x) + \text{c2}\sin(\beta x) + \text{c3}\cosh(\beta x) + \text{c4}\sinh(\beta x) \tag{2.4}$$

where $\beta = \sqrt{\frac{\omega}{c}}$ is the angular frequency in space domain while ω is the angular frequency in time domain. The first thing to do is to impose the boundary conditions for $x = 0$ and $x = L$ where there are the two pinned ends.

1. $x = 0$

   - $W(0) = 0 \Longrightarrow c_1 + c_3 = 0$
   - $\frac{d^2}{dx^2}[W(0)] = 0 \Longrightarrow -\beta^2 c_1 + \beta^2 c_3 = 0$

   It results that both c_1 and c_3 are equal to zero.

2. $x = L$

   - $W(L) = 0 \Longrightarrow c_2 \sin(\beta L) + c_4 \sinh(\beta L) = 0$
   - $\frac{d^2}{dx^2}[W(L)] = 0 \Longrightarrow -\beta^2 c_2 \sin(\beta L) + \beta^2 c_4 \sinh(\beta L) = 0$

   It can be rewritten as: $\begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2 \sin(\beta L) & \beta^2 \sinh(\beta L) \end{bmatrix} \cdot \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   Computing $\det \begin{bmatrix} \sin(\beta L) & \sinh(\beta L) \\ -\beta^2 \sin(\beta L) & \beta^2 \sinh(\beta L) \end{bmatrix} = 0$, it results in $2\beta^2 \sin(\beta L)\sinh(\beta L) = 0$. To have real solutions $\sin(\beta L) = 0$ is solved, resulting in:

$$\beta = \frac{n \cdot \pi}{L} \qquad \text{with } n\text{=0, 1, 2, ...} \tag{2.5}$$

2.1.1 Natural frequencies

From equation 2.5 is possible to compute the first three natural frequencies of the system. In fact we have that:

$$\begin{cases} n = 1 \longrightarrow \beta_1 = \frac{\pi}{L} \Longrightarrow \omega_1 = \frac{\pi^2 \cdot c}{L^2} = 781.647 \text{ rad/s} = 124.466 \text{ Hz} \\ n = 2 \longrightarrow \beta_2 = \frac{2\pi}{L} \Longrightarrow \omega_2 = \frac{4\pi^2 \cdot c}{L^2} = 3126.59 \text{ rad/s} = 497.864 \text{ Hz} \\ n = 3 \longrightarrow \beta_3 = \frac{3\pi}{L} \Longrightarrow \omega_3 = \frac{9\pi^2 \cdot c}{L^2} = 7034.82 \text{ rad/s} = 1120.19 \text{ Hz} \end{cases}$$

2.1.2 Mode shapes

At this point it is necessary to find the remaining constants c_2 and c_4 for each mode.

1. Mode 1 $\Longrightarrow \begin{bmatrix} \cancelto{0}{\sin(\pi)} & \sinh(\pi) \\ -\beta^2 \cancelto{0}{\sin(\pi)} & \beta^2 \sinh(\pi) \end{bmatrix} \cdot \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow c_4 = 0$
2. Mode 2 $\Longrightarrow \begin{bmatrix} \cancelto{0}{\sin(2\pi)} & \sinh(2\pi) \\ -\beta^2 \cancelto{0}{\sin(2\pi)} & \beta^2 \sinh(2\pi) \end{bmatrix} \cdot \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow c_4 = 0$
3. Mode 3 $\Longrightarrow \begin{bmatrix} \cancelto{0}{\sin(3\pi)} & \sinh(3\pi) \\ -\beta^2 \cancelto{0}{\sin(3\pi)} & \beta^2 \sinh(3\pi) \end{bmatrix} \cdot \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow c_4 = 0$

After the computation of the constants, from equation 2.4 it results that

$$W(x) = \mathrm{c}_2 \sin(\beta x) \tag{2.6}$$

From 2.6 the three mode shapes can be defined as follows:

$$\begin{cases} W_1(x) = \mathrm{c}_{21} \sin(\frac{\pi}{L} x) \\ W_2(x) = \mathrm{c}_{22} \sin(\frac{2\pi}{L} x) \\ W_3(x) = \mathrm{c}_{23} \sin(\frac{3\pi}{L} x) \end{cases}$$

Figure 2.2 shows the plot of the first three mode shapes of the beam, placing the constants c_{21}, c_{22} and c_{23} equal to one.

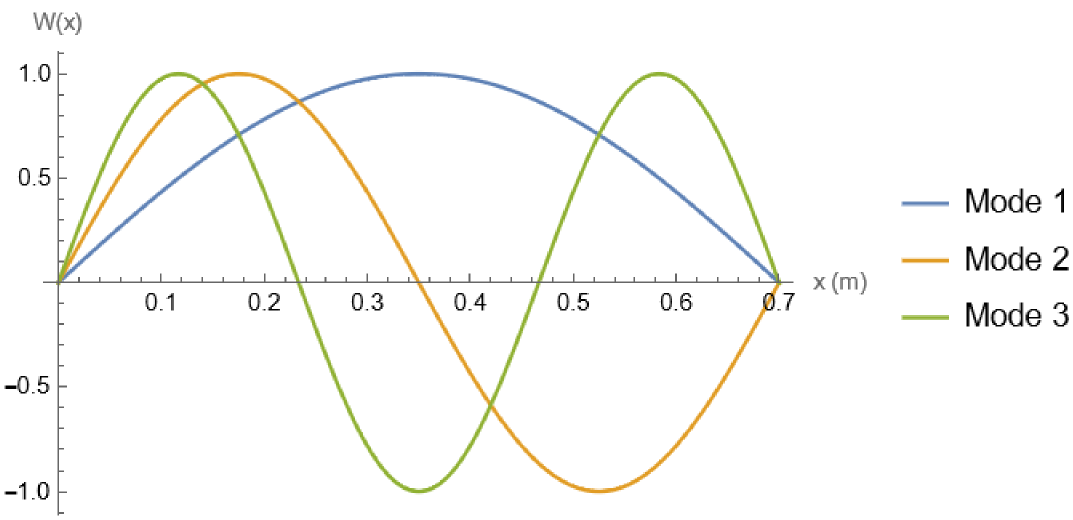


Figure 2.2: First three mode shapes of the beam

2.2 Damped system

In order to calculate the modal projection of the impulse force and find the analytical transfer function of the damped system is necessary to start from the differential equation of the general modal oscillator with dissipation and find the solution in time domain $q_{\mathrm{n}}(t)$:

$$\frac{d^2}{dt^2}[q_{\mathrm{n}}(t)] + 2\xi_{\mathrm{n}}\omega_{\mathrm{n}}\frac{d}{dt}[q_{\mathrm{n}}(t)] + {\omega_{\mathrm{n}}}^2 q_{\mathrm{n}}(t) = \frac{1}{\rho A b_{\mathrm{n}}} \cdot Q_{\mathrm{n}}(t) \tag{2.7}$$

where $b_{\mathrm{n}} = \int_0^L W_{\mathrm{n}}(x)^2\,dx$ is the modal mass and $Q_{\mathrm{n}}(t) = \int_0^L f(t,x) \cdot W_{\mathrm{n}}(x)\,dx$ is the modal projection of the force.

2.2.1 Modal projection of the impulse force

Equation 2.8 provides a general method to determine the modal force for all the mode shapes considered:

$$Q_{\mathrm{n}}(t) = \int_0^L f(t)\delta(x-x_0)W_{\mathrm{n}}(x)\,dx = f(t)\cdot W_{\mathrm{n}}(x_0) \tag{2.8}$$

where $x_0 = \frac{L}{8}$ is the point of application of the force with respect to the reference frame, while f(t) is the impulsive force generated by the hammer.
The following system collects the modal projection of the force for the first three mode shapes of the beam already presented in Section 2.1.2.

$$\begin{cases} n=1 \longrightarrow W_1(\frac{L}{8}) = \mathrm{c}_{21}\sin(\frac{\pi}{8}) \Longrightarrow Q_1(t) = f(t)\cdot \mathrm{c}_{21}\sin(\frac{\pi}{8}) \\ n=2 \longrightarrow W_2(\frac{L}{8}) = \mathrm{c}_{22}\sin(\frac{\pi}{4}) \Longrightarrow Q_2(t) = f(t)\cdot \mathrm{c}_{22}\sin(\frac{\pi}{4}) \\ n=3 \longrightarrow W_3(\frac{L}{8}) = \mathrm{c}_{23}\sin(\frac{3\pi}{8}) \Longrightarrow Q_3(t) = f(t)\cdot \mathrm{c}_{23}\sin(\frac{3\pi}{8}) \end{cases}$$

2.2.2 Analytical transfer function

To determine the transfer function between the impulse force and the beam acceleration, equation 2.7 can be rewritten as:

$$\frac{d^2}{dt^2}[q_{\mathrm{n}}(t)] + 2\xi_{\mathrm{n}}\omega_{\mathrm{n}}\frac{d}{dt}[q_{\mathrm{n}}(t)] + {\omega_{\mathrm{n}}}^2 q_{\mathrm{n}}(t) = \frac{1}{\rho A b_{\mathrm{n}}} \cdot f(t)\cdot W_{\mathrm{n}}(x_0) \tag{2.9}$$

Using *Wolfram Mathematica* to solve Equation 2.9 and substituting the beam data is possible to find a general solution in time domain, which is the sum of the free-response and the forced response. The general solution of the homogeneous part of 2.9, i.e. the free-response of the system is the following:

$$\mathrm{q}_{\mathrm{n,free}}(t) = e^{-t\left(\sqrt{({\xi_{\mathrm{n}}}^2-1){\omega_{\mathrm{n}}}^2}+\xi_{\mathrm{n}}\omega_{\mathrm{n}}\right)}\left(\mathrm{C_n} + \mathrm{D_n}e^{2t\sqrt{({\xi_{\mathrm{n}}}^2-1){\omega_{\mathrm{n}}}^2}}\right) \tag{2.10}$$

while the forced response of the system is the below:

$$\mathrm{q}_{\mathrm{n,forced}}(t) = \frac{\sqrt{\left({\xi_{\mathrm{n}}}^2-1\right){\omega_{\mathrm{n}}}^2}\mathrm{W_n}(\mathrm{x_0})e^{-t\left(\sqrt{({\xi_{\mathrm{n}}}^2-1){\omega_{\mathrm{n}}}^2}+\xi_{\mathrm{n}}\omega_{\mathrm{n}}\right)}\left(e^{2t\sqrt{({\xi_{\mathrm{n}}}^2-1){\omega_{\mathrm{n}}}^2}}-1\right)}{2A\mathrm{b_n}\left({\xi_{\mathrm{n}}}^2-1\right)\rho{\omega_{\mathrm{n}}}^2} \tag{2.11}$$

Since we are interested in finding the transfer function between the applied force and the resulting beam acceleration we only consider the forced response. The next step is to find the time-specific solution for the first three mode shapes which are associated to different damping ratios and natural frequencies. Computing the modal mass for the different mode shapes it can be seen that the results are the same and particularly it holds that

$$b_{\mathrm{n}} = 0.35 \cdot c_{2\mathrm{n}} \qquad \text{with n=1, 2, 3} \tag{2.12}$$

The damping ratios associated to the different modes are the following:

$$\begin{cases} \text{Mode 1} \longrightarrow \xi_1 = 0.05 \\ \text{Mode 2} \longrightarrow \xi_2 = 0.01 \\ \text{Mode 3} \longrightarrow \xi_3 = 0.01 \end{cases}$$

With this knowledge is possible to compute the solution in time of equation 2.11 for the first three modes, i.e. $q_1(t)$, $q_2(t)$ and $q_3(t)$. To find the analytical transfer function between the applied force and the resulting beam acceleration the next steps have to be followed:

1. Define the lateral vibrations of the beam up to the third mode:

$$w(t,x) = \mathrm{W_1(x) \cdot q_1(t) + W_2(x) \cdot q_2(t) + W_3(x) \cdot q_3(t)} \tag{2.13}$$

   considering only the forced response 2.11

2. Compute $w(t, x = \frac{L}{2})$ since the accelerometers are located in the mid-length of the beam

3. Derive $w(t, x = \frac{L}{2})$ twice in time to find the acceleration of the beam $\Longrightarrow \ddot{w}(t, x = \frac{L}{2})$

4. Find the Laplace Transform of $\ddot{w}(t, x = \frac{L}{2}) \Longrightarrow \mathscr{L}\{\ddot{w}(t, x = \frac{L}{2})\}$

$$\mathscr{L}\{\ddot{w}(t, x = \frac{L}{2})\} = \frac{\frac{\omega_3 \mathrm{W}_3\left(\frac{L}{8}\right)(2\xi_3 s + \omega_3)}{\mathrm{b}_3(s^2 + 2\xi_3 s\omega_3 + \omega_3{}^2)} - \frac{\omega_1 \mathrm{W}_1\left(\frac{L}{8}\right)(2\xi_1 s + \omega_1)}{\mathrm{b}_1(s^2 + 2\xi_1 s\omega_1 + \omega_1{}^2)}}{A\rho} \tag{2.14}$$

5. Substitute the complex variable $s \rightarrow (i\omega)$ to be in the frequency domain

Figure 2.3 shows the log-log plot of the magnitude of the transfer function of the damped system up to 2000 Hz.

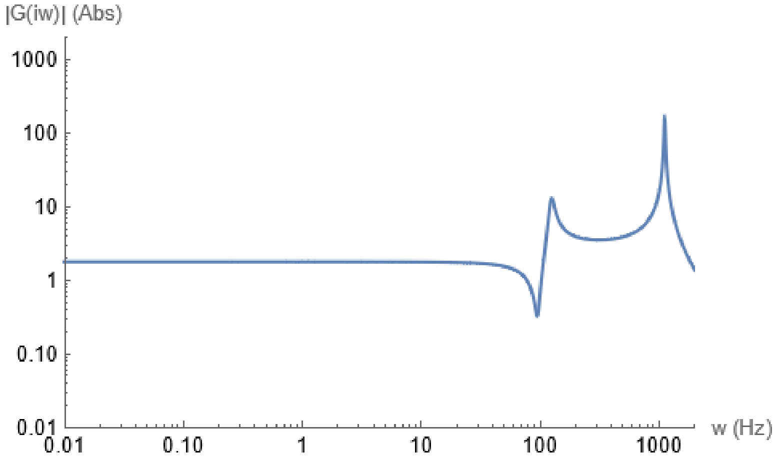


Figure 2.3: Analytical transfer function. Damped system

From Figure 2.3 it can be seen that the transfer function has two peaks which are in the proximity of the poles which can be computed from equation 2.14:

$$\begin{cases} s^2 + 2\xi_1 s\omega_1 + {\omega_1}^2 = 0 \longrightarrow s_{1,2} = -\omega_1\xi_1 \pm \sqrt{{\omega_1}^2\left({\xi_1}^2 - 1\right)} \\ s^2 + 2\xi_3 s\omega_3 + {\omega_3}^2 = 0 \longrightarrow s_{3,4} = -\omega_3\xi_3 \pm \sqrt{{\omega_3}^2\left({\xi_3}^2 - 1\right)} \end{cases}$$

Substituting the data we obtain the following poles in the complex domain and in the frequency domain respectively:

$$\begin{cases} s_{1,2} = -39.0824 \pm 780.669i \quad || \quad \omega_{\text{poles}_{1,2}} = 39.0824i \pm 780.669 \text{ rad/s} \\ s_{3,4} = -70.3482 \pm 7034.47i \quad || \quad \omega_{\text{poles}_{3,4}} = 70.3482i \pm 7034.47 \text{ rad/s} \end{cases}$$

As can be seen, although the system is defined by three natural frequencies, there are only two poles. Indeed, when computing $w(t, x = \frac{L}{2})$ with equation 2.13 the term $W_2(x = \frac{L}{2}) \cdot q_2(t)$ goes to zero since $x = \frac{L}{2}$ is a node for the second mode shape, hence $W_2(x = \frac{L}{2}) = \mathrm{c}_{22}\sin(\frac{2\pi}{L}\frac{L}{2}) = \mathrm{c}_{22}\sin(\pi) = 0$. Since the poles are complex and not purely imaginary (indeed there is damping), the system does not show infinite gain at the first and third natural frequencies, but a finite gain corresponding to the peaks of the transfer function.

2.2.3 Comparison between damped and undamped transfer functions

Starting from the differential equation of the general modal oscillator without damping:

$$\frac{d^2}{dt^2}[q_{\mathrm{n}}(t)] + {\omega_{\mathrm{n}}}^2 q_{\mathrm{n}}(t) = \frac{1}{\rho A b_{\mathrm{n}}} \cdot Q_{\mathrm{n}}(t) \tag{2.15}$$

and following the same steps followed to calculate the transfer function of the damped system is possible to compute the transfer function for the undamped system, whose log-log plot is shown in Figure 2.4.

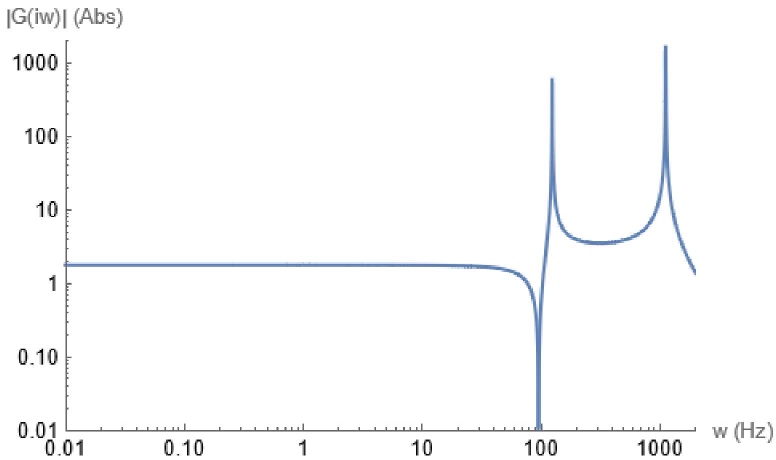


Figure 2.4: Analytical transfer function. Undamped system

Comparing the transfer functions for the damped system and the undamped system (2.3 and 2.4 respectively) it can be seen that in the undamped case the peaks of the magnitude of the transfer function reach much higher values.

Chapter 3

Experimental methodology

3.1 Data from the laboratory

In the laboratory, the acceleration values recorded by the two accelerometers placed in the middle of the beam and the values of the impulsive force generated by the impact of a hammer in a specific point of the beam were collected. Specifically, 204800 samples per accelerometer collected with a sampling frequency of 51200 Hz for a total of 4 seconds. Figure 3.1 shows respectively the plots of the accelerations recorded by the two accelerometers and the mean acceleration of the beam. To find the mean acceleration of the beam it is necessary to equalize the acquisition direction of the two accelerometers which is otherwise opposite.

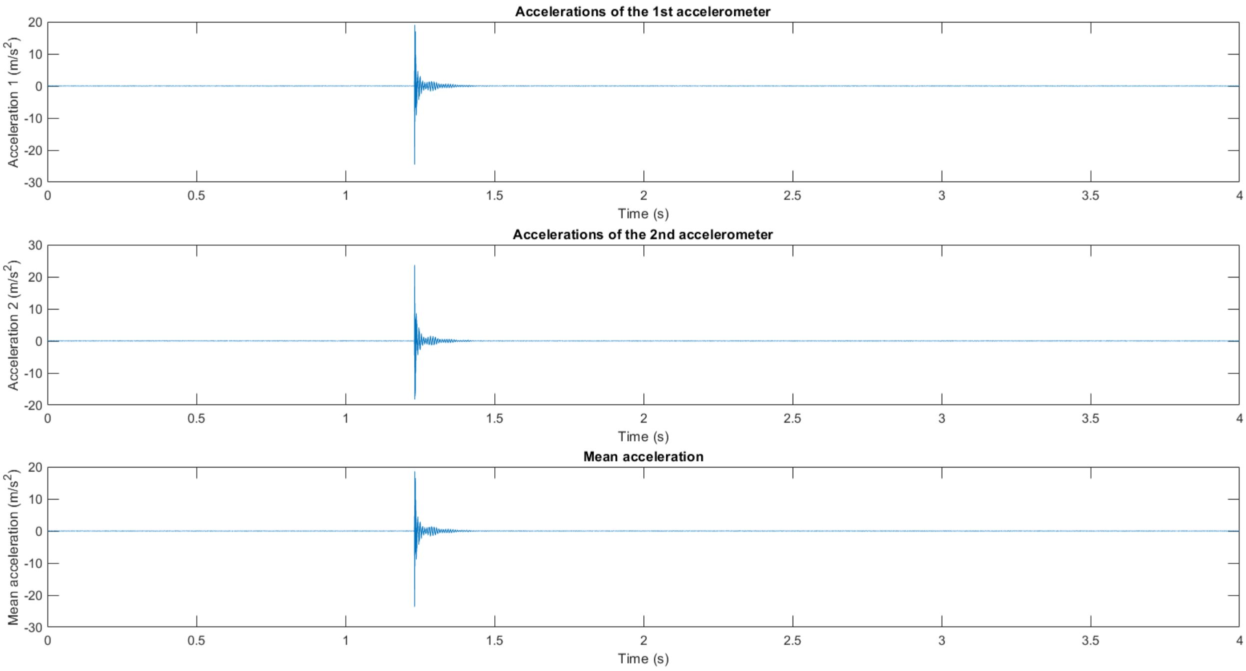


Figure 3.1: Accelerations recorded by the accelerometers and mean acceleration of the beam for the total acquisition time

As can be seen from Figure 3.1 not all the acquired samples are significant, in fact some are noise.

Figure 3.2 shows the accelerations plotted considering only the meaningful time window, which is approximately between 1.23 s and 1.45 s, when the acceleration settles to zero.

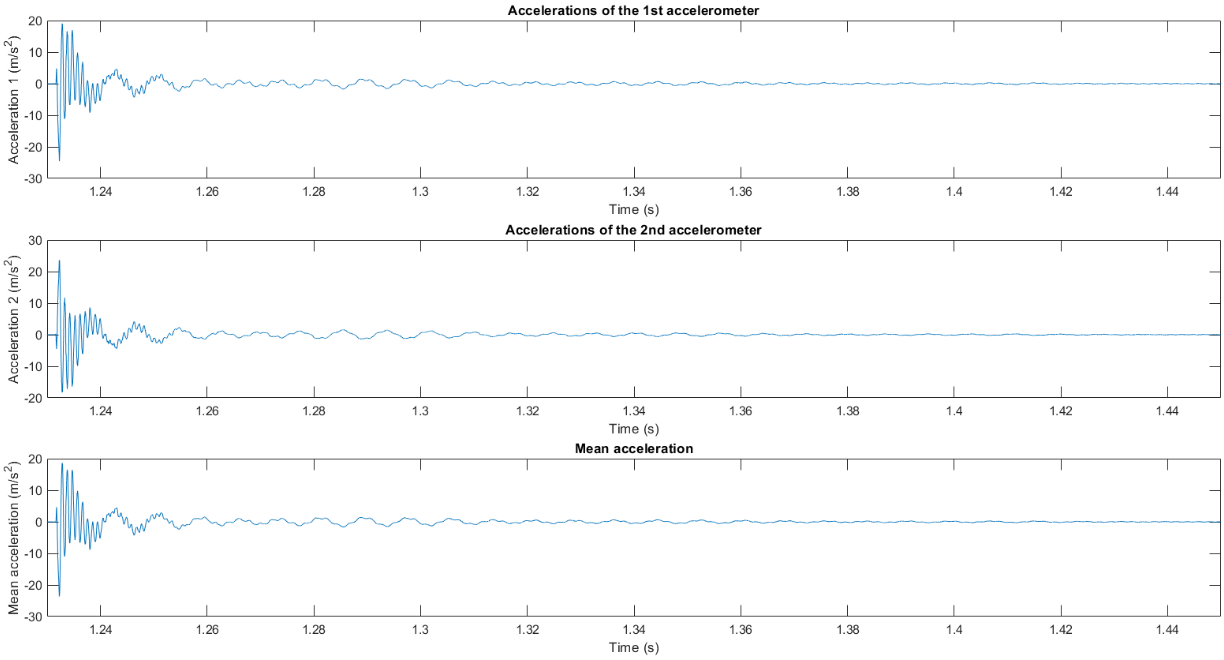
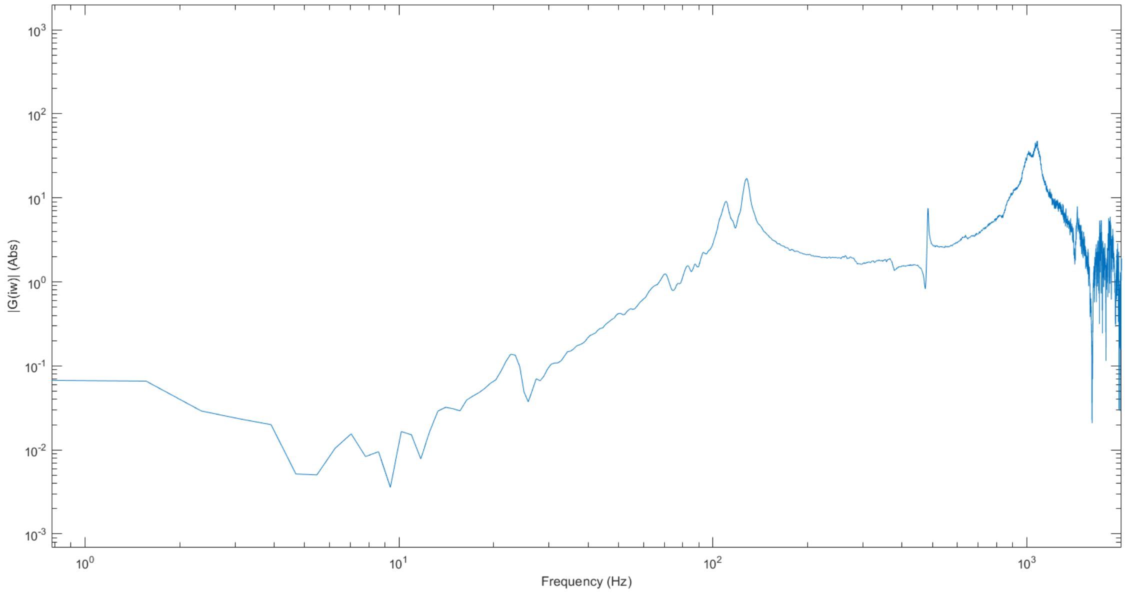


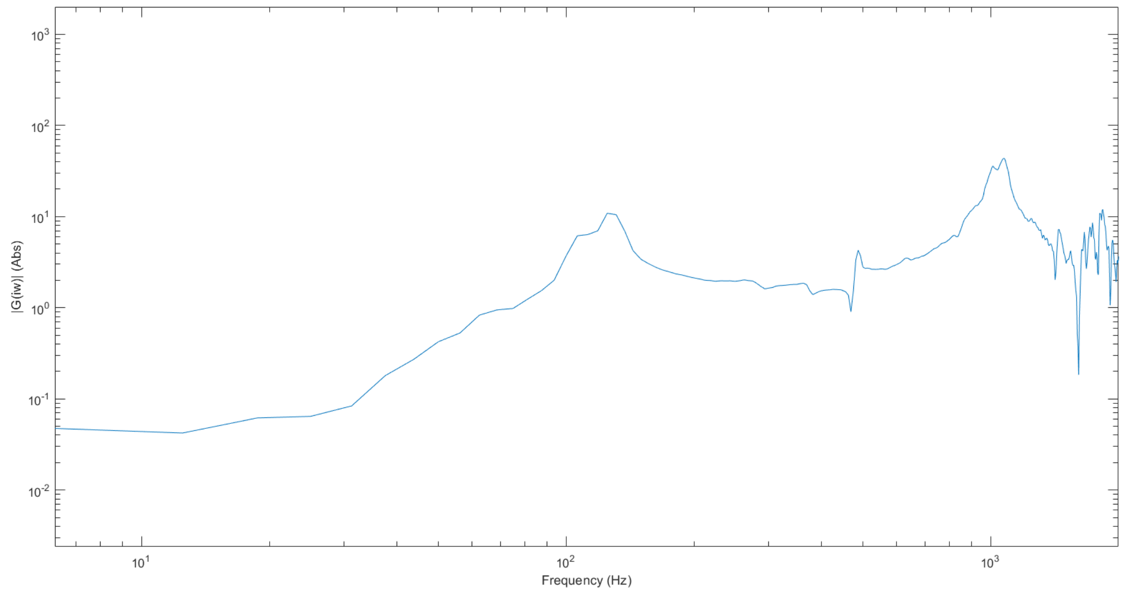
Figure 3.2: Accelerations recorded by the accelerometers and mean acceleration of the beam only for the significant time window

3.2 Experimental transfer function

Figure 3.3a shows the log-log plot of the transfer function computed between the experimental values of force and acceleration found in the laboratory, considering the whole time scale and therefore also the noise, while Figure 3.3b shows the transfer function only for the meaningful time window t ∈ [1.23, 1.45]s where the noise is minimized. The transfer functions are portrayed up to a frequency of 2000 Hz.



(a) Experimental transfer function for the total duration of acquisition



(b) Experimental transfer function only for the significant time window

Figure 3.3: Experimental transfer function

Comparing the analytical transfer function (2.3) and the experimental transfer function (3.3b) it can be seen that they are quite similar, except for the starting value of the magnitude and the value of the peaks, which are lower for the experimental case.

3.3 Half power points method

The half power points method is a method to estimate the damping ratios of the natural frequencies of the system. It consists in determining the points at which the output power has dropped to half of its peak value; that is, at a level of approximately -3 dB. The idea is to find the frequencies which correspond to the intersection points between the experimental transfer function of Figure 3.3b and the RMS values of the peaks of the transfer function that appear when the frequency is equal to the natural frequencies of the system. Since there are only two peaks, because the second natural frequency provides for a zero-pole cancellation, only the damping ratios for the first and third natural frequencies can be found with this method. The formula to compute the damping ratios is the following:

$$\xi_n = \frac{\omega_2 - \omega_1}{2\omega_n} \tag{3.1}$$

where ω_1 and ω_2 are the frequencies at the intersection points ($\omega_2 - \omega_1$ corresponds to the bandwidth) while ω_n is the considered natural frequency. Equation 3.1 can also be simplified considering, for small values of the damping ratios, $\omega_n = \frac{\omega_1 + \omega_2}{2}$:

$$\xi_n = \frac{\omega_2 - \omega_1}{\omega_1 + \omega_2} \tag{3.2}$$

3.3.1 Damping ratio for the first natural frequency

Figure 3.4 shows the intersections between the transfer function and the RMS value of the first peak. The peak value is about 10.8903 in magnitude and corresponds to Q_1 in the Figure.

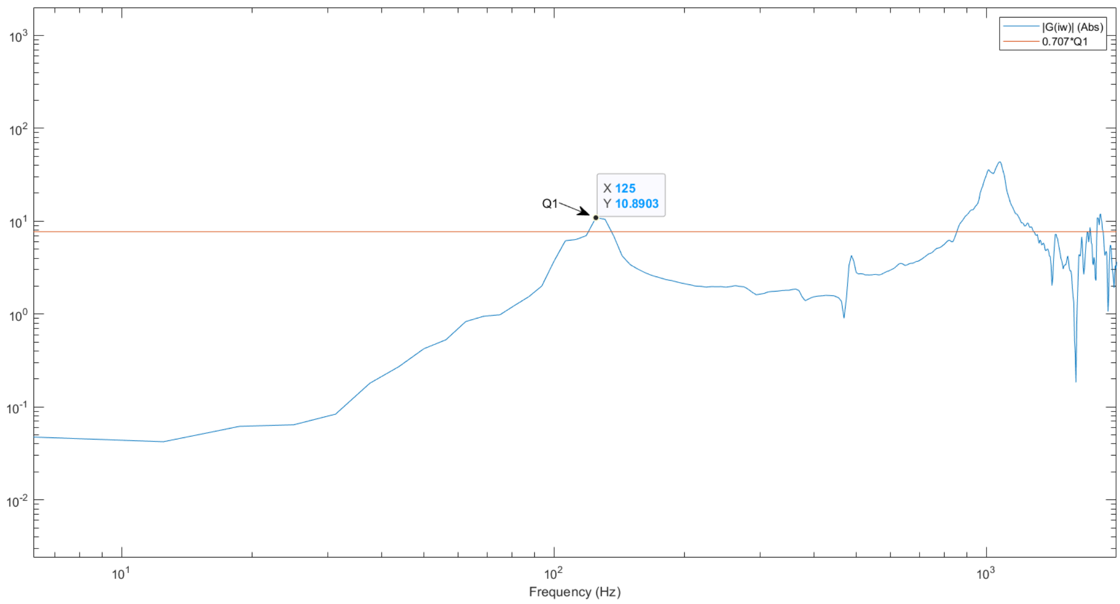


Figure 3.4: Intersections between the transfer function and the RMS value of Q1

The frequencies relative to the points of intersection are, starting from the left:

$$\begin{cases} \omega_1 = 119.8948\text{Hz} \\ \omega_2 = 136.1159\text{Hz} \end{cases}$$

Using the formula in equation 3.2 it is possible to compute the first damping ratio as $\xi_1 = 0.0634$.

3.3.2 Damping ratio for the third natural frequency

Figure 3.5 shows the intersections between the transfer function and the RMS value of the second peak. The peak value is about 43.6708 in magnitude and corresponds to Q_3 in the Figure.

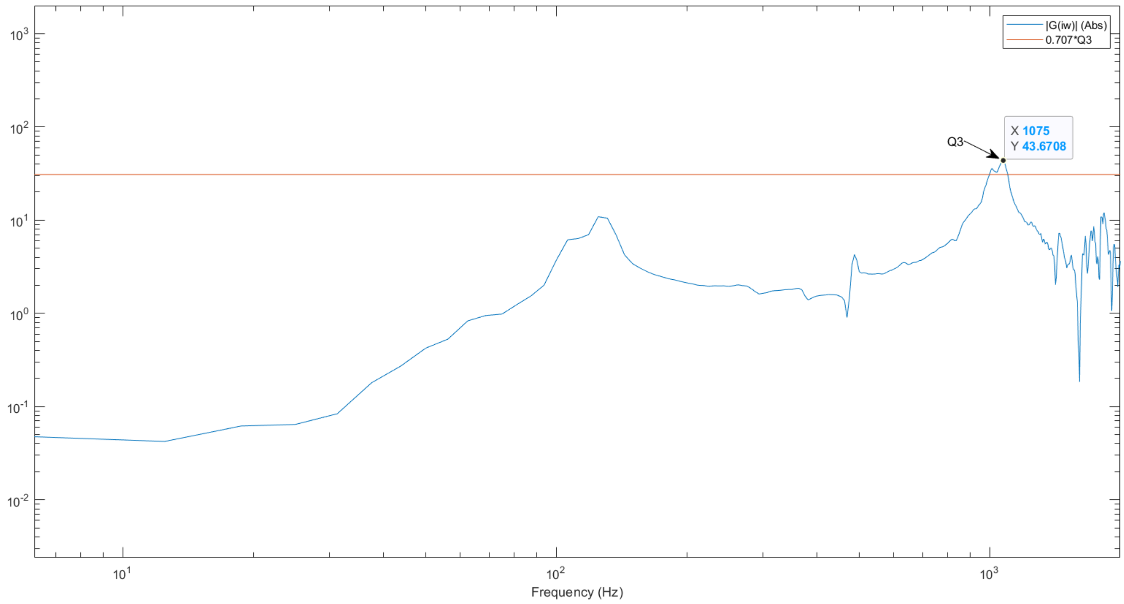


Figure 3.5: Intersections between the transfer function and the RMS value of Q3

The frequencies relative to the points of intersection are, starting from the left:

$$\begin{cases} \omega_1 = 999.5119\text{Hz} \\ \omega_2 = 1101.0072\text{Hz} \end{cases}$$

Using the formula in equation 3.2 it is possible to compute the third damping ratio as $\xi_3 = 0.0483$. After the computation of the damping ratio associated to the third natural frequency it is possible to explain the lower value of the second peak of the experimental transfer function with respect to the analytical one, a comparison already done in Section 3.2. Being the experimental damping ratio higher than the theoretical one, which is $\xi_3 = 0.01$, the damping in the real case is greater, with a consequent greater decrease of the magnitude.

Chapter 4

Conclusions

In conclusion, a system consisting of a beam pinned at both ends was analyzed both in the analytical (theoretical) and in the experimental (practical) case. In the analytical case the system without damping was first considered and subsequently it was demonstrated how the damped system, considering specific damping ratios, present peaks in the transfer function at lower values. A comparison was then made between the main results obtained in the two different cases, finding a certain consistency in the trend of the transfer function, demonstrating that the experimental peaks move to lower values than the theoretical ones because in the real case the damping is greater.