University of Trento
Department of Industrial Engineering

Automatic Control

Project Work

Frahm Absorber

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Chapter 1

Introduction

The aim of the project is to study an augmented version of the Frahm absorber shown in Figure 1.1.

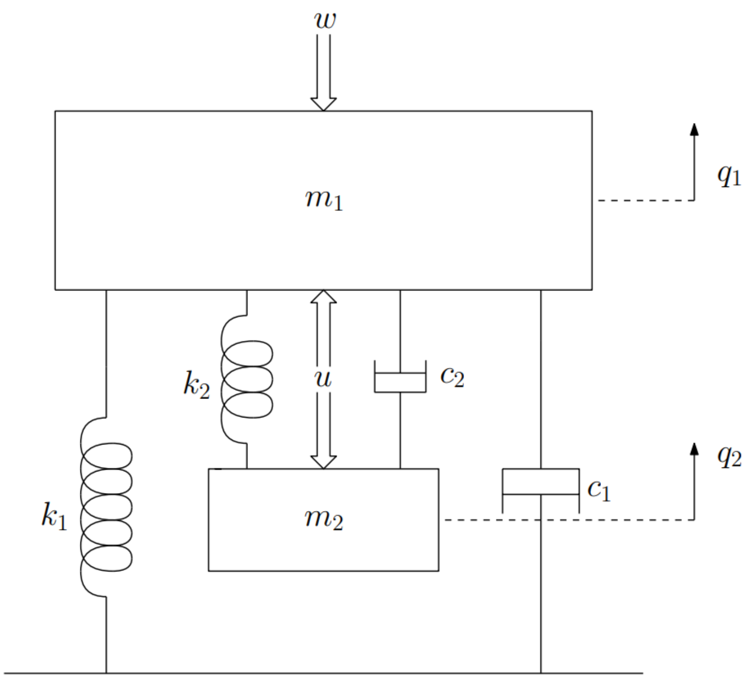

Figure 1.1: Frahm absorber

The equations of motion of the system are reported in Equation 1.1:

$$\begin{cases} \mathrm{m}_1\ddot{q}_1 = -\mathrm{c}_1\dot{q}_1 + \mathrm{c}_2\left(\dot{q}_2 - \dot{q}_1\right) - \mathrm{k}_1 q_1 + \mathrm{k}_2(q_2 - q_1) + u - w \\ \mathrm{m}_2\ddot{q}_2 = \mathrm{c}_2\left(\dot{q}_1 - \dot{q}_2\right) + \mathrm{k}_2(q_1 - q_2) - u \end{cases} \tag{1.1}$$

where u is a force input acting between the two masses and w is a disturbance signal (also a force), hence they do not directly affect the performance output, which is q_1.

Chapter 2

State-space representation

The state-space representation of the system is made starting from the equations of motion (1.1) and is presented in the following form:

$$\begin{cases} \dot{x} = Ax + Bu + Ew \\ z = Cx + Du + Fw \end{cases}$$

where the matrices are the following:

$$\begin{cases} A_{n\times n} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & \frac{-c_1-c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & \frac{-k_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \\ B_{n\times p} = \begin{pmatrix} 0 \\ 1/m_1 \\ 0 \\ -1/m_2 \end{pmatrix} \\ C_{m\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ D_{m\times d} = 0 \\ E_{n\times p} = \begin{pmatrix} 0 \\ -1/m_1 \\ 0 \\ 0 \end{pmatrix} \\ F_{m\times d} = 0 \end{cases}$$

To obtain the matrices of the state space-representation the following states were taken with their respective derivatives:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ \dot{x_1} \\ q_2 \\ \dot{x_3} \end{pmatrix} \xrightarrow{\mathrm{d/dt}} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} \dot{q_1} \\ \ddot{q_1} \\ \dot{q_2} \\ \ddot{q_2} \end{pmatrix}$$

considering q_1 as the performance output z.

Chapter 3

$\mathcal{L}_2$-gain of the passive system

Considering only the passive system ($u = 0$) and assigning the following values to some of its parameters

Parameter	Value
m_1 (kg)	4.7764
m_2 (kg)	0.5155
k_1 (N/m)	27950
c_1 (kg/s)	7.152

Table 3.1: System parameters

is possible to plot the values of the $\mathcal{L}_2$-gain of the passive system as a function of k_2, for constant values of c_2 in the interval $[1, 2]\,(kg/s)$ with a step of 0.2 between each value. Figure 3.1 shows the plotted values of the $\mathcal{L}_2$-gain for k_2 varying over the range $[1500, 3500]\,(N/m)$ and for the different values of c_2.

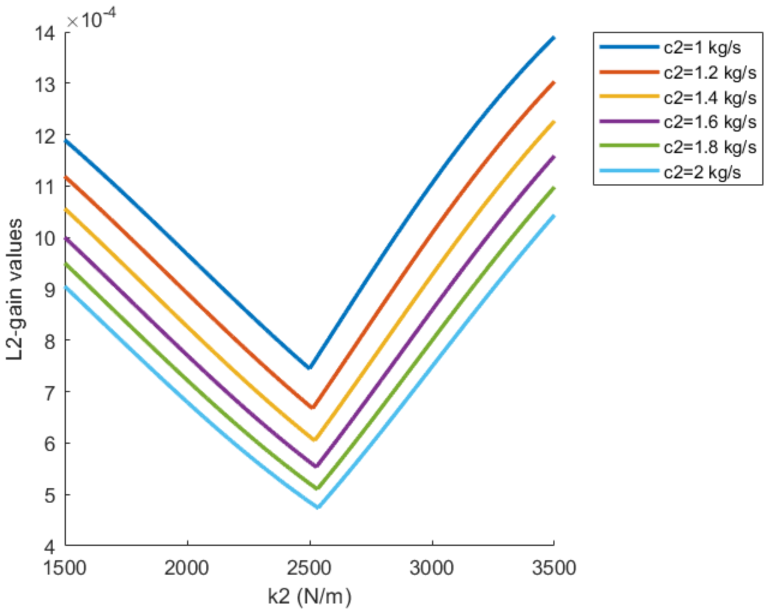

Figure 3.1: $\mathcal{L}_2$-gain of the passive system

The method used to obtain the $\mathcal{L}_2$-gain consists in determining the peak of the transfer function between w and z for the different values of the parameters k_2 and c_2. From Figure 3.1 it can be seen that the curves follow a V shape with the minimum value located approximately at k_2=2500 (N/m). It can also be noted that increasing c_2 the curves tend to shift downward with the consequent decrease in the values of the $\mathcal{L}_2$-gain.

Chapter 4

Static full-state feedback design

Considering the parameters of Table 3.1 and assigning the following parameters to the passive elements of the Frahm absorber

Parameter	**Value**
k_2 (N/m)	1619
c_2 (kg/s)	1.717

Table 4.1: Parameters values of the passive elements of the Frahm absorber

is possible to design a full-state feedback gain matrix κ that minimizes the $\mathcal{L}_2$-gain from the disturbance w to the performance output z. Because of limitations given by the producer of the actuator, the gain matrix should have a bounded norm $\|\kappa\| \leq \overline{k}$, with $\overline{k} = 5 \cdot 10^3$. The overall idea is to solve a linear problem minimizing the $\mathcal{L}_2$-gain (γ) with the following constraints:

$$\begin{cases} \begin{bmatrix} He(Aw + Bx) & E & (Cw + Dx)^T \\ E^T & -\gamma I_d & F^T \\ Cw + Dx & F & -\gamma I_m \end{bmatrix} < 0 \\ \\ \begin{bmatrix} \rho \overline{k} I_n & x^T \\ x & \rho \overline{k} I_p \end{bmatrix} \geq 0 \\ \\ w \geq \rho I_n \\ \\ \rho > 0 \end{cases}$$

where, in general, I_a is the identity matrix with $a \times a$ dimensions and ρ is a coefficient that relates the norm of κ with the norm of x in the form $\|\kappa\| \leq \frac{1}{\rho}\|x\|$.
In this type of problem the optimization variables are w, x, ρ and γ. Performing the optimization the following value for the $\mathcal{L}_2$-gain is obtained:

$$\gamma = 0.00041462$$

It is also possible to compute the matrices x and w:

- $x = \begin{bmatrix} -15.8007 & -1.5473 & -0.2317 & 0.1603 \end{bmatrix}$

- $w = \begin{bmatrix} 0.0050 & -0.0305 & 0.0255 & 0.2975 \\ -0.0305 & 25.9979 & -0.6072 & 31.0647 \\ 0.0255 & -0.6072 & 0.3769 & -0.7851 \\ 0.2975 & 31.0647 & -0.7851 & 1084.4246 \end{bmatrix}$

With the knowledge of x and w the gain matrix κ can be computed by applying the following formula:

$$\kappa = x \cdot w^{-1}$$

This results in

$$\kappa = \begin{bmatrix} -4947.0566 & 0.0949 & 337.2916 & 1.5987 \end{bmatrix}$$

With the limitation on the norm of κ imposed by the producer of the actuator it is possible to guarantee a better noise rejection than the one observed without actuation. In fact it can be seen that with the designed matrix κ the value of the $\mathcal{L}_2$-gain is γ= 0.00041462, a value which is lower with respect to the minimum value of the $\mathcal{L}_2$-gain obtained in the plots of Chapter 3 where the open-loop system was considered, which is about $5{\cdot}10^{-4}$. Furthermore, considering Figure 3.1 and taking the same values of k_2 and c_2 of Table 4.1, the value of the $\mathcal{L}_2$-gain is between $9{\cdot}10^{-4}$ and $10{\cdot}10^{-4}$ which is higher than γ= 0.00041462.

Chapter 5

Simulation and plotting

The aim of this final chapter is to show the results of the simulations of the evolution of both the open-loop (u = 0) and closed-loop (u = $\kappa\ x$) system performed in Simulink. Figure 5.1 shows the model created in Simulink of both the open-loop and closed-loop system.

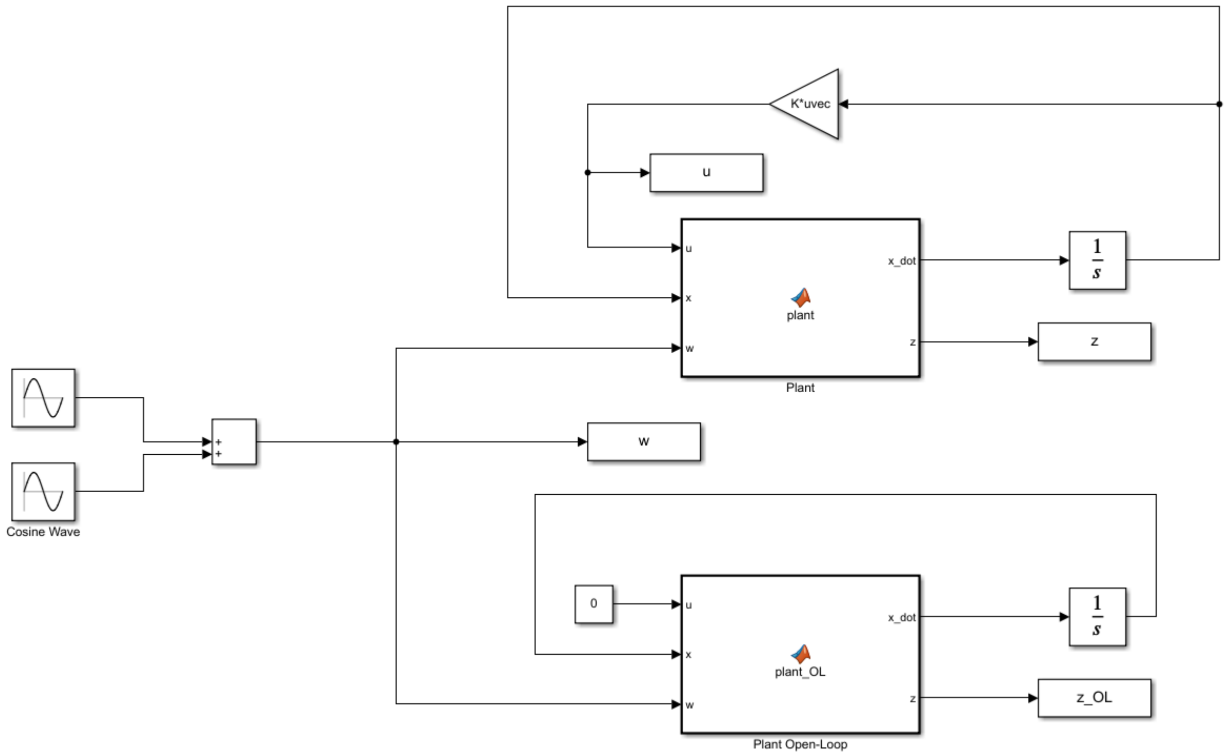

Figure 5.1: Simulink model of both the open-loop and closed-loop system

The systems are subjected to a disturbance input signal $w(t) = 0.05\left(\sin(80t) + \cos\left(30t + \frac{\pi}{3}\right)\right)$ and they are simulated for a total time of 4 seconds with a maximum step size of the solver of 10^{-3}. The following initial conditions for the states are considered:

$$\begin{cases} q_1(0) = 0 \quad ; \quad \dot{q_1}(0) = 0 \\ q_2(0) = 0 \quad ; \quad \dot{q_2}(0) = 0 \end{cases}$$

Figures 5.2, 5.3 and 5.4 show respectively the evolution of the disturbance signal w, the evolution of the performance output z for both systems and the evolution of the control input u in the closed-loop system.

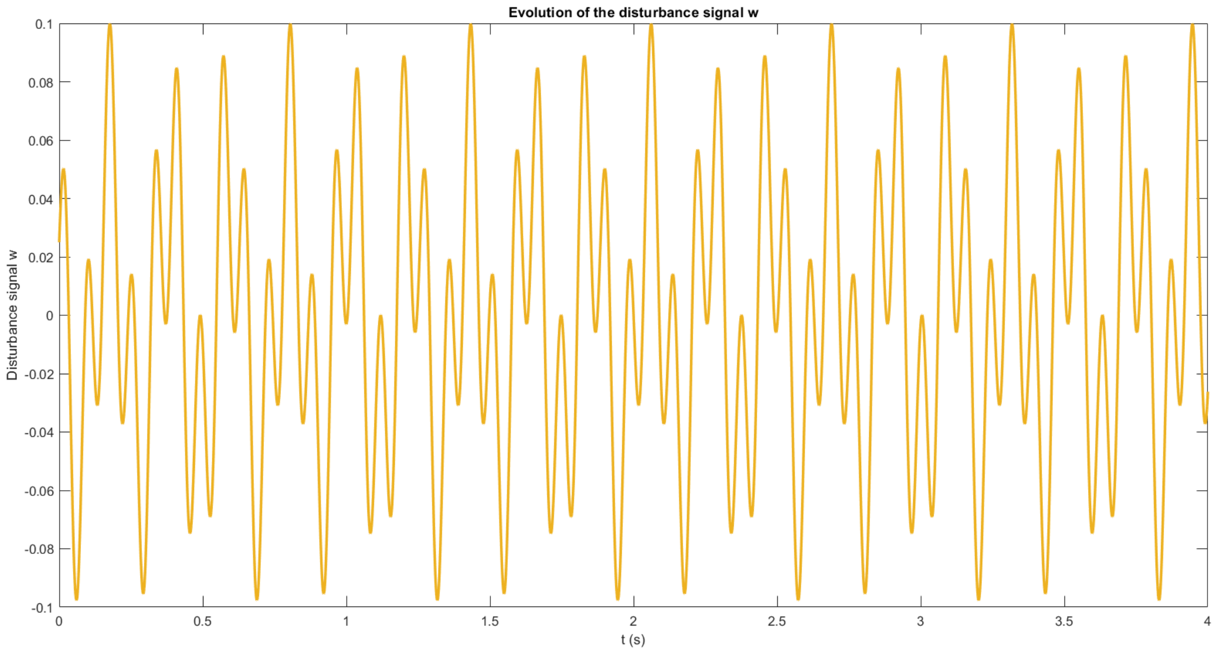

Figure 5.2: Evolution of the disturbance signal w

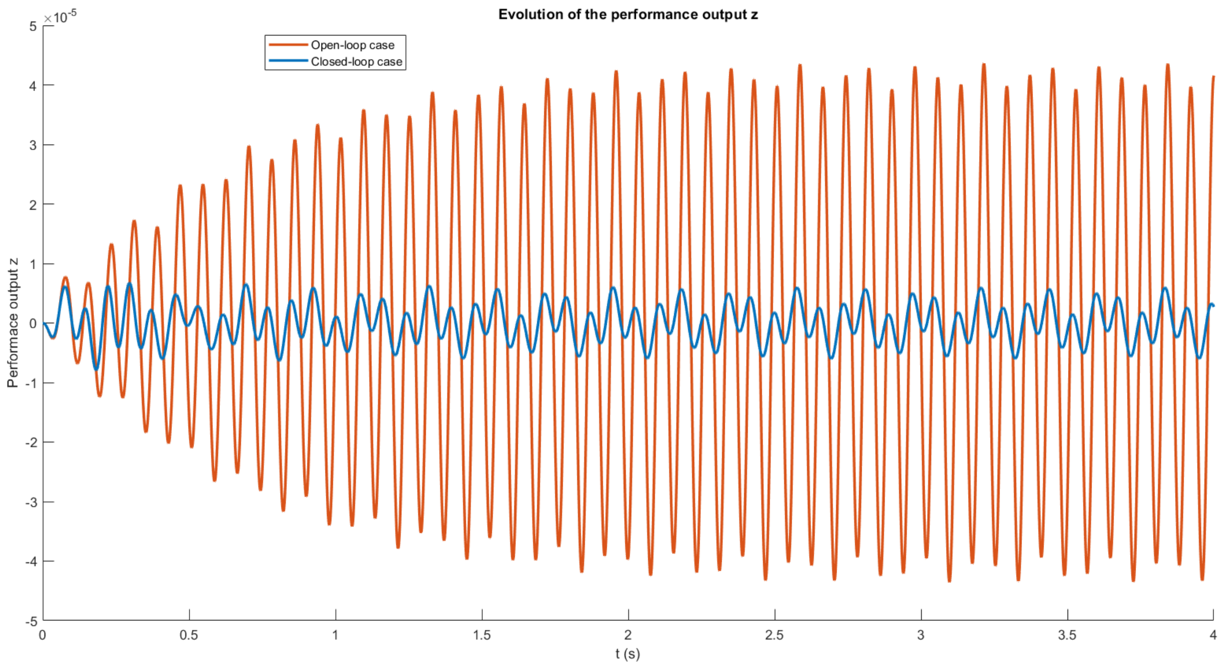

Figure 5.3: Evolution of the performance output z

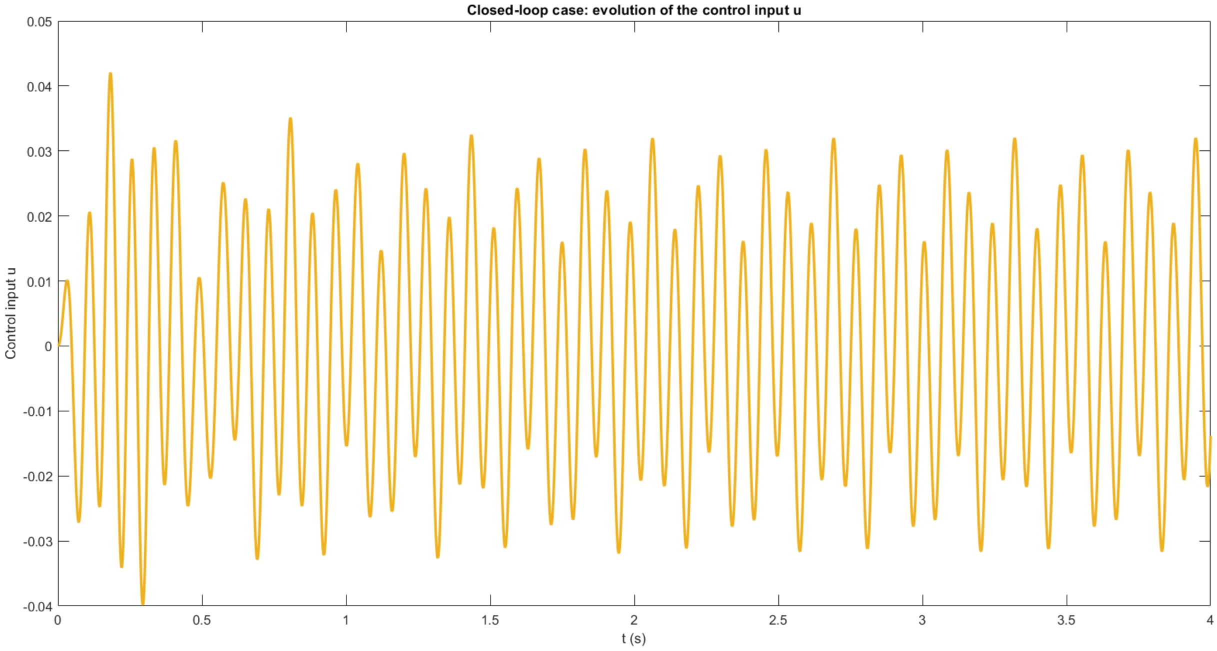

Figure 5.4: Evolution of the control input u

As can be seen from Figure 5.3 the amplitude of the oscillations in output gets significantly reduced in the closed-loop system, which is the main function of the Frahm absorber.