# Lab 6a
# Pole Placement
# for the Inverted Pendulum
# ELENG128


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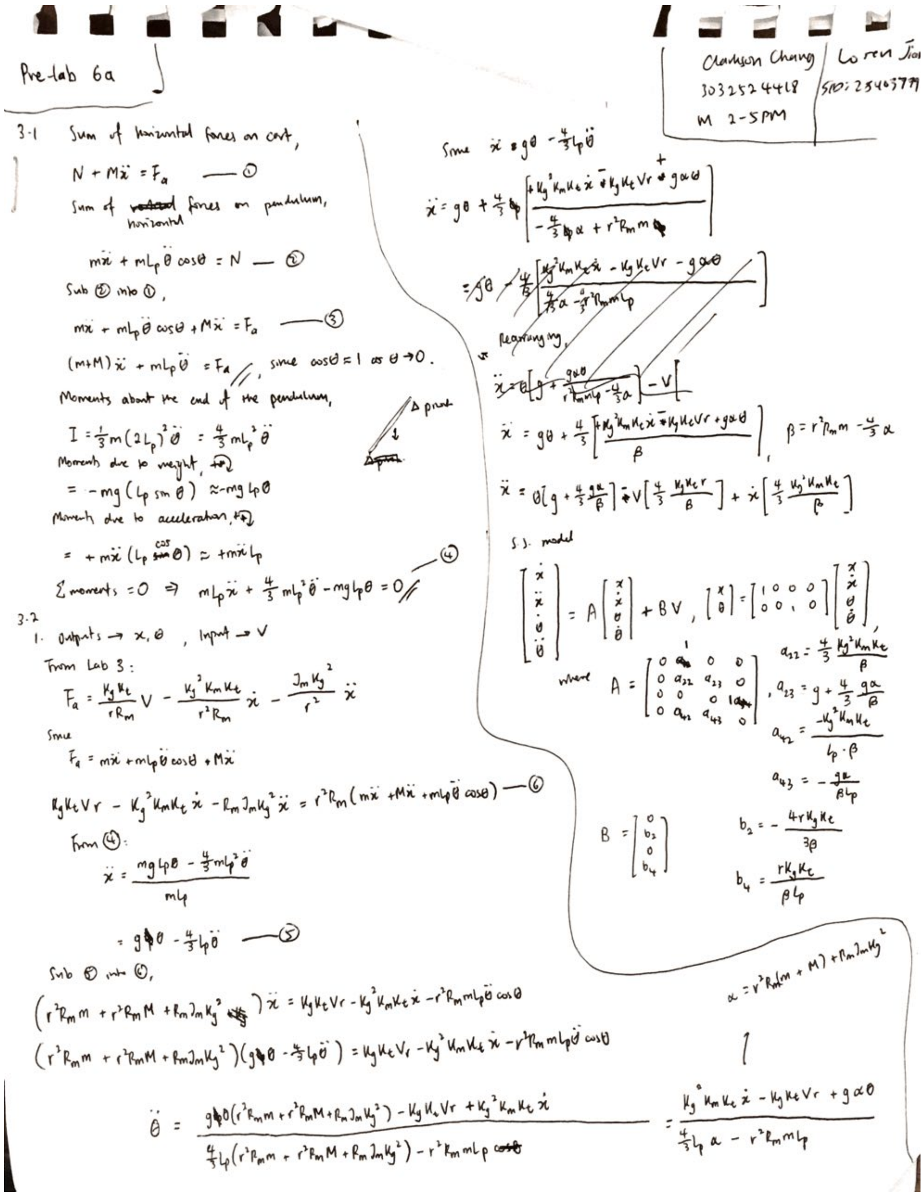
| | |
|---|---|
| Date Performed: | April 3, 2017 |
| Lab Instructor: | Eric CHOI |

# 1 Objective

The objective of this lab is to achieve simultaneous control of both the angular position of the pendulum and horizontal position of the cart on the track using full-state feedback. We will be considering small angle perturbations and sine wave reference tracking of the cart position. Note that the system is a SIMO a Single Input Multiple Output system, since we are trying to control both the position of the cart and the angle of the pendulum by using only the motor voltage.

## 2 Pre-lab assignments



Pre-lab 6a

Clarkson Chung | Loren Jian
303252 4418 | SID: 2540377?
M 2-5PM

3.1 Sum of horizontal forces on cart,

$N + M\ddot{x} = F_a$ — ①

Sum of ~~vertical~~ horizontal forces on pendulum,

$m\ddot{x} + mL_p\ddot{\theta}\cos\theta = N$ — ②

Sub ② into ①,

$m\ddot{x} + mL_p\ddot{\theta}\cos\theta + M\ddot{x} = F_a$ — ③

$(m+M)\ddot{x} + mL_p\ddot{\theta} = F_a$ //, since $\cos\theta = 1$ as $\theta \to 0$.

Moments about the end of the pendulum,

Δ pivot

$I = \frac{1}{3}m(2L_p)^2\ddot{\theta} = \frac{4}{3}mL_p^2\ddot{\theta}$

Moments due to weight, (+↻)

$= -mg(L_p \sin\theta) \approx -mgL_p\theta$

Moment due to acceleration, (+↻)

$= +m\ddot{x}(L_p \cos\theta) \approx +m\ddot{x}L_p$

$\sum \text{moments} = 0 \Rightarrow mL_p\ddot{x} + \frac{4}{3}mL_p^2\ddot{\theta} - mgL_p\theta = 0$ // — ④

3.2

1. Outputs → $x, \theta$, Input → $V$

From Lab 3:

$F_a = \frac{K_gK_t}{rR_m}V - \frac{K_g^2K_mK_t}{r^2R_m}\dot{x} - \frac{J_mK_g^2}{r^2}\ddot{x}$

Since

$F_a = m\ddot{x} + mL_p\ddot{\theta}\cos\theta + M\ddot{x}$

$K_gK_tVr - K_g^2K_mK_t\dot{x} - R_mJ_mK_g^2\ddot{x} = r^2R_m(m\ddot{x} + M\ddot{x} + mL_p\ddot{\theta}\cos\theta)$ — ⑥

From ④:

$\ddot{x} = \frac{mgL_p\theta - \frac{4}{3}mL_p^2\ddot{\theta}}{mL_p}$

$= g\theta - \frac{4}{3}L_p\ddot{\theta}$ — ⑤

Sub ⑤ into ⑥,

$(r^2R_mm + r^2R_mM + R_mJ_mK_g^2)\ddot{x} = K_gK_tVr - K_g^2K_mK_t\dot{x} - r^2R_mmL_p\ddot{\theta}\cos\theta$

$(r^2R_mm + r^2R_mM + R_mJ_mK_g^2)(g\theta - \frac{4}{3}L_p\ddot{\theta}) = K_gK_tVr - K_g^2K_mK_t\dot{x} - r^2R_mmL_p\ddot{\theta}\cos\theta$

$\ddot{\theta} = \frac{g\theta(r^2R_mm + r^2R_mM + R_mJ_mK_g^2) - K_gK_tVr + K_g^2K_mK_t\dot{x}}{\frac{4}{3}L_p(r^2R_mm + r^2R_mM + R_mJ_mK_g^2) - r^2R_mmL_p\cancel{\cos\theta}} = \frac{K_g^2K_mK_t\dot{x} - K_gK_tVr + g\alpha\theta}{\frac{4}{3}L_p\alpha - r^2R_mmL_p}$

$\alpha = r^2R_m(m + M) + R_mJ_mK_g^2$

Since $\ddot{x} = g\theta - \frac{4}{3}L_p\ddot{\theta}$

$\ddot{x} = g\theta + \frac{4}{3}\left[\frac{-K_g^2K_mK_t\dot{x} + K_gK_tVr - g\alpha\theta}{-\frac{4}{3}\alpha + r^2R_mm}\right]$

Rearranging,

$\ddot{x} = g\theta + \frac{4}{3}\left[\frac{-K_g^2K_mK_t\dot{x} + K_gK_tVr + g\alpha\theta}{\beta}\right]$, $\beta = r^2R_mm - \frac{4}{3}\alpha$

$\ddot{x} = \theta\left[g + \frac{4}{3}\frac{g\alpha}{\beta}\right] + V\left[\frac{4}{3}\frac{K_gK_tr}{\beta}\right] + \dot{x}\left[\frac{4}{3}\frac{K_g^2K_mK_t}{\beta}\right]$

S.S. model

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\theta}\\ \ddot{\theta}\end{bmatrix} = A\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix} + BV, \quad \begin{bmatrix}x\\ \theta\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix},$$

where $A = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & a_{22} & a_{23} & 0\\ 0 & 0 & 0 & 1\\ 0 & a_{42} & a_{43} & 0\end{bmatrix}$, $a_{22} = \frac{4}{3}\frac{K_g^2K_mK_t}{\beta}$, $a_{23} = g + \frac{4}{3}\frac{g\alpha}{\beta}$

$a_{42} = \frac{-K_g^2K_mK_t}{L_p\cdot\beta}$

$a_{43} = -\frac{g\alpha}{\beta L_p}$

$B = \begin{bmatrix}0\\ b_2\\ 0\\ b_4\end{bmatrix}$, $b_2 = -\frac{4rK_gK_t}{3\beta}$, $b_4 = \frac{rK_gK_t}{\beta L_p}$

Figure 1

Using MATLAB,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -6.812 & 7.497 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 15.473 & 25.683 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1.523 \\ 0 \\ -3.4583 \end{bmatrix}$$

3.3.

1. Using MATLAB,

The eigenvalues $-7.56, -4.13, 0, 4.87$.

The system is unstable as there are RHP poles.

2. $x$ increases without bounds, so the cart moves infinitely along the track. $\theta$ remains constant. The system will not behave this way in real life as the track is not of infinite length, and there are external forces such as friction and motor resistance

should be increasing in your step plot.

No output response?

3. a) $K = [k_1 \ k_2 \ k_3 \ k_4]$

$$A_k = A - BK$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} [k_1 \ k_2 \ k_3 \ k_4]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_2k_1 & b_2k_2 & b_2k_3 & b_2k_4 \\ 0 & 0 & 0 & 0 \\ b_4k_1 & b_4k_2 & b_4k_3 & b_4k_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_2k_1 & a_{22}-b_2k_1 & a_{23}-b_2k_3 & -b_2b_4 \\ 0 & 0 & 0 & 1 \\ -b_4k_1 & a_{42}-b_4k_2 & a_{43}-b_4k_3 & -b_4k_4 \end{bmatrix} //$$

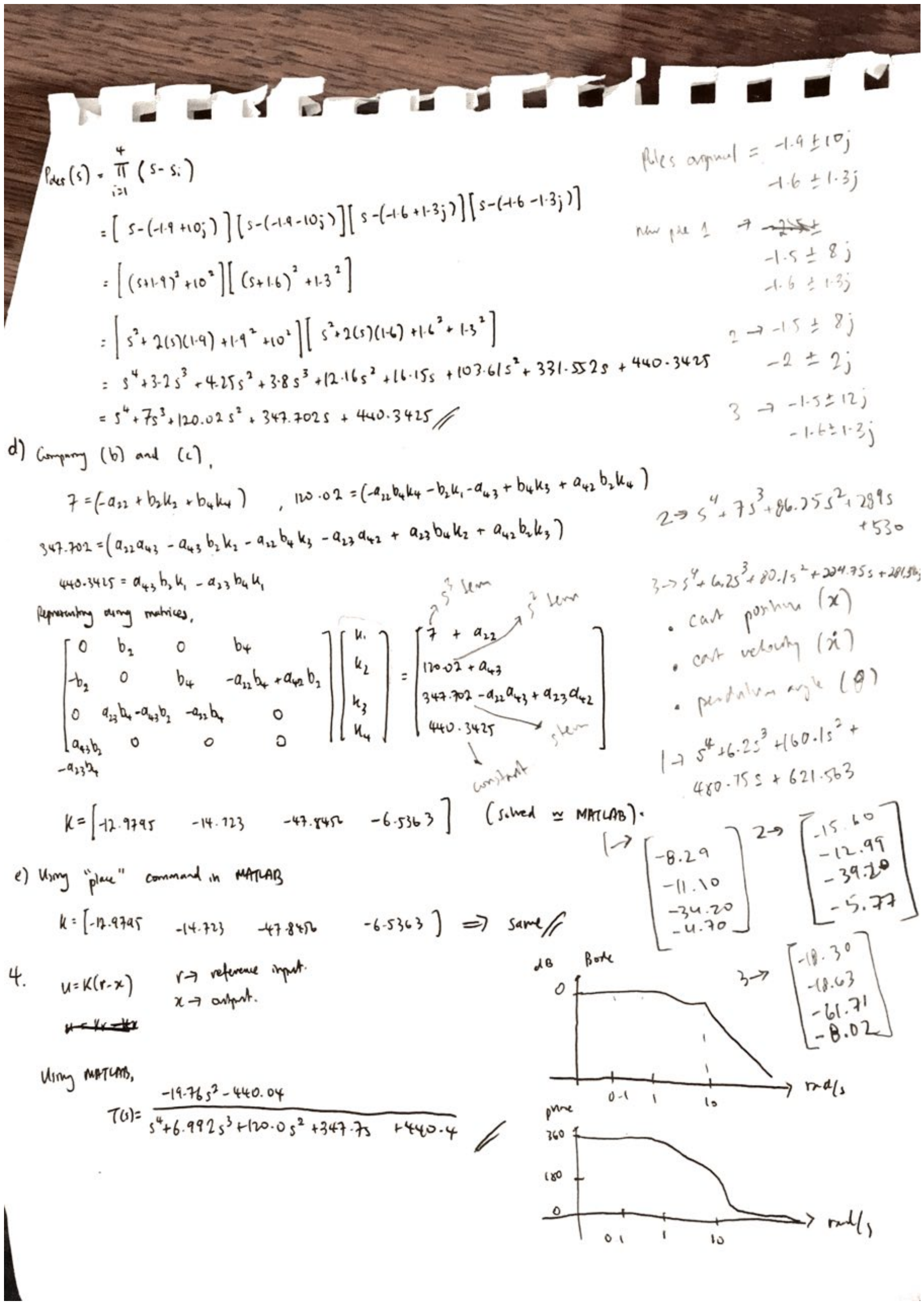
b)

$$sI - A_k = \begin{bmatrix} s & -1 & 0 & 0 \\ +b_2k_1 & s-a_{22}+b_2k_1 & b_2k_3-a_{23} & b_2b_4 \\ 0 & 0 & s & -1 \\ b_4k_1 & b_4k_2-a_{42} & b_4k_3-a_{43} & s+b_4k_4 \end{bmatrix}$$

$$\begin{aligned} \det(sI - A_k) = {} & s(s-a_{22}+b_2k_1)(s^2+sb_4k_4+b_4k_3-a_{43}) \\ & + s(-a_{42}+b_4k_2)(a_{23}-b_2k_3-sb_2b_4) \\ & - (b_2k_1)(s^2+sb_4k_4-a_{43}+b_4k_3) - (b_4k_1)(a_{23}-b_2k_3-sb_2k_4) \end{aligned}$$

$$\begin{aligned} p(k;s) = {} & (s^2-sa_{22}+sb_2k_1)(s^2+sb_4k_4+b_4k_3-a_{43}) \\ & + (sb_4k_2-sa_{42})(a_{23}-b_2k_3-sb_2b_4) \\ & - (b_2k_1)(s^2+sb_4k_4-a_{43}+b_4k_3) \\ & - (b_4k_1)(a_{23}-b_2k_3-sb_2k_4) \\ = {} & \ldots \\ = {} & s^4 + s^3(-a_{22}+b_2k_2+b_4k_4) \\ & + s^2(-a_{22}b_4k_4 - b_2k_1 - a_{43} + b_4k_3 + a_{42}b_2k_4) \\ & + s(a_{22}a_{43} - a_{43}b_2k_2 - a_{22}b_4k_3 - a_{23}a_{42} + a_{23}b_4k_2 + a_{43}b_2 \\ & + (a_{43}b_2k_1 - a_{23}b_4k_1) // \end{aligned}$$

Figure 2



$$P_{des}(s) = \prod_{i=1}^{4}(s - s_i)$$

$$= [s-(-1.9+10j)][s-(-1.9-10j)][s-(-1.6+1.3j)][s-(-1.6-1.3j)]$$

$$= [(s+1.9)^2+10^2][(s+1.6)^2+1.3^2]$$

$$= [s^2+2(s)(1.9)+1.9^2+10^2][s^2+2(s)(1.6)+1.6^2+1.3^2]$$

$$= s^4+3.2s^3+4.25s^2+3.8s^3+12.16s^2+16.15s+103.61s^2+331.552s+440.3425$$

$$= s^4+7s^3+120.02s^2+347.702s+440.3425$$

Poles original $= -1.9 \pm 10j$, $-1.6 \pm 1.3j$

new pole 1 $\rightarrow$ $-1.5 \pm 8j$, $-1.6 \pm 1.3j$

$2 \rightarrow -1.5 \pm 8j$, $-2 \pm 2j$

$3 \rightarrow -1.5 \pm 12j$, $-1.6 \pm 1.3j$

d) Comparing (b) and (c),

$$7 = (-a_{22} + b_2k_2 + b_4k_4), \quad 120.02 = (-a_{22}b_4k_4 - b_2k_1 - a_{43} + b_4k_3 + a_{42}b_2k_4)$$

$$347.702 = (a_{22}a_{43} - a_{43}b_2k_2 - a_{22}b_4k_3 - a_{23}a_{42} + a_{23}b_4k_2 + a_{42}b_2k_3)$$

$$440.3425 = a_{43}b_2k_1 - a_{23}b_4k_1$$

$2 \rightarrow s^4+7s^3+86.25s^2+289s+530$

$3 \rightarrow s^4+6.2s^3+80.1s^2+224.75s+281.5$

Representing using matrices,

$$\begin{bmatrix} 0 & b_2 & 0 & b_4 \\ -b_2 & 0 & b_4 & -a_{22}b_4+a_{42}b_2 \\ 0 & a_{23}b_4-a_{43}b_2 & -a_{22}b_4 & 0 \\ a_{43}b_2-a_{23}b_4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 7 + a_{22} \\ 120.02 + a_{43} \\ 347.702 - a_{22}a_{43} + a_{23}a_{42} \\ 440.3425 \end{bmatrix}$$

$s^3$ term, $s^2$ term, $s$ term, constant

- cart position ($x$)
- cart velocity ($\dot{x}$)
- pendulum angle ($\theta$)

$1 \rightarrow s^4+6.2s^3+160.1s^2+480.75s+621.563$

$$K = [-12.9795 \quad -14.723 \quad -47.8452 \quad -6.5363] \quad \text{(solved w MATLAB)}.$$

$1 \rightarrow \begin{bmatrix} -8.29 \\ -11.10 \\ -34.20 \\ -4.70 \end{bmatrix}$ $2 \rightarrow \begin{bmatrix} -15.60 \\ -12.99 \\ -39.20 \\ -5.77 \end{bmatrix}$ $3 \rightarrow \begin{bmatrix} -18.30 \\ -18.63 \\ -61.71 \\ -8.02 \end{bmatrix}$

e) Using "place" command in MATLAB

$$K = [-12.9795 \quad -14.723 \quad -47.8456 \quad -6.5363] \Rightarrow \text{same}$$

4. $u = K(r - x)$ $\quad r \rightarrow$ reference input. $\quad x \rightarrow$ output.

~~$u = Kr - Kx$~~

Using MATLAB,

$$T(s) = \frac{-19.76s^2 - 440.04}{s^4 + 6.992s^3 + 120.0s^2 + 347.7s + 440.4}$$



Figure 3

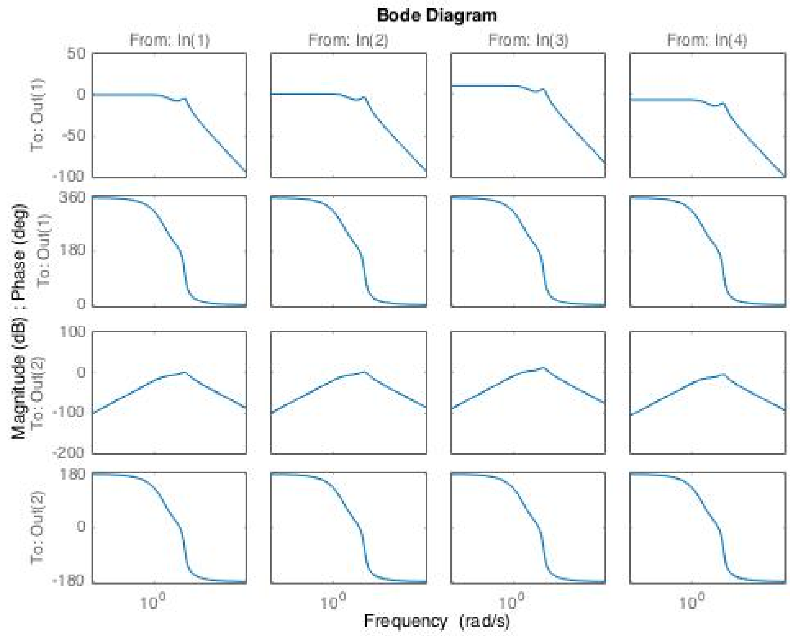



Figure 4

# 3 Lab Results

## 3.1 Implementing the Controller in Simulink

We implemented the state-feedback controller with the Quanser I/O blocks in Simulink as shown in Figure 5. In order to prevent the gear from slipping, we put a saturation block before the Analog Output block, set to -6V to 6V. The top gain block was set to a value of $\frac{1}{4.396}$ m/counts. The bottom gain block was set to a value of $\frac{1}{666.7}$ rad/counts.

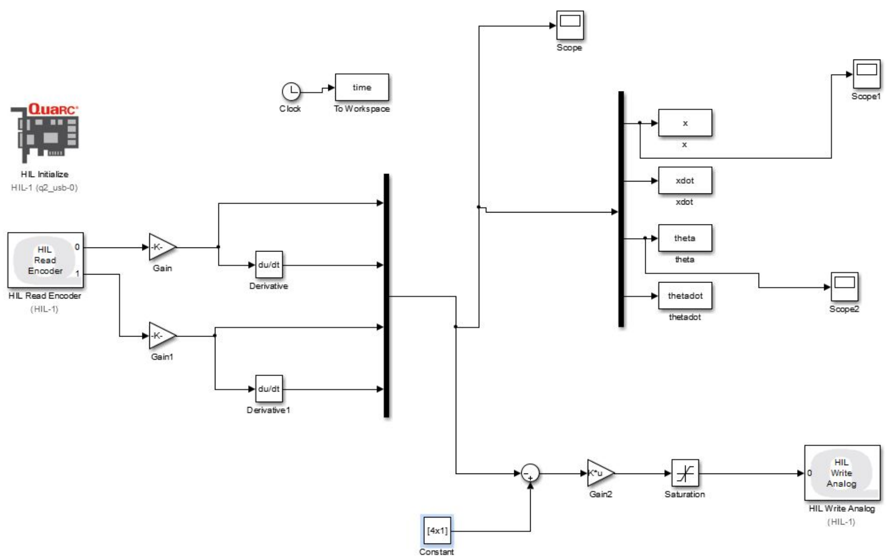



Figure 5: Simulink block diagram.

## 3.2 Running the Controller on the Hardware

### 3.2.1 Initially balancing the pendulum

We ran the controller on the hardware (with reference $r$ set to $r = [0\ 0\ 0\ 0]^T$) and made sure it balanced. We held the pendulum exactly vertical before connecting to the real time target (our controller stabilized the unstable equilibrium $\theta = 0°$). If we try to connect the hardware when the pendulum is in its stable equilibrium position ($\theta = 180°$), nothing should happen since the pendulum will always be in its desired position.

### 3.2.2 System response to small perturbations

With the pendulum balancing, we manually applied small perturbations to the pendulum and checked the response.

When the pendulum was perturbed in one direction, the cart responded by moving the same direction in order to balance the torque applied by the perturbation.

We then plotted the variation of the cart position (Figure 6) and pendulum angle (Figure 7) with time for small perturbations (manually induced) about the equilibrium value. The spikes in the plots represent the manual perturbations. Overall, the response time to correct the perturbations is decent, but we noticed that the hardware continued to oscillate about the equilibrium point even when undisturbed. This is because when the pendulum will always be slightly left or right of the equilibrium point. When the hardware attempts to correct the pendulum angle, there will be overshoot, and this will repeat in an oscillatory fashion.

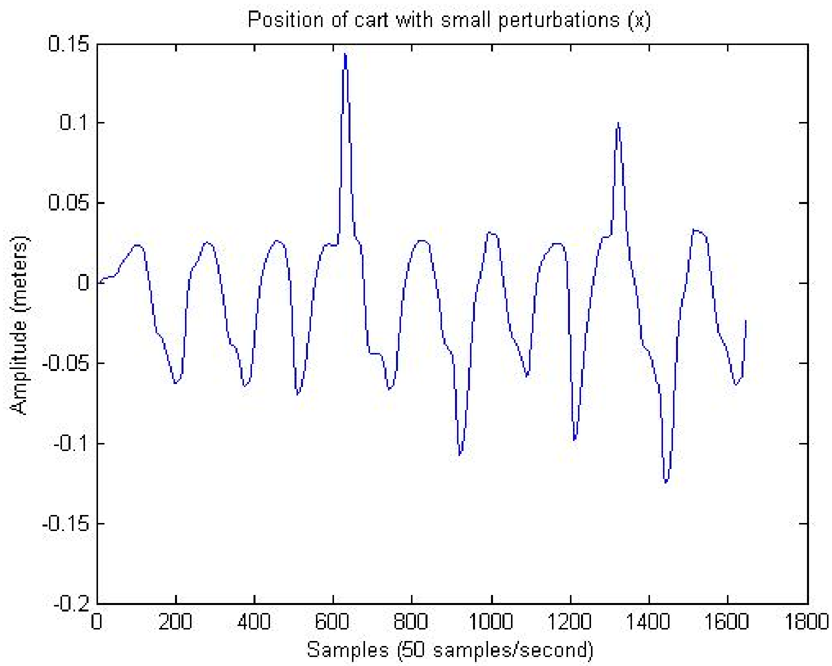



Figure 6: Cart position with small perturbations.

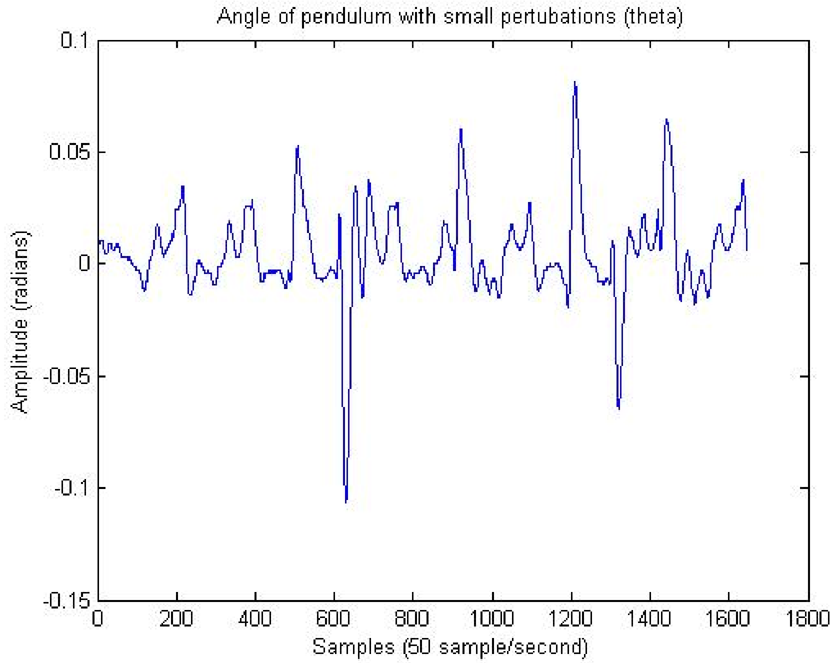



Figure 7: Pendulum angle with small perturbations.

### 3.2.3 Introducing a sine wave reference signal

We introduced a sine wave reference signal by changing the constant block in Figure 1 to be of the form $[M sin\omega t, 0, 0, 0]^T$, with $M = 0.1$m and $\omega$ being varied at $\omega = 1, 2, 5 rad/s$. We then ran the simulation and analyzed the results.
The plots for $M = 0.1$m and $\omega = 1$ rad/s are shown in Figure 8, 9, 10 below:

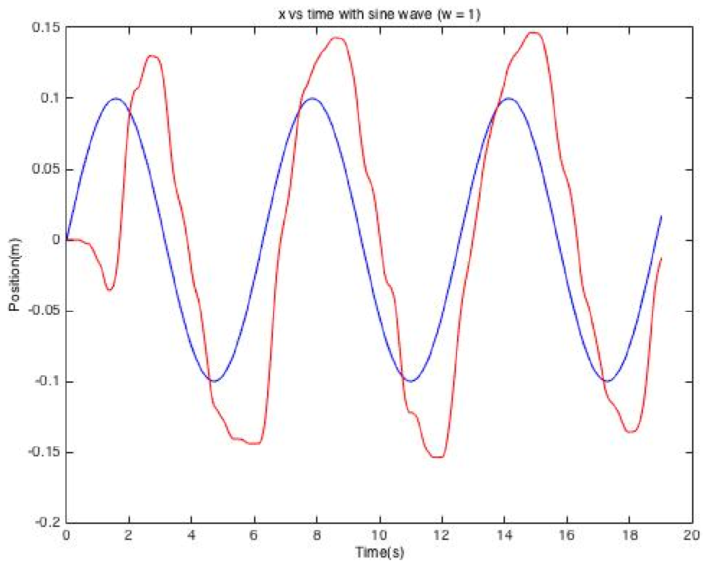



Figure 8: Position vs time graph

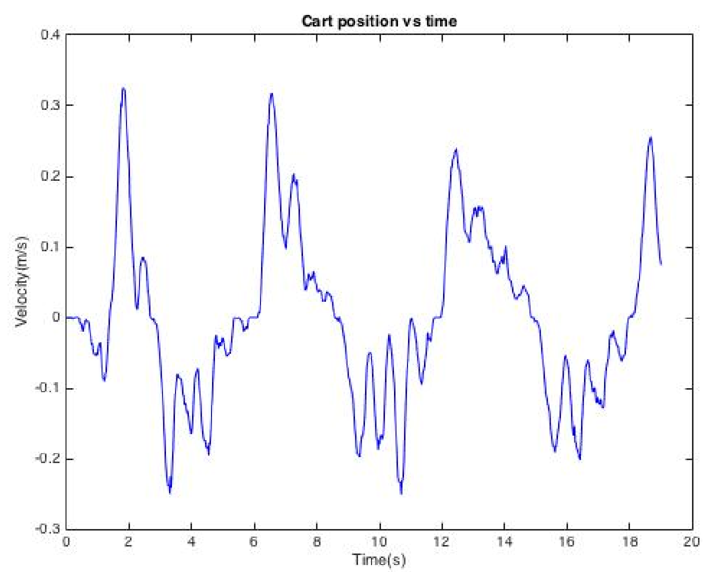



Figure 9: Velocity vs time graph

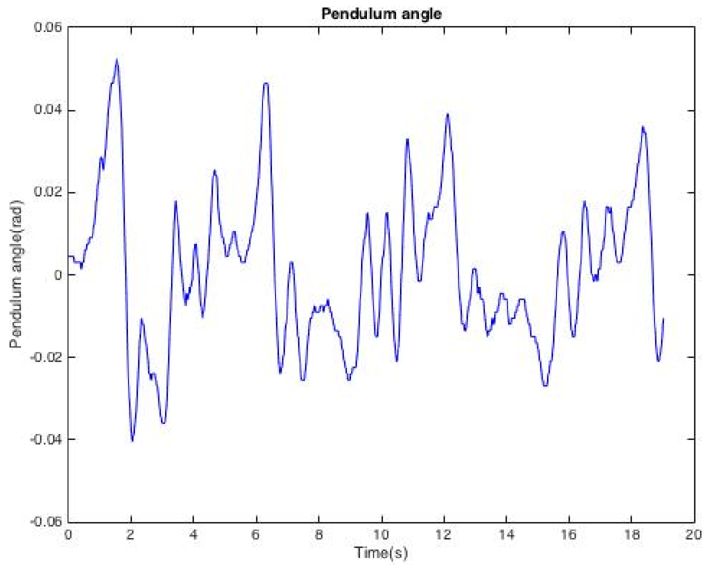



Figure 10: Pendulum angle vs time graph

The plots for $M = 0.1$m and $\omega = 2$rad/s are shown in Figure 11, 12, 13 below:

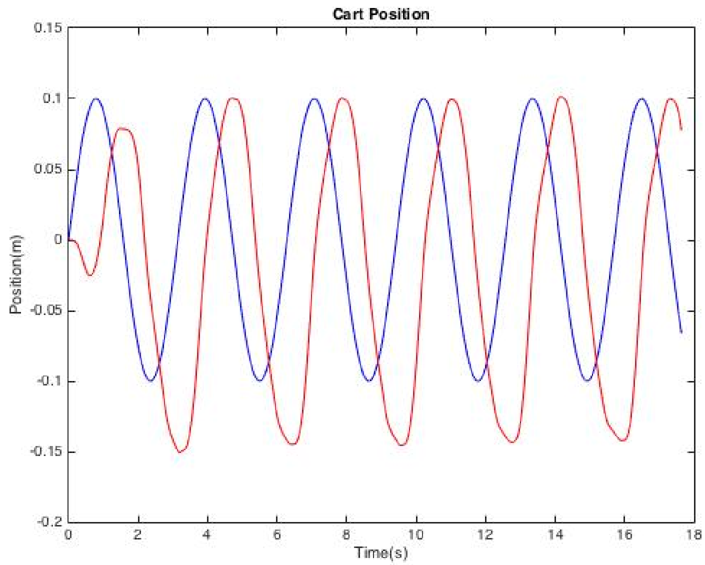



Figure 11: Position vs time graph

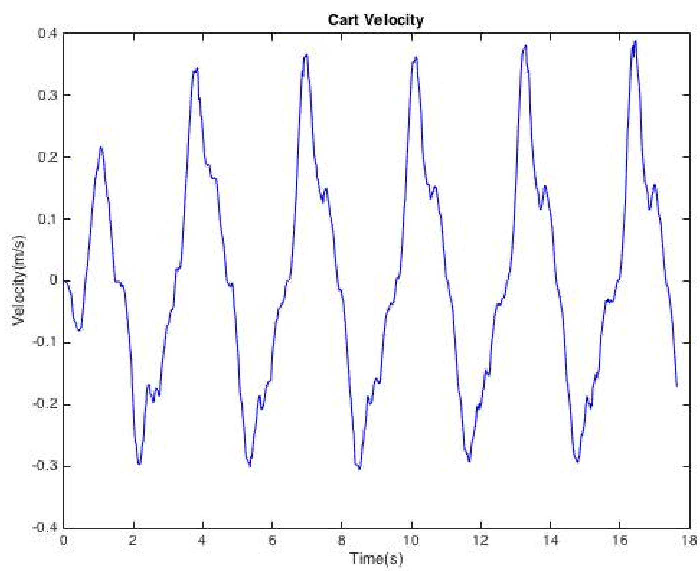



Figure 12: Velocity vs time graph

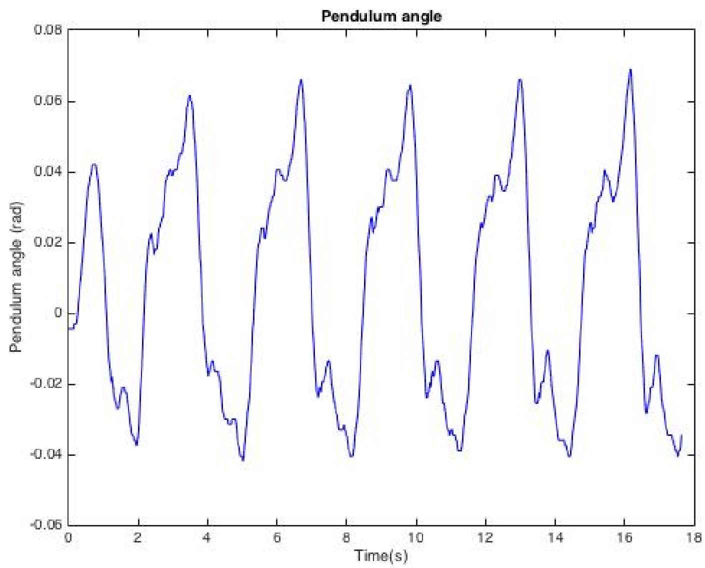



Figure 13: Pendulum angle vs time graph

The plots for $M = 0.1$m and $\omega = 5$rad/s are shown in Figure 14, 15, 16 below:

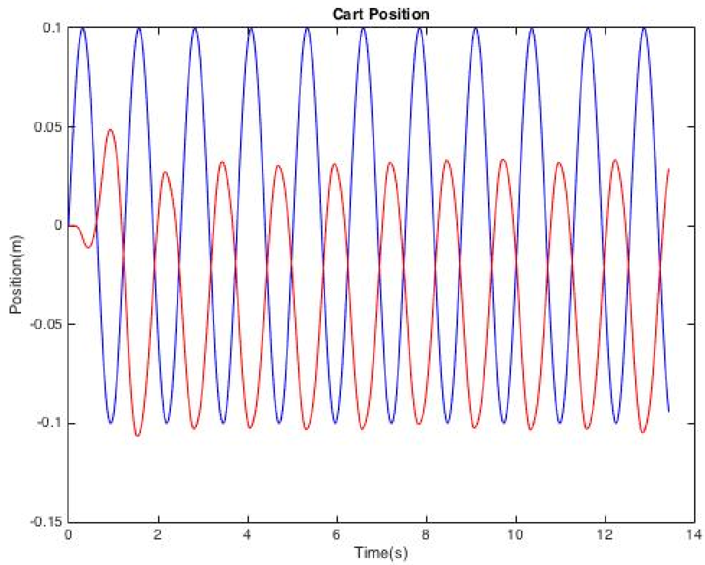



Figure 14: Position vs time graph

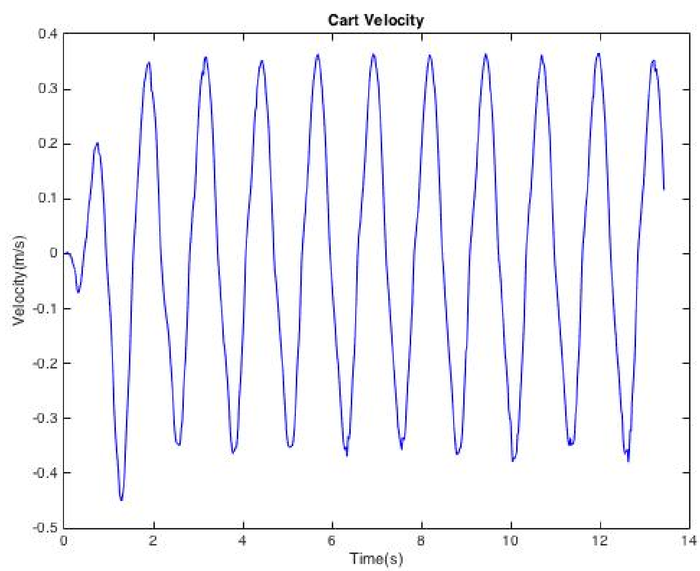



Figure 15: Velocity vs time graph

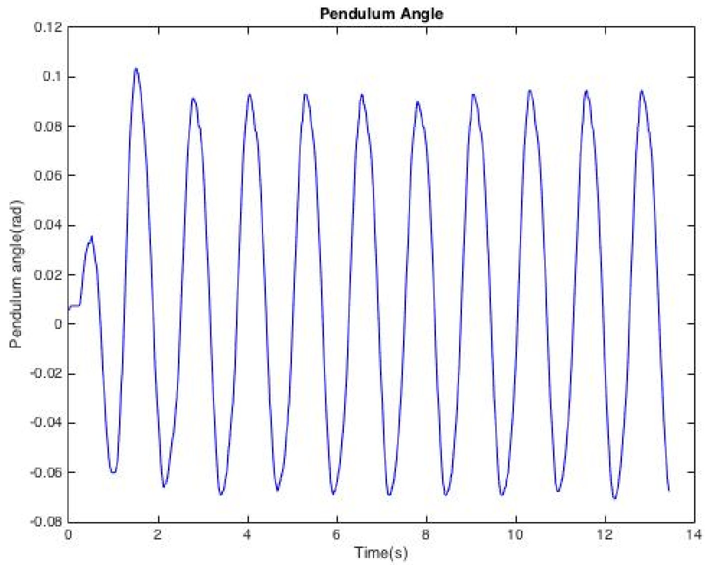



Figure 16: Pendulum angle vs time graph

### 3.2.4 Calculate the gain and phase for each of the frequencies in your frequency response (ignoring the offset from the hardware response). Locate these frequencies on the Bode plot from Part 3.3.4 of your Pre-Lab and compare the results. Do your values match for each frequency? If not, explain possible causes for the difference.

| Frequency (rad/s) | Actual Gain | Actual Phase(rad) | Bode Gain | Bode Phase(rad) |
|---|---|---|---|---|
| 1 | 1.400 | 1.13 | 0.980 | 0.820 |
| 2 | 1.125 | 1.76 | 0.806 | 1.61 |
| 5 | 0.675 | 2.14 | 0.445 | 2.74 |

Table 1: Table of actual gain and phase and bode gain and phase

The values do not match. The modelled system does not factor in the non linearities, which are definitely present in the actual system. Also, the model ignores friction and internal motor resistance.

### 3.2.5 Slightly change the position of the desired closed-loop poles. Try a couple of different values and run the resulting controllers on the hardware. Again include plots of cart position, cart velocity and pendulum angle in your report. Discuss how the changes in the position of the poles affect the behavior of the system. You do not need to repeat part 3a.

The position of the closed loop poles were changed as follows:

| Pole Values | K Values |
|---|---|
| -1.5 $\pm j8.0, -1.6 \pm j1.3$ | [-8.29 -11.10 -34.20 -4.70] |
| -1.5 $\pm j8.0, -2.0 \pm j2.0$ | [-15.60 -12.99 -39.20 -5.77] |
| -1.5 $\pm j12.0, -1.6 \pm j1.3$ | [-18.30 -18.63 -61.71 -8.02] |

Table 2: Table of actual gain and phase and bode gain and phase

The resulting plots of the first pole change are shown below:

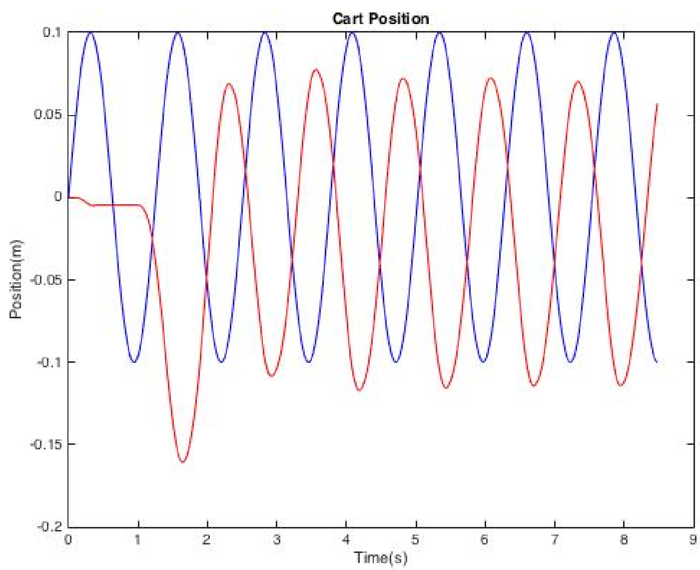



Figure 17: Position vs time graph

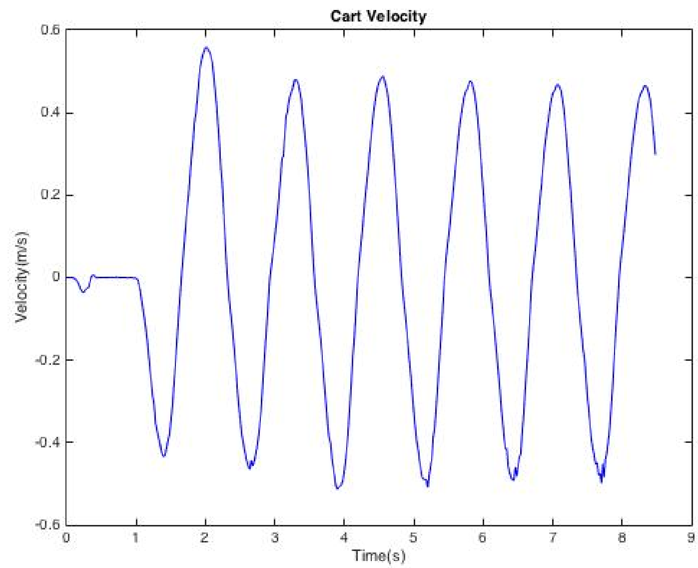



Figure 18: Velocity vs time graph

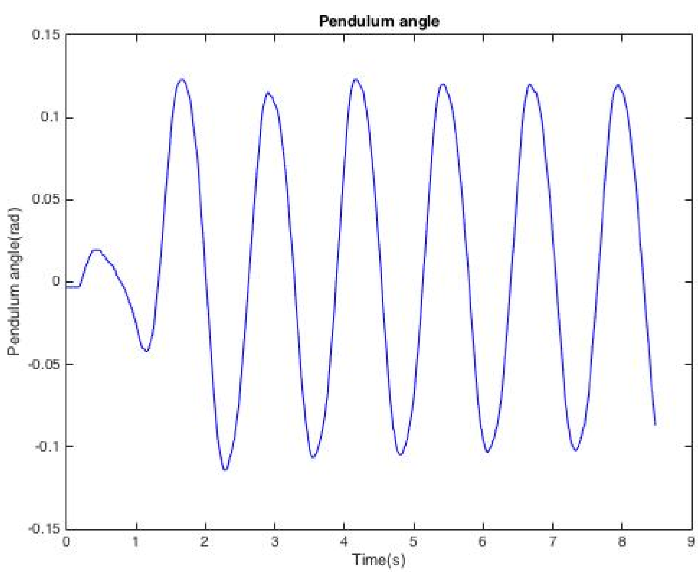



Figure 19: Pendulum angle vs time graph

The resulting plots of the second pole change are shown below:

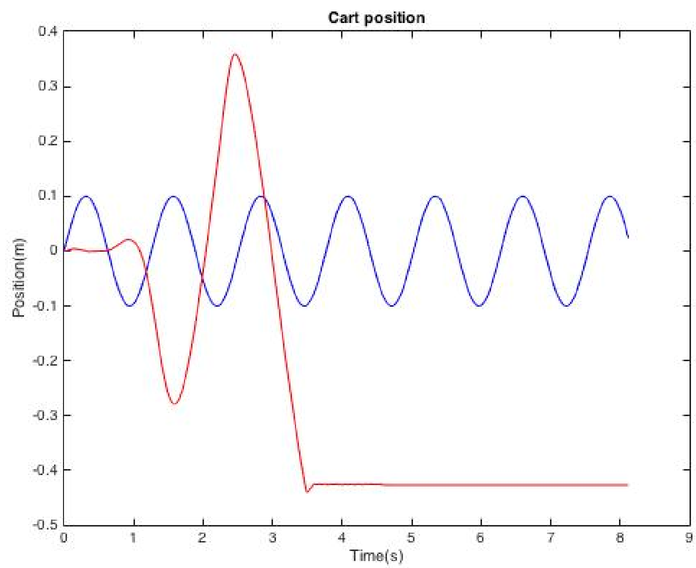



Figure 20: Position vs time graph

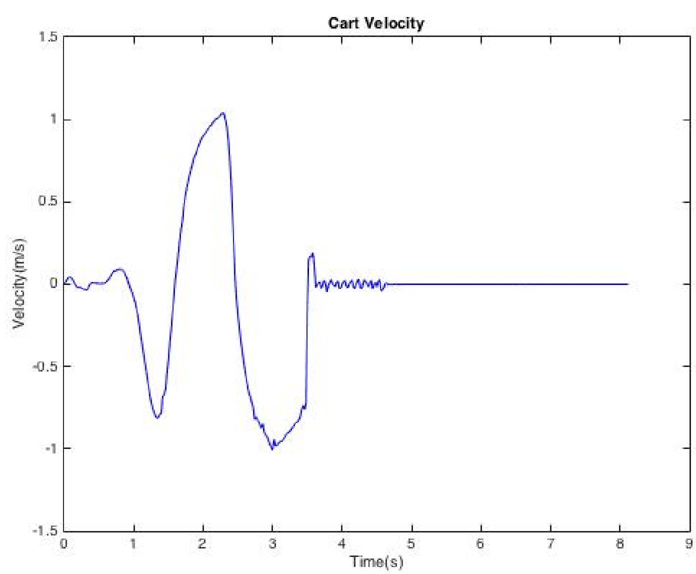



Figure 21: Velocity vs time graph

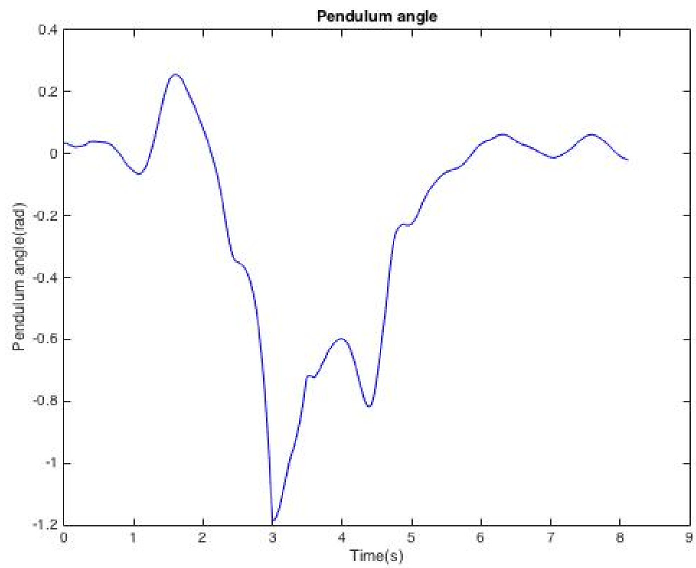



Figure 22: Pendulum angle vs time graph

The resulting plots of the third pole change are shown below:

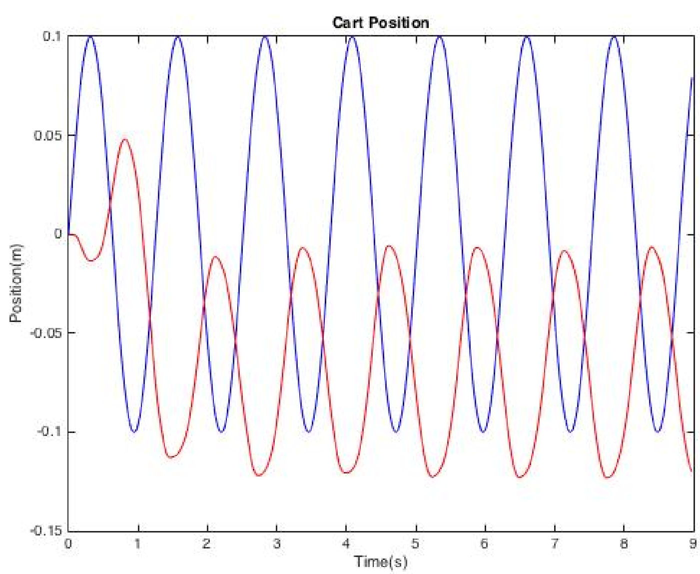



Figure 23: Position vs time graph

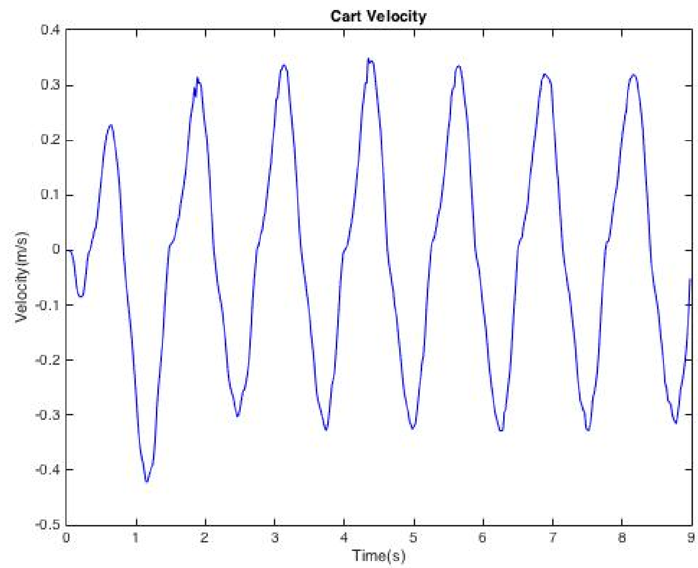



Figure 24: Velocity vs time graph

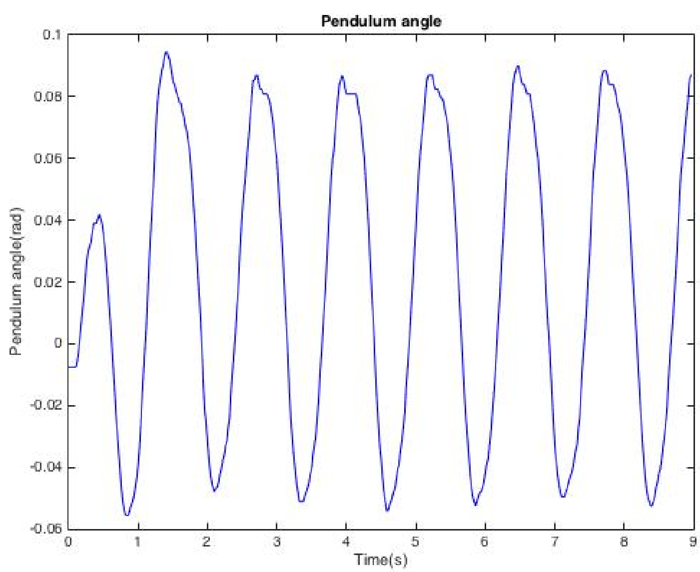



Figure 25: Pendulum angle vs time graph

The system will be stable as long as no right hand poles are present. As we changed the poles to be closer to the imaginary axis, the system turned out to be less stable, as seen from the larger fluctuations in position, velocity and pendulum angle. For the second pole pairing that we tried, the fluctuations got too large and the system failed (the pendulum crashed).

### 3.3 Plot the cart velocity and the pendulums angular velocity , which are obtained by numerically differentiating the signals $x$ and $\theta$, respectively. Comment on the quality of the obtained signals.

The plots were derived from numerically differentiating the signals.

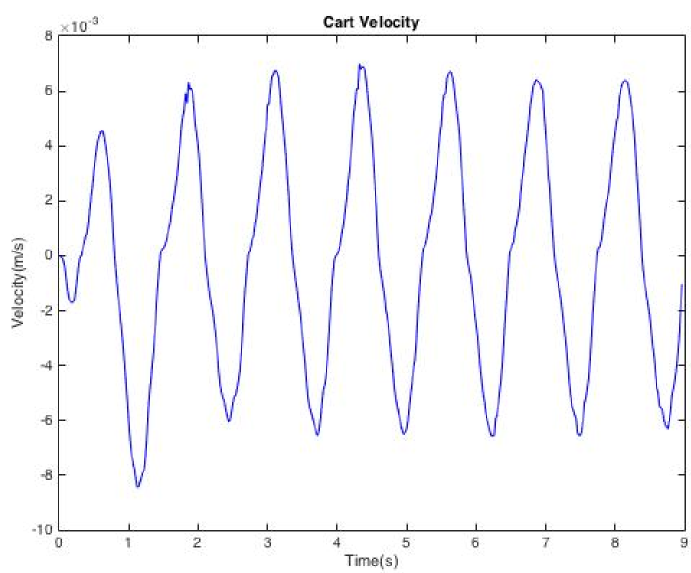



Figure 26: Derivative graph of cart position

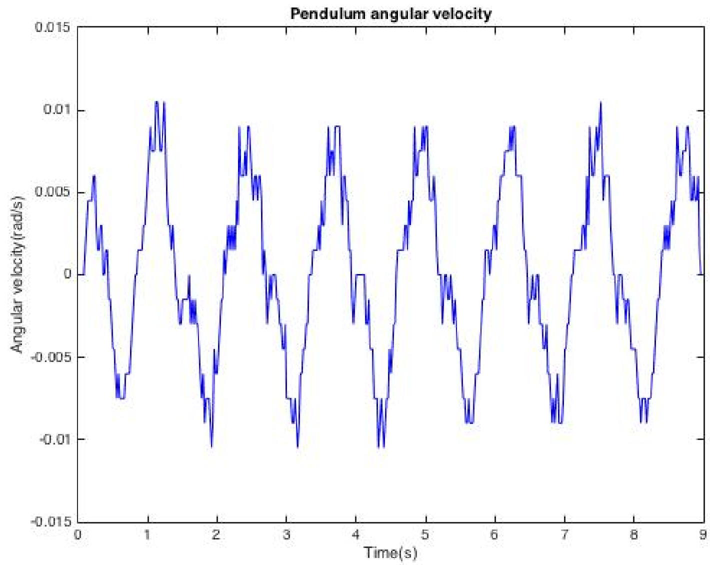



Figure 27: Derivative graph of pendulum angle

A lot of noise was observed from the derivative plots. This is possibly because the read encoder samples the outputs x and theta at discrete intervals. The derivatives for the time period inbetween these discrete intervals would fluctuate between infinity and zero, hence resulting in the noise we see from the plots.