Oxford Lecture Series in
Mathematics and its Applications

Series Editors
John Ball Dominic Welsh

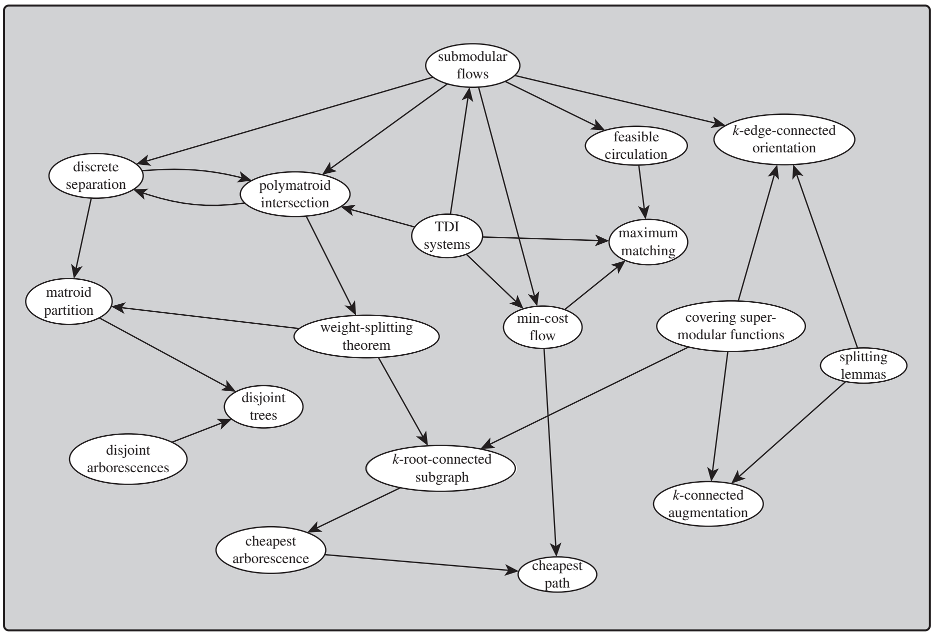
OXFORD LECTURE SERIES IN MATHEMATICS AND ITS APPLICATIONS

For a full list of titles, please visit:
http://ukcatalogue.oup.com/nav/p/category/academic/series/mathematics/olsma.do

10. Pierre-Louis Lions: *Mathematical Topics in Fluid Mechanics, Vol. 2: Compressible Models*
11. W.T. Tutte: *Graph Theory As I Have Known It*
12. Andrea Braides and Anneliese Defranceschi: *Homogenization of Multiple Integrals*
13. Thierry Cazenave and Alain Haraux: *An Introduction to Semilinear Evolution Equations*
14. Jean-Yves Chemin: *Perfect Incompressible Fluids*
15. Giuseppe Buttazzo, Mariano Giaquinta, and Stefan Hildebrandt: *One-Dimensional Variational Problems: An Introduction*
16. Alexander I. Bobenko and Ruedi Seiler: *Discrete Integrable Geometry and Physics*
17. Doina Cioranescu and Patrizia Donato: *An Introduction to Homogenization*
18. E.J. Janse van Rensburg: *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*
19. Sergei B. Kuksin: *Analysis of Hamiltonian PDEs*
20. Alberto Bressan: *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*
21. Benoit Perthame: *Kinetic Formulation of Conservation Laws*
22. Andrea Braides: *Gamma-Convergence for Beginners*
23. Robert Leese and Stephen Hurley: *Methods and Algorithms for Radio Channel Assignment*
24. Charles Semple and Mike Steel: *Phylogenetics*
25. Luigi Ambrosio and Paolo Tilli: *Topics on Analysis in Metric Spaces*
26. Eduard Feireisl: *Dynamics of Viscous Compressible Fluids*
27. Antonín Novotný and Ivan Straškraba: *Introduction to the Mathematical Theory of Compressible Flow*
28. Pavol Hell and Jaroslav Nešetřil: *Graphs and Homomorphisms*
29. Pavel Etingof and Frederic Latour: *The Dynamical Yang-Baxter Equation, Representation Theory, and Quantum Integrable Systems*
30. Jorge L. Ramírez Alfonsín: *The Diophantine Frobenius Problem*
31. Rolf Niedermeier: *Invitation to Fixed-Parameter Algorithms*
32. Jean-Yves Chemin, Benoit Desjardins, Isabelle Gallagher, and Emmanuel Grenier: *Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier-Stokes Equations*
33. Juan Luis Vázquez: *Smoothing and Decay Estimates for Nonlinear Diffusion Equations*
34. Geoffrey Grimmett and Colin McDiarmid: *Combinatorics, Complexity, and Chance*
35. Alessio Corti: *Flips for 3-folds and 4-folds*
36. Andreas Kirsch and Natalia Grinberg: *The Factorization Method for Inverse Problems*
37. Marius Ghergu and Vicentiu Radulescu: *Singular Elliptic Problems: Bifurcation & Asymptotic Analysis*
38. András Frank: *Connections in Combinatorial Optimization*

Connections in Combinatorial Optimization

András Frank
MTA–ELTE Egerváry Research Group
Institute of Mathematics
Eötvös Loránd University, Budapest



OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi
Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan Poland Portugal Singapore
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Published in the United States
by Oxford University Press Inc., New York

First published 2011

British Library Cataloguing in Publication Data

Data available

Library of Congress Cataloging in Publication Data

Data available

Typeset by SPI Publisher Services, Pondicherry, India
Printed in Great Britain
on acid-free paper by
CPI Antony Rowe, Chippenham, Wiltshire

ISBN 978–0–19–920527–1

1 3 5 7 9 10 8 6 4 2

Preface

Topics and goals

Combinatorial optimization deals with algorithmic approaches to finding specified configurations or objects in finite structures such as directed and undirected graphs, hypergraphs, networks, matroids, partially ordered sets, and so forth. In particular, the subject requires structural investigations to explore—possibly all—obstacles that prevent the existence of the requested configurations, as well as the construction of efficient algorithms for computing an object in question. Often we need to find an optimal solution with respect to a given objective function. Some well-known and typical problems are computing a cheapest (or shortest) spanning tree of a graph, a shortest st-path, a perfect matching, a maximum flow, a chain decomposition of a partially ordered set, a Hamilton circuit, an optimal graph colouring, a set covering, the clique number, and so forth. **NP**-hard problems constitute a huge class of problems for which neither a polynomial time algorithm nor a good characterization can be expected to exist. In these cases, the typical approaches are using heuristics for getting solutions acceptable from a practical point of view, or creating approximation algorithms along with a certificate for the obtained solution being not far from the optimal one, or exploring special cases that admit exact and efficient solution algorithms.

A fundamental trend of combinatorial optimization is concerned with problems where polynomial time algorithms or good characterizations (including min-max theorems) are available. These problems form the classes denoted by **P** and **NP**∩co-**NP**, respectively. Quite often, solutions and algorithms for these problems help in attacking **NP**-complete problems, too. For example, N. Christofides proposed one of the earliest and best-known approximation algorithms, which computes a Hamilton circuit of a complete graph the cost of which is at most 3/2-times larger than the optimal one, provided the cost function obeys the triangle inequality. The method and its verification are based on two polynomially solvable problems (J.B. Kruskal's greedy algorithm for cheapest spanning trees and J. Edmonds' weighted perfect matching algorithm).

The bulk of combinatorial optimization problems in **NP**∩co-**NP** can be divided into two main branches. One branch is about packing, covering, and optimization problems concerning paths, trees, flows, and matroid union and intersection. Typical results are the Max-flow Min-cut theorem, the greedy algorithm for cheapest trees, the Hungarian method for weighted bipartite matching, Dijkstra's shortest path algorithm, and Edmonds' matroid intersection theorem and algorithm. A unifying feature of this class is its strong and intimate relationship, explicit or implicit, to sub- or supermodular functions. In several situations, submodular functions take the center stage while a large number of other applications need supermodular functions. Therefore we sometimes use the unifying term semimodular optimization to refer to an optimization problem or method involving submodular or supermodular functions. This book concentrates, almost exclusively, on this branch, which forms a compact, homogeneous body of material in which a great majority of problems can be treated with unified methods based on semimodular functions and polyhedral techniques.

In the other big branch of tractable combinatorial optimization problems, the role of parity, implicit or explicit, becomes central. For example, the answer to the very first problem in combinatorial optimization—deciding if a graph has an Euler tour—involves parity. Further typical problems are non-bipartite matchings, T-joins and T-cuts, and matroid parity. A prime example is a theorem of W.T. Tutte on the existence of perfect matchings in general graphs, where the notion of odd components proves fundamental. Parity-related problems often need different, significantly more diverse and complicated techniques, and this topic alone could easily fill a complete book. Parity may come into the picture at different levels. Sometimes it appears in the answer to a problem, as exemplified by Tutte's matching theorem. At other times, parity shows up already in the problem formulation: the optimal T-joins form a prime example, as does the deep theory of matroid parity. And there is a third case, when neither the problem formulation nor the answer requires parity, yet the use of parity becomes unavoidable during the proof. The strong form of the orientation theorem of C.St.J.A. Nash-Williams belongs to this apparently strange category, and it will be fully discussed in Section 9.8.

The word 'connections' in the title suggests two different meanings. On the one hand it refers to, in broadest sense, the mathematical theory of graph connectivity, including results on paths, cuts, trees, flows, arborescences, bipartite matchings, and on the several variations of higher-order connections like node-, edge-, partition-, mixed- and rooted-connections of graphs, digraphs, and hypergraphs. The title, on the other hand, refers to one of the main intentions of the book: revealing and exploring often unexpected connections and links between apparently unrelated areas. To single out one, it is really stunning that E. Győri's difficult theorem on finding a minimum covering of a planar rectilinear region with rectangles has anything to do with connectivity augmentation problems of digraphs.

The most important connection, constituting the backbone of the book, is the one between connectivity and semimodular functions. Their interaction occurs on two levels. In the lower one, semimodular functions are used to prove results on connectivity. For example, P. Hall's theorem on perfect bipartite matching can be derived by using the submodularity of the neighbourhood function $|\Gamma(X)|$. Similarly, the proof of L. Lovász' splitting-off theorem relies on the submodularity of the degree-function of a graph. With the semimodular technique, often deep and otherwise difficult theorems can be proved relatively easily. Typical examples we shall consider are theorems of C.L. Lucchesi and D.H. Younger on optimal dijoins, of W. Mader on undirected splitting-off, and of T. Watanabe and A. Nakamura on optimal edge-connectivity augmentation. One of the main goals of this book is to convey the striking efficiency of semimodular proof techniques.

A higher level of interaction is the one between 'specific' connectivity results and their 'abstract' counterparts concerning semimodular functions. For example, though it is not difficult to verify the validity of Kruskal's greedy algorithm for computing a cheapest spanning tree, it has been a great revelation to realize that the greedy algorithm works properly in this case because there is a matroid in the background: the correctness of the greedy algorithm is an essential feature of matroids.

The general scheme of the interaction between specific connectivity results and abstract semimodular ones works as follows: An initial result on connectivity makes one look for possible generalizations. This gives rise to problem formulations in an abstract setting concerning semimodular functions, and an answer here may then be specialized back to (extensions of) the initial connectivity problem. For example, H.E. Robbins' early theorem on strong orientability of a 2-edge-connected graph was generalized by Nash-Williams to k-edge-connected orientations. As Robbins' theorem easily extends to mixed graphs, one naturally wants to generalize Nash-Williams' result to mixed graphs, too. This problem forms then a motivation for investigating the feasibility problem of submodular flows, and a solution to this problem specializes to the graph-orientation problem. Another beautiful realization of this scheme is how Edmonds extended the Hungarian method to matroid intersection and specialized then the general result back to solve algorithmically the problem of finding a cheapest rooted k-edge-connected subgraph of a digraph. This kind of mechanism shows up in many other cases, and we are going to explore its nature to the reader as much as possible. To this end, we shall show how abstract notions like matroids and polymatroids, crossing submodular functions, and submodular flows arise naturally, and why it is worth to becoming familiar with them.

Exploring and exhibiting the intimate relationship between connectivity and submodularity is the major objective of the book. By using this book to look behind the scenes and understand properly what had made a specific proof work, the reader is not only exposed to a list of results and proofs, but can also learn proof methods and algorithmic tools as well. Another important objective is to keep track of the development of ideas and proof techniques from their very first simple appearance to their use in more and more complicated circumstances. These specific approaches constitute the most important features that separate this book from all other available books on related topics.

Structure of the book

The book consists of three parts. Part I discusses, in a relatively concise form, classical connectivity-related optimization results on paths, trees, cuts, flows, and bipartite matchings. Also, the necessary background of polyhedral combinatorics and matroid theory is outlined here. In order to sweeten this part for the more experienced reader, I have included some interesting applications which are not so well-known. It was also a major goal to make the discussion easily digestible even for newcomers.

Part II covers more recent topics such as connectivity orientations, connectivity augmentations, and constructive characterizations of connectivity properties. Packing and covering with trees and branchings are also discussed here. Among the several methods, special emphases will be put on the splitting-off technique, the uncrossing procedure, polyhedral approaches, greedy algorithms, and the push–relabel algorithm. Many of the results of this part serve as a main motivation to investigate sub- and supermodular frameworks.

Part III treats several aspects of submodular optimizations and polyhedral methods. Quite often these 'abstract' results have been motivated by theorems on graph connection from

Part II, and they give rise to new, concrete connectivity-type special cases. The approach exemplifies nicely an essential feature of mathematics showing that well-conceived concepts may give rise to short and transparent proofs of quite deep (and otherwise difficult) theorems.

A glimpse at the origins of combinatorial optimization problems discussed

Readers interested in a detailed account of the history of combinatorial optimization are advised to consult Schrijver's book [340], an extremely rich source. See also the survey by Schrijver [341] on the history of combinatorial optimization until 1960. In what follows, I outline in a nutshell a personal view of the chronology of the progress of the basic notions, ideas, and results considered in this book, without touching other areas of combinatorial optimization. For example, no comments are made on the history of such fundamental results as non-bipartite matchings, submodular function minimization, matroid parity, or regular matroids. There are three main sources: graph theory, linear programming, and the theory of submodular functions including matroids. The main unifying feature of combinatorial optimization is its algorithmic nature.

In graph theory, the theorem of L. Euler on traversing edges and the result of J. Petersen on matchings were known before the twentieth century. D. Kőnig's bipartite matching theorem [246] and K. Menger's result [288] on disjoint st-paths appeared by 1930, as well as two basic algorithmic techniques: alternating paths, due to Petersen, and the greedy algorithm for cheapest trees, first worked out by O. Bôruvka [37]. In 1938, T. Gallai [178] gave a new proof of Menger's theorem. His method is algorithmic and can be considered as a forerunner of the augmenting paths method of R.L. Ford and D.R. Fulkerson for maximum flows. In a tiny footnote Gallai also points out that the same method actually proves the directed counterpart of Menger's theorem. J. Egerváry [88] proved his min-max theorem on weighted bipartite matchings in 1931. This result is the first instance of the linear programming duality theorem. It should be noted that already in 1912 F.G. Frobenius [157] proved a theorem on matrices that is equivalent to Kőnig's theorem. A recent historical discovery by F. Ollivier shows that a variation of the weighted bipartite matching algorithm (the Hungarian method) developed by H.W. Kuhn [252] in 1955 had been invented a century earlier by C.G.J. Jacobi [226].

The fundamentals of linear programming (including the Farkas lemma, the duality theorem, optimality criteria, and the equivalence of polytopes and bounded polyhedra) and network flows (shortest paths, assignment and transportation problems, optimal flows and circulations) were established by 1960. A far-reaching discovery by A.J. Hoffman and J.B. Kruskal [209] from 1956 is worthy of special mention. They found a fundamental link, totally unimodular (TU) matrices, between linear programming and network flows, providing in this way a coherent view of a great number of combinatorial min-max theorems. They pointed out first that a linear program with a TU constraint matrix and an integral bounding vector always has an optimum solution that is integer-valued. Then they applied this result to TU-matrices arising from bipartite graphs and digraphs. In this view, for example, both the Max-flow Min-cut theorem and Egerváry's perfect matching theorem are consequences of the linear programming duality theorem, and Hoffman's circulation theorem follows immediately from the Farkas lemma. In 1958, Gallai [179] also used linear programming techniques and proved min-max theorems in graph theory. These results could be seen as the foundations of a new era, polyhedral combinatorics.

Kuhn's Hungarian method [252] from 1955 served as a starting point of several later extensions like min-cost flows or weighted matroid intersections. The algorithmic work on network flows by Ford and Fulkerson [104, 105, 106, 168, 169] during the period of 1955–61, including the Max-flow Min-cut theorem and the minimum cost flow algorithm, had made a far-reaching and lasting impact on the later development of combinatorial optimization. The results were summarized in their book [107] in 1962. Due to its crystal-clear style, this book has remained refreshing reading even after five decades.

Matroids were introduced by H. Whitney [383] as early as 1935. Their role in combinatorial optimization was indicated by R. Rado, who discovered a matroidal extension of Hall's theorem in 1942 [326] and the matroid greedy algorithm in 1957 [327]. A next milestone was the matroid partition theorem for linear matroids by A. Horn [211] from 1955. Rado slightly later observed, by a simple elementary construction, that his result from 1942 implies Horn's theorem for arbitrary matroids.

The papers of O. Ore [313, 314] from 1955–6 deserve special mention because these works were among the very first ones where submodular functions are explicitly used in deriving graph-theoretical results. Nash-Williams' deep and difficult orientation theorem from 1960 will be of central importance to our discussions. The theorem's weak form, stating that every $2k$-edge-connected graph has a k-edge-connected orientation, has been a major inspiration for a great number of extensions to be discussed in this book. It is, however, pretty mysterious that the strong form of Nash-Williams' orientation theorem has so far resisted every attack trying to relate it with other parts of combinatorial optimization and has remained quite isolated. The paper of Nash-Williams relies heavily on submodular ideas. Yet another basic result of Nash-Williams [306] established the rank formula for the homomorphic image of a matroid. In the developments of matroid min-max theorems, the paper of Tutte [368] from 1961 was of fundamental importance. It contains not only a necessary and sufficient condition for graphs having k-edge-disjoint spanning trees but the matroid partition theorem for distinct graphic matroids is also established and proved. The chain decomposition theorem of R.P. Dilworth [68] from 1950 and its significant extensions in the works of C. Greene and D.J. Kleitman [192, 193] in 1976 were also outstanding contributions to the topic. In 1965, Y.-J. Chu and T.-H. Liu [50] published the first polynomial algorithm for finding a cheapest arborescence of a digraph.

The view reflected in this book has been founded in large part on the work and vision of Edmonds. His contributions proved fundemental in several areas of combinatorial optimization. Basic notions like good characterization, polynomial algorithms, the polyhedral approach (without total unimodularity), polymatroids, and submodular flows come from his work and form the framework for this entire book. In 1965, Edmonds and Fulkerson [85] proved the matroid partition theorem. In 1970, Edmonds discovered the matroid and polymatroid intersection theorem [80], along with its weighted extension. In 1965, he developed polynomial algorithms for matroid partition [76], while his weighted matroid intersection algorithm [82] appeared in 1979. His theorem on disjoint arborescences [83] from 1973 also remains of central importance. Extending the earlier work of Hoffman [208] from 1974, Edmonds and Giles [86] introduced in 1977 the pervasive concept of total dual integrality (TDI) of linear inequality systems and proved the integrality of the polyhedron described by a TDI system. The polyhedral descriptions of several combinatorial objects (branchings, non-bipartite matchings, matroid intersections) paved the way for a great number of extensions and generalizations. The fundamental theorem of Edmonds and Giles on submodular flows implies the Max-flow Min-cut theorem, the matroid (and even the polymatroid) intersection theorem, the theorem of Lucchesi and Younger on minimum dijoins, and also the weak form of Nash-Williams' orientation result.

It was 1976 when K.P. Eswaran and R.E. Tarjan [92] published the first results on connectivity augmentation. This was followed in 1987 by the fundamental work of T. Watanabe and A. Nakamura [375] on optimal edge-connectivity augmentation, which became a starting point of a new research area. Interestingly, though this area relies heavily on submodular functions, the corresponding polyhedra are not integral, and submodular flows do not help.

The significance of the work of W. Mader on graph connectivity cannot be overestimated. His undirected splitting-off theorem [278] from 1978 as well as the directed splitting-off theorem [281] from 1983 became major tools in handling connectivity problems. His constructive characterization of k-edge-connected digraphs [280] was one of the prototypes of several later extensions.

Though the disjoint paths problem is **NP**-complete in general, P. Seymour explored several interesting special cases when it is nicely tractable. See his papers [345], [347], [311], and [346] from 1980–1.

Another major player during the last forty years of combinatorial optimization was L. Lovász. Beside his breakthrough results on perfect graphs, he developed new techniques for applying semimodular functions in proving graph-theoretic results [263, 264]. In addition, his paper [263] was an early appearance of the scheme outlined above: motivated by a specific graph result (Kőnig's theorem), he proved an abstract result on supermodular functions from which he obtained, in special cases, new graph results. There are two far-reaching proof techniques pioneered by Lovász. The uncrossing technique showed up in a paper by Younger [389] and in a report of the solutions of a mathematical contest [262]. Later, it was successfully applied to obtain a relatively simple proof of the optimum dijoin theorem of Lucchesi and Younger [275]. The splitting-off theorem of Lovász also proved to be fundamental and has appeared in a great number of generalizations and extensions.

Submodular optimization was extensively developed by A. Schrijver and S. Fujishige. For example, Schrijver [337] proved his supermodular colouring theorem in 1985 and used it to derive a conjecture of Edmonds and Giles for source-sink connected digraphs (while he disproved the conjecture in the general case). Fujishige [158] developed algorithms for submodular frameworks on graphs as early as in 1978. His results on the structure of base polyhedra [159, 160] became a major tool in handling graph and hypergraph orientation problems.

Two algorithmic breakthroughs occurred during the 1980s, which are the following. É. Tardos [359] was the first who developed a strongly polynomial algorithm for min-cost flows. A.V. Goldberg and E.R. Tarjan [186] invented the elegant push–relabel algorithm for maximum flows. Both papers triggered a new wave of research. We shall discuss several variations and extensions of the latter method. In 1992, H. Nagamochi and T. Ibaraki [296] published their astonishingly simple min-cut algorithm, which was later extended and applied in various settings. A recent book of Nagamochi and Ibaraki [300] covers all these results.

There are several outstanding works—monographs, textbooks, handbooks—which at least partly cover combinatorial optimization topics. The earliest one, Kőnig's book on Graph Theory [247], appeared in 1931. We mentioned already the seminal book of Ford and Fulkerson from 1962. Ore's book [315] on graph theory from the same year is also worth mentioning, since it exhibits the basic elements of the submodular technique in matching theory. The excellent work of L. Mirsky [291] entitled *Transversal Theory* was published in 1971 and appears to be the first book studying matroids from a combinatorial optimization aspect. C. Berge's *Graphs and Hypergraphs* [21] from 1973 is also a rich source.

In 1976 three important books appeared from the area. J.A. Bondy and U.S.R. Murty: *Graph Theory with Applications* [35] was an excellent textbook. D.J.A. Welsh: *Matroid Theory* [379] was the first systematic study of matroids. E.L. Lawler's work entitled *Combinatorial Optimization: Networks and Matroids* [256] was the first to have combinatorial optimization in its title. The book provided the outstanding service of popularizing the subject as a single, coherent discipline.

Lovász' problem book [267] from 1979 (*Combinatorial Problems and Exercises*) was the first source, among others, of several matching and connectivity problems, including his splitting-off lemma, which became the starting point for several far-reaching extensions. As far as matchings are concerned, Lovász and Plummer: *Matching Theory* [273] from 1986 included not only the most important results on matchings but also covered network flows and matroid optimization as well.

There are numerous other excellent monographs and textbooks focusing on special aspects of combinatorial optimization, such as

R.K. Ahuja, T.L. Magnanti, and J.B. Orlin: *Network Flows—Theory, Algorithms, and Applications* [2]
J. Bang-Jensen and G. Gutin: *Digraphs—Theory, Algorithms and Applications* [10]
J.A. Bondy and U.S.R. Murty: *Graph Theory* [36]
R. Diestel: *Graph Theory* [67]
S. Fujishige: *Submodular Functions and Optimization* [161]
H. Nagamochi and T. Ibaraki: *Algorithmic Aspects of Graph Connectivities* [300]
K. Murota: *Discrete Convex Analysis* [294]
J.G. Oxley: *Matroid Theory* [316]
A. Recski: *Matroid Theory and its Applications in Electric Network Theory and in Statics* [328].

A crowning achievement is the encyclopaedic, 3-volume book on combinatorial optimization by A. Schrijver [340]. This monumental work provides a comprehensive overview of efficient algorithms and polyhedral combinatorics. It also contains a plethora of invaluable historical notes.

Target audience

While planning and writing the book, I imagined an audience familiar with the fundamentals: elements of graph theory, network flows, linear programming, and matroids. J. Lee's [259] introductory book is an excellent choice for beginners. Two other exceptionally well-written advanced textbooks on combinatorial optimization, by Cook, Cunningham, Pulleyblank, and

Schrijver [56] and by Korte and Vygen [248], are also highly recommended. I assume some knowledge from complexity theory such as the concept of polynomial running time of an algorithm, the notions of **NP**-completeness, and the problem classes **NP** and co-**NP**. For this reason, the first part of this book, though self-contained, is relatively concise.

Since graduate students from mathematics, computer science, and operations research form one part of the targeted readership, I included not only exercises and problems, but 'questions' as well, which appear when a solution to a current problem raises natural questions concerning possible extensions and generalizations. Answers to these questions will appear in the text of later chapters. Readers are not really expected to figure out these answers themselves. But trying to find an answer and meditating a bit on possible solutions may serve as a means to understand better the gradual developments of ideas and the motivations behind those later generalizations. I also followed a practice that several smaller observations needed and proved at a later point are posed as problems, to be solved by the reader, where the necessary tools have just been developed for their solution. The last section of the book includes solutions to some selected problems. These problems are indicated by $(*)$ in the text.

Another target audience consists of specialists working in related areas who seek a relatively coherent exposition. The semimodular technique presented here proves to be powerful in several applications, and I hope that these readers will find it worthwhile to achieve some familiarity with it, even if their main research interest does not revolve around connectivity issues. Also, those working on approximation algorithms for **NP**-complete problems (concerning connectivity) may profit from specific aspects of this book. Yet another goal is to widen the standard curriculum of combinatorial optimization courses. The simple proofs make it possible even for early M.S. students to understand such fundamental results as the theorems of Lucchesi and Younger on minimum dijoins, of Nash-Williams on orientations, of Tutte on disjoint trees, or the weighted matroid intersection algorithm. Some research problems and conjectures will also be mentioned. For a richer selection, along with links, references, and partial results, see Egres Open (`http://lemon.cs.elte.hu/egres/open`), a homepage of the Egerváry Research Group.

A glimpse at characteristic problems

Since connectivity is the central theme of this book, let us take a brief glimpse at what this notion is about in the first place. A beginner intuitively perceives an undirected graph $G = (V, E)$ to be connected if there is no way to separate it into two non-empty pieces with no link between them. Formally, $G = (V, E)$ is **connected** if its node-set V has no non-empty proper subset X with $d_G(X) = 0$, where the **degree** $d_G(X)$ of X denotes the number of edges of G connecting X and $V - X$. A graph that is not connected is called **disconnected**. Another, no less natural, approach for defining connectivity is to require the existence of a path between each pair of nodes.

Although it is well known and straightforward that these definitions are equivalent, they are conceptually quite different. While the first one is a co-**NP** property in the sense that it provides an easily verifiable certificate for the lack of connectivity (namely, a subset X with $d_G(X) = 0$), the second is an **NP** property since it provides an easily verifiable certificate for the existence of connectivity (namely, a list of $\binom{n}{2}$ paths).

There is an analogous result for digraphs too. A digraph $D = (V, A)$ is **strongly connected** if it has no non-empty proper subset X with $\varrho_D(X) = 0$, where the in-degree $\varrho_D(X)$ of X denotes the number of edges entering X. Another possibility for defining strong connectivity is to require that each node can be reached along a directed path from every other node. Again, a well-known and easy result asserts the equivalence of these properties.

For higher-order connections there are also two different ways to capture the intuitive feeling that a graph or digraph is 'pretty much' connected: a co-**NP** and an **NP** property. The first one requires that the graph or digraph cannot be dismantled into small pieces by leaving out only a few edges. Here are four possible definitions to make this intuition formal:

(A1) A graph $G = (V, E)$ is k**-edge-connected** if discarding fewer than k of its edges leaves a connected graph. (This is essentially a reformulation of the requirement that $d_G(X) \geq k$ whenever $\emptyset \subset X \subset V$.)

(A2) A digraph $D = (V, A)$ is k**-edge-connected** if discarding fewer than k of its edges leaves a strongly connected digraph. (This is essentially a reformulation of the requirement that $\varrho_D(X) \geq k$ whenever $\emptyset \subset X \subset V$.) For $k = 1$, k-edge-connectivity is just strong connectivity.

(A3) G is k**-partition-connected** if discarding fewer than kq edges leaves a graph with at most q connected components for every $q = 1, 2, \ldots, |V| - 1$. Equivalently, there are at least kq edges connecting distinct parts for every partition of V into $q + 1$ non-empty parts for every q, $1 \leq q \leq |V| - 1$. Note that for $k = 1$, partition connectivity is equivalent to connectivity.

(A4) A digraph $D = (V, A)$ with a specified root-node r is r**-rooted** k**-edge-connected** if $\varrho_D(X) \geq k$ whenever $\emptyset \subset X \subset V - r$. For $k = 1$, we simply say that D is **root-connected**.

The second possible approach to capture the notion of higher-order edge-connections is requiring the graph or digraph to include several edge-disjoint 'simple' connected constituents. Here are four possibilities.

(B1) In G there are k edge-disjoint paths between every pair of nodes.

(B2) In D there are k edge-disjoint directed paths from every node to every other.

(B3) G includes k edge-disjoint spanning trees (in which case G is called k**-tree-connected**).

(B4) D includes k edge-disjoint spanning r-rooted arborescences.

Some fundamental results of graph theory assert the equivalence of the corresponding definitions. Namely, by the edge-versions of Menger's theorem [107], the definitions **(A1)** and **(B1)** [and, respectively, **(A2)** and **(B2)**] are equivalent.

Also, theorems of Tutte [368] and Edmonds [83] assert the equivalence of **(A3)** and **(B3)** and the equivalence of **(A4)** and **(B4)**, respectively. Nash-Williams' orientation theorem [303] provides a link between $2k$-edge-connected undirected graphs and k-edge-connected digraphs asserting that an undirected graph has a k-edge-connected orientation if and only if G is $2k$-edge-connected.

The vast majority of the problems we are going to investigate can be divided into six categories.

1. Subgraph problems. Given an undirected or directed graph, find a subgraph with prescribed connectivity properties. Moreover, if there are several possible choices, find the best one with respect to a cost (or weight) function. For example, finding a cheapest spanning tree of a graph is a well-known subgraph problem, as is the one of finding a cheapest path from s to t. A significantly more complex, but nicely tractable, subgraph problem is about finding a cheapest subgraph of a digraph in which every node is reachable from a root-node along k edge-disjoint paths. One may also be interested in the largest part of a partially ordered set (poset) that includes no long chains or antichains.

A great number of subgraph problems are **NP**-complete, most notably the Hamilton circuit problem, where a connected spanning subgraph having all degrees exactly 2 is sought.

2. Augmentation or supergraph problems. There are problems, in a sense opposite to subgraph problems, where one is interested in the minimum number of new edges to be added to an initial graph so as to achieve a target connectivity. For example, how many new edges must be added to a digraph to obtain a strongly connected digraph? Interestingly, this particular problem can be solved in polynomial time, while its subgraph counterpart, in which the construction of a smallest strongly connected subgraph of a digraph is requested, is **NP**-complete.

3. Orientation problems. Here one wants to orient the edges of an undirected graph so that the resulting digraph fulfils a target connectivity requirement. For example, how should we orient a graph so as to obtain a strongly connected digraph? A naturally related problem is the reorientation problem when an initial digraph has to be reoriented by reversing a minimum number of edges so as to obtain a digraph that is strongly connected. This problem is still tractable, though the answer derived from an equivalent formulation of a theorem of Lucchesi and Younger [275] is much harder, especially from an algorithmic point of view.

4. Packing, covering, and colouring problems. In a packing problem, one is expected to select a maximum number of disjoint members of a set of given objects. For example, find k disjoint spanning arborescences of a digraph, or k disjoint paths from s to t. Or, what is the maximum number of disjoint dicuts in a digraph?

In a covering problem, we want to cover the ground-set by a minimum number of given objects. For example, when is it possible to cover a digraph by k branchings? Or, what is the minimum number of chains covering the ground-set of a partialy ordered set?

Finally, in a colouring problem, we want to partition (colour) the ground-set into given objects. For example, when is it possible to partition the edge-set of a bipartite graph into k matchings? Or, when is it possible to partition a poset into k antichains?

5. Constructive characterizations. Certificates obtained constructively for the existence of a certain property are called constructive characterizations. For example, an undirected graph is connected if and only if it can be built up from a node by adding edges one by one in such a way that at least one end-node of the newly added edge belongs to the already constructed subgraph. Slightly more complicated constructions are the ear-decomposition theorems for 2-edge-connected graphs and strongly connected digraphs. The advantage of such characterizations is nicely illuminated by a well-known folklore result, asserting that a directed graph is acyclic if and only if it can be built up from a node by adding new nodes one by one in such a way that a new node v is added along with some new directed edges with head v such that the tails are existing nodes. This constructive characterization is the only polynomially verifiable certificate for the non-existence of directed circuits.

6. Polyhedral descriptions. In order to optimize a linear objective function over some given objects, it can be extremely useful to describe the convex hull (of incidence vectors) of these objects by linear inequalities. Such a description is often indispensable for an efficient algorithm since, through the linear programming duality theorem, the description provides an optimality condition and stopping rule. For example, the problem of finding the cheapest subgraph of a digraph in which there are k openly disjoint paths from a root-node to every other node can hardly be solved without polyhedral ideas.

This categorization of problems does not mean a disjoint partition. It may be the case that the same problem belongs to more than one class. For example, perfect matching can be considered as a subgraph problem, and also as a packing problem. Or the orientation problem can be considered as a subgraph problem: replace each edge of the initial undirected graph by two directed parallel edges oriented oppositely and delete exactly one member of these pairs. Conversely, a subgraph problem, too, can often be formulated with the help of orientations. We shall see that the subgraph and the augmentation problems are also strongly related.

Acknowledgements

Eötvös Loránd University of Budapest (ELTE) has been an ideal place for studying as well as for teaching mathematics. The Institute of Mathematics of the university provided a supportive environment for conducting research and for completing this book. The contribution of the Hungarian Academy of Sciences (MTA) by supporting the Egerváry Research Group on Combinatorial Optimization (EGRES), a joint group of MTA and ELTE, provided a significant improvement in the research environment during the past decade. The support of the Hungarian Research Foundation (OTKA, No. K60802 and No. CK80124) was also decisive. During my research career, I enjoyed the support of several other institutions: University of Bonn, Fourier University Grenoble, EPFL Lausanne, and Ericsson Hungary.

A large number of people helped me during the long process of bringing this book to life. I am extremely grateful to Imre Rábai for his fundamental contribution to the polishing of my mathematical thinking during high school. As a graduate student, I was particularly lucky to meet Lajos Pósa, who not only taught a lot of basic proof techniques but encouraged me to take the first steps toward become a researcher. It was András Recski from whom I learnt the technique of writing a paper or preparing a talk. His help in facilitating my early participation on international conferences is also gratefully acknowledged. Without the years I spent at the Research Institute for Discrete Mathematics, founded and directed by Bernhard Korte in Bonn, I would not have been able to write this book. He has served as a role model by providing the most efficient environment for researchers, and I was among the fortunate ones who could enjoy his hospitality for long periods. Jean Fonlupt in France, Tom Liebling in Switzerland, and Satoru Fujishige in Japan also have made invaluable contributions by providing me with mathematical shelter several times.

Special thanks are due to Vera Sós, who not only gave the first course in graph theory in which I participated but who also has closely followed my path during the past forty years. Her constant support and encouragement have been indispensable.

In 1980 J. Edmonds invited me to the University of Waterloo, Canada. This first visit, as well as the numerous later occasions since then, when he shared his penetrating ideas with me, have been a lasting inspiration.

I have been so fortunate to meet personally a great number of major players of the field whose results have become classic and are included in the book: A.V. Goldberg, J. Edmonds, S. Fujishige, T. Gallai, A.J. Hoffman, T. Ibaraki, H.W. Kuhn, L. Lovász, C.L. Lucchesi, W. Mader, H. Nagamochi, C.St.J.A. Nash-Williams, R. Rado, A. Schrijver, P. Seymour, R.E. Tarjan, W.T. Tutte, D.H. Younger, and T. Watanabe.

I feel that the collaboration with my former students Éva Tardos, András Sebő, Tibor Jordán, Zoltán Szigeti, László Szegő, Tamás Király, Alpár Jüttner, Gyula Pap, Jácint Szabó, and László Végh has been particularly enjoyable and fruitful. The several discussions I made with Tamás Fleiner and Zoltán Király were also very helpful.

Several people helped me in correcting the book, and their help is heartily acknowledged. In particular, much energy was invested to check the mathematical content of the book by Kristóf Bérczi, Attila Bernáth, Balázs Fleiner, Tamás Király, Erika Kovács, Péter Kovács, Zoltán Miklós, Gyula Pap, Júlia Pap, Jácint Szabó, Zoltán Szigeti, Tibor Jordán, and László Végh. I am also grateful to David Kratz, who helped to polish the English of the text. I also thank my son Péter for giving me Kőnig's classic book as a birthday present.

Two close and longtime friends, György Elekes and Péter Kas, asked me often about the book. I strongly wanted to show them the final version but this is no longer possible.

The single most important influence in my entire mathematical career has been provided by László Lovász. I first heard many of the basic results and techniques of this book from him. His ideas and the spirit of his results have been extremely motivating, and their impact to my mathematical taste and orientation cannot be overestimated. His constant interest in my work has been a major driving force.

During the last two decades, my wife Daisy (whom I happen to have known from the very day of her birth) not only provided an intimately warm, loving, and cheerful family background, but her never-fading support, patience, and persuasion were also the most indispensable factors during the entire period of carrying out this project.

The book is dedicated to the memory of my dear parents, Márta Kemény and László Frank.

Contents

978–0–19–920527–1 00-Andras-Frank-Prelims-drv Andras Frank (Typeset by SPi, Publisher Services, India) xix of 640 December 11, 2010 15:14

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

PART III SEMIMODULAR OPTIMIZATION

Notation

Set-functions Let b and p be set-functions on a ground-set S. Let $m : S \to \mathbf{R}$ be a function on S.

$\overline{b}$, the complement of b, defined by $\overline{b}(X) := b(S) - b(S - X)$.

$\dot{b} : S \to \mathbf{R}$, defined by $\dot{b}(v) := b(v)$ for $v \in S$.

$\ddot{b} : 2^S \to \mathbf{R}$, a modular set-function, defined by $\ddot{b}(X) := \sum[b(v) : v \in X]$.

b_{imin}, inner minimization of b, defined by $b_{imin}(Z) := \min\{b(X) : X \subseteq Z\}$.

b_{omin}, outer minimization of b defined by $b_{omin}(Z) := \min\{b(X) : X \supseteq Z\}$.

$b^\vee$, lower truncation of b, defined by $b^\vee(Z) := \min\{\sum_i^t b(X_i) : \{X_1, \dots, X_t\}$ a partition of Z $(t \geq 1)\}$.

$p^\wedge$, upper truncation of p, defined by $p^\wedge(Z) := \max\{\sum_i^t p(X_i) : \{X_1, \dots, X_t\}$ a partition of Z, $(t \geq 1)\}$.

$\widetilde{m} : 2^S \to \mathbf{R}$, a modular set-function defined by $\widetilde{m}(X) = \sum[m(v) : v \in X]$.

$b \bigtriangledown m$, lower convolution of b and m, defined by $(b \bigtriangledown m)(Z) := \min\{b(X) + \widetilde{m}(Z - X) : X \subseteq Z\}$.

Functions on undirected graphs Let $G = (V, E)$ be an undirected graph, X a subset of nodes, and $x : E \to \mathbf{R} \cup \{\pm\infty\}$ a function.

$d(X) = d_G(X) = d_E(X)$, the number of edges connecting X and $V - X$.

$d_x(X) := \sum[x(e) : e \in \Delta_E(X)]$.

$i_G(X) = i_E(X)$, the number of edges induced by X.

$e_G(X)$, the number of edges with at least one end-node in X.

$d(X, Y) = d_G(X, Y) = d_E(X, Y)$, the number of edges connecting $X - Y$ and $Y - X$.

$d(u, v) = d_G(u, v)$, the number of parallel uv-edges.

$\overline{d}(X, Y) = \overline{d}_G(X, Y) = \overline{d}_E(X, Y)$, the number of edges connecting $X \cap Y$ and $V - (X \cup Y)$.

$c_G(X)$, the number of components of the subgraph induced by X.

$\sigma_G(X)$, the number of components of $G - X$ when $X \neq \emptyset$ and $\sigma_G(\emptyset) = 0$.

$\lambda_G(X, Y)$, the maximum number of edge-disjoint paths from X to Y.

$\lambda_G(u, v)$, maximum number of edge-disjoint uv-paths

$\kappa_G(u, v)$, maximum number of openly disjoint uv-paths

Functions on directed graphs Let $D = (V, A)$ be a directed graph, X a subset of nodes, and $x : A \to \mathbf{R} \cup \{\pm\infty\}$ a function.

$\varrho_D(X) = \varrho_A(X)$, the number of arcs entering X.

$\delta_D(X) = \delta_A(X)$, the number of arcs leaving X.

$\varrho_x(X) := \sum \big[x(e) : \ e \in \Delta^-_D(X)\big]$.

$\delta_x(X) := \sum \big[x(e) : \ e \in \Delta^+_D(X)\big]$.

$\Psi_x(X) := \varrho_x(X) - \delta_x(X)$.

$i_D(X) = i_A(X)$, the number of arcs induced by X.

$c_D(X)$, the number of components of the subgraph induced by X in the underlying undirected graph.

$\sigma_D(X)$, the number of components of $D - X$ when $X \neq \emptyset$ and $\sigma_D(\emptyset) = 0$.

$\sigma_D(X) := c_D(V - X)$ when $X \neq \emptyset$, and $\sigma_D(\emptyset) := 0$,

$d_D(X, Y)$, the number of edges connecting $X - Y$ and $Y - X$ in the underlying undirected graph.

$\lambda_D(X, Y)$, maximum number of edge-disjoint paths from X to Y.

$\lambda_D(u, v)$, maximum number of edge-disjoint uv-paths.

$\kappa_D(u, v)$, maximum number of openly disjoint uv-paths.

Functions on hypergraphs and dypergraphs Let $H = (V, \mathcal{E})$ be hypergraph or a dypergraph, and X a subset of nodes.

$\underline{\chi}_{\mathcal{E}}(Z)$, indicator function of $\mathcal{E}$, the number of copies of Z occurring in $\mathcal{E}$.

$\underline{\chi}_Z(v)$, characteristic function of Z, 1 or 0 depending on v is in Z or not.

$d(X) = d_H(X)$, the number of hyperedges intersecting both X and $V - X$.

$i_H(X)$, the number of hyperedges or dyperedges induced (= included) by X.

$e_H := i_H + d_H$.

$p_H(X)$, the number of hyperedges or dyperedges disjoint from X, (i.e., $p_h(X) = i_H(V - X)$).

$d_H(X, Y)$, the number of hyperedges $Z \in \mathcal{E}$ for which $Z \subseteq X \cup Y$, $Z \cap (X - Y) \neq \emptyset$, and $Z \cap (Y - X) \neq \emptyset$.

$d^*_H(X, Y)$, the number of hyperedges $Z \in \mathcal{E}$ for which $Z \cap X \cap Y = \emptyset$, $Z \cap (X - Y) \neq \emptyset$, and $Z \cap (Y - X) \neq \emptyset$.

$\varrho_H(X)$, the number of dyperedges entering X.

$\delta_H(X)$, the number of dyperedges leaving X.

Bi-set functions Let $D = (V, A)$ be a digraph, $X = (X_O, X_I)$ a bi-set, and $x : A \to \mathbf{R} \cup \{\pm\infty\}$ a function.

$i_D(X)$: the number of edges induced by X, that is, the number of edges with head in X_I and tail in X_O.

$\varrho_D(X)$: the number of arcs entering X, that is, the number of edges with head in X_I and tail in $V - X_O$.

$\delta_D(X)$: the number of arcs leaving X.

$\varrho_x(X) := \sum[x(a) : \text{a enters } X]$.

$\delta_x(X) := \sum[x(a) : \text{a leaves } X]$.

$W(X) := X_O - X_I$, the wall of a bi-set X.

$w(X) := |W(X)|$, the wall-size of X.

$\overline{X} := (V - X_I, V - X_O)$, the complement of a bi-set $X = (X_O, X_I)$.

$X \sqcap Y := (X_O \cap Y_O, X_I \cap Y_I)$, the intersection of two bi-sets.

$X \sqcup Y := (X_O \cup Y_O, X_I \cup Y_I)$, the union of two bi-sets.

$X \sqsubseteq Y$: if $X_O \subseteq Y_O$ and $X_I \subseteq Y_I$.

Set-to-set functions

$\Gamma_G(X) := \{u \in V - X$: there is an edge $uv \in E$ for which $v \in X\}$.

$\Gamma^-_D(X) := \{u \in V - X$: there is an arc $uv \in A$ for which $v \in X\}$.

$V(F)$, set of nodes covered by the edge-set F.

$E_G(X, Y) = E(X, Y)$, set of edges connecting $X - Y$ and $Y - X$.

$E_G(u, v) = E(u, v)$, set of parallel edges connecting u and v.

$E_G(X) = E(X)$, set of edges with at least one end-node in X.

$I_G(X)$, set of edges induced by X.

$\Delta_G(X) = \Delta_E(X)$, a cut of G: set of edges connecting X and $V - X$.

$\Delta_H(X) = \Delta_{\mathcal{E}}(X)$, a cut of H: set of hyperedges intersecting X and $V - X$.

$\Delta^+_D(X)$, out-cut, set of arcs leaving X and entering $V - X$.

$\Delta^-_D(X)$, in-cut, set of arcs entering X and leaving $V - X$.

$\text{cl}(X)$, closure of a subset $X \subseteq S$ in a matroid.

Graph, hypergraph, and digraph functions

$\alpha(G)$, stability number.

$\omega(G)$, clique number.

$\chi(H)$, chromatic number.

$\chi'(H)$, chromatic index.

$\nu(H)$, matching number.

$\tau(H)$, transversal number.

$\Delta(H)$, maximum degree.

$\kappa(G)$ and $\kappa(D)$, node-connectivity.

$\lambda(G)$ and $\lambda(D)$, edge-connectivity.

Operations

$X \ominus Y$, symmetric difference of sets.

$\lceil x \rceil$, upper integer part of a number x.

$\lfloor x \rfloor$, lower integer part of a number x.

$\lceil f \rceil$, upper integer part of a vector or a function f.

$\lfloor f \rfloor$, lower integer part of a vector or a function f.

$(f)^+$, non-negative part of a number or function f.

$M_1 \circ M_2$, composition of matroids M_1 and M_2.

$M_1 + M_2$, sum of matroids M_1 and M_2.

$A - B := \{z : \ z \in A, z \notin B\}$.

$X + s := X \cup \{s\}$.

$X - s := X - \{s\}$.

$H - e$, deleting an edge e of a graph or digraph H.

H/e, contracting an edge e.

$H - v$, deleting a node v of a graph or digraph H.

$H - F$, deleting a subset F of edges of a graph or a digraph H.

H/F, contracting a subset f of edges.

$H - Z$, deleting a subset Z of nodes of a graph or digraph H.

H/Z, shrinking a subset Z of nodes of a graph or digraph H.

978–0–19–920527–1 01-Andras-Frank-c01-drv Andras Frank (Typeset by SPi, Publisher Services, India) 1 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Part I
Basic Combinatorial Optimization

1

Elements of graphs and hypergraphs

This chapter is a summary of fundamental concepts, terminology, notation, and basic properties used throughout the book, and it is provided primarily for reference. The reader may find it helpful to read through this chapter quickly at first, returning to the appropriate sections only when a systematic study of a later topic requires the exact concepts and notation.

1.1 Fundamentals

We start with an overview of some central notions of the book: sets, bi-sets, graphs, hypergraphs, partially ordered sets, and reachability.

1.1.1 Sets and functions

Sets and multi-sets

A set X is a **subset** of Y if every element of X belongs to Y. This is denoted by $X \subseteq Y$. Throughout the book, we will say that a set Y **includes** a set X if $X \subseteq Y$ and that Y **contains** an element v if $v \in Y$. If, in addition, $X \neq Y$, then Y is said to **include properly** X or that X is a **proper subset** of Y, which is denoted by $X \subset Y$. When $X \subseteq Y$, we also say that Y is a **superset** of X. For two distinct elements s and t of a finite ground-set V, we say that a subset X of V is an $s\bar{t}$**-set** if $s \in X \subseteq V - t$. For the difference of two sets X and Y, we use the notation $X - Y$, that is, $X - Y := \{v : v \in X, v \notin Y\}$. The **symmetric difference** of two sets is defined by $X \ominus Y := (X - Y) \cup (Y - X)$. The **complement** of a subset $X \subseteq V$ is defined by $V - X$. A subset X of V and its complement $\overline{X}$ form a **bipartition** of V.

Let V be a ground-set. For a subset $Z \subseteq V$, the **characteristic** or **indicator function** (or vector) $\underline{\chi}_Z : V \to \{0, 1\}$ of Z is defined by

$$\underline{\chi}_Z(v) := \begin{cases} 1 & \text{if } v \in Z \\ 0 & \text{if } v \in V - Z. \end{cases} \tag{1.1}$$

By a **multi-set** Z, we mean a collection of elements of V where an element of V may occur in more than one copy. The indicator function $\underline{\chi}_Z : V \to \{0, 1, 2, \dots\}$ of Z tells that $\underline{\chi}_Z(v)$ copies of an element v of V occurs in Z. By the **multi-union** of multi-sets $Z_1, \dots, Z_k$, we mean a multi-set for which the indicator function is $\underline{\chi}_{Z_1} + \cdots + \underline{\chi}_{Z_k}$.

A set of cardinality 1 (that is, a set consisting of a single element) is called a **singleton**. Although formally not precise, in most cases we will not distinguish a singleton from its only element: this practice causes no ambiguity. In the few cases when confusion may arise, we will make the distinction. (For example, the degree of a node v of a graph is not the same as the degree of the singleton $\{v\}$, since the loops sitting at v are taken into consideration only in the first case.) The union of a set X and a singleton $\{s\}$ will be denoted by $X + s$, while the difference $X - \{s\}$ will be abbreviated to $X - s$. A set of two elements is sometimes called a **doubleton**.

If X is a set and v is an element of X, then we say that v **covers** X and that X **covers** v. Two sets are **disjoint** if they have no element in common. Sets $X_1, X_2, \dots, X_t$ are called disjoint if they are pairwise disjoint. Two subsets of a ground-set are called **co-disjoint** if their complements are disjoint. Two subsets X and Y of V are called **intersecting** if $X \cap Y$ is non-empty. They are **properly intersecting** if none of the subsets $X \cap Y$, $X - Y$, or $Y - X$ is empty. If, in addition, $V - (X \cup Y)$ is also non-empty, we speak of **crossing** subsets. The subsets X and Y are **comparable** if $X \subseteq Y$ or $Y \subseteq X$, and they are non-comparable or **unrelated** otherwise. A set X is said to **separate** another set Y (or Y is separated by X) if neither $X \cap Y$ nor $Y - X$ is empty. In particular, two elements s and t are **separated** by a subset X if $|X \cap \{s, t\}| = 1$.

For a given property P, a subset X of V is called **maximal** if X admits this property but no proper superset of X does. X is called **maximum** with respect to P if its cardinality is the largest one among the subsets of V having this property. The distinction between the terms minimal and minimum is defined analogously.

Set systems and families

The power set 2^V of V consists of all subsets of V. A subset of 2^V, that is, a set of subsets of V, is called a **set-system** of V or a **system of subsets** of V. By this definition, a set-system consists of distinct sets. A set-system $\mathcal{P}$ is a **partition** of V if every element of V belongs to exactly one member of $\mathcal{P}$. A partition of a subset of V is called a **subpartition** of V. That is, a subpartition is a set of disjoint sets. A set-system consisting of pairwise co-disjoint sets is called a **co-subpartition** of V. A partition having t members is said to be t-**partite**, or just called a t-**partition**. A partition is **proper** if its members are non-empty. A partition is **trivial** if it consists of the single class $\{V\}$. The other extreme, when each member of $\mathcal{P}$ is a singleton, is called the **point-partition** of V. A t-partition $\mathcal{P}$ of V is sometimes called a t-**colouring** of V the members of which are the **colour-classes**. A **Sperner family** is a set-system in which no member includes another member.

A **family** (sometimes **collection**) $\mathcal{F}$ of subsets of V is a multi-set of subsets of V, that is, $\mathcal{F}$ consists of subsets of V where a subset may occur in more than one copy. The **indicator function** $\underline{\chi}_{\mathcal{F}} : 2^V \to \mathbf{Z}_+$ of $\mathcal{F}$ is a set-function where $\underline{\chi}_{\mathcal{F}}(X)$ denotes the number of copies of $X \subseteq V$ occurring in $\mathcal{F}$. One may also think of $\mathcal{F}$ as a list of subsets of V where the actual ordering of the subsets is irrelevant. By the **multi-union** of families (and, in particular, set-systems) $\mathcal{F}_i$ on a common ground-set V $(i = 1, \dots, k)$, we mean a family $\mathcal{F}$ of subsets of V for which the indicator function is $\underline{\chi}_{\mathcal{F}_1} + \cdots + \underline{\chi}_{\mathcal{F}_k}$.

Let $\mathcal{F}$ be a family of subsets of V. Call two elements $u, v \in V$ **separated by** $\mathcal{F}$ if there is a set $Z \in \mathcal{F}$ separating u and v. If no such set exists, the two elements are non-separated or equivalent. This is clearly an equivalence relation. Each equivalence class is called an **atom**

978–0–19–920527–1 01-Andras-Frank-c01-drv Andras Frank (Typeset by SPi, Publisher Services, India) 5 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

of $\mathcal{F}$, and also an **atom of the hypergraph** $(V, \mathcal{F})$. In other words, an atom defined by $\mathcal{F}$ is a maximal subset Z of V that is not separated by any member of $\mathcal{F}$. The atoms of a family of sets form a partition of V.

Partially ordered sets

A **partially ordered set** or, for short, a **poset** is a pair $P = (S, \preceq)$ consisting of a ground-set S and a reflexive, antisymmetric, and transitive binary relation $\preceq$ on S. That is, $s \preceq s$ for every $s \in S$; at most one of $x \preceq y$ and $y \preceq x$ holds for every pair of distinct $x, y \in S$; and $x \preceq y$ and $y \preceq z$ imply $x \preceq z$. The notation $x \prec y$ means that $x \preceq y$ and $x \neq y$. Two distinct elements x and y of S are called **comparable** if $x \prec y$ or $y \prec x$. Otherwise they are **incomparable**. A poset is a **chain** (also a **totally** or **linearly ordered set**) if its elements are pairwise comparable. When the elements are pairwise incomparable we speak of an **antichain**.

A **subposet** of P consists of a subset S' of S with the restriction of the relation $\preceq$ to the pairs of elements of S'. If this is a chain (antichain), then we say that S' is a **chain** (**antichain**) **of** P. The **height** of a poset is the maximum cardinality of a chain while the **width** is the maximum cardinality of an antichain of P.

A subset X of S in a poset P is called a **lower ideal** or just an **ideal** if $y \preceq x \in X$ implies that $y \in X$. A subset $X \subseteq S$ is an **upper ideal** if $x \preceq y$ and $x \in X$ imply $y \in X$. The set-system of ideals are obviously closed under taking union and intersection.

Bi-sets

Given a ground-set V, by a **bi-set** on V we mean a pair $X = (X_O, X_I)$ of subsets of V for which $X_I \subseteq X_O \subseteq V$. Here X_O is the **outer** member and X_I is the **inner** member of X. The notion of bi-sets is not standard but its use will prove particularly convenient in several places. The reader will find it useful to become familiar with bi-sets at this stage. We call a bi-set $X = (X_O, X_I)$ **simple** if $X_O = X_I$ and will tacitly identify a simple bi-set with the set X_I. A bi-set $X = (X_O, X_I)$ is **trivial** if $X_I = \emptyset$ or $X_O = V$. When $X_I = \emptyset$, we say that X is **void**. For a bi-set X, define the **wall** $W(X)$ of X by the subset $W(X) := X_O - X_I$ of V and let the **wall-size** $w(X)$ of X be defined by

$$w(X) := |W(X)|.$$

The **complement** $\overline{X}$ of a bi-set $X = (X_O, X_I)$ is defined by $\overline{X} = (V - X_I, V - X_O)$. For simple bi-sets, $w(X) = 0$. For two elements s and t of the ground-set, we say that $X = (X_O, X_I)$ is a $t\bar{s}$**-bi-set** if $t \in X_I$ and $s \notin X_O$. A bi-set X **separates** s and t if $|X_I \cap \{s, t\}| = 1 = |(V - X_O) \cap \{s, t\}|$. A bi-set $X = (X_O, X_I)$ is called a $t\bar{s}$**-bi-set** if $t \in X_I$ and $s \in V - X_O$.

The set of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. A multi-set $\mathcal{F}$ of bi-sets is often called a family of bi-sets, that is, $\mathcal{F}$ is a collection of bi-sets where a bi-set may occur in more than one copy. The **indicator function** $\underline{\chi}_{\mathcal{F}} : \mathcal{P}_2 \to \mathbf{Z}_+$ of a family $\mathcal{F}$ of bi-sets and **the multi-union** of families of bi-sets are defined analogously to those concerning families of subsets.

The intersection $\sqcap$ and the union $\sqcup$ of bi-sets is defined in a straightforward manner: for $X, Y \in \mathcal{P}_2$ let $X \sqcap Y := (X_O \cap Y_O, X_I \cap Y_I)$, $X \sqcup Y := (X_O \cup Y_O, X_I \cup Y_I)$. We write $X \sqsubseteq Y$ if $X_O \subseteq Y_O$, $X_I \subseteq Y_I$ and this relation is a partial order on $\mathcal{P}_2$. Accordingly, when

$X \sqsubseteq Y$ or $Y \sqsubseteq X$, we call X and Y **comparable**. A family of pairwise comparable bi-sets is called a **chain**. Two bi-sets X and Y are **inner disjoint** if $X_I \cap Y_I = \emptyset$ and **independent** if $X_I \cap Y_I = \emptyset$ or $V = X_O \cup Y_O$. A set of bi-sets is independent if its members are pairwise independent. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$ and **properly intersecting** if, in addition, they are not comparable. Note that $X_O \cup Y_O = V$ is allowed for two intersecting bi-sets. Two properly intersecting bi-sets X and Y are called **crossing** if $X_O \cup Y_O \subset V$.

Functions

For a real number x, let $x^+ := \max\{0, x\}$. For a function $f : V \to \mathbf{R}$, let the non-negative part $f^+ : V \to \mathbf{R}_+$ of f be defined by

$$f^+(v) := (f(v))^+ \text{ for } v \in V.$$

For a vector $z \in \mathbf{R}^n$, let $z^+ \in \mathbf{R}^n$ be defined by $z(i) = (z(i))^+$ for $i = 1, \ldots, n$. For a real x, let $\lfloor x \rfloor$ denote the largest integer z for which $z \leq x$ and let $\lceil x \rceil$ denote the smallest integer z with $z \geq x$. A function on the edge-set or on the node-set of a graph or digraph often represents an (upper or lower) bound in which case we use the intuitive name (upper or lower) **capacity function**. Typically, f is used for lower bound and g is used for upper bound. Also, a function $c \in \mathbf{R}^n$ determines a linear function on $\mathbf{R}^n$ defined by the inner product cz ($z \in \mathbf{R}^n$). Typically, when we are to minimize cx, we speak of a **cost function**, while if maximization is the goal, the intuitive name **weight function** is used.

By a **set-function** on ground-set V, we mean a function $b : 2^V \to \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$. For example, the indicator function of a family of subsets of V is a non-negative, integer-valued set-function. For a singleton $\{v\}$, instead of the precise notation $b(\{v\})$, we use $b(v)$. A set-function b is **subcardinal** if $b(X) \leq |X|$ for every $X \subseteq S$, and **non-decreasing** if $b(X) \leq b(Y)$ whenever $X \subseteq Y$ and $b(X)$ is finite.

Convention on $b(\emptyset) = 0$. We always assume that the value of a set-function is 0 for the empty set. In particular, if a concrete definition of a set-function b gives rise to a non-zero value for $b(\emptyset)$, we automatically define $b(\emptyset) = 0$. For example, for a real k, we will say that b is identically k, using the notation $b :\equiv k$, without explicitly mentioning the following formal definition of b:

$$b(X) := \begin{cases} k & \text{if } \emptyset \subset X \subseteq V \\ 0 & \text{if } X = \emptyset. \end{cases}$$

By a **bi-set-function**, we mean a function defined for the set of bi-sets of V. Since a simple bi-set (X, X) may be identified with the set X, bi-set functions may be considered as straight generalizations of set-functions. We will assume for bi-set functions that their value on trivial bi-sets is 0.

A set-function $b : 2^V \to \mathbf{R} \cup \{\infty\}$ is said to satisfy the **submodular inequality** on $X, Y \subseteq V$ if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \tag{1.2}$$

By convention, we consider the inequality automatically satisfied if at least one of the two terms on the left-hand side is ∞, even if both terms on the right-hand side are ∞. Similarly, the submodular inequality for a bi-set-function $b : \mathcal{P}_2(V) \to \mathbf{R} \cup \{\infty\}$ on bi-sets $X, Y \in \mathcal{P}_2(V)$ is as follows:

$$b(X) + b(Y) \geq b(X \sqcap Y) + b(X \sqcup Y).$$

A set-function $p : 2^V \to \mathbf{R} \cup \{-\infty\}$ is said to satisfy the **supermodular inequality** on $X, Y \subseteq V$ if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{1.3}$$

For a bi-set-function $p : \mathcal{P}_2(V) \to \mathbf{R} \cup \{-\infty\}$, the supermodular inequality is analogous:

$$p(X) + p(Y) \leq p(X \sqcap Y) + p(X \sqcup Y).$$

By convention, we consider the inequality satisfied if at least one of the two terms of the left-hand side is $-\infty$. A function p is said to be **fully supermodular** or just supermodular if it satisfies the supermodular inequality for every pair of sets X and Y. If (1.3) holds for intersecting (crossing) pairs, we speak of **intersecting** (**crossing**) **supermodular** functions. Sometimes (1.3) is required only for pairs with $p(X) > 0$ and $p(Y) > 0$, in which case we speak of **positively supermodular** functions. **Positively intersecting** or **crossing** supermodular functions are defined analogously. A typical way of constructing a positively intersecting supermodular function is to replace each negative value of an intersecting supermodular function by 0. The analogous notions of **fully**, **intersecting**, or **crossing** submodular functions are introduced in a similar way, with the difference that the submodular inequality is required for all pairs, for intersecting pairs, or for crossing pairs of sets X and Y. Typically, the term submodular function will refer to full submodularity, and the adjective *full* is used only when the distinction from weaker submodularity is to be emphasized.

A set-function b on V is said to be **modular** if

$$b(X) + b(Y) = b(X \cup Y) + b(X \cap Y) \text{ for every } X, Y \subseteq V.$$

If b is a finite-valued modular function with $b(\emptyset) = 0$, then $b(X) = \sum[b(v) : v \in X]$, that is, the function is determined by its values on singletons. A bi-set-function b is said to be **modular** if

$$b(X) + b(Y) = b(X \sqcup Y) + b(X \sqcap Y) \text{ for every pair of bi-sets } X, Y.$$

Proposition 1.1.1 *The wall-size of bi-sets determine a modular function, that is,*

$$w(X) + w(Y) = w(X \sqcap Y) + w(X \sqcup Y) \textit{ for every pair of bi-sets } X \textit{ and } Y. \bullet$$

Important: three unusual notational conventions!

We introduce three non-conventional, perhaps strange, notational conventions that will be used throughout this book.

Dot. Given a set-function b on V, let $\dot{b} : V \to \mathbf{R}$ be defined by

$$\dot{b}(v) := b(v) \ (= b(\{v\})) \quad \text{for } v \in V.$$

Obviously, this convention has nothing to do with derivation. The dot notation simply refers to the fact that a function on the elements of the ground-set V is created from a set-function on V. It will not be used when the variable is specified; we shall write throughout $b(v)$

instead of $\dot{b}(v)$. However this notation is quite useful when we have to refer to the function $\dot{b}$ itself in order to distinguish it from the set-function b.

Double dot. For the set-function b on V, let $\ddot{b} : 2^V \to \mathbf{R}$ be the modular set-function defined by

$$\ddot{b}(X) := \sum[b(v) : v \in X].$$

Tilde. For a function $m : V \to \mathbf{R}$ (or vector $m \in \mathbf{R}^V$), we define a modular set-function $\widetilde{m}$ by

$$\widetilde{m}(X) = \sum[m(v) : v \in X].$$

In the same spirit, a set-function b on V can be extended to the families $\mathcal{F}$ of subsets of V by

$$\widetilde{b}(\mathcal{F}) := \sum[b(X) : X \in \mathcal{F}].$$

When the $\sim$ (tilde) notation is applied to a function with a subscript or a superscript, like m_1 or m', we use the slightly sloppy notation $\widetilde{m}_1$ for $\widetilde{m_1}$ and $\widetilde{m}'$ for $\widetilde{m'}$.

The **modulus** of a vector m is the sum of its components, that is, $\widetilde{m}(V)$. The vector $(1, 1, \ldots, 1)$ in $\mathbf{R}^V$ is denoted by $\underline{1}$ (without referring to V or to the dimension). Therefore $\widetilde{m}(V)$ is the inner product $m\underline{1} = \underline{1}m$.

A useful lemma

The following simple observation will be used several times for proving the finiteness of what is called the uncrossing procedure, a fundamental proof technique in semimodular optimization.

Lemma 1.1.2 *Let $r_1, \ldots, r_n$ be a sequence of non-negative rational numbers. As long as possible, apply the following 4-change step. Select four distinct members for which the two middle ones are positive. Let α denote the minimum of the two middle elements. Decrease by α the value of the two middle elements and increase by α the value of the first and fourth ones. Then, after a finite number of 4-change steps, the procedure terminates.*

Proof. By multiplying through with the least common denominator, if necessary, we can assume that the sequence consists of integers. Observe that the first member of the sequence never decreases, each member remains non-negative, and the total sum remains constant. Therefore, the first member becomes fixed after a finite number of 4-change steps, and the lemma follows by induction on n. •

Problem 1.1.1 *Show that the procedure described in the lemma may not terminate in a finite number of steps if the members of the initial sequence are non-negative irrational numbers.*

1.1.2 Undirected graphs

A **simple undirected graph** G (or, for short, a **simple graph**) is a pair (V, E) where V is the set of nodes and E is the set of edges of G. Each edge e is a two-element subset $\{u, v\}$ of V, in other words, an unordered pair where u and v are called the **end-nodes** of the edge. We often use the abbreviation $e = uv$ and say that e is a uv**-edge**. Note that uv and vu denote the same edge. We say that an edge **connects** or **joins** its end-nodes. Also, an edge e is **incident to** a node v if v is an end-node of e. Two nodes u and v of a graph G are **neighbours** or said to be **adjacent** if there is a uv-edge in G. The set of neighbours of a node v is denoted by $\Gamma_G(v) = \Gamma(v)$.

A simple graph is called **complete** if u and v are adjacent for every $u, v \in V$. A complete graph on n nodes is denoted by K_n. The **complement** $\overline{G} = (V, \overline{E})$ of a simple graph $G = (V, E)$ is a graph in which two nodes are adjacent if and only if they are not adjacent in G. Evidently, the complement of $\overline{G}$ is G itself.

Intuitively, a uv-edge can be represented by a 'line' connecting u and v. There are situations when two nodes are joined by more than one line or when a node is joined by a line to itself. These motivate the following formal definition of a graph.

An **undirected graph** G (or, for short, a **graph**) is a triplet (V, E, φ) where V is the set of nodes, E is the set of edges of G, and φ is a function assigning an unordered pair of (not-necessarily-distinct) nodes to every edge. When the two end-nodes of an edge coincide, we say that the edge is a **loop**. We say that two or more non-loop edges are **parallel** if they have the same end-nodes. The set of parallel edges of G connecting u and v is denoted by $E_G(u, v) = E(u, v)$. An element of $E(u, v)$ is a uv**-edge**.

It is a standard (though slightly sloppy) custom to avoid this formally precise notation and use only (V, E) to denote a graph, even if it is not simple. We follow this practice throughout and think of E as a family of unordered pairs of (not-necessarily distinct) nodes, thereby allowing loops and parallel edges as well. When special care is needed to distinguish parallel edges connecting u and v, we will say that e and f are two uv-edges or that $e, f \in E(u, v)$.

An edge is said to **connect** or to **join** two disjoint subsets X and Y of V if it connects an element of X and an element of Y. An edge connecting X and $V - X$ is said to **leave** or **enter** X. For two arbitrary subsets X and Y of nodes, let $E(X, Y) = E_G(X, Y)$ denote the set of edges connecting $X - Y$ and $Y - X$ while

$$d_G(X, Y) := |E_G(X, Y)|.$$

The number of edges connecting $X \cap Y$ and $V - (X \cup Y)$ is denoted by $\bar{d}(X, Y) = \bar{d}_G(X, Y)$, that is,

$$\bar{d}_G(X, Y) = d_G(X, V - Y).$$

The **incidence matrix** Q of a graph $G = (V, E)$ is a $(0, 1)$-matrix, the rows of which correspond to the nodes of G and the columns of which correspond to the edges of G. An entry of Q corresponding to a node $v \in V$ and an edge $e \in E$ is 1 precisely if e is incident to v.

Some operations

Deleting an edge from an undirected graph simply means the removal of the edge (leaving alone its end-nodes). The operation of **multiplicating** a uv-edge e consists of replacing e by $k \geq 1$ parallel uv-edges. When $k = 2$, we use the term **duplicating** e.

Deleting a node means the removal of the node along with all edges incident to it. A **subgraph** of a graph $G = (V, E)$ is obtained by deleting some edges and nodes of G. When only a subset F of edges is deleted and the node-set is kept unchanged, we speak of a **spanning subgraph** of G and denote it by $G - F$. If H is a subgraph of G, then G is said to be a **supergraph** of H. If G is a spanning subgraph of H, then we say that H is an **augmentation of** G or that H is obtained by augmenting G.

When a subset Z of nodes is deleted, the resulting subgraph $G - Z$ on node-set $X := V - Z$ is called an **induced subgraph** of G. The number of its edges is denoted by $i(X) = i_G(X) = i_E(X)$, while the set of its edges is $I(X) = I_G(X) = I_E(X)$. A **hole** of a graph G is an induced subgraph of G which is a circuit with at least four edges. A graph is said to be **chordal** if it includes no hole.

Let Z be a non-empty subset of nodes of $G = (V, E)$. The operation of **shrinking** Z consists of replacing Z by a new node v_Z which is connected to u by $d_G(Z, u)$ parallel edges for every $u \in V - Z$. The graph obtained from G in this way is denoted by G/Z. **Contracting** a non-loop edge $e = uv$ of G means that we shrink the set $\{u, v\}$. The resulting graph is denoted by G/e. Evidently, we can contract more than one edge, and the final graph does not depend on the order of contractions. Note that if Z induces a connected graph, then the operations of shrinking Z and contracting the set of edges induced by Z are equivalent. In such situations, we sometimes say that Z is contracted.

Degrees

The number of edges connecting X and $V - X$ is denoted by $d(X) = d_G(X) = d_E(X)$ and is called the **degree** of X. The function d_G is called the **degree function** of G. For a node v, the degree $d(v)$ of v denotes the number of edges ending at v, where the loops sitting at v are counted twice. Therefore, $d(v)$ is equal to $d(\{v\})$ plus twice the number of loops at v. Since typically we do not use loops, it will not give rise to ambiguity when the degree of a node is denoted simply by $d(v)$.

A graph $G = (V, E)$ is **connected** if $d_G(X) > 0$ for every non-empty, proper subset X of V. A **connected component** or just a **component** of a graph G is a maximal induced subgraph of G the node-set of which is not separated by any set X of degree 0. That is, connected components are the atoms of the set-system $\{X \subseteq V : d_G(X) = 0\}$.

We say that an edge $e = uv$ **covers** or **enters** a bi-set $X = (X_O, X_I)$ if it enters both X_O and X_I, that is, if one of its end-nodes is in X_I while the other one is outside X_O. The number of edges of G covering X is also denoted by $d(X) = d_G(X) = d_E(X)$ and is called the **degree** of bi-set X.

A node of degree 0 is said to be **isolated**. A node v of degree 1 is called a **leaf node** while the only edge incident to v is a **leaf** or **leaf edge**. An undirected graph is said to be **regular** if the degree of every node is the same number r. In this case, we also speak of r**-regular** graphs.

A **matching** of a graph is a subgraph in which every degree is at most one while a **perfect matching** is one where every degree is exactly one. A **circuit** is a connected graph in which every degree is exactly two. A loop is considered a one-element circuit, while a graph on two nodes connected by two parallel edges is a 2-element circuit. An edge e of a graph G is said to be **cyclic** if e belongs to a circuit of G. A **path** is a simple connected graph in which every degree is at most two and there is a node with degree less than two. In particular, a graph consisting of a single node and no edge is a path. A uv-edge e can also be considered as a path, which is formally the graph $(\{u, v\}, \{e\})$. It is a simple exercise to see that each path having at least one edge has exactly two nodes of degree one, and these are called the

978–0–19–920527–1 01-Andras-Frank-c01-drv Andras Frank (Typeset by SPi, Publisher Services, India) 11 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

two **end-node**s or **ending nodes** of the path. The other nodes of the path are called **inner** nodes.

An undirected graph (connected or not) is said to be **Eulerian** or called an **Euler graph** if the degree of each node is even. For example, a circuit is Eulerian, or more generally, the union of edge-disjoint circuits is Eulerian.

By a **bipartite graph** $G = (V, E)$, we mean a graph in which the node-set can be partitioned into two subsets S and T (sometimes called **colour classes** of G) so that the two end-nodes of each edge belong to two different classes. Accordingly, a bipartite graph is typically denoted by $G = (S, T; E)$. If $E = \{st : s \in S, t \in T\}$, we speak of a **complete bipartite** graph. A complete bipartite graph with $|S| = k$, $|T| = \ell$ is denoted by $K_{k,\ell}$.

Cuts

The set $\Delta(X) = \Delta_G(X)$ of edges connecting a set $X \subseteq V$ of nodes and its complement $V - X$ is called a **cut**. X and $V - X$ are the **shore**s of the cut. A cut $\Delta(X)$ is **trivial** if X or $V - X$ is empty, otherwise the cut is non-trivial. We are going to show that each cut of a connected graph uniquely determines its shores.

Proposition 1.1.3 *Let G be a connected graph. If X and Y are subsets of nodes for which $\emptyset \neq \Delta(X) = \Delta(Y)$ and $X \cap Y \neq \emptyset$, then $X = Y$.*

Proof. If indirectly $X \neq Y$, then the symmetric difference $Z := X \ominus Y$ is a non-empty proper subset of V. Since G is connected, there is an edge leaving Z. But such an edge leaves exactly one of the two sets X and Y, contradicting the hypothesis $\Delta(X) = \Delta(Y)$. •

In a connected graph, a cut $\Delta(X)$ **separates** two nodes s and t if $|X \cap \{s, t\}| = 1$. By Proposition 1.1.3, this definition makes sense, since a cut determines its shores. A cut is called a **star-cut** if one of its shores is a singleton. The single edge of a one-element cut is called a **cut-edge** or **isthmus**. The deletion of any edge can increase the number of components by at most one, and an edge is a cut-edge if and only if its deletion increases the number of components of G by exactly one. A **forest** is an undirected graph in which all edges are cut-edges. A connected forest is called a **tree**. In other words, a tree is a minimally connected graph. By a **spanning tree** of a graph G, we mean a spanning subgraph of G that is a tree.

A non-empty cut is **minimal** or **elementary** if it does not include properly a non-empty cut. An elementary cut is sometimes called a **bond**.

Proposition 1.1.4 *A non-trivial cut $B = \Delta(X)$ of a connected undirected graph $G = (V, E)$ is minimal if and only if both of its shores induce a connected graph.*

Proof. Suppose first that $\Delta(X)$ is minimal. Suppose indirectly that the subgraph induced by X has components $X_1, \ldots, X_t$, where $t \geq 2$. Then B partitions into the cuts $\Delta(X_i)$, showing that B is not a minimal cut. Conversely, suppose that both X and $V - X$ induce a connected graph. Then a proper subset B' of B cannot disconnect the graph, since for an edge $e \in B - B'$ the subgraph $I_G(X) \cup I_G(V - X) + e$ of G is connected and spanning. Therefore, B cannot include a cut, so B is minimal. •

Further notions and parameters of graphs

For a graph $G = (V, E)$, a subset $X \subseteq V$ is said to be **stable** if no two elements of X are adjacent. The maximum cardinality $\alpha(G)$ of a stable set of G is called the **stability number** of G. A subset $X \subseteq V$ inducing a complete subgraph is called a **clique**. The maximum cardinality $\omega(G)$ of a clique of G is called the **clique number** of G. Note that a stable subset of G is a clique of the complement $\overline{G}$ of G, and hence $\alpha(G) = \omega(\overline{G})$.

The **chromatic number** $\chi(G)$ of a graph $G = (V, E)$ is the minimum number of classes in a partition of the node-set V into stable sets. A proper partition of V into stable sets is sometimes called a **proper colouring** of G. The **chromatic index** $\chi'(G)$ of a graph G is the minimum number of classes in a partition of the edge-set of G into matchings.

1.1.3 Directed and mixed graphs

A **directed edge** $e = uv$ is an ordered pair of nodes where v is the **head-node** or just the **head**, while u is the **tail-node** or **tail** of e. We also say that e is a directed uv**-edge** or that e is a directed edge from u to v. For two disjoint subsets S and T of nodes, a directed edge st is called an (S, T)**-edge** or simply an ST**-edge** if $s \in S$ and $t \in T$. If no confusion arises, we use the term edge of a directed graph to mean a directed edge. That is, an edge may refer to a directed edge or to an undirected edge, depending on the context. Sometimes we also call a directed edge an **arc**. A uv-arc is said to **enter** v and to **leave** u. If the head and the tail coincide, we speak of a directed loop, or just a loop. Two non-loop edges are **parallel** if their heads are the same and their tails are the same. A uv-edge and a vu-edge are said to be **opposite**.

Orienting an undirected uv-edge means that we replace e by one of the two possible directed edges uv or vu. **Deorienting** a directed edge means the inverse operation. A **directed graph** or **digraph** D is created from an undirected graph by orienting each edge of G, where G is called the **underlying** undirected graph of D. That is, in a digraph $D = (V, A)$, the elements of A are directed edges. A digraph without loops and parallel edges is said to be **simple**. Note that we allow opposite edges in a simple digraph. **Reorienting** (or **reversing**) a directed uv-edge means its replacement by a vu-edge. One gets a **reorientation** of a digraph by reorienting some of its edges. The **orientation of an undirected graph** G is a digraph arising from G by orienting all of its edges. Therefore, the underlying graph of a digraph D is obtained by deorienting all edges of D. A **tournament** is an oriented complete graph, as distinguished from a **complete digraph** in which uv is an edge for every ordered pair $\{u, v\}$ of distinct nodes. Therefore, a tournament on n nodes has $n(n-1)/2$ edges, while a complete digraph has $n(n-1)$.

A **mixed graph** $H = (V, A + F)$ can have both directed and undirected edges. Here A denotes the set of directed edges, while F is the set of undirected edges. A mixed graph is called a **partial orientation** of G if it is obtained from G by orienting some of the edges of G. By an **orientation of a mixed graph** M, we mean the digraph arising from M by orienting all the undirected edges (without changing the directed ones). The **underlying undirected** graph of M arises from M by deorienting each directed edge.

The **incidence matrix** Q of a directed graph $D = (V, A)$ is a $(0, \pm 1)$-matrix, the rows of which correspond to the nodes of G and the columns of which correspond to the edges of G. An entry of Q corresponding to a node $v \in V$ and an edge $e \in A$ is $+1$ if e enters v, -1 if e leaves v, and 0 otherwise.

A directed edge e is said to **enter** a subset Z of nodes if its head is in Z and its tail is in $V - Z$. An edge **leaves** Z if it enters $V - Z$. In a digraph $D = (V, A)$, for a subset $X \subseteq V$, the **in-degree** $\varrho(X) = \varrho_D(X) = \varrho_A(X)$ of X denotes the number of edges entering X. The **out-degree** $\delta(X) = \delta_D(X) = \delta_A(X)$ of X is the number of edges leaving X. The functions ϱ_D and δ_D are called the **in-degree** and the **out-degree functions** of D.

Let

$$\Psi_D := \varrho_D - \delta_D.$$

The set of edges entering X is called an **in-cut** (determined by X) and is denoted by $\Delta^-(X) = \Delta^-_D(X)$ while $\Delta^+(X) = \Delta^+_D(X)$ denotes the set of edges leaving X and is called an **out-cut** (determined by X). The **entrance** $\Gamma^-(X)$ of a subset $X \subseteq V$ of nodes is defined by

$$\Gamma^-(X) := \{v \in X\text{: there is an edge } uv \in A \text{ for which } u \in V - X\}).$$

For a node v, the in-degree $\varrho(v)$ denotes the number of edges with head v. Therefore, $\varrho(v)$ is equal to $\varrho(\{v\})$ plus the number of directed loops at v. Similarly, for a node v, the **out-degree** $\delta(v)$ denotes the number of edges with tail at v. For a digraph D, we also use the functions $d_D(X, Y) := d_G(X, Y)$ and $\bar{d}_D(X, Y) := \bar{d}_G(X, Y)$, where G denotes the underlying undirected graph of D.

We say that a directed edge e **enters** or **covers** a bi-set $X = (X_O, X_I)$ if it enters both X_O and X_I. The **in-degree** $\varrho_D(X)$ of a bi-set X is the number of edges of D entering X. A uv-edge e is **induced** by X if $u \in X_O$ and $v \in X_I$. The set of edges induced by X is denoted by $I_D(X) = I(X)$. Note that for a simple bi-set $X = (X_O, X_O)$, $I(X)$ is just the set of edges induced by the subset X_O.

Notions like deleting edges or nodes, shrinking a subset, and augmenting a digraph, can be defined for digraphs analogously to the way in which we did it for undirected graphs. Similarly, we may also speak of spanning or induced subgraphs of a directed or mixed graph.

We shall call a digraph **weakly connected** (or just connected) if its underlying undirected graph is connected. If no edge of D leaves a subset $\emptyset \subseteq X \subseteq V$, that is, $\delta_D(X) = 0$, then the set $B := \Delta^-(X)$ of edges entering X is called a **one-way cut** or a **directed cut**, or, for short, a **dicut**. The set X is the **in-shore** of B, while $V - X$ is its **out-shore**. The dicut B is **trivial** if X or $V - X$ is empty. A dicut is **minimal** or **elementary** if it does not include properly a non-empty dicut. Since a dicut is a cut of the underlying graph, Proposition 1.1.4 implies that a minimal dicut of a connected digraph uniquely determines its in-shore and out-shore. In a mixed graph M, the set of directed edges entering a subset X of nodes is called a **dicut** of M if neither directed nor undirected edges leave X.

A digraph is said to be **strongly connected** or just **strong** if $\varrho(X) \geq 1$ for every proper, non-empty subset X of nodes. This condition is obviously equivalent to requiring $\delta(X) \geq 1$ for every proper, non-empty subset X.

A digraph D is called **root-connected** (from s) if it has a special node s so that $\varrho_D(X) \geq 1$ for all non-empty subsets of V not containing s. An **arborescence of root** s, or more concisely, an s**-arborescence** is a directed tree in which there is a special node s, called **root-node** or root, of in-degree 0 while the in-degree of all other nodes is 1. By reorienting each edge of an arborescence, one obtains a **reverse arborescence**. A **branching** is a directed forest B so that every in-degree is at most 1, or equivalently, every component of B is an arborescence. The set of roots of these arborescences is called the **root-set** of the branching.

A **directed circuit** (**di-circuit**) is a strongly connected digraph in which $\varrho(v) = \delta(v) = 1$ for every node v. A **directed path** (**di-path**) is a digraph for which the underlying graph is a path and each in-degree and out-degree is at most 1. The number of edges of a path or circuit R (directed or undirected) is called the **length** of R. In particular, a digraph consisting of a single node and no edge is a di-path of length 0 while a (directed) uv-edge can be identified with a (directed) path of length 1. A (directed) loop is a (directed) circuit of length 1. Each di-path P of length at least 1 has exactly one node with out-degree 1 and in-degree 0. This node is called the **starting** or **initial** node of P. Also, P has exactly one node with in-degree 1 and out-degree 0, called the **terminal** node of P. All other nodes of P, called **inner** nodes, have in-degree and out-degree 1.

A digraph is **Eulerian** or **directed Eulerian** if the in-degree is equal to the out-degree at every node. Sometimes an Euler graph (respectively, digraph) is called a **cycle** (respectively, a directed cycle). A digraph is **near-Eulerian** or **smooth** if the in-degree and the out-degree of every node differ by at most 1. The digraph is **cyclic** if every edge belongs to a di-circuit. A digraph with no directed circuit is **acyclic**.

A digraph is **transitively closed** if $xy \in A$ and $yz \in A$ imply $xz \in A$. A partially ordered set $(S, \preceq)$ defines a digraph on node-set S in which st is an edge if $s \succ t$. This digraph is acyclic and transitively closed. Its underlying graph is called a **comparability graph**.

Some operations

Subdividing a directed or undirected edge $e = uv$ by a new node z means that we replace e by a directed or undirected path $P = (uz, zv)$ of two edges, respectively. The inverse operation is called **suppressing** z. In the undirected case, this is equivalent to remove a node z of degree 2 and adding an edge connecting the two neighbours of z. Suppose for a given node z in a digraph that uz and zv are the only arcs entering and leaving z, respectively. The operation of **suppressing** z, consists of removing z and adding a new uv-arc. More generally, **subdividing** a (directed) edge uv means that we replace uv by a (directed) path P from u to v where the inner nodes of P are new nodes of the graph.

Pinching k edges together with a node z is an operation in which we first subdivide each of k given edges by a node and then identify the k subdividing nodes with node z.

Paths and walks: reachability

In an undirected graph $G = (V, E)$, by a **walk** W, we mean a sequence $(v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ consisting of not necessarily distinct nodes and edges where e_i is a $v_{i-1}v_i$-edge. Note that a path, defined earlier, may be interpreted as a special walk for which the nodes are distinct. The nodes v_0 and v_k are called the **end-nodes** (sometimes ending nodes) or the **terminal nodes** of W, while the other nodes of W are its **inner nodes**. Sometimes we say that W is a walk between v_0 and v_k or we say that W is a walk between v_0 and v_k or that W is a walk from v_0 to v_k (or from v_k to v_0). Similarly, in a digraph $D = (V, A)$, by a **directed walk** (**diwalk**) W, we mean a sequence $(v_0, e_1, v_1, e_2, \dots, e_k, v_k)$, where e_i is a directed $v_{i-1}v_i$-edge. In a mixed graph M, a sequence $(v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ is a **forward** walk if each edge e_i is either a directed or undirected $v_{i-1}v_i$-edge.

In each case, we say that W is a walk from v_0 to v_k or that v_k is **reachable** from v_0 by W. The first node v_0 is the **starting** or **initial node** of the walk, while the last node v_k is its **end-node** (sometimes ending node) or **terminal node**. The walk is **closed** if $v_0 = v_k$. A walk for which the members are distinct can be identified with a path defined above. The number of edges is its **length**. For the sake of brevity, a walk (in particular, a path) from s to t will be called an st**-walk** (st**-path**). The smallest length of an st-path in a graph or a digraph is called the **distance** of t from s. A set of k edge-disjoint st-walks (directed or undirected) is called a k**-braid**. In the directed case, we call a k-braid **acyclic** if the union does not include any di-circuits. We call k st-paths **openly disjoint** if, apart from s and t, they are pairwise disjoint. A set of k openly disjoint st-paths (directed or undirected) is called a k**-bundle**.

We say that a walk is **edge-simple** if its edges are distinct and **simple** if all of its defining terms are distinct. An edge-simple walk using all edges of the (di)graph is called an **Euler walk**. A closed Euler walk is called an **Euler tour**. Note that a simple (directed) walk is a (directed) path, and a (directed) circuit is a special closed walk in which the starting node and the ending node coincide but in which nodes are distinct otherwise. A circuit of a graph is called **Hamiltonian** if it contains all nodes of the graph. A path is Hamiltonian if it contains all nodes. Directed Hamilton paths and circuits of a digraph can be defined analogously.

Let $W = (v_0, e_1, \dots, e_k, v_k)$ be a forward walk for which $v_0 \neq v_k$. Suppose that $v_i = v_j$ for some i, j where $0 \leq i < j \leq k$, and the subsequence $C = (v_i, e_{i+1}, \dots, e_{j-1}, v_j)$ is a circuit. **Reducing** W by circuit C means that we define a new walk $W' := (v_0, e_1, \dots, v_i, e_{j+1}, \dots e_k, v_k)$. **Simplifying** W means that one reduces W as much as possible. The final walk is a path from v_0 to v_k (that may depend on the order of reductions). Thus we have the following propositions:

Proposition 1.1.5 *In a mixed graph, if there is a forward uv-walk, then there is a forward uv-path. In particular, if there is a uv-walk in a graph, then there is a uv-path, and if there is a directed uv-walk in a digraph, then there is a directed uv-path.* •

Proposition 1.1.6 *Let s and t be two specified nodes of a graph or digraph $H = (V, F)$.*

(A) *If H is undirected, there is an st-path if and only if for all $s\bar{t}$-sets $S \subset V$ there is an edge leaving S, that is, $d_H(S) \geq 1$.*

(B) *If H is directed, there is a directed st-path if and only if for all $s\bar{t}$-sets S there is an edge leaving S, that is, $\delta_H(S) \geq 1$.*

Proof. The necessity of the conditins is straightforward. Sufficiency follows in both cases by observing that, if there is no path from s to t, the set S of nodes reachable from s has no leaving edges. •

We call a mixed graph M **traversable** if there is a forward uv-path for every ordered pair $\{u, v\}$ of nodes. If M has no undirected edges, then traversibility is just strong connectivity of a digraph. If M has no directed edges, then traversability is equivalent to connectivity.

We outline a well-known and simple device, called labelling technique, to determine the set S of nodes reachable from s, along with an s-arborescence of D that spans S.

We use a label called R-label for every node v to show if v has already been reached or not. If not, the label has entry **NON-REACHED**. If v is reached, its R-label signals **REACHED** and contains the arc $uv \in A$ along which v has been reached. The only exception is the source node s: the entry of its label is always REACHED. At the beginning, every node but s has NON-REACHED in its R-label. We also use another label called an S-label for every node to indicate whether v is SCANNED or UNSCANNED.

At the beginning, the entry in the S-label for every node is UNSCANNED. At a general step, we pick up an unscanned node u that has already been reached (at the beginning, only the source is such) and decide if there is a non-reached node v such that uv is an arc of D. If there is none, declare u SCANNED and repeat. If there is one, declare v REACHED, put uv into its R-label, and repeat. The algorithm terminates if no more unscanned nodes are reached.

Proposition 1.1.7 *The set S of nodes that have REACHED in their R-labels has no leaving arcs and consists precisely of those nodes of D that are reachable from s. The set of arcs occurring in their R-labels forms an s-arborescence and has node-set S.* •

In this algorithm, there is much freedom in choosing a node that is both reached and unscanned. One possible strategy is to choose each time the unscanned node u which had been reached earliest. In this case, the procedure is called **breadth-first search** (**BFS**).

An application of **BFS** is to compute the distance of the nodes in S from s. The only modification needed in the algorithm above is the introduction of a third variable $\mathrm{dist}(v)$ at every node v to store the distance of v from s. At the beginning, this is 0 at s and ∞ at all other nodes. When a node v is reached from u, define $\mathrm{dist}(v)$ to be $\mathrm{dist}(u) + 1$.

Another natural strategy is to choose each time an unscanned node that has been reached latest. In this case the procedure is called **depth-first search** (**DFS**). Depth-first search has a great number of important applications, and some of them will be mentioned in the next section.

Note that the procedure can be applied to undirected or mixed graphs as well. With the help of careful data structures, the complexity of these algorithms can be made linear. (For details, see [361, 362, 300].)

Let F be a spanning tree of an undirected graph with a special node s. We call a non-tree edge uv a **cross-edge** with respect to s if the closest node to s along the unique path P connecting u and v is distinct from u and from v.

Problem 1.1.2 *Prove the following proposition.*

Proposition 1.1.8 *If F is a DFS tree of a connected graph $G = (V, E)$ rooted at s, then G has no cross-edge to F.* •

1.1.4 Set systems and hypergraphs

The notion of hypergraphs is closely related to that of set systems and families of sets. In this context, a **hyperedge** is a subset of V. Formally, a **hypergraph** is a subset of V. Formally, a **hypergraph** H is a triplet $(V, \mathcal{E}, \varphi)$, where V is the set of nodes, $\mathcal{E}$ is the set of hyperedges, (more concisely, edges), and $\varphi : \mathcal{E} \to 2^V$ is a mapping that shows the elements of a hyperedge.

With a slight abuse of terminology, we simply denote a hypergraph by $H = (V, \mathcal{E})$ and think of a hyperedge as a subset of V. In this sense, a hyperedge may occur in several copies. The empty set is not allowed to be a hyperedge. A hypergraph is **simple** if it has no parallel edges, that is, simple hypergraphs and set systems are the same. A set-system is said to be **hereditary** if every subset of a member is also a member of the set-system. For example, the set of chains of a partially ordered set is hereditary, as is the set of antichains. A pair $(S, \mathcal{F})$ is called a **matroid** if $\mathcal{F}$ is a non-empty hereditary set-system such that $X, Y \in \mathcal{F}$ and $|X| < |Y|$ imply the existence of an element $s \in Y - X$ for which $X + s \in \mathcal{F}$.

Deleting a hyperedge Z of H means that we remove Z from $\mathcal{E}$. **Deleting a node** v means that we remove v from V and delete from $\mathcal{E}$ all hyperedges containing v. A **subhypergraph** (sometimes a **partial** hypergraph) of H is obtained from H by deleting some hyperedges and some nodes of H. In other words, a hypergraph $H' = (V', \mathcal{E}')$ is a sub-hypergraph of $H = (V, \mathcal{E})$ if $V' \subseteq V$, $\mathcal{E}' \subseteq \mathcal{E}$, and each member of $\mathcal{E}'$ is a subset of V'. H' is a **spanning subhypergraph** of H if $V' = V$. We say that a hyperedge Z is **induced by** a subset $Y \subseteq V$ if $Z \subseteq Y$. The number of hyperedges induced by $Y \subseteq V$ is denoted by $i(X) = i_H(X)$, while the set of these hyperedges is $I(X) = I_H(X)$. An **induced subhypergraph** of H is defined on a subset Y of V and consists of all the hyperedges of H induced by Y. In other words, an induced subhypergraph arises from H by deleting some nodes of H along with all hyperedges containing at least one of the deleted nodes.

The **degree** $d_H(v)$ of a node is the number of hyperedges containing v. The maximum degree of a node of H is denoted by $\Delta(H)$. The hypergraph is **regular** if the degree of each node is the same. For example, a partition of the ground-set is a 1-regular hypergraph.

For a subset $X \subseteq V$, the set $\Delta(X) = \Delta_H(X)$ of hyperedges of H intersecting both X and $V - X$ is called a **cut** of the hypergraph. A cut $\Delta(X)$ is **trivial** if X or $V - X$ is empty, The **degree** $d_H(X)$ of a subset X is the number of hyperedges intersecting both X and $V - X$. When every hyperedge has at least two elements, then $d_H(v) = d_H(\{v\})$, and typically we do not distinguish between a singleton and its only element. The number of hyperedges of H intersecting X is denoted by $e(X) = e_H(X)$. Obviously,

$$e_H = i_H + d_H.$$

A hypergraph H is called **connected**, if for every non-empty proper subset $X \subset V$, there is a hyperedge intersecting both X and $V - X$, that is, $d_H(X) \geq 1$.

A **connected component** or simply a **component** of a hypergraph H is a maximal induced subhypergraph of G for which node-set is not separated by any set $X \subset V$ of degree 0. That is, connected components are the atoms of the set-system $\{X \subseteq V : d_H(X) = 0\}$.

The **rank** of a hypergraph is the maximum size of its hyperedges. If each edge has the same cardinality, the hypergraph is **uniform**. Therefore, loopless graphs and 2-uniform hypergraphs are the same. There is only one marginal aspect by which hypergraphs are not an exact generalization of graphs. In graphs, a loop is an edge for which the two end-nodes coincide, and a loop at node v contributes 2 to the degree of v. In hypergraphs, we do not use this notion. There may be single-element hyperedges, but for such a hyperedge, its contribution to the degree of its element is just 1.

The **line-graph** of a hypergraph $H = (V, \mathcal{F})$ is a simple undirected graph for which the nodes correspond to the hyperedges and two nodes are adjacent if the corresponding hyperedges are intersecting.

A **transversal** X of a hypergraph H is a subset of nodes intersecting all hyperedges. The **transversal number** $\tau(H)$ is the minimum cardinality of a transversal. A **matching** of a hypergraph is a subset of pairwise disjoint hyperedges. If, in addition, the edges cover all nodes, we speak of a **perfect matching**. The **matching number** $\nu(H)$ is the maximum cardinality of a matching of H. It is evident that $\tau(H) \geq \nu(H)$ for every hypergraph. A hypergraph is **normal** if $\tau(H') = \nu(H')$ for every subhypergraph H' of H.

A **system of distinct representatives** of a hypergraph is an assignment of nodes to hyperedges in such a way that the node z assigned to a hyperedge Z is in Z and the nodes assigned to distinct hyperedges are distinct.

The **chromatic number** $\chi(H)$ of a hypergraph H is the minimum cardinality of a partition $\mathcal{P}$ of the node-set V so that every hyperedge of H intersects at least two members of $\mathcal{P}$.

The **chromatic index** $\chi'(H)$ of a hypergraph H is the minimum number of classes in a partition of the edge-set of H into matchings. Obviously, $\Delta(H) \leq \chi'(H)$. Note that the chromatic index of a hypergraph is the chromatic number of its line-graph.

Hypergraphs versus bipartite graphs and (0, 1)*-matrices*

With every hypergraph $H = (V, \mathcal{E})$, we can associate a simple bipartite graph $G = (V, U; F)$, where there is a one-to-one correspondence between the elements of U and $\mathcal{E}$, and where nodes $v \in V$ and $u \in U$ are adjacent in G precisely when v belongs to the hyperedge corresponding to u. We will say that G is the **bipartite graph belonging to** (or **associated with**) **the hypergraph**. Obviously, the degree $d_G(t)$ of any node $t \in U$ is the cardinality of the hyperedge T corresponding to t, while the degree $d_G(v)$ of any node $v \in V$ is $d_H(v)$.

The **incidence matrix** Q of a hypergraph H is a $(0, 1)$-matrix, the rows of which correspond to the nodes and the columns of which correspond to the hyperedges. The entry of Q corresponding to node u and hyperedge F is 1 if and only if $u \in F$. Conversely, every $(0, 1)$-matrix is the incidence matrix of a hypergraph. The **transpose** of a hypergraph H is the hypergraph associated with the transpose of the incidence matrix of H. In other words, in the transpose H' to $H = (V, \mathcal{E})$, the nodes correspond to the hyperedges of H and the hyperedges correspond to the nodes of H. An equivalent interpretation of the transpose is that we interchange the role of V and U in the bipartite graph $G = (V, U; F)$ associated with H. Obviously, a hypergraph is uniform if and only if its transpose is regular.

A hypergraph having no empty hyperedges is connected if and only if its associated bipartite graph is connected. The components of the associated bipartite graph determine a decomposition of H into induced subhypergraphs, which are easily seen to be the components of H.

Directed hypergraphs

The notion of directed graphs can be extended to hypergraphs in various ways. We adopt one for which results on connectivity of digraphs can be generalized nicely. By a **directed hyperedge**, or a **dyperedge**, for short, we mean a pair (X, v) where X is a set and v is an element of X. The node v is called the **head** of X. (A regrettable drawback of this definition of dyperedges is the loss of symmetry between head and tail that exists for ordinary directed graphs.) A **dypergraph** $D = (V, \mathcal{D})$ consists of a set V of nodes and a set $\mathcal{D}$ of dyperedges. Subdypergraphs and induced subdypergraphs are defined analogously to those for hypergraphs. A dyperedge (Z, z) is said to **enter** a subset X (and to leave $V - X$) if $z \in X$ and $Z - z$ intersects $V - X$. The number of dyperedges entering a subset X is denoted by $\varrho(X) = \varrho_D(X)$, while $\delta(X) = \delta_D(X)$ denotes the number of dyperedges leaving X. A dyperedge (Z, z) **leaves** X if it enters $V - X$. The set of dyperedges entering X is denoted by $\Delta_D^-(X)$. The **in-degree** $\varrho_D(X)$ is the cardinality of $\Delta_D^-(X)$. The set of edges leaving Z and the cardinality of Z are defined by $\Delta_D^+(X) := \Delta_D^-(V - X)$ and $\delta_D(X) := \varrho_D(V - X)$, respectively.

A dypergraph D is **strongly connected** if $\varrho_D(X) \geq 1$ for every proper non-empty subset $X \subseteq V$. D is **root-connected** with respect to a specified root-node $r_0 \in V$ if $\varrho_D(X) \geq 1$ for every non-empty subset $X \subseteq V - r_0$. The **underlying hypergraph** of a dypergraph H is obtained ignoring the heads and tails of the dyperedges.

By **orienting a hyperedge** Z, we mean the operation of designating an element z of Z to be the head of Z and replacing Z by the dyperedge (Z, z). An orientation of a hypergraph H is a dypergraph resulting from H by orienting each hyperedge. Note that finding a system of distinct representatives of a hypergraph H in which every hyperedge has at least two elements is equivalent to requiring the existence of an orientation of H so that the in-degree of each node is at most 1.

With every dypergraph $\vec{H} = (V, \vec{\mathcal{E}})$, we can associate a simple directed bipartite graph $\vec{G} = (V, U; \vec{F})$ which is obtained from the graph $G = (V, U; F)$ associated with the undirected underlying hypergraph of $\vec{H}$ as follows: For each dyperedge (Z, z) of $\vec{H}$, let u_Z denote the element of U associated with Z. For each $v \in Z - z$, orient the edge $u_Z v$ toward u_Z, and orient the edge $u_Z z$ toward z. Conversely, for every simple directed bipartite graph $\vec{G}$ in which the out-degree of each node of one of the colour classes is exactly 1, there is a (unique) dypergraph such that the associated directed bipartite graph is $\vec{G}$.

A dypergraph $\vec{H}$ is easily seen to be strongly connected if and only if its associated directed bipartite graph $\vec{G}$ is strongly connected. Also, providing that every dyperedge of $\vec{H}$ has at least two elements, $\vec{H}$ is root-connected if and only if $\vec{G}$ is root-connected.

Trimming a hyperedge Z of at least size three is the operation in which Z is replaced by a subset of it. By trimming a dyperedge (Z, v) with at least three elements, we mean that Z is replaced by a subset of Z that contains v. Trimming a hypergraph or a dypergraph consists of trimming its hyperedges or dyperedges.

The hypergraph $(\{a, b, c\}, \{\{a, b, c\}\})$ shows that a connected hypergraph, in general, cannot be trimmed to a connected graph.

Problem 1.1.3 *Prove that a root-connected dypergraph can be trimmed to a root-connected digraph. Show a strongly connected dypergraph that cannot be trimmed to a strongly connected digraph.*

A solution to an extension of the first part will be presented in Theorem 7.4.9.

1.2 Cut functions and connectivity concepts

This section is a collection of the most important connectivity-type functions and concepts that occur in the book.

1.2.1 Identities and inequalities

Let $G = (V, E)$ be an undirected graph and $D = (V, A)$ a digraph. We have already defined the set-functions $d_G, i_G,\ e_G,\ \varrho_D,\ \delta_D$, and Ψ_D. Also, $d_G(X, Y)$ denoted the number of edges joining $X - Y$ and $Y - X$. Let z be a function on E or on A. Define $d_z,\ i_z,\ \varrho_z$, and δ_z as follows. For a subset $X \subseteq V$ or for a bi-set X, let

$$d_z(X) := \sum[z(f) : f \in E,\ f \text{ enters } X],$$

$$i_z(X) := \sum[z(f) : f \in E,\ f \text{ is induced by } X],$$

$$e_z(X) := \sum[z(f) : f \in E,\ \text{ at least one end-node of } f \text{ is in } X].$$

For a subset $X \subseteq V$ or for a bi-set X of V, let

$$i_z(X) := \sum[z(f) : f \in A,\ f \text{ is induced by } X],$$

$$\varrho_z(X) := \sum[z(f) : f \in A,\ f \text{ enters } X],$$

$$\delta_z(X) := \sum[z(fe) : f \in A,\ f \text{ leaves } X],$$

$$\Psi_z := \varrho_z - \delta_z.$$

For two subsets $X, Y \subseteq V$, let

$$d_z(X, Y) := \sum[z(e) : e \in E, e \text{ joins } X - Y \text{ and } Y - X],$$

$$\overline{d}_z(X, Y) := \sum[z(e) : e \in E, e \text{ joins } X \cap Y \text{ and } V - (X \cup X)].$$

The following identities will be used throughout the book.

Proposition 1.2.1 *For two subsets X and Y of nodes of an undirected graph G, the following identities hold:*

$$d_z(X) + d_z(Y) = d_z(X \cap Y) + d_z(X \cup Y) + 2d_z(X, Y), \tag{1.4}$$

$$d_z(X) + d_z(Y) = d_z(X - Y) + d_z(Y - X) + 2\bar{d}_z(X, Y), \tag{1.5}$$

$$i_z(X) + i_z(Y) = i_z(X \cap Y) + i_z(X \cup Y) - d_z(X, Y), \tag{1.6}$$

$$e_z(X) + e_z(Y) = e_z(X \cap Y) + e_z(X \cup Y) + d_z(X, Y). \tag{1.7}$$

For two subsets X and Y of nodes of a digraph D, the following identities hold:

$$\varrho_z(X) + \varrho_z(Y) = \varrho_z(X \cap Y) + \varrho_z(X \cup Y) + d_z(X, Y), \tag{1.8}$$

$$\delta_z(X) + \delta_z(Y) = \delta_z(X \cap Y) + \delta_z(X \cup Y) + d_z(X, Y), \tag{1.9}$$

$$\begin{aligned}\varrho_z(X) + \varrho_z(Y) = {} & \varrho_z(X - Y) + \varrho_z(Y - X) + \bar{d}_z(X, Y) \\ & + [\varrho_z(X \cap Y) - \delta_z(X \cap Y)],\end{aligned} \tag{1.10}$$

$$\Psi_z(X) + \Psi_z(Y) = \Psi_z(X \cap Y) + \Psi_z(X \cup Y). \bullet \tag{1.11}$$

The proof of each identity consists of showing, by simple case-checking, that each single edge contributes the same amount to the two sides of the indentities. We often apply these identities to $z \equiv 1$. For example, (1.4) transforms into

$$\varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) + d_D(X, Y).$$

Proposition 1.2.2 *When z is non-negative, the functions d_z, e_z, ϱ_z, and δ_z are submodular, while i_z is supermodular.* •

The following is an extension of (1.11).

Proposition 1.2.3 *For functions $f : A \to \mathbf{R} \cup \{-\infty\}$, $g : A \to \mathbf{R} \cup \{+\infty\}$, let $p_{fg}(X) := \varrho_f(X) - \delta_g(X)$. Then*

$$p_{fg}(X) + p_{fg}(Y) = p_{fg}(X \cap Y) + p_{fg}(X \cup Y) - d_{g-f}(X, Y) \tag{1.12}$$

for every $X, Y \subseteq V$. In particular, if $f \le g$, then p_{fg} is supermodular. •

The proof again is based on the observation that it suffices to prove (1.12) for digraphs having a single edge, and for such digraphs the identity follows by simple case-checking according to the possible locations of the only edge.

Exercise 1.2.1 *If $\varrho_z(v) \ge \delta_z(v)$ for every node $v \in X \cap Y$, then*

$$\varrho_z(X) + \varrho_z(Y) \ge \varrho_z(X - Y) + \varrho_z(Y - X) + \bar{d}_z(X, Y).$$

If $\varrho_z(v) \ge \delta_z(v)$ for every node $v \in V - (X \cup Y)$, then

$$\delta_z(X) + \delta_z(Y) \ge \delta_z(X - Y) + \delta_z(Y - X) + \bar{d}_z(X, Y).$$

If $\delta_z(X \cup Y) = \varrho_z(X \cup Y)$, then

$$\varrho_z(X) + \varrho_z(Y) = \varrho_z(X - Y) + \varrho_z(Y - X) + \bar{d}_z(X, Y).$$

Proposition 1.2.4 (Triple inequality) *Let A, B, and C be three subsets of nodes in an undirected graph $G = (U, E)$. Then*

$$d(A) + d(B) + d(C) \ge d(A \cap B \cap C) + d(A - (B \cup C)) + d(B - (A \cup C))$$
$$+ d(C - (A \cup B)) + 2d(A \cap B \cap C, U - (A \cup B \cup C)).$$

Proof. It suffices to prove the inequality for graphs having exactly one edge. For such a graph, in turn, the inequality follows by checking the possible positions of the end-nodes of the single edge. •

Proposition 1.2.5 *In a digraph D,*

$$\varrho_z(X) + \varrho_z(Y) \ge \varrho(X \sqcap Y) + \varrho_z(X \sqcup Y) \tag{1.13}$$

and

$$i_z(X) + i_z(Y) \le i_z(X \sqcap Y) + i_z(X \sqcup Y) \tag{1.14}$$

hold for every pair of bi-sets $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$.

Number of components
Let $G = (V, E)$ be an undirected graph. For every subset X of nodes, let $c(X) = c_G(X)$ denote the number of components induced by X.

Proposition 1.2.6 *The set-function c satisfies the following inequality.*

$$c(X) + c(Y) \leq c(X \cap Y) + c(X \cup Y) + d_G(X, Y). \tag{1.15}$$

Proof. We proceed by induction on the number of edges. If G has no edges at all, then $c(X) = |X|$, and hence (1.15) holds with equality. The deletion of any node outside $X \cup Y$ does not affect any term of (1.15), so we may assume that $V = X \cup Y$. The contraction of an edge induced by any of the subsets $X - Y$, $Y - X$, and $X \cap Y$ does not affect the terms of (1.15) either, so we may assume that these subsets are stable.

Let $e = uv$ be an edge of G. Let $G' = G - e$ and $c' := c_{G'}$. It is evident that $c' \geq c$. If e joins $X - Y$ and $Y - X$, then $d_{G'}(X, Y) = d_G(X, Y) - 1$, $c'(X) = c(X)$, $c'(X \cap Y) = c(X \cap Y)$, and $c'(X \cup Y) \geq c(X \cup Y) - 1$, and by induction we are done. Therefore, we may assume that $d_G(X, Y) = 0$.

Suppose now that e joins $X \cap Y$ and either of $X - Y$ and $Y - X$, say $X - Y$. Then $c'(X \cap Y) = c(X \cap Y)$ and $c'(Y - X) = c(Y - X)$. By induction, we have $c'(X) + c'(Y) \leq c'(X \cap Y) + c'(X \cup Y)$, from which

$$c'(X) + c(Y) \leq c(X \cap Y) + c'(X \cup Y). \tag{1.16}$$

Furthermore, if the deletion of e from G increases the number of components, that is, if $c'(X \cup Y) > c(X \cup Y)$, then e is a cut-edge of G, and hence e must be a cut-edge of the subgraph induced by X, too, from which

$$c'(X \cup Y) - c(X \cup Y) \leq c'(X) - c(X). \tag{1.17}$$

By combining (1.16) and (1.17), we obtain $c(X) + c(Y) \leq c(X \cap Y) + c(X \cup Y)$, which is just (1.15) with $d_G(X, Y) = 0$. •

Sometimes we need the related set-function σ, which is defined by $\sigma(X) := c(V - X)$ if $X \neq \emptyset$ and $\sigma(\emptyset) := 0$. It follows from (1.15) that

$$\sigma(X) + \sigma(Y) \leq \sigma(X \cap Y) + \sigma(X \cup Y) + d_G(X, Y)$$

holds for intersecting sets X and Y.

Bipartite graphs and hypergraphs

Let $G = (S, T; E)$ be a bipartite graph and define a set-valued set-function Γ on S by

$$\Gamma(X) := \{v \in T\text{: there is an edge } uv \in E \text{ with } u \in T\} \text{ for } X \subseteq S, \tag{1.18}$$

that is, $\Gamma(X)$ denotes the set of neighbours of X. We will often use the more specific notation Γ_G or Γ_E for Γ. For a non-negative function $m : S \to \mathbf{Z}_+$, let a set-function γ_m on T be defined by $\gamma_m(X) := \widetilde{m}(\Gamma(X))$.

Proposition 1.2.7 *The set-function γ_m is submodular. In particular, the set-function γ defined by $\gamma(X) := |\Gamma(X)|$ is submodular.*

Proof. Since $\Gamma(X) \cap \Gamma(Y) \supseteq \Gamma(X \cap Y)$ and $\Gamma(X) \cup \Gamma(Y) = \Gamma(X \cup Y)$, the non-negativity of m implies at once the submodular inequality. •

Let $H = (V, \mathcal{E})$ be a hypergraph. For a subset $\mathcal{F} \subseteq \mathcal{E}$ of hyperedges, let $V(\mathcal{F}) := \cup(Z : Z \in \mathcal{F})$. For a non-negative function $m : V \to \mathbf{R}_+$, let a function $n_m : 2^{\mathcal{E}} \to \mathbf{R}_+$ be defined by $n_m(\mathcal{F}) := \widetilde{m}(V(\mathcal{F}))$. In particular (when $m \equiv 1$), let $n(\mathcal{F}) := |V(\mathcal{F})|$.

Proposition 1.2.8 *The function n_m, and in particular n, on the power set of $\mathcal{E}$ is submodular.*

Proof. Consider the bipartite graph $(V, U; F)$ associated with the hypergraph H where U corresponds to $\mathcal{E}$. Observe that, by using the notation $S = V$ and $T = U$, the function γ_m in Proposition 1.2.7 can be identified with n_m, and hence the submodularity of n_m follows from Proposition 1.2.7. •

For a hypergraph or a dypergraph H, let $d_H(X, Y)$ denote the number of hyperedges Z for which $Z \subseteq X \cup Y$, $Z \cap (X - Y) \neq \emptyset$, and $Z \cap (Y - X) \neq \emptyset$. Let $d^*_H(X, Y)$ denote the number of hyperedges Z for which $Z \cap X \cap Y = \emptyset$, $Z \cap (X - Y) \neq \emptyset$, and $Z \cap (Y - X) \neq \emptyset$. Note that if H is a graph or a digraph, then $d_H(X, Y) = d_H(\overline{X}, \overline{Y})$, but for general hyper- or dypergraphs, $d_H(X, Y)$ and $d_H(\overline{X}, \overline{Y})$ may differ.

Recall the functions d_H, i_H, e_H associated with a hypergraph H and the functions ϱ_D, δ_D associated with a dypergraph D. Let p_H be a set-function for which $p_H(X)$ is the number of hyperedges disjoint from X, that is, $p_h(X) = i_H(V - X)$. Again, a simple case-checking on the possible positions of hyperedges gives rise to the following identities:

Proposition 1.2.9 *For a hypergraph H,*

$$i_H(X) + i_H(Y) = i_H(X \cap Y) + i_H(X \cup Y) - d_H(X, Y), \tag{1.19}$$

$$p_H(X) + p_H(Y) = p_H(X \cap Y) + p_H(X \cup Y) - d^*_H(X, Y). \tag{1.20}$$

For a dypergraph D,

$$\varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) + d_D(X, Y). \tag{1.21}$$

The functions d_H, e_H, ϱ_D, and δ_D are submodular, while i_H and p_H are supermodular. •

1.2.2 Various notions for higher-order connections

Undirected graphs

For a positive integer k, an undirected graph $G = (V, E)$ is **k-edge-connected** if every non-trivial cut contains at least k edges, that is, $d_G(X) \geq k$ whenever $\emptyset \subset X \subset V$. A simple observation yields the following.

Proposition 1.2.10 *A graph G is k-edge-connected if and only if the removal of any j edges $(0 \leq j \leq k - 1)$ leaves a connected graph.* •

The minimum cardinality $\lambda(G)$ of a non-trivial cut of G is called the **edge-connectivity** of G. This cardinality is the largest value k such that the deletion of any subset of less than k edges leaves a connected graph.

More generally, for a non-empty subset T of nodes, G is said to be **k-edge-connected within** T if $d_G(X) \geq k$ for every subset X separating T. We say that X **separates** T if neither $T \cap X$ nor $T - X$ is empty. Note that this property differs from the one that requires the subgraph induced by T to be k-edge-connected.

978–0–19–920527–1 01-Andras-Frank-c01-drv Andras Frank (Typeset by SPi, Publisher Services, India) 24 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

A hyperedge Z of a hypergraph H is called a **cross-hyperedge** to a partition $\mathcal{P}$ of V if Z intersects at least two members of $\mathcal{P}$. The number of cross-hyperedges is denoted by $e_H(\mathcal{P})$. When H is a graph, we speak of **cross-edge**s. The set of all cross-hyperedges to $\mathcal{P}$ is called the **border** of the partition. Observe that if each class of $\mathcal{P}$ induces a connected subgraph of G, then $\mathcal{P}$ is uniquely determined by its border B as follows: the members of $\mathcal{P}$ are the node-sets of the connected components of $G - B$.

A graph $G = (V, E)$ is k**-partition-connected** if the number $e_G(\mathcal{P})$ of cross-edges to every proper partition $\mathcal{P}$ of V is at least $k(|\mathcal{P}| - 1)$. A simple observation is that 1-partition-connectivity is equivalent to connectivity. For $k \geq 2$, a k-partition-connected graph is obviously k-edge-connected, but the reverse implication fails to hold, as is shown by a triangle (a complete graph on 3 nodes), which is 2-edge-connected but not 2-partition-connected. More generally, for non-negative integers k, ℓ, we call G (k, ℓ)**-partition-connected** if $e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1) + \ell$ for every partition $\mathcal{P}$.

Proposition 1.2.11 *For $k \leq \ell$, (k, ℓ)-partition-connectivity of a graph G is equivalent to $(k + \ell)$-edge-connectivity.*

Proof. If G is (k, ℓ)-partition-connected, then for any 2-partition $\{X, V - X\}$ the number of edges connecting X and $V - X$ is at least $k(2 - 1) + \ell = k + \ell$, and hence G is $(k + \ell)$-edge-connected. Conversely, if G is $(k + \ell)$-edge-connected and $\mathcal{P}$ is a partition of V, then $e_G(\mathcal{P}) = \sum[d_G(X) : X \in \mathcal{P}]/2 \geq (k + \ell)|\mathcal{P}|/2 \geq (k - 1)|\mathcal{P}| + \ell$ where the last inequality easily follows from $k \leq \ell$ and $|\mathcal{P}| \geq 2$. •

We call a graph G k**-tree-connected** if G includes k edge-disjoint spanning trees. More generally, G is (k, ℓ)**-tree-connected** if G includes k edge-disjoint spanning trees after deleting any set of at most ℓ edges. A graph G is readily seen to be $(1, \ell)$-tree-connected if and only if G is $(\ell + 1)$-edge-connected.

The standard definition for a graph G being k-node-connected requires that G has at least $k + 1$ nodes and the removal of any j nodes $(0 \leq j \leq k - 1)$ leaves a connected graph. We use a slightly more general definition (which is actually equivalent in the relevant cases) because it fits better formally to that of k-edge-connectivity and leaves room for further generalizations. Let us call an undirected graph $G = (V, E)$ k**-node-connected** or just k**-connected** if

$$d_G(X) + w(X) \geq k \text{ for every non-trivial bi-set } X = (X_O, X_I) \tag{1.22}$$

where $w(X) = |X_O - X_I|$ is the wall-size of X. Since $w(X) = 0$ for simple bi-sets, k-edge-connectivity of G is equivalent to requiring $d_G(X) + w(X) \geq k$ for every non-trivial simple bi-set.

Proposition 1.2.12 *A graph G with at most k nodes is k-connected if and only if there are at least $k + 2 - |V|$ parallel uv-edges for every pair $\{u, v\}$ of nodes. A graph G with at least $k + 1$ nodes is k-connected if and only if the removal of any j nodes $(0 \leq j \leq k - 1)$ leaves a connected graph.* •

The **node-connectivity** $\kappa(G)$ of a graph is the largest integer k for which G is k-connected. Proposition 1.2.12 shows that k-connectivity is not a particularly interesting property when the graph has at most k nodes. For this reason, the standard definition of k-connectivity assumes that the graph has more than k nodes and consists of the equivalent property formulated in the second part of the proposition.

We introduce the concept of hybrid-connectivity which takes into account both edge- and node-connectivity. Call an undirected graph (k, ℓ)**-hybrid-connected** if

$$d_G(X) + \ell w(X) \geq k\ell \text{ for every non-trivial bi-set } X. \tag{1.23}$$

For $k = 1$, this is just ℓ-edge-connectivity, since the inequality automatically holds when $w(X) \geq 1$ and for bi-sets with $w(X) = 0$ (1.23) transforms to $d_G(X') \geq k$ for every subset X' of nodes. When $\ell = 1$, we are at k-node-connectivity.

Proposition 1.2.13 *A graph is (k, ℓ)-hybrid-connected if and only if the removal of any j $(0 \leq j \leq k-1)$ nodes leaves an $\ell(k-j)$-edge-connected graph.* •

Digraphs

A digraph $D = (V, A)$ is said to be **globally k-edge-connected** or just **k-edge-connected** if $\varrho_D(X) \geq k$ whenever $\emptyset \subset X \subset V$. The minimum of $\varrho_D(X)$ over all non-empty proper subsets X of V is called the **edge-connectivity** D and is denoted by $\lambda(D)$. This minimum is the largest value k such that the deletion of any subset of less than k edges leaves a strongly connected digraph. More generally, given two non-empty subsets S and T of nodes, D is **k-ST-edge-connected** if $\varrho_D(X) \geq k$ for every subset X for which $X \cap T \neq \emptyset$ and $S - X \neq \emptyset$. In the special case when S consists of a single node r_0 and $T = V - r_0$, we say that D is **out-rooted** or just **rooted k-edge-connected** (with respect to root-node r_0), while if $S = V - r_0$ and $T = \{r_0\}$, then D is **in-rooted k-edge-connected**. A rooted 1-connected digraph is also called **root-connected**. It follows at once that D is k-edge-connected if and only if it is both out-rooted and in-rooted k-edge-connected with respect to the same root-node. When we want to emphasize the root-node, we say that D is **r_0-rooted** k-edge-connected.

Proposition 1.2.14 *Let $D = (V, A)$ be an r_0-rooted k-edge-connected digraph. Let $u \in V - r_0$ be a node and let $P_1, \dots, P_k$ be k edge-disjoint dipaths from r_0 to u. Then the digraph D' arising from D by reorienting each P_i is rooted k-edge-connected with respect to root-node u.*

Proof. Let Ψ_1 be a set-function on V defined by

$$\Psi_1(X) := \begin{cases} -1 & \text{if } u \in X \subseteq V - r_0 \\ +1 & \text{if } r_0 \in X \subset V - u \\ 0 & \text{otherwise.} \end{cases}$$

The reorientation of one r_0u-path changes the in-degree of a subset $X \subseteq V$ from $\varrho_D(X)$ to $\varrho_D(X) + \Psi_1(X)$. Therefore, $\varrho_{D'}(X) = \varrho_D(X) + k\Psi_1(X)$, which implies that, if $X \subseteq V - \{r_0, u\}$, then $\varrho_{D'}(X) = \varrho_D(X) \geq k$, while if $r_0 \in X \subseteq V - u$, then $\varrho_{D'}(X) = \varrho_D(X) + k\Psi_1(X) \geq k$. Therefore, D' is u-rooted k-edge-connected. •

Note here that the directed edge version of Menger's theorem (Theorem 2.5.1) will imply that an r_0-rooted k-edge-connected digraph includes k edge-disjoint paths from r_0 to u for every $u \in V$.

Another special case of k-ST-edge-connectivity is when $S = T$, in which case we say that D is k**-edge-connected within** T. Note that this property is NOT the same as requiring that the digraph induced by T is k-edge-connected. There is a natural common generalization of global and rooted k-edge-connectivity. Let k and ℓ be non-negative integers. We say that a digraph is (k, ℓ)**-edge-connected** if it has a node r_0 so that the in-degree and the out-degree of every non-empty subset $X \subseteq V - r_0$ is at least k and ℓ, respectively. In the special case $\ell = 0$, we are back at rooted k-edge-connectivity, while the case $k = \ell$ corresponds to k-edge-connectivity. Clearly, a digraph is (k, ℓ)-edge-connected if and only if the digraph arising by reorienting each edge is (ℓ, k)-edge-connected.

Problem 1.2.2 *Let D be an r_0-rooted (k, ℓ)-edge-connected digraph (where $0 \leq \ell \leq k$), and let r_1 be a node of D. Prove that the digraph arising from D by reorienting $k - \ell$ edge-disjoint dipaths from r_0 to r_1 is r_1-rooted (k, ℓ)-edge-connected.*

The standard definition for a digraph D being k-node-connected requires that D has at least $k + 1$ nodes and that the removal of any j nodes $(0 \leq j \leq k - 1)$ leaves a strongly connected digraph. As in the undirected case, we introduce a formally slightly more general but basically equivalent definition. Call a directed graph $D = (V, A)$ k**-node-connected** or just k**-connected** if

$$\varrho_D(X) + w(X) \geq k \text{ for every non-trivial bi-set } X.$$

Proposition 1.2.15 *A digraph D with at most k nodes is k-connected if and only if there are at least $k + 2 - |V|$ parallel uv-edges for every ordered pair $\{u, v\}$ of nodes. A digraph D with at least $k + 1$ nodes is k-connected if and only if the removal of any j nodes $(0 \leq j \leq k - 1)$ leaves a strongly connected digraph.* •

The **node-connectivity** $\kappa(D)$ of a digraph is the largest integer k for which D is k-connected. Analogously to the undirected case, we call a digraph $D = (V, A)$ (k, ℓ)**-hybrid-connected** if

$$\varrho_D(X) + \ell w(X) \geq k\ell \text{ for every non-trivial bi-set } X. \tag{1.24}$$

For $k = 1$, (k, ℓ)-edge-connectivity is just ℓ-edge-connectivity, while for $\ell = 1$ we are at k-node-connectivity.

Proposition 1.2.16 *A digraph is (k, ℓ)-hybrid-connected if and only if the removal of any j $(0 \leq j \leq k - 1)$ nodes leaves an $\ell(k - j)$-edge-connected digraph.* •

We call a digraph $D = (V, A)$ **rooted** (k, ℓ)**-hybrid-connected** with respect to a root-node r_0 if

$$\varrho_D(X) + \ell w(X) \geq k\ell \text{ for every non-trivial bi-set } X = (X_O, X_I) \text{ with } X_O \subseteq V - r_0. \tag{1.25}$$

For $k = 1$, this property is just rooted ℓ-edge-connectivity of D, while in the $\ell = 1$ we simply say that the digraph is **rooted** k**-node-connected**.

Proposition 1.2.17 *A digraph is rooted (k, ℓ)-hybrid-connected if and only if the removal of any j $(0 \leq j \leq k - 1)$ nodes distinct from r_0 leaves a rooted $\ell(k - j)$-edge-connected digraph.* •

A mixed graph $H = (V, A + E)$ consisting of an undirected graph $G = (V, E)$ and a digraph (V, A) is k**-edge-connected** (respectively, r_0-**rooted** k**-edge-connected**) if $\varrho_D(X) + d_G(X) \geq k$ whenever $\emptyset \subset X \subset V$ (or, respectively, whenever $r_0 \notin X \subset V$).

Hypergraphs and dypergraphs

A hypergraph is k**-edge-connected** if $d_H(X) \geq k$ for for every non-empty proper subset $X \subset V$. The minimum cardinality $\lambda(H)$ of a non-trivial cut of H is called the **edge-connectivity** of H. A hypergraph H is k**-partition-connected** if the number $e_H(\mathcal{P})$ of cross-hyperedges to every proper partition $\mathcal{P}$ of V is at least $k(|\mathcal{P}| - 1)$. When $k = 1$, we simply speak of a **partition-connected** hypergraph. More generally, for non-negative integers k, ℓ, H is (k, ℓ)**-partition-connected** if the number of cross-hyperedges to every proper partition $\mathcal{P}$ of V is at least $k(|\mathcal{P}| - 1) + \ell$. An interesting special case is (k, k)-partition-connectivity, where the defining inequality $e_H(\mathcal{P}) \geq k|\mathcal{P}|$ for every partition $\mathcal{P}$ is equivalent to requiring that the number of components of the hypergraph resulting from H by deleting at most j hyperedges is at most j/k.

A dypergraph D is k**-edge-connected** if the in-degree of every proper non-empty subset $X \subseteq V$ is at least k. In the special case $k = 1$, we speak of a **strongly connected** dypergraph. D is **rooted** k**-edge-connected** with respect to a specified root-node $r_0 \in V$ if the in-degree of every non-empty subset $X \subseteq V - r_0$ is at least k. In the special case $k = 1$, we speak of a **root-connected** dypergraph. More generally, a dypergraph is (k, ℓ)**-edge-connected** if it has a node r_0 so that the in-degree and the out-degree of every non-empty subset $X \subseteq V - r_0$ is at least k and ℓ, respectively.

1.2.3 *NP-characterizations of simple connectivity properties*

A characteristic feature shared by the various connectivity definitions is that each of them is a co-**NP** property in the sense that there is a polynomially verifiable certificate for their failure. For example, if a digraph is not k-edge-connected, we can exhibit a proper subset X of nodes with less than k entering edges. Or, if a hypergraph is not k-partition-connected, we can exhibit a proper partition $\mathcal{P}$ of V to which the number of cross-hyperedges is less than $k(|\mathcal{P}| - 1)$.

We are going to prove that each of these properties belongs to **NP** as well. The proof is sometimes easy, while in other cases only a relatively deep theorem helps. Proposition 1.1.6, for example, immediately implies that an undirected graph $G = (V, E)$ is connected if and only if there is a uv-path for every pair $\{u, v\}$ of nodes. Here the certificate for the connectivity of G consists of a list of $n(n - 1)/2$ paths. Such a list is clearly polynomially verifiable, and hence it is completely satisfactory from the point of view of undirected connectivity belonging to **NP**. It is a different matter (important from an aesthetic or an efficiency aspect) to find simpler or more succint certificates. For example, it is obvious that a list of only $n - 1$ paths (each connecting a fixed node u with the other $n - 1$ nodes) already certifies the connectivity of G. We present some other simple and well-known characterizations for which the proofs are elementary and left to the reader.

Proposition 1.2.18 *For a graph $G = (V, E)$ the following are equivalent:*

(A) *G is a tree.*

(B) *G is a connected graph including no circuit.*

(C) *In G, there is a unique path between any pair of nodes.*

(D) *G is connected and $|E| = |V| - 1$.*
(E) *G can be built up from any of its nodes by consecutively adjoining edges so that exactly one end-node of the currently added edge belongs to the graph already constructed.* •

Proposition 1.2.19 *For a graph G, the following are equivalent.*

(A) *G is connected.*
(B) *There is a path between each pair of nodes of G.*
(C) *G includes a spanning tree.*
(D) *G can be built up from any of its nodes by consecutively adjoining edges so that at least one end-node of the currently added edge belongs to the graph already constructed.* •

Let $G = (V, E)$ be a connected graph and $T = (V, F)$ a spanning tree of G where $F \subseteq E$. For a non-tree edge $e = uv \in E - F$, the unique path in T connecting u and v and the edge e forms a circuit, called the **fundamental circuit belonging to** e. For a tree edge $f \in F$, the **fundamental cut belonging to** f is the cut of G for which the shores are the two components of the forest $T - f$.

Let us call two nodes of a graph $G = (V, E)$ **path-equivalent** if there is a path connecting them. This is an equivalence relation: from the definition of paths it is symmetric and reflexive, and by Proposition 1.1.5, it is transitive.

Proposition 1.2.20 *The equivalence classes of the binary relation of path-equivalence are exactly the components of G. In other words, for nodes u and v, there is a uv-path in G if and only if u and v belong to the same component of G.* •

Proposition 1.2.21 *For a digraph D, the following are equivalent:*

(A) *D is an arborescence.*
(B) *D contains a node r_0 such that every node can be reached from r_0 by a unique directed walk.*
(C) *D contains a node r_0 such that every node can be reached from r_0 and deleting any edge yields a node that is not reachable from r_0. That is, D is a minimally root-connected digraph.*
(D) *D can be built up from a node r_0 by adjoining arcs sequentially so that the tail of the currently added new arc belongs to the digraph already constructed, while the head is a new node.* •

Corollary 1.2.22 *For a digraph D the following are equivalent:*

(A) *D is root-connected from a root r_0.*
(B) *There is a path from r_0 to every node of D.*
(C) *D includes a spanning r_0-arborescence.*
(D) *D can be built up from a node r_0 by adjoining arcs sequentially so that the tail of the currently added new arc belongs to the digraph already constructed.* •

Lemma 1.2.23 (Reachability lemma) *Let S denote the set of nodes of a digraph $D = (V, A)$ that are reachable from a node r_0. Then every proper non-empty subset S' of S containing r_0 admits an edge leaving S' while no edge leaves S. Moreover, there is an r_0-arborescence spanning S.*

Proof. If indirectly an edge uv leaves S, then v would also be reachable from r_0, since $u \in S$ is reachable by definition. In other words, there is a path P from r_0 to u. Then $P + uv$ would be a path from r_0 to v, showing that v is reachable, too.

If indirectly no edge leaves S', then no node outside S' is reachable, contradicting the definition of S. Finally, let F be a maximal r_0-arborescence. We claim that its node-set S' is exactly S. Since every node of F is reachable form r_0, we have $S' \subseteq S$. If indirectly $S' \subset S$, then there is an edge $e = uv$ leaving S'. But then by adding e to F we would get a larger arborescence, contradicting the maximal choice of F. •

Root-connected orientation

In later chapters, we will show that graph orientations often provide a link between directed and undirected connectivity properties. The first result of this type is so trivial that it hardly deserves to be called a theorem. The only reason we still mention it is that it serves as a starting point for several exciting extensions.

Theorem 1.2.24 *A graph $G = (V, E)$ with a special root-node r_0 has a root-connected orientation from r_0 if and only if G is connected.* •

Actually, this observation can easily be generalized to mixed graphs.

Theorem 1.2.25 *A mixed graph $H = (V, A + E)$ has an orientation so that the resulting digraph is root-connected from r_0 if and only if $\varrho_A(X) + d_E(X) \geq 1$ holds for every non-empty subset $X \subseteq V - r_0$.*

Proof. Necessity is straightforward. To see sufficiency, replace each undirected edge by two directed parallel edges oriented oppositely. By the specified conditions, the resulting digraph is root-connected from r_0. Hence it contains a spanning r_0-arborescence F. This arborescence determines an orientation of its originally undirected edges, while the other undirected edges may be arbitrarily oriented. •

1.3 Special graphs and digraphs

The concepts of Eulerian and bipartite graphs have already been introduced. In this section, we provide an overview of some other important classes of graphs and list their elementary properties.

1.3.1 Bipartite and Euler graphs

Theorem 1.3.1 *A graph G is bipartite if and only if G includes no odd circuits.* •

Theorem 1.3.2 *Let B be a cut of a bipartite graph G. Contracting the edges in B results in a bipartite graph.* •

Theorem 1.3.3 *For a graph $G = (V, E)$, the following are equivalent.*

(A) *G is Eulerian.*
(B) *The edge-set of G can be partitioned into circuits.*
(C) *$d(X)$ is even for every subset $X \subseteq V$.*

Proof. (A)→(B) If an Euler graph has an edge, then it includes a circuit C, since a forest has a node of degree one. By deleting the edges of C, we obtain an Euler graph, and by induction we are done.

(B)→(C) Since a circuit and a cut have an even number of edges in common, (C) follows.

Property (A) requires that the degree of each node is even, a special case of (C). •

Theorem 1.3.4 *For a digraph $D = (V, A)$, the following are equivalent.*

(A) *D is Eulerian.*
(B) *The edge-set of D can be partitioned into directed circuits.*
(C) *$\varrho(X) = \delta(X)$ for every subset $X \subseteq V$.*

Proof. (A)→(B) If an Euler digraph has an edge, then it includes a directed circuit C, since an acyclic digraph has a node of in-degree zero. By deleting the edges of C, we obtain an Euler digraph and by induction we are done.

(B)→(C) Walking along a directed circuit C, we alternately enter and leave X and hence $\varrho_C(X) = \delta_C(X)$. Hence (B) implies (C).

Property (A) requires that $\varrho(v) = \delta(v)$ for single nodes, a special case of (C). •

Exercise 1.3.1 *Show that a digraph is Eulerian if and only if $\varrho(v) \geq \delta(v)$ for every node v.*

Problem 1.3.2 *Prove that a strongly connected digraph can be made Eulerian by adding parallel edges.*

Theorem 1.3.5 *A digraph D is the union of edge-disjoint st-paths and di-circuits if and only if $\delta(s) \geq \varrho(s)$ and $\varrho(v) = \delta(v)$ holds for every node $v \in V - \{s, t\}$. The number of st-paths in the partition is $\delta(s) - \varrho(s)$.*

Proof. Let D' be the digraph obtained from D by adding $\delta(s) - \varrho(s)$ parallel arcs from t to s. D' is Eulerian and it has a decomposition into di-circuits. Among these circuits, the ones using a new edge correspond to st-paths in D. The second part of the theorem follows by observing that the number of new edges is $\delta(s) - \varrho(s)$. •

In Section 1.1, we called a set of k edge-disjoint st-walks a k-braid from s to t.

Theorem 1.3.6 *A connected digraph D is the edge-set of a k-braid from s to t if and only if $k = \delta(s) - \varrho(s) \geq 0$ and $\varrho(v) = \delta(v)$ holds for every node $v \in V - \{s, t\}$.* •

Theorem 1.3.7 *A graph $G = (V, E)$ has an Euler orientation if and only if G is Eulerian.*

Proof. Necessity is straightforward. Sufficiency follows from Theorem 1.3.3, since every undirected circuit has an Euler orientation. •

Theorem 1.3.8 *Every graph $G = (V, E)$ has a smooth* (= near-Eulerian) *orientation.*

Proof. Let T denote the set of nodes of odd degree. Since the sum of the degrees of all nodes is twice the number of edges, the cardinality of T is even. Let M be a matching on T. By adding M to G one obtains an Euler graph. This has an Euler orientation, for which the restriction to E is a smooth orientation of G. •

Problem 1.3.3 $(*)$ *Let $G = (V, E)$ be an undirected graph and let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2), \ldots, G_t = (V_t, E_t)$ be subgraphs of G so that $\{E_1, \ldots, E_t\}$ is a partition of E. Then G has a smooth orientation the restriction of which to E_j is a smooth orientation of G_j for each $j = 1, \ldots, t$.*

The following problem emerges quite naturally. Suppose we have oriented some of the edges of an Euler graph and our task is to complete this partial orientation so as to get an Euler digraph.

Question 1.3.1 *What is a necesary and sufficient condition for a mixed graph to have an Euler orientation?*

At this point it may be instructive if the reader tries to figure out some necessary conditions. A complete answer will be provided in Theorem 2.3.4.

Theorem 1.3.9 *Every connected Euler graph has an Euler tour. Every connected Euler digraph has a directed Euler tour.*

Proof. By Theorem 1.3.7, it suffices to prove only the directed version. We describe an algorithm which computes in linear time an Euler tour of a connected Euler digraph $D = (V, A)$. Select an arbitrary node s and find a spanning reverse s-arborescence F of D. Starting at s, construct a walk by traversing edges one by one so that an edge can be used only if it was not previously traversed and an edge uv of F can be traversed only when all other edges leaving u has been traversed. Since the digraph is Eulerian, the algorithm terminates when we are at s and all edges leaving s have been traversed.

Lemma 1.3.10 *At termination, all edges of D have been traversed.*

Proof. Since D is Eulerian, it suffices to show for every node that all the edges leaving u are traversed. By the property of termination, this is true for s. By the rule of the algorithm, it suffices to show that the edges of F are traversed. If, indirectly, there is an edge uv of F which is not traversed, select one in such a way that v is as close in F to s as possible. By the termination property, $v \neq s$, and hence F has a unique edge vz. By the choice of uv, this edge is traversed, but then all edges entering v must be traversed, too, contradicting the indirect hypothesis that uv is not traversed. • •

1.3.2 Planar graphs

A graph $G = (V, E)$ is said to be **planar** if it can be embedded into a plane so that the nodes of G correspond to distinct points, while an edge of G joining u and v is represented by a simple curve of the plane joining u and v in such a way that the curves representing e and f are disjoint apart from their end points. Such an embedding is called a plane representation of G. A graph with a plane representation is called a **plane graph**. One can speak analogously of a directed planar graph, where an orientation is assigned to the representing curves. An embedding of a connected planar graph divides the plane into regions. These regions are called the **faces** of the planar embedding of G. There is exactly one unbounded face.

Theorem 1.3.11 *Each face of a 2-connected planar graph is bounded by a circuit of the graph.* •

With each plane graph, we can associate a **dual** plane graph $G^* = (V^*, E^*)$ (along with an embedding) as follows:. The nodes of G^* correspond to the regions of the given embedding of G, while the edges of G^* are in a one-to-one correspondence with the edges of G. Specifically, place a node of G^* into each region of G. G^* is called a **planar dual** of G. If two regions of G share at least one edge, then join the two corresponding nodes of G^* by as many edges as the number of common edges of the two regions. For example, if G is a tree, then there is one region, and the planar dual consists of loops sitting at a node. If G is a circuit, then the planar dual is a graph with two nodes connected by parallel edges. We emphasize that the dual is assigned to a plane embedding of a planar graph G and not to G itself. When D is a plane embedding of a directed planar graph, we define its **directed planar dual** D^* as follows: Orient the dual of the underlying undirected graph of G so that the orientation of e^* is obtained from the orientation of the corresponding e by rotating e clockwise. (See Figure 1.1.)

The connection between cuts and circuits of the planar graph and its dual is rather straightforward.

Theorem 1.3.12 *Let F be a subset of edges of a plane graph G and let F^* be the corresponding subset of edges of its planar dual G^*. F is a circuit of G if and only if F^* is a bond of G^*. F is a bond of G if and only if F^* is a circuit of G^*.* •

This immediately implies the following theorem.

Theorem 1.3.13 *A connected plane graph is bipartite if and only if its planar dual is Eulerian.* •

Theorem 1.3.14 *Let F be a subset of edges of a plane digraph D and let F^* be the corresponding subset of edges of its directed planar dual D^*. F is a di-circuit of G if and only if F^* is a minimal dicut of G^*. F is a minimal dicut of D if and only if F^* is a di-circuit of D^*.* •

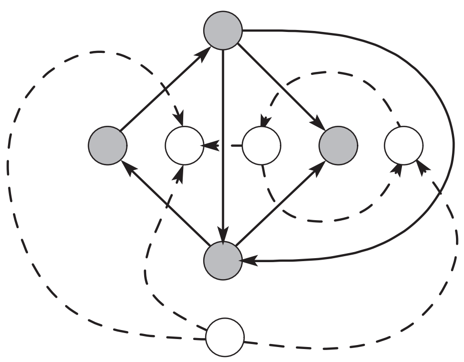


Figure 1.1 Directed dual graph of planar digraph

This immediately implies:

Theorem 1.3.15 *A connected planar digraph is strongly connected if and only if its planar dual digraph is acyclic.* •

Theorem 1.3.16 (Euler's formula) *For a connected plane graph $G = (V, E)$,*

$$|V| + |V^*| = |E| + 2$$

where V^* denotes the node-set of the dual plane graph $G^* = (V^*, E^*)$ (that is, $|V^*|$ is the number of faces of G).

Proof. Let F be a spanning tree of G and let F^* denote the subset of edges of E^* corresponding to the elements of $E - F$. Then $|E| = |F| + |F^*|$. Theorem 1.3.12 implies that F^* is a spanning tree of G^*. Therefore, $|F| = |V| - 1$ and $|F^*| = |V^*| - 1$, from which Euler's formula follows at once. •

Euler's formula implies that the number of faces does not depend on the embedding.

Theorem 1.3.17 (Fáry [98]) *A simple planar graph has a plane representation in which the curves representing the edges are straight lines.* •

Theorem 1.3.18 (Tutte [370]) *A simple planar 3-connected graph has a plane representation in which the curves representing the edges are straight lines and each bounded face is a convex region.* •

Euler's formula implies that neither a complete graph K_5 nor a complete bipartite graph $K_{3,3}$ is planar and hence their subdivisions are not planar either.

Theorem 1.3.19 (Kuratowski [253]) *A graph G is planar if and only if it includes no subdivided K_5 and $K_{3,3}$.* •

1.3.3 Perfect graphs

A graph $G = (V, E)$ is said to be **perfect** if $\chi(G') = \omega(G')$ holds for every induced subgraph G' of G. The following theorem is a fundamental result of Lovász [265].

Theorem 1.3.20 (Weak Perfect Graph theorem) *A graph G is perfect if and only if its complement is perfect.*

Proof. A basic ingredient of the proof is the following lemma.

Lemma 1.3.21 (Replication lemma) *Let $H = (V, F)$ be a perfect graph. For a function $\beta : V \to \mathbf{Z}_+$, let $H^+ := (V^+, F^+)$ denote the graph arising from H by replacing each node v of H by a clique of $\beta(v)$ nodes so that two new nodes u' and v' corresponding to original nodes u and v, respectively, are adjacent in H^+ precisely if u and v are adjacent in H. Then H^+ is also perfect.*

Proof. By repeated applications, the lemma immediately follows from its special case when $\beta(z) = 2$ for a node z and $\beta(v) = 1$ for $v \in V - z$, so we concentrate on proving only this. By induction, we can assume that each induced proper subgraph of H^+ is perfect, so it suffices to show that $\chi(H^+) = \omega(H^+)$. For notational convenience, we assume that the graph H^+ arises from H by adding a new node z', which is connected to z and to all original neighbours of z.

If z belongs to a largest clique K (one with cardinality $\omega(H)$) of H, then $K + z'$ is a clique of H^+, from which $\omega(H^+) = \omega(H) + 1$. In this case, a proper colouring of H with $\omega(H)$ colours can be extended with the colour class $\{z'\}$ to obtain a proper colouring of H^+ with $\omega(H^+)$ colours, as required.

Suppose now that z does not belong to any largest clique of H. Then $\omega(H^+) = \omega(H)$. Since H is perfect, it admits a proper colouring $\{S_1, \dots, S_{\omega(H)}\}$ with $\omega(H)$ colours. Suppose that z belongs to colour class S_1. Consider the induced subgraph H' arising from H by removing the set $S_1 - z$. Since each largest clique of H must intersect each colour class S_i, and in particular S_1, and since z does not belong to any largest clique of H, we conclude that $\omega(H') = \omega(H) - 1$. By the perfectness of H, there is a proper colouring of H' with $\omega(H) - 1$ colours. By adjoining the stable set $S'_1 := S_1 - z + z'$ as an additional colour class, one obtains a proper colouring of H^+ with $\omega(H^+)(= \omega(H))$ colours. •

Let us turn to the proof of the theorem. By symmetry, it suffices to prove that G is perfect if $\overline{G}$ is perfect. Since a subgraph of G induced by a subset $X \subseteq V$ is the complement of the subgraph of $\overline{G}$ induced by X, we can assume by induction that all induced proper subgraphs G' of G are perfect, and thus $\chi(G') = \omega(G')$, so it suffices to prove that $\chi(G) = \omega(G)$.

Assume first that there is a maximal stable set S of G intersecting all largest cliques of G. Then, for the induced subgraph $G' = G - S$, we have $\chi(G) \leq \chi(G') + 1$ and $\omega(G') \leq \omega(G) - 1$. Therefore, $\chi(G') = \omega(G') \leq \omega(G) - 1 \leq \chi(G) - 1 \leq \chi(G')$ from which $\chi(G) = \omega(G)$ follows.

Second, let $\{S_1, S_2, \dots, S_q\}$ be the set of all maximal stable sets of G and suppose that none of them intersects all largest cliques of G, that is, there is a list $\mathcal{K} = \{K_1, \dots, K_q\}$ of not-necessarily-distinct largest cliques of G so that $S_i \cap K_i = \emptyset$ for $i = 1, \dots, q$.

For each node v, let $\beta(v)$ denote the number of subscripts i for which K_i contains v. Let $G^+ := (V^+, E^+)$ denote the graph arising from G by replacing each node v of G by a stable set of $\beta(v)$ nodes so that two new nodes u' and v' corresponding to original nodes u and v, respectively, are adjacent in G^+ precisely if u and v are adjacent in G. (When $\beta(v) = 0$, this simply means the deletion of v.)

By this construction, we have $|V^+| = \sum[|K_i| : i = 1, \dots, q] = q\omega(G)$. Furthermore, the cardinality $\alpha(G^+)$ of the largest stable set of G^+ is $\max\{\widetilde{\beta}(S_i) : i = 1, \dots, q\}$ (where $\widetilde{\beta}(X) = \sum[\beta(v) : v \in X]$). Since each S_i is intersected by at most $q - 1$ members of $\mathcal{K}$ and each of these intersections can have at most one element, we obtain that $\widetilde{\beta}(S_i) \leq q - 1$ and hence $\alpha(G^+) \leq q - 1$.

Note that the complement of G^+ arises from the complement $\overline{G}$ of G by replacing each node v by a clique of $\beta(v)$ elements. By applying the Replication Lemma to the complement of G^+, we obtain that $\overline{G^+}$ is perfect and hence its node-set can be partitioned into $\omega(\overline{G^+})$ stable sets of $\overline{G^+}$. Consequently, the node-set of G^+ can be partitioned into $\omega(\overline{G^+}) = \alpha(G^+) \leq q - 1$ cliques of G^+. Since $|V^+| = q\omega(G)$, we obtain that the largest of these cliques must have strictly more than $\omega(G)$ elements, contradicting the definition of $\omega(G)$. • •

The following graphs and their complements are basic examples of perfect graphs: bipartite graphs, comparability graphs, the line-graph of a bipartite graph, chordal graphs.

Note that every graph on at most four nodes is perfect while an odd hole never is. The following extremely difficult result of Chudnovsky, Robertson, Seymour, and Thomas [51] offers an affirmative answer to a longstanding conjecture of Berge.

Theorem 1.3.22 (Strong Perfect Graph theorem [51]) *A graph is perfect if and only if it includes neither an odd hole nor the complement of an odd hole.* •

1.4 Special hypergraphs and bi-set systems

A partition is a special hypergraph in which the degree of each node is 1. A subpartition was defined as a partition of a subset of V. A **co-partition** of V is a hypergraph arising from a partition of V by complementing its hyperedges. A hypergraph is a **chain of subsets**, or simply a **chain**, if its hyperedges are pairwise comparable (with respect to inclusion). A common generalizaton of chains and subpartitions is the notion of laminar hypergraphs. A hypergraph H is **laminar** if it has no two properly intersecting members, that is, every two hyperedges are comparable or disjoint. Let Z be a proper subset of the ground-set V. For a partition $\{Z_1, \dots, Z_t\}$ of $V - Z$, we call the set-system $\mathcal{F} = \{V - Z_1, V - 1Z_2, \dots, V - Z_t\}$ a **co-partition** of Z. Note that $\mathcal{F}$ forms a co-subpartition of V. H is **cross-free** if it has no two crossing members. Cross-free hypergraphs form a generalization of laminar hypergraphs and co-partitions. A family $\mathcal{F}$ of bi-sets is **laminar** if it has no two properly intersecting members. $\mathcal{F}$ is **cross-free** if it has no two crossing members.

We call a family $\mathcal{F}$ of sets or bi-sets a **ring-family** (or a ring, for short) if $\mathcal{F}$ is closed under taking union and intersection. $\mathcal{F}$ is an **intersecting** family if both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{F}$ whenever X and Y are intersecting. $\mathcal{F}$ is a **crossing** family if both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{F}$ whenever X and Y are crossing.

A hypergraph $H = (V, \mathcal{F})$ is **intersecting** or **crossing** if $\mathcal{F}$ is intersecting or crossing, respectively.

Problem 1.4.1 *Prove that a hypergraph is cross-free if and only if complementing each of its hyperedges containing a specified node yields a laminar hypergraph.*

Problem 1.4.2 *If a hypergraph H is the union of two partitions of V, then the transpose of H is a bipartite graph. If a hypergraph H is the union of two chains of subsets, then the transpose of H is isomorphic to a hypergraph on a path consisting of subpaths.*

1.4.1 Representing chains and laminar and cross-free hypergraphs

Chains

Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq V$ be the hyperedges of a chain of subsets. Such a chain can be encoded by a function $\pi : V \to \mathbf{Z}_+$, where $\pi(v)$ denotes the degree of v for $v \in V$. Every non-negative, integer-valued function $\pi : V \to \mathbf{Z}_+$ can be obtained in this form, as follows. Let $p_1 < p_2 < \cdots < p_h$ denote the distinct values of π. Consider the hypergraph in which the set $X_1 := \{v : \pi(v) \geq p_1\}$ occurs in p_1 copies (meaning, in particular, that X_1 does not occur when $p_1 = 0$) and, for $i = 2, \dots, h$, the set $X_i := \{v : \pi(v) \geq p_i\}$ occurs in $p_i - p_{i-1}$ copies. The degree function of the chain of subsets obtained in this way is easily verified to be π.

Laminar hypergraphs

There is a simple way to construct laminar hypergraphs. Consider an s-arborescence $H = (V, F)$. For every edge $e = uv \in F$, let V_e denote the set of nodes reachable from v in H. Then the family of these sets forms a laminar hypergraph. The following useful observation of Edmonds and Giles [86] shows that essentially every laminar family can be obtained in this form.

Theorem 1.4.1 *For every laminar hypergraph $L = (V, \mathcal{L})$, there exists an arborescence $H = (U, F)$ along with a map $\varphi : V \to U$ so that the hyperedges in $\mathcal{L}$ and the edges of H are in a one-to-one correspondence as follows. For every edge $e \in F$, the hyperedge of L corresponding to e is $\varphi^{-1}(V_e)$, where V_e denotes the component of $H - e$ that is entered by e.*

Proof. We can assume that $\mathcal{L}$ is simple, since if a simple laminar hypergraph has the requested representation, then the subdivision of an edge e of the arborescence by a node correponds to the addition of a parallel copy of the hyperedge corresponding to e.

We may also assume that every element $v \in V$ belongs to a member of $\mathcal{L}$. Denote the smallest of these hyperedges by $\sigma(v)$. With every hyperedge $X \in \mathcal{F}$, associate a new node $f(X)$, and let s be a further additional node. These nodes will form the node-set U of the arborescence (and hence U has one more element than $\mathcal{F}$).

Construct an arborescence F on U as follows. Every maximal member X of $\mathcal{F}$ defines an edge from s to $f(X)$. For a non-maximal member X of $\mathcal{L}$, there is a unique smallest member $Y \in \mathcal{F}$ including X. In this case, X defines an edge from $f(Y)$ to $f(X)$. Through this approach, we obtain an s-arborescence H. Finally, for every node $v \in V$, let $\varphi(v) := f(\sigma(v))$. It is immediately seen from the construction that the arborescence H and the map φ meet the requirements of the theorem. •

Cross-free hypergraphs

How can we represent cross-free hypergraphs? If F is a directed tree (not necessarily an arborescence) and, among the two components of $F - e$, V_e denotes the one entered by e, then these sets V_e form a cross-free hypergraph. Again, the following result from [86] describes a kind of converse:

Theorem 1.4.2 *For every cross-free hypergraph $C = (V, \mathcal{C})$, there exists a directed tree $H = (U, F)$, along with a map $\varphi : V \to U$, so that the hyperedges in $\mathcal{C}$ and the edges of H are in a one-to-one correspondence, as follows. For every edge $e \in F$, the corresponding hyperedge of L is $\varphi^{-1}(V_e)$.*

Proof. Let z be an arbitrary element of V. Let L be the hypergraph arising from C by complementing each hyperedge containing z. The resulting $\mathcal{C}'$ is laminar. Theorem 1.4.1 ensures the existence of an arborescence and a map representing $\mathcal{C}'$. By reorienting each edge of the arborescence that corresponds to a complemented hyperedge of the original $\mathcal{C}$, we obtain the requested representation of $\mathcal{C}$. •

Note that we can allow that the preimage $\varphi^{-1}(w)$ in V of a node w of the directed tree to be empty. In this case, w is called an **empty node**.

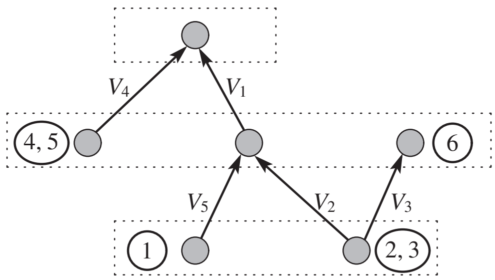


Figure 1.2 Tree-representation of cross-free hypergraph

A tree-representation of the cross-free hypergraph $\mathcal{C}$ on node-set $V = \{1, 2, 3, 4, 5, 6\}$ with hyperedges $V_1 = \{4, 5\}$, $V_2 = \{1, 4, 5\}$, $V_3 = \{6\}$, $V_4 = \{1, 2, 3, 6\}$, $V_5 = \{2, 3, 4, 5, 6\}$ is shown in Figure 1.2. The encircled numbers show the image of V, while the letters V_i on the tree edges show the hyperedge of $\mathcal{C}$ corresponding to the edge. There are two empty nodes in the representation.

It can be proved by a simple induction that any directed tree (U, F) admits a function $\pi : U \to \{0, 1, \dots\}$ so that $\pi(v) - \pi(u) = 1$ for each $uv \in F$. We call π a **level function** and $\pi(v)$ the **level** of v. The following lemma will be useful in Section 15.3.

Lemma 1.4.3 *Let (F, φ) be a tree-representation of a cross-free hypergraph $C = (V, \mathcal{C})$ and let π be a level function of F. For any two elements s, s' of V with $d_C(s) \geq d_C(s')$, the difference of the levels of $\varphi(s)$ and $\varphi(s')$ is $d_C(s) - d_C(s')$.*

Proof. Let P denote the unique path in F (in the undirected sense) connecting $\varphi(s)$ and $\varphi(s')$. If $f \in F - P$, then the member of $\mathcal{C}$ corresponding to f does not separate s and s', and hence it does not affect the difference $d_C(s) - d_C(s')$. There is a one-to-one correspondence between the members of $\mathcal{C}$ separating s and s' and the edges of P, namely, $s\overline{s'}$-sets in $\mathcal{C}$ correspond to the forward edges of P when we start from s', while $s'\overline{s}$-sets in $\mathcal{C}$ correspond to the backward edges of P. Therefore, $d_C(s) - d_C(s')$ is the difference of the number of forward and backward edges of P, which is exactly the difference of the level of $\varphi(s)$ and the level of $\varphi(s')$. •

Exercise 1.4.3 *Prove that if $\mathcal{F}$ is a cross-free hypergraph in which there are no two comparable members, then $\mathcal{F}$ is a subpartition or a co-subpartition.*

Problem 1.4.4 *Prove that the edge-set of a regular and cross-free hypergraph can be decomposed into partitions and co-partitions of V. (See the proof of Lemma* 15.3.1.*)*

1.4.2 Representing ring-families and encoding intersecting and crossing set systems

Representing ring-families

The intersection R_0 (respectively, the union V') of all members of a ring-family $\mathcal{R}$ is the unique smallest (largest) member of $\mathcal{R}$. By replacing each member X of $\mathcal{R}$ with $X - R_0$

and deleting $V - V'$ from V, we obtain a ring-family $\mathcal{R}'$ containing the empty set and the (modified) ground-set. It suffices to represent only such ring-families, since $\mathcal{R}$ arises from $\mathcal{R}'$ by extending each member with a specified subset R_0 of new elements and extending the ground-set further with other new elements.

Theorem 1.4.4 *Let $\mathcal{R}$ be a simple hypergraph containing $\emptyset$ and V. The following are equivalent.*

(A) *$\mathcal{R}$ is a ring-family.*
(B) *There is a unique simple transitively closed digraph $D = (V, A)$ so that $\mathcal{R} = \{X \subseteq V : \varrho_D(X) = 0\}$.*
(C) *There is a digraph $D' = (V, A')$ so that $\mathcal{R} = \{X \subseteq V : \varrho_{D'}(X) = 0\}$.*

Proof. (A)→(B) Let $A := \{uv$: there is no $v\bar{u}$-set in $\mathcal{R}\}$. Then D is transitively closed, and $X \in \mathcal{R}$ implies $\varrho_D(X) = 0$. Conversely, let X be a set with $\varrho_D(X) = 0$. For every element $v \in X$, the intersection $P(v)$ of the members of $\mathcal{R}$ containing v is in $\mathcal{R}$. We claim that $P(v) \subseteq X$, for if there is an element $u \in P(v) - X$, then uv does not enter any member of $\mathcal{R}$, that is, uv would belong to A, contradicting the assumption $\varrho_D(X) = 0$. It follows that $X = \cup\{P(v) : v \in X\}$, from which $X \in \mathcal{R}$, since R is closed under union.

To see that D is unique, let D_1 be another simple transitively closed digraph on node-set V. Let uv be an edge belonging to exactly one of the two digraphs, say to D. Then the set X of nodes from which v is reachable in D_1 does not contain u. Hence X is in $\mathcal{R}$ but $\varrho_D(X) \geq 1$.

Property (C) is a special case of (B), and (C) clearly implies (A). •

The unique digraph in property (B) is called the **representing digraph** of the ring-family.

Corollary 1.4.5 *The atoms of a simple ring-family are exactly the strongly connected components of its representing digraph.* •

Theorem 1.4.6 *The ideals of a poset P form a simple ring-family $\mathcal{R}$. A simple ring-family $\mathcal{R}$ on V is the set of ideals of a poset if and only if $\emptyset, V \in \mathcal{R}$ and every atom of $\mathcal{R}$ is a singleton.*

Proof. The first part of the theorem follows directly from the defining properties of a poset. It is also clear that the empty set, as well as the ground-set, is an ideal. Next, we show that in a poset there is an ideal separating any two distinct elements u and v. Indeed, consider the ideals $I_u := \{z : z \preceq u\}$, $I_v := \{z : z \preceq v\}$. Clearly, $u \in I_u$ and $v \in I_v$, and it is not possible that $v \in I_u$ and $u \in I_v$, since then we would have $u \preceq v$ and $v \preceq u$. Hence at least one of I_u and I_v is an ideal separating u and v. It follows that every atom in the ring-family of the ideals is a singleton.

For the last part of the theorem, consider a ring-family $\mathcal{R}$ with the given property. By Theorem 1.4.4, there is a simple transitively closed digraph $D = (V, A)$ so that $\mathcal{R} = \{X \subseteq V : \varrho_D(X) = 0\}$. Since every atom is a singleton, this digraph is acyclic, and it determines a poset $P = (V, \preceq)$ in which $u \succeq v$, by definition, if vu is an arc of D. By this definition, the ideals of P are exactly those subsets X of V for which $\varrho_D(X) = 0$. •

Theorem 1.4.7 *Let $\mathcal{R}$ be a ring-family with $\emptyset \in \mathcal{R}$ on a ground-set V and $h : \mathcal{R} \to \mathbf{R}$ a finite-valued modular set-function on $\mathcal{R}$ for which $h(\emptyset) = 0$. Then there is a function $m : V \to \mathbf{R}$ for which*

$$h(X) = \widetilde{m}(X) \text{ for every } X \in \mathcal{R}.$$

Proof. We can assume that the largest member of $\mathcal{R}$ is V. It can also be assumed that each atom of $\mathcal{R}$ is a singleton, since any atom can be shrunk into a node. By Theorem 1.4.6, there is a poset $P = (V, \preceq)$ so that $\mathcal{R}$ is the set of ideals of P. Let v_1 be a smallest element of P, v_2 a smallest element of $P - v_1$, v_3 a smallest element of $P - \{v_1, v_2\}$, and so forth. In this way we define a linear ordering $v_1, \ldots, v_n$ of P so that $1 \le i < j \le n$ implies that $v_j \not\preceq v_i$. Equivalently, every starting segment $V_i := \{v_1, \ldots, v_i\}$ is an ideal of P. Define $m : V \to \mathbf{R}$ by

$$m(v_i) := \begin{cases} h\{v_1\} & \text{if } i = 1 \\ h(V_i) - h(V_{i-1}) & \text{if } i = 2, \ldots, n \end{cases} \tag{1.26}$$

Claim 1.4.8 $h(X) = \widetilde{m}(X)$ *for every ideal of P.*

Proof. The proof proceedes by induction on the cardinality of ideals. There is nothing to prove for the empty ideal, and from the definition of m we also have $h(V_i) = \widetilde{m}(V_i)$ for $i = 1, \ldots, n$. Consider a non-empty ideal X and assume inductively that $h(X') = \widetilde{m}(X')$ holds for every proper subset X' of X that is an ideal. Let v_i be an element of X for which i is maximal. Then $X' := X - v_i$ is an ideal. If $i = 1$, then $X = \{v_1\}$ and $h(X) = m(v_1) = \widetilde{m}(X)$. Suppose that $i \ge 2$. By applying the modularity of h, we get

$$\begin{aligned} \widetilde{m}(V_{i-1}) + h(X) &= h(V_{i-1}) + h(X) = h(V_{i-1} \cap X) + h(V_{i-1} \cup X) \\ &= h(X') + h(V_i) = \widetilde{m}(X') + \widetilde{m}(V_i) \end{aligned}$$

from which

$$h(X) = \widetilde{m}(X') + \widetilde{m}(V_i) - \widetilde{m}(V_{i-1}) = \widetilde{m}(X') + m(v_i) = \widetilde{m}(X).$$

completing the proof of the claim and the theorem. • •

Encoding intersecting and crossing set systems

Let $\mathcal{F}$ be an intersecting set-system. No general construction is known to represent $\mathcal{F}$, but at least there is a way to encode them in polynomial space. Namely, for each node v, $\mathcal{F}_v := \{X \in \mathcal{F}, v \in X\}$ is a ring-family. It follows from Theorem 1.4.4 that $\mathcal{F}_v$ can be represented by a digraph. Therefore, $\mathcal{F}$ can be encoded with the help of these n digraphs.

There is another way to encode an intersecting set-system $\mathcal{F}$ even more concisely: with $2n - 2$ appropriate subsets of V. The maximal members of $\mathcal{F}$ not containing an element $s \in V$ form a subpartition $\mathcal{F}_s$ of $V - s$ for every $s \in V$. Consider the set-system $\mathcal{F}^*$ consisting of the distinct members of $\cup(\mathcal{F}_s : s \in V)$. Obviously, $\mathcal{F}^*$ has at most $n(n-1)$ members. Bernáth [25] proved that $\mathcal{F}^*$ has at most $2n - 2$ members. (He actually proved this bound even for weaker systems in which only the union of two intersecting members is required to belong to $\mathcal{F}$.) The nice thing is that $\mathcal{F}^*$ is not only small but it encodes $\mathcal{F}$ as well, in the following sense.

Lemma 1.4.9 *A subset $X \subset V$ is a member of an intersecting set-system $\mathcal{F}$ if and only if X is the intersection of the members of $\mathcal{F}^*$ including X.*

Proof. Suppose first that X is in $\mathcal{F}$ and let X' denote the intersection of the members of $\mathcal{F}^*$ including X. Clearly, $X \subseteq X'$, and here we must have equality, since if there is an $s \in X' - X$, then there is a member of $\mathcal{F}_s$ that includes X but does not contain s, contradicting the definition of X'.

Conversely, if $X \subset V$ is the intersection of some members of $\mathcal{F}^*$, then $X \in \mathcal{F}$, since $\mathcal{F}$ is intersecting. •

Let $\mathcal{F}$ be a crossing family. Select an arbitrary node s. Then the members of $\mathcal{F}$ not containing s forms an intersecting family. Similarly, the complements of the members of $\mathcal{F}$ containing s form another intersecting family. Therefore, a crossing family can be encoded by $2n$ digraphs, as well as by $2(2n-2)$ subsets of V.

Theorem 1.4.10 *Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_k$ be a chain of set-systems* (or simple hypergraphs) *on a ground-set V of n elements such that each $\mathcal{F}_i$ contains the empty set and V. If each $\mathcal{F}_i$ is a ring-family, then $k \leq n(n-1)$. If each $\mathcal{F}_i$ is an intersecting family, then $k \leq n(n-1)(n-2)$. If each $\mathcal{F}_i$ is a crossing family, then $k \leq 2n(n-1)(n-2)$.*

Proof. By Theorem 1.4.4, there are transitively closed simple digraphs $D_i = (V, A_i)$ representing ring-families $\mathcal{F}_i$. The assumption $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_k$ implies that $A_1 \supset A_2 \supset \cdots \supset A_k$. Since $|A_1| \leq n(n-1)$, we have $k \leq n(n-1)$.

Suppose now that each F_i is an intersecting family. Let $\mathcal{F}_i^v$ denote the subsystem of $\mathcal{F}_i$ consisting of sets containing v. Then $\mathcal{F}_i^v$ is a ring-family. By the first part of the theorem, a chain of distinct ring-families of sets containing a given node can have at most $(n-1)(n-2)$ members. We say that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ is caused by v if $\mathcal{F}_i^v \subset \mathcal{F}_{i+1}^v$. (Naturally, $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ may be caused by several nodes v.) Since one node may cause $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for at most $(n-1)(n-2)$ distinct subscripts i, we can conclude that $k \leq n(n-1)(n-2)$.

Finally, suppose that each F_i is a crossing family. For a specified node s, the members of $\mathcal{F}_i$ not containing s form an intersecting family $\mathcal{F}_i'$, while the complements of the members of $\mathcal{F}_i$ containing s form another intersecting family $\mathcal{F}_i''$. It follows from the second part of the theorem that $k \leq 2n(n-1)(n-2)$. •

1.4.3 Totally unimodular and normal hypergraphs

Normal hypergraphs

A hypergraph H is said to admit the **Helly property** if every set of pairwise intersecting hyperdges can be covered by one node. This is equivalent to requiring that $\tau(H') = \nu(H')$ holds for every subhypergraph H' of H for which $\nu(H') = 1$. A hypergraph H is **τ-normal** if $\tau(H') = \nu(H')$ holds for every subhypergraph H' of H. A hypergraph H is **Δ-normal** if $\Delta(H') = \chi'(H')$ for every subhypergraph H' of H. Lovász' Perfect graph theorem (Theorem 1.3.20) is equivalent to the following result.

978–0–19–920527–1 01-Andras-Frank-c01-drv Andras Frank (Typeset by SPi, Publisher Services, India) 41 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 1.4.11 (Lovász [265]) *A hypergraph H is τ-normal if and only if H is Δ-normal.* •

This theorem allow the notions of τ-normal and Δ-normal to be combined into one, and we simply speak of **normal** hypergraphs. A **subtree hypergraph** H is defined on the node-set of a tree T so that each hyperedge is a subtree of T. Such a tree is called a **basic tree** of H. In Section 3.2, we will show how Kruskal's algorithm for computing a cheapest spanning tree can be used to test whether a hypergraph is a subtree hypergraph. It can be proved that a subtree hypergraph is normal.

Theorem 1.4.12 *A hypergraph $H = (V, \mathcal{E})$ is a subtree hypergraph if and only if H admits the Helly property and the line-graph $L(H)$ of H is chordal.* •

Totally unimodular hypergraphs

A matrix Q is **totally unimodular** (TU) if each subdeterminant of Q is of value 0, 1, or -1. A hypergraph is **totally unimodular** if its incidence matrix is totally unimodular. It follows from the definition that the transpose of a TU hypergraph is also TU. For example, if the node-set of H is the edge-set of a directed tree T and the hyperedges are directed subpaths of T, then T can be proved to be totally unimodular. For details, see Section 4.2.

1.5 Complexity of problems and algorithms

We assume familiarity with such basic notions as polynomial time algorithms, **NP**-complete problems, and the problem classes of **NP** and co-**NP**. This section sketches an intuitive reminder, as well as a list of some basic **NP**-complete problems (see [6, 177]).

1.5.1 Basic notions

The class of properties for which existence can be decided in polynomial time is denoted by **P**. Also, the problem of seeking to decide if such a property holds or not is said to be in **P**.

A property belongs to **NP** if there is a configuration, often called a certificate, which verifies the existence of the property and is verifiable in polynomial time. For example, the property of a graph being Hamiltonian is in **NP**, since a Hamilton circuit is an appropriate certificate: a proposed circuit of the graph G can easily be tested in polynomial time whether it contains each node of G. Often the problem concerning the given property is said to be in **NP**. A property (or problem) belongs to co-**NP** if its negation is in **NP**, that is, if there is a polynomially verifiable certificate verifying the non-existence of the property. For example, the Euler tour problem is in co-**NP**, since a node of odd degree is a polynomially verifiable certificate for the non-existence of an Euler tour. From the definitions, $\mathbf{P} \subseteq \mathbf{NP} \cap$ co-$\mathbf{NP}$, and it is a fundamental open problem whether equality holds. Without further reference, we shall assume throughout this book the hypothesis that $\mathbf{P} \neq \mathbf{NP} \cap$ co-$\mathbf{NP}$.

A property in **NP** is said to be **NP**-complete if any other problem from **NP** has a polynomial reduction to it. Cook [54] proved that there are **NP**-complete problems. Karp [232] showed, among others, that deciding if a graph has a Hamilton circuit is an **NP**-complete problem. This result implies that if there is a polynomial time algorithm for finding a Hamilton circuit, then there is one for every problem in **NP**. The class of **NP**-complete problems is denoted by **NPC**. The hypothesis $\mathbf{P} \neq \mathbf{NP} \cap \text{co-}\mathbf{NP}$ implies that $\mathbf{P} \cap \mathbf{NPC} = \emptyset$. An interesting example is the property of perfectness of a graph. From its definition, it is not clear if this property belongs to **NP** or to co-**NP**. It actually belongs to both, but these theorems are difficult to prove, particularly the one asserting that perfectness is in **NP**.

For a property A in **NP**, a **good characterization** is another property B in **NP** along with a theorem stating that A holds if and only if B does not. A simple example for such a good characterization is the theorem stating that a graph G is bipartite if and only if G has no odd circuits. Sometimes a good characterization can be rephrased in the form of a min-max theorem. For example, a basic theorem of Kőnig can be formulated so as to state that a bipartite graph has a k-element matching if and only if it is not possible to cover all edges by less than k nodes, while the min-max version asserts that the maximum number of disjoint edges is equal to the minimum number of nodes covering all edges. (Concisely, in a bipartite graph the matching number and the transversal number are equal.) The two forms are obviously equivalent. In this sense, a min-max theorem is also considered a good characterization.

Sometimes we investigate weighted, capacitated, or min-cost versions of problems. For example, one may be interested in deciding if there is a directed circuit of negative total weight in an edge-weighted directed graph. A result of Gallai [179] asserts that there is such a negative circuit if and only if there is no feasible potential π (see Theorem 3.1.1) where π is a function on the node-set and feasibility means that $\pi(v) - \pi(u) \leq c(uv)$ holds for every edge uv of the digraph. Apparently this is a good characterization, since the existence of a feasible potential certifies at once that the total weight of every di-circuit C is non-negative: $\widetilde{c}(C) = \sum_{i=1}^{k} c(v_i v_{i+1}) \geq \sum[\pi(v_{i+1}) - \pi(v_i)] = 0$. Strictly speaking, however, Gallai's theorem in this form cannot be considered as a good characterization since the theorem does not explicitly exclude the troublesome theoretical possibility that the values of the feasible π may be so extremely large that they cannot be written down in polynomial space and time even if the components of the input cost function c are small integers. In order to have a genuine good characterization, a theorem should explicitly state that, for rational c, the size of the feasible π is polynomial in the size of c. It turns out that this is indeed the case and the proof of the theorem in Section 3.1 actually does provide such a 'small' π. It turns out that the situation is similar in many other good characterizations and min-max theorems where weight or cost functions appear, as well. Since this issue is outside the main course of the book, after this warning it will not be considered anymore.

There is yet another subtle point to be touched upon concerning min-max theorems when a weight function is involved. The weighted extension of Kőnig's theorem, found by Egerváry (see Theorem 3.3.2), asserts that the maximum weight of a matching in a bipartite graph with respect to a weight function w on the edge-set is equal to $\min\{\sum \pi(v) \in V : \pi(u) + \pi(v) \geq w(uv)$ for every edge $uv\}$ where the minimum is taken over all non-negative functions π on the node-set V. The difficulty here is that the existence of the minimum is not a priori true because we minimize over an infinite set. (The existence of the maximum here is not a problem, since there are only a finite number of matchings.) Therefore, in this min-max theorem as well as in all others in this book (most importantly in the duality theorem of linear

978–0–19–920527–1 01-Andras-Frank-c01-drv Andras Frank (Typeset by SPi, Publisher Services, India) 43 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

programming), the statement that a certain maximum is equal to a minimum of something else always implicitly includes the tacit additional statement that the extrema in question do exist and that they are equal to each other.

The complexity of an algorithm measures how the number of steps in the worst case depends on the size of the problem instance. We shall consider combinatorial algorithms working on such structures as graphs, hypergraphs, posets, and so forth, where the input includes a description of the structures in question as well as accompanying data like capacity or cost functions that include numbers. A subtle point here is that the size of an integer should not be confused with its value. The size of an integer K is the number of its digits, which is roughly the logarithm of K. In this sense, the value of K depends exponentially on the size of K. Therefore, an algorithm for which its complexity is proportional to the value of an input number is not polynomial if this number can be large.

One may feel that these concerns are mostly of theoretical importance, as they may cause problems only if the input numbers are so big that their logarithm is an exponential function of the size of the combinatorial structure. But this view might be misleading. Imagine a situation in which a minimum capacitated cut is to be computed in a small network—on ten nodes, say—but the capacities fall between one and two billions. No one would be content with an algorithm that requires a billion steps. It is quite typical that even if a polynomial algorithm is available for the uncapacitated version of an optimization problem, developing one for its capacitated extension demands new ideas.

A polynomial algorithm is called **strongly polynomial** if its complexity does not depend at all on the size of the input numbers in the computational model when each basic operation (addition, subtraction, multiplication, division, comparison) on integers is considered as a single step. This model often makes sense, since in order to judge properly the theoretical efficiency of a combinatorial algorithm, it is quite natural to disregard the complexity of subroutines needed to carry out elementary operations on integers. Throughout this book, all complexity estimations are understood within the use of this model: the basic operations with numbers are assumed to take one step independent of the size of the numbers or the way in which they are given.

Yet another fine distinction is in order. Finding a k-element stable subset of a graph on n nodes is known to be **NP**-complete. There is, however, a simple brute force algorithm that enumarates all k-element subsets of nodes and tests each individually for being stable. Since there are $\binom{n}{k}$ k-element subsets, this algorithm is clearly exponential. However, if k is considered fixed, then the algorithm depends polynomially on n (since the power of n in the complexity is a fixed constant).

*1.5.2 A list of basic **NP**-complete problems*

Within combinatorial opimization, there is a multitude of **NP**-complete problems [177], [6]. Below we list only a few of them.

Graphs and digraphs

The edges of a graph can be covered by k nodes.

There is a stable set (or a clique) of size k.

The chromatic index of a 3-regular graph is 3. [210]

The chromatic number of a graph is 3.

There is a Hamilton circuit (or path) in a graph or a digraph.

There is a circuit in a graph of size at least k.

Given a $(0, \pm 1)$-valued (or an arbitrary) cost function on the edge-set of a digraph, determine the cheapest di-circuit.

There is a cut of size at least k (:the max cut problem).

A graph has a 2-edge-connected spanning subgraph with at most k edges.

A graph includes an st-path and a spanning tree (alternatively: a spanning connected subgraph) that are edge-disjoint. [29]

The edge-set of a graph partitions into a tree and a spanning tree. [29]

A digraph has a strongly connected spanning subgraph with at most k edges.

There are k nodes of a digraph covering all di-circuits (the feedback problem). There are k edges of a digraph covering all di-circuits (the feedback arc-set problem). The problem is **NP**-complete even for tournaments. [4, 45]

There is a dipath P_1 from s_1 to t_1 and a dipath P_2 from s_2 to t_2 in a digraph so that P_1 and P_2 are edge-disjoint (node-disjoint).

There are dipaths from s_i to t_i for $i = 1, \ldots, k$ in an acyclic digraph that are edge-disjoint (node-disjoint).

There are paths from s_i to t_i for $i = 1, \ldots, k$ in an undirected graph that are edge-disjoint (node-disjoint).

There is a tree of size at most k covering a specified subset of nodes (:the Steiner tree problem).

There is an arborescence in a digraph of size at most k covering a specified subset of nodes (:the directed Steiner tree problem).

There are 2 edge-disjoint arborescences covering a specified subset of nodes (problem of disjoint Steiner arborescences).

A bipartite $G = (V, U; E)$ graph has a subgraph in which the degree of every node z belongs to a specified set $F(z)$ of feasible degrees assigned to every node z. The problem is **NP**-complete even in the special case when $d_G(u) = 3$ for every $u \in U$, $F(u) = \{0, 3\}$ for every $u \in U$, and $F(v) = \{1\}$ for every $v \in V$.

Hypergraphs

The chromatic number of a hypergraph is 2.

There are two disjoint transversals of a hypergraph.

There is a perfect matching in a 3-uniform hypergraph.

The hyperedges can be covered by k nodes.

The node-set can be covered by k hyperedges.

There are two edge-disjoint connected spanning sub-hypergraphs. [144]

2

Connectivity, paths, and matchings

In this chapter, we provide an overview of some basic results on bipartite matchings and disjoint paths, as well as on graph connectivity.

2.1 2-connection of graphs

2.1.1 2-edge-connected undirected graphs

As we already defined, a graph G is 2-edge-connected if every non-trivial cut has at least two edges, that is, if G is connected and contains no cut-edges.

Theorem 2.1.1 *For a graph $G = (V, E)$, the following are equivalent:*

(A) *G is 2-edge-connected.*
(B) *For any pair of nodes, there are two edge-disjoint paths connecting them.*
(C) *G is connected and cyclic* (where cyclic means that every edge is contained in a circuit).
(D) *There is a sequence $G_0, G_1, \ldots, G_q = G$ of subgraphs of G, to be called an* **ear-decomposition** *of G, where G_0 consists of one node and no edge, and each G_i arises from G_{i-1} by adding a path P_i* (called an **ear**) *for which the* (not-necessarily-distinct) *end-nodes belong to G_{i-1} while the* (possibly empty) *set of inner nodes of P_i do not.*
(D') *G can be built up from a node by sequentially adding edges* (allowing loops) *connecting two existing nodes and subdividing edges.*

Remark In the definition of an ear decomposition, we slightly abused the formal terminology, since an ear with coinciding end-nodes is not a path but a circuit. When the end-nodes are distinct, we speak of an **open ear**. The equivalence of (A) and (D) is implicitly in Robbins' paper [332], whereas the other pairs are folklore.

Proof. (A)→(B) Let s and t be two nodes. There is a circuit C containing s, since the deletion of any edge $e = sx$ leaves a connected graph which, therefore, includes an sx-path P; $P + e$ is then the requested circuit. We are done if C contains t as well, so suppose this is not the case. Contract C into a single node s'. The resulting graph is also 2-edge-connected and, by induction, contains 2 edge-disjoint $s't$-paths P_1' and P_2'. These correspond to edge-disjoint paths P_1 and P_2 in the original graph G, each connecting t and C, and therefore they can be extended with two suitable subpaths of C to 2 edge-disjoint st-paths.

(B)→(C) Straightforward.

(C)→(D) Starting at any node, add ears arbitrarily as long as possible. We claim that the final graph $G' = (V', E')$ obtained in this way is G itself. This clearly holds if $V' = V$, since an edge connecting two existing nodes forms a path which could have been added as the next ear. Suppose indirectly that $V' \subset V$. Since G is connected, there is an edge $e = uv$ leaving V'. This edge belongs to a circuit C, because G is cyclic. Starting with u and e, traverse C until a node x of V' is reached. The segment of C from u to x forms a path which could have been added as an ear to G', contradicting the maximality of G'.

(D)→ (D′) Adding an ear of length one is the same as adding an edge. Adding an ear of length $k \geq 2$ can be reformulated as adding an edge and subdividing it by $k - 1$ inner nodes.

(D′) →(A) Neither the addition nor the subdivision of an edge can create a one-element cut. •

The constructive characterization in (D) and (D′) is a prototype of ear-decomposition theorems. We shall see several other ear-decomposition theorems, as well, such as for 2-connected graphs, strong digraphs, and elementary bipartite graphs.

2.1.2 2-edge-connected subgraphs and supergraphs

Proposition 2.1.2 *The problem of finding a smallest spanning 2-edge-connected subgraph of a graph G is* **NP***-complete.*

Proof. The minimum in question is exactly the number of nodes of G if and only if the graph includes a Hamilton circuit. Therefore, if the minimum could be computed in polynomial time, then we would have an algorithm for Hamilton circuits which is an **NP**-complete problem. •

The graph arising from a tree on n nodes by replacing each edge by two parallel edges is minimally 2-edge-connected and has exactly $2n - 2$ edges.

Proposition 2.1.3 *Every minimally 2-edge-connected graph G on n nodes has at most $2n - 2$ edges.*

Proof. By the minimality of G, every ear has at least two edges in an ear-decomposition of G, from which the estimation follows. •

How many edges must be added to a graph to make it 2-edge-connected? When the set of usable edges is prescribed, the problem is **NP**-complete, even in the special case when the graph to be augmented is a tree.

Proposition 2.1.4 (Frederickson and Ja'Ja' [109]) *Let $T = (V, E)$ be a tree and $H = (V, F)$ a given graph. The problem of finding a minimum number of edges of H such that the addition to T results in a 2-edge-connected graph is* **NP***-complete.* •

In this light, it is somewhat surprising that Eswaran and Tarjan [92] proved that the augmentation problem is tractable when there is no restriction for the usable set of new edges. Call a non-empty subset U of nodes **2-solid** (or **2-extreme**) if $d_G(U) \leq 1$

and $d_G(X) \geq 2$ holds for every non-empty, proper subset X of U. The subgraph G' induced by U is 2-edge-connected, for if there is a non-empty subset $X \subset U$ for which $d_{G'}(X) \leq 1$, then $2 + 2 \leq d_G(X) + d_G(U - X) = d_{G'}(X) + d_{G'}(U - X) + d_G(U) \leq 1 + 1 + 1$, which is not possible. Furthermore, 2-solid sets are pairwise disjoint, since $X \cap Y \neq \emptyset$ would imply $1 + 1 \geq d_G(X) + d_G(Y) \geq d_G(X - Y) + d_G(Y - X) \geq 2 + 2$. Let t_0 and t_1 denote the number of 2-solid sets of degree 0 and 1, respectively.

Theorem 2.1.5 (Eswaran and Tarjan [92]) *The minimum number γ of new edges the addition of which to an undirected graph $G = (V, E)$ results in a 2-edge-connected graph is $t_0 + \lceil t_1/2 \rceil$.*

Proof. Shrinking a 2-solid subset into a single node does not affect the values t_0 and t_1, and the minimal γ remains unchanged, as well. Therefore, we can assume that every 2-solid set is a singleton, and hence G is a forest. In this case, t_0 is the number of isolated nodes, while t_1 is the number of leaf nodes.

In a 2-edge-connected augmentation of G, there are at least 2 new edges incident to an isolated node of G, and at least 1 new edge incident to a leaf node of G. Therefore, the number of new edges is at least $\lceil 2t_0 + t_1/2 \rceil = t_0 + \lceil t_1/2 \rceil$.

To see that the graph can be made 2-edge-connected by adding $t_0 + \lceil t_1/2 \rceil$ new edges, it suffices to show by induction that there is a new edge e such that the addition of e to G decreases the value of $t_0 + \lceil t_1/2 \rceil$. Such an edge is said to be **reducing**.

Assume first that G is disconnected. Let u and v be two nodes of degree at most one belonging to distinct components. A simple case-checking—depending on the degrees of u and v—shows that the new edge $e = uv$ is reducing.

Therefore, we can assume that G is actually a tree (and hence $t_0 = 0$) which has at least 2 nodes (and hence $t_1 \geq 2$). When $t_1 = 2$, the tree is a path, and we obtain a 2-edge-connected graph (namely, a circuit) by adding one edge connecting the end-nodes of the path. In this case, $\gamma = 1 = t_0 + \lceil t_1/2 \rceil$.

If $t_1 = 3$, then the tree consists of three paths ending at a common node. Let a, b, and c denote the other end-nodes of these paths. By adding the two new edges ab and ac to the tree, we obtain a 2-edge-connected graph, and hence $\gamma = 2 = 0 + 2 = t_0 + \lceil t_1/2 \rceil$.

The remaining case is when $t_1 \geq 4$. There is a path P in the tree connecting two leaf nodes such that at least two edges leave $V(P)$. (For example, a longest path including a node of degree at least four or a longest path including two nodes of degree three will suffice.) By adding the new edge e between the two end-nodes of P, we obtain a graph G' in which the value of t_0 continues to be 0. Furthermore, the node-set of $P + e$ is not 2-solid in G', since its degree is at least 2. Thus, the addition of e to G reduces the value of t_1 by exactly 2. Consequently, e is reducing. •

Theorem 2.1.5 tells us how many new edges are required to make a given graph 2-edge-connected, and the proof efficiently finds an optimal augmentation. What if one is interested in higher-order edge-connectivity?

Question 2.1.1 *Given a graph G, what is the minimum number of new edges the addition of which to G results in a k-edge-connected graph?*

Although the above proof of Theorem 2.1.5 is pretty simple, its case-checking nature does not really lend itself to extension to k-edge-connection. In Section 11.1, a radically different approach will give rise to a theorem of Watanabe and Nakamura (Theorem 11.1.3) that contains a complete answer to the question.

Sometimes we are interested in augmentation where, instead of minimizing the number of new edges, a degree specification is to be met.

Problem 2.1.1 *Let $G = (V, E)$ be an undirected graph and $m : V \to \mathbf{Z}_+$ a degree specification with $\widetilde{m}(V)$ even. There is a graph $H = (V, F)$ for which $G + H = (V, E \cup F)$ is 2-edge-connected and $d_H(v) = m(v)$ for every $v \in V$ if and only if $\widetilde{m}(X) + d_G(X) \geq 2$ for every non-empty proper subset X of V.*

For a solution to a more general problem, see Theorem 11.1.1 and its proof.

2.1.3 2-node-connectivity

Let $G = (V, E)$ be a graph with no loops. A node of G is called a **cut-node** if its deletion increases the number of components. In particular, a node of a connected graph is a cut-node if its deletion disconnects the graph. It follows from Proposition 1.2.12 that a graph with at least three nodes is 2-connected if and only if it has no cut-nodes. A graph having two nodes joined by at least two parallel edges is also 2-connected. Note that $G = (\{u, v\}, \{uv\})$ is not 2-connected and it has no cut-node.

Exercise 2.1.2 *The subdivision of an edge of a 2-connected graph preserves 2-connectivity.*

The next characterization was found by Whitney [382].

Theorem 2.1.6 *For a connected loopless graph $G = (V, E)$ with $|V| \geq 2$, the following are equivalent:*

(A) *G is 2-connected.*
(B) *For every pair of a node z and a uv-edge $e \in E$, there is a circuit passing through z and e.*
(C) *G has a circuit and can be built up from any of its circuits by sequentially adjoining open ears joining existing nodes.*
(C′) *G has a circuit and it can be built up from any of its circuits by sequentially adjoining edges* (loops are not allowed) *and subdividing edges* (in any order).

Proof. (A)→(B) For the edge $e = uv$, let V' denote the union of the node-sets of circuits passing through e. We claim that $V' = V$. Suppose indirectly that $V' \subset V$ and let $f = xy$ be an edge with $x \in V'$, $y \notin V'$. There is a circuit C passing through e and x, and by the 2-connectivity of G, there is a path P connecting y and $V(C) - x$. Then the union of $P + f$ and C includes a circuit through e and y, contradicting the definition of V'.

(B)→(C) Let C be a circuit and let $G' = (V', E')$ be a maximal subgraph of G which can be built from C by adding sequentially open ears. We claim that $G' = G$. To see this, it suffices to show that $V = V'$, since any non-loop edge can be considered to be an open ear. Suppose indirectly that $V' \subset V$. By (B) there is a circuit passing through both an edge induced by V' and a node y not in V', and the subpath of this circuit lying outside V' determines an open ear that could be added to G', contradicting its maximality.

(C) and (C′) are clearly reformulations of each other, and the implication (C)→(A) is also trivial because the operation of adding an open ear does not destroy 2-connectivity. •

Theorem 2.1.7 *For a connected loopless graph $G = (V, E)$ with $|V| \geq 2$, the following are equivalent:*

(A) *G is 2-connected.*
(B) *For every pair of nodes v and z, there is a circuit passing through v and z.*
(C) *For every pair of edges e and f, there is a circuit passing through e and f.*

Proof. (A)→(B) Consider an edge e incident to v and apply Property (B) of Theorem 2.1.6. Conversely, (B) implies that G can have no cut-node, that is, that (A) holds.

(A)→(C) Subdivide f by a node z and apply Property (B) of Theorem 2.1.6. Conversely, (C) implies that G can have no cut-node, that is, that (A) holds. •

Theorem 2.1.8 *Let s and t be two nodes of a 2-connected graph $G = (V, E)$ with $n \geq 3$ nodes. Then the nodes of G admit a linear ordering $v_1 = s, v_2, \ldots, v_n = t$ such that every node v_i $(2 \leq i \leq n-1)$ has a neighbour with smaller subscript and another one with larger subscript.*

Proof. By Theorem 2.1.6, there is an ear-decomposition $G_0, G_1, \ldots, G_q = G$ of G. Here G_0 is a circuit, and the proof of the theorem actually showed that G_0 can be chosen to be an arbitrary circuit of G. Since there is a circuit C containing s and t, we can assume that G_0 contains s and t. We apply induction on the number q of ears. If $q = 0$ – that is, if G itself is a circuit – then the statement is straightforward. Suppose now that $q \geq 1$, and let $G' = G_{q-1}$. Then G arises from the 2-connected G' by adding a path P connecting two existing nodes u and v so that the inner nodes of P are new. By induction, there is a requested ordering of the nodes of G'. By assuming that u precedes v in this ordering, we can insert the inner nodes of P right after u in the order determined by P, which will clearly satisfy the requirements. •

The following result is a useful reduction property of 2-connected graphs.

Proposition 2.1.9 *For every edge $e = uv$ of a 2-connected graph, the deletion or the contraction of e results in a 2-connected graph.*

Proof. Suppose that the contraction of e destroys 2-connectivity of G, that is, $G - \{u, v\}$ is not connected. Let $K_1, \ldots, K_k$ denote the components of $G - \{u, v\}$. We prove the 2-connectivity of $G' := G - e$ by showing that every two nodes s and t of G' lie in a circuit of G'. Let C be a circuit of G passing through s and t. If C does not use e, we are done, so suppose that e belongs to C. Then there is a component K_1 such that $C \subseteq K_1 \cup \{u, v\}$. Since K_2 induces a connected graph and both u and v have neighbours in K_2 (as G has no cut-nodes), there is a uv-path P not using e for which the inner nodes are in K_2. By replacing the edge e in C with this path P, we obtain a circuit of G' passing through s and t. •

Based on this result, we obtain another constructive characterization of 2-connected graphs.

Theorem 2.1.10 *Starting from a triangle, a loopless 2-connected graph G with at least three nodes can be built up by applying the following two operations.*

(A) *Add an edge connecting two distinct existing nodes.*

(B) *Take a node v, partition the set of edges incident to v into two non-empty parts F' and F'', split v into two copies v' and v'' with a new edge connecting them, and finally replace each edge $e = uv$ incident to v by uv' or uv'', according to whether e belongs to F' or F''.* •

Exercise 2.1.3 *Prove that, in a 2-connected graph, there are two disjoint paths connecting A and B for any two sets $A, B \subseteq V$ with $|A|, |B| \geq 2$.*

Block structure of connected graphs

Lemma 2.1.11 *Let X and Y be two subsets of nodes of a graph $G = (V, E)$ such that $X \cup Y = V$ and $|X \cap Y| \geq 2$. If both X and Y induce a 2-connected graph, then G itself is 2-connected.*

Proof. G is obviously connected. Suppose indirectly that G has a cut-node s. Let C_1 and C_2 be two components of $G - s$. Let G_X and G_Y denote the subgraphs induced by X and Y, respectively. Since G_X and G_Y are 2-connected, both X and Y can intersect at most one of C_1 and C_2. Since $X \cup Y = V$, one of X and Y intersects C_1 and the other one intersects C_2, but in this case s can be the only common element of X and Y, a contradiction. •

Corollary 2.1.12 *Let e, f, and h be three edges of a graph G. If there is a circuit C_1 through e and f and there is circuit C_2 through f and h, then there is a circuit through e and h.*

Proof. Consider the graph $G' = (V', E')$ formed by $C_1 \cup C_2$. By the hypothesis, the end-nodes of f belong to both of C_1 and C_2. By Lemma 2.1.11, G' is 2-connected, and we can apply Theorem 2.1.7. •

Let $G = (V, E)$ be a 2-edge-connected graph that is not necessarily 2-node-connected. A **block of graph** G is a maximal induced 2-connected subgraph of G. The blocks of G form a tree-like structure in the following sense. (See Figures 2.1 and 2.2.) Let $B_1, B_2, \ldots, B_k$

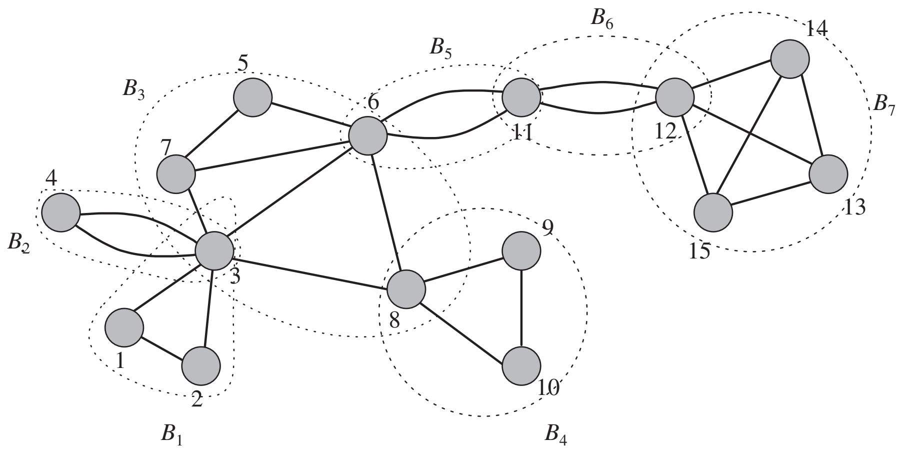


Figure 2.1 Blocks of graph

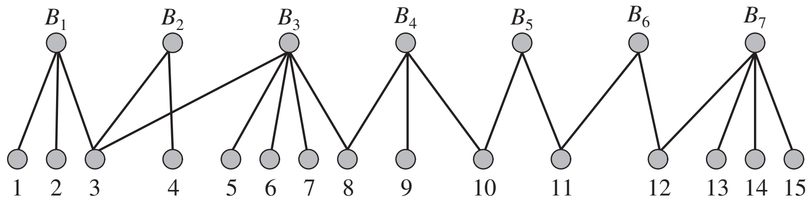


Figure 2.2 Tree of blocks

be the blocks of G. Form a bipartite graph $T = (V, B; F)$ so that the elements of $B = \{b_1, b_2, \ldots, b_k\}$ correspond to the blocks of G, and an edge vb_j is an edge of T if $v \in V$ and $v \in B_j$.

Theorem 2.1.13 *The blocks of a 2-edge-connected graph $G = (V, E)$ partition the set E of edges. Two edges belong to the same block if and only if there is a circuit containing both. Any two blocks have at most one node in common, and the nodes belonging to more than one block are cut-nodes. The graph $T = (V, B; F)$ is a tree.*

Proof. Since G is 2-edge-connected, every edge belongs to a circuit, so every edge belongs to a block. Lemma 2.1.11 implies that two blocks can have at most one node in common, and hence each edge belongs to exactly one block.

Lemma 2.1.11 also implies that if a block B_i and a circuit C have at least two nodes in common, then $C \subseteq B_i$. This implies at once that if the intersection of blocks B_i and B_j is a singleton $\{s\}$, then s is a cut-node of G. Furthermore, if two edges belong to a circuit, then they belong to the same block. Conversely, since a block B_i induces a 2-connected graph, by Theorem 2.1.7 every two edges of B_i belong to a circuit.

Finally, T is clearly connected. In seeking a contradiction, suppose T has a circuit such that its nodes in B are $b_1, \ldots, b_k$. If $k = 2$, then the corresponding blocks B_1 and B_2 would have at least two elements in common, but this possibility has already been ruled out. Therefore, $k \geq 3$ and $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, k$, where $B_{k+1} = B_1$. Let v_{i+1} denote the (single) common node of B_i and B_{i+1}. Then $v_i \neq v_{i+1}$, there is a path P_i in B_i joining v_i and v_{i+1}, and the union of these k paths forms a circuit of G. But this is impossible again since a circuit either belongs to a block or has at most one element in common with it. •

By using depth-first search, the blocks of an undirected graph can be computed in linear time [361, 362]. The block structure of a connected graph G can be defined by that of the graph obtained from G by duplicating each cut-edge.

We close this section by mentioning that the augmentation problem in which we want to make an initial graph 2-connected by adding a minimum number of new edges was solved independently by Plesník [321] and by Eswaran and Tarjan [92] in the same year.

2.2 Strong connectivity

By definition, a digraph D is strongly connected if every non-empty proper subset of nodes admits an entering edge.

2.2.1 Characterizations of strong connectivity

Theorem 2.2.1 *For a digraph $D = (V, A)$, the following are equivalent:*

(A) *D is strongly connected.*

(B) *Every node is reachable from every other node.*

(C) *D is weakly connected and cyclic, where cyclic means that every edge is in a di-circuit.*

(D) *There is a sequence $D_0, D_1, \ldots, D_t = D$ of subgraphs of D* (called an **ear-decomposition** of D) *where D_0 consists of one node and no edge, and each D_i arises from D_{i-1} by adding a directed path P_i* (called a **directed ear**) *for which the* (not-necessarily-distinct) *initial and terminal nodes belong to D_{i-1} while the* (possibly empty) *set of inner nodes of P_i does not.*

(D′) *D can be built up from a node by sequentially adding arcs* (allowing loops) *and subdividing arcs.*

Proof. (A)→(B) Since the set S of nodes reachable from any given node admits no leaving edges, the strong connectivity of D implies that $S = V$.

(B)→(C) The weak connectivity of D is obvious. For any uv-edge e, there is a path P from v to u and hence $P + e$ is a di-circuit including e, that is, D is cyclic.

(C)→(D) Starting at any node, add ears arbitrarily as long as possible. We claim that the final digraph $D' = (V', A')$ obtained in this way is D itself. This is clear if $V' = V$, since an edge connecting two existing nodes forms a path which could be added as the next ear. Suppose indirectly that $V' \subset V$. Since D is connected and cyclic, there is an edge $e = uv$ leaving V' and a di-circuit containing e. In traversing C from u, there will be a first node x belonging to V'. The segment of C from u to x forms a path which could have been added as an ear to D', contradicting the maximality of D'.

(D)→(D′) Adding an ear of length one is the same as adding an edge. Adding an ear of length $k \geq 2$ can be reformulated as adding an edge and subdividing it by $k - 1$ inner nodes.

(D′)→(A) Neither the addition nor the subdivision of an edge can create a dicut. •

Flat covers of di-circuits

As an application of ear-decomposition, we derive an interesting result of Knuth [242]. The proof below comes from Iwata and Matsuda [221]. Given a strong digraph $D = (V, A)$, we say that a subset B of edges intersecting each di-circuit is a **flat cover** of di-circuits if each edge of D belongs to a di-circuit covered exactly once by B.

Theorem 2.2.2 (Knuth's lemma) *In every strongly connected digraph $D = (V, A)$, there is a flat cover of di-circuits.*

Proof. By Theorem 2.2.1, there is an ear-decomposition of D. Let P denote the last ear for which the initial node is s and the terminal node is t, and let D' denote the subgraph of D for which D arises from D' by adding P. By induction, D' admits a flat cover B' of its di-circuits. Let b be an arbitrary edge of P. If $s = t$ – that is, if P is a di-circuit, – then $B' + b$ is a flat cover of the di-circuits of D, in which case we are done. Suppose now that s and t are distinct.

Lemma 2.2.3 *D' has a flat cover B^* of di-circuits such that s is reachable from t in $D' - B^*$.*

Proof. If s belongs to the set Z of nodes reachable from t in $D' - B'$, then $B^* = B'$ suffices. If s is not in Z, then $Z \subset V$, and each edge of D' leaving Z is in B'. It follows from the flatness of B' that no edge entering Z can belong to B', since any di-circuit containing such an edge leaves Z along an edge which is in B'. Revise B' by removing all the edges leaving Z and adding all the edges entering Z. Let B'' denote the resulting set. We have $|B'' \cap C| = |B' \cap C|$ for every di-circuit C, implying that B'' is also a flat cover of D'. Since the set of nodes reachable from t in $D' - B''$ properly includes Z, after at most n revisions, we obtain a flat cover B^* so that s is reachable from t in $D' - B^*$, as required. •

Now $B := B^* + b$ covers each di-circuit of D and we claim that B is flat. Indeed, consider a path P' of $D' - B^*$ from t to s ensured by the lemma. Then $C := P' \cup P$ is a di-circuit and b is the only element of B belonging to C, implying that the edges of D not in D' also belong to a di-circuit covered exactly once. • •

In Section 4.3.4, Theorem 2.2.2 will find an interesting application in the proof of a theorem of Bessy and Thomassé (Theorem 4.3.27), which states that the node-set of a strong digraph D can be covered by as many directed circuits as the maximum cardinality of a stable set of D.

Problem 2.2.1 $(*)$ *Prove that a strongly connected digraph D has a flat cover B of di-circuits such that each node is reachable in $D - B$ from a specified root-node r_0.*

Problem 2.2.2 (Fukuda, Prodon, and Sakuma [166]) $(*)$ *Let $D = (V, A)$ be a strongly connected orientation of a 3-edge-connected undirected graph G. Let $F \subseteq A$ be a subset of edges such that the digraph obtained from D by reversing all elements of F is strongly connected. Then there is an element f of F such that reversing only f results in a strong digraph.*

2.2.2 Strong components and acyclic digraphs

We say that two nodes u and v of a digraph $D = (V, A)$ are equivalent if there is a directed path from u to v and one from v to u. This is an equivalence relation. An equivalence class is called a **strong atom** or just an atom of D. An atom is also called a **strongly connected** (or briefly **strong**) **component** of D. A clever use of DFS helps in computing the atoms of a digraph in linear time; see Tarjan's works [361, 362] or a more recent book by Sedgewick [344].

Theorem 2.2.4 *Let $D = (V, A)$ be a digraph. Let $C_1, C_2, \dots, C_k$ denote the atoms of D. The digraph D' obtained by shrinking each atom C_i into a single node is acyclic.*

Proof. Suppose indirectly that D' includes a directed circuit $K = (c_1, e_1, c_2, e_2, \dots, c_h = c_1)$, and let $C_1, \dots, C_h$ be the atoms in D corresponding to the nodes of K. Let u_i and v_i denote the initial and the terminal node of e_i in D. Then both u_i and v_{i-1} (where $v_0 = v_h$) belong to C_i for each $i = 1, \dots, h$, and since C_i induces a strongly connected subgraph, there is a directed path P_i from v_{i-1} to u_i. These paths P_i along with the edges e_i show that u_1 is reachable from v_1 in D, implying that u_1 and v_1 belong to the same atom, a contradiction. •

An atom Z is called a **source-atom** (or source-component) if no arc enters it, and Z is a **sink-atom** (or sink-component) if no arc leaves it. (A strong atom with neither entering nor leaving arcs is both a source- and a sink-atom.)

A directed ear-decomposition is a possible certificate for strong connectivity. What can a polynomially verifiable certificate for a digraph being acyclic look like? A linear ordering $\{v_1, v_2, \dots, v_n\}$ of the node-set of a digraph $D = (V, A)$ is called a **topological ordering** if the head of each edge has higher a subscript than its tail.

Theorem 2.2.5 *For a digraph $D = (V, A)$, the following are equivalent.*

(A) *D is acyclic.*
(B) *Every edge of D belongs to a dicut.*
(C) *The node-set of D has a topological ordering.*
(C′) *D can be built up from a node by sequentially adding new nodes and, with each node, some new arcs for which their head is the new node while their tails are old nodes.*

Proof. Since a di-circuit and a dicut cannot share an edge, (B) implies (A). Suppose now that D is acyclic. To derive (C) from (A), it suffices to show that an acyclic digraph contains a sink-node since this property can be used inductively to obtain a topological ordering. If, indirectly, every node has a leaving edge, then starting at any node we could construct a directed walk of any length, and a walk of length n certainly includes a di-circuit. Consequently, (A) implies (C). The constructive characterization in (C′) is just a reformulation of (C). Finally, if $v_1, \dots, v_n$ is a topological ordering, then the sets $V_i := \{v_1, \dots, v_i\}$ $(i = 1, \dots, n-1)$ are out-shores of dicuts and these cuts cover all edges. Consequently, (C) implies (B). •

DFS can be used to compute in linear time a topological ordering of the nodes of an acyclic digraph (in which every edge goes forward) [361, 362]. By combining these results, we obtain the following.

Corollary 2.2.6 *The strong atoms of a digraph can be ordered so that there is no edge from a later atom to an earlier one.*

Problem 2.2.3 (Király [237]) $(*)$ *Prove that an acyclic digraph has a flat cover of directed cuts.*

2.2.3 Making a digraph strongly connected

Suppose we want to make a digraph $D = (V, A)$ strongly connected by adding a minimum number of new edges. Let $H = (V, F)$ be the digraph of possible new edges. In order to have a solution at all, we assume that the union of H and D is strong. The problem is **NP**-complete, even in the special case when D has no edges at all, since in this case the value of the optimum is $n = |V|$ if and only if H contains a directed Hamilton circuit. However, Eswaran and Tarjan [92] showed that the augmentation problem is tractable when no restriction is imposed on the set of usable new edges. To formulate their result, let us consider the atoms of D and the acyclic digraph obtained by shrinking each atom into a single node. The number of source- and sink-atoms are denoted by c_{so} and c_{si}, respectively.

Theorem 2.2.7 (Eswaran and Tarjan) *The minimum number γ of new directed edges whose addition makes a digraph $D = (V, A)$ strongly connected is* $\max\{c_{so}, c_{si}\}$.

Proof. In a strongly connected augmentation, every source-atom will have a new entering edge and every sink-atom will have a new leaving edge, and hence $\gamma \geq c_{so}$ and $\gamma \geq c_{si}$. By shrinking an atom into a single node, neither the minimum nor the maximum is changed, hence it suffices to prove the theorem for acyclic digraphs.

We can also assume that there are no isolated nodes. Indeed, an isolated node v can be replaced by a single edge vv', where v' is a new node. This operation, too, leaves the value of min and max unchanged.

If a sink-node t is not reachable from a source node s, then by adding the new edge ts to D we obtain a digraph that is still acyclic. Moreover, the value of max is reduced this way by one, and hence by induction we are done.

Therefore it can be assumed that each sink-node is reachable from each source node. For simplifying the notation, let $k := c_{so}$ and $\ell = c_{si}$. Let $s_1, \ldots, s_k$ denote the source nodes and $t_1, \ldots, t_\ell$ the sink-nodes.

By symmetry we can assume that $k \leq \ell$. By adding the edges $t_1 s_1, \ldots, t_k s_k$ along with the edges $t_{k+1}s_1, t_{k+2}s_1, \ldots, t_\ell s_1$ (altogether ℓ new edges), we obtain a strongly connected augmentation of D. •

Finally, we note that there is another augmentation problem which is nicely tractable. Here not every possible edge uv is usable in the augmentation, only those for which u is reachable from v in the initial digraph. The answer will be included in Theorem 9.7.2.

Question 2.2.1 *What can be a good candidate for a necessary and sufficient condition for making a given digraph D k-edge-connected by adding at most γ new arcs?*

A necessary and sufficient condition will be presented in Section 11.2 (Theorem 11.2.2).

2.2.4 Strongly connected orientation

Theorem 2.2.1 and 2.1.1 indicate that the analogous concepts for graphs and digraphs are 2-edge-connectivity and strong connectivity. Actually, parts (D) of these theorems immediately imply a theorem of Robbins [332].

Theorem 2.2.8 (Robbins) *An undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected.* •

Alternative proof. Let s be a specified node and compute a DFS tree F of root s. Define an arborescence $\vec{F}$ by orienting the edges of F away from s. By Proposition 1.1.8, the unique path connecting the end-nodes of every non-tree edge determines a directed path in $\vec{F}$. Orient all the non-tree edges so as to form a directed circuit with this path (that is, toward s). We claim that each arc of the digraph D obtained in this way belongs to a di-circuit, and hence D is strongly connected. As a result of this construction, each non-tree edge belongs to a di-circuit. Let $f = uv$ be an arc of $\vec{F}$ and let X denote the subset of nodes reachable in $\vec{F}$ from v. Since G is 2-edge-connected, there is an edge e in G leaving X. Since there is no cross-edge to F, f belongs to the di-circuit defined by e. •

An advantage of this proof is that it gives rise to a linear-time algorithm for finding a strong orientation of G [361]. Given Robbins' theorem, it is natural to pose the following question.

Question 2.2.2 *What is a necessary and sufficient condition for a graph to have a k-edge-connected orientation?*

Exercise 2.2.4 *Show that the underlying graph of a k-edge-connected digraph is $(2k)$-edge-connected.*

By a **dijoin**, we mean a subset of arcs covering all non-trivial dicuts.

Corollary 2.2.9 *A digraph D with no cut-edge includes two disjoint dijoins.*

Proof. We may assume that D is weakly connected. By Robbins' theorem there is a strongly connected reorientation D' of D. Let F_1 denote the set of edges of D which have been reoriented and F_2 the set of edges which have not. We claim that both F_1 and F_2 are dijons. Let X be the in-shore of a dicut B. In D' at least one edge enters X, so F_2 covers B. Similarly, at least one edge leaves X, so F_1 also covers B. •

This easy corollary naturally gives rise to the following conjecture.

Conjecture 2.2.10 (Woodall [386]) *If every dicut of a digraph contains at least k edges, then there are k disjoint dijoins.*

This conjecture is one of the most challenging open problems of the area. The answer is not known even for planar digraphs when $k = 3$. The following weaker version is also open.

Conjecture 2.2.11 (Guenin) *Let $D = (V, A)$ be an acyclic digraph in which there is a supersink* (a sink-node which is reachable along a dipath from each source node) *and there is supersource* (a source node from which each sink-node is reachable along a dipath). *If every dicut of D contains at least k edges, then there are k disjoint dijoins.*

Woodall's conjecture has been proved by Schrijver [335] under the hypothesis that every sink is reachable from every source.

The following pretty extension of Robbins' theorem to mixed graphs was observed by Boesch and Tindell [34]. Here we present a greedy-type proof that appeared in [110]. In a mixed graph M, if neither directed nor undirected edges leave a subset X of nodes, then the set of directed edges entering X was called a dicut of M.

Theorem 2.2.12 *The undirected edges of a mixed graph $M = (V, A + E)$ can be oriented in such a way that the resulting digraph is strongly connected if and only if the underlying graph of M is 2-edge-connected and there is no dicut.*

Proof. Necessity is straightforward. To demonstrate sufficiency, we proceed by induction on the number of undirected edges. There is nothing to prove if this number is 0, so suppose that $e = uv \in E$ is an undirected edge. If orienting e from u to v does not produce a dicut, by induction we are done. So we can suppose that there is a $v\bar{u}$-set X for which, apart from e, each edge between X and $V - X$ is directed and is oriented toward X. Similarly, we can suppose that there is a $u\bar{v}$-set Y for which, apart from e, each edge between Y and $V - Y$ is directed and is oriented toward Y.

But then neither directed nor undirected edges can leave $X \cap Y$, and the same holds for $X \cup Y$, too. By the hypothesis, we must have $X \cap Y = \emptyset$ and $X \cup Y = V$, that is,

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 57 of 640 December 11, 2010 15:15

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$Y = V - X$. Therefore, the only edge between X and $V - X$ is e, contradicting the assumption that there is no cut-edge. •

Question 2.2.3 *Given a digraph $D = (V, A)$, what is the minimum number of arcs whose reorientation makes D strongly connected?*

The answer requires much deeper ideas and will be given by a theorem of Lucchesi and Younger [275] (Theorem 9.7.2).

Problem 2.2.5 *Let D be a digraph whose underlying graph is 2-edge-connected. Prove that if F is a minimal set of edges covering all dicuts, then the reorientation of the elements of F leaves a strongly connected digraph.* (For a solution, see Proposition 9.7.1.)

We have seen that every graph has a smooth orientation and also that every 2-edge-connected graph has a strong orientation.

Question 2.2.4 *What is a necessary and sufficient condition for a 2-edge-connected graph to have a strongly-connected and smooth orientation? Can you find* any *2-edge-connected graph having no such orientation?*

A somewhat surprising answer will be provided in Section 9.2.

2.2.5 Strong blocks

A strongly connected loopless digraph with at least two nodes is called a **strong block** if it has no cut-node. An ear P_i of an ear-decomposition of D is said to be **open** if its initial and terminal nodes are distinct.

Theorem 2.2.13 *For a loopless digraph $D = (V, A)$ with $|V| \geq 3$, the following are equivalent:*

(A) *D is a strong block.*
(B) *D can be built up from any di-circuit of D by sequentially adding open ears.*
(B′) *D can be built up from any di-circuit of D by sequentially adding arcs* (excluding loops) *and subdividing arcs.*

Proof. (A)→(B) Since D is strongly connected, it certainly has a di-circuit. Let C_0 be any di-circuit of D. Let $D' = (V', A')$ denote a maximal subgraph of D which can be built up from C_0 by sequentially adding open ears. If $V' = V$, then $A' = A$, since an edge from $A - A'$ could be added as an open ear. Therefore, $V - V'$ is non-empty.

There is a node $u \in V'$ which is the tail of some edges leaving V'. Let Y denote the set of nodes in $V - V'$ reachable from u by a path whose first edge leaves V'. We claim that the head t of every edge st leaving Y is u. Indeed, t must be in V' by the definition of Y. Furthermore, $t \neq u$ would imply that a path from u to s plus the edge st would form an open ear that could be added to D', contradicting the maximality of D'.

Since the underlying graph of D is a block, there is an edge xy (in the undirected sense) such that $y \in Y,\ x \in V - (Y + u)$. By the claim, xy, as a directed edge, enters Y. Since D is strong, there exists a di-circuit C containing xy.

Since the head of every edge leaving Y is u, the di-circuit C has exactly one such edge. Therefore, xy is the only edge of C entering Y by which the edge uu' of C does not enter Y.

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 58 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

By the definition of Y, this edge uu' cannot leave V', implying that C has at least two nodes in V'. It follows that there is a subpath of C (through xy) that could be added as an open ear to D', contradicting the maximality of D'.

Properties (B) and (B′) are clearly equivalent, and (B)→(A) is also straightforward since the addition of an open ear preserves the 2-connectivity of the underlying graph, and the addition of a directed ear preserves strong connectivity, too. •

Note that it is not true that every pair of nodes of a strong block lies on a di-circuit.

Proposition 2.2.14 *It is* **NP***-complete to decide whether a digraph includes a di-circuit containing two specified nodes.*

Proof. Reduction to the directed two disjoint paths problem, a basic **NP**-complete problem. •

2.3 Degree-constrained orientations

We have seen in Theorem 1.3.8 that every undirected graph has a smooth orientation. The question naturally arises: When does a graph have an orientation in which the in-degree of each node belongs to a prescribed interval? A special case is when each interval has exactly one element, that is, when the in-degree of each node is specified. Note that one could consider out-degrees as well. However, the in-degree plus the out-degree of a node is its undirected degree, which is independent of the orientation, and therefore a constraint for the out-degree can easily be transformed to a constraint for the in-degree.

2.3.1 In-degree specification

The earliest result of this type is by Landau [255], who solved it for complete graphs. By the **in-degree sequence** (or **vector**) of a directed graph with n nodes, we mean a sequence consisting of the in-degrees of the nodes of the digraph.

Theorem 2.3.1 (Landau) *A sequence $m_1 \geq m_2 \geq \cdots \geq m_n$ is the in-degree sequence of a tournament if and only if*

$$\sum_{i=1}^{n} m_i = \binom{n}{2} \tag{2.1}$$

and

$$\sum_{i=1}^{k} m_i \leq k(k-1)/2 + k(n-k) \quad (k = 1, \ldots, n) \tag{2.2}$$

hold, which are equivalent to requiring (2.1) *and*

$$\sum_{i=n-h+1}^{n} m_i \geq \binom{h}{2} \quad (h = 1, \ldots, n). \tag{2.3}$$

We prove the theorem in a more general form.

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 59 of 640 December 11, 2010 15:15

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 2.3.2 (Orientation lemma, Hakimi [200]) *For an undirected graph $G = (V, E)$ and a function $m : V \to \mathbf{Z}$ satifying $\widetilde{m}(V) = |E|$, the following are equivalent.*

(A) *G has an orientation so that $\dot{\varrho} = m$, that is,*

$$\varrho(v) = m(v) \text{ for every node } v, \tag{2.4}$$

(B) *$e_G \geq m$, that is,*

$$e_G(X) \geq \widetilde{m}(X) \text{ for every subset } X \subseteq V, \tag{2.5}$$

(C) *$i_G \leq m$, that is,*

$$i_G(Y) \leq \widetilde{m}(Y) \text{ for every subset } Y \subseteq V. \tag{2.6}$$

Proof. Since $e_G(X) + i_G(V - X) = |E| = \widetilde{m}(V) = \widetilde{m}(X) + \widetilde{m}(V - X)$, the equivalence of (2.5) and (2.6) is evident.

Suppose now that there exists a requested orientation. Then $e_G(X) = \sum[\varrho(v) : v \in X] + \delta(X) \geq \sum[m(v) : v \in X] = \widetilde{m}(X)$ holds for any subset $X \subseteq V$, and hence (A) implies (B).

Finally, suppose that (2.5) is met. The function e_G can easily be seen to be submodular, that is, $e_G(X) + e_G(Y) \geq e_G(X \cap Y) + e_G(X \cup Y)$. Call a subset X **tight** if $\widetilde{m}(X) = e_G(X)$. The empty set is tight and so is V by the hypothesis $\widetilde{m}(V) = |E|$.

Proposition 2.3.3 *The intersection and the union of two tight sets X and Y are also tight.*

Proof. $\widetilde{m}(X) + \widetilde{m}(Y) = e_G(X) + e_G(Y) \geq e_G(X \cap Y) + e_G(X \cup Y) \geq \widetilde{m}(X \cap Y) + \widetilde{m}(X \cup Y) = \widetilde{m}(X) + \widetilde{m}(Y)$ and the proposition follows. •

We proceed by induction on $\widetilde{m}(V)$. The statement is straightforward when $\widetilde{m}(V) = |E| = 0$, so we can assume that there is a node s for which $m(s) > 0$. The proposition implies that there is a unique largest tight set Z not containing s. There exists an edge $f = us$ for which $u \notin Z$, for otherwise $e_G(Z + s) = e_G(Z) = \widetilde{m}(Z) = \widetilde{m}(Z + s) - m(s) < \widetilde{m}(Z + s)$, and hence $Z + s$ would violate condition (2.5). Delete f and reduce the value of $m(s)$ by one. We claim that condition (2.5) also holds for the resulting graph G' and for the revised in-degree specification m'. Indeed, if a subset X would violate (2.5), then X would originally be a tight $u\bar{s}$-set. From the maximal choice of Z we would have $X \subseteq Z$, contrary to the assumption $u \notin Z$.

By induction, G' has an orientation of in-degree vector m', from which we obtain an orientation of G with in-degree vector m by adding the directed edge us. • •

This is the first occasion where we encounter the use of submodular functions in a proof. Submodularity will play a central role throughout this book.

Theorem 2.3.1 follows immediately from the Orientation lemma, since in a complete graph the number $i_G(X)$ of edges induced by a subset $X \subseteq V$ is the same for each h-element subset, namely, $h(h - 1)/2$. Therefore, it suffices to require the condition $i_G(X) \leq \widetilde{m}(X)$ only for the h smallest values m_i.

Problem 2.3.1 $(*)$ *At a chess tournament, the winner of a game gets one point, the loser no points, while both players get half a point for a draw. In order to avoid fractions, multiply everything by two. Then the winner, for example, gets 2 points. Under this assumption, when can a sequence $m_1 \geq m_2 \geq \cdots \geq m_n$ be the final score of a chess tournament?*

Research problem 2.3.1 (Iványi [218]) *Decide if a sequence of n integers can be the final score of a football tournament of n teams. The winner of a game gets 3 points, the loser no point, while both terms get 1 point for a draw.*

Euler orientations

As mentioned above, an undirected Euler graph always has an Euler orientation. The Orientation lemma allows one to extend this observation to mixed graphs.

Theorem 2.3.4 (Ford and Fulkerson) *Let $M = (V, A + E)$ be a mixed graph consisting of an undirected graph $G = (V, E)$ and a digraph $D = (V, A)$. It is possible to orient the edges of E in such a way that the resulting directed graph is Eulerian if and only if every node of M is incident to an even number of* (directed or undirected) *edges, that is,*

$$\delta_D(v) + \varrho_D(v) + d_G(v) \text{ is even} \tag{2.7}$$

and

$$d_G(X) \geq \varrho_D(X) - \delta_D(X) \text{ holds for every subset } X \subseteq V. \tag{2.8}$$

Proof. Let $\varrho_{\vec{G}}$ and $\delta_{\vec{G}}$ denote, respectively, the in-degree and the out-degree functions of an orientation $\vec{G} = (V, \vec{E})$ of G. The digraph $D + \vec{G}$ is Eulerian if and only if $\varrho_D(v) + \varrho_{\vec{G}}(v) = \delta_D(v) + \delta_{\vec{G}}(v)$ for every node v. This equality is equivalent, via $\varrho_{\vec{G}}(v) + \delta_{\vec{G}}(v) = d_G(v)$, to $\varrho_{\vec{G}}(v) = (\delta_D(v) - \varrho_D(v) + d_G(v))/2$. Denote the right-hand side by $m(v)$. By (2.7), m is integer-valued. Apply theorem 2.3.2 and observe that the requirements (2.8) and (2.5) are equivalent for the specified m. •

2.3.2 Upper and lower bounds

The Orientation lemma follows immediately from a more general result of Hakimi [200], in which lower bounds, rather than exact values, are given for the in-degrees. This problem is equivalent to that of imposing upper bounds on the out-degree of the nodes. We now combine these two and solve the orientation problem when both lower and upper bounds are prescribed for the in-degree of the nodes. To this end, let $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ be two functions for which $f \leq g$. (The lower bound $-\infty$ on a node means that there is no actual lower bound. We could have replaced $-\infty$ by zero, but $-\infty$ shows more directly that these nodes do not play any role. Analogous is the situation with the upper bound ∞.) Parts (A) and (B) of the next theorem were found by Hakimi [200], while Part (C) seems to have appeared explicitly first in [137].

Theorem 2.3.5 *An undirected graph $G = (V, E)$ has an orientation for which*

(A) $\dot{\varrho} \geq f$ (that is, $\varrho(v) \geq f(v)$ for every node v) *if and only if*

$$e_G \geq \widetilde{f} \quad \text{(that is, } e_G(X) \geq \widetilde{f}(X) \text{ for every subset } X \subseteq V\text{)}, \tag{2.9}$$

(B) $\dot{\varrho} \leq g$ *if and only if*

$$i_G \leq \widetilde{g}, \tag{2.10}$$

(C) $f \leq \dot{\varrho} \leq g$ *if and only if both $e_G \geq \widetilde{f}$ and $i_G \leq \widetilde{g}$ hold.*

Proof. If a requested orientation exists, then $\widetilde{f}(X) \leq \sum[\varrho(v) : v \in X] \leq e_G(X)$, and the necessity of (2.9) follows.

For proving sufficiency, assume (2.9). In an orientation of G we define a node s **deficient** if $\varrho(s) < f(s)$. Let us choose an orientation of G in which the total deficiency defined by the sum $\sum[f(v) - \varrho(v) : v$ deficient] is minimum. If this sum is positive, then there is a deficient node s. Let X denote the set of nodes reachable from s in the given orientation. No directed edge leaves X, hence $\sum[\varrho(v) : v \in X] = e_G(X)$. Now X must contain a node t, for which $\varrho(t) > f(t)$. Otherwise $\widetilde{f}(X) > \sum[\varrho(v) : v \in X] = e_G(X)$, contradicting (2.9). By reorienting a directed path from s to t, we obtain another orientation of G in which the total deficiency is smaller, contradicting the choice of the original orientation. Therefore, the total deficiency must be 0, that is, there is no deficient node, and we are done.

Part (B) can be proved analogously (with the difference that a node t is deficient now if in the current orientation $\varrho(t) > g(t)$ and X denotes the set of nodes from which t is reachable). Alternatively, the second part is formally equivalent to that version of the first one when the out-degree of a node v is at least $f(v) := d_G(v) - g(v)$.

Finally, to see sufficiency in Part (C), let us start with an orientation of G for which $(*)\ \ \varrho(v) \leq g(v)$ holds for every node v. Apply the algorithm of Part (A) and observe that the in-degree of a node s can increase only if $\varrho(s) < f(s) \leq g(s)$, and hence $(*)$ remains automatically valid. •

Note that the Part (C) involves the first two parts.

Corollary 2.3.6 *Let $G = (V, E)$ be an undirected graph with a specified subset $U \subseteq V$ of nodes and let $m : U \to \mathbf{Z}_+$ be an in-degree specification on U. There exists an orientation of G such that $\varrho(v) = m(v)$ for every $v \in U$ if and only if $i_G(X) \leq \widetilde{m}(X) \leq e_G(X)$ holds for every subset $X \subseteq U$.*

Proof. Let $f(v) := g(v) := m(v)$ if $v \in U$, and $f(v) := -\infty$ and $g(v) := \infty$ if $v \in V - U$. Apply Theorem 2.3.5. •

It is worth emphasizing the following interesting consequence.

Corollary 2.3.7 *Let f and g be integer-valued functions on E for which $f \leq g$. If the graph $G = (V, E)$ has an orientation for which $\varrho_1(v) \geq f(v)$ for every node v, and G has an orientation for which $\varrho_2(v) \leq g(v)$ for every node v, then there is an orientation of G meeting both requirements.* •

This property is called the **linking** property. One of its earliest occurrences appeared in a paper of Mendelsohn and Dulmage [287]. It was formulated by Ford and Fulkerson [107, p. 49] in a related theorem on the existence of integral matrices for which the row-sums and the column-sums lie between specified bounds. The concept was investigated in detail in the book of Mirsky [291]. The linking property will show up under much more complicated circumstances, too.

Note that the Orientation lemma follows immediately from Part (A) of Theorem 2.3.5.

Alternative proof of (the non-trivial part of) the Orientation lemma. Observe that for $f := m$, (2.5) and (2.9) are the same. Therefore, Theorem 2.3.5 implies the existence of an orientation of G for which $\varrho(v) \geq m(v)$ for every node v. Since $|E| = \sum[\varrho(v) : v \in$

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 62 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$V] \geq \sum[m(v) : v \in V] = \widetilde{m}(V) = |E|$, we must have equality for every node v, that is, $\varrho(v) = m(v)$.•

Exercise 2.3.3 *Prove that if D_1 and D_2 are two orientations of the same undirected graph such that $\varrho_1(v) = \varrho_2(v)$ holds for each node v, then it is possible to get from D_1 to D_2 by a sequence of reorienting directed circuits.*

Exercise 2.3.4 *Show that if ϱ and ϱ' denote the in-degree functions of two orientations of G for which $\varrho(v) = m(v) = \varrho'(v)$ for every node v, then $\varrho(X) = \varrho'(X)$ for every subset $X \subseteq V$.*

Problem 2.3.5 *Develop a necessary and sufficient condition for the existence of an orientation where lower and uppers bounds are given for the in-degrees as well as for the out-degrees.*

Alternative algorithmic proof: a push–relabel approach

Consider again Part (A) of Theorem 2.3.5, where the goal was to find an orientation for which the in-degrees obey lower bounds on the nodes. Suppose for a moment that we do not know yet the path reorientation technique used above and that we want to solve the orientation problem from scratch. A naive approach would be to start with an arbitrary orientation, reorient any edge leaving an arbitrary node with $\varrho(v) < f(v)$, and repeat these edge reorientations as long as there are deficient nodes. Not surprisingly, such a primitive procedure need not terminate: For example, if e is an uv-edge and both u and v are deficient, then the algorithm may choose to reorient e every time, in which case there is no progress. However, if we introduce a clever control parameter (to be called level) on the nodes to select the proper deficient node and the proper leaving edge, then this approach does work. Precisely this approach is the idea of the push–relabel algorithm of Goldberg and Tarjan [186] (see Section 6.1 in Part II) that was developed for computing a maximum flow (and beat the alternating path method). The algorithm below is merely an adaptation of the algorithm of Goldberg and Tarjan for orientations. It may help the reader to capture the very essence of this technique in the present, particularly simple setting. In Part III, we shall see that the push–relabel approach extends well beyond network flows.

The procedure starts with an arbitrary orientation and works throughout with a non-negative integer-valued level function $\Theta : V \to \mathbf{Z}_+$ on the nodes. The following two level properties will be maintained:

(L1) Every oversaturated node v (that is, one with $\varrho(v) > f(v)$) is on level 0.
(L2) $\Theta(v) \geq \Theta(u) - 1$ holds for every directed edge uv.

The algorithm terminates when one of the following stopping rules holds.

(A) There are no more in-deficient nodes where v is in-deficient if $\varrho(v) < f(v)$.
(B) There exists an in-deficient node z and an empty level j under the level of z, that is, $j < \Theta(z)$ and $\{u \in V : \Theta(u) = j\} = \emptyset$.

Stopping rule (A) means that the current orientation satisfies the requirement $\varrho(v) \geq f(v)$ for every node v. We claim that Stopping rule (B) implies that $e_G(X) < \widetilde{f}(X)$ for subset $X := \{u : \Theta(u) \geq j\}$, and hence X violates (2.9). Indeed, on the one hand, X contains no oversaturated node by Property (L1), whereas it contains the in-deficient z, from which $\sum[\varrho(u) : u \in X] < \sum[f(u) : u \in X] = \widetilde{f}(X)$ follows. On the other hand, no edge leaves X by Property (L2), from which $e_G(X) = \sum[\varrho(u) : u \in X] < \widetilde{f}(X)$.

The algorithm runs as follows. At the beginning $\Theta \equiv 0$. As long as there are in-deficient nodes, select arbitrarily one, to be denoted by z. If there is an arc $e = zv$ with $\Theta(v) = \Theta(z) - 1$, then reorient e. When this operation makes Stopping rule (A) valid, the algorithm terminates by returning a requested orientation.

If there is no such an arc, then increase $\Theta(z)$ by one. When this operation makes Stopping rule (B) valid, the algorithm terminates by returning a subset X violating (2.9). Note that both operations maintain properties (L1) and (L2).

The level of a node can be increased at most n times, since once a node reaches level $n = |V|$ there must be an empty level under it, in which case (B) certainly holds. Therefore, the total number of level increases is at most n^2.

Since an edge is reoriented only if its head has a lower level than its tail, the sum $\Theta(u) + \Theta(v)$ increases by at least 2 between two consecutive reorientations of uv. Since the level of every node is less than n, the sum $\Theta(u) + \Theta(v)$ is at most $2n$, and hence every edge can be reoriented at most $n = 2n/2$ times. Therefore, the total number of reorientations is at most mn, implying that the overall complexity of the algorithm is $O(nm)$. •

An application to finding degree-constrained subgraphs

Let $G = (V, E)$ be an undirected graph. At every node v of G, we are given a set $F(v) \subseteq \{0, 1, \ldots, d_G(v)\}$ of forbidden degrees, where $d_G(v)$ denotes the degree of v. A subgraph $G' = (V, E')$ of G is called F**-avoiding** if $d_{G'}(v) \notin F(v)$ for every $v \in V$. The problem of deciding if there is an F-avoiding subgraph is **NP**-complete in general. Indeed, the hypergraph perfect matching problem (that seeks to find a partition of the node set consisting of hyperedges), which is known to be **NP**-complete even for 3-uniform hypergraphs, can easily be formulated this way. To this end, consider the bipartite graph $G = (V, U; E)$ associated with hypergraph $H = (V, \mathcal{F})$. Note that the degree of every node $u \in U$ is 3. At every node $v \in V$, let $F(v) := \{0, 2, 3, \ldots, d_G(v)\}$. This means that for $v \in V$ the set $F(v)$ the complementary set of $\{1\}$. At every node $u \in U$, let $F(u) = \{1, 2\}$. Clearly, there is a one-to-one correspondence between the perfect matchings of H and the F-avoiding subgraphs of G.

Intuition suggests that there must be an F-avoiding subgraph if each forbidden set $F(v)$ is sufficiently small and this natural feeling is formulated in the next result of Shirazi and Verstraëte [351].

Theorem 2.3.8 *If*

$$|F(v)| \leq \lfloor d_G(v)/2 \rfloor \text{ for every } v \in V, \tag{2.11}$$

then G admits an F-avoiding subgraph.

Interestingly, the original proof of this disarmingly simple-sounding statement made use of a fundamental combinatorial result concerning polynomials, a result by Alon [3]. The simple combinatorial proof below neatly exemplifies the paradigm that it can be highly rewarding to figure out a proper extension of the statement in order to get a short and simple proof.

We have seen in Theorem 1.3.8 that every graph G has a smooth orientation $D = (V, \vec{E})$. In such an orientation, $\varrho_D(v) \geq \lfloor d_G(v)/2 \rfloor$ for every node v. Therefore, the following result of Frank, Lao, and Szabó [146] implies Theorem 2.3.8.

Theorem 2.3.9 *If a graph G has an orientation $D = (V, \vec{E})$ in which $\varrho_D(v) \geq |F(v)|$ for every node v, then G has an F-avoiding subgraph.*

Proof. We proceed by induction on the number of edges. For an edge $e \in E$ of G, let $\vec{e}$ denote the corresponding directed edge of D. If 0 does not occur at any node as a forbidden degree, then the graph $(V, \emptyset)$ is an F-avoiding subgraph of G. Suppose now that $0 \in F(t)$ for some node t. Then $\varrho_D(t) \geq |F(t)| \geq 1$, and therefore there is an edge $e = st$ of G for which $\vec{e}$ is directed toward t.

Let $G^- := G - e$ and $D^- := D - \vec{e}$. Define F^- as follows.

$$F^-(v) := \begin{cases} \{i - 1 : i \in F(t) - \{0\}\} & \text{if } v \in \{s, t\}, \\ F(v) & \text{if } v \in V - \{s, t\}. \end{cases} \tag{2.12}$$

Since $|F^-(t)| = |F(t)| - 1$, we have $\varrho_{D^-}(v) \geq |F^-(v)|$ for every node v. By induction, G^- has an F^--avoiding subgraph G''. It follows from the definition of F^- that the subgraph $G' := G'' + e$ of G is F-avoiding. •

This approach shows that the hypothesis (2.11) in Theorem 2.3.8 can be made more flexible. By combining Theorem 2.3.9 with Part (A) of Theorem 2.3.5, one obtains the following.

Theorem 2.3.10 *Suppose that $e_G(X) \geq \sum[|F(v)| : v \in X]$ holds for every subset $X \subseteq V$ of nodes of an undirected graph G. Then G has an F-avoiding subgraph.* •

An application to recognizing k-sparse graphs

A interesting feature of graph orientation problems is that, in several applications, our primary interest is not the existence of a certain orientation. Instead, we want to check, for example, the validity of criterion (2.9) or (2.10). We call a subset $Z \subseteq V$ of nodes of an undirected graph $G = (V, E)$ k-**sparse** if

$$i_G(X) \leq k(|X| - 1) \tag{2.13}$$

holds for every non-empty subset $X \subseteq Z$. The graph G itself is also said to be k-**sparse** if V is k-sparse. In Section 10.5, we shall prove a beautiful theorem of Nash-Williams (Theorem 10.4.3) which states the edge-set of G can be partitioned into k forests if and only if G is k-sparse. Here we show how it is possible to test a graph for k-sparsity with the help of degree-constrained orientations.

It follows from Part (B) of Theorem 2.3.5 that there is an orientation of G so that the in-degree of a specified node s is 0 while all other in-degrees are at most k if and only if $i_G(X) \leq k(|X| - 1)$ holds for every subset $X \subseteq V$ containing s and $i_G(X) \leq k|X|$ holds for every subset $X \subseteq V$ not containing s. Therefore all we need to do is to check if there is such an orientation for every possible choice of $s \in V$. If the answer is yes for every s, then the graph is k-sparse, while if the answer is no for at least one $s \in V$, then the algorithm returns a subset X violating (2.13), showing that the graph is not k-sparse.

In Section 13.5, orientations will be used in a more sophisticated and more efficient way to check if a graph satisfies k-sparsity or other related properties.

Problems

2.3.6 *Let ℓ and k be non-negative integers where $\ell \leq k$. Extend the approach above to test, for a graph G, whether $i_G(X) \leq k|X| - \ell$ holds for every non-empty subset $X \subseteq V$.*

2.3.7 (Kampen [230]) *Let G be a simple planar graph.* (A) *Relying on the Euler's formula, prove that G has an orientation in which every in-degree is at most three.* (B) *If, in addition, G is bipartite, then prove that it has an orientation in which every in-degree is at most two.*

2.3.8 $(*)$ *Let $G = (V, E)$ be a simple maximal* (that is, triangulated) *planar graph on at least five nodes. Prove that E can be partitioned into claws, where a claw is the complete bipartite graph $K_{1,3}$.*

2.3.9 *Adapt the graph-orientation idea above to test in a graph if $i_G(X) \leq \widetilde{m}(X) - \ell$ holds for every non-empty subset X of V where $m : V \to \mathbf{Z}_+$ is a non-negative integer-valued function and ℓ is a non-negative integer for which $\ell \leq m(v)$ for every $v \in V$.*

2.3.10 *Develop a* (slightly more sophisticated) *approach to test in a graph if $i_G(X) \leq 2|X| - 3$ holds for every subset $X \subseteq V$ with at least 2 elements.*

The graphs in Problem 2.3.10 are important in studying rigid graphs: see Section 8.3. An algorithm for even more general problems of this type will be described in Section 13.5.

2.4 Bipartite matchings

2.4.1 Largest matchings and perfect matchings

Bipartite matchings form one of the most fundamental objects of combinatorial optimization. There are no books on the subject, introductory or advanced, that do not discuss bipartite matchings in some detail. A starting task is to characterize bipartite graphs, admitting perfect matching as a spanning subgraph. These graphs are said to be **perfectly matchable**. A more general question seeks to find a matching in a bipartite graph of largest cardinality. One may also be interested in packings and coverings with matchings. In this section, we summarize answers to the most important questions of this type.

The earliest min-max theorem in graph theory is due to Kőnig [246]. Its constructive proof relies on the notion of alternating paths. Given a matching M, a path P is said to be **M-alternating**, or just **alternating**, if every second edge of P belongs to M.

Theorem 2.4.1 (Kőnig) *In a bipartite graph $G = (S, T; E)$, the maximum cardinality $\nu = \nu(G)$ of a matching is equal to the minimum number $\tau = \tau(G)$ of nodes covering all the edges.*

Proof. In order to cover a matching of ν elements, one needs ν nodes. Hence $\nu \leq \tau$.

To prove the non-trivial direction $\nu \geq \tau$, we construct a matching $M \subseteq E$ and a covering $L \subseteq S \cup T$ for which $|M| = |L|$. The procedure starts with an arbitrary matching of G (which may consist of a single edge or can be the empty set). At a general stage, either

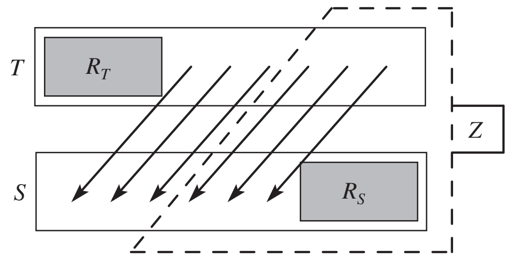


Figure 2.3 Termination of matching algorithm

we find a one-larger matching and iterate the procedure, or else we find a covering L with the desired $|M| = |L|$, in which case the algorithm terminates.

Orient the edges of M from T towards S while orienting all the other edges from S to T. Let R_S and R_T denote the subsets of S and of T, respectively, that consist of nodes exposed by M. In the resulting digraph D_M, let Z denote the set of nodes reachable from R_S.

There can be two cases. If R_T and Z are not disjoint, then we have obtained a directed path P from R_S to R_T which is alternating in M. Therefore, the symmetric difference of M and P is a matching M' having one more edge than M. (Technically, it is easy to implement this change in the auxiliary digraph: simply reorient the edges of P.)

In the second case, R_T and Z are disjoint. By the definition of Z, no directed edge leaves Z. Furthermore, no directed edge from M can enter Z, either, for if $uv \in M$ were such an edge, then v could be reached only through u, contradicting the fact that u is not reachable. It follows, on the one hand, that the set $L := (T \cap Z) \cup (S - Z)$ covers all the edges of G, while on the other hand L contains exactly one end-node of each edge in M. Therefore $|M| = |L|$, and the proof is complete. (See Figure 2.3.) •

The proof provides an algorithm to compute the extrema in question in time $O(nm)$ (where n denotes the number of nodes while m is the number of edges). We say that a subset X of nodes is **matching-covered** if there is a matching covering all nodes in X. Although the matchings during the run of the algorithm change in such a way that an edge may enter and leave the current matching several times, the sets of nodes covered by the current matching form an ever increasing sequence. Therefore, the proof above has the following interesting consequence.

Corollary 2.4.2 *Every matching-covered subset of S is included in a matching-covered subset of cardinality $\nu(G)$.* •

Theorem 2.4.3 (Mendelsohn and Dulmage [287]) *If both $X \subseteq S$ and $Y \subseteq T$ are matching-covered, then so is $X \cup Y$.*

Proof. Let M_Y be a matching that covers Y and as many elements of X as possible, and let M_X be a matching covering X. We are done if M_Y covers each element of X, so suppose that $x \in X$ is not covered. The symmetric difference of two matchings consists of disjoint alternating paths and circuits. Let P denote the path ending at x. Since M_X covers X and M_Y covers Y, the other end-node of P is in $S - X$ or in $T - Y$. Therefore, the symmetric difference $M_Y \ominus P$ is a matching which covers Y, as well, and also covers one more element of X, contradicting the maximal choice of M_Y. •

The theorem of Mendelsohn and Dulmage has been generalized by Perfect [317] and by Pym [324, 325].

It is worth formulating Kőnig's theorem in an equivalent form. To this end, define the **deficiency** $h(X)$ of a subset $X \subseteq S$ as follows.

$$h(X) := |X| - |\Gamma(X)| \tag{2.14}$$

where $\Gamma(X) = \Gamma_G(X) = \Gamma_E(X)$ denotes the set of neighbours of X, where $\Gamma(X) := \{v \in T :$ there is an edge $uv \in E$ with $u \in X\}$. Let $\mu = \mu(G, S)$ denote the maximum deficiency of the subsets of S. We say that a subset X of S is **max-deficient** if $h(X) = \mu$. Let $\varphi = \varphi(G, S)$ denote the minimum number of nodes exposed by a matching.

Obviously $\varphi + \nu = |S|$ and it also follows rather easily that $\tau + \mu = |S|$. Hence, we get an equivalent form of Kőnig's theorem, which was formulated by Ore [313, 314].

Theorem 2.4.4 (Kőnig–Ore) *In a biparite graph $G = (S, T; E)$, $\varphi = \mu$. In other words, the minimum number of nodes in S left exposed by a matching is equal to the maximum deficiency of a subset of S.*

(Kőnig–Hall) *In particular, there exists a matching covering S if and only if there is no deficient set. The last condition can be rephrased as the Hall condition which requires that*

$$|\Gamma_G(X)| \geq |X| \textit{ for every subset } X \subseteq S. \bullet \tag{2.15}$$

Every theorem on bipartite graphs can be reformulated for hypergraphs and vice versa. For example, Hall's theorem for hypergraph is as follows. Given a hypergraph, a **system of distinct representatives** is a map μ of the set of hyperedges into the set of nodes so that $\mu(Z) \in Z$ for every hyperedge Z and $\mu(Z) \neq \mu(Z')$ for distinct hyaperedges Z and Z'.

Theorem 2.4.5 (Hall [201]) *A hypergraph has a system of distinct representatives if and only if the union of any j hyperedges has at least j elements ($j \geq 1$) or, equivalently, any subset of h nodes induces at most h hyperedges.* •

Note that the reformulation of Hall's theorem in terms of hypergraph orientations asserts that *a hypergraph has an orientation such that the in-degree of every node is at most one if and only if the Hall condition holds.*

Remark 2.4.1 The Kőnig–Hall theorem is equivalent to a result of Frobenius [157] concerning matrices, and thus it could have been called Frobenius' theorem. For a historical overview, see the books of Lovász and Plummer [273] and of Schrijver [340].

With a simple elementary construction, one easily obtains the following extension of the Kőnig–Hall theorem.

Theorem 2.4.6 *Let $G = (S, T; E)$ be a bipartite graph and $z : S \to \mathbf{Z}_+$ a degree specification. There exists a subgraph $(S, T; F)$ of G such that $d_F(v) = z(v)$ for each $v \in S$ and*

$d_F(v) \leq 1$ for each $v \in T$ if and only if

$$\widetilde{z}(X) \leq |\Gamma_G(X)| \text{ for every } X \subseteq S. \tag{2.16}$$

Proof. Replace each node $v \in S$ by $z(v)$ copies and join each of these copies to the neighbours of v in G. In the resulting graph $G^+ = (S^+, T; E^+)$, there is a matching covering S^+ if and only if G has the requested degree-constrained subgraph. Furthermore, the Hall condition holds for G^+ if and only if (2.16) holds. Therefore, the result follows at once from the Kőnig–Hall theorem. •

The submodular technique

The submodular technique for proving structural properties of bipartite matchings was originated by Ore [313, 314]. Here we show how this approach can be used to derive (the non-trivial direction) of the Kőnig–Hall theorem. Suppose that the Hall condition holds. Call a subset $X \subseteq S$ **tight** if $|\Gamma_G(X)| = |X|$. For brevity, we shall denote $|\Gamma_G(X)|$ by $\gamma(X)$. Since $\Gamma_G(X) \cup \Gamma_G(Y) = \Gamma_G(X \cup Y)$ and $\Gamma_G(X) \cap \Gamma_G(Y) \supseteq \Gamma_G(X \cap Y)$, we see that γ is submodular.

Lemma 2.4.7 *The intersection and the union of two tight sets are tight.*

Proof. Let X and Y be tight. The Hall condition implies that $\gamma(X \cap Y) \geq |X \cap Y|$ and $\gamma(X \cup Y) \geq |X \cup Y|$. From the submodularity of γ, we have

$$|X| + |Y| = \gamma(X) + \gamma(Y) \geq \gamma(X \cap Y) + \gamma(X \cup Y) \geq |X \cap Y| + |X \cup Y| = |X| + |Y|.$$

Therefore we must have equality throughout, and in particular, $\gamma(X \cap Y) = |X \cap Y|$ and $\gamma(X \cup Y) = |X \cup Y|$. •

By the repeated application of the lemma, we obtain the observation that the intersection $B(z)$ of all tight sets containing an element $z \in S$ is tight.

Turning to the proof of the Kőnig–Hall theorem, we can assume that the removal of any edge would destroy the Hall condition, implying that every node in S belongs to a tight set.

Claim 2.4.8 *Every node in S is of degree one.*

Proof. Suppose indirectly that there are two edges $e = zu$ and $f = zv$ $(u \neq v)$ incident to a node $z \in S$. Since the removal of e destroys the Hall condition, there is a tight set X containing z for which z is the only neighbour of u in X. Since $B(z) \subseteq X$, we can assume that $X = B(z)$. It follows analogously that z is the only neighbour of v in $B(z)$. In this case, however, $\Gamma_G(B(z) - z)$ contains neither u nor v and hence $|B(z)| - 1 = |B(z) - z| \leq \gamma(B(z) - z) \leq \gamma(B(z)) - 2 = |B(z)| - 2$, a contradiction. •

It follows from Claim 2.4.8 that E is itself a matching covering S, since the Hall condition implies that every two-element subset of S has at least two neighbours. • •

This proof is certainly not the simplest one, but it serves as a prototype for an approach that works powerfully in many other situations. To present one more, consider a bipartite graph $G = (S, T; E)$ in which Hall's condition does not hold for S. Recall the deficiency $h(X) := |X| - |\Gamma_G(X)|$ of a subset X and the parameter $\mu = \mu(G, S)$, which denoted the maximum deficiency of a subset of S. A subset X of S was called max-deficient if $h(X) = \mu$.

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Consider the constructive proof of Theorem 2.4.1 above. The procedure terminated when the set Z of nodes reachable from R_S in the directed graph associated with the current matching M was disjoint from R_T.

Theorem 2.4.9 (Ore [313]) *In a bipartite graph $G = (S, T; E)$, the set of max-deficient subsets of S is closed under taking intersection and union. The set $Z \cap S$* (in the constructive proof of Kőnig's theorem) *is the unique smallest max-deficient set.*

Proof. For two max-deficient sets X and Y we have $\mu + \mu = h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y) \geq \mu + \mu$ from which we must have $h(X \cap Y) = \mu$ and $h(X \cup Y) = \mu$, that is, both the intersection and the union of X and Y are max-deficient sets.

To see the second part, recall the algorithmic proof of Kőnig's theorem given above. It assigned an orientation D_M of G to the current matching M and computed the set Z of nodes reachable from the R_S in D_M where R_S denoted the set of nodes in S left exposed by M. The algorithm terminated when Z did not contain any exposed element of T.

In such a situation, the deficiency of any subset X of S is at most $|R_S|$, and we claim that $X_0 := Z \cap S$ is a set for which the deficiency is exactly $|R_S|$. Indeed, $|\Gamma_G(X_0)| = |\Gamma_M(X_0)| = |X_0 - R_S| = |X_0| - |R_S|$, from which $|X_0| - |\Gamma_G(X_0)| = |R_S|$, as required.

Moreover, we claim that the deficiency of every proper subset X of X_0 is smaller than $|R_S|$, the deficiency of $|X_0|$. Indeed, since Z is the set of nodes reachable from R_S, every proper subset of Z admits an arc of D_M leaving it. Therefore $\Gamma_G(X) \supset \Gamma_M(X)$, from which $h(X) = |X| - |\Gamma_G(X)| < |X| - |\Gamma_M(X)| \leq |R_S| = h(X_0)$. •

Orientations versus matchings

There is a close relationship between in-degree constrained orientations of a graph and degree-constrained subgraphs of a bipartite graph.

Let $G = (A, B; E)$ be a bipartite graph, in which $|A| = |B|$. Let $V = A \cup B$ and define $m : V \to \mathbf{Z}$ by $m(v) = 1$ for every $v \in B$ and $m(v) = d_G(v) - 1$ for every $v \in A$. By orienting the edges in a perfect matching of G towards B while orienting all the other edges towards A, one obtains an orientation of G meeting the in-degree specification m. Conversely, in any orientation of G satisfying this in-degree specification, the set of edges oriented towards B forms a perfect matching of G.

This link allows one to derive the non-trivial direction of the Kőnig–Hall theorem. To this end, suppose that the Hall condition holds, which means that $|\Gamma(Z)| \geq |Z|$ for every subset Z of A. Obviously $\widetilde{m}(V) = |E|$, and we claim that $\widetilde{m}(X) \geq i_G(X)$ holds for every $X \subseteq V$. Suppose, indirectly, that there is a set X for which $\widetilde{m}(X) < i_G(X)$, and assume that $|X|$ is maximum. Then $\Gamma(X \cap A) \subseteq X \cap B$, for if there is an edge $uv \in E$ with $u \in X \cap A$, $v \in B - X$, then for $X' := X + v$ we have $\widetilde{m}(X') = \widetilde{m}(X) + 1 < i_G(X) + 1 \leq i_G(X')$, contradicting the maximal choice of X. Hence the Hall condition implies $|X \cap B| \geq |X \cap A|$, and we have $i_G(X) > \widetilde{m}(X) = \widetilde{m}(X \cap A) + \widetilde{m}(X \cap B) = |X \cap B| + d_G(X \cap A) - |X \cap A| \geq d_G(X \cap A)$, a contradiction. By the Orientation lemma, there is an orientation of G with in-degree vector m and hence a perfect matching of G.

Notice that the alternating-path method, when specialized to find a perfect matching, is nothing but a reformulation of the path-reversing method given in the proof of Theorem 2.3.5.

With an analogous construction, one can see that the degree-specified subgraph problem of a bipartite graph can be formulated as a degree-specified orientation, and the Orientation lemma immediately gives rise to the following.

Theorem 2.4.10 *Let $G = (S, T; E)$ be a bipartite graph and let $z : S \cup T \to \mathbf{Z}_+$ be a degree-specification. G has a spanning subgraph in which the degree of every node v is $z(v)$ if and only if $\widetilde{z}(S) = \widetilde{z}(T)$ and*

$$\widetilde{z}(X \cap S) \leq \widetilde{z}(X \cap T) + d_G(X, T) \text{ holds for every } X \subseteq S \cup T. \tag{2.17}$$

Proof. Suppose first that there exists a requested subgraph $G' = (S, T; F)$ of G and let $X \subseteq S \cup T$ be a subset. Then $\widetilde{z}(X \cap S) = d_F(X \cap S, X \cap T) + d_F(X, T) \leq \widetilde{z}(X \cap T) + d_G(X, T)$.

For sufficiency, let $V := S \cup T$ and define $m : V \to \mathbf{Z}_+$ by

$$m(v) := \begin{cases} z(v) & \text{if } v \in T \\ d_G(v) - z(v) & \text{if } v \in S. \end{cases} \tag{2.18}$$

Note that m is non-negative, since if we had $z(v) > d_G(v)$ for some node $v \in S$, then $X := \{v\}$ would violate (2.17). We have $\widetilde{m}(V) = \widetilde{z}(T) + d_G(S) - \widetilde{z}(S) = d_G(S) = |E|$. Also, for every subset $X \subseteq V$, (2.17) implies $\widetilde{m}(X) = \widetilde{m}(X \cap S) + \widetilde{m}(X \cap T) = d_G(X \cap S) - \widetilde{z}(X \cap S) + \widetilde{z}(X \cap T) \geq d_G(X \cap S) - d_G(X, T) = i_G(X)$. By the Orientation lemma (Theorem 2.3.2), there exists an orientation of G such that $\varrho(v) = m(v)$ for every node $v \in V$. Let F denote the set of edges oriented towards T. It follows from the definition of m that F satisfies the degree specification, that is, $d_F(v) = z(v)$ for every $v \in V$. •

It is useful to observe that in the special case $z \equiv 1$ Theorem 2.4.10 provides a characterization of perfectly matchable bipartite graphs by requiring that $|S| = |T|$ and

$$|X \cap S| \leq |X \cap T| + d_G(X, T) \text{ for every subset } X \text{ of } S \cup T. \tag{2.19}$$

How is this condition related to that of Hall? If Z is a subset of S, then for $X := Z \cup \Gamma(Z)$ we have $d_G(X, T) = 0$, and thus (2.19) transforms to $|Z| \leq |\Gamma(Z)|$. In this sense, (2.19) is a redundant form of the Hall condition.

Problem 2.4.1 *Prove* (without using the Kőnig–Hall theorem) *that the Hall condition implies* (2.19).

Not only the bipartite perfect matching problem can be reduced to orientations but, conversely, the degree-specified orientation problem can also be formulated as a matching problem with the help of a simple elementary construction. Namely, first subdivide each edge of G by a node. There is a one-to-one correspondence between the orientations of G and those subgraphs of the resulting bipartite graph G' for which the degree of every subdividing node is exactly one. Namely, if uv is a directed edge in an orientation of G and s_{uv} denotes its subdividing node, then the edge of G' joining $s_{uv}v$ belongs to the subgraph of G' and, conversely, if $s_{uv}v$ is an edge in the subgraph of G', then we can orient the original undirected uv-edge of G from u to v. Based on this correspondence, Theorem 2.4.6 implies at once the Orientation lemma.

Theorem 2.4.11 *Let $G = (S, T; E)$ be a bipartite graph, $V := S \cup T$, and let $f : V \to \mathbf{Z}_+$ and $g : V \to \mathbf{Z}_+$ be two functions with $f \leq g$. G has a subgraph $H = (S, T; F)$ for which $f(v) \leq d_F(v) \leq g(v)$ holds for every $v \in V$ if and only if*

$$\widetilde{f}(X \cap S) - \widetilde{g}(X \cap T) \leq d_G(X, T) \text{ for every } X \subseteq S \cup T, \tag{2.20}$$

and

$$\widetilde{f}(X \cap T) - \widetilde{g}(X \cap S) \leq d_G(X, S) \text{ for every } X \subseteq S \cup T. \tag{2.21}$$

Proof. Define $f' : V \to \mathbf{Z}_+$, $g' : V \to \mathbf{Z}_+$ as follows.

$$f'(v) := \begin{cases} f(v) & \text{if } v \in T \\ d_G(v) - g(v) & \text{if } v \in S. \end{cases} \tag{2.22}$$

$$g'(v) := \begin{cases} g(v) & \text{if } v \in T \\ d_G(v) - f(v) & \text{if } v \in S. \end{cases} \tag{2.23}$$

On the one hand, since $\widetilde{g'}(X) = \widetilde{g'}(X \cap S) + \widetilde{g'}(X \cap T) = d_G(X \cap S) - \widetilde{f}(X \cap S) + \widetilde{g}(X \cap T) = i_G(X) + d_G(X, T) - \widetilde{f}(X \cap S) + \widetilde{g}(X \cap T)$, we see that (2.20) is equivalent to requiring that $i_G(X) \leq \widetilde{g'}(X)$ for every $X \subseteq S \cup T$, and an analogous argument shows that (2.4.11) is equivalent to requiring that $e_G(X) \geq \widetilde{f'}(X)$ for every $X \subseteq S \cup T$.

On the other hand, there is a one-to-one correspondence between the orientations of G and the subsets F of edges of G in which F is the set of edges oriented towards T. By virtue of the definitions of f' and g', an orientation of G satisfies $f'(v) \leq \varrho(v) \leq g'(v)$ for every $v \in V$ if and only if the corresponding subgraph $H = (S, T; F)$ of G satisfies $f(v) \leq d_F(v) \leq g(v)$. The result, therefore, is a direct consequence of Theorem 2.3.5. •

Corollary 2.4.12 *If there is a subgraph of G such that degrees meet the lower bounds on S and the upper bounds on T, and if there is a subgraph of G such that the degrees satisfy the upper bounds on S and the lower bounds on T, then there is a subgraph satisfying all degree constraints.* •

Exercise 2.4.2 *Show that the Mendelsohn–Dulmage theorem is a special case of Corollary* 2.4.12.

2.4.2 Excessive bipartite graphs

Theorem 2.4.4 characterized the maximum deficiency of the subsets of S in a bipartite graph $G = (S, T; E)$. How can we characterize bipartite graphs in which the Hall condition holds with strict inequality everywhere apart from the empty set? We define such a graph as **excessive** (in S).

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 72 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 2.4.13 (Lovász [263]) *In a bipartite graph $G = (S, T; E)$, there exists a forest for which the degree of every node $s \in S$ is exactly 2 if and only if G is excessive in S, that is, if*

$$|\Gamma_G(X)| \geq |X| + 1 \tag{2.24}$$

for every non-empty subset $X \subseteq S$.

Proof. To see necessity, consider a forest F in the graph induced by $X \cup \Gamma_G(X)$ where $X \subseteq S$. Then F can contain at most $|X| + |\Gamma_G(X)| - 1$ edges, that is, $i_F(X \cup \Gamma_G(X)) \leq |X| + |\Gamma_G(X)| - 1$. On the other hand, if every node in S is incident to two edges from F, then $i_F(X \cup \Gamma_G(X)) = 2|X|$, from which $|\Gamma_G(X)| \geq |X| + 1$ follows, that is, (2.24) is indeed necessary.

To see sufficiency, we first observe that there is a matching M of G covering S by the Kőnig–Hall theorem. Let R denote the set of nodes in T which are left exposed by M. Shrink R into a single new node r. Orient the elements of M towards T and all the other edges towards S. We claim that every node in S is reachable from r in the resulting digraph D. Indeed, if the set X of nodes in S which are not reachable were non-empty, then $\Gamma_G(X) = \Gamma_M(X)$, that is, X would violate (2.24). It follows from the construction of D that every node in T is also reachable from r, and hence D has a spanning r-arborescence F. The edges in the original G corresponding to the edges in F form a forest in which the degree of every node in S is two. •

Corollary 2.4.14 *Let S be a stable subset of nodes of a connected graph $G = (V, E)$. G has a spanning tree F such that $d_F(s) \geq 2$ for every node $s \in S$ if and only if $|\Gamma_G(X)| \geq |X| + 1$ holds for every non-empty subset X of S.*

Proof. Let $T := \Gamma_G(X)$ and consider the bipartite subgraph G' of G on node-set $S \cup T$ consisting of the edges G incident to S. The required tree exists in G if and only if there exists a spanning forest F' of G' for which $d_{F'}(s) = 2$ for every $s \in S$. Hence by Theorem 2.4.13 we are done. •

Problem 2.4.3 ($*$) *Derive the following extension* [263] *of Theorem* 2.4.13.

Theorem 2.4.15 *Let $m : S \to \mathbf{Z}_+$ be a function. In a bipartite graph $G = (S, T; E)$, there exists a forest for which the degree of every node $s \in S$ is exactly $m(s)$ if and only if*

$$|\Gamma_G(X)| \geq \widetilde{m}(X) - |X| + 1 \tag{2.25}$$

for every non-empty subset $X \subseteq S$. •

Question 2.4.1 *What is a necessary and sufficient condition for a graph to include a spanning tree the degee of which at every node of a given stable set S meets a prescribed upper and lower bound?* (For an answer, see Section 9.1.5.)

Theorem 2.4.13 of Lovász can be rephrased in terms of hypergraphs. We call a hypergraph **wooded** if it can be trimmed to a graph which is a forest, that is, if it is possible to select two distinct elements from each hyperedge in such a way that the selected pairs, as graph edges, form a forest.

Theorem 2.4.16 *A hypergraph is wooded if and only if the union of every j hyperedges has at least $j+1$ elements* ($j \geq 1$). •

By definition, a wooded hypergraph on n nodes can have at most $n-1$ hyperedges. A hypergraph $H = (V, \mathcal{F})$ is called a **hypercircuit** if $|V| = |\mathcal{F}|$ and the union of every j distinct hyperedges ($1 \leq j \leq |V|-1$) has at least $j+1$ elements. The bipartite graph associated with a hypercircuit is said to be 'elementary,' an interesting property to be defined and discussed below. A hypergraph is a **hyperforest** if it includes no hypercircuit as a subhypergraph or, equivalently, the union of every j distinct hyperedges ($1 \leq j$) has at least $j+1$ elements. Rephrasing it yet another way, a hyperforest is a hypergraph in which the strong Hall condition is fulfilled. A **spanning hypertree** on node-set V is a hyperforest having $|V|-1$ hyperedges. In these terms, Lovász' theorem asserts that a hypergraph is wooded if and only if it is a hyperforest. Note, however, that a hypercircuit cannot be trimmed to a graph which is a circuit: consider the hypergraph on $\{a, b, c, d\}$ with hyperedges $\{ab, ac, ad, bcd\}$.

Question 2.4.2 *When does a hypergraph include a spanning hypertree?* (For an answer, see p. 297.)

Elementary bipartite graphs

One arrives at another interesting class of bipartite graphs when the strong Hall condition is required for all non-empty *proper* subsets of S but $|T| = |\Gamma_G(S)| = |S|$.

Theorem 2.4.17 *For a perfectly matchable bipartite graph $G = (S, T; E)$, the following properties are equivalent.*

(A) $|\Gamma_G(X)| \geq |X| + 1$ *for every non-empty proper subset X of S.*
(B) *For every pair $\{s, t\}$ of nodes with $s \in S$ and $t \in T$, the subgraph $G - \{s, t\}$ is perfectly matchable.*
(C) *G is connected and every edge of G belongs to a perfect matching.*
(D) *For any perfect matching M of G, G can be obtained from an edge in M by successively adding M-alternating ears connecting distinct colour classes.*
(E) *G can be obtained from an edge by successively adding odd-length ears connecting distinct colour classes.*

Proof. (A)→(B) follow immediately from the Kőnig–Hall theorem. Since (C) is just a special case of (B), we have (B)→(C).

(C)→(D). There is nothing to prove if $|M| = 1$, so suppose that $|M| \geq 2$. Orient the edges in M towards S and all other edges of G towards T. We claim that the resulting digraph D_M is cyclic, that is, every edge belongs to a directed circuit. Indeed, for a non-matching edge $e = st$, there is a perfect matching M' containing e. The symmetric difference $M \ominus M'$ consists of disjoint M-alternating circuits, and these correspond to disjoint directed circuits in D. By the construction of these circuits, one of them contains e. For every matching edge f, there is an edge e having the same end-node. We have seen that e belongs to a circuit and that circuit must contain f by the construction of D_M. Since D_M is cyclic and connected, Theorem 2.2.1 implies the existence of an ear-decomposition of D. By the construction of D_M, the ears of D correspond to M-alternating ears of G.

(D)→(E) Obvious, since an M-alternating ear connecting S and T is of odd length.

(E)→(A) Observe first that by taking every second edge of each ear in an odd ear-decomposition of G, one obtains a perfect matching M of G. Hence the Hall condition is satisfied. Suppose indirectly that there is a proper non-empty subset $X \subset S$ with $|\Gamma_E(X)| = |X|$. The ear-decomposition implies that G is connected and hence there exists an edge e joining $\Gamma_E(X)$ and $S - X$. But such an edge cannot belong to any ear of the ear-decomposition, a contradiction. •

The graphs characterized in Theorem 2.4.17 are called **elementary** while the construction described in in (E) is referred to as a **bipartite ear-decomposition**. The next two results show that elementary bipartite graphs and strongly connected directed graphs are intimately related.

Theorem 2.4.18 *Let M be a perfect matching of a bipartite graph $G = (S, T; E)$, and consider the directed graph D_M obtained from G by orienting each element of M towards S and the other edges towards T. Then G is elementary if and only if D_M is strongly connected.*

Proof. Suppose first that G is elementary. In the proof of Theorem 2.4.17, we have already seen (Part (C)→(D)) that D_M is strongly connected. Conversely, if D_M is strongly connected, then it has a directed ear-decomposition, which determines an odd ear-decomposition of G, showing that G is elementary. •

Theorem 2.4.19 *Let $D = (V, A)$ be a directed graph, and consider a bipartite graph $G = (V', V''; E)$ in which V' and V'' are copies of V and $E := \{v'v'' : v \in V\} \cup \{u'v'' : uv \in A\}$. Then D is strongly connected if and only if G is elementary bipartite.*

Proof. Let M denote the perfect matching $M := \{v'v'' : v \in V\}$ of G. There is a one-to-one correspondence between the ear-decompositions of D and the M-alternating ear-decompositions of G. Hence Theorems 2.4.17 and 2.2.1 imply the theorem. •

2.4.3 Edge-colourings

Theorem 2.4.20 (Kőnig's edge-colouring) *The chromatic index $\chi'(G)$ of a bipartite graph $G = (S, T; E)$ is equal to the maximum degree $\Delta = \Delta(G)$ of G. In particular, the edge-set of a Δ-regular graph G partitions into Δ perfect matchings.*

Proof. Obviously $\chi' \leq \Delta$, so we deal only with the other direction, stating that E can be partitioned into Δ matchings. We prove this for the regular case first and derive then the general case with a simple construction. Suppose that G is Δ-regular. There is nothing to prove when $\Delta = 0$, so suppose that $\Delta \geq 1$. By induction, it suffices to show that G admits a perfect matching M, since the removal of M leaves a $(\Delta - 1)$-regular bipartite graph. For a subset X of S, consider the subgraph $G' = (X, \Gamma_G(X); E')$ induced by $X \cup \Gamma_G(X)$.

By the regularity of G, we have $\Delta|X| = |E'| \leq \Delta|\Gamma_G(X)|$, and hence there is a perfect matching by the Kőnig–Hall theorem, as required.

For the general case, make the two node-classes of G equal in size by adding $||S| - |T||$ new (isolated) nodes to the smaller class. Second, as long as possible, add new (possibly parallel) edges connecting nodes belonging to distinct classes for which the degree is smaller than Δ. In the resulting graph G', the degree of every node in at least one of the two node-classes is Δ, implying that the graph is actually Δ-regular, since the two classes have the same cardinality. Therefore there is a Δ-colouration of G' and this colouration can be restricted to G.•

We also exhibit another proof given by Rizzi [329] because it is a direct inductive proof that does not rely on the Kőnig–Hall theorem.

Alternative proof for the Δ-regular case of Theorem 2.4.20. By induction on the number of nodes we are going to show that there is a perfect matching when $\Delta \geq 1$. Let s and t be two adjacent nodes which are connected by $\beta > 0$ parallel edges. Let S_t denote the set of neighbours of t distinct from s. Let T_s denote the set of neighbours of s distinct from t. Delete s and t and add $\Delta - \beta$ new edges connecting S_t and T_s in such a way that the resulting graph G' is Δ-regular. By induction, the edge-set of G' can be partitioned into Δ perfect matchings. Since the number of newly added edges in G' is $\Delta - \beta < \Delta$, at least one of these matchings does not contain any new edge. This matching plus an edge between s and t is a perfect matching of G. •

Problem 2.4.4 *Prove that the edge-set of a bipartite graph of maximum degree Δ can be partitioned into Δ matchings of cardinality $\lfloor |E|/\Delta \rfloor$ or $\lceil |E|/\Delta \rceil$.*

A non-negative square matrix is said to be **bi-stochastic** if every row-sum and every column-sum is one. A (0, 1)-valued bi-stochastic matrix is called a **permutation matrix**.

Theorem 2.4.21 (Birkhoff [33], Neumann [309]) *A square matrix B is bi-stochastic if and only if it is the convex combination of permutation matrices.*

Proof. Let $G = (S, T; E)$ be a bipartite graph in which the elements of S and T correspond to the rows and columns of B, respectively, and ij is an edge of G for $i \in S$ and $j \in T$ if the corresponding entry b_{ij} of B is positive.

We claim that G has a perfect matching. For a subset X of S, consider the subgraph $G' = (X, Y; E')$ induced by $X \cup Y$, where $Y := \Gamma_G(X)$. This subgraph corresponds to a submatrix B' of B defined by the rows and columns corresponding to the elements of X and Y, respectively, and B' has the property that each of its row sums is exactly one and each of its column sums is at most one. Hence $|X| \leq |Y| = |\Gamma_G(X)|$, and by Hall's theorem there is a perfect matching M of G.

Let P denote the permutation matrix corresponding to the elements of M. Let α be the minimum of the entries of B corresponding to the elements of M. By the definition of M, $\alpha > 0$. If $\alpha = 1$, then $B = P$, and we are done. Suppose now that $\alpha < 1$. A simple calculation shows that $B_1 := (B - \alpha P)/(1 - \alpha)$ is also a bi-stochastic matrix. Since B_1 has fewer positive entries than B, by induction B_1 is a convex combination of permutation matrices. Moreover, $B = (1 - \alpha)B_1 + \alpha P$, and hence B is a convex combination of B_1 and P, implying that B is a convex combination of permutation matrices. •

Problem 2.4.5 *Show that Theorem* 2.4.21 *for rational matrices is a direct consequence of Theorem* 2.4.20.

Equitable edge-colourings

An **equitable** k**-edge-colouring** of a graph $G = (V, E)$ is a partition $\{E_1, E_2, \ldots, E_k\}$ of E such that

$$\lfloor d_G(v)/k \rfloor \leq d_i(v) \leq \lceil d_G(v)/k \rceil \tag{2.26}$$

holds for every node v of G where d_i denotes the degree function of the graph $G_i = (V, E_i)$. In other words, for each node v, the numbers of edges of the same colour incident to v are essentially the same in the sense that they can differ by at most one. The colouring is **strongly equitable** if, in addition, the colour classes are uniform in the sense of having almost the same size in the sense that $\lfloor |E|/k \rfloor \leq |E_i| \leq \lceil |E|/k \rceil$ for each $i = 1, \ldots, k$. When k is the maximum degree Δ of G, an equitable k-edge-colouring is a partition of E into matchings. Therefore the following theorem of de Werra [64] is a straight generalization of Kőnig's edge-colouring theorem.

Theorem 2.4.22 (de Werra) *Every bipartite graph $G = (S, T; E)$ admits a strongly equitable k-edge-colouring for each integer $k \geq 2$.*

Proof. We prove first the special case $k = 2$. There is a pairing M of the odd-degree nodes of G such that at most one pair meets both S and T. By adding M as a matching to G, we obtain an Euler graph. This Euler graph has an Euler orientation D', which defines an orientation D of G.

Any orientation of G determines a 2-edge-colouring according to whether an edge is oriented towards T or S. The 2-edge-colouring arising from D is strongly equitable, since D is smooth and $|\varrho_D(S) - \delta_D(S)| \leq 1$ also holds, since $\varrho_{D'}(S) = \delta_{D'}(S)$ and since at most one edge from M connects S and T.

For the general case $k \geq 3$, first observe that a k-edge-colouring $\{E_1, \ldots, E_k\}$ is strongly equitable if and only if $\{E_i, E_j\}$ is a strongly equitable 2-edge-colouring of the graph $(S, T; E_{ij})$ for every i, j $(1 \leq i < j \leq k)$ where $E_{ij} := E_i \cup E_j$.

Start with an arbitrary k-edge-colouring. As long as there are two colour-classes $\{E_i, E_j\}$ which are not strongly equitable on the subgraph $(S, T; E_{ij})$, replace them by a strongly equitable 2-colouring $\left\{E_i', E_2'\right\}$. Since this operation strictly increases the value $\sum_{l=1}^{k} |E_l|^2 + \sum_{v \in S \cup T} \sum_{l=1}^{k} d_l(v)^2$, the process is finite and the final colouring is strongly equitable. •

Question 2.4.3 *How can a strongly equitable k-edge-colouring be found in polynomial time?*

Note that Kőnig's edge-colouring theorem is obtained in the special case when k is the maximum degree of the graph. At the other extreme, when k is the minimum degree of G, one gets the following.

Corollary 2.4.23 (Gupta [197]) *In a bipartite graph G, the maximum number of disjoint subsets of edges so that each set covers all nodes is equal to the minimum degree of G.* •

In Section 2.5 we will derive the following interesting result on packing matchings of given size.

Theorem 2.4.24 (Folkman and Fulkerson [103]) *In a bipartite graph $G = (S, T; E)$ there are ℓ edge-disjoint matchings of size k if and only if*

$$i_G(Z) \geq \ell(k + |Z| - |U|) \tag{2.27}$$

holds for every subset Z of $U := S \cup T$ where $i_G(Z)$ denotes the number of edges induced by Z.

2.4.4 Chains and antichains of posets

An important consequence of Kőnig's theorem is the following result of Dilworth [68]. The elegant proof is provided by Fulkerson [167].

Theorem 2.4.25 (Dilworth) *In a partially ordered set $(P, \preceq)$, the minimum number of chains covering P is equal to the maximum cardinality $a(P)$ of an antichain.*

Proof. Since a chain and an antichain can have at most one element in common, the inequality max $\leq$ min is obvious. To see the other direction, construct a bipartite graph $G = (X, Y; E)$ as follows. Both X and Y correspond to P, and an element $x_i \in X$ is connected with $y_j \in Y$ if $p_i \succ p_j$. (Note that x_i and y_i are NOT connected). Let n denote the cardinality of P. •

Lemma 2.4.26 *For an arbitrary matching M of G, there is a chain decomposition of P consisting of $n - |M|$ chains.*

Proof. Consider the subset $B \subseteq X$ left exposed by M. Clearly, $|B| = n - |M|$. For every element $x_i \in B$ we construct a chain C_i of P. If y_i is exposed, then let C_i consist of the single element p_i. If y_i is covered by an edge $x_j y_i \in M$, then $p_j > p_i$. Let p_j be the next element of the chain. If y_j is covered by an edge $x_k y_j \in M$, then let p_k be the third element of the chain. Continuing in this way, we increase the chain until y_m is exposed by M, where p_m denotes the last element of the chain.

It follows easily from this construction that the chains assigned this way to the exposed elements of X are pairwise disjoint and cover P. •

Lemma 2.4.27 *Let $L \subseteq X \cup Y$ be a minimal covering of the edges of G. Then there is an antichain $A \subseteq P$ for which $|L| + |A| = n$.*

Proof. We claim first that $x_i \in L$ implies $y_i \notin L$. If indirectly both elements belong to L, then the minimality of L implies that there are edges $x_i y_j$ and $x_k y_i$ of G for which $y_j, x_k \notin L$. In this case, $p_k \succ p_i \succ p_j$, from which $p_k \succ p_j$ follows, and hence $x_k y_j$ must also be an edge of G. But this edge is not covered by L, and this contradiction shows our claim.

Let $A := \{p_i : x_i \notin L,\ y_i \notin L\}$. It follows that A is an antichain and $|L| + |A| = n$. •

By combining the two lemmas with Kőnig's theorem, the theorem immediately follows. • •

Remark. An important advantage of this derivation is that both a largest antichain and a minimum chain decomposition of a poset can be computed in time polynomial in the size of P.

The proof above relied on the node duplicating technique. A similar approach will be used in Section 2.5.2 to derive a variation of Menger's theorem from Kőnig's theorem. In Section 3.6.2, node duplication will also be used in obtaining a considerable generalization of Dilworth's theorem.

In honor of Dilworth, we call an antichain of maximum size a **D-antichain**.

Problems

2.4.6 *A D-antichain A is* **higher** *than a D-antichain A' if there is a one-to-one mapping $\varphi : A \to A'$ so that $p \succeq \varphi(p)$ for every $p \in A$. Prove that every poset has a unique highest D-antichain.*

2.4.7 *Prove that the maximum number of disjoint D-antichains is equal to the minimum number of elements covering all D-antichains.*

2.4.8 $(*)$ *Consider the partial order on the power set of a set S defined by set inclusion. An antichain corresponds to a Sperner family of S. Let $\mathcal{D}$ denote the highest antichain, ensured by Problem* 2.4.6. *Prove that if j denotes the largest cardinality of a subset of S belonging to $\mathcal{D}$, then every j-element subset of S must belong to $\mathcal{D}$. Derive Sperner' theorem* [352, 90] *which states that $\mathcal{D}$ can have at most $\binom{n}{\lceil n/2 \rceil}$ members.*

Problem 2.4.9 *Derive Kőnig's theorem from Dilworth's theorem.*

Theorem 2.4.28 (weighted Dilworth) *Given an integer-valued weight function $w : P \to \mathbf{Z}_+$ on a partially ordered set $(P, \preceq)$, the minimum number of chains covering each element $p \in P$ at least $w(p)$ times is equal to the maximum total weight of an antichain.*

Proof. First remove each element of weight zero, then replace each element p by an antichain of $w(p)$ elements, and finally apply the unweighted Dilworth's theorem to the resulting poset. •

There is an interesting counterpart to Dilworth's theorem, one which is obtained formally by interchanging the terms antichain and chain.

Theorem 2.4.29 (polar-Dilworth) *In a partially ordered set $(P, \preceq)$, the minimum number of antichains covering P is equal to the maximum cardinality $c(P)$ of a chain.*

Proof. Since a chain and an antichain can have at most one element in common, the inequality max $\leq$ min is obvious.

To see the other direction, let A_1 be the set of minimal elements of P. This is clearly an antichain. Let A_2 be the set of minimal elements in $P - A_1$. Continuing in this way, we can construct a decomposition of P into non-empty antichains $A_1, A_2, \ldots, A_c$. Proceeding backward, we can construct greedily a chain of c elements. To this end, let p_c be an arbitrary element of A_c. Since p_c has not been placed in A_{c-1}, there must be an element p_{c-1} in A_{c-1} which is smaller than p_c. This element has not been placed in A_{c-2}, therefore there is an element p_{c-2} in A_{c-2} which is smaller than p_{c-1}. Continuing in this way, the procedure terminates by providing a chain of c elements. •

This proof can be considered as a two-phase greedy procedure. In the first phase, an antichain decomposition of P is constructed greedily, in the sense that the antichains are determined by following a simple selection rule and the selected antichains are not later changed. In the second phase, relying on the antichain decomposition computed in the first phase, we compute greedily a maximum chain.

The polar-Dilworth theorem can also be extended to the weighted case.

Theorem 2.4.30 (weighted polar-Dilworth) *Given an integer-valued weight function $w : P \to \mathbf{Z}_+$ on a partially ordered set $(P, \preceq)$, the minimum number of antichains covering each element $p \in P$ at least $w(p)$ times is equal to the maximum total weight of a chain.*

Proof. Remove each element of weight zero. Replace each element p by a chain of $w(p)$ new elements. An element p' of this chain is smaller by definition than an element q' from the chain assigned to $q \in P$ if $p \prec q$. The polar-Dilworth theorem for the new poset proves the weighted version. •

This reduction does not yield a polynomial time algorithm, but in Section 3.1 we describe an algorithm which is a refinement of the unweighted case above.

2.5 Disjoint paths

We have seen characterizations for 2-edge- and 2-node-connected graphs and also for strongly connected digraphs. A fundamental question concerning higher-order connections of graphs and digraphs is to characterize situations where there are k edge-disjoint or k openly disjoint st-paths. A set of st-paths is **openly disjoint** if every node $v \in V - \{s, t\}$ belongs to at most one of these st-paths.

2.5.1 Menger's theorems for directed and undirected graphs

In a directed or undirected graph $H = (V, F)$, let $\lambda(s, t) = \lambda_H(s, t)$ denote the maximum number of edge-disjoint st-paths and let $\kappa(s, t) = \kappa_H(s, t)$ denote the maximum number of openly disjoint st-paths. How can these parameters be characterized and computed? The theoretical answer is given by Menger's theorem, which has several versions according to whether the graph is directed or undirected and the paths are edge-disjoint or openly disjoint. There are further versions as well. One may, for example, be interested in (completely) disjoint paths from a subset S of nodes to another subset T.

Originally, Menger [288] proved the undirected openly disjoint version. Later Gallai [178] found a constructive proof and noted that the result holds for directed graphs as well. Ford and Fulkerson [107] derived the edge-disjoint forms from the Max-flow Min-cut (MFMC) theorem; see Theorem 3.4.9. Actually, all these variations can be derived from the others by using simple elementary constructions. In addition, both Kőnig's theorem and Dilworth's theorem are also equivalent to Menger's theorem. We begin our investigations with the directed edge version.

Edge-disjoint st*-paths*

Theorem 2.5.1 (directed edge-Menger) *In a digraph D, there are $k \geq 1$ edge-disjoint st-paths if and only if*

$$\delta_D(S) \geq k \text{ for every } s\bar{t}\text{-set } S, \text{ (or, equivalently, } \varrho_D(T) \geq k \text{ for every } t\bar{s}\text{-set } T). \quad (2.28)$$

The k edge-disjoint paths can be selected in such a way that their union forms an acyclic digraph.

Proof. Since any st-path uses at least one edge leaving S, the necessity of the condition is obvious.

For proving sufficiency, observe first that if t is not reachable from s, then the set S of nodes reachable from s is of out-degree $0 < k$, contrary to (2.28). If there is an st-path P of one or two edges, then the deletion of its edges reduces the out-degree of each $s\bar{t}$-set by exactly one. By induction, there are $k - 1$ edge-disjoint st-paths in the resulting digraph, and these paths along with P form k edge-disjoint st-paths in D.

Therefore, every st-path has at least three edges, so there is an edge $e = uv$ incident to neither s nor t. If the out-degree of every $s\bar{t}$-set in $D - e$ is at least k, by induction we are done. We can thus assume that e leaves an $s\bar{t}$-set of out-degree exactly k. First shrink S into a single node s', and let D' denote the resulting digraph. By induction, there are k edge-disjoint $s't$-paths in D'.

Second, let D'' denote the digraph arising from D by shrinking $T := V - S$ into a single node t'. By induction there are k edge-disjoint st'-paths in D''.

Since there are exactly k edges of D leaving S, the two path systems, each consisting of k paths, can be glued together to obtain k edge-disjoint st-paths in D, proving the first part of the theorem. Since this gluing operation preserves acyclicity, the second part also follows. •

Theorem 2.5.2 (undirected edge-Menger) *In an undirected graph, there are $k \geq 1$ edge-disjoint st-paths if and only if the degree of every $s\bar{t}$-set S is at least k.*

Proof. Replace each edge by two oppositely oriented parallel edges and apply the directed edge-version of Menger's theorem. Since it ensures an acyclic solution, at most one of the two oppositely oriented copies of an original undirected edge will be used. •

Remark 2.5.1 The directed and the undirected edge-versions of Menger's theorem above have been formulated in a feasibility form, where a necessary and sufficient condition was given for the existence of the k edge-disjoint st-paths. These theorems, as well as all the forthcoming variations of Menger's theorem, can be reformulated in an equivalent min-max form. We carry out this reformulation only for the two theorems above but emphasize that the feasibility results below also admit an equivalent min-max version.

Theorem 2.5.3 (edge-Menger, min-max version) *In an undirected graph, the maximum number of edge-disjoint st-paths is equal to the minimum cardinality of a cut separating s and t. In a directed graph D, the maximum number of edge-disjoint st-paths is equal to the minimum value of $\delta_D(S)$ over all $s\bar{t}$-sets $S \subset V$.*

Proof. We consider only the undirected case; the directed one follows analogously. The inequality max $\leq$ min clearly holds, since every st-path uses at least one edge of a cut separating s and t. To see the non-trivial max $\geq$ min direction, let k denote the minimum in question. Then every cut separating s and t has at least k edges, and Theorem 2.5.2 implies that G has k edge-disjoint st-paths, as required for max $\geq$ min. •

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 81 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

The reader can readily check that, conversely, Theorem 2.5.3 implies at once Theorems 2.5.1 and 2.5.2. Here is yet another equivalent version of the edge-Menger theorem.

Theorem 2.5.4 *In a directed or undirected graph, the maximum number of edge-disjoint st-paths is equal to the minimum number of edges covering all st-paths.*

Proof. Obviously, max $\leq$ min. We only prove the reverse inequality for digraphs because the undirected case is analogous. Let k denote the minimum in question. Then $\varrho(T) \geq k$ for every $t\bar{s}$-set T, for if $\varrho(T) < k$, then the set of edges entering T covers all st-paths, contradicting the hypothesis. By Theorem 2.5.1, there are k edge-disjoint st-paths. •

Alternative proof of Theorem 2.5.1 using submodularity

To gain more familiarity with the submodular technique, we exhibit an alternative proof for the sufficiency of (2.28) in Theorem 2.5.1. We use induction on the number of edges. Define an $s\bar{t}$-set X as tight if $\delta_D(X) = k$.

Lemma 2.5.5 *The intersection and the union of tight sets are tight.*

Proof. Let X and Y be tight. By (2.28), $\delta_D(X \cap Y) \geq k$ and $\delta_D(X \cup Y) \geq k$. The submodularity of δ_D implies $k + k = \delta_D(X) + \delta_D(Y) \geq \delta_D(X \cap Y) + \delta_D(X \cup Y) \geq k + k$. Hence each inequality holds with equality, in particular $\delta_D(X \cap Y) = k$ and $\delta_D(X \cup Y) = k$. •

If t is the only neighbour of s, then $\delta_D(s) \geq k$ implies that there are at least k parallel edges from s to t and these form the requested k st-paths. Therefore, there is an edge $e = sz$ with $z \neq t$. We can assume that e leaves a tight set, beacuse otherwise e can be left out from D without destroying (2.28), and by induction we are done.

It follows from the lemma that there is a unique largest tight set S not containing z. There must be an edge $f = zu$ with $u \in V - S$, for otherwise $\delta_D(S + z) < \delta_D(S) = k$, contradicting (2.28). Let D' denote the digraph arising from D by replacing e and f by a new edge $h = su$. (This operation is called **splitting-off** e and f.) We claim that D' satisfies (2.28). Indeed, the out-degree of an $s\bar{t}$-set X can decrease only if $u \in X$ and $z \notin X$, in which case $\delta_D(X)$ decreases by 1. So if $\delta_{D'}(X) < k$, then $\delta_D(X) = k$, showing that X is tight. But this contradicts the assumption that S is the unique maximal tight set not containing z.

By induction, there are k edge-disjoint st-paths in D'. If one of them contains h, then h can be replaced in the path by e and f. The resulting st-walk of D can be simplified, and we obtain the requested k st-paths in D. • •

Algorithmic proof of Theorem 2.5.1 using orientations

Neither of the proofs above gives rise to a polynomial time algorithm. Here we show how the non-trivial direction of Theorem 2.5.1 can be proved with the help of the Orientation lemma (Theorem 2.3.2). Since the proof of that result was algorithmic (indeed, two such methods were described: path-reversing and push–relabel), this way one will have a polynomial algorithm that either finds k edge-disjoint st-paths or finds an $s\bar{t}$-set X with $\delta_D(X) < k$. To this end, observe first that it suffices to construct a subgraph $D' = (V, A')$ of D in which

$$\delta_{D'}(s) = k, \varrho_{D'}(s) = 0 \text{ and } \varrho_{D'}(v) = \delta'(v) \text{ holds for every } v \in V - \{s, t\} \qquad (2.29)$$

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 82 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

since in such a D' one can find in a greedy way the k edge-disjoint st-walks. (See Theorem 1.3.5.)

We can assume that $\varrho_D(s) = 0 = \delta_D(t)$. Let G denote the underlying undirected graph of D and define $m : V \to \mathbf{Z}$ by

$$m(v) := \begin{cases} \varrho_D(v) & \text{if } v \in V - \{s, t\} \\ k & \text{if } \ v = s \\ \varrho_D(t) - k & \text{if } v = t. \end{cases} \tag{2.30}$$

Claim 2.5.6 $\widetilde{m}(X) \le e_G(X)$ *for every* $X \subseteq V$, *with equality for* $X = V$.

Proof. First, $\widetilde{m}(V) = \sum[m(v) : v \in V] = \sum[\varrho_D(v) : v \in V] + k - k = e_D(V) = e_G(V)$. Second, observe that (2.28) is equivalent to requiring that $\delta_D(X) \ge k(|X \cap \{s\}| - |X \cap \{t\}|)$ for *every* subset X of V. We have

$$\widetilde{m}(X) = \sum[\varrho_D(v) : v \in X] + k(|X \cap \{s\}| - |X \cap \{t\}|)$$
$$= e_G(X) - \delta_D(X) + k(|X \cap \{s\}| - |X \cap \{t\}|)$$

from which $\widetilde{m}(X) \le e_G(X)$ follows. •

By the Orientation lemma (Theorem 2.3.2), there is an orientation $\vec{G}$ of G with in-degree specification m. Let D' denote the subgraph of D consisting of those edges which have been reversed in $\vec{G}$. Because of the definition of m, D' satisfies the properties given by (2.29). • •

By adapting the path-reversing proof technique described in the proof of Theorem 2.3.5, one obtains the following simple algorithm for finding the requested degree-specified reorientation of D. Start with D. As long as there is a directed st-path in the current reorientation of D, find one and reorient its edges. Since this operation reduces the out-degree of each $s\bar{t}$-set by one, after k reorientations we shall have arrived at the one satisfying the requested in-degree specification m.

2.5.2 Node-disjoint st-paths

Recall that, for a bi-set $X = (X_O, X_I)$, $W(X) := X_O - X_I$ denoted the wall of X, and the wall size was defined by $w(X) = |W(X)|$. Also, $\varrho_D(X)$ denoted the number of edges entering both X_O and X_I. A bi-set $X = (X_O, X_I)$ was called a $t\bar{s}$-bi-set if $t \in X_I$ and $s \in V - X_O$.

Theorem 2.5.7 (directed node-Menger) *For a digraph* $D = (V, A)$, *the following are equivalent.*

(A) *There are k openly disjoint st-paths.*

(B1)

$$\varrho_D(X) + w(X) \ge k \text{ for every } t\bar{s}\text{-bi-set } X = (X_O, X_I). \tag{2.31}$$

(B2) *It is not possible to cover all st-paths of D^- by less than $k - \alpha$ nodes distinct from s and t, where α is the number of parallel st-edges of D and D^- denotes the digraph arising from D by deleting these α edges.*

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 83 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proof. (A)→(B1). Suppose that there are k openly disjoint st-paths, and let X be a bi-set with $t \in X_I,\ s \in V - X_O$. Among the k paths, at most $w(X)$ can use an element from $W(X)$, and hence at least $k - w(X)$ must use an edge from $V - X_O$ to X_I. Hence $\varrho_D(X) \geq k - w(X)$ and (B1) follows.

(B1)→ (B2). Suppose indirectly that (B2) does not hold, and there is a subset $Z \subseteq V - \{s, t\}$ of less than $k - \alpha$ nodes such that $D^- - Z$ includes no directed st-path. Let X_I denote the set of nodes from which t is reachable in $D^- - Z$ and let $X_O := X_I \cup Z$. Then $s \in V - X_O$, and for the bi-set $X = (X_O, X_I)$ we have $\varrho_D(X) = \alpha$ and $w(X) = |Z|$, from which $\varrho_D(X) + w(X) < \alpha + (k - \alpha) = k$, contradicting (2.31).

(B2)→(A). It suffices to prove that there are $k - \alpha$ openly disjoint st-paths in D^-, since the α parallel edges can be considered as α st-paths, obtaining in this way k openly disjoint paths of D. Therefore we can actually assume that $\alpha = 0$.

Construct a new digraph D' as follows. Replace every node u by two new nodes u' and u'' but remove immediately s'' and t'. For every edge $uv \in A$, put $k + 1$ parallel copies of edge $u'v''$ into D'. Furthermore for every node $u \in V - \{s, t\}$, put an edge $u''u'$ into D'.

If D' includes k edge-disjoint paths from s' to t'', then these paths correspond to k openly disjoint st-paths in D.

If D' does not include k edge-disjoint $s't''$-paths, then the directed edge-Menger theorem implies that there are $k - 1$ edges covering all paths from s' to t''. These edges are necessarily of type $u''u'$ and thus they correspond to $k - 1$ nodes of D from $V - \{s, t\}$, which cover all st-paths, contradicting the assumption. •

This reduction method can be called the **node duplicating** (or splitting) technique. A variation of it was already used in the derivation of the Dilworth theorem from Kőnig's theorem. Since the edge-disjoint version was seen to be tractable algorithmically, this approach enables one to compute efficiently k openly disjoint st-paths or a bi-set violating (2.31).

Remark 2.5.2 In the literature, the standard form of the directed node-Menger theorem states the equivalence of (A) and (B2). We proved above that (B2) easily implies (B1) and, in fact, we pointed out that it suffices to assume (B2) for bi-sets X entered only by st-edges. The point in considering such redundant necessary conditions as (B1) is that its greater flexibility often makes a proof more fluent. See, for example, Section 11.3.

Problem 2.5.1 ($*$) *Suppose that a digraph $D = (V, A)$ includes no st-edges. Prove that there are two edge-disjoint st-paths after deleting any node from $V - \{s, t\}$ if and only if there are two openly disjoint st-paths after deleting any edge.*

Theorem 2.5.8 (undirected node-Menger) *For an undirected graph $G = (V, E)$, the following are equivalent.*

(A) *There are k openly disjoint st-paths.*

(B1)

$$d_G(X) + w(X) \geq k \text{ for every } t\bar{s}\text{-bi-set } X = (X_O, X_I). \tag{2.32}$$

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 84 of 640 December 11, 2010 15:15

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

(B2) *It is not possible to cover all st-paths of G^- by less than $k - \alpha$ nodes distinct from s and t where α is the number of parallel st-edges of G and G^- denotes the graph arising from G by deleting these α edges.*

Proof. Replace each edge by two directed parallel edges oriented oppositely and apply the directed node-Menger theorem. •

The original version of Menger's theorem states the equivalence of (A) and (B2) in Theorem 2.5.8.

Kőnig from Menger

We present first a standard derivation of the non-trivial part of Kőnig's theorem (Theorem 2.4.1) from the directed node-Menger theorem. Let $G = (S, T; E)$ be the bipartite graph in question and let $U := S \cup T$. Construct a digraph $D = (V, A)$ as follows. Let s and t be two new nodes and let $V := U \cup \{s, t\}$. For every node $a \in S$, add an sa-arc, and for every node $b \in T$ add a bt-arc. Moreover, replace each ab-edge of G by an ab-arc where $a \in S$.

For this digraph, Theorem 2.5.7 is equivalent to the following: *the maximum number of openly disjoint directed st-paths in D is equal to the minimum number of nodes from U covering all directed st-paths.* Since every directed st-path of D is of length three, there is a one-to-one correspondence between the sets of k openly disjoint st-paths of D and the k-element matchings of G. Similarly, a subset $Z \subseteq U$ covers all directed st-paths of D if and only if Z covers all edges of G. Hence the directed node-Menger theorem indeed implies Kőnig's theorem. •

Menger from Kőnig

We show conversely that an equivalent version of Menger's theorem also follows readily from Kőnig's theorem. For subsets S and T of nodes, we say that a path is an ST**-path** if its first node is in S and its last node is in T. In particular, any node in $S \cap T$ forms an ST-path (of zero length).

Theorem 2.5.9 *In a digraph $D = (V, A)$, we are given two subsets S and T of nodes, each having k elements. There are k* (completely) *disjoint ST-paths if and only if the ST-paths cannot be covered by less than k nodes. Equivalently, the maximum number of disjoint ST-paths is equal to the minimum number of nodes covering all ST-paths.*

Proof. The theorem is a direct consequence of Theorem 2.5.7: add a new node s to D and an edge sv for every $v \in S$. Analogously, add a new node t to D and an edge vt for every $v \in T$.

Nevertheless, we exhibit another proof relying on Hall's theorem. To this end, construct a bipartite graph $G = (A', B''; E)$ as follows. Replace every node u by two new nodes u' and u'', remove the node s'' for every $s \in S$, and remove the node t' for every node $t \in T$. Let A' denote the set of nodes with one prime and B'' those with two primes. Every edge $uv \in A$, where $u \in V - T$ and $u \in V - S$, defines an undirected edge $u'v''$ of G. Furthermore, for every node $u \in V - S - T$, let $u''u'$ be an edge of G.

If there is a perfect matching M of G, it determines k disjoint paths of D from S to T. Indeed, for each node $s \in S$, let $s'u_1'' \in M, u_1'u_2'' \in M, \ldots, u_j't'' \in M$, where $t \in T$. Then $s, u_1, u_2, \ldots, u_j, t$ is the node-set of a directed path in D and these paths are disjoint.

If there is no perfect matching in G, then by Hall's theorem there is a subset $X' \subseteq A'$ having less than $|X'|$ neighbours. Let $Y' := S' - X'$ and $Z'' := \Gamma(X') - (X - S)''$. Then $|\Gamma(X')| < |X'|$ implies $|Y'| + |Z''| < k$. By the construction, the subset $Y \cup Z$ in D covers all the paths from S to T, contradicting the assumption. •

Since the alternating path technique for proving Kőnig's theorem was algorithmic, this way we have a more direct polynomial time algorithm for finding k openly disjoint st-paths. (The earlier one used the reduction to edge-Menger and the algorithmic approach of the Orientation lemma.)

Theorem 2.5.10 *In a digraph $D = (V, A)$, the maximum number of disjoint ST-paths is equal to the minimum number of nodes covering all ST-paths.*

Proof. As max $\leq$ min is obvious, we deal only with max $\geq$ min. Let k denote the minimum in question. We are going to show that there are k disjoint ST-paths. To this end, add a set S' of k new nodes to D along with an $s's$-edge for each $s' \in S'$ and $s \in S$, and add a set T' of k new nodes to D along with a tt'-edge for each $T \in T$ and $t' \in T'$. Let $D' = (V', A')$ denote the resulting digraph.

Suppose first that there is a subset $X \subseteq V'$ of less than k nodes covering all $S'T'$-paths of D'. We can assume that X is minimal with respect to this property. Since $|S'| = k > |X|$, there is a node $s'_1 \in S' - X$. We claim that X is actually disjoint from S', for if there is an $s'_2 \in S' \cap X$, then the minimality of X implies the existence of a path P from s'_2 to T' such that s'_2 is the only element of X covering P. But then if we replace s'_2 to s'_1 in P, we obtain an $S'T'$-path not covered by X. A similar argument shows that $X \cap T' = \emptyset$. It follows that X covers all ST-paths of D, contradicting the definition of k.

By Theorem 2.5.9, there are k disjoint $S'T'$-paths in D' and the restriction of these to V determines the requested k disjoint ST-paths in D. •

Theorem 2.5.11 *Let s and t be two specified nodes of a directed or undirected graph in which there is no st-edge. The maximum number of openly disjoint st-paths is equal to the minimum number of nodes distinct from s and t which cover all st-paths.*

Proof. The undirected case is an immediate consequence of the directed one, since we can replace each undirected edge by two oppositely oriented parallel edges. Let D' denote the digraph in question and let k denote the minimum in question. Let $D = D' - \{s, t\}$, $S := \{v : \text{there is an } sv\text{-edge}\}$, and $T := \{v : \text{there is a } vt\text{-edge}\}$. Theorem 2.5.10 ensures k disjoint ST-paths in D, and these determine k openly disjoint st-paths in D'. •

Problem 2.5.2 *Derive Theorem* 2.5.7 *from Theorem* 2.5.9.

2.5.3 Hybrid connectivity

The similarity between the edge-disjoint and the node-disjoint versions of Menger's theorem naturally raises the question whether these results can be combined into a single result, and this question motivates the introduction of a kind of hybrid notion of edge-disjoint and openly disjoint paths. Let $g : V \to \{1, \ldots, h\}$ be a function. A set of h (directed or undirected) st-paths is said to be g**-bounded** if the paths are edge-disjoint and they use every node $v \in V - \{s, t\}$ at most $g(v)$ times. When $g \equiv 1$, we are back at the notion of openly disjoint st-paths whereas in the case of $g \equiv h$ the g-boundedness of paths simply requires that the paths are edge-disjoint. For a bi-set $X = (X_O, X_I)$, let

$$w_g(X) := \widetilde{g}(X_O - X_I) \ \ (:= \sum[g(v) : v \in X_O - X_I]).$$

Hybrid connectivity in digraphs

The following result is still considered to be a variation of Menger's theorem.

Theorem 2.5.12 *In a digraph $D = (V, F)$, there are h g-bounded st-paths if and only if*

$$\varrho_D(X) + w_g(X) \geq h \text{ for every } t\bar{s}\text{-bi-set } X. \tag{2.33}$$

The h paths can be selected in such a way that their union forms an acyclic digraph.

Proof. Suppose there are h g-bounded st-paths, and consider a bi-set $X = (X_O, X_I)$ with $t \in X_I$ and $s \in V - X_O$. Among these h paths, at most $w_g(X)$ use a node from $X_O - X_I$, and hence at least $j - w_g(X)$ of them must use an edge entering X, from which the necessity of (2.33) follows.

Conversely, suppose that (2.33) holds. We can assume that there is no edge entering s and no edge leaving t. Define a new digraph $D' := (V' \cup V'', F' \cup E' \cup E'')$, as follows. V' and V'' are disjoint copies of V. For each edge $uv \in F$, let $u'v''$ be a member of F'. For each node $v \in V - \{s, t\}$, put $g(v)$ parallel directed edges from v'' to v' and h parallel edges from v' to v''. The edges from v'' to v' form E', while the edges from v' to v'' form E''.

By this construction, if D' includes h edge-disjoint $s't''$-paths, then these paths correspond to h g-bounded st-paths in D. Suppose now indirectly that no such paths exist in D'. By the directed edge version of Menger's theorem (Theorem 2.5.1), there is a subset Z of nodes of D' so that $\varrho_{D'}(Z) < h$ and $t'' \in Z \subseteq V' \cup V'' - s'$. Define $X_O := \{v \in v' \in Z\}$ and $X_I := \{v \in v'' \in Z\}$. By the construction of D', $u'' \in Z$ implies $u' \in Z$, from which $X_I \subseteq X_O$, and for the bi-set $X = (X_O, X_I)$ we have $h > \varrho_{D'}(Z) = \varrho_D(X) + w_g(X)$, contradicting (2.33). •

The second part follows from the second part of Theorem 2.5.1.

Theorem 2.5.13 (directed hybrid Menger) *Let h and ℓ be positive integers for which $h \geq \ell$ and let $D = (V, A)$ be a digraph. The following are equivalent.*

(A) *There are h edge-disjoint st-paths such that every node distinct from s and t is used by at most ℓ paths.*

(B1)

$$\varrho_D(X) + \ell w(X) \geq h \text{ for every } t\bar{s}\text{-bi-set } X. \tag{2.34}$$

(B2) *After removing any subset $Z \subseteq V - \{s, t\}$ of j nodes, the in-degree of every $t\bar{s}$-set in the rest is at least $h - \ell j$.*

Proof. When applied to h and $g \equiv \ell$, Theorem 2.5.12 immediately implies the equivalence of the first two conditions. Furthermore, (B2) is nothing but a reformulation of (B1) that avoids the use of bi-sets. •

In the case where $\ell = 1$, we are back to the directed node-Menger theorem. In the case where $h = \ell$, Condition (2.34) holds automatically for non-simple bi-sets, while for simple ones it is equivalent to requiring $\varrho_D(T) \geq \ell$ for every $t\bar{s}$-set T. In this case, we obtain the edge-Menger theorem (with h in place of k).

Previously, we defined a k-bundle from s to t as a set of k openly disjoint st-paths.

Theorem 2.5.14 (Egawa, Kaneko, and Matsumoto [87]) *Let k and ℓ be positive integers. In a directed graph $D = (V, A)$, there are ℓ edge-disjoint k-bundles from s to t if and only if*

$$\varrho_D(X) + \ell w(X) \geq k\ell \text{ for every } t\bar{s}\text{-bi-set } X. \tag{2.35}$$

Proof. Let $h := k\ell$. Observe that in this case conditions (2.34) and (2.35) are equivalent. If the ℓ k-bundles exist, then there are h edge-disjoint st-paths such that each node $v \in V - \{s, t\}$ belongs to at most ℓ of them, and hence (2.34) follows from (the trivial direction of) Theorem 2.5.13.

To prove sufficiency, observe by Theorem 2.5.13 that there are h edge-disjoint st-paths using every node in $V - \{s, t\}$ at most ℓ times. We can assume that there are no other edges in the digraph. Then the in-degree of each node $v \in V - \{s, t\}$ agrees with its out-degree (and is at most ℓ). Furthermore the out-degree of s is $k\ell$ and the in-degree of t is also $k\ell$.

Construct a bipartite graph as follows. Replace every node $v \in V - \{s, t\}$ with two new nodes denoted by v' and v''. Replace s by new nodes $s'_1, \ldots, s'_k$ and t by $t''_1, \ldots, t''_k$. Add $\ell - \varrho(v) = \ell - \delta(v)$ parallel edges between v' and v''. For every edge $uv \in A$, where $u, v \in V - \{s, t\}$, let $u'v''$ be an edge. Finally, divide the $k\ell$ edges leaving s into k groups of size ℓ. With each sv-edge in group i, associate an edge between s'_i and v'. Analogously, divide the $k\ell$ edges entering t into k groups of size ℓ. With each ut-edge in the group i, associate an edge between u'' and t''_i.

In the resulting bipartite graph, the degree of every node is ℓ. By the edge-colouring theorem of Kőnig (Theorem 2.4.20), the edge-set partitions into ℓ perfect matchings. It can easily be checked that each perfect matching determines k openly disjoint st-paths in D. In other words, we obtained in this way an ℓ-colouration of the edge-set of D such that each colour-class includes a k-bundle from s to t. •

Corollary 2.5.15 *A digraph $D = (V, A)$ is (k, ℓ)-hybrid-connected if and only if, for every pair of nodes s and t, there are ℓ edge-disjoint k-bundles form s to t.* •

Problem 2.5.3 *Suppose that a digraph $D = (V, A)$ includes no st-edges. Prove that there are two edge-disjoint st-paths after removing any node from $V - \{s, t\}$ if and only if there are 3 2-bundles from s to t so that each edge belongs to at most two of them.*

Problem 2.5.4 *Prove the following slight extension of Theorem* 2.5.14.

Theorem 2.5.16 *Let $Z \subseteq V - \{s, t\}$ be a non-empty subset of nodes of a digraph $D = (V, A)$. Let k and ℓ be positive integers. It is possible to colour the edges of D by ℓ colours so that each colour-class includes k edge-disjoint st-paths that are node-disjoint in Z if and only if $\varrho_D(X) + \ell(|Z \cap w(X)| \geq k\ell$ for every $t\bar{s}$-bi-set X.* •

Problem 2.5.5 *Derive the following result of Egawa et al.* [87] *from Theorem* 2.5.16.

Recall that a k-braid was previously defined as a set of k edge-disjoint paths from s to t.

978–0–19–920527–1 02-Andras-Frank-c02-drv Andras Frank (Typeset by SPi, Publisher Services, India) 88 of 640 December 11, 2010 15:15

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 2.5.17 *Let k and ℓ be positive integers and let $D = (V, A)$ be a directed graph. Moreover, let $g : A \to \{1, 2, \ldots, \ell\}$ be a capacity function on the edges. There are ℓ k-braids from s to t such that each arc $e \in A$ belongs to at most $g(e)$ of them if and only if*

$$\varrho_g(X) \geq k\ell \text{ for every } t\bar{s}\text{-set } X. \bullet \tag{2.36}$$

Not only can matchings help in deriving results on disjoint paths, but the opposite is also possible, as is demonstrated by the proof of the following theorem.

Theorem 2.5.18 (Folkman and Fulkerson [103]) *In a bipartite graph $G = (S, T; E)$, there are ℓ edge-disjoint matchings of size k if and only if*

$$i_G(Z) \geq \ell(k + |Z| - |U|) \tag{2.37}$$

holds for every subset Z of $U := S \cup T$, where $i_G(Z)$ denotes the number of edges induced by Z.

Proof. Since a matching M contains at most $|U| - |Z|$ edges for which at least one of their end-nodes is not in Z, M must contain at least $|M| - (|U| - |Z|)$ edges induced by $|Z|$. Therefore, if there are ℓ disjoint matchings of size k, then Z induces at least $\ell(k + |Z| - |U|)$ edges, and thus (2.37) is necessary.

To see sufficiency, construct a digraph $D = (V, A)$ as follows. Let s and t be two new nodes and let $V := U \cup \{s, t\}$. For every node $a \in S$, add ℓ parallel sa-arcs, and for every node $b \in T$, add ℓ parallel bt-arcs. Moreover, replace each ab-edge of G by an ab-arc where $a \in S$. (See Figure 2.4.)

Claim 2.5.19 *D satisfies the conditions of Theorem* 2.5.14.

Proof. Let $X = (X_O, X_I)$ be a $t\bar{s}$-bi-set. Let $Z := (S - X_O) \cup (T \cap X_I)$. Then $\varrho_D(X) = \ell|S \cap X_I| + \ell|T - X_O| + i_G(Z)$. Hence $\varrho_D(X) + \ell w(X) = \ell(|S \cap X_I| + |T - X_O| + w(X)) + i_G(Z) = \ell(|U| - |Z|) + i_G(Z) \geq \ell k$, where the last inequalilty is just (2.37). $\bullet$

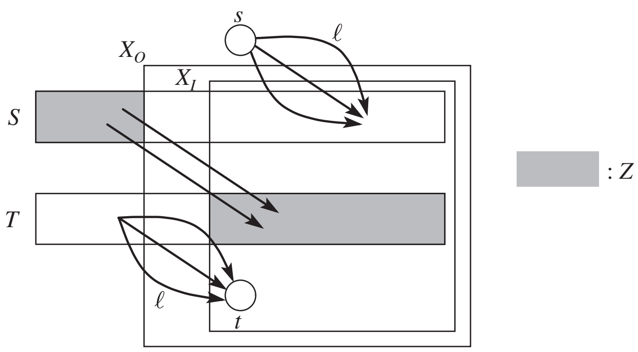


Figure 2.4 $\varrho_D(X) = \ell|S \cap X_I| + i_G(Z)$

By Theorem 2.5.14, D includes ℓ edge-disjoint k-bundles from s to t, and hence the theorem follows because a k-bundle restricted to the edges of G is just a matching of k elements. • •

Hybrid connectivity in undirected graphs

Theorem 2.5.12 implies its undirected counterpart.

Theorem 2.5.20 *In an undirected graph $G = (V, E)$, there are h g-bounded st-paths if and only if*

$$d_G(X) + w_g(X) \geq h \text{ for every } t\bar{s}\text{-bi-set } X. \tag{2.38}$$

Proof. Replace each edge by two oppositely oriented arcs and apply Theorem 2.5.12 to the resulting digraph D. Clearly, (2.33) for D is equivalent to (2.38). Furthermore, the h g-bounded paths in G determine h g-bounded paths in D, and, conversely, the second part of Theorem 2.5.12 ensures that the existence of h g-bounded paths in D determine h g-bounded paths in G. • •

In the special case when the function g is identically the same value ℓ, we have the following.

Theorem 2.5.21 (undirected hybrid Menger) *Let $h \geq \ell$ be positive integers. In an undirected graph $G = (E, A)$ there are h edge-disjoint st-paths such that every node distinct from s and t is used by at most ℓ paths if and only if*

$$d_G(X) + \ell w(X) \geq h \text{ for every } t\bar{s}\text{-bi-set } X. \bullet \tag{2.39}$$

The undirected counterpart of Theorem 2.5.14 [87] can be obtained in a similar way.

Theorem 2.5.22 *Let k and ℓ be positive integers. In an undirected graph $G = (V, E)$, there are ℓ edge-disjoint k-bundles from s to t if and only if $d_G(X) + \ell w(X) \geq k\ell$ for every $t\bar{s}$-bi-set X* (or equivalently, after removing any subset Z of at most $k - 1$ nodes distinct from s and t, every cut of the rest separating s and t has at least $\ell(k - |Z|)$ edges). •

Corollary 2.5.23 *In an undirected graph $G = (V, E)$, there are ℓ edge-disjoint circuits passing through two specified nodes s and t if and only if $d_G(X) + \ell w(X) \geq 2\ell$ for every $t\bar{s}$-bi-set X. The condition is equivalent to requiring that G include 2ℓ edge-disjoint st-paths and $G - v$ include ℓ edge-disjoint st-paths for every $v \in V - \{s, t\}$.* •

We remark that there is a deep theorem of Mader [279] characterizing graphs in which there are k edge-disjoint circuits passing through *one* specified node. Mader's characterization relies on parity, and therefore it is outside the scope of this book. In Euler graphs, however, parity does not come in and in Section 8.4 we will derive a characterization (see Corollary 8.4.10) for Euler graphs.

*2.5.4 **NP**-characterizations of connectivity properties*

In Section 1.2.3, we have already speculated on the need for **NP**-characterization of the various connection properties (such as k-edge-connectivity of a graph).

By their definition, introduced in Section 1.2, they belong to co-**NP**. For ordinary connectivity of a graph and for rooted or strong connectivity of a digraph, such **NP**-characterizations were easy to find. Having obtained a rich set of variations of Menger's theorem above, we are in a position to characterize higher-order connections of (directed or undirected) graphs and hypergraphs, as well. We emphasize, however, that the question of finding **NP**-characterizations for partition-type connections continues to remain unanswered until Section 9.3.

The next result follows immediately from the directed and undirected edge-Menger theorems.

Theorem 2.5.24 (A) *A directed or undirected graph $M = (V, F)$ is k-edge-connected if and only if*

$$\lambda(u, v) \geq k \text{ for every } u, v \in V,$$

that is, there are k edge-disjoint paths from every node to every other.

(B) *A digraph $D = (V, A)$ is rooted k-edge-connected from a designated root-node r_0 if and only if*

$$\lambda(r_0, v) \geq k \text{ for every node } v \in V - r_0. \bullet$$

Since shrinking a subset of nodes does not destroy k-edge-connectivity, we obtain the following.

Corollary 2.5.25 *Let $M = (V, F)$ be a k-edge-connected directed or undirected graph with two specified non-empty subsets S and T of nodes. Then there are k edge-disjoint paths from S to T.* •

In Theorem 10.1.1, we shall discuss a significant sharpening of Part (B) of Theorem 2.5.24, proved by Edmonds [83], asserting that a rooted k-edge-connected digraph includes k edge-disjoint spanning arborescences of the given root.

Theorem 2.5.26 (A) *A directed or undirected graph $M = (V, F)$ is k-node-connected if and only if*

$$\kappa(s, t) \geq k \text{ for every ordered pair of nodes } s, t \in V, \tag{2.40}$$

that is, there are k openly disjoint paths from every node to every other.

(B) *A digraph $D = (V, A)$ is rooted k-node-connected from a designated root-node r_0 if and only if*

$$\kappa(r_0, t) \geq k \text{ for every } t \in V - r_0.$$

Proof. (A) We consider only the directed case. The undirected case can be treated analogously, or it also follows at once from the directed version. Suppose first that (2.40) holds. Let $X = (X_O, X_I)$ be a non-trivial bi-set of V and choose a node t from X_I and a node s from $V - X_O$. By (2.40), there are k openly disjoint st-paths. Among them, at most $w(X) = |X_O - X_I|$ can use an element of $X_O - X_I$, so at least $k - w(X)$ paths must use an edge entering X, from which $\varrho_D(X) + w(X) \geq k$ holds, and hence D is indeed k-node-connected. Conversely, assume that D is k-node-connected, that is, $\varrho_D(X) + w(X) \geq k$

holds for every non-empty bi-set. Then Theorem 2.5.7 implies the existence of the requested k openly disjoint st-paths.

The proof of Part (B) is similar. •

Hypergraphs and dypergraphs

The graph notions of walk and path can be extended to (directed) hypergraphs in a natural way. In a hypergraph $H = (V, \mathcal{E})$, a **hyperwalk** (or simply a **walk**) W is a sequence $(v_0, E_1, v_1, E_2, \ldots, E_k, v_k)$ consisting of not necessarily distinct nodes and hyperedges where E_i is a hyperedge of H containing v_{i-1} and v_i. In a dypergraph, we additionally require that v_i be the head of E_i for each $i = 1, \ldots, k$. A directed (respectively, undirected) walk with distinct members is called a directed (respectively, undirected) **hyperpath** (or path for short). As with the (di)graph case, it is an easy exercise to show that if there is a (di)walk connecting two nodes in a hypergraph, then there is a (di)path as well.

Let $H = (V, \mathcal{E})$ be a (directed) hypergraph and let $G = (V, U; F)$ denote its associated (directed) bipartite graph. Define $g : V \cup U \to \mathbf{Z}_+$ by

$$g(v) := \begin{cases} 1 & \text{if } v \in U \\ k & \text{if } v \in V. \end{cases} \tag{2.41}$$

The definition of the associated bipartite graph immediately gives rise to the following observation.

Lemma 2.5.27 *In a (directed) hypergraph H, there are k edge-disjoint (directed) hyperpaths from s to t if and only if there are k g-bounded st-paths in the associated (directed) bipartite graph G.*

Theorem 2.5.28 *For a hypergraph $H = (V, \mathcal{E})$, the following are equivalent.*

(A) *H is k-edge-connected.*

(B1) *There are k edge-disjoint hyperpaths between s and t for every pair of nodes $s, t \in V$.*

(B2) *There are k g-bounded st-paths in the associated bipartite graph for every pair of nodes $s, t \in V$.*

Proof. The equivalence of (B1) and (B2) was observed in Lemma 2.5.27. Suppose now that (B1) holds. Consider a non-empty proper subset Z of nodes and k edge-disjoint hyperpaths between s and t for $s \in Z$ and $t \in V - Z$. Since each of these hyperpaths must include a hyperedge intersecting both Z and $V - Z$, we infer $d_H(Z) \geq k$, that is, H is k-edge-connected.

Suppose now that (A) holds and select two arbitrary nodes s and t. Assume first that there is a bi-set $X = (X_O, X_I)$ of $U \cup V$ with $t \in X_I$ and $s \in V - X_O$ such that $d_G(X) + w_g(X) < k$. Then $W(X) \cap V = \emptyset$ by the definition of g. Let $Z := X_I \cap V$. Since H is k-edge-connected, there are k hyperedges of H intersecting both X and $V - X$. Consider the elements $u_1, \ldots, u_k$ of U corresponding to these k hyperedges.

If u_i is in $W(X)$, then it contributes 1 to $w(X)$. If u_i is in $U - X_O$, then there is an edge uv of G for which $v \in X_I$ and for which u_i contributes at least 1 to $d_G(X)$. Finally, if u_i is in X_I, then there is an edge uv of G for which $v \in V - X_O$ and u_i contributes at least 1 to $d_G(X)$, showing that we cannot have $d_G(X) + w_g(X) < k$. Theorem 2.5.20 (with k in place of h) implies that there are k g-bounded st-paths in G, and hence (B2) holds. •

By using the same approach and Theorem 2.5.12 instead of Theorem 2.5.20, one can derive the following directed version. The details are left to the reader.

Theorem 2.5.29 *For a dypergraph $D = (V, \mathcal{A})$, the following are equivalent.*

(A) *D is k-edge-connected.*

(B1) *There are k edge-disjoint directed hyperpaths from s to t for every ordered pair of nodes $s, t \in V$.*

(B2) *There are k g-bounded directed st-paths in the associated directed bipartite graph for every ordered pair of nodes $s, t \in V$.* •

Theorem 2.5.30 *For a dypergraph $D = (V, \mathcal{A})$ with a specified root-node r_0, the following are equivalent.*

(A) *D is rooted k-edge-connected.*

(B1) *There are k edge-disjoint directed hyperpaths from r_0 to t for every node $t \in V$.*

(B2) *There are k g-bounded directed r_0t-paths in the associated directed bipartite graph for every node $t \in V$.* •

2.5.5 The edge-disjoint paths problem

In a directed or undirected graph M, we are given k pairs of not-necessarily distinct nodes: $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$. The **edge-disjoint paths problem** consists of finding k pairwise edge-disjoint paths P_i so that P_i is a path from s_i to t_i $(i = 1, \ldots, k)$. It is useful to mark the pairs to be connected by what is called a **demand edge**. In the directed case the demand edge is also directed, namely, it is $t_i s_i$. The (di)graph $H = (U, F)$ formed by the demand edges and their end nodes is called a **demand graph**, while the initial (di)graph M is the **supply graph**. The edge-disjoint paths problem is equivalent to finding $|F|$ edge-disjoint (di)circuits in $M + H$ so that each of these circuits contains exactly one demand edge. Such a circuit is called F**-good**.

Some NP-complete disjoint paths problems

In Section 1.5, we briefly listed some **NP**-complete disjoint paths problems. Here are some others. For an undirected graph G, the edge-disjoint paths problem is known to be **NP**-complete even if $G + H$ is Eulerian [283] or if $G + H$ is planar [290]. For $k = 2$, Thomassen [365] found an elegant characterization, while for fixed $k \geq 2$ Robertson and Seymour [331] proved that it is polynomally solvable. The directed edge-disjoint paths problem is **NP**-complete even for $k = 2$ [108]. For acyclic digraphs, the problem is **NP**-complete for general k, but there is a polynomial time algorithm for fixed k (based on dynamic programming ideas). For a survey of the status of edge-disjoint paths problems, see the work of Naves and Sebő [308].

Some tractable disjoint paths problems

There is a natural necessary condition in both cases:

$$\text{undirected cut-inequality:} \quad d_G(X) \geq d_H(X), \tag{2.42}$$

$$\text{directed cut-inequality:} \quad \varrho_{\vec{G}}(X) \geq \delta_{\vec{H}}(X). \tag{2.43}$$

The (**directed**) **cut condition** (or criterion) requires the (directed) cut-inequality for every $X \subseteq V$. This condition is not sufficient, in general, but there are important special cases when it is. For example, the edge-Menger theorem implies that one such case is when $s_1 = \cdots = s_k$ and $t_1 = \cdots = t_k$. A simple elementary construction shows that the cut condition is sufficient even in the case when only $s_1 = \cdots = s_k$ is required (or only $t_1 = \cdots = t_k$).

Two terminal pairs

Suppose that H consists of two sets of parallel edges, that is, H has k_i edges from t_i to s_i $(i = 1, 2)$. This is one of the smallest demand graphs for which the edge-disjoint paths problem cannot be handled with Menger's theorem. (The other one is when the demand edges form a triangle.)

Theorem 2.5.31 *In the directed case, if the demand graph $\vec{H}$ consists of two sets of parallel edges and $\vec{G} + \vec{H}$ is Eulerian, then the directed cut condition is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof. By applying the directed edge-Menger theorem to the restricted demand graph consisting of the demand edges from t_1 to s_1, we conclude that there are k_1 edge-disjoint directed s_1t_1-paths in $\vec{G}$. By removing the edges of these k_1 paths as well as the k_1 demand edges from $\vec{G} + \vec{H}$, we obtain an Euler digraph which partitions into edge-disjoint circuits. Each of these circuits contains at most one F-edge, so exactly k_2 of them is an F-good circuit. •

Theorem 2.5.32 (Rothschild and Whinston [333]) *In the undirected case, if a demand graph H consists of two sets of parallel edges and $G + H$ is Eulerian, then the undirected cut condition is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof. First orient the k_i parallel demand edges from t_i to s_i $(i = 1, 2)$. By the cut condition, $d_G(X) \geq d_H(X) = \varrho_{\vec{H}}(X) + \delta_{\vec{H}}(X) \geq \varrho_{\vec{H}}(X) - \delta_{\vec{H}}(X)$, meaning that (2.8) holds, and hence Theorem 2.3.4 implies that G has an orientation such that $\vec{G} + \vec{H}$ is Eulerian. We claim that the undirected cut condition for $G + H$ implies the directed one for $\vec{G} + \vec{H}$. Indeed, we have $\varrho_{\vec{G}}(X) + \varrho_{\vec{H}}(X) = \delta_{\vec{G}}(X) + \delta_{\vec{H}}(X)$, that is

$$\varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X) = \delta_{\vec{H}}(X) - \varrho_{\vec{H}}(X). \tag{2.44}$$

Then (2.42) is equivalent to

$$\varrho_{\vec{G}}(X) + \delta_{\vec{G}}(X) \geq \varrho_{\vec{H}}(X) + \delta_{\vec{H}}(X) \tag{2.45}$$

and adding (2.45) to (2.44), we obtain (2.43) twice. The theorem follows from Theorem 2.5.31. •

Note that the theorem of Rothschild and Whinston is a strengthening of an earlier result of Hu stating that if the demand graph H, as before, consists of two sets of parallel edges but $G + H$ is not necessarily Eulerian, then the cut condition is necessary and sufficient for a relaxation of the edge-disjoint paths problem when each path can be chosen with half-capacity. (For a proof, simply duplicate each edge of $G + H$ and then apply Theorem 2.5.32.) For example, if G is a circuit on four nodes s_1, s_2, t_1, t_2 (in this cyclic order) and the demand graph consists of the edges s_1t_1 and s_2t_2, then the cut condition clearly holds but the requested two edge-disjoint paths do not exist. On the other hand, by taking both of the two paths in G between s_1 and t_1 with half capacity, as well as both of the two paths between s_2 and t_2 with half capacity, each edge of G is used altogether exactly once, and in this way we have obtained a half-integral solution to the edge-disjont paths problem.

3
Elements of network optimization

In this chapter, we briefly review some of the most important optimization problems and algorithms in network flow theory: cheapest paths, maximum weight bipartite matchings, cheapest trees and arborescences, maximum flows, and minimum cost flows.

3.1 Cheapest paths and feasible potentials

3.1.1 Conservative cost functions

Let $D = (V, A)$ be a digraph with n nodes and m edges, and let $c : A \to \mathbf{R}$ be a cost function on its edge-set. By a **negative circuit** we mean a directed circuit of negative total cost. When c is a cost function on the edge-set of an undirected graph G, by a negative circuit we mean an undirected circuit of negative total cost. In both cases, the cost function c is said to be **conservative** if there is no negative circuit. One of the basic problems in network optimization is finding a characterization for conservative cost functions, and we analyse this problem for directed graphs below.

For undirected graphs, the problem is significantly more difficult, especially from an algorithmic point of view, because of its strong relationship to non-bipartite matchings. Since the latter topic is outside the scope of this book, conservativity in undirected graphs will not be studied in detail either. However, we will touch on the problem in Section 9.6 and shall provide a characterization in a major special case when the graph is bipartite and the cost function is ±1-valued (see Theorem 9.6.15).

Note that the more general problem of computing the most negative circuit of a graph or digraph is **NP**-complete even for $c \equiv -1$ since it includes the Hamilton circuit problem.

A function $\pi : V \to \mathbf{R}$ on the node-set is often called a **potential** of D. It is said to be c-**feasible** or **feasible** (with respect to c) if

$$\pi(v) - \pi(u) \leq c(uv) \text{ holds for every edge of } D. \tag{3.1}$$

Theorem 3.1.1 (Gallai [179]) *A cost function $c : A \to \mathbf{R}$ on the edge-set of a digraph $D = (V, A)$ is conservative if and only if there is a feasible potential. Moreover, if c is integer-valued, then π can also be selected to be integer-valued.*

Proof. If there is a feasible potential, then for every directed circuit C one has $\widetilde{c}(C) = \sum_{i=1}^{k} c(v_i v_{i+1}) \geq \sum[\pi(v_{i+1}) - \pi(v_i)] = 0$, implying that c is conservative. (Here $v_1, \ldots, v_k$ denote the nodes of C with the notational convention $v_{k+1} = v_1$.)

To see sufficiency, let $W(v)$ denote the cheapest walk terminating at v $(v \in V)$ and having at most n edges. Let $\pi(v)$ be its cost. Clearly, π is integer-valued if c is. We claim that π is a feasible potential. To see this, let uv be an arbitrary edge. Since c is conservative, $W(u)$ induces no negative circuit. $W(u)$ cannot induce a positive circuit either since by removing the edges of such a circuit we would obtain a walk terminating at u with cost strictly smaller than $\pi(u)$. Finally, we can assume that $W(u)$ does not induce any 0-circuit since the removal of its edges would yield another cheapest walk terminating at u.

It follows that $W(u)$ is actually a path and hence it has at most $n-1$ edges. Therefore $W(u) + uv$ is a walk of length at most n and thus $\pi(v) \leq \widetilde{c}(W(u)) + c(uv) = \pi(u) + c(uv)$, that is, $\pi(v) - \pi(u) \leq c(uv)$, as required. •

Algorithm for finding a feasible potential or a negative circuit

We describe an algorithm for an arbitrary cost function c, which finds either a feasible potential or a negative circuit. The method is a variant of the Bellman–Ford algorithm which will also be outlined below. For every node v and for $i = 0, 1, \ldots, n$, let $\pi_c^{(i)}(v)$ denote the minimum cost of walks terminating at v and having at most i edges (but with no restriction made on its initial node). By increasing the counting paramater i one by one, these values can easily be computed for each $i = 0, 1, 2, \ldots, n$ since a walk terminating at v and having at most $i+1$ edges consists of either exactly $i+1$ edges or at most i edges. This straightforward observation verifies the following recursion.

Let $\pi_c^{(0)}(v) \equiv 0$. Assume that, for a given $i \geq 0$, the values $\pi_c^{(i)}(v)$ for all nodes v of D are already available. Set

$$\pi_c^{(i+1)}(v) = \min\left\{\pi_c^{(i)}(v),\ \min\left\{\pi_c^{(i)}(u) + c(uv) : uv \in A\right\}\right\}. \tag{3.2}$$

The same recursion can be used to construct the optimizer walks $W_c^{(i)}(v)$, as well. Namely, for $i = 0$, let $W_c^{(0)}(v)$ be the walk consisting of the single node v and no edge. Assuming that a walk $W_c^{(i)}(v)$ for a given $i \geq 0$ have already been constructed for all nodes v, define $W_c^{(i+1)}(v)$ as follows.

$$W_c^{(i+1)}(v) := \begin{cases} W_c^{(i)}(v), & \text{if } \pi_c^{(i+1)}(v) = \pi_c^{(i)}(v) \\ W_c^{(i)}(u) + uv, & \text{if } \pi_c^{(i+1)}(v) = \pi_c^{(i)}(u) + c(uv) \text{ for some edge } uv \in A. \end{cases} \tag{3.3}$$

From this recursion it follows that $W_c^{(i+1)}(v)$ is a cheapest walk of length at most $i+1$ terminating at v. For every i, the algorithm considers every edge only once so its complexity is $O(nm)$. By combining Gallai's theorem with the recursions above, we obtain the following sharpening of Gallai's theorem. (The letters **P** and **D** refer to the words primal and dual.)

Theorem 3.1.2 *For a cost function $c : A \to \mathbf{R}$ on the edge-set of a digraph $D = (V, A)$ the following are equivalent.*

(P1) *c is conservative.*

(P2) *The walks $W_c^{(n)}(v)$ $(v \in V)$ constructed by recursion* (3.3) *induce no negative circuits.*

(D1) *There is a feasible potential.*
(D2) *The potential* $\pi_c^{(n)}(v)$ $(v \in V)$ *defined by recursion* (3.2) *is feasible.* •

Problem 3.1.1 *Prove* $\pi_c^{(n-1)} = \pi_c^{(n)}$ *if and only if* $\pi_c^{(n)}$ *is feasible.*

The proof of Gallai's theorem is quite simple and is algorithmic, too, but it has a somewhat ad hoc flavour: we came up with a certain function $\pi_c^{(n)}$ which 'miraculously' turned out to be feasible when c is conservative. We present now another proof, which relies on a more natural approach.

Alternative proof for Gallai's theorem.

Start with an arbitrary potential π that is integer-valued when c is. Define $c_\pi : A \to \mathbf{R}$ by $c_\pi(uv) := c(uv) - \pi(v) + \pi(u)$ $(uv \in A)$. If c_π is non-negative, we are done, since then π is feasible. Suppose therefore that there are **deficient** edges, that is, edges with negative c_π-value. The procedure attempts to heal the deficient edges without creating new ones. If the attempt eventually fails, the algorithm returns a negative circuit.

Consider a deficient edge $st \in A$, that is, we have $c_\pi(st) < 0$. Let $A_\pi := \{e \in A : c_\pi(e) \leq 0\}$ and consider the subset Z of nodes reachable from t in the subgraph $D_\pi = (V, A_\pi)$ of D. If s is in Z, that is, if there is a directed ts-path P in D_π, then $K := P + st$ is a directed circuit for which $\widetilde{c}(K) = \widetilde{c}_\pi(K) = \widetilde{c}_\pi(P) + c_\pi(st) < 0$, that is, K is a negative circuit.

If s is not in Z, then revise π as follows.

$$\pi'(v) := \begin{cases} \pi(v) - \varepsilon, & \text{if } v \in Z \\ \pi(v), & \text{if } v \in V - Z, \end{cases} \tag{3.4}$$

where $\varepsilon := \min\{|c_\pi(st)|,\ \varepsilon_1\}$ and

$$\varepsilon_1 := \min\{c_\pi(e) : e \in A, e \text{ leaves } Z\}. \tag{3.5}$$

Here ε_1 is defined to be ∞ if no edge of D leaves Z. The definition of ε shows that ε is positive, and the revision above does not create a new deficient edge. If $\varepsilon = |c_\pi(st)|$, then st is not deficient with respect to π', that is, the set of deficient edges has become smaller. If $\varepsilon = \varepsilon_1 < |c_\pi(st)|$, then repeat the procedure with the same (still) deficient edge st and with the revised potential π'.

Observe that Z induces the same set of edges in A_π and in $A_{\pi'}$ while the edge e of D leaving Z where the minimum in (3.5) is attained gets into $A_{\pi'}$ since $\varepsilon = \varepsilon_1 = c_\pi(e)$ implies $c_{\pi'}(e) = 0$. Therefore the set of nodes reachable from t in $D_{\pi'}$ is strictly larger than Z, showing that such a case can occur at most $n - 1$ times in a row. In other words, after at most $n - 1$ revisions of π either a negative circuit is found or else the edge st ceases to be deficient.

Finally, observe that the integrity of π is preserved throughout provided that c is integer-valued. •

Tensions, neutral cost functions, reduced costs

For an arbitrary potential $\pi : V \to \mathbf{R}$, the function $\Delta_\pi : A \to \mathbf{R}$ defined by

$$\Delta_\pi(uv) := \pi(v) - \pi(u) \quad \text{for } uv \in A \tag{3.6}$$

is called a **potential difference** or **tension** determined by π. The set of tensions is a subspace of $\mathbf{R}^A$. Tensions form a very special class of cost functions as is shown by the next proposition.

Proposition 3.1.3 *For a tension $h = \Delta_\pi$ the cost of every directed circuit is zero, and the cost of every st-path is the same number, namely $\pi(t) - \pi(s)$.*

Proof. For a directed circuit C, we have $\widetilde{\Delta}_\pi(C) = \sum_{i=1}^{k} \Delta_\pi(v_i v_{i+1}) = \sum[\pi(v_{i+1}) - \pi(v_i)] = 0$. Similarly, for an st-path P having nodes $s = v_1, \dots, v_k = t$, we have $\widetilde{\Delta}_\pi(P) = \sum_{i=1}^{k-1} \Delta_\pi(v_i v_{i+1}) = \sum[\pi(v_{i+1}) - \pi(v_i)] = \pi(t) - \pi(s)$. •

Proposition 3.1.3 shows that tensions are neutral with respect to the problem of finding negative circuits and cheapest paths. Therefore, for a given cost function c the **reduced** (or translated) **cost function** c_π defined by

$$c_\pi(uv) := c(uv) - \Delta(uv) = c(uv) - \pi(v) + \pi(u) \tag{3.7}$$

is conservative if and only if c is conservative, and, furthermore, an st-path is of minimum c-cost if and only if it is of minimum c_π-cost. It follows directly from the definitions that a potential π is c-feasible if and only if the reduced cost c_π is non-negative.

Let C be a circuit of D in the undirected sense. We say that C is **c-balanced** or balanced with respect to c if the total cost of edges in one direction along the circuit is the same as the total cost of edges in the opposite direction. In particular, a directed circuit is balanced precisely if its cost is 0. A cost function c is said to be **neutral** if every circuit is balanced.

Proposition 3.1.4 *Let $x : A \to \mathbf{R}$ be a function on the edge-set of a digraph $D = (V, A)$ and let $F \subseteq A$ be a subset of edges. Suppose that $D' = (V, A')$ arises from D by reversing the elements of F and that $x' : A' \to \mathbf{R}$ arises from x by negating the components of x corresponding to the elements of F. Then x' is a tension on A if and only if x is a tension on A'.*

Proof. Since x arises from x' the same way as x' arises from x, it suffices to show that x' is a tension if x is. Suppose that x is a tension, that is, $x = \Delta_\pi$ for some function $\pi : V \to \mathbf{R}$. The same π shows that x' is a tension on A'. Indeed, if an edge uv is in F, then its opposite edge vu is in A' and then $x'(vu) = -x(uv) = -(\pi(v) - \pi(u)) = \pi(u) - \pi(v)$, while if $uv \in A - F$, then $uv \in A'$ and then $x'(vu) = x(uv) = \pi(v) - \pi(u)$. •

Theorem 3.1.5 *Let T be a spanning tree of a digraph D. For a function $x : A \to \mathbf{R}$, the following are equivalent.*

(A) *x is a tension.*
(B) *Every circuit is balanced* (that is, x is neutral).
(C) *Every fundamental circuit belonging to T is balanced.*

Proof. (A)→(B) Let x be a tension, that is, $x = \Delta_\pi$ for some function $\pi : V \to \mathbf{R}$. Let C be a circuit with nodes $v_1, \dots, v_q$ (in this cyclic order). Let C_1 denote the set of edges of C of type $v_i v_{i+1}$ (where $v_{q+1} = v_1$) and let $C_2 := C - C_1$. Then $\widetilde{x}(C_1) - \widetilde{c}(C_2) = \widetilde{\Delta}_\pi(C_1) - \widetilde{\Delta}_\pi(C_2) = \widetilde{\Delta}_\pi(C) = 0$, that is, C is x-balanced.

(B)→(C) is evident.

(C)→(A) Let s be an arbitrary node of T. For every node $v \in V$, let P_v be the unique sv-path in T with node-set $s = u_1, u_2, \dots, u_q = v$. Let P_1 denote the set of edges in P_v of type $u_i u_{i+1}$. Let $P_2 := P_v - P_1$ and define $\pi(v) := \widetilde{x}(P_1) - \widetilde{x}(P_2)$.

We claim that $x = \Delta_\pi$. The definition of π immediately implies that $x(uv) = \pi(v) - \pi(u)$ holds for every tree edge $uv \in T$. Our goal is to show that $x(e) = \pi(b) - \pi(a)$ holds for every non-tree edge $e = ab \in A - T$. Let P denote the unique path in T connecting a and b, and let $v_0 = b, v_1, \dots, v_q = a$ denote the nodes of P. Let $C \in P + ab$ be the fundamental circuit belonging to ab. Let K'_1 denote the set of edges of P of type $v_{i-1}v_i$. Let $K_2 := P - K'_1$ and $K_1 := K'_1 + e$. Now $\widetilde{x}(K_1) = \widetilde{x}(K_2)$ since K is x-balanced. Hence

$$-x(e) = \widetilde{x}\left(K'_1\right) - \widetilde{x}(K_2) = \widetilde{\Delta}_\pi\left(K'_1\right) - \widetilde{\Delta}_\pi(K_2) =$$

$$\sum_{v_{i-1}v_i \in K'_1} [\pi(v_i - \pi(v_{i-1})] - \sum_{v_i v_{i-1} \in K_2} [\pi(v_{i-1}) - \pi(v_i)] = \sum_{i=1}^{q} [\pi(v_i) - \pi(v_{i-1})] = \pi(a) - \pi(b),$$

that is, $x(e) = \pi(b) - \pi(a)$, as required. •

3.1.2 Cheapest sv-walks and sv-paths

Suppose now, given again a cost function $c : A \to \mathbf{R}$, we are interested in finding a cheapest st-path of D where s and t are two designated nodes. When there is no restriction on c, the problem is **NP**-complete even for $c \equiv -1$. (This is the problem of longest st-paths, a close relative to the problem of Hamilton circuits.) For conservative cost functions (and, in particular, for non-negative ones), the following min-max theorem is available.

Theorem 3.1.6 (Duffin [74]) *For a conservative cost function c, the cost $\mu_c(t)$ of a cheapest st-path is equal to the maximum of $\pi(t) - \pi(s)$ taken over all feasible potentials π. When c is integer-valued, the optimal π can also be selected to be integer-valued.*

Proof. Let π be a feasible potential and P an arbitrary st-path with nodes $v_1 = s, v_2, \dots, v_k = t$. Then

$$\widetilde{c}(P) = \textstyle\sum_i c(v_i v_{i+1}) \geq \sum_i [\pi(v_{i+1}) - \pi(v_i)] = \pi(t) - \pi(s), \tag{3.8}$$

that is, min ≥ max follows.

For the converse, we need to find a feasible potential for which equality holds in (3.8). Let α denote the cost of the cheapest st-path. Add a new edge $a = ts$ to D with cost $-\alpha$. This extended cost function remains conservative by the definition of α. By Gallai's theorem there is a feasible potential π which is integral if c is integral. Then $\pi(s) - \pi(t) \leq c(a) = -\alpha$, that is, $\pi(t) - \pi(s) \geq \alpha$, from which $\alpha = \widetilde{c}(P) \geq \pi(t) - \pi(s) \geq \alpha$ for a cheapest st-path P and hence $\widetilde{c}(P) = \pi(t) - \pi(s)$, as required. •

Given a feasible potential π, we say that an edge $uv \in A$ is **tight** if $\pi(v) - \pi(u) = c(uv)$. The proof of Duffin's theorem immediately implies the following optimality criterion.

Corollary 3.1.7 *An st-path P is of minimum cost if and only if there is a feasible potential so that each edge of P is tight.* •

How can one algorithmically construct a cheapest st-path and an optimal potential for a conservative cost function? One possibility is as follows. By Gallai's theorem, we first compute a feasible potential π. Since the potential difference is neutral, it suffices to find a cheapest st-path with respect to the reduced cost function c_π. But the feasiblity of π means that $c_\pi \geq 0$ and for non-negative cost functions the well-known Dijkstra algorithm is available (see below).

The other way is a direct approach not relying on Dijkstra's algorithm. Since it is better to treat the cheapest st-path problem simultaneously for all $t \in V$, we assume that the digraph is root-connected from s. We are interested in finding a cheapest sv-walk $P_c^{(i)}(v)$ of length at most i for $i = 0, 1, \ldots, n$ and for each node $v \in V$. The cost of $P_c^{(i)}(v)$ will be denoted by $\mu_c^{(i)}(v)$. If no sv-walk of length at most i exists, then let $\mu_c^{(i)}(v) := \infty$. By a simple reduction to (3.2), the values $\mu_c^{(i)}(v)$ as well as the sv-walks $P_c^{(i)}(v)$ can be computed. To this end, add a new node s' to the digraph along with a new edge $s's$ for which the cost is defined to be $-M$ where M is an appropriately large number. In the resulting digraph on $n + 1$ nodes, the cheapest walk terminating at v will automatically start at s' for every node v. Therefore $\mu_c^{(i)}(v)$ is exactly $\pi_c^{(i+1)}(v) + M$ for $i \leq n$ where $\pi_c^{(i+1)}(v)$ denotes the cost of a cheapest walk in D' having at most $i + 1$ edges that end at v.

This reduction can be avoided and the following direct recursion, due to Bellman [16] and Ford [106], can be given. For $i = 0, 1, \ldots, n$, let $\mu_c^{(0)}(s) = 0$ and let $\mu_c^{(0)}(v) \equiv \infty$ for every $v \in V - s$. If the values $\mu_c^{(i)}(v)$ are already available, then, for every node v, let

$$\mu_c^{(i+1)}(v) = \min\left\{\mu_c^{(i)}(v),\ \min\left\{\mu_c^{(i)}(u) + c(uv) : uv \in A\right\}\right\}. \tag{3.9}$$

As before, the same recursion can be used to compute a cheapest sv-walk $P_c^{(i)}(v)$ $(v \in V)$. The complexity is again $O(mn)$.

Corollary 3.1.8 *If the cheapest sv-walk $P_c^{(n)}(v)$ $(v \in V)$ induces no negative circuits, then $\mu_c^{(n)}$ is a feasible potential, that is, for every edge uv,*

$$\mu_c^{(n)}(v) - \mu_c^{(n)}(u) \leq c(uv). \bullet \tag{3.10}$$

Suppose now that c is conservative. A cheapest st-walk of length at most n can always be chosen to be a path since by removing the circuits included in the walk the total cost does not decrease so the st-path obtained this way is also cheapest. Therefore (3.9) may be used to compute a cheapest st-path.

Corollary 3.1.7 asserted that for a cheapest st-path P there is a feasible potential π for which each edge of P is tight. The next result shows that μ_c is a universal feasible potential that will suffice for all possible cheapest st-paths. Let $D_0 = (V, A_0)$ denote the subgraph of tight edges with respect to the feasible potential μ_c.

Theorem 3.1.9 *Given a root-connected digraph D and a conservative cost function c, an st-path P is a cheapest st-path* (that is, of cost $\mu_c(t)$) *if and only if all of its edges are in D_0.*

Proof. Suppose first that P is an st-path in D_0. Let $c_1(uv) := c(uv) - \mu_c(v) + \mu_c(u)$ for $uv \in A$. Since μ_c is feasible, $c_1 \geq 0$ and hence the c_1-cost of every st-path in D is non-negative. Since P consists of tight edges, $\widetilde{c}_1(P) = 0$, and hence P is a cheapest st-path with respect to c_1. But then P is cheapest with respect to the equivalent cost function c, as well.

Conversely, suppose that P is a cheapest st-path in D and let $s = v_0, v_1, \cdots, v_k = t$ denote the nodes of P. Then $\mu_c(t) = \widetilde{c}(P) = \sum[c(v_{i-1}v_i) : i = 1, \dots, k] \geq \sum[\mu_c(v_i) - \mu_c(v_{i-1}) : i = 1, \dots, k] = \mu_c(t) - \mu_c(s) = \mu_c(t)$ from which we must have $c(v_{i-1}v_i) = \mu_c(v_i) - \mu_c(v_{i-1})$ for each $i = 1, \dots, k$, showing that each edge of P belongs to D_0. •

A (not necessarily spanning) s-arborescence F is called an **arborescence of cheapest paths** if the unique sv-path in F is a cheapest sv-path in D for every node v of F.

Corollary 3.1.10 *Under the hypothesis of Theorem* 3.1.9*, there is a spanning s-arborescence of cheapest paths.* •

Problems

3.1.2 *Show that $\pi_c^{(i)}$ can be computed with the help of $\mu_c^{(i)}$.*

3.1.3 *Suppose that D is root-connected from s and c is a conservative cost function. Prove that $\pi \leq \mu_c$ for every feasible potential π with $\pi(s) = 0$.*

3.1.4 *Develop an algorithm to decide if there exists a 0-circuit for a conservative cost function.*

3.1.5 *Given an arbitrary weight function, determine an st-path for which the weight of its heaviest edge is as small as possible.*

3.1.6 *We are given two conservative cost functions c_1 and c_2. Develop an algorithm which finds a minimum c_2-cost st-path among the minimum c_1-cost st-paths.*

3.1.7 *If each edge of an st-path P belongs to a cheapest st-path, then P itself is a cheapest st-path.*

Non-negative costs: the idea of Dijkstra's algorithm

Suppose that c is a non-negative cost function. As before, let $\mu_c(v)$ denote the cost of a cheapest sv-path.

Lemma 3.1.11 *Let T be an s-arborescence of cheapest paths. Suppose that the minimum in $m_T := \min\{\mu_c(u) + c(uv) : uv$ leaves $V(T)\}$ is attained at an edge $a = u_a v_a$. Then $T' := T + a$ is also an arborescence of cheapest paths.*

Proof. It suffices to prove that the unique sv_a-path P' of T' is cheapest in D. From the definitions, $\widetilde{c}(P') = m_T$. Let P be an arbitrary sv_a-path in D. Let $e = u_e v_e$ be the first edge of P (starting from s) that leaves $V(T)$ and let P'' be its subpath from s to u_e. By the non-negativity of c, we have $\widetilde{c}(P') = m_T \leq \mu_c(u_e) + c(e) \leq \widetilde{c}(P'') + c(e) \leq \widetilde{c}(P)$. •

Based on Lemma 3.1.11, Dijkstra's method consists of $n - 1$ phases. Starting at s, we build up an s-arborescence of cheapest paths by adding new edges one by one. With an appropriate data structure, the algorithm is of complexity $O(n^2)$.

Acyclic digraphs

Suppose that D is an acyclic root-connected digraph. Then an arbitrary cost function c is automatically conservative. This has the benefit that, by negating c, the *maximum* cost of an st-path can also be determined. In general digraphs this problem is **NP**-complete.

Let $v_1 = s, v_2, \dots, v_n$ be a topological order of the nodes. (Such an ordering can be computed with depth-first search in linear time.) The computation of an s-arborescence of cheapest paths is even simpler than in the case of non-negative costs since the order of nodes in which they are added to T is determined beforehand by the topological order. If F_{j-1} and the values $\mu_c(v_i)$ $(1 \le i \le j-1)$ are already computed for the subgraph induced by $\{v_1, \dots, v_{j-1}\}$, then $\mu_c(v_j) = \min\{\mu_c(v_i) + c(v_i v_j) : v_i v_j \in A\}$. Furthermore if $v_i v_j$ denotes the minimizer edge, then $F_j := F_{j-1} + v_i v_j$ is an arborescence of cheapest paths on the first j nodes.

Theorem 3.1.6 can be specialized to arbitrary cost functions on acyclic digraphs. Since in applications (like PERT) it is typical that one is interested in the heaviest paths, we formulate a min-max result for that version.

Theorem 3.1.12 *Let $c : A \to \mathbf{R}$ be an arbitrary weight function on the edge-set of an acyclic digraph $D = (V, A)$. Then*

$$\max\{\widetilde{c}(P) : P \text{ an } st\text{-path}\} = \min\{\pi(t) - \pi(s) : \pi(v) - \pi(u) \ge c(uv),\ uv \in A\}. \bullet \tag{3.11}$$

Note that the algorithm outlined above can be used to compute the heaviest st-path. This algorithm can also be used to compute a maximum weight st-path if the weight function is defined on the node-set: define the weight of an arc uv to be the weight of v. In particular, a heaviest chain of a weighted partially ordered set can also be computed. Due to its elegance, we work out explicitly the resulting algorithm for this case.

Recall the two-phase greedy-type algorithm in Section 2.4 that was used to prove the polar-Dilworth theorem (Theorem 2.4.29). That approach was also used for proving the more general weighted polar-Dilworth theorem (Theorem 2.4.30) but it did not give rise to a polynomial algorithm. The present two-phase algorithm is a refinement and computes in strongly polynomial time a maximum weight chain as well as the optimal antichain packing. An additional feature of the algorithm is that the algorithm also works for real weights and proves the validity of an extension of Theorem 2.4.29 that was formulated only for integer weights.

Algorithmic proof for weighted polar-Dilworth

In the first phase, we compute at most n distinct antichains A_i along with a positive integer y_i which will show how many copies of A_i are to be put in the antichain family. At the same time, we suitably reduce the weight function. The first phase terminates when every revised weight is zero.

More specifically, in step i first remove all elements for which the current weight is zero. Let A_i be the set of minimal elements in the rest. Then A_i is clearly an antichain. Define y_i to be the minimum of current weight of the elements of A_i. Reduce the weight of each element of A_i by y_i. This increases the number of elements with weight zero (namely, an element of A_i where the minimum is attained becomes zero) while the revised weight function remains non-negative.

Suppose that the first phase terminates after k steps, that is, the first phase produces k distinct antichains $A_1, \dots, A_k$ along with the associated coefficients $y_1, \dots, y_k$. Consider the antichain family $\mathcal{A}$ consisting of y_i copies of A_i for $i = 1, \dots, k$. By this construction, each element p of P belongs to exactly $w(p)$ members of $\mathcal{A}$. More precisely, there is a smallest index h and a largest index $\ell \geq h$ such that p belongs to $A_h, \dots, A_\ell$, and hence $w(p) = \sum_{i=h}^{\ell} y_i$.

In Phase 2, we proceed backward on the antichains $A_k, A_{k-1}, \dots, A_1$. Pick up first an arbitrary element p_1 of A_k. Let i be the largest subscript for which A_i does not contain p_1 (if there is any). Since p_1 is not in A_i but it is in A_{i+1}, there is an element p_2 of A_i which is smaller than p_1. Continuing this way, we are building a chain $C = \{p_1, p_2, \dots, p_t\}$ until the construction cannot be continued since p_t is in A_1. It follows that the total weight of C is exactly $\sum_{i=1}^{k} y_i$. •

Problems

3.1.8 *Develop an algorithm to select disjoint members of a given family of subpaths of an underlying path such that its total length is maximum.*

3.1.9 *Develop an algorithm that, given two sequences of letters, selects a maximum common subsequence. Consider also the generalization when each letter has a weight and a common subsequence of maximum weight is to be found.*

3.1.10 *Determine algorithmically a longest monotone increasing subsequence of a sequence of numbers.*

3.1.3 Making a cost function conservative

Suppose that a cost function c on the edge-set of a digraph $D = (V, A)$ is not conservative. Lifting c by a value α means that the cost of each edge is increased uniformly by α. The **deficit** $\varepsilon = \varepsilon(c)$ is the smallest positive number such that the lift of c by ε results in a conservative cost function c^+. In other words, ε is the number so that after lifting c by ε the cheapest circuit is of cost zero. By Theorem 3.1.1, this latter property is equivalent to the existence of a potential π for which $c_\pi(e) \geq -\varepsilon$ for each edge e of D and there is a di-circuit of D consisting of edges with $c_\pi(e) = -\varepsilon$. We describe a method of Karp [234] to compute ε.

The **mean** of a di-circuit C is defined to be $\widetilde{c}(C)/|C|$. Note that lifting c uniformly by α increases the mean of each di-circuit by α. This immediately implies the following.

Proposition 3.1.13 *Suppose that c is not conservative. The deficit $\varepsilon(c)$ is the negative of the minimum mean $\mu = \mu(c)$ of a di-circuit.* •

Therefore it suffices to compute the minimum circuit mean $\mu = \mu(c)$. We do this for arbitrary cost functions, conservative or not. It is convenient to suppose that there is a node s from which every node is reachable. This can be achieved by adding a new node s and an sv-edge of zero cost for every node v. Such an extension does not affect the minimum circuit mean. Let $p_c^{(k)}(v)$ $(k = 0, 1, \ldots, n)$ denote the minimum cost of an sv-walk having exactly k edges. (When no such a walk exists, then $p_c^{(k)}(v) := \infty$.) The values $p_c^{(k)}(v)$ can be computed by the following recursion:

$$p_c^{(k+1)}(v) = \min\left\{p_c^{(k)}(u) + c(uv) : uv \in A\right\}.$$

Lemma 3.1.14 *If c is conservative, then for every $v \in V$,*

$$\max_k \left\{p_c^{(n)}(v) - p_c^{(k)}(v) : 0 \leq k \leq n-1\right\} \geq 0. \tag{3.12}$$

Proof. The maximum is attained where $p_c^{(k)}(v)$ takes its minimum. Since c is conservative, this minimum is just $\mu_c^{(n)}(v)$, which is at most $p_c^{(n)}(v)$. •

Lemma 3.1.15 *If c is conservative and there is a 0-circuit C, then there is a node t for which*

$$\max_k \left\{p_c^{(n)}(t) - p_c^{(k)}(t) : 0 \leq k \leq n-1\right\} = 0. \tag{3.13}$$

Proof. Since max ≥ 0 by Lemma 3.1.14, we only need to show that there is a node t and an st-walk P_n of n edges such that $\widetilde{c}(P_n)$ is the minimum cost of an st-path.

Let P be a cheapest path from s to C. The reduced cost function $\bar{c}(uv) := c(uv) - \mu_c^{(n)}(v) + \mu_c^{(n)}(u)$ is non-negative and, for any $v \in V$, an arbitrary sv-walk is of minimum cost if and only if $\bar{c}(uv) = 0$ holds for all of its edges. Since the $\bar{c}$-cost of every edge of a 0-circuit is 0, the path P can be extended by going around C as long as we get a walk P_n of exactly n edges. The terminal node t of this walk satisfies the requirement of the lemma. •

Theorem 3.1.16 (Karp [234])

$$\mu(c) = \min_{v \in V} \max_{k=0}^{n-1} \frac{p_c^{(n)}(v) - p_c^{(k)}(v)}{n-k}. \tag{3.14}$$

Proof. In the case $\mu = 0$, (3.14) is equivalent to (3.13). This implies the general case once one observes that lifting c by any value α increases the ratio

$$\frac{p_c^{(n)}(v) - p_c^{(k)}(v)}{n-k}$$

by α. •

By the proof method of Karp's theorem, both the minimum circuit mean or the deficit $\varepsilon(c)$ and a feasible potential π with respect to c^+ can be computed in $O(mn)$, where c^+ arises from c by lifting c with $\varepsilon(c)$. As mentioned above, a circuit consisting of edges with $c^+(e) = 0$ is a minimum mean circuit. We will refer to this method as Karp's algorithm for the minimum circuit mean. This will be a main ingredient of a strongly polynomial algorithm for computing cheapest feasible circulations and flows (see Section 6.4).

3.2 Cheapest trees and arborescences

One of the oldest combinatorial optimization problems involves computing a cheapest spanning tree of a connected graph. It is well known that such a computation can be carried out with the help of a simple greedy algorithm. The directed counterpart of this problem, finding a cheapest arborescence, is not so simple but is still nicely tractable. In this section we briefly overview some basic properties of trees and arborescences and outline the corresponding optimization algorithms.

3.2.1 Cheapest trees

Let T be the edge-set of a spanning tree of $G = (V, E)$. The **fundamental cut** belonging to a tree edge $e \in T$ means the cut of G detemined by the two components of $T - e$. The **fundamental circuit** belonging to a non-tree edge $f = uv \in E - T$ means the circuit obtained from the unique path of T connecting u and v by adding the edge f. The following simple observations describe when a tree edge and a non-tree edge can be exchanged to obtain a tree.

Proposition 3.2.1 *Let T be a spanning tree T of a graph. For a tree edge e and non-tree edge f the following are equivalent.*

(A) *$T - e + f$ is a spanning tree.*
(B) *e is in the fundamental circuit of f.*
(C) *f is in the fundamental cut of e.* •

An even more important exchange property of trees follows.

Lemma 3.2.2 (symmetric exchange) *Let T_1 and T_2 denote two trees having the same node-set V. For every edge $e_1 \in T_1$ there is an edge $e_2 \in T_2$ such that both $T_1 - e_1 + e_2$ and $T_2 - e_2 + e_1$ are spanning trees.*

Proof. If $e_1 \in T_2$, then $e_2 := e_1$ will suffice. Suppose that $e_1 = st \notin T_1$. The forest $T_1 - e_1$ has two components for which their node-sets are K_1 and K_2. In T_2 there is a unique path P connecting s and t. Let e_2 be an edge of the path P connecting K_1 and K_2. Then $T_1 - e_1 + e_2$ is a spanning tree since e_2 is an edge connecting the two components of the forest $T_1 - e_1$. Since e_2 is an element of the unique circuit of $T_2 + e_1$, it follows that $T_2 + e_1 - e_1$ is also a spanning tree. •

Remark The following much stronger symmetric exchange property is also true. Let T_1 and T_2 denote two trees having the same node-set V. For every subset $E_1 \subseteq T_1$ there is a subset $E_2 \subseteq T_2$ in such a way that both $(T_1 - E_1) \cup E_2$ and $(T_2 - E_2) \cup E_1$ are spanning trees. The proof, however, requires tools from matroid theory [191], see Theorem 13.3.4.

Problem 3.2.1 $(*)$ *Suppose that a graph $G = (V, E)$ can be partitioned into k spanning trees and that $F \subseteq E$ is an arbitrary subset of at most k edges. Then G can be partitioned into k spanning trees $T_1, \ldots, T_k$ in such a way that the trees are equitable in F in the sense that $|T_i \cap F| \leq 1$ for every $1 \leq i \leq k$.*

Problem 3.2.2 *Suppose that a graph G can be partitioned into k spanning trees. Let s be a specified node of G. Then G can be partitioned into k edge-disjoint spanning trees*

$T_1, \dots, T_k$ in such a way that the trees are equitable at s in the sense that $|d_i(s) - d_j(s)| \leq 1$ where $d_i(s)$ denotes the number of edges of F_i incident to s.

Let $c : E \to \mathbf{R}$ denote a cost function. There are several variations of the greedy algorithm and here we outline three of them. Lemma 3.2.2 immediately implies the following optimality criteria.

Theorem 3.2.3 *For a spanning tree T of a graph the following are equivalent:*

(A) *T is a cheapest spanning tree.*

(B) *$c(e) \leq c(f)$ holds for every edge $e \in T$ where f is an element of the fundamental cut of e,*

(C) *$c(f) \geq c(e)$ holds for every edge $f \notin T$ where e is an element of the fundamental circuit of f.*

Proof. (A)→(B) By Proposition 3.2.1 $T' = T - e + f$ is a tree and hence $c(e) > c(f)$ would imply that T is not cheapest. (B)→(C) also follows from Proposition 3.2.1.

(C)→(A) Let T' be a cheapest tree having as many edges in common with T as possible. We are done if $T' = T$ so suppose that T' has an edge f which is not in T. By Lemma 3.2.2 there is an edge $e \in T - T'$ for which both $T' - f + e$ and $T - e + f$ are spanning trees. Since T' is cheapest spanning tree, we have $c(f) \leq c(e)$. On the other hand, Property (C) implies that $c(f) \geq c(e)$ from which $c(f) = c(e)$ follows. This means that $T'' := T' - f + e$ is another spanning tree of minimum cost but $|T'' \cap T| = |T' \cap T| + 1$ contradicting the maximal choice of T'. •

Greedy algorithm [Bo̊ruvka [37], Kruskal [251]] The procedure builds up a forest by adding new edges of G one by one. It starts with a forest on V with no edges and terminates with a spanning tree. At a general step, a cheapest edge connecting two distinct components of the current forest is added to the forest where ties are broken arbitrarily if there is more than one cheapest edge.

Algorithm of Prim and Dijkstra [Prim [323], Dijkstra [66]] Starting at an arbitrary node x_0, a tree is built up by adding edges one by one as long as the current tree is not spanning. At a general step a cheapest edge uv is added to the current tree T for which $u \in V(T)$ and $v \in V - V(T)$.

The following algorithm can be called cautious or reverse greedy. Instead of selecting cheap edges for building up the tree, one gets rid of the expensive ones while taking care to preserve connectivity.

Reverse greedy (cautious) algorithm At a general step, select the most expensive edge which is not a cut-edge and remove it.

With the help of Lemma 3.2.2, the correctness of each of the three algorithms can easily be verified. Actually there is a generic form of the greedy algorithm [362] that includes each of these algorithms as a special case.

Generic algorithm The algorithm consists of applications of the following two operations in an arbitrary order. The first operation builds a spanning forest F by adding edges one by one while the second operation deletes edges one by one. Specifically, let F denote the forest already constructed (where the beginning F is a forest of no edges.) If F is a spanning tree, STOP.

Step A Choose an arbitrary cut B of G disjoint from F and select a cheapest edge e from B. Add e to F.

Step B Choose an arbitrary circuit C of the current graph G' (if there is none, Step B no longer applies), and select the most expensive edge e from $C - F$. Delete e from G'.

Theorem 3.2.4 *The final tree of the generic algorithm is of minimum cost.*

Proof. Any stage of the algorithm can be specified by a pair (F, D) of disjoint subsets of E where F denotes the forest constructed so far and D denotes the set of edges deleted so far.

We prove by induction that at each stage (F, D) of the algorithm there is a minimum cost tree T of G for which $F \subseteq T \subseteq E - D$. Any min-cost tree will suffice when $F = D = \emptyset$. Suppose we have already proved the statement for (F, D), that is, there is a min-cost tree T with $F \subseteq T \subseteq E - D$.

Assume first that Step A is applied and let $e \in B$ be the newly added edge and $F' := F + e$. If $e \in T$, we are done. Otherwise let C_e be the fundamental circuit of e with respect to T. Edge e belongs to both cut B and circuit C_e, so there must be another edge f in $B \cap C_e$. Since $e, f \in B$, the rule in Step A implies $c(e) \leq c(f)$. Since $e, f \in C_e$ and T is of minimum cost, we have $c(f) \geq c(e)$. Hence $c(e) = c(f)$ and $T' := T - e + f$ is another min-cost tree for which $F' \subseteq T' \subseteq E - D$.

Second, assume that Step B is applied and let $e \in C$ be the newly deleted edge. If T does not contain e, we are done. Otherwise let B_e be the fundamental cut of e belonging to T. There is an edge $f \neq e$ with $f \in C \cap B_e$. Since $e, f \in C$, by the rule in Step B we have $c(e) \geq c(f)$. Since $e, f \in B_e$ and T is of minimum cost we have $c(e) \leq c(f)$. Hence $c(e) = c(f)$ and $T' := T - f + e$ is another min-cost tree for which $F \subseteq T' \subseteq E - (D + e)$. •

An application for recognizing subtree hypergraphs

Recall the definition of a subtree hypergraph $H = (V, \mathcal{E})$ in Section 1.4. As a pretty application of the greedy algorithm, we describe a simple method [12] for constructing a basic tree. Define a weight function c on the edge-set of the complete graph on V as follows. For every unordered pair $\{u, v\}$ of nodes,

let $c(uv)$ be the number of hyperedges containing both u and v.

Theorem 3.2.5 *A hypergraph H admits a basic tree* (that is, H is a subtree hypergraph) *if and only if a spanning tree of maximum c-weight is a basic tree for H.*

Proof. To see the non-trivial direction, let Z be a hyperedge and let T be an arbitrary spanning tree. Then Z induces at most $|Z| - 1$ edges of T. Therefore

$$\widetilde{c}(T) := \sum_{uv \in E(T)} c(uv) = \sum_{uv \in E(T)} \sum_{Z \in \mathcal{E}, \{u,v\} \subseteq Z} 1 \leq \sum_{Z \in \mathcal{E}} (|Z| - 1) \tag{3.15}$$

and equality holds if and only if every hyperedge Z induces precisely $|Z| - 1$ edges of T, that is, if T is basic for H. •

It follows that Kruskal's algorithm can be used to compute a basic tree for a subtree hypergraph and this algorithm is polynomial in the number of hyperedges.

Further problems

Are there other situations when a Kruskal-type greedy algorithm computes the optimum in question? It can be shown, for example, for a given a matrix with real weights assigned to its columns that the set of linearly independent columns with maximum total weight can be computed with the greedy algorithm (relying on Gauss elimination). A basic feature of both the forests of graphs and the linearly independent columns of a matrix is that they form the set of independent sets of a *matroid*. For an introduction to matroids, see Chapter 5. With the help of matroids, we shall be able to answer the following natural questions concerning trees.

Questions

3.2.1 *When and how is it possible to find k edge-disjoint spanning trees of a graph?*

3.2.2 *Given a cost function on the edges, how can we find a cheapest subset of edges which can be partitioned into k edge-disjoint spanning trees?*

3.2.3 *In a digraph, how can we find a cheapest subgraph that includes k edge-disjoint* (or openly disjoint) *paths from a root-node r_0 to every other node?*

Good luck! The last question for $k = 1$ is so simple that the answer does not require matroids, and this special case is the subject of the next section.

3.2.2 Cheapest r_0-arborescences

Let $D = (V, A)$ be a directed graph in which every node is reachable by a directed path from a designated root-node r_0, that is, D contains a spanning r_0-arborescence. Let $c : A \to \mathbf{R}_+$ be a non-negative cost function. The problem is to construct a cheapest spanning r_0-arborescence, that is, one of minimum total cost. This problem can be considered as a common generalization of the cheapest r_0t-paths (for non-negative costs) in digraphs and the cheapest spanning trees in undirected graphs. Indeed, in the former case add a new edge tv of zero cost to the digraph for every node v. The unique r_0t-path of the cheapest r_0-arborescence in the extended digraph is a cheapest r_0t-path in D.

In order to formulate the cheapest tree problem, replace each undirected edge by two oppositely directed edges. The cheapest r_0-arborescence of the digraph obtained in this way corresponds to a cheapest spanning tree of the initial graph.

A recursive algorithm

The first algorithm for the cheapest arborescence problem is due to Chu and Liu [50]. We can assume that $\varrho(r_0) = 0$, because an edge entering r_0 never belongs to an r_0-arborescence. Observe that by reducing the cost of all edges entering a node v by the same number α, the cost of each r_0-arborescence is decreased by α. Therefore it can be assumed that there is an edge of zero cost entering every node v distinct from r_0. We call such edges **0-edges**. If the subgraph D_0 of 0-edges includes a spanning r_0-arborescence T, then the total cost of

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 109 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

T is zero, that is, T is cheapest since c is non-negative. Assume now that D_0 includes no spanning r_0-arborescence.

Claim 3.2.6 *D_0 includes a directed circuit C.*

Proof. Let S denote the set of nodes of D_0 reachable from r_0. Then no 0-edge leaves S and, by the assumption, $V - S$ is non-empty. Since every node $v \in V - S$ admits an entering 0-edge uv and u is not in S we conclude that $V - S$ includes a di-circuit consisting of 0-edges. •

Claim 3.2.7 *The contraction of a di-circuit C of zero cost into a single node does not change the minimum cost of r_0-arborescences.*

Proof. By contracting any subset of edges, the minimum certainly cannot increase. In order to see that it does not decrease either when C is contracted, let F' be a cheapest r_0-arborescence in the resulting digraph D'. Let $e' = uv_C$ denote its unique edge entering the contracted node v_C and let $e = uv$ denote the original edge of D corresponding to e' where v is a node of C. Let F denote the subset of A corresponding to the edges of F'. The removal of the edge of C that enters v results in a path P of 0-edges with starting node v. Then the union $F \cup P$ forms a spanning r_0-arborescence of D having the same cost as F'. •

This last operation to obtain an r_0-arborescence of D from that of D' is referred to as **blowing up** circuit C. By Claim 3.2.7 it suffices to compute a cheapest r_0-arborescence of the contracted digraph D' and by recursively applying this procedure one obtains a cheapest r_0-arborescence of D.

A direct algorithm

The recursive algorithm above can also be described in a direct way without refering to recursion. The direct approach consists of two phases. The first one reduces the cost of edges entering nodes and contracts 0-circuits while the second phase serves to build up a cheapest r_0-arborescence.

Phase 1 (A) For every node v distinct from r_0, reduce the cost of the edges entering v by the minimum of these costs.
(B) Compute the set S of nodes reachable from r_0 using 0-edges only. If every node is reachable, then turn to Phase 2. If $S \neq V$, then find a directed circuit C of 0-edges in $V - S$, contract it and, with the contracted digraph, return to Step (A).

Phase 2 Compute a spanning r_0-arborescence of the contracted digraph consisting of 0-edges. In a reverse order of the circuit-contractions in Phase 1, construct the cheapest r_0-arborescence of D by blowing up the contracted circuits.

The procedure above can easily be seen to be of polynomial time but it is not particularly economic. For example, in Step (B) of Phase 1, when a di-circuit C is contracted it is possible that there is a 0-edge entering the contracted node in which case nothing happens in Step (A) and we return immediately to (B). These superfluous steps can be eliminated by contracting not just one single circuit of D_0 at a time but an entire strongly connected component of 0-edges which is not entered by any 0-edge. Such a set is called a **source-component** of

D_0. Note that there is a one-to-one correspondence between the source-components of D_0 and the source-nodes of the acyclic digraph that arises from D_0 by contracting each strong component.

Claim 3.2.8 *During Phase 1, the current digraph D_0 of 0-edges includes a source-component not containing r_0.*

Proof. The set S of nodes reachable from r_0 in D_0 is not the entire V. By adding a new vr_0-edge for each $v \in S$, the set S becomes a sink-component containing r_0. But then a source-component of the enlarged digraph is a source-component of D_0 as well and it cannot contain r_0. •

Revised Phase 1 In Step (B), contract a source-component K of D_0 not containing r_0. Notice that K is not a single node since Step (A) ensures that every node admits an entering 0-edge, that is, a single node cannot form a source-component.

The advantage of this aggregation is that the strong components of a digraph (as well as a source-node of an acyclic digraph) can be computed in linear time with the help of a depth-first search [362]. In other words, the source-components of D_0 can be computed in linear time.

A displeasing side effect, however, is that the r_0-arborescence building procedure in Phase 2 using blow-up operations becomes a bit messy from a computational point of view. In order to avoid this difficulty, observe that Claim 3.2.7 remains true when, instead of a di-circuit, a source-component is contracted. It follows that what we need in Phase 2 is a spanning r_0-arborescence F consisting of 0-edges and having the property that

$$F \text{ enters each contracted subset of } V \text{ exactly once.} \tag{3.16}$$

This can be achieved as follows [133].

Revised Phase 2 Starting at r_0, build up an r_0-arborescence F of D by adding 0-edges one by one so as to obey the rule that, at every step, the new edge added is that for which the revised cost became zero earliest during Phase 1.

Lemma 3.2.9 *The final spanning r_0-arborescence constructed in this way meets the requirement of* (3.16)*, and hence it is of minimum cost.*

Proof. Let K be a source-component that has been contracted sometime during Phase 1. Suppose, indirectly, that F enters K more than once. Let $e = uv$ be the second edge of F entering K (second, in the ordering of the arborescence building in Phase 2). Let K' denote the subset of K which belongs to the current arborescence F' at the moment preceding the addition of e.

At the moment of Phase 1 preceding the contraction of K no 0-edge enters K and K is a strong component of the digraph of current 0-edges. It follows that e became a 0-edge strictly later than the 0-edges induced by K. Therefore in the revised Phase 2 the adding of e to F' contradicts the rule of earliest choice since there is a 0-edge from K' to $K - K'$ which can be added to F' and became 0-edge earlier than e. This contradiction proves the lemma. •

Concerning Duffin's theorem (Theorem 3.1.6), the following question naturally arises:

Question 3.2.1 *Does there exist a min-max theorem on the minimum cost of spanning r_0-arborescences similar to the one of Duffin on cheapest st-paths?*

Theorem 11.4.1, due to Fulkerson, offers a positive answer in Section 11.4.

3.3 Weighted matchings of bipartite graphs

In Section 2.4, we saw Kőnig's min-max theorem on bipartite matchings as well as the alternating paths algorithm for computing a largest matching. Let c be a cost (or weight) function on the edge-set of a bipartite graph $G = (S, T; E)$ and let $V := S \cup T$. There are various versions of the weighted matching problem. We may be interested in a maximum-weight matching, in a maximum- (or minimum-) weight perfect matching or in a maximum-weight matching of k elements.

3.3.1 The Hungarian method

Let us start with the problem of maximum weight perfect matchings which is sometimes called **the assignment problem**. The min-max theorem was developed by Egerváry [88]. The technique of his proof was transformed to a strongly polynomial algorithm by Kuhn [252] who called the algorithm the Hungarian method.

Suppose that $G = (S, T; E)$ has a perfect matching. We are interested in finding a maximum-weight perfect matching of G. As a generalization of a node-set covering all edges, call a function $\pi : S \cup T \to \mathbf{R}$ on the node-set $V = S \cup T$ a **weighted covering** of c if $\pi(u) + \pi(v) \geq c(uv)$ for every edge $uv \in E$. An edge uv is called **tight** with respect to π if $\pi(u) + \pi(v) = c(uv)$. The **total value** of π is $\widetilde{\pi}(V) = \sum[\pi(v) : v \in V]$.

Theorem 3.3.1 (Egerváry) *In a perfectly matchable bipartite graph $G = (S, T; E)$, the maximum weight ν_c of a perfect matching with respect to a weight function c is equal to the minimum total value τ_c of a weighted covering π of c. When c is integer-valued, the optimal π can also be selected to be integer-valued. If G is a complete bipartite graph and c is non-negative, then the optimal weighted covering can be selected to be non-negative.*

Proof. We start by proving the last part of the theorem and describe a simple tool to show that in the given case an arbitrary weighted covering π can be converted into a non-negative one with the same total value. Let $-K$ denote the smallest value of π where $K > 0$ and assume that there is a node s, say in S, with $\pi(s) = -K$. Since G is a complete bipartite graph, c is non-negative, and π is a weighted covering we have $\pi(v) \geq K$ for every node $v \in T$. Modify π so as to increase the π-values uniformly by K on the elements of S and to decrease them by K on the elements of T. The revised function π' is also a non-negative weighted covering for which $\widetilde{\pi}(V) = \widetilde{\pi}'(V)$ since $|S| = |T|$.

To prove the min-max part, we first show that max $\leq$ min. Let $M := \{u_1v_1, u_2v_2, \dots, u_nv_n\}$ be an arbitrary perfect matching in G and π a weighted covering of c. Then $\widetilde{c}(M) := \sum c(u_iv_i) \leq \sum[\pi(u_i) + \pi(v_i) : i = 1, \dots, n] = \widetilde{\pi}(V)$ from which $\nu_c \leq \pi_c$ follows.

Here equality holds if and only if every edge of M is tight. Hence, in order to prove the non-trivial direction max $\geq$ min, we need to find a suitable weighted covering π along with a perfect matching consisting of tight edges. Equivalently, such a π is sought for which the subgraph of tight edges on V has a perfect matching.

To this end, we start with an arbitrary weighted covering π that is integer-valued if c is. At a general step, consider the subgraph $G_\pi = (S, T; E_\pi)$ of G consisting of tight edges. Suppose that a matching M of G_π is already available (which can be empty at the beginning).

Orient the edges in M towards S and all other tight edges towards T. Let R_S and R_T denote the subset of nodes in S and in T, respectively, exposed by M. Let Z be the set of nodes reachable from R_S in the digraph obtained in this way. (Z can be computed by a breadth-first search, for example.) If the intersection of R_T and Z is non-empty, then we have obtained a path P from R_S to R_T which is alternating in M. The symmetric difference of M and P is a matching M' having one more edge than M. Iterate the procedure with M'. Note that the modified orientation of G defined by M' is easily obtained by reorienting the path P.

Suppose now that R_T and Z are disjoint. By the definition of Z, no directed edge leaves Z. No directed edge $f = uv$ of M can enter Z either because f is the only edge with head v, and therefore v is reachable only through u. Since G has a perfect matching by the hypothesis, it satisfies the Hall condition. This criterion and $|S \cap Z| = |T \cap Z| + |R_S| > |T \cap Z|$ imply that there is an edge e of G connecting $S \cap Z$ and $T - Z$. Such an edge cannot be tight since it would define a directed edge leaving Z. Hence the value

$$\delta := \min\{\pi(u) + \pi(v) - c(uv) : uv \in E, u \in Z \cap S, v \in T - Z\} \tag{3.17}$$

is strictly positive. Revise π as follows.

$$\pi'(v) = \begin{cases} \pi(v) - \delta, & \text{if } v \in Z \cap S, \\ \pi(v) + \delta, & \text{if } v \in Z \cap T, \\ \pi(v) & \text{otherwise.} \end{cases} \tag{3.18}$$

The choice of δ ensures that π' is also a weighted covering which is integer-valued if c and π are. The subgraph $G_{\pi'}$ of tight edges with respect to π' induces the same set of edges in Z as G_π, moreover, $G_{\pi'}$ has at least one edge (where the minimum in the definition of δ is attained) connecting $Z \cap S$ and $T - Z$. Therefore the set of nodes reachable from R_S in the orientation of G is strictly larger than Z. (See Figure 3.1.)

It follows that between two consecutive matching augmentations the second case may occur at most $|S|$ times and this completes the proof of the min-max relation. The second statement of the theorem follows from the observation that if c is integer-valued, then the current weighted covering π is throughout integer-valued. •

Since a BFS can be carried out in linear time, the complexity of the algorithm is $O(|E||S|^2)$.

Problem 3.3.1 *Let π be any weighted covering of minimum total value and let $G_\pi = (S, T; E_\pi)$ be the subgraph of tight edges. Prove that a perfect matching M of G is of maximum weight if and only if $M \subseteq E_\pi$.*

Problem 3.3.2 *Prove that a perfect matching M is of maximum weight if every element of M belongs to a perfect matching of maximum weight.*

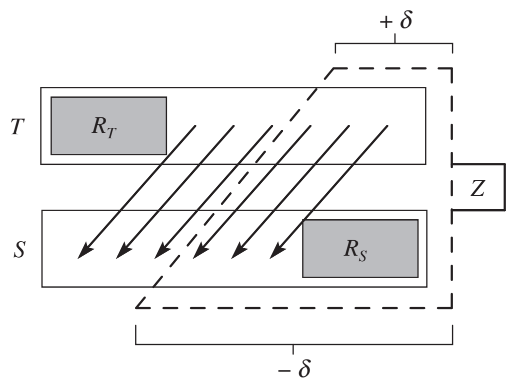


Figure 3.1 Changing π

Remark 3.3.1 Egerváry originally considered the case when $G = (S, T; E)$ is a complete bipartite graph and the weight function c is integer-valued and non-negative. For the proof, he started with an optimal dual solution π which is integral and non-negative. When G_π has a perfect matching M, then it is of maximum weight. If G_π has no perfect matching, then there is a deficient subset $X \subseteq S$ in Γ_{G_π} by Hall's theorem, that is, $|\Gamma_{G_\pi}(X)| < |X|$. Egerváry used this X to modify π in the way described in (3.18) so as to get a better dual solution, contradicting the optimal choice of π. For non-integral c he used continuity arguments. The proof technique gives rise to an algorithm that uses an arbitrary deficient set X for improving π. It can be shown, however, that this generic algorithm is not polynomial even if X is chosen to be a most deficient set (that is one for which $|X| - |\Gamma_{G_\pi}(X)|$ is maximum). Note that Kuhn's Hungarian method tacitly uses the unique smallest most deficient set X for improving π.

Remark 3.3.2 The proof above of the correctness of Kuhn's algorithm showed immediately that the algorithm is strongly polynomial. That was first stated explicitly by Munkres [293] and therefore the Hungarian method is sometimes called the Kuhn–Munkres algorithm. Perhaps both names will be considered inappropriate in the future since a recent discovery revealed that a posthumous paper of Jacobi [226] from the nineteenth century includes an algorithm which is essentially equivalent to the Kuhn–Munkres algorithm. Since that was just an auxiliary tool for Jacobi, no wonder the min-max theorem of Egerváry was not formulated in his paper. It is not yet clear how this discovery will affect the name of the 'Hungarian' method in the future. The situation has two irreconcilable aspects: the chronological priority of Jacobi and the widespread impact of the Hungarian method on the development of combinatorial optimization.

Alternative proof via negative circuits

We exhibit another technique for proving the non-trivial max $\geq$ min direction of Egerváry's theorem.

Let M be a perfect matching of maximum weight. Our goal is to find a function π for which

$$\pi(u) + \pi(v) = c(uv) \text{ for every edge } uv \in M \tag{3.19}$$

and

$$\pi(u) + \pi(v) \geq c(uv) \text{ for every edge } uv \in E - M. \tag{3.20}$$

In fact, we will find a slightly weaker π for which

$$\pi(u) + \pi(v) \leq c(uv) \text{ for every edge } uv \in M \tag{3.21}$$

and

$$\pi(u) + \pi(v) \geq c(uv) \text{ for every edge } uv \in E - M. \tag{3.22}$$

By increasing such a π appropriately, we can easily obtain one satisfying (3.19) and (3.20).

Orient the edges in M towards S, all other edges towards T and negate the weight of these latter edges. Let D' denote the resulting digraph and c' the revised weight function.

We claim that c' is conservative. For if K' is a negative circuit of D', then it corresponds to an M-alternating circuit K of G for which $\widetilde{c}(K - M) > \widetilde{c}(K \cap M)$ follows from $\widetilde{c'}(K') < 0$. Hence the symmetric difference $M' := M \ominus K$ is a perfect matching for which its weight is larger than the maximum $\widetilde{c}(M)$, a contradiction.

Since c' is conservative, there is feasible potential π' by Gallai's theorem (Theorem 3.1.1). This means that $\pi'(y) - \pi'(x) \leq c(xy)$ for each edge xy of M ($x \in T$, $y \in S$) and $\pi'(v) - \pi'(u) \leq c'(uv) = -c(uv)$ for each edge uv of $E - M$ ($u \in S, v \in T$).

Let π denote the function arising from π' by negating its values on the elements of T. Then the inequality $\pi'(y) - \pi'(x) \leq c(xy)$ transforms to $\pi(y) + \pi(x) \leq c(xy)$ while $\pi'(v) - \pi'(u) \leq -c(uv)$ transforms to $-\pi(v) - \pi(u) \leq -c(uv)$, that is $\pi(v) + \pi(u) \geq c(uv)$ and we conclude that both (3.21) and (3.22) are met. •

3.3.2 Maximum-weight matchings

Let us consider the problem of finding and characterizing matchings of maximum weight that are necessarily perfect. With the help of a simple trick, this problem can be reduced to that of perfect matchings. Since an edge of negative weight is never contained in a matching of maximum weight, we can assume that c is non-negative.

Theorem 3.3.2 *Let c be a non-negative weight function on the edge-set of a bipartite graph $G' = (S', T'; E')$. The maximum weight ν'_c of a matching is equal to the minimum total value τ'_c of a non-negative* (!) *weighted covering π of c. If c is integer-valued, the optimal π can also be selected to be integer-valued.*

Proof. The inequality $\nu'_c \leq \tau'_c$ is obvious so we deal only with the converse direction. By adding new nodes, if necessary, we may assume that the two colour-classes of G have the same size. Moreover, add new edges of zero weight to obtain a complete bipartite graph G. The weight function on G will be denoted by the same term c.

By (the third part of) Theorem 3.3.1 there is a perfect matching M in G and a non-negative weighted covering π of c for which $\widetilde{c}(M) = \widetilde{\pi}(V)$. Since the weight of the new edges is zero, the matching M' of G' obtained from M by removing the new edges has the same weight as M, that is, $\widetilde{c}(M') = \widetilde{c}(M)$.

Furthermore, the edges in M are tight and π is non-negative implying that $\pi(u) = \pi(v) = 0$ for every new edge $uv \in M$. Therefore, if we restrict π to the original nodes, the resulting function π' satisfies $\sum[\pi'(v) : v \in V] = \sum[\pi(v) : v \in V]$ and $\sum[\pi'(v) : v \in V] = \widetilde{c}(M')$. •

There are other versions of the weighted matching problem. For example, one may be interested in a maximum-weight matching of k edges, where k is a prescribed integer. It turns out that this problem is easier to discuss in the broader framework of network flows; this is the topic of the next section.

3.4 Flows and circulations

One motivation for investigating flows in networks comes from Menger's theorem. Its directed edge-version characterizes digraphs $D = (V, A)$ in which there are k edge-disjoint st-paths. It is a task of central importance to develop an algorithm for finding these paths if they exist or for finding an $s\bar{t}$-subset $S \subset V$ of out-degree less than k, verifying that the k st-paths do not exist. (Actually, we have already exhibited such an algorithm: one of the proofs of Menger's theorem was based on the Orientation lemma, and the proof of this latter result was algorithmic.) Recall the definition of a k-braid and Theorem 1.3.5. This theorem shows that in order to find k-edge-disjoint st-paths it suffices to find a subgraph D' of D in which

$$\varrho'(s) = \delta'(t) = 0, \tag{3.23}$$

$$\delta'(s) = k, \tag{3.24}$$

$$\varrho'(v) = \delta'(v) \text{ for every node } v \in V - \{s, t\} \tag{3.25}$$

where ϱ' and δ' denote the in-degree and the out-degree functions of D'. Therefore the problem reduces to finding a subgraph satisfying these requirements. Instead of a subgraph, it is simpler to work with its incidence vector and this gives rise to the concept of flows. It is worth introducing immediately a more general concept.

Let $D = (V, A)$ be a digraph endowed with two functions $f : A \to \mathbf{R}_+ \cup \{-\infty\}$ and $g : A \to \mathbf{R}_+ \cup \{\infty\}$ on its edge-set for which $f \leq g$. The functions are sometimes called lower and upper **bounds** or **capacities**. Often $f \equiv 0$, when g is referred to as a **capacity function**. We say that a function (or vector) $x : A \to \mathbf{R}$ is **feasible** if $f \leq x \leq g$. Let $\varrho_x(Z) := \sum[x(e) : e \in A \text{ enters } Z]$, $\delta_x(Z) := \varrho_x(V - Z)$, and $\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z)$ $(Z \subseteq V)$. The restriction of Ψ_x to singletons, in notation $\dot{\Psi}_x$, is called the **net in-flow vector** of x.

Proposition 3.4.1 *Ψ_x is modular in the sense that*

$$\Psi_x(Z) = \sum[\dot{\Psi}_x(v) : v \in Z]. \tag{3.26}$$

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 116 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proof.

$$\sum[\dot{\Psi}_x(v) : v \in Z] = \sum[\varrho_x(v) : v \in Z] - \sum[\delta_x(v) : v \in Z] =$$

$$\left\{\sum[\varrho_x(v) : v \in Z] - i_x(Z)\right\} - \left\{\sum[\delta_x(v) : v \in Z] - i_x(Z)\right\} = \varrho_x(Z) - \delta_x(Z) = \Psi_x(Z). \bullet$$

A subset Z of nodes and the cut determined by Z is said to be x-**balanced**, or simply **balanced**, if $\Psi_x(Z) = 0$. We say that x meets the **conservation rule** at a node v if v is balanced, that is, if $\dot{\Psi}_x(v) = 0$. A function $x : A \to \mathbf{R}$ is called a **circulation** if the conservation rule is satisfied at every node.

Exercises

3.4.1 *Show that if x is balanced at $|V| - 1$ nodes, then x is a circulation.*

3.4.2 *Show that if $\dot{\Psi}_x \leq 0$, then x is a circulation.*

3.4.3 *Show that circulations and tensions form complementary orthogonal subspaces of $\mathbf{R}^A$.*

Proposition 3.4.2 *Let T be a spanning tree of a digraph D. For a function $x : A \to \mathbf{R}$, the following are equivalent.*

(A) *x is a circulation.*
(B) *Every cut is balanced.*
(C) *Every fundamental cut belonging to T is balanced.*

Proof. (A)→(B) Let x be a circulation. For an arbitrary set $Z \subseteq V$, (3.26) implies $\Psi_x(Z) = \sum[\Psi_x(v) : v \in Z] = 0$. (B)→(C) is evident. To see (C)→(A), let v be an arbitrary node. Let $e_1, \dots, e_q$ denote the edges of T incident to v and let Z_i denote the set of nodes of the component of $T - e_i$ not containing v $(i = 1, \dots, q)$. Since $\{\{v\}, Z_1, \dots, Z_q\}$ is a partition of V and $\Psi_x(Z_i) = 0$, we obtain $0 = \Psi_x(V) = \Psi_x(v) + \sum[\Psi_x(Z_i) : i = 1, \dots, q] = \Psi_x(v)$, that is, v is balanced. $\bullet$

We speak of a **flow** from s to t (or an st-flow) if x is non-negative and satisfies the conservation rule at every node apart, possibly, from two designated nodes s and t. The value $\dot{\Psi}_x(t)$ is required to be non-negative and is called the **flow-amount** of x. Note that the modularity of Ψ implies that $\dot{\Psi}_x(t) = -\dot{\Psi}_x(s)$. The node s is called the **source** of the flow while t is its **target** or **sink**. We shall always assume that

$$\varrho_D(s) = \delta_D(t) = 0$$

in which case the flow-amount of x is $\delta_x(s) = \varrho_x(t)$.

There is a natural common generalization of flows and circulations. Suppose that we are given a function $m : V \to \mathbf{R}$ for which $\widetilde{m}(V) = 0$. We say that $x : A \to \mathbf{R}$ is a **modular flow**, an m-**flow** for short, if

$$\dot{\Psi}_x(v) = m(v) \text{ for every node } v \in V, \tag{3.27}$$

that is, if its net in-flow vector is the prescribed m. If $m \equiv 0$, we are back at circulations. If m is defined for a given $k \geq 0$ by

$$m(v) := \begin{cases} k & \text{if } v = t \\ -k & \text{if } v = s \\ 0 & \text{if } v \in V - \{s, t\} \end{cases} \tag{3.28}$$

where s and t are two specified nodes, the m-flow is an st-flow of amount k.

There are two fundamental problems concerning modular flows. The first one is the **modular flow feasibility** problem which is about finding characterizations for the existence of a feasible m-flow (with respect to capacity functions f and g) as well as algorithms for its computation. The second one is the **minimum cost modular flow** problem when a cheapest feasible m-flow is to be found with respect to a (linear) cost function $c : A \to \mathbf{R}$, that is, we want to minimize cx over all feasible m-flows x. We shall be particularly interested in integer-valued m-flows. In the literature the min-cost modular flow problem is referred to as a trans-shipment problem. We prefer the term modular flows since it serves as a starting point to submodular flows, a much more general combinatorial framework to be discussed in Part III. The general form of modular flows has three important special cases.

Non-negative m-flows: find a non-negative m-flow. (Here m is arbitrary, $f \equiv 0$, and $g \equiv \infty$.)

Feasible circulations: find a circulation x for which $f \leq x \leq g$. (That is, $m \equiv 0$.)

Feasible st-flows of given amount: find an st-flow x of a given flow-amount for which $x \leq g$. (That is, $f \equiv 0$, and m is defined by (3.28)). A close variation seeks a feasible st-flow of maximum flow-amount which will be called a **maximum flow**.

It will turn out that each of these special cases are equivalent to the general case in the sense that feasibility and min-cost problems for modular flows can be transformed into those concerning any of the three special cases.

Let P be a directed st-path of D and let C be a directed circuit of D. For some non-negative α, define $x_1 : A \to \mathbf{R}_+$ and $x_2 : A \to \mathbf{R}_+$ by

$$x_1(e) := \begin{cases} \alpha & \text{if } e \in P \\ 0 & \text{if } e \in A - P, \end{cases} \tag{3.29}$$

$$x_2(e) := \begin{cases} \alpha & \text{if } e \in C \\ 0 & \text{if } e \in A - C. \end{cases} \tag{3.30}$$

Then x_1 is a special st-flow called a **path-flow**, while x_2 is a special circulation, called a **cycle-flow**.

Lemma 3.4.3 (flow decomposition) *Every non-negative circulation x is the sum of at most $|A|$ non-negative cycle-flows. If x is integral, the cycle-flows can also be selected to be integral. Every non-negative st-flow x is the sum of at most $|A| + 1$ path-flows and cycle-flows. If x is integral, the path- and cycle-flows can also be selected to be integral.*

Proof. Let x be a non-negative circulation. We use induction on the number of edges on which x is positive. There is nothing to prove if this number is 0, that is, if $x \equiv 0$. Suppose that x has a positive component. Then the conservation rule and the non-negativity of x

imply that there exists a directed circuit C of D for which $\alpha := \min\{x(e) : e \in C\}$ is positive. Let x_1 be the cycle-flow defined by (3.30). Then $x' := x - x_1$ is a non-negative circulation having fewer positive edges than x has, from which by induction we are done.

Suppose now that x is an st-flow. Let k denote the flow-amount of x. Add an additional ts-edge e_0 and define x' on the enlarged digraph $D' = (V, A')$ as follows.

$$x'(e) := \begin{cases} x(e) & \text{if } e \in A \\ \varrho_x(t) & \text{if } e = e_0. \end{cases} \tag{3.31}$$

Then x' is a circulation and the second part follows from the first part. •

3.4.1 Feasibility and Max-flow Min-cut

We shall prove the following simple observation.

Proposition 3.4.4 *Suppose that $\widetilde{m}(V) = 0$. Then x is an m-flow if and only if $\varrho_x(v) - \delta_x(v) \le m(v)$ holds for every node v. If x is an m-flow, then $\varrho_x(Z) - \delta_x(Z) = \widetilde{m}(Z)$ for every subset $Z \subseteq V$.*

Proof. By definition, $\varrho_x(v) - \delta_x(v) = m(v)$ holds for an m-flow x. Suppose now that $\varrho_x(v) - \delta_x(v) \le m(v)$ holds for every node $v \in V$. Then $0 = \varrho_x(V) - \delta_x(V) = \sum[\varrho_x(v) - \delta_x(v) : v \in V] \le \sum[m(v) : v \in V] = \widetilde{m}(V) = 0$ from which we must have $\varrho_x(v) - \delta_x(v) = m(v)$ for every $v \in V$. The second statement follows from: $\varrho_x(Z) - \delta_x(Z) = \sum[\varrho_x(v) - \delta_x(v) : v \in Z] = \sum[m(v) : v \in Z] = \widetilde{m}(Z)$. •

Theorem 3.4.5 (Hoffman [207, 289]) *There is a feasible m-flow if and only if $\widetilde{m}(V) = 0$ and*

$$\varrho_f(X) - \delta_g(X) \le \widetilde{m}(X) \textit{ for every subset } X \subseteq V. \tag{3.32}$$

Moreover, if f, g, and m are integer-valued and (3.32) *holds, then there is an integer-valued feasible m-flow as well.*

Proof. If there is a feasible m-flow x, then $\varrho_f(X) - \delta_g(X) \le \varrho_x(X) - \delta_x(X) = \widetilde{m}(X)$, from which (3.32) follows for every $X \subseteq V$, while for V we have $0 = \varrho_x(V) - \delta_x(V) = \widetilde{m}(V)$. To see sufficiency, define p_{fg} by

$$p_{fg}(X) := \varrho_f(X) - \delta_g(X) \text{ for } X \subseteq V.$$

Recall Proposition 1.2.3 which stated

$$p_{fg}(X) + p_{fg}(Y) = p_{fg}(X \cap Y) + p_{fg}(X \cup Y) - d_{g-f}(X, Y) \text{ for every } X,\ Y \subseteq V. \tag{3.33}$$

Note that condition (3.32) can be concisely written in the form $p_{fg} \le \widetilde{m}$. We say that a subset Z of nodes is **tight** if $p_{fg}(Z) = \widetilde{m}(Z)$, and defines an edge e of D as **tight** if $f(e) = g(e)$.

Suppose indirectly that the theorem fails for D, and choose a counterexample (D is given) in which the total number of tight edges and tight sets is maximum. It cannot be the case that every edge is tight since then (3.32) would imply for $x := f(= g)$ that $\varrho_x(v) - \delta_x(v) = \varrho_f(v) - \delta_g(v) \le m(v)$ holds for every $v \in V$, and hence x would be a feasible m-flow by Proposition 3.4.4.

Let $h = st$ be an edge of D for which $f(h) < g(h)$.

Claim 3.4.6 *The edge h enters a tight set T and leaves a tight set S.*

Proof. We shall prove only the first statement as the second one is analogous. If h did not enter any tight set, then $f(h)$ could be increased so that the revised f' also satisfies $f' \leq g$ and $\varrho_{f'}(Z) - \delta_g(Z) \leq \widetilde{m}(Z)$ holds for every $Z \subseteq V$. Furthermore, either the edge h becomes tight or a subset entered by h becomes tight. Because this modification keeps every tight set tight, the maximality assumption made about the total number of tight edges and sets implies that there is a feasible m-flow x' with respect to f' and g. But then x' is feasible with respect to f and g, a contradiction. •

Since $f(h) < g(h)$ and h connects $S - T$ and $T - S$, it follows from (3.33) and from (3.32) that $\widetilde{m}(S) + \widetilde{m}(T) = p_{fg}(S) + p_{fg}(T) < p_{fg}(S \cap T) + p_{fg}(S \cup T) \leq \widetilde{m}(S \cap T) + \widetilde{m}(S \cap T) = \widetilde{m}(S) + \widetilde{m}(T)$, and this contradiction shows that no counterexample can exist.

The same argument also shows that there is an integer-valued feasible m-flow provided that f, g, and m are all integer-valued. • •

Remark 3.4.1 A disadvantage of this proof is that it does not yield a polynomial algorithm. The next section will outline the classic algorithmic approach of augmenting paths. In Part II of the book, we present the push–relabel technique of Goldberg and Tarjan which will provide an alternative proof for Theorem 3.4.5. But if we are interested only in a short proof, then the one above is certainly among the most compact proofs. In addition, it has another great advantage. By following closely the steps of the proof, one can realize that we have not fully exploited the fact that $\widetilde{m}$ is modular and that if $\widetilde{m}$ is replaced by a submodular function b, then the same proof technique still works.

Problem 3.4.4 *Using the approach suggested in Remark* 3.4.1*, derive the following result.*

Theorem 3.4.7 *Let $f \leq g$ be functions on the edge-set of a digraph $D = (V, A)$ and let b be a submodular function for which $b(V) = 0$. There is a vector $x : A \to \mathbf{R}$ for which $f \leq x \leq g$ and $\varrho_x(Z) - \delta_x(Z) \leq b(Z)$ for every $Z \subseteq V$* (called a feasible submodular flow) *if and only if*

$$\varrho_f(Z) - \delta_g(Z) \leq b(Z) \text{ for every } Z \subseteq V. \tag{3.34}$$

If f, g and b are all integral, then (3.34) *implies the existence of an integral x.*

This result will be repeated in Part III as Theorem 16.1.5 where its proof is presented along with far-reaching consequences and applications such as the matroid or polymatroid intersection theorem. Yet, it is quite possible that the reader who understood well the proof of Theorem 3.4.5 will not find it difficult to solve Problem 3.4.4.

In the special case of $m \equiv 0$, Hoffman's theorem is as follows.

Theorem 3.4.8 (Hoffman) *There is a feasible circulation if and only if and $\varrho_f \leq \delta_g$. Moreover, if f and g are integer-valued and $\varrho_f \leq \delta_g$, then there is an integer-valued feasible circulation, too.* •

By specializing Theorem 3.4.5 to st-flows, one obtains a version of the Max-flow Min-cut (MFMC) theorem of Ford and Fulkerson [107].

Theorem 3.4.9 *There is feasible st-flow of flow-amount k if and only if $\delta_g(S) \geq k$ holds for all $s\bar{t}$-sets S. When g is integer-valued, the flow can also be selected to be integer-valued.*

Note that for an integer-valued function g this result is a simple consequence of the directed edge-Menger theorem since we can replace each edge a by $g(a)$ parallel edges. The theorem is sometimes formulated in the following equivalent form.

Theorem 3.4.10 (Max-flow Min-cut) *Given a capacity function g on the edge-set of a digraph $D = (V, A)$, the maximum amount of a feasible flow from s to t is equal to the minimum of $\delta_g(S)$ over all $s\bar{t}$-sets S. When g is integer-valued, the maximum flow may also be chosen integer-valued.*

Proof. Let k denote the minimum in question and apply Theorem 3.4.9. •

Problem 3.4.5 $(*)$ *Hoffman's theorem has a self-refining nature: derive the following extension.*

Theorem 3.4.11 (Hoffman [207, 289]) *Suppose that in addition to the bounds f and g on the edge-set of D, we are given bounds $f_V : V \to \mathbf{R} \cup \{-\infty\}$ and $g_V : V \to \mathbf{R} \cup \{\infty\}$ on the node-set, as well, for which $f_V \leq g_V$. There exists an $m : V \to \mathbf{R}$ for which $f_V \leq m \leq g_V$ so that there exists a feasible m-flow if and only if*

$$\varrho_f(X) - \delta_g(X) \leq \min\{-\widetilde{f}_V(V - X), \widetilde{g}_V(X)\} \text{ for every } X \subseteq V. \tag{3.35}$$

3.4.2 Equivalence of modular flows, circulations, and st-flows

Fulkerson [168] recognized that various network models are equivalent in the sense that an algorithm for one framework can be transformed to an algorithm for another one. Here we outline these reductions for modular flows, circulations, and st-flows.

Modular flows from circulations

We show how the feasibility and min-cost problems of m-flows reduce to those of circulations. Extend D by adding a new node t and a ut-edge for each $u \in V$. Let $D' = (V', A')$ denote the resulting digraph. Extend f and g to the new edges by $f(ut) := g(ut) := m(u)$ for $u \in V$. For the min-cost version, extend c to the new edges by $c(ut) := 0$. A simple observation shows that the restriction of a circulation in D' to A is an m-flow and conversely, that an m-flow in D extends to a circulation of D' by defining its value on a new edge ut to be $m(u)$. Therefore, there is a feasible m-flow in D if and only if there is a feasible circulation in D'. Moreover the costs of the corresponding m-flow and circulation coincide.

Circulations from st*-flows*

Our next goal is to discuss how the feasibility and min-cost problems for circulations can be reduced to those for st-flows. To this end, let $\Psi_f(v) := \varrho_f(v) - \delta_f(v)$ for every node $v \in V$. (For simplicity, we assume now that both f and g are finite-valued. It is a little exercise to derive the general case from this.) If Ψ_f is identically zero, then f is a feasible circulation and we are done.

If Ψ_f is not identically zero, than the sets $S = \{v : \Psi_f(v) > 0\}$ and $T = \{v : \Psi_f(v) < 0\}$ are non-empty. Construct a digraph $D' = (V', A')$ such that $V' = V \cup \{s, t\}$ and $A' = A \cup \{sv : v \in S\} \cup \{vt : v \in T\}$ and define a capacity function $g' : A' \to \mathbf{R}_+$ as follows.

$$g'(a) := \begin{cases} \Psi_f(v) & \text{if } a = sv, v \in S \\ -\Psi_f(v) & \text{if } a = vt \\ g(a) - f(a) & \text{if } a \in A, v \in T \end{cases} \tag{3.36}$$

In the min-cost circulation problem, extend c to the new edges a' by $c(a') := 0$. Finally, let $M = \sum[\Psi_f(v) : v \in S]$. The next lemma describes the link between the feasible circulations of D and the feasible flows of D' from s to t on the one hand and between the cuts of D' with value less than M and the subsets of D violating (3.32) for $m \equiv 0$, on the other. Let ϱ' and δ' denote the in-degree and out-degree functions of D'.

Lemma 3.4.12 (A) *Let x' be a g'-feasible st-flow in D' with flow-amount M and let $x : A \to \mathbf{R}$ denote the restriction of $f + x'$ to the edge-set A of D. Then x is a feasible circulation.* (B) *If $\delta_{g'}(X + s) < M$ for some subset $X \subseteq V$, then X violates* (3.32) *for $m \equiv 0$.*

Proof. (A) The feasibility of x, that is, $f(a) \leq f(a) + x'(a) \leq g(a)$ for every $a \in A$, follows from the construction of D'.

The conservation rule holds for every node v in $V - (S \cup T)$ since $\varrho_x(v) = \varrho'_{x'}(v) = \delta'_{x'}(v) = \delta_x(v)$. Since the flow-amount of x' is M, every edge leaving s is saturated. Therefore for each node $v \in S$ we have

$$\varrho'_{x'}(v) = x'(sv) + \varrho_{x'}(v) = \Psi_f(v) + \varrho_{x-f}(v) = \varrho_x(v) - \delta_f(v)$$

and

$$\delta'_{x'}(v) = \delta_{x'}(v) = \delta_{x-f}(v) = \delta_x(v) - \delta_f(v).$$

Since $\varrho'_{x'}(v) = \delta'_{x'}(v)$ we infer $\varrho_x(v) = \delta_x(v)$, that is, the conservation rule holds for the nodes of S. The proof for the nodes in T is analogous.

(B) Combining $\Psi_f(S) = M > \delta'_{g'}(X + s) = \delta_{g-f}(X) + \Psi_f(S - X) - \Psi_f(T \cap X)$ and $\Psi_f(S - X) = \Psi_f(S) - \Psi_f(S \cap X)$ we get

$$0 > \delta_{g-f}(X) - \Psi_f(S \cap X) - \Psi_f(T \cap X) = \delta_{g-f}(X) - \Psi_f(X) =$$
$$\delta_{g-f}(X) + \delta_f(X) - \varrho_f(X),$$

that is, $\delta_g(X) < \varrho_f(X)$. •

This reduction shows that an algorithm for finding a feasible st-flow of maximum flow-amount can be used to decide if (3.32) holds, and if it does, a feasible circulation can be computed. Similarly, with the help of an algorithm for finding a cheapest st-flow of maximum amount, it is possible to solve algorithmically the cheapest feasible circulation problem as well.

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 122 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Feasible* m*-flows from non-negative* m*-flows

Finally we exhibit how the general feasibility and min-cost problem of modular flows can be reduced to the special case when $f \equiv 0$ and $g \equiv \infty$.

First, we can get rid of the finite values of g as follows. Replace each arc uv having $g(uv) < \infty$ with two new arcs uw and vw where w is a new node, and let D' denote the resulting digraph. Define $f'(uw) := f(uv)$, $f'(vw) := -g(uv)$, $g'(uw) := \infty$, $g'(vw) := \infty$, $c'(uw) := c(uv)$, $c'(vw) := 0$, and $m'(w) := 0$. If x is a feasible m-flow in D, then x' is a feasible m'-flow in D' where x' is defined by $x'(uv) = x(uv)$ when $g(uv) = \infty$ while $x'(uw) := x(uv)$ and $x'(vw) = -x(uv)$ when $g(uv) < \infty$. Conversely, if x' is a feasible m'-flow in D', then x is a feasible m-flow in D where x is defined by $x(uv) := x'(uv)$ when $g(uv) = \infty$ and $x(uv) := x'(uw)$ when $g(uv) < \infty$. The costs of the corresponding m-flows and m'-flows are the same.

Therefore we assume that $g \equiv \infty$. Second, we get rid of the $-\infty$ values of f as follows. If $f(st) = -\infty$ for an edge st, then add the reverse edge ts to D and define $f'(st) := f(ts) := 0$ and $c'(ts) := -c(st)$. Finally, we may assume that $f \equiv 0$. For if $f(st) \neq 0$ for an edge st, then let

$$m'(v) := \begin{cases} m(t) - f(st) & \text{if } v = t \\ m(s) + f(st) & \text{if } v = s \\ m(v) & \text{otherwise} \end{cases} \tag{3.37}$$

and define $f'(st) := 0$. An easy exercise shows that each transformation gives rise to an equivalent problem.

3.5 Computing maximum flows

We describe the augmenting path algorithm of Ford and Fulkerson to compute a flow from s to t of maximum flow-amount. This algorithm is a constructive proof of the MFMC theorem for integer-valued capacities.

We assume throughout that there is no st-path consisting of edges with infinite capacity (which is equivalent to the existence of an $s\bar{t}$-set S with $\delta_g(S) < \infty$) since such a path would define a flow of arbitrarily large amount.

3.5.1 *Algorithmic proof of the MFMC theorem*

Without loss of generality, we can assume that no edge of D enters s and no edge leaves t.

In order to establish the optimality conditions, recall the proof of the max $\leq$ min inequality. For a feasible flow x and for an $s\bar{t}$-set S, we have the following upper bound for the flow-amount.

$$\delta_x(s) = \delta_x(s) - \varrho_x(s) = \sum[\delta_x(v) - \varrho_x(v) : v \in S] = \delta_x(S) - \varrho_x(S) \leq \delta_g(S)$$

from which the requested max $\leq$ min follows.

The estimate also shows that a flow x is certainly of maximum flow-amount if there is an $s\bar{t}$-set S for which the following **optimality criteria** hold.

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 123 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

(A) $x(a) = g(a)$ for every edge a leaving S,
(B) $x(a) = 0$ for every edge a entering S.

Our goal is to describe an algorithm to compute such a flow x and set S. First, we consider only the special case when g is integer-valued. The algorithm of Ford and Fulkerson starts with an arbitrary feasible flow x (which can be, for example, $x \equiv 0$) and improves it iteratively. Construct an auxiliary digraph $D_x = (V, A_x)$ as follows. An edge uv belongs to A_x if

$uv \in A$ and $x(uv) < g(uv)$, in which case uv is called a **forward** edge of D_x, or
$vu \in A$ and $x(vu) > 0$, in which case uv is called a **backward** edge of D_x.

Let S denote the set of nodes reachable from s in D_x.

Case 1: Termination. *$t \notin S$, that is, t is not reachable from s.* Since no edge of D_x leaves S, each edge of D leaving S must be saturated in the sense that $x(uv) = g(uv)$, and each edge of D entering S must have $x(uv) = 0$. This means that the optimality criteria (A) and (B) are met and the algorithm terminates by returning x as a maximum flow and S defining a minimum cut.

Case 2: Augmentation. *$t \in S$, that is, t is reachable from s.* Let P be an arbitrary st-path in D_x. Let $\Delta_1 := \min\{g(uv) - x(uv) : uv$ a forward edge of $P\}$ and $\Delta_2 := \min\{x(vu) : uv$ a backward edge of $P\}$. Let $\Delta := \min\{\Delta_1, \Delta_2\}$. Then Δ is positive. Call an edge of P **critical** if Δ is attained on this edge.

Revise x as follows. If uv is a forward edge of P, increase the flow value $x(uv)$ on the edge uv of D by Δ. If uv is a backward edge of P, decrease the flow value $x(vu)$ on the edge vu of D by Δ. It is straightforward to check that the revised function x' is a feasible flow and that its flow-amount is Δ larger than that of x.

Consequently, if g is integer-valued, then Case 2 can occur only a finite number of times, that is, Case 1 occurs after a finite number of flow augmentations and the algorithm terminates. This observation completes the proof of the MFMC theorem for integer capacities. •

For a rational g, by multiplying the capacities with a common multiple of the denominators we are at the integer-valued case. Note, however, that for an irrational g the generic procedure above need not terminate, as was shown by Ford and Fulkerson.

Another drawbacks is apparent already for integer capacities since the number of iterations can be proportional to the largest value of g which is an exponential function of the size of the largest value. (The size of an integer N is the number of its digits which is proportional to the logarithm of N.) Both disadvantages can be eliminated by appropriately restricting the selections of the augmenting paths.

3.5.2 Shortest augmenting paths

Edmonds and Karp [84] and Dinits [69] suggested that in each iteration the st-path of D_x used for the augmentation should be chosen to be of minimum length. Note that a BFS algorithm automatically provides such a path. This natural restriction allows us to bound the complexity of the algorithm of Ford and Fulkerson by a polynomial of $|V| + |A|$,

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 124 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

independent of the magnitude of the capacity function (where each basic operation of numbers, such as addition, subtraction, or comparison, is considered a single step).

Theorem 3.5.1 (Edmonds–Karp, Dinits) *If a shortest augmenting path is always used in the algorithm of Ford and Fulkerson, then the algorithm terminates after $O(|V||A|)$ augmentations for an arbitrary capacity function g.*

Proof. Let $\mathrm{dist}_x(v)$ denote the distance of v in D_x from s. (When v is not reachable from s, then $\mathrm{dist}_x(v) = \infty$.) Let P be a shortest st-path in D_x. For every edge uv of P, we have $\mathrm{dist}_x(v) = \mathrm{dist}_x(u) + 1$.

Lemma 3.5.2 *Let x' denote the flow obtained from x after an augmentation along P. Then $dist_{x'}(v) \geq dist_x(v)$ for every $v \in V$ where $dist_{x'}(v)$ denotes the distance of v in $D_{x'}$ from s.*

Proof. Consider the effect of the augmentation on the auxiliary digraph D_x. Obviously, a change can occur only on the edges along P. Specifically, all the new edges in D'_x are edges of P with opposite orientation. Furthermore, at least one edge of P (namely, the critical edges) will certainly disappear from D_x. The distance of v from s could decrease only if a new edge uw with $\mathrm{dist}_x(w) > \mathrm{dist}_x(u) + 1$ were to enter the auxiliary digraph, from which the lemma follows. •

The series of consecutive augmentations is divided into phases. During one phase $\mathrm{dist}_x(t)$ remains unchanged. Hence there are at most $|V| - 1$ phases.

Lemma 3.5.3 *Within one phase, at most $|A|$ augmentations can occur.*

Proof. Let $\mathrm{dist}_i(v)$ denote the distance of a node v from s at the beginning of Phase i in the current auxiliary digraph. We say that an edge uv is ***i*-tight** if $\mathrm{dist}_i(v) = \mathrm{dist}_i(u) + 1$. During the course of phase i, only i-tight edges are used. We noted that an augmentation kills an i-tight edge and does not create any new i-tight edge. Since the auxiliary digraph has at most $|A|$ i-tight edges, the lemma follows. •

By combining these observations, we can conclude that the algorithm terminates after at most $|V||A|$ augmentations and thus the overall complexity of the algorithm of Edmonds–Karp and of Dinits is $O(|V||A|^2)$ since one augmentation needs $O(|A|)$ steps. • •

Remark 3.5.1 We proved that if consistently shortest augmenting paths are used in the augmenting path algorithm, then the number of flow augmentations is a polynomial of the size of the graph. Since the algorithm uses only addition and subtraction but not multiplication or division, the size of the numbers occurring during the run of the algorithm is also a polynomial of $|A|$ provided that the capacity function g is integer-valued. For a meticulous analysis of complexity, in general, one must be careful when multiplications can occur since the size of the numbers can become unacceptably large. This danger is the reason why the Gaussian elimination for computing the rank of an integer matrix is not a priori polynomial even if the entries are small integers, even though the number of pivoting steps is clearly polynomial in the size of the matrix. (For a polynomial algorithm, see Edmonds' paper [79].) It is a bit peculiar, but the same difficulty can show up in connection with the Edmonds–Karp–Dinits algorithm, as well. To see how, suppose that each capacity $g(e)$ is given in the form $\log q(e)$ where $q(e)$ is a positive integer. Since the algorithm uses only addition,

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 125 of 640 December 11, 2010 15:15

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

subtraction, and comparison, each number arising will remain in the same form. Yet, for this input format, the Edmonds–Karp–Dinits algorithm is NOT polynomial. It is possible to construct an example where the size of the numbers appearing in the course of the algorithm is exponential (see [153]).

Structure of minimum cuts

The flow algorithm above is more restricted than the original algorithm of Ford and Fulkerson but it still offers freedom in choosing the subsequent augmenting paths. Accordingly, the final maximum flow may depend on the actual run of the algorithm. Surprisingly, the minimum cut determined by the algorithm depends only on D and g. The following result is due to Picard and Queyranne [320].

Theorem 3.5.4 *Let $D = (V, A)$ be a digraph endowed with a non-negative capacity function g and let k denote the maximum amount of a feasible st-flow. Then the set-system $\mathcal{S} := \{Z : s \in Z \subseteq V - t,\ \delta_g(Z) = k\}$ is closed under intersection and union. Furthermore, the set S produced by the augmenting path flow algorithm is the unique smallest member of $\mathcal{S}$.*

Proof. For $X, Y \in \mathcal{S}$, we have $k + k = \delta_g(X) + \delta_g(Y) \geq \delta_g(X \cap Y) + \delta_g(X \cup Y) \geq k + k$ from which $k = \delta_g(X \cap Y)$ and $k = \delta_g(X \cup Y)$ follow, and hence both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{S}$.

The set S produced by the algorithm consisted of the nodes reachable from s in the auxiliary digraph D_x defined by the final (maxiumum) st-flow x. We have already proved that $S \in \mathcal{S}$. Suppose now that $s \in Z$ and Z is a proper subset of S. Then D_x has an edge uv leaving Z. If uv is a forward edge, then $uv \in A$ and $x(uv) < g(uv)$, while if uv is a backward edge, then $vu \in A$ and $x(vu) > 0$. In both cases, $\delta_g(Z) > \delta_x(Z) - \varrho_x(Z) = k$, and hence $Z \notin \mathcal{S}$. •

3.6 Cheapest flows

Let $D = (V, A)$ be a digraph of $n \geq 2$ nodes endowed with a lower bound $f : A \to \mathbf{R} \cup \{-\infty\}$ and an upper bound $g : A \to \mathbf{R} \cup \{\infty\}$ for which $f \leq g$. A function $x : A \to \mathbf{R}$ is called **feasible** if $f \leq x \leq g$. Let $m : V \to \mathbf{R}$ be a vector for which there is a feasible m-flow and let $c : A \to \mathbf{R}$ be a cost function. We are interested in finding the cheapest feasible m-flow. In Section 3.4.2 we pointed out that an algorithm for the special case of st-flows can be used for the general min-cost m-flow problem. In Section 6.4 a direct, strongly polynomial algorithm will be presented for m-flows.

We call a positive integer k feasible if there are k edge-disjoint st-paths in D. By relying on the augmenting path algorithm, one is able to compute efficiently k edge-disjoint st-paths. As an extension of the cheapest st-path problem, we analyse for a given cost function $c : A \to \mathbf{R}_+$ how k edge-disjoint st-paths of minimum total cost can be computed. What we need is a minimum cost integer-valued flow of amount k which is feasible with respect to the capacity function $g \equiv 1$. In fact, the algorithm can easily be extended to general g but for the sake of simplicity we assume first that $g \equiv 1$. In this case an integer-valued feasible flow is (0, 1)-valued. Recall that a k-braid is defined as the set of k edge-disjoint st-walks. In Theorem 1.3.5 we pointed out that the (0, 1)-valued st-flows of flow-amount k can be identified with the edge-set of a k-braid. With a slight abuse of notation, in what follows we also call such a flow a k-braid. A k-braid z is **cheapest** if its cost defined by the scalar product $cz = \sum[c(e)z(e) : e \in A]$ is minimum among k-braids. The algorithm constructs a cheapest k-braid for every feasible k.

Let $\pi : V \to \mathbf{Z}_+$ be a function on V for which $\pi(s) = 0 \leq \pi(v) \leq \pi(t)$ for every $v \in V$. We call such a function a **potential**. Consider the tension $\Delta_\pi : A \to \mathbf{R}$ defined by π as follows:

$$\Delta_\pi(uv) := \pi(v) - \pi(u). \tag{3.38}$$

Due to Proposition 3.1.3, the Δ_π-cost of every st-path is $\pi(t) - \pi(s) = \pi(t)$ and the Δ_π-cost of every di-circuit is zero. Therefore the Flow decomposition lemma (Lemma 3.4.3) implies that the Δ_π-cost of every flow of flow-amount k is $k\pi(t)$. This implies that the reduced cost c_π defined by

$$c_\pi(uv) := c(uv) - \Delta_\pi(uv) \text{ for } uv \in A \tag{3.39}$$

is equivalent to c in the sense that a flow is cheapest with respect to c if and only if it is cheapest with respect to c_π.

Theorem 3.6.1 (Ford and Fulkerson) *Let $c : A \to \mathbf{R}_+$ be a non-negative cost function on the edge-set of a digraph $D = (V, A)$. The minimum cost of a k-braid from s to t* (that is, the minimum total cost of k edge-disjoint st-paths) *is equal to the maximum of*

$$k\pi(t) + \sum[c_\pi(uv) : uv \in A, c_\pi(uv) < 0] \tag{3.40}$$

where the maximum is taken over all potentials $\pi : V \to \mathbf{R}_+$. A k-braid z is of minimum cost among k-braids if and only if there is a potential π for which the following optimality criteria hold:

$$\text{For every edge } uv \in A : \quad \begin{cases} c_\pi(uv) > 0 \Rightarrow z(uv) = 0 & \text{(i)} \\ c_\pi(uv) < 0 \Rightarrow z(uv) = 1. & \text{(ii)} \end{cases} \tag{3.41}$$

When c is integer-valued, the optimal π can also be selected to be integer-valued.

Proof. For the cost cz of a braid z we have the following lower bound.

$$\sum c(uv)z(uv) = \sum \Delta_\pi(uv)z(uv) + \sum c_\pi(uv)z(uv) = k\pi(t) + \sum[c_\pi(uv)z(uv) : c_\pi(uv) > 0] + \sum[c_\pi(uv)z(uv) : c_\pi(uv) < 0] \geq k\pi(t) + 0 + \sum[c_\pi(uv) : c_\pi(uv) < 0].$$

This immediately implies the inequality min $\geq$ max. Moreover, we observe that a k-braid is certainly of minimum cost among all k-braids if there is a potential π for which we have equalities throughout in the estimation above. But this is just equivalent to the optimality criteria (i) and (ii) given in the theorem. Therefore both parts of the theorem follow once we prove for every feasible k that there is a k-braid and a potential (which is integer-valued when c is) satisfying the optimality criteria. Such a k-braid and potential can be constructed by the min-cost flow algorithm of Ford and Fulkerson. This algorithm can be considered as a refinement of their maximum flow algorithm.

978–0–19–920527–1 03-Andras-Frank-c03-drv Andras Frank (Typeset by SPi, Publisher Services, India) 127 of 640 December 11, 2010 15:15
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

The procedure starts with the identically zero braid z and identically zero potential π. The flow-amount is increased one by one, but between two augmentations there can be potential changes. The optimality criteria are preserved throughout. The algorithm terminates when it finds a maximum flow and a minimum cut. In the case when c is integer-valued, the algorithm preserves the integrality of π throughout.

Iterative step At a general stage, we are given a braid z and a potential π which satisfy both optimality criteria (i) and (ii) in (3.41). Construct an auxiliary digraph $D' = (V, A')$ (depending on z and π) as follows. D' has two types of edges: forward and backward.

An edge $uv \in A'$ is a **forward edge** if $uv \in A$, $c_\pi(uv) = 0$ and $z(uv) = 0$. An edge $uv \in A'$ is a **backward** edge if $vu \in A$, $c_\pi(vu) = 0$ and $z(vu) = 1$. Let S denote the set of nodes reachable from s in D'. There can be two cases.

Case 1 *$t \notin S$, or informally, t is not reachable from s.*

Let $\varepsilon_1 = \min\left\{c_\pi(uv) : uv \in \Delta^+_D(S),\ z(uv) = 0\right\}$ and $\varepsilon_2 = \min\left\{-c_\pi(uv) : uv \in \Delta^-_D(S),\ z(uv) = 1\right\}$, where the minimum on the empty set is defined to be ∞. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. It follows from the optimality criteria and from the definition of S, that ε is positive.

If $\varepsilon = \infty$, then the algorithm terminates. In this case all the edges leaving S in the original digraph D are saturated and the flow value on all the edges entering S in D is zero. Therefore $\delta_D(S) = \delta_z(S) - \varrho_z(S) = \delta_z(s)$, and hence the current flow z is of maximum flow-amount and $\Delta^+_D(S)$ is a minimum cut.

Suppose now that $\varepsilon < \infty$, and revise π by inceasing $\pi(v)$ for each node $v \in V - S$ by ε. From the defintition of ε, we obtain the following observation.

Claim 3.6.2 *The revised potential π' and the unchanged braid z satisfy the optimality criteria.* •

Construct the auxiliary digraph with respect to π' and z and iterate the procedure. Observe that in the new auxiliary digraph the set of edges induced by S remains the same, but that there is at least one new edge leaving S. Therefore the set of nodes reachable from s becomes strictly larger than S, implying that after at most $(|V| - 1)$ consecutive occurrences of Case 1 either $\varepsilon = \infty$ and termination or else Case 2 will take place.

Case 2 *$t \in S$, that is, t is reachable from s.*

Let P be a directed st-path in D'. Revise z as follows. Let $z'(uv) = 1$ if uv is a forward edge of P, and let $z'(uv) = 0$ if vu is a backward edge of P. Then z' is a $(k + 1)$-braid. It follows from this modification that:

Claim 3.6.3 *The revised braid z' and the unchanged potential π satisfy the optimality criteria.* •

Now the description of the algorithm is complete. Since the algorithm provides, in a finite number of steps, a k-braid for each feasible k and a potential satisfying the optimality criteria, the proof of the theorem is also complete. •

Since there are at most $|A|$ flow augmentations, and between any two of them there are at most $|V| - 1$ potential changes, the algorithm is of polynomial time.

Remark 3.6.1 The algorithm above can be used even if the non-negativity of the cost function c is weakened to conservativity. In this case we can work with an equivalent non-negative cost function arising from c by translating it with the potential difference of a feasible potential. Therefore the algorithm can be used to compute k edge-disjoint paths of an acyclic digraph for which the total weight is as large as possible. In particular, one can compute k disjoint chains of a poset for which the total length is as large as possible. As another application, one can compute a maximum weight matching of k elements in an edge-weighted bipartite graph.

The general case

Consider now the general case, when there is an integer-valued capacity function g on the edge-set. Let M denote the maximum flow-amount of a feasible st-flow. When we are interested in a cheapest flow of flow-amount k for *every* possible $k = 1, \dots, M$, then each edge e can by multiplied $g(e)$ times and we can apply the algorithm described above. Also, Theorem 3.6.1 immediately implies the following.

Theorem 3.6.4 *Let $g : A \to \mathbf{Z}_+$ be an integer-valued capacity function and let $c : A \to \mathbf{R}_+$ be a cost function on the edge-set of a digraph $D = (V, A)$. The minimum cost of a feasible st-flow of flow-amount k is equal to the maximum of*

$$k\pi(t) + \sum [c_\pi(uv)g(uv) : uv \in A, c_\pi(uv) < 0] \tag{3.42}$$

where the maximum is taken over all potentials π. A feasible flow z of amount k is of minimum cost among the feasible flows of amount k if and only if there is a potential for which the following optimality criteria hold for every edge $uv \in A$.

$$\begin{cases} c_\pi(uv) > 0 \Rightarrow z(uv) = 0 & \text{(i)} \\ c_\pi(uv) < 0 \Rightarrow z(uv) = g(uv). & \text{(ii)} \end{cases} \tag{3.43}$$

When c is integer-valued the optimal π can also be selected to be integer-valued.

Note that the complexity of the algorithm depends polynomially on the size of the graph and on the value of M (but not on the size of M). It is, however, a typical situation that one needs a cheapest flow only for one specified flow-amount K, for example $K = M$. In this case an algorithm is considered polynomial only if its complexity can be bounded by a polynomial of $O(|\lceil \log M \rceil|)$, the size of M. Making use of the strongly polynomial algorithm of Edmonds and Karp, and of Dinits, we outline a variation that is polynomial at least for 'small' integer-valued cost functions c. In this case we can drop the integrality assumption about the capacity function g. We hasten to indicate, however, that this approach does not give rise to a polynomial (let alone a strongly polynomial) algorithm when no restriction is made on the cost function. In Part II, we describe a strongly polynomial algorithm, invented Goldberg and Tarjan [187], based on significant new ideas.

Consider an intermediate stage of the algorithm, when an integer-valued feasible flow z of flow-amount k and a potential π which satisfy the optimality criteria (3.43) are available. Define an auxiliary digraph $D' = (V, A')$ as follows. An edge $uv \in A'$ is a forward edge if $uv \in A,\ c_\pi(uv) = 0$ and $z(uv) < g(uv)$. An edge $uv \in A'$ is backward edge if $vu \in A,\ c_\pi(vu) = 0$ and $z(vu) > 0$.

If the set S of nodes reachable from s in D' does not contain t, then the definition of ε is modified as follows. Let $\varepsilon_1 = \min\left\{c_\pi(uv) : uv \in \Delta_D^+(S),\ z(uv) < g(uv)\right\}$ and $\varepsilon_2 = \min\left\{-c_\pi(uv) : uv \in \Delta_D^-(S),\ z(uv) > 0\right\}$, where the minimum over the empty set is defined to be ∞. Due to the optimality criteria, $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ is strictly positive.

If $\varepsilon = \infty$, the algorithm terminates: the edges of D leaving S are all saturated while $z(e) = 0$ for every edge $e \in A$ entering S. In this case $\delta_g(S) = \delta_z(S) - \varrho_z(S) = \delta_z(s)$, that is the current st-flow is of maximum flow-amount and the set S defines a minimum cut.

When $\varepsilon < \infty$, we revise π in such a way that the value of $\pi(v)$ is increased by ε for every $v \in V - S$. As in the special case $g \equiv 1$ above, the optimality criteria continue to hold, and after at most $(|V| - 1)$ potential changes either $\varepsilon = \infty$ or Case 2 must occur.

If $t \in S$, choose a shortest directed st-path P of D'. Let $\Delta_1 := \min\{g(uv) - x(uv) : uv$ is a forward edge of $P\}$ and $\Delta_2 = \min\{x(vu) : uv$ is a backward edge of $P\}$. Let $\Delta = \min\{\Delta_1, \Delta_2\}$ and revise z as follows. Let $z'(uv) = z(uv) + \Delta$ if uv is a forward edge of P and let $z'(uv) = z(uv) - \Delta$ if vu is a backward edge of P. By the proof of Theorem 3.5.1, these flow changes can occur $O(|V||A|)$ times in a row.

We can conclude that both potential changes and flow augmentations can occur only a polynomial number of times in a row. But this algorithm is not polynomial in general since the number of changes between the two types of modifications may be exponential. There are, however, two special cases when this difficulty does not occur.

In the first one, the cost function $c \geq 0$ is arbitrary, while g is integer-valued and small in the sense that its largest value is bounded by a polynomial of $|V|$. Since the maximum flow-amount is equal to $\min\{\delta_g(S) : s \in S \subseteq V - t\}$ and this minimum is now a polynomial of $|V|$, the total number of flow augmentations is also bounded by a polynomial of $|V|$.

In the other special case the capacity function $g \geq 0$ is arbitrary while c is integer-valued and small in the sense that its largest value is bounded by a polynomial of $|V|$. Since each augmenting path consists of edges for which $c_\pi(uv) = 0$, the value $\pi(t)$ is always the total cost of an st-path of the underlying undirected graph of D. Therefore the distinct values of $\pi(t)$, that is, the total number of potential changes can be bounded from above by the total cost of the edges and hence by a polynomial of $|V|$.

3.6.1 Cheapest degree-specified reorientations

Assume that an in-degree specification m satisfies the conditions of the Orientation lemma (Theorem 2.3.2) and hence that there is an orientation of $G = (V, E)$ for which $\varrho(v) = m(v)$ for every node v. In the cheapest orientation problem both of the two possible orientations of an edge have a cost and we are interested in a cheapest orientation of given in-degree function m. By adding the same value to the costs of the two possible orientations of an edge, we obtain an equivalent problem, so we can assume that the smaller of the costs of the two possible orientations is zero. In this way, we arrive at the reorientation problem in which we start from a given orientation D and want to reorient a set of edges with minimum total cost so as to obtain an in-degree specified orientation. Suppose that $c : A \to \mathbf{R}_+$ is a cost function for which $c(a)$ indicates the cost of reorienting edge a. We show that this problem can be reduced to a minimum cost st-flow problem and hence that the min-cost flow algorithm can be applied.

For each node v, let $b(v) := \varrho_D(v) - m(v)$ and let $M := \sum[b(v) : b(v) > 0]$. Add new nodes s and t to D. Furthermore, add $b(v)$ parallel vt-edges if $b(v) > 0$ and $|b(v)|$ parallel sv-edges if $b(v) < 0$.

An easy consideration shows that an M-braid from s to t determines a reorientation of D satisfying the in-degree specification m and therefore the cheapest reorientation problem can be solved algorithmically by applying the algorithm above for finding a minimum cost M-braid from s to t.

3.6.2 Chains and antichains of posets

In this section, we discuss a surprising application of the min-cost flow algorithm described in the previous section. By an analysis of the run of the algorithm, we derive an important *theoretical* result on chain and antichain decompositions of posets.

Let $(P, \preceq)$ be a partially ordered set with ground-set $P = \{p_1, \dots, p_n\}$. The cardinality of a chain is called its **length** and the length of the longest chain is the **height** of the poset denoted by $c(P)$. The cardinality $a(P)$ of the largest antichain is called the **breadth** of P.

In Section 2.4 we proved Dilworth's theorem (Theorem 2.4.25) stating that the breadth of a poset is equal to the minimum number of chains covering P. We proved the polar-Dilworth theorem (Theorem 2.4.29) asserting that the height of a poset is equal to the minimum number of antichains covering P. Since the subset of a chain is also a chain, we could speak of minimum decompositions of a poset into chains, and analogously, into antichains.

As a natural extension of these theorems, we investigate the problem of finding the largest subset of P which is the union of $\alpha \geq 1$ antichains and the largest subset of P which is the union of $\gamma \geq 1$ chains where α and γ are given integers.

For a set-system $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$, we use the abbreviation $\cup\mathcal{B} = \cup(B_i : i = 1, \dots, k)$. By a **chain family** $\mathcal{C}_\gamma = \{C_1, C_2, \dots, C_\gamma\}$, we mean a set of γ disjoint chains. Let $\mathbf{C}_\gamma$ denote the set of chain families consisting of γ chains and let $\mathbf{C}$ denote the set of all chain families.

Let $c_\gamma = \max\{|\cup\mathcal{C}_\gamma| : \mathcal{C}_\gamma \in \mathbf{C}_\gamma\}$, or informally, c_γ is the cardinality of the largest set which is the union of γ chains (and hence, by Dilworth, c_γ is the maximum cardinality of a set including no antichains of more than γ elements).

By an **antichain family** $\mathcal{A}_\alpha = \{A_1, A_2, \dots, A_\alpha\}$, we mean a set of α disjoint antichains. Let $\mathbf{A}_\alpha$ denote the set of antichain families consiting of α antichains. Let $\mathbf{A}$ denote the set of all antichain families. Let $a_\alpha = \max\{|\cup\mathcal{A}_\alpha| : \mathcal{A}_\alpha \in \mathbf{A}_\alpha\}$, or informally, a_α is the cardinality of the largest set which is the union of α antichains (and hence, by polar-Dilworth, a_α is the maximum cardinality of a set including no chains of more than α elements).

Dilworth's theorem implies $c_{a(P)} = n$, while its polar form implies $a_{c(P)} = n$. What can be said about c_γ $(1 \leq \gamma \leq a(P))$ and a_α $(1 \leq \alpha \leq c(P))$? The answer is given by the following two results.

Theorem 3.6.5 (Greene and Kleitman [193]) *In a partially ordered set P,*

$$a_\alpha = \min\{q\alpha + |P - \cup\mathcal{C}_q| : \mathcal{C}_q \in \mathbf{C}\}$$

holds for every integer $1 \leq \alpha \leq c(P)$.

Theorem 3.6.6 (Greene [192]) *In a partially ordered set P,*

$$c_\gamma = \min\{q\gamma + |P - \cup\mathcal{A}_q| : \mathcal{A}_q \in \mathbf{A}\}$$

holds for every integer $1 \le \gamma \le a(P)$.

Since a chain and an antichain can have at most one element in common, any antichain family, given q disjoint chains, can cover at best all elements outside the q chains and at most $q\alpha$ elements from the union of the chains. Therefore a_α is at most the minimum in question. An analogous argument shows that c_γ is also at most the minimum in question. Therefore the main content of the theorems is that these bounds can actually be achieved.

Definition We say that an antichain family $\mathcal{A}_\alpha = \{A_1, A_2, \ldots, A_\alpha)$ and a chain-family $\mathcal{C}_\gamma = \{C_1, C_2, \ldots, C_\gamma\}$ are **orthogonal** if

$$P = (\cup\mathcal{A}_\alpha) \cup (\cup\mathcal{C}_\gamma) \tag{3.44}$$

and

$$A_i \cap C_j \neq \emptyset \text{ whenever } 1 \le i \le \alpha,\ 1 \le j \le \gamma, \tag{3.45}$$

that is, the chain family and the antichain family together cover P and each antichain A_i intersects each chain C_j.

Based on this definition, the non-trivial part of Theorems 3.6.5 and 3.6.6 can be reformulated as follows.

Theorem 3.6.7 *For every α, $1 \le \alpha \le c(P)$, there exists an antichain-family $\mathcal{A}_\alpha \in \mathbf{A}_\alpha$ and a chain family $\mathcal{C}_\gamma \in \mathbf{C}$, for some appropriate γ, that are orthogonal.*

Theorem 3.6.8 *For every γ, $1 \le \gamma \le a(P)$, there exists a chain family $\mathcal{C}_\gamma \in \mathbf{C}_\gamma$ and an antichain family $\mathcal{A}_\alpha \in \mathbf{A}$, for some appropriate α, that are orthogonal.*

The following theorem of Frank [115] is a common generalization of Theorems 3.6.7 and 3.6.8. An interesting feature of the proof is that a careful analysis of the run of the min-cost flow algorithm of Ford and Fulkerson yields the requested result. Similarly to the Fulkerson derivation of Dilworth's theorem from Kőnig's, the present approach also relies on the node duplication technique.

Theorem 3.6.9 *Let $a = a(P)$ and $c = c(P)$ denote the breadth and the height, respectively, of a partially ordered set $(P, \preceq)$. There exists a sequence $\mathcal{C}_a|\ \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{i_1}|\ \mathcal{C}_{a-1}, \mathcal{C}_{a-2}, \ldots, \mathcal{C}_{a-j_1}|\ \ \mathcal{A}_{i_1+1}, \ldots, \mathcal{A}_{i_2}|\ \ \mathcal{C}_{a-j_1-1}, \ldots, \mathcal{C}_{a-j_2}| \ldots$ which is obtained by combining the sequences $\mathcal{C}_a, \mathcal{C}_{a-1}, \ldots, \mathcal{C}_1$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_c$, where $\mathcal{C}_j \in \mathbf{C}_j$, $\mathcal{A}_i \in \mathbf{A}_i$, and every member* (apart from the first one) *of the sequence* (whether a $\mathcal{C}_j$ or an $\mathcal{A}_i$) *is orthogonal to the last preceding member of opposite type.* (That is, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{i_1}$ are all orthogonal to $\mathcal{C}_a$, furthermore $\mathcal{C}_{a-1}, \mathcal{C}_{a-2}, \ldots, \mathcal{C}_{a-j_1}$ are all orthogonal to $\mathcal{A}_{i_1}$, and so on.)

Proof. Assign a digraph $D = (V, A)$ to $(P, \preceq)$ where

$$V := \{s, t, x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n\},$$

and

$$A := \{x_i y_j : \text{if } p_i \succ p_j\} \cup \{x_i y_i : i = 1, 2, \dots, n\} \cup$$
$$\{s x_i : i = 1, 2, \dots, n\} \cup \{y_i t : i = 1, 2, \dots, n\}.$$

Let the capacity $g(e)$ of every edge be 1. Let $c(e) = 1$ if $e = x_i y_i$, and 0 otherwise. Apply the minimum cost flow algorithm of Ford and Fulkerson as described in Section 3.6 for braids. Let z and π be the flow and potential, respectively, belonging to an intermediate stage of the algorithm. Recall that z is (0, 1)-valued (and hence z is a braid), $\pi \geq 0$ and $\pi(s) = 0$, and that they meet the following optimality criteria.

$$\text{For every edge } uv \in A: \begin{cases} \pi(v) - \pi(u) < c(uv) \Rightarrow z(uv) = 0 & \text{(i)} \\ \pi(v) - \pi(u) > c(uv) \Rightarrow z(uv) = 1. & \text{(ii)} \end{cases} \tag{3.46}$$

There are four types of edges: $x_i y_j$, $x_i y_i$, $s x_i$, $y_i t$. Writing out in detail the criteria for these types, we get the following.

$$\text{For every edge } x_i y_j \text{ (i.e., } p_i \succ p_j\text{):} \begin{cases} \pi(y_j) - \pi(x_i) < 0 \Rightarrow z(x_i y_j) = 0 & \text{(i)} \\ \pi(y_j) - \pi(x_i) > 0 \Rightarrow z(x_i y_j) = 1. & \text{(ii)} \end{cases} \tag{3.47}$$

$$\text{For every edge } x_i y_i \ (i = 1, \dots, n)\text{:} \begin{cases} \pi(y_i) - \pi(x_i) < 1 \Rightarrow z(x_i y_i) = 0 & \text{(i)} \\ \pi(y_i) - \pi(x_i) > 1 \Rightarrow z(x_i y_i) = 1. & \text{(ii)} \end{cases} \tag{3.48}$$

$$\text{For every edge } s x_i \ (i = 1, \dots, n): \quad \pi(x_i) > 0 \Rightarrow z(s x_i) = 1. \tag{3.49}$$

$$\text{For every edge } y_i t \ (i = 1, \dots, n): \quad \pi(y_i) < \pi(t) \Rightarrow z(y_i t) = 1. \tag{3.50}$$

Note that in the last two cases we left out the implications $\pi(x_i) < 0 \Rightarrow z(s x_i) = 0$ and $\pi(y_i) > \pi(t) \Rightarrow z(y_i t) = 0$ corresponding to Part (i) of (3.46) since their premisses can never occur.

Claim 3.6.10 *For every* $i = 1, \dots, n$

$$\pi(y_i) \leq \pi(x_i) + 1. \tag{3.51}$$

Proof. (3.51) holds at the start of the algorithm. A flow augmentation does not affect (3.51). A potential change can destroy it only if $\pi(y_i) = \pi(x_i) + 1$ before the change and x_i is reachable from s in the auxiliary digraph D', while y_i is not. Then $\pi'(y_i) = \pi'(x_i) + 2$ holds for the revised potential π'. Property (ii) of (3.48), when applied to π', implies that $z(x_i y_i) = 1$. Hence $z(s x_i) = 1$ from which $y_i x_i$ is the only edge of D' entering x_i. But this contradicts the indirect assumption that x_i is reachable while y_i is not. •

Claim 3.6.11 *If* $p_i \succ p_j$ *and* $z(x_i y_j) = 1$, *then* $\pi(x_i) = \pi(y_j)$.

Proof. If a flow augmentation increases the value of $z(x_i y_j)$ from 0 to 1, then $x_i y_j$ is an edge of D' and hence $\pi(y_i) - \pi(x_j) = c(x_i y_j) = 0$, from which $\pi(y_i) = \pi(x_j)$. During the subsequent potential changes, due to implication (i) in (3.47), the inequality $\pi(y_j) \geq \pi(x_i)$ continues to hold as long as $z(x_i y_j) = 1$.

Hence the equality $\pi(y_i) = \pi(x_j)$ can break down at a potential change only if $\pi(y_i)$ is increased by 1 while $\pi(x_j)$ is not changed, which appears when x_i is reachable from s while y_j is not. This, however, is impossible since $y_j x_i$ is the only edge in D' entering x_i, and hence x_i can be reached from s only through y_j. •

Let P' denote the set of elements p_h for which $z(x_h y_h) = 0$. With the help of the construction of chains in the algorithmic proof of Dilworth's theorem, we can define a chain family $\mathcal{C}_\gamma$ (where $\gamma = n - \delta_z(s)$) from the set of edges $x_i y_j$ $(i < j)$ with $z(x_i y_j) = 1$. This $\mathcal{C}_\gamma$ forms a partition of P'. For $\alpha := \pi(t)$, define a set-system $A_\alpha = \{A_1, A_2, \ldots, A_\alpha\}$ by taking $A_i := \{p_j : \pi(x_j) + 1 = \pi(y_j) = i\}$.

Claim 3.6.12 *$\mathcal{A}_\alpha$ is an antichain family which is orthogonal to $\mathcal{C}_\gamma$.*

Proof. Let us show first that each A_i is an antichain. If indirectly $p_m \succ p_j$ for some elements $p_m, p_j \in A_i$, then $\pi(y_j) - \pi(x_m) = 1$, and hence (ii) of (3.47) implies $z(x_m, y_j) = 1$, in contradiction to Claim 3.6.11.

In order to prove (3.44), let p_m be an element not in $P' = \cup\mathcal{C}_\gamma$, which is equivalent to $z(x_m, y_m) = 1$. By implication (i) of (3.48), $\pi(y_m) - \pi(x_m) \geq 1$, from which Claim 3.6.10 implies $\pi(y_m) - \pi(x_m) = 1$. That is, for $i := \pi(y_m)$ we have $p_m \in A_i$ and hence (3.44) holds.

Finally, we prove that $A_i \cap C_j \neq \emptyset$ $(1 \leq i \leq \alpha,\ 1 \leq j \leq \gamma)$. Suppose that the elements of chain C_j are $p_1 \succ \cdots \succ p_k$ $(k \geq 1)$. Now $z(y_1 t) = 0$ from which (3.50) implies $\pi(y_1) \geq \pi(t)$ and hence $\pi(y_1) = \pi(t) = \alpha$. Similarly, we get $z(s x_k) = 0$ and hence (3.49) implies $\pi(x_k) \leq 0$, from which $\pi(x_k) = 0$. From Claim 3.6.11 we obtain $\pi(x_1) = \pi(y_2),\ \pi(x_2) = \pi(y_3), \ldots, \pi(x_{k-1}) = \pi(y_k)$. Combining this observation with Claim 3.6.10, one concludes that each member of the sequence $0 = \pi(x_k), \pi(y_k), \pi(y_{k-1}), \pi(y_{k-2}), \ldots, \pi(y_1) = \alpha$ consisting of integers can be larger than the preceding member by at most one.

Therefore there must exist a subscript m for each i $(1 \leq i \leq \alpha)$ such that $\pi(x_m) = \pi(y_m) + 1 = i$. That is, $p_m \in C_j \cap A_i$. •

Suppose now that the minimum cost flow algorithm ran as follows. Starting with the identically zero flow z and zero potential π, the flow-amount is increased one by one to k_0; the largest potential value (which is the value of $\pi(t)$) is then increased one by one to i_1; and these steps is followed by gradual increases of the flow-amount to k_1, and so on. Finally, the flow-amount reaches its maximum value k_q and the largest potential value reaches its maximum value i_q. In this case, the maximum flow-amount is n, and hence $k_q = n$.

Let $a := n - k_0$ and $j_i := k_i - k_0$ $(i = 1, 2, \ldots, q)$. Consider the stage of the algorithm characterized by the couple $(k_0, 1)$: at this stage the flow-amount is k_0 and $\pi(t) = 1$. As described before Claim 3.6.12, we can construct a chain family $\mathcal{C}_a$ consisting of a chains and this chain family is orthogonal to the antichain family $\mathcal{A}_1$ consisting of the single antichain A_1.

As mentioned above, the largest potential value is then increased one by one to i_1. We assigned the antichain families $\mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_{i_1}$ to the intermediate stages, and these families are all orthogonal to the unchanged $\mathcal{C}_a$. After this, the flow-amount increases one by one to k_1. The chain families $\mathcal{C}_{a-1}, \mathcal{C}_{a-2}, \ldots, \mathcal{C}_{a-j_1}$ belonging to the intermediate stages are orthogonal to the unchanged $\mathcal{A}_{i_1}$, and so on.

We can conclude that the sequence of flows and potentials occurring the process define the combined sequence of chain and antichain families as described in the theorem. • •

Problem 3.6.1 $(*)$ *Call an antichain of maximum cardinality a* **D-antichain**. *Prove that the maximum number of disjoint D-antichains is equal to the minimum number of elements covering all D-antichains. Can you find these quantities algorithmically?*

4

Elements of polyhedral combinatorics

One of the major tools of combinatorial optimization is polyhedral combinatorics. Its fundamental idea is based on the duality theorem of linear programming and on a theorem of Minkowski and Weyl which implies that the convex hull of a finite set of points in $\mathbf{R}^n$ can always be obtained as the solution set of a finite number of linear inequalites. These results inspire the following strategy for finding the best one among a given set of combinatorial objects: first take the polytope R determined by the convex hull of their incidence vectors, then find explicitly the set of linear inequalities (ensured by the Minkowski–Weyl theorem) the solution set of which is R, and finally apply the linear programming duality theorem to obtain a min-max result for the optimum. The characterization obtained in this way opens the door for an attempt to find a direct and algorithmic proof of the min-max theorem. This approach is implicitly in the background of results concerning linear programs with totally unimodular (TU) constraint matrices. The point is that a linear program constrained with a TU-matrix (when the bounding vector is integral) always has an integral optimum solution. Here and througout the book the integrality of an optimum solution vector z means that the *components* of z are integer-valued.

It turns out that a large part of bipartite matching and network optimization problems behave so nicely just because the describing matrix in question is totally unimodular. For example, Kőnig's matching theorem, as well as its weighted extension by Egerváry, can be viewed as the duality theorem applied to the incidence matrix of a bipartite graph, which turns out to be totally unimodular. Likewise, Hoffman's theorem on feasible circulations can be viewed as the Farkas lemma when applied to the incidence matrix of the digraph. (See below for details.) It was Edmonds who first realized that the approach outlined above may work even in cases when TU matrices do not help (directly) anymore. This conscious effort allowed him to describe, for example, the convex hull of non-bipartite matchings to establish a min-max theorem and develop a polynomial algorithm. Another great success of Edmonds has been the solution of the matroid and polymatroid intersection problem. Again, he first found the polyhedral description, then applied duality theorem of linear programming, and finally developed a direct algorithm. The very same approach led to an algorithmic solution of a theorem of Lucchesi and Younger on finding a cheapest subset F of edges of a digraph such that the contraction of F results in a strongly connected digraph.

There is here a subtle point to be considered. It may be, and in fact often is, the case that the number of inequalities in a particular polyhedral description is exponential in the size of the underlying graph. Edmonds recognized that, in spite of the extraordinarily large number of constraints in the linear progam, it is quite possible to develop a stongly polynomial algorithm. (Compare this with the Kőnig–Hall theorem where the Hall condition included an exponential number of inequalities, and yet it was possible to check all of them in polynomial time.)

In this chapter, we first summarize the basics of linear optimization (Farkas lemma, duality, optimality criteria), then give a brief overview of the structure of polyhedra, next introduce total dual integrality, and finally summarize the major properties of TU matrices along with several applications in network optimization. A detailed account can be found in Schrijver [338].

4.1 Linear inequality systems and polyhedra

Though we assume that the reader is familiar with basic linear programming to some extent, in order to have a unified terminology and view, we use this section to summarize the most important notions and results of linear optimization.

4.1.1 Feasibility, boundedness, and optimality

In what follows, we use the notational convention that the j'th column of a matrix Q is denoted by q_j while the i'th row of Q is denoted by ${}_iq$. In our notation, we do not distinguish between row and column vectors. The scalar (or inner) product of two vectors x and y is denoted by xy or by yx, without using the sign of the transpose of a vector. When we write Qx for a matrix Q, x is meant to be a column vector while y denotes a row vector in yQ. (This will not cause any ambiguity, since we never need the diadic product of a column vector and a row vector, where the product is a square matrix.)

For a vector $a \in \mathbf{R}^n$ and a number β, the solution set of a single inequality $ax \leq \beta$ is called a **half-space**. It is **homogeneous** if $\beta = 0$. By a **polyhedron** R, we mean the intersection of a finite number of half-spaces. In other words, a polyhedron is the solution set of a linear inequality system: $R = \{x \in \mathbf{R}^n : Qx \leq b\}$. Here Q is an m by n matrix, called the **constraint matrix**, while $b \in \mathbf{R}^m$ is the **bounding vector**. In what follows, we typically do not refer explicitly to dimensions and assume that the matrices and vectors are of suitable size. We often write $\underline{1}$ to mean an appropriately dimensioned identically 1 vector. For example, if B is an m by n matrix, then in the equation $Bx = \underline{1}$ x is m-dimensional and the right-hand side has n components, while in $yB = \underline{1}$ the right-hand side is n-dimensional. The same convention will be applied to the identically 0 vector. Hence, 0 can mean a number as well as a vector of the appropriate dimension.

For an element z of $R = \{x \in \mathbf{R}^n : Qx \leq b\}$, we say that an inequality ${}_iqx \leq b(i)$ is z-**active** (or just active) if ${}_iqz = b(i)$. The submatrix of Q formed by the active rows is denoted by $Q^{=}_{z}$.

Since an equality $bx = \beta$ can be replaced by inequalities $bx \leq \beta$ and $-bx \leq -\beta$, a polyhedron can be given in the form $\{x : Px = b_0,\ Qx \leq b_1\}$. It is also typical that the non-negativity of the variables are handled separately from the other inequalities. In this case, a polyhedron is described in the form:

$$\begin{cases} Px_0 + Ax_1 = b_0 \\ Qx_0 + Bx_1 \leq b_1 \\ x_1 \geq 0. \end{cases} \tag{4.1}$$

We shall refer to the solution set of (4.1) as the **primal polyhedron**. The linear system obtained from (4.1) by replacing the righ-hand side vector b by zero is called the **homogenization** of the system.

Obviously, one could consider even more general forms, when non-positivity is required on some variables or the inequality sign $\leq$ is reversed but these are easily reduced to the form in (4.1) by negating the variable or the inequality in question. In several cases, the polyhedron is defined in the column space. For example, one can consider the solution set of the following linear system.

$$\begin{cases} y_0P + y_1Q = c_0 \\ y_0A + y_1B \geq c_1 \\ y_1 \geq 0. \end{cases} \tag{4.2}$$

Given $c = (c_0, c_1)$, the linear system (4.2) is called the **dual** of (4.1) while its solution set is the **dual polyhedron**. The special polyhedron $\{x : Px = b\}$ is called an **affine subspace**. When $b = 0$, we speak of a **null-space**. The subspace $\{x : Px = 0\}$ is also called the null-space of P. If P has only one row, the affine subspace is a **hyperplane**. Hence an affine subspace is the intersection of hyperplanes. A **polyhedral cone** is a polyhedron defined by $\{x : Qx \leq 0\}$. Note that both a null-space and a homogeneous half-space are polyhedral cones.

In a linear combination $x_1b_1 + \cdots + x_nb_n$ of vectors $b_1, \ldots, b_n$, the coefficients x_i are arbitrary reals. In a **non-negative combination** each x_i is non-neagtive. In an **affine** combination, the sum of the x_i's is 1. Finally, a non-negative and affine combination is said to be **convex**. The set of all {linear, affine, non-negative, convex} combinations of a finite number of points (or vectors) is called the {linear, affine, non-negative, convex} hull of these points. The linear hull of some vectors is called a **generated subspace**. The affine hull of some points is called a **generated affine subspace**. If the number of (distinct) generating elements is two, we speak of a **line**. The non-negative hull of finitely many vectors is called a **generated cone**. By a **direction** $\vec{z}$, we mean a cone generated by a single non-zero vector z. The convex hull of a finite number of points (possibly zero) is called a **polytope**. Note that a generated subspace is a generated cone.

Translating a subset X of $\mathbf{R}^n$ means that we add a given vector a to every element of X. The sum $X + Y$ (sometimes called the Minkowski sum) is defined by $\{x + y : x \in X, y \in Y\}$. Note that the translation is a special sum when Y is a singleton.

Inner and outer descriptions

A basic result of linear algebra is as follows.

Theorem 4.1.1 *Every subspace of* $\mathbf{R}^n$ *can be given both as a null-space and as a generated subspace. In particular, a subset* R *is a null-space if and only if it is a generated subsapce.* •

A null-space can be viewed as an outer description of a subspace R, while a generated subspace can be viewed as an inner description of R. Dual descriptions of this sort extend to affine spaces, cones, and polyhedra as well.

Theorem 4.1.2 *A subset R of $\mathbf{R}^n$ is an affine subspace if and only if R is the translation of a subspace. The subspace is uniquely determined by R.* •

We call the subspace in the theorem the **translation subspace** of R. If an affine subspace is given in the form $R = \{x : Px = b\}$, then its translation subspace is the null-space of P, that is, $\{x : Px = 0\}$. It follows that a hyperplane of $\mathbf{R}^n$ can be obtained as the affine hull of $n - 1$ linearly independent vectors. In addition, a line can be obtained as the solution set of a system $Ax = b$ where A consists of $n - 1$ linearly independent rows.

Theorem 4.1.3 (Minkowski and Weyl) *A subset R of $\mathbf{R}^n$ is a polyhedron if and only if R is the sum of a polytope and a generated cone. In particular, R is a bounded polyhedron if and only if R is a polytope. Furthermore, R is a polyhedral cone if and only if it is a generated cone.* •

Feasibility

A linear system is said to be **feasible** if it has a solution, or equivalently, if the polyhedron defined by the system is non-empty. For linear equalities, a characterization is given by the following classic result.

Theorem 4.1.4 *An affine subspace $\{x : Px = b\}$ is non-empty if and only if there is no y for which $yP = 0$, and $yb \neq 0$.* •

Theorem 4.1.5 (Farkas lemma, general form) *The linear system defined in* (4.1) *has a solution if and only if there is no $y = (y_0, y_1)$ for which*

$$yb := y_0b_0 + y_1b_1 < 0 \tag{4.3}$$

and

$$\begin{cases} y_0P + y_1Q = 0 \\ y_0A + y_1B \geq 0 \qquad \bullet \\ y_1 \geq 0. \end{cases} \tag{4.4}$$

Note that by normalizing y in Theorem 4.1.4 (which means multiplying y with an appropriate constant), the condition $yb \neq 0$ can be replaced by $yb = \alpha$ for an arbitrary non-zero value α. Also, (4.3) can be replaced by $yb = -1$. Hence the special case of Theorem 4.1.5 when only P is non-empty is just Theorem 4.1.4. (See Figure 4.1)

It is worth formulating the other three special cases separately.

Theorem 4.1.6 (Farkas lemma, special forms) *There is a solution to the linear systems*

(A) $\{Ax = b, x \geq 0\}$ *if and only if there is no y for which $yA \geq 0$ and $yb = -1$,*
(B) $\{Bx \leq b, x \geq 0\}$ *if and only if there is no $y \geq 0$ for which $yB \geq 0$ and $yb = -1$,*
(Q) $\{Qx \leq b\}$ *if and only if there is no $y \geq 0$ for which $yQ = 0$ and $yb = -1$.* •

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 139 of 640 December 11, 2010 15:16

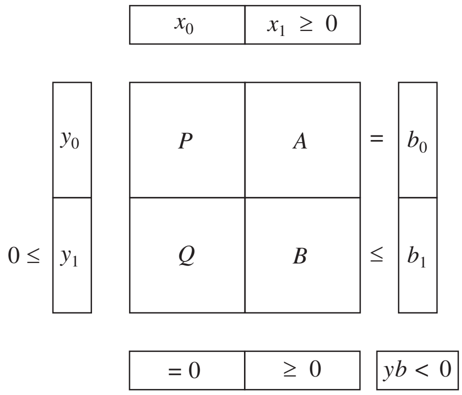


Figure 4.1 Farkas lemma: general form

In fact, the original, standard form of the Farkas lemma is Case (A). Since the general form can easily be deduced from any of these three special cases, we took the liberty of calling Theorem 4.1.5 the Farkas lemma as well. Case (A) immediately implies the following intuitive formulation.

Theorem 4.1.7 (Farkas lemma, geometric form) *A generated cone $C \subseteq \mathbf{R}^k$ contains a given vector $b \in \mathbf{R}^k$ if and only if there is no homogeneous halfspace including C but not containing b. A polytope P contains a point b if and only if there is no halfspace including P but not containing b.* •

Directional boundedness

Given a non-empty polyhedron $R \subseteq \mathbf{R}^n$ and a vector $c \in \mathbf{R}^n$, a characterization for the boundedness of a linear objective function cx over the elements of R requires an easily verifiable property to certify that $\{cx : x \in R\}$ is bounded from above (say) and also an easily verifiable property to certify that $\{cx : x \in R\}$ is not bounded.

Theorem 4.1.8 (directional boundedness) *Suppose that the polyhedron R defined by* (4.1) *is non-empty. For a given vector $c = (c_0, c_1)$ the following are equivalent.*

(1) *$\{cx : x \in R\}$ is bounded from above.*

(2) *There is no $z = (z_0, z_1)$ for which $cz > 0$ and z is a solution to the homogenization of* (4.1):

$$\begin{cases} Pz_0 + Az_1 = 0 \\ Qz_0 + Bz_1 \leq 0 \\ z_1 \geq 0. \end{cases} \tag{4.5}$$

(3) *There is a solution $y = (y_0, y_1)$ to the dual linear system* (4.2).

Proof. To see (1)→(2), let x^* be an element of R. If there is a z satisfying (4.5), then $x^* + \lambda z$ is in R for for any big λ, and hence $\{cx : x \in R\}$ is not bounded from above. The implication (2)→(3) is the non-trivial direction of the general form of the Farkas lemma.

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 140 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Finally, to derive (3)→(1), we prove for a y in Property (3), that the value yb is an upper bound for $\{cx : x \in R\}$. Indeed

$$cx = c_0x_0 + c_1x_1 \le \left[(y_0, y_1)\begin{pmatrix} P \\ Q \end{pmatrix}\right] x_0 + \left[(y_0, y_1)\begin{pmatrix} A \\ B \end{pmatrix}\right] x_1 =$$

$$(yM)x = y(Mx) = y_0[Px_0 + Ax_1] + y_1[Qx_0 + Bx_1] \le y_0b_0 + y_1b_1 = yb$$

where $M = \begin{pmatrix} P & A \\ Q & B \end{pmatrix}$. •

Note that the existence of z in Property (2) is an co-**NP**-characterization for (1) in the sense that a z satisfying (4.5) is an easily verifyable certificate for the non-boundedness of $\{cx : x \in R\}$. Property (3) on the other hand is a **NP**-characterization for (1).

Of course, this result can also be specialized to the cases when only A or B or Q is non-empty.

Theorem 4.1.9 *Let R be a non-empty polyhedron. Then $\{cx : x \in R\}$ is bounded from above,*

(A) *when $R = \{x : Ax = b, x \ge 0\}$, if and only if there is no $z \ge 0$ for which $cz > 0$ and $Az = 0$, and if and only if there is a y for which $yA \ge c$,*
(B) *when $R = \{x : Bx \le b, x \ge 0\}$, if and only if there is no $z \ge 0$ for which $cz > 0$ and $Bz \le 0$, and if and only if there is a $y \ge 0$ for which $yB \ge c$,*
(Q) *when $R = \{x : Qx \le b\}$, if and only if there is no z for which $cz > 0$ and $Qz \le 0$, and if and only if there is a $y \ge 0$ for which $yQ = c$.* •

Optimality criteria and duality

Let x^* be an element of a polyhedron R and let $c \in \mathbf{R}^n$ be a vector for which $\{cx : x \in R\}$ is bounded from above. In order to have appropriate stopping rules for an algorithm, it is useful to formulate optimality criteria for x^*. By definition, we call x^* maximal if $cx^* \ge cx$ for every $x \in R$, that is, x^* maximizes the objective function cx over R. We seek for finding an easily verifiable property to show that x^* is maximal and also, in the case when x^* is not maximal, an easily verifiable tool to find a better element x of R in the sense that $cx > cx^*$.

Theorem 4.1.10 (optimality criteria) *Let x^* be an element of a polyhedron R given by (4.1) and suppose that $\{cx = c_0x_0 + c_1x_1 : (x_0, x_1) \in R\}$ is bounded from above. Let $\left(Q^=_{x^*}, B^=_{x^*}\right)$ denote x^*-active submatrix of (Q, B). The following are equivalent.*

(1) *x^* is optimal, meaning that $cx^* \ge cx$ for every $x \in R$.*
(2) *There is no $z = (z_0, z_1)$ for which $cz > 0$ and*

$$\begin{cases} Pz_0 + Az_1 = 0 \\ Q^=_{x^*}z_0 + B^=_{x^*}z_1 \le 0 \\ x_1^*(i) = 0 \Rightarrow z_1(i) \ge 0. \end{cases} \tag{4.6}$$

(3) *There is a solution* $y^* = (y_0^*, y_1^*)$ *to* (4.2) *for which the following optimality criteria hold*

$$\begin{cases} x_1^*(j) > 0 \Rightarrow y_0^* a_j + y_1^* b_j = c_1(j) \\ y_1^*(i) > 0 \Rightarrow {}_i q x_0^* + {}_i b x_1 = b_1(i) \end{cases}$$

where a_j *and* b_j *denote the* j*'th column of* A *and* B*, respectively, while* ${}_i q$ *and* ${}_i b$ *denote the* i*'th row of* Q *and* B*, respectively.* •

In words, the optimality criteria require that a sign-constrained primal variable $x_1^*(j)$ or dual variable $y_1^*(i)$ can be strictly positive only if the corresponding dual or primal inequality, respectively, is met with equality. Property (3) can equivalently be formulated so as to require that there is an element y^* of the dual polyhedron (for short, a dual solution) for which x^* and y^* satisfy the optimality criteria.

Theorem 4.1.11 *Let* x^* *be an element of a polyhedron* R *and suppose that* $\{cx : x \in R\}$ *is bounded from above. Then the optimality of* x^* *is equivalent:*

(A) *when* $R = \{x : Ax = b, x \geq 0\}$*, to* (2A): *there is no* z *for which* $cz > 0$*,* $Az = 0$*, and* $x^*(i) = 0 \Rightarrow z(i) \geq 0$*, and to* (3A): *there is a* y *for which* $yA \geq c$*, and* $x(j) > 0 \Rightarrow ya_j = c(j)$*.*

(B) *when* $R = \{x : Bx \leq b, x \geq 0\}$*, to* (2B): *there is no* z *for which* $cz > 0$*,* $B^=_{x^*} z \leq 0$*, and* $x^*(i) = 0 \Rightarrow z(i) \geq 0$*, and to* (3B): *there is a* y *for which* $yB \geq c$*,* $y \geq 0$*, and* $x(j) > 0 \Rightarrow yb_j = c(j)$*.*

(Q) *when* $R = \{x : Qx \leq b\}$*, to* (2Q): *there is no* z *for which* $cz > 0$*,* $Q^=_{x^*} z \leq 0$*, and to* (3Q): *there is a* y *for which* $yQ = c$*,* $y \geq 0$*, and* $y^*(i) > 0 \Rightarrow {}_i q x^* = b(i)$*.* •

Theorem 4.1.12 (duality theorem) *Suppose that the primal polyhedron* R *defined by* (4.1) *is non-empty and* $\{cx : x \in R\}$ *is bounded from above for a given vector* $c = (c_0, c_1)$ (or equavalently, the dual polyhedron R^* defined by (4.2) is non-empty). *Then the extrema*

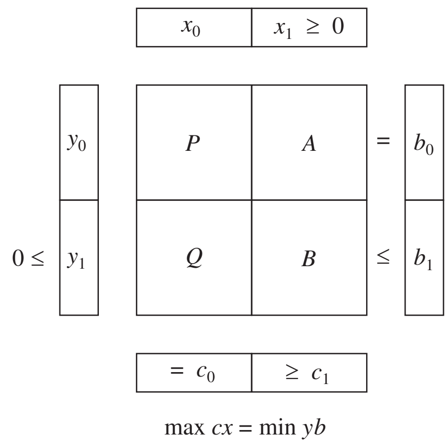


Figure 4.2 Duality theorem: general form

$\max\{cx : x \in R\}$ *and* $\min\{by : y \in R^*\}$ *exist and are equal. In the special cases when only* A *or* B *or* Q *is non-empty, one has*

$$\max\{cx : Ax = b, x \geq 0\} = \min\{by : yA \geq c\}$$
$$\max\{cx : Bx \leq b, x \geq 0\} = \min\{by : yB \geq c, y \geq 0\}$$
$$\max\{cx : Qx \leq b\} = \min\{by : yQ = c, y \geq 0\}. \bullet \tag{4.7}$$

4.1.2 Faces, vertices, and basic solutions

It follows from the duality theorem that if $\{cx : x \in R\}$ is bounded from above for a non-empty polyhedron $R \subseteq \mathbf{R}^n$, then $\delta = \max\{cx : x \in R\}$ exists. By a **face** F of R defined by a vector c, we mean the set of maximizing elements of R, that is,

$$F := \{x \in R : cx = \delta\}. \tag{4.8}$$

It follows that a face itself is a polyhedron. For $c \equiv 0$, the definition shows that R itself is a face. By a **proper face** of R we mean a face distinct from R. Geometrically, a proper face is a part of the polyhedron touched by a hyperplane (of normal c). When $c \neq 0$, the hyperplane $H = \{x : cx = \delta\}$ is called a **supporting hyperplane** of R.

A face consisting of a single point is called a **vertex**. That is, a point $z \in R$ is a vertex of R if there is a vector c for which $cz > cx$ for every $x \in R - z$. A **mimimal face** (with respect to inclusion) of a polyhedron is one including no proper face. A maximal proper face is called a **facet**. A polyhedron is said to be **pointed** if it has a vertex. An element $z \in R$ is called **extreme** if it is not the convex combination of other elements of R. This is equivalent to requiring that z is not an inner point of a line segment belonging to R or that there is no $x' \neq 0$ for which both $z + x'$ and $z - x'$ are in R. It is not difficult to prove that an element of a polyhedron is a vertex if and only if it is an extreme point.

In linear optimization it is quite typical for the investigated polyhedron to be pointed. This is the case, for example, if the non-negativity restriction is imposed on the variables. A **translation vector** q of a polyhedron R is one for which $z + \lambda q \in R$ for every $z \in R$ and every real λ. It is a simple exercise to show that q is a translation vector of $R = \{x \in \mathbf{R}^n : Qx \leq b\}$ if and only if $Qq = 0$. Therefore the set of translation vectors is the null-space of Q and is called the **translation subspace** of R.

Our way of defining a face has the advantage that it depends on the polyhedron itself and not on the concrete inequality system defining R. With this approach, however, it is not straightforward to prove simple expected properties like the one stating that a face of R is itself a face of R, or that there is a finite number of faces. These properties are implied immediately by the following result which provides a useful link between the geometric definition and the concrete inequality system.

Theorem 4.1.13 *A non-empty subset* F *of a polyhedron* $R = \{x : Qx \leq b\}$ *is a face of* R *if and only if there is a submatrix* Q' *consisting of some rows of* Q *for which* $F = \{x \in R : Q'x = b'\}$*, where* b' *denotes the subvector of* b *corresponding to the rows of* Q'.

Proof. Suppose first that F is a face defined by (4.8). Consider an optimal solution y' to the dual linear program $\min\{yb : yQ = c, y \geq 0\}$. Let Q' be the submatrix of Q consisting of those rows ${}_iq$ for which the corresponding component $y'(i)$ is positive. Then $cx = (y'Q)x =$

$y'(Qx) \leq y'b$ for an arbitrary $x \in R$ and it follows from the optimality criteria (Theorem 4.1.11) that $x^* \in R$ is a primal optimum (that is, x^* is in F) if and only if $y'(i) > 0$ implies ${}_iqx = b(i)$. Therefore $F = \{x \in R : Q'x = b'\}$.

Conversely, let $F = \{x \in R : Q'x = b'\}$ be a non-empty subset of Q where Q' has k rows. Let $\underline{1}$ denote an identically 1 vector of k components. Furthermore, let c denote the sum of the rows of Q' (that is, $c = \underline{1}Q'$), while $\delta := \underline{1}b'$. For $x \in R$, we have $cx = (\underline{1}Q')x = \underline{1}(Q'x) \leq \underline{1}b' = \delta$ implying that $Q'x = b'$ holds if and only if $cx = \delta$, from which the theorem follows. •

Suppose that a non-empty polyhedron is given in the form $R = \{x : Px = b_0, Qx \leq b_1\}$. An inequality $qx \leq \beta$ defined by a row of Q is said to be **essential** if its removal changes (increases) the polyhedron. The inequality is **genuine** if its replacement by equality changes (decreases) the polyhedron.

Theorem 4.1.14 *A non-empty polyhedron R has no proper face if and only if R is an affine subspace.*

Proof. If $R = \{x : Qx = b\}$ is an affine subspace, then by Theorem 4.1.13 it cannot have a proper face.

Conversely, assume that R has no proper face. Consider a description $R = \{x : Px = b_0,\ Qx \leq b_1\}$ of R in which every inequality is genuine and essential. (Such a description can be obtained from an arbitrary description of R by first removing the current non-essential inequalities one by one and then replacing each non-genuine inequality with an equality.) The theorem follows from the following claim.

Claim 4.1.15 *Q is empty.*

Proof. Suppose indirectly that there is a genuine and essential inequality $qx \leq \beta$. Let Q' denote the submatrix arising from Q by removing the row q and let b_1' denote the corresponding right-hand side. Then, on the one hand, R has an element x' for which $Px' = b_0,\ Q'x' \leq b_1'$ and $qx' > \beta$. On the other hand, R has an element x'' for which $qx'' < \beta$. Hence the line segment $x'x''$ has an element z for which $qz = \beta$ and $z \in R$, that is, $\{x : Px = b_0, Q'x \leq b_1',\ qx = \beta\}$ is a proper non-empty face of R contradicting the hypothesis. •

Theorem 4.1.16 *Any minimal face F of a polyhedron $R = \{x : Qx \leq b\}$ is an affine subspace. The translation subspaces of F and of R are the same, namely, the null-space of Q. In particular, every minimal face is of dimension $n - r(Q)$.*

Proof. Theorem 4.1.13 implies that a face of a face of a polyhedron R is itself a face of R. Therefore F is a polyhedron having no proper face and hence Theorem 4.1.14 shows that F is indeed an affine subspace. Therefore there is a submatrix (Q', b') of (Q, b) for which $F = \{x \in \mathbf{R}^n : Q'x = b'\}$. Let N' and N be the translation subspaces of F and R, respectively. $F \subseteq R$ implies that $N' \subseteq N$. Since $N' = \{x : Q'x = 0\}$ and $N = \{x : Qx = 0\}$, we have $N \subseteq N'$ from which $N = N'$. •

Corollary 4.1.17 *A polyhedron is pointed if and only if it includes no straight lines.* •

An immediate corollary is the following useful result.

Theorem 4.1.18 *For a non-empty polyhedron $R = \{x : Qx \leq b\}$ the following are equivalent.*

(A) *R is pointed.*
(B) *R includes no straight lines.*
(C) *$r(Q) = n$.* •

For a linear system $\{Qx \leq b\}$, we say that a solution z is a **basic solution** if $r(Q) = r\left(Q_z^=\right)$. The following theorem provides a nice geometric meaning of this notion incidentally implying that if two linear systems have the same solution set, then their basic solutions coincide. In particular, the notion depends only on the polyhedron R and not on the concrete description of R.

Theorem 4.1.19 *A solution z to $\{Qx \leq b\}$ is basic if and only if z belongs to a minimal face of the polyhedron $R = \{x : \ Qx \leq b\}$.*

Proof. Suppose first that z is a basic solution. We claim that $F \subseteq R$ for the affine space $F := \left\{x : Q_z^= x = b_z^=\right\}$. Indeed, let ${}_jq$ be any row of Q not in $Q_z^=$. Since $r(Q) = r\left(Q_z^=\right)$, the row ${}_jq$ is a linear combination of the rows of $Q_z^=$, that is, there is a y for which ${}_jq = yQ_z^=$. For every $x \in F$ we have ${}_jqx = \left(yQ_z^=\right)x = y\left(Q_z^= x\right) = yb_z^=$. In particular, $yb_z^= = {}_jqz \leq b(j)$. Therefore ${}_jqx \leq b(j)$ and thus $x \in R$, that is, $F \subseteq R$. It follows that F is a face of R and in fact F is a minimal face since its dimension is $n - r\left(Q_z^=\right) = n - r(Q)$.

Conversely, let $F' = \{x : Q'x = b'\}$ be a minimal face of R where Q' is a submatrix of Q and let $z \in F'$. Then $r(Q') = r(Q)$ and $Q_z^=$ includes Q'. Hence $r(Q) \geq r\left(Q_z^=\right) \geq r(Q')$ from which $r(Q) = r\left(Q_z^=\right)$, that is, z is a basic solution. •

Problem 4.1.1 *Derive the following theorem.*

Theorem 4.1.20

(A) *A solution z to the linear system $\{Px = b_0, \ Qx \leq b_1\}$ is basic if and only if $r(M) = r\left(M_z^=\right)$ where $M = \begin{pmatrix} P \\ Q \end{pmatrix}$.*
(B) *A solution z to $\{Ax = b, x \geq 0\}$ is basic if and only if the columns of A corresponding to positive components of z are linearly independent.*
(C) *A solution z to $\{Bx \leq b, \ x \geq 0\}$ is basic if and only if there is a non-singular square submatrix B' of B so that z arises from the unique solution to $B'x' = b'$ by extending it with zero components.* •

Remark 4.1.1 In the standard literature, the notion of a basic solution is introduced only for a linear system $\{Ax = b, x \geq 0\}$ by *defining* a solution z to be basic if the columns of A corresponding to positive components of z are linearly independent. The approach above in which this property arises as a statement to be proved, is more general. We also note that for pointed polyhedra the property described in Part (A) of the theorem was used by Bertsimas and Tsitsiklis [31] to define the notion of a basic solution.

If the polyhedron is not pointed, then it has an infinite number of basic solutions. For example, every solution to a system of linear equations is basic. This fact motivates the introduction of a stronger notion.

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 145 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

We call a basic solution z of the linear inequality system $Qx \leq b$ **strong** if the columns of Q corresponding to non-zero components of z are linearly independent. In particular, if the columns of Q are linearly independent (which is equivalent to assuming that the polyhedron $R = \{x : Qx \leq b\}$ is pointed), then every basic solution is strong and these solutions are the vertices of R.

If the solution sets of the linear systems $\{Qx \leq b\}$ and $\{Q'x \leq b'\}$ are the same non-empty polyhedron R, then the null-spaces of Q and Q' are the same since each of them is equal to the translation subspace of R. Therefore a subset of columns of Q is linearly independent if and only if the corresponding subset of columns of Q' is linearly independent and hence the notion of strong basic solution depends only on R. It follows that a basic solution z to a linear system $\{Px = b_0, Qx \leq b_1\}$ is strong if and only if the columns of M corresponding to the non-zero components of z are linearly independent where $M = \begin{pmatrix} P \\ Q \end{pmatrix}$.

Theorem 4.1.21 *A solution z of the system $Qx \leq b$ is a strong basic solution if and only if there is an $r(Q)$ by $r(Q)$ non-singular submatrix Q' of Q such that z arises from the unique solution to $Q'x' = b'$ by extending it with zero components* (where b' denotes the subvector of b for which the components correspond to the rows of Q').

Proof. If z is obtained in the given way, then $Q_z^=$ includes Q' from which $r\left(Q_z^=\right) = r(Q)$. Furthermore, the columns of Q corresponding to the non-zero components of z are linearly independent since these columns are extensions of the columns of the non-singular Q'. Hence z is indeed a strong basic solution.

Conversely, let z be a strong basic solution. Then $r\left(Q_z^=\right) = r(Q)$. Select $r(Q)$ linearly independent rows of $Q_z^=$ and select $r(Q)$ linearly independent colums of Q which include all columns correponding to non-zero components of z. By elementary linear algebra, the $r(Q)$-by-$r(Q)$ submatrix of Q determined by these rows and columns is non-singular, and hence z arises in the requested way. •

It follows that a linear system has a finite number of strong basic solutions. For example, the strong basic solutions to the single linear equality $\{x_1 + 2x_2 = 4\}$ are (4, 0) and (0, 1).

Theorem 4.1.22 *Let $R = \{x : Qx \leq b\}$ be a non-empty polyhedron for which $\{cx : x \in R\}$ is bounded from above. Then the maximum of $\{cx : x \in R\}$ is always attained at a strong basic solution. In particular, if R is pointed* (equivalently, straight line free)*, then the maximum is attained at a vertex of R.*

Proof. Let F be a minimal face consisting of the maximizing elements of R. Let z be an element of F having the minimum number of non-zero components. If $z = 0$, then it is a strong basic solution, so suppose that $z \neq 0$. We showed already that every element of F is a basic solution, and we claim now that z is a strong one. Suppose indirectly that the columns of Q corresponding to the non-zero components of z are linearly dependent. Then there is a vector $q \neq 0$ for which $Qq = 0$ and a component $q(j)$ can be non-zero only if $z(j) \neq 0$. Then $z' := z + \lambda q$ belongs to F for every real λ and for an appropriate choice of λ, z' has more zero components than z, a contradiction. •

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 146 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

4.1.3 Integral solutions

A point, vector, or matrix is said to be integral if each of its components or entries is an integer. We say that a polyhedron is **integral** if each of its faces contains an integral point. It suffices to require integrality only for minimal faces, so for pointed polyhedra, the vertices are required to be integral. It is **NP**-complete to decide if a polyhedron contains an integral point, but for affine subspaces there is a good characterization.

Theorem 4.1.23 (Hermite) *Let A be an integral matrix and let b be an integral vector. The system of equations $Ax = b$ has an integral solution if and only if yb is integral whenever yA is integral. If there is a vector y for which yA is integral but yb is not, then there is a non-negative one.*

Proof. If there is an integral solution x and yA is integral for some y, then $(yA)x = y(Ax) = yb$, that is, yb is integral.

Conversely, if $Ax = b$ has no solution at all, then there exists a y for which $yA = 0$ and $yb \neq 0$. But then y can be chosen in such a way that yb is not integral. Therefore $Ax = b$ can be supposed to be solvable. We can also assume that the rows of A are linearly independent since if a row depends on other rows, then it can be left out (along with the corresponding component of b). A solution to the new system is a solution to the original one while if the new system has no solution, then there is a y' by induction for which $y'A'$ is integral and $y'b'$ is not. By extending y' with a zero component, we obtain a vector y for which yA is integral and yb is not.

By negating a column or adding it to another column of A, we obtain an equivalent problem from the point of view of both x and y. By the repeated application of these operations and with possible column exchanges, we can transform A into a matrix of the form $[B, 0]$ where B is an integral lower triangular matrix (meaning that each element above the main diagonal is 0) and the entries of the main diagonal are non-zero.

Since $B^{-1}(B, 0) = (I, 0)$ is an integral matrix, the condition in the theorem implies that $B^{-1}b$ is an integral vector. Then $x^* := \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ is integral and $(B, 0)x^* = b$, and hence $Ax^* = b$.

To see the second part of the theorem, assume that $y'A$ is integral but $y'b$ is not for some y'. Let y'' be an integral vector for which $y^* := y' + y''$ is non-negative. Since $y''A$ and $y''b$ are integral, we conclude that y^*A is integral and y^*b is not. •

The following fundamental result was first obtained by Hoffman [208] for bounded polyhedra, while the general case was found by Edmonds and Giles [86].

Theorem 4.1.24 *A non-empty polyhedron* $R := \{x : Qx \leq b\}$ (where $Q,\ b$ are integral) *is integral if and only if the value of* $\max\{cx : x \in R\}$ *is an integer for every integral vector c for which* $\{cx : x \in R\}$ *is bounded from above.*

Proof. The necessity of the condition is evident so we focus on its sufficiency. Consider the face of R consisting of the points of R attaining $\max\{cx : x \in R\}$ and let R' be a minimal face of this maximizing face.

By Corollary 4.1.16, R' is an affine subspace, that is it can be given in form $\{x : Q'x = b'\}$ where Q' is a matrix consisting of some rows of Q.

We want to show that R' contains an integral point. If it does not, then Theorem 4.1.23 implies the existence of a vector $y' \geq 0$ for which $c' := y'Q'$ is integral while $y'b'$ is not. Then $c'x' = y'(Q'x') = y'b'$ for every $x' \in R'$ and $c'x = y'(Q'x) \leq y'b'$ for every $x \in R$. Hence $\max\{c'x : x \in R\}$ is attained at the elements of R' implying that the value of the maximum is $y'b'$ contradicting the hypothesis of the theorem since $y'b'$ is not integral. •

Totally dual integral systems

A typical way of ensuring that the value $\max\{cx : Qx \leq b\}$ is an integer for every integral c is to require that the dual linear program has an optimum solution which is integer-valued for every integral c for which the dual program has an optimum solution at all. In such a case the linear system $Qx \leq b$ is said to be **totally dual integral** (TDI). We emphasize that this is a property of the linear system and not of the polyhedron defined by the system. In fact, a theorem of Giles and Pulleyblank [183] states that every integral polyhedron admits a TDI-describing system. Note that Theorem 4.1.24 immediately implies the following basic result.

Corollary 4.1.25 *A non-empty polyhedron R defined by a TDI system is integral. In particular, the optimum value in the primal linear programming problem, if finite, is attained at an integral element of R for every real vector c.* •

The following useful result states that replacing an inequality by an equality in a TDI system preserves TDI-ness. This obviously implies that any number of inequalities can be replaced by equality without destroying TDI-ness.

Theorem 4.1.26 (Cook [55]) *If a linear system $\{Qx \leq b, qx \leq \beta\}$ is TDI* (where Q, q, b, β are integral) *and the polyhedron defined by*

$$\{Qx \leq b, qx = \beta\} \tag{4.9}$$

is non-empty, then the system (4.9) *is TDI.*

Proof. Let $P := \{x : Qx \leq b, qx \leq \beta\}$ and $D = \{(y, \pi) : yQ + \pi q = c, \quad (y, \pi) \geq 0\}$ denote the original primal and dual polyhedra. Let c be an integral vector for which neither the revised primal polyhedron $P' = \{x : Qx \leq b, qx = \beta\}$ nor the revised dual polyhedron $D' = \{(y, \pi) : yQ + \pi q = c, y \geq 0\}$ is empty. Clearly, $P' \subseteq P$ and $D \subseteq D'$. Now P is non-empty and $m_c \geq m'_c$ where m_c (respectively, m'_c) denotes the common optimum value in the original (respectively, revised) pair of linear programs. (In this definition it is possible that D is empty, in which case $m_c = \infty$.)

We need to show that the dual linear program

$$\min\{yb + \pi\beta : (y, \pi) \in D'\} \tag{4.10}$$

admits an optimum (y, π) which is integer-valued.

First, we verify this property for vectors c for which (4.10) has a non-negative optimal solution (y', π'). In this case $(y', \pi') \in D$ and hence $m'_c \geq m_c$. Since $m_c \geq m'_c$, we have $m_c = m'_c$, and hence any optimal element of D is an optimal element of D' as well. But the hypothesis of the theorem implies that D admits an optimum solution (y, π) which is integer-valued and hence so does D'.

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 148 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

The general case reduces to this special case as follows. Let (y', π') be an optimal solution to D'. Let ℓ be an integer for which $\lambda' := \ell + \pi' \geq 0$ and let $c_\ell := c + \ell q$. Consider the dual polyhedron $D'_\ell := \{(y, \lambda) : yQ + \lambda q = c_\ell,\ y \geq 0\}$. Clearly, $(y, \pi) \in D'$ if and only if $(y, \pi + \ell) \in D'_\ell$. Furthermore, $yb + (\pi + \ell)\beta = yb + \pi\beta + \ell\beta$ and hence $(y, \pi) \in D'$ is optimal in D' if and only if $(y, \pi + \ell)$ is optimal in D'_ℓ.

Therefore (y', λ') is an optimal element of D'_ℓ which is non-negative, and hence, by the first part of the proof, there is an optimal solution (y^*, λ^*) of D'_ℓ which is integer-valued. But then, (y^*, π^*) is an integral optimum of D' where $\pi^* := \lambda^* - \ell$. •

TDI systems are often used in situations where the constraint matrix has exponentially many rows in comparison with the number n of columns. Apparently this is a real problem from an algorithmic point of view since, without devoting any further care, it is not even possible to write down the optimal dual solution in polynomial time. When only a fractional optimal solution is required, Theorem 4.1.20 ensures the existence of an optimal dual solution in which at most n components are non-zero. The following theorem shows that the situation, though not exactly the same, is similarly favourable for dually optimal integral solutions.

Theorem 4.1.27 (Cook, Fonlupt, and Schrijver [57]) *Suppose that the polyhedron R defined by the TDI linear system $\{x : Qx \leq b\}$ is non-empty and of full dimension. Then the dual linear program* $\min\{by : yQ = c,\ y \geq 0\}$ *has an integral optimal solution y in which the number of non-zero components is at most $2n - 1$ for every integral c for which $\{by : yQ = c,\ y \geq 0\}$ is bounded from below.* •

4.2 Total unimodularity

Totally unimodular matrices form an important class of constraint matrices which define integral polyhedra for every integral right-hand side. They have plenty of applications in network optimization.

4.2.1 Basic properties and examples

A matrix Q is said to be **totally unimodular** (TU for short) if each of its subdeterminants is of value $(0, \pm 1)$. In particular, each entry of Q is $0, +1$, or -1. Clearly, the transpose of a TU-matrix is also TU. By multiplying a row or a column by -1, we again get a TU-matrix. A submatrix of a TU-matrix is also TU.

When a TU-matrix Q is extended by adding a unit vector (having all but one element zero) as a row or a column, then the resulting matrix is TU. Hence if Q is extended by the identity matrix I, then the resulting (Q, I) is TU. Also, if Q is TU, then so is $(Q, -Q)$. Note, however, that adding a column of all 1's can destroy total unimodularity: let Q be the 4×3 incidence matrix of a tree with node-set $\{1, 2, 3, 4\}$ and edge-set $\{12, 13, 14\}$.

Theorem 4.2.1 *Let Q be an $m \times n$ TU-matrix and q_1 a column of Q for which $q_1(1) \neq 0$. Let $e_i \in \mathbf{R}^m$ be the i'th unit vector $(i = 1, \ldots, m)$. Let Q_1 denote the matrix arising from Q by expressing the columns of Q in the basis formed by the vectors $q_1, e_2, \ldots, e_m$. Then Q_1 is totally unimodular.*

Proof. We can assume that q_1 is the first column of Q. By negating rows, if necessary, we can assume that $q_1(1) = 1$ and $q_1(j)$ is 0 or -1 for $j \geq 2$. Then Q_1 is obtained from Q in such a way that the first row of Q is added to each row of Q for which the first entry is -1.

If the theorem is indirectly not true, then Q_1 has a square submatrix B for which $|\det B| \geq 2$. At this basis exchange the value of a subdeterminant does not change if it contains elements from the first row, so suppose that B does not. Since the first column of Q_1 is zero apart from its first component, we can assume that B contains no elements from the first column. Finally, we can assume that B, apart from the first row and first column, includes each other row and column of Q_1 since a row or column not included by B could be left out from Q and from Q_1. But now $\det B = \det Q_1 = \det Q \in \{0, \pm 1\}$, a contradiction. •

Corollary 4.2.2 *If Q is a rank-m TU-matrix of size $m \times n$ and B is a non-singular $m \times m$ submatrix of Q, then $B^{-1}Q$ is TU. In particular, the inverse of a non-singular* (square) *TU-matrix is also TU.* •

In the incidence matrix of a digraph $D = (V, A)$, the rows correspond to the nodes, the columns to the edges of D, and an entry $q_{v,e}$ is $+1$ or -1 according to whether the edge e enters or leaves v, respectively, and $q_{v,e} = 0$ otherwise. In the incidence matrix of an undirected graph an entry $q_{v,e}$ is 1 or 0 depending on whether e is incident to v or not.

Theorem 4.2.3 (A) *The incidence matrix of a digraph D is totally unimodular.* (B) *The incidence matrix of a bipartite graph G is totally unimodular.*

Proof. (A) Let Q be the incidence matrix of D and let Q' be a square submatrix of Q. If Q' has a column in which at most one element is non-zero, then by expanding Q' along this column, by induction we are done. Hence we can assume that each column contains exactly two non-zero elements, one of which is $+1$, the other of which is -1. Hence the sum of the rows is the 0 vector, and so the determinant is 0.

(B) Negate each row of the incidence matrix of G corresponding to an element of one of the two colour classes of G. In this way we obtain the incidence matrix of a digraph which is TU by the first part of the theorem. •

We say that a hypergraph H is **totally unimodular** if its incidence matrix is TU. Graphs are special hypergraphs and we saw that bipartite graphs are TU. On the other hand no other graphs can be TU since the determinant of the incidence matrix of an odd circuit is ± 2.

Problem 4.2.1 *Prove that columns of the incidence matrix of a digraph D are linearly independent if and only if D is a directed forest.*

Network matrices

We have seen that the incidence matrix of a digraph is TU. This can be generalized to what is called network matrices. Let D be a weakly connected digraph and let T be a spanning directed tree. A **network matrix** H_T associated with D and T is defined as follows. The rows of H_T correspond to the edges of T, while the columns of H_T correspond to the edges not in T, called non-tree edges. For every non-tree edge uv of D, there is a unique (but not necessarily directed) path P in the tree T from v to u. For an element f of P, we define an entry $a_{f,e}$ of the matrix H_T to be 1 if the direction of f coincides with that of P and -1 if they are opposite. Every other entry of the matrix is 0.

Lemma 4.2.4 *A submatrix of a network matrix Q is also a network matrix. Negating a row or a column of Q also results in a network matrix.*

Proof. The removal of a column from Q corresponds to the removal of the corresponding non-tree edge from the digraph. The removal of a row from Q corresponds to the contraction of the corresponding tree edge. Negating a row or a column of Q corresponds to the reorientation of the corresponding edge (whether a tree edge or a non-tree edge). •

The following important result appeared implicitly in a paper of Tutte [371].

Theorem 4.2.5 (Tutte) *A network matrix $Q = H_T$ is totally unimodular.*

Proof. By Lemma 4.2.4, it suffices to prove that the determinant of a square network matrix is 0, 1, or -1. Consider a leaf-node v of the tree T. Let α denote the number of non-zero elements in the row of Q corresponding to the single edge of T incident to v. If $\alpha \leq 1$, then by the rule of determinant expansion we are done by induction.

Suppose that $\alpha > 1$, which means that v is incident to at least two non-tree edges. By reorienting edges, if necessary, we can assume that exactly one of these α edges is oriented towards v. Let sv be this unique non-tree edge. Let vt be another non-tree edge. If the column of Q corresponding to sv is added to the column of vt, then the determinant does not change and the resulting matrix is again a network matrix which is associated with a digraph arising from D by replacing vt with a new edge st. With such a transformation, the value of α decreases and hence we are done by induction. •

A special case of Theorem 4.2.5 occurs when the fundamental circuit of each non-tree edge with respect to the given tree is a directed circuit. In this case the network matrix is non-negative and the theorem reads as follows.

Corollary 4.2.6 *Let H be a hypergraph defined on the edge-set of a directed tree T such that each hyperedge is a directed subpath of T. Then the transpose of the incidence matrix of H is a network matrix and* (hence) *H is a TU hypergraph. In particular, the subpaths of a path form a totally unimodular hypergraph.* •

The **extended incidence matrix** of a bipartite graph is obtained from its incidence matrix by adjoining a new row in which every element is 1.

Theorem 4.2.7 *The extended incidence matrix of a bipartite graph $(S, T; E)$ is a network matrix and* (hence) *totally unimodular.*

Proof. We can assume that G is simple. Let F be a directed tree on node-set $S \cup T \cup \{s, t\}$ in which there is an edge st, there are edges us for all $u \in S$, and there are edges tv for all $v \in T$. For each edge uv of G with $u \in S$ and $v \in T$, let vu be a directed non-tree edge. Then the corresponding network matrix is identical to the extended incidence matrix of G and hence it is TU. •

Problem 4.2.2 *Show that the following matrix is TU but that it is not a network matrix, nor is its transpose:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

4.2.2 Laminar and cross-free hypergraphs

Recall the definition of a laminar hypergraph and its characterization in Theorem 1.4.1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two laminar hypergraphs on a ground-set S. Let A_i $(i = 1, 2)$ denote the transpose of the incidence matrix of $\mathcal{F}_i$. In A_i, the columns correpond to the elements of S, while the rows correspond to the members of $\mathcal{F}_i$. Let $M := \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$.

Theorem 4.2.8 *M is a network matrix and* (hence) *the hypergraph $H = (S, \mathcal{F}_1 \cup \mathcal{F}_2)$ is totally unimodular.*

Proof. Let $F_i = (V_i, E_i)$ be the arborescence and φ_i the mapping ensured by Theorem 1.4.1 for representing $\mathcal{F}_i$ $(i = 1, 2)$. Let r_i be the root of F_i. Assume that the two arborescences are disjoint. Shrink r_1 and r_2 into a new single node r and reverse the edges of F_2. In this way we obtain a directed tree F in which the unique path $P(v)$ between $\varphi_2(v)$ and $\varphi_1(v)$ is a directed path for each $v \in S$.

It follows from the construction that a hyperedge Z of H contains $v \in S$ if and only if Z corresponds to an edge of $P(v)$. Therefore Corollary 4.2.6 implies the theorem. •

Let $\mathcal{F}$ be a laminar family of bi-sets and let $D = (V, A)$ be a digraph. A directed edge e **covers** (or **enters**) a bi-set X if e enters both members of X. Associate a (0, 1)-matrix $A_{\mathcal{F}}$ with $\mathcal{F}$ in such a way that the rows and columns correspond to the members of $\mathcal{F}$ and to the edges of D, respectively, and that an entry $a_{X,e}$ corresponding to a bi-set X and an edge e is 1 if e covers X, and 0 otherwise.

Theorem 4.2.9 *The matrix $A_{\mathcal{F}}$ associated with a laminar family $\mathcal{F}$ of bi-sets* (in particular, with a laminar family of sets) *is a network matrix and* (hence) *$A_{\mathcal{F}}$ is totally unimodular.*

Proof. The inner members of the bi-sets in $\mathcal{F}$ form a laminar family of sets. By Theorem 1.4.1, this family can be represented with the help of an arborescence F and a mapping φ. For an edge e of D, consider the family $\mathcal{F}_e$ consisting of those members of $\mathcal{F}$ which are covered by e. The edges of F corresponding to the inner members of the elements of $\mathcal{F}_e$ form a directed path. Therefore Corollary 4.2.6 implies the theorem. •

Let $D = (V, A)$ be a directed graph and let $\mathcal{F}$ be a cross-free family of subsets of V. Let $B_{\mathcal{F}}$ be a (0, ±1)-valued matrix the rows of which correspond to the members of $\mathcal{F}$ while its columns correspond to the edges of D. An entry corresponding to edge e and subset Z is 1 if e enters Z, −1 if e leaves Z and 0 otherwise.

Theorem 4.2.10 *The matrix $B_{\mathcal{F}}$ is a network matrix and* (hence) *it is totally unimodular.*

Proof. The result directly follows from Theorem 1.4.2. •

4.2.3 Feasibility, optimality, and duality for TU-matrices

Lemma 4.2.11 *Every strong basic solution to a linear inequality system defined by a TU-matrix M and an integral bounding vector b is integral.*

Proof. Let $M = \binom{P}{Q}$ and consider the following linear system.

$$\begin{cases} Px = b_0 \\ Qx \leq b_1. \end{cases} \tag{4.11}$$

Theorem 4.1.21 stated that every strong basic solution can be obtained by extending the unique solution of $M'x' = b'$ with zero components where M' is an appropriate $r(M) \times r(M)$ non-singular submatrix of M and b' denotes the subvector of b correponding to the rows of M'.

Since M is a TU-matrix, the determinant of the non-singular submatrix M' is $+1$ or -1. By Cramer's rule, as b' is supposed to be integral, the unique solution x' is also integral. •

Lemma 4.2.12 *Let c be an arbitrary* (not necessarily integral) *vector. If there is an element x' of a polyhedral cone K defined by a TU-matrix M for which $cx' > 0$, then K also has such an element with* $(0, \pm 1)$ *components.*

Proof. Let $M = \binom{P}{Q}$ and let $K := \{x : Px = 0,\ Qx \leq 0\}$. Since a positive multiple of x' is also in K, we can assume that each component of x' belongs to the closed interval $[-1, +1]$. That is, the bounded polyhedron determined by the system

$$(-1, \ldots, -1) \leq x \leq (1, \ldots, 1),\ Px = 0,\ Qx \leq 0 \tag{4.12}$$

has an element x' for which $cx' > 0$. By Theorem 4.1.21 there is a strong basic solution x^* to (4.12) for which $cx^* \geq cx'$. By Lemma 4.2.11, x^* is integral, and hence each of its components is $(0, \pm 1)$-valued. •

Theorem 4.2.13 (Farkas lemma for TU-matrices) *Suppose that a matrix $M = \binom{P}{Q}$ is totally unimodular. If the primal system* (4.11) *is feasible and b is integral, then* (4.11) *has an integral solution. If the dual system*

$$\begin{cases} y_1 \geq 0 \\ yM = 0 \\ yb < 0 \end{cases} \tag{4.13}$$

is feasible, where $y = (y_0, y_1)$, then there is $(0, \pm 1)$*-valued solution y as well* (regardless of the integrality of b).

Proof. The first half of the theorem follows from Lemma 4.2.11 and from the earlier result that a feasible system always has a strong basic solution. The second half is a direct consequence of Lemma 4.2.12. •

The theorems below can be derived similarly from their respective continuous counterpart. In each theorem, the constraint matrix $M = \begin{pmatrix} P \\ Q \end{pmatrix}$ is totally unimodular and b is an integral vector.

Theorem 4.2.14 *If the linear programming problem* $\max\{cx : Px = b_0,\ Qx \le b_1\}$ *has a solution, then the optimum is attained at an integral point* (regardless of the integrality of c). *Equivalently, a polyhedron defined by a totally unimodular constraint matrix and an integral bounding vector is integral.*

Proof. By Theorem 4.1.22, the optimum is taken at a strong basic solution and by Lemma 4.2.11 every strong basic solution is integral whenever M is TU and b is integral. •

Theorem 4.2.15 (directional boundedness for TU-matrices) *Suppose that the polyhedron* $R = \{x : Px = b_0,\ Qx \le b_1\}$ *is non-empty. The following are equivalent.*

(A) $\{cx : x \in R\}$ *is bounded from above.*

(B) *There is no* $(0, \pm 1)$*-valued vector* x' *for which* $Px' = 0,\ Qx' \le 0$*, and* $cx' > 0$.

(C) *There is a vector* $y = (y_0, y_1)$ *for which* $y_1 \ge 0$ *and* $yM = c$*. Moreover,* y *can be selected to be integral whenever* c *is integral.*

Proof. The implications (A)→(B) and (C)→(A) follow from Theorem 4.1.8. Furthermore Properties (B) and (C) are equivalent by Theorem 4.2.13. •

Theorem 4.2.16 (optimality criteria for TU-matrices) *Let* x^* *be an element of* $R := \{x : Px = b_0, Qx \le b_1)\}$*. Let* $Q^{=}_{x^*}$ *denote an* x^**-active submatrix of* Q*. The following are equivalent.*

(A) $cx^* \ge cx$ *for every* $x \in R$.

(B) *There is no* $(0, \pm 1)$*-valued vector* x' *for which* $Px' = 0,\ Q^{=}_{x^*}x' \le 0$*, and* $cx' > 0$.

(C) *There is a vector* $y = (y_0, y_1)$ *for which* $y_1 \ge 0$*,* $yM = c$*, amd* $y(b - Mx^*) = 0$*, and* y *can be selected to be integral whenever* c *is integral.*

Proof. The implications (A)→(B) and (C)→(A) follow from Theorem 4.1.10. Furthermore, the equivalence of Properties (B) and (C) can be established by using Theorem 4.2.13. •

TU-matrices and total dual integrality

Since the transpose of a TU matrix is also totally unimodular, Theorem 4.2.14 implies the following result.

Theorem 4.2.17 *A linear system defined by a TU-matrix and an integral bounding vector is totally dual integral.* •

In several applications, however, the entire constraint matrix is not TU and yet the TDI-ness of the entire system can be derived from the total unimodularity of some submatrices.

Lemma 4.2.18 *Suppose that a feasible linear system* $\{Px = b_0,\ Qx \le b_1\}$ *has the property that for an integral vector c for which cx is bounded from above, the dual linear program*

$$\min\{b_0 y_0 + b_1 y_1 :\ y_1 \ge 0,\ y_0 P + y_1 Q = c\} \tag{4.14}$$

has an optimal solution y^* *for which* $M' = \begin{pmatrix} P' \\ Q' \end{pmatrix}$ *is totally unimodular where* M' *is a matrix consisting of the rows of* P *and* Q *corresponding to non-zero components of* y^**. Then the system* $\{Px = b_0,\ Qx \le b\}$ *is TDI.*

Proof. The total unimodularity of M' implies that the linear program

$$\min\left\{b_0' y_0' + b_1' y_1' :\ y_1' \ge 0,\ y_0' P' + y_1' Q' = c\right\}$$

has an integral optimum. By extending this vector with 0 components, one obtains an integral optimum of (4.14). •

4.2.4 Equitable colourings

In Section 2.4 we proved that every bipartite graph G has an equitable k-edge-colouring. We are going to show that this is, in fact, a characteristic feature of TU-matrices. We say that a k-partition (k-colouring) $\{A_1, A_2, \dots, A_k\}$ of the columns of an integral matrix A is **equitable** if, for each row a of A, the sums of the components of a in each colour class are essentially the same value: each of these sums is $\lfloor \underline{1}a/k \rfloor$ or $\lceil \underline{1}a/k \rceil$, where $\underline{1}a$ is the modulus of a. In preparation, we prove that each polyhedron defined by a TU-matrix admits what is called the integer decomposition property, formulated in the next theorem.

Theorem 4.2.19 (Baum and Trotter [14]) *Let A be a TU-matrix, b an integral vector, and k a positive integer. Let z be an integral element of the polyhedron* $R_k = \{x : Ax \le kb\}$*. Then z can be expressed as the sum of some integral vectors* $z_1, z_2, \dots, z_k$ *of* $R_1 = \{x : Ax \le b\}$*.*

Proof. We use induction on k. We claim that there is an integral vector z_1 in R_1 for which $A(z - z_1) \le (k-1)b$, that is, the polyhedron $\{x : Az - (k-1)b \le Ax \le b\}$ has an integral point. The polyhedron is non-empty since z/k certainly belongs to it. Moreover the constraint matrix is TU, so the requested z_1 indeed exists.

Since $z' := z - z_1$ belongs to R_{k-1}, it can be expressed, by induction, as the sum of $k-1$ integral elements $z_2, \dots, z_k$ of R_1. Therefore $z = z_1 + \dots + z_k$, as required. •

The result easily extends to the case when the non-negativity of z is also required and not only an upper bound restriction is imposed on Ax but a lower bound as well. Indeed, since A is a TU-matrix so is $(A, -A, I)$. Hence one has the following.

Corollary 4.2.20 *If* $z \ge 0$ *is an integral vector for which* $kb_1 \le Az \le kb_2$*, then z is the sum of some integral vectors* $z_1, z_2, \dots, z_k$ *for which* $z_i \ge 0$ *and* $b_1 \le Az_i \le b_2$*.* •

A polyhedron $R_1 = \{x : Ax \le b\}$ is said to admit the **integer decomposition property** if every integral element z of $R_k = \{x : Ax \le kb\}$ and for every positive integer k, z can be expressed as the sum of integral vectors $z_1, z_2, \dots, z_k$ of R_1. There are important classes of polyhedra with this property which cannot be described by a TU matrix. For example, the

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 155 of 640 December 11, 2010 15:16

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

base polyhedron of a matroid can be proved to have the integer decomposition property: see Corollary 13.3.17.

Theorem 4.2.21 *The columns of a TU-matrix A have an equitable k-colouring.*

Proof. Let d denote the sum of the columns of A and let $b_1 := \lfloor d/k \rfloor$ and $b_2 := \lceil d/k \rceil$. Then $z :\equiv \underline{1}$ is in $\{x : \ kb_1 \leq Ax \leq kb_2, \ x \geq 0\}$. By Corollary 4.2.20, z is the sum of integral vectors $z_1, z_2, \ldots, z_k$ for which $z_i \geq 0$ and $b_1 \leq Az_i \leq b_2$. It follows that each z_i is a $(0, 1)$-vector and that the z_i's define a partition of the columns of A. By the properties of z_i's ensured by Corollary 4.2.20, this partition is an equitable k-colouring of the columns of A. •

The following result shows that equitable 2-colourability in a sense characterizes TU-matrices.

Theorem 4.2.22 (Ghouila-Houri [182]) *A $(0, \pm 1)$-valued matrix Q is totally unimodular if and only if any subset of its columns has an equitable 2-colouring.*

Proof. The equitable 2-colourability of a TU-matrix was proved above so we concentrate on the reverse implication. Suppose for a contradiction that Q is a counterexample of minimal size. Then Q is not totally unimodular but every proper submatrix of it is TU. Therefore Q is a square matrix for which $K := \det Q \notin \{0, \pm 1\}$. Hence Q has more than one entry and every proper subdeterminant of it is of value $\{0, \pm 1\}$. Consider the inverse matrix Q^{-1} of Q. By Cramer's rule

$$\text{every non-zero entry of } Q^{-1} \text{ is of the form } \pm 1/K. \tag{4.15}$$

Let q_1^* denote the first column of Q^{-1} and let R denote the set of those subscripts j for which $q_1^*(j) \neq 0$. As usual, let ${}_iq$ denote the i'th row of Q and let $e_1 = (1, 0, \ldots, 0)$ be the first unit vector. Since ${}_iqq_1^* = 0$ for every $i \geq 2$, (4.15) implies that ${}_iq$ has an even number of non-zero entries q_{ij} for which $j \in R$.

The columns of Q belonging to R admits an equitable 2-colouring, that is, there is a $\{0, \pm 1\}$-vector z for which Qz is $\{0, \pm 1\}$-valued and $z(j)$ is non-zero if and only if $j \in R$. By the observation on the parity above, ${}_iqz = 0$ holds for every $i \geq 2$. But then ${}_1qz \neq 0$, for otherwise we would have $Qz = 0$ in contradiction with the assumption $|\det Q| \geq 2$. By negating z if necessary, we can assume that ${}_1qz = 1$, that is, $Qz = e_1$ and hence $z = q_1^*$, contradicting (4.15). •

An application to dipaths of a directed tree

Corollary 4.2.23 *Let F be a directed tree and $\mathcal{P} := \{P_1, \ldots, P_t\}$ a system of directed subpaths of F. Then the members of $\mathcal{P}$ can be coloured by k colours* (for every positive integer k) *in such a way that every edge e of F belongs to essentially the same number of paths from each colour class in the sense that the difference is at most* 1 *for any two colour classes.* •

Corollary 4.2.24 *Let F and $\mathcal{P}$ be the same as before. Then the edges of F can be coloured by k colours* (for every positive integer k) *in such a way that each path in $\mathcal{P}$ contains essentially the same number of colours.* •

Problem 4.2.3 $(*)$ *Prove directly Corollary* 4.2.24 *by devising a simple greedy algorithm.*

4.2.5 A characterization of TU-matrices

We have proved that a polyhedron described by a TU-matrix is integral whenever the bounding vector is integral. Since integral polyhedra play a central role in combinatorial optimization problems, it is natural to raise the question whether there are classes of matrices with the same integrality property that are more general than TU-matrices. The next theorem shows that in a sense the answer is no.

Theorem 4.2.25 (Hoffman and Kruskal) *An integral matrix Q is totally unimodular if and only if the polyhedron*

$$R_b := \{x : x \geq 0, \ Qx \leq b\} \text{ is integral for every integral vector } b. \tag{4.16}$$

Proof. Let b be a vector for which R_b is non-empty. We have seen the 'only if' part already in Theorem 4.2.14. To see the converse, let Q be a matrix satisfying (4.16). Suppose indirectly that Q has a square submatrix Q' with $|\det Q'| \geq 2$. Assume that Q' is minimal in the sense that every proper subdeterminant of Q' is $0, \pm 1$.

Proposition 4.2.26 *For every integral b', the unique solution $x_{b'}$ to $Q'x' = b'$ is integral.*

Proof. First, we prove the proposition only for such vectors b' for which $x_{b'} \geq 0$. Let x^* denote the vector obtained from $x_{b'}$ by extending it with zero components. We can also extend b' with appropriately large integral components so as to obtain a vector b for which x^* satisfies the linear system $\{Qx \leq b, \ x \geq 0\}$, that is, x^* is in R_b. The construction implies that x^* is actually a vertex of R_b. By (4.16), x^* is integral and hence $x_{b'}$ is also integral.

For a general b', let z' be an integral vector for which $x_{b'} + z' \geq 0$. Then we see for $b'' := b' + Q'z' = Q'(x_{b'} + z')$ that the unique solution $x_{b'} + z'$ of $Q'x' = b''$ is non-negative and hence it is integral by the first part of the proof. But then the integrality of z' implies that $x_{b'}$ is also integral. •

Since Q' is non-singular, if we expand it along the first row at least one expansion term is not zero. We can assume that it belongs to $q_{1,1}$. Let $b' = (1, 0, \ldots 0)$ be a unit vector. By Cramer's rule, the first component of $x_{b'}$ is obtained as the ratio of a subdeterminant of Q' and $\det Q'$. Since the subdeterminant is ± 1 while $|\det Q'| \geq 2$, this ratio is not an integer and hence $x_{b'}$ is not integral, contradicting the proposition. • •

Remark 4.2.1 One can rightly think that the theorem of Hoffman and Kruskal is intuitive. Its true value is perhaps even better understood if we consider the following, apparently no less natural statement: *An integral square matrix Q is totally unimodular if and only if the unique solution to $Qx = b$ is integral for every integral vector b.* This statement is, however, false as demonstrated by the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

4.3 Applications of totally unimodular matrices

What can be the driving force behind the several elegant min-max and feasibility theorems concerning bipartite matchings, paths, flows, and circulations? Also, how could such results be figured out? In this section, by following the approach of Hoffman and Kruskal [209], we show that many of the network optimization problems in question can be considered as linear programming problems (feasibility or optimality) with a totally unimodular constraint matrix. It turns out that these results can be viewed as a linear programming theorem (such as the Farkas lemma, theorems on directional boundedness, optimality criteria, and duality) formulated for TU-matrices. The benefit of this revelation is not only that new proofs can be provided for existing theorems, but also an efficient tool is obtained in this way for formulating and proving more general results as well.

Linear programs constrained by network matrices

In each of the following applications, the constraint matrix is a network matrix or its transpose. We start our discussion by pointing out that any linear programming problem with such a constraint matrix is equivalent to a circulation problem or to its dual.

Proposition 4.3.1 *Let $D = (V, A)$ be a weakly connected digraph, let T be the edge-set of a spanning tree of D, and let $N := A - T$ be the set of non-tree edges. Let B denote the network matrix belonging to T.*

(A) *A vector $x = (x_N, x_T) \in \mathbf{R}^A$ is a circulation if and only if $x_T = Bx_N$.*
(B) *A vector $y = (y_N, y_T) \in \mathbf{R}^A$ is a tension if and only if $y_N = -y_T B$.*

Proof. (A) Let I denote the unit matrix of $|V| - 1$ columns. Then $x_T = Bx_N$ can be rewritten as $(B, -I)x = 0$. Let $e = uv$ be a directed or undirected edge of T and let Z_e denote the set of nodes of the component of $T - e$ containing v. If b_e denotes the row of $(B, -I)$ corresponding to e, then $b_e x = \varrho_x(Z_e) - \delta_x(Z_e)$. Therefore $(B, -I)x = 0$ is equivalent to requiring that $\varrho_x(Z_e) - \delta_x(Z_e) = 0$ for every edge e of T, that is, every fundamental cut belonging to T is balanced. By Proposition 3.4.2, this latter property holds if and only if x is a circulation.

The proof of the second part is left to the reader. •

Problem 4.3.1 *Derive Part* (B) *of Proposition* 4.3.1.

Let s be an element of a set S. By the **projection** of a vector $v \in \mathbf{R}^S$ along s, we mean a vector of $\mathbf{R}^{S-s}$ arising from v by removing its component corresponding to s. The **projection of a subset** Q **of** $\mathbf{R}^S$ **along** s arises by projecting each element of Q. More generally, for a subset $Z \subset S$, the **projection** of a vector $v \in \mathbf{R}^S$ along Z (to $\mathbf{R}^{S-Z}$) we mean a vector of $\mathbf{R}^{S-Z}$ arising from v by projecting the elements of Z.

On the one hand, projection is a friendly operation from the optimization point of view, in the following sense. If we can efficiently optimize any linear function cx over the elements of a given polyhedron Q, then we can efficiently optimize over a projection of Q, too. On the other hand, projection is not so friendly if we want to figure out an explicit polyhedral description of the projection of Q. It may be the case that Q is the solution set of n linear inequalities, while the number of inequalitites in the polyhedral description of a projection

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 158 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

of Q is exponential in n. This is the situation, for example, when Q is the polyhedron of feasible circulations.

Problem 4.3.2 *Let $D = (V, A)$ be a digraph, f an g two functions on A with $f \leq g$, and $F \subset A$ a subset of edges. Based on Hoffman's feasibility theorem for circulations, develop a necessary and sufficient condition for a vector $z \in R^F$ to be the projection* (along A–F) *of a feasible circulation.*

In the special case, however, when we project the circulation polyhedron along the elements of a spanning tree T, there is a simple description of the projection using the network matrix.

Theorem 4.3.2 *Let $D = (V, A)$ be a weakly connected digraph, let T be a spanning tree of D, and let $N := A - T$ be the set of non-tree edges. Let Q denote the incidence matrix of D and let B denote the network matrix belonging to T. Let $f = (f_T, f_N)$ and $g = (g_T, g_N)$ be given bounds for which $f \leq g$. Then the polyhedron $R \subseteq \mathbf{R}^N$ defined by*

$$R = \{x_N : f_T \leq Bx_N \leq g_T \text{ and } f_N \leq x_N \leq g_N\} \tag{4.17}$$

is the projection of the polyhedron C of feasible circulations in $\mathbf{R}^A$ (along T into $\mathbf{R}^N$) *where*

$$C = \{x = (x_N, x_T) : \ Qx = 0, \ f \leq x \leq g\} .$$

Proof. By Proposition 4.3.1, $x = (x_T, x_N)$ is a circulation if and only if $x_T = Bx_N$. Therefore for such an x, $f \leq x \leq g$ is equivalent to the constraints in (4.17). Hence the projection x_N of $x \in C$ is in R, and conversely every element x_N of R is the projection of $x = (x_N, x_T) \in C$ where $x_T = Bx_N$. •

Corollary 4.3.3 *Every linear programming problem that is constrained by a network matrix can be reformulated as an optimal circulation problem. Every linear programming problem that is constrained by a transposed network matrix can be reformulated as an optimal tension problem.* •

It follows that the algorithms developed for flows and circulations can be applied to solve linear programming problems constrained by network matrices. For example, in this way one can construct an algorithm for finding a strongly equitable k-edge-colouring of a bipartite graph and for finding a k-colouring of a given system of directed subpaths of a directed tree. Similarly, the rounding results below can also be handled with the help of feasible circulations.

4.3.1 Rounding numbers, vectors, and matrices

We say that an integer z is a **rounding** of a number x if $|x - z| < 1$. For example, 1.001 has two roundings: 1 and 2. A vector z is a rounding of vector x if each of its components is a rounding of the corresponding component of x. The rounding of a matrix or a function is defined analogously. As usual, the lower integer part $\lfloor x \rfloor$ is the largest integer not larger than x and upper integer part $\lceil x \rceil$ is the smallest integer not smaller than x. For an integer x, clearly $\lfloor x \rfloor = \lceil x \rceil = x$. When x is a vector, $\lfloor x \rfloor$ denotes the vector obtained from x by taking the lower integer part componentwise, and $\lceil x \rceil$ is defined analogously.

Lemma 4.3.4 *Let A be a totally unimodular matrix and x_0 a vector of appropriate dimension. Then there is an integral vector q for which $\lfloor x_0 \rfloor \le q \le \lceil x_0 \rceil$ and $\lfloor Ax_0 \rfloor \le Aq \le \lceil Ax_0 \rceil$. In other words, there is a rounding q of x_0 such that aq is a rounding of ax_0 for every row a of A.*

Proof. By the assumption the linear system $\{\lfloor x_0 \rfloor \le z \le \lceil x_0 \rceil,\ \lfloor Ax_0 \rfloor \le Az \le \lceil Ax_0 \rceil\}$ is feasible and hence it has an integral solution by Theorem 4.2.13. •

Corollary 4.3.5 *If $(S, \mathcal{F})$ is a totally unimodular hypergraph, then every function $x_0 : S \to$ $\mathbf{R}$ has a rounding q such that the value $\sum[q(v) : v \in A]$ is a rounding of $\sum[x_0(v) : v \in A]$ for every hyperedge $A \in \mathcal{F}$.* •

Theorem 4.3.6 *Every $m \times n$ matrix B has a rounding such that each of the following quantities changes less than 1: every row sum, every column sum, the sum of the elements in the first j rows $(j = 1, 2, \dots, m)$, and the sum of the elements in the first i columns $(i = 1, 2, \dots, n)$.*

Proof. Let S be the set of positions of matrix B (and hence $|S| = mn$). Define a laminar hypergraph $H_1 = (S, \mathcal{F}_1)$ such that for each row the set of its positions forms a hyperedge and the union of the positions in the first i rows forms a hyperedge for each i $(2 \le i \le m)$. $H_2 = (S, \mathcal{F}_2)$ is defined in a similar way, with columns in place of rows. Then $\mathcal{F}_i$ is laminar and hence the result follows by Theorem 4.2.8 and Lemma 4.3.4. •

Theorem 4.3.7 *There are roundings $z_1, \dots, z_n$ of the members of a sequence $x_1, \dots, x_n$ such that the sum $z_i + \dots + z_j$ is a rounding of $x_i + \dots + x_j$ for each pair of indices i, j with $1 \le i \le j \le n$.*

Proof. Let $\{v_1, \dots, v_n\}$ be the node-set of a hypergraph for which the hyperedges are subsets of type $\{v_i, \dots, v_j\}$ for every $1 \le i \le j \le n$. This hypergraph is isomorphic to the hypergraph of subpaths of an underlying path which was shown to be totally unimodular. Hence Lemma 4.3.4 applies. •

4.3.2 Bipartite graphs and linear programming

In the following discussion, a vector $(1, 1, \dots, 1)$ is denoted by $\underline{1}$. We do not specify the dimension and assume that in a system like $Ax \le \underline{1}$ the vector $\underline{1}$ is of appropriate dimension. For a vector $x \in \mathbf{R}^S$ and for a subset $Z \subseteq S$, the scalar product $\underline{\chi}_Z \cdot x = \underline{1}x$ will be denoted by $\widetilde{x}(Z)$ where $\underline{\chi}_Z$ denotes the characteristic function of Z.

Optimal subgraphs

As a first application of TU matrices, we derive Kőnig's theorem (Theorem 2.4.1).

Theorem 4.3.8 (Kőnig) *In a bipartite graph $G = (S, T; E)$ the maximum cardinality $\nu = \nu(G)$ of a matching is equal to the minimum number $\tau = \tau(G)$ of nodes covering all the edges.*

Proof. Let A denote the incidence matrix of G and let $V := S \cup T$ be its node-set. Consider the following pair of primal and dual linear programs.

$$\max\{\widetilde{x}(E) : Ax \leq \underline{1},\ x \geq 0\}, \tag{4.18}$$

$$\min\{\widetilde{y}(V) : yA \geq \underline{1},\ y \geq 0\}. \tag{4.19}$$

Note that the dimension of $\underline{1}$ is $|V|$ in (4.18) and $|E|$ in (4.19). By Theorem 4.2.14, both programs have optimal solutions that are integral. Let us denote them by x_0 and y_0, respectively. Because of the bounding vector $\underline{1}$, every integral solution to (4.18) is (0, 1)-valued, and it also follows immediately that every optimal integral solution to (4.19) is also (0, 1)-valued. Let M denote the set of edges e of G for which $x_0(e) = 1$, and let L denote the set of nodes v for which $y_0(v) = 1$. The constraint $Ax \leq \underline{1}$ implies that M is a matching of G, while $yA \geq \underline{1}$ implies that L covers all the edges of G. By the duality theorem, the primal and dual optimum values are equal, and thus $|M| = |L|$. Hence $\nu(G) \geq |M| = |L| \geq \tau(G) \geq \nu(G)$ from which $\nu(G) = \tau(G)$ follows. •

In light of this proof, we can say that Kőnig's theorem is nothing but the integral version of the linear programming duality theorem for TU-matrices when the constraint matrix is the incidence matrix of G and both the bounding vector and the objective function are identically 1 vectors (of appropriate dimension). Even more, in the primal program, we can use an arbitrary c in place of the identically 1 cost function. Then the same approach gives rise to the following variation of Egerváry's theorem which was already formulated and proved in Section 3.3 as Theorem 3.3.2.

Theorem 4.3.9 *Let c be a weight function on the edge-set of a bipartite graph $G = (S, T; E)$ and let $V = S \cup T$. The maximum weight of a matching of G is equal to*

$$\min\left\{\sum_{v \in V} \pi(v) : \pi \geq 0, \pi(u) + \pi(v) \geq c(uv) \text{ for every edge } uv\right\}.$$

If c is integral, the optimal π can also be selected to be integral. •

As a by-product, we obtain a polyhedral description of matchings.

Theorem 4.3.10 *Let A be the incidence matrix of a bipartite graph G. The polyhedron*

$$\{x : Ax \leq \underline{1},\ x \geq 0\} \tag{4.20}$$

is integral and its vertices are the incidence vectors of the matchings of G. •

By the **matching polytope** of a graph, we mean the convex hull of incidence vectors of matchings. Theorem 4.1.3 of Minkowsky and Weyl asserts that every polytope is a bounded polyhedron. In particular, the matching polytope is also a polyhedron, or in other words, the solution set of a system of linear inequalities. An interpretation of Theorem 4.3.10 is that it specifically describes this linear system (4.20). (For this reason, the matching polytope is often called matching polyhedron.)

The same approach provides a description of the perfect matching polytope as well.

Theorem 4.3.11 *Let G be a perfectly matchable bipartite graph. The polyhedron*

$$R := \{x : Ax = \underline{1}, x \geq 0\} \tag{4.21}$$

is integral and its vertices are the incidence vectors of perfect matchings of G. Concisely, R is the convex hull of the incidence vectors of perfect matchings of G. •

Observe that Theorem 2.4.21 of Birkhoff and Neumann on bistochastic matrices is just an equivalent formulation of Theorem 4.3.11. Not surprisingly, Egerváry's theorem (Theorem 3.3.1) is also a consequence.

Theorem 4.3.12 (Egerváry) *In a perfectly matchable bipartite graph $G = (S, T; E)$, the maximum weight ν_c of a perfect matching with respect to a non-negative weight function c is equal to the minimum total value τ_c of a weighted covering π of c. When c is integer-valued, the optimal π can also be selected to be integer-valued. If G is a complete bipartite graph and c is non-negative, then the optimal weighted covering can be selected to be non-negative.*

Proof. Consider the perfect matching polyhedron $R := \{x : Ax = \underline{1}, x \geq 0\}$. Then the primal linear program is $\max\{cx : x \in R\}$ while its dual program is

$$\min\left\{\sum_{v\in V}\pi(v) : \pi(u) + \pi(v) \geq c(uv) \text{ for every edge } uv\right\}.$$

(Note that there is no sign restriction on π.) The first part of the theorem follows from the duality theorem and the integrality of the optima. In the second part, the optimal dual solution can be selected to be non-negative since the non-negativity of c and the completeness of G imply that the optimum of the linear program $\{\max cx : \; Ax \leq \underline{1}, \; x \geq 0\}$ is attained at a perfect matching, and since the dual variables in the dual of this linear program are non-negative. •

What happens if we are interested in a maximum-weight matching of exactly k edges? Theorem 4.2.7 asserted that the extended incidence matrix of a bipartite graph is totally unimodular. Therefore the duality theorem for the following pair of primal–dual programs provides an answer:

$$\max\left\{cx : Ax \leq \underline{1}, \; \widetilde{x}(E) = k\right\}$$

and

$$\min\left\{\widetilde{\pi}(V) + k\alpha : \pi A + \alpha \cdot \underline{1} \geq c, \; \pi \geq 0\right\}.$$

Here the primal optimal vector is integral and hence (0, 1)-valued, that is, it is the incidence vector of a k-element matching. The dual optimal vector (π, α) is also integral whenever c is. Clearly, the same approach works if one is interested in optimal matchings for which the cardinality is at least k_1 and at most k_2.

One can develop further extensions for the case when the right-hand side is an arbitrary non-negative integral vector b rather than just $\underline{1}$. The combinatorial meaning of this is that one is interested in an optimal subgraph of a bipartite graph for which there is a degree specification $b(v)$ at every node v. More generally, one can impose lower and upper bounds on the degrees of the subgraph, on the total number of edges used by the subgraph, and also on the number of copies showing how many times an edge is allowed to be used. (This is the most general form of the transportation problem.) However, it is not worth formulating the corresponding min-max theorem for all these cases since the duality theorem and the total unimodularity of the incidence matrix already include all the necessary information.

Problem 4.3.3 *Let $G = (S, T; E)$ be a bipartite graph. Prove that there are two disjoint subsets K and N of edges such that $d_N(v) = d_K(v) + 1$ for every node v of G if and only if $|S| = |T|$ and $d_G(X) \geq ||X \cap S| - |X \cap T||$ holds for every subset $X \subseteq S \cup T$. Construct an example to demonstrate that the following necessary condition is not sufficient in general: $|\Gamma(X)| + d(\Gamma(X), S - X) \geq |X|$ holds for every $X \subseteq S$.*

Edge-colourings

Recall that in Section 2.4 we proved Kőnig's edge-colouring theorem as well as its large-scale extension to strongly equitable k-edge-colourings of bipartite graphs (Theorem 2.4.22). This result can be obtained as a special case of Theorem 4.2.21 when it is applied to the extended incidence matrix of the graph.

With the help of linear programming techniques, we can derive Theorem 2.4.24 of Folkman and Fulkerson which was formulated in Section 2.4 and proved in Section 2.5. The non-trivial direction of the theorem states that *there are ℓ edge-disjoint k-element matchings in a bipartite graph $G = (S, T; E)$ if*

$$i_G(Z) \geq \ell(k + |Z| - |U|) \tag{4.22}$$

holds for every subset Z of $U := S \cup T$ where $i_G(Z)$ denotes the number of edges induced by Z.

To prove this, let A denote the incidence matrix of G. Consider the primal program

$$\max \{\widetilde{x}(E) : \ x \geq 0, \ Ax \leq \ell\underline{1}, \ x \leq \underline{1}\} \tag{4.23}$$

and the dual program

$$\min \{\ell\widetilde{\pi}(U) + \widetilde{y}(E) : \ (\pi, y) \geq 0, \ \pi A + y \geq \underline{1}\} \tag{4.24}$$

where $\pi : U \to \mathbf{R}_+$ is the vector of dual variables associated with the rows of A, while $y : E \to \mathbf{R}_+$ is the vector of those variables associated with the inequalies $x \leq \underline{1}$. As A is TU, both the primal and the dual programs have integral optimal solutions. Actually, these solutions are (0, 1)-valued since in the primal program we explicitly have the constraint $0 \leq x \leq 1$, while in the dual program the bounding vector is identically 1, so in an optimal vector (π, y) every component is at most 1.

By the duality theorem, the optima in the primal and dual programs coincide. If this common value is at least $k\ell$, then a (0, 1)-valued primal optimal vector determines a subgraph $G' = (U, E')$ of G having at least $k\ell$ edges such that the degree of every node is at most ℓ. By deleting edges if necessary, we can assume that G' has exactly $k\ell$ edges. By Theorem 2.4.22 E' can be partitioned into ℓ matchings, each of size k.

Suppose now that the common optimum value is less than $k\ell$. Then there is a (0, 1)-valued dual optimal solution (π, y) for which $\ell\widetilde{\pi}(U) + \widetilde{y}(E) < k\ell$. Let Z denote the subset of nodes v for which $\pi(v) = 0$. The dual constraints imply $y(e) = 1$ for every edge induced by Z from which $i_G(Z) \le \widetilde{y}(E)$. Since $\widetilde{\pi}(U) = |U| - |Z|$ we have $\ell(|U| - |Z|) + i_G(Z) \le \ell\widetilde{\pi}(U) + \widetilde{y}(E) < k\ell$, contradicting (4.22). •

4.3.3 Networks and linear programming

The goal of this section is to show that—like bipartite matching problems—the driving force behind many results on network optimization is linear programming and total unimodularity. Throughout the section, $D = (V, A)$ is a digraph for which its $(0, \pm 1)$-valued incidence matrix is denoted by Q.

Feasible potentials

In Section 3.1, we called a function $\pi : V \to \mathbf{R}$ a feasible potential with respect to a cost function $c : A \to \mathbf{R}$ if $\pi(v) - \pi(u) \le c(uv)$ for every edge $uv \in A$. Note that the feasibility of π can be concisely written as $\pi Q \le c$. A function $x : A \to \mathbf{R}$ was called a circulation if $Qx = 0$. Recall Gallai's theorem (Theorem 3.1.1):

Theorem 4.3.13 (Gallai) *A cost function $c : A \to \mathbf{R}$ on the edge-set of $D = (V, A)$ is conservative if and only if there is a feasible potential. Moreover, if c is integer-valued, then π can also be selected to be integer-valued.*

Proof. Since Q is totally unimodular, so is its transpose and hence Theorem 4.2.13 implies that either the linear system $\pi Q \le c$ has a solution (integral when c is integral) or the dual linear system $\{Qx = 0, x \ge 0, cx < 0\}$ has a $(0, \pm 1)$-valued solution.

The first case just means the existence of a feasible potential while the second one provides, by virtue of $x \ge 0$, a $(0, 1)$-valued circulation of negative cost. Such a circulation decomposes into edge-disjoint di-circuits, and one of these di-circuits must also be negative. •

Feasible circulations and flows

Let $D = (V, A)$ be a digraph. In order to handle circulations more easily, it is useful to observe that the conservation rule can be formally relaxed to the inequality $\varrho_x(v) \le \delta_x(v)$ for every node v. Indeed, these inequalities automatically imply equality everywhere because

$$\sum[\delta_x(v) : v \in V] = \widetilde{x}(A) = \sum[\varrho_x(v) : v \in V] \le \sum[\delta_x(v) : v \in V].$$

This means that the solution set to the linear system $Qx \le 0$ is exactly the set of circulations.

Theorem 4.3.14 *If $f \le g$ are integer-valued bounding functions on A, then the polyhedron $\{x \in \mathbf{R}^A : f \le x \le g,\ \varrho_x(v) = \delta_x(v)$ for every $v \in V\}$ of feasible circulations, when non-empty, is integral.*

Proof. Since Q is a TU-matrix, it remains so after adjoining a negative or positive identity matrix. Hence Theorem 4.2.14 applies. •

A similar argument shows the following.

Theorem 4.3.15 *Let $g : A \to \mathbf{Z}_+$ be a capacity function and s and t two designated nodes for which $\varrho_D(s) = 0 = \delta_D(t)$. The polyhedron $\{x \in \mathbf{R}^A : 0 \le x \le g,\ \varrho_x(v) = \delta_x(v)$ for $v \in V - \{s, t\}$ and $\delta_x(s) = k\}$ of the feasible st-flows of amount k, when non-empty, is integral.* •

Next, we show that Hoffman's feasibility theorem is nothing but the Farkas lemma for TU-matrices as formulated in Theorem 4.2.13 when the TU-matrix is the incidence matrix of a digraph.

Theorem 4.3.16 (Hoffman [207, 289]) *In a digraph $D = (V, A)$, there is a feasible circulation with respect to bounding functions $f \leq g$ if and only if*

$$\varrho_f(X) \leq \delta_g(X) \text{ for every } X \subseteq V. \tag{4.25}$$

Moreover, if f and g are integer-valued and (4.25) *holds, then there is an integer-valued feasible circulation.*

Proof. We omit the proof of necessity. For sufficiency, consider the linear system $Qx \leq 0,\ x \leq g,\ -x \leq -f$. A solution to this system is a feasible circulation. By Theorem 4.2.13, if the system has no solution, then there is a (0, 1)-valued vector (y, u, v) for which $(*)$ $yQ + u - v = 0$ and $(**)$ $ug - vf < 0$. Since $f \leq g$, we can assume that at least one of the variables $u(e)$ and $v(e)$ is 0 for every edge e since if both of them were 1, then we could replace both by 0.

Let Z denote the set of nodes z for which $y(z) = 1$. By virtue of $(*)$, we have $u(e) = v(e) = 0$ for every edge e with both end-nodes either in Z or in $V - Z$. Furthermore, $v(e) = 1$ and $u(e) = 0$ for every edge e entering Z, and $v(e) = 0$ and $u(e) = 1$ for every edge leaving Z. Since $ug = \delta_g(Z)$ and $vf = \varrho_f(Z)$, we conclude that $(**)$ contradicts (4.25). •

Optimality criteria and duality for circulations

Let c be a cost function and suppose that z is a feasible circulation. Construct a digraph $D_z = (V, A_z)$ and a cost function c_z as follows. An edge uv belongs to A_z if $uv \in A$ and $z(uv) < g(uv)$ or if $vu \in A$ and $z(vu) > f(vu)$. In the first case, let $c_z(uv) := c(uv)$ while in the second, let let $c_z(uv) := -c(vu)$.

By customizing Theorem 4.2.16 one gets the following optimality criteria but for the sake of completeness, we repeat the proof specialized for the given case. (Recall that a negative circuit is defined as a directed circuit of negative total cost.)

Theorem 4.3.17 *For a feasible circulation z the following are equivalent.*

(A) *z is a cheapest feasible circulation.*
(B) *There is no negative circuit in D_z with respect to c_z.*
(C) *There is a potential $\pi : V \to \mathbf{R}$ for which*

$$\pi(v) - \pi(u) \leq c(uv) \text{ whenever } uv \in A \text{ and } z(uv) < g(uv), \tag{4.26}$$

$$\pi(v) - \pi(u) \geq c(uv) \text{ whenever } uv \in A \text{ and } z(uv) > f(uv). \tag{4.27}$$

If c is integral, then π can also be selected to be integral.

Proof. (A)→(B) Suppose indirectly that there is a negative circuit C in D_z. It can have two types of edges. The first one corresponds to an edge $uv \in A$ for which $z(uv) < g(uv)$, while the second one corresponds to an edge $vu \in A$ for which $z(vu) < f(vu)$. Increase or decrease the value of z by a small positive Δ on edges of the first and second type, respectively. In this way we get another feasible circulation for which the cost is smaller than cz since C is a negative circuit of D_z, contradicting (A).

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 165 of 640 December 11, 2010 15:16

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

(B)→(C) If there is no negative circuit in D_z, then by Theorem 4.3.13 there is a function $\pi : V \to \mathbf{R}$ for which $\pi(v) - \pi(u) \leq c_z(uv) = c(uv)$ holds whenever $uv \in A$ and $z(uv) < g(uv)$ (that is, when $uv \in A_z$), and $\pi(u) - \pi(v) \leq c_z(vu) = -c(uv)$ or equivalently $\pi(v) - \pi(u) \geq c(uv)$ holds whenever $uv \in A$ and $z(uv) > f(uv)$ (that is, when $vu \in A_z$). In other words, (4.26) and (4.27) hold in these cases.

(C)→(A) The cost of an arbitrary circulation x is zero with respect to a node-induced cost function $\Delta_\pi(uv) := \pi(v) - \pi(u)$. Furthermore (4.26) is equivalent to requiring that $z(uv) = g(uv)$ whenever $c_\pi(uv) > 0$ where $c_\pi(uv) := c(uv) - \pi(v) + \pi(u)$ while (4.27) is equivalent to requiring that $z(uv) = f(uv)$ whenever $c_\pi(uv) < 0$.

It follows for a feasible circulation x and a π satisfying (C) that

$$cx = \sum_{uv\in A} c_\pi(uv)x(uv) =$$

$$\sum_{uv\in A}[c_\pi(uv)x(uv) : c_\pi(uv) > 0] + \sum_{uv\in A}[c_\pi(uv)x(uv) : c_\pi(uv) < 0] \geq$$

$$\sum_{uv\in A}[c_\pi(uv)g(uv) : c_\pi(uv) > 0] + \sum_{uv\in A}[c_\pi(uv)f(uv) : c_\pi(uv) < 0] =$$

$$\sum_{uv\in A}[c_\pi(uv)z(uv) : c_\pi(uv) > 0] + \sum_{uv\in A}[c_\pi(uv)z(uv) : c_\pi(uv) < 0] = cz,$$

that is, z is indeed a cheapest feasible circulation. •

Theorem 4.3.18 *Given D, f, g, and c as before, for the minimum cost of a feasible circulation the following min-max result holds.*

$$\min\{cx : f \leq x \leq g,\ x \text{ a circulation}\} = \max_\pi \left\{ \sum_{c_\pi(uv)>0} f(uv)c_\pi(uv) + \sum_{c_\pi(uv)<0} g(uv)c_\pi(uv) \right\}$$

where the maximum is taken over all $\pi : V \to \mathbf{R}_+$ and $c_\pi(uv) = c(uv) - \pi(v) + \pi(u)$. If f, g are integral, then the primal optimum x can be selected to be integral while if c is integral, then the dual optimum π can be selected to be integral.

Proof. Consider the primal program

$$\min\{cx :\ Qx \geq 0,\ x \geq f,\ -x \geq -g\}$$

and its dual

$$\max\{yf - zg :\ \pi Q + y - z = c,\ (\pi, y, z) \geq 0\}.$$

Here an arbitrary π uniquely determines the positive components of y and z as follows. $y(uv) = c_\pi(uv)$ when $\pi(v) - \pi(u) < c(uv)$ and $z(uv) = -c_\pi(uv)$ when $\pi(v) - \pi(u) > c(uv)$. Therefore, the theorem is just the duality theorem for the given pair of linear programs, while the integrity result follows from the total unimodularity of Q. •

Min-cost flows

Using a similar approach, we can derive Theorem 3.6.4 as well.

Theorem 4.3.19 *Let $g : A \to \mathbf{R}_+$ be a capacity function and $c : A \to \mathbf{R}$ a cost function on the edge-set of a digraph $D = (V, A)$. Suppose that there is a feasible st-flow of amount k. The minimum cost of feasible st-flows of amount k is equal to the maximum of the value*

$$k\pi(t) + \sum \left[c_\pi(uv)g(uv) : uv \in A, c_\pi(uv) < 0\right] \tag{4.28}$$

where the maximum is taken over all functions $\pi : V \to \mathbf{R}$ for which $\pi(s) = 0 \leq \pi(v) \leq \pi(t)$ for every $v \in V$. If in addition g is integer-valued, the optimal flow can also be selected to be integral. If c is integer-valued, the optimal π can be selected to be integral.

Proof. Suppose that in the incidence matrix Q of D the first two rows correspond to s and t. Consider the primal linear program $\min\{cx : 0 \leq x \leq g,\ \ Qx = (-k, +k, 0, 0, \ldots, 0)\}$. The inequality can be rewritten in the form $-x \geq -g$ and therefore the dual problem is as follows.

$$\max\left\{k[\pi(t) - \pi(s)] - gz : \pi Q - z \leq c, z \geq 0\right\}.$$

The elements of the primal polyhedron are the feasible st-flows of amount k. By Theorem 4.2.14, this polyhedron is integral whenever g is integral (regardless of the integrality of c). Similarly, the dual polyhedron is integral when c is integer-valued. Observe that an arbitrary π uniquely determines a best z belonging to it: $z(uv) := \pi(v) - \pi(u) - c(uv)$ if $\pi(v) - \pi(u) > c(uv)$, and $z(uv) := 0$ if $c(uv) \leq \pi(v) - \pi(u)$. Therefore the value $k(\pi(t) - \pi(s)) - gz$ of the objective function belonging to π is exactly the formula in (4.28) since $\pi(s) = 0$ can be achieved by translating π. •

The directed Chinese postman problem

Let $D = (V, A)$ be a strongly connected digraph. The (unweighted version of the) directed Chinese postman problem consists of finding a closed directed walk of minimum length such that every edge is used at least once. If D happens to be Eulerian, then there is an Euler tour, which is a closed directed walk using each edge exactly once, in which case there is nothing to be optimized. The general case is equivalent to finding a minimum number of new edges which are parallel to existing ones and their addition to D results in an Euler digraph. As the next application of TU-matrices and linear programming, we derive a min-max formula for the minimum number of necessary new edges.

Theorem 4.3.20 *In a strongly connected digraph $D = (V, A)$, the minimum number of new edges, which are parallel to existing ones and the addition of these new edges to D results in an Euler digraph, is equal to the following maximum:*

$$\max\left\{\sum_{i=1}^{q}\left[\delta(V_i) - \varrho(V_i)\right]\right\} \tag{4.29}$$

where the maximum is taken over all chains $V_1 \supset V_2 \supset \cdots \supset V_q$ of subsets of V for which $\delta(V_i) - \varrho(V_i) \geq 0$ and for which no edge of D enters more than one V_i.

Proof. A chain of sets is said to be feasible if it satisfies the conditions in the theorem. Since in an Euler digraph the in-degree of a set of nodes is the same as its out-degree, in an (augmented) Euler digraph at least $\delta(V_i) - \varrho(V_i)$ new edges must enter every set V_i. Since

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 167 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

only parallel edges are allowed to be added to D and no edge of D enters more than one V_i, the number of new edges is at least $\sum[\delta(V_i) - \varrho(V_i) : i = 1, \ldots, q]$, and hence max $\leq$ min.

This estimate also shows that for a given Euler augmentation of D and for a feasible chain of sets $V_1 \supset V_2 \supset \cdots \supset V_q$, equality holds precisely if only those edges are multiplied that enter some V_i. We call this property the optimality criterion. To prove the direction max $\geq$ min, we are going to prove the existence of a chain and an augmentation satisfying the optimality criterion.

To this end, consider an integral feasible circulation z of minimum cost with respect to the bounding functions $f \equiv 1$, $g \equiv +\infty$ and cost function $c \equiv 1$. Such a z determines the edges to be multiplied: add $z(uv) - 1$ parallel copies of each edge $e = uv$ for which $z(uv) \geq 2$.

By Theorem 4.3.17, there is a function $\pi : V \to \mathbf{Z}$ for which $\pi(v) - \pi(u) \leq c(uv) = 1$ if $uv \in A$ and $z(uv) < g(uv)$, and for which $\pi(v) - \pi(u) \geq c(uv) = 1$ if $uv \in A$ and $z(uv) > f(uv)$. Since $g(uv) \equiv +\infty$, we always have $z(uv) < g(uv)$, and hence $\pi(v) - \pi(u) \leq 1$ holds for every edge $uv \in A$. It follows from the second condition that $\pi(v) - \pi(u) \geq 1$ for every multiplied edge uv (where uv is multiplied if $z(uv) \geq 2$) and hence $\pi(v) - \pi(u) = 1$.

By translating π if necessary, we can assume that the smallest value of π is zero. Let q denote its largest value and let $V_i := \{v \in V : \pi(v) \geq i\}$ for $i = 1, \ldots, q$.

Claim 4.3.21 *The edge multiplication defined by z and the chain of sets V_i satisfy the optimality criterion.*

Proof. $\pi(v) - \pi(u) \leq 1$ implies that every edge uv of D enters at most one V_i. Hence $V_i \supset V_j$ for $i < j$. Since $\pi(v) - \pi(u) = 1$ holds whenever $z(uv) \geq 2$, it follows that only those edges can be multiplied that enter some V_i. Finally, $\delta(V_i) \geq \varrho(V_i)$ holds for each V_i since if we had $\delta(V_i) < \varrho(V_i)$ for some i, then at least $\varrho(V_i) - \delta(V_i)$ new edges must enter $V - V_i$ and if uv is such a new edge, then $\pi(v) - \pi(u) \leq -1$, in contradiction with $\pi(v) - \pi(u) = 1$ which holds for new edges. • •

4.3.4 Covering a digraph by directed circuits

Let $D = (V, A)$ be a digraph in which every node belongs to a directed circuit. Suppose that D is endowed with an integer-valued cost function $c : A \to \mathbf{Z}_+$ on the edge-set A and with an integer-valued weight function $w : V \to \mathbf{Z}_+$ on the node-set V. We say that a non-negative circulation x **covers** w if $\delta_x(u) \geq w(u)$ holds for every node $u \in V$. For a directed circuit K, we use the same letter K to denote the edge-set of K while the node-set of K is denoted by $V(K)$. Define the cost $\widetilde{c}(K)$ of K by $\widetilde{c}(K) := \sum[c(e) : e \in K]$. A non-negative integral vector $z : V \to \mathbf{Z}_+$ is said to be ***c*-independent** if $\widetilde{z}(V(K)) \leq \widetilde{c}(K)$ holds for every di-circuit K of D where $\widetilde{z}(V(K)) := \sum[z(v) : v \in V(K)]$. Such a vector z can be interpreted as a family of not-necessarily-distinct nodes of D such that each di-circuit K contains at most $\widetilde{c}(K)$ of these nodes. The following result is due to Gallai [179].

Theorem 4.3.22 *The maximum total w-weight of a family of c-independent nodes of D is equal to the minimum c-cost of a non-negative integral circulation covering w. More formally,*

978–0–19–920527–1 04-Andras-Frank-c04-drv Andras Frank (Typeset by SPi, Publisher Services, India) 168 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$$\max\{wz : z \geq 0 \ \textit{a c} - \textit{independent integral function on} V\} =$$
$$\min\{cx : x \geq 0 \ \textit{an integral circulation covering } w\}. \tag{4.30}$$

Proof. The idea of the proof is quite simple. With the node-duplication technique, which was used to derive Dilworth's theorem from Kőnig's theorem and also to derive the theorems of Greene and Kleitman and of Greene from min-cost flows, we first convert the problem of covering w by a circulation of D to that of finding a feasible circulation in the duplicated digraph. We then apply the duality theorem for circulations and establish what the primal and the dual optima mean in the original digraph.

Claim 4.3.23 *Let $z \geq 0$ be a c-independent integral function on V and $x \geq 0$ an integral circulation. Then*

$$\sum_{v \in V} z(v)\delta_x(v) \leq cx. \tag{4.31}$$

Proof. Recall that a cycle-flow is defined as a circulation that is a positive constant on the edges of a directed circuit and zero on all other edges. By Lemma 3.4.3, x can be expressed as the sum of non-negative integral cycle-flows, that is, there are directed circuits $K_1, \dots, K_q$ and positive integer coefficients $\alpha_1, \dots, \alpha_q$ such that $x = \sum_{i=1}^q \alpha_i \cdot \underline{\chi}_{K_i}$. Then $\delta_x(v) = \sum[\alpha_i : v \in V(K_i)]$ from which

$$\sum_{v \in V} z(v)\delta_x(v) = \sum_{v \in V} z(v) \sum_{i=1}^{q} [\alpha_i : v \in V(K_i)] = \sum_{i=1}^{q} \alpha_i \sum_{v \in V(K_i)} z(v) \leq$$
$$\sum_{i=1}^{q} \alpha_i \widetilde{c}(K_i) = \sum_{i=1}^{q} \alpha_i \sum_{e \in K_i} c(e) = \sum_{e \in A} c(e) \sum_{e \in K_i} \alpha_i = \sum_{e \in A} c(e)x(e) = cx,$$

completing the proof of the claim. •

If x covers w, then (4.31) implies $wz = \sum_{v \in V} w(v)z(v) \leq \sum_{v \in V} z(v)\delta_x(v) \leq cx$ from which max $\leq$ min follows in (4.30).

To prove the reverse direction, construct a digraph $D_2 = (V_2, A_2)$ from D as follows (see Figure 4.3). Let $V_2 = V' \cup V''$ where V' and V'' are two disjoint copies of V. We shall use the notational convention that elements of V' and V'' corresponding to $v \in V$ are denoted by v' and v'', respectively. Let

$$A_2 := \{u''u' : u \in V\} \cup \{u'v'' : uv \in A\}.$$

Define a lower bound $f : A_2 \to \mathbf{Z}_+$ by

$$f(u''u') := w(u) \text{ for } u \in V \text{ and } f(u'v'') := 0 \text{ for } uv \in A.$$

Furthermore, define a cost function $c_2 : A_2 \to \mathbf{Z}_+$ by

$$c_2(u''u') := 0 \text{ for } u \in V \text{ and } c_2(u'v'') := c(uv) \text{ for } uv \in A.$$

The construction of D_2 immediately implies the following.

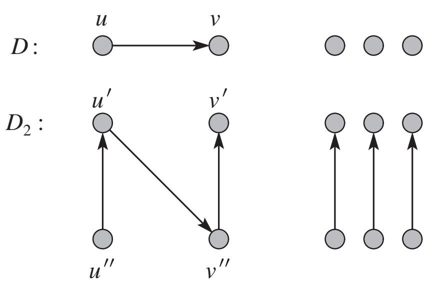


Figure 4.3 Constructing D_2 from D

Claim 4.3.24 *For a circulation $x_2 \geq 0$ of D_2, define $x : A \to \mathbf{R}_+$ by $x(uv) := x_2(u'v'')$ for $uv \in A$. Then x is a circulation of D for which $cx = c_2x_2$. If $x_2 \geq f$, then x covers w.* •

Let Q denote the incidence matrix of D_2. Then x_2 is a circulation of D_2 if and only if $Qx_2 \geq 0$ (since $Qx_2 \geq 0$ implies $Qx_2 = 0$). Consider the primal linear program

$$\min\{c_2x_2 : Qx_2 \geq 0, x_2 \geq f\}$$

and its dual program:

$$\max\{fy : \pi Q + yI = c_2, (\pi, y) \geq 0\}$$

where I is an identity matrix the rows and columns of which correspond to the edges of D_2, $\pi : V_2 \to \mathbf{R}_+$ denotes the dual variables assigned to the rows of Q while $y : A_2 \to \mathbf{R}_+$ denotes the dual variables assigned to the rows of I (where these rows correspond to the edges of D_2). The dual constraints for the two types of edges of D_2 are as follows.

$$\pi(u') - \pi(u'') + y(u''u') = 0 \text{ for every } u \in V,$$
$$\pi(v'') - \pi(u') + y(u'v'') = c(uv) \text{ for every } uv \in A.$$

Since Q is totally unimodular, there is an optimal integral primal solution x_2 and an optimal integral dual solution (π, y) for which $c_2x_2 = fy$ holds by the duality theorem. In Claim 4.3.24 we noted that x_2 defines an integral circulation x in D covering w for which $c_2x_2 = cx$.

Let $z : V \to \mathbf{Z}_+$ be defined by

$$z(u) := y(u''u') \text{ for every } u \in V.$$

Then we have $wz = fy$ from which $cx = c_2x_2 = fy = wz$. Hence the proof of the theorem becomes complete once we prove the following claim.

Claim 4.3.25 *z is c-independent.*

Proof. Let K be a di-circuit of D and let $u_1, u_2, \ldots, u_q$ be the nodes of K in this order. We shall use the notation $u_{q+1} = u_1$. We have

$$0 = \left[\pi\left(u_1'\right) - \pi\left(u_1''\right)\right] + \left[\pi\left(u_2''\right) - \pi\left(u_1'\right)\right] + \left[\pi\left(u_2'\right) - \pi\left(u_2''\right)\right] + \cdots$$

$$+\left[\pi\left(u_1''\right) - \pi\left(u_q'\right)\right] = \sum_{i=1}^{q}\left[\pi\left(u_i'\right) - \pi\left(u_i''\right)\right] + \sum_{i=1}^{q}\left[\pi\left(u_{i+1}''\right) - \pi\left(u_i'\right)\right] \leq \sum_{i=1}^{q}[-z(u_i)]$$

$$+\sum_{i=1}^{q} c(u_i u_{i+1}) = -\widetilde{z}(V(K)) + \widetilde{c}(K),$$

that is, $\widetilde{z}(V(K)) \leq \widetilde{c}(K)$, as required. • •

We say that a family of not-necessarily-distinct di-circuits of D **covers** a weight function w if every node $u \in V$ belongs to at least $w(u)$ of these di-circuits. Gallai [179] (p. 402, Theorem 2.1.7) originally proved his result in the following equivalent form.

Theorem 4.3.26 (Gallai) *The maximum total w-weight of a family of c-independent nodes of D is equal to the minimum total c-cost of a family of di-circuits of D covering w.*

Proof. Consider the optimal integral circulation x ensured by Theorem 4.3.22. The Flow decomposition lemma (Lemma 3.4.3) implies that x is the sum of integral cycle-flows: $x = \sum_{i=1}^{q} \alpha_i \cdot \underline{\chi}_{K_i}$. Therefore by taking each K_i in α_i copies, we obtain the requested optimal family of di-circuits covering w. •

Remark 4.3.1 Gallai's theorem can be formulated in terms of total dual integrality. The theorem is equivalent to stating that the linear system $\{x \geq 0, \widetilde{x}(C) \leq \widetilde{c}(C)$ for every di-circuit of $D\}$ is TDI. This result was generalized by Cameron and Edmonds [41] who proved that the system $\{f \leq x \leq g,\ \widetilde{x}(C) \leq \widetilde{c}(C)$ for every di-circuit of $D\}$ is also TDI. In the proof above, we avoided using the theory of TDI-ness. If we rely on Theorem 4.1.24, then the second part of the proof, in which we constructed an integral solution z, can be left out.

The story of a conjecture of Gallai

Determining the stability number α of a graph is a well-known **NP**-complete problem. It is also **NP**-complete to find the minimum number γ of directed circuits covering all nodes of a digraph since just deciding if this minimum is 1 is equivalent to the directed Hamilton circuit problem. Bessy and Thomassé [32], answering a longstanding conjecture of Gallai [180] in the affirmative, proved that these 'difficult' parameters are related as follows.

Theorem 4.3.27 (Bessy and Thomassé) *For every strongly connected digraph D, $\gamma(D) \leq \alpha(D)$ where $\alpha(D)$ denotes the stability number of the underlying undirected graph. In other words, the node-set of D can be covered by $\alpha(D)$ directed circuits.*

Proof. Recall that a flat covering of di-circuits is a subset F of edges intersecting all di-circuits such that every edge of the digraph belongs to a di-circuit intersected exactly once. Knuth's lemma (Theorem 2.2.2) asserted that every strongly connected digraph has a flat covering F.

Claim 4.3.28 *A family of $\underline{\chi}_F$-independent nodes consists of distinct nodes that form a stable set of D.*

Proof. The $\underline{\chi}_F$-cost of a di-circuit K is equal to the number of F-edges of K. Since F is a flat covering, every edge of D belongs to a di-circuit of cost 1.

Let v be any node of D and e an edge leaving v. Since e belongs to a di-circuit K of cost 1, at most one copy of v can occur in a $\underline{\chi}_F$-independent family of nodes. Similarly, $\underline{\chi}_F$-independent nodes cannot induce any edge e since then e belongs to a di-circuit of cost 1 contradicting the $\underline{\chi}_F$-independence. •

Refer to a stable set S of D as F**-stable** if every di-circuit K contains at most $|F \cap K|$ elements of S. This is equivalent to requiring S to be $\underline{\chi}_F$-independent. Theorem 4.3.26, when applied to cost function $c := \underline{\chi}_F$ and to weight function $w \equiv 1$, implies that the maximum cardinality of an F-stable set of D is equal to the minimum total $\underline{\chi}_F$-cost of directed circuits covering each node. Since the $\underline{\chi}_F$-cost of every di-circuit is at least 1, we can conclude that the stability number of D is at least the minimum number of di-circuits covering V. • •

Problems

4.3.4 *Prove that in Theorem* 4.3.27*, strong connectivity of D can be weakened to the assumption that each node belongs to a directed circuit.*

4.3.5 *Let $D = (V, A)$ be a strongly connected digraph and let $w \geq 0$ be an integral weight function on the node-set. Prove that the maximum total weight of a stable set is at most the minimum number of directed circuits covering each node u at least $w(u)$ times. In particular, given a subset U of nodes, prove that the maximum cardinality of a stable subset of U is at most the minimum number of directed circuits of D covering U.*

4.3.6 *Prove that any clique C of a strongly connected digraph D with at least two nodes can be covered by $|C|$ directed circuits. In particular, prove the following theorem of Camion* [42]*: every strongly connected tournament with at least two nodes has a Hamilton circuit.*

Remark 4.3.2 Gallai's conjecture had been a well-known open problem for four decades before Bessy and Thomassé solved it in 2007. Their original proof followed a different line relying on the notion of coherent cyclic ordering of nodes and applying Dilworth's theorem. Sebő [343] also used coherent cyclic orderings but relied on circulations that enabled him to establish several extensions including a weighted case. In 2008, Iwata and Matsuda [221] recognized that one of the main ingredients in the proof of Bessy and Thomassé (the existence of a coherent cyclic ordering) is equivalent to Knuth's lemma from 1974 (Theorem 2.2.2) for which they provided a simple proof (the one presented in Section 2.2). Cameron and Edmonds [41] proved an extension of Gallai's theorem in 1992, and their main result, in fact, had already appeared in the Ph.D. thesis of Cameron [40] (supervised by Edmonds) in 1982. All in all, by combining Gallai's theorem and Knuth's lemma, one obtains the answer to the conjecture of Gallai (= the theorem of Bessy and Thomassé). The story nicely exemplifies how sometimes a good knowledge of the literature and a sharp eye could give rise (or, in this case, could have given rise) to the solution of an apparently difficult open problem. Gallai's conjecture could well have been proved 30 years earlier without devising difficult mathematical ideas. It should, however, be admitted that Gallai's long paper from 1958, because of its somewhat peculiar terminology, is not particularly easy reading.

5
Elements of matroid theory

A matroid is an abstract structure given by a pair $(S, \mathcal{F})$ where S is a finite ground-set and $\mathcal{F}$ is a system of subsets of S satisfying certain axioms. The concept of matroids, introduced by Whitney [383] in 1935, embeds 'independence'—in particular linear independence—into an abstract framework, analogously to the paradigm of how fundamental structures like groups, rings, and fields intend to capture essential features of operations, or of how a metric space generalizes the concept of distance.

Another possible perspective introduces matroids as structures for which the greedy algorithm works correctly for every possible objective function. In Section 3.2, we outlined how a maximum-weight spanning tree of a graph can be computed greedily by selecting at every step the heaviest available edge that does not form a circuit with the edges already selected. The greedy algorithm, on the other hand, fails to produce an optimal matching or arborescence. The question therefore naturally emerges: which are the essential features of a structure that ensure the success of a greedy algorithm? This question can, of course, be answered only if the notion of a greedy algorithm is precisely formulated. One possibility is to let w be a weight function on S and let $\mathcal{F}$ be a hereditary set-system. In each step, select the heaviest element such that the set of selected elements belong to $\mathcal{F}$. Then matroids may be defined as hereditary set systems for which this greedy algorithm provides a heaviest member of $\mathcal{F}$ for every weight function w. Although from the aspect of combinatorial optimization this property of matroids is the significant one, for the introductory treatment we follow the path laid down by Whitney (and then the statement that the greedy algorithm works correctly for matroids will become a theorem). The method is standard: we pick up some simple features of linear independence (which are proved statements in linear algebra) and declare them to be axioms. Actually, there are several equivalent axiom systems for matroids, but these are not superfluous, since working with any one of them may be easier or more difficult depending on the specific applications.

The usefulness of matroids (like any other successful mathematical structure) is based on two features. On the one hand, matroids are quite general for finding a wide range of applications and special cases, and they are, on the other hand, sufficiently special for deriving deep theorems and algorithms.

This chapter reviews only elements of the theory. More serious results and applications will be discussed in Part III.

5.1 Independence and rank

5.1.1 Independence axioms

A set-system $M = (S, \mathcal{F})$ is called a **matroid** if it satisfies the following properties, called **independence axioms**.

(I1) $\emptyset \in \mathcal{F}$.
(I2) *If* $X \subseteq Y \in \mathcal{F}$, *then* $X \in \mathcal{F}$.
(I3) *For every subset* $X \subseteq S$, *the maximal subsets of* X *which are in* $\mathcal{F}$ *have the same cardinality.*

The members of $\mathcal{F}$ are called M**-independent**, or simply **independent**, sets; all other subsets of S are **dependent**. Axiom (I1) requires the empty set to be independent, (I2) means that the subset of an independent set is independent, while (I3) is another way of saying that the maximal independent subsets of each subset of S are equicardinal. This maximum number is called the **rank of** X and is denoted by $r(X)$, where r is the **rank function** of the matroid and $r(S)$ is the **rank of the matroid**.

Two matroids are **isomorphic** if they are isomorphic as set systems, or more specifically, if there is a bijection between their ground-sets such that a subset is independent in the first matroid if and only if its map is independent in the second.

There are variations of the independence axioms. The proof of their equivalence is a simple exercise and is left to the reader. For example, (I1) can be replaced by:

(I1$'$) $\mathcal{F}$ *is non-empty.*

(I3) is equivalent to requiring that the greedy algorithm provides a heaviest independent set for every (0, 1)-valued weight function on S. As we shall prove, the greedy algorithm works correctly for any weight function. Often Property (I3) is replaced by:

(I3$'$) *Let* $K, N \in \mathcal{F}$ *with* $|K| < |N|$. *Then there is an element* $x \in N - K$ *for which* $K + x \in \mathcal{F}$.

A weaker form of (I3) is as follows:

(I3$''$) *If* $I_k, I_{k+1} \in \mathcal{F}$, $|I_k| = k$, *and* $|I_{k+1}| = k + 1$, *then there is an element* $s \in I_{k+1} - I_k$ *for which* $I_k + s \in \mathcal{F}$.

Exercise 5.1.1 *Prove that the system* {(I1),(I2),(I3)} *of axioms is equivalent to* {(I1), (I2), (I3$'$)}.

Exercise 5.1.2 *Prove that* {(I1),(I2),(I3)} *is equivalent to* {(I1),(I2), (I3$''$)}.

The definition immediately implies, for a matroid $M = (S, \mathcal{F})$ and for a subset $S' \subseteq S$, that $M' := (S', \mathcal{F}')$ is a matroid, where $\mathcal{F}' := \{F : F \subseteq S', F \in \mathcal{F}\}$. M' is called the **sub-matroid** of M. It is also said that M' arises from M by **deleting** $Z := S - S'$, or that M' is the **restriction** of M to S'. We denote this restriction of M by $M' = M - Z$ or by $M' = M|S'$.

The third independence axiom can be relaxed further.

(I3$'''$) *The maximal independent subsets of S have the same cardinality* r, *and if* $I_{r-1}, I_r \in \mathcal{F}$, $|I_r| - 1 = |I_{r-1}| = r - 1$, *then there is an element* $s \in I_r - I_{r-1}$ *for which* $I_{r-1} + s \in \mathcal{F}$.

978–0–19–920527–1 05-Andras-Frank-c05-drv Andras Frank (Typeset by SPi, Publisher Services, India) 174 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proposition 5.1.1 *The axiom systems* {(I1),(I2),(I3′)} *and* {(I1),(I2),(I3′′′)} *are equivalent.*

Proof. Since (I3′′′) is a special case of (I3′), we only need to show that (I3′) follows from the second system of axioms. To this end, let $K, N \subseteq S$ be two members of $\mathcal{F}$ for which $|K| < |N|$. Then (I3′′′) implies the existence of B_K and $B_N \in \mathcal{F}$ from which $K \subseteq B_K$, $N \subseteq B_N$ and $|B_K| = |B_N| = r$. Assume that these are selected in such a way that $|B_K \cap B_N|$ is maximum.

Claim 5.1.2 $(B_K - K) \cap N$ *is non-empty.*

Proof. Suppose indirectly that $(*)$ there is no $x \in (B_K - K) \cap N$. Since $|K| < |N|$ and $|B_K| = |B_N|$, it follows that $|B_K - K| > |B_N - N|$ and thus there is an $x_1 \in B_K - K$ which is not in $B_N - N$. Assumption $(*)$ implies $x_1 \notin B_N$, therefore, by (I3′′′), there is an element $x_2 \in B_N - B_K$ for which $B'_K := B_K - x_1 + x_2$ is in $\mathcal{F}$. But the existence of such a B'_K contradicts the maximality of $|B_K \cap B_N|$. •

Therefore there is an element x in $(B_K - K) \cap N$ and then $K + x \subseteq B_K$, that is, $K + x \in \mathcal{F}$ and hence (I3′) does indeed hold. • •

Another kind of weakening of (I3′) is the following.

(I3′′′′) *Let* $K, N \in \mathcal{F}$ *with* $|K - N| = 1$ *and* $|N - K| = 2$. *There exists an element* $x \in N - K$ *for which* $K + x \in \mathcal{F}$.

Problem 5.1.3 *Prove that* {(I1),(I2),(I3)} *is equivalent to* {(I1),(I2), (I3′′′′)}.

An advantage of having the axioms in weaker and stronger forms is that it is easier to prove results on matroids relying on stronger properties while it is easier to prove the validity of a weaker property when we want to show that a concrete set-system is a matroid. The same reason explains why the introduction of axiom systems focusing on other notions will prove useful.

5.1.2 Examples

Uniform and partition matroids

Let $\{S_1, \dots, S_t\}$ be a partition of S and let $g_1, \dots, g_t$ be non-negative integers. Declare a subset $I \subseteq S$ independent if $|I \cap S_i| \leq g_i$ holds for each i. The independence axioms are readily seen to hold. The matroid obtained in this way is called a **partition matroid**. Its rank function r is given by $r(X) := \sum_i [\min\{g_i, |X \cap S_i|\}]$.

In the special case of $t = 1$, we speak of a **uniform matroid**, which is denoted by $U_{n,k}$, where $n = |S|$ and $k = g_1$. If $k = n$, that is, if every subset is independent, the matroid is **free**. If $k = 0$, that is, if only the empty set is independent, the matroid is **trivial**.

Laminar matroid

It is not difficult to check that if $\{S_1, \dots, S_t\}$ is a laminar family, rather than just a partition, the set-system

$$\mathcal{I} := \{I : |I \cap S_i| \leq g_i, i = 1, \dots, t\} \tag{5.1}$$

also satisfies the independence axioms. The matroid arising this way is called a **laminar matroid**.

Vector or linear matroid

Let S denote a finite subset of a vector space over an arbitrary field T. Define $\mathcal{F}$ to consist of the linearly independent subsets of S. By elementary linear algebra, $(S, \mathcal{F})$ is a matroid, called a **vector** or **linear** matroid.

Often the elements of a linear matroid are represented by the columns of a matrix. The field T plays an important role. For example, the uniform matroid $U_{4,2}$ can be shown to be linear over the rationals but not linear over $GF(2)$. Linear matroids over $GF(2)$ are said to be **binary** while **regular** matroids are those representable over all fields. There is an extensive and deep theory concerning representability of matroids: see [316] for an exhaustive account.

Affine matroid

Let S be a finite subset of points in an n-dimensional linear space. Declare a subset to be independent if it is affinely independent. Again, by elementary linear algebra, affine independence satisfies the matroid axioms. An easy exercise shows that every affine matroid can be represented as a linear matroid, and vice versa. An advantage of the affine matroid representation is that vectors in the plane form rank-2 matroids, while points in the plane form rank-3 matroids. For example, in the plane the one- and two-element sets are always independent, while a set of three elements is independent if the three points are not collinear. Hence $U_{4,2}$ is represented by four distinct, collinear points.

Graphic or circuit matroid

Let $G = (V, E)$ be an undirected graph and let the edge-set E be the ground-set of the matroid. Declare a subset F of edges independent if it contains no circuit, that is, if F is a forest.

The first two independence axioms trivially hold, while the third one follows from the well-known fact that the cardinality of a spanning forest of a graph is the number of nodes minus the number of components. The matroid arising this way is called the **circuit matroid** (or sometimes forest matroid) of G. A matroid isomorphic to a circuit matroid is said to be **graphic**. Note that graphic matroids are not new, in the sense that each of them is isomorphic to a vector matroid over an arbitrary field T. Indeed, take an arbitrary orentation $\vec{G}$ of G and let Q denote the $(0, \pm 1)$-valued node-edge incidence matrix of $\vec{G}$.

Exercise 5.1.4 *Prove that a subset F of edges of G is a forest if and only if the corresponding columns of Q are linearly independent over T.*

The circuit matroids of two equicardinal trees are isomorphic, and even two non-isomorphic 2-connected graphs can have isomorphic circuit matroids. A deep theorem of Whitney [384] states that the circuit matroids of two non-isomorphic 3-connected graphs are not isomorphic.

5.1.3 Rank, circuit, and further notions

What can be said about the rank function r of a matroid? It is clearly non-negative, integer-valued, subcardinal (that is, $r(X) \leq |X|$), and non-decreasing in the sense that $r(X) \leq r(Y)$ whenever $X \subseteq Y$. Its most important property is as follows.

Lemma 5.1.3 *The rank function r is submodular, that is, for every two subsets $X, Y \subseteq S$*

$$r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y). \tag{5.2}$$

Proof. Let F be a maximal independent subset of $X \cap Y$. Then $|F| = r(X \cap Y)$ and, by the independence axiom (I3), F can be extended to a maximal independent subset N of $X \cup Y$. Then $|N| = r(X \cup Y)$, and the maximality of F in $X \cap Y$ implies that $N \cap X \cap Y = F$, from which $|N \cap X| + |N \cap Y| = |F| + |N|$.

Since $N \cap X$ is an independent subset of X, we have $r(X) \geq |N \cap X|$ and similarly $r(Y) \geq |N \cap Y|$, from which $r(X) + r(Y) \geq |N \cap X| + |N \cap Y| = |F| + |N| = r(X \cap Y) + r(X \cup Y)$. •

A maximal independent subset of S is called a **basis** or a **base** of the matroid. A subset $X \subseteq S$ is said to **span** or to **generate** the subset $Y \subseteq S$ if $r(X \cup Y) = r(X)$. A subset X is a **generator** of the matroid if it generates S or, equivalently, if X is the superset of a basis.

A subset $X \subseteq S$ is called a **circuit** of the matroid if X is dependent but every proper subset of X is independent. The name of a one-element circuit is a **loop**. The two elements of a 2-element circuit are **parallel**. Matroids with no loops or parallel elements are **simple**. A simple matroid is sometimes called a **combinatorial geometry**.

Lemma 5.1.4 *Suppose that e and f are parallel and f and g are parallel. Then e and g are also parallel and $r(\{e, f, g\}) = 1$.*

Proof. Let $X := \{e, f\}$ and $Y := \{f, g\}$. From the submodularity of r we get $1 + 1 = r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y) = 1 + r(X \cup Y)$ from which $r(X \cup Y) \leq 1$. Furthermore, r is non-decreasing, and hence $r(X \cup Y) \geq 1$, that is, $r(\{e, f, g\}) = r(X \cup Y) = 1$. It follows that $r(\{e, g\}) = 1$, from which e and g are parallel. •

The notion of a cut in a graph can also be extended to matroids. In fact, the right notion is the elementary cut (bond). Recall that a bond is a minimal subset of edges the removal of which increases the number of components.

A **bond** (often in the literature a **cut** or a **co-cycle**) of a matroid is a minimal subset of S intersecting every basis. A one-element bond is called a **cut-element** or **bridge**. This definition is equivalent to requiring that the element not be included in any circuit. More generally, for a subset $X \subseteq S$, an element $t \in X$ is said to be a **cut-element** or a **bridge** of X if t belongs to every maximal independent subset of X. Notice that if t is a bridge of X, then t is a bridge of every subset of X containing t.

5.2 Circuits and connectivity

Since every hereditary set-system H is uniquely determined by the minimal subsets not belonging to H, every matroid is determined by the sets of its circuits. The independent sets are those not including any circuit, that is,

$$\mathcal{F} = \{F : \text{ there is no } C \in \mathcal{C}, C \subseteq F\} \tag{5.3}$$

978–0–19–920527–1 05-Andras-Frank-c05-drv Andras Frank (Typeset by SPi, Publisher Services, India) 177 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

where $\mathcal{C}$ denotes the set of circuits. Suppose now that we start with a set-system $\mathcal{C}$ and want to determine some basic properties so that $\mathcal{F}$ as defined in (5.3) satisfies the independence axioms.

5.2.1 Circuit axioms

Obviously, the empty set is not a circuit and a circuit does not include any other circuit.

Theorem 5.2.1 *Let $\mathcal{C}$ be the set of circuits of a matroid. Let C_1 and C_2 be two distinct members of $\mathcal{C}$ and let $e \in C_1 \cap C_2$. Then there exists a member $C \in \mathcal{C}$, for which $C \subseteq C_1 \cup C_2 - e$.*

Proof. Suppose indirectly that there are two circuits C_1, C_2 that violate the statement. Let K denote their union. Then $K - e$ is independent while K is not from which $r(K) = |K| - 1$. On the other hand $C_1 \cap C_2$ is independent so it can be extended to a maximal independent subset F of K which is of cardinality $r(K) = |K| - 1$. But F includes neither C_1 nor C_2 so its cardinality is at most $|K| - 2$, a contradiction. •

Theorem 5.2.1 can be given in the following equivalent form.

Theorem 5.2.2 *If F is independent and $e \in S$, then $F + e$ contains at most one circuit.* •

When F is independent, $e \in S - F$, and $F + e$ is dependent, the unique circuit in $F + e$ is called the **fundamental circuit** of e belonging to F and is denoted by $C(F, e)$. Note that by removing any element of $C(F, e)$ from $F + e$ we obtain another independent set which is a basis when F is.

Theorem 5.2.3 *Let C_1 and C_2 be two distinct circuits, $e \in C_1 \cap C_2$, $e_1 \in C_1 - C_2$. Then there is a circuit $C \in \mathcal{C}$ for which $e_1 \in C \subseteq C_1 \cup C_2 - e$.*

Proof. Suppose, indirectly, that there are circuits violating the assertion. Select C_1, C_2 in such a way that their union K is of minimum size. By Theorem 5.2.1 there is a circuit C_3 for which $C_3 \subseteq C_1 \cup C_2 - e$. Then $e_1 \notin C_3$ since C_1, C_2 form a counterexample.

Since C_3 is not a subset of C_1, there is an element $f \in C_3 - C_1$ which is in C_2. By the minimality of K the assertion of the theorem holds for circuits C_2 and C_3 (because their union is smaller than K). Hence there exists a circuit $C_4 \subseteq C_2 \cup C_3 - f$ which contains e. Note that $e_1 \in C_1$ and $e_1 \notin C_4$ from which $C_1 \neq C_4$.

Now the union of circuits C_4 and C_1 is a proper subset of K so the assertion of the theorem holds for them, and hence there is a circuit $C \subseteq C_1 \cup C_4 - e \subseteq K - e$ which contains e_1, contradicting the indirect assumption. •

For a set-system $\mathcal{C}$, consider the following properties, called **circuit axioms**.

(C1) $\emptyset \notin \mathcal{C}$.
(C2) *If $C_1, C_2 \in \mathcal{C}$, then $C_1 \not\subset C_2$.*
(C3) (Weak circuit axiom) *If C_1 and C_2 are two distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there exists a $C \in \mathcal{C}$ for which $C \subseteq C_1 \cup C_2 - e$.*

We proved above that the circuits of a matroid satisfy these properties. Notice that in the proof of Theorem 5.2.3 we used only these three properties and hence we can conclude that the following property, called the strong circuit axiom, follows from the three properties {(C1), (C2), (C3)}.

(C3′) (Strong circuit axiom) *Let C_1 and C_2 be two distinct members of $\mathcal{C}$ and let $e \in C_1 \cap C_2$, $e_1 \in C_1 - C_2$. Then there is a $C \in \mathcal{C}$ for which $e_1 \in C \subseteq C_1 \cup C_2 - e$.*

In other words the axiom systems {(C1), (C2), (C3)} and {(C1), (C2), (C3′)} are equivalent.

Theorem 5.2.4 *If $\mathcal{C}$ satisfies the three circuit axioms, then $\mathcal{F}$ defined by* (5.3) *satisfies the independence axioms and the set of circuits of the matroid $M = (S, \mathcal{F})$ is $\mathcal{C}$.*

Proof. Axioms (I1) and (I2) hold trivially. Suppose indirectly that (I3) does not hold, and let K and N form a counterexample for which $|K| < |N|$ and $|K \cap N|$ is as large as possible.

There must be an element $e \in K - N$ since if $K \subset N$, then we would have $K + f \in \mathcal{F}$ for any $f \in N - K$. Let f be an arbitrary element of $N \cup K$ if $N + e$ is in $\mathcal{F}$, and let f be an element of the unique circuit in $F + e$ if $F + e$ is not in $\mathcal{F}$. In both cases $N' := N + f - e$ is in $\mathcal{F}$. Since $|N' \cap K| < |N \cap K|$, there must be an element h of $N' - K$ such that $K + h$ is in $\mathcal{F}$. But $h \in N - K$, a contradiction.

Finally, if C' is a circuit of M, then C' includes a member C of $\mathcal{C}$ and C' is minimal for this property, and hence $C' = C$. Conversely, if $C \in \mathcal{C}$, then C is dependent in M so it includes a circuit C' of M. But $C' \in \mathcal{C}$, and axiom (C2) therefore implies that $C = C'$. •

Exercise 5.2.1 *Prove that the bonds of an undirected graph satisfy the circuit axioms.*

More generally, we will see soon that the bonds of a matroid satisfy the circuit axioms.

Problem 5.2.2 (∗) *In Proposition* 1.2.6*, we proved for a graph $G = (V, E)$ that $c(X) + c(Y) \le c(X \cap Y) + c(X \cup Y) + d_G(X, Y)$ holds whenever $X, Y \subseteq V$, where $c(Z)$ denotes the number of components induced by Z and $d_G(X, Y)$ is the number of edges connecting $X - Y$ and $Y - X$. Derive this inequality from the submodularity of the rank function of the circuit matroid of G.*

There is yet another important property of circuits.

Theorem 5.2.5 *For two given elements x and y, if there is a circuit C_1 containing x and there is a circuit C_2 containing y so that $C_1 \cap C_2 \neq \emptyset$, then there is a circuit containing both x and y.*

Proof. Choose a counterexample in which $K = C_1 \cup C_2$ is minimal. Let $c \in C_1 \cap C_2$. By the strong circuit axiom there is a circuit $C_1' \subset K$ for which $c \notin C_1'$, $x \in C_1'$. We then have $C_1' \cup C_2 = K$ for if $C_1' \cup C_2 \subset K$ then the minimality of K implies that the circuits C_1' and C_2 would not be a counterexample and there would exist a circuit containing both x and y.

Similarly there is a circuit C_2' for which $c \notin C_2'$, $y \in C_2'$ and $C_1 \cup C_2' = K$. Now C_1' and C_2' must intersect and their union is a proper subset of K (as c is not in the union). Therefore C_1', C_2' do not form a counterexample, that is, there is a circuit containing x and y, a contradiction. •

Hypergraphic matroids

Lorea [261] extended the notion of the circuit matroid of graphs to hypergraphs. Recall that a hypergraph is called wooded or forest representable if it was possible to select two distinct elements of every hyperedge in such a way that the selected pairs as graph edges form a forest. Theorem 2.4.16 of Lovász asserted that a hypergraph is wooded if and only if the strong Hall condition holds, that is, the union of any j $(j \geq 1)$ hyperedges has at least $j+1$ elements.

Theorem 5.2.6 (Lorea, [261]) *The wooded subhypergraphs of a hypergraph $H = (V, \mathcal{E})$ form the independent sets of a matroid on the ground-set $\mathcal{E}$ of hyperedges.*

Proof. For a set $\mathcal{K}$ of hyperedges, let $\gamma(\mathcal{K})$ denote the cardinality of the union of hyperedges in $\mathcal{K}$. Call a subhypergraph $(V, \mathcal{C})$ of H a hypercircuit if it is not wooded but each of its proper subhypergraphs is wooded. By Theorem 2.4.16, $\gamma(\mathcal{C}) = |\mathcal{C}|$, and $\emptyset \subset \mathcal{C}' \subset \mathcal{C}$ implies $\gamma(\mathcal{C}') \geq |\mathcal{C}'| + 1$.

We claim that hypercircuits satisfy the circuit axioms. Obviously a hypercircuit does not include another hypercircuit. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two hypercircuits and let Z be a member of their intersection. We have $|\mathcal{C}_1| + |\mathcal{C}_2| = \gamma(\mathcal{C}_1) + \gamma(\mathcal{C}_2) \geq \gamma(\mathcal{C}_1 \cap \mathcal{C}_2) + \gamma(\mathcal{C}_1 \cup \mathcal{C}_2) \geq |\mathcal{C}_1 \cap \mathcal{C}_2| + 1 + \gamma(\mathcal{C}_1 \cup \mathcal{C}_2)$, from which $\gamma(\mathcal{C}_1 \cup \mathcal{C}_2) \leq |\mathcal{C}_1| + |\mathcal{C}_2| - |\mathcal{C}_1 \cap \mathcal{C}_2| - 1 = |\mathcal{C}_1 \cup \mathcal{C}_2| - 1$ and hence $\mathcal{C}_1 \cup \mathcal{C}_2 - \{Z\}$ violates the strong Hall condition, so it is not independent and thus contains a hypercircuit. •

The matroid in the theorem is called the **circuit matroid** of the hypergraph. Matroids arising in this way are called **hypergraphic matroid**s.

5.2.2 Connectivity of matroids

Direct sum

For $i = 1, 2, \ldots, k$, let $M_i = (S_i, \mathcal{F}_i)$ be matroids on pairwise disjoint ground-sets S_i. Let $S := \cup_i S_i$ and $\mathcal{F} := \{F_1 \cup \cdots \cup F_k : F_i \in \mathcal{F}_i\}$. $\mathcal{F}$ is readily seen to satisfy the independence axioms. The matroid $(S, \mathcal{F})$ is called the **direct sum** of matroids $M_1, \ldots, M_k$. Obviously each circuit of M is a circuit of one of the direct summands M_i. Furthermore, the rank of a set $X \subseteq S$ is $\sum_i r_i(S_i \cap X)$. A matroid M is **separable** if it arises as the direct sum of smaller submatroids and **non-separable** if it does not. We call a subset $X \subseteq S$ **non-separable** if the submatroid $M|X$ is non-separable.

Theorem 5.2.7 *Let $\{S_1, S_2\}$ be a bipartition of S into non-empty subsets. M is the direct sum of its submatroids $M|S_i$ if and only if*

$$r(S_1) + r(S_2) = r(S). \tag{5.4}$$

Proof. It follows from the definition that if M is the direct sum, then $r(S_1) + r(S_2) = r(S)$.

Conversely, if M is not the direct sum of $M|S_1$ and $M|S_2$, then it has a circuit C intersecting both S_1 and S_2. Then $r(C) = |C| - 1$ and $r(S_1 \cap C) + r(S_2 \cap C) = |C|$. From the submodularity of r, we get $r(S_i) + r(C) \geq r(S_i \cap C) + r(S_i \cup C)$ for $i = 1, 2$. Adding these up and applying submodularity again, we have $r(S_1) + r(S_2) + 2r(C) \geq r(S_1 \cap C) + r(S_1 \cup C) + r(S_2 \cap C) + r(S_2 \cup C) \geq r((S_1 \cup C) \cap (S_2 \cup C)) + r((S_1 \cup C) \cup (S_2 \cup C)) + r(S_2 \cap C) + r(S_2 \cap C) = r(C) + r(S) + |C|$ from which $r(S_1) + r(S_2) \geq r(S) + |C| - r(C) > r(S)$, as required. •

A matroid M is called **connected** if it has at most one element or the hypergraph C_M of its circuits is connected.

Theorem 5.2.8 *For a matroid $M = (S, r)$, the following are equivalent.*

(A) *M is non-separable,*
(B) *M is connected,*
(C) *$r(X) + r(S - X) > r(X)$ for every non-empty proper subset X of S.*

Proof. By Theorem 5.2.7, (A) and (C) are equivalent while the equivalence of (A) and (B) follows from the definitions. •

Blocks of a matroid

If the hypergraph C_M of the circuits of a matroid M is disconnected, then its components define a partition $\{S_1, \dots, S_k\}$ $(k \geq 2)$ of S. It follows immediately from the definition that a matroid arises as the direct sum of its submatroids M_i on S_i. These connected matroids M_i are called the **blocks** of M.

It follows from Theorem 5.2.5 that a matroid is connected if and only if every two distinct elements belong to a circuit. Hence the circuit matroid of a graph G with at least 2 edges is connected if and only if G is 2-connected.

Theorem 5.2.9 *A matroid $M = (S, \mathcal{F})$ with $|S| \geq 2$ is connected if and only if the hypergraph C_B of fundamental circuits with respect to a basis B is connected.*

Proof. Obviously if C_B is connected then so is the larger C_M. Conversely, suppose there is a bipartition of S into two non-empty subsets S_1 and S_2 in such a way that every fundamental circuit with respect to B is completely included in S_1 or in S_2. Then $S_i \cap B$ is a maximal independent subset of S_i, that is, $r(S_i) = |B \cap S_i|$ $(i = 1, 2)$ from which $r(S_1) + r(S_2) = |B \cap S_1| + |B \cap S_2| = |B| = r(S)$. By Theorem 5.2.7, M is not connected. •

5.3 Bases, rank, and co-rank

A maximal independent set of a matroid is called a basis and its cardinality is the rank of the matroid. It is a simple exercise to see that set of bases uniquely determines the matroid.

5.3.1 Basis axioms

Given a set-system $\mathcal{B}$ on ground-set S, consider the following properties, called **basis axioms**.

(B1) $\mathcal{B}$ is non-empty.
(B2) If $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 - B_2$, then there is an element $x_2 \in B_2 - B_1$ for which $B_1 - x_1 + x_2 \in \mathcal{B}$.

Property (B2) is called the **exchange axiom**.

Theorem 5.3.1 *The bases of a matroid satisfy Properties* (B1) *and* (B2)*. If $\mathcal{B}$ satisfies the basis axioms, then*

$$\mathcal{F} := \{F : \text{there is a } B \in \mathcal{B},\ F \subseteq B\} \tag{5.5}$$

satisfies the independence axioms.

Proof. The first part is a direct consequence of the independence axioms. For the converse, it is obvious that $\mathcal{F}$ satisfies the first two independence axioms. Let us prove (I3$'''$). Its first half requires that the cardinalities of every two members $B_1, B_2 \in \mathcal{B}$ be the same.

Suppose, indirectly, that $|B_2| < |B_1|$ and select these sets so that $|B_2 - B_1|$ is minimum. By axiom (B2), for any $x_1 \in B_1 - B_2$ there is an element $x_2 \in B_2 - B_1$ for which $B_1' := B_1 - x_1 + x_2 \in \mathcal{B}$. But then $|B_2| < |B_1| = |B_1'|$ and $|B_2 - B_1'| < |B_2 - B_1|$, contradicting the choice of B_1 and B_2. Let r denote the common cardinality of the members of $\mathcal{B}$.

To see the second half of (I3$'''$), let $K, N \subseteq S$ be two members of $\mathcal{F}$ with cardinality $r - 1$ and r, respectively. Then $B_2 := N$ belongs to $\mathcal{B}$ and by definition there is a $B_1 \in \mathcal{B}$ and $x_1 \in B_1$ for which $K = B_1 - x_1$. If $x_1 \in B_2$, then $K + x_1 \in \mathcal{F}$. If $x_1 \notin B_2$, then axiom (B2) implies the existence of $x_2 \in B_2 - B_1$ for which $B_1 - x_1 + x_2 \in \mathcal{B}$, that is, K can indeed be extended with an element of N to get a member of $\mathcal{F}$. •

The exchange axiom has a mirror version.

Proposition 5.3.2 *If $B_1, B_2 \in \mathcal{B}$ and $x_2 \in B_2 - B_1$, then there is an element $x_1 \in B_1 - B_2$ for which $B_1 - x_1 + x_2 \in \mathcal{B}$.*

Proof. Consider the fundamental circuit C of x_2 with respect to B_1. This circuit cannot lie completely in B_2, and hence there is an element $x_1 \in C - B_2$ and this element satisfies the statement of the Proposition. •

The property formulated in Proposition 5.3.2 will be called the co-exchange axiom and denoted by (B2*). There is a common generalization of the two variations and this was already formulated for graphic matroids in Lemma 3.2.2.

Theorem 5.3.3 (symmetric basis exchange) *If $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 - B_2$, then there is an element $x_2 \in B_2 - B_1$ for which $B_1 - x_1 + x_2 \in \mathcal{B}$ and $B_2 - x_2 + x_1 \in \mathcal{B}$.*

We shall say that the elements x_1 and x_2 are mutually exchangeable. The property formulated in the theorem is called the **symmetric basis exchange property**.

Proof. Let C_2 denote the fundamental circuit of x_1 with respect to B_2. Consider a circuit C for which

$$x_1 \in C \subseteq B_1 \cup B_2 \text{ and } C - B_1 \subseteq C_2 - B_1 \tag{5.6}$$

and $|C - B_1|$ is minimal. (There is a circuit satisfying (5.6), for example, C_2 is such a circuit.) The minimum cannot be 0 because B_1 includes no circuit.

We claim that $|C - B_1| = 1$. Indeed, if indirectly $|C - B_1| > 1$, then consider the fundamental circuit C_1 of element $x \in C - B_1$ with respect to B_1. By the minimality of C, this circuit does not contain x_1. By the strong circuit axiom, there is a circuit $C' \subseteq C_1 \cup C_2 - x$ containing x_1 and the existence of such a circuit contradicts the minimal choice of C.

We have obtained that $C - B_1$ consists of a single element, denoted by x_2. Hence the circuit C is the fundamental circuit of x_2 with respect to B_1 and C contains x_1. Furthermore, x_2 is in the fundamental circuit of x_1 with respect to B_2. Therefore these elements are mutually exchangeable. •

The following theorem is another extension of axiom (B2).

Theorem 5.3.4 *For bases B_1 and B_2 there is a bijection $f : (B_1 - B_2) \to (B_2 - B_1)$ in such a way that $B_1 - x + f(x)$ is a basis for every element $x \in B_1 - B_2$.*

Proof. $B_1 - x + f(x)$ is a basis if and only if $x \in C(B_1, f(x))$ where $C(B_1, f(x))$ is the fundamental circuit of $f(x)$ with respect to basis B_1. Consider the set-system $\{C(B_1, z) - B_2 : z \in B_2 - B_1\}$. We claim that this satisfies the Hall condition, that is, that the union of any j sets has at least j elements. Indeed, choose j elements from $B_2 - B_1$ and consider their fundamental circuits $C_1, \ldots, C_j$ with respect to B_1. Let $K := \cup C_i$. What we need to show is that $|K - B_2| \geq |K - B_1|$. On the one hand, $r(K) \geq r(K \cap B_2) = |K \cap B_2|$. On the other, $K \cap B_1$ cannot be enlarged in K so as to maintain independence and hence $|K \cap B_1| = r(K) \geq |K \cap B_2|$ from which $|K - B_2| \geq |K - B_1|$ and hence Hall's condition does indeed hold.

By the Kőnig–Hall theorem, there is a bijection f such that each element $x \in B_1 - B_2$ is in the fundamental circuit of element $f(x)$ which means that $B_1 - x + f(x)$ is a basis. •

Problems

5.3.1 *Let $x_1, x_2, \ldots, x_k$ be elements of a basis B and let $y_1, y_2, \ldots, y_k$ be elements not in B. Suppose that each x_i belongs to the fundamental circuit $C(B, y_i)$ of y_i with respect to B but that $x_h \notin C(B, y_j)$ whenever $h > j$. Prove that $B - \{x_1 \ldots, x_k\} \cup \{y_1, \ldots, y_k\}$ is a basis.*

5.3.2 *Let B be a basis of matroid $M = (S, \mathcal{B})$. Suppose that we are given a 'level function' $\Theta : S \to \{0, 1, \ldots, n\}$ for which* $(*)$ $\Theta(v) \leq \Theta(u) + 1$ holds for every pair $\{u, v\}$ of elements with $u \in S - B$, $v \in C(B, u)$. *Let s and t be two elements for which $\Theta(t) = \Theta(s) + 1$, $s \in S - B$, $t \in C(B, s)$. Prove that Property $(*)$ holds with respect to basis $B' := B - t + s$, as well.* (For an application to a matroid push–relabel algorithm, see Lemma 13.3.8.)

5.3.3 *Prove that a bond and a circuit never intersect in exactly one element.*

Generalized partition matroid

Let $\{S_1, \ldots, S_t\}$ be a partition of S. Let $g_1, \ldots, g_t$ and $f_1, \ldots, f_t$ be non-negative integers for which $0 \leq f_i \leq g_i \leq |S_i|$. Furthermore, let k be a positive integer for which $\sum_i f_i \leq k \leq \sum_i g_i$. An easy exercise shows that the set-system $\mathcal{B} := \{X : |X| = k,\ f_i \leq |S_i \cap X| \leq g_i$ for $i = 1, \ldots, t\}$ satisfies the basis axioms. The resulting matroid is called a **generalized partition matroid**. When $f_i :\equiv 0$ and $k := \sum_i \min\{g_i, |S_i|\}$, we are back to partition matroids.

Problem 5.3.4 *Prove that a subset F is independent in the generalized partition matroid if and only if $|F \cap S_i| \leq g_i$ for every i and $\sum_i \max\{f_i, |F \cap S_i|\} \leq k$.*

Problem 5.3.5 *Show that if $\{S_1, \ldots, S_t\}$ is a cross-free set-system rather than just a partition, then $\mathcal{B} := \{X : |X| = k,\ f_i \leq |S_i \cap X| \leq g_i$ for $i = 1, \ldots, t\}$, if non-empty, satisfies the basis axioms.*

Question 5.3.1 *What is a necessary and sufficient condition for $\mathcal{B}$ to be non-empty?*

This question will be answered in a much more general form in Section 15.3 by a theorem of Fujishige.

Paving matroid

A different generalization of partition matroids is as follows.

Theorem 5.3.5 *Let $r \geq 2$ be an integer and S a set of size at least r. Let $\mathcal{H} := \{H_1, \dots, H_t\}$ be a* (possibly empty) *set-system of proper subsets of S in which every set H_i has at least r elements and the intersection of any two of them has at most $r-2$ elements. Let*

$$\mathcal{B}_{\mathcal{H}} = \{B \subseteq S,\ |B| = r,\ B \text{ is not a subset of any } H_i\}.$$

Then $\mathcal{B}_{\mathcal{H}}$ satisfies the basis axioms. •

Problem 5.3.6 *Prove Theorem* 5.3.5. (A more general matroid construction will be proved in Section 5.5.6.)

The matroid in the theorem is called a **paving matroid** [204, 379].

Problem 5.3.7 *Prove that a rank-r matroid is a paving matroid if and only if every circuit has at least r* (that is, r or $r+1$) *elements.*

Exercise 5.3.8 *Prove that in a paving matroid a subset I is independent if and only if $|I| \leq r$ and $|I \cap H_i| \leq r-1$ $(i = 1, \dots, t)$.*

Theorem 5.3.6 (Knuth [243]) *Let S be a ground-set of $n \geq 5$ elements and let $r = 2\lfloor n/4 \rfloor$. There are double-exponentially many* (namely, at least $2^{(2^r)}$) *paving matroids on S.*

Proof. It suffices to prove the theorem when $|S|$ is a multiple of 4 since if $|S| = 4k + \ell$ for some $0 < \ell < 4$, then removing ℓ elements from S does not change r. Partition S is into r disjoint pairs. We call a subset $X \subseteq S$ a twin set if X is the union of some of the given pairs. The intersection of two twin sets has even cardinality, which therefore cannot be $r-1$, an odd number.

The set-system $\mathcal{H}$ determines a paving matroid $M_{\mathcal{H}}$ for an arbitrary set-system $\mathcal{H}$ of twin subsets. The bases of $M_{\mathcal{H}}$ are precisely those r-element (not necessarily twin) sets which are not included in any members of $\mathcal{H}$. Hence distinct set systems define distinct matroids, and the number of set systems of this type is $2^{\binom{2r}{r}} > 2^{(2^r)}$. Therefore there are at least $2^{(2^r)}$ paving rank-r matroids on S. •

Theorem 5.3.6 has the important consequence that it is not possible to encode matroids with code words of size polynomial in $|S|$. For a comparison, note that rank-r graphic matroids can be encoded in polynomial space with the help of an undirected graph on $r+1$ nodes.

5.3.2 Rank and co-rank

It follows from the definition of the rank function that a subset is independent if and only if its size is equal to its rank, that is,

$$\mathcal{F} = \{X \subseteq S, r(X) = |X|\}. \tag{5.7}$$

This implies that distinct matroids on S have distinct rank functions. For the sake of a later generalization, we remark that, because $r(X) \leq |X|$, (5.7) is equivalent to the following:

$$\mathcal{F} = \{X \subseteq S, r(X) \geq |X|\}. \tag{5.8}$$

How can we recognize that a given set-function is the rank function of a matroid?

Theorem 5.3.7 *A non-negative, integer-valued set-function $r : 2^S \to \mathbf{Z}_+$ is the rank function of a matroid if and only if r satisfies the following properties, called* **rank axioms***.*

(R1) $r(\emptyset) = 0$ (zero on the empty set),
(R2) $r(X) \geq r(Y)$ *when* $Y \subset X \subseteq S$ (non-decreasing),
(R3) $r(X) \leq |X|$ *for every* $X \subseteq S$ (subcardinal),
(R4) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ *for every* $X, Y \subseteq S$ (submodular).

Proof. For a matroid rank function r, the first three properties directly follow from the definitions while the last one was proved in Lemma 5.1.3.

Conversely, suppose that r satisfies the axioms above. We show that the set-system $\mathcal{F}$ defined in (5.7) satisfies the independence axioms. To this end, first we prove the following.

(R3′) $r(A + e) \leq r(A) + 1$ whenever $A \subseteq S$ and $e \in S - A$.

Indeed, from submodularity, $r(A) + 1 \geq r(A) + r(e) \geq r(A \cap \{e\}) + r(A \cup \{e\}) \geq r(A + e)$, and hence (R3′) holds.

Lemma 5.3.8 *Let $A \subseteq S$ and $e_1, \dots, e_k \in S - A$. If $r(A + e_1) = \cdots = r(A + e_k) = r(A)$, then $r(A \cup \{e_1, \dots, e_k\}) = r(A)$.* (That is, if the addition of elements cannot individually increase the rank of A, then neither can their simultaneous addition.)

Proof. By induction. The lemma is trivial for $k = 1$ so suppose that $k \geq 2$ and also that the lemma holds for $k - 1$. That is, $r(A') = r(A)$ where $A' := A \cup \{e_1, \dots, e_{k-1}\}$. From (R2) and (R4), we get $r(A) + r(A) = r(A + e_k) + r(A') \geq r((A + e_k) \cap A') + r((A + e_k) \cup A') = r(A) + r(A \cup \{e_1, \dots, e_k\}) \geq r(A) + r(A)$ from which the lemma follows. •

(R1) implies that $\mathcal{F}$ satisfies the first independence axiom (I1). Let $X \subseteq Y \in \mathcal{F}$. Then $r(Y) = |Y|$. By the repeated application of Property (R3′), we have $r(Y) \leq r(X) + |Y - X|$, and hence $r(X) \geq |X|$. By (R3), $r(X) = |X|$, that is, $X \in \mathcal{F}$ and (I2) holds.

To show that (I3) holds for a subset $X \subseteq S$, consider a maximal subset F of X which is a member of $\mathcal{F}$. We claim that $|F| = r(X)$. Indeed, the maximality of F implies that $F + v \notin \mathcal{F}$ for every element $v \in X - F$, and hence $r(F) \leq r(F + v) \leq |F + v| - 1 = |F| = r(F)$. Here equality must hold everywere and by Lemma 5.3.8, we obtain $|F| = r(F) = r(X)$, showing that the cardinality of F depends only on X. Therefore $(S, \mathcal{F})$ is indeed a matroid and its rank function is r. •

Exercise 5.3.9 *Prove that* (R3) *can be replaced by* (R3′)*. Also prove that* (R3) *can be replaced by* (R3″)*: $r(s) \leq 1$ for every element $s \in S$.*

The rank $r(X)$ of a subset X can be interpreted as the maximum size of the intersection of X with a basis. The minimum size $t(X)$ of the intersection of X and a basis is the **co-rank** of X. Obviously $r(S) = t(S)$, and it can be checked easily that $t(X) = r(S) - r(S - X)$. This implies that t is supermodular (that is, $t(X) + t(Y) \leq t(X \cap Y) + t(X \cup Y)$).

A subset $X \subseteq S$ is said to be **closed** if $r(X + x) > r(X)$ for every element $x \in S - X$. A closed set is called a **flat**. The complement of a closed set is **open**. The ground-set is closed. A flat of rank $r(S) - 1$ is a **hyperplane**.

The **closure** $\mathrm{cl}(X)$ of a subset X consists of those elements whose addition to X does not increase its rank. The closure of X is sometimes called the set **generated** or **spanned** by X. By Lemma 5.3.8, $\mathrm{cl}(X)$ is the unique largest superset of X with the same rank. Also, $\mathrm{cl}(X)$ is the intersection of flats including X.

Problems

5.3.10 *Show that flats are closed under intersection. Show that a subset is a hyperplane if and only if it is the complement of a cut.*

5.3.11 (∗) *Let X be a closed set for which $r(X) < r(S)$. Show that X can be obtained as the intersection of $r(S) - r(X)$ hyperplanes. Show that a non-empty set is open if and only if it is the union of cuts.*

5.3.12 *Let F be a maximal independent subset of X. Prove that X is closed if and only if $F + s$ includes no circuits for every $s \in S - X$.*

5.3.13 *Prove that in the circuit matroid of a graph a subset of edges is open if and only if it is the border of a partition of V.*

5.4 Constructing matroids

5.4.1 Operations on matroids

Parallel multiplication

For a non-loop element s of a matroid M, the parallel multiplication of s means the following. Let k be a non-negative integer. Replace s by a set $S' := \{s_1, \ldots, s_k\}$ disjoint from S (where $S' = \emptyset$ when $k = 0$) and declare a subset X of the resulting ground-set $(S - s) \cup S'$ to be independent if either $X \subseteq S - s$ and X is independent in M or $|X \cap S'| = 1$ and $(X - S') + s$ is independent in M. In this way, we get a matroid, in which any two new elements form a circuit. In the circuit matroid of a graph this operation corresponds to replacing an edge with k parallel copies. Note that multiplicating an element $k = 0$ times is the same as deleting it.

It is useful to consider the following interpretation. Let $(S, T; E)$ be a bipartite graph and let M be a loopless matroid on S. Then M can be 'placed on' the edge-set E by declaring a subset $F \subseteq E$ to be independent if the end-nodes of F are distinct in S and form an independent set of M. This matroid M' is isomorphic to the one obtained from M by multiplying each node $s \in S$ in parallel $d_G(s)$ times.

Serial multiplication

As before, replace s by a set $S' := \{s_1, \ldots, s_k\}$ disjoint from S and declare a subset X of the resulting ground-set $(S - s) \cup S'$ to be a circuit if either $X \subseteq S - S'$ and X is a circuit of M or else $S' \subseteq X$ and $X - S' + s$ is a circuit of M. This is a matroid (as its circuits satisfy the

978–0–19–920527–1 05-Andras-Frank-c05-drv Andras Frank (Typeset by SPi, Publisher Services, India) 186 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

circuit axioms). In the circuit matroid of a graph, this construction corresponds to replacing an edge by a path of length k or, in other words, subdividing an edge by $(k-1)$ new nodes.

Shortening

Let $M = (S, \mathcal{F})$ be a matroid and $g \geq 0$ an integer. By a g-**shortening** or, for short, **shortening** of M, we mean a matroid M_g in which a set X is independent if it belongs to $\mathcal{F}$ and its size is at most g. This construction defines a matroid for which the rank function is given by $r_g(X) = \min\{r(X), g\}$. In the literature this operation is often called truncation but we reserve this term for Dilworth truncation, a far more relevant matroid construction to be introduced in Chapter 12.

Elongation

Let $M = (S, \mathcal{F})$ be a matroid and let $f \geq 0$ be an integer. By the f-**elongation** or, for short, **elongation** of M, we mean the matroid M^f in which a subset X is independent if it is obtained from an independent set of M by adding at most f elements of S. This is a matroid for which the rank function is given by $r^f(X) = \min\{r(X) + f, |X|\}$.

Adjoint

Let $M = (S, \mathcal{F})$ be a matroid, $Z \subseteq S$, and z a new element. In the **adjoint of** z **along** Z a subset X of the ground-set, $S + z$ is independent if $X \in \mathcal{F}$ or if $z \in X$ and Z has an element $z' \notin X$ for which $X - z + z' \in \mathcal{F}$. If Z consists of the single node z', then the adjoint of z is just parallel duplication. In an affine matroid, the adjoint operation corresponds to adding a new element which belongs to the flat spanned by Z but is otherwise in general position.

Exercise 5.4.1 *Prove that the adjoint is indeed a matroid whose rank function r' is determined by $r'(X) = r(X)$, when $X \subseteq S$ and $r'(X) = \min\{r(X \cup Z - z), r(X - z) + 1)\}$ when $z \in X$.*

Composition

Let $M_i = (S_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two matroids whose ground-sets are disjoint. We define the set $\mathcal{B}$ of bases of a matroid on the ground-set $S := S_1 \cup S_2$ as follows. For every pair of sets $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ with $|F_1| = |F_2|$, let $(S_1 - F_1) \cup F_2$ be a basis.

Exercise 5.4.2 *Show that $\mathcal{B}$ satisfies the basis axioms.*

The matroid defined in this way is denoted by $M_1 \circ M_2$ and is called the **composition** of M_1 and M_2. Note that the role of M_1 and M_2 is not symmetric, since $M_1 \circ M_2$ and $M_2 \circ M_1$ are distinct matroids. A simple relation between them is that a subset is a basis of $M_1 \circ M_2$ if and only if its complement is a basis of $M_2 \circ M_1$.

Exercise 5.4.3 *Prove that the rank function r of $M_1 \circ M_2$ is given by $r(X) = |S_1 \cap X| + \min\{r_2(S_2 \cap X), r_1(S_1 - X)\}$.*

Homomorphic image

Let $M = (S, \mathcal{F})$ be a matroid, T a set (not necessarily disjoint from S), and $\varphi : S \to T$ a map. Consider the set $\mathcal{F}' := \{X' \subseteq T :$ there is a set $X \in \mathcal{F}$ for which $\varphi(X) = X'\}$.

Informally, $\mathcal{F}'$ consists of those subsets of T which are obtained as the image of an independent set of M.

Theorem 5.4.1 (Nash-Williams [306]) *$\mathcal{F}'$ satisfies the independence axioms.*

Proof. The first two independence axioms obviously hold. In order to prove Axiom (I3′), let $K', N' \in \mathcal{F}'$ be subsets of T for which $|K'| < |N'|$. Then there are subsets $K, N \in \mathcal{F}$ for which $\varphi(K) = K'$ and $\varphi(N) = N'$. We can assume that $|K| = |K'|$ and $|N| = |N'|$, since in the case $|K| > |K'|$, say, K would have two elements e and f with $\varphi(e) = \varphi(f) \in K'$, and then e could be left out from K.

Let us select K and N in such a way that $|K \cap N|$ is as large as possible. By applying Axiom (I3′) to K and N, we conclude that there is an element $e \in N - K$ for which $K + e \in \mathcal{F}$. We claim that $e' := \varphi(e)$ is not in K'. Indeed, if $e' \in K'$, then there is an element k of K for which $\varphi(k) = e'$ and hence $|N| = |N'|$ implies $k \notin N$. But then $K_1 := K + e - k \in \mathcal{F}$, $\varphi(K_1) = K'$ and $|K_1 \cap N| > |K \cap N|$, contradicting the maximality of $|K \cap N|$. It follows that $K' + e'$ is the image of the independent set $K + e$, and hence (I3′) indeed holds. •

The matroid $(S, \mathcal{F}')$ arising in Theorem 5.4.1 is called the **homomorphic image** of M and is denoted by $\varphi(M)$. A formula for its rank function r_φ is equivalent to a basic result of Rado and will be discussed in Section 13.1. See Theorem 13.1.6 which asserts that

$$r_\varphi(Z) = \min\{r(\varphi^{-}(X)) + |Z - X| : X \subseteq Z\} \tag{5.9}$$

for every subset $Z \subseteq T$.

Sum of matroids

The direct sum operation was seen to be a simple construction to produce a new matroid from k matroids given on disjoint ground-sets. The sum operation is more sophisticated (and more applicable): it produces a matroid from matroids on the same ground-set. Let $M_1, \dots, M_k$ be matroids on a common ground-set S. A subset of S is said to be **partitionable** if it can be obtained as the union of sets $F_1, F_2, \dots, F_k$ where each F_i is independent in M_i.

Theorem 5.4.2 (Edmonds and Fulkerson [85]) *Given matroids $M_1, M_2, \dots, M_k$ on a common ground-set S, the set $\mathcal{F}_\Sigma$ of partitionable subsets of S satisfies the independence axioms.*

Proof. For $i = 1, \dots, k$, let S_i be a copy of S such that the sets S_i are pairwise disjoint. Let M'_i denote a copy of M_i on S_i and let M_{big} be the direct sum of these matroids. Consider the map φ of $S_1 \cup S_2 \cup \cdots \cup S_k$ to S in which the image of every element of S_i is the corresponding element of S. It follows from this construction that the images of the independent sets of M_{big} are exactly the members of $\mathcal{F}_\Sigma$ and hence Theorem 5.4.1 implies the validity of the independence axioms. •

A rank formula for the sum will also be derived in Section 13.1.

Relating homomorphic image and adjoint

Consider a matroid M on S and a map $\varphi : S \to T$ where $T = \{t_1, \dots, t_k\}$. The preimage of T determines a partition $\{S_1, \dots, S_k\}$ of S. It follows from the definitions that the homomorphic image $\varphi(M)$ can be viewed as a matroid arising from M by first adjoining the elements $t_1, \ldots, t_k$ along the sets $S_1, \ldots, S_k$, respectively, and then restricting the resulting matroid to T.

Conversely, the adjoint can also be obtained from the homomorphic image. To see this, suppose that a matroid M' arose from M by adjoining a new element z along $Z \subseteq S$. Let M^+ be a matroid arising from M by adding a parallel copy of each element of Z. Let Z^+ denote the set of newly added elements, let $T := S + z$ and define a map $\varphi : S \cup Z^+ \to T$ as follows. For $x \in S$, let $\varphi(x) = x$, while for $x \in Z^+$, let $\varphi(x) = z$. Then the adjoint matroid M' is just the homomorphic image $\varphi(M^+)$.

5.4.2 Dual matroid

Suppose that a matroid $M = (S, \mathcal{B})$ is given by its bases. The **dual of M** means the matroid $M^* = (S, \mathcal{B}^*)$ where $\mathcal{B}^* := \{X : S - X \in \mathcal{B}\}$. Informally, the bases of M^* are the complements of the bases of M. Obviously, the dual of the dual matroid is the original.

Theorem 5.4.3 *$\mathcal{B}^*$ satisfies the basis axioms. The rank function r^* of the dual matroid is given by*

$$r^*(X) = |X| + r(S - X) - r(S). \tag{5.10}$$

Proof. The first basis axiom clearly holds. To see (B2) let $B_1^* = S - B_1$ and $B_2^* = S - B_2$ be two members of $\mathcal{B}^*$, that is, B_1 and B_2 are bases of M. For every element $x \in B_1^* - B_2^*$, we need to show that there is an element $y \in B_2^* - B_1^*$ for which $B_1^* - x + y \in \mathcal{B}^*$. This is equivalent to requiring that there is an element $y \in B_1 - B_2$ for $x \in B_2 - B_1$ such that $B_1 + x - y$ is a basis of M, but this last assertion is just Proposition 5.3.2.

To prove (5.10), note that $r^*(X)$ is the minimum size of the intersection of X and a dual basis. This is determined by the minimum size of the intersection of a basis of M and $S - X$ from which an easy calculation shows that $r^*(X) = |X| - r(S) + r(S - X)$. •

It follows from the definition of the dual matroid that a set $X \subseteq S$ is independent in M if and only if its complement is a generator of the dual matroid.

Exercises

5.4.4 *Prove that $r^*(X) + t(X) = |X|$ where t is the co-rank function.*

5.4.5 *Prove that a matroid is connected if and only if its dual is.*

5.4.6 *Prove that the circuits of a matroid are exactly the bonds of the dual matroid.*

Problem 5.4.7 *Prove that the circuit matroid of a planar graph and the circuit matroid of its planar dual are dual matroids.*

5.4.3 Minors: deletion and contraction

Let M be a matroid of rank function r on ground-set S. We have already become familiar with the operation of deletion, which gave rise to a submatroid of M. Contraction is a kind of dual operation of deletion. Let Z be a proper non-empty subset of S and let $S' := S - Z$. Define the set-function $r' : 2^{S'} \to \mathbf{Z}_+$ as follows

$$r'(X) := r(X \cup Z) - r(Z). \tag{5.11}$$

978–0–19–920527–1 05-Andras-Frank-c05-drv Andras Frank (Typeset by SPi, Publisher Services, India) 189 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Exercise 5.4.8 *Prove that r' satisfies the rank axioms.*

Let M' be the matroid on S' determined by r'. We say that M' arises from M by **contracting** Z or that M' is a contraction of M to $(S - Z)$. In notation, $M' = M/Z$ or $M' = M \cdot (S - Z)$.

Theorem 5.4.4 *The following properties are equivalent.*

(A) *$F \subseteq S'$ is independent in M'.*
(B) *For every maximal subset I of Z which is independent in M, the set $I \cup F$ is independent in M.*
(C) *There is a maximal subset I of Z that is independent in M such that $I \cup F$ is independent in M.*

Proof. (A)→(B). Let I be a maximal subset of Z which is independent in M. I can be extended to a maximal M-independent subset $F' \cup I$ of $F \cup Z$. By Property (A), F is independent in M' implying that $r(F \cup Z) - r(Z) = |F|$, and hence $|F \cup I| \geq |F' \cup I| = r(F \cup Z) = |F| + r(Z) = |F| + |I|$. Here we must have equality and thus $F' = F$, from which Property (B) follows.

The direction (B)→(C) is trivial. Let us assume now that (C) holds. Now $r(F \cup I) = |F \cup I|$ and $r(Z) = |I|$, therefore $r'(F) = r(F \cup Z) - r(Z) \geq r(F \cup I) - r(Z) = |F \cup I| - |I| = |F| \geq r'(F)$. We must have equality throughout and thus F is independent in M', and hence Property (A) holds. •

It follows from the definitions that, given two disjoint subsets Z_1, Z_2 of S, the same matroid M' arises if we first contract Z_1 and then delete Z_2 or if we first delete Z_2 and then contract Z_1. The resulting matroid M' is called a **minor** of M.

Problems

5.4.9 *Prove that matroid contraction in the circuit matroid of a graph corresponds to the graph operation of contracting some edges of the graph. In a vector matroid, the contraction of a non-null vector z is equivalent to projecting the other vectors onto the hyperplane orthogonal to z.*

5.4.10 *Let $Z_1 \subset Z \subset S$ and $Z_2 := Z - Z_1$. Prove that $M/Z = (M/Z_1)/Z_2 = (M/Z_2)/Z_1$ and $(M - Z_1)/Z_2 = (M/Z_2) - Z_1$.*

5.4.11 *Prove that $(M/Z)^* = M^* - Z$ and $(M - Z)^* = M^*/Z$.*

5.4.12 *Prove that for every element s of a connected matroid M with at least two elements, the contraction or the deletion of s results in a connected matroid.*

5.4.4 Matroids from matchings and paths

Transversal matroids and deltoids

Let $G = (S, T; E)$ be a bipartite graph. A subset $I \subseteq S$ is said to be **matchable** if there is a matching of G covering I. The matchable sets satisfy the independence axioms as the first two are trivial while (I3) follows from Corollary 2.4.2. A matroid obtained in this way is called a **transversal matroid**. Its rank function can be obtained from Theorem 2.4.4:

$$r(S') = \min\{|S' - X| + |\Gamma(X)| : X \subseteq S'\}. \tag{5.12}$$

Because of the correspondence between hypergraphs and bipartite graphs, transversal matroids can be obtained in an alternative way. Let $\mathcal{T} := \{A_1, A_2, \ldots, A_t\}$ be a family of subsets of S. A subset I of S is called a **subtransversal** if it is possible to assign a member A_i to each element x of I such that every set is assigned to at most one element. Then the subtransversals form the independent sets of a transversal matroid. Yet another equivalent version is as follows. Let $(S, \mathcal{T})$ be a hypergraph. A subset $\mathcal{F}$ of hyperedges is said to be representable if it has a system of distinct representatives. Defined on the ground-set $\mathcal{T}$, the representable subhypergraphs form the independent sets of a transversal matroid.

The circuit matroid of K_4 (the complete graph on 4 nodes) is not a transversal matroid. It follows from Theorem 2.4.3 of Mendelsohn and Dulmage that a rank-r transversal matroid on S can be given by a bipartite graph $(S, T'; E')$ with $|T'| = r$. Namely, the subgraph induced by $S \cup T'$ of G suffices where T' denotes the set of nodes of T covered by an arbitrarily chosen maximum matching of G.

A related construction, due to Ingleton and Piff [216] is as follows. Let $\mathcal{B}$ be a set-system consisting of subsets B of $S \cup T$ for which the size is $|S|$ and arise as the symmetric difference of S and the node-set covered by a matching. A simple exercise shows that $\mathcal{B}$ satisfies the basis axioms. The resulting matroid on the ground-set $S \cup T$ is called a **deltoid of base** S. The dual of a deltoid is also a deltoid since the deltoid of base S and the deltoid of base T are the duals of each other. Also, the transversal matroid on S (defined by G) is a submatroid of the deltoid of base T. On the other hand, a deltoid of base S is a transversal matroid defined by the hypergraph $(S \cup T, \{\Gamma_G(s) + s : s \in S\})$.

Corollary 5.4.5 *A matroid is transversal if and only if it is a submatroid of a deltoid.* •

Exercise 5.4.13 *Prove that a matroid is transversal if and only if it is the homomorphic image of a partition matroid.*

Matching matroids

Let $G = (V, E)$ be a simple graph. Using alternating paths, it is not difficult to see that the subsets of nodes which can be covered by a matching of G form the independent sets of a matroid, called the **matching matroid** of G. Determining the rank function of the matching matroid is more complicated since it needs the knowledge of the Berge–Tutte formula on the maximum cardinality of a matching in a graph. In this light, it is surprising that every matching matroid is isomorphic to a transversal matroid, as proven in [85] by Edmonds and Fulkerson.

Gammoids

In Section 2.5, we showed how an equivalent version of Menger's theorem (Theorem 2.5.9) can be derived with the help of the node-duplicating technique from the Kőnig–Hall theorem. That construction naturally makes one feel that transversal matroids should somehow be converted to a matroid on digraphs defined by disjoint paths. Indeed, this idea can be worked out nicely.

Let $D = (V, A)$ be a directed graph and $R \subseteq V$ a non-empty subset of nodes. We say that R is **linked to** a subset Z of nodes if $|R| = |Z|$ and there are $|R|$ disjoint dipaths with initial nodes in R and terminal nodes in Z. By this definition, R is always linked to itself. Let

$$\mathcal{B}_R = \{Z \subseteq V : R \text{ linked to } Z\}.$$

The following result is due to Mason [284].

Theorem 5.4.6 *The set-system $\mathcal{B}_R$ satisfies the basis axioms.*

Proof. Let V' and V'' be disjoint copies of V and $V - R$, respectively. We use the notational convention that for any element v or subset X of V, the corresponding element and subset of V' will be denoted by v' and X', respectively. Similarly, we denote by v'' and X'' the element and subset of V'' corresponding to $v \in V$ and $X \subseteq V - R$, respectively.

Let $G' = (V', V''; E)$ be a bipartite graph in which $u' \in V'$ and $v'' \in V''$ are adjacent if $u = v \in V - R$ or if uv is an edge of D. Consider the transversal matroid M' on V' determined by G'. The maximum cardinality of a matching in G is $|V - R|$ since $\{v'v'' : v \in V - R\}$ is a matching of $|V - R|$ elements and the edges of G can be covered by a set of $|V - R|$ nodes since V'' is such a covering. Therefore, the rank of M' is $|V - R|$.

Proposition 5.4.7 *A set $B' \subseteq V'$ is a basis of M' if and only if R is linked to $V - B$.*

Proof. Suppose first that B' is a basis of M' and let N be a matching of G covering B'. From the construction, $|B| = |V - R| = |N|$.

For an arbitrary node $v' \in V' - B'$, we construct a dipath $P(v)$ in D starting in R and terminating in v. If $v' \in R'$, then $P(v)$ consists of the single node v and no edges. Suppose now that $v' \notin R'$. Since N covers V'', there is an edge $e_1 \in N$ covering v''. Let v'_1 denote the other end-node of e_1. If v_1 is not in R, then there is an element e_2 of N covering v''_1. Let v'_2 denote the other end-node of e_2. Continuing this procedure, we eventually arrive at a stage when the currently chosen matching edge is $e_k = v''_{k-1}v'_k$ and v_k is in R. The selected matching edges determine the requested dipath $P(v)$ of D from $v_k \in R$ to v. It is easy to check that the dipaths $P(v)$ for $v \in V - B$ are pairwise disjoint implying that R is indeed linked to $V - B$.

Conversely, consider a system of $|R|$ disjoint dipaths of D from R to some set $X \subseteq V$. Consider those edges of G' corresponding to the edges of the path system along with $v'v''$-edges for which $v \in V - R$ does not belong to any of the $|R|$ paths. These edges of G' form a matching N of G' that covers V''. Moreover, the set of nodes in V' exposed by N is $V' - X'$, and hence $V' - X'$ is a basis of the transversal matroid. •

By the proposition, a subset Z is in $\mathcal{F}_R$ if and only if $V' - Z'$ is a basis of the transversal matroid M'. • •

The matroid in the theorem is called a **strict gammoid**. A submatroid of a strict gammoid is a **gammoid**.

Theorem 5.4.8 (Ingleton and Piff [216]) *A matroid is a strict gammoid if and only if its dual is transversal.*

Proof. The proof of Theorem 5.4.6 showed that a strict gammoid is the dual of a transversal matroid. To see the converse, consider a bipartite graph $G = (V', V''; E)$ and the transversal matroid M on V' defined by G. As we already noted above, it can be assumed by the theorem of Mendelsohn and Dulmage (Theorem 2.4.3) that $r(M) = |V''|$, and hence there is a matching N of G covering V''.

Let R' denote the set of nodes in V' exposed by N. Construct a digraph $D = (V, A)$ in which V is a copy of V' and xy is an edge of D if $x' \in V'$, $y'' \in V''$ and $x'y'' \in E - N$. Let R denote the subset of V corresponding to R'. Using the proof technique of Theorem 5.4.6, we obtain that the strict gammoid defined by D and that R is the dual of M. •

Since strict gammoids are exactly the dual of transversal matroids, transversal matroids are submatroids of deltoids, and deltoids are closed under taking dual, we obtain the following corollaries.

Corollary 5.4.9 *Gammoids are closed under taking dual and minors.* •

Corollary 5.4.10 *Gammoids are exactly the minors of deltoids and exactly the contractions of transversal matroids.* •

Research problem 5.4.1 (Bárász [11]) *Suppose that $M = (S, \mathcal{F})$ is a gammoid on a ground-set S. This means that there is a digraph $D = (V, A)$ with $S \subseteq V$ and a subset R of V such that $F \subseteq S$ is independent in M precisely if there are $|F|$ pairwise disjoint dipaths ending at F for which the initial nodes belong to R. Is it true that D can always be chosen to be relatively small? More formally, does there exist a bound for $|V|$ which is polynomial in the size of S? Intuitively, it is rather unlikely that a gammoid on ten nodes, for example, can only be defined with the help of a digraph with a million nodes.*

5.5 Matroid algorithms and polyhedra

5.5.1 Oracles

In order to speak at all of matroid algorithms and of their complexity, the form in which a matroid is given should be made clear. For example, listing the independent sets is not satisfactory, since their number is typically exponential in the size of the ground-set S, while we want algorithms to be polynomial in $|S|$. A graphic matroid M_G of a graph G, for example, can be specified by the incidence matrix of G, and we do not have to list all forests of G in order to define M_G. In this sense, the graph is a concise way to encode M_G. With this analogy in mind, it would be highly desirable to develop a general result on matroids stating intuitively that every matroid on S can be encoded in such a way that the size of the code is polynomial in $|S|$.

Unfortunately, there is no way to carry out this program since, as we briefly indicated in Section 5.3, there is a double-exponential number of matroids. To overcome this difficulty, Edmonds suggested the striking idea that, instead of giving the matroid in any explicit form, it suffices to have an independence oracle which tells for any input subset whether it is independent or not. The complexity for a matroid algorithm is then measured by the number of oracle calls and other conventional elementary steps. It is a different matter, independent of the matroid algorithm in question, how the independence oracle can be realized for specific matroids.

Of course, a matroid can be given by other oracles as well. For example, the rank oracle tells the rank of any input subset $X \subseteq S$, while the strong basis oracle admits a specified a basis and tells whether or not an input subset is a basis of the matroid. We have seen that the various axiom systems (independence, basis, rank, circuit, and so forth) are equivalent in the sense that any of them defines a matroid. The corresponding oracles, however, are not necessarily polynomially equivalent. The following proposition is not difficult to prove.

Proposition 5.5.1 *The rank oracle, the independence oracle, and the strong basis oracle are polynomially equivalent.* •

5.5.2 The matroid greedy algorithm

Let $c : S \to \mathbf{R}$ be a weight function (or cost function). We are interested in constructing a basis of maximum weight, or a heaviest basis for short. Note that with the help of such an algorithm a maximum weight independent set can also be computed. Indeed, delete the elements of negative weight and construct a maximum weight basis of the resulting submatroid.

Let $\hat{r}(c)$ denote the maximum weight of a basis. Function $\hat{r}$ is called the **vector-rank** function of M. The name is motivated by the observation that in the special case when c is the characteristic vector of a subset $Z \subseteq S$, $\hat{r}(c)$ is just $r(Z)$. That is, the vector-rank function can be considered as an extension of the rank function.

The greedy algorithm considers the elements one by one in an order in which the weights are non-increasing. The algorithm selects the current element if it is independent from the set of elements selected already. By the independence axioms of matroids, the final set B_{alg} is a basis.

Theorem 5.5.2 *The basis B_{alg} computed by the greedy algorithm is a basis of maximum weight.*

Proof. Let B_{max} be a heaviest basis for which its intersection with B_{alg} is as large as possible. We claim that $B_{alg} = B_{max}$. Suppose indirectly that this is not the case and let f be the first element in the given order for which $f \in B_{alg} - B_{max}$.

By Theorem 5.3.3 there is an element $e \in B_{max} - B_{alg}$ which is mutually exchangeable with f. By the optimality of B_{max}, $c(e) \geq c(f)$ and here $c(e) = c(f)$ cannot actually occur since then $B_{max} - f + e$ would be another basis of maximum weight having a larger intersection with B_{alg}. But $c(e) > c(f)$ implies that e precedes f in the ordering and since $B_{alg} - f + e$ is independent, the algorithm should have selected e, a contradiction. •

Obviously, the greedy algorithm is suitable for computing a basis of minimum weight: apply the original algorithm to $-c$. The greedy algorithm was first introduced for the circuit matroid of graphs. Its cautious version described in Section 3.2 can now be interpreted as the ordinary greedy algorithm applied to the dual matroid. In terms of matroids this means that we drop out the subsequent cheapest element provided the rank does not decrease. The final basis is then one of maximum weight.

Problem 5.5.1 *Suppose that there are two weight functions c_1 and c_2. How can a basis of a matroid be constructed that is of maximum c_1-weight and, with respect to this, has maximum c_2-weight?*

Optimality criteria

Theorem 5.5.3 *A basis B is of maximum weight if and only if $c(y) \leq c(x)$ for every pair of elements $y \in S - B$ and $x \in C(B, y)$.*

Proof. If x is in the fundamental circuit $C(B, y)$, then $B - x + y$ is a basis from which $c(x) \geq c(y)$ follows when B is a heaviest basis.

Conversely, let B' be a heaviest basis and apply Theorem 5.3.4 with the choices $B_1 = B$ and $B_2 = B'$. By the hypothesis, $c(f(x)) \leq c(x)$ for every $x \in B - B'$ from which $\widetilde{c}(B') \leq \widetilde{c}(B)$. Since B' is heaviest, we must have $\widetilde{c}(B) = \widetilde{c}(B')$, and hence B is also a basis of maximum weight. •

Theorem 5.5.4 *An independent set F is of maximum weight if and only if*
(A) $c(x) \geq 0$ *for* $x \in F$,
(B) $c(y) \leq 0$ *whenever* $F + y$ *is independent and* $y \in S - F$, *and*
(C) $c(y) \leq c(x)$ *holds whenever* $F + y$ *is dependent and* $x \in C(F, y)$.

Proof. The conditions are obviously necessary. To see sufficiency, let S' denote the set of elements with strictly positive weight and let $F' := S' \cap F$. Since the elements in F are of non-negative weight, we have $\widetilde{c}(F') = \widetilde{c}(F)$ from which F is of maximum weight precisely if F' is heaviest in the submatroid $M' := M|S'$. Because the elements of S' are positive, the latter property is equivalent to requiring that F' is a heaviest basis of M'. The conditions on F in the theorem imply that the conditions in Theorem 5.5.3 hold and thus F' is a heaviest basis of M', that is, F is a heaviest independent set of M. •

Theorem 5.5.5 *Let $M = (S, r)$ be a matroid and let $c : S \to \mathbf{R}$ be a weight function. The maximum weight $\hat{r}(c)$ of a basis is equal to*

$$r(S)c(s_n) + \sum_{i=1}^{n-1} r(S_i)[c(s_i) - c(s_{i+1})], \tag{5.13}$$

where $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$ and $S_i := \{s_1, \ldots, s_i\}$.

Proof. By running the greedy algorithm on this ordering of the elements, one obtains a basis B for which $|B \cap S_i| = r(S_i)$ for every i. From this we have

$$r(S)c(s_n) + \sum_{i=1}^{n-1} r(S_i)[c(s_i) - c(s_{i+1})]$$

$$= |B \cap S|c(s_n) + \sum_{i=1}^{n-1} |B \cap S_i|[c(s_i) - c(s_{i+1})] = \sum_{s \in B} c(s) = \widetilde{c}(B) = \hat{r}(c). \bullet$$

Due to the formula in (5.13), the vector-rank function $\hat{r}$ of M is also called the **linear extension** of r.

Problems

5.5.2 *Prove that the set of heaviest bases satisfies the basis axioms.*

5.5.3 *With the help of the greedy algorithm, prove for an integer-valued c that* $\hat{r}(c + \underline{\chi}_Z) = \hat{r}(c) + r_c(Z)$ *for* $Z \subseteq S$ *where* $r_c(Z)$ *denotes the maximal size of the intersection of Z and a heaviest basis* (that is, the rank of Z in the matroid defined in Problem 5.5.2).

5.5.4 *For an integer-valued c, prove that* $\hat{r}(c - \underline{\chi}_Z) = \hat{r}(c) + r_c(S - Z) - r(S)$.

5.5.5 *Let* $c : S \to \mathbf{Z}_+$ *be integer-valued and let* $T \subseteq S$ *be a subset for which* $x \in T$ *and* $y \in S - T$ *imply that* $c(x) > c(y)$. *Then* $\hat{r}(c - \underline{\chi}_T) = \hat{r}(c) - r(T)$.

5.5.6 *Prove that* $\hat{r}$ *is subadditive, that is, that* $\hat{r}(c_1) + \hat{r}(c_2) \geq \hat{r}(c_1 + c_2)$.

5.5.7 *Determine algorithmically if an independent set can be extended to a heaviest basis.*

5.5.8 *Determine algorithmically if there is a basis which is simultaneously heaviest with respect to given weight functions* $c_1, \ldots, c_k$.

5.5.9 *Prove the following statment. Let B be a heaviest basis with respect to a weight function c. Suppose that* $x_1, x_2, \ldots, x_k$ *are in B, while* $y_1, y_2, \ldots, y_k$ *are outside, such that* $x_i \in C(B, y_i)$ *and* $c(x_i) = c(y_i)$ *for* $i = 1, \ldots, k$. *Furthermore,* $h > j$ *and* $c(x_h) = c(y_j)$ *imply that* $x_h \notin C(B, y_j)$. *Then* $B' := B - \{x_1 \ldots, x_k\} \cup \{y_1, \ldots, y_k\}$ *is also a heaviest basis.*

5.5.3 Polyhedra of matroids

Suppose that we are given again a weight funtion c and that the elements of S are indexed in such a way that $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$. The following three theorems are due to Edmonds [81].

Theorem 5.5.6 *The linear program*

$$\max \left\{ cx : x \in \mathbf{R}^S, x \geq 0, \widetilde{x}(Z) \leq r(Z) \text{ for every } Z \subset S \text{ and } \widetilde{x}(S) = r(S) \right\} \tag{5.14}$$

has an integer-valued optimum solution (which is automatically (0, 1)-valued). *The dual linear program*

$$\min \left\{ \sum_{Z \subseteq S} y(Z) r(Z) : \sum_{s \in Z} y(Z) \geq c(s) \text{ for every } s \in S, \text{ and } y(Z) \geq 0 \text{ for every } Z \subset S \right\} \tag{5.15}$$

has an optimal solution y for which the set-system $\{Z : y(Z) > 0\}$ *is a chain. Moreover, for an integer-valued c, the optimal y can also be chosen to be integer-valued. Concisely, the linear system in* (5.14) *is totally dual integral.*

Proof. Let B be a basis of maximum weight and let x be its characteristic vector. For $i = 1, \dots, n$, let $S_i := \{s_1, \dots, s_i\}$. Furthermore, let y be defined by

$$y(S_n) := c(s_n) \tag{5.16}$$

and

$$y(S_i) = c(s_i) - c(s_{i+1}) \text{ for } i = 1, \dots, n-1. \tag{5.17}$$

Then x is a (0, 1)-valued element of the primal polyhedron and y is in the dual polyhedron. By its very definition, y has the chain property and is integer-valued if c is. Moreover, by Theorem 5.5.5, we have $cx = yr$ $(:= \sum[y(Z)r(Z) : Z \subseteq S])$, that is, x is primal optimum and y is dual optimum. •

Theorem 5.5.7 *The linear program*

$$\max\{cx : x \in \mathbf{R}^S, x \geq 0, \widetilde{x}(X) \leq r(X) \textit{ for every } X \subseteq S\}$$

has an integer-valued optimum solution (which is automatically (0, 1)-valued). *The dual linear program*

$$\min\left\{\sum[y(Z)r(Z) : Z \subseteq S] : \; y \geq 0, \; \sum[y(Z) : s \in Z] \geq c(s) \textit{ for every } s \in S\right\}$$

has an optimal solution y for which the set-system $\{Z : y(Z) > 0\}$ is a chain. Moreover, for an integer-valued c, the optimal y can also be chosen to be integer-valued. Concisely, the linear system $\{x \geq 0, \widetilde{x}(X) \leq r(X)$ for every $X \subseteq S\}$ is totally dual integral.

Proof. For a non-negative c, the primal and dual optima in Theorem 5.5.6 will serve here as the corresponding optima since in this case the value $y(S) = c(s_n)$ is non-negative. If c is non-positive, then $x :\equiv 0$ is a primal, while $y :\equiv 0$ is a dual solution and they are optimal. Suppose now that c has both positive and negative components and let i be the largest subscript for which $c(s_i) > 0 \geq c(s_{i+1})$. Let $S' := S_i$ and $M' := M|S'$. Let x' and y' be the primal and the dual solution ensured by Theorem 5.5.6 with respect to the matroid M'. Then the vector x obtained by extending x' with zero components is a primal optimal solution and the unchanged y' is a dual optimal solution to the pair of linear programs in Theorem 5.5.7. •

For a matroid M with rank function r, the **matroid polytope** (or **independence polytope**) $P(r)$ is defined as the convex hull of the incidence vectors of independent sets of M. Similarly, we can consider the **base-polytope** $B(r)$ of M defined as the convex hull of the incidence vectors of the bases of M.

By Theorem 4.1.3, every polytope is a polyhedron so it is right to call $P(r)$ the **matroid** (or **independence**) **polyhedron** and $B(r)$ the **base-polyhedron** of M. How can $P(r)$ and $B(r)$ be explicitly given as polyhedra? Let

$$P' := \{x \in \mathbf{R}^S : x \geq 0, \; \widetilde{x}(Z) \leq r(Z) \text{ for every } Z \subseteq S\} \tag{5.18}$$

and let

$$B' := \{x \in \mathbf{R}^S : x \geq 0, \; \widetilde{x}(Z) \leq r(Z) \text{ for every } Z \subset S \text{ and } \widetilde{x}(S) = r(S)\}. \tag{5.19}$$

Theorem 5.5.8 $B(r) = B'$ *and* $P(r) = P'$.

Proof. Clearly $B(r) \subseteq B'$ and $P(r) \subseteq P'$. By Theorems 5.5.6 and 5.5.7, the optimum over both B' and P' is attained at a (0, 1)-vector. In other words, the vertices of B' and P' are (0, 1)-valued. Since such a vertex is the characteristic vector of a basis (respectively, of an independent set), that is, it belongs to $B(r)$ (respectively, $P(r)$), it follows that $B(r) = B'$ and $P(r) = P'$. •

An application to finding a basis spanning given sets

We illustrate the benefit of the polyhedral view by an application of the greedy algorithm. Let M be a matroid on ground-set S and let $\{S_1, \ldots, S_k\}$ be a family of subsets of S. The problem is to develop an algorithm for deciding whether the matroid has a basis B for which

$$B \cap S_i \text{ spans } S_i \text{ for every } i. \tag{5.20}$$

In order to find the requested basis, consider the weight function $c := \sum_i \underline{\chi}_{S_i}$ and a heaviest basis B^* with respect to c, which is computed by the greedy algorithm.

Note that in the special case when the matroid is the circuit matroid of a complete graph, we arrive at the algorithm exhibited in Section 3.2 for computing a basic tree of a subtree hypergraph (Theorem 3.2.5).

Theorem 5.5.9 *There exists a basis satisfying* (5.20) *if and only if* B^* *satisfies* (5.20).

Proof. Sufficiency is evident. For necessity, assume that there is a basis B' satisfying (5.20). Consider the dual linear program in Theorem 5.5.6, and let $y'(Z) := 1$ if $Z = S_i$ for some $i = 1, \ldots, k$ while $y'(Z) := 0$ otherwise. By the definition of c, y' is a dual feasible solution. Hence the maximum c-weight of a basis is at most $\sum[r(S_i) : i = 1, \ldots, k]$ where equality holds for a basis B exactly when $r(S_i) = |B \cap S_i|$ for every i, which is just (5.20). Since there is such a basis B', we conclude that every basis of maximum weight satisfies (5.20). •

5.5.4 Polymatroids and polymatroidal sets

The vertices of a matroid polyhedron are the incidence vectors of independent sets. As a generalization, Edmonds [80] introduced the notion of polymatroids. A set-function b on S is called a **polymatroid function** if $b(\emptyset) = 0$, b is non-decreasing, and b is submodular. We shall almost exclusively work with integer-valued polymatroid functions, so unless otherwise stated, b is tacitly assumed to be integer-valued throughout.

Although some of the results (in particular, the polymatroid greedy algorithm below) extend to real-valued set-functions as well, and hence it would not be necessary to include a priori the integrality of b in the definition, we do so because all applications to be dealt with in this book need only integer-valued set-functions.

Note that, among the four basic properties of a matroid rank function formulated in Theorem 5.3.7, (R1), (R2), and (R4) continue to be required and only subcardinality is dropped.

By a **polymatroid** $P(b)$ we mean the polyhedron

$$P(b) := \{x \in \mathbf{R} : \; x \geq 0, \; \widetilde{x}(X) \leq b(X) \text{ for every } X \subseteq S\}. \tag{5.21}$$

By a **polymatroidal set** we mean the set of integral points of an integral polymatroid. When b is the rank function of a matroid M, the polymatroid $P(b)$ is just the matroid polyhedron while the polymatroidal set defined by b is the set of incidence vectors of indpendent sets of M.

Problem 5.5.10 *Let $G = (S, T; E)$ be a bipartite graph. Prove that the set-function γ defined by $\gamma(X) := |\Gamma_G(X)|$ for $X \subseteq S$ is a polymatroid function. Prove that an integral vector $m \in \mathbf{Z}_+^S$ belongs to the polymatroidal set determined by γ if and only if there is a subset $F \subseteq E$ of edges so that $d_F(t) \leq 1$ for every $t \in T$ and $d_F(s) = m(s)$ for every $s \in S$.*

Exercise 5.5.11 *For a hypergraph H, let $e_H(X)$ denote the number of hyperedges having a non-empty intersection with X. Show that e_H is a polymatroid function.*

As the base-polyhedron of a matroid can be described with the help of its rank function, it is natural to introduce the notion of a **base-polyhedron** $B(b)$ defined by a polymatroid function b.

$$B(b) := \{x \in \mathbf{R} : \widetilde{x}(S) = b(S) \text{ and } \widetilde{x}(X) \leq b(X) \text{ for every } X \subset S\}. \tag{5.22}$$

Note that in this definition of $B(b)$ we do not require explicitly the non-negativity of x since any member x of $B(b)$ is automatically non-negative: $x(s) = \widetilde{x}(S) - \widetilde{x}(S - s) = b(S) - \widetilde{x}(S - s) \geq b(S) - b(S - s) \geq 0$ for every $s \in S$. We also call $B(b)$ a **base-polymatroid** while the integral elements of $B(b)$ form a **base-polymatroidal set**. (Note that a base-polymatroid is not a polymatroid.) A base-polymatroid $B(b)$ is just a facet of the polymatroid $P(b)$. Observe that in the matroid case, $P(r)$ was defined as a polytope and was then proved in Theorem 5.5.8 actually to be a polymatroid. A polymatroid (as well as a base-polymatroid) is a polyhedron by definition, and therefore it is not a priori true that its vertices are integral. We are going to prove this latter property with the help of the polymatroid greedy algorithm.

Problem 5.5.12 *Let $G = (V, E)$ be a graph. By relying on the Orientation lemma* (Theorem 2.3.2)*, prove that $Z_{in} := \{m \in \mathbf{Z}^V$: there is an orientation of G in which the in-degree of every node v is $m(v)\}$ is a base-polymatroidal set, namely, Z_{in} is the set of integral elements of the base-polyhedron $B(e_G)$.* •

5.5.5 Polymatroid greedy algorithm

Edmonds [80] observed that the greedy algorithm described above for matroids can be extended to polymatroids as well. Let b be a polymatroid function and for a vector $c : S \to \mathbf{R}$ consider the optimization problem

$$\max\{cx : x \in P(b)\}. \tag{5.23}$$

If a component $c(s)$ of c happens to be negative, then $z(v) = 0$ holds for every optimal solution z to (5.23) since $0 \leq x \leq z$ implies that $x \in P(b)$. Therefore, we can assume that c is non-negative, since a negative component can be deleted from S, by extending an optimal solution to the reduced problem with zero components, one gets an optimal solution to the original problem. Writing out (5.23), we get the primal linear program:

$$\max\{cx : x \geq 0,\ \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subseteq S\} \tag{5.24}$$

that has the following dual:

$$\min\{yb : y \in D(c)\}, \tag{5.25}$$

where $yb := \sum_{Z\subseteq S} y(Z)b(Z)$ and

$$D(c) := \left\{ y \in \mathbf{R}^{2^S} : y(Z) \geq 0 \text{ for every } Z \subseteq S \text{ and } \sum_{Z\subseteq S} y(Z)\underline{\chi}_Z \geq c \right\} \tag{5.26}$$

is the dual polyhedron.

We describe a greedy algorithm providing solutions to the primal as well as to the dual problem. In this way, we offer a constructive proof of the property that the linear system in (5.24) is totally dual integral.

We may assume that the components of c are arranged in a decreasing order, that is, $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$. Let $S_i := \{s_1, \ldots, s_i\}$ for $i = 1, .., n$. Define a vector x_{alg} as follows.

$$x_{alg}(s_1) := b(s_1) \text{ and } x_{alg}(s_i) := b(S_i) - b(S_{i-1}) \text{ for } i = 2, \ldots, n. \tag{5.27}$$

It follows from the definition that $\widetilde{x}_{alg}(S_i) = b(S_i)$ for every $i = 1, \ldots, n$ where $\widetilde{x}_{alg}(Z) := \sum_{z\in Z} x_{alg}(z)$. Also, x_{alg} is integral.

Lemma 5.5.10 *The vector x_{alg} defined by* (5.27) *is in $P(b)$.*

Proof. We need to show that $x_{alg} \geq 0$ and

$$\widetilde{x}_{alg}(Z) \leq b(Z) \text{ for every } Z \subseteq S. \tag{5.28}$$

As the non-negativity of x_{alg} immediately follows from the assumption that b is non-decreasing, we concentrate on proving (5.28).

Let $m(Z)$ denote the largest subscript i for which $s_i \in Z$. We use induction on $m(Z)$. If $m(Z) = 1$, then $Z = \{s_1\}$ and hence $\widetilde{x}_{alg}(Z) = b(Z)$. Let $i := m(z) \geq 2$. Relying on the submodularity of b, we obtain $b(Z) + b(S_{i-1}) \geq b(Z \cap S_{i-1}) + b(Z \cup S_{i-1})$. By applying the inductive hypothesis to $Z \cap S_{i-1}$ and observing that $Z \cup S_{i-1} = S_i$, we obtain that

$$b(Z \cap S_{i-1}) + b(Z \cup S_{i-1}) \geq \widetilde{x}_{alg}(Z \cap S_{i-1}) + b(S_i)$$

from which

$$b(Z) \geq \widetilde{x}_{alg}(Z \cap S_{i-1}) + b(S_i) - b(S_{i-1}) =$$

$$\widetilde{x}_{alg}(Z \cap S_{i-1}) + x_{alg}(s_i) = \widetilde{x}_{alg}(Z),$$

as required. •

Define a vector y^* as follows.

$$y^*(S_n) := c(s_n) \text{ and } y^*(S_i) := c(s_i) - c(s_{i+1}) \text{ for } i = 1, 2, \ldots, n-1 \tag{5.29}$$

and $y^*(Z) := 0$ for any other subset of S. It is evident that y^* is a feasible solution to (5.25) and that y^* is integral whenever c is integral.

Theorem 5.5.11 (Edmonds [80]) *x_{alg} is an optimal solution to the primal program* (5.24) *and y^* is an optimal solution to the dual program* (5.25)*. The linear system in* (5.24) *is totally dual integral.*

Proof. Since x_{alg} is a primal solution and y^* is a dual solution, we have max $\leq$ min, and all we need to show is that $cx_{alg} = y^*b$. But this follows immediately from the optimality conditions of linear programming since a dual variable $y(Z)$ can only be positive if $Z = S_i$ and for such a set Z the corresponding primal inequality $\widetilde{x}(Z) \leq b(Z)$ is indeed met by equality due to the definition of x_{alg}. The optimality can also be obtained without referring to linear programming:

$$cx_{alg} = c(s_1)b(S_1) + \sum_{i=2}^{n} c(s_i)[b(S_i) - b(S_{i-1})]$$

$$= c(s_n)b(S_n) + \sum_{i=1}^{n-1}[c(s_i) - c(s_{i+1})]b(S_i) = \sum_{Z \subseteq S} y^*(Z)b(Z) = y^*b. \bullet$$

Remark 5.5.1 It is worth mentioning that the way in which the vector x_{alg} was defined in the polymatroid greedy algorithm above is even 'more greedy' than the one seen for matroids, in the sense that the matroid greedy algorithm took care of choosing an independent set of the matroid, while in defining x_{alg}, among the inequalities $\widetilde{x}(Z) \leq b(Z)$, only those were taken into consideration for which $Z = S_i$.

It follows from the greedy algorithm above that the vector x_{alg} is not only in $P(b)$ but it also belongs to the base-polyhedron $B(b)$ of b. When this observation is applied to the incidence vector $c := \underline{\chi}_A$ of a subset $A \subseteq S$, one obtains the following important result of Edmonds.

Theorem 5.5.12 *For every subset $A \subseteq S$, there is an element x_0 of a polymatroid $P(b)$ for which $\widetilde{x}_0(A) = b(A)$. In addition, x_0 can be chosen to be integral.* $\bullet$

Corollary 5.5.13 *A polymatroid P uniquely determines its defining polymatroid function, namely $b(A) = \max\{\widetilde{x}(A) : x \in P\}$.* $\bullet$

The unique polymatroid function defining P is called the **border** function of P.

5.5.6 Matroids versus polymatroids

Let us explore further the relationship of matroids and polymatroids.

Matroids from polymatroid functions

Our first goal is to show that not only matroid rank functions determine a matroid but that polymatroid functions also do. A difference, however, is that distinct polymatroidal functions can define the same matroid, which cannot be the case for two matroid rank functions.

978–0–19–920527–1 05-Andras-Frank-c05-drv Andras Frank (Typeset by SPi, Publisher Services, India) 201 of 640 December 11, 2010 15:16
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

For a set-function b on S, let

$$\mathcal{I}(b) := \{I \subseteq S : b(Y) \geq |Y \cap I| \text{ for every } Y \subseteq S\}. \tag{5.30}$$

When b is non-decreasing, then it suffices to require the inequalities in (5.30) only for subsets of I, that is,

$$\mathcal{I}(b) = \{I \subseteq S : b(Y) \geq |Y| \text{ for every } Y \subseteq I\}. \tag{5.31}$$

Theorem 5.5.14 *For a polymatroid function b on S, the family $\mathcal{I}(b)$ defined in* (5.30) *satisfies the independence axioms. The rank function r_b of the matroid $M_b = (S, \mathcal{I}(b))$ is given by the following formula.*

$$r_b(Z) = \min\{b(X) + |Z - X| : X \subseteq Z\}. \tag{5.32}$$

Proof. The first two independence axioms clearly hold. A nice trick of the proof is that the truth of independence axiom (I3) and formula (5.32) will be proved simultaneously. Observe first that the cardinality of any member $I \subseteq Z$ of $\mathcal{I}(b)$ is at most $b(X) + |Z - X|$ for every $X \subseteq Z$. Indeed, (5.31) implies $|I \cap X| \leq b(I \cap X) \leq b(X)$ and hence

$$|I| = |I \cap X| + |I - X| \leq b(X) + |Z - X|. \tag{5.33}$$

Here one has equality precisely if

$$|I \cap X| = b(X) \text{ and } Z - X \subseteq I. \tag{5.34}$$

Therefore if I and a certain subset X satisfy (5.34), then $|I|$ is equal to the minimum given in (5.32). Let $I \subseteq Z$ be a maximal member of $\mathcal{I}(b)$ in the sense that I cannot be extended by any element of Z to a member of $\mathcal{I}(b)$. For any subset $X \subseteq Z$, let $m(X) := |I \cap X|$. Obviously, $m(X) + m(Y) = m(X \cap Y) + m(X \cup Y)$. The assumption $I \in \mathcal{I}(b)$ means that $m(X) \leq b(X)$ holds for every subset X of I. We define a subset X as **tight** (with respect to I) if $m(X) = b(X)$. Clearly, the empty set is tight.

Lemma 5.5.15 *Tight sets are closed under taking intersection and union.*

Proof. Let X and Y be tight sets. Then $m(X) + m(Y) = b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \geq m(X \cap Y) + m(X \cup Y) = m(X) + m(Y)$ from which the lemma follows. •

Since I is maximal in Z, there is a subset $X_z \subseteq I + z$ for every element z of $Z - I$ for which $b(X_z) \leq |X_z| - 1 = m(X_z)$. This X_z necessarily contains z and hence $b(X_z) \geq b(X_z - z) \geq |X_z - z| = |X_z| - 1 \geq b(X_z)$ which implies that X_z is tight.

The repeated application of the lemma shows that there is a unique largest tight set X which contains each element of $Z - I$, that is, X meets (5.34). As mentioned above, this implies that (5.33) is satisfied with equality, and hence the cardinality of I depends only on X. Therefore the third independence axiom holds, and formula (5.32) is also proved. • •

The matroid $M_b = (S, \mathcal{I}(b))$ defined by (5.30) is said **to belong to** b.

Paving matroids again

With the help of Theorem 5.5.14, we show that paving matroids are indeed matroids. Let $r \geq 2$ be an integer and let S be a set of size at least r. Let $\mathcal{H} := \{H_1, \dots, H_t\}$ be a set-system of proper subsets of S in which every set H_i has at least r elements and the intersection of any two of them has at most $r-2$ elements. Define a set-function b as follows.

$$b(X) := \begin{cases} \min\{|X|, r-1\} & \text{if } X \subseteq H_i \text{ for some } i \\ \min\{|X|, r\} & \text{otherwise.} \end{cases} \tag{5.35}$$

It can easily be checked (by exploiting the properties of $\mathcal{H}$) that b is a polymatroid function. Therefore $M = (S, \mathcal{I}_b)$ is a matroid by Theorem 5.5.14. Now a subset I is in $\mathcal{I}_b$ if and only if $|I| \leq r$ and $|I \cap H_i| \leq r-1$ $(i = 1, \ldots, t)$ and hence, by Exercise 5.3.8, M is the paving matroid defined by $\mathcal{H}$.

Polymatroid functions from matroids

Not only polymatroid functions determine a matroid but conversely every polymatroid function arises from a matroid. Let M be a matroid on a ground-set T with rank function r. Furthermore, let S be a set and let $\varphi : T \to S$ be a mapping. Define a set-function b_r on S by $b_r(X) := r(\varphi^-(X))$, where $\varphi^-(X)$ denotes the preimage of $X \subseteq S$.

Exercise 5.5.13 *Prove that b_r is a polymatroid function.*

The polymatroid function b_r is called an **aggregate** of r. There is an equivalent way to define the aggregate based on the observation that a mapping of T determines a partition of T. Let $\mathcal{P} = \{S_1, \ldots, S_q\}$ be a partition of T into non-empty subsets. Furthermore let $S := \{s_1, \ldots, s_q\}$ be a set whose elements correspond to the parts S_i of $\mathcal{P}$. Then the aggregate of M is given by $b_r(\{s_{i_1}, \ldots, s_{i_h}\}) = r(S_{i_1} \cup \cdots \cup S_{i_h})$.

Exercise 5.5.14 *Show that e_G can be obtained as the aggregate of a matroid rank function.*

Theorem 5.5.16 (Helgason [205]) *Let b be a polymatroid function on a ground-set S. Then there exists a matroid (T, r) and a mapping $\varphi : T \to S$ for which $b(X) = r(\varphi^-(X))$ for $X \subseteq S$* (that is, every polymatroid function arises as an aggregate of a matroid rank function).

Proof. For every element s of S, consider $b(s)$ copies of s and let T denote the set of these new elements (that is, $|T| = \sum[b(s) : s \in S]$). Define $\varphi : T \to S$ such that $\varphi(t) = s$ if t is a copy of s. Define a set-function on ground-set T by $b_T(X) := b(\varphi(X))$. This is clearly a polymatroid function. Consider the matroid M on T belonging to b_T and let r denote its rank function. It follows from Theorem 5.5.14 that

$$r(Z) = \min\{b_T(X) + |Z - X| : X \subseteq Z\}. \tag{5.36}$$

Let $Y \subseteq S$ and $Z := \varphi^-(Y)$. We claim that, for this choice of Z, the minimum in (5.36) is attained on $X := Z$, Let X_m denote the largest minimizer. Then there cannot be elements $u \in X_m$ and $v \in Z - X_m$ for which $\varphi(u) = \varphi(v) \in Y$, for otherwise $X' := X_m + v$ would be a set satisfying $b_T(X') + |Z - X'| = b_T(X_m) + |Z - X'| < b_T(X_m) + |Z - X_m|$, contradicting the assumption that X_m is a minimizer. Therefore $\varphi^-(s) \subseteq Z - X_m$ for any element $s \in Y - \varphi(X_m)$. Let $X' := X_m \cup \varphi^-(t)$. By submodularity, $b_T(X') \leq b_T(X_m) + b_T(\varphi^-(s)) = b_T(X_m) + b(s) = b_T(X_m) + |\varphi^-(s)|$. Hence $b_T(X') + |Z - X'| = b_T(X') + |Z - X_m| - |\varphi^-(s)| \leq b_T(X_m) + |\varphi^-(s)| + |Z - X_m| - |\varphi^-(s)| =$

978–0–19–920527–1 05-Andras-Frank-c05-drv Andras Frank (Typeset by SPi, Publisher Services, India) 203 of 640 December 11, 2010 15:16

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$b_T(X_m) + |Z - X_m|$. It follows that X' is also a minimizer, contradicting the maximal choice of X_m.

We have obtained that $r(Z) = b_T(Z)$. The definition of b_T implies that $b(Y) = b_T(Z)$ and hence $b(Y) = r(Z) = r(\varphi^-(Y))$, that is, the matroid (S, r) and the mapping $\varphi : S \to T$ satisfy the requirements of the theorem. •

Polymatroidal sets

Not only polymatroid functions can be obtained from matroid aggregates, but polymatroidal sets can as well. Let $M = (T, \mathcal{F})$ be a matroid with rank function r. Let b_r be the aggregate of the r defined by the partition $\{S_1, \dots, S_q\}$ of T, where the ground-set of b_r is $S := \{s_1, \dots, s_q\}$.

Theorem 5.5.17 *The set* $Q = \{z \in \mathbf{Z}^S : z(s_i) = |F \cap S_i|$ *for some* $F \in \mathcal{F},\ i = 1, \dots, q\}$ *of integral vectors is polymatroidal. Specifically,* Q *is the set of integral points of the polymatroid* $P(b_r)$.

Since the proof relies on a classical result of Rado, it is deferred until Section 13.1.

Problem 5.5.15 *Let* $G = (S, T; E)$ *be a bipartite graph and let* $g : S \cup T \to \mathbf{Z}_+$ *be a function. Prove that the set* $\{z \in \mathbf{Z}^S :$ *there is a subgraph* G' *of* G *so that* $z(v) \le d_{G'}(v)$ *for every* $v \in S$ *and* $d_{G'}(v) \le g(v)$ *for every* $v \in S \cup T\}$ *is a polymatroidal set.*

Research problem 5.5.2 *However elegant Theorem* 5.5.16 *is for providing a means to encode a polymatroid function b with the help of a matroid and a mapping, it is not particularly useful when b can have extremely large values compared with the cardinality of the ground-set S, because the size of the matroid ensured by the theorem can be exponential in* $|S|$. *Is it possible to encode an polymatroid function with the help of a matroid and a mapping or digraph, so that the size of the matroid and the digraph is polynomial in* $|S|$ *and in* $\log \max\{b(s) : s \in S\}$*?*

Part II
Higher-Order Connections

The major goal of the first part of this book was to summarize classic results concerning connectivities, flows, bipartite matchings, and matroids. We also introduced basic methods like the polyhedral and the submodular techniques. Part II examines more advanced topics, such as connectivity orientations, augmentations, and constructive characterizations. Of course, no clear borderline can be drawn between the material of the first two parts, and the logic of the treatment dictated some deviation from this arrangement. For example, several exciting applications already discussed in Part I (such as theorems of Greene and Kleitman, and the theorem of Bessy and Thomassé) can hardly be considered basic or elementary, but they could be obtained from the fundamentals relatively easily. On the other hand, some basic results, such as the existence of Gomory–Hu trees, are included in Part II.

This middle part of the book is intended to explore more recent and significantly deeper topics concerning connectivities. One of the main motifs is that the submodular technique is used again and again to derive relatively easily, otherwise difficult results. This approach permits us to obtain quick proofs for the cactus representation of minimum cuts and for the splitting-off theorems of Lovász and of Mader. A digestable algorithmic proof of the theorem of Lucchesi and Younger will also be discussed. Nash-Williams' theorem on k-edge-connected orientatibility, as well as its strong form on the existence of well-balanced orientations are also treated, along with various generalizations. Starting with Edmonds' theorem on disjoint arborescences, the most recent developments on packing arborescences and trees will also be discussed. It will turn out that several connectivity augmentation results, such as the one of Watanabe and Nakamura, can be handled with no technical difficulties.

In this part, semimodular functions are merely tools, but their startling success in making so many difficult results accessible naturally gives rise to the need to study these functions separately, and this will be the major topic of Part III. Also, a number of specific connectivity results in Part II serve as a starting point and motivation for the study of semimodular functions.

6

Efficient algorithms for flows and cuts

To make the transition smooth, we continue our exploration with topics familiar from Part I, and present efficient algorithms for computing feasible and min-cost flows as well as for finding minimum cuts of directed and undirected graphs.

6.1 Push–relabel algorithms

6.1.1 Flows, preflows, and modular flows

Let $D = (V, A)$ be a digraph on n nodes with two specified nodes s and t for which $\varrho_D(s) = 0 = \delta_D(t)$. Let $g : A \to \mathbf{R}_+$ be a non-negative capacity function. In Section 3.4, we outlined the max-flow min-cut algorithm of Ford and Fulkerson as well as its strongly polynomial version using shortest augmenting paths. That algorithm computed a feasible flow x of maximum flow-amount and an $s\bar{t}$-subset S for which $\delta_g(S)$ is as small as possible. The MFMC theorem asserted that $\delta_x(s) = \delta_g(S)$. Here $\delta_x(s)\ \ (= \varrho_x(t))$ is by definition the flow-amount of x while $\delta_g(S)$ is the total capacity of the out-cut determined by S. Goldberg and Tarjan [186] developed another algorithm, called push–relabel, for the same purposes and in this section we are going to exhibit variations of their method. A relation of these versions to the original algorithm of Goldberg and Tarjan will be discussed at the end of the section. The basic ideas of the method were already exhibited in the simple and transparent environment of degree-constrained orientations on Page 62. In that algorithm, we used a level function $\Theta : V \to \{0, 1, \ldots, n\}$. The variation we exhibit for computing a feasible modular flow (and, in particular, an st-flow of *prescribed* amount) also uses such a level function. On the other hand, in the original algorithm of Goldberg and Tarjan for computing an st-flow of *maximum* amount, the level function Θ can take values from the larger interval $\{0, 1, \ldots, 2n - 1\}$. We will explain the exact reason why it is sufficient in some cases to work with a level function with the smaller range $\{0, 1, \ldots, n\}$ but not sufficient in other cases.

Flows and preflows

A crucial idea of the push–relabel algorithm of Goldberg and Tarjan is that it works with preflows, a relaxation of flows. This finer notion provides a more flexible setting in which only tiny local changes are made at a time rather than the large-scale modification of a flow along an entire augmenting path, as required by the Ford and Fulkerson algorithm.

In what follows, we continue to assume that no edge enters the source-node s but for the sake of a later application (namely, the algorithm of Hao and Orlin [203] to compute a minimum cut of D), edges leaving the target node t are now allowed. A vector (or a function) $x : A \to \mathbf{R}$ is a **preflow** (from s to t) if $\varrho_x(v) \geq \delta_x(v)$ for every node $v \in V - s$. Obviously, every flow is a preflow. The set-function Ψ_x was defined by $\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z)$ $(Z \subseteq V)$ and was observed to be modular, and hence $\Psi_x(Z) = \sum[\Psi_x(v) : v \in Z]$. The **flow-amount**, or for short the amount of a preflow, x is defined by $\Psi_x(t) = \varrho_x(t) - \delta_x(t)$. Recall that the flow-amount of an st-flow x was defined to be $\delta_x(s)$ and we noticed that $\delta_x(s) = \delta_x(S) - \varrho_x(S)$ holds whenever $s \in S \subseteq V - t$. In the special case $S := V - t$, we have $\delta_x(s) = \varrho_x(t) - \delta_x(t) = \Psi_x(t)$.

For general preflows x, $\delta_x(s)$ cay be strictly larger than $\Psi_x(t)$ (and it is actually larger precisely when x is not a flow). The following link between flows and preflows shows that in order to find a maximum st-flow it suffices to find a preflow of maximum amount.

Proposition 6.1.1 *For every preflow x, there is an st-flow $x_0 \leq x$ with the same flow-amount.*

Proof. Add a new vt-edge for every node $v \in V - \{s, t\}$ and extend the given preflow x to the enlarged digraph $D' = (V, A')$ as follows.

$$x'(e) := \begin{cases} x(e) & \text{if } \ e \in A \\ \varrho_x(v) - \delta_x(v) & \text{if } \ e \text{ is a new } vt\text{-edge.} \end{cases} \tag{6.1}$$

Then x' is an st-flow in D' and the Flow decomposition lemma (Lemma 3.4.3) implies that x' can be expressed as the sum of cycle-flows and paths-flows. By leaving out the cycle-flows and those path-flows using new edges, we are left with a flow x_0 of D whose flow-amount is the same as the flow-amount of x. •

Of course, by applying the simple inductive proof of Lemma 3.4.3, one can compute x_0 directly as follows. Using a search procedure, construct first an st-path P_1 consisting of positive edges. The preflow property implies that there is such a path if $\Psi_x(t) > 0$. Let $\alpha := \min\{x(e) : e \in P_1\}$ and let

$$x_1(e) := \begin{cases} \alpha & \text{if } \ e \in P_1 \\ 0 & \text{if } \ e \in A - P_1. \end{cases} \tag{6.2}$$

Then x_1 is a path-flow and hence $x - x_1$ is a preflow. We can iterate this procedure with $x - x_1$ as long as there are any positive edges entering t. Let $x_1, x_2, \dots, x_h$ be the path-flows obtained in this way and let $x_0 := x_1 + \cdots + x_h$. Then $h \leq |A|$, $x_0 \leq x$ and x_0 is an st-flow for which $\Psi_{x_0}(t) = \Psi_x(t)$.

We emphasize that this procedure for constructing a flow $x_0 \leq x$ from an existing preflow x with the same flow-amount is significantly simpler than the augmenting path algorithm for maximum flows since the $h \leq |A|$ path-flows for which the sum is x_0 are constructed essentially in a greedy fashion.

978–0–19–920527–1 06-Andras-Frank-c06-drv Andras Frank (Typeset by SPi, Publisher Services, India) 209 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proposition 6.1.1 and its simple constructive proof show that in order to compute a feasible flow of maximum amount, it suffices to compute a feasible preflow of maximum amount.

For later purposes (namely, for computing a minimum capacity in-cut of a digraph in Section 6.2), we formulate the Max-flow Min-cut (MFMC) theorem for preflows.

Theorem 6.1.2 (MFMC for preflows) *The maximum flow-amount of a feasible preflow from s to t is equal to*

$$\min\left\{\varrho_g(Z) : t \in Z \subseteq V - s\right\}.$$

A subset Z is a minimizer if and only if there is a feasible preflow x for which

$$\varrho_x(Z) = \varrho_g(Z),\ \delta_x(Z) = 0, \textit{ and } \varrho_x(v) = \delta_x(v) \textit{ for every } v \in Z - t. \tag{6.3}$$

Proof. Let Z be a set for which $t \in Z \subseteq V - s$ and let x be a feasible preflow. Then

$$\varrho_g(Z) - 0 \geq \varrho_x(Z) - \delta_x(Z) = \sum[\varrho_x(v) - \delta_x(v) : v \in Z] \geq \varrho_x(t) - \delta_x(t) = \Psi_x(t) \tag{6.4}$$

from which max $\leq$ min follows. The other direction is a direct consequence of the MFMC theorem since a flow is a preflow. The last part of the theorem follows by observing that $\Psi_x(t) = \varrho_g(Z)$ holds if and only if both inequalities in (6.4) are tight. •

Modular flows

The original push–relabel algorithm of Goldberg and Tarjan constructed a feasible st-flow of maximum amount. In the next subsections, we shall describe a variant of the push–relabel algorithm [148] for computing a feasible m-flow and for providing an alternative proof of the non-trivial direction of Hoffman's feasibility theorem (Theorem 3.4.5). In Section 6.1.3, we will return to maximum st-flows. Let $D = (V, A)$ be a digraph endowed with bounding functions $f : A \to \mathbf{R} \cup \{-\infty\}$ and $g : A \to \mathbf{R} \cup \{\infty\}$ for which $f \leq g$. Let $m : V \to \mathbf{R}$ be a function for which $\widetilde{m}(V) = 0$. Recall that a function $x : A \to \mathbf{R}$ was called a modular flow or an m-flow if

$$\Psi_x(v) = m(v) \text{ for every node } v \in V \tag{6.5}$$

and feasible if $f \leq x \leq g$. Hoffman's theorem stated the following.

There is a feasible m-flow if and only if $\widetilde{m}(V) = 0$ and

$$\varrho_f(Z) - \delta_g(Z) \leq \widetilde{m}(Z) \textit{ for every subset } Z \subseteq V. \tag{6.6}$$

Moreover, if f, g, and m are integer-valued and (6.6) *holds, then there is an integer-valued feasible m-flow.*

In order to gain a more accurate bound on the complexity, we can assume that there are no parallel edges and hence $|A| \leq n^2$. For if e' and e'' are two parallel uv-edges, then they can be replaced by a single uv-edge e for which $f(e) := f(e') + f(e'')$ and $g(e) := g(e') + g(e'')$. In the digraph obtained in this way, (6.6) continues to hold, and there is a feasible m-flow if and only if there is one in D.

6.1.2 A push–relabel algorithm for finding a feasible m-flow

Before describing the steps of the push–relabel algorithm, we examine the basic properties of the level function Θ that are maintained throughout the algorithm, and we introduce the stopping rules: two specific situations when the algorithm terminates.

Level properties and stopping rules

The algorithm finds either a feasible m-flow or else a subset Z of nodes violating (6.6). It maintains a feasible vector $x : A \to \mathbf{R}$ and tries to achieve (6.5). We say that a node $v \in V$ is **Ψ-larger**, **Ψ-smaller**, or **neutral** (with respect to x) according to whether $\Psi_x(v) - m(v)$ is positive, negative, or zero. An edge e is **decreasable** if $x(e) > f(e)$ and **increasable** if $x(e) < g(e)$.

In addition to a feasible x, the algorithm maintains a level function $\Theta : V \to \{0, 1, \dots, n\}$ where $\Theta(v)$ is the level of node v. For a level $j \in \{0, 1, \dots, n\}$, the set $L_j := \{v \in V : \Theta(v) = j\}$ is called a **level set**. Consider the following **level properties**.

- **(LP1)** Every Ψ-smaller node is on the lowest level, that is, in L_0.
- **(LP2$'$)** $\Theta(v) \geq \Theta(u) - 1$ for every increasable edge uv. (Intuitively, each increasable edge steps down at most one level.)
- **(LP2$''$)** $\Theta(v) \leq \Theta(u) + 1$ for every decreasable edge uv. (Intuitively, each decreasable edge steps up at most one level.)

We will refer to the twin Properties (LP2$'$) and (LP2$''$) by (LP2). The algorithm terminates when one of the following two **stopping rules** occurs.

- **(A)** There are no more Ψ-larger nodes.
- **(B)** There exists a Ψ-larger node z and an empty level set L_ℓ under z (where $\ell < \Theta(z)$).

Lemma 6.1.3 *Suppose that x and Θ meet the level properties. Then* **(A)** *implies that x is a feasible m-flow while* **(B)** *implies that the set $Z := \{v \in V : \Theta(v) > \ell\}$ violates* (6.6).

Proof. If there are no Ψ-larger nodes, then, by virtue of $\widetilde{m}(V) = 0 = \Psi_x(V)$, there cannot be Ψ-smaller nodes either and hence x is a feasible m-flow.

Suppose now that **(B)** holds. Since all the Ψ-smaller nodes are in level set L_0 by (LP1), Z contains no Ψ-smaller nodes. But Z does contain the Ψ-larger z and hence $\Psi_x(Z) = \sum[\Psi_x(v) : v \in Z] > \widetilde{m}(Z)$. On the other hand, the emptiness of level set L_ℓ implies that every edge e leaving Z steps down at least two levels and hence (LP2$'$) implies $x(e) = g(e)$ from which $\delta_x(Z) = \delta_g(Z)$ follows. Similarly, every edge e entering Z steps up at least two levels and hence (LP2$''$) implies $x(e) = f(e)$ from which $\varrho_x(Z) = \varrho_f(Z)$ follows. By combining these observations, we get $\varrho_f(Z) - \delta_g(Z) = \varrho_x(Z) - \delta_x(Z) = \Psi_x(Z) > \widetilde{m}(Z)$ showing that Z violates (6.6). •

Two basic operations at a Ψ-larger node **z**

At an intermediate stage of the algorithm, a feasible vector $x : A \to \mathbf{R}$ is available along with a level function Θ so that they satisfy the level properties but neither of the stopping rules holds.

1. **Edge-push at** z changes $x(e)$ on an edge e at z as follows.
   (Increasing) If $e = zu$ is an increasable edge stepping down from z, then increase $x(e)$ by
   $$\alpha := \min\{g(e) - x(e), \Psi_x(z) - m(z)\}.$$
   (Decreasing) If $e = uz$ is a decreasable edge stepping up to z, then decrease $x(e)$ by
   $$\alpha := \min\{x(e) - f(e), \Psi_x(z) - m(z)\}.$$
2. **Node-lift of** z increases $\Theta(z)$ by 1 provided that no edge-push operation is possible at z.

We call an edge-push at z **neutralizing** if z becomes neutral (which occurs when $\alpha = \Psi_x(z) - m(z)$). Otherwise (when z remains Ψ-larger), the push is **non-neutralizing**. The core subroutine of the algorithm is as follows.

The core subroutine: treating a Ψ-larger node z

While z stays Ψ-larger and there is an increasable edge stepping down from z or a decreasable edge stepping up to z, apply an edge-push operation at z. When z is still Ψ-larger but no more edge-pushes are possible at z, apply the node-lift operation at z. That is, a treatment of z is completed either when z becomes neutral or when z is lifted.

Description of the algorithm

The algorithm starts with an arbitrary $x : A \to \mathbf{R}$ for which $f \leq x \leq g$ and with the identically zero level function Θ. At an intermediate stage, a feasible x and a Θ are available which satisfy the level properties. Suppose that neither of the two stopping rules holds. Then there are Ψ-larger nodes but none of them can be in L_n since $\Theta(z) = n$ for a Ψ-larger node z would imply that at least one level set under z is empty, that is, stopping rule **(B)** would occur.

As long as there are Ψ-larger nodes, the algorithm selects one for which the level is as high as possible and treats it. This rule of selecting a node for treatment is called the **highest level rule**.

One type of termination of the current treatment occurs when we neutralize z and no more Ψ-larger node remains. In this case, stopping rule **(A)** holds and hence the resulting x is a feasible m-flow by Lemma 6.1.3. The other type of termination of the current treatment occurs when we lift z and its (original) level set becomes empty. In this case, stopping rule **(B)** holds and hence the set $Z = \{v : \Theta(v) \geq \Theta(z)\}$ violates (6.6) by Lemma 6.1.3 showing that no feasible m-flow exists.

Correctness and complexity

Consider a transition from a stage to the subsequent one which transforms the current functions x and Θ into x' and Θ', respectively.

Lemma 6.1.4 *The basic operations preserve feasibility and the level properties.*

Proof. The definition of α implies that an edge-push preserves feasibility. Since an edge-push does not create new Ψ-smaller nodes, Property (LP1) also remains valid. Because an edge-push operates on an edge uv for which $|\Theta(u) - \Theta(v)| = 1$, Property (LP2) cannot break down either.

978–0–19–920527–1 06-Andras-Frank-c06-drv Andras Frank (Typeset by SPi, Publisher Services, India) 212 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

A node-lift of z keeps (LP1) intact since $\Theta(z)$ is increased only when z is a Ψ-larger node. It preserves (LP2), too, since a node-lift was applied only when there were no increasable edges stepping down from z and no decreasable edges stepping up to z. •

The non-trivial direction of Theorem 3.4.5 follows once we show that one of Stopping rules **(A)** and **(B)** occurs after a finite number of steps. In fact, we prove the following polynomial bound.

Theorem 6.1.5 *The total number of basic operations is at most* n^3.

Proof. By a **phase** of the algorithm we mean the set of operations carried out between two consecutive node-lifts. Since the level of a single node can increase at most n times, the total number of phases is at most n^2. Since an edge-push at a node z' can turn a node to Ψ-larger only if this node is under z', the highest level rule implies that after a neutralizing edge-push makes z neutral, z cannot become Ψ-larger again within the same phase. Therefore in one phase there can be at most n neutralizing edge-pushes and hence their total number is at most n^3.

A non-neutralizing edge-push on e either increases $x(e)$ to $g(e)$ making e non-increasable, or decreases $x(e)$ to $f(e)$ making e non-decreasable. Therefore, after a non-neutralizing edge-push is carried out on e, the next edge-push on e can occur only when the sign of $\Theta(z) - \Theta(u)$ has changed. By that time the sum $\Theta(z) + \Theta(u)$ must have been increased by at least two. Hence the number of non-neutralizing edge-pushes on a single edge e is at most n, and thus the total number of non-neutralizing edge-pushes is at most $|A|n \leq n^3$. •

In conclusion, after at most n^3 edge-push and node-lift operations, the algorithm terminates by returning either a feasible m-flow or a violating subset Z, and this completes the proof of Theorem 3.4.5. It should be noted that with the help of a suitable data structure the basic operations can be carried out in constant time and therefore the complexity of the overall algorithm is $O(n^3)$. With a clever counting method, Cheriyan and Maheswari [46] proved that the complexity of the push–relabel algorithm with the highest level rule is actually $O(n^{5/2})$. Cheriyan and Mehlhorn [48] greatly simplified the proof.

A special case: computing an st-flow of a given amount k

As a special case of the push–relabel algorithm for m-flows, we can compute a feasible flow from s to t having a specified amount k if one exists. As we have seen earlier, Theorem 3.4.5, with the choices $f \equiv 0 \leq g$ and

$$m(v) := \begin{cases} k & \text{if } v = t \\ -k & \text{if } v = s \\ 0 & \text{if } v \in V - \{s, t\} \end{cases} \tag{6.7}$$

specializes to Theorem 3.4.9. We assume that no edge of D enters s or leaves t. In this case a feasible m-flow is a flow from s to t of amount k. Furthermore, if a subset S violates (6.6), that is, if $-\delta_g(S) = \varrho_f(S) - \delta_g(S) > \widetilde{m}(S)$, then $0 \leq \delta_g(S) < \widetilde{m}(S)$ and hence $s \in S \subseteq V - t$ and $\widetilde{m}(S) = k$. When the flow-amount k is specified, the algorithm above either returns an st-flow of amount k or finds an $s\bar{t}$-set S, for which $\delta_g(S) < k$.

We emphasize that in this approach each node, including the source-node s, is on level 0 at the beginning and the range of the level function is $\{0, 1, \ldots, n\}$. The original algorithm of Goldberg and Tarjan, as we shall soon see, initializes Θ so that $\Theta(s) = n$. In what follows, we show how to modify the algorithm if the flow-amount is not specified in advance and we are interested in a feasible st-flow of *maximum* amount.

6.1.3 Variants of the algorithm

In this section we describe two variants of the algorithm above along with their relationship to the original form of Goldberg and Tarjan.

Variant A: computing a most violating set and a feasible vector of minimum excess
For a subset $X \subseteq V$, we call the value $(\varrho_f(X) - \delta_g(X) - \widetilde{m}(X))^+$ the **deficit** of X. For an arbitrary vector $x : A \to \mathbf{R}$, we call the value $\sum[(\Psi_x(v) - m(v))^+ : v \in V]$ the m**-excess** of x. Note that a vector is an m-flow exactly if its m-excess is zero. Furthermore, the maximum deficit is zero if and only if (6.6) holds.

The algorithm above, in the case when no feasible m-flow existed, found a subset Z violating (6.6). With a slight modification of the procedure even a feasible vector of minimum m-excess can be computed as well as a most violating subset, that is one of maximum deficit. To this end, we revise the algorithm a bit so that it does not terminate when a node-lift operation leaves a level set empty. Therefore in this version there can be nodes in level set L_n and this motivates the other modification: a node z for treatment is chosen to be a Ψ-larger node of highest level *under* n. The algorithm terminates when there are no more Ψ-larger nodes under level n. This can occur after either an edge-push or a node-lift operation.

Let x^* denote the feasible vector available at termination. When there are Ψ-larger nodes, then all of them are in L_n. In this case there is an empty level set since there are n nodes and $n + 1$ levels. Let Z^* denote the set of nodes above the highest empty level set. When there are no Ψ-larger nodes at all, then x^* is a feasible m-flow. In this case, let $Z^* := \emptyset$.

Theorem 6.1.6 *Let $D = (V, A)$ be a digraph endowed with bounding functions $f : A \to \mathbf{R} \cup \{-\infty\}$ and $g : A \to \mathbf{R} \cup \{\infty\}$ for which $f \le g$. Let $m : V \to \mathbf{R}$ be a function for which $\widetilde{m}(V) = 0$. Then the minimum m-excess of a feasible vector is equal to the maximum deficit of a subset of V. When f, g, and m are integral, the optimum feasible vector can also be chosen to be integral. Moreover, Z^** (defined above) *is a subset of maximum deficit and the feasible vector x^* found by the algorithm is of minimum m-excess.*

Proof. For any feasible vector x and any subset $Z \subseteq V$, we have

$$\begin{aligned} &\varrho_f(Z) - \delta_g(Z) - \widetilde{m}(Z) \le \varrho_x(Z) - \delta_x(Z) - \widetilde{m}(Z) \\ &= \sum[\Psi_x(v) - m(v) : v \in Z] \le \sum[(\Psi_x(v) - m(v))^+ : v \in V] \end{aligned} \tag{6.8}$$

from which max $\le$ min follows. Here equality holds if and only if (i) $\varrho_f(Z) = \varrho_x(Z)$, $\delta_g(Z) = \delta_x(Z)$ and (ii) Z contains every Ψ-larger node but no Ψ-smaller nodes.

To prove max $=$ min, one must find a feasible x and a $Z \subseteq V$ meeting (6.8) with equality, that is, x and Z satisfy (i) and (ii). We are going to show that x^* and Z^* will suffice, proving this way the last part of the theorem as well.

When there are no Ψ-larger nodes (with respect to x^*), then x^* is a feasible m-flow and $Z^* = \emptyset$ and these obviously satisfy (i) and (ii). Suppose now that there are Ψ-larger nodes. By the definition of Z^* and by level Property (LP2), x^* and Z^* satisfy (i). Furthermore, Z^* fulfils (ii), too, since every Ψ-larger node is in L_n and every Ψ-smaller node is in L_0 while $L_n \subseteq Z^* \subseteq V - L_0$. (Here we use only that Z^* consists of all nodes above *any* empty level set and not that the empty level set is the highest one.) •

In the sequel, this version of the algorithm will be referred to as **Variant A**.

Variant B: computing a minimum st-cut and a maximum st-preflow

Our next goal is to show how Variant A of the algorithm for computing a feasible vector x^* of minimum m-excess and a most violating set Z^* can be specialized to prove Theorem 6.1.2 algorithmically. We assume again that no edges of D enter s but edges leaving t are now allowed. Let $f \equiv 0 \leq g$ and for $k := \delta_g(s) + 1$ let m be defined by (6.7).

The algorithm starts with $x :\equiv 0$ which is certainly feasible since $f \equiv 0$. For this x, t is the only Ψ-smaller node. Because throughout the process the algorithm never creates a new Ψ-smaller node, t remains the only Ψ-smaller node throughout. Hence the feasible vector x available at any stage of the algorithm is automatically a preflow from s to t. Furthermore, the m-excess of x is $\sum[(\Psi_x(v) - m(v))^+ : v \in V] = -[\Psi_x(t) - m(t)] = -[\varrho_x(t) - \delta_x(t) - k] = k - \Psi_x(t)$.

Let x^* and Z^* denote the output of Variant A. Note that x^* has a positive deficit due to the choice of k. We observed that x^* is an st-preflow for which the flow-amount, by definition, is $\Psi_{x^*}(t)$. Furthermore, beacuse Z^* contains no Ψ-positive nodes, Z^* is an $s\bar{t}$-set whose deficit is $\varrho_f(Z^*) - \delta_g(Z^*) - \widetilde{m}(Z^*) = 0 - \delta_g(Z^*) - (-k) = k - \delta_g(Z^*)$. By Theorem 6.1.6, the m-excess of x^* is equal to the deficit of Z^* and hence $k - \Psi_{x^*}(t) = k - \delta_g(Z^*)$. Therefore $\Psi_{x^*}(t) = \delta_g(Z^*)$, showing that x^* is a maximum st-preflow and the set of edges leaving Z^* is a minimum st-cut, as required.

In order to clarify the relation of Variant B to the original push–relabel algorithm of Goldberg and Tarjan, we make a little observation by following closely what the algorithm does with the initial $x \equiv 0$. By the definition of m, the only Ψ-larger node is s. Since no edge-push is possible at s, the algorithm lifts s to level 1. After that, s is treated and this operation results in a feasible vector x (that is, a preflow) which is given for $uv \in A$ by

$$x(uv) := \begin{cases} g(uv) & \text{if } u = s \\ 0 & \text{if } u \in V - s. \end{cases} \tag{6.9}$$

Since $k > \delta_g(s)$, after its treating, s remains Ψ-larger and the algorithm keeps lifting s until the level of s becomes n.

In order to avoid these initial steps, we can initialize the algorithm so that $\Theta(s) := n$ and the initial preflow x is defined for every edge $uv \in A$ by (6.9). We will refer to this version of the algorithm as **Variant B**. In Section 6.2, Variant B will be developed further to obtain an efficient algorithm computing a minimum cut of a digraph. In the next subsection, we show that Variant B can be interpreted as a slight modification of the original algorithm of Goldberg and Tarjan.

Computing a maximum st-flow: a relation to the original algorithm of Goldberg and Tarjan

We have already indicated in the proof of Proposition 6.1.1 how a greedy-type approach can be used to compute an st-flow from an st-preflow with the same flow-amount. An alternative approach is that we consider the minimum cut value k' computed by Variant B and run the algorithm a second time with k' in place of k. Since there is an st-flow of amount k', the algorithm will terminate by returning a feasible m-flow which is in this case a maximum st-flow.

The original push–relabel algorithm (with the highest level rule) of Goldberg and Tarjan computes a maximum st-flow in one run but it must allow the levels of nodes to increase up to $2n - 1$. More specifically, the levels of s and t are fixed throughout the algorithm by $\Theta(s) = n$ and $\Theta(t) = 0$, and the initial preflow x is defined by (6.9). The steps of the algorithm are the same as those in Variant B the only difference being that nodes are allowed to be lifted above level n. Namely, at a general step, we pick up a Ψ-larger node $z \in V - \{s, t\}$ of highest level and treat z. The algorithm terminates when there are no more Ψ-larger nodes in $V - \{s, t\}$. In this case, the current vector x is a feasible st-flow and the set $S := \{v \in V : \Theta(v) > j\}$ of nodes satisfies the optimality criteria: $\delta_x(S) = \delta_g(S)$ and $\varrho_x(S) = 0$. Here j is any index for which $0 < j < n$ and the level set $L_j = \{v \in V : \Theta(v) = j\}$ is empty. Therefore x is a maximum st-flow and S determines a minimum cut.

Problem 6.1.1 $(*)$ *Prove that during the Goldberg–Tarjan algorithm the maximum level of every node is at most* $2n - 1$.

When the algorithm of Goldberg and Tarjan is modified so that z is always selected as the highest level Ψ-larger node *under* level n (meaning that all nodes but s are kept under level n), then the algorithm terminates with a maximum preflow and a minimum cut, that is, in this case we are back at Variant B.

Finally, we remark that Goldberg and Tarjan considered more generic forms of the algorithm and proved, for example that the algorithm is polynomial even without the highest level rule.

Problem 6.1.2 $(*)$ *In Section* 3.4.2*, we showed that the m-flow feasiblity problem is equivalent to its special case when* $f \equiv 0$ *and* $g \equiv \infty$*. Work out the details of the push-relabel algorithm for this special case along with possible simplifications.*

6.2 Computing a minimum cut of a digraph

Let $D = (V, A)$ be a digraph endowed with a non-negative capacity function $g : A \to \mathbf{R}_+$. Consider the following three closely related min-cut type problems. In each of them, we want to find a non-empty proper subset $Z \subset V$ for which $\varrho_g(Z)$ is minimum, where:

(1) given a non-empty subset R and a target node $t \in V - R$, $Z \subseteq V - R$ is a subset containing t,
(2) given a root-node r_0, Z is a subset of $V - r_0$,
(3) Z is arbitrary.

In Problem (1), R can be shrunk into a single source-node and hence any MFMC computation can be applied, for example, the push–relabel algorithm discussed in Section 6.1. Problem (2) can be solved by $|V| - 1$ applications of Problem (1), namely, for each $t \in V - r_0$, we run an MFMC algorithm for source-node r_0 and target node t.

Finally, Problem (3) can be reduced to two applications of Problem (2). Namely, we select a node r_0 of D, and solve separately Problem (2) both for D and for the digraph D' arising from D by reorienting every edge of D.

Considering the suggested approach to Problem (2), one has the impression that the $|V| - 1$ individual MFMC computations for the distinct target nodes $t \in V - r_0$ are not at all independent. Indeed, Hao and Orlin [203] developed an algorithm having the same complexity as a single push–relabel MFMC algorithm. Here we exhibit a conceptually slightly simpler variant of their algorithm [148]. While the original algorithm of Hao and Orlin is based on the use of the concept of dormant sets, the present version avoids this concept.

Let $U := V - r_0$ and $n := |U|$. For any non-empty subset $U' \subseteq U$, let

$$\mu_g(U') := \min\{\varrho_g(X) : \emptyset \subset X \subseteq U'\}.$$

Therefore Problem (2) is equivalent to computing $\mu = \mu_g(U)$ along with a minimizer set X.

Instead of applying the n MFMC computations completely separately to the n distinct target nodes in the original digraph, we will rely on the following tiny improvement. For a given target node $t_1 \in U$, let $Z_1 \subseteq U$ denote a set containing t_1 for which $\varrho_g(Z_1)$ is minimum. Let $Z \subseteq U$ be a non-empty set for which $\varrho_g(Z)$ is minimum (that is, $\mu_g(U) = \varrho_g(Z)$). Then Z may or may not contain t_1 and hence $\mu_g(U) = \min\{\varrho_g(Z_1), \mu_g(U - t_1)\}$. This implies the validity of the following strategy.

First, select a target node $t_1 \in U$ and compute a set $Z_1 \subseteq U$ containing t_1 for which $\varrho_g(Z_1)$ is minimum. Next, form a root-set $R_1 := \{r_0, t_1\}$, select a new target node $t_2 \in U' := U - R_1$, and compute a set $Z_2 \subseteq U'$ containing t_2 for which $\varrho_g(Z_2)$ is minimum. (This is a special case of Problem (1).) Next, form a root-set $R_2 := \{r_0, t_1, t_2\}$, select a new target node $t_3 \in U' := U - R_2$, and compute a set $Z_3 \subseteq U'$ containing t_3 for which $\varrho_g(Z_3)$ is minimum. By repeating this procedure n times, we obtain a sequence $r_0, t_1, \dots, t_n$ of the nodes of D and a sequence of n subsets $Z_1, \dots, Z_n$ for which $\mu_g(U) = \min\{\varrho_g(Z_j) : j = 1, \dots, n\}$.

Accordingly, the algorithm will consist of n stages, where each one is a push–relabel computation for the current root-set R_i and the current target node t_i. Each stage is a modification of Variant B of the push-relabel algorithm. A small difference is that the digraph now has $n + 1$ nodes while the range of the level function Θ remains $\{0, 1, \dots, n\}$. An essential difference will be that we prevent the node-lift operation from producing any empty level sets. In this way, we ensure that Θ is **contiguous** in the sense that there are no indices i, j, and k for which $0 \le i < j < k < n$ and $L_i \neq \emptyset$, $L_j = \emptyset$, and $L_k \neq \emptyset$, where $L_h = \{v : \Theta(v) = h\}$. At the end of each stage, we place the current target node on level n and select any node from the lowest non-empty level set for as being the subsequent node.

Optimality criteria

Let $r_0, t_1, \dots, t_n$ be an ordering of the nodes of D and let $U' = U_j := \{t_j, t_{j+1}, \dots, t_n\}$ for $j = 1, \dots, n$. We call a set Z_j a **minimizer** for t_j if $t_j \in Z_j \subseteq U_j$ and $\varrho_g(Z_j) = \min\{\varrho_g(Z) : t_j \in Z \subseteq U_j\}$.

Proposition 6.2.1 (optimality criteria) *For a specified subscript $j \in \{1, \dots, n\}$, let $Z_j \subseteq U_j$ be a set containing t_j and $x : A \to \mathbf{R}_+$ a feasible vector* (i.e., $0 \le x \le g$) *for which*

$$\varrho_x(v) \ge \delta_x(v) \text{ for every } v \in U_j, \tag{6.10}$$

$$\varrho_x(v) = \delta_x(v) \text{ for every } v \in Z_j - t_j, \tag{6.11}$$

$$\varrho_x(Z_j) = \varrho_g(Z_j) \text{ and } \delta_x(Z_j) = 0. \tag{6.12}$$

Then Z_j is a minimizer for t_j.

Proof. Shrink $V - U_j$ into a node s. Let D_j denote the resulting digraph, let x_j be the restriction of x to the edges of D_j, and let $t := t_j$. Then x_j is a preflow from s to t in D_j and Theorem 6.1.2 applies. •

Level properties

At a general stage j, we are given a root-set $R = R_j$, a feasible vector $x : A \to \mathbf{R}_+$, and a level function $\Theta : V \to \{0, 1, \dots, n\}$. We say that a a node $v \in V - R$ is **Ψ-positive**, **Ψ-negative**, or **neutral** according to whether $\Psi_x(v)$ is positive, negative, or zero. Let $U' := V - R$ and let $I_D(U')$ denote the set of edges of D induced by U'. The level properties are as follows.

(LP1) $R = L_n$, every Ψ-negative node is in R, $\delta_x(R) = \delta_g(R)$ and $\varrho_x(R) = 0$.
(LP2′) $\Theta(v) \ge \Theta(u) - 1$ for every increasable edge $uv \in I_D(U')$.
(LP2″) $\Theta(v) \le \Theta(u) + 1$ for every decreasable edge $uv \in I_D(U')$.
(LP3) Θ is contiguous, $|R| = j + 1$ where L_j is the lowest non-empty level set, and $|L_j| = 1$.

We emphasize that Properties (LP2′) and (LP2″) are required only for the edges induced by U'.

6.2.1 Description of a stage of the algorithm

The input $\{R, x, \Theta\}$ of Stage 1 is defined by $R := \{r_0\}$,

$$x(uv) := \begin{cases} g(uv) & \text{if } u = r_0 \\ 0 & \text{otherwise,} \end{cases} \tag{6.13}$$

$$\Theta(v) := \begin{cases} n & \text{if } v = r_0 \\ 0 & \text{if } v = t_1 \\ 1 & \text{otherwise.} \end{cases} \tag{6.14}$$

For $j \ge 2$, let x and Θ denote the output of Stage $j - 1$ satisfying the level properties. We initialize Stage j as follows. Let $\Theta(t_{j-1}) := n$, that is, we move t_{j-1} from L_{j-1} to $R = L_n$. Let $U' := U' - t_{j-1}$. Modify x for each edge $e = t_{j-1}v$ with $v \in U'$ by letting $x(e) := g(e)$, and for each edge $e = ut_{j-1}$ with $u \in U'$ by letting $x(e) := 0$. In this way L_j becomes the lowest non-empty level and we ensure that **(LP1)** holds at the beginning of Stage j. Select an arbitrary element of L_j to be the next target node t_j. Increase the level of each element of $L_j - t_j$ by one. In this way we ensure that $|L_j| = 1$ and $|R| = j + 1$ and hence **(LP3)** continues to hold. The initialization does not effect **(LP2)**.

Stage j revises x and Θ, and returns a subset $Z_j \subseteq U'$ containing t_j such that the current x and Θ meet both the level properties and the optimality criteria.

Two basic operations at a Ψ-positive node z

1. **Edge-push at** z changes $x(e)$ on an edge $e \in I_D(U')$ at z as follows.
   (Increasing) If $e = zu$ is increasable and steps down ($\Theta(u) = \Theta(z) - 1$), then increase $x(e)$ by
   $$\alpha := \min\{g(e) - x(e), \Psi_x(z)\}.$$
   (Decreasing) If $e = uz$ is decreasable and steps up ($\Theta(z) = \Theta(u) + 1$), then decrease $x(e)$ by
   $$\alpha := \min\{x(e), \Psi_x(z)\}.$$
2. **Node-lift of** z increases $\Theta(z)$ by 1.

Treating a Ψ-positive node z

Apply edge-pushes at z as long as possible. If no more edge-pushes are possible at z but z is still Ψ-positive and z is not the only element on its level, then apply the node-lift operation at z. A treatment of z is completed either when z becomes neutral, when z remains Ψ-positive and gets lifted, or when z remains Ψ-positive and z is the only element on its level.

We say that a Ψ-positive node $z \in U'$ is **singular** if no more edge-pushes are possible at z and z is the only node on its level, that is, $L_\ell = \{z\}$ for $\ell = \Theta(z)$.

Steps of Stage j

As long as possible, select a non-singular Ψ-positive node $z \in U' - t_j$ of the highest level and treat it. Consider now the situation when no such a z exists anymore, that is, when each Ψ-positive node in $U' - t_j$ is singular.

If there are no Ψ-positive nodes in $U' - t_j$, then let $Z_j := U'$. In this case $\varrho_x(Z_j) = \varrho_g(Z_j)$ and $\delta_x(Z_j) = 0$ hold by the initialization of Stage j. Therefore the optimality criteria described in Proposition 6.2.1 hold and Stage j terminates.

Suppose now that there are Ψ-positive nodes in $U' - t_j$ distinct from t_j. Since no treatment is possible, these nodes are singular. Let $z' \in U' - t_j$ be a singular node of lowest level j'. For the subset $Z_j := \{v : j \leq \Theta(v) < j'\}$, the only possible Ψ-positive node in Z_j is t_j. Since z' is singular, **(LP2)** implies that $x(uv) = 0$ for every uv-edge with $u \in Z_j$, $v \in U' - Z_j$, and $x(uv) = g(uv)$ for every uv-edge with $u \in U' - Z_j$, $v \in Z_j$. In addition, **(LP1)** implies that $x(uv) = g(uv)$ for every uv-edge with $u \in R$, $v \in Z_j$, and $x(uv) = 0$ for every uv-edge with $u \in Z_j$, $v \in R$. Hence $\varrho_x(Z_j) = \varrho_g(Z_j)$ and $\delta_x(Z_j) = 0$, that is, x and Z_j satisfy the optimality criteria, and Stage j is completed.

6.2.2 Correctness and complexity

Recall that an edge-push at z is said to be neutralizing if it converts z to neutral. Otherwise (when z remains Ψ-positive) the push is non-neutralizing.

978–0–19–920527–1 06-Andras-Frank-c06-drv Andras Frank (Typeset by SPi, Publisher Services, India) 219 of 640 December 11, 2010 15:17

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Lemma 6.2.2 *The basic operations preserve feasibility and the level properties.*

Proof. Since only non-singular nodes are lifted, the level function Θ remains contiguous throughout and hence **(LP3)** remains intact.

Consider a moment of Stage j when we are about to lift a node $z \in U' - t_j$. Let $h := \Theta(z)$. Since z is non-singular, $|L_h| \geq 2$. We claim that $h < n - 1$. For if $\Theta(z) = n - 1$, then every level set $L_j, \ldots, L_{n-1}$ contains at least one element of U' and hence $|U'| \geq n - j = n - (|R| - 1) = |U'|$ follows from **(LP1)**. Therefore $|U'| = n - j$ from which we must have $|L_j| = \cdots = |L_{n-1}| = 1$ contradicting $|L_h| \geq 2$. We obtained that lifting z does not destroy $R = L_n$. Lifting z also preserves **(LP2)**, since a node-lift was applied only when there were no increasable edges stepping down from z and no decreasable edges stepping up to z.

The definition of α implies that an edge-push preserves feasibility. Since it does not create a Ψ-negative node in U', Property **(LP1)** also remains valid. Because an edge-push operates on an edge uv for which $|\Theta(u) - \Theta(v)| = 1$, Property **(LP2)** cannot break down either. •

Theorem 6.2.3 *The total number of basic operations is at most n^3.*

Proof. By a phase of the algorithm we mean again the set of operations carried out between two consecutive node-lifts. Since the level of any node can increase at most n times, the total number of phases is at most n^2. Since an edge-push at a node z' can turn another node to Ψ-positive only if this node is under z', the highest selection rule implies that after an edge-push makes z neutral, z cannot become Ψ-positive again within the same phase. Therefore in one phase there can be at most n neutralizing edge-pushes and hence their total number is at most n^3.

A non-neutralizing edge-push on e increases $x(e)$ to $g(e)$ making e non-increasable, or decreases $x(e)$ to 0 making e non-decreasable. Therefore, after a non-neutralizing edge-push is carried out on e, the next edge-push on e can occur only when the sign of $\Theta(z) - \Theta(u)$ has changed. By that time the sum $\Theta(z) + \Theta(u)$ must have increased by at least two. Hence the number of non-neutralizing edge-pushes on a single edge e is at most n and thus the total number of non-neutralizing edge-pushes is at most $|A|n \leq n^3$. •

Remark 6.2.1 In the original algorithm of Hao and Orlin, when a singular node z arises, the set X of nodes above the level of z is declared 'dormant', meaning that all operations inside a dormant set will be blocked until all nodes outside X have put into the current root-set R. Although the proof of its correctness is longer, the original algorithm of Hao and Orlin behaves better experimentally than the version above.

6.3 Computing a minimum cut of an undirected graph

In this section we exhibit an elegant algorithm of Nagamochi and Ibaraki [300, 296] for computing the edge-connectivity $\lambda(G)$ of an undirected graph G.

6.3.1 Max adjacency orderings

Let $G = (V, E)$ be a loopless undirected graph in which parallel edges are allowed. Let $\{v_1, \ldots, v_n\}$ be an ordering of the nodes of G and let V_i denote the set of the first i nodes

$(i = 1, \dots, n)$. We say that $\{v_1, \dots, v_n\}$ is a **max adjacency** ordering, an **MA ordering** for short, if

$$d_G(v_i, V_{i-1}) \geq d_G(v_j, V_{i-1}) \tag{6.15}$$

holds for every pair $\{i, j\}$ $(2 \leq i < j \leq n)$. More generally, we speak of an MA ordering with respect to a capacity function $g : A \to \mathbf{R}_+$ if

$$d_g(v_i, V_{i-1}) \geq d_g(v_j, V_{i-1}) \tag{6.16}$$

holds for every pair $\{i, j\}$ $(2 \leq i < j \leq n)$. We shall refer to (6.15) and (6.16) as the **MA rule**.

With the help of an **MA search**, an MA ordering can be determined easily. The search starts with an arbitrary node v_1. When nodes $v_1, \dots, v_{i-1}$ have already been determined, choose v_i to be a node z from the remaining set of nodes that has a maximum number of incident edges going to V_{i-1} (and in the capacitated case, $d_g(z, V_{i-1})$ is to be maximum). Nagamochi and Ibaraki [297] showed how an MA ordering can be computed in $O(m + n \log n)$ steps which is actually $O(m + n)$ in the uncapacitated case. For details about the theorem and for applications of MA orderings, see the book by Nagamochi and Ibaraki [300].

The concept of MA orderings was introduced by Tarjan and Yannakakis [364]. They used it, among others, for testing in linear time if a graph is chordal. This application will be discussed in Section 7.4. The major applications of MA orderings are due to Nagamochi and Ibaraki. Their striking discovery [296] has been that an MA ordering can be used to compute the edge-connectivity of an undirected graph in a much simpler and more efficient way than the algorithms relying on Max-flow Min-cut computations.

Consider first the uncapacited case. As usual, $\lambda_G(x, y)$ denotes the maximum number of edge-disjoint xy-paths, and by Menger this is equal to the minimum cardinality of a cut separating x and y. The algorithm is based on the crucial observation that any loopless graph G with at least two nodes admits two nodes u, v with $\lambda_g(u, v) = d_G(u)$, and an MA ordering provides at once such a pair. The existence of u and v actually follows from the more general device of Gomory–Hu trees (see Corollary 7.2.3 in Section 7.2). The point here is that they can be found at once with the help of an MA ordering. The following result is the heart of the algorithm. The proof is taken from [129].

Theorem 6.3.1 (Nagamochi and Ibaraki) *If $v_1, \dots, v_n$ is an MA ordering of the nodes of a loopless undirected graph $G = (V, E)$ with $n \geq 2$, then $\lambda_G(v_n, v_{n-1}) = d_G(v_n)$.*

Proof. We proceed by induction on $|V| + |E|$. The theorem is evident if $n = 2$ so we can assume that $n \geq 3$. Suppose first that v_n and v_{n-1} are connected by $\alpha > 0$ parallel edges. Then $v_1, \dots, v_n$ is an MA ordering for the graph G', too, arising from G by deleting these α edges. By induction, we have $d_{G'}(v_n) = \lambda_{G'}(v_n, v_{n-1})$ from which $d_G(v_n) - \alpha = d_{G'}(v_n) = \lambda_{G'}(v_n, v_{n-1}) = \lambda_G(v_n, v_{n-1}) - \alpha$ and thus $d_G(v_n) = \lambda_G(v_n, v_{n-1})$, as required.

Suppose now that v_n and v_{n-1} are not adjacent in G. Let $G' = G - v_n$ and $G'' = G - v_{n-1}$. Note that $v_1, \dots, v_{n-1}$ is an MA ordering of G' and that $v_1, \dots, v_{n-2}, v_n$ is an MA ordering of G''. Also, by the MA rule, $d_G(v_{n-1}) \geq d_G(v_n)$. By induction, we have

$$\lambda_{G'}(v_{n-1}, v_{n-2}) = d_{G'}(v_{n-1}) \text{ and } \lambda_{G''}(v_n, v_{n-2}) = d_{G''}(v_n).$$

Since v_n and v_{n-1} are not adjacent in G, we have

$$\lambda_G(v_{n-1}, v_{n-2}) \geq \lambda_{G'}(v_{n-1}, v_{n-2}) = d_{G'}(v_{n-1}) = d_G(v_{n-1}) \geq d_G(v_n)$$

and

$$\lambda_G(v_n, v_{n-2}) \geq \lambda_{G''}(v_n, v_{n-2}) = d_{G''}(v_n) = d_G(v_n)$$

from which

$$\lambda_G(v_n, v_{n-1}) \geq \min\{\lambda_G(v_{n-1}, v_{n-2}),\ \lambda_G(v_n, v_{n-2})\} \geq d_G(v_n)$$

and hence $\lambda_G(v_n, v_{n-1}) = d_G(v_n)$. •

Given Theorem 6.3.1, the algorithm is based on the following simple observation. Let G' be a graph arising from G by shrinking (= identifying) two nodes x and y (that is, shrinking the set $\{x, y\}$). Clearly,

$$\lambda(G) = \min\{\lambda_G(x, y), \lambda(G')\} \tag{6.17}$$

since the cuts of G can be partitioned into two classes depending on whether they separate x and y or not. Therefore if we have $d_G(x) = \lambda_G(x, y)$, then $\lambda(G) = \min\{\lambda(G'), d_G(x)\}$, and hence in order to determine $\lambda(G)$, it suffices to compute the edge-connectivity $\lambda(G')$ of the smaller graph G'. Theorem 6.3.1 tells us how to find such a pair of nodes x, y. These arguments have thereby verified the correctness of the following algorithm.

6.3.2 The algorithm of Nagamochi and Ibaraki

Let $G_1 := G$ and iterate $n - 1$ times the following procedure. Suppose that graph G_i has already been constructed by shrinking $i - 1$ pairs of nodes ($i = 1, \ldots, n - 1$). Determine an MA ordering of the nodes of G_i and put the last node of this ordering, denoted by x_i, into a list along with its degree $d_{G_i}(x_i)$. Construct G_{i+1} from G_i by shrinking the current last two nodes of G_i. (Loops possibly arising as a result of a shrinking operation are left out.)

After completing the $n - 1$ iterations, choose an element x_j from the list for which $d_{G_j}(x_j)$ is smallest. Then $\lambda(G)$ is equal to $d_{G_j}(x_j)$ and for the subset $X_j \subseteq V$ of nodes corresponding to the node x_j of G_j, we have $d_G(X_j) = \lambda(G)$, that is, $\Delta(X_j)$ is a minimum cut of G. The equality $\lambda(G) = d_{G_j}(x_j)$ follows by the repeated application of (6.17): $\lambda(G) = \lambda(G_1) = \min\{\lambda(G_2), d_{G_1}(x_1)\} = \min\{\lambda(G_3), d_{G_1}(x_1), d_{G_2}(x_2)\} = \cdots = \min\{d_{G_1}(x_1), d_{G_2}(x_2), \ldots, d_{G_{n-1}}(x_{n-1})\}$.

Since an MA ordering can be computed in $O(m)$ time, the complexity of the entire algorithm is $O(mn)$.

Remark 6.3.1 Arikati and Mehlhorn [5] described a simple method for finding a maximum flow between the last two nodes of an MA ordering.

The capacitated case

The algorithm can be extended to the more general situation when a non-negative capacity function g is given on the edge-set of the graph and the objective is to find a cut of minimum total capacity. The only difference is that we need the following extension of Theorem 6.3.1 Let $\lambda_g(x, y; G)$ denote the minimum g-capacity of a cut separating x and y.

Theorem 6.3.2 $\lambda_g(v_n, v_{n-1}; G) = d_g(v_n, V_{n-1}; G) \;\; (= d_g(v_n))$. •

Proof. Obviously the left-hand side cannot exceed the right-hand side, so we only need to prove the $\geq$ direction. Assume indirectly that the theorem is false and let G be a minimal counterexample. Then, clearly, $n \geq 3$.

We claim that $d_g(v_n) = d_g(v_n, V_{n-2})$, which is equivalent to stating that *there is no edge connecting* v_n *and* v_{n-1}. If there were such an edge e, then the same ordering is an MA ordering with respect to $G - e$. $G - e$ is smaller than G, hence the theorem holds for $G - e$ and then it would hold for G, as well, a contradiction.

The theorem holds for graph $G - v_n$ so

$$\lambda_g(v_{n-1}, v_{n-2}; G) \geq \lambda_g(v_{n-1}, v_{n-2}; G - v_n) \geq d_g(v_{n-1}, V_{n-2}) \geq d_g(v_n, V_{n-2}) = d_g(v_n)$$

where the last inequality follows from the MA rule. The theorem holds for $G - v_{n-1}$ as well, therefore

$$\lambda_g(v_n, v_{n-2}; G) \geq \lambda_g(v_n, v_{n-2}; G - v_{n-1}) \geq d_g(v_n, V_{n-2}) = d_g(v_n).$$

Hence

$$\lambda_g(v_n, v_{n-1}; G) \geq \min\{\lambda_g(v_n, v_{n-2}; G), \lambda_g(v_{n-1}, v_{n-2}; G)\} \geq d_g(v_n),$$

contradicting that G is a counterexample. •

Computing a minimum cut distinct from $\boldsymbol{\Delta(\{z\})}$

In Section 8.1.3 we will introduce the notion of undirected splitting. For efficiently computing a splittable pair of edges at a designated node z, we shall need to be able to compute a minimum cut of G which is distinct from the cut $\Delta(\{z\})$. The algorithm of Nagamochi and Ibaraki can be slightly modified so as to compute such a cut and its cardinality (or capacity in the capacitated case) denoted by $\lambda'(G)$. To this end, in each stage of the algorithm we select an MA ordering in which $v_1 = z$. (Note that any node of G can be chosen to be the first node of an MA ordering). Like before, we have $\lambda'(G) = \lambda'(G_1) = \min\{\lambda'(G_2), d_{G_1}(x_1)\} = \min\{\lambda'(G_3), d_{G_1}(x_1), d_{G_2}(x_2)\} = \cdots = \min\{d_{G_1}(x_1), d_{G_2}(x_2), \ldots, d_{G_{n-2}}(x_{n-2})\}$, that is, the only change is that $d_{G_{n-1}}(x_{n-1})$ is not taken into consideration.

6.4 Strongly polynomial algorithm for cheapest *m*-flows

The algorithm of Ford and Fulkerson for computing a cheapest flow of a given amount from s to t, discussed in Section 3.6, is not polynomial for general capacity functions. Here we consider the general minimum cost m-flow problem. The first strongly polynomial time algorithm was devised by Tardos [359]. Since then a great number of other algorithms have been developed: for extensive overviews, see [2, 340, 248]. Perhaps the most elegantly formulated algorithm, though not the most efficient one, is due to Goldberg and Tarjan [187]. As a straight generalization of the maximum flow algorithm of Edmonds and Karp and of Dinits, the algorithm of Goldberg and Tarjan make improvements along minimum mean circuits. Actually, they developed two variants of the algorithm and it is the first one that uses minimum mean circuit for every individual improvement. We are going to describe the second one since its proof is slightly simpler and its complexity is also better. The main difference is that here several subsequent flow improvements are grouped together so that only one minimum mean circuit computation is needed for the an entire group.

In Section 3.4.2 it was pointed out that the cheapest feasible circulation problem with general bounding functions $f \leq g$ is equivalent to a cheapest feasible m-flow problem with $f \equiv 0$ and $g \equiv \infty$ and here we describe the algorithm for that case. This is technically simpler and easier to understand, but in practical applications the reduction to m-flows increases the complexity, and the original version of the algorithm of Goldberg and Tarjan is more efficient.

We assume henceforth that $\widetilde{m}(V) = 0$, $f \equiv 0$, and $g \equiv \infty$, in which case the feasibility of a function $x : A \to \mathbf{R}$ simply means that $x \geq 0$. Note that in this case, the necessary and sufficient condition in Hoffman's feasibility theorem requires

$$\widetilde{m}(X) \geq 0 \text{ whenever } \delta_D(X) = 0$$

so we assume this. We can also assume that the cost function $c : A \to \mathbf{R}$ is conservative. Indeed, $g \equiv \infty$ implies that $x_0 + \alpha \cdot \underline{\chi}_C$ is a feasible m-flow whenever x_0 is a feasible m-flow, C is a di-circuit, and α is an arbitrarily large positive number. Hence if $\widetilde{c}(C)$ is negative, then cx is not bounded from below over feasible m-flows x. We can even assume that c is non-negative since there is a feasible potential π for c and then c_π is a non-negative and equivalent cost function. Here $c_\pi(uv) := c(uv) - \pi(v) + \pi(u)$ is the standard notation used for any potential $\pi : V \to \mathbf{R}$.

Optimality criteria and optimality deficit

Let $D^* = (V, A^*)$ denote the digraph obtained from D by adding the reverse h' of each edge h of D. We extend c to the new edges by letting $c(h') := -c(h)$. For an m-flow $x \geq 0$, let the **residual graph** $D_x = (V, A_x)$ be the subgraph of D^* in which every edge $uv \in A$ belongs to A_x (forward edge) and, furthermore, vu also belongs to A_x whenever $uv \in A$ and $x(uv) > 0$ (backward edge).

Theorem 6.4.1 *Assume that $f \equiv 0$ and $g \equiv \infty$. For an m-flow $x \geq 0$, the following are equivalent.*

(A) *x is a cheapest feasible m-flow.*
(B) *There are no negative di-circuits in D_x.*
(C) *There is a potential $\pi : V \to \mathbf{R}$ for which $c_\pi(uv) \geq 0$ for every edge $uv \in A$ and moreover $c_\pi(uv) = 0$ whenever $x(uv) > 0$.*

Proof. (A)$\Rightarrow$(B). If there is a negative di-circuit C in D_x, then increase or decrease $x(e)$ by a sufficiently small value $\alpha > 0$ on edges of D corresponding to forward or backward edges of C, respectively. The cost of the arising non-negative m-flow is smaller than that of x since $\widetilde{c}(C) < 0$.

(B)$\Rightarrow$(C). By Gallai's theorem (Theorem 3.1.1), if there is no negative di-circuit, then there is a feasible potential. But the feasibility of π in digraph D_x is just equivalent to the properties in (C). (This actually proves the equivalence of (B) and (C).)

(C)$\Rightarrow$(A). Let $z \geq 0$ be an m-flow. Since the difference of two m-flows is a circulation, we have $\Delta_\pi x = \Delta_\pi z$ where the potential difference Δ_π is defined by $\Delta_\pi(uv) := \pi(v) - \pi(u)$.

Property (C) implies $c_\pi x = 0$ and $c_\pi z \geq 0$ from which $cz = c_\pi z + \Delta_\pi z \geq c_\pi x + \Delta_\pi x = cx$. •

A non-negative m-flow x is said to be **optimal** or **non-optimal** according to whether or not x satisfies the optimality criteria above. Consider now a subgraph $D' = (V, A')$ of D^* for which the cost function c is restricted to D'. We say that D' is **deficient** if it has a negative circuit. By the **deficit** of $D' = (V, A')$ (with respect to c), we mean the smallest value $\varepsilon = \varepsilon_{D'} \geq 0$ such that lifting $c(e)$ uniformly by ε on every $e \in A'$ results in a cost function c^+ which is conservative on D'. (In particular, this means that the deficit is zero if c is conservative on D'.) By Theorem 3.1.1, there is a potential π which is feasible with respect to c^+, that is, $c_\pi(e) \geq -\varepsilon_{D'}$ holds whenever $e \in A'$. When D' is deficient, the minimality of ε implies that there is a di-circuit C of D' such that $\widetilde{c}^+(C) = 0$ and hence $c^+_\pi(e) = 0$, that is, $c_\pi(e) = -\varepsilon_{D'}$ follows for every edge e of C. In this case we say that π **verifies** ε. Note that in Section 3.1 we described Karp's algorithm to compute $\varepsilon_{D'}$ for the case when c is not conservative on D'. Recall that the minimum circuit mean of a deficient D' with respect to c_π (and hence also for c) is $-\varepsilon_{D'}$.

For a non-optimal non-negative m-flow x, we define its **optimality deficit** $\varepsilon(x)$ to be the deficit of its residual digraph D_x. Let π_x denote a potential verifying $\varepsilon(x)$. For an optimal x, the optimality deficit $\varepsilon(x)$ is defined to be 0. By this, the optimality criterion (B) in Theorem 6.4.1 is equivalent to requiring that $\varepsilon(x) = 0$. The underlying idea of the algorithm is that we start with a feasible m-flow x. If c is conservative on D_x, then x is of minimum cost by Theorem 6.4.1 and the algorithm terminates in this case. If there is a negative circuit C in D_x, then the algorithm makes an improvement of x along C (to be described below) and iterates the procedure for the resulting m-flow. This change is called a cancellation along C. The first algorithm of Goldberg and Tarjan chooses C for cancellation to be a minimum mean circuit at every step. Below we describe their second algorithm, which uses a different selection rule. In order to have a polynomial bound for the number of cancellations we need to make some preparations.

Proposition 6.4.2 *Let y and z be two non-negative m-flows for which $z(h) < y(h)$ where $h = st \in A$. Let D' be a digraph in which uv is an edge if $uv \in A$ and $z(uv) < y(uv)$, while vu is an edge if $uv \in A$ and $z(uv) > y(uv)$. Then there is a di-circuit of D' passing through h.*

Proof. Let T denote the set of nodes reachable from t in D'. If s is in T, then a ts-path along with the edge $h = st$ form the requested di-circuit. If s is not reachable, then $z(e) \geq y(e)$ for every edge e of D leaving T, that is, $\delta_z(T) \geq \delta_y(T)$. Moreover, $z(e) \leq y(e)$ for every edge e of D entering T, and hence $\Psi_z(T) < \Psi_x(T)$. But this is not possible since $\Psi_z(T) = \widetilde{m}(X) = \Psi_x(T)$. •

Fixed edges

For a given non-optimal m-flow $z \geq 0$, we say that an edge $h \in A$ is z-**fixed** if $y(h) = 0$ holds for every m-flow $y \geq 0$ for which $\varepsilon(y) \leq \varepsilon(z)$. The following proposition formalizes the intuitive feeling that if the cost of a given edge h is very high (relative to the current optimality deficit), then $y(h) = 0$ must hold for every feasible m-flow y for which the deficit is at most $\varepsilon(z)$. This property will be used for obtaining a polynomial upper bound for the number of circuit improvements.

Proposition 6.4.3 *Let $z \geq 0$ be a non-optimal m-flow and let $h = st \in A$ be an edge of D for which $c_{\pi_z}(h) \geq 2n\varepsilon(z)$ holds. Then $z(h) = 0$ and h is z-fixed.*

Proof. Recall that $\varepsilon(z) > 0$ since z is non-optimal. We see immediately that $z(st) = 0$ since $z(st) > 0$ would imply that ts is in D_z, from which $c_{\pi_z}(ts) = -c_{\pi_z}(st) \leq -2n\varepsilon(z) < -\varepsilon(z)$ would follow contradicting that $c_{\pi_z}(e) \geq -\varepsilon(z)$ holds for every edge e of D_z.

Second, let $y \geq 0$ be an m-flow for which $\varepsilon(y) \leq \varepsilon(z)$, and suppose indirectly that $y(h) > 0 = z(h)$. Consider the di-circuit C ensured by Proposition 6.4.2. Let C' denote the di-circuit consisting of the reverse edges of C. It follows from the definitions that C is in D_z while C' is in D_y. Therefore

$$\widetilde{c}_{\pi_z}(C) = c_{\pi_z}(h) + \sum[c_{\pi_z}(e) : e \in C - h] \geq 2n\varepsilon(z) + (|C| - 1)(-\varepsilon(z)) \geq 2n\varepsilon(z) + (n-1)(-\varepsilon(z)) = (n+1)\varepsilon(z)$$

and hence

$$\widetilde{c}(C') = \widetilde{c}_{\pi_z}(C') = -\widetilde{c}_{\pi_z}(C) \leq -(n+1)\varepsilon(z).$$

This means, in particular, that $\widetilde{c}(C')$ is negative from which

$$\widetilde{c}(C')/|C'| < \widetilde{c}(C')/(n+1) \leq -\varepsilon(z).$$

We conclude that the mean of C' is smaller than $-\varepsilon(z)$, contradicting the hypothesis $\varepsilon(y) \leq \varepsilon(z)$ • •

Improving along one di-circuit and overhauls

Suppose that $x \geq 0$ is a non-optimal m-flow, that is, there is a negative di-circuit C in D_x. Let $\varepsilon(x)$ denote the optimality-deficit of x and let π_x denote the potential verifying $\varepsilon(x)$. By an **improvement along** C, we mean the following revision of x: increase or decrease $x(e)$ by Δ on edges of D corresponding to forward or backward edges of C, respectively, where Δ denotes the minimum of the $x(e)$ values on the edges $e \in A$ corresponding to the backward edges of C. Such a backward edge of C certainly exists since we assumed that c is non-negative on A. This operation is often called the cancelling of a di-circuit.

By an **overhaul**, we mean a sequence of di-circuit cancellations, defined as follows. The input of an overhaul is a non-optimal m-flow $x \geq 0$. With the help of Karp's algorithm, we start by computing the deficit $\varepsilon(x)$ of x as well as a potential π_x verifying $\varepsilon(x)$ along with a di-circuit C of D_x for which $\widetilde{c}(X) = -\varepsilon(x)$. Note that $c_{\pi_x}(e) = -\varepsilon(x)$ holds for every edge e of C. Next, we carry out an improvement along C and continue with the resulting m-flow x'.

At an intermediate stage of the overhaul, the previously computed m-flow $x' \geq 0$ is considered along with its residual graph $D_{x'}$. If there are no negative di-circuits in $D_{x'}$, then x' is optimal, and the overhaul, as well as the entire algorithm, terminates by returning x' as an optimal feasible m-flow. If $D_{x'}$ is deficient but each di-circuit of $D_{x'}$ contains an edge e for which $c_{\pi_x}(e) \geq 0$, then the current overhaul terminates and the next one starts with input x'.

Suppose now that there is di-circuit C of $D_{x'}$ so that $c_{\pi_x}(e) < 0$ for each edge e of C. Carry out an improvement along C and continue the overhaul with the resulting m-flow. Note that during one overhaul there can be several circuit improvements but that Karp's algorithm is needed only once. (Warning: during the entire overhaul, it is the unchanged cost function c_{π_x} determined by the initial m-flow x of the overhaul that matters in selecting the appropriate di-circuit for improvement.)

Claim 6.4.4 *If an overhaul starts with a non-optimal m-flow x and returns a non-optimal m-flow x^*, then $\varepsilon(x^*) \leq \varepsilon(x)(1 - 1/n)$.*

Proof. In an overhaul starting with x, every improvement is made along a di-circuit for which the edges are all negative with respect to c_{π_x} and hence no new negative edges can arise. Therefore if the c_{π_x}-cost of an edge e of D_{x^*} is negative, then e belongs to D_x, too. Every di-circuit K of D_{x^*} has an edge h for which $c_{\pi_x}(h) \geq 0$. Therefore $\widetilde{c}_{\pi_x}(K) \geq -\varepsilon(x)(|K| - 1)$ from which

$$\widetilde{c}_{\pi_x}(K)/|K| \geq -\varepsilon(x)(1 - 1/|K|) \geq -\varepsilon(x)(1 - 1/n),$$

that is, the minimum circuit mean in D_{x^*} is at least $-\varepsilon(x)(1 - 1/n)$, and hence the optimality deficit of x^* is at most $\varepsilon(x)(1 - 1/n)$. •

Claim 6.4.5 *Let z denote the non-optimal m-flow obtained from a non-optimal m-flow $x \geq 0$ after $N := n\lceil\log(2n)\rceil$ overhauls. Then $\varepsilon(z) < \varepsilon(x)/2n$.*

Proof. By Claim 6.4.4, during one overhaul the optimality deficit decreases to $(1 - 1/n)$ times the initial one. Therefore it is less than $\varepsilon(x)(1 - 1/n)^n$ after n overhauls. Since $[1 + 1/(n - 1)]^n > 1 + n \cdot \frac{1}{n-1} > 2$, we have $(1 - 1/n)^n = 1/[1 + 1/(n - 1)]^n < 1/2$ and therefore after n overhauls the optimality deficit decreases below its half. Hence nr overhauls reduce the optimality deficit below $\varepsilon(x)/2^r$ for every integer $r > 0$ and, in particular, below $\varepsilon(x)/2n$ for $r = \lceil\log(2n)\rceil$. •

Claim 6.4.6 *Let x and z be the same as in Claim* 6.4.5. *Let ts be an edge of the minimum mean di-circuit C in D_x used for an improvement at the overhaul of D_x for which the c_{π_z}-cost is smallest. Then $st \in A$. Furthermore st is z-fixed but not x-fixed.*

Proof. Since the c_{π_x}-cost of every edge of C is $-\varepsilon(x)$, the c_{π_z}-mean of C is the same $-\varepsilon(x)$ and hence $c_{\pi_z}(ts) \leq -\varepsilon(x)$. By Claim 6.4.5, $\varepsilon(z) \leq \varepsilon(x)/2n$ and thus $c_{\pi_z}(ts) \leq -2n\varepsilon(z)$, that is, $c_{\pi_z}(st) = -c_{\pi_z}(ts) \geq 2n\varepsilon(z)$. If indirectly st is not in A, then $ts \in A$ from which $ts \in A_z$ and $c_{\pi_z}(ts) \geq -\varepsilon(z)$, that is, $c_{\pi_z}(st) = -c_{\pi_z}(ts) \leq \varepsilon(z)$ would follow contradicting the inequality $c_{\pi_z}(st) \geq 2n\varepsilon(z) > \varepsilon(z)$. Therefore $st \in A$ and Proposition 6.4.3 implies that st is z-fixed. •

The algorithm starts with an arbitrary feasible m-flow and carries out overhauls as long as the current residual graph is deficient. Since $O(n \log n)$ overhauls fix an edge that was not fixed or results in an optimal m-flow, the m-flow obtained after at most $O(|A|n \log n)$ overhauls is a cheapest one. The description of the second algorithm of Goldberg and Tarjan as well as the proof of its strong polynomiality is now complete. • •

The first version of their algorithm (the one typically discussed in the literature) considers separately the improvements along minimum mean circuits. Therefore Karp's algorithm is called as many times as the total number of circuit improvements, that is, $O(|A|^2 n \log n)$ times. The variant above needs Karp only once at every overhaul, that is, only $O(|A| n \log n)$ times. Since an overhaul can be carried out in $O(|A| n)$ time, the overall complexity of the algorithm is $O(|A|^2 n^2 \log n)$. By applying dynamic trees, the complexity of the algorithm of Goldberg and Tarjan reduces to $O(|A|^2 n \log^2 n)$.

For the sake of completeness, we formulate the optimality criteria for the general case when $f \leq g$ can be arbitrary. Let z be a feasible m-flow. Construct a digraph $D_z = (V, A_z)$ along with cost function c_z on its edge-set as follows. uv is an edge of A_z if either $uv \in A$ and $z(uv) < g(uv)$, in which case let $c_z(uv) := c(uv)$ or else $vu \in A$ and $z(vu) > f(vu)$, in which case let $c_z(uv) := -c(vu)$.

Theorem 6.4.7 *For a feasible m-flow z the following are equivalent.*

(A) *z is a cheapest feasible m-flow.*
(B) *There are no negative di-circuits in D_z.*
(C) *There is a function $\pi : V \to \mathbf{R}$ for which*

$$\begin{cases} \pi(v) - \pi(u) \leq c(uv), \text{ if } uv \in A \text{ and } z(uv) < g(uv), \\ \pi(v) - \pi(u) \geq c(uv), \text{ if } uv \in A \text{ and } z(uv) > f(uv). \end{cases}$$

If c is integer-valued, then π can also be chosen to be integer-valued. •

Problems

6.4.1 *Prove Theorem* 6.4.7.

6.4.2 *Develop a good characterization for boundedness in the general case.*

6.4.3 *Formulate the algorithm of Goldberg and Tarjan for the general case.*

7

Structure and representations of cuts

In Theorem 3.5.4, we already became familiar with a result concerning the structure of minimum cuts separating two specified nodes. In this chapter we discuss further structural properties of minimum cuts.

7.1 Cactus representation of minimum cuts

Let $G = (V, E)$ be a connected graph. Recall from Section 1.1 that for a subset $\emptyset \subset X \subset V$ of nodes, the set $\Delta(X)$ of edges connecting X and $V - X$ is called a cut while X and $V - X$ are the shores of the cut. Proposition 1.1.3 implies that a cut uniquely determines its shores in the sense that if $\Delta(X) = \Delta(Y)$, then $X = Y$ or $X = V - Y$. A star-cut is a cut for which one of its shores consists of a single node; otherwise the cut is **proper**. Let $d(x, y)$ denote the number of parallel xy-edges. Two subsets X and Y of nodes are said to be crossing if none of $X - Y,\ Y - X,\ X \cap Y,\ V - (X \cup Y)$ is empty. Two cuts $\Delta(X)$ and $\Delta(Y)$ are crossing if X and Y are crossing. We call a cut of a k-edge-connected graph G a **min-cut** if it has exactly k edges. A subset $T \subset V$ is **tight** if $d(T) = k$. A **proper tight** set is one for which $|T| \neq 1 \neq |V - T|$, that is, (proper) tight sets are the shores of (proper) min-cuts. A tight set is said to cross another tight set if they are crossing. A min-cut $\Delta(X)$ crosses another min-cut $\Delta(Y)$ if X crosses Y. A family $\mathcal{F}$ of sets is cross-free if no two members of $\mathcal{F}$ cross each other.

Proposition 7.1.1 *Let $G = (V, E)$ be a k-edge-connected graph. If X and Y are two crossing tight sets, then each of $X - Y, Y - X, X \cap Y$, and $X \cup Y$ is tight. Moreover, $d(X, Y) = 0 = \overline{d}(X, Y)$.*

Proof. It follows from

$$k + k = d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y) \geq k + k + 2d(X, Y)$$

that $d(X \cap Y) = k,\ d(X \cup Y) = k$, and $d(X, Y) = 0$. Similarly,

$$k + k = d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\overline{d}(X, Y) \geq k + k + 2\overline{d}(X, Y)$$

implies that $d(X - Y) = k$, $d(Y - X) = k$, and $\overline{d}(X, Y) = 0$. •

The structure of min-cuts when k is odd is particularly simple since it turns out that in this case no two min-cuts can cross each other. This observation, combined with the tree-representation of cross-free families, gives rise to the following result.

Theorem 7.1.2 *If the minimum cardinality k of a cut of a graph $G = (V, E)$ is odd, then there is a tree $H = (U, F)$ along with a map $\varphi : V \to U$ such that the min-cuts of G and the edges of H are in a one-to-one correspondence: for every edge $e \in F$, the pre-images of the two components of $H - e$ are the shores of the corresponding min-cut.*

Proof. We claim that there are no two crossing min-cuts. Indeed, if indirectly there are two crossing tight sets X and Y, then $\overline{d}(X, Y) = 0$ by Proposition 7.1.1 and hence $k = d(X \cap Y) = d(X - Y, X \cap Y) + d(Y - X, X \cap Y)$. Since k is odd, the two summands are not equal. We can assume that $d(X - Y, X \cap Y) > d(Y - X, X \cap Y)$ but then $k = d(X) = d(X - Y) - d(X - Y, X \cap Y) + d(Y - X, X \cap Y) < k$, a contradiction. The respresentation theorem of cross-free families (Theorem 1.4.2) implies the result. •

When k is even, the structure of min-cuts is more complicated. For example, if $k = 2$ and G is a circuit, then every pair of edges forms a min-cut. The next lemma, which is the heart of the main result of the section, is a sort of a converse.

Lemma 7.1.3 *Let $k \geq 2$ be an even integer. Let $G = (V, E)$ be a k-edge-connected graph in which there is a proper min-cut and every proper min-cut is crossed by some proper min-cuts. Then G can be obtained from a circuit by replacing each edge with $k/2$ parallel edges.*

Proof. Since the complement of a tight set is also tight, the hypothesis implies that

$$\text{for any proper tight set } T \text{ and node } v \in V, \text{ there is a tight set crossing } T \text{ and containing } v. \tag{7.1}$$

Claim 7.1.4 *The degree of every node of G is k.*

Proof. Suppose indirectly that $d(v) > k$ for a node v. Consider a minimal proper tight set T containing v. By (7.1), there is a tight set X crossing T that contains v. But then $T \cap X$ is tight by Proposition 7.1.1, and $|T \cap X|$ is proper as $d(v) > k$. This contradicts the minimality of T. •

Claim 7.1.5 *If $T = \{x, y\}$ is a tight set, then the number $d(x, y)$ of parallel xy-edges is $k/2$.*

Proof. $k = d(T) = d(x) + d(y) - 2d(x, y) = k + k - 2d(x, y)$, that is, $d(x, y) = k/2$. •

Claim 7.1.6 *Let v be a node and T a proper tight set containing v. Then T includes a 2-element tight set containing v.*

Proof. We proceed by induction on the cardinality of T. Since T itself will suffice if $|T| = 2$, we assume that $|T| \geq 3$. By (7.1), there is a tight set X crossing T and containing v. Then $T \cap X$ is also tight and in the case $|T \cap X| \geq 2$ by induction we are done. Suppose now that $T \cap X = \{v\}$. By Proposition 7.1.1, $T' := T - X$ $(= T - v)$ is a proper tight set. By (7.1),

978–0–19–920527–1 07-Andras-Frank-c07-drv Andras Frank (Typeset by SPi, Publisher Services, India) 230 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

there is a proper tight set X' crossing T' and containing v. Then either $X' \subset T$ or else X' and T are crossing. In both cases, $T \cap X'$ is tight and $|T| > |T \cap X'| \geq 2$, and by induction we are done again. •

Claim 7.1.7 *For every node v, there are two 2-element tight sets containing v.*

Proof. It follows from Claim 7.1.6 that there is a 2-element tight set $T_1 = \{v, x\}$. By (7.1), there is a tight set T' crossing T_1 that contains v. A second application of Claim 7.1.6 (with T' in place of T) implies that there is a 2-element tight subset T_2 of T' which contains v, and that this differs from T_1. •

To complete the proof of the lemma, suppose that $\{v, x\}$ and $\{v, y\}$ are tight sets. Since there are exactly $k/2$ parallel vx-edges and $k/2$ parallel vy-edges, we conclude that every node v of G has exactly two distinct neighbours. Since G is connected, it arises from a circuit by replacing each edge with $k/2$ parallel copies. • •

We call a 2-edge-connected, loopless graph $C = (U, F)$ a **cactus** if each edge belongs to exactly one circuit. This criterion is equivalent to saying that all blocks are circuits (allowing 2-element circuits). For example, a special cactus can be obtained by duplicating in parallel each edge of a tree. A more general cactus is shown in Figure 7.1.

Note that the min-cuts of a cactus C are exactly those pairs of edges that belong to the same circuit of C. The following result states that the mincuts of an arbitrary graph have the same structure as the mincuts of a cactus. The present proof is due to Fleiner and Frank [99]; for algorithmic aspects and related results, see the book by Nagamochi and Ibaraki [300].

Theorem 7.1.8 (Dinits, Karzanov, and Lomonosov [70]) *Let $k \geq 1$ be an integer and $G = (V, E)$ a loopless graph for which the minimum cardinality of a cut is k. There is a cactus $C = (U, F)$ and a mapping $\varphi : V \to U$ such that the preimages $\varphi^{-1}(U_1)$ and $\varphi^{-1}(U_2)$ are the two shores of a min-cut of G for every 2-element cut of C with shores U_1 and U_2. Moreover, every min-cut of G arises this way. Concisely: X is a tight set of G if and only if $\varphi(X)$ is a tight set of C.*

Proof. Since a tree-representation of cross-free cuts can be made into a cactus representation by doubling each edge of the representing tree in parallel, Theorem 7.1.2 implies the theorem when k is odd. Henceforth we assume that k is even and proceed by induction on $|V|$. Since the theorem is trivial when $|V| \leq 2$, we assume that $|V| \geq 3$.

Suppose first that each min-cut is a star-cut and let $v_1, \dots, v_h$ denote the nodes of degree k. Let $U = \{u_0, u_1, \dots u_h\}$ be the node-set of cactus C in which u_0 and u_i are connected by two parallel edges for each $i = 1, \dots, h$. Let $\varphi : V \to U$ be defined by $\varphi(v_i) = u_i$ for $i = 1, \dots, h$ and $\varphi(v) = u_0$ for $v \in V - \{v_1, \dots, v_h\}$. Then C and φ satisfy the requirements of the theorem.

Suppose now that there is a proper min-cut, so $|V| \geq 4$. If every proper min-cut is crossed by a min-cut, then Lemma 7.1.3 implies that G can be obtained from a circuit $C = (V, F)$ by replacing each edge of C with $k/2$ parallel edges. In this case, C is a cactus which forms (along with the identity map $\varphi : V(G) \to V(C)$ the requested representation of min-cuts. Therefore we can assume that there is a min-cut B with shores V_1 and V_2 which is not crossed by any other min-cut.

978–0–19–920527–1 07-Andras-Frank-c07-drv Andras Frank (Typeset by SPi, Publisher Services, India) 231 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

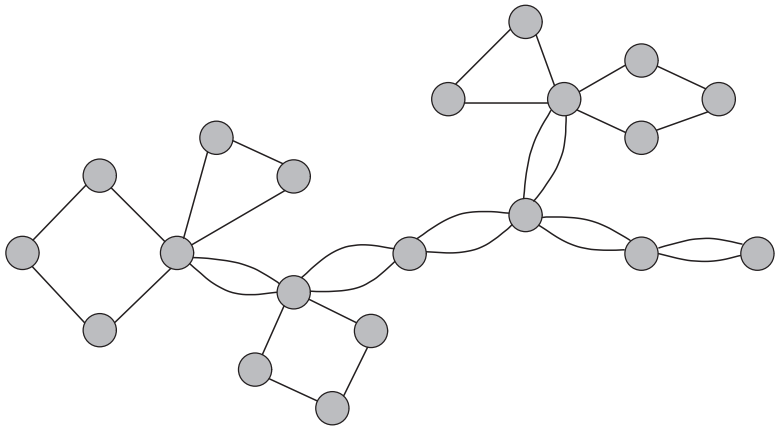


Figure 7.1 Cactus

For $j = 1, 2$, let G_j denote the graph arising from G by shrinking V_j into a single new node v_j in the sense that V_j is replaced by v_j in such a way that there are as many parallel uv_j-edges in G_j as the number of edges in G connecting V_j and u for every node $u \in V - V_j$. By induction, the min-cuts of G_j can be represented by a cactus $C_j = (U_j, F_j)$ and a mapping φ_j. We assume that U_1 and U_2 are disjoint. Since $d_{G_j}(v_j) = k$, the node $u_j := \varphi_j(v_j)$ is of degree 2 in C_j and there is no other node v of G_j with $\varphi_j(v) = u_j$.

Let $C = (U, F)$ be a cactus arising from C_1 and C_2 by identifying u_1 and u_2. Define $\varphi : V \to U$ by $\varphi(v) := \varphi_1(v)$ if $v \in V_2$ and $\varphi(v) := \varphi_2(v)$ if $v \in V_1$. Since no min-cut crosses B, each min-cut of G is either a min-cut of G_1 or a min-cut of G_2 and hence C and φ provide the requested cactus representation of the min-cuts of G. •

The proof of Theorem 7.1.8 extends essentially word for word to the capacitated version of the theorem. In this case, a strictly positive capacity function $g : E \to \mathbf{R}_+$ is given on the edge-set E and k denotes the minimum total capacity of a cut.

Dinits and Vainshtein extended Theorem 7.1.8, as follows. Let $G = (V, E)$ be a graph with a terminal set $S \subseteq V$ of at least two elements. We say that a cut B of G separates S if both shores of B intersect S. Suppose that the minimum cardinality of a cut separating S is k. A subset $\emptyset \subset T \subset S$ is S-tight if there is a subset $X \subset V$ for which $d(X) = k$ and $T = S \cap X$.

Theorem 7.1.9 (Dinits and Vainshtein [71]) *The S-tight sets admit a cactus representation.* •

The proof is a natural extension of the proof of Theorem 7.1.8. Note that the family of all minimum cuts separating S cannot be represented by a cactus if $|S| = 2$ since the number of minimum cuts separating nodes s and t can be exponential in $|V|$ while the number of cuts represented by a cactus is always less than $|V|^2$.

978–0–19–920527–1 07-Andras-Frank-c07-drv Andras Frank (Typeset by SPi, Publisher Services, India) 232 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

7.2 Local edge-connectivities

7.2.1 Flow-equivalent trees

Let $G = (V, E)$ be an undirected graph endowed with a non-negative capacity function $g : E \to \mathbf{R}_+$. Let $\lambda_g(u, v)$ denote the maximum amount of a feasible flow from u to v. This parameter called the local edge-connectivity between u and v and, by the Max-flow Min-cut (MFMC) theorem, it is equal to $\min\{d_g(X) : X \subset V \text{ separates } u \text{ and } v\}$.

Theorem 7.2.1 *Let F be a maximum-weight spanning tree of the complete graph $G^* = (V, E^*)$ with respect to the weight function $w :\to E^*$ defined by $w(uv) := \lambda_g(u, v)$ $(u, v \in V)$. Then F encodes each possible local edge-connectivity values of G in the sense that*

$$\lambda_g(u, v) = \min\{w(e) : e \in P,\ P \text{ is the unique } uv\text{-path in } F\}. \tag{7.2}$$

The heaviest tree can be chosen as a Hamilton path.

Proof. It follows from the definition of λ_g that $\lambda_g(u, v) \geq \min\{w(e) : e \in P'\}$, where P' is an arbitrary uv-path, since a cut separating u and v certainly separates at least one pair of consecutive nodes of the path. On the other hand, we must have $\lambda_g(u, v) \leq \min\{w(e) : e \in P,\ P \text{ is the unique } uv\text{-path in } F\}$ since F was chosen to be a maximum-weight spanning tree. Hence (7.2) follows.

For the second part, let F be a maximum-weight tree in which the number of leaf nodes is minimum and, subject to this, the smallest distance of two leaf nodes of F is minimum. Let s and t be two leaf nodes for which their distance in F is minimum. If F is not a path, then the unique path P of F connecting s and t has a node z of degree at least 3. Let u and v be the two neighbours of z along P. It follows from the first part of the theorem, that $w(uv) = \min\{w(zu), w(zv)\}$. Suppose that $w(uv) = w(zu)$. Then the tree $F' := F - uv + zu$ is also a maximum-weight tree. But then F' has either fewer leaf nodes or else has the same number of leaf nodes as F but there are two leaf nodes for which the their distance in F' is smaller than the distance of s and t in F, contradicting the choice of F. •

A straightforward consequence of the theorem is that the number of distinct $\lambda_g(u, v)$ values is at most $n - 1$. The tree in the theorem is called a **flow-equivalent tree**. It is useful because any local edge-connectivity value can readily be determined from F. It would be similarly useful to encode economically a smallest cut separating u and v for each possible pair of nodes. The content of the next section is that such a concise encoding is also possible.

7.2.2 Gomory–Hu trees

Let $G = (V, E)$ be a connected undirected graph and $g \to \mathbf{R}_+$ a non-negative capacity function. We say that a set $Z \subset V$ separating u and v is uv**-minimal** if $d_g(Z)$ is minimal over all subsets separating u and v. By Menger's theorem, Z is uv-minimal if and only if $d_g(Z) = \lambda_g(u, v)$.

A set X is said to be **critical** if X is uv-minimal for some u and v. In order to have some insight into the structure of critical sets, one can be interested in a list of critical sets that contains a uv-minimal set for each pair $\{u, v\}$. How short can this list be? By choosing a separate uv-minimal set for each pair, one gets a list of $n(n-1)/2$ sets. But we can do much better.

Let $G_T = (V, F)$ be any tree on node-set V (not necessarily a subgraph of G). For every edge e in F let $m(e) := d_g(X_e)$ where X_e and $V - X_e$ are the two components of $G_T - e$. G_T is called a **Gomory–Hu tree** of G (with respect to the given capacity function g) if (A) for every pair $\{s, t\}$ of nodes $\lambda_g(s, t)$ is the minimum of m-values over the edges of the unique path in G_T connecting s and t and (B) if e is an edge where the minimum is attained, then X_e is st-minimal. For example, if $G = K_{3,3}$ and $g \equiv 1$, then a star of five edges forms a Gomory–Hu tree (and there is no other one showing that a Gomory–Hu tree cannot be chosen, in general, as a subgraph of G).

Theorem 7.2.2 (Gomory and Hu [188]) *Every graph possesses a Gomory–Hu tree.*

Proof. We are going to consider only the case $g \equiv 1$ since the proof goes along the same lines for general g. We need the following terminology. We say that a family $\mathcal{F}$ of subsets of nodes **separates** nodes u and v if at least one member of $\mathcal{F}$ separates u and v. Let $\mathcal{F}$ be a laminar family and $\{u, v\}$ a pair of nodes not separated by $\mathcal{F}$. We say that a subset X of nodes separating u and v is uv**-minimal** with respect to $\mathcal{F}$ if $\mathcal{F} \cup \{X\}$ is laminar and $d(X)$ is as small as possible. Note that such an X can be computed by one MFMC computation in a graph obtained from G by shrinking the complement of the smallest member Y of $\mathcal{F}$ containing u and v and shrinking the maximal members of $\mathcal{F}$ included in Y.

Let us construct a laminar family $\mathcal{F}$ of $n - 1$ sets as follows. Let $\mathcal{F}_0$ be empty. Suppose we have constructed a laminar family $\mathcal{F}_{k-1} = \{V_1, \dots, V_{k-1}\}$ for some $k = 1, \dots, n - 1$. Let $\{u_k, v_k\}$ be any pair of nodes not separated by $\mathcal{F}_{k-1}$. Determine a set V_k that is $u_k v_k$-minimal with respect to $\mathcal{F}_{k-1}$ and let $\mathcal{F}_k := \mathcal{F}_{k-1} \cup \{V_k\}$.

Let $\mathcal{F} := \mathcal{F}_{n-1}$ and let $E_1 := \{u_i v_i : i = 1, \dots, n - 1\}$. It follows from this construction that $T_1 := (V, E_1)$ is a tree. (This is just an auxiliary tree for the proof.)

Claim 1 *Let X be an xx'-minimal set and Y a critical set. If Y does not contain x and x', then either $X - Y$ or $X \cup Y$ is xx'-minimal. If Y contains x and x', then either $X \cap Y$ or $Y - X$ is xx'-minimal.*

Proof. To prove the first statement suppose that Y is yy'-minimal. Then either $Y - X$ or $Y \cap X$ separates $\{y, y'\}$. In the first case one has

$$\lambda(x, x') + \lambda(y, y') = d(X) + d(Y) \geq d(X - Y) + d(Y - X) \geq \lambda(x, x') + \lambda(y, y').$$

Hence equality follows everywhere showing that $X - Y$ is xx'-minimal. In the second case

$$\lambda(x, x') + \lambda(y, y') = d(X) + d(Y) \geq d(X \cup Y) + d(X \cap Y) \geq \lambda(x, x') + \lambda(y, y').$$

Hence equality follows everywhere showing that $X \cup Y$ is xx'-minimal, as required for the first statement.

The second statement follows from the first one if we replace Y by its complement. •

Claim 2. *$V_i \in \mathcal{F}$ is $u_i v_i$-minimal for each $i = 1, \dots, n - 1$.*

Proof. The claim is evident for $i = 1$ and we suppose that it has already been proved for $1, 2, \dots, i - 1$. Let $x := u_i$ and $x' := v_i$. The claim follows once we show for $j = 0, 1, \dots, i - 1$ that

$$\text{there is an } xx'\text{-minimal set } X \text{ for which } \mathcal{F}_j \cup \{X\} \text{ is laminar.} \tag{7.3}$$

The statement in (7.3) obviously holds for $j = 0$. Suppose we have already shown (7.3) for some $0 \leq j < i - 1$. Let X' be the xx'-minimal set assured by Claim 1 when applied to X and $Y := V_j$. Then $\mathcal{F}_j \cup \{X'\}$ is laminar, as required. •

Claim 3 *For every pair $\{s, t\}$ of nodes there is an st-minimal member of $\mathcal{F}$.*

Proof. Let P be the unique path in T_1 connecting s and t and let $M := \min\{\lambda(u_i, v_i) : u_i v_i$ an edge of $P\}$. By the MFMC theorem $\lambda(s, t) \geq M$. Let j be the smallest subscript for which $u_j v_j$ is an edge of P and $\lambda(u_j, v_j) = M$. We claim that V_j does not separate any other edge of P. Indeed, if V_j separates an edge $u_i v_i$ of P, then $i < j$ by the construction of $\mathcal{F}$. Furthermore, $M \leq \lambda(u_i, v_i) \leq d(V_j) = M$ and hence $\lambda(u_i, v_i) = M$, contradicting the minimal choice of j.

Therefore V_j must separate $\{s, t\}$ and hence we have $M \geq \lambda(u_j, v_j) = d(V_j) \geq \lambda(s, t) \geq M$. We can conclude that V_j is an st-minimal set. •

Let $V_0 := V$ and $\mathcal{F}' := \mathcal{F} \cup \{V\}$. For each $V_i \in \mathcal{F}'$, the union of maximal sets of $\mathcal{F}$ included in V_i is precisely one element smaller than V_i. Let t_i denote this element and for $i \geq 1$ let $s_i := t_j$ where V_j is the unique minimal element of $\mathcal{F}'$ including V_i. Let $F' := \{(s_i, t_i) : i = 1, \ldots, n - 1\}$. Then $G'_T = (V, F')$ is an arborescence such that each arc of it enters one member of $\mathcal{F}$. Let G_T denote the underlying (undirected) tree. By this construction and by Claim 3, G_T is a Gomory–Hu tree. • •

The following result was already proved with the help of MA orderings. (See Theorem 6.3.1.)

Corollary 7.2.3 *Every loopless undirected graph $G = (V, E)$ with at least two nodes has two nodes u and v so that $\lambda(u, v) = d(u)$.*

Proof. Let u be a leaf node of Gomory–Hu tree T and let v be its only neighbour. By the properties of Gomory–Hu trees, it follows that the cut $\Delta(u)$ is a minimum cut of G separating u and v. Therefore $\lambda(u, v) = |\Delta(u)| = d(u)$. •

7.2.3 A synthesis problem

So far we have investigated properties of minimum cuts of an *existing* graph. The following problem is somewhat the converse: we are given some expected properties of minimum cuts, and the goal is to construct an optimal graph having these properties. More specifically, let V be a ground-set and r a non-negative **demand function** on the pairs of nodes which is symmetric in the sense that $r(u, v) = r(v, u)$ for every pair u, v of nodes. Let $G^* = (V, E^*)$ denote the complete (undirected) graph on node-set V. We say that a capacity function $g : E^* \to \mathbf{R}_+$ is r**-feasible**, or **feasible**, if

$$\lambda_g(u, v) \geq r(u, v) \text{ whenever } u, v \in V, \tag{7.4}$$

where $\lambda_g(u, v)$ denotes the minimum capacity of a cut separating u and v. By the MFMC theorem, this is equivalent to requiring that the maximum flow-amount between u and v be at least $r(u, v)$ for every pair of nodes.

The synthesis problem we consider consists of constructing a feasible capacity function g for which the total value $\widetilde{g}(E^*) = \sum[g(e) : e \in E^*]$ is as small as possible. For each node v, let $R(v) := \max\{r(v, x) : x \in V - v\}$.

Theorem 7.2.4 (Gomory and Hu [189]) *Given a demand function r, the minimum total value $\widetilde{g}(E^*)$ of feasible capacity functions g is*

$$\widetilde{R}(V)/2 := \sum_{v \in V} R(v)/2. \tag{7.5}$$

If r is integer-valued, the optimal g can be chosen to be half-integral.

Proof. If g is feasible, then $d_g(v) \geq r(v, x)$ for every $x \in V - r$ and hence $d_g(v) \geq R(v)$ from which $\widetilde{g}(E^*) = \sum[d_g(v) : v \in V]/2 \geq \sum[R(v) : v \in V]/2 = \widetilde{R}(V)/2$, and hence $\widetilde{R}(V)/2$ is a lower bound for the total value of a feasible capacity.

We are going to prove by induction on $|V|$ plus the number of positive components of r that there is a g attaining this lower bound. Let $\alpha := \min\{R(v) : v \in V\}$. Suppose first that $\alpha = 0$ and let s be a node with $R(s) = 0$. Consider the synthesis problem on $V' := V - s$ with respect to the demand function r' which is the restriction of r to the pairs of nodes in V'. By induction, there is an r'-feasible g' with total value $\widetilde{R}'(V')/2$. Define g as follows.

$$g(xy) := \begin{cases} g'(xy) & \text{if } x, y \in V' \\ 0 & \text{if } x = s,\ y \in V'. \end{cases} \tag{7.6}$$

Then g is an r-feasible capacity function for which the total value is $\widetilde{R}'(V')/2 = \widetilde{R}(V)/2$, as required.

Suppose now that α is positive. Define $r'(u, v) := \max\{r(u, v) - \alpha, 0\}$. Then $R'(v) = R(v) - \alpha$ for every $v \in V$ and r' has strictly less positive components than r has. By induction, there is an r'-feasible capacity function g' with total value $R'(V)/2$. Let C be an arbitrary Hamilton circuit on V and let

$$g(e) := \begin{cases} g'(e) + \alpha/2 & \text{if } e \in C \\ g'(e) & \text{if otherwise.} \end{cases} \tag{7.7}$$

Then g is r-feasible since $\lambda_g(u, v) \geq \lambda_{g'}(u, v) + 2\alpha/2 \geq r'(u, v) + \alpha \geq r(u, v)$ for every $u, v \in V$. Moreover, its total value is $\widetilde{g}(E^*) = \widetilde{g}'(E^*) + \alpha|V|/2 = \widetilde{R}'(V)/2 + \alpha|V|/2 = \widetilde{R}(V)/2$, as required.

The half-integrality of g, in the case when r is integral, is a direct consequence of the proof above. •

Note that the integrality of the optimal g cannot be expected in general: take the identically 1 demand function on three or more nodes. In Section 11.1.4 it will be shown that if r is integer-valued and has no component of value 1, then the optimum of the integral solution can be at most one-half bigger than that of the fractional (that is, half-integral) solution. In fact, this result will be derived in a significantly more general framework when the capacities of an already existing graph are to be optimally augmented. (See Corollary 11.1.15 in Chapter 11.)

Problem 7.2.1 *Prove that there is an optimal half-integral g in Theorem* 7.2.4 *so that the edges e for which $g(e)$ is non-integral belong to a circuit.*

Problem 7.2.2 *Prove Theorem* 7.2.5.

Theorem 7.2.5 (Szigeti) *Let $G = (S, T; E)$ be a k-regular bipartite graph $(k \geq 1)$. Prove that there is a pairing $\{s_i t_i : s_i \in S,\ t_i \in T\}$ so that $\lambda_G(s_i, t_i) = k$ for each of the $|S|$ pairs.* •

Theorem 7.2.5 is a generalization of a theorem of Hamidoune and Las Vergnas [202].

7.2.4 Local edge-connectivities in digraphs

The structure of local edge-connectivities in a directed graph is far more complex than that in undirected graphs. For example, the number of distinct values of local edge-connectivity values in a graph is at most $n - 1$ while in a digraph it is $O(n^2)$. In what follows, $\lambda_D(u, v)$ will be abbreviated to $\lambda(u, v)$. The following result of Lovász [266] reveals a non-trivial relationship of local edge-connectivities of a digraph.

Theorem 7.2.6 *Let $D = (V, A)$ be a digraph with a specified root-node r_0. If $\delta(r_0) > \varrho(r_0)$, then there is a node $v \in V - r_0$ for which $\lambda(r_0, v) > \lambda(v, r_0)$.*

Proof. Suppose indirectly that

$$\lambda(r_0, v) \leq \lambda(v, r_0) \text{ for every node } v \in V - r_0. \tag{7.8}$$

Call a subset $X \subseteq V - r_0$ **in-tight** with respect to a node $v \in V - r_0$ if $v \in X$ and $\varrho(X) = \lambda(r_0, v)$. By Theorem 3.5.4, these sets are closed under intersection and union. Therefore there is a unique maximal in-tight set $T(v)$ containing v.

Claim 7.2.7 *If $u \in T(v)$, then $T(u) \subseteq T(v)$.*

Proof. $\lambda(r_0, u) + \lambda(r_0, v) = \varrho(T(u)) + \varrho(T(v)) \geq \varrho(T(u) \cap T(v)) + \varrho(T(u) \cup T(v)) \geq \lambda(r_0, u) + \lambda(r_0, v)$ from which we must have equality throughout, implying that $T(u) \cup T(v)$ is in-tight with respect to v. Because of the maximality of $T(v)$, we conclude that $T(u) \subseteq T(v)$. •

Let $\mathcal{T} = \{T_1, \ldots, T_t\}$ be the set-system of maximal members of the family $\{T(v) : v \in V - r_0\}$ and suppose that $T_i = T(v_i)$. The members of $\mathcal{T}$ cover $V - r_0$. Let S_i denote the set of elements of T_i which do not belong to other members of $\mathcal{T}$. Clearly, $\mathcal{S} = \{S_1, \ldots S_t\}$ is a subpartition. By Claim 7.2.7, $v_i \in S_i$ and hence

$$\delta(S_i) \geq \lambda(v_i, r_0) \geq \lambda(r_0, v_i) = \varrho(T_i). \tag{7.9}$$

Consider the set F of all edges leaving r_0 or some members of $\mathcal{S}$. We claim that each element f of F enters either r_0 or a member of $\mathcal{T}$. Indeed, this is clear if f leaves r_0. Since the union of the members of $\mathcal{T} - \{T_i\}$ is $(V - r_0) - S_i$, any edge leaving S_i enters r_0 or a member of $\mathcal{T} - \{T_i\}$. Therefore

$$\delta(r_0) + \sum_i \delta(S_i) = |F| \leq \varrho(r_0) + \sum_i \varrho(T_i),$$

from which $\varrho(r_0) - \delta(r_0) \geq \sum_i \delta(S_i) - \sum_i \varrho(T_i) \geq 0$ holds by (7.9), contradicting the hypothesis of the theorem. • •

Corollary 7.2.8 *If $\delta(r_0) \geq \varrho(r_0)$ holds for a node r_0 of a digraph D, then there is a node $v \in V - r_0$ for which $\lambda(r_0, v) \geq \lambda(v, r_0)$.*

Proof. By adding an arbitrary edge leaving r_0 to D, the resulting digraph D' satisfies the hypothesis of Theorem 7.2.6. Therefore there is a node $v \in V - r_0$, for which $\lambda_{D'}(r_0, v) > \lambda_{D'}(v, r_0')$ from which $\lambda_D(r_0, v) \geq \lambda_{D'}(r_0, v) - 1 \geq \lambda_{D'}(v, r_0) = \lambda_D(v, r_0)$. •

Corollary 7.2.9 *If $\lambda_D(u, v) = \lambda_D(v, u)$ holds for every pair $\{u, v\}$ of nodes of a digraph D, then D is Eulerian, that is, $\varrho(v) = \delta(v)$ for every node v.* •

Problem 7.2.3 (∗) *Prove the following sharpening of Theorem* 7.2.6*, devised by Szigeti* [356].

Theorem 7.2.10 *If $\delta(r_0) > \varrho(r_0)$ for a node r_0 of a digraph D, then there is a node $v \in V - r_0$ for which $\lambda_D(r_0, v) > \lambda_D(v, r_0)$ and $\delta(v) < \varrho(v)$.* •

Exercise 7.2.4 (∗) Decide whether the following statement is true or not. *If a digraph admits k edge-disjoint spanning arborescences of root r_0 and $\varrho(r_0) = \ell \leq k$, then there is a node v for which $\lambda(v, r_0) \geq \ell$.*

7.3 Solid sets

7.3.1 Solid sets of undirected graphs

Let $G = (V, E)$ be an undirected graph. We say that a non-empty subset $Z \subseteq V$ is **solid** (or **extreme**) if $d_G(X) > d_G(Z)$ for every non-empty $X \subset Z$. In particular, every singleton is solid, while V is solid precisely if G is connected. A solid set X is **proper** if $X \subset V$.

Proposition 7.3.1 *The set-system $\mathcal{S}_G$ of solid sets is laminar.*

Proof. Suppose indirectly that two solid sets X and Y are properly intersecting. Then $X - Y$ and $Y - X$ are proper non-empty subsets of X and Y, respectively, from which $d_G(X) + d_G(Y) \geq d_G(X - Y) + d_G(Y - X) > d_G(X) + d_G(Y)$, a contradiction. •

Proposition 7.3.2 *Each non-emtpy subset $X \subseteq V$ includes a solid set Z with $d(Z) \leq d(X)$.*

Proof. If X itself is solid, then $Z := X$ will suffice. If X is not solid, then there is a non-empty subset $Z \subset X$ with $d(Z) \leq d(X)$. Assume that Z is minimal with respect to this property. Then Z is solid, for if there is a non-empty subset $Z' \subset Z$ with $d(Z') \leq d(Z)$, then $d(Z') \leq d(X)$, contradicting the minimal choice of Z. •

We call a non-empty subset $K \subseteq V$ a **λ-component** of G if

$$\min\{\lambda_G(u, v) : u, v \in K\} > \max\{\lambda_G(u, v) : u \in K,\ v \in V - K\} \tag{7.10}$$

and the minimum is called the edge-connectivity of K. By convention, the node-set V of a connected graph is considered to be a λ-component.

978–0–19–920527–1 07-Andras-Frank-c07-drv Andras Frank (Typeset by SPi, Publisher Services, India) 238 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proposition 7.3.3 *The set-system $\mathcal{L}_G$ of λ-components of G is laminar.*

Proof. Assume indirectly that two λ-components X and Y are properly intersecting. For nodes $x \in X - Y$, $y \in Y - X$, and $z \in X \cap Y$, we have $\lambda_G(x,z) > \lambda_G(z,y)$ since X is a λ-component, and $\lambda_G(z,y) > \lambda_G(x,z)$ since Y is a λ-component, a contradiction. •

With the help of a Gomory–Hu tree T of G, the λ-components of G can be computed as follows. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_t$ denote the distinct min-cut values assigned to the edges of T. For each i, let $\mathcal{P}_i$ denote the partition of V defined by the components of the forest arising from T by deleting all the edges of T for which the edge-value is strictly smaller than λ_i. For $i = 1$, for example, $\mathcal{P}_1 = \{V\}$. It is evident that the members of $\mathcal{P}_i$ and $\mathcal{P}_j$ are distinct for $i \neq j$. Let $\mathcal{L} := \cup_{i=1}^{t} \mathcal{P}_i$.

Proposition 7.3.4 $\mathcal{L} = \mathcal{L}_G$.

Proof. Suppose first that $K \in \mathcal{P}_i$ for some $i = 1, \dots, t$. Then the properties of Gomory–Hu trees imply that $\lambda_G(u,v) \geq \lambda_i$ for every $u, v \in K$ and $\lambda_G(u,v) < \lambda_i$ whenever $u \in K$, $v \in V - K$, that is, K is a λ-component.

Conversely, let K be a λ-component. Then its edge-connectivity is λ_i for some $i = 1, \dots, t$ and hence K is a member of $\mathcal{P}_i$. •

Theorem 7.3.5 (Naor, Gusfield, and Martel [302]) $\mathcal{S}_G \subseteq \mathcal{L}_G$, *that is, every solid set is a λ-component. A λ-component K is a solid set if and only if $d(K') > d(K)$ for every λ-component $K' \subset K$.*

Proof. Let Z be a solid set. If $Z = V$, then G is connected and hence Z is a λ-component. Suppose now that $Z \subset V$ and let $\ell := \min\{\lambda_G(u,v) : u, v \in Z\}$. If $\ell > d(Z)$, then $\lambda_G(u,v) \leq d(Z) < \ell$ whenever $u \in Z$ and $v \in V - Z$, that is, Z is a λ-component in this case. Suppose now that $\ell \leq d(Z)$ and let u_1 be an arbitrary node in $V - Z$. Let X be a set separating Z for which $d(X) = \ell$. By complementing X, if necessary, we can assume that $u_1 \in X$. Since Z is solid and $Z - X$ is a non-empty proper subset of Z, we have $d(Z - X) > d(Z)$ from which $d(X) + d(Z) \geq d(X - Z) + d(Z - X) > d(X - Z) + d(Z)$ and hence $d(X - Z) < d(X) = \ell$, that is, every node $u_1 \in V - Z$ belongs to a subset of $V - Z$ of degree less then ℓ, implying that Z is a λ-component.

For the second part, suppose first that a λ-component K is solid and let $K' \subset K$ be a λ-component. Since K is solid, $d(K') > d(K)$. Conversely, let K be a λ-component that is not solid. Then there is a no-empty set $X \subset K$ for which $d(X) \leq d(K)$. By Proposition 7.3.2, there is a solid subset $K' \subseteq X$ for which $d(K') \leq d(X)$ from which $d(K') \leq d(K)$. •

We showed above that the laminar set-system $\mathcal{L}_G$ of all λ-components can be computed by using a Gomory–Hu tree. Relying on Theorem 7.3.5, we can select all solid members of $\mathcal{L}$ in polynomial time.

In Section 11.1 we shall show that solid sets play a major role in computing a cheapest set of γ new edges the addition of which to a given graph results in a k-edge-connected graph, provided that the cost function is node-induced.

978–0–19–920527–1 07-Andras-Frank-c07-drv Andras Frank (Typeset by SPi, Publisher Services, India) 239 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

7.3.2 Solid sets of directed graphs

Given a digraph $D = (V, A)$, we define a non-empty subset Z of V as **in-solid** (respectively, **out-solid**) if $\varrho(X) > \varrho(Z)$ (respectively, $\delta(X) > \delta(Z)$) for every non-empty proper subset X of Z. An in- or out-solid set is said to be **solid**. Singletons are always in- and out-solid, and a minimal k-in-deficient set is in-solid (for any k). Let $H_D = (V, \mathcal{E}_D)$ denote the hypergraph of all solid sets.

Proposition 7.3.6 *If X is in-solid and Y is out-solid, then at least one of the subsets $A := X - Y$, $B := Y - X$ and $C := X \cap Y$ is empty.*

Proof. Let α, β, γ, and γ' denote, respectively, the number of edges from C to A, from B to C, from $V - (X \cup Y)$ to C, and from C to $V - (X \cup Y)$. If, indirectly, none of A, B and C is empty, then $\varrho(A) > \varrho(X)$ and $\delta(B) > \delta(Y)$. Therefore $\alpha > \beta + \gamma$ and $\beta > \alpha + \gamma'$ from which the impossible $0 > \gamma + \gamma'$ would follow. •

The following result is by Bárász, Becker, and Frank [12].

Theorem 7.3.7 *The hypergraph $H_D = (V, \mathcal{E}_D)$ of solid sets of a digraph $D = (V, A)$ is a subtree hypergraph, that is, there is a spanning tree on the ground-set V such that each solid set of D induces a subtree.*

Proof. By Theorem 1.4.12, H_D is a subtree hypergraph if and only if it has the Helly property and its line graph is chordal. Suppose indirectly that H_D is not chordal, that is, it induces a chordless circuit of length at least 4. Then there are solid sets $X_1, \ldots, X_h$ $(h \geq 4)$ for which $X_i \cap X_j \neq \emptyset$ holds if and only if i and j are consecutive integers where we use the notational convention $X_{h+1} = X_1$. Proposition 7.3.6 implies that either all X_i's are in-solid or all X_i's are out-solid. By symmetry, we can assume that the first case occurs. It follows that the h intersections $X_i \cap X_{i+1}$ are pairwise disjoint and hence

$$\sum_{i=1}^{h} \varrho(X_i \cap X_{i+1}) \leq \sum_{i=1}^{h} \varrho(X_i). \tag{7.11}$$

Since X_i is in-solid, $\varrho(X_i) < \varrho(X_i \cap X_{i+1})$ for $i = 1, \ldots, h$ and hence $\sum_i \varrho(X_i) < \sum_i \varrho(X_i \cap X_{i+1})$, contradicting (7.11).

We claim that H_D admits the Helly property. If it does not, then there is a smallest number $h \geq 3$ along with h solid sets $X_1, \ldots, X_h$ such that any two of these sets intersect each other while the intersection $M = X_1 \cap \cdots \cap X_h$ is empty. Suppose first that $X = X_i$ is in-solid and $Y = X_j$ is out-solid $(i \neq j)$. By Proposition 7.3.6, one of the sets $X - Y$, $Y - X$, and $X \cap Y$ is empty. By the assumptions about the h sets, we cannot have $X \cap Y = \emptyset$. We cannot have $X - Y = \emptyset$ either since in this case $X \subseteq Y$ and this would contradict the minimality of h since Y could be left out from the h sets. The last case of $Y - X = \emptyset$ can be exluded analogously.

Therefore either all the sets $X_1, \ldots, X_h$ are in-solid or they are all out-solid. By symmetry, we can assume that each X_i is in-solid. Let $Y_i = X_1 \cap X_2 \cap \cdots \cap X_{i-1} \cap X_{i+1} \cap \cdots \cap X_h$ $(i = 1, \ldots, h)$. By the minimal choice of h, $Y_i \neq \emptyset$, while $M = \emptyset$ implies that $Y_i \cap Y_j = \emptyset$ $(1 \leq i < j \leq h)$. If an edge enters one of the sets Y_i, then it enters at least one of the sets X_j. Therefore $\sum_i \varrho(Y_i) \leq \sum_i \varrho(X_i)$. On the other hand $\varrho(Y_i) > \varrho(X_{i+1})$ for each

i since X_{i+1} is in-solid and $Y_i \subset X_{i+1}$. Hence $\sum_i \varrho(Y_i) > \sum_i \varrho(X_{i+1}) = \sum_i \varrho(X_i)$, a contradiction. •

An application to source location

Suppose we are given a digraph $D = (V, A)$ and two non-negative integers k and ℓ. The Flow-constrained Directed Source Location (FDSL) problem consists of finding a subset S of nodes of smallest size so that by shrinking S into a node s we obtain a rooted (k, ℓ)-edge-connected digraph with respect to root-node s. A non-empty proper subset X of nodes is **k-in-deficient** or simply **in-deficient** if $\varrho(X) < k$ and **ℓ-out-deficient** or simply **out-deficient** if $\delta(X) < \ell$. An in- or out-deficient set is said to be **deficient**. A deficient set Z is **minimal** if no proper subset of Z is deficient. The hypergraph of minimal deficient sets will be denoted by $H_{k\ell}$. Therefore, the FDSL problem is tantamount to finding a smallest transversal of deficient sets. Ito et al. [217] proved the following.

Theorem 7.3.8 *The hypergraph $H_{k\ell}$ of minimal deficient sets forms a subtree hypergraph and hence*

$$\tau(H_{k\ell}) = \nu(H_{k\ell}), \tag{7.12}$$

where $\tau(H_{k\ell})$ denotes the minimum cardinality of a (k, ℓ)-source, while $= \nu(H_{k\ell})$ is the maximum number of pairwise disjoint deficient sets.

Proof. Observe that every minimal deficient set is solid. By Theorem 7.3.7, $H_{k\ell}$ is a normal hypergraph. •

Based on additional properties of solid sets, a polynomial time algorithm has been developed in [12] for computing the smallest transversal and the largest matching of deficient sets. The algorithm is fully described in the book of Nagamochi and Ibaraki [300].

Problem 7.3.1 $(*)$ *If the intersection of two in-solid (out-solid) sets X and Y is non-empty, then $X \cup Y$ is in-solid (out-solid).*

7.4 Sparse certificates for directed graphs and hypergraphs

The connectivity of an undirected graph G on n nodes can be certified by producing a spanning tree of G while a small certificate for root-connectivity of a digraph is a spanning arborescence. It is a natural demand to develop a similar type of results which say that a graph with a specified connectivity property always includes a relatively small spanning graph, the sparse certificate, admitting the same connectivity property. For example, the ear-decomposition theorem (Theorem 2.2.1) immediately implies that a strongly connected digraph on n nodes includes a strong spanning subgraph of at most $2(n-1)$ edges or, putting it another way, a strong digraph that is minimal with respect to edge-deletion has at most $2(n-1)$ edges.

Exercise 7.4.1 *Construct a minimally strong digraph on n nodes that has exactly* $2(n-1)$ *edges.*

In this section we present sparse certificates for several connectivity properties.

Lemma 7.4.1 *Suppose that a digraph $D = (V, A)$ is $(*)$ k-edge-connected from a root-node r_0 to a terminal set $T \subseteq V - r_0$. If $\varrho(t) > k$ for some $t \in T$, then a suitable edge entering t can be left out without destroying $(*)$. In particular, in a minimally rooted k-edge-connected digraph, $\varrho(r_0) = 0$ and $\varrho(v) = k$ hold for every $v \in V - r_0$.*

Proof. Consider k edge-disjoint r_0t-paths and an edge $e = ut$ entering t which is not used by these paths. We claim that e satisfies the requirement of the theorem. Let $D' := D - e$. For every node $v \in T$, the k selected r_0t-paths ensure that $\varrho_{D'}(X) \geq k$ for every $v\bar{r}_0$-set X, that is, D' satisfies $(*)$. •

Corollary 7.4.2 *Let $D = (V, A)$ be an r_0-rooted k-edge-connected digraph which is minimal with respect to edge-deletion. Then D has exactly $k(n-1)$ edges.* •

This result can be used to derive the following extension.

Theorem 7.4.3 *A minimally (k, ℓ)-edge-connected digraph D has at most $(k + \ell)(n - 1)$ edges and this bound is sharp.*

Proof. Let r_0 be a root-node of D such that the in-degree and the out-degree of every non-empty subset $X \subseteq V - r_0$ is at least k and ℓ, respectively. By Corollary 7.4.2, D includes a minimally rooted k-edge-connected digraph D_1 which has $k(n-1)$ edges. Analogously, D includes a digraph D_2 having $\ell(n-1)$ edges such that reversing all edges of D_2 results in a rooted ℓ-edge-connected digraph. Since the union of D_1 and D_2 is (k, ℓ)-edge-connected and D is minimal, it follows that D has at most $(k + \ell)(n - 1)$ edges.

To see that this bound is sharp, consider a digraph in which there are k parallel r_0v-edges and ℓ parallel vr_0-edges for every $v \in V - r_0$. •

Corollary 7.4.4 (Dalmazzo [63]) *A minimally k-edge-connected digraph D has at most $2k(n-1)$ edges and this bound is sharp.* •

The idea in the proof of Lemma 7.4.1 can be used in more complicated situations. Let $D = (V, A)$ be a digraph endowed with an integer-valued capacity function $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ on the node-set of D. In Section 2.5 we called a set of st-paths g-bounded if they are edge-disjoint and every node $v \in V - \{s, t\}$ is used by at most $g(v)$ of these paths. A hybrid version of Menger's theorem (Theorem 2.5.12) stated that there are h g-bounded st-paths if and only if

$$\varrho_D(X) + w_g(X) \geq h \text{ for every } t\bar{s}\text{-bi-set } X = (X_O, X_I) \tag{7.13}$$

where $w_g(X) := \widetilde{g}(X_O - X_I)$. Let $\lambda_g(u, v; D)$ denote the maximum number of g-bounded uv-paths in D.

Theorem 7.4.5 *Let r_0 be a root-node of a digraph $D = (V, A)$ and $t \in V - r_0$ a node for which $\varrho_D(t) > \lambda_g(r_0, t; D)$. Let $\mathcal{P}$ be a set of $\lambda_g(r_0, t; D)$ g-bounded r_0t-paths and $e = ut$ an edge not used by these paths. Then*

$$\lambda_g(r_0, v; D - e) = \lambda_g(r_0, v; D) \text{ holds for every node } v \in V. \tag{7.14}$$

Proof. In the proof we use the abbreviations $\lambda(v) := \lambda_g(r_0, v; D)$. By the hybrid version of Menger's theorem, there is a $t\overline{r}_0$-bi-set $T = (T_O, T_I)$ for which $\varrho_D(T) + w_g(T) = \lambda(t)$. Among the $\lambda(t)$ members of $\mathcal{P}$, exactly $w_g(T)$ paths use nodes from $T_O - T_I$ and hence the other $\varrho_D(t)$ paths in $\mathcal{P}$ use the $\varrho_D(t)$ edges of D entering T. Therefore e does not enter T, and similarly it cannot enter any other $t\overline{r}_0$-bi-set T' for which $\varrho_D(T') + w_g(T') = \lambda(t)$.

Suppose indirectly that there is a node z for which $\lambda_g(r_0, z; D - e) < \lambda(z)$. By the hybrid Menger again, there is a $z\overline{r}_0$-bi-set Z for which $\varrho_D(Z) + w_g(Z) = \lambda(z)$ and e enters Z. Then e enters $T \sqcap Z$ and hence $\varrho_D(T \sqcap Z) + w_g(T \sqcap Z) > \lambda(t)$. Since $T \sqcup Z$ is a $z\overline{r}_0$-bi-set, we have $\varrho_D(T \sqcup Z) + w_g(T \sqcup Z) \geq \lambda(z)$. By combining these inequaliltites with the submodularity of ϱ_D and the modularity of w_g, we obtain

$$\lambda(t) + \lambda(z) = [\varrho_D(T) + w_g(T)] + [\varrho_D(Z) + w_g(Z)] \geq [\varrho_D(T \sqcap Z) + w_g(T \sqcap Z)] + [\varrho_D(T \sqcup Z) + w_g(T \sqcup Z)] > \lambda(t) + \lambda(z),$$

a contradiction. •

Let D, r_0, and g be the same as before. We say that D is a **rooted λ_g-minimal** if for every edge e of D there is a node $v \in V - r_0$ for which $\lambda_g(r_0, v; D - e) < \lambda_g(r_0, v; D)$. In the special case when $g \equiv \infty$, we speak of a **rooted λ-minimal** digraph while if $g \equiv 1$, D is **rooted κ-minimal**. The next result is an immediate consequence of Theorem 7.4.5.

Theorem 7.4.6 *If D is a rooted λ_g-minimal digraph, then $\varrho_D(v) = \lambda_g(r_0, v; D)$ for every node $v \in V - r_0$. In particular, if D is rooted λ-minimal, then $\varrho_D(v) = \lambda(r_0, v; D)$, while if D is rooted κ-minimal, then $\varrho_D(v) = \kappa_D(r_0, v)$ for every node $v \in V - r_0$.* •

The special case of Theorem 7.4.6 concerning rooted λ-minimal digraphs was observed by Lovász [266]. Theorem 7.4.6 implies that if D is a λ_g-minimal digraph, then for any edge $e = ut$, $\lambda_g(r_0, t; D - e) < \lambda_g(r_0, t; D)$.

Theorem 7.4.7 *Let $D = (V, A)$ be a digraph with a root-node r_0 and a terminal set $T \subseteq V - r_0$. Let $g : (V - r_0) \to \{1, \ldots, k\}$ be a capacity function such that there are k g-bounded r_0t-paths for every $t \in T$ but the removal of any edge destroys this property. Then $\varrho_D(t) = k$ for every $t \in T$.*

Proof. Suppose indirectly that $\varrho_D(t) > k$ for some $t \in T$. We can assume that D is rooted λ_g-minimal and hence $\lambda_g(t) = \varrho_D(t)$ follows from Theorem 7.4.6. Let $e = ut \in A$ be an edge. By the hypothesis of the theorem, there is a node $z \in T$ for which $\lambda_g(r_0, z; D - e) \leq k - 1$. By Theorem 2.5.12, there is a bi-set X for which $z \in X_I$ and $\varrho_{D-e}(X) + w_g(X) \leq k - 1$. Hence $\varrho_D(X) + w_g(X) \leq k$ and e enters X, implying $t \in X_I$. But then $k < \varrho_D(t) = \lambda_g(t) \leq \varrho_D(X) + w_g(X) \leq k$, a contradiction. •

Corollary 7.4.8 *Let $D = (V, A)$ be an r_0-rooted k-node-connected digraph which is minimal with respect to edge-deletion. Then $\varrho_D(r_0) = 0$ and $\varrho_D(v) = k$ for every $v \in V - r_0$. In particular, D has exactly $k(n - 1)$ edges.* •

This result will be particularly useful in developing an algorithm for finding a cheapest rooted k-node-connected spanning subgraph of a digraph in Part III.

7.4.1 Trimming a dypergraph

Theorem 7.4.9 *Let $D = (V, \mathcal{A})$ be a rooted k-edge-connected dypergraph with respect to a root-node r_0 and suppose that every dyperedge has at least two elements. Then D can be trimmed to a digraph which is rooted k-edge-connected.*

Proof. We use induction on the sum $\sum[|Z| - 2 : Z \in \mathcal{A}]$. If this sum is zero, then every dyperedge is of two elements, that is, D itself is a digraph.

Suppose now that (Z, z) is a dyperedge with $|Z| \geq 3$. Let u and v be two elements of $Z - z$. Call a set $X \subset V - r_0$ **tight** if $\varrho_D(X) = k$. If replacing Z by $Z - u$ does not destroy rooted k-edge-connectivity, then by induction we are done again. So assume that it does and hence that there is a tight set X (with respect to D) for which $u \notin X$ and $Z - u \subseteq X$. Similarly for v, we can assume that there is a tight set Y for which $v \notin Y$ and $Z - v \subseteq Y$. But then the hyperedge Z shows that $d_D(X, Y) \geq 1$, and hence Proposition 1.2.9 and rooted k-edge-connectivity imply $k + k = \varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) + d_D(X, Y) \geq k + k + 1$, a contradiction. •

Corollary 7.4.10 *Let $D = (V, \mathcal{A})$ be a rooted k-edge-connected dypergraph which is minimal in the sense that leaving out any dyperedge destroys this property. Then D has $k(|V| - 1)$ dyperedges.*

Proof. By Theorem 7.4.9, D can be trimmed to a rooted k-edge-connected digraph D'. By the minimality of D, D' is minimally rooted k-edge-connected, and hence by Corollary 7.4.2 we are done.

There are several other results in the literature on sparse certificates of connectivity properties. See, for example, Mader's paper [282] on minimally k-node-connected digraphs, or the paper of Berg and Jordán [23] on minimally k-edge-connected simple digraphs.

7.5 Sparse certificates for undirected graphs

Before turning to sparse certificates for connectivity properties of undirected graphs, we introduce a simple tool.

7.5.1 Forest decompositions

By a **maximal forest decomposition** of a loopless undirected graph $G = (V, E)$, we mean a partition of E into non-empty forests $F_1, F_2, \ldots$ obtained by the following procedure. First choose a maximal forest F_1 of G (which is, of course, a spanning tree if G is connected). Then choose a maximal forest F_2 of $G - F_1$, and continue in this way until all the edges of G are assigned to a forest. In spite of its unsophisticated character, a maximal forest decomposition will have startlingly good connectivity properties. For example, we shall prove soon that if G is k-edge-connected, then so is the union of the first k forests.

There can be several ways to construct a maximal forest of a graph. For example, a search procedure such as depth-first search (DFS) or breadth-first search (BFS) provide one. A third type of search algorithms is **scan-first search** (SFS), introduced by Cheriyan, Kao, and Thurimella [47], will be particularly useful.

978–0–19–920527–1 07-Andras-Frank-c07-drv Andras Frank (Typeset by SPi, Publisher Services, India) 244 of 640 December 11, 2010 15:17
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

An SFS for a digraph consists of stages, where each stage determines a maximal arborescence. The first stage of SFS starts from an arbitrary root s, determines the set S of nodes of the digraph reachable from s, and computes an arborescence of root s spanning S. Then, as long as $V - S$ is non-empty, it considers the digraph induced by $V - S$ and iterates the first stage. At a general step, SFS selects any node u which has already been reached from s (that is, visited) but not yet scanned and scans u by adding every such edge uv of the digraph to the arborescence for which the head v has not yet been reached. It follows from this rule that the node u chosen for being scanned is necessarily an end-node of the current arborescence (or, at the very beginning, its root). Note that SFS can be interpreted as a relaxation of BFS in the sense that BFS selects the node u for scanning to be the one of visited first but not yet scanned.

An SFS, when applied to an undirected graph G, produces a maximal forest F_1 of G. A next SFS applied to $G - F_1$ produces a forest F_2, and continuing in this way we obtain a **scan-first forest decomposition** of G.

There is another, though strongly related, way to obtain a maximal forest. Let $v_1, \dots, v_n$ be an arbitrary ordering of the nodes of G. For any edge $e = v_i v_j$ with $i < j$, v_i is the **first node** of e and v_j is its **last node**. The ordering is said to be **contiguous** if for every component C of G,

(A) $v_j \in C$ holds whenever $i < j < k$ are indices for which $v_i, v_k \in C$, and
(B) for every node u of C distinct from the first one, there is a node of C preceding u in the ordering.

A simple but useful observation is that an MA ordering (introduced in Section 6.3.1) is always contiguous. With every ordering we can associate a forest F as follows: for every node u for which there is an edge going to a node preceding u in the ordering, choose a uv-edge e whose first end-node v is the earliest node and put e into F. This subgraph F cannot include any circuits since then the latest node v of this circuit would have two neighbours preceding v, contradicting the construction of F. We also claim that F is a maximal forest. It suffices to prove this for connected graphs and in this case, apart from the first node, every other node v has a neighbour preceding v and hence F is a spanning tree. By repeating this procedure, we obtain the **forest decomposition belonging to the ordering**. This can be given explicitly, as follows. For each pair of adjacent nodes u and v, fix an ordering of the parallel uv-edges. This ordering determines an ordering $e_1^j, \dots, e_{h(j)}^j$ of the edges connecting a given node v_j with nodes preceding v_j and then $F_i = \left\{e_i^j : h(j) \geq i,\ j = 2, \dots, n\right\}$.

During a scan-first search, we considered the nodes one by one for scanning. Therefore a run of SFS determines an ordering of the nodes, which will be called a **scan-first ordering**.

Theorem 7.5.1 *An ordering $v_1, \dots, v_n$ of nodes of a graph is contiguous if and only if it is a scan-first ordering.*

Proof. Consider first a scan-first ordering. Let $K_1, \dots, K_t$ denote the components of G. SFS proceeds to a new component only when it has found a spanning tree of the current component. That is, the components of G occur contiguously in the scan-first order. Let $v_1, \dots, v_\ell$ denote the scan-first order of the nodes of one component. Apart from v_1, an arbitrary node v_k gets scanned only when v_k is an end-node of a currently existing tree F. Let $e = v_j v_k$ be the unique edge in F incident to v_k. Then v_j precedes v_k in the ordering. We claim that v_j is the first node in the scan-first ordering which is adjacent with v_k. Indeed, if there were a node v_i preceding v_j which is adjacent with v_k, then, by the rule of SFS, edge $v_i v_k$ would have been added to the tree, contradicting the choice of e.

Conversely, suppose that $v_1, \dots, v_n$ is contiguous. We can assume that G is connected. Every node, and in particular v_1, can be chosen as the first node of an SFS. Let j be the largest index for which there is a run of SFS in which the scan-first ordering of the first j nodes is $\{v_1, \dots, v_j\}$. We are done if $j = n$, so suppose that $j < n$. Since the ordering is contiguous, there is an edge from v_{j+1} going back. Let v_i be a smallest index neighbour of v_{j+1}. When v_i is being scanned, then v_{j+1} has been added to the tree, that is, there is a run of SFS which takes v_{j+1} for scanning right after v_j has been scanned, a contradiction to the maximal choice of j. •

The proof of the next proposition follows directly from the definition and is left to the reader.

Proposition 7.5.2 *Let $v_1, \dots, v_n$ be an MA ordering of a graph G. If F is the forest associated with the ordering, then the same ordering $v_1, \dots, v_n$ is an MA ordering of the graph $G - F$. For arbitrary indices i and j $(1 \leq i < j \leq n)$ $v_1, \dots, v_i, v_j$ is an MA ordering of the graph induced by these nodes.* •

It follows that the forest decomposition belonging to an MA ordering is a scan-first forest decomposition of G.

7.5.2 k-edge-connectivity

Theorem 7.5.3 *Every k-edge-connected graph $G = (V, E)$ includes a spanning k-edge-connected subgraph having at most $k(n-1)$ edges.*

Proof. Let G be minimally k-edge-connected. Replace each edge by a pair of opposite directed edges. The resulting digraph D is k-edge-connected. Since every edge of G belongs to a k-element cut, every arc of D belongs to a k-element in-cut. By Corollary 7.4.4, D has at most $2k(n-1)$ arcs, so G has at most $k(n-1)$ edges. •

We describe a simple and elegant method of Nishizeki and Poljak [310] to find a sparse certificate for a k-edge-connected graph.

Proposition 7.5.4 *Let $F_1, \dots, F_k$ be the first k forests of a maximal forest decomposition of G. Then $E_k := F_1 \cup \dots \cup F_k$ contains at least $\min\{k, |B|\}$ elements of cut B.*

Proof. There is nothing to prove if E_k includes the entire cut B. So assume that E_k does not contain an element $e = xy$ of B. What we need to show is that $|B| \geq k+1$. We claim that the nodes x and y belong to the same component of F_i for each i $(1 \leq i \leq k)$. Indeed, if this is not true and i is the smallest subscript for which x and y belong to different components of F_i, then e could have been added to F_i, contradicting the maximality of F_i in the graph $G - \{F_1 \cup \dots \cup F_{i-1}\}$. Therefore x and y belong to the same component of F_i, showing that there is an xy-path in F_i. Consequently each F_i contains an element of B. Because e is also in B, we obtain $|B| \geq k + 1$. •

Proposition 7.5.4 implies at once the following.

Theorem 7.5.5 (Nishizeki and Poljak) *If $G = (V, E)$ is k-edge-connected and $F_1, \ldots, F_k$ are the first k forests of a maximal forest decomposition of G, then the subgraph $G_k = (V, F_1 \cup \cdots \cup F_k)$ is also k-edge-connected.* •

Since the union of k forests has at most $n(k-1)$ elements, Theorem 7.5.3 follows from Theorem 7.5.5 as well.

Corollary 7.5.6 *If $G = (V, E)$ is k-edge-connected and $F_1, \ldots, F_k$ are the first k forests of a forest decomposition belonging to an MA ordering, then the subgraph $G_k = (V, F_1 \cup \cdots \cup F_k)$ is also k-edge-connected.* •

It follows that a sparse certificate for k-edge-connectivity can be found in linear time.

Application to speeding up connectivity augmentation algorithms

There is an interesting algorithmic application of Proposition 7.5.4. Suppose that we want to make a graph G k-edge-connected by adding a minimum number of new edges, a problem to be explored in Section 11.1. Proposition 7.5.4 implies that the set of cuts of graph $G_k = (V, E_k)$ with cardinality smaller than k is the same as the set of cuts of G with cardinality smaller than k. Therefore it suffices to solve the augmentation problem for the subgraph G_k, which has only a few edges (namely, at most $k(n-1)$).

7.5.3 k-node-connectivity

Recall that we defined a simple graph $G = (V, E)$ as k-node-connected if $d_G(X) + w(X) \geq k$ for every non-trivial bi-set $X = (X_O, X_I)$ and proved that this is equivalent to requiring $\kappa_G(u, v) \geq k$ for every pair of nodes u, v where $\kappa_G(u, v; G)$ denotes the maximum number of openly disjoint uv-paths in G (Theorem 2.5.8). Our next goal is to find a sparse certificate for k-connected graphs. The next result (in a slightly more general form) was proved by Even, Itkis, and Rajsbaum [93]. The present proof appeared in a paper of Fülöp [171].

Theorem 7.5.7 *Let F_k be the k'th forest of a scan-first forest decomposition of G* (in particular, the k'th forest belonging to an MA ordering). *Let x and y be two nodes belonging to the same component of $G_k = (V, F_k)$. Then $\kappa_{G_k}(x, y) \geq k$.*

Proof. We use induction on k. The theorem is trivial for $k = 1$. Suppose that $k \geq 2$ and consider a bi-set $X = (X_O, X_I)$ for which $x \in X_I$, $y \in V - X_O$. By the construction of F_i's (namely, by the fact that they are maximal), x and y belong to the same components of F_i for each i, $1 \leq i \leq k$. Let $G' := G - F_1$ and $J := F_2 \cup \cdots \cup F_k$.

If F_1 contains an edge covering X, then by applying the inductive hypothesis (with $k-1$ in place of k) to G' and to forests $F_2, \ldots, F_k$, we obtain that $d_{E_k}(X) + w(X) \geq 1 + d_J(X) + w(X) \geq 1 + (k-1) = k$, as required.

Suppose now that F_1 contains no edge covering X. Let C_1 denote the component of F_1 containing both x and y. By replacing X with the bi-set $\overline{X} = (V - X_O, V - X_I)$ if necessary, we can assume that the root of C_1 is in X_O. Let z_1 be the node of $(X_O - X_I) \cap V(C_1)$

which has been scanned first while constructing F_1. Such a z_1 exists since no edge of F_1 covers X. Therefore the rule of SFS and the choice of z_1 imply that J contains no edge between z_1 and X_I. This implies that $d_J(X) = d_J(X')$ where $X' := (X_O - z_1, X_I)$.

By applying the inductive hypothesis to graph G' and to forests $F_2, \ldots, F_k$, we obtain that $w(X) + d_{E_k}(X) = 1 + w(X') + d_J(X) = 1 + w(X') + d_J(X') \geq 1 + (k-1) = k$, as required. •

Exercise 7.5.1 *Show by an example that the following statement is false.* Let x and y be two nodes belonging to the same components of F_k. Then the k paths between x and y determined by forests $F_1, \ldots, F_k$ are openly disjoint.

Corollary 7.5.8 *If E_k does not contain every edge of G covering a non-trivial bi-set X, then $w(X) + d_{E_k}(X) \geq k$.*

Proof. Let $e = xy \in E - E_k$ be an edge not covering X. By the maximality of F_k, x and y belong to the same component of F_k and hence Theorem 7.5.7 implies the result. •

With the help of Corollary 7.5.8, we obtain the following sparse certificate for k-connectivity.

Theorem 7.5.9 (Mader, [277]) *Every k-connected graph $G = (V, E)$ on n nodes contains a k-connected spanning subgraph $G_k = (V, E_k)$ for which $|E_k| \leq k[n - (k+1)/2]$.*

Proof. We can assume that G is simple. Consider the maximal SFS forests $F_1, F_2, \ldots, F_k$ defined before Theorem 7.5.7 and let $E_k := F_1 \cup \cdots \cup F_k$. By Corollary 7.5.8, $G_k = (V, E_k)$ is k-connected. Let s_i denote the first root-node of forest F_i which is incident to an edge of F_i. By the rule of SFS, if $j > i$, then F_j contains no edge incident to s_i and therefore F_j can have at most $n - j$ edges, that is, $|E_k| \leq (n-1) + (n-2) + \cdots + (n-k) = k[n - (k+1)/2]$. •

The counterpart of Theorem 6.3.1 is valid, as well.

Theorem 7.5.10 *Every simple graph with at least two nodes has at least two nodes x, y for which $\kappa(x, y) = d(x)$, namely, the last two nodes $y = v_{n-1}$ and $x = v_n$ of an MA ordering will suffice.*

Proof. Let $k := d(v_n)$ and consider the forests $F_1, \ldots, F_k$ belonging to the ordering. In graph $G_k = (V, E_k)$ (where $E_k := F_1 \cup \cdots \cup F_k$) by Corollary 7.5.8 the last two nodes belong to the same component of each forest. •

Corollary 7.5.11 *Suppose graph G has at least two nodes. If G is minimally k-node-connected or minimally k-edge-connected* (with respect to edge deletion)*, then G has a node of degree k.*

Proof. The last node of an MA ordering will siffice. Indeed, by assumption, the degree of every node is at least k. If, indirectly, the degree of v_n were greater than k, then the last edge e incident with v_n (where $e = v_i v_n \in E$ and i is maximum) would not belong to F_k. On the other hand, we have seen earlier that graph $(V, F_1 \cup \cdots \cup F_k)$ is k-connected (resp.,

k-edge-connected) if G is, and hence $G - e$ is also k-connected (resp., k-edge-connected), contradicting the assumption made about the minimality of G. •

Let $g : V \to \mathbf{Z}_+$ be an integer-valued function on the node-set of a graph $G = (V, E)$. Let $\lambda_g(x, y; G)$ denote the maximum number of g-bounded st-paths (see p. 85).

Theorem 7.5.12 *Let $G_k = (V, F_1 \cup \cdots \cup F_k)$ be a subgraph of G defined by the first k forests in a forest decomposition of G belonging to an MA ordering. Suppose that there are at most* $\min\{g(u), g(v)\}$ *parallel uv-edges for each pair u, v of nodes. If $\lambda_g(x, y; G) \geq k$ for every pair x, y of nodes, then $\lambda_g(x, y; G_k) \geq k$ for every pair x, y of nodes.* •

This result was obtained by Nagamochi [295] by applying a method of Berg and Jordán [24] who proved the special case when g is constant.

Problems

7.5.2 *Every k-connected bipartite graph $G = (V, E)$ contains a k-connected subgraph $G' = (V, E')$ for which $|E'| \leq k(n - k)$. Show that the bound is sharp. Does the same estimate hold for triangle-free graphs?*

7.5.3 *Show that for every graph G the subgraph G_k defined above satisfies the inequality* $\kappa_{G_k}(x, y) \geq \min\{k, \kappa \in G(x, y)\}$ *for every pair x, y of nodes.*

7.5.4 *Prove that in Theorem* 7.5.10 *x and y can be chosen to be adjacent.*

A constructive characterization of chordal graphs

An undirected graph $G = (V, E)$ is defined as chordal if every circuit of length at least 4 admits a chord (an edge connecting two non-consecutive nodes of the circuit). This is a co-**NP** property since a circuit of the graph of length at least 4 with no chord is an easily verifiable certificate. What kind of witness can we imagine for a graph being chordal? An ordering of the nodes of a graph is said to be **simplicial** if the neighbours of each node v that follow v in the ordering form a clique. If there is a simplicial ordering of the nodes, then G is surely chordal. Indeed, let C be any circuit of length at least 4 and let u be its earliest node on the simplicial ordering. Then the two neighbours v and z of u along C follow u and therefore vz is an edge of G forming a chord of C. The main content of the next result is that a chordal graph always admits a simplicial ordering.

Theorem 7.5.13 (Dirac [72]) *G is chordal if and only if the nodes of G have a simplicial ordering. Equivalently, G is chordal if and only if it can be built up from a node by adding new nodes consecutively in such a way that each new node is adjacent to the elements of an existing clique.* •

The existence of a simplicial ordering follows from the following.

Theorem 7.5.14 (Tarjan and Yannakakis [364]) *An MA ordering of a chordal graph is a reverse simplicial ordering.*

Proof. Since an induced subgraph of a chordal graph is also chordal, it suffices to show that the last node of an MA ordering $v_1, \ldots, v_n$ is simplicial. We can assume that G is simple.

Let v_j be a neighbour of v_n and let α denote the number of those neighbours of v_n which (strictly) precede v_j in the given ordering. Let G_j denote the subgraph of G induced by nodes $v_1, v_2, \ldots, v_j, v_n$. We noted in Proposition 7.5.2 that $v_1, v_2, \ldots, v_j, v_n$ is an MA ordering of G_j. Therefore by Theorem 7.5.10, we have $\kappa_{G_j}(v_n, v_j) = d(v_n, V_j) = \alpha + 1$ where $V_j = \{v_1, \ldots, v_j\}$. By Menger's theorem there are $\alpha + 1$ openly disjoint paths in graph G_j between v_n and v_j. We can choose these paths to be chordless. Since G_j is chordal, by the existence of edge $v_n v_j$, the length of all of these paths is at most 2 which is equivalent to stating that v_j is adjacent to every neighbour v_h of v_n for which $h < j$. Therefore we have obtained that every two neighbours of v_n are adjacent, and hence v_n is indeed simplicial. •

Since an MA ordering is computable in linear time, so is a simplicial ordering of a chordal graph. This, in turn, can be used to compute a maximum-weight clique and a stable set of a chordal graph.

Problem 7.5.5 ($*$) *Let F be a tree and let $F_1, F_2, \ldots, F_n$ be a family of subtrees of F. Let G be a simple graph with node-set $\{v_1, \ldots, v_n\}$ in which v_i and v_j are adjacent if and only if the subtrees F_i and F_j have a node in common. Prove that G is chordal and that every chordal graph can be obtained in this way.*

8

The splitting-off operation and constructive characterizations

In order to prove graph-theoretical results inductively, it is often helpful to use simple reductions such as deleting an edge or a node. Sometimes, however, the change in the graph caused by these operations is too drastic to maintain the investigated property for inductive purposes. In this chapter, we introduce a more subtle operation, splitting off a pair of edges, which preserves connectivity properties more truly and hence will be stunningly efficient in several applications to graph orientation and connectivity augmentation problems. Results concerning splitting-off will also help in obtaining constructive characterizations for properties like k-edge-connectivity of directed and undirected graphs or rooted k-edge-connectivity of digraphs. These are much deeper than their special cases seen in Part I (see, for example, the ear-decomposition theorems). In addition, the splitting-off operation will serve as a basis for far-reaching extensions, to be explored in Part III.

Let $H = (U, F)$ be an undirected graph in which $e = zu$ and $f = zv$ are incident edges. We say that the graph $G^{ef} = (U, F - \{e, f\} + h)$ arises from H by **splitting off** e and f where h is a new edge between u and v (perhaps parallel to existing uv-edges). Analogously, in a directed graph $H = (U, F)$ if $e = uz$ and $f = zv$ are directed edges while h is a new edge from u to v, then $H^{ef} = (U, F - \{e, f\} + h)$ is the digraph arising from H by splitting off e and f. It is straightforward that in both the directed and the undirected cases $\lambda_H(x, y) \geq \lambda_{H^{ef}}(x, y)$ holds for every pair $\{x, y\}$ of nodes (where $\lambda(x, y)$ is equal to the minimum (in-)degree of a $y\bar{x}$-set). Therefore the splitting-off operation certainly does not increase edge-connectivity. The proper goal therefore is to explore situations where edge-connectivity does not drop. Because the degree of node z reduces anyway, our only hope can be in preserving edge-connectivity between pairs of nodes distinct from z. Note that the operation of directed splitting-off has already been used in a proof of the directed edge-Menger theorem (see the alternative proof on Page 81).

8.1 Undirected splitting

In this section, we investigate how global and local edge-connection of undirected graphs can be preserved by the splitting-off operation.

8.1.1 Preserving global edge-connectivity of graphs

A fundamental result is due to Lovász [267].

Theorem 8.1.1 (Lovász' undirected splitting lemma) *Let $G = (V + z, E)$ be an undirected graph with a designated node z for which the degree $d(z)$ is even and positive. Let $k \geq 2$ be an integer and suppose that*

$$\lambda_G(x, y) \geq k \text{ for every pair } x, y \in V \text{ of nodes.} \tag{8.1}$$

Then there is an edge $f = zv$ for every edge $e = zt$ such that $\lambda_{G^{ef}}(x, y) \geq k$ holds for every pair $x, y \in V$ of nodes.

Proof. By Menger's theorem, Condition (8.1) is equivalent to

$$d(X) \geq k \text{ for every subset } \emptyset \neq X \subset V, \tag{8.2}$$

where $d(X)$ denotes the degree of X. If the degree of a set drops at a splitting, then it drops by 2. Call a non-empty subset $X \subset V$ **dangerous** if $d(X) \leq k + 1$.

Let S denote the set of nodes adjacent to z. An edge $f = zv$ cannot be split off with the selected edge $e = zt$ if and only there is a dangerous set containing both v and t. Suppose, indirectly, that S can be covered by a family of dangerous sets containing t and let $\mathcal{F}$ denote a family of maximal dangerous sets containing t for which $|\mathcal{F}|$ is as small as possible.

The set $\mathcal{F}$ must have at least two members for if there is a dangerous set X including the entire S, then $d(V - X) = d(X) - d(z) \leq (k + 1) - 2 = k - 1$, contradicting (8.2). We cannot have $\mathcal{F} = \{X, Y\}$ either since if we had, then

$$k + 1 + k + 1 \geq d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X, Y) \geq k + k + 2$$

from which it follows that there is only one edge between z and $X \cap Y$. Using the notation $\alpha := d(z, X - Y)$ and $\beta := d(z, Y - X)$, we get $d(z) = 1 + \alpha + \beta$. Since $d(z)$ is even by hypothesis, $\alpha \neq \beta$, and we can assume that $\alpha > \beta$. But then $d(V - X) = d(X + z) = d(X) - (\alpha + 1) + \beta \leq d(X) - 2 \leq (k + 1) - 2 \leq k - 1$, and hence $V - X$ violates (8.2).

Therefore $|\mathcal{F}| \geq 3$. Let A, B and C be three members of $\mathcal{F}$. By the triple inequality given in Proposition 1.2.4, we have $3(k + 1) \geq d(A) + d(B) + d(C) \geq d(A \cap B \cap C) + d(A - (B \cup C)) + d(B - (A \cup C)) + d(C - (A \cup B)) + 2d(A \cap B \cap C,\ V - (A \cup B \cup C)) \geq k + k + k + k + 2$, contradicting the assumption $k \geq 2$. • •

The star of 4 edges (which is the complete bipartite graph $K_{4,1}$) shows that the theorem does not hold for $k = 1$. Furthermore the complete graph K_4 shows that the assumption about the evenness of $d(z)$ cannot be left out either. By the repeated application of Theorem, 8.1.1 one immediately obtains the following result. For later applications, we formulate it for an integer K in place of k.

Theorem 8.1.2 (Lovász) *In an undirected graph $G = (V + z, E)$, let the degree $d(z)$ of a special node z be positive and even. Suppose that $K \geq 2$ and $\lambda_G(x, y) \geq K$ for every pair $x, y \in V$ of nodes. Then the edges incident to z can be paired in such a way that splitting off simultaneously the $d(z)/2$ pairs results in a K-edge-connected graph $G' = (V, E')$.* •

The simultaneous splittings of the disjoint $d(z)/2$ pairs of edges in the theorem will be called a **complete splitting** (at z). Therefore the theorem states that a K-edge-connected

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 252 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

graph always has a complete splitting at any of its nodes with even degree such that the resulting graph (having one less node) is K-edge-connected.

Algorithmic aspects

There is a natural way to decide algorithmically whether two given edges $e = zt$ and $f = zu$ can be split off without violating (8.1). To this end, split them off and, with at most n Max-flow Min-cut (MFMC) computations, check whether (8.1) holds. This approach gives rise to a strongly polynomial time algorithm for finding a complete splitting at z. Note, however, that there are significantly more efficient approaches based on MA orderings (Section 6.3). The works of Nagamochi and Ibaraki [296, 298, 299, 300] provide a detailed treatment. The work [184] of Goemans explores other algorithmic aspects of the splitting-off operation.

8.1.2 Constructive characterization of K-edge-connected graphs

Recall the simple constructive characterization of connected graphs given in Proposition 1.2.19. Theorem 2.1.1 stated that a graph is 2-edge-connected if and only if it can be built up from a node by sequentially adding edges and subdividing edges. With the help of Theorem 8.1.2, we will be able to extend these characterizations for general K-edge-connected graphs.

It turns out that the situation is different for even K and for odd K, and hence we study the two cases separately. In particular, the building procedure for $2k$-edge-connected graphs is simpler and has several far-reaching consequences. The following lemma will be useful in both cases. A K-edge-connected graph is said to be **minimal** if deleting any of its edges destroys K-edge-connectivity. The following lemma is a direct consequence of the existence of Gomory–Hu trees (Theorem 7.2.2), but we provide a direct proof.

Lemma 8.1.3 *Every minimally K-edge-connected graph with at least two nodes has a node of degree K $(K \geq 1)$.*

Proof. The deletion of an edge e destroys K-edge-connectivity if and only if there is a subset X of nodes with degree K for which $e \in \Delta(X)$. Let X be a smallest subset of nodes with $d(X) = K$. We are done if $|X| = 1$, so assume that this is not the case and let $s \in X$. Then $d(s) > K$, so there is an edge $e = st$ for which $t \in X$. There is an $s\bar{t}$-set Y for which $d(Y) = K$. By the minimality of X neither Y nor $V - Y$ can be a subset of X, and hence X and Y are crossing. But then we have $K + K \geq d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \geq K + K$ from which $d(X \cap Y) = K$ follows, contradicting the minimality of X. •

Problem 8.1.1 $(*)$ *Prove that in Lemma* 8.1.3 *there actually are at least two nodes of degree K.*

Suppose that $K = 2k$. The inverse operation of a complete splitting at a node z of degree $2j$ is as follows: select j existing edges, subdivide each by a new node and identify the j subdividing nodes with a new node z. We shall refer to this operation as **pinching** together j edges (with a node z). Note that for $j = 1$, the pinching operation is simply subdividing an edge by a new node.

Exercise 8.1.2 *Prove that pinching together $j \geq K$ edges of a K-edge-connected graph results in a K-edge-connected graph.*

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 253 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 2.1.1 described a constructive characterization of 2-edge-connected graphs. The following theorem is a straight generalization.

Theorem 8.1.4 (Lovász) *An undirected graph $G = (V, E)$ is $2k$-edge-connected if and only if it can be constructed from an initial node, by the repeated application of the following two operations.*

(A) *Add a new edge connecting two existing nodes.*
(B) *Pinch together k existing edges with a new node.*

Proof. Operation (A) obviously preserves $2k$-edge-connectivity and so does Operation (B) by Exercise 8.1.2. To prove the non-trivial part, we use induction on $|E|$. The theorem is void if $|V| = 1$, so let $|V| \geq 2$. By induction, it suffices to show that a $2k$-edge-connected graph G arises from a $2k$-edge-connected graph by Operations (A) or (B).

If there is an edge e for which $G' := G - e$ is $2k$-edge-connected, then G arises from G' by Operation (A). Suppose now that G is minimally $2k$-edge-connected and let $K = 2k$. By Lemma 8.1.1, there is a node of degree $2k$. Apply Theorem 8.1.2. The resulting graph G' is $2k$-edge-connected and G can be obtained from G' by applying Operation (B) to the k split-off edges of G'. • •

The constructive characterization of K-edge-connected graphs is more complex when K is odd. We postpone the general formulation and proof to the next section. For $K = 3$, however, the situation is better and we treat it here. The next proposition will be needed for the general case as well.

Proposition 8.1.5 *Let $K \geq 1$ be odd and G a minimally K-edge-connected graph with at least two nodes. Then G has an edge $e = uv$ such that the only possible minimum cut containing e is $\Delta(u)$ or $\Delta(v)$, showing that each K-cut containing e is a star-cut* (where a K-cut is a cut of K elements).

Proof. If every K-cut is a star-cut, then e can be chosen arbitrarily. Suppose now that there is a set $Z \subseteq V$ for which $2 \leq |Z| \leq |V| - 2$, $d(Z) = K$, and assume that $|Z|$ is minimal. By Lemma 7.1.3, the K-cuts are cross-free. This implies that every K-cut containing an arbitrary edge induced by Z is a star-cut. •

Theorem 8.1.6 *A graph $G = (V, E)$ is 3-edge-connected if and only if it can be built up from an initial node by applying the following operations.*

(A) *Add a new edge connecting two existing nodes.*
(B) *Subdivide an existing edge by a node z and add an edge connecting z and an existing node.*
(C) *Take two existing edges, subdivide each by a node, and add a new edge connecting the two subdividing nodes.*

Proof. An easy analysis shows that each operation preserves 3-edge-connectivity. To prove conversely that every 3-edge-connected graph can be obtained in this way, we can assume that G has more than one node. By induction, it suffices to show that G arises from a 3-edge-connected graph by one of the three operations.

If $G' = G - e$ is 3-edge-connected for some edge e of G, then G can be obtained from G' by (A). Suppose now that G is minimally 3-edge-connected. By Proposition 8.1.5, there is an edge $e = uv$ such that each 3-cut containing e is a star-cut. Since G is minimal, e does belong to a 3-cut. Therefore there are two cases depending on whether one or both end-nodes of e are of degree 3.

If $d(u) = d(v) = 3$, then let G'' denote the graph obtained from G by first, deleting e, then splitting off the two remaining edges incident to u, and finally splitting off the two remaining edges incident to v. Then G'' is 3-edge-connected since no minimal non-star-cut of G contains e. Moreover, G arises from G'' by Operation (C).

Assume now that node u, say, is of degree 3 while $d(v) \geq 4$. Let G''' denote the graph obtained from G by deleting first e and then splitting off the two remaining edges incident to u. Then G''' is 3-edge-connected since no minimal non-star-cut of G contains e. Moreover, G arises from G''' by Operation (B). •

8.1.3 Preserving local edge-connectivity

Let $G = (V + z, E)$ be a connected undirected graph with $|V| \geq 2$ in which $\lambda_G(u, v)$ denotes the size of a minimum cut separating u and v. By the undirected edge-version of Menger's theorem, $\lambda_G(u, v)$ is equal to the maximum number of edge-disjoint uv-paths. This number is sometimes called the **local edge-connectivity** of the pair $\{u, v\}$. We say that a pair of edges incident to the special node z is **splittable** if splitting off the edges preserves the local edge-connectivity for every pair of nodes $u, v \in V$.

We noted already that Lovász' splitting lemma does not hold for $k = 1$, and the evenness of $d_G(z)$ could not be left out from the hypotheses of the theorem. Therefore, in order to generalize the splitting lemma for local edge-connectivity, some restrictions must be imposed. Before presenting the fundamental result of Mader [278], consider the following property of G.

$$\text{The local edge-connectivity between the neighbours of } z \text{ is at least 2.} \tag{8.3}$$

This is equivalent to requiring that

$$\text{no cut-edge of } G \text{ is incident to } z. \tag{8.4}$$

Theorem 8.1.7 (Mader) *Let $G = (V + z, E)$ be a connected undirected graph in which $d_G(z) \neq 3$ and* (8.4) *holds. Then there is a splittable pair of edges incident to z.*

It will be more convenient to prove the following equivalent version.

Theorem 8.1.8 *Let $G = (V + z, E)$ be a connected undirected graph in which $d_G(z)$ is even and* (8.4) *holds. Then the edges incident to z can be matched into $d_G(z)/2$ pairs in such a way that splitting off these pairs simultaneously preserves the local edge-connectivity for every pair of nodes $u, v \in V$.*

Before proving this result, we show that the last two theorems are actually equivalent. Suppose first that Theorem 8.1.7 is true and consider a splittable pair of edges. Their splitting preserves (8.4) due to its equivalence to (8.3). Therefore Theorem 8.1.7 can be consecutively applied $d_G(z)/2$ times and Theorem 8.1.8 follows.

Conversely, supposing that Theorem 8.1.8 is true, we want to derive Theorem 8.1.7. There is nothing to prove if $d_G(z)$ is even, so suppose that $d_G(z)$ is odd. By the hypothesis, $d_G(z) \geq 5$. Adjoin a new node x to G along with three parallel edges between z and x.

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 255 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Then condition (8.4) holds for the resulting graph G' as well, so Theorem 8.1.8 applies to G' and ensures $(d_G(z)+3)/2 \geq 4$ splittable pairs. At most three of these can contain a new edge so at least one of them consists of original edges. This pair is clearly splittable in the original G as well.

Skew supermodular functions

Sub- and supermodular functions proved to be useful in Lovász' splitting lemma. In the present more general setting, we need a more general class of functions. A set-function R is called **skew supermodular** if, for every pair of subsets X and Y, at least one of the following two inequalities holds:

$$\begin{cases} R(X)+R(Y) \leq R(X\cap Y)+R(X\cup Y) & (\alpha) \\ R(X)+R(Y) \leq R(X-Y)+R(Y-X) & (\beta) \end{cases} \tag{8.5}$$

We introduce a class of skew supermodular functions which will be useful not only in the proof of Mader's theorem but also in Theorem 11.1.13 on optimal augmentations of a graph meeting local edge-connectivity prescriptions. Let $G=(V,E)$ be an undirected graph. Let $r(x,y)$ $(x,y\in V)$ be a non-negative function which is symmetric, that is, $r(x,y)=r(y,x)$. Define a set-function R_r as follows.

$$R_r(X) := \begin{cases} 0 & \text{if } X=\emptyset \text{ or } X=V \\ \max\{r(x,y): x\in X, y\in V-X\} & \text{if } \emptyset\subset X\subset V. \end{cases} \tag{8.6}$$

Lemma 8.1.9 *The function R_r is skew supermodular.*

Proof. For convenience, we abbreviate R_r to R. Observe first that if Y is replaced by $V-Y$, then (8.5α) and (8.5β) transform into each other. Let $\{z,z'\}$ be a pair of nodes maximizing the value of $r(z,z')$ over all pairs separated by at least one of X and Y. (A set Z is said to separate z and z' if $|Z\cap\{z,z'\}|=1$.) We can assume by symmetry that $z\in X$ and $z'\in V-X$. By replacing Y with $V-Y$, if necessary, we can assume that $z\notin Y$, in which case $z\in Y-X$.

If $z'\in Y$, then $r(z,z')=R(X)=R(Y)=R(X-Y)=R(Y-X)$ and hence (8.5β) holds (actually with equality). If $z'\notin Y$, then $r(z,z')=R(X)=R(X\cup Y)=R(X-Y)$. Since obviously $R(Y)\leq R(X\cap Y)$ or $R(Y)\leq R(Y-X)$, we have accordingly (8.5α) or (8.5β). •

Proof of Theorem 8.1.8

We assume that the theorem holds for every graph smaller than G. It suffices to prove that there is a splittable pair since by the repeated application of this statement one obtains Theorem 8.1.8. In what follows R_{λ_G} will be abbreviated to R_G. It is clear that $d_G(X)\geq R_G(X)$ holds for every subset $X\subseteq V$.

Claim 8.1.10 *A pair of edges is splittable if and only if in the graph G' arising by their splitting off one has $d_{G'}(X)\geq R_G(X)$ for every $X\subseteq V$.*

Proof. If the pair of edges is splittable, then $d_{G'}(X)\geq R_G(X)$ follows directly from the definition of R. Conversely, if the splitting off decreases the local edge-connectivity of u

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 256 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

and v, then there is a set X for which $|X \cap \{u, v\}| = 1$ and $d_{G'}(X) < \lambda_G(u, v)$. By taking the complement, if necessary, we can assume that $X \subseteq V$. Now $d_{G'}(X) < \lambda_G(u, v) \leq R_G(X)$ and the claim follows. •

We say that a subset $X \subseteq V$ is **tight** if $d_G(X) = R_G(X)$, **near-tight** if $d_G(X) = R_G(X) + 1$, and **dangerous** if X is tight or near-tight, that is, if $R_G(X) \leq d_G(X) \leq R_G(X) + 1$. Note that $d_G(V) = d_G(z) \geq 2$ implies that V is never dangerous. Since a splitting-off can decrease the degree of a set by 2, a pair $\{e = zt, f = zu\}$ of edges is splittable if and only if there is no dangerous set containing both u and t. For a set $X \subseteq V$, let $s(X) := d_G(X) - R_G(X)$ denote the **surplus** of X. As we noted above, the surplus is non-negative. X is tight if $s(X) = 0$ and dangerous if $s(X) \leq 1$.

By the identities concerning d_G (see Proposition 1.2.1), we obtain from Lemma 8.1.9 the following.

Claim 8.1.11 *For every $X, Y \subseteq V$ at least one of the following inequalities holds.*

$$\begin{cases} s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) & (\alpha) \\ s(X) + s(Y) \geq s(X - Y) + s(Y - X) & (\beta) \end{cases} \tag{8.7}$$

Claim 8.1.12 *Let T be a tight set ($\emptyset \subset T \subset V$) and let $e = zu$, $f = zv$ be two edges. Shrink T into a single node and let G' denote the resulting graph. Let e' and f' denote the edges of G' corresponding to the edges e and f, respectively. If $\{e', f'\}$ is splittable in G', then $\{e, f\}$ is splittable in G.*

Proof. Let Z be a set for which $Z \subseteq V - T$ or $T \subseteq Z \subseteq V$ and let Z' denote the subset of nodes of G' which arises from Z after shrinking T. Then $R_{G'}(Z') \geq R_G(Z)$ and $d_{G'}(Z') = d_G(Z)$. Therefore if Z is dangerous in G, then Z' is dangerous in G'.

If $\{e, f\}$ were not splittable in G, then there would be a dangerous set X of G containing u and v. The set $Z := X \cup T$ cannot be dangerous in G since then Z' would be dangerous in G' contradicting the hypothesis that $\{e', f'\}$ is splittable in G'. This we have $s(X \cup T) \geq 2$. Apply the preceding claim to X and T. Now alternative (8.7α) cannot occur since it would imply

$$0 + 1 \geq s(T) + s(X) \geq s(X \cap T) + s(X \cup T) \geq 0 + 2,$$

which is impossible. Therefore we must have (8.7β), that is,

$$0 + 1 \geq s(T) + s(X) \geq s(X - T) + s(T - X) + 2\overline{d}_G(X, T) \geq 0 + 0 + 2\overline{d}_G(X, T).$$

This implies that $2\overline{d}_G(X, T) = 0$ and $s(D) \leq 1$ where $D := X - T$. It follows from the equality that $u, v \in D$, while the inequality means that D is dangerous in G. But then D' is dangerous in G' showing that $\{e', f'\}$ is not splittable in G', contradicting the hypothesis of the claim. •

Suppose, indirectly, that the theorem does not hold for G. Since we assumed its truth for graphs smaller than G, it follows from Claim 8.1.12 that every tight set is a singleton.

Claim 8.1.13 *If every tight set is a singleton, then $\lambda_G(x, y) = \min\{d_G(x), d_G(y)\}$ for every $x, y \in V$.*

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 257 of 640 December 11, 2010 15:18

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proof. The claim follows immediately once we observe that a set $X \subset V$ is tight provided that $\lambda_G(x, y) = d_G(X)$ and X separates x and y. •

Let S denote the set of neighbours of z and let $t \in S$ be a node of minimum degree.

Claim 8.1.14 *$R_G(X - t) \geq R_G(X)$ holds for every subset $X \subset V$ for which $t \in X$ and $|S \cap X| \geq 2$.*

Proof. Let $u \in S \cap (X - t)$. By the choice of t, we have $d_G(u) \geq d_G(t)$. By the definition of $R_G(X)$, there exist $v \in X$ and $v' \in V - X$ for which $R_G(X) = \lambda_G(v, v')$. If $v \neq t$, then $R_G(X - t) \geq \lambda_G(v, v') = R_G(X)$. If $v = t$, then Claim 8.1.13 implies $R_G(X) = \lambda_G(t, v') = \min\{d_G(t), d_G(v')\} \leq \min\{d_G(u), d_G(v')\} = \lambda_G(u, v') \leq R_G(X - t)$. •

Claim 8.1.15 *If X is dangerous, then $d_G(z, X) \leq d_G(z, V - X)$.*

Proof. Let $\alpha := d_G(z, X)$ and $\beta := d_G(z, V - X)$. Then $R_G(V - X) = R_G(X) \geq d_G(X) - 1 = d_G(V - X) - \beta + \alpha - 1 \geq R_G(V - X) - \beta + \alpha - 1$ from which $\alpha \leq \beta + 1$. We are done if $\alpha \leq \beta$. But $\alpha = \beta + 1$ cannot occur since then we would have $d_G(z) = 2\beta + 1$ contradicting the hypothesis of the theorem that $d_G(z)$ is even. •

Since G is a counterexample, no pair $\{zt, zu\}$ is splittable and hence every neighbour of z belongs to a dangerous subset containing t. Let $\mathcal{L}$ be a smallest set of dangerous sets containing t such that the members of $\mathcal{L}$ cover S.

Claim 8.1.16 $|\mathcal{L}| \geq 3$.

Proof. It follows from Claim 8.1.15 that $|\mathcal{L}| \geq 2$. Suppose, indirectly, that $|\mathcal{L}| = 2$ and let $\mathcal{L} = \{X, Y\}$. Then $S \subseteq X \cup Y$, and again using Claim 8.1.15, we obtain that $d_G(z, X) \leq d_G(z, V - X) < d_G(z, Y) \leq d_G(z, V - Y) < d_G(z, X)$, a contradiction. (The first strict inequality holds since $(S - Y) \cup \{t\} \subseteq X$ and the second one is analogous.) •

Let X_1, X_2, X_3 be three members of $\mathcal{L}$ and let $\mathcal{F} := \{X_1, X_2, X_3\}$. By the minimal choice of $\mathcal{L}$, each X_i contains an element $s_i \in S$ which is not in the two others.

Claim 8.1.17 *For each two members X, Y of $\mathcal{F}$, the alternative* (8.7β) *holds.*

Proof. Suppose that (8.7α) holds. By the minimality of $\mathcal{L}$ we have $s(X \cup Y) \geq 2$ and hence $1 + 1 \geq s(X) + s(Y) \geq s(X \cap Y) + s(X \cup Y) \geq 0 + 2$ from which $s(X \cap Y) = 0$ follows, and hence $X \cap Y$ is tight. Since each tight set is a singleton, we have $X \cap Y = \{t\}$. Then $X - Y = X - \{t\}$ and $Y - X = Y - \{t\}$ and hence Claim 8.1.14 implies $R_G(X) \leq R_G(X - Y)$ and $R_G(Y) \leq R_G(Y - X)$. Therefore $R_G(X) + R_G(Y) \leq R_G(X - Y) + R_G(Y - X)$ from which $s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2\overline{d}_G(X, Y)$, and hence (8.7$\beta$) is valid. In the case when (8.7α) does not hold, then (8.7β) follows from Claim 8.1.11. •

Claim 8.1.18 *For any two members X, Y of $\mathcal{F}$, $|X - Y| = |Y - X| = 1$ and $\overline{d}_G(X, Y) = 1$.*

Proof. By Claim 8.1.17 $1 + 1 \geq s(X) + s(Y) \geq s(X - Y) + s(Y - X) + 2\overline{d}_G(X, Y) \geq 0 + 0 + 2$. This implies $\overline{d}_G(X, Y) = 1$ and that both $X - Y$ and $Y - X$ are tight and hence are singletons. •

To complete the proof of the theorem, let $M := X_1 \cap X_2 \cap X_3$. Claim 8.1.18 and the minimality of $\mathcal{L}$ imply that $X_i = M + s_i$ $(i = 1, 2, 3)$, and that $\overline{d}_G(X_i, X_j) = 1$ $(1 \leq i < j \leq 3)$. This, in turn, means that the edge zt is the only edge of G leaving M, and hence zt is a cut-edge contradicting the hypothesis of the theorem. • •

Constructive characterization of* $(2k+1)$*-edge-connected graphs

In Theorem 8.1.6, we already studied the case of 3-edge-connected graphs. The next result for general odd K is due to Mader [278].

Theorem 8.1.19 (Mader) *Let $K = 2k + 1$. A graph G is K-edge-connected if and only if it can be built up from a node by applying the following Operations.*

(A) *Add a new edge connecting two existing nodes.*
(B) *Pinch together k existing edges with a new node z and connect z with an existing node.*
(C) *First pinch together k existing edges with a new node s. Then select k edges in the resulting graph such that at least one of them is an original edge of G and pinch these k edges with a new node z. Finally, add a new edge connecting z and s.*

Proof. It is left to the reader to check that each operation maintains K-edge-connectivity, noting that (C) would not maintain K edge-connectivity without the assumption that in the second pinching operation at least one of the pinched edges is an original edge.

For the converse, it suffices to prove that a K-edge-connected graph G can be obtained from a smaller K-edge-connected graph by applying one of the three operations. If there is an edge e for which $G' := G - e$ is K-edge-connected, then G arises from G' by Operation (A). If G is minimally K-edge-connected, that is, if every edge belongs to a K-cut, then by Proposition 8.1.5 there is an edge $e = zs$ such that every K-cut containing e is a star-cut and at least one of the end-nodes of e, say, z is of degree K.

First suppose that $d_G(s) \geq K + 1$. Then $G' = G - e$ is a graph in which the degree of z is $K - 1 = 2k$ and $(*)$ $\lambda_{G'}(x, y) \geq K$ holds for every $x, y \in V - z$. By applying Theorem 8.1.8, we obtain a graph G'' on node-set $V - z$ which is K-edge-connected and G can be obtained from G'' by applying Operation (B): pinch the k split-off edges of G'' and then add edge $e = zs$.

Second, suppose that $d_G(s) = K = d_G(z)$. Then $(*)$ holds for every $x, y \in V - \{z, s\}$. By Theorem 8.1.8, there is a complete splitting at z which results in a graph G'' on node-set $V - z$ for which $\lambda_{G''}(x, y) \geq K$ for every $x, y \in V - \{z, s\}$. Note that in G' there are less than k parallel edges between z and s for otherwise $d_G(\{s, z\}) = 2k < K$. Therefore at least one of the split-off edges is not incident to s. In G'' the degree of s is $K = 2k$. A second application of Theorem 8.1.8 implies that in G'' there is a complete splitting at s such that the resulting graph G''' is K-edge-connected on node-set $V - \{z, s\}$. It follows from this construction that G arises from G''' by Operation (C). •

8.2 Directed splitting

The following easy identities from Proposition 1.2.1 will be useful for studying results on directed splitting-off.

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 259 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$$\varrho(X) + \varrho(Y) = \varrho(X \cap Y) + \varrho(X \cup Y) + d(X, Y). \tag{8.8}$$

$$\varrho(X) + \varrho(Y) = \varrho(X - Y) + \varrho(Y - X) + \overline{d}(X, Y) + \varrho(X \cap Y) - \delta(X \cap Y). \tag{8.9}$$

We say that a digraph $D = (V, A)$ is k**-edge-connected** in a subset V' of its nodes if

$$\varrho(X) \geq k \text{ and } \delta(X) \geq k \text{ for every subset } X \subset V \text{ separating } V' \tag{8.10}$$

where X is said to **separate** V' if $V' \cap X \neq \emptyset$ and $V' - X \neq \emptyset$. By Menger's theorem, (8.10) is equivalent to requiring that there be k edge-disjoint paths (in D) from every node of V' to each other node of V'. Note that this notion is weaker than requiring the k-edge-connectivity of the subgraph induced by V'.

More generally, for two (not-necessarily-disjoint) subsets S and T of nodes of a digraph D we say that D is k**-edge-connected from** S **to** T if $\varrho_D(X) \geq k$ for every subset X of nodes for which $X \cap T \neq \emptyset$ and $S - X \neq \emptyset$. Again by Menger, this is equivalent to requiring that there are k edge-disjoint st-paths for every pair $s \in S$, $t \in T$ of nodes.

8.2.1 Preserving global and rooted edge-connectivity of digraphs

Preserving k-edge-connectivity
The directed counterpart of Lovász' splitting lemma is due to Mader [280].

Theorem 8.2.1 (Mader's directed splitting lemma) *Let $D = (V + z, A)$ be a digraph that is k-edge-connected ($k \geq 1$) in V, and for the special node z suppose that $\varrho(z) = \delta(z)$. Then there is an edge $f = uz$ for every edge $e = zt$ for which the digraph D^{ef} arising by splitting of $\{e, f\}$ is k-edge-connected in V.*

Proof. We say that a subset $X \subset V$ is **in-tight** if $\varrho(X) = k$ and **out-tight** if $\delta(X) = k$. X is **tight** if it is in-tight or out-tight.

Lemma 8.2.2 *Let X and Y be two tight sets containing t. Then $X \cup Y$ is tight.*

Proof. There is nothing to prove if $X \subseteq Y$ or $Y \subseteq X$ so we assume this is not the case. It is not possible that one of X and Y is out-tight and the other one is in-tight since if $\varrho(X) = k$ and $\delta(Y) = k$, say, then $\overline{Y} := V + z - Y$ and for X we have $2k = \varrho(X) + \varrho(\overline{Y}) = \varrho(X \cup \overline{Y}) + \varrho(X \cap \overline{Y}) + d(X, \overline{Y}) \geq 2k + 1$.

Suppose that both X and Y are out-tight. (The proof is analogous if both of them are in-tight.) We cannot have $X \cup Y = V$ for otherwise $\overline{X} \cap \overline{Y} = \{z\}$, and then, by using $\varrho(z) = \delta(z)$ and (8.9), we would obtain $2k = \varrho(\overline{X}) + \varrho(\overline{Y}) = \varrho(\overline{X} - \overline{Y}) + \varrho(\overline{Y} - \overline{X}) + \overline{d}(\overline{X}, \overline{Y}) \geq k + k + 1$. Therefore $X \cup Y \subset V$ and then (8.10) implies that $2k = \delta(X) + \delta(Y) \geq \delta(X \cap Y) + \delta(X \cup Y) \geq k + k$, from which $\delta(X \cup Y) = k$, as required. •

If there is no tight set at all containing t, then an arbitrary edge $f = uz$ can be split off with $e = zt$. If there is such a tight set, then the lemma ensures a unique largest tight set M containing t.

We claim that there is an edge $f = uz$ entering z that does not leave M. Indeed, if no such an edge exists, then in the case when $\varrho(M) = k$ we have $\delta(V - M) = \varrho(M + z) \leq \varrho(M) - 1 = k - 1$, contradicting (8.10), while in the case of $\delta(M) = k$ we have

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 260 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$\varrho(V - M) = \delta(M + z) \leq \delta(M) - \varrho(z) + \delta(z) - 1 \leq k - 1$, again contradicting (8.10). Since there is no tight set containing both t and u, the edges e and f can be split off. • •

Exercise 8.2.1 *Show that the assumption $\varrho(z) = \delta(z)$ in Mader's theorem cannot be left out even in the special case $k = 1$.*

Problem 8.2.2 *Suppose that digraph $D = (V + z, A)$ is k-edge-connected in V $(k \geq 1)$ and $\delta(z) \leq \varrho(z) < 2\delta(z)$. Then there is an entering and a leaving edge at z such that their splitting-off preserves k-edge-connectivity in V.*

By the repeated application of Theorem 8.2.1 one immediately gets the following result.

Theorem 8.2.3 (Mader) *Let $D = (V + z, A)$ be a digraph that is k-edge-connected in V $(k \geq 1)$, and suppose that $\varrho(z) = \delta(z)$. Then the edges entering z can be paired with those leaving z in such a way that splitting-off all the $\varrho(z)$ pairs results in a k-edge-connected digraph on V.* •

The simultaneous splittings of the pairs of edges in the theorem will be called a **complete splitting** (at z).

Preserving rooted k-edge-connectivity

By a simple construction, Theorem 8.2.3 can easily be used to obtain an analogous result concerning rooted k-edge-connectivity.

Theorem 8.2.4 *Let $D = (V + z, A)$ be a rooted k-edge-connected digraph from a designated root-node $r_0 \in V$. Suppose that $\varrho_D(z) \geq \delta_D(z)$. Then the edges leaving z can be paired with appropriate $\delta_D(z)$ edges entering z in such a way that by splitting off simultaneously these pairs first and then removing z, one obtains a digraph on node-set V which is rooted k-edge-connected.*

Proof. For every node $v \in V - r_0 + z$ for which $\varrho(v) > \delta(v)$, add $\varrho(v) - \delta(v)$ new parallel edges from v to r_0. Let D' denote the resulting digraph.

Claim 8.2.5 *D' is k-edge-connected in V.*

Proof. By the hypothesis of the theorem, we have $\varrho_{D'}(X) = \varrho_D(X) \geq k$ for every non-empty $X \subseteq V - r_0$. Furthermore, in D' the out-degree of every node of X is at least its in-degree from which $\delta_{D'}(X) \geq \varrho_{D'}(X) \geq k$. •

By Theorem 8.2.3, D' admits a complete splitting at z which results in a k-edge-connected digraph on V. By removing from this digraph the edges entering r_0, we obtain a rooted k-edge-connected digraph which arises from D as described in the theorem. Namely, among the $\varrho_D(z)$ pairs of edges which are split off there are $\delta_D(z)$ consisting only of original edges while each of the remaining $\varrho_D(z) - \delta_D(z)$ pairs contains one newly added edge from z to r_0. The other members of these pairs (entering z) are those edges which are to be removed in the construction described in the theorem. • •

8.2.2 An extension of the directed splitting lemma

In the undirected case, Mader's theorem (Theorem 8.1.7) is an extension of Lovász' undirected splitting lemma. Unfortunately, no analogous extension of Mader's directed splitting lemma is known. Still, we describe a generalization of Mader's splitting lemma and the proof is only slightly more complicated.

Theorem 8.2.6 *Let $D = (V + z, A)$ be a digraph with a special node z for which $\varrho(z) = \delta(z)$ and let $T \subseteq V$ be a subset for which $\varrho(v) = \delta(v)$ holds for every node not in T. Suppose that D is k-edge-connected in T ($k \geq 1$). Then for every edge $e = zt$ there is an edge $f = uz$ for which the digraph D^{ef} arising from D by splitting off $\{e, f\}$ is k-edge-connected in T.*

Proof. We defined a subset $X \subseteq V$ separating T as in-tight if $\varrho(X) = k$ and out-tight if $\delta(X) = k$. A pair $\{uz, zt\}$ of edges meets the requirements of the theorem if there is no tight set containing both t and u.

Lemma 8.2.7 *Let X and Y be two tight sets containing t. Then $X \cup Y$ is tight.*

Proof. There is nothing to prove if $X \subseteq Y$ or $Y \subseteq X$ so suppose that X and Y are properly intersecting. Consider the case when one of them is in-tight and the other is out-tight: $\varrho(X) = k$ and $\delta(Y) = k$. Let $\overline{Y} := V + z - Y$. Assume first that $X - Y = X \cap \overline{Y}$ does not separate T. It is not possible that $T \subseteq X - Y$ since then Y would not separate T. Therefore no element of $X - Y$ belongs to T. By the hypothesis of the theorem, the in-degree and the out-degree of each node outside T are equal, from which $\varrho(X \cap \overline{Y}) = \delta(X \cap \overline{Y})$ follows. At most one of $X \cap Y$ and $X \cup Y$ can separate T and hence (8.9) implies

$$2k = \varrho(X) + \varrho(\overline{Y}) = \varrho(X - \overline{Y}) + \varrho(\overline{Y} - X) + \overline{d}(X, \overline{Y}) \geq \varrho(X \cap Y) + \delta(X \cup Y) \geq k + k.$$

We must have equality throughout and hence $X \cup Y$ is out-tight.

Therefore we can assume that $X - Y$ separates T and, analogously, that $Y - X = V - (X \cup \overline{Y})$ separates T. It follows that $\varrho(X - Y) \geq k$ and $\varrho(Y - X) \geq k$ and hence

$$2k = \varrho(X) + \varrho(\overline{Y}) = \varrho(X \cup \overline{Y}) + \varrho(X \cap \overline{Y}) + d(X, \overline{Y}) \geq 2k + 1$$

which cannot be the case.

Suppose now that X and Y are out-tight. (The proof is analogous when X, Y are in-tight.) It is not possible that $T \subseteq X \cup Y$, since then $\varrho(\overline{X} \cap \overline{Y}) = \delta(\overline{X} \cap \overline{Y})$ follows from the hypothesis and then (8.9) would imply $2k = \varrho(\overline{X}) + \varrho(\overline{Y}) = \varrho(\overline{X} - \overline{Y}) + \varrho(\overline{Y} - \overline{X}) + \overline{d}(\overline{X}, \overline{Y}) \geq k + k + 1$. Analogously, it is not possible that $T \cap (X \cap Y) = \emptyset$. Therefore both $X \cup Y$ and $X \cap Y$ separate T and (8.8) implies that $2k = \delta(X) + \delta(Y) \geq \delta(X \cap Y) + \delta(X \cup Y) \geq k + k$, and hence $X \cup Y$ is out-tight, and the lemma follows. •

If there is no tight set containing t, then an arbitrary edge $f = uz$ can be split off with $e = zt$. If there is such a tight set, then the lemma implies that there is a unique largest one, denoted by M. We claim that there is an edge uz with $u \in V - M$. Suppose indirectly that the tail of each edge with head z is in M. If $\varrho(M) = k$, then $\delta(V - M) = \varrho(M + z) \leq \varrho(M) - 1 = k - 1$, contradicting (8.10). If $\delta(M) = k$, then $\varrho(V - M) = \delta(M + z) \leq \delta(M) - \varrho(z) + \delta(z) - 1 = k - 1$, again contradicting (8.10). • •

Next we show that Theorem 8.2.6 can also be used for preserving rooted k-edge-connectivity, a result to be used in proving Theorem 10.1.3 on disjoint arborescences.

Theorem 8.2.8 *Let $D = (V, A)$ be a digraph with a root-node r_0 and a terminal set $T \subseteq V - r_0$ for which*

$$\varrho(v) \geq \delta(v) \text{ for every } v \in V - (T + r_0). \tag{8.11}$$

Suppose that

$$\lambda(r_0, u) \geq k \text{ for every } u \in T, \tag{8.12}$$

that is, there are k edge-disjoint paths from r_0 to every element of T. If $\varrho(z) > \delta(z)$ for a node $z \in V - (T + r_0)$, then it is possible to remove $\varrho(z) - \delta(z)$ edges entering z without destroying property (8.12)*. If $\varrho(z) = \delta(z)$, then there is a splitting at z preserving* (8.12)*.*

Proof. For each node $v \in V - r_0$ with $\varrho(v) > \delta(v)$, add $\varrho(v) - \delta(v)$ parallel vr_0-edges. Then $\varrho'(v) \leq \delta'(v)$ holds for every node $v \in V - r_0$ of the resulting digraph D' from which $\delta'(X) \geq \varrho'(X)$ follows for every subset $X \subseteq V - r_0$. We claim that D' is k-edge-connected in $T' := T + r_0$. Indeed, by the hypothesis of the theorem $\delta'(X) \geq \varrho'(X) = \varrho(X) \geq k$ holds for every $X \subseteq V$ intersecting T.

By Theorem 8.2.6, the edges of D' entering and leaving z can be paired in such a way that splitting off any number of these pairs preserves k-edge-connectivity in T. It follows that in the case when $\varrho(z) = \delta(z)$, the same splittings in D preserve (8.12). If, in turn, $\varrho(z) > \delta(z)$, then there are $\varrho(z) - \delta(z)$ edges entering z which are split off in D' with edges entering r_0. Therefore these $\varrho(z) - \delta(z)$ edges can be removed from D without destroying (8.12). •

There is another extension of Mader's directed splitting lemma which is not comparable with Theorem 8.2.8. It is a special case of a result on covering supermodular functions, to be discussed in Part III.

Theorem 8.2.9 *Let $D = (V + z, A)$ be a digraph with a special node z for which $\varrho(z) = \delta(z)$. Let $T \subseteq V$ be a subset of nodes containing all neighbours of z and suppose that D is k-edge-connected in T $(k \geq 1)$. Then, for every edge $e = zt$, there is an edge $f = uz$ for which the digraph D^{ef} obtained by splitting off the pair $\{e, f\}$ is k-edge-connected in T.*

Problem 8.2.3 *Prove Theorem* 8.2.9 *by extending the proof method of Theorem* 8.2.1.

Problem 8.2.4 *Prove that in an Euler digraph, there is a splitting at a node z which preserves local edge-connectivity between all pairs of nodes distinct from z.*

Research problem 8.2.1 *Let $D = (V + z, A)$ be a digraph which is k-edge-connected in V. Determine a maximum number of disjoint pairs of edges for which one member of each pair enters z, while the other member leaves z in such a way that splitting these pairs simultaneously preserves k-edge-connectivity in V.* (Note that Mader's splitting-off lemma shows that this number is $\varrho(z)$ if $\varrho(z) = \delta(z)$.)

When $\delta(z) \leq \varrho(z)$ then the maximum is clearly at most $\delta(z)$. Berg, Jackson, and Jordán [30] characterized digraphs (and dypergraphs) for which the maximum in question is $\delta(z)$.

8.2.3 Constructing global and rooted k-edge-connected digraphs

The undirected splitting lemma could be used for characterizing $2k$-edge-connected graphs. Its directed counterpart will help to describe an analogous construction for digraphs. We start with rooted k-edge-connectivity because it is much simpler.

Rooted k-edge-connectivity

We say that a digraph D is **minimally rooted k-edge-connected** if it is rooted k-edge-connected but deleting any edge destroys this property.

Theorem 8.2.10 *A digraph $D = (V, A)$ is minimally rooted k-edge-connected* ($k \geq 1$) *with respect to a root-node r_0 if and only if it can be built up from r_0 by applying consecutively the following two operations.*

(B) *Add a new node z along with k* (possibly parallel) *edges entering z for which their tails are existing nodes.*

(C) *Pinch j existing edges* ($0 < j < k$) *with a new node z and add $k - j$* (possibly parallel) *edges entering z for which the tails are existing nodes.*

Proof. It can readily be checked that the given operations preserve rooted k-edge-connectivity. Furthermore, for a digraph D obtained in this way, $\varrho(r_0) = 0$ and $\varrho(v) = k$ for each node $v \in V - r_0$ and hence D is minimally rooted k-edge-connected.

To see the converse, we proceed by induction on the number of edges. Assume that $|V| \geq 2$. By Lemma 7.4.1, the minimality of D implies that $\varrho(v) = k$ if $v \neq r_0$ and $\varrho(r_0) = 0$. Since $\varrho(r_0) < \delta(r_0)$, there must be a node z for which $k = \varrho(z) > \delta(z)$. Theorem 8.2.4 implies that there are $k - \delta(z)$ edges entering z which can be left out and the remaining $j := \delta(z)$ edges entering z can be paired and simultaneously split off with the $\delta(z)$ edges leaving z in such a way that the resulting digraph D' is rooted k-edge-connected. But then the original D can be obtained from D' by operation (B) in the case when $\delta(z) = 0$ and by Operation (C) in the case when $\delta(z) \geq 1$ from which the theorem follows by induction. •

Note that Operation (B) could be interpreted so as to pinch no edge with a new node z and add k new edges entering z. Hence (B) and (C) can be united in the following form.

(BC) Pinch j existing edges ($0 \leq j < k$) with a new node z and add $k - j$ (possibly parallel) edges entering z for which the tails are existing nodes.

An immediate corollary of Theorem 8.2.10 is the following.

Theorem 8.2.11 *A digraph $D = (V, A)$ is rooted k-edge-connected* ($k \geq 1$) *with respect to a root-node r_0 if and only if it can be built up from r_0 by applying consecutively the following operations.*

(A) *Add a new directed edge connecting two existing nodes.*

(B) *Add a new node z along with k* (possibly parallel) *edges entering z for which the tails are existing nodes.*

(C) *Pinch j existing edges* ($0 < j < k$) *together with a new node z and add $k - j$* (possibly parallel) *edges entering z for wchich the tails are existing nodes.* •

As a nice application, we shall show in Section 10.1.1 how Theorem 8.2.11 immediately implies a fundamental result of Edmonds on disjoint arborescences.

Global k-edge-connectivity

For the constructive characterization of K-edge-connected graphs, we made use of an easy lemma (Lemma 8.1.3) which stated that every minimally K-edge-connected graph on at least two nodes has a node of degree K. The following directed counterpart, proved by Mader [278], is much deeper and will play an analogous role for digraphs. The relatively simple proof below appeared in [126].

Theorem 8.2.12 (Mader) *Let $D = (V, A)$ be a minimally k-edge-connected digraph with at least two nodes ($k \geq 1$). Then D has a node v for which $\varrho(v) = \delta(v) = k$.*

Proof. A set $X \subset V$ is **in-tight** if $\varrho(X) = k$ and **out-tight** if $\delta(X) = k$. By the minimality of D, every edge enters an in-tight set. Let $\mathcal{L}$ be a system of in-tight sets such that every edge of D enters at least one of its members, $|\mathcal{L}|$ is minimal and with respect to this, $\sum[|Z|^2 : Z \in \mathcal{L}]$ is maximal.

Claim 8.2.13 *$\mathcal{L}$ is cross-free.*

Proof. Suppose indirectly that X and Y are two crossing members of $\mathcal{L}$. Then we have $k + k = \varrho(X) + \varrho(Y) = \varrho(X \cap Y) + \varrho(X \cup Y) + d(X, Y) \geq k + k + 0$ which implies that both the intersection and the union of X and Y are in-tight and that $d(X, Y) = 0$. But in this case we can replace X and Y in $\mathcal{L}$ by $X \cap Y$ and $X \cup Y$ and the resulting $\mathcal{L}'$ also consists of in-tight sets for which every edge of D enters one of its members, $|\mathcal{L}'| = |\mathcal{L}|$ and $\sum[|Z|^2 : Z \in \mathcal{L}'] > \sum[|Z|^2 : Z \in \mathcal{L}]$, contradicting the choice of $\mathcal{L}$. •

Let s be an arbitrary element of V. Let $\mathcal{F}_{in} := \{X \subseteq V - s, X \in \mathcal{L}\}$ and let $\mathcal{F}_{out} := \{V - X : s \in X \in \mathcal{L}\}$. Then $\mathcal{F} := \mathcal{F}_{in} \cup \mathcal{F}_{out}$ is a laminar family such that every edge of D

$$\text{enters an in-tight member or leaves an out-tight member of } \mathcal{F}. \tag{8.13}$$

Suppose now that $\mathcal{F}$ is a family satisfying (8.13) and, in addition, that $\sum[|Z| : Z \in \mathcal{F}]$ is as small as possible.

Case 1 *Every member of $\mathcal{F}$ is a singleton.* We are done if there is a node z for which the singleton $\{z\}$ belongs to both $\mathcal{F}_{in}$ and $\mathcal{F}_{out}$ since in this case $\varrho(z) = k = \delta(z)$. So suppose that this is not the case. If an edge f leaves s, then f cannot leave any member of $\mathcal{F}$, so f must enter an in-tight member of $\mathcal{F}$. Therefore the set $Z := \{v : \{v\} \in \mathcal{F}_{in}\}$ is non-empty. But in this case an edge leaving Z violates (8.13).

Case 2 *$\mathcal{F}$ has a non-singleton member.* Let Z be a minimal non-singleton member of $\mathcal{F}$. Let Z_{in} consist of those nodes z of Z for which $\{z\} \in \mathcal{F}_{in}$ and let Z_{out} consist of those nodes z of Z for which $\{z\} \in \mathcal{F}_{out}$. By symmetry, we can assume that Z is in-tight. We are obviously done if $Z_{in} \cap Z_{out} \neq \emptyset$, so suppose that $Z_{in} \cap Z_{out} = \emptyset$.

We claim that Z induces a strongly connected subgraph. For if this is not the case, then there is a proper non-empty subset X of Z which is not entered by any edge leaving $Z - X$. In this case $k \leq \varrho(X) \leq \varrho(Z) = k$ from which $\varrho(X) = k$ and hence $\mathcal{F} - \{Z\} + \{X\}$ would also be a system with the properties of $\mathcal{F}$, contradicting the minimality of $\sum[|Z| : Z \in \mathcal{F}]$.

We cannot have $Z_{in} = Z$ since then Z could simply be removed from $\mathcal{F}$. In fact, Z_{in} must be empty for otherwise the strong connectivity of the digraph induced by Z implies

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 265 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

that there is an edge $uv \in A$ with $u \in Z_{in}$ and $v \in Z - Z_{in}$ but such an edge would violate (8.13).

Every node $u \in Z$ must be in Z_{out} for otherwise any edge uv with $v \in Z$ violates (8.13). Therefore $Z_{out} = Z$. But then $k = \varrho(Z) = \ddot{\varrho}(Z) - i_D(Z) \geq k|Z| - i_D(Z) = \ddot{\delta}(Z) - i_D(Z) = \delta(Z) \geq k$, and hence each inequality must be satisfied by equality from which $\varrho(v) = \delta(v) = k$ for every node v of Z (where $i_D(Z)$ denotes the number of edges of D induced by Z). • •

Problem 8.2.6 *Show that under the hypothesis of the preceding theorem there are actually two nodes with the required property.*

Exercise 8.2.7 *Prove that the operation of pinching k existing edges together with a new node z preserves k-edge-connectivity of a digraph.*

We are now in a position to exihibit a constructive characterization of k-edge-connected digraphs. The result is a straight generalization of Theorem 2.2.1 concerning strongly connected digraphs.

Theorem 8.2.14 (Mader [280]) *A digraph $D = (V, A)$ is k-edge-connected if and only if it can be built up from an initial node by consecutively applying the following two operations.*

(A) *Add a new directed edge connecting two existing nodes.*
(B) *Pinch k existing edges together with a new node z.*

Proof. To prove the non-trivial direction, we proceed by induction on the number of edges. There is nothing to prove when $|V| = 1$ so suppose that $|V| \geq 2$. By assumption, D is k-edge-connected. If there is an edge e for which $D' := D - e$ is k-edge-connected, then by induction D' has the required construction and then e can be added back with Operation (A), obtaining in this way D.

Suppose now that D is a minimally k-edge-connected digraph. By Theorem 8.2.12, there is a node z with $\delta(z) = \varrho(z) = k$. By Theorem 8.2.3, there is a complete splitting at z such that the resulting digraph D' is k-edge-connected. By induction, D' can be built up in the required way and D can be obtained from D' by the Operation (B). •

Conjecture 8.2.15 (Mader) *A minimally k-node-connected digraph with at least 2 nodes has a node v with $\varrho_D(v) = k = \delta_D(v)$.* (Mader proved this for $k = 2$.)

8.3 Further constructive characterizations

In the previous section, we demonstrated how the splitting-off operation could be used to provide constructive characterizations of graph classes with specified connectivity properties. Here we describe further characterizations for some other properties.

8.3.1 (k, ℓ)-edge-connected digraphs

We have seen in Theorems 8.2.11 and 8.2.14 how rooted and global k-edge-connected digraphs can be built up from a node. It is tempting to try to find a common generalization of these results for the construction of (k, ℓ)-edge-connected digraphs. Recall (p. 26) that a digraph is by definition (k, ℓ)-edge-connected for given integers $0 \le \ell \le k$ if it has a node r_0 such that the in-degree and the out-degree of every non-empty subset $X \subseteq V - r_0$ is at least k and ℓ, respectively.

Below we shall formulate the general result, found by Kovács and Végh [245], but its proof is quite complex and we do not involve it. Beside the special cases when $\ell = 0$ (rooted k-edge-connectivity) and when $\ell = k$ (global k-edge-connectivity), we shall discuss the next extreme, when $\ell = 1$. The approach of using the splitting-off technique is the same as in the special case when $\ell = k$ but there is a significant difference. In the proof of Theorem 8.2.14, for example, after proving that a minimally k-edge-connected digraph always has a node of in-degree and out-degree k, it would have sufficed to show that among these nodes there is *at least one* that admits a complete splitting. It was a lucky fact that actually *any* of these nodes have a complete splitting. The proof of the theorems below is more complicated since finding the right node where an appropriate splitting can be carried out requires special care.

$(k, 1)$*-edge-connected digraphs*

When $k = 1$, $(k, 1)$-edge-connectivity is just strong connectivity, and then the ear-decomposition theorem of strongly connected digraphs (Theorem 2.2.1) provides the requested constructive characterization. Therefore we can assume that $k \ge 2$. The next result was proved by Frank and Szegő [150].

Theorem 8.3.1 *Given an integer $k \ge 2$, a digraph $D = (V, A)$ is $(k, 1)$-edge-connected with respect to a root-node r_0 if and only if D can be built up from r_0 by consecutively applying the following operations.*

(A) *Add a new directed edge connecting two existing nodes.*
(B) *Pinch j existing edges $(1 \le j \le k - 1)$ together with a new node z and add $k - j$* (possibly parallel) *edges entering z for which their tails are existing nodes.* •

Proof. By definition, a digraph D_0 is $(k, 1)$-edge-connected with respect to a node r_0 if

$$\varrho_{D_0}(X) \ge k \text{ for every non-empty } X \subseteq V - r_0 \tag{8.14}$$

and

$$\delta_{D_0}(X) \ge 1 \text{ for every non-empty } X \subseteq V - r_0. \tag{8.15}$$

This is equivalent to requiring that D is strongly connected and rooted k-edge-connected.

Lemma 8.3.2 *Let $D = (V, A)$ be a $(k, 1)$-edge-connected digraph which is minimal with respect to edge-deletion* (where $k \ge 2$, $|V| \ge 2$). *Then there is a node $z \in V - r_0$ for which $k = \varrho_D(z) > \delta_D(z)$ and for which that it is possible to pair the $\delta_D(z)$ edges leaving z with $\delta_D(z)$ appropriate edges entering z in such a way that the digraph $D' = (V - z, A')$ obtained from D by splitting off these $\delta_D(z)$ pairs and deleting z is $(k, 1)$-edge-connected.*

Proof. We need some simple observations.

Claim 1 *There is a node $z \in V - r_0$ with $\varrho_D(z) > \delta_D(z)$.*

Proof. By (8.15), there is an edge e entering r_0. Then (8.14) holds for $D_0 := D - e$ and hence the minimality of D implies that e leaves a set $X \subset V - r_0$ for which $\delta_D(X) = 1$. Since $\varrho_D(X) \geq k \geq 2$, there must be a node z in X for which $\varrho_D(z) > \delta_D(z)$. •

Select a node $z \in V - r_0$ in such a way that $\varrho_D(z) > \delta_D(z)$ and the distance of r_0 from z is as large as possible. (Here the distance is the number of edges of a shortest zr_0-path in D.)

Claim 2 *Let $F \subseteq A$ be an arbitrary subset of at most $k - 1$ edges entering z. Then $D_0 := D - F$ satisfies* (8.15).

Proof. Suppose indirectly that there is a non-empty set $X \subseteq V - r_0$ for which $\delta_{D-F}(X) = 0$. Since $\delta_D(X) \geq 1$, all the edges of D leaving X must belong to F and hence $\delta_D(X) \leq |F| < k$. Since $\varrho_D(X) \geq k$, the subset X must contain a node z' for which $\varrho_D(z') > \delta_D(z')$. Since the head of all edges leaving X is z, each $z'r_0$-path must go through z, contradicting the selection rule of z. •

Claim 3 $\varrho_D(z) = k$.

Proof. By applying Claim 2 to a singleton $F = \{e\}$, we obtain that $D_0 := D - e$ satisfies (8.15) for each edge e entering z.

Hence the minimality of D implies that each edge entering z must enter a subset $X \subseteq V - r_0$ for which $\varrho_D(X) = k$. Such a set is said to be **in-tight**. If X and Y are two in-tight sets containing z, then $k + k = \varrho_D(X) + \varrho_D(Y) \geq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \geq k + k$, from which $\varrho_D(X \cap Y) = k$, and hence there is a unique smallest in-tight set Z containing z. Moreover every edge entering z must enter Z and thus $\varrho_D(z) = k$. •

By Theorem 8.2.4, it is possible to pair the $\delta_D(z)$ edges leaving z with $\delta_D(z)$ edges entering z in such a way that the digraph $D' = (V', A')$ obtained from D by splitting off these $\delta_D(z)$ pairs and deleting z is $(k, 0)$-edge-connected. We are going to show that D' is automatically $(k, 1)$-edge-connected. Suppose indirectly that there is a non-empty subset $X \subseteq V' - r_0$ for which $\delta_{D'}(X) = 0$. It follows that every edge of D leaving X must enter z and hence every $z'r_0$-path ($z' \in X$) must go through z. Therefore $\delta_D(X)$ is at least k since $\delta_D(X) < k$ and $\varrho_D(X) \geq k$ would imply the existence of a node $z' \in X$ for which $\delta_D(z') < \varrho_D(z')$, contradicting the selection rule of z. It follows that $\delta_D(X) = k$ and that the k edges of D leaving X are those entering z. This in turn implies that all the edges of D leaving z must enter X for otherwise at least one of the split-off edges of D' would leave X. But then we have $\delta_D(X + z) = 0$, contradicting (8.15), and the proof of the lemma is complete. • •

Returning to the proof of the theorem, observe first that the two operations clearly preserve $(k, 1)$-edge-connectivity. We prove the converse by induction on the number of edges. If there is an edge e of D such that $D - e$ is $(k, 1)$-edge-connected, then by induction $D - e$ admits the required construction. By adding back the edge e with Operation (A), we obtain the construction of D.

Therefore we can assume that D is minimally $(k, 1)$-edge-connected with respect to edge-deletion. We are done if $|V| = 1$, so assume that $|V| \geq 2$. Consider the node z and the digraph D' ensured by Lemma 8.3.2. By induction, D' can be built from r_0 by the given operations and hence D is obtained from D' by Operation (B) where $j = \delta_D(z)$. • • •

The general case

Frank and Király [150] proved a constructive characterization for $\ell = k-1$, while the general case is settled by Kovács and Végh [245].

Theorem 8.3.3 (Kovács and Végh) *Let $0 \leq \ell \leq k-1$ be integers. A digraph D is (k,ℓ)-edge-connected with respect to a root-node r_0 if and only if D can be built up from r_0 by consecutively applying the following operations.*

(A) *Add a new directed edge connecting two existing nodes.*
(B) *Pinch j existing edges ($\ell \leq j \leq k-1$) together with a new node z and add $k-j$* (possibly parallel) *edges entering z for which their tails are existing nodes.* •

Note that Theorems 8.2.11 and 8.3.1 are special cases of this theorem for $\ell = 0$ and $\ell = 1$, respectively. It is a simple exercise to see that the two operations in Theorem 8.3.3 preserve (k,ℓ)-edge-connectivity of the digraph. The proof of converse direction, however, is difficult and we omit it.

Problem 8.3.1 *Let D be a (k,ℓ)-edge-connected digraph with respect to a root-node r_0 and suppose that D is minimal in the sense that the removal of any edge destroys (k,ℓ)-edge-connectivity. Prove that if $|V| \geq 2$, then D has a node $v \in V - r_0$ for which $k = \varrho_D(v) \geq \delta_D(v)$.*

8.3.2 Rigid graphs

Let $G = (V, E)$ be a loopless undirected graph. Place its nodes in the plane in general position (which means that no algebraic dependence occurs among the coordinates) and represent each edge by a rigid bar which can rotate around its end-nodes (in the plane). G is said to be **generically rigid** (in the plane), or for our purposes just rigid, if the physical framework obtained in this way is rigid. The graph is minimally rigid if it is rigid but removing any of its edges destroys rigidity. This is definition is merely intuitive but it can be made precise by requiring the rank of a certain matrix of indeterminates to be $2|V| - 3$.

Remark Based on the formal definition, it can be shown that rigidity is a property of the graph and does not depend on the concrete embedding. However the assumption made about the general position of the nodes is essential. The complete bipartite graph $K_{3,3}$, for example, is rigid but if we build a plane structure consisting of a regular polygon of 6 sides along with the three main diagonals, then the resulting structure, which is an embedding of $K_{3,3}$, is physically not rigid.

For our present purposes, however, we do not need the exact definition of rigidity since the following characterization of Laman [254] translates it into pure graph-theoretic properties.

Theorem 8.3.4 (Laman) *A loopless undirected graph $G = (V, E)$ with n nodes is minimally rigid if and only if*

$$|E| = 2n - 3 \tag{8.16}$$

and

$$i_G(Z) \leq 2|Z| - 3 \text{ for every } Z \subseteq V,\ |Z| \geq 2 \tag{8.17}$$

where $i_G(Z)$ denotes the number of edges induced by Z.

This result is a co-**NP** characterization of minimal rigidity: if the graph is not minimally rigid, there is an easily verifiable certificate (a subset Z violating (8.16) or (8.17)). The next theorem of Henneberg [206] provides an **NP**-characterization by describing an easily verifiable certificate for a graph to be minimally rigid.

Theorem 8.3.5 (Henneberg) *A graph $G = (V, E)$ is minimally rigid if and only if it can be built from the graph K_2 by a sequence of the following operations.*

(A) *Add a new node and connect it with two distinct existing nodes.*
(B) *Subdivide an existing edge $e = s_1 s_2$ with a new node z and connect z with an existing node s_3 which is distinct from s_1 and s_2.*

Proof. Suppose first that G arises in this way. By induction it is evident that G satisfies (8.16) and one can check readily that both operations preserve (8.17).

Conversely, suppose that these two properties hold for G. There is nothing to prove when $|V| = 2$ so assume that $|V| \geq 3$. Let b be a set-function on V defined by $b(X) := 2|X| - 3 - i_G(X)$. Note that $b(\{v\}) = -1$ for singletons and that (8.17) is equivalent to requiring that $b(Z) \geq 0$ whenever $|Z| \geq 2$. Such a set Z is said to be **tight** if $b(Z) = 0$. Identity (1.6) implies that b is submodular.

Proposition 8.3.6 *If X and Y are tight sets for which $|X \cap Y| \geq 2$, then $X \cup Y$ is tight. If $X_1,\ X_2,\ X_3$ are tight sets for which $X_1 \cap X_2 \cap X_3 = \emptyset$ and $|X_i \cap X_j| = 1$ for $1 \leq i < j \leq 3$, then $X_1 \cup X_2 \cup X_3$ is tight.*

Proof. The first part follows from the submodularity of b. For the second part we have

$$0 \leq b(\cup X_j) = 2|\cup X_j| - 3 - i_G(\cup X_j) \leq$$

$$2\left(\sum |X_j| - 3\right) - 3 - \sum i_G(X_j) = \sum [2|X_j| - 3 - i_G(X_j)] = \sum b(X_j) = 0$$

from which $b(X_1 \cup X_2 \cup X_3) = 0$. •

There is a node z of degree at most 3 for otherwise there would be at least $4|V|/2$ edges. The degree of z cannot be 1 since then for $Z := V - z$ we would have $i_G(Z) = |E| - 1 = 2|V| - 4 = 2|Z| - 2$ contradicting (8.17). If $d_G(z) = 2$, then $G' := G - z$ satisfies (8.16) and (8.17). By induction G' can be constructed in the required way and G is then obtained from G' by Operation (A).

Suppose now that $d_G(z) = 3$ and let s_1, s_2, s_3 be the three neighbours of z. There is no tight set Z' for which $\{s_1, s_2, s_3\} \subseteq Z' \subseteq V - z$ since then $Z := Z' + z$ would violate (8.17). This and Proposition 8.3.6 imply that there are two neighbours of z, say s_1 and s_2, such that there is no tight set Z for which $\{s_1, s_2\} \subseteq Z \subseteq V - z$. Therefore the graph G' arising from $G - z$ by adding a new edge $e = s_1 s_2$ satisfies (8.16) and (8.17). By induction, G' can be constructed in the required way and G is then obtained from G' by Operation (B). • •

Problem 8.3.2 *Develop a necessary and sufficient condition for a graph to be rigid.*

This problem shall be answered in Section 13.5 with the help of matroids (see Theorem 13.5.2). It will turn out that the minimal rigid subgraphs of a rigid graph not only have the same number of edges but also form the set of bases of a matroid, called the rigidity matroid. At this point, it is a nice challenge for the reader to formulate just the characterization even without a proof.

Research problem 8.3.2 *When and how is it possible to destroy rigidity of a graph by deleting at most γ edges?*

Strictly k-sparse graphs

Let $k \geq 2$ be an integer and $G = (V, E)$ a loopless undirected graph. Wa say that a subset $Z \subseteq V$ of nodes is **strictly k-sparse** if

$$i_G(X) \leq k(|X| - 1) - 1 \tag{8.18}$$

holds for every subset X of Z with $|X| \geq 2$. The graph G itself is also said to be **strictly k-sparse** if V is strictly k-sparse. Evidently, a graph G is rigid if and only if G is strictly 2-sparse and has exactly $2(|V| - 3)$ edges. Therefore the following result of Frank and Szegő [150] is a plain extension of Theorem 8.3.5. Its proof, however, is significantly more difficult and we omit it.

Theorem 8.3.7 *A loopless graph $G = (V, E)$ is strictly k-sparse $(k \geq 2)$ if and only if it can be built up from an initial node by the following operations.*

(A) *Add a new node and at most k new* (possibly parallel) *edges ending at z in such a way that the resulting graph icludes no set of k parallel edges.*
(B) *Pinch j existing edges $(1 \leq j \leq k - 1)$ together with a new node z and add $k - j$* (possibly parallel) *new edges connecting z with other nodes in such a way that the resulting graph icludes no set of k parallel edges.* •

A useless characterization

It is important to realize that all the proofs above for constructive characterizations give rise to polynomial algorithms by which we can actually build up the graph or digraph with the given property. We close this section by noting that a constructive characterization in itself can be useless. This is demonstrated by the following problem.

Problem 8.3.4 (∗) *Prove that an undirected graph G admits a Hamilton circuit if and only if G can be obtained from a 2-element circuit by the following operations.*

(A) *Add an edge connecting two existing nodes.*
(B) *Subdivide an existing edge incident to a node of degree 2.*

This characterization, though formally pretty similar to that of 2-edge-connected graphs, is hardly more than a rephrasing of the Hamiltonian definition of a graph. Without knowing a priori a Hamilton circuit of G, there is no way to find the construction of G in the given form.

8.3.3 Rooted 2-connected digraphs

No general constructive characterization is known for k-node-connected digraphs or for rooted k-node-connected digraphs. In Section 2.1, we proved the open ear-decomposition theorem for 2-connected graphs (Theorem 2.1.10). Here we describe characterizations for rooted 2-node-connected digraphs. Let $D = (V, A)$ be a digraph with a two-element root-set $R = \{r_1, r_2\} \subseteq V$. We say that D is 2-connected from R if

$$|\{u : \text{there is a } uv\text{-edge entering } X\}| \geq 2 \text{ for every non-empty subset } X \subseteq V - R.$$

By Menger's theorem this is equivalent to requiring that

$$D \text{ has two dipaths from } R \text{ to every } v \in V - R \text{ that are disjoint in } V - v. \tag{8.19}$$

The following result can be considered as a directed counterpart of Theorem 2.1.8 concerning 2-connected undirected graphs.

Theorem 8.3.8 (Whitty [385]) *A digraph $D = (V, A)$ is 2-connected from a root-set $R = \{r_1, r_2\} \subseteq V$ if and only if there is an ordering $r_1 = v_1, v_2, \ldots, v_n = r_2$ of the nodes of D such that, for each node $v_i \in V - R$, there is an edge $v_h v_i$ with $h < i$ and an edge $v_j v_i$ with $i < j$.*

Proof. If the ordering in question exists, then (8.19) obviously holds. The proof of sufficiency proceeds by induction on $|V|$. There is nothing to prove if $V = \{r_1, r_2\}$ so we assume that $|V| \geq 3$.

Lemma 8.3.9 *There is an edge $e = r_1 u \in A$ $(u \in V - \{r_1, r_2\})$ the contraction of which preserves* (8.19) *in the sense that*

$$D' \text{ has two dipaths from } R' = \left\{r_1', r_2\right\} \text{ to every } v \in V' - R' \text{ that are disjoint in } V - v. \tag{8.20}$$

(Here $D' = (V', A')$ denotes the digraph obtained from D by contracting e, and r_1' denotes the node of D' arising from the contraction of e.)

Proof. In the subgraph $D - r_1$, let F be a spanning arborescence of root r_2. Let $e = r_1 u$ be an edge of D for which u is as far from r_2 in F as possible. We claim that e will suffice.

By the directed node-version of Menger's theorem, if (8.20) fails to hold for some $v \in V' - R'$, then there exists a subset $X \subseteq V' - R'$ such that the tail of every edge of D entering X is either r_1 or u. Moreover (8.19) actually implies that both types of edges must occur. Let $r_1 u'$ be an edge of D entering X. Since every node of X can only be reachable from r_2 (in $D - r_1$) through u, the distance of u' from r_2 in F is larger than that of u, contradicting the maximal choice of u. •

Returning to the proof of the theorem, let D' denote the digraph obtained from D by contracting the edge $e = r_1 u$ ensured by the lemma and let r_1' denote the contracted node. By induction, there is a requested ordering of the nodes of D', in which r_1' is the first node. By replacing r_1' with r_1 and u (in this order) we obtain an ordering of the nodes of D which satisfies the requirements of the theorem. • •

Corollary 8.3.10 *Let $D = (V, A)$ be a digraph which is 2-connected from $R = \{r_1, r_2\} \subseteq V$ and suppose that D is minimal in the sense that the deletion of any edge destroys* (8.19). *Then $\varrho_D(u) = 2$ for every $u \in V - R$ and $\varrho_D(r_1) = \varrho_D(r_2) = 0$.*

Proof. The second statement is evident since an edge entering r_1 or r_2 has nothing to do with (8.19). Consider the ordering of nodes ensured by Theorem 8.3.8. If indirectly $\varrho(u) \geq 3$ for a node $u \in V - R$, then there are two edges entering u such that both of their tails precede u or both follow u. Therefore if we remove any of these two edges, the ordering continues to satisfy the properties described in Theorem 8.3.8, contradicting the minimality of D. •

Let $z \in V - R$ be a node for which

$$\varrho_D(z) = 2 \text{ and } \delta_D(z) = 1, \tag{8.21}$$

and let $f = zv$ denote the only edge leaving z while $e_1 = u_1z$ and $e_2 = u_2z$ are the two edges entering z. Splitting off e_1 and f means the operation that removes z and adds a u_1v-edge to $D - z$.

Lemma 8.3.11 *Let $D = (V, A)$ be a digraph which is 2-connected from $R = \{r_1, r_2\} \subseteq V$ and let $z \in V - R$ be a node with $\varrho_D(z) = 2$ and $\delta_D(z) = 1$. Then there is a splitting at z that preserves* (8.19).

Proof. Consider the ordering of the nodes of D ensured by Theorem 8.3.8. By symmetry we can assume that u_1 precedes z while u_2 and v are after z. Then splitting off e_1 and f preserves the property of the ordering, so the resulting digraph fulfils (8.19). •

Theorem 8.3.12 *A digraph $D = (V, A)$ is 2-connected from $R = \{r_1, r_2\}$ if and only if D can be obtained from R by the following operations.*

(A) *Add a new edge joining existing nodes.*
(B) *Add a new node z and two edges entering z for which their tails are distinct existing nodes.*
(C) *Subdivide an existing uv-edge by a node z and add an edge entering z for which its tail is distinct from u.*

Proof. It is a simple exercise to check that the given operations preserve (8.19). To see the converse, we can assume that D has at least three nodes. By induction it suffices to prove that there is a digraph D' which is 2-connected from R such that D arises from D' by applying one of the given operations.

If there is an edge e such that $D' = D - e$ is 2-connected from R, then D arises from D' by (A) and we are done. Therefore we can assume that D is minimal which implies that no edge enters r_1 and r_2, and hence there must be a node $z \in V - R$ for which $2 = \varrho_D(z) > \delta_D(z)$.

If $\delta_D(z) = 0$, then $D' := D - z$ is 2-connected from R and D can be obtained from D' by Operation (B). Suppose now that $\delta_D(z) = 1$. By Lemma 8.3.11 it is possible to remove z and add a new edge u_1v or u_2v in such a way that the resulting digraph D' on $V - z$ is 2-connected from R. Therefore D can be obtained from D' by Operation (C). •

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 273 of 640 December 11, 2010 15:18

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Detaching a node r_0 (of in-degree zero) into two nodes r_1 and r_2 means that we replace r_0 by r_1 and r_2 and replace each r_0v-edge by either r_1v or r_2v.

Theorem 8.3.13 *Let $D = (V, A)$ be a rooted 2-node-connected digraph with respect to a root-node r_0. Then r_0 can be detached into two nodes r_1 and r_2 in such a way that* (8.19) *holds.*

Proof. Subdivide each edge e leaving r_0 by a new node, and let V_0 denote the set of subdividing nodes. Furthermore, replace r_0 by a set $R = \{r_1, r_2\}$ of nodes and replace each r_0v-edge by the two edges r_1v and $r_2v\}$ for every $v \in V_0$. The resulting digraph is rooted 2-connected in which the in-degree of each subdividing node is 2 and the out-degree is 1. By applying Lemma 8.3.11 individually to each subdividing node, we obtain the requested detachment of r_0. •

Theorem 8.3.14 *A digraph $D = (V, A)$ is rooted 2-connected with respect to a root-node r_0 if and only if it can be obtained from r_0 by the following operations.*

(A) *Add a new edge joining existing nodes.*
(B) *Add a new node z and two edges entering z for which their tails are distinct existing nodes.*
(B') *Add a new node z and two parallel r_0z-edges.*
(C) *Subdivide an existing uv-edge by a node z and add an edge entering z for which its tail is distinct from u.*
(C') *Subdivide an existing r_0v-edge by a node z and add an edge entering z for which its tail is any existing node.*

Proof. Again, the operations preserve rooted 2-connection. For the sufficiency, apply Theorem 8.3.13 which implies that r_0 can be detached into r_1 and r_2 in such a way that the resulting digraph D_1 is 2-connected from $R = \{r_1, r_2\}$. Apply Theorem 8.3.12 to D_1 and observe that the operations in Theorem 8.3.12 transform to the operations in the present theorem. •

In Section 10.1, we shall derive yet another characterization of rooted 2-connected digraphs in terms of independent arborescences (Theorem 10.1.17).

Detachment of the root of a rooted k-node-connected digraph

Though it will not be used later, we derive the following pretty generalization of Theorem 8.3.13. Detaching a root-node r_0 into a set $R = \{r_1, \dots, r_k\}$ of k nodes means that we replace r_0 by R and replace each r_0v-edge by one of $r_1v, \dots, r_kv$.

Theorem 8.3.15 *Let $D = (V, A)$ be a rooted k-node-connected digraph with respect to a root-node r_0. Then r_0 can be detached into a k-element node-set R in such a way that the resulting digraph is k-connected from R, meaning that there are r_iu-paths $(i = 1, \dots, k)$ for every node $u \in V - r_0$ which are disjoint in $V - u$.*

We shall prove this result in the following equivalent form.

Theorem 8.3.16 *Let $D = (V, A)$ be a rooted k-node-connected digraph with respect to a root-node r_0. The set A_0 of edges leaving r_0 can be coloured by k colours in such a way* *that there are k openly disjoint r_0u-paths for every node $u \in V - r_0$ for which the k starting edges have different colours.*

Proof. Let $\{1, \dots, k\}$ denote the set of possible colours. By a partial colouring of A_0 we mean a subset C of A_0 along with a colouring of C. The partial colouring is **full** or **empty** if $C = A_0$ or $C = \emptyset$, respectively. A partial colouring is said to be **good** if there are k openly disjoint r_0u-paths for every node $u \in V - r_0$ such that each colour is used by at most one of these paths. Our goal is to show that there is a good full colouring.

For a partial colouring of A_0, we say that a **colour** i **enters** a bi-set X if there is an edge of colour i entering X. The number of distinct colours entering X will be denoted by $\varrho^c(X)$. The following characterization is a consequence of Menger's theorem.

Lemma 8.3.17 *A partial colouring of A_0 is good if and only if*

$$\varrho'(X) + w(X) + \varrho^c(X) \geq k \text{ for every non-trivial bi-set } X = (X_O, X_I) \text{ with } X_O \subseteq V - r_0 \tag{8.22}$$

where $\varrho'(X)$ denotes the number of uncoloured edges entering X and $w(X) = |X_O - X_I|$. •

Since D is rooted k-node-connected, the empty colouring of A_0 (meaning that no element of A_0 is coloured) is good. Suppose that we have a good partial colouring and e is an uncoloured r_0v-edge. We prove that e can be coloured such that the resulting partial colouring continues to be good. We say that a bi-set X is **tight** (with respect to the given partial colouring) if it meets (8.22) with equality.

Claim 8.3.18 *If X and Y are intersecting tight bi-sets, then $X \sqcup Y$ is also tight.*

Proof. It can readily be checked that ϱ^c is submodular on bi-sets. Since ϱ' is also submodular and w is modular, the standard submodular technique shows that $X \sqcup Y$ is tight. •

Suppose now that assigning a colour i to e destroys (8.22). This implies that there is a bi-set X such that, before colouring e, the bi-set X is tight, the colour i enters X, and e enters X.

Therefore if there is no tight bi-set X with $v \in X_I$, then e can be coloured arbitrarily. Suppose now that there is such a tight set. By the claim, there is a unique largest tight set $Z = (Z_O, Z_I)$ for which $v \in Z_I$.

We cannot have $\varrho^c(Z) = k$ since this and the tightness of Z would imply $\varrho'(Z) = 0$ contradicting the fact that e is an uncoloured edge entering Z. Let i be a colour not entering Z. We claim that edge e can be coloured by i without destroying (8.22), for otherwise there is a tight bi-set X for which $v \in X_I$ and that colour i enters X. By the maximality of Z, we have $X \sqsubseteq Z$. But this implies that colour i enters Z as well in a contradiction to our assumption. • •

It can be shown that Theorem 8.3.16 also follows from the main result of the paper of Wu, Jain, and Kung [387]. The following extension can also be proved with similar techniques.

Theorem 8.3.19 *Let $D = (V, A)$ be a digraph, r_0 a specified root-node, and let A_0 denote the set of edges leaving r_0. Furthermore, let $T \subseteq V - r_0$ be a set of terminals such that $\kappa_D(r_0, t) \geq k$ holds for every $t \in T$ and $(*)$ T contains the head of each element of A_0.*

Then the elements of A_0 can be coloured by k colours such that there are k openly disjoint r_0t-paths for every $t \in T$ for which the k starting edges have different colours.

Problem 8.3.5 *Prove Theorem* 8.3.19. *Show by an example that the assumption* $(*)$ *cannot be left out.*

8.3.4 3-connected undirected graphs

For 2-connected graphs, we proved the open ear-decomposition theorem in Section 2.1 (Theorem 2.1.10). Here we describe characterizations for 3-node-connected undirected graphs. We shall use the operation of contracting an edge, which always means that the arising loops are deleted and only one copy of the resulting parallel edges is kept. This implies that the graph obtained by contracting an edge is simple. Recall that a connected undirected graph G is said to be 3-connected if it has at least four nodes and it is not possible to disconnect G by deleting at most two of its nodes. Menger's theorem implies that there are 3 openly disjoint uv-paths for every pair $\{u, v\}$ of nodes. An edge e of a 3-connected graph G is **contractible** if its contraction results in a 3-connected graph.

Lemma 8.3.20 (Tutte [369]) *A 3-connected graph $G = (V, E)$ with at least five nodes has a contractible edge.*

Proof. (Thomassen [366]) If an edge xy is not contractible, then there is a node z such that $G - \{x, y, z\}$ is disconnected. Suppose indirectly that no edge of G is contractible and select a tree-element cut-set $\{x, y, z\}$ of G in such a way that xy is an edge of G and the largest component K of $G' := G - \{x, y, z\}$ is as large as possible.

Claim 8.3.21 *The subgraph G_1 of G induced by the node-set $K \cup \{x, y\}$ is 2-connected.*

Proof. Since G is 3-connected and $\{x, y, z\}$ is a cut-set of G, there must be an edge between x and K and an edge between y and K. Therefore G_1 is certainly connected and so are both $G_1 - x$ and $G_1 - y$. It follows that a cut node t of G_1 must be distinct from x and y. Since xy is an edge, the nodes x and y must belong to the same component of $G_1 - t$. But then every path of G between another component of $G_1 - t$ and x must use z or t, contradicting the 3-connectivity of G. •

Let K' be another component of G'. Since G is 3-connected, there is an edge uz for some $u \in K'$. Since the contraction of uz destroys 3-connectivity, there is a node v for which $G'' = G - \{u, v, z\}$ is disconnected. Let K'' denote a component of G'' which intersects $K \cup \{x, y\}$. By Claim 8.3.21, $K \cup \{x, y\} - \{v\}$ induces a connected graph and it completely belongs to K'' and this contradicts the maximal choice of K. • •

Theorem 8.3.22 (Tutte [369]) *A graph $G = (V, E)$ is 3-connected if and only if it can be obtained from K_4* (the complete graph on four nodes) *by the following operations.*

(A) *Add a parallel copy of an existing edge.*
(B) *Select a node z of degree at least four. Partition the set of edges incident to z into two groups F_1 and F_2 in such a way that $|F_i| \geq 2$ and F_i has two non-parallel edges. Replace z by two nodes z_1 and z_2, connect them by an edge, and replace each element uz of F_i by an edge uz_i for $i = 1, 2$.*

Proof. It is a simple exercise to check that both operations preserve 3-connectivity. The converse is a direct consequence of Lemma 8.3.20. •

There is another interesting way to build up 3-connected graphs. The next result is also due to Tutte [372]. We say that a graph G is **subdivided 3-connected** if it can be obtained from a 3-connected graph by subdividing some of its edges. Define an edge e as **removable** (from G) if $G - e$ is subdivided 3-connected. Note that if G is 3-connected and $G - e$ is subdivided 3-connected, then only the end-nodes of e can be of degree 2 in $G - e$.

Theorem 8.3.23 *Let $e = st$ be an edge of a 3-connected graph $G = (V, E)$ with at least five nodes. Then e is contractible or removable, that is, G/e is 3-connected or $G - e$ is subdivided 3-connected.*

Proof. Suppose that G/e is not 3-connected. Then there is a node z for which $G' = G - \{s, t, z\}$ is disconnected. We are going to show that there are 3 openly disjoint uv-paths in $G - e$ for every pair $\{u, v\}$ of nodes for which $\{u, v\} \cap \{s, t\} = \emptyset$.

Since G is 3-connected, it certainly includes 3 openly disjoint uv-paths, denoted by P_1, P_2, P_3. We are done if none of these paths passes through e, so assume that P_1 contains e. It is not possible that u and v belong to distinct components of G' since P_1 uses two members of the cut set $\{s, t, z\}$ hence both P_2 and P_3 should use the third member. Clearly at least one of u and v, say u, does not belong to $\{s, t, z\}$. Let K denote the component of G' containing u. Then either v is in K or $v = z$. In both cases the inner nodes of P_2 and P_3 are completely in K, as P_1 contains both s and t. Let K' denote another component of G'.

Since G is 3-connected, there must be an edge from K' both to s and to t. Therefore there is an st-path P' (not using e) for which the inner nodes belong to K'. We can then replace the edge e of path P_1 by path P'. The resulting uv-walk can be simplified to a uv-path P_1' which avoids e and is openly disjoint from P_2 and P_3. •

Theorem 8.3.24 (Barnette and Grünbaum [13]) *A 3-connected graph $G = (V, E)$ with at least 5 nodes has an edge e for which $G - e$ is subdivided 3-connected.*

Proof. We proceed by induction on the number of nodes. If an edge e is not removable, then there are nodes x and y for which e is such a cut-edge in $G - \{x, y\}$ that both components of $G - \{x, y\} - e$ have at least two nodes. Hence we can assume that G has at least 6 nodes. We can suppose that G is simple. If, indirectly, there is no removable edge, then Theorem 8.3.23 implies that each edge can be contracted without destroying 3-connectivity.

Let $f = st$ be an arbitrary edge. Since the contracted graph $G' = G/f$ is 3-connected, by induction it has an edge $e = uv$ for which $G' - e$ is subdivided 3-connected. Since e is not removable from G, there are nodes x and y for which $G - \{x, y\} - e$ has two components K_1 and K_2 both having at least two nodes. It is not possible that both K_1 and K_2 have at least 3 nodes, since then e would not be removable from G'. Assume that K_1 has exactly 2 nodes and let u be that end-node of e belonging to K_1. Since $d_G(u) \geq 3$, K_1 has two elements, and because G is simple, we can conclude that there must be at least one edge between u and $\{x, y\}$. But if, for example, xu is an edge of G, then xu is not contractible in G since $\{x, y, u\}$ is a cut set of G contradicting the statement above that every edge of G is contractible. •

A direct consequence of Theorem 8.3.24 is the following constructive characterization of 3-connected graphs.

Theorem 8.3.25 *A simple graph G with at least four nodes is 3-connected if and only if it can be built up from K_4 by the following operations.*

(A) *Add a new edge connecting two existing non-adjacent nodes.*

(B) *Subdivide an existing edge xy with a new node v and add a new edge connecting v and an existing node distinct from x and y.*

(C) *Subdivide two existing edges e end f by nodes u and v, respectively, and add a new edge uv.* •

8.4 Packing paths

In Section 2.5.5 we introduced the edge-disjoint paths problem and formulated the cut criterion, a basic necessary condition of solvability. In the undirected case, the cut criterion requires the cut inequality $d_G(X) \geq d_H(X)$ to hold for every subset X of nodes (see (2.42)). Here $G = (V, E)$ and $H = (V, F)$ denote the supply graph and the demand graph, respectively. The undirected edge-version of Menger's theorem is a special case (in which F is a set of parallel edges) when the cut criterion is sufficient, while Theorem 2.5.32 (where F is two sets of parallel edges and $G + H$ is Eulerian) described another similar situation. In Section 2.5, we already indicated that the splitting-off technique can be used for proving results of these types (see p. 81). In this section, we discuss more applications of the splitting-off technique concerning edge-disjoint paths in undirected graphs.

8.4.1 Paths connecting terminal pairs when $G+H$ is Eulerian

When one pair of edges of G is split off, the degree of a subset of nodes can decrease by two. Therefore the splitting-off operation destroys the cut inequality for a subset X if and only if $\Delta_G(X)$ contains both split-off edges and $d_G(X) = d_H(X)$ or $d_G(X) = d_H(X) + 1$. If the graph $G + H$ is Eulerian, then the latter alternative cannot occur. For this reason, the edge-disjoint paths problem is often easier when the graph $G + H$ is Eulerian. Although the problem remains **NP**-complete in the Eulerian case, there are non-trivial situations in which the cut criterion proves to be sufficient. In this section, we outline some of them.

We make preparations that are useful for each case. In Section 2.5, a circuit of $G + H$ was defined to be good if it contained exactly one demand edge. A partition of the edge-set of $G + H$ into circuits is **good** if each circuit contains at most one demand edge. Note that with this terminology the edge-disjoint paths problem for Eulerian $G + H$ is equivalent to finding a good circuit-partition. Given G and H, we say that a subset X of nodes is **tight** if $d_G(X) = d_H(X)$.

Lemma 8.4.1 *Suppose that $d_G \geq d_H$.*

(A) *If A and B are tight and $d_H(A, B) = 0$, then both $A \cup B$ and $A \cap B$ are tight and $d_G(A, B) = 0$.*

(B) *If A and B are tight and $\overline{d}_H(A, B) = 0$, then both $A - B$ and $B - A$ are tight and $\overline{d}_G(A, B) = 0$.*

Proof. By applying Proposition 1.2.1, we have $d_H(A) + d_H(B) = d_G(A) + d_G(B) = d_G(A \cap B) + d_G(A \cup B) + 2d_G(A, B) \geq d_H(A \cap B) + d_H(A \cup B) + 2d_G(A, B) = d_H(A) + d_H(B) + 2(d_G(A, B) - d_H(A, B))$, from which Part (A) follows. We obtain Part (B) when (A) is applied to A and $V - B$. •

Recall the definition of a bond and its characterization in Proposition 1.1.4.

Lemma 8.4.2 *The cut criterion holds if and only if the cut inequality holds for every bond of G, and hence $d_G(X) \geq d_H(X)$ holds whenever $\Delta_G(X)$ is a bond.*

Proof. The 'only if' part is evident since the cut criterion requires the inequality for every cut. To see the converse, let $B = \Delta_G(X)$ be a smallest cut violating the cut inequality. Then B is not elementary, and hence one of its shores, say X, induces more than one component: $X_1, \ldots, X_t$. Since B partitions into the cuts $\Delta_G(X_i)$, these cuts are smaller than B and hence we have $d_G(X) = \sum_i d_G(X_i) \geq \sum_i d_H(X_i) \geq d_H(X)$, contradicting the assumption. •

Planar supply graphs

By a **terminal node** or simply a **terminal** we mean a node which is a incident to a demand edge.

Theorem 8.4.3 (Okamura and Seymour [311]) *Suppose that G is planar, $G + H$ is Eulerian, and each terminal is on one face of the planar representation of G. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof. We proceed by induction on the number of edges of G. We can assume that G is 2-connected. Consider a given embedding of G into the plane and let C denote the circuit bounding the infinite region. Assume that the subscripts of the nodes $v_1, \ldots, v_h$ of C reflect their cyclic order and that the terminals are on C. Note first of all that the assumption made about the location of terminals implies that every tight cut intersects C and second that the planarity implies that every elementary cut of G can intersect C in at most two edges.

We can assume that every edge of C belongs to a tight cut for if e does not, then by removing e from G and adding it to H we obtain G' and H' which satisfy both the hypotheses of the theorem as well as the cut criterion. By induction there is a good partition of $G' + H'$ into circuits and this is good with respect to $G + H$, as well.

Consider the edge $e = v_h v_1$ of C and let A be a minimal $v_1\overline{v}_h$-set which is tight. Choose a demand edge $f = v_i v_j$ $(i < j)$ for which $v_i \in A, v_j \notin A$, and j is as big as possible. Delete e from G and replace f in H by two demand edges $f' = v_1 v_i$ and $f'' = v_j v_h$.

Claim 8.4.4 *The cut criterion holds for the resulting graphs G' and H'.*

Proof. By Lemma 8.4.2, it suffices to show the cut inequality for a subset $B \subseteq V$ which intersects C in a subpath of C. Suppose indirectly that B violates the cut inequality. Then B must be tight with respect to $G + H$. Moreover, by complementing B if necessary, we can assume that, among the four nodes v_1, v_i, v_j, and v_h, B contains (i) v_1, (ii) v_h, (iii) v_1 and v_h.

Due to the choice of f, we have $d_H(A, B) = 0$ in each case. Part (A) of Lemma 8.4.1 implies that $A \cap B$ is tight and $d_G(A, B) = 0$. But in the cases of (i) and (iii) the tightness of $A \cap B$ contradicts the minimality of A while in the case of (ii) $d_G(A, B) = 0$ contradicts the presence of edge e. •

Since the hypotheses of the theorem also hold for $G' + H'$, there is a requested set $\mathcal{P}'$ of edge-disjont paths in G'. By pasting together the v_1v_i-path P_1 and the v_jv_h-path P_2 in $\mathcal{P}'$ with the edge e, we obtain a v_1v_h-path P of G, and $\mathcal{P} := \mathcal{P}' - \{P_1, P_2\} \cup \{P\}$ forms the requested system of edge-disjoint paths in G. • •

If we drop the restriction that each terminal is on one face, then the edge-disjoint paths problem is **NP**-complete even if the supply graph G is a grid graph, a result proved by Marx [283].

Special demand graphs

Let H' denote the graph obtained from H by replacing each (maximal) set of parallel edges by one edge. We call H' the **pattern** of H. Let us call a graph a **double star** if there are at most two nodes that cover all the edges. As usual, let K_n denote a complete graph on n nodes and C_5 a circuit on 5 nodes. Let $K_2 + K_3$ denote a graph on 5 nodes with components K_2 and K_3. Similarly, $3K_2$ is a graph consisting of three disjoint edges. The case in Theorem 8.4.5, when the pattern H' is a double star is due to Rothschild and Whinston [333]. The theorem for $H' = K_4$ was proved by Lomonosov [260] and by Seymour [345], while the case when $H' = C_5$ is due to Lomonosov [260].

Theorem 8.4.5 *Suppose that $G + H$ is Eulerian and the pattern H' of H is either a double star, or K_4, or C_5. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof. For the special case when the double star H consists of two sets of parallel edges, the theorem was proved in Section 2.5 (see Theorem 2.5.32). By an elementary construction, this implies the theorem for arbitrary double stars. Namely, let s_1 and s_2 denote the two nodes covering the edges of H. First subdivide each demand edge s_1t_i with a new node t_i' in such a way that s_1t_i' belongs to the demand graph while $t_i't_i$ belongs to the supply graph. Second, shrink the new nodes t_i' into one node. Finally, do the same with the demand edges incident to s_2. This way we obtain a new problem which is equivalent to the original one and the demand graph consists of two sets of parallel edges. Furthermore the cut criterion continues to hold.

Turning to the two other cases, let $G + H$ be a counterexample with a minimum number of edges.

Claim 8.4.6 *There are no edges $e \in E$ and $f \in F$ that are parallel.*

Proof. Deleting e and f does not destroy the cut criterion (and the evenness condition) and then a good circuit partition of the reduced graph along with the circuit $\{e, f\}$ would form a good circuit-partition of $G + H$. •

Let z be a non-terminal node and $e = vz$ an edge of G.

Claim 8.4.7 *Let A and B be two distinct maximal tight $v\overline{z}$-sets. Then $\overline{d}_H(A, B) > 0$ and $d_H(A, B) > 0$.*

Proof. By Part (A) of Lemma 8.4.1, if $d_H(A, B) = 0$, then $A \cup B$ is tight contradicting the maximality of A and B. By Part (B) of Lemma 8.4.1, if $\overline{d}_H(A, B) = 0$, then $\overline{d}_G(A, B) = 0$ contradicting the presence of e. •

Let $A_1, A_2, \ldots, A_k$ denote the maximal tight $v\overline{z}$-sets. Suppose that there is an edge $f = zu$ which does not belong to any tight cut. Then splitting off e and f does not destroy the cut criterion and hence the resulting graph has a good circuit partition. But this defines a good circuit partition of $G + H$, which is impossible. Therefore every edge zu enters some A_i.

Claim 8.4.8 $k \geq 3$.

Proof. If $k = 1$, then every neighbour of z is in A_1 in which case $\Delta_G(A_1 + z)$ would violate the cut-inequality. If $k = 2$ and $d_G(z, A_1) \geq d_G(z, A_2)$, for example, then $d_G(A_1 + z) < d_G(A_1)$ (due to the edge e) and therefore $\Delta_G(A_1 + z)$ would violate the cut-inequality. •

Suppose first that the pattern of H is K_4 and let s_1, s_2, s_3, s_4 be the four nodes of K_4. By Claim 8.4.6, no edge of G connects two terminal nodes. If there is no tight set containing exactly s_1 and s_2 among the terminal nodes, then let $e = s_1 z$ be any edge of G incident to s_1. If there is such a tight set, then by Lemma 8.4.1 there is a unique smallest one, denoted by Z. We claim that there is a node $z \in Z - \{s_1, s_2\}$ adjacent with s_1. For otherwise, $d_G(Z) = d_G(s_1) + d_G(Z - s_1) \geq d_H(s_1) + d_H(Z - s_1) = d_H(Z) + 2d_H(s_1, Z - s_1) = d_G(Z) + 2d_H(s_1, Z - s_1) > d_G(Z)$, a contradiction. We conclude that in both cases $e = s_1 z$ is such an edge of G for which there is no tight $s_1\overline{z}$-set containing s_2 and not containing s_3, s_4.

By Claim 8.4.8, there are three maximal tight $z\overline{s}_1$-sets A_1, A_2, A_3. By Claim 8.4.7, $d_H(A_i, A_j) > 0$ and $\overline{d}_H(A_i, A_j) > 0$ for $1 \leq i < j \leq 3$. But this is possible only if each A_i contains a terminal node which is not in the union of the two others. Assume that A_2 contains s_2. Then A_2 is a tight set containing s_1 and s_2 and not containing s_3, s_4, z, a contradiction. Therefore the case $H' = K_4$ is settled.

Suppose now that $H' = C_5$. If $|V| = 5$, then, by Claim 8.4.6, G is a subgraph of a 5-circuit with possible parallel edges. But in this case the theorem of Okamura and Seymour shows that $G + H$ cannot be a counterexample. Therefore $|V| > 5$ and there is an edge $e = zv$ where z is not a terminal node while v is.

By Claim 8.4.8, there are three maximal tight $v\overline{z}$-sets A_1, A_2, A_3. By Claim 8.4.7, we have $(*)$ $d_H(A_i, A_j) > 0$ and $\overline{d}_H(A_i, A_j) > 0$ for $1 \leq i < j \leq 3$. The complement of A_i is also tight so we can assume that there are tight sets B_1, B_2, B_3 (where B_i is either A_i or $\overline{A_i}$) for which $(*)$ holds (with B_h in place of A_h) and each B_i contains exactly two terminals. If $B_1 \cap B_2 \cap B_3$ contains a terminal, then each B_i contains a terminal which is not in the union of the two others. But then $(*)$ implies that these three terminal nodes form a triangle in H (where the nodes are z_1, z_2 and z_3), which is impossible.

Suppose now that $B_1 \cap B_2 \cap B_3$ contains no terminal node. Since $B_1 \cap B_1$ contains a terminal node z_3, it must be in $(B_1 \cap B_2) - B_3$. Analogously, there is a terminal node z_1 in $(B_2 \cap B_3) - B_1$, and there is a terminal node z_2 in $(B_1 \cap B_3) - B_2$. Then the remaining three terminal nodes must be outside $B_1 \cup B_2 \cup B_3$, and hence there is again a triangle in H, a contradiction. • •

8.4.2 *T*-paths

There is another type of interesting path packing problems in which not the terminal pairs to be connected are specified. For a terminal set T, we say that a path is a T**-path** if it connects two (distinct) nodes of T. We say that the pair (G, T) is **inner Eulerian** if $d_G(v)$ is even for every node $v \in V - T$.

Theorem 8.4.9 (Lovász [269] and Cherkassky [49]) *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ a set of terminals so that (G, T) is inner Eulerian. Then the maximum number of edge-disjoint T-paths is equal to*

$$\sum[\lambda_t : t \in T]/2 \tag{8.23}$$

where λ_t denotes the maximum number of edge-disjoint paths connecting t and $T - t$. Equivalently, there is a set $\mathcal{F}$ of edge-disjoint T-paths such that, for each element $t \in T$, the paths in $\mathcal{F}$ ending at t form a maximum set of edge-disjoint $(t, T - t)$-paths.

Proof. In a system of edge-disjoint T-paths, the number of paths ending at t is at most λ_t so the total number of paths is at most $\sum[\lambda_t : t \in T]/2$.

To see the other direction, we can use induction on $|E|$. It can be assumed that G is connected. The statement is trivial if $\Delta_G(T) = \emptyset$. Define a subset $X \subseteq V$ as **tight** with respect to a terminal node $t \in T$ if $X \cap T = \{t\}$ and $d(X) = \lambda_t$. It is **non-trivial** if $|X| \geq 2$.

If there is a non-trivial tight X set with respect to a terminal node s, then shrink X into a node s'. In the shrunk graph G' for the revised terminal set $T' := T - s + s'$, we have $\sum\left[\lambda'_t : t \in T'\right]/2 \geq \sum[\lambda_t : t \in T]/2$ and hence, by induction, there are $\sum[\lambda_t : t \in T]/2$ edge-disjoint T'-paths in G'. In the subgraph of G induced by X there are $d_G(X)$ edge-disjoint paths from s to the $d_G(X)$ edges between X and $V - X$. By pasting together these two systems of paths, we obtain $\sum[\lambda_t : t \in T]/2$ edge-disjoint T-paths of G.

Suppose now that there is no non-trivial tight set with respect to any terminal node. Choose arbitrarily two edges e and f which are incident to a non-terminal node z and split them off. Let h denote the new edge. In the resulting graph G^{ef}, the λ_t-values are unchanged by the assumption that every tight set is trivial. Therefore there are $\sum[\lambda_t : t \in T]/2$ edge-disjoint T-paths in G^{ef}. If one of the paths contains h, then replacing h with e and f and simplifying the resulting walk, we obtain the requested T-paths in G. •

It is interesting to observe that the theorem itself implies an algorithm to find the paths. To see how, observe that it follows from the theorem that taking any non-terminal node z, there are two edges incident to z such that splitting them off does not reduce the λ_t-values. With the help of $|T|$ Max-flow Min-cut computations, we can test a pair of edges incident to z whether it is splittable. To get a complete splitting at z preserving the λ_t-values, one needs $O(d_G(z)|T|)$ MFMC computations and hence the total number of MFMC computations is $O(|E||T|)$. Clearly, this is a polynomial-time algorithm, even though its complexity is not particularly spectacular.

Corollary 8.4.10 *In an Euler graph $G = (V, E)$, there are k edge-disjoint circuits of length at least 3 passing through a specified node s if and only if $\sum[\lambda'(t) : t \in \Gamma(s)] \geq 2k$ where $\Gamma(s)$ denotes the set of neighbours of s and $\lambda'(t) := \min\{d_G(s, t), \min\{d_{G-s}(X) : X \subset V,\ X \cap (\Gamma_G(s) + s) = \{t\}\}\}$.*

Proof. First subdivide each edge incident to s by a node, then delete s, and finally shrink each set of subdividing nodes corresponding to parallel sz-edges ($z \in \Gamma(s)$). Let T denote the set of shrunk nodes. Then $|T| = |\Gamma(s)|$ and the result is a direct consequence of Theorem 8.4.9. •

Karzanov and Lomonosov [235] extended Theorem 8.4.9. Suppose again that (G, T) is inner Eulerian. For a subset $A \subseteq T$, $\lambda_G(A) = \lambda(A)$ denotes the maximum number of edge-disjoint paths connecting A and $T - A$. We say that a set $\mathcal{F}$ of edge-disjoint T-paths **locks a subset** $A \subseteq T$ if $\mathcal{F}$ contains $\lambda(A)$ $(A, T - A)$-paths. Furthermore, $\mathcal{F}$ **locks a family** $\mathcal{L}$ of subsets of T if $\mathcal{F}$ locks all members of $\mathcal{L}$. In these terms, Theorem 8.4.9 asserts that there is a set of edge-disjoint T-paths locking all singletons of T. The question naturally arises whether it is always possible to find a set of edge-disjoint T-paths that locks a specified family $\mathcal{L}$ of subsets of T. The answer, in general, is no, as is shown by the following example. Let $G = (V, E)$ be a graph with node-set $V = \{a, b, c, d, s\}$ and edge-set $E = \{sa, sb, sc, sd\}$. Let $T := \{a, b, c, d\}$ and $\mathcal{L} := \{\{a, b\}, \{a, c\}, \{a, d\}\}$. Now $\lambda_G(X) = 2$ for each $X \in \mathcal{L}$ and there is no way two choose 2 edge-disjoint T-paths which lock each member of $\mathcal{L}$. Note that the 3 members of $\mathcal{L}$ are pairwise crossing. In this light, it is rather natural to require $\mathcal{L}$ to be 3-cross-free. A family $\mathcal{L}$ of subsets of T is said to be **3-cross-free** if it does not contain three pairwise crossing members.

Theorem 8.4.11 (Locking theorem: Karzanov and Lomonosov [235]) *Let (G, T) be inner Eulerian and $\mathcal{L}$ a 3-cross-free family of subsets of T. Then there is a set of edge-disjoint T-paths that locks $\mathcal{L}$.*

Proof. We can assume that $T - A \in \mathcal{L}$ whenever $A \in \mathcal{L}$ because adding $T - A$ to $\mathcal{L}$ affects neither 3-cross-freeness nor lockability. Also, assume that G is connected.

We proceed by induction on the number of edges incident to the elements of $V - T$. If this number is zero, then the statement is trivial. Therefore there is an edge $e = st$ with $t \in T$, $s \notin T$. We are going to show that there is an edge $f = sx$ for which

$$\lambda_G(A) = \lambda_{G^{ef}}(A) \text{ for every } A \in \mathcal{L}. \tag{8.24}$$

The existence of such an f implies the theorem at once, since by induction there is a set $\mathcal{F}$ of edge-disjoint T-paths of G^{ef} locking $\mathcal{L}$. If a path $P \in \mathcal{F}$ uses the new edge h of G^{ef} having arisen from splitting off e and f, then we can revise $\mathcal{F}$ by replacing h in P by e and f. By (8.24), the revised $\mathcal{F}$ locks $\mathcal{L}$ in G.

Claim 8.4.12 *Suppose for $X, Y \subseteq V$ that $X \cap T \subseteq Y \cap T$ and that $d(X) = \lambda(X \cap T)$ and $d(Y) = \lambda(Y \cap T)$. Then $d(X \cap Y) = \lambda(X \cap T)$, $d(X \cup Y) = \lambda(Y \cap T)$, and $d(X, Y) = 0$.*

Proof. Since $X \cap T \subseteq Y \cap T$ we have $(X \cap Y) \cap T = X \cap T$ and hence $d(X \cap Y) \geq \lambda(X \cap T)$. Analogously, $(X \cup Y) \cap T = Y \cap T$ and $d(X \cup Y) \geq \lambda(Y \cap T)$. Therefore, by (1.4) we have $\lambda(X \cap T) + \lambda(Y \cap T) = d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y) \geq \lambda(X \cap T) + \lambda(Y \cap T) + 2d(X, Y)$, from which the claim follows. •

Define a set $X \subseteq V$ as **tight** if $X \cap T \in \mathcal{L}$ and $d(X) = \lambda(X \cap T)$. Since $\mathcal{L}$ is closed under complementation, $V - X$ is tight if X is tight. Because (G, T) is inner Eulerian, a pair of

978–0–19–920527–1 08-Andras-Frank-c08-drv Andras Frank (Typeset by SPi, Publisher Services, India) 283 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

edges $e = st$ and $f = sx$ will satisfy (8.24) precisely if

$$\text{there is no tight set } X \text{ with } t, x \in X \subseteq V - s. \tag{8.25}$$

Claim 8.4.13 *There cannot be three maximal tight $t\overline{s}$-sets.*

Proof. Let X, Y, and Z be maximal tight $t\overline{s}$-sets. Since $\mathcal{L}$ is 3-cross-free, two of the three sets $X \cap T$, $Y \cap T$, and $Z \cap T$, for example $X \cap T$ and $Y \cap T$, are non-crossing. Then either $X \cap T \subseteq Y \cap T$ or $Y \cap T \subseteq X \cap T$ or $T \subseteq X \cup Y$. In the first two cases Claim 8.4.12 implies that $X \cup Y$ is tight contradicting the maximality of X and Y. In the last case, by applying Claim 8.4.12 to $X' = V - X$ and Y, we obtain $d(X', Y) = 0$, contradicting the existence of edge st. •

Let S denote the set of neighbours of s.

Claim 8.4.14 *It is not possible to cover S by two tight $t\overline{s}$-sets.*

Proof. Suppose that $S \subseteq X \cup Y$ where X and Y are tight $t\overline{s}$-sets. Let $\alpha := d(s, X - Y)$, $\beta := d(s, Y - X)$, $\gamma := d(s, X \cap Y)$. By symmetry we can assume that $\alpha \geq \beta$. $(X + s) \cap T = X \cap T$ implies that $d(X + s) \geq \lambda(X \cap T)$. On the other hand, since γ is positive, we have $d(X + s) = d(X) - \alpha - \gamma + \beta < d(X) = \lambda(X \cap T)$, a contradiction. •

Finally, observe that a single tight set X cannot cover S either, since if it does, then $\lambda(X \cap T) = \lambda((X + s) \cap T) \leq d(X + s) < d(X) = \lambda(X \cap T)$, a contradiction. By Claims 8.4.13 and 8.4.14 there is an edge $f = sx$ satisfying (8.25) and then (8.24) holds. The proof of the Locking theorem is complete. • •

Remark 8.4.1 The proof above is taken from [140], where additional results concerning edge-disjoint T-paths are discussed. One may be interested in other possible locking theorems when, rather than 3-cross-freeness, some other property is assumed for the family $\mathcal{L} \subseteq 2^T$ to be locked. For example, let G be a planar Euler graph and let $T := \{t_1, \ldots, t_k\}$ denote the nodes of its outer face in the cyclic order. If $\mathcal{L}$ consists of all subsets of T of the form $\{t_i, \ldots, t_j\}$ $(1 \leq i \leq j \leq k)$ then, though $\mathcal{L}$ is not 3-cross-free when $k \geq 6$, the statement of the Locking theorem holds. This is an equivalent formulation, obtained by planar dualization, of a result by Hurkens, Schrijver, and Tardos [214].

9

Orientations of graphs and hypergraphs

In Part I, we already encountered situations where graph orientations proved useful. For example, the construction of M-alternating paths in the bipartite matching problem was made especially easy by using orientations. Finding an Euler orientation was also helpful since constructing an Euler tour is easier for directed Euler graphs. Yet another example was the short derivation of Theorem 2.3.8 on the existence of a subgraph avoiding forbidden degrees. We have also considered orientation problems for their own sake and characterized, for example, those mixed graphs which have root-connected and strongly connected orientations.

In a general graph orientation problem, one is interested in the existence of an orientation of an undirected graph $G = (V, E)$ covering a set-function $h : 2^V \to \mathbf{Z} \cup \{-\infty\}$ in the sense that $\varrho \geq h$, where ϱ denotes the in-degree function of the orientation. Such an orientation is said to **cover** h. In other words, this requires that the in-degree of every subset $X \subseteq V$ is at least $h(X)$. We deal exclusively with properties concerning the in-degrees. Naturally, one could consider out-degrees as well, but a requirement for the out-degree $\delta(X)$ of a subset $X \subset V$ can be reformulated as one for the in-degree $\varrho(V - X)$ of the complementary set $V - X$.

The orientation problem is apparently pretty simple when h is (0, 1)-valued. In this case, the orientation of G is expected to cover a given family $\mathcal{F}$ of subsets of the nodes, in the sense that each member of $\mathcal{F}$ admits at least one entering edge. It turns out, however, that the well-known **NP**-complete problem of 2-colouring the node-set of a hypergraph H in such a way that no unicoloured hyperedge arises (a proper 2-colouring) can be formulated as an orientation problem of this type, and therefore that this special orientation problem is already **NP**-complete. Indeed, suppose that we want to decide if a hypergraph $H = (U, \mathcal{H})$ is 2-colourable. Form an isomorphic copy $(U', \mathcal{H}')$ of H, with U' disjoint from U. Let $V = U \cup U'$ and let the graph $G = (V, E)$ consist of edges uu' where $u \in U$ and $u' \in U'$ are corresponding elements. Note that G is a perfect matching on V. Let $\mathcal{F} := \mathcal{H} \cup \mathcal{H}'$. It is straightforward to see that there is a one-to-one correspondence between the proper 2-colourations of H and the orientation of G covering $\mathcal{F}$.

In this sense, the graph orientation problem is too general. This chapter explores some cases for which good characterizations and efficient algorithms do exist. We shall also consider orientations of mixed graphs and hypergraphs. An interesting common feature of problems below will be that, apart from mixed graph orientations, each of them can be handled by using the path reorientation technique that has already provided good service in the proof of the Orientation lemma (Theorem 2.3.2).

More complex questions arise when a minimum number of edges have to be reoriented in order to reach the target property in an already directed graph. For example, what is the minimum number of edges in a digraph the reorientation of which results in a strongly connected digraph? This kind of optimization problem will be discussed in Section 9.7. In particular, we shall describe an algorithmic proof of a fundamental theorem of Lucchesi and Younger.

In Part I, we became familiar with two types of orientation problems: in-degree constrained and connection constrained (like strong or root-connected) orientations. It is natural to ask how these extend to global and to rooted k-edge-connectivity, and how the degree and the connection constraints can be combined.

9.1 Rooted k-edge-connected orientations and reorientations

In Section 1.2, we made the obvious observation (Theorem 1.2.24) that a graph G has a root-connected orientation if and only if G is connected. Our present goal is to extend this triviality to higher-order root-connections and to combine connectivity requirements with degree constraints.

What is a necessary and sufficient condition for a graph to have a rooted k-edge-connected orientation? A natural necessary condition, that the graph itself be k-edge-connected, is obviously sufficient in the case when $k = 1$, but not for $k = 2$: a triangle has no such orientation. In a rooted k-edge-connected orientation, apart from the root, the in-degree of each node is at least k; therefore, such a graph must have at least $k(n-1)$ edges, another necessary condition. However, this and k-edge-connectivity together are not yet sufficient, as is shown by a graph consisting of three large disjoint cliques that are pairwise connected by one edge. Motivated by this example, we can formulate a general necessary condition that requires that number $e_G(\mathcal{P})$ of the cross-edges be at least $k(|\mathcal{P}|-1)$ for every partition $\mathcal{P}$ of V into (non-empty) subsets. Such a graph is said to be k**-partition-connected**. We shall prove soon that this property characterizes graphs admitting a rooted k-edge-connected orientation. Since the main device, the path reorientation technique, appeared already in the proof of the Orientation lemma, we study first reorientations of directed graphs.

9.1.1 Rooted k-edge-connected reorientations of digraphs

Let k be a positive integer and $D = (V, A)$ a rooted k-edge-connected digraph with respect to a root-node r_0. By definition, this requires the inequality $\varrho_D(X) \geq k$ for every non-empty subset $X \subseteq V - r_0$, from which

$$e_D(\mathcal{P}) = \sum[\varrho_D(V_i) : i = 1, \ldots, q] \geq k(|\mathcal{P}|-1) \tag{9.1}$$

follows for every partition $\mathcal{P} := \{V_1, \ldots, V_q\}$ of V $(q \geq 1)$. We say that a subset $X \subseteq V - r_0$ is k**-tight** or simply **tight** if $\varrho_D(X) = k$, while a subset X containing r_0 is **tight** if $\varrho_D(X) = 0$. A **partition** $\mathcal{P}$ of V is k**-tight** if equality holds in (9.1). This is clearly equivalent to requiring that each member V_i of $\mathcal{P}$ be k-tight. The partition consisting of the single set

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 286 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

V is always k-tight. We define an an edge f of $D = (V, A)$ as **removable** if $D - f$ has a rooted k-edge-connected reorientation. A partition $\mathcal{P}$ of V is said to **separate** r_0 and a subset $R \subseteq V - r_0$ if the member of $\mathcal{P}$ containing r_0 is disjoint from R.

Proposition 9.1.1 *Let s and t be two arbitrary nodes of an r_0-rooted k-edge-connected digraph D and let P be a directed st-path. If there is no tight $t\overline{s}$-set, then the digraph D' obtained from D by reorienting P is also rooted k-edge-connected.*

Proof. Suppose indirectly that there is a non-empty subset $Z \subseteq V - r_0$ for which $\varrho_{D'}(Z) \leq k - 1$. Then $\varrho_{D'}(Z) \leq \varrho_D(Z) - 1$ and hence Z must be a $t\overline{s}$-set, from which $\varrho_{D'}(Z) = \varrho_D(Z) - 1$ follows. Therefore $k - 1 \geq \varrho_{D'}(Z) = \varrho_D(Z) - 1 \geq k - 1$ and hence $\varrho_D(Z) = k$, contradicting the hypothesis. •

Proposition 9.1.2 *A cross-edge f to a k-tight partition $\mathcal{P}$ is not removable.*

Proof. For any reorientation D' of $D - f$, we have $e_{D'}(\mathcal{P}) = e_D(\mathcal{P}) - 1 = k(|\mathcal{P}| - 1) - 1$, showing that D' violates (9.1), and hence f is not removable. •

Theorem 9.1.3 *Suppose that $D = (V, A)$ is a rooted k-edge-connected digraph with respect to r_0. Let $F \subseteq A$ be a subset of edges with tail r_0 and let R denote the set of head-nodes of the elements of F. There is a removable edge $f \in F$ if and only if there is no k-tight partition $\mathcal{P}$ separating r_0 and R.*

Proof. Necessity follows from Proposition 9.1.2. To prove sufficiency, we can assume that $\varrho_D(r_0) = 0$ since any edge entering r_0 can be reversed. Let S denote the set of nodes reachable from R along a dipath. Then $r_0 \notin S$ and $\varrho_D(V - S) = 0$.

Case 1 For every $v \in S$, $\lambda_D(r_0, v) = k$, implying that there is a tight $v\overline{r}_0$-set (by Menger).

Claim 9.1.4 (A) *The intersection $T(v)$ of the tight $v\overline{r}_0$-sets is tight.* (B) *If a set-system $\mathcal{T}$ of tight subsets of $V - r_0$ forms a connected hypergraph $(Z, \mathcal{T})$, then the union Z of these tight sets is also tight.*

Proof. If X and Y are two intersecting tight subsets of $V - r_0$, then $k + k = \varrho_D(X) + \varrho_D(Y) \geq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \geq k + k$ from which $\varrho_D(X \cap Y) = k$ and $\varrho_D(X \cup Y) = k$, and hence both the intersection and the union are tight. This immediately implies (A). To see (B), let X be a maximal tight subset of Z. If, indirectly, X is a proper subset of Z, then the connectedness of $(Z, \mathcal{T})$ implies that there is a member $Y \in \mathcal{T}$ properly intersecting X. In this case, however, $X \cup Y$ is tight again, contradicting the maximal choice of X. •

Claim 9.1.5 *$T(v) \subseteq S$ for every $v \in S$.*

Proof. Suppose indirectly that $T(v) \not\subseteq S$ for some $v \in S$. Then for $\overline{S} = V - S$ we have

$$\begin{aligned} 0 + k &= \varrho_D(\overline{S}) + \varrho_D(T(v)) \\ &= \varrho_D(\overline{S} \cap T(v)) + \varrho_D(\overline{S} \cup T(v)) + d_D(\overline{S}, T(v)) \geq k + 0 + d_D(\overline{S}, T(v)) \geq k. \end{aligned}$$

Hence $\varrho_D(\overline{S} \cup T(v)) = 0$ and $d_D(\overline{S}, T(v)) = 0$ follow. Since there is an edge from r_0 to every element of R, $d_D(\overline{S}, T(v)) = 0$ implies that $R \cap T(v) = \emptyset$. But in this case, $\varrho_D(\overline{S} \cup T(v)) = 0$ implies that v is not reachable from R, contradicting the definition of S. •

Let $V_1 := V - S$ and let $V_2, \dots, V_q$ denote the node-sets of connected components of the hypergraph $(S, \{T(v) : v \in S\})$. These components are tight by Claim 9.1.4, and $\mathcal{P} := \{V_1, \dots, V_q\}$ forms a partition of V by virtue of to Claim 9.1.5. Now $r_0 \in V_1$ and $R \subseteq S = V - V_1$. Furthermore, $\varrho_D(V_1) = 0$ and $\varrho_D(V_i) = k$ for $i = 2, \dots, q$, and hence $\mathcal{P}$ is a k-tight partition separating r_0 and R, and this contradiction to the hypothesis of the theorem demonstrates that Case 1 cannot occur.

Case 2 There is an element $t \in S$ for which $\lambda_D(r_0, t) \geq k + 1$.

Let P' be an arbitrary dipath from R to t. Let r be its first node and $f \in F$ an r_0r-edge. Then $P := P' + f$ is an r_0t-dipath, and by Proposition 9.1.1 the digraph D' obtained from D by reorienting P is rooted k-edge-connected. By leaving out the reversed f from D', we obtain a rooted k-edge-connected reorientation of $D - f$. • •

The proof gives rise to a polynomial algorithm that either finds a k-tight partition separating r_0 and R or finds a rooted k-edge-connected reorientation of $D - f$ for an appropriate element f of F. To this end, we need a subroutine that can decide if a node $v \in S$ belongs to a tight $v\bar{r}_0$-set, and if it does, the subroutine can compute the smallest tight set $T(v)$ containing v. With an Max-flow Min-cut (MFMC) computation the maximum number $\lambda_D(r_0, v)$ of edge-disjoint r_0v-paths can be determined. If $\lambda_D(r_0, v) > k$, then v does not belong to any tight subset of $V - r_0$. If $\lambda_D(r_0, v) = k$, then the MFMC routine can be used to compute the smallest $v\bar{r}_0$-set of in-degree k, and this set is exactly $T(v)$.

The finest k-tight partition

A partition $\mathcal{P}'$ of V is called a **refinement** of another partition $\mathcal{P}''$ of V, or that $\mathcal{P}'$ is **finer** than $\mathcal{P}''$, if $\mathcal{P}'$ arises from $\mathcal{P}''$ by replacing some members V_i of $\mathcal{P}''$ by a partition of V_i.

Theorem 9.1.6 *Let $\mathcal{P}_0$ denote a k-tight partition of a rooted k-edge-connected digraph $D = (V, A)$ for which $|\mathcal{P}_0|$ is maximum. An edge f of D is non-removable if and only if f is a cross-edge to $\mathcal{P}_0$. Furthermore, $\mathcal{P}_0$ is the unique finest k-tight partition of D.*

Proof. The core of the proof is the following lemma.

Lemma 9.1.7 *Suppose that $D = (V, A)$ is a rooted k-edge-connected digraph with respect to r_0. Let $\mathcal{P} = \{V_1, \dots, V_q\}$ be a k-tight partition for which $r_0 \in V_1$, and let $f = r_0v$ be a non-removable edge for which $v \in V_1$. Then there is a partition $\mathcal{F} = \{U_1, \dots, U_p\}$ of V_1 such that $r_0 \in U_1$, $v \notin U_1$, and the partition $\mathcal{P}' := \{U_1, \dots, U_p, V_2, V_3, \dots, V_q\}$ of V is k-tight.*

Proof. Consider the subgraph $D_1 = (V_1, A_1)$ of D induced by V_1. The k-tightness of $\mathcal{P}$ implies that $\varrho_D(V_1) = 0$. Therefore $\varrho_{D_1}(X) = \varrho_D(X) \geq k$ for every $X \in V_1 - r_0$, and hence D_1 is rooted k-edge-connected.

We claim that f is not removable from D_1. Indeed, if indirectly $D_1 - f$ has a rooted k-edge-connected reorientation, then this along with the unchanged arcs of D not induced by V_1 determine a reorientation D' of $D - f$. Since both the subgraph of D' induced by V_1 and the digraph arising from D' by shrinking V_1 are rooted k-edge-connected, so is D', contradicting the assumption that f is not removable from D.

By applying Theorem 9.1.3 to D_1, we obtain that there is a k-tight partition $\mathcal{F} := \{U_1, \dots, U_p\}$ of V_1. Now the partition $\mathcal{P}'$ of V is indeed k-tight since $|\mathcal{P}'| = q + p - 1$ and $e_D(\mathcal{P}') = e_D(\mathcal{P}) + e_D(\mathcal{F}) = k(p - 1) + k(q - 1) = k(|\mathcal{P}'| - 1)$. •

We turn to the proof of the first part of the theorem. By Proposition 9.1.2, a cross-edge to $\mathcal{P}_0$ is not removable. Suppose now that an edge $f = uv$ is not removable. Since D is rooted k-edge-connected, there are k edge-disjoint r_0u-paths. Proposition 1.2.14 states that reorienting these k paths results in a rooted k-edge-connected digraph with respect to root-node u. Therefore we can assume that f is an r_0v-edge.

Now f must be a cross-edge to $\mathcal{P}_0$, for if r_0 and v belong to the same member of $\mathcal{P}_0$, then Lemma 9.1.7 implies the existence of a k-tight partition $\mathcal{P}'$ of V which is a refinement of $\mathcal{P}_0$, and this contradiction to the maximal choice of $|\mathcal{P}_0|$ proves the first half of the theorem.

To prove the second half, we show that $\mathcal{P}_0$ is a refinement of any k-tight partition $\mathcal{P}$. Suppose indirectly that there is a member P_0 of $\mathcal{P}_0$ and a member P of $\mathcal{P}$ such that neither of $X := P_0 \cap P$ and $Y := P_0 - P$ is empty. If $r_0 \in P_0$, then $0 = \varrho_D(P_0) = \varrho_D(X) + \varrho_D(Y) - d_D(X, Y) \geq 0 + k - d_D(X, Y)$, while if $r_0 \notin P_0$, then $k = \varrho_D(P_0) = \varrho_D(X) + \varrho_D(Y) - d_D(X, Y) \geq k + k - d_D(X, Y)$. In both cases $d_D(X, Y) \geq k$ holds and hence there is a cross-edge f to $\mathcal{P}$. Then f is not removable. On the other hand, f is induced by the member P_0 of $\mathcal{P}_0$, so it is removable, a contradiction. • •

An algorithm to construct $\mathcal{P}_0$ starts with an arbitrary k-tight partition $\mathcal{P}$ and considers the edges induced by a member of $\mathcal{P}$ in an arbitrary order. For such an edge $f = uv$, find k edge-disjoint r_0u-paths. By reorienting these paths, D becomes a u-rooted k-edge-connected digraph. With the algorithmic proof of Theorem 9.1.3, decide if f is removable or not. If so, turn to the next edge induced by a member of $\mathcal{P}$. If f is not removable, then turn to the next edge induced by the k-tight partition $\mathcal{P}'$ ensured by Lemma 9.1.7. The algorithm terminates with a k-tight partition $\mathcal{P}_0$ for which every edge induced by a member of $\mathcal{P}_0$ is removable.

9.1.2 Rooted k-edge-connected orientations of undirected graphs

We shall describe two different approaches for finding rooted k-edge-connected orientations of an undirected graph. The first one is based on Theorem 9.1.3. The idea of the second proof is that we first determine the in-degree vector of a rooted k-edge-connected orientation and then apply the Orientation lemma. Actually, this second approach will be exhibited only later (Section 9.4) in the more general context of hypergraph orientations. The following result appeared in [113].

Theorem 9.1.8 *Let $G = (V, E)$ be an undirected graph and $r_0 \in V$ a designated root-node. There is a rooted k-edge-connected orientation of G if and only if G is k-partition-connected, that is,*

$$e_G(\mathcal{P}) \geq k(q-1) \text{ holds for every partition } \mathcal{P} := \{V_1, \ldots, V_q\} \text{ of } V. \tag{9.2}$$

More generally, the minimum number γ of new edges for which their addition to G results in a graph having a rooted k-edge-connected orientation is equal to

$$M := \max\{k(q-1) - e_G(\mathcal{P}) : \text{ for every partition } \mathcal{P} := \{V_1, \ldots, V_q\} \text{ of } V\}. \tag{9.3}$$

The γ new edges can be chosen to be adjacent to r_0.

Proof. If there is a rooted k-edge-connected orientation, then $\varrho(V_i) \geq k$ for every subset $V_i \subseteq V - r_0$ and hence $e_G(\mathcal{P}) = \sum_i \varrho(V_i) \geq k(q-1)$, and hence (9.2) is indeed necessary. The same argument shows that $\gamma \geq M$.

In order to prove that $\gamma \le M$, we need to show that there is a partition $\mathcal{P}$ and there are $k(q-1) - e_G(\mathcal{P})$ new edges for which their addition results in a graph having a rooted k-edge-connected orientation. To this end, add a minimum number of (possibly parallel) new edges to G in such a way that each new edge is incident to r_0 and the resulting graph G^+ has a rooted k-edge-connected orientation $D = (V, A)$. (This is certainly possible since adding k parallel r_0v-edges for each $v \in V - r_0$ results in a graph having a rooted k-edge-connected orientation.)

Let $F \subseteq A$ denote the set of directed edges corresponding to the new edges. By the minimality of $|F|$, Theorem 9.1.3 implies that there is a partition $\mathcal{P}$ for which $|F| = k(q-1) - e_G(\mathcal{P})$, as required. •

Since the proof of Theorem 9.1.3 was algorithmic, there is an algorithm for computing the extrema in Theorem 9.1.8 as well. The proof technique above immediately implies the following.

Corollary 9.1.9 *If a graph G can be made k-partition connected by adding at most γ new edges, then the γ new edges can be chosen in such a way that each of them is incident to a given node r_0.* •

Remark In Section 10.5, we shall prove a fundamental theorem of Edmonds stating that a rooted k-edge-connected digraph includes k disjoint spanning arborescences of a given root. This result and Theorem 9.1.8 will at once imply Tutte's theorem (Theorem 10.5.1) on the existence of k disjoint spanning trees in a k-partition-connected undirected graph. On the other hand, Tutte's theorem immediately implies Theorem 9.1.8.

9.1.3 Degree-constrained rooted k-edge-connected orientations

Our next goal is to combine degree constraints with rooted k-edge-connectivity. It will turn out that the linking property shows up again. In order to formulate the theorem, we introduce a set-function h_k as follows.

$$h_k(X) := \begin{cases} k & \text{if } \emptyset \subset X \subseteq V - r_0 \\ 0 & \text{if } r_0 \in X \text{ or } X = \emptyset. \end{cases} \tag{9.4}$$

Note that a digraph is rooted k-edge-connected (with respect to root-node r_0) if and only if $\varrho \ge h_k$. The next result is a direct consequence of a general orientation theorem [114].

Theorem 9.1.10 *Let $G = (V, E)$ be a graph with a root-node r_0. Let $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbf{Z}_+ \cup \{+\infty\}$ be two functions for which $f \le g$. G has a rooted k-edge-connected orientation with respect to r_0 :*

(A) *for which* $\dot{\varrho} \ge f$ (that is, if $\varrho(v) \ge f(v)$ for every node $v \in V$) *if and only if G is k-partition-connected,*

$$\widetilde{f}(Z) \le e_G(Z) \textit{ for every } Z \subseteq V, \tag{9.5}$$

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 290 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

and

$$e_G(\mathcal{P}) + i_G(V_0) \geq \widetilde{f}(V_0) + kq - h_k(V_0)$$

$$\text{holds for every partition } \mathcal{P} := \{V_0, V_1, \ldots, V_q\} \text{ of } V \tag{9.6}$$

where $q \geq 1$,

(B) *for which* $\dot{\varrho} \leq g$ *if and only if G is k-partition-connected and*

$$\widetilde{g}(X) \geq i_G(X) + h_k(X) \text{ for every } X \subseteq V, \tag{9.7}$$

(C) *for which* $f \leq \dot{\varrho} \leq g$ *if and only if there is a rooted k-edge-connected orientation of G for which* $f \leq \dot{\varrho}$ *and there is also one for which* $\dot{\varrho} \leq g$.

Proof. Necessity. We have seen in Theorem 2.3.5 that (9.5) is necessary. Also the necessity of (9.2) was shown in Theorem 9.1.8. Suppose now that there is an orientation for which $\dot{\varrho} \geq f$ and $\varrho \geq h_k$. Let $\mathcal{P} := \{V_0, V_1, \ldots, V_q\}$ be a partition of V. Then $e_G(\mathcal{P}) + i_G(V_0) = \ddot{\varrho}(V_0) + \sum[\varrho(V_i) : i = 1, \ldots, q] \geq \widetilde{f}(V_0) + \sum[h_k(V_i) : i = 1, \ldots, q] \geq \widetilde{f}(V_0) + kq - h_k(V_0)$, and hence (9.6) is indeed necessary for (A).

For the necessity of (9.7) in (B), consider an orientation of G for which $\dot{\varrho} \leq g$ and $\varrho \geq h_k$. Then $\widetilde{g}(X) \geq \ddot{\varrho}(X) = i_G(X) + \varrho(X) \geq i_G(X) + h_k(X)$ for any subset X of V, and hence (9.7) is indeed necessary.

Sufficiency. Both in (A) and in (B), we start with a rooted k-edge-connected orientation of G and, while maintaining this property, we reorient paths so as to improve gradually the orientation until the degree constraints are met.

(A) Assume that G is k-partition-connected, and that (9.5) and (9.6) hold. By Theorem 9.1.8, there is a rooted k-edge-connected orientation. Define the **total deficiency** of such an orientation to be the amount $H = \sum[(f(v) - \varrho(v))^+ : v \in V]$. We are done if H is zero, since then $\varrho(v) \geq f(v)$ for every node v.

Lemma 9.1.11 *If H is positive, then there is a dipath the reorientation of which preserves rooted k-edge-connectivity and reduces the total deficiency by* 1.

Proof. Let s be a deficient node (allowing $s = r_0$), that is, $\varrho(s) < f(s)$, and let S denote the set of nodes reachable from s. There must be a node $t \in S$ for which $\varrho(t) > f(t)$, for otherwise we would have $e_G(S) = \delta_G(S) + \ddot{\varrho}(S) = 0 + \ddot{\varrho}(S) < \widetilde{f}(S)$ and this would contradict (9.5). If there is no tight $t\bar{s}$-set, then by Proposition 9.1.1 we would obtain an orientation of G by reorienting a directed st-path P such that this reorientation is rooted k-edge-connected and its total deficiency is $H - 1$, as required for the lemma.

We assume now that every $t \in S$ with $\varrho(t) > f(t)$ belongs to a tight $t\bar{s}$-set and show that this assumption leads to a contradiction. Let $V_1, \ldots, V_p$ denote the maximal tight subsets of $V - s$ which intersect S. By Claim 9.1.4, these are pairwise disjoint and cover all nodes $t \in S$ for which $\varrho(t) > f(v)$.

Claim 9.1.12 $V_i \subseteq S$.

Proof. The claim is obvious if $S = V$, so suppose that $Y := V - S$ is non-empty in which case $r_0 \in Y$ since every node is reachable from r_0. If $V_i \not\subseteq S$, then V_i and Y are properly intersecting, and hence $0 + k = \varrho(Y) + \varrho(V_i) \geq \varrho(Y \cap V_i) + \varrho(Y \cup V_i) \geq k + 0$

from which $\varrho(Y \cup V_i) = 0$. But this is impossible since every node in $S \cap V_i$ is reachable from s. •

Let $V_0 := S - (V_1 \cup \cdots \cup V_p)$. If $S = V$, then let $q := p$ and consider the partition $\mathcal{P} = \{V_0, V_1, \ldots, V_q\}$ of V. In this case $r_0 \in V_0$ for if $r_0 \in V_1$, say, then $\varrho(V_1) = h_k(V_1) = 0$ contradicting the assumption that the elements of S are reachable from t. Now we have $e_G(\mathcal{P}) + i_G(V_0) = \ddot{\varrho}(V_0) + \sum[\varrho(V_i) : i = 1, \ldots, q] = \ddot{\varrho}(V_0) + kq < \widetilde{f}(V_0) + kq = \widetilde{f}(V_0) + kq - h_k(V_0)$, contradicting (9.6). If $S \subset V$, then let $q := p + 1$ and $V_q := V - S$. Consider the partition $\mathcal{P} = \{V_0, V_1, \ldots, V_q\}$ of V. In this case $r_0 \in V_q$ and we have $e_G(\mathcal{P}) + i_G(V_0) = \ddot{\varrho}(V_0) + \sum[\varrho(V_i) : i = 1, \ldots, q] = \ddot{\varrho}(V_0) + k(q-1) < \widetilde{f}(V_0) + k(q-1) = \widetilde{f}(V_0) + kq - h_k(V_0)$, contradicting (9.6). These contradictions complete the proof of the lemma. • •

Lemma 9.1.11 implies that, after reorienting at most $H \leq |E|$ paths, we arrive at a rooted k-edge-connected orientation of G for which $\dot{\varrho} \geq f$, completing the proof of Case (A).

(B) For proving sufficiency of (9.7), we start again from a rooted k-edge-connected orientation of G. Define the **total excess** of such an orientation to be the amount $H' := \sum[\varrho(v) - g(v))^+ : v \in V]$. We are done if H' is zero, since then $\varrho(v) \leq g(v)$ for every node v.

Lemma 9.1.13 *If H' is positive, then there is a dipath the reorientation of which preserves rooted k-edge-connectivity and reduces the total excess by one.*

Proof. Let t be a node with $\varrho(t) > g(t)$ and let T denote the set of nodes from which t is reachable. Obviously, $\varrho(T) = 0$ and $r_0 \in T$.

Suppose first that there is a tight set containing t and let $Z := T(t)$ denote the unique smallest tight set containing t.

Claim 9.1.14 $Z \subseteq T$.

Proof. The claim is obvious if $T = V$, so suppose that $T \subset V$ and assume, indirectly, that Z and T are properly intersecting. Then $k + 0 = \varrho(Z) + \varrho(T) \geq \varrho(Z \cap T) + \varrho(Z \cup T) \geq k + 0$ from which $\varrho(Z \cap T) = k$, that is, $Z \cap T$ is also tight contradicting the definition of Z. •

Now Z must contain a node s for which $\varrho(s) < g(s)$, for otherwise $\widetilde{g}(Z) < \ddot{\varrho}(Z) = \varrho(Z) + i_G(Z) = k + i_G(Z) = h_k(Z) + i_G(Z)$, contradicting (9.7). By Proposition 9.1.1, reorienting any directed st-path P results in a rooted k-edge-connected digraph in which the total excess is $H' - 1$, proving the lemma in this case.

Next, suppose that no tight set contains t. We claim that T contains a node s for which $\varrho(s) < g(s)$, for otherwise $\widetilde{g}(T) < \ddot{\varrho}(T) = \varrho(T) + i_G(T) = 0 + i_G(T) = h_k(T) + i_G(T)$, contradicting (9.7). By reorienting a directed st-path we obtain again a rooted k-edge-connected orientation in which the total excess is $H' - 1$, completing the proof of the lemma. • •

Lemma 9.1.13 implies that after reorienting at most $H' \leq |E|$ paths, we arrive at a rooted k-edge-connected orientation satisfying $\dot{\varrho} \leq g$, completing the proof of Part (B).

Finally, Part (C) follows immediately from the proof above, provided that we start at Part (B) from a rooted k-edge-connected orientation for which $\dot{\varrho} \geq f$. One only needs to observe that the reorientation of the path in the proof decreases the in-degree of t by one, but that this indegree was larger than $g(t)$ $(\geq f(t))$, so the in-degree of t cannot get below $f(t)$. ● ● ●

Note that Part (C) in Theorem 9.1.10 shows that the linking property found for degree-constrained orientation (see Corollary 2.3.7) continues to hold when the additional property of rooted k-edge-connectivity is imposed.

Problems

9.1.1 *Let s and t be two distinct nodes of G. Describe a method to obtain a t-rooted k-edge-connected orientation from an s-rooted k-edge-connected orientation.*

9.1.2 *Characterize graphs G having a rooted k-edge-connected orientation from r_0 such that the out-degree of r_0 is at most γ.*

9.1.3 *Suppose that a graph has a rooted k-edge-connected orientation in which $\dot{\varrho} \geq f$ where $f(v) \geq k$ for every node $v \in V - r_0$. Prove that G can be partitioned into two subgraphs G_1 and G_2 on node-set V in such a way that G_1 has a rooted k-edge-connected orientation and G_2 has an orientation in which $\varrho_2(v) \geq f(v) - k$ for every $v \in V$.*

Theorem 9.1.10 will find an interesting application in Section 10.5 where it will be used to derive a theorem of Whiteley on packing trees and 1-orientable graphs in an undirected graph.

9.1.4 Simplification for $k = 1$

The root-connected orientability of a graph G depended only on the connectivity of G while the characterization in Theorem 9.1.8 for $k \geq 2$ included a considerably more sophisticated partition-type inequality. Next, we are going to show that Case (A) in Theorem 9.1.10 can also be simplified in the case when $k = 1$. To this end, define a set-function σ_0 on V in such a way that $\sigma_0(X)$ is the number of components of $G - X$ not containing r_0 for a non-empty set $X \subseteq V$ and $\sigma_0(\emptyset) = 0$. Let the set-function σ be defined in such a way that $\sigma(X)$ is the number of components of $G - X$ when $X \subseteq V$ is non-empty and let $\sigma(\emptyset) := 0$.

Theorem 9.1.15 (Frank and Gyárfás [137]) *We are given two functions $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ for which $f \leq g$. A connected undirected graph $G = (V, E)$ with a designated root-node r_0 has a root-connected orientation:*

(A) *for which $\dot{\varrho} \geq f$ if and only if $\widetilde{f} \leq e_G - \sigma_0$, that is,*

$$\widetilde{f}(X) \leq e_G(X) - \sigma_0(X) \text{ holds for every } X \subseteq V, \tag{9.8}$$

(B) *for which $\dot{\varrho} \leq g$ if and only if*

$$\widetilde{g}(X) \geq \begin{cases} i_G(X) + 1 & if\ \emptyset \neq X \subseteq V - r_0 \\ i_G(X) & if\ r_0 \in X \subseteq V, \end{cases} \tag{9.9}$$

(C) *for which $f \leq \dot{\varrho} \leq g$ if and only if there is a rooted-connected orientation of G for which $f \leq \dot{\varrho}$ and there is also one for which $\dot{\varrho} \leq g$.*

Proof. Cases (B) and (C) are just special cases of those in Theorem 9.1.10, so we deal only with (A).

Suppose first that there is a root-connected orientation with $\dot{\varrho} \geq f$. Consider a non-empty subset X of V. If r_0 is in X, then at least one edge enters every component of $G - X$, because of root-connectivity, and hence $\delta(X) \geq \sigma(X) = \sigma_0(X)$. If r_0 is not in X, then $\delta(X) \geq \sigma(X) - 1 = \sigma_0(X)$. Hence $e_G(X) = i_G(X) + \varrho(X) + \delta(X) = \ddot{\varrho}(X) + \delta(X) \geq \widetilde{f}(X) + \sigma_0(X)$, as required for (9.8).

For sufficiency, observe first that (9.5) follows from (9.8). Also, for $k = 1$, a k-partition-connectivity is equivalent to the connectivity of G. Therefore we only need to prove that (9.8) implies (9.6) in the case when $k = 1$. Suppose indirectly that a partition $\mathcal{P} = \{V_0, V_1, \ldots, V_q\}$ of V violates (9.6) and choose such a $\mathcal{P}$ which minimizes q.

We claim that there is no edge connecting V_i and V_j for $1 \leq i < j \leq q$. Indeed, if there were an edge between V_i and V_j, then by replacing V_i and V_j with their union, since $k = 1$, we would obtain from $\mathcal{P}$ another partition of V which also violates (9.6), contradicting the minimality of $|\mathcal{P}|$. Therefore, if $r_0 \in V_0$, then $\sigma_0(V_0) \geq q = q - h_1(V_0)$ while if $r_0 \notin V_0$, then $\sigma_0(V_0) \geq q - 1 = q - h_1(V_0)$. This and (9.8) imply that $e_G(\mathcal{P}) + i_G(V_0) = e_G(V_0) \geq \widetilde{f}(V_0) + \sigma_0(V_0) \geq \widetilde{f}(V_0) + q - h_1(V_0)$, contradicting the indirect assumption that $\mathcal{P}$ violates (9.6). •

Problem 9.1.4 *Prove that an undirected graph $G = (V, E)$ with a root-node r_0 has an in-root-connected orientation* (meaning that r_0 is reachable from every other node) *for which $\dot{\varrho} \geq f$ if and only if*

$$\widetilde{f}(X) \leq \begin{cases} e_G(X) & if\ \emptyset \neq X \subseteq V - r_0 \\ e_G(X) - 1 & if\ r_0 \in X \subseteq V. \end{cases} \tag{9.10}$$

9.1.5 Three applications of root-connected orientations

Degree-constrained trees

Given a connected graph, we are interested in finding a spanning tree meeting degree constraints on the nodes. This problem formulation is too general since the problem of Hamilton paths, one of the basic **NP**-complete problems, can be formulated in this way by imposing uniformly the upper bound 2 on every node. However, if the bounds are given only on the elements of a stable set of nodes, the problem becomes tractable.

Theorem 9.1.16 *Let G be a connected undirected graph and $S \subset V$ a stable subset of nodes of G. Let $f_S : S \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g_S : S \to \mathbf{Z}_+ \cup \{\infty\}$ be two functions for which $f_S \leq g_S$. There exists a spanning tree F of G:*

(A) *for which $d_F(v) \geq f_S(v)$ for every node $v \in S$ if and only if*

$$\widetilde{f_S}(Y) \leq |Y| + |\Gamma(Y)| - 1 \ \textit{for every subset} \ \emptyset \subset Y \subseteq S, \tag{9.11}$$

(B) *for which $d_F(v) \leq g_S(v)$ for every node $v \in S$ if and only if*

$$\widetilde{g}_S(Y) \geq |Y| + \sigma(Y) - 1 \ \textit{for every subset} \ \emptyset \subset Y \subseteq S, \tag{9.12}$$

(C) *for which $f_S(v) \leq d_F(v) \leq g_S(v)$ for every $v \in S$ if and only if both* (9.11) *and* (9.12) *hold.*

Proof. Necessity. Let F be a spanning tree and let $Y \subseteq S$ be a non-empty subset. Let F_Y denote the subforest of F consisting of the edges incident to the nodes in Y. If $d_F(V) \geq f_S(v)$ for every node $v \in S$, then clearly $|F_Y| \geq \widetilde{f_S}(Y)$. On the other hand, since the number of edges of a forest is less than the number of its nodes, $|F_Y| \leq |V(F_Y)| - 1 = |Y| + |\Gamma_{F_Y}(S)| - 1 \leq |Y| + |\Gamma(Y)| - 1$, and the combination of these two inequalities implies (9.11).

Suppose now that $d_F(v) \leq g_S(v)$ for every $v \in S$. A spanning tree has at least $t-1$ cross-edges to every t-partition of V. Applying this to the value $t := |Y| + \sigma(Y)$ and to the t-partition consisting of the singletons formed by the elements of Y and by the components of $G - Y$, we obtain that F has at least $|Y| + \sigma(Y) - 1$ cross-edges. These edges are all incident to the elements of Y since there can be no edges at all between components of $G - Y$. On the other hand, the number of edges of F incident to Y is at most $\widetilde{g}_S(Y)$, from which (9.12) follows.

To prove sufficiency, one can assume that G is bipartite. Indeed, if $T := V - S$ were to span some edges, then first subdivide each by a node, then add the new nodes to S and finally assign lower bound 0 and the upper bound ∞ to each subdividing node. The revised problem obtained in this way is equivalent to the original one, both on the primal side and on the dual side.

The connection between orientations and degree-specified trees is described by the following lemma.

Lemma 9.1.17 *Let $G = (S, T; E)$ be a connected bipartite graph and $m : S \to \mathbf{Z}^+$ a function for which $\widetilde{m}(S) = |V| - 1$. Let r_0 be an arbitrary element of T. There exists a spanning tree F of G such that $d_F(v) = m(v)$ for every node $v \in S$ if and only if there is a root-connected orientation of G for which the in-degree of every node $v \in S$ is $d(v) - m(v) + 1$.*

Proof. Suppose first that a required tree F exists. Orient the edges of F so as to obtain an r_0-rooted arborescence, and orient all other edges toward S. In the resulting orientation of G, every node is reachable from r_0. Furthermore, the out-degree of every node $v \in S$ is exactly $m(v) - 1$, showing that its in-degree is $d(v) - m(v) + 1$.

Conversely, let us consider an orientation of G in which every node is reachable from r_0 and the in-degree of every node $v \in S$ is $d(v) - m(v) + 1$, or equivalently, the out-degree $\delta(v)$ of v is $m(v) - 1$. Let $\vec{F}$ be an arbitrary spanning r_0-arborescence. We claim that the spanning tree F obtained from $\vec{F}$ by deorienting its edges satisfies the degree specification $d_F(v) = m(v)$ for every node $v \in S$. Indeed, $d_F(v) = 1 + \delta_{\vec{F}}(v) \leq 1 + \delta(v) = m(v)$, and hence $d_F(v) \leq m(v)$ for every $v \in S$. Hence $|V| - 1 = |F| = \sum_{v \in S} d_F(v) \leq \widetilde{m}(S) = |V| - 1$, from which we must have $d_F(v) = m(v)$ for every node $v \in S$. •

Define functions $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ and $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ as follows.

$$g(v) := \begin{cases} 1 & \text{if } v \in T - r_0 \\ d(v) - f_S(v) + 1 & \text{if } v \in S \\ \infty & \text{if } v = r_0, \end{cases}$$

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 295 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$$f(v) := \begin{cases} -\infty & \text{if } v \in T \\ d(v) - g_S(v) + 1 & \text{if } v \in S. \end{cases}$$

We claim that (9.8) holds. Since f is $-\infty$ on the elements of T, we can assume that $X \subseteq S$. In this case, $d_G(X) = e_G(X)$ and $\sigma_0(X) = \sigma(X) - 1$. By applying (9.12) to X in place of Y, we obtain $\widetilde{f}(X) = \sum[d_G(v) - g_S(v) + 1 : v \in X] = d_G(X) - \widetilde{g}_S(X) + |X| \leq d_G(X) + \sigma(X) - 1 = e_G(X) + \sigma_0(X)$, as required.

Claim 9.1.18 *Condition* (9.9) *holds.*

Proof. Suppose indirectly that a subset X violates (9.9). Since $g(r_0) = \infty$, X does not contain r_0 and hence $\widetilde{g}(X) < i_G(X) + 1$. Assume that $i_G(X) - \widetilde{g}(X)$ is as large as possible (indicating that X is a most violating set), and with respect to this, $|X|$ is as large as possible. Let $Y := X \cap S$. Every node $v \in X \cap T$ has a neighbour in Y for if not, then $i_G(X - v) = i_G(X)$ and $\widetilde{g}(X - v) = \widetilde{g}(X) - 1$, contradicting the assumption that X is most violating. Every neighbour of Y is in X for if v is a neighbour in $T - X$, then $i_G(X + v) \geq i_G(X) + 1$ and $\widetilde{g}(X + v) = \widetilde{g}(X) + 1$, contradicting again the special choice of X. We conclude that $\Gamma_G(Y) = X \cap T$ and $i_G(X) = d_G(Y)$. Hence $d_G(Y) + 1 = i_G(X) + 1 > \widetilde{g}(X) = \widetilde{g}(Y) + \widetilde{g}(\Gamma(Y)) = \sum[d_G(v) - f_S(v) + 1 : v \in Y] + |\Gamma(Y)| = d_G(Y) - \widetilde{f}_S(Y) + |Y| + |\Gamma(Y)|$ from which $\widetilde{f}_S(Y) > |Y| + |\Gamma(Y)| - 1$, contradicting (9.11). •

Theorem 9.1.15 implies the existence of a root-connected orientation of G satisfying the given upper and/or lower bounds on the in-degrees, and hence Lemma 9.1.17 implies the theorem. • •

Problem 9.1.5 *Prove that a 2-edge-connected loopless graph G has a spanning tree F for which $d_F(v) \leq \lfloor (d_G(v) + 3)/2 \rfloor$ for every node v. Show that the number 3 in the inequality cannot be lowered to 2.*

Detachment of nodes

The 'first' theorem of graph theory states that, in a connected graph, there is an Euler tour if and only if the degree of every node is even. An Euler tour can be interpreted in such a way that each node v is detached into $d(v)/2$ nodes of degree 2 and that the resulting graph is connected (and therefore a circuit). What can be said if more general degree constraints are given for the detached nodes?

Let $G = (V, E)$ be an undirected graph and let $m : V \to \mathbf{Z}_+$ be a strictly positive integer-valued function. By an m-**detachment** of G, we mean a graph arising from G in such a way that each node v of G is replaced by $m(v)$ new nodes and every edge uv of G is replaced by an edge $u'v'$, where u' is any one of the $m(u)$ new nodes assigned to u, and v' is any one of the $m(v)$ new nodes assigned to v.

Theorem 9.1.19 (Nash-Williams [307]) *A connected graph G has a connected m-detachment if and only if*

$$e_G(X) \geq \widetilde{m}(X) + \sigma(X) - 1 \tag{9.13}$$

holds for every non-empty subset $X \subseteq V$.

Proof. Suppose first that G has a connected m-detachment G'. For a subset $X \subseteq V$, consider the partition $\mathcal{P}$ of the node-set of G' consisting of the detached nodes of the elements of X as singletons and of the subsets of G' corresponding to the components of $G - X$. Then $|\mathcal{P}| = \widetilde{m}(X) + \sigma(X)$. Since G' is connected there are at least $\widetilde{m}(X) + \sigma(X) - 1$ cross-edges to $\mathcal{P}$. These cross-edges correspond to edges of G for which at least one end-node is in X, so (9.13) is indeed necessary.

To prove sufficiency, let r_0 be an arbitrary node of G. Case (A) of Theorem 9.1.15 states that, given a lower bound function $f : V \to \mathbf{Z}_+$, G has an orientation in which every node is reachable from r_0 and the in-degree of every node v is at least $f(v)$ if

$$e_G(X) \geq \widetilde{f}(X) + \sigma_0(X) \tag{9.14}$$

for every subset X.

Let $f(v) := m(v)$ if $v \in V - r_0$ and $f(r_0) := m(r_0) - 1$. It is clear that (9.14) follows from (9.13) and hence G admits the requested orientation. Let F be a spanning r_0-arborescence in the given orientation. Detach each node v as follows: Select $m(v) - 1$ edges entering v that are not in F. Each of these edges determines a one-element part in the detachment, while the remaining edges incident to v form another part of the detachment. The arborescence F ensures the connectedness of the detachment of G obtained in this way. •

Corollary 9.1.20 *Suppose that m satisfies* (9.13). *If, in addition to the number $m(z)$ assigned to node z, we are given $m(z)$ positive integers, serving as degree specifications of the detached nodes, for which their sum is $d_G(z)$, then there is an m-detachment in which the degree of every node is equal to the specification.*

Proof. By Theorem 9.1.19 there is an m-detachment G'. If this does not meet the degree specification, then there are two nodes z' and z'' that arises from the detachment of node z in such a way that the degree of z'' is too large while the degree of z' is too small. Let F be a spanning tree of the detachment. Let uz'' be an edge of G' which is not the first edge of the unique path of F between z'' and z'. By replacing uz'' with the edge uz', we preserve connectedness and the degrees are closer to the specification. •

Partition-connected hypergraphs

The last goal of this section is to show that degree-constrained root-connected orientations of graphs can be used to decide if a hypergraph is partition-connected.

Theorem 9.1.21 *Let $H = (S, \mathcal{E})$ be a hypergraph in which every hyperedge has at least two elements. H is partition-connected if and only if H has a root-connected orientation.*

Proof. Let D be a root-connected orientation of H with respect to a specified root-node r_0 and let $\mathcal{P} = \{S_1, \dots, S_q\}$ be a partition of S. Suppose that $r_0 \in S_1$. Since there is a dyperedge entering S_i for each $i = 2, 3, \dots, q$, the number of cross-hyperedges of H to $\mathcal{P}$ is at least $q - 1$, and hence H is partition-connected.

To see the reverse implication, suppose that H is partition-connected and consider the bipartite graph $G = (S, U; E)$ associated with H, where the elements of U correspond to the hyperedges of H. Let $V := S \cup U$ and define $f : V \to \mathbf{Z} \cup \{-\infty\}$ as follows.

$$f(v) := \begin{cases} -\infty & \text{if } v \in S \\ d_G(v) - 1 & \text{if } v \in U. \end{cases} \tag{9.15}$$

We claim that (9.8) holds. Suppose indirectly that there is a subset X for which $\widetilde{f}(X) > e_G(X) - \sigma_0(X)$. Then X cannot contain any node v with $f(v) = -\infty$. Hence $X \subseteq U$ from which $\sigma_0(X) > e_G(X) - \widetilde{f}(X) = d_G(X) - \sum[d_G(v) - 1 : v \in X] = |X|$ follows.

Let $S_1, \ldots, S_q$ denote the partition of S determined by the components of the subhypergraph arising from H be deleting the hyperedges corresponding to the elements of X. Then $q - 1 = \sigma_0(X) > |X|$, contradicting the partition-connectivity of H.

By Part (A) of Theorem 9.1.15 there is a root-connected orientation $\vec{G}$ of G in which $\varrho(v) \geq d_G(v) - 1$ for every $v \in U$. We can actually assume that $\varrho(v) = d_G(v) - 1$ for if $\varrho(v) = d_G(v)$, then reversing an arbitrary edge entering v could destroy root-connectivity only if $d_G(v) = 1$ but this danger was excluded by the hypothesis that every hyperedge of H has at least two elements and hence $d_G(v) \geq 2$.

We obtained a root-connected orientation $\vec{G}$ of G in which $\delta(v) = 1$ for every $v \in U$. Such an orientation corresponds to a root-connected orientation of H. •

Theorem 9.1.22 *Let $H = (S, \mathcal{E})$ be a hypergraph in which every hyperedge has at least to elements. H is partition-connected if and only if H has a spanning hypertree.*

Proof. Suppose first that H has a spanning hypertree. By Theorem 2.4.16 a hyperforest can be trimmed to a forest and, in particular, a spanning hypertree can be trimmed to a spanning tree. A tree clearly has a root-connected orientation and this determines a root-connected orientation of H. By Theorem 9.1.21, H is partition-connected.

Suppose now that H is partition-connected. We can assume that H is minimal in the sense that leaving out any hyperedge destroys partition-connectivity. By Theorem 9.1.21, H has a root-connected orientation $\vec{H}$. By Theorem 7.4.9, $\vec{H}$ can be trimmed to a root-connected digraph D. By the minimality of H, D is a minimal root-connected digraph, which is an arborescence, and hence H is a spanning hypertree. •

Since the proof of Theorem 9.1.15 was algorithmic, through this we have obtained an algorithm for deciding if a hypergraph is partition-connected. Theorem 9.4.4 will generalize Theorem 9.1.21 by characterizing k-partition-connected hypergraphs as exactly those admitting a rooted k-edge-connected orientation. One of the two proofs will be algorithmic, providing in this way an algorithm to decide if a hypergraph is k-partition-connected.

9.2 Global k-edge-connection

Having explored rooted k-edge-connected orientability, we now turn our interest to global k-edge-connected orientations. One goal is to derive the promised generalization of Robbins' orientation theorem, which was proved by Nash-Williams. The proof is based on Lovász' splitting lemma, proved in Section 8.1. We also consider degree-constrained k-edge-connected orientations.

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 298 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

A quite natural question, however, remains unanswered in this section. Finding a k-edge-connected orientation of a mixed graph will need the theory of submodular flows to be discussed in Part III.

9.2.1 k-edge-connected orientations

One of the proofs of Robbins' theorem was based on ear-decomposition of 2-edge-connected graphs. Its extension, the constructive characterization of $2k$-edge-connected graphs (Theorem 8.1.4) gives rise to the following important result on k-edge-connected orientations.

Theorem 9.2.1 (Nash-Williams, weak orientation theorem) *An undirected graph $G = (V, E)$ has a k-edge-connected orientation if and only if G is $2k$-edge-connected.*

Proof. Necessity is straightforward. To prove sufficiency, we use induction on $|V| + |E|$. The case $|V| = 1$ is trivial so assume that $|V| \geq 2$. Apply Theorem 8.1.1. If G is obtained from a $2k$-edge-connected graph G' by adding an edge e, then G' has a k-edge-connected orientation by induction, so orienting e arbitrarily yields a k-edge-connected orientation of G. If G arises from G' by operation (B) in Theorem 8.1.1, that is, if G arises by pinching a subset F of k edges of G' together, then a k-edge-connected orientation of G', ensured by the inductive hypothesis, determines one for G by pinching the k directed edges obtained from the orientation of F. •

Question 9.2.1 *Given an arbitrary orientation of a $2k$-edge-connected graph, what is the minimum number of edges whose reorientation results in a k-edge-connected digraph?*

For $k = 1$, this will be answered by a theorem of Lucchesi and Younger in Section 9.6. For higher k, the answer relies on the theory of submodular flows to be studied in Part III.

With a tiny refinement of the proof technique above, we derive the following sharpening of Nash-Williams' theorem.

Theorem 9.2.2 *A $2k$-edge-connected graph has a smooth k-edge-connected orientation.*

Proof. If G has a node z of even degree, then Theorem 8.1.2 ensures a complete splitting at z which preserves $2k$-edge-connectivity. In this case, we are done by induction. If the degree of every node is odd, then Lemma 8.1.3 implies that there is a uv-edge e which can be removed from the graph without destroying $2k$-edge-connectivity.

By induction, $G - e$ has a smooth k-edge-connected orientation. Since the degrees of u and v are even in $G - e$, by adding any orientation of e, one obtains a smooth digraph. •

When the graph G itself is Eulerian, Nash-Williams' orientation theorem becomes trivial: every di-Euler orientation of G will suffice, since then $\varrho(X) = d_G(X)/2 \geq k$. This observation suggests an apparently natural approach for finding a k-edge-connected orientation for general G as well: first take an arbitrary maximal Euler subgraph G' of G, then consider a di-Euler orientation of G', and finally find an appropriate orientation of the remaining forest. Although intuition suggests that such preorientation of an Euler subgraph cannot endanger our chances to complete the orientation, the proof of this fact needs other tools. Here we only formulate the result, its proof is postponed until Section 15.3.

Theorem 9.2.3 *Let $G = (V, E)$ be a $2k$-edge-connected undirected graph. Then a di-Euler orientation of any Euler subgraph of G can always be completed to a k-edge-connected orientation of G.* •

This result immediately gives rise to the following extension of Theorem 9.2.1.

Corollary 9.2.4 *Let $G = (V, E)$ be an arbitrary undirected graph and F a subset of edges. There is an orientation of the elements of F such that the resulting mixed graph M is k-edge-connected if and only if*

$$d_F(X) + 2d_{E-F}(X) \geq 2k \text{ for every subset } \emptyset \subset X \subset V. \tag{9.16}$$

Proof. In the requested orientation of M, $\varrho(X) + d_{E-F}(X) \geq k$ and $\delta(X) + d_{E-F}(X) \geq k$, from which $d_F(X) + 2d_{E-F}(X) = \varrho(X) + d_{E-F}(X) + \delta(X) + d_{E-F}(X) \geq 2k$, and hence (9.16) is necessary.

Conversely, duplicate in parallel each element of $E - F$. The resulting graph G^+ is $2k$-edge-connected by (9.16) and the subset of duplicated edges forms an Euler subgraph G' of G^+. Consider a di-Euler orientation of G' in which every pair of parallel edges is oriented oppositely. By Theorem 9.2.3, any di-Euler orientation of G' can be completed to a k-edge-connected orientation of G^+. Let $\vec{F}$ denote the orientation of F obtained in this way. It follows from this construction that the mixed graph $(V, \vec{F} + E - F)$ is k-edge-connected. •

Conjecture 9.2.5 *An undirected graph $G = (V, E)$ has a k-node-connected orientation if and only if G is $(k, 2)$-hybrid-connected.*

9.2.2 Degree-constrained k-edge-connected orientation

Next we show how degree constraints and k-edge-connectivity can be combined. The result appears in [114] in a more general form.

Theorem 9.2.6 *Let $G = (V, E)$ be a $2k$-edge-connected graph for $k \geq 1$. Let $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbf{Z}_+\{\infty\}$ be two functions for which $f \leq g$. There is a k-edge-connected orientation of G in which*

(A) $\dot{\varrho} \geq f$ *if and only if* $\widetilde{f}(V) \leq |E|$ *and*

$$i_G(V_0) + e_G(\mathcal{P}) \geq \widetilde{f}(V_0) + kq \text{ for every partition } \mathcal{P} = \{V_0, V_1, \ldots, V_q\} \text{ of } V, \tag{9.17}$$

(B) $\dot{\varrho} \leq g$ *if and only if* $\widetilde{g}(V) \geq |E|$ *and*

$$e_G(V_0) - e_G(\mathcal{P}) \leq \widetilde{g}(V_0) - kq \text{ for every partition } \mathcal{P} = \{V_0, V_1, \ldots, V_q\} \text{ of } V, \tag{9.18}$$

(C) $f \leq \dot{\varrho} \leq g$ *if and only if both* (9.17) *and* (9.18) *hold.*

Proof. Suppose first that there is a k-edge-connected orientation in which $\dot{\varrho} \geq f$. Then the sum $i_G(V_0) + e_G(\mathcal{P})$ counts the edges entering the nodes of V_0 plus those entering one of the sets $V_1, \ldots, V_q$. Therefore this sum is at least $\widetilde{f}(V_0) + kq$, and hence (9.17) is necessary.

Now suppose that (9.17) is met. Start from a k-edge-connected orientation of G for which the total deficiency $H := \sum[(f(v) - \varrho(v))^+ : v \in V]$ is minimal (where $x^+ := \max\{0, x\}$). If $H = 0$, then $\varrho(v) \geq f(v)$ holds for every node v, and then we are done. So we assume, indirectly, that $H > 0$, in which case there is a node s with $\varrho(s) < f(s)$.

If X and Y are two intersecting subsets of $S - s$ with in-degree k, then $k + k = \varrho(X) + \varrho(Y) \geq \varrho(X \cup Y) + \varrho(X \cap Y) \geq k + k$ from which we must have $\varrho(X \cup Y) = k = \varrho(X \cap Y)$. It follows that the maximal subsets $V_1, \ldots, V_q$ of $S - s$ with in-degree k are pairwise disjoint.

Let $V_0 := V - \cup(V_i : i = 1, \ldots, q)$ if there is any subset $X \subseteq V - s$ with in-degree k and let $V_0 := V$ otherwise. We claim that there is a node t in V_0 with in-degree excess, which means that $\varrho(t) > f(t)$. Suppose, indirectly, that no such a node exists. In the case when $V_0 = V$, we have $|E| = \sum[\varrho(v) : v \in V] < \sum[f(v) : v \in V] = \widetilde{f}(V)$ contradicting the first assumption in (A). In the case when $V_0 \subset V$, we have $\widetilde{f}(V_0) + kq > \ddot{\varrho}(V_0) + \sum[\varrho(V_i) : i = 1, \ldots, q] = i_G(V_0) + e_G(\mathcal{P})$, contradicting (9.17). Let P be an arbitrary st-path. We claim that reorienting P yields a k-edge-connected orientation. Indeed, if the reorientation decreases the in-degree of a set Z below k, then $t \in Z, s \notin Z$, and $\varrho(Z) = k$. But the existence of such a Z contradicts the maximal choice of V_i's.

The reorientation of P, in addition, decreases the total in-deficiency by one, contradicting the minimal choice of the initial orientation.

Part (B) can be proved with an analogous argument, but it can be derived directly from the first part. To this end, notice that imposing the upper bound $g(v)$ on the in-degree is equivalent to imposing $f'(v) := d_G(v) - g(v)$ as a lower bound on the out-degree of v. It is easy to check that condition (9.17), when applied to f' in place of f, is just equivalent to (9.18). Therefore, there is a k-edge-connected orientation in which the in-degree of every node v is at least $f'(v)$. By reorienting each edge, we obtain a k-edge-connected orientation in which the out-degree of each node v is at least $f'(v)$, and hence the in-degree is at most $g(v)$.

In order to prove (C), start with a k-edge-connected orientation of G for which $\varrho(v) \leq g(v)$ for every node v and apply the proof of the first part to this initial orientation. All we need to observe is that the in-degree of a node s can increase only if $\varrho(s) < f(s)$. Because of assumption $f \leq g$, the inequality $\varrho(v) \leq g(v)$ will never break down. •

Note that Part (C) in Theorem 9.2.6 shows that the linking property found for degree-constrained orientation (see Corollary 2.3.7) continues to hold when the additional property of k-edge-connectivity is imposed. The proof of the first part of Theorem 9.2.6 immediately implies the following.

Corollary 9.2.7 *Provided that G has a k-edge-connected orientation in which $\varrho(v) \geq f(v)$ for every $v \in V$, such an orientation can be obtained from an arbitrary k-edge-connected orientation of G by reorienting dipaths one by one such that each intermediate orientation is k-edge-connected.* •

The next result appears in [120].

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 301 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 9.2.8 *If D_1 and D_2 are two k-edge-connected orientations of a graph G, then it is possible to reach D_2 from D_1 by successively reorienting current dipaths and di-circuits in such a way that each intermediate orientation is k-edge-connected.*

Proof. Let $f(v) := \varrho_2(v)$ for every $v \in V$. Since $\widetilde{f}(V)$ is just the number of edges of G, it follows for any orientation in which $\dot{\varrho} \geq f$ that equality must actually hold here.

Corollary 9.2.7 implies that, starting from D_1, it is possible to get a k-edge-connected orientation D_1' by reorienting paths in which the in-degree of each node v is at least $f(v)$ and hence exactly $f(v)$, and, in addition, every intermediate orientation is k-edge-connected. By Theorem 2.3.3, we can obtain D_2 from D_1' by reorienting di-circuits. Obviously the reorientation of a di-circuit does not affect the in-degree of any subset, so it preserves k-edge-connectivity. •

9.2.3 Simplification for $k = 1$

Nash-Williams' orientation theorem is significantly more difficult for $k \geq 2$ than for $k = 1$ when it specializes to Robbins' theorem (Theorem 2.2.8). As for rooted edge-connectivity, we pointed out earlier that Theorem 9.1.10 could be simplified when $k = 1$. In light of this observation, it is quite natural that the situation with Theorem 9.2.6 is similar. Recall that for an undirected graph $G = (V, E)$, $\sigma(X) = \sigma_G(X)$ denotes the number of components of the subgraph induced by $G - X$ when $X \neq \emptyset$ and $\sigma(\emptyset) = 0$. The following result was found by Frank and Gyárfás [137].

Theorem 9.2.9 *Let $f : V \to \mathbf{Z}_+ \cup \{-\infty\}$ and $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ be two functions for which $f \leq g$. A 2-edge-connected undirected graph $G = (V, E)$ has a strongly connected orientation in which:*

(A) $\dot{\varrho} \geq f$ *if and only if* $\widetilde{f} \leq e_G - \sigma$, *that is, if*

$$\widetilde{f}(X) \leq e_G(X) - \sigma(X) \text{ for every subset } X \subseteq V, \tag{9.19}$$

(B) $\dot{\varrho} \leq g$ *if and only if* $\widetilde{g} \geq e_G + \sigma$, *that is, if*

$$\widetilde{g}(X) \geq i_G(X) + \sigma(X) \text{ for every subset } X \subseteq V, \tag{9.20}$$

(C) $f \leq \dot{\varrho} \leq g$ *if and only if both* (9.19) *and* (9.20) *hold.*

Proof. Since the situation in Parts (B) and (C) is analogous, we deal only with (A).

Suppose that $\dot{\varrho} \geq f$ holds in a strong orientation of G. We claim that the out-degree (and also the in-degree) of a subset $X \subseteq V$ in a strong orientation of G is at least $\sigma(X)$. Indeed, every component of $G - X$ admits at least one entering edge, so there are at least $\sigma(X)$ edges leaving X. Hence $\widetilde{f}(X) \leq \ddot{\varrho}(X) = e_G(X) - \delta(X) \leq e_G(X) - \sigma(X)$, and hence (9.19) is indeed necessary.

For sufficiency, assume that (9.19) holds. Then $\widetilde{f}(V) \leq e_G(V) - \sigma(V) = |E|$. We are going to prove that (9.17) also holds for $k = 1$. Suppose indirectly that a partition $\mathcal{P} = \{V_0, V_1, \ldots, V_q\}$ of V violates (9.17). Choose such a $\mathcal{P}$ for which q is minimum. We claim that there is no edge connecting V_i and V_j for $1 \leq i < j \leq q$. Indeed, if there were an edge between V_i and V_j, then by replacing V_i and V_j with their union, since $k = 1$, we would obtain from $\mathcal{P}$ another partition of V which also violates (9.17), contradicting the minimality of $|\mathcal{P}|$. It follows that $\sigma(V_0) \geq q$ and $e_G(\mathcal{P}) = d_G(V_0)$. By (9.19), we have $e_G(\mathcal{P}) + i_G(V_0) = d_G(V_0) + i_G(V_0) = e_G(V_0) \geq \widetilde{f}(V_0) + \sigma(V_0) \geq \widetilde{f}(V_0) + q$, contradicting the indirect assumption that $\mathcal{P}$ violates (9.17) for $k = 1$. Therefore Case (A) of Theorem 9.2.6 applies and ensures the existence of the requested orientation. •

9.3 Common generalization: (k, ℓ)-edge-connected orientations

Despite the differences between rooted and global k-edge-connectivity orientation problems, one may legitimately feel that there should be a common ground. Indeed, in this section we describe an orientation theorem which involves, as a special case, both Nash-Williams' theorem and Theorem 9.1.8. The proof will be a slightly trickier application of the path-reorientation technique that was successfully used earlier. Recall that for non-negative integers ℓ and k, a digraph is (k, ℓ)-edge-connected if it has a node r_0 such that the in-degree and the out-degree of every non-empty subset $X \subseteq V - r_0$ are at least k and ℓ, respectively. By Menger's theorem, this is equivalent to requiring that there are k edge-disjoint paths from r_0 to every other node and that there are ℓ edge-disjoint paths from every node to r_0. In the special case when $\ell = 0$, we are back at rooted k-edge-connectivity, while the case $k = \ell$ corresponds to k-edge-connectivity.

Note that if $\ell \leq k$ and a digraph is (k, ℓ)-edge-connected with respect to a root-node r_0, then, by taking $k - \ell$ edge-disjoint paths from r_0 to another node r_1 and reorienting the edges of these paths, one obtains a (k, ℓ)-edge-connected orientation of G with respect to r_1. Therefore, the choice of the root does not make any difference.

As a natural generalization of the partition type condition in Theorem 9.1.8, one can formulate the following condition: *the number of cross-edges to every partition of $G = (V, E)$ into $q \geq 2$ parts is at least $k(q-1) + \ell$.* Such a graph is said to be (k, ℓ)-**partition-connected**. The following result and proof technique were developed in [114] where a more general orientation result concerning supermodular functions was proved. Here we exhibit only the special case of (k, ℓ)-edge-connected orientations while the general result will be discussed (with a different proof) in Theorem 15.4.1.

Theorem 9.3.1 (Frank [114]) *Let $G = (V, E)$ be an undirected graph with a designated root-node r_0 and let $0 \leq \ell \leq k$ be integers. G has a (k, ℓ)-edge-connected orientation if and only if G is (k, ℓ)-partition-connected. G has an (ℓ, k)-edge-connected orientation if and only if G is (k, ℓ)-partition-connected.*

Proof. The two halves of the theorem are equivalent since an orientation is (ℓ, k)-edge-connected if and only if its reverse orientation is (k, ℓ)-edge-connected. We are going to prove the first part.

In a (k, ℓ)-edge-connected orientation of G (with respect to r_0), there are at least k edges entering each of the $p - 1$ members of a p-partition $\mathcal{P}$ not containing r_0 and at least ℓ edges entering the member of $\mathcal{P}$ containing r_0. Hence the number of cross-edges to $\mathcal{P}$ is indeed at least $k(p-1) + \ell$, showing that (k, ℓ)-partition-connectivity of G is indeed necessary for a (k, ℓ)-edge-connected orientation.

To see sufficiency, define a set-function $h_{k\ell}$ as follows.

$$h_{k\ell}(X) := \begin{cases} k & \text{if } \emptyset \subset X \subseteq V - r_0 \\ \ell & \text{if } r_0 \in X \subset V \\ 0 & \text{if } X = \emptyset \text{ or } X = V. \end{cases} \tag{9.21}$$

Note that (k, ℓ)-partition connectivity is equivalent to requiring

$$e_G(\mathcal{P}) \geq \sum_i h_{k\ell}(V_i) \tag{9.22}$$

for every partition $\mathcal{P} = \{V_1, \ldots, V_p\}$ of V where $e_G(\mathcal{P})$ denotes the number of cross-edges to $\mathcal{P}$.

It follows from the definition that $h_{k\ell}$ is crossing modular. All that we will use is that $h_{k\ell}$ is crossing supermodular, that is, $h_{k\ell}(X) + h_{k\ell}(Y) \leq h_{k\ell}(X \cap Y) + h_{k\ell}(X \cup Y)$ whenever X and Y are crossing. We say that an orientation D of G **covers** $h_{k\ell}$ if $\varrho_D(X) \geq h_{k\ell}(X)$ for every $X \subseteq V$. Clearly, an orientation is (k, ℓ)-edge-connected if and only it covers $h_{k\ell}$. For brevity, we call an orientation covering $h_{k\ell}$ **good** while a non-good orientation is **bad**. A bad orientation is **near-good** with respect to nodes s and t if adding a new directed st-edge results in a (k, ℓ)-edge-connected digraph.

In what follows, we abbreviate $h_{k\ell}$ to h. In a near-good orientation, call a subset X **in-deficient** or **in-tight** according to whether $\varrho_D(X) = h(X) - 1$ or $\varrho_D(X) = h(X)$. The complement of X is called **out-deficient** or **out-tight**, respectively. Suppose now that G is (k, ℓ)-partition-connected but, indirectly, that G admits no good orientation.

Claim 9.3.2 *Let D be a orientation of G that is near-good with respect to nodes s and t. Then the set-system of in-deficient sets is closed under taking union, and there is a unique largest in-deficient set.*

Proof. Since every in-deficient set is a $t\bar{s}$-set, for two in-deficient sets X and Y we have

$$\begin{aligned} \varrho_D(X) + \varrho_D(Y) = h(X) - 1 + h(Y) - 1 \leq h(X \cap Y) - 1 + h(X \cup Y) \\ - 1 \leq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \leq \varrho_D(X) + \varrho_D(Y) \end{aligned}$$

from which $h(X \cap Y) - 1 = \varrho_D(X \cap Y)$ and $h(X \cup Y) - 1 = \varrho_D(X \cup Y)$ must hold. Since there is no good orientation of G, there is at least one in-deficient set. Since each in-deficient set is a $t\bar{s}$-set, it follows that the union of all in-deficient sets is in-deficient. •

Lemma 9.3.3 *Let D be a near-good orientation of G with respect to nodes s and t, and let N denote the unique largest in-deficient set* (ensured by Claim 9.3.2). *Then there is a dipath the reorientation of which results in an orientation D' of G such that D' is either good or else D' is near-good* (with respect to s and t) *and the unique largest in-deficient set in D' is a proper subset of N.*

Proof. Define a partition $\mathcal{P}$ as **in-deficient** (or respectively, **out-deficient**) if one of its classes is in-deficient and the others are in-tight (if one of its classes is out-deficient and the others are out-tight). The partition is **deficient** if it is in-deficient or out-deficient.

Claim A *There are no deficient partitions.*

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 304 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proof. The number α of cross-edges to a partition $\mathcal{P} = \{V_1, \dots, V_p\}$ is $\sum_i \varrho_D(V_i) = \sum_i \delta_D(V_i)$. If $\mathcal{P}$ is in-deficient, then $\alpha = \sum_i \varrho_D(V_i) = \sum_i h(V_i) - 1 = k(p-1) + \ell - 1$, contradicting (k, ℓ)-partition-connectivity. If $\mathcal{P}$ is out-deficient, then $\alpha = \sum_i \delta_D(V_i) = \sum_i h(V - V_i) - 1 = \ell(p-1) + k - 1 \leq k(p-1) + \ell - 1$, contradicting (k, ℓ)-partition-connectivity. •

Claim B *If X and Y are two crossing in-tight sets for which $X \cap Y \not\subseteq N$, then both $X \cap Y$ and $X \cup Y$ are in-tight.*

Proof. Since $X \cap Y \not\subseteq N$, neither $X \cap Y$ nor $X \cup Y$ is in-deficient, from which $h(X) + h(Y) = \varrho_D(X) + \varrho_D(Y) \geq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \geq h(X \cap Y) + h(X \cup Y)$ and therefore both $X \cap Y$ and $X \cup Y$ must be in-tight. •

Claim C *There is a node $b \in V - N$ for which every in-tight set containing b intersects N.*

Proof. By Claim B, the maximal in-tight sets disjoint from N are pairwise disjoint. If the requested node b does not exist, then these maximal in-tight sets along with N form a deficient partition of V, contradicting Claim A. •

Claim D *Let B be an in-tight set containing the node b* (ensured by Claim B)*, and let N' be an in-deficient set crossing B. Then $B \cap N'$ is in-deficient and $t \in B \cap N'$.*

Proof. Since $B \cup N'$ contains b, we have $\varrho_D(B \cup N') \geq h(B \cup N')$ and therefore $h(B) + h(N') - 1 = \varrho_D(B) + \varrho_D(N') \geq \varrho_D(B \cap N') + \varrho_D(B \cup N') \geq h(B \cap N') - 1 + h(B \cup N') \geq h(B) + h(N') - 1$ from which $\varrho_D(B \cap N') = h(B \cap N') - 1$. Therefore $B \cap N'$ is in-deficient and hence $t \in B \cap N'$. •

It follows from Claim B that the maximal out-tight sets not containing b but intersecting N are pairwise disjoint. Among them, let $K_1, K_2, \dots, K_p$ $(p \geq 0)$ denote those intersecting $V - N$ while $L_1, L_2, \dots, L_q$ $(q \geq 0)$ are those lying in N.

Suppose first that there is a node $a \in N$ not belonging to any K_i or L_j. Then there exists an ab-path P, for otherwise there would be an $a\bar{b}$-set of out-degree zero. But such a set is necessarily out-tight contradicting the assumption that a does not belong to any out-tight $a\bar{b}$-set. The reorientation of P can decrease the in-degree of any subset X of nodes by at most one and it does decrease the in-degree precisely if X is a $b\bar{a}$-set. Moreover, the reorientation of P increases the in-degree of N. Since there are no in-tight $b\bar{a}$-sets, the reorientation of P does not create any new in-deficient set and ends the in-deficiency of N. Therefore, in this case P satisfies the requirements of the lemma.

Suppose now that the sets K_i's and L_j's cover N. There must exist a set K_i, in other words, p must be positive, for otherwise $\{L_1, \dots, L_q, V - N\}$ would form an out-deficient partition and this would contradict Claim A. Each $B_i := V - K_i$ is in-tight and crosses N since $B_i \not\subseteq V - N$. Therefore Claim D implies that each B_i contains t, and hence none of the K_i's contains t, implying that there must exist a set L_j, that is, $q \geq 1$.

By the repeated application of Claim D, we obtain inductively that $N_1 := B_1 \cap N$ is in-deficient and that each of the sets $N_2 := B_2 \cap N_1, \dots, N_p := B_p \cap N_{p-1}$ is in-deficient. Since $N_p = (B_1 \cap \dots \cap B_p) \cap N$, we conclude that $V - N_p$ along with the sets $L_1, \dots, L_q$ form a partition which is deficient contradicting Claim A. Therefore this case cannot occur and the proof of the lemma is complete. • •

Returning to the proof of the theorem, recall the indirect assumption that no (k, ℓ)-edge-connected orientation of G exists. By adding sufficiently many new edges to G, we obtain a graph having a (k, ℓ)-edge-connected orientation. Therefore there exists a counterexample which does have a (k, ℓ)-edge-connected orientation after adding just one new edge. By removing the new edge, we obtain a near-good orientation of G which can be made good by applications of Lemma 9.3.3, contradicting the indirect assumption. • •

Remark 9.3.1 There is a useful interpretation of Theorem 9.3.1. By definition, the property of (k, ℓ)-partition-connectivity is in co-**NP** in the sense that its absence can be demonstrated by exhibiting a deficient p-partition (one admitting less than $k(p-1)+\ell$ cross-edges). But how can one certify that a given graph is (k, ℓ)-partition-connected? More exactly, what would be an **NP**-characterization of (k, ℓ)-partition-connectivity? Theorem 9.3.1 ensures a (k, ℓ)-edge-connected orientation of G and such an orientation serves the requested certificate since a digraph can be tested for (k, ℓ)-edge-connectivity in polynomial time via MFMC computations.

Algorithmic aspects

For the sake of brevity, we organized the proof above in an indirect scheme. The proof, however, can be transformed into a polynomial time algorithm. To this end, we can start with a rooted k-edge-connected orientation of G since the proof of Theorem 9.1.8 provided an algorithm to construct such an orientation.

Next, add some new directed edges entering r_0 in such a way that the enlarged digraph is (k, ℓ)-edge-connected. (At most ℓn new edges will suffice.) In the main course of the algorithm, we try to remove the newly added edges one by one in an arbitrary order. One phase consists of getting rid of one new edge $e = sr_0$ in such a way that the reduced graph continues to have a (k, ℓ)-edge-connected orientation. After at most ℓn phases all new edges will be eliminated and the resulting digraph will be a good orientation of G. Let D denote a digraph that is near-good with respect to s and r_0 and investigate the steps of the proof above from an algorithmic point of view. Note that in the present approach the root-node r_0 takes the role of node t in the proof above.

Let $\lambda(u, v) = \lambda_D(u, v)$ denote the minimum in-degree of a $v\bar{u}$-set. Standard submodularity shows that the set-system $\mathcal{F} := \{X : \varrho_D(X) = \lambda(u, v), v \in X \subseteq V - u\}$ is closed under taking intersection and union, and hence there is a unique smallest and a largest member of $\mathcal{F}$. By Menger's theorem, there are $\lambda(u, v)$ edge-disjoint paths and with the help of an MFMC computation, these paths can be computed as can the unique smallest and the unique largest $v\bar{u}$-sets of in-degree $\lambda(u, v)$. Based on such a subroutine, the sets $K_1, \dots, K_p$ as well as the sets $L_1, \dots, L_q$ appearing in the proof of the lemma above are all computable. Therefore we are able to construct the requested orientation if it exists or a partition violating the requirement for (k, ℓ)-partition-connectivity.

A constructive characterization for (k, ℓ)-partition-connected graphs

By combining Theorems 8.3.3 and 9.3.1, we obtain the following result.

Theorem 9.3.4 *Let $G = (V, E)$ be an undirected graph and $0 \le \ell \le k$ integers. G is (k, ℓ)-partition-connected if and only if it can be built up from a node by consecutively applying the following operations.*

(A) *Add a new edge connecting two existing nodes.*
(B) *Pinch j existing edges ($\ell \le j \le k$) together with a new node z and add $k - j$* (possibly parallel) *edges connecting z with existing nodes.* •

9.4 Orientation of hypergraphs

Several graph orientation results discussed earlier can be extended to hypergraphs. In some cases this is quite straightforward while new techniques are required for others. The results of this section are from Frank, Király, and Király [145].

9.4.1 Degree-constrained orientation

A particularly convenient situation is when the hypergraph can be trimmed to a graph that has the requested orientation.

Theorem 9.4.1 *Let $H = (V, \mathcal{E})$ be a hypergraph in which every hyperedge is at least of size two. Let $m : V \to \mathbf{Z}_+$ be a function for which $i_H \le \widetilde{m}$ where $i_H(X)$ denotes the number of hyperedges induced by X. Then it is possible to trim H in such a way that the resulting graph G satisfies $i_G \le \widetilde{m}$.*

Proof. By induction, it suffices to prove that a hyperedge Z with $|Z| \ge 3$ can be trimmed in such a way that $i_{H'} \le \widetilde{m}$ holds for the resulting hypergraph H'. Let u and v be two elements of Z. Define a set $X \subseteq V$ as **tight** if $i_H(X) = \widetilde{m}(X)$.

If the replacement of Z by $Z - u$ yields a hypergraph H' for which $i_{H'} \not\le \widetilde{m}$, then there is a tight set X (with respect to H) for which $u \notin X$ and $Z - u \subseteq X$. Similarly for v, we can assume that there is a tight set Y for which $v \notin Y$ and $Z - v \subseteq Y$. But then the hyperedge Z shows that $d_H(X, Y) \ge 1$, and hence Proposition 1.2.9 and the assumption $i_H \le \widetilde{m}$ imply that $\widetilde{m}(X) + \widetilde{m}(Y) = i_H(X) + i_H(Y) = i_H(X \cap Y) + i_H(X \cup Y) - d_H(X, Y) < \widetilde{m}(X \cap Y) + \widetilde{m}(X \cup Y) = \widetilde{m}(X) + \widetilde{m}(Y)$, a contradiction. (Here $d_H(X, Y)$ denotes the number of hyperedges Z' for which $Z' \subseteq X \cup Y$, $Z' \cap (X - Y) \ne \emptyset$ and $Z' \cap (Y - X) \ne \emptyset$.) •

Problem 9.4.1 *Develop an algorithm to test $i_H \le \widetilde{m}$.*

Theorem 9.4.2 (Hypergraph orientation lemma) *Let $H = (V, \mathcal{E})$ be a hypergraph in which every hyperedge is at least of size two. Let $m : V \to \mathbf{Z}_+$ be a degree specification. H has an orientation in which the in-degree of every node v is $m(v)$ if and only if*

$$\widetilde{m}(V) = |\mathcal{E}| \text{ and } i_H \le \widetilde{m} \tag{9.23}$$

where $i_H(X)$ denotes the number of hyperedges induced by X.

Proof. Proving necessity is a simple exercise and is left to the reader. The sufficiency follows directly from Theorem 9.4.1 and from the Orientation lemma (Theorem 2.3.2). •

The proof of Theorem 9.4.1 is algorithmic (in fact, it is a greedy-type algorithm) provided a subroutine is available to test whether a given hypergraph H' satisfies $i_{H'} \le \widetilde{m}$. Although such a subroutine can be constructed by using, for example, an MFMC algorithm, we describe an alternative, more direct way to reduce Theorem 9.4.2 algorithmically to the graph orientation lemma. In addition, we show how Theorem 2.3.5 on in-degree constrained orientations of graphs can be used to derive its extension to hypergraphs. Note that the problem of finding a system of distinct representatives of a hypergraph H (which was answered by Hall's theorem, see Theorem 2.4.5) can be interpreted as one concerning the existence of an orientation of H in which the in-degree of every node is at most 1.

Theorem 9.4.3 *A hypergraph $H = (V, \mathcal{E})$ has an orientation for which $f \leq \dot{\varrho} \leq g$ if and only if both $e_H \geq \widetilde{f}$ and $i_H \leq \widetilde{g}$ hold.*

Proof. Let $G = (V, U; F)$ denote the bipartite graph associated with hypergraph $H = (V, \mathcal{E})$. Recall that there is a one-to-one correspondence between the hyperedges of H and the elements of U. Hence $d_G(t) = |T|$ for $t \in U$ where $T \in \mathcal{E}$ is the hyperedge corresponding to t. Extend f and g to $V \cup U$ by $f(t) := g(t) := d_G(t) - 1$ for $t \in U$.

We are going to show that the conditions in Part (C) of Theorem 2.3.5 are fulfilled. To this end, consider a subset $Z = X \cup Y$ where $X \subseteq V$, $Y \subseteq U$. Let j denote the number of nodes in Y having at least one neighbour in $V - X$. The construction of the associated graph G implies that $j \geq |Y| - i_H(X)$ and we have $\widetilde{g}(Z) = \widetilde{g}(X) + \widetilde{g}(Y) = \widetilde{g}(X) + (d_G(Y) - |Y|) \geq i_H(X) + d_G(Y) - |Y| \geq d_G(Y) - j \geq i_G(Z)$.

Let l denote the number of edges of G connecting Y and $V - X$, that is, $l = e_G(Z) - d_G(Y)$. Then we have $e_H(X) = |\Gamma_G(X)| \leq |Y| + l = |Y| + e_G(Z) - d_G(Y)$ and hence $\widetilde{f}(Z) = \widetilde{f}(X) + \widetilde{f}(Y) = \widetilde{f}(X) + [d_G(Y) - |Y|] \leq e_H(X) + (d_G(Y) - |Y|) \leq e_G(Z)$.

By Theorem 2.3.5 there is an orientation of G for which the in-degree of every node $t \in U$ is exactly $d_G(t) - 1$ while the in-degree of a node $v \in V$ falls between $f(v)$ and $g(v)$. Such an orientation of G determines an orientation of the hypergraph H for which $f(v) \leq \varrho(v) \leq g(v)$ for every node $v \in V$. •

Note that the proof of Theorem 2.3.5 was algorithmic and hence that a degree-constrained orientation of a hypergraph can also be computed in polynomial time.

9.4.2 Rooted k-edge-connected orientation

Relying on Theorem 9.4.2, we can extend Theorem 9.1.8 to hypergraphs too, but the approach of the proof, as indicated earlier, will differ from the one we applied for Theorem 9.1.8. We shall assume that the cardinality of every hyperedge is at least two. For a given partition $\mathcal{F}$ of V, we say that a hyperedge is a **cross-hyperedge** to $\mathcal{F}$ if it intersects at least two members of the partition.

Theorem 9.4.4 *Let $H = (V, \mathcal{E})$ be a hypergraph in which every hyperedge has at least two elements, and let $r_0 \in V$ be a designated root-node. There is a rooted k-edge-connected orientation of H if and only if H is k-partition-connected, that is,*

$$e_H(\mathcal{F}) \geq k(t-1) \text{ holds for every } t\text{-partite partition } \mathcal{F} \text{ of } V \qquad (9.24)$$

where $e_H(\mathcal{F})$ denotes the number of cross-hyperedges to $\mathcal{F}$.

Proof. If there is a good orientation and $\{V_1, \ldots, V_t\}$ is a partition of V, then $\varrho(V_i) \geq k$ for every subset $V_i \subseteq V - r_0$. Therefore $e_H(\mathcal{F}) = \sum_i \varrho(V_i) \geq k(t-1)$, end hence (9.24) holds.

For sufficiency, define a set-function p_k as follows.

$$p_k(X) := \begin{cases} k + i_H(X) & \text{if } \emptyset \subset X \subseteq V - r_0 \\ i_H(X) & \text{otherwise.} \end{cases} \tag{9.25}$$

Then p_k is intersecting supermodular. Let $m : V \to \mathbf{Z}$ be a function for which $m(r_0) = 0$ and $\widetilde{m} \geq p_k$. Supposing that m is minimal in the sense that reducing the value of $m(v)$ by 1 (for any $v \in V - r_0$), we have $\widetilde{m}' \not\geq p_k$ for the resulting m'. The minimality of m implies that every node $v \in V$ belongs to a tight set X where tightness means that $\widetilde{m}(X) = p_k(X)$. For two intersecting tight sets X and Y, we have

$$\begin{aligned} \widetilde{m}(X) + \widetilde{m}(Y) &= p_k(X) + p_k(Y) \leq p_k(X \cap Y) + p_k(X \cup Y) \leq \widetilde{m}(X \cap Y) + \widetilde{m}(X \cup Y) \\ &= \widetilde{m}(X) + \widetilde{m}(Y) \end{aligned}$$

from which it follows that both $X \cap Y$ and $X \cup Y$ are also tight. Therefore the maximal tight sets $V_1, \ldots, V_t$ are pairwise disjoint and cover all nodes. In other words, $\mathcal{F} := \{V_1, V_2, \ldots, V_t\}$ is a partition of V into tight sets. Suppose that $r_0 \in V_1$. We know from (9.2) that $e_H(\mathcal{F}) \geq k(t-1)$ and hence

$$\begin{aligned} i_H(V) = p_k(V) \leq \widetilde{m}(V) &= \widetilde{m}(V_1) + \sum_{j=2}^{t} \widetilde{m}(V_j) = i_H(V_1) + \sum_{j=2}^{t} [k + i_H(V_j)] \\ &= k(t-1) + \sum_{j=1}^{t} i_H(V_j) = k(t-1) + |\mathcal{E}| - e_H(\mathcal{F}) \leq |\mathcal{E}| = i_H(V), \end{aligned}$$

implying that $\widetilde{m}(V) = |\mathcal{E}|$. Since $\widetilde{m} \geq p_k \geq i_H$, we can apply the Hypergraph orientation lemma and conclude that H has an orientation for which $\dot{\varrho} = m$. This orientation is rooted k-edge-connected since $\varrho(Z) = \ddot{\varrho}(Z) - i_H(Z) = \widetilde{m}(Z) - i_H(Z) \geq p_k(Z) - i_H(Z) = k$ for every non-empty $Z \subseteq V - r_0$. •

Theorem 9.4.5 *A k-partition-connected hypergraph $H = (V, \mathcal{E})$ can be trimmed to a k-partition-connected graph. If H is minimal in the sense that removing any hyperedge destroys k-partition-connectivity, then H has exactly $k(|V| - 1)$ hyperedges.*

Proof. By Theorem 9.4.4, H has a rooted k-edge-connected orientation $\vec{H}$. By Theorem 7.4.9, $\vec{H}$ can be trimmed to a rooted k-edge-connected digraph D. The underlying undirected graph G of D is a k-partition-connected graph that is obtained from H by trimming. If H is minimally rooted k-partition-connected, then D is minimally rooted k-edge-connected and by Corollary 7.4.2, D has $k(|V| - 1)$ edges. •

In order to have an algorithmic approach for finding a rooted k-edge-connected orientation of a hypergraph H, if one exists, and also for finding a t-partition of V violating (9.24) in the case when no such orientation exists, we present an alternative proof of Theorem 9.4.4, which is an extension of the algorithmic proof of Theorem 9.1.8. From another perspective, the algorithm determines if a given hypergraph is k-partition-connected or not.

Algorithmic proof of the sufficiency of (9.24). Consider the bipartite graph $G = (V, U; F)$ associated with H. We need to find an orientation of G which is k-edge-connected from r_0 to

V (that is, $\varrho(X) \geq k$ for every non-empty subset $X \subseteq (V \cup U) - r_0$ intersecting V) in such a way that the in-degree of every node $t \in U$ is exactly $d_G(t) - 1$. Temporarily, we define such an orientation of G as **good**.

Add a minimum number of new edges between r_0 and $V - r_0$ in such a way that the enlarged graph has a good orientation. (Such an orientation certainly exists if k parallel r_0v-edges are added for every node v.) Let γ denote this minimum. Our goal is to prove that $\gamma = 0$.

Let ϱ^+ denote the in-degree function of the current orientation of the enlarged graph. We can assume $\varrho^+(r_0) = 0$. A subset $X \subseteq (V \cup U) - r_0$ with $X \cap V \neq \emptyset$ is **tight** if $\varrho^+(X) = k$.

Claim 9.4.6 *If X and Y are two tight sets for which $X \cap Y \cap V \neq \emptyset$, then both $X \cap Y$ and $X \cup Y$ are tight.*

Proof. We have $k + k = \varrho^+(X) + \varrho^+(Y) \geq \varrho^+(X \cap Y) + \varrho^+(X \cup Y) \geq k + k$ from which $\varrho^+(X \cap Y) = k$ and $\varrho^+(X \cup Y) = k$ follow. •

The claim implies that if there are tight sets containing a node $v \in V$, then their intersection $T(v)$ is also tight. Let $z \in V$ be the head of a newly added edge e^+ and let Z denote the set of nodes reachable from z in the given orientation of G. Then $r_0 \notin Z$ and $\varrho^+((U \cup V) - Z) = 0$.

Suppose first that there is a node $v \in Z \cap V$ that is not contained in any tight set. Consider a directed zv-path P' and let $P = P' + e^+$. The reorientation of P can decrease the in-degree of any subset by at most one and if it does decrease the in-degree of X, then X is a $v\bar{r}_0$-set. Since v is not in any tight set, the reorientation of P does not destroy rooted k-edge-connectivity. But in this case the reverse of the new edge e^+ can be left out contradicting the minimal choice of γ.

Second, suppose that each element v of $Z \cap V$ belongs to a tight set.

Claim 9.4.7 *$T(v) \subseteq Z$ for every $v \in Z$.*

Proof. Suppose indirectly that $T(v) \not\subseteq Z$ for some $v \in Z$. Then for $Y := V - Z$ we have $0 + k = \varrho^+(Y) + \varrho^+(T(v)) = \varrho^+(Y \cap T(v)) + \varrho^+(Y \cup T(v)) + d^+(Y, T(v)) \geq k + 0 + d^+(Y, T(v)) \geq k$. Hence $\varrho^+(Y \cup T(v)) = 0$ and $d^+(Y, T(v)) = 0$ follow. Due to the new r_0z-edge e^+ and to $d^+(Y, T(v)) = 0$, we have $z \notin T(v)$. In this case, however, $\varrho^+(Y \cup T(v)) = 0$ implies that v is not reachable from z, contradicting the definition of Z. •

Let $V'_2, \ldots, V'_t$ denote the maximal tight subsets of Z and let $V'_1 := (V \cup U) - Z$. By Claims 9.4.6 and 9.4.7, the sets $V_1 := V'_1 \cap V, V_2 := V'_2 \cap V, \ldots, V_t := V'_t \cap V$ form a partition $\mathcal{F}$ of V.

Lemma 9.4.8

$$e_H(\mathcal{F}) < \textstyle\sum_{i=1}^{t} \varrho^+\left(V'_i\right). \tag{9.26}$$

Proof. Observe first that the new r_0z-edge e^+ contributes only to the right-hand side.

For each cross-hyperedge X of H to $\mathcal{F}$, let u_X denote the element of U corresponding to X. We are going to show that at least one of the edges incident to u_X (whether leaving or entering it) enters one of the sets $V'_1, \dots, V'_t$ and in this will imply the claim. Since we are done if the only edge e_X leaving u_X enters some V'_i, we can suppose that this is not the case. Since X intersects one of $V_2, \dots, V_t$ and no directed edge leaves Z, we see that u_X is in Z. Since e_X does not enter any V'_i, u_X must be in one of the sets $V'_2, \dots, V'_t$, say in V'_2, moreover, e_X cannot leave V'_2. Since $X \not\subseteq V_2$, there is an element $v \in X \cap V_i$ for some $i \neq 2$ and hence the directed edge vu_X enters V'_2. •

By the lemma we have $e_H(\mathcal{F}) < \sum_{i=1}^t \varrho^+\left(V'_i\right) = k(t-1)$, contradicting (9.24). • •

9.4.3 Strongly connected orientation and an outlook

Recall Robbins' orientation theorem (Theorem 2.2.8) stating that a graph has a strong orientation if and only if the graph is 2-edge-connected. Based on this, it is quite enticing to conjecture that a similar statement extends to hypergraphs. Although the 2-edge-connectivity of H is indeed necessary for strong orientability, it is not sufficient. Actually, this is not so surprising once we observe that a strong dypergraph on node-set V must have at least $|V|$ dyperedges (as every node is the head of at least one dyperedge) while a hypergraph having V in two parallel copies as hyperedges is 2-edge-connected. The right hypergraphic extension of Robbins' orientation theorem is as follows.

Theorem 9.4.9 *A hypergraph $H = (V, \mathcal{E})$ has a strongly connected orientation if and only if H is $(1,1)$-partition-connected, that is,*

$$e_H(\mathcal{P}) \geq |\mathcal{P}| \text{ for every partition } \mathcal{P} \text{ of } V \tag{9.27}$$

where $e_H(\mathcal{P})$ denotes the number of hyperedges intersecting at least two members of $\mathcal{P}$.

Proof. In a strongly connected orientation of H, the in-degree of each part of a partition is at least one showing the necessity of (9.27). For sufficiency, consider the bipartite graph $G = (V, U; F)$ associated with H. Define $g : V \cup U \to \mathbf{Z}_+ + \{\infty\}$ by $g(t) = 1$ for $t \in U$ and $g(v) = \infty$ for $v \in V$.

Claim 9.4.10 $i_G(X) \leq \widetilde{g}(X) - \sigma_G(X)$ *for every non-empty subset X of $V \cup U$.*

Proof. The inequality clearly holds if $\widetilde{g}(X) = \infty$ so we assume that $\widetilde{g}(X) < \infty$ and hence $X \subseteq U$. In this case, $\widetilde{g}(X) = |X|$ and $i_G(X) = 0$ and the inequality turns to $\sigma_G(X) \leq |X|$. But this is equivalent to requiring that the number of components of the hypergraph arising by leaving out $j = |X|$ hyperedges from H is at most j which is a reformulation of (9.27). •

By Part (B) of Theorem 9.2.9, there is a strongly connected orientation of G in which the in-degree of every node on U is at most 1, and, in fact, exactly 1 because of strong connectivity. By reorienting each edge, we obtain an orientation of G in which the out-degree of every node in H is one. Such an orientation of G determines a strongly connected orientation of H. • •

There is another way to look at Theorem 9.4.9. By its definition, the $(1,1)$-partition-connectivity of a hypergraph is a co-**NP** property in the sense that its absence can be certified by displaying a partition $\mathcal{P}$ of V violating (9.27). Theorem 9.4.9 can be interpreted as providing an **NP**-characterization since the $(1,1)$-partition-connectivity of a hypergraph H

can be certified by displaying a strongly connected orientation of H. Note that the strong connectivity of a dypergraph (with dyperedges of cardinality at least two) is equivalent to the strong connectivity of its associated directed bipartite graph, a readily verifiable property.

We also remark that a $(1, 1)$-partition-connected hypergraph need not be trimmed to a $(1, 1)$-partition-connected graph (which is just the same as a 2-edge-connected graph): consider the hypergraph on node-set $\{a, b, c, d\}$ for which the hyperedges are $\{ab, ac, ad, bcd\}$.

The following consequence of Theorem 9.4.9 sheds some light on why in the graph case the simple 2-edge-connectivity was sufficient for strong orientability.

Corollary 9.4.11 *Let $H = (V, \mathcal{E})$ be a $(2q)$-edge-connected hypergraph in which the cardinality of every hyperedge is at most q. Then H has a strongly connected orientation.*

Proof. By Theorem 9.4.9, it suffices to prove that H is $(1, 1)$-partition-connected. For a t-partition $\mathcal{P} = \{V_1, \dots, V_t\}$ of V we have $e_H(\mathcal{P}) \geq [\sum_i d_H(V_i)]/q \geq tq/q = t$, as required. •

One may be wondering if Nash-Williams' theorem on k-edge-connected orientation of graphs could also be extended to hypergraphs. In Section 15.4, we shall show that such an orientation exists if and only if the hypergraph is (k, k)-partition-connected. The proof, however, requires polyhedral techniques, so we postpone it to Section 15.4 (see Theorem 15.4.3). Among others, we will prove the following more general result.

Theorem 9.4.12 *Let $\ell \leq k$ be non-negative integers. A hypergraph $H = (V, \mathcal{E})$ has a (k, ℓ)-edge-connected orientation if and only if H is (k, ℓ)-partition-connected, that is,*

$$e_H(\mathcal{P}) \geq k(|\mathcal{P}| - 1) + \ell \text{ for every partition } \mathcal{P} \text{ of } V. \bullet \tag{9.28}$$

There are two interpretations of this theorem. On the one hand, it can be viewed (and the theorem is actually formulated from this angle) as a co-**NP**-characterization of hypergraphs admitting a (k, ℓ)-edge-connected orientation. Indeed, a partition violating (9.28) is a polynomially verifiable certificate for the non-existence of the requested orientation. On the other hand, the theorem can be interpreted as an **NP**-characterization of (k, ℓ)-partition-connectivity of a hypergraph. Indeed, a (k, ℓ)-edge-connected orientation is verifiable in polynomial time by using appropriate MFMC computations in the bipartite graph associated with H.

In this light, if one is interested in an extension of Theorem 9.4.12 to the case when $k < \ell$, there are two different questions to be answered: one for finding a co-**NP**-characterization of hypergraphs admitting a (k, ℓ)-edge-connected orientation, and one for finding an **NP**-characterization for (k, ℓ)-partition-connectivity.

Consider first the (k, ℓ)-edge-connected orientability of H in the case when $k \leq \ell$. When H is an ordinary graph, this problem easily reduces to the case when $\ell \leq k$ since an orientation of a graph is (k, ℓ)-edge-connected if and only if the reverse orientation is (ℓ, k)-edge-connected. For general hypergraphs, however, Theorem 9.4.12 does not help and we need a separate characterization. (For a proof, see Theorem 15.4.4.)

Theorem 9.4.13 *Let $k \le \ell$ be non-negative integers. A hypergraph $H = (V, \mathcal{E})$ has a (k, ℓ)-edge-connected orientation if and only if* (9.28) *holds and*

$$\textstyle\sum_{Z \in \mathcal{E}} [e_{\mathcal{P}}(Z) - 1] \ge \ell(|\mathcal{P}| - 1) + k \text{ for every partition } \mathcal{P} \text{ of } V \qquad (9.29)$$

where $e_{\mathcal{P}}(Z)$ denotes the number of members of $\mathcal{P}$ intersecting Z. •

Second, we present an **NP**-characterization of (k, ℓ)-partition-connected hypergraphs. For graphs the notion of (k, ℓ)-partition-connectivity when $k \le \ell$ was shown to be equivalent to the $(k + \ell)$-edge-connectivity of the graph (see Proposition 1.2.11) and this property is clearly in **NP**. For general hypergraphs, however, (k, ℓ)-partition-connectivity has its own meaning for $k \le \ell$, as well. In this case the requested **NP**-characterization is as follows.

Theorem 9.4.14 ([145]) *Let $0 \le k \le \ell$ be integers. A hypergraph H is (k, ℓ)-partition-connected if and only if, for every pair of distinct nodes s and t, H has a k-edge-connected orientation in which $\varrho(Z) \ge \ell$ holds for every $s\bar{t}$-set Z.* •

With the help of MFMC computations, a given orientation of H can be tested in polynomial time whether the orientation does in fact satisfy the requirements given in the theorem. T. Király [238] described an algorithm for a specified integer k to compute the largest integer ℓ for which H is (k, ℓ)-partition-connected.

Conjecture 9.4.15 *Let H be a 3-uniform hypergraph. H has three head-disjoint strongly connected orientations if and only if the hypergraph is 3-edge-connected.* (Three orientations are said to be **head-disjoint** if the heads of a hyperedge are distinct in the three orientations for each hyperedge of H. If H is only 2-edge-connected, then the existence of just one strongly connected orientation does not follow: consider the hypergraph on three elements $\{a, b, c\}$ in which the hyperedges are $\{a, b, c\}$ and $\{a, b, c\}$ hyperedges.)

9.5 Mixed graphs and orientations

In Part I, we made the trivial observation that a graph G has a root-connected orientation if and only if G is connected. As an extension, Theorem 1.2.25 provided an easy characterization of mixed graphs having a root-connected orientation. Similarly, Robbins' theorem (Theorem 2.2.8) on strong orientability could rather easily be extended to mixed graphs (Theorem 2.2.12). These examples may suggest that theorems concerning orientations of undirected graphs can be extended to mixed graphs without any further effort. It will turn out, however, that unlike the two examples above, the situation for mixed graph orientations is far more complex.

To see a significant difference, recall Theorem 9.2.9 which was an occurrence of the linking propery. Perhaps surprisingly, the linking property for mixed graphs is not valid anymore, that is, the following statement is **false**:

If a mixed graph has a strong orientation in which the in-degree of every node v is at least $f(v)$, and it has a strong orientation in which the in-degree of every node v is at most $g(v)$ (where $f \le g$), then there is a strong orientation meeting both the upper and the lower bound constraints.

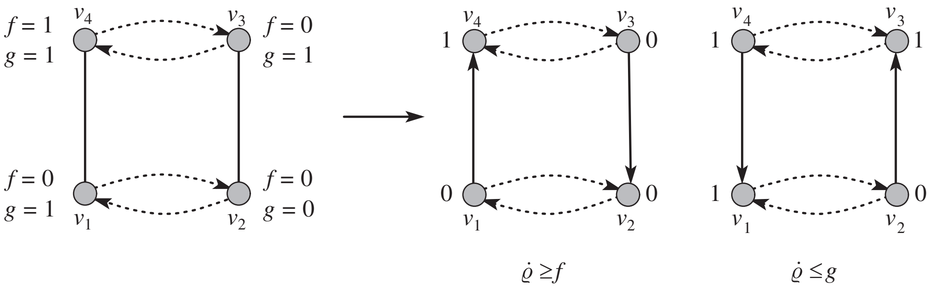


Figure 9.1 Linking property fails

A counterexample disproving the statement consists of four nodes and 6 edges. It has two undirected edges v_1v_4, v_2v_3, furthermore two oppositely directed parallel edges between v_3 and v_4, and between v_1 and v_2. The bounding functions (on the in-degrees of the subgraph of newly oriented edges) are $f = (1, 0, 0, 0)$ and $g = (1, 1, 0, 1)$. (See Figure 9.1.)

We can certainly be content by detecting and documenting that the linking property holds in some cases while not in others, but it would also be desirable to have a deeper insight.

Question 9.5.1 *What is in the background of the linking property?*

As we will see in Subsection 14.3.2, the linking property holds for generalized polymatroids, a special class of integral polyhedra, while it fails to hold for the intersection of two g-polymatroids. It turns out that the in-degree vectors of strong orientations of an undirected graph induce a g-polymatroid while in the case of mixed graphs the corresponding polyhedron is the intersection of two g-polymatroids. See Section 15.4 for further orientation problems where the linking property holds.

9.5.1 Rooted k-edge-connected orientation of mixed graphs

In this subsection, we show that Theorem 9.1.8 can be extended to mixed graphs, too. Although the characterization below for rooted k-edge-connected orientability of mixed graphs is formally quite similar to that of undirected graphs, the linking property is no more true for mixed graphs. To see this, consider the mixed graph on node-set $\{r_0, v_1, v_2, v_3, v_2\}$ with undirected edges $\{r_0v_1,\ r_0v_2,\ v_3v_4\}$ and with directed edges $\{v_1v_4,\ v_4v_1,\ v_2v_3,\ v_3v_4\}$, and let $f = (1, 1, 0, 0, 0),\ g = (1, 1, 1, 0, 1)$. (See Figure 9.2.)

Another difference is that the relatively simple path reorienting technique does not work anymore for mixed graphs. This is why we have chosen below a relatively short non-algorithmic proof based on the uncrossing technique. In Part III, a push–relabel algorithm will be presented for an even more general problem.

Before formulating the extension of Theorem 9.1.8 to mixed graphs, we introduce a useful notation. For a set-function $h : 2^V \to \mathbf{Z}_+$ and for a family $\mathcal{F}$ of sets, we use again the notation $\widetilde{h}(\mathcal{F}) := \sum_{X \in \mathcal{F}} h(X)$. For a regular hypergraph $(V, \mathcal{F})$ and for an edge $f = uv$, let $\beta(f, \mathcal{F})$ denote the number of members of $\mathcal{F}$ which contain v and do not contain u. By the regularity of $\mathcal{F}$, the roles of u and v are interchangeable. When $\mathcal{F}$ is a partition, $\beta(e, \mathcal{F})$ is 1

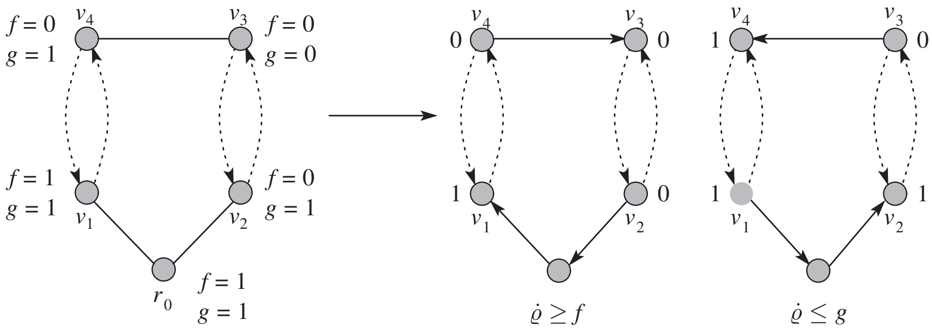


Figure 9.2 Linking property fails

or 0 according to whether e is a cross-edge to $\mathcal{F}$ or not. Therefore $e_G(\mathcal{F}) = \sum_{e \in E} \beta(e, \mathcal{F})$ for a graph $G = (V, E)$ and a partition $\mathcal{F}$ of V.

Theorem 9.5.1 *Let $M = (V, A + E)$ be a mixed graph consisting of an undirected graph $G = (V, E)$ and a directed graph $D = (V, A)$, and let $r_0 \in V$ be a designated root-node. M has a rooted k-edge-connected orientation with respect to r_0 if and only if*

$$e_G(\mathcal{P}) \geq \sum_{i=1}^{q} [k - \varrho_D(V_i)] \text{ holds for every partition } \mathcal{P} := \{V_0, V_1, \ldots, V_q\} \text{ of } V \tag{9.30}$$

where $r_0 \in V_0$ and $e_G(\mathcal{P})$ denotes the number of cross-edges of G to partition $\mathcal{P}$.

Proof. To see the necessity of (9.30), consider a rooted k-edge-connected orientation $D^+ = (V, A + \vec{E})$ of M and a partition $\mathcal{P} := \{V_0, V_1, \ldots, V_q\}$ of V for which $r_0 \in V_0$. Then $\varrho_D(V_i) + \varrho_{\vec{E}}(V_i) \geq k$ for $i = 1, \ldots, q$ and hence $e_G(\mathcal{P}) = \sum[\varrho_{\vec{E}}(V_i) : i = 1, \ldots, q] \geq \sum[k - \varrho_D(V_i)] : i = 1, \ldots, q]$, as required.

For the sufficiency of (9.30), assume indirectly that there is a mixed graph $M = (V, A + E)$ admitting no required orientation and suppose that $|E|$ is minimum. Define a set-function h as follows.

$$h(X) := \begin{cases} (k - \varrho_D(X))^+ & \text{if } \emptyset \subset X \subseteq V - r_0 \\ 0 & \text{otherwise.} \end{cases} \tag{9.31}$$

We claim that (9.30) implies the following condition:

$$e_G(\mathcal{P}) \geq \widetilde{h}(\mathcal{P}) \text{ holds for every partition } \mathcal{P} \text{ of } V. \tag{9.32}$$

To see this, suppose that (9.32) fails and let $\mathcal{P} := \{V_0, V_1, \ldots, V_q\}$ be a partition violating (9.32) for which $|\mathcal{P}|$ is minimum. We can assume that $r_0 \in V_0$. Then $k > \varrho_D(V_i)$ for $i = 1, \ldots, q$ for if $k \leq \varrho_D(V_q)$, say, then the partition $\mathcal{P}' := \{V_0 \cup V_q, V_1, \ldots, V_{q-1}\}$ would also violate (9.32) contradicting the minimal choice of $|\mathcal{P}|$. Therefore $k - \varrho_D(V_i) = (k - \varrho_D(V_i))^+$ for $i = 1, \ldots, q$, and hence $\mathcal{P}$ violates (9.30), too.

Due to the submodularity of ϱ_D, function h is **positively intersecting supermodular** in the sense that the supermodular inequality holds for X and Y whenever $X \cap Y$ is non-empty and both $h(X)$ and $h(Y)$ are positive.

By our assumption, there is no orientation of G which covers h in the sense that $\varrho \geq h$, that is, $\varrho(X) \geq h(X)$ for every subset $X \subseteq V$. Let $e = uv$ be an arbitrary edge of G. Orient first e toward v. This means that the directed edge uv is added to D while e is removed from G. Let $D' = (V, A')$, $G' = (V, E')$, and h' denote, respectively, the arising digraph, undirected graph, and set-function. Note that

$$h'(X) := \begin{cases} h(X) - 1 & \text{if } h(X) > 0,\ v \in X,\ u \notin X \\ h(X) & \text{otherwise.} \end{cases} \tag{9.33}$$

Since $|E'| = |E| - 1$, the mixed graph $M' := (V, A' + E')$ is not a counterexample. Hence if (9.32) holds for h', then M' has a rooted k-edge-connected orientation. But this is also an orientation of M, contradicting the indirect assumption that M is a counterexample. Therefore (9.32) fails to hold, that is, there is a violating partition $\mathcal{P}_1$ with respect to h'. This means that $e_G(\mathcal{P}_1) = \widetilde{h}(\mathcal{P}_1)$ and the member X_1 of $\mathcal{P}_1$ containing v cannot contain u, and $h(X_1) = 0$.

Next, by orienting e toward u, we obtain analogously that there is a partition $\mathcal{P}_2$ of V for which $e_G(\mathcal{P}_2) = \widetilde{h}(\mathcal{P}_2)$, the member X_2 of $\mathcal{P}_2$ containing u cannot contain v, and $h(X_2) = 0$.

Consider now the family $\mathcal{F} := \mathcal{P}_1 \cup \mathcal{P}_2$ of sets in the sense that a set X occurs in $\mathcal{F}$ in two copies provided X belongs to both $\mathcal{P}_1$ and $\mathcal{P}_2$. Starting with X_1 and X_2, apply the uncrossing procedure to the members of $\mathcal{F}$ as follows.

(A) Replace two disjoint sets X and Y by $X \cup Y$ if $h(X) = h(Y) = 0$.
(B) Replace two properly intersecting sets X and Y by $X \cap Y$ and $X \cup Y$ if $h(X) = h(Y) = 0$.
(C) Replace two properly intersecting sets X and Y by $X \cap Y$ and $X \cup Y$ if $h(X) > 0$ and $h(Y) > 0$.

In the first uncrossing step, we apply (A) or (B) to X_1 and X_2 according to whether $X_1 \cap X_2$ is empty or not. Each of the three operations preserves the regularity of the family. Also, since h is positively intersecting supermodular, the sum of the h-values over the family does not decrease.

Neither step increases the cardinality of the family and at Step (A), the number of sets in the family decreases by one. Therefore Step (A) can occur at most $|\mathcal{F}| \leq 2n$ times. Between two applications of Step (A) the number of sets is fixed but the sum of the square of their cardinalities strictly increases. It follows that the uncrossing procedure above terminates after a finite number of steps. Let $\mathcal{L}$ denote the laminar family obtained finally. As mentioned, this covers each element of V exactly twice and $\widetilde{h}(\mathcal{L}) \geq \widetilde{h}(\mathcal{F})$.

Claim 9.5.2 *$\mathcal{L}$ can be decomposed into two partitions of V.*

Proof. Let $\mathcal{L}_1$ consist of the minimal members (with respect to inclusion) of $\mathcal{L}$ in the sense that if a set X occurs with two copies in $\mathcal{L}$, then we put one copy into $\mathcal{L}_1$. By the laminarity of $\mathcal{L}$, $\mathcal{L}_1$ forms a subpartition which is actually a partition of V since $\mathcal{L}$ is 2-regular. It follows that $\mathcal{L}_2 := \mathcal{L} - \mathcal{L}_1$ is also a partition. •

For an arbitrary edge $f = xy$ of G, an uncrossing step either keeps the value $\beta(f, \mathcal{F})$ unchanged or decreases it by one. The latter case occurs if and only if one of the two sets in the uncrossing step is an $x\bar{y}$-set while the other one is a $y\bar{x}$-set (whether they are disjoint or not). Correspondingly, at the very first uncrossing step the value $\beta(e, \mathcal{F})$ certainly decreases. We thus obtain $\sum_{f\in E} \beta(f, \mathcal{F}) > \sum_{f\in E} \beta(f, \mathcal{L})$.

By applying condition (9.32) to $\mathcal{L}_1$ and to $\mathcal{L}_2$, we get $\widetilde{h}(\mathcal{L}_1) + \widetilde{h}(\mathcal{L}_2) = \widetilde{h}(\mathcal{L}) \geq \widetilde{h}(\mathcal{F}) = \sum_{f\in E} \beta(f, \mathcal{F}) > \sum_{f\in E} \beta(f, \mathcal{L}) = \sum_{f\in E} \beta(f, \mathcal{L}_1) + \sum_{f\in E} \beta(f, \mathcal{L}_2) \geq \widetilde{h}(\mathcal{L}_1) + \widetilde{h}(\mathcal{L}_2)$ and this contradiction shows the impossibility of the indirect assumption. •

9.5.2 On k-edge-connected orientation of mixed graphs

In Subsection 9.5.1, we characterized mixed graphs admitting a rooted k-edge-connected orientation. The characterization given in Theorem 9.5.1 is a straight extension of Theorem 9.1.8 concerning undirected graphs, at least in the sense that both characterizations required the number of cross-edges of G to each partition of V to be sufficiently large. Also, as already mentioned, Robbins' orientation theorem could easily be extended in Theorem 2.2.12 to mixed graphs. In this light, one can expect that there is a natural characterization of mixed graphs having a k-edge-connected orientation.

It is clearly necessary that $d_G(X) \geq (k - \varrho_D(X))^+ + (k - \delta_D(X))^+$ for every $X \subset V$, or more generally, the number of cross-edges of G to every partition $\{V_1, \ldots, V_q\}$ of V is at least $\sum_i [k - \varrho_D(V_i)]$. The following example, however, shows that this is not sufficient even for $k = 2$.

Let $G = (V, E)$ be an undirected graph with node-set $V = \{v_1, v_2, v_3, v_4\}$ and edge-set $E = \{v_1v_2,\ v_3v_4\}$. Let $D = (V, A)$ be a digraph where A consists of the following 9 edges: $v_1v_4, v_1v_4, v_4v_1,\ v_1v_3, v_1v_3, v_3v_1,\ v_2v_3, v_2v_3$, and v_3v_2.

The goal is to find a 2-edge-connected orientation of the mixed graph $M = G + D$. (See Figure 9.3.) Now D is strongly connected and the non-empty proper subsets of V with in-degree exactly one are: $V_1 = \{v_2\}$, $V_2 = \{v_1, v_2, v_3\}$, and $V_3 = \{v_1, v_4\}$. In order to have a 2-edge-connected orientation of M, one must find an orientation of the two edges of G so that each V_i will admit at least one newly oriented entering edge. Neither of the four possible cases yields a 2-edge-connected orientation of M. On the other hand, for any two of the sets V_1, V_2, V_3, the graph G does have an orientation so that each of the two selected V_i's will have a newly oriented entering edge. This demonstrates that any certificate for the non-existence of a 2-edge-connected orientation of M must include each of V_1, V_2, and

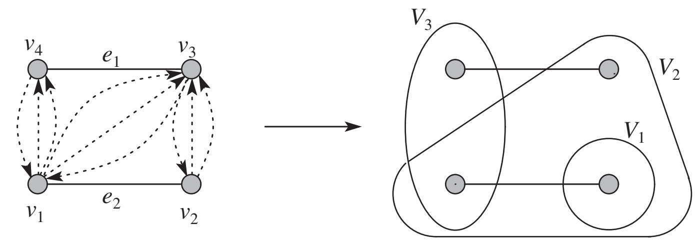


Figure 9.3 No 2-edge-connected orientation exists

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 317 of 640 December 11, 2010 15:18

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

V_3. However, these three sets form neither a partition nor a co-partition of V showing that any characterization must involve a family of subsets more complicated than partitions or co-partitions. Note that $\{V_1, V_2, V_3\}$ is cross-free.

Question 9.5.2 *Figure out a necessary and sufficient condition for a mixed graph to have a k-edge-connected orientation.*

This question will be answered with the help of submodular flows in Section 16.1.5.

9.6 Optimal orientations and reorientations

After exploring problems on the *existence* of orientations with specified connectivity properties, we turn our attention to problems when the goal is to find the best orientation. In one formulation of the problem, the two possible orientations of each edge have a cost, and the goal is to find a cheapest orientation meeting the requested connectivity requirements. Here one can assume that the smaller of the two costs is zero, in which case the optimization problem can be formulated as follows. Given a directed graph and a non-negative cost function c on its edge-set, find a minimum cost reorientation of edges resulting in a requested orientation. In section 3.6.1, it was noted that a minimum cost circulation algorithm can be used to construct a cheapest degree-specified reorientation of a digraph. Our next goal is to present a solution to the optimal reorientation problem where the target is root-connectivity.

9.6.1 Cheapest root-connected reorientations

In order to have a root-connected orientation at all, we assume that D is weakly connected. Let r_0 denote the root-node and let $c : A \to \mathbf{R}_+$ be a cost function. We reduce the cheapest root-connected reorientation problem to that of cheapest arborescences. To this end, denote the reverse of a directed edge $e \in A$ by $\bar{e}$ and, for each edge of D, add its reverse to D. In the resulting digraph D', let $c'(e) := 0$ and $c'(\bar{e}) := c(e)$ for each original edge $e \in A$.

Claim 9.6.1 *The c'-cost of a cheapest r_0-arborescence is equal to the minimum c-cost of edges of D to be reoriented in order to get a root-connected digraph.*

Proof. For an arbitrary spanning r_0-arborescence F of D', its cost $\widetilde{c'}(F)$ is just the c-cost of its reverse edges. Therefore it is possible to make D root-connected by reorienting edges of total cost at most $\widetilde{c'}(F)$. Conversely, if J is a cheapest subset of edges of D for which the reorientation of J yields a spanning r_0-arborescence F, then the minimality of J implies that F contains the reverse of all edges in J, and thus D' admits an r_0-arborescence of cost $\widetilde{c}(J)$. •

It follows from the claim that one can apply any algorithm for computing a cheapest spanning arborescence of root r_0. (One was described in Section 3.2.2, while a more efficient version will be presented in Subsection 11.4.1.)

Unfortunately, this simple approach does not work anymore when a cheapest reorientation of a digraph is sought which is not only root-connected but in-degree constrained as well.

In Chapter 16, however, we shall show that even a generalization of this problem, when root-connectivity is replaced by rooted k-edge-connectivity, can be handled algorithmically via submodular flows.

Problem 9.6.1 *Prove the following result.*

Theorem 9.6.2 *The minimum number of edges for which their reorientation makes a weakly connected digraph $D = (V, A)$ root-connected is equal to the maximum number of disjoint dicuts for which their in-shores contain r_0.* •

9.6.2 Strong orientation minimizing $\varrho(T)$

One can be interested in a strongly connected orientation of a 2-edge-connected graph $G = (V, E)$ in which the in-degree of a specified non-empty, proper subset T of nodes is as small as possible. Denote this minimum by $\mu_G(T) = \mu(T)$.

Recall that $\sigma_G(X)$ denotes the number of components of $G - X$ when $\emptyset \subset X \subseteq V$, and $\sigma_G(\emptyset) = 0$. Obviously, both $\sigma_G(T)$ and $\sigma_G(V - T)$ are lower bounds for $\mu(T)$. If G is obtained from a path on $2n$ nodes by duplicating each edge in parallel and T consists of the nodes at an odd distance from the first node, then $\sigma_G(T) = \sigma_G(V - T) = |T| = n$ while T is entered by $2n - 1$ edges in every strong orientation showing that $\mu(T)$ can be as large as $\sigma_G(T) + \sigma_G(V - T) - 1$.

Lemma 9.6.3 *Let G' denote the bipartite graph obtained from G in such a way that each component induced by T and each component induced by $V - T$ is shrunk into a single node. Let T' be the subset arising from T* (which will be one of the colour classes of G'). *Then $\mu_{G'}(T') = \mu_G(T)$.*

Proof. Shrinking a subset of nodes in a strong digraph preserves strong connectivity from which $\mu_{G'}(T') \leq \mu_G(T)$. Conversely, a strong orientation of G' determines a partial orientation of G. This satisfies the hypotheses of Theorem 2.2.12 and hence it can be extended to a strong orientation of G from which $\mu_{G'}(T') \geq \mu_G(T)$. •

By the claim it suffices to solve the problem for bipartite graphs. Proposition 1.2.6 immediately implies for intersecting X and Y that,

$$\sigma_G(X) + \sigma_G(Y) \leq \sigma_G(X \cap Y) + \sigma_G(X \cup Y) + d(X, Y). \tag{9.34}$$

Theorem 9.6.4 (Frank, Sebő, and Tardos [149]) *Let $G = (S, T; E)$ be a 2-edge-connected bipartite graph. The minimal number $\mu(G)$ of edges in a strongly connected orientation of G entering T is equal to*

$$\max \left\{ \sum_i \sigma_G(T_i) : \{T_1, \ldots, T_t\} \text{ a partition of } T \right\}. \tag{9.35}$$

Proof. (max $\leq$ min) For any strong orientation with in-degree function ϱ and for a non-empty subset X, we have $\varrho(X) \geq \sigma_G(X)$ from which $\varrho(T) = \sum_i \varrho(T_i) \geq \sum_i \sigma_G(T_i)$.

(max $\geq$ min) We describe an algorithm which, starting from an arbitrary strong orientation of G, either finds another strong orientation which is better in the sense that fewer edges enter T, or else finds a partition $\{T_1, \dots, T_t\}$ of T for which $\varrho(T_i) = \sigma_G(T_i)$ $(i = 1, \dots, t)$, where ϱ denotes the in-degree function of the initial orientation. Since the first possibility can take place at most $|E|$ times, the second one will certainly occur, and when it occurs we have $\mu(G) \le \varrho(T) = \sum_i \sigma_G(T_i)$ from which min $\le$ max follows.

Define a non-empty subset X as **tight** (with respect to the given strong orientation) if $\varrho(X) = \sigma_G(X)$. In particular, $S \cup T$ is always tight.

Claim 9.6.5 *If A and B are intersecting tight sets, then $A \cup B$ and $A \cap B$ are also tight.*

Proof. From 9.34 we get $\varrho(A) + \varrho(B) = \sigma_G(A) + \sigma_G(B) \le \sigma_G(A \cap B) + \sigma_G(A \cup B) + d(A, B) \le \varrho(A \cap B) + \varrho(A \cup B) + d(A, B) = \varrho(A) + \varrho(B)$ from which $\sigma_G(A \cup B) = \varrho(A \cup B)$ and $\sigma_G(A \cap B) = \varrho(A \cap B)$. •

This immediately implies the following.

Claim 9.6.6 *If a family of tight sets form a connected hypergraph, then the union of these sets is tight. The intersection $P(v)$ of tight sets containing a node v is tight.* •

Let ϱ be the in-degree function of a strong orientation of G. There can be two cases.

Case 1 *$P(v) \subseteq T$ for every node $v \in T$.* Let $T_1, \dots, T_t$ denote the components of hypergraph $\{P(v) : v \in T\}$. Then $\{T_1, \dots, T_t\}$ forms a partition of T and each T_i is tight by Claim 9.6.6. Therefore the given orientation satisfies the requested equality $\varrho(T) = \sum_i \varrho(T_i) = \sum_i \sigma_G(T_i)$.

Case 2 *There are nodes $t \in T$ and $s \in S$ for which $s \in P(t)$.* Let P be a directed st-path and reorient the edges of P. In the resulting orientation of G, the number of edges entering T is one less than $\varrho(T)$. We claim that the new orientation is also strongly connected. Indeed, if this were not the case, then there would exist a subset Z of nodes admitting no entering edge in the revised orientation. But then the path P cannot contain any edge leaving Z, and Z is a $t\bar{s}$-set for which $\varrho(Z) = 1$. That is, Z is a tight $t\bar{s}$-set, contradicting the assumption that $s \in P(t)$. By repeating this procedure, after at most $|E|$ path reorientations we arrive at Case 1. • •

Algorithmic aspects

The proof technique of Theorem 9.6.4 gives rise to a polynomial time algorithm for computing the strong orientation minimizing $\varrho(T)$ as well as the optimal partition of T, provided a subroutine is available, given a strong orientation of G, to compute the unique smallest tight set $P(t)$ containing an element $t \in T$. The next claim helps in constructing such an oracle.

Claim 9.6.7 *$v \in P(t)$ if and only if there are two edge-disjoint vt-paths in the given strong orientation.*

Proof. If the two vt-paths do not exist, then there is a $t\bar{v}$-set Z for which $\varrho(Z) = 1$. Then Z is tight and hence $P(t) \subseteq Z$ from which $v \notin P(t)$. Suppose now that $v \notin P(t)$. Then v belongs to one of the components of $G - P(t)$. Since $P(t)$ is tight, the out-degree of each of the components $G - P(t)$ is 1 and therefore there cannot be two edge-disjoint vt-paths. •

Based on this claim, the algorithm runs as follows. Start with any strong orientation of G and decide, for example, with the help of repeated MFMC computations, if there are 2 edge-disjont st-paths for some $s \in S$ and $t \in T$. If so, reorient one of these two paths. The resulting digraph continues to be strongly connected and the in-degree of T drops by one. Repeat this procedure as long as there are two edge-disjoint st-paths for some pair $\{s, t\}$ of nodes with $s \in S,\ t \in T$.

When this is not possible anymore, then the in-degree of T in the given strong orientation is minimum. Moreover, by Claim 9.6.7, we can determine $P(t)$ for every $t \in T$. The components of the hypergraph $\{P(t) : t \in T\}$, as indicated in the proof above, form the optimal partition of T, as required for the theorem.

Extension

For extensions and applications, it is worthwhile to formulate the following consequence of Theorem 9.6.4. We call a subset F of edges of a digraph a **dijoin** if F intersects all dicuts.

Theorem 9.6.8 *Let $\vec{G} = (V, \vec{E})$ be a directed graph obtained from a bipartite graph $G = (S, T; E)$ by orienting each edge toward T. The minimum cardinality of a dijoin of $\vec{G}$ is equal to the maximum number $\nu(\vec{G})$ of disjoint dicuts. Moreover, there is an optimal family of disjoint dicuts which is cross-free.*

Proof. The max $\leq$ min direction is straightforward. To see the reverse inequality, we can assume that G is 2-edge-connected. Consider a strong orientation of G in which $\varrho(T)$ is minimum and let $\{T_1, \dots, T_t\}$ be the partition of T provided by Theorem 9.6.4 for which $\varrho(T) = \sum_i \sigma_G(T_i)$.

Since this orientation is strong, the edges entering T cover each dicut of $\vec{G}$. On the other hand, for each T_i, the $\sigma_G(T_i)$ components of $\vec{G} - T_i$ determine edge-disjoint dicuts of $\vec{G}$ which are also edge-disjoint for distinct T_i and T_j. Therefore we have $\sum_i \sigma_G(T_i)$ disjoint dicuts of $\vec{G}$ and the same number of edges covering all dicuts.

To see the second part, first observe that if X_1 and X_2 are crossing in-shores of two disjoint dicuts B_1 and B_2, then the dicuts B_1' and B_2' determined by $X_1 \cap X_2$ and $X_1 \cup X_2$ are also disjoint and their union is $B_1 \cup B_2$. Furthermore, $|X_1|^2 + |X_2|^2 < |X_1 \cap X_2|^2 + |X_1 \cup X_2|^2$. Therefore if we choose a maximum number of disjoint dicuts in such a way that the sum of the squares of sizes of their in-shores is as large as possible, then this set is cross-free. •

Observe that the disjoint directed cuts are obtained in polynomial time from the optimal partition of T, which was computed previously. Lucchesi and Younger [275] proved (Theorem 9.7.2) that the statement of Theorem 9.6.8 is actually true for any directed graph. But this is more complicated, especially from an algorithmic point of view (see Section 9.7).

9.6.3 Tree-compositions of bipartite graphs

Let $\vec{G} = (V, \vec{E})$ again be the digraph arising from $G = (S, T; E)$ by orienting each edge toward T. Theorem 9.6.4 described one way to represent an optimal family of disjoint dicuts (in terms of a partition of T). One drawback was that the structure of the actual dicuts in this representation is rather messy, furthermore, there is an undue asymmetry between the role of T and S. Theorem 9.6.8 described another way to represent an optimal family of disjoint

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 321 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

dicuts (in term of cross-free dicuts). Our next goal is to show that the advantages of the two representations can be integrated into a single theorem. We describe a structure, called a tree-composition, which simultaneously involves an optimal partition of T, an optimal partition of S, and an optimal cross-free family of disjoint dicuts.

Largest tree-compositions

Let $\mathcal{P}$ be a partition of $V := S \cup T$ consisting of a partition $\mathcal{S} := \{S_1, \dots, S_p\}$ of S and a partition $\mathcal{T} := \{T_1, \dots, T_q\}$ of T where the S_i's and T_j's are non-empty and $p \geq 1, q \geq 1$. We say that $\mathcal{P}$ is a **tree-composition** of $\vec{G}$ (and of G) if shrinking first each class of $\mathcal{P}$ into a single node and replacing then each set of the resulting parallel edges by one edge we obtain a (directed) tree. The number of edges of this tree (which is $p + q - 1$) is called the **size** of the tree-composition and is denoted by $s(\mathcal{P})$. For example, if $\mathcal{S} = \{S\}$ and $\mathcal{T}$ consists of the singletons of T, then $\mathcal{P}$ is a tree-composition and its size is $|T|$. A tree-composition $\mathcal{P}$ of $\vec{G}$ determines a cross-free set of $s(\mathcal{P})$ disjoint dicuts of $\vec{G}$.

Theorem 9.6.9 *Let $\vec{G} = (V, \vec{E})$ be a directed graph obtained from a bipartite graph $G = (S, T; E)$ by orienting each edge toward T. Then $\vec{G}$ admits a tree-composition of size $\nu(\vec{G})$.*

Proof. We proceed by induction on $\nu(\vec{G})$.

Claim 9.6.10 *There is a node z of $\vec{G}$ and $\nu(\vec{G}) - 1$ disjoint dicuts of $\vec{G}$ not using any edge incident to z.*

Proof. By Theorem 9.6.8, there is a cross-free set $\mathcal{F}$ consisting of the in-shores of $\nu(\vec{G})$ disjoint dicuts of $\vec{G}$. Consider a tree-representation of $\mathcal{F}$ provided by Theorem 1.4.2. A directed tree has a node of in-degree one or a node of out-degree one, and by symmetry, we can assume that the first alternative holds. The member Z of $\mathcal{F}$ corresponding to this in-degree one node has the property that every other member of $\mathcal{F}$ is either disjoint from Z or includes Z, furthermore $\delta_{\vec{G}}(Z) = 0$. Therefore Z contains an element z of T and z has the required property. •

By symmetry, we can assume that the node z ensured by the claim is in T. Let G' denote the bipartite graph arising from G by contracting z and its neighbours into a new node z' and let $\vec{G}'$ be the digraph arising from $\vec{G}$ in the same way. By the definition of z, $\vec{G}'$ includes $\nu(\vec{G}) - 1$ disjoint dicuts. By induction, $\vec{G}'$ admits a tree-composition $\mathcal{P}'$ of size $\nu(\vec{G}) - 1$. By blowing up z' and adding the singleton $\{z\}$ to $\mathcal{P}'$, we obtain a tree-composition $\mathcal{P}$ of $\vec{G}$ for which the size is $\nu(\vec{G})$, as required. • •

By the **tree-index** of G we mean the largest size of a tree-composition of G. By combining Theorems 9.6.4 and 9.6.9, one obtains the following symmetric form of Theorem 9.6.4.

Theorem 9.6.11 *Let $G = (S, T; E)$ be a 2-edge-connected bipartite graph. The minimum number $\mu(G)$ of edges entering T in a strong orientation of G is equal to the tree-index of G.*

Proof. Since a tree-composition $\mathcal{P}$ determines $s(\mathcal{P})$ disjoint dicuts of $\vec{G}$, in a strong orientation of G at least $s(\mathcal{P})$ edges must enter T. Conversely, it follows from Theorem 9.6.4 that there are $\mu(G)$ disjoint dicuts of $\vec{G}$. By Theorem 9.6.9 there is a tree-composition of size $\mu(G)$. •

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 322 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Application to disjoint cuts in bipartite graphs and disjoint circuits in planar Euler graphs

As a pretty application of the algorithmic proof of Theorem 9.6.4, we show that the problem of finding a maximum number of edge-disjoint circuits in a planar Euler graph is doable in polynomial time. By planar dualization, this problem is equivalent to the one of finding edge-disjoint cuts in a biparite planar graph. This latter problem turns out to be tractable even if the planarity assumption is dropped, and we are going to show that the problem of finding a maximum number of disjoint cuts in an arbitrary bipartite graph is polynomially solvable. Note that this problem is **NP**-complete for general graphs [52] and even for planar (non-bipartite) graphs [43, 44].

Theorem 9.6.12 *In a bipartite graph $G = (S, T; E)$ the maximum number $\nu_{cut}(G)$ of disjoint cuts is the same as the maximum number $\nu(\vec{G})$ of disjoint dicuts of the digraph $\vec{G}$ obtained from G by orienting each edge toward T.*

Proof. Obviously, $\nu_{cut}(G) \geq \nu(\vec{G})$ and we prove the equality by induction on $\nu_{cut}(G)$.

Let us consider a set $\mathcal{C}$ of $\nu_{cut}(G)$ disjoint cuts of G. Let r be an arbitrary node of G, and identify each cut in $\mathcal{C}$ by its shore not containing r. Suppose that $\mathcal{C}$ is chosen in such a way that the sum of squares of the size of shores is as large as possible. Then the set $\mathcal{F}$ of these $\nu_{cut}(G)$ shores is laminar. Indeed, if X and Y were two properly intersecting shores, then the cuts $\Delta(X)$ and $\Delta(Y)$ could be replaced by cuts $\Delta(X \cap Y)$ and $\Delta(X \cup Y)$ since these are disjoint and $\Delta(X) \cup \Delta(Y) = \Delta(X \cap Y) \cup \Delta(X \cup Y)$. But this contradicts the choice of $\mathcal{C}$ since $|X|^2 + |Y|^2 < |X \cap Y|^2 + |X \cup Y|^2$.

Let Z be a minimal shore in $\mathcal{F}$. We can assume that Z is a singleton. Indeed, if $|Z| \geq 2$ and $z \in Z$ then each other shore of $\mathcal{F}$ is disjoint from Z and hence Z could be replaced by the singleton $\{z\}$. Note that the cut $\Delta(z)$ corresponds to a dicut of $\vec{G}$.

Let G' denote the graph obtained from G by contracting the cut $\Delta(z)$. Then $G' = (S', T'; E')$ is bipartite and includes $\nu(G) - 1$ disjoint cuts, and hence $\nu_{cut}(G') \geq \nu_{cut}(G) - 1$. By induction, these cuts can be chosen in such a way that they are dicuts in $\vec{G}'$. These $\nu(G) - 1$ dicuts correspond to dicuts of $\vec{G}$ which are disjoint from the dicut corresponding to $\Delta(z)$ in $\vec{G}$. Therefore $\nu(\vec{G}) \geq \nu(\vec{G}') + 1 = \nu_{cut}(G') + 1 \geq \nu_{cut}(G) \geq \nu(\vec{G})$ and hence $\nu(\vec{G}) = \nu_{cut}(G)$. •

Note that in the bipartite graph C_4 the maximum number of disjoint cuts is 2 while the minimum number of edges covering all cuts is 3.

Feasible tree-compositions

We shall now make a little detour from orientations, and investigate a variation of the largest tree-composition problem. Suppose that we are given a specified subset F of edges of a bipartite graph $G = (S, T; E)$. We say that a tree-composition $\mathcal{P}$ of G is F**-feasible** if each cut of G determined by $\mathcal{P}$ contains at most one element of F. What is a necessary and sufficient condition for the existence of an F-feasible tree-composition? In order to answer this question, we need the following fascinating result of Sebő [149]. Recall (p. 95) that a cost function is conservative if the graph includes no circuit of negative total cost.

Lemma 9.6.13 (Sebő) *Let $G = (S, T; E)$ be a simple bipartite graph with at least three nodes, and let $w : E \to \{+1, -1\}$ be a conservative cost function for which there is a negative path connecting any two nodes lying in the same colour class of G. Then G is a tree and $w \equiv -1$.*

Proof. Proceed by induction on the number of nodes. The lemma is straightforward when $|S \cup T| = 3$, so we assume that $|S \cup T| \geq 4$. Let P be a path for which the $\widetilde{w}(P)$ is minimum and, with respect to this, P has a minimum number of edges. Since there are negative paths in G, P has at least one edge. Let t denote one of the end-nodes of P and let xt be the first edge of P. By the choice of P, every subpath $P[y, t]$ of P starting at t must be of negative cost, and in particular, $w(xt) = -1$.

Proposition 9.6.14 *The edge xt is the only edge of G incident to t.*

Proof. Suppose indirectly that there is an edge, denoted by tz, which is incident to t and distinct from xt. We claim that $w(tz) = +1$. Indeed, if we had $w(tz) = -1$, then in the case when $z \in P$, the circuit $P[z, t] + tz$ would be negative since $\widetilde{w}(P[z, t]) < 0$, while in the case when $z \notin P$, $P' := P + tz$ would be a path for which $\widetilde{w}(P') < \widetilde{w}(P)$ contradicting the choice of P.

The nodes x and z belong to the same colour-class of G since both are neighbours of t. By the hypothesis of the lemma, there is a negative xz-path R. Assume that $\widetilde{w}(R)$ is minimal. Then R must use t for otherwise $R + xt + tz$ would be a negative circuit. We claim that R actually uses the edge xt. For otherwise $R[x, t] + xt$ would form a circuit and hence $w(xt) = -1$ implies $\widetilde{w}(R[x, t]) \geq 1$. But in this case by replacing the segment $R[x, t]$ of R with the edge xt we would obtain an xz-path R' for which $\widetilde{w}(R') < \widetilde{w}(R)$, contradicting the minimal choice of R.

Since G is bipartite, $\widetilde{w}(R)$ is even and hence at most -2. Therefore $\widetilde{w}(R[t, z]) \leq -1$ and hence $C := R[t, z] + tz$ is a circuit of cost 0. If t is the only node of P and $R[t, z]$ in common, then $P + R[t, z]$ forms a path P' for which $\widetilde{w}(P') = \widetilde{w}(P) + \widetilde{w}(R[t, z]) < \widetilde{w}(P)$, contradicting the choice of P. Therefore P and $R[t, z]$ must have a node in common distinct from t. Let y be the first such node along P when we start from t. Since $\widetilde{w}(P[t, y]) < 0$, both subpaths of C between t and y must be positive, and this contradiction to $\widetilde{w}(C) = 0$ completes the proof of the proposition. •

Let $G' := G - t$. Since exactly one edge of G is incident to t, the hypothesis of the lemma holds for G' and by induction we are done. • •

Theorem 9.6.15 *Let $G = (S, T; E)$ be a bipartite graph with at least one edge. A cost function $w : E \to \{+1, -1\}$ is conservative if and only if there is a tree-composition of G such that each of the $|\mathcal{P}| - 1$ cuts of G determined by $\mathcal{P}$ contains at most one negative edge.*

Proof. Suppose first that $\mathcal{P}$ is a tree-composition with the requested property. Since a circuit C and a cut has an even number of edges in common and each of the $|\mathcal{P}| - 1$ cuts determined by $\mathcal{P}$ contains at most one negative edge, the cost of C is non-negative.

Conversely, suppose that w is conservative. We can suppose that G is simple. Indeed, we arrive at an equivalent problem if a set B of parallel uv-edges is replaced by one uv-edge for which the cost is -1 or $+1$ according to whether B contains a negative edge or not.

The statement is trivial if $|S \cup T| = 2$, so we assume that $|S \cup T| \geq 3$. If there is a negative path between any two nodes of G in the same colour class, then Sebő's lemma shows that G itself is a tree. Suppose now that there are two nodes u and v in the same colour class such that every uv-path is non-negative. Then shrinking the doubleton $\{u, v\}$ does not create a negative circuit. By induction, the resulting graph G' has a tree-composition $\mathcal{P}'$ with the required properties. The tree-composition of G arising from $\mathcal{P}'$ by blowing up the shrunk node will suffice for G. •

Problem 9.6.2 *Show that in Theorem* 9.6.15 *if w is conservative, then there is a tree-composition $\mathcal{P}$ such that each of the $|\mathcal{P}| - 1$ cuts of G determined by $\mathcal{P}$ contains exactly one negative edge.*

Theorem 9.6.16 *Let $G = (S, T; E)$ be a bipartite graph and $F \subseteq E$ a non-empty subset of edges. There is an F-feasible tree-composition of G if and only if*

$$|C \cap F| \leq |C - F| \text{ for every circuit } C \text{ of } G. \tag{9.36}$$

Proof. Apply Theorem 9.6.15 to the cost function w_F defined by

$$w_F(e) := \begin{cases} -1 & \text{if } e \in F \\ +1 & \text{if } e \in E - F \end{cases} \tag{9.37}$$

and observe that (9.36) is equivalent to requiring w_F to be conservative. •

Note that the proof of Theorem 9.6.9 above is algorithmic while the proof of Theorem 9.6.16 does not give rise to a polynomial-time algorithm.

Application to edge-disjoint paths in planar graphs

Theorem 9.6.17 (Seymour [347]) *Let $G = (V, E)$ and $H = (V, F)$ be graphs such that $G + H = (V, E + F)$ is planar and Eulerian. Then the edge-disjoint paths problem with demand graph H and supply graph G has a solution if and only if the cut condition which requires $d_G(X) \geq d_H(X)$ for every $X \subseteq V$ holds.*

Proof. Consider the planar dual of $G + H$. This is bipartite since $G + H$ is Eulerian. In the dual graph, define a cost function w in such a way that the cost of the edges corresponding to E and F, respectively, are $+1$ and -1. The cut condition is clearly equivalent to the requirement that w is conservative.

By Theorem 9.6.16, there are $|F|$ disjoint cuts in the dual graph such that each contains one negative edge. This implies that $G + H$ includes $|F|$ edge-disjoint circuits such that each contains exactly one element of F, and hence the edge-disjoint paths problem in question indeed has a solution. •

9.7 Optimal strongly connected reorientation

Recall Robbins' theorem (Theorem 2.2.8) stating that a 2-edge-connected graph always has a strong orientation. In Question 2.2.3 the problem was posed to characterize the minimum number of edges of a digraph the reorientation of which results in a strong digraph. Finding algorithmically an optimal strongly connected reorientations is significantly more complicated even in the minimum cardinality case. The more general minimum cost version needs the theory of submodular flows and will be discussed in Part III. Note that in a weakly connected digraph D a subset F of edges is a dijoin if and only if the addition of the reverse of F yields a strong digraph. Minimal dijoins have a stronger property.

Proposition 9.7.1 *Let $D = (V, A)$ be a digraph with a 2-edge-connected underlying graph. If $\vec{F} \subset A$ is a minimal dijoin* (with respect to inclusion)*, then the reorientation of the elements of $\vec{F}$ leaves a strongly connected digraph.* •

Proof. Consider the mixed graph M obtained from D by deorienting the elements of $\vec{F}$ and let F denote the corresponding set of undirected edges. By Theorem 2.2.12, M has a strongly connected orientation $\vec{M}$. We claim that in $\vec{M}$ the orientation of every element of F is opposite to its original orientation in $\vec{F}$. Indeed, the minimality of $\vec{F}$ means that there is a dicut B_f of D for every $f \in \vec{F}$ for which $B_f \cap \vec{F} = \{f\}$. Therefore f must be oppositely oriented in M. •

Therefore our problem is determining the minimum number of edges covering all dicuts, or equivalently, finding a dijoin of smallest cardinality.

9.7.1 Minimum dijoins and packing dicuts

Let $D = (V, A)$ be a weakly connected digraph. Let $\emptyset \subset X \subset V$ be a subset of nodes of out-degree 0. The set of edges entering X previously was called a dicut and X is the in-shore of the cut. Two dicuts are **crossing** if their in-shores are crossing. A family of dicuts is **cross-free** if their in-shores are cross-free. The 'uncrossing' proof technique appearing in the next proof is due to Lovász [268].

Theorem 9.7.2 (Lucchesi and Younger) *Let $D = (V, A)$ be a digraph. The minimum cardinality $\tau = \tau(D)$ of a dijoin is equal to the maximum number $\nu = \nu(D)$ of disjoint dicuts.*

Proof. Since the inequality $\nu \leq \tau$ is clear, we prove only the reverse direction. We proceed by induction on $\nu(D)$. If this number is 0, that is, if there is no dicut in D, then D is strongly connected and hence τ is also 0. Suppose now that $\nu(D) > 0$. Since the theorem clearly holds for digraphs with one edge, we can assume that $|A| > 1$.

Case 1 *D has an edge e for which $\nu(D_e) < \nu(D)$* where D_e denotes the digraph arising from D by contracting e. By applying the inductive hypothesis to $\nu(D_e)$ and observing the straightforward inequality $\tau(D) \leq \tau(D_e) + 1$, we obtain $\tau(D) \leq \tau(D_e) + 1 = \nu(D_e) + 1 \leq \nu(D)$ from which the requested $\tau(D) \leq \nu(D)$ follows.

Case 2 *$\nu(D_e) = \nu(D)$ for every edge e of D*. Therefore there is a set $\mathcal{I}_e$ of $\nu(D)$ disjoint dicuts of D such that none of them contains e. Let $\mathcal{J}'$ denote the multi-union of sets $\mathcal{I}_e$ over $e \in A$. Then $\mathcal{J}'$ has $|A|\nu(D)$ members and

$$\text{every edge of } D \text{ is in at most } k := |A| - 1 \text{ members.} \qquad (*)$$

Apply the following uncrossing technique. If $\mathcal{J}'$ has two crossing members, then replace these two by the dicuts belonging to the intersection and to the union of their in-shores. This exchange preserves $(*)$. Since the sum of the squared cardinality of the in-shores strictly increases, after a finite number of uncrossing steps, we obtain a cross-free family $\mathcal{J}$ consisting of $|A|\nu(D)$ (not necessarily distinct) dicuts for which $(*)$ holds.

Claim 9.7.3 *$\mathcal{J}$ can be coloured by k colours such that each colour-class consists of disjoint dicuts.*

Proof. By Theorem 1.4.2, $\mathcal{J}$ can be represented by a directed tree $B = (U, F)$ and a mapping $\varphi : V \to U$ in such a way that every edge $e = uv$ of B uniquely corresponds to the member $\varphi^{-1}(U_e)$ of $\mathcal{J}$ where, among the two components of $F - e$, U_e denotes the one entered by e. Since the members of $\mathcal{J}$ are dicuts, the unique path in B from $\varphi(x)$ to $\varphi(y)$ is a directed path whenever xy is an edge of D. Moreover, $(*)$ implies that the length of this path is at most k. Therefore it suffices to show that the edges of B can be k-coloured such that every directed subpath of length at most k consists of edges of distinct colours. We can arrange the nodes of tree B into disjoint levels $U_1, \dots, U_t$ in such a way that the level of the head of each edge of B is one higher than the level of its tail. Let the colour of an xy-edge $e \in B$ be $i \pmod k$ when $y \in U_i$. Because of the construction, this colouring satisfies the requirement. •

The largest colour class in the k-colouring of $\mathcal{J}$ includes at least $\lceil |\mathcal{J}|/k \rceil = \lceil |A|\nu(D)/(|A|-1) \rceil > \nu(D)$ disjoint dicuts contradicting the definition of $\nu(D)$. Therefore Case 2 cannot occur. • •

By serial multiplication of edges, one readily obtains the following weighted version of the theorem of Lucchesi and Younger.

Theorem 9.7.4 *Let $D = (V, A)$ be a digraph endowed with a non-negative integer-valued cost function $c : A \to \mathbf{Z}_+$ on its edge-set. The minimum cost of a dijoin is equal to the maximum number of c-independent dicuts where a family of not necessarily distinct dicuts is c-independent if every edge e of D belongs to at most $c(e)$ of them.* •

A closer look at the proof of Theorem 9.7.2 shows that we have not fully exploited that all dicuts are to be covered. The proof carries over almost word for word for the extension when a given crossing family of dicuts must be covered. Therefore we can formulate the following generalization.

Theorem 9.7.5 *Let $D = (V, A)$ be a digraph and $\mathcal{B}$ a crossing family of dicuts. The minimum number of edges covering all members of $\mathcal{B}$ is equal to the maximum number of disjoint members of $\mathcal{B}$.* •

Corollary 9.7.6 *If every dicut in D has at least k elements, then the minimum number of edges covering all k-element dicuts is equal to the maximum number of disjont k-element dicuts.*

Proof. If X and Y are two crossing in-shores of k-element dicuts, then $k + k = \varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \geq k + k$ from which $\varrho_D(X \cap Y) = k$ and $\varrho_D(X \cup$

$Y) = k$, showing that the k-element dicuts form a crossing family. Theorem 9.7.5 can be applied. •

Application to edge-disjoint paths in planar digraphs

Consider the edge-disjoint paths problem in a digraph $D = (V, A)$ in which we want to find k edge-disjoint directed paths $P_1, \ldots, P_k$ such that P_i is a path from s_i to t_i $(i = 1, \ldots, k)$. Let $H = (V, F)$ denote the demand graph where $F := \{t_1s_1, \ldots, t_ks_k\}$. Then the edge-disjoint paths problem is equivalent to requiring the existence of k edge-disjoint F-good di-circuits in the digraph $(V, A + F)$ where a di-circuit is said to be F**-good** if it contains exactly one demand edge.

As was mentioned in Section 1.5, this problem is **NP**-complete even if D is acyclic. However, for planar digraphs the situation is different. Consider the following condition.

Covering condition It is not possible to cover the F-good di-circuits with less than k elements of $A \cup F$.

This condition is clearly necessary for the edge-disjoint paths problem for arbitrary digraphs.

Theorem 9.7.7 *If D is acyclic and $D + H$ is planar, then the covering condition is necessary and sufficient for the solvability of the directed edge-disjoint paths problem.*

Proof. Consider the planar dual digraph of $D + H$ and denote it by $D' + H'$. We claim that $D' + H'$ includes k edge-disjoint dicuts. If this failed, then, by virtue of the theorem of Lucchesi and Younger, there would be at most $k - 1$ edges of $D' + H'$ covering all dicuts. This would mean that there are at most $k - 1$ edges of the original digraph $D + H$ covering all di-circuits of $D + H$, contradicting the covering condition.

Therefore $D' + H'$ includes k edge-disjoint dicuts and hence $D + H$ includes k edge-disjoint di-circuits $C_1, \ldots, C_k$. Since D is acyclic by the hypothesis, each C_i must contain at least one demand edge. But there are exactly k demand edges implying that each C_i is F-good, as required. •

Exercise 9.7.1 *Suppose that there is a subset Z of nodes violating the directed cut condition, which means that $\varrho_F(Z) > \delta_A(Z)$. Show that in this case the covering condition also fails to hold.*

Application to edge-disjoint circuits in planar digraphs

By planar dualization, Corollary 9.7.6 immediately gives rise to the following pretty result which is interesting even in the special case when $k = 3$.

Theorem 9.7.8 *Let $D = (V, A)$ be a planar digraph in which every di-circuit has at least k edges. Then the minimum number of edges covering all k-element di-circuits is equal to the maximum number of edge-disjoint k-element di-circuits.*•

9.7.2 Algorithmic proof for the theorem of Lucchesi and Younger

Let $D = (V, A)$ be a weakly connected digraph. For a subset $X \subseteq V$, let $\Delta^-(X)$ denote the set of edges entering X. For a constructive proof of the non-trivial direction $\nu \geq \tau$ of the theorem of Lucchesi and Younger, we describe an algorithm that constructs a dijoin F and finds $|F|$ disjoint dicuts. The first polynomial algorithm appeared in [117] and treated the more general minimum cost version as well. Its extension to (0, 1)-valued submodular flows [119] will be discussed in Section 16.3. The algorithmic proof below is based on the ideas of the negative circuit method developed for submodular frameworks by Fujishige [158] and Zimmermann [388].

First observe that adding a new edge parallel to an existing one does not affect anything. Therefore we can assume that

$$\text{each edge of } D \text{ occurs in two parallel copies.} \tag{9.38}$$

This technical assumption will slightly simplify the description and the proof of the algorithm.

The set-system of in-shores is closed under taking union and intersection, and also $d(X, Y) = 0$ holds for in-shores X, Y. For digraph D, let $\sigma = \sigma_D$ be σ_G where G denotes the underlying undirected graph of D. The inequality 9.34 immediately implies:

Lemma 9.7.9 *The function σ restricted to in-shores is intersecting supermodular.* •

If $F \subseteq A$ is a dijoin and X is an in-shore, then $\varrho_F(X) \geq \sigma(X)$, where $\varrho_F(X)$ denotes the number of elements of F entering X. A non-empty in-shore X and its dicut $\Delta^-(X)$ are **tight** (with respect to F) if $\sigma(X) = \varrho_F(X)$. Since $\sigma(V) = 0 = \varrho_F(V)$, the ground-set V is also considered a tight in-shore although it does not determine a non-empty dicut. When $\varrho_F(X) = 1$, we speak of a **1-tight** in-shore or dicut. Every non-trivial tight dicut partitions into 1-tight dicuts: the dicuts determined by the components of $D - X$ define such a partition. Moreover, every non-trivial tight in-shore is the intersection of 1-tight in-shores.

Lemma 9.7.10 *Both the union and the intersection of two intersecting tight in-shores are tight.*

Proof. We have $\varrho_F(X) + \varrho_F(Y) = \sigma(X) + \sigma(Y) \leq \sigma(X \cap Y) + \sigma(X \cup Y) \leq \varrho_F(X \cap Y) + \varrho_F(X \cup Y) = \varrho_F(X) + \varrho_F(Y)$ from which equality follows in the estimations and hence $\sigma(X \cap Y) = \varrho_F(X \cap Y)$ and $\sigma(X \cup Y) = \varrho_F(X \cup Y)$. •

It follows that the intersection $T(v) = T_F(v)$ of all tight in-shores containing a given node v is also tight. Therefore $T(v)$ is the unique smallest tight in-shore containing v. In order to run the algorithm, one must be able to compute $T(v)$. This will be described separately after completing the proof. It also follows from the lemma that the node-set U of a connected hypergraph consisting of tight in-shores is also tight. Indeed, let X be a maximal tight in-shore of U and suppose indirectly that $X \subset U$. By the connectivity, the hypergraph has a hyperedge Y intersecting both X and $U - X$, but then $X \cup Y$ is also tight contradicting the maximal choice of X.

The algorithm starts with an arbitrary dijoin F (which can be, for example, a spanning tree). At a general phase, the algorithm either finds a dijoin of smaller cardinality or else it finds $|F|$ disjoint dicuts. We will tacitly assume that the current dijoin F never includes parallel edges since if it does, then removing a parallel copy clearly results in a smaller dijoin. Given a dijoin F, we construct an auxiliary digraph $D' = (V, A')$ along with a cost function c' on its edge-set, as follows.

Every element of $A - F$ belongs to A' with cost $+1$. The reverse of every element of F belongs to A' with cost -1. Finally, add a uv-edge to A' for every $v \in V$ and $u \in T(v)$ and define its cost to be 0. Such an edge is called a a **jumping edge**. Note that uv is a jumping edge if and only if it does not enter any tight in-shore. The reverse e' of each edge e of D is always a jumping edge since e' cannot enter any in-shore of D. (Notions like in-shore, dicut, and tightness concern the original digraph D.)

By Gallai's theorem (Theorem 3.1.1), c' is conservative if and only if there is a feasible potential with respect to c'. In addition, the algorithmic proof of Gallai's theorem given in Section 3.1 computes either a negative di-circuit in D' or an integer-valued feasible potential.

Case 1 *There is no negative di-circuit in D', and hence there is an integer-valued feasible potential π.*

We are going to show how such a π can be used to construct $|F|$ disjoint dijoins of D.

Claim 9.7.11 *If there is a jumping edge from u to v, then*

$$\pi(v) \leq \pi(u).$$

If there is an edge in $A - F$ from u to v, then

$$\pi(v) \leq \pi(u) + 1.$$

If there is an edge in F from u to v, then

$$\pi(v) = \pi(u) + 1.$$

Proof. If there is a jumping edge from u to v, then $\pi(v) - \pi(u) \leq c'(uv) = 0$, and hence $\pi(v) \leq \pi(u)$. If there is an edge e in $A - F$ from u to v, then e is an edge of D', and hence $\pi(v) - \pi(u) \leq c'(uv) = 1$, from which $\pi(v) \leq \pi(u) + 1$. Finally, if there is an edge f in F from u to v, then the reverse of f is in D', from which $\pi(u) - \pi(v) \leq c'(vu) = -1$ follows, and hence $\pi(v) \geq \pi(u) + 1$. By assumption (9.38), there is a parallel uv-edge which is in $A - F$ and hence we must have $\pi(v) = \pi(u) + 1$. •

We can assume that the minimal value of π is 0 since adding a constant to π does not affect the feasibility. Let t denote the maximal value of π and let $V_i := \{v : \pi(v) \geq i\}$ $(i = 1, 2, \ldots, t)$. No jumping uv-edge can enter any V_i since then we would have $\pi(v) > \pi(u)$, contradicting Claim 9.7.11. Since the reverse of each edge of D is a jumping edge, no edge of D can leave any V_i, and hence each V_i is an in-shore. Claim 9.7.11 also implies that every edge $uv \in A - F$ enters at most one V_i and every edge $uv \in F$ enters exactly one. Since the graph is connected, we obtain that $V \supset V_1 \supset \cdots \supset V_t$.

Claim 9.7.12 *A dicut $\Delta^-(V_i)$ defined by an in-shore V_i can be partitioned into 1-tight dicuts.*

Proof. Since V_i is not entered by any jumping edge, we have $T(v) \subseteq V_i$ for every node $v \in V_i$, and hence the node-sets $Z^i_1, \ldots, Z^i_l$ of the components of hypergraph $\{T(v) : v \in V_i\}$ form a partition of V_i. Since these sets are in-shores, there is no edge in D between two such sets. As we noted above, each Z^i_j is a tight in-shore, moreover, the dicuts $\Delta^-(Z^i_j)$ $(j = 1, \ldots, l)$ form a partition of $\Delta^-(V_i)$. Since every non-trivial tight dicut partitions into 1-tight dicuts, the claim follows. •

In conclusion, in Case 1 we have found a set of disjoint dicuts such that each of them is covered exactly once by F and every element of F is contained in exactly one of them. That is, we have obtained the desired $|F|$ disjoint dicuts.

Case 2 *There is a negative di-circuit in D'.*

We are going to describe how such a circuit can be used to obtain another dijoin of less than $|F|$ elements. Let C be a negative di-circuit of D' which is minimal in the sense that there is no jumping ab-edge j for which a and b are two non-consecutive nodes of C and the directed subpath P of C from b to a has negative cost. If an initial negative di-circuit had such a jumping edge, then $P + j$ would determine another negative di-circuit of fewer edges, and after at most $|C|$ applications of this reduction step we would arrive at a minimal negative di-circuit.

Lemma 9.7.13 *The jumping edges of C can be arranged in a linear order $e_1 = u_1v_1, e_2 = u_2v_2, \ldots, e_k = u_kv_k$ in such a way that there is no jumping edge from u_j to v_i for $1 \le i < j \le k$.*

Proof. Associate a new node with each jumping edge of C. Let j_1 and j_2 be two (distinct) jumping edges of C, and let x_1 and x_2 denote the nodes associated, respectively, with j_1 and j_2. We defined a directed edge from x_1 to x_2 if there is a jumping edge from the tail of j_1 to the head of j_2. The lemma is equivalent to requiring that the associated digraph H obtained in this way has a topological ordering (which is an ordering of the nodes with no edges going back). If no topological ordering exists, then there is a directed circuit of H. In terms of jumping edges of D', this means that some jumping edges of C have a cyclic order $f_1 = s_1t_1, f_2 = s_2t_2, \ldots, f_m = s_mt_m$ for which that s_it_{i+1} is a jumping edge (where $t_{m+1} = t_1$). For $i = 1, \ldots, m$, let P_i denote the subpath of C from t_{i+1} to s_i. By the minimality of C, the cost of each of these subpaths is non-negative.

It can be checked inductively that every non-jumping edge of C belongs to the same number $q > 0$ of P_i's. This implies that the cost of C is exactly q times the total cost of the subpaths P_i, but this sum is non-negative contradicting the assumption that C is a negative di-circuit. •

Revise F as follows. Remove an edge f from F whenever the reverse of f is an element of C, and add an uv-edge h from $A - F$ to F whenever the corresponding uv-edge is an element of C. Let F' denote the resulting subset of edges of D. Since C is a negative di-circuit, we certainly have $|F'| < |F|$. We are going to prove that F' is a dijoin. For a subset $X \subset V$, let $\delta^j(X)$ and $\delta^n(X)$ denote the number of jumping edges and non-jumping edges of C leaving X, respectively. For entering edges, we define $\varrho^j(X)$ and $\varrho^n(X)$ analogously.

Claim 9.7.14 *For an arbitrary non-empty in-shore X, $\varrho_{F'}(X) = \varrho_F(X) + \delta^j(X) - \varrho^j(X)$.*

Proof. From the definitions, we get $\varrho^j(X) + \varrho^n(X) = \varrho_C(X) = \delta_C(X) = \delta^j(X) + \delta^n(X)$, and hence $\varrho^n(X) - \delta^n(X) = \delta^j(X) - \varrho^j(X)$. Since X is an in-shore, a non-jumping edge f of C leaving X cannot be an edge of D, and hence the reverse of f belongs to F. Similarly, a non-jumping edge of C entering X is in D and belongs to $A - F$. Therefore the definition of F' implies that $\varrho_{F'}(X) = \varrho_F(X) + \varrho^n(X) - \delta^n(X) = \varrho_F(X) + \delta^j(X) - \varrho^j(X)$. •

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 331 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Claim 9.7.15 *For a non-empty in-shore X,*

$$\varrho_F(X) - \sigma(X) \geq \varrho^j(X). \tag{9.39}$$

Proof. We proceed by induction on $\varrho^j(X)$. Let $b_F(X) := \varrho_F(X) - \sigma(X)$. This function is non-negative as F is a dijoin, furthermore $b_F(X)$ is 0 exactly when X is tight. Moreover, b_F is submodular on intersecting subsets with $\delta_D(X) = 0 = \delta_D(Y)$. The inequality is evident when $\varrho^j(X) = 0$, so assume that $\varrho^j(X) > 0$. Let e_i be the jumping edge of C from u_i to v_i which enters X and which is the first in the linear order ensured by Lemma 9.7.13. Let $B := T(v_i)$. If $u_j v_j$ is another jumping edge of C entering X, then $j > i$. We claim that $u_j \notin B$. If indirectly we had $u_j \in B$, then $u_j v_i$ is also a jumping edge, contradicting the property of the linear order. Since no jumping edge enters any tight set, we have $\varrho^j(X \cup B) = \varrho^j(X) - 1$. Since B is the smallest tigth set containing v_i and $B \cap X \subset B$, we conclude that $B \cap X$ is not tight, and hence $b_F(X \cap B) \geq 1$. Now

$$\begin{aligned} b_F(X) + 0 = b_F(X) + b_F(B) \geq b_F(X \cap B) + b_F(X \cup B) \geq 1 \\ + b_F(X \cup B) \geq 1 + \varrho^j(X \cup B) = \varrho^j(X), \end{aligned}$$

where the last inequality follows from the inductive statement. •

Combining Claims 9.7.14 and 9.7.15, we obtain $\varrho_{F'}(X) = \varrho_F(X) - \varrho^j(X) + \delta^j(X) \geq \varrho_F(X) - \varrho^j(X) \geq \sigma(X)$, and hence F' is indeed a dijoin.

Summing up, in Case 2 we have found a dijoin F' smaller than the initial F. By iterating the procedure with F', we arrive at Case 1 after at most $n - 1$ occurrences of Case 2. • •

Computing $T(v)$

Let F be a dijoin. The smallest tight set $T(v)$ containing a node v can be computed as follows. If $T(v)$ is not the entire V, then $T(v)$ is the intersecion of all 1-tight sets containing t. Recall the assumption made in (9.38). Construct an auxiliary digraph D^+ from D by adding the reverse of every edge in F. The resulting D^+ is strongly connected.

Claim 9.7.16 *$T(v)$ consists of those nodes u for which there are two edge-disjoint vu-paths in D^+, or formally, $\lambda_{D^+}(v, u) \geq 2$.*

Proof. If there is a 1-tight $v\bar{u}$-set X which is an in-shore of D, then exactly one edge of D^+ leaves X (namely, the reverse of the only element of F entering X), and hence in this case we have $\lambda_{D^+}(v, u) \leq 1$. Conversely, if $\lambda_{D^+}(v, u) = 1$, then the directed edge-Menger theorem implies that there is a $v\bar{u}$-set X which is left by only one edge xy of D^+. By the construction, X is an in-shore for which yx is the only element of F entering X, and hence X is 1-tight. •

There are more efficient algorithms for computing a minimum dijoin: see the works of Gabow [175], Shepherd and Vetta [348], and Iwata and Kobayashi [222].

Possible generalizations

After we are able to find algorithmically a minimum-cardinality strongly connected reorientation of a digraph, at least two questions emerge naturally. First, what can be said about finding a minimum-cost strongly connected reorientation for a general non-negative cost function? This question does not seem particularly easy since solving minimum-cost optimization problems often needs new ideas even if an algorithm for the cardinality version is available. (For a comparison, relate the problems of maximum cardinality matching and maximum weighted matching in a bipartite graph.) Second, how can the algorithm above be extended to solve the minimum cardinality reorientation problem when the target is k-edge-connectivity for general $k \geq 1$? In light of the fact that Nash-Williams' theorem on k-edge-connected orientability is significantly more difficult for general k than for $k = 1$ (Robbins' theorem), one may be afraid that finding algorithmically an optimal reorientation is also much more difficult for general k. In Part III, however, we shall show that, after introducing the right concept of submodular flows, even the minimum cost k-edge-connected reorientation problem can be treated smoothly.

9.8 Well-balanced orientations

The weak orientation theorem of Nash-Williams (Theorem 9.2.1) stated that an undirected graph $G = (V, E)$ has a k-edge-connected orientation if and only if G is $2k$-edge-connected. In Theorem 9.2.2, we pointed out that the k-edge-connected orientation of G can be chosen to be smooth, where a digraph is said to be smooth (or near-Eulerian) if the in-degree and the out-degree of every node differ by at most one. In this section, we prove a generalization, called the Strong orientation theorem of Nash-Williams [303].

The approach of the proof differs significantly from other techniques used in this book. In the Preface, we indicated that this book does not touch on results related to parity considerations (apart from the use of Euler graphs and digraphs). We make an exception with the Strong orientation theorem which has the peculiar feature that parity is required neither in the problem formulation nor in the answer, but it is central in the proof.

We say that an orientation $D := \vec{G}$ of an undirected graph $G = (V, E)$ is **well-balanced** if

$$\lambda_D(x, y) \geq \lfloor \lambda_G(x, y)/2 \rfloor \text{ holds for every pair } x, y \in V \text{ of nodes.} \tag{9.40}$$

If, in addition, D is smooth, we speak of a **best-balanced** orientation.

Theorem 9.8.1 (Nash-Williams' Strong orientation theorem) *Every undirected graph $G = (V, E)$ has a best-balanced orientation.*

We will prove this fundamental result after proving the main tool of the proof, the odd-node pairing theorem which needs some preparations. For an integer f, let $\hat{f} := 2\lfloor f/2 \rfloor$. For any integer-valued function $h : S \to \mathbf{Z}$, let $\hat{h}$ be defined by $\hat{h}(s) := 2\lfloor h(s)/2 \rfloor$ for every $s \in S$. For a non-negative, symmetric function r on the pairs of nodes of G, we introduced the set-function R_r in (8.6) as follows.

$$R_r(X) := \begin{cases} 0, & \text{if } X \in \{\emptyset, V\} \\ \max\{r(x, y) : X \text{ separates } x \text{ and } y\}, & \text{if } \emptyset \subset X \subset V. \end{cases} \tag{9.41}$$

It follows from this definition that $\hat{R}_r = R_{\hat{r}}$.

Lemma 9.8.2 *Let $R := R_r$ and let $\hat{R} := \hat{R}_r$. The function $\hat{R}$ is skew supermodular, that is, for every pair of subsets $X, Y \subseteq V$, at least one of the following two inequalities holds.*

$$\begin{cases} \hat{R}(X) + \hat{R}(Y) \leq \hat{R}(X \cap Y) + \hat{R}(X \cup Y) & (\alpha) \\ \hat{R}(X) + \hat{R}(Y) \leq \hat{R}(X - Y) + \hat{R}(Y - X) & (\beta) \end{cases} \tag{9.42}$$

Moreover, if

$$R(X) \leq \min\{R(X \cap Y), R(X \cup Y)\}, \tag{9.43}$$

then (9.42α) *holds.*

Proof. By Lemma 8.1.9, $R_{\hat{r}}$ is skew supermodular. Since $\hat{R} = R_{\hat{r}}$, the first part follows. To see the second part, let $z \in Y$ and $z' \in V - Y$ be nodes for which $\hat{R}(Y) = \hat{r}(z, z')$. For positive integers, $f \leq g$ implies $\hat{f} \leq \hat{g}$, and hence Condition (9.43) implies that $(*)$ $\hat{R}(X) \leq \min\{\hat{R}(X \cap Y), \hat{R}(X \cup Y)\}$. We claim that $(**)$ $\hat{R}(Y) \leq \max\{\hat{R}(X \cap Y), \hat{R}(X \cup Y)\}$. For if $\hat{R}(Y) > \max\{\hat{R}(X \cap Y), \hat{R}(X \cup Y)\}$, then $z \in Y - X$ and $z' \in X - Y$, from which $\hat{R}(X) \geq \hat{r}(z, z') = \hat{R}(Y) > \hat{R}(X \cap Y)$, contradicting $(*)$. But $(*)$ and $(**)$ imply (9.42α). •

In what follows, R_{λ_G} will be abbreviated to R_G. We will also need the set-function b_G defined by

$$b_G(X) := d_G(X) - \hat{R}_G(X). \tag{9.44}$$

Since $d_G(X) \geq \lambda_G(x, y)$ for every pair of nodes x, y separated by X, it follows that $d_G(X) \geq \max\{\lambda_G(x, y) : x, y \in V, |X \cap \{x, y\}| = 1\} = R_G(X)$ and hence $b_G \geq 0$.

The basic idea of Nash-Williams to prove his theorem is as follows. If G is Eulerian, then an arbitrary Euler orientation of G is automatically best-balanced since the in-degree of every subset $X \subset V$ is exactly $d_G(X)/2$. Suppose now that G is not Eulerian. We will say that a matching M of the odd-degree nodes of G is an **odd-node pairing** for G. (The elements of M are not from E.) First we make G Eulerian by adding a 'suitable' odd-node pairing M, take then an Euler orientation of the augmented graph, and finally remove the newly added and oriented edges. The resulting orientation of G is clearly smooth, but it is well-balanced only if the pairing M fulfils certain requirements. We say that an odd-node pairing M for G is **admissible** if

$$d_M(X) \leq b_G(X) \tag{9.45}$$

holds for every $\emptyset \subset X \subset V$. Sometimes an admissible odd-node pairing will be called an **admissible pairing** for G. We say that a subset $X \subset V$ is **M-violating** or just **violating** if $d_M(X) > b_G(X)$. Since both b_G and d_M are symmetric set-functions, a subset is violating if and only if its complement is violating. We use tacitly this property thoughout. Since $b_G \geq 0$ and $d_M(X) \equiv b_G(X)$ (mod 2) for every $X \subseteq V$, (9.45) automatically holds when the cardinality of $|X|$ or $|V - X|$ is at most one. In the proof, such an X is said to be **trivial**. We will see that the following result of Nash-Williams immediately implies his Strong orientation theorem.

Theorem 9.8.3 (Nash-Williams' Odd-node pairing theorem) *Every undirected graph admits an admissible odd-node pairing.*

Proof. Suppose indirectly that the Odd-node pairing theorem fails. We define a graph as **good** or **bad** according to whether it has an admissible pairing or not. Let $G = (V, E)$ be a bad graph for which $|V| + |E|$ is minimal. Then G is loopless and has odd-degree nodes. We will use throughout that every smaller graph is good.

Claim 9.8.4 *The degree of every node of G is at least 3.*

Proof. Suppose indirectly that there is a node z for which $d_G(z) \leq 2$. If $d_G(z) = 0$, then an admissible pairing for $G - z$ is admissible for G as well. If $d_G(z) = 2$, then the two edges incident to z can be split off. The resulting graph G' admits an admissible pairing, and this is admissible for G as well. This contradiction to the badness of G implies that $d_G(z) = 1$.

Let $e = uz$ be the edge incident to z and let $G' = G - z$. G' has an admissible pairing M'. If $d_G(u)$ is odd, let $M := M' + uz$. If $d_G(u)$ is even, then $d_{G'}(u)$ is odd and there is an element uv of M'. Let $M := M' - uv + vz$. In both cases, M is an odd-node pairing of G. For an arbitrary subset X of $V - z$, we have $\hat{R}_{G'}(X) = \hat{R}_G(X)$ and hence

$$d_G(X) - d_M(X) \geq d_{G'}(X) - d_{M'}(X) \geq \hat{R}_{G'}(X) = \hat{R}_G(X).$$

Therefore X is not M-violating and hence M is admissible contradicting the badness of G. •

Claim 9.8.5 $b_G(X) > 0$ *holds for every non-trivial subset $X \subset V$.*

Proof. Suppose indirectly that there is a non-trivial subset $X \subset V$ for which $b_G(X) = 0$. Let G_1 and G_2 denote the graphs arising from G by shrinking X and $V - X$, respectively. For any subset $Z \subseteq V - X$, $R_{G_1}(Z) \geq R_G(Z)$ and hence $b_{G_1}(Z) \leq b_G(Z)$. Similarly, $b_{G_2}(Z) \leq b_G(Z)$ also holds for $Z \subseteq X$. Now G_i admits an admissible pairing M_i $(i = 1, 2)$. Since $d_G(X)$ is even, the shrunk nodes are of even degree, and thus $M := M_1 + M_2$ is an odd-node pairing for G.

We claim that M is admissible, that is, $d_M(Y) \leq b_G(Y)$ holds for every $Y \subseteq V$. To prove this, we can assume that (9.42α) holds, for otherwise Y can be replaced by its complement and then (9.42β) transforms into (9.42α). By (9.42α), b_G satisfies the submodular inequality for X and Y. Since d_M is submodular and $d_M(X, Y) = 0$, we get

$$\begin{aligned} d_M(Y) = d_M(X \cap Y) + d_M(X \cup Y) = d_{M_2}(X \cap Y) + d_{M_1}(V - (X \cup Y)) \leq b_{G_2}(X \cap Y) \\ + b_{G_1}(V - (X \cup Y)) \leq b_G(X \cap Y) + b_G(X \cup Y) \leq b_G(X) + b_G(Y) = b_G(Y), \end{aligned}$$

and hence M is indeed an admissible pairing for G. But this is impossible since G is bad. •

Let X be a non-trivial set for which $b_G(X) = 1$. Let G_1 and G_2 denote the graphs arising from G by shrinking X or $V - X$, respectively. Since X is non-trivial, the minimality of G implies that there is an admissible pairing M_i for G_i $(i = 1, 2)$. The degree of the shrunk node of G_i is $d_G(X)$. This is an odd number since $b_G(X) = 1$. Therefore there is an element $e_i \in M_i$ covering the shrunk node. Let v_i be the other end-node of e_i $(i = 1, 2)$. Then $M := M_1 + M_2 - e_1 - e_2 + v_1v_2$ is an odd-node pairing for G for which $d_M(X) = 1$. We will say that M is an odd-node pairing of G **associated with** X.

Claim 9.8.6 *Let X be a non-trivial set for which $b_G(X) = 1$ and let M be an odd-node pairing of G associated with X. Every M-violating set crosses X. If Y is an M-violating set for which $d_M(X, Y) = 0$, then (9.42α) does not hold.*

Proof. For every $Z \subseteq V - X$, we have $d_{G_1}(Z) = d_G(Z)$ and $R_{G_1}(Z) \geq R_G(Z)$ from which $b_{G_1}(Z) \leq b_G(Z)$. Similarly, $b_{G_2}(Z) \leq b_G(Z)$ holds for every $Z \subseteq X$. Let Y be an M-violating set. We cannot have $Y \subseteq X$, since then $d_M(Y) = d_{M_2}(Y) \leq b_{G_2}(Y) \leq b_G(Y)$, and then Y would not be M-violating. Analogously, $Y \not\subseteq V - X$. Therefore X and Y are indeed crossing. Suppose indirectly that (9.42α) holds and $d_M(X, Y) = 0$. Then

$$\begin{aligned}1 + d_M(Y) &= d_M(X) + d_M(Y) = d_M(X \cap Y) + d_M(X \cup Y)\\ &= d_{M_2}(X \cap Y) + d_{M_1}(V - (X \cup Y)) \leq b_{G_2}(X \cap Y) + b_{G_1}(V - (X \cup Y))\\ &\leq b_G(X \cap Y) + b_G(X \cup Y) \leq b_G(X) + b_G(Y) = 1 + b_G(Y).\end{aligned}$$

That is, $d_M(Y) \leq b_G(Y)$, contradicting the assumption that Y is M-violating. •

Claim 9.8.7 *G is 2-edge-connected. In particular, $\hat{R}_G(X) \geq 2$ for every non-empty subset $X \subset V$.*

Proof. By Claim 9.8.4, the degree of every node is at least 3. If $d_G(X) = 0$ for a non-trivial subset X, then $\hat{R}_G(X) = 0$ and hence $b_G(X) = 0$, but this would contradict Claim 9.8.5, and hence G is connected. Suppose now indirectly that G has a cut-edge. Since $d_G(v) \geq 3$ for every node, there is a non-trivial subset X for which $d_G(X) = 1$. Clearly $R_G(X) = 1$, $\hat{R}_G(X) = 0$, and hence $b_G(X) = 1$.

Let M be an odd-node pairing of G associated with X. There is an M-violating set Y and, by complementing it if necessary, we can assume that $d_M(X, Y) = 0$. By Claim 9.8.6, X and Y are crossing. Now $R_G(X) = 1 \leq R_G(Z)$ holds for every subset $Z \subseteq V$, and hence (9.43) holds, implying (9.42α) for X and Y, but this contradicts Claim 9.8.6. •

Claim 9.8.8 *Let $f = uv \in E$ be an edge for which $d_G(u)$ and $d_G(v)$ are odd and let $G' := G - f$. Then $\hat{R}_{G'} = \hat{R}_G$.*

Proof. What we prove is that $\hat{\lambda}_{G'}(x, y) = \hat{\lambda}_G(x, y)$ for every pair $x, y \in V$ of nodes and this implies the claim. If indirectly $\hat{\lambda}_{G'}(x, y) < \hat{\lambda}_G(x, y)$ for some nodes x and y, then $\lambda_{G'}(x, y) = \lambda_G(x, y) - 1$ and $\lambda_G(x, y)$ is even. By Menger's theorem, there is an $x\overline{y}$-set X separating u and v for which $d_G(X) = \lambda_G(x, y)$. We can assume that $|X| \leq |V - X|$ for otherwise X can be replaced by its complement. Now $b_G(X) = 0$ and by Claim 9.8.5, we have $|X| = 1$, and hence $X = \{u\}$ or $X = \{v\}$. But this contradicts the assumption that both $d_G(u)$ and $d_G(v)$ are odd. •

Claim 9.8.9 *The degree of at least one end-node of every edge of G is even.*

Proof. Suppose indirectly that there is an edge $f = uv \in E$ for which $d_G(u)$ and $d_G(v)$ are odd. There exists an admissible pairing M' for $G' := G - f$. Claim 9.8.8 implies for every subset $X \subseteq V$ that $b_{G'}(X) = b_G(X) - 1$ or $b_{G'}(X) = b_G(X)$ according to whether X separates u and v or not, and hence $M' + uv$ is an admissible pairing for G. But this is impossible since G is bad. •

Let T_3 denote the set of nodes of degree 3 and let $S = V - T_3$. By Claim 9.8.9, T_3 is stable. A non-trivial subset $X \subset V$ is **essential** if $d_G(v, X - v) \geq 2$ for every $v \in T_3 \cap X$ and $d_G(v, X) \leq 1$ for $v \in T_3 - X$. By Claim 9.8.9, $X \not\subseteq T_3$ and $S \not\subseteq X$.

Claim 9.8.10 *For any odd-node pairing M of G, there is an M-violating essential set.*

Proof. By Claim 9.8.7, $\hat{R}_G(X) \geq 2$ whenever $\emptyset \subset X \subset V$. This implies for any node v of degree 3 and for any non-trivial X that $\hat{R}_G(X + v) = \hat{R}_G(X)$ when $v \in V - X$, and $\hat{R}_G(X - v) = \hat{R}_G(X)$ when $v \in X$.

Since G is bad, there is an M-violating set X. Suppose that $d_G(X)$ is as small as possible. Then X must be essential for if not, then either there is a node $v \in T_3 \cap X$ with $d_G(v, X - v) \leq 1$ or there is a node $v \in T_3 - X$ with $d_G(v, X) \geq 2$. In the first case, let $X' := X - v$, while $X' := X + v$ in the second. Since X is non-trivial, $\emptyset \subset X' \subset V$. Furthermore $d_G(X') \leq d_G(X) - 1$, $\hat{R}_G(X') = \hat{R}_G(X)$ and hence $b_G(X') + 1 \leq b_G(X)$. By the minimality of $d_G(X)$, X' is not M-violating and thus $d_M(X) \leq d_M(X') + 1 \leq b_G(X') + 1 \leq b_G(X)$, contradicting the assumption that X is M-violating. •

Claim 9.8.11 $|S| \geq 2$.

Proof. Suppose indirectly that S is a singleton $\{s\}$. Then s is connected to every node $t \in T_3$ with three parallel edges and then G has no any other edges. Claim 9.8.9 implies that $d_G(s)$ is even. Let M be an arbitrary odd-node pairing. For every non-empty subset $X \subseteq V - s$, we have $R_G(X) = 3$ and $d_G(X) = 3|X|$ from which $d_M(X) \leq |X| \leq 3|X| - 2 = b_G(X)$. That is M is admissible, contradicting the badness of G. •

Let s be an element of S with minimum degree. Let $\lambda_G(S) := \min\{\lambda_G(x, y) : x, y \in S\}$. Clearly, $\lambda_G(S) \leq d_G(s)$.

Lemma 9.8.12 $\lambda_G(S) < d_G(s)$.

Proof. Suppose indirectly that $\lambda_G(S) = d_G(s)$. By Claim 9.8.4, $d_G(s) \geq 3$, and actually $d_G(s) \geq 4$ since the nodes of degree 3 belong to T_3. Since G is 2-edge-connected, no edge incident to s is a cut-edge. By the undirected splitting theorem of Mader (Theorem 8.1.7), there is a pair of edges $e = su$ and $f = st$ for which their splitting off results in a graph G' for which $\lambda_{G'}(x, y) = \lambda_G(x, y)$ holds for every $x, y \in V - s$.

Claim 9.8.13 $\hat{R}_{G'}(X) = \hat{R}_G(X)$ *for every set X that is essential in G.*

Proof. Let X be an essential set. Evidently $R_{G'}(X) \leq R_G(X)$ since the splitting off operation does not increase the local edge-connectivity. We can assume that $s \in X$, for otherwise X can be replaced by its complement. Since $\lambda_G(S) \geq 4$ and every essential set separates S, $\hat{R}_G(X) \geq 4$. Let $u \in X$ and $v \in V - X$ be nodes for which $R_G(X) = \lambda_G(u, v)$. $\hat{R}_G(X) \geq 4$ implies that $u, v \in S$. If $R_G(X) > \lambda_G(S) = d_G(s)$, then $u, v \neq s$, and thus $R_{G'}(X) \geq \lambda_{G'}(u, v) = \lambda_G(u, v) = R_G(X)$, that is, the claim holds in this case.

Therefore we can assume that $R_G(X) = \lambda_G(S) = d_G(s)$. Then we must have $X \cap S \neq \{s\}$. Indeed, if $X \cap S = \{s\}$, then every element of $X - s$ is of degree 3 and, by Claim 9.8.9, these nodes induce no edge. As X is essential, for every node u in $X - s$ there are at least two us-edges and at most one edge connecting u and $V - X$ from which $d_G(s) > d_G(X)$, but this is not possible since $d_G(X) \geq R_G(X) = d_G(s)$. Therefore there is a node $x \in (S \cap X) - s$ and hence $R_{G'}(X) \geq \lambda_{G'}(x, v) = \lambda_G(x, v) \geq \lambda_G(S) = R_G(X)$, from which the claim follows. •

As G' is smaller than G, there exists an admissible pairing M for G'. Since $d_{G'} \leq d_G$, we obtain from Claim 9.8.13 that $b_{G'}(X) \leq b_G(X)$ for every essential set X of G, and hence M is an admissible pairing for G. This contradiction to the badness of G proves the lemma. • •

Let $x, y \in S$ be two nodes for which $\lambda_G(x, y) = \lambda_G(S)$ and let X be an $x\overline{y}$-set for which $d_G(X) = \lambda_G(x, y)$. Now X is not trivial since if X or $V - X$ is a singleton, say $X = \{x\}$, then the minimality of $d_G(s)$ and Lemma 9.8.12 would imply that $d_G(s) \leq d_G(x) = d_G(X) = \lambda_G(S) < d_G(s)$. Furthermore, $R_G(X) \geq \lambda_G(x, y) = d_G(X) \geq R_G(X)$ and hence $d_G(X) = R_G(X)$. By Claim 9.8.5, $0 < b_G(X) = d_G(X) - \hat{R}_G(X) \leq d_G(X) - R_G(X) + 1 = 1$ and thus $b_G(X) = 1$. Moreover, X is essential.

Let M be an odd-node pairing of G associated with X. There is an M-violating essential set Y and, by complementing it if necessary, we can assume that $d_M(X, Y) = 0$. By Claim 9.8.6, X and Y are crossing. Since both X and Y are essential, Claim 9.8.9 implies that $S \cap X \cap Y \neq \emptyset$ and $S \cap (V - (X \cup Y)) \neq \emptyset$. It follows from the choice of X that $R_G(X) \leq R_G(Z)$ for every set Z separating S. This implies (9.43) from which (9.42α) follows, contradicting Claim 9.8.6. This final contradiction completes the proof of the Odd-node pairing theorem. • • •

Proof of the Strong orientation theorem (Theorem 9.8.1). Let M be an admissible odd-node pairing for G ensured by the Odd-node pairing theorem. Let D' be an Euler orientation of the Euler graph $G + M$. Let $D := D' - \vec{M}$ where $\vec{M}$ is the set of arcs of D' corresponding to M. (Note that D is an orientation of G.) For $X \subseteq V$ we have

$$\varrho_D(X) \geq \varrho_{D'}(X) - d_M(X) = (d_G(X) + d_M(X))/2 - d_M(X)$$
$$= (d_G(X) - d_M(X))/2 \geq \hat{R}_G(X)/2,$$

where the last inequality is implied by (9.45). By the directed edge-version of Menger's theorem, (9.40) and thus the theorem follows. •

The present proof of the Odd-node pairing theorem is based on the approach of [127]. In Part III, we will discover that the weak form of Nash-Williams' orientation theorem can be embedded into the theory of submodular flows. With the help of that approach, several extensions of the weak orientation theorem become tractable, such as the degree-constrained, the minimum cost, and the mixed graph k-edge-connected orientation problems. It is perhaps disappointing that for well-balanced orientations, each of these three problems is **NP**-complete. This was proved for the first two problems by Bernáth et al. [26], and for the third one by Bernáth and Joret [27]. In [26], it was also pointed out that the problem of finding a minimum-cost admissible odd-node pairing is **NP**-complete.

Research problem 9.8.1 *Given an odd-node pairing M of a graph G, decide algorithmically whether M is admissible.*

Research problem 9.8.2 *Given a subset T of odd-degree nodes of G, find a characterization and/or an algorithm for deciding whether G has a best-balanced orientation in which $\varrho(t) = \lfloor d_G(t)/2 \rfloor$ for every $t \in T$.*

978–0–19–920527–1 09-Andras-Frank-c09-drv Andras Frank (Typeset by SPi, Publisher Services, India) 338 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

In this light, Theorem 9.8.1, however beautiful, remains pretty isolated in combinatorial optimization, and it is a fascinating challenge to find links to other connectivity problems. One initial step toward this goal is an elegant paper by Király and Szigeti [240]. By relying on the Odd-node pairing theorem, they proved, among others, the following surprising extension of Theorem 9.8.1.

Theorem 9.8.14 (Király and Szigeti) *Let $G = (V, E)$ be an undirected graph, $\{E_1, \dots, E_t\}$ an arbitrary partition of E, and let $G_j := (V, E_j)$ $(1 \leq j \leq t)$. Then G has a best-balanced orientation such that its restriction to E_j is a best-balanced orientation of G_j for each j.* •

The special case of this theorem, when all but one E_i are singletons, was already observed by Nash-Willams in his original paper. The latter result has the following stunning consequence.

Corollary 9.8.15 *If $G_1 = (V, E_1)$ is a maximal Euler subgraph of G, then any Euler orientation of G_1 can be completed, by suitably orienting the forest $E - E_1$, to a best-balanced orientation of G.* •

The same way as the strong form of Nash-Williams' orientation theorem implies its weak form, Theorem 9.8.14 gives rise to the following consequence.

Corollary 9.8.16 *Let $G = (V, E)$ be a $2k$-edge-connected graph and let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \dots, G_t = (V_t, E_t)$ be subgraphs of G such that $\{E_1, \dots, E_t\}$ is a partition of E and G_j is $2k_j$-edge-connected for $j = 1, \dots, t$. Then G has a k-edge-connected orientation for which the restriction to E_j is a k_j-edge-connected orientation of G_j for each $j = 1, \dots, t$.* •

This result will be derived in Section 15.4 by using submodular flows (see Theorem 15.4.10).

10

Trees and arborescences: packing and covering

Paths form one class of the basic building blocks of connectivity problems, and variations of Menger's theorem characterize situations when k edge- or node-disjoint (directed or undirected) st-paths exist. Trees and arborescences are no less important constituents of the area, and this chapter is devoted to packing and covering problems concerning these objects. A typical approach will be that we first derive a result on directed graphs which is later combined with an orientation theorem, worked out in Chapter 9, in order to get the undirected counterpart.

10.1 Packing arborescences

10.1.1 Packing arborescences of fixed root

The starting point is Edmonds' seminal result on disjoint arborescences [83]. Recall that a digraph D is termed as rooted k-edge-connected with respect to a root-node r_0 if the in-degree of every non-empty subset of $V - r_0$ is at least k. By the directed edge-version of Menger's theorem this was shown to be equivalent to the existence of k edge-disjoint dipaths of D from r_0 to every node of D.

Theorem 10.1.1 (Edmonds' disjoint arborescences: weak form) *Let $D = (V, A)$ be a directed graph with a designated root-node r_0. D has k disjoint spanning arborescences of root r_0 if and only if D is rooted k-edge-connected, that is,*

$$\varrho_D(X) \geq k \text{ whenever } X \subseteq V - r_0 \text{ and } X \neq \emptyset. \tag{10.1}$$

Proof. Necessity is evident. To prove sufficiency, consider Theorem 8.2.11 which provides a constructive characterization of rooted k-edge-connected digraphs asserting that D can be obtained from a smaller rooted k-edge-connected digraph D' by one of the elementary operations (A), (B), (C). Therefore it suffices to show that if D' includes k disjoint spanning r_0-arborescences $F'_1, \ldots, F'_k$, then so does D. This is straightforward for (A) and for (B), so we focus our attention only to (C) when D is obtained from D' by pinching together j edges with a new node z and adding then $k - j$ new edges entering z.

If no edge belonging to any F'_i is pinched, then the k new edges entering z can be distributed evenly among the k arborescences F'_i and we obtain the requested k disjoint spanning arborescences of D. So suppose that there are $\alpha \geq 1$ arborescences, say $F'_1, \ldots, F'_\alpha$, having at least one edge pinched in (C). For each of these arborescences F'_i, consider a pinched u_iv_i-edge $e_i \in F'_i$ which is closest in F'_i to r_0. Revise F'_i by replacing e_i with the edges u_iz and zv_i and, for every other pinched uv-edge h of F'_i, replace h by zv. In this way we obtain spanning arborescences $F_1, \ldots, F_\alpha$ of D. Furthermore, there are $k - \alpha$ unused edges entering z which can be evenly distributed among the $k - \alpha$ arborescences $F'_{\alpha+1}, \ldots, F'_k$ so as to obtain the missing $k - \alpha$ spanning arborescences of D. •

This derivation, found by Mader [280], shows that Edmonds' theorem is a simple consequence of the constructive characterization of rooted k-edge-connected digraphs, a result which followed also rather easily from Mader's directed splitting-off lemma (Theorem 8.2.1). The proof of this latter result was not particularly demanding either. Yet, it is a natural expectation to have a concise, direct proof of Theorem 10.1.1. The following simple proof is due to Lovász [266].

Direct proof of the 'if part' of Theorem 10.1.1. The theorem is evident if $k = 1$ and by induction it follows for $k \geq 2$ if we are able to show that there is a spanning arborescence F_1 (of root r_0) such that

$$\varrho_{A-F_1}(X) \geq k - 1 \text{ for every } \emptyset \subset X \subseteq V - r_0 \tag{10.2}$$

Let F_1 be a maximal arborescence of root r_0 satisfying (10.2), and suppose indirectly that F_1 is not spanning. Let $V_1 := V(F_1)$ and let ϱ' denote the in-degree function of $D' := D - F_1$. We say that a subset $Z \subseteq V - r_0$ is **dangerous** if $\varrho'(Z) = k - 1$ and $Z - V_1 \neq \emptyset$. Since $\varrho_D(X) = \varrho'(X) \geq k$ holds for $X \subseteq V - V_1$, every dangerous set intersects V_1.

Let $e = uv \in A$ be an arbitrary edge leaving V_1. If e does not enter any dangerous set, then $F'_1 := F_1 + e$ is an arborescence satisfying (10.2) with F'_1 in place of F_1, and this contradicts the maximal choice of F_1. Therefore $(*)$ each edge leaving V_1 must enter a dangerous set.

Let Z be a minimal dangerous set. There must be an edge $e = uv \in A$ from $Z \cap V_1$ to $Z - V_1$, for otherwise $\varrho_D(Z - V_1) = \varrho'(Z - V_1) \leq \varrho'(Z) = k - 1$, contradicting the hypothesis of the theorem. By $(*)$, e enters a dangerous set X. But then

$$(k-1) + (k-1) = \varrho'(X) + \varrho'(Z) \geq \varrho'(X \cap Z) + \varrho'(X \cup Z) \geq (k-1) + (k-1)$$

from which $\varrho'(X \cap Z) = k - 1$ follows. Since v is in $X \cap Z$ while u is not, $X \cap Z$ is a dangerous set which is a proper subset of Z, contradicting the minimal choice of Z. This contradiction proves the theorem. •

Although we described the proof in an indirect form, by relying on Max-flow Min-cut (MFMC) computations, it can easily be turned to a polynomial algorithm. For more efficient algorithms, see the work of Gabow and Manu [172], [176]. Both proof techniques above can be extended to obtain generalizations of Edmonds' theorem. In Section 10.1.2, sharpenings of the splitting off approach will be explored, while Sections 10.2 and 10.3 include extensions based on Lovász's approach.

Problems

10.1.1 $(*)$ *Let $D = (V, A)$ be a root-connected digraph from a root-node r_0. Suppose that every dicut of D has k elements. Prove that A can be coloured by k colours such that each colour class covers every dicut of D.*

10.1.2 *With a simple elementary construction, derive the directed edge-Menger theorem from Edmonds' theorem.*

10.1.3 $(*)$ *Let $\{s_i, t_i\}$ be k ordered pairs of not-necessarily-distinct nodes in a k-edge-connected digraph D. Prove Schiloach's theorem* [350] *stating that there are $s_i t_i$-paths $(i = 1, 2, \ldots, k)$ which are edge-disjoint.*

10.1.4 *A digraph $D' = (V, B)$ is the union of k edge-disjoint spanning arborescences of root r_0 if and only if $\varrho'(r_0) = 0$ and $\varrho'(v) = k$ for every $v \in V - r_0$ and the underlying undirected graph of D' is the union of k edge-disjoint spanning trees.*

The statement of Problem10.1.4 will be used in Section 13.3.4 to find algorithmically a cheapest subgraph of a rooted k-edge-connected digraph that partitions into k spanning arborescences of root r_0.

Research problem 10.1.5 *Let $D = (V, A)$ be a digraph and $g : V \to \mathbf{Z}_+$ a function. A directed spanning tree F of the underlying undirected graph is g-**bounded** if $\varrho_F(v) \leq g(v)$ for every node $v \in V$. Develop a necessary and sufficient condition for the existence of k disjoint g-bounded spanning trees. Or, prove that the problem is **NP**-complete. Edmonds' arborescence theorem provides an answer in the special case when g is 1 everywhere apart from a node r_0 where $g(r_0) = 0$.*

Arborescences in directed hypergraphs

With the help of trimming, Edmonds' theorem can readily be extended to hypergraphs [144].

Theorem 10.1.2 *Suppose every dyperedge of a dypergraph $D = (V, \mathcal{A})$ has at least two elements. Let r_0 be a given root-node. Then D can be decomposed into k edge-disjoint spanning rooted k-edge-connected dypergraphs if and only if*

$$\varrho_D(X) \geq k \text{ for every non-empty } X \subseteq V - r_0. \tag{10.3}$$

Proof. Necessity is evident. Sufficiency follows directly from Theorems 7.4.9 and 10.1.1. •

Problem 10.1.6 $(*)$ *Construct a strongly connected dypergraph which cannot be trimmed to a strongly connected digraph.*

10.1.2 Packing Steiner-arborescences

Given the directed edge-Menger theorem and Theorem 10.1.1 on disjoint arborescences, the following possible common generalization naturally emerges. Suppose that beside a designated root-node r_0, we are given a set $T \subseteq V - r_0$ of terminals, as well. An arborescence of root r_0 is said to be a **Steiner arborescence** (with respect to T) if every node of T belongs to F (but it may have other nodes, too). It is clearly necessary for the existence of k disjoint Steiner arborescences that $\varrho(X) \geq k$ for every $X \subseteq V - r_0$ intersecting T. This condition is sufficient as well in the special case when $T = V - r_0$ (by Edmonds' theorem) or when $|T| = 1$ (by Menger's theorem). For general T, however, the condition is not sufficient as is shown by the following simple example of Lovász [266]. (See Figure 10.1.)

Here $k = 2$, T consists of three nodes and there are no two disjoint arborescences both containing every element of T. The existence of such a counterexample is not so surprising since the problem of finding two disjoint Steiner arborescences can be shown to be **NP**-complete. In this light, additional special cases in which the disjoint Steiner arborescence problem is tractable are of interest.

Observe that in the example above, $\varrho(x) = 1 < \delta(x) = 2$ holds for the only node x not in T. The next result of Bang-Jensen, Frank, and Jackson [8] shows that if we exclude such nodes, the situation gets brighter.

Theorem 10.1.3 *Let $D = (V, A)$ be a digraph, r_0 a root-node, and $T \subseteq V - r_0$ a set of terminals such that*

$$\varrho(X) \geq k \text{ whenever } X \subseteq V - r_0 \text{ and } X \cap T \neq \emptyset. \tag{10.4}$$

Suppose that $\varrho(v) \geq \delta(v)$ for every $v \in (V - T) - r_0$. Then there are k disjoint arborescences of root r_0 such that every node of T belongs to each of them.

Proof. By Lemma 7.4.1, we can assume that the in-degree of each node in T is exactly k and that $\varrho(r_0) = 0$.

Suppose first that there is a node z in $V - T - r_0$. By Theorem 8.2.8, if $\varrho(z) > \delta(z)$, then there is an edge entering z which can be left out without destroying (10.4) and by induction, we are done; while if $\varrho(z) = \delta(z)$, then there is an edge entering z and one leaving z such that their splitting off preserves (10.4), in which case by induction we are also done.

Suppose now that $T = V - r_0$. (Note that this is the case in Theorem 10.1.1, but we reprove it since the inductive proof in the present, stronger environment is simpler.) Since $\varrho(r_0) = 0 < \delta(r_0)$ there is a node z for which $\varrho(z) > \delta(z)$. By induction, (when the theorem is applied to the smaller $T' = T - z$) there are k edge-disjoint r_0-arborescences containing all nodes in $T - z$. If j of them contains an edge entering z, then the remaining $k - j$

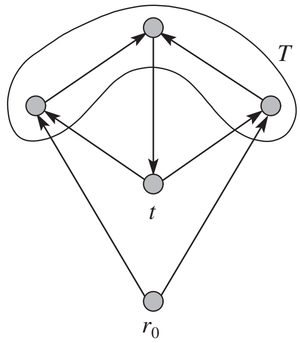


Figure 10.1 Digraph not including 2 Steiner arborescences

edges entering z can be assigned to the remaining $k - j$ arborescences and all the resulting arborescences will be spanning. •

Corollary 10.1.4 *Suppose that the edges of a digraph $D = (V, A)$ are coloured in such a way that each edge of the same colour has the same head. There are k disjoint spanning r_0-arborescences in such a way that each colour-class is used by at most one of them if and only if there are at least k distinct colours among the edges entering X for every $X \subseteq V - r_0$.*

Proof. Necessity is evident. To see sufficiency, add first $\sum p(v)$ new nodes to D for every $v \in V - r_0$ where $p(v)$ denotes the number of distinct colours of the edges entering v. Furthermore, replace each edge of colour i and head v with an edge entering v_i where v_i is the node corresponding to colour i.

Let $T := V - r_0$. Now each node outside $T + r_0$ has out-degree 1 and hence the hypothesis of Theorem 10.1.3 holds. Therefore the new digraph includes k edge-disjoint r_0-arborescences such that each of them contains every node in T. It follows that the k corresponding arborescences of D satisfy the requirements. •

Theorem 10.1.3 can be further sharpened [8].

Theorem 10.1.5 *Let $D = (V, A)$ be a digraph with a special root-node r_0 and let $T' := \{x \in V - r_0 : \varrho(x) < \delta(x)\}$ be a set of terminals. Assume that $\lambda(r_0, x) \geq k$ (≥ 1) for every $x \in T'$. Then there is a set $\mathcal{F}$ of k edge-disjoint arborescences rooted at r_0 such that every node $x \in V$ belongs to at least $r(x) := \min\{k, \lambda(r_0, x)\}$ members of $\mathcal{F}$.* •

The proof of this result, though it follows the same lines as that of Theorem 10.1.3, is technically more complicated, and we skip it. Two consequences, however, are worth mentioning. Call a digraph $D = (V, A)$ with root r_0 a **pre-flow digraph** if $\varrho(x) \geq \delta(x)$ holds for every $x \in V - r_0$. Recall the Flow decomposition lemma (Lemma 3.4.3) which stated that any flow from s to t can be decomposed into path-flows. The following corollary can be interpreted as a non-trivial generalization of Lemma 3.4.3.

Corollary 10.1.6 *In a pre-flow digraph $D = (V, A)$, there is a set $\mathcal{F}$ of k edge-disjoint arborescences of root r_0 for every integer k (≥ 1) such that every node v belongs to $\min\{k, \lambda(r_0, v)\}$ members of $\mathcal{F}$. In particular, if $k := \max\{\lambda(r_0, v) : v \in V - r_0\}$, then every v belongs to $\lambda(r_0, v)$ members of $\mathcal{F}$.* •

The theorem of Shiloach mentioned in Problem10.1.3 can also be extended.

Corollary 10.1.7 *Let $(s_1, t_1), \ldots, (s_k, t_k)$ be k ordered pairs of nodes in a digraph $D = (V, A)$. Suppose that there are edge-disjoint directed $s_i v$-paths $(i = 1, \ldots, k)$ for every node v for which $\varrho(v) < \delta(v)$ or $v = t_i$ holds. Then there are edge-disjoint directed $s_i t_i$-paths $(i = 1, \ldots, k)$.*

Proof. Extend the digraph by a new node r_0 and an $r_0 s_i$-edge for each $i = 1, \ldots, k$. By Theorem 10.1.5 there are k disjoint arborescences $F_1, \ldots, F_k$ rooted at r_0 such that each contains every t_i. Since there are exactly k edges leaving r_0, each $r_0 s_i$-edge belongs to one of F_i's. Therefore F_i includes a path P_i from s_i to t_i $(i = 1, \ldots, k)$ and these paths are edge-disjoint. •

10.1.3 Packing arborescences of free roots

What can we say when the roots of the k spanning arborescences are not fixed? We are going to answer a more general question. Call a vector $z : V \to \{0, 1, \ldots, k\}$ a **root vector** if there are k edge-disjoint spanning arborescences such that each node v is the root of $z(v)$ arborescences. From Edmonds' theorem it is easy to get a characterization of root vectors.

Theorem 10.1.8 *Given a digraph $D' = (V', A')$, a vector $z : V' \to \{0, 1, \ldots, k\}$ is a root vector if and only if $\widetilde{z}(V') = k$ and $\widetilde{z}(X) \geq k - \varrho'(X)$ for every non-empty subset $X \subseteq V'$ where ϱ' is the in-degree function of D'.*

Proof. The necessity of both conditions is evident. To prove sufficiency, extend D' with a node r_0 and $z(v)$ parallel r_0v-edges for each $v \in V'$. In the resulting digraph $D = (V, A)$, the out-degree of r_0 is exactly k and $\varrho_D(X) = \widetilde{z}(X) + \varrho'(X) \geq k$ holds for every non-empty $X \subseteq V'$. By Edmonds' theorem D includes k disjoint spanning arborescences of root r_0. Since $\delta_D(r_0) = k$, each of these arborescences must have exactly one edge leaving r_0, and therefore their restrictions to A' form the k disjoint spanning arborescences of D' with root vector z. •

The following result is a bit technical as it plays mostly a preparatory role. It will be used in the proof of the main result of the subsection.

Theorem 10.1.9 *Let $D = (V, A)$ be a rooted k-edge-connected digraph with respect to a root-node r_0. Suppose that $\gamma \geq k$ is an integer and $f : (V - r_0) \to \mathbf{Z}_+$ is a function such that*

$$\textit{the number of parallel } r_0v\textit{-edges is at least } f(v). \tag{10.5}$$

It is possible to delete some edges leaving r_0 in such a way that the resulting digraph $\hat{D}$ is rooted k-edge-connected, satifies (10.5)*, and the out-degree of r_0 is at most γ if and only if*

$$\widetilde{f}(V - r_0) \leq \gamma \tag{10.6}$$

and

$$\widetilde{f}(X_0) + kt - \sum_{i=1}^{t} \varrho_{D-r_0}(X_i) \leq \gamma \ \textit{ holds for every partition } \{X_0, X_1, \ldots, X_t\} \textit{ of } V - r_0 \tag{10.7}$$

where $t \geq 1$ and only the set X_0 can be empty.

Proof. Suppose that the desired subgraph $\hat{D}$ exists and let $\{X_0, X_1, \ldots, X_t\}$ be a partition of $V - r_0$. By (10.5) there are at least $\widetilde{f}(V - r_0)$ edges leaving r_0 that enter $V - r_0$, showing that (10.6) is necessary. Furthermore there are at least $\widetilde{f}(X_0)$ edges from r_0 to X_0. Since the in-degree of X_i in $\hat{D}$ is at least k for $i \geq 1$, there must be at least $k - \varrho_{D-r_0}(X_i)$ edges from r_0 to X_i. Therefore the out-degree of r_0 in $\hat{D}$ is at least $\widetilde{f}(X_0) + \sum_{i=1}^{t}[k - \varrho_{D-r_0}(X_i)]$ on the one hand, and at most γ on the other. Hence (10.7) is necessary.

To prove sufficiency, remove as many edges of D leaving r_0 as possible, in an arbitrary order, without violating (10.5) and rooted k-edge-connectivity.

Claim 10.1.10 *In the resulting digraph $\hat{D}$, the out-degree of r_0 is at most γ.*

Proof. A subset $X \subseteq V - r_0$ is **tight** if its in-degree in $\hat{D}$ is exactly k. By the minimality of $\hat{D}$ every edge $e = r_0 v$ enters a tight set or there are exactly $f(v)$ parallel copies of e in $\hat{D}$. In the special case when every edge $e = r_0 v$ has exactly $f(v)$ parallel copies in $\hat{D}$, (10.6) implies $\delta_{\hat{D}}(r_0) = \widetilde{f}(V - r_0) \leq \gamma$ and we are done. So we can assume that there is an edge $e = r_0 v$ which has strictly more than $f(v)$ parallel copies in $\hat{D}$ and therefore it enters a tight set.

It follows from the submodularity of the in-degree function that the union and the intersection of two intersecting tight sets are tight again. Therefore the maximal tight sets form a subpartition $\{X_1, \dots, X_t\}$ of $V - r_0$. Let $X_0 := (V - r_0) - \cup_{i=1}^{t} X_i$. Then $\delta_{\hat{D}}(r_0) = \widetilde{f}(X_0) + \sum_{i=1}^{t}[k - \varrho_{D-r_0}(X_i)]$ and here the right-hand side is at most γ by (10.7). •

The following result is due to Cai [39] and Frank [113].

Theorem 10.1.11 *Let $f : V^* \to \mathbf{Z}_+$ be a lower and $g : V^* \to \mathbf{Z}_+ \cup \{\infty\}$ an upper bound function for which $f \leq g$. A digraph $D^* = (V^*, A^*)$ includes k edge-disjoint spanning arborescences for which their root vector z satisfies $f \leq z \leq g$ if and only if*

$$\widetilde{f}(V^*) \leq k, \tag{10.8}$$

$$\sum_{i=1}^{t} \varrho_{D^*}(X_i) \geq k(t-1) + \widetilde{f}(X_0) \text{ holds for every partition } \{X_0, X_1, \dots, X_t\} \text{ of } V^* \tag{10.9}$$

where $t \geq 1$ and only X_0 can be empty, and

$$\widetilde{g}(X) \geq k - \varrho_{D^*}(X) \text{ for every } \emptyset \subset X \subseteq V^*. \tag{10.10}$$

Proof. To see the necessity, suppose that the k required arborescences exist. Since $\widetilde{f}(V^*)$ is a lower bound for the number k of arborescences, (10.8) follows. Let $\{X_0, X_1, \dots, X_t\}$ be a partition of V^*. Among the k given arborescences, $\alpha \geq \widetilde{f}(X_0)$ has its root in X_0 and therefore each of these α arborescences enter each of the sets $X_1, \dots, X_t$, while the other $k - \alpha$ arborescences enter at least $t - 1$ of them. Therefore, the total number $\sum_{i=1}^{t} \varrho_D(X_i)$ of edges entering the sets $X_1, X_2, \dots, X_t$ is at least least $\alpha t + (k - \alpha)(t - 1) = \alpha + k(t-1) \geq \widetilde{f}(X_0) + k(t-1)$ from which (10.9) follows. Moreover, among the k arborescences, at most $\varrho(X)$ can enter X which implies that the roots of those at least $k - \varrho(X)$ arborescences not entering X are necessarily in X. On the other hand the number of these arborescences is at most $\widetilde{g}(X)$ from which (10.10) follows.

The see sufficiency, extend D^* by a new node r_0 and by $\min\{g(v), k\}$ parallel edges from r_0 to v for every node v. The resulting digraph D is rooted k-edge-connected by virtue of (10.10). Observe that for $\gamma = k$, the conditions (10.7) and (10.9) are equivalent and so are (10.6) and (10.8). By applying Theorem 10.1.9 to D and to $\gamma = k$, we conclude that D has a rooted k-edge-connected subgraph $\hat{D}$ for which $\varrho_{\hat{D}}(V^*) = k$, furthermore every edge $r_0 v$ occurs in at least $f(v)$ and at most $g(v)$ copies.

By Theorem 10.1.1, $\hat{D}$ has k edge-disjoint spanning arborescences of root r_0. Since $\varrho_{\hat{D}}(V^*) = k$, each of these arborescence contains exactly one edge leaving r_0. Therefore, after removing these k edges, we are left with k edge-disjoint spanning arborescences of D^* for which their root vector z satisfies $f \leq z \leq g$. •

Corollary 10.1.12 *A digraph $D = (U, A)$ includes k disjoint spanning arborescences if and only if $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ for every subpartition $\{X_1, \ldots, X_t\}$ of U.* •

Exercise 10.1.7 *What is a necessary and sufficient condition for the existence of k disjoint spanning arborescences for which the roots are distinct?*

The following corollary shows that the linking property is valid for spanning arborescences.

Corollary 10.1.13 *In a digraph $D = (U, A)$ there exist k edge-disjoint spanning arborescences such that*

(A) *each node v is the root of at most $g(v)$ of them if and only if*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) \text{ holds for every subpartition } \{X_1, \ldots, X_t\} \text{ of } U \tag{10.11}$$

and

$$\widetilde{g}(X) \geq k - \varrho_D(X) \text{ for every } \emptyset \subset X \subseteq U, \tag{10.12}$$

(B) *each node v is the root of at least $f(v)$ of them if and only if $\widetilde{f}(U) \leq k$ and* (10.9) *holds,*

(C) *each node v is the root of at least $f(v)$ and at most $g(v)$ of them if and only if both the lower bound problem and the upper bound problem have separately solutions.* •

Problem 10.1.8 *Given a digraph D, find k edge-disjoint spanning branchings of D such that the total number of their components is as small as possible. Show that this problem is equivalent to the one of finding a minimum number of new edges the addition of which to the digraph gives rise to the existence of k edge-disjoint spanning arborescences.*

Remark 10.1.1 The proof of Theorem 10.1.11 was based on the fortunate circumstance that it did not matter which new r_0v-arcs were used to extend the digraph, only (10.9) was important to be satisfied. Moreover, during the phase of removing new edges, the order of deletions was not essential either, only the preservation of (10.9) was important. The reader rightfully has the impression that this looks like a greedy algorithm and a matroid-like structure must be in the background. Indeed, we shall prove in Part III that the root vectors of k disjoint spanning arborescences form a base-polymatroidal set and the approach used in the proof of Theorem 10.1.11 is nothing but a variation of the greedy algorithm for base-polyhedra.

10.1.4 Independent arborescences

We close this section by presenting some results on independent arborescences. While studying the various forms of Menger's theorem, we realized that the edge- and the node-versions are pretty close to each other. This similarity between edge-disjoint and openly disjoint st-paths gives rise to the following conjecture that might seem a natural counterpart of Edmonds' theorem on edge-disjoint arborescences.

If D admits k openly disjoint r_0v-paths for every node v, then there are k spanning arborescences of root r_0 such that the k r_0v-paths determined by the k arborescences are openly disjoint for every node $v \in V$.

We say that such a set of arborescences is **independent**. In other words, the conjecture states that every rooted k-node-connected digraph includes k independent spanning arborescences of root r_0. Huck [213], however, disproved this conjecture for every $k \geq 3$. In what follows, we show that the conjecture does hold when D is acyclic and k is arbitrary, and also when D is arbitrary and $k = 2$.

Acyclic digraphs

The following result is due to Huck [212].

Theorem 10.1.14 (Huck) *Let $D = (V, A)$ be a simple acyclic digraph in which $R = \{r_1, \dots, r_k\}$ denotes the set of source-nodes* (where a source-node is one with in-degree 0) *while $U := V - R$ is the rest. Suppose that the in-degree of each node $u \in U$ is at least k. Then there are r_i-arborescences F_i spanning $U + r_i$ for $i = 1, \dots, k$ such that the k unique r_iu-paths in the arborescences F_i are openly disjoint for every node $u \in U$.*

Proof. The next lemma is the core of the proof.

Lemma 10.1.15 *Suppose that $D' = (U + r, A')$ is a simple acyclic digraph in which r is a source-node and the in-degree of each other node is at least one. Then there is an ordering of the elements of U in such a way that the set of all edges going forward can be completed with some edges leaving r so as to obtain a spanning r-arborescence.*

Proof. The lemma is clear when U is a singleton so we can assume that $|U| \geq 2$. Then there is a sink-node z. By induction, there is a requested ordering of the elements of $U - z$ with respect to the digraph $D' - z$. If there is an rz-edge of D', then by putting z at the beginning of the existing ordering of $U - z$ we are done.

If there is no rz-edge, then there is a node in U from which there is an edge of D' entering z. Let u_i denote the earliest one of these nodes in the given ordering of $U - z$. Insert z between u_i and u_{i+1}. The resulting ordering of U satisfies the requirements of the lemma, since the only new edge going forward created by the insertion of z is u_iz. •

Since the theorem is obvious for $k = 1$, we can assume that $k \geq 2$. Apply the lemma to the subgraph $D' = (U + r_k, A')$ of D induced by $U + r_k$. Let $u_1, \dots, u_p$ be the ordering of the elements of U ensured by the lemma and let F_k denote the arborescence belonging to it. Let D'' be a subgraph of D obtained by deleting node r_k and the edges of F_k. By induction, D'' admits the requested independent arborescences $F_1, \dots, F_{k-1}$ for $i = 1, \dots, k - 1$. Since all the edges of these arborescences go backward in the ordering $u_1, \dots, u_p$ while all the edges of F_k go forward, it follows that the unique r_ku-path in F_k and the unique r_iu-path of F_i $(i = 1, \dots, k - 1)$ have the only node u in common for every $u \in U$. • •

Theorem 10.1.16 (Huck) *Let $D = (V, A)$ be an acyclic digraph with a designated root-node r_0. There are k independent spanning arborescences of root r_0 if and only D is rooted k-node-connected.*

Proof. Necessity is evident from the definition of independence. For sufficiency, subdivide first each edge e leaving r_0 by a new node. Let V_0 denote the set of subdividing nodes. Replace then r_0 by a set $\{r_1, \dots, r_k\}$ of nodes and replace each edge r_0v by edges $\{r_1v, \dots, r_kv\}$ for every $v \in V_0$. Let $U' := (V - r_0) \cup V_0$. The hypotheses of Theorem 10.1.14 hold for the resulting digraph $D' = (V', A')$. Therefore there are r_i-arborescences F'_i in D' spanning $U' + r_i$ for $i = 1, \dots, k$ such that $(*)$ *the k r_iu-paths determined by these arborescences are openly disjoint for each $u \in U'$*. Each F'_i, when restricted to the original edges of D, determines a spanning arborescence F_i of root r_0 and these arborescences are independent because of $(*)$. •

Two independent arborescences

Our next goal is to derive a result of Whitty [385] stating that a rooted 2-node-connected digraph includes two independent arborescences.

Theorem 10.1.17 (Whitty) *Let $D = (V, A)$ be a digraph with a root-node $r_0 \in V$. There are two independent spanning arborescences of root r_0 if and only if D is rooted 2-node-connected.*

Proof. The necessity is evident. For proving sufficiency, apply Theorem 8.3.13. It implies that r_0 can be detached into two nodes r_1 and r_2 in such a way that (8.19) holds. Consider the ordering of nodes ensured by Theorem 8.3.8. For each node $v_i \in V - R$, choose an edge v_hv_i for which $h < i$. These edges define an arborescence F'_1 rooted at r_1 and spanning $V - r_2$. Similarly, for each node $v_i \in V - R$, choose an edge v_jv_i for which $j > i$. These edges define an arborescence F'_2 rooted at r_2 and spanning $V - r_1$. By the construction, both F'_1 and F'_2 determine a spanning arborescence of D rooted at r_0 and these arborescences are independent by virtue of the property of the ordering in question. •

Finally, we note that Georgiadis and Tarjan [181] generalized Theorem 10.1.17 as follows.

Theorem 10.1.18 *Let $D = (V, A)$ be a digraph with a root-node $r_0 \in V$. Then there are two* (possibly not edge-disjoint) *spanning arborescences F_1 and F_2 of root r_0 such that the two r_0v-paths in F_1 and F_2 are openly disjoint for every node $v \in V - r_0$ for which $\kappa(r_0, v; D) \geq 2$.* •

Conjectures

10.1.19 *Let $D = (V, A)$ be a rooted $k\ell$-edge-connected digraph with root-node r_0. Suppose that the set of edges entering each non-root node v is coloured in such a way that each colour class has at most ℓ elements. Then there are ℓ edge-disjoint rooted k-edge-connected spanning subgraphs $(V, F_1), \dots, (V, F_\ell)$ of D in such a way that the edges in each F_i entering v are of different colours for every $v \in V - r_0$.*

10.1.20 *Let $g : A \to \mathbf{Z}_+$ be an integer-valued capacity function for which $g(e) \leq \ell$ for every edge e of a digraph $D = (V, A)$. There are ℓ rooted k-edge-connected spanning subgraphs of D such that each edge e belongs to at most $g(e)$ of these subgraphs if and only if $\varrho_g(Z) \geq k\ell$ for every non-empty subset $Z \subseteq V - r_0$ where $\varrho_g(Z)$ denotes the total capacity of edges entering Z.* (This is formally a special case of Conjecture 10.1.19 but it can be shown that the two conjectures are actually equivalent. Note that the corresponding statements hold for k-braids from s to t and also for k-trees in undirected graphs, where a k-tree is the union of k edge-disjoint spanning trees.)

10.1.21 *Let D be a rooted (k, ℓ)-hybrid-connected digraph with respect to a root-node r_0. Then the edge-set of D can be partitioned into ℓ rooted k-node-connected parts.*

10.2 Packing branchings

Edmonds actually proved his theorem in a stronger form where the goal was to pack k edge-disjoint spanning branchings of given root-sets. Recall from Section 1.1 that a branching is a directed forest in which the in-degree of each node is at most 1 and the root-set of the branching is the set of nodes of in-degree 0. Note that a branching with root-set R is the union of $|R|$ node-disjoint arborescences. Often we identify a branching with its edge-set. For example, by disjoint branchings we mean edge-disjoint branchings. For a digraph $D = (V, A)$ and root-set $\emptyset \subset R \subseteq V$ a branching (V, B) is called a **spanning R-branching** of D if its root-set is R. In particular, if R is a singleton consisting of the element r_0, then a spanning branching is a spanning arborescence of root r_0.

10.2.1 Packing branchings spanning V

The proof technique of Lovász [268] for Theorem 10.1.1 can be extended to derive the following result of Edmonds [83]. Its power will be further exploited in Section 10.3.

Theorem 10.2.1 (Edmonds' disjoint branchings) *In a digraph $D = (V, A)$, let $\mathcal{R} = \{R_1, \dots, R_k\}$ be a family of k non-empty* (not necessarily disjoint) *subsets of V and let $H = (V, \mathcal{R})$ denote the hypergraph of the given root-sets. There are k disjoint spanning branchings of D with root-sets $R_1, \dots, R_k$, respectively, if and only if*

$$\varrho_D(X) \geq p_H(X) \text{ whenever } \emptyset \subset X \subseteq V \tag{10.13}$$

where $p_H(X)$ denotes the number of root-sets R_i disjoint from X.

Proof. Necessity. Since a spanning branching B_i of D with root-set R_i contains an edge entering a non-empty $X \subset V$ whenever X and R_i are disjoint, we have $\varrho_D(X) \geq \sum_i \varrho_{B_i}(X) \geq p_H(X)$, and hence (10.13) is indeed necessary.

Sufficiency. There is nothing to prove if $R_i = V$ for each $i = 1, \dots, k$, so we can suppose that $R_1 \neq V$. A non-empty subset $X \subseteq V$ is **tight** if $\varrho_D(X) = p_H(X)$. A tight set that properly intersects R_1 is **dangerous**.

Claim 10.2.2 *There is an edge $e = uv \in A$ leaving R_1 that does not enter any dangerous set.*

Proof. If there is no dangerous set at all, then any edge e leaving R_1 will suffice, and there exists such an edge since (10.13) implies that $\varrho_D(V - R_1) \geq p_H(V - R_1) \geq 1$. Suppose now that there are dangerous sets and let M be a minimal one (with respect to inclusion). There must be an edge $e = uv$ of D for which $u \in M \cap R_1$ and $v \in M - R_1$, for otherwise $p_H(M - R_1) > p_H(M) = \varrho_D(M) \geq \varrho_D(M - R_1)$, and hence $M - R_1$ would violate (10.13).

We show now that e does not enter any dangerous set. Suppose indirectly that e enters a dangerous X. By Proposition 1.2.9,

$$p_H(X) + p_H(M) = p_H(X \cup M) + p_H(X \cap M) - d_H^*(X, M)$$

where $d_H^*(X, M)$ denotes the number of hyperedges which intersect both $X - M$ and $M - X$ but are disjoint from $X \cap M$. Hence

$$p_H(X) + p_H(M) = \varrho_D(X) + \varrho_D(M) \geq \varrho_D(X \cup M) + \varrho_D(X \cap M) \geq$$

$$p_H(X \cap Y) + p_H(X \cup Y) = p_H(X) + p_H(M) + d_H^*(X, M)$$

from which equality must hold everywhere, implying that $M \cap X$ is tight and $d_H^*(X, M) = 0$. Since both X and M intersect R_1, $d_H^*(X, M) = 0$ implies that $X \cap M \cap R_1 \neq \emptyset$ from which we can conclude that $M \cap X$ is also dangerous, contradicting the minimal choice of M. (Note that node u is in $M - X$ and hence $M \cap X$ is a proper subset of M.) •

Claim 10.2.3 *Condition* (10.13) *holds with respect to* $D' := D - e$ *and to* $H' := \left(V, \left\{R_1', R_2, \ldots, R_k\right\}\right)$ *where* $R_1' := R_1 + v$.

Proof. Suppose indirectly that there is a set $X \subset V$ for which $\varrho_{D'}(X) \leq p_{H'}(X) - 1$. In this case, $\varrho_D(X) \leq \varrho_{D'}(X) + 1 \leq p_{H'}(X) \leq p_H(X) \leq \varrho_D(X)$ from which equality holds everywhere. In particular, $\varrho_D(X) = \varrho_{D'}(X) + 1 = p_H(X)$, which means that X is tight and e enters X. Furthermore, $p_{H'}(X) = p_H(X)$ and hence X must intersect R_1 for otherwise X would be disjoint from R_1 but not from R_1' (as $v \in X$), and then we would have $p_{H'}(X) < p_H(X)$. Therefore X is dangerous, which is impossible since e does not enter a dangerous set. •

By induction, $D - e$ contains k edge-disjoint spanning branchings $B_1', B_2, \ldots, B_k$ of root-sets $\left\{R_1', R_2, \ldots, R_k\right\}$, respectively, and then $B_1, B_2, \ldots, B_k$ are the requested branchings of D where $B_1 := B_1' + e$. • •

Remark 10.2.1 In the special case of Theorem 10.2.1 when each root-set R_i is the same singleton $\{r_0\}$, we are back at Theorem 10.1.1. Conversely, when all the R_i's are singletons but they may be distinct, then it was shown in the proof of Theorem 10.1.8 that Theorem 10.2.1 follows from Theorem 10.1.1. However, for general R_i's no reduction is known to derive Theorem 10.2.1 from Theorem 10.1.1.

It is interesting to formulate Edmonds' theorem (Theorem 10.2.1) in an equivalent form.

Theorem 10.2.4 (Edmonds' disjoint arborescences: strong form) *Let* $D = (V, A)$ *be a digraph for which the node-set* V *is partitioned into a root-set* $R = \{r_1, \ldots, r_k\}$ (of distinct roots) *and a terminal set* $T = V - R$. *Suppose that no edge of* D *enters any node of* R. *There are* k *disjoint arborescences* $F_1, \ldots, F_k$ *in* D *such that* F_i *is rooted at* r_i *and spans* $T + r_i$ *for each* $i = 1, \ldots, k$ *if and only if* $\varrho_D(X) \geq |R - X|$ *for every subset* $X \subseteq V$ *for which* $X \cap T \neq \emptyset$.

Proof. By applying Theorem 10.2.1 to the subgraph D' of D induced by T with the choice $R_i = \{v : \text{there is an edge } r_i v \in A\}$ $(i = 1, \dots, k)$ (and the same construction shows the reverse implication, too). •

Exercise 10.2.1 *Show that Theorem* 10.1.1 *follows directly from Theorem* 10.2.4.

There is yet another interesting reformulation of Theorem 10.2.1.

Theorem 10.2.5 *Let $D' = (V, A')$ be a digraph with a root-node r_0. Let $F_1, \dots, F_k$ be edge-disjoint arborescences of root r_0* (where any edge-set can be empty but the node-set always contains r_0) *and let $D = (V, A)$ denote the subgraph of D' consisting of edges not used by any F_i. These arborescences can be extended into k edge-disjoint spanning arborescences of D' if and only if*

$$\varrho_D(X) \geq p(X) \text{ holds for every subset } \emptyset \neq X \subseteq V - r_0 \tag{10.14}$$

where $p(X)$ denotes the number of arborescences for which $V(F_i) \cap X = \emptyset$. •

Problem 10.2.2 (∗) *Prove that Theorems* 10.2.1 *and* 10.2.5 *are equivalent.*

Theorem 10.1.8 characterized root vectors of k disjoint spanning arborescences. Though this result formally is more general than the weak form of Edmonds' theorem, yet the latter one immediately implied it. In this light, one may naturally feel that there should exist an extension of Theorem 10.1.8 to include the strong form of Edmonds' theorem, and indeed the following theorem can be proved.

Theorem 10.2.6 *Let $D = (V, A)$ be a digraph for which the node-set V is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set $T = V - R$. Suppose that no edge of D enters any node of R. Let $m : R \to \mathbf{Z}_+$ be a function and let $k = \widetilde{m}(R)$. There are k disjoint arborescences in D such that $m(r)$ of them are rooted at r and span $T + r$ for each $r \in R$ if and only if*

$$\varrho_D(X) \geq \widetilde{m}(R - X) \text{ for every subset } X \subseteq V \text{ for which } X \cap T \neq \emptyset. \bullet \tag{10.15}$$

This result, however, does not seem to follow from Theorem 10.2.1, and its proof needs new ideas. Actually, we shall derive a proper generalization of Theorem 10.2.1 (see Theorem 10.2.14) which includes Theorem 10.2.6 as a special case.

The situation turns much worse if we want to generalize Theorem 10.1.12 in which lower and upper bounds are imposed on the root vector. A special case of Theorem 10.1.12 (when $f \equiv 0$, $g \equiv 1$) characterizes digraphs in which there are k disjoint spanning arborescences such that their roots are distinct. The next result indicates that no such type of characterization can be expected concerning the strong form of Edmonds' theorem.

Theorem 10.2.7 *Let $D = (V, A)$ be a digraph for which the node-set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R. The problem of deciding if there are k disjoint arborescences such that they are rooted at distinct elements of R and each of them spans T is* **NP***-complete.* •

For a proof, see [20].

Algorithmic aspects

Since the proof of Theorem 10.1.1 relied on Mader's directed splitting lemma, it can be converted into a polynomial time algorithm that uses an MFMC algorithm as a subroutine. But how can we compute the disjoint branchings in Theorem 10.2.1? The following natural approach was observed by Tarjan [363]. First of all, we have to be able to test the truth of (10.13). To this end, extend D as follows. Add $k+1$ new nodes $r_0, r_1, \ldots, r_k$, and for each $i = 1 \ldots, k$, add an edge $r_0 r_i$ and edges $r_i v$ for every $v \in R_i$. Let $D' = (V', A')$ denote the resulting digraph.

Proposition 10.2.8 *The condition* (10.13) *holds if and only if*

$$\varrho'(X) \geq k \text{ for every } X \subseteq V' - r_0,\ X \cap V \neq \emptyset \tag{10.16}$$

where ϱ' denotes the in-degree function of D'.

Proof. Recall that $p_H(X)$ denoted the number of root-sets R_i which are disjoint from X. Suppose first that $X \subseteq V' - r_0$ is a maximal set violating (10.16). If $X \cap R_i \neq \emptyset$ for some $i = 1, \ldots, k$, then $r_i \in X$ for otherwise $\varrho'(r_i) = 1$ would imply $\varrho'(X + r_i) \leq \varrho'(X) < k$ contradicting the maximality of X. It follows that the number of $r_0 r_i$-edges entering X is the number of R_i'-s intersecting X. Therefore $\varrho'(X) = \varrho(X) + k - p_h(X)$ and hence X violates (10.13).

Conversely, suppose that a subset $X \subseteq V$ is a set violating (10.13). Let $\hat{X} := X \cup \{r_i : X \cap R_i \neq \emptyset\}$. Then $\varrho'(\hat{X}) = \varrho(X) + k - p_H(X) < p_H(X) + k - p_H(X) = k$, and hence $\hat{X}$ violates (10.16). •

Since (10.16) can be tested by $|V|$ MFMC computations, we have obtained the requested subroutine for testing (10.13). This subroutine can be applied to check if there is an edge leaving a root-set R_i which is usable in the sense that it does not enter any dangerous set. Since Theorem 10.2.1 ensures the existence of a usable edge leaving $R_i \subset V$, by testing each edge leaving R_i, we will find a usable one (or find a subset violating (10.13)) and by repeatedly adding usable edges, eventually we obtain the requested k spanning branchings.

This is a strongly polynomial algorithm though its complexity is not particularly spectacular. To get a bound, notice that if an edge leaving an R_i turns out to be not usable at a certain stage of the algorithm, then it will no longer be usable for branching B_i. Therefore each edge needs to be tested for usability at most once for a fixed i and at most k times altogether. Since one test needs $O(|V|)$ MFMC computations, the overall algorithm requires $O(k|V||A|)$ MFMC computations.

An interesting feature of Theorem 10.2.1 is that this algorithmic approach relies only on the theorem itself and not on its proof.

Problem 10.2.3 *Let c be a non-negative real-valued weight function on the edge-set of D for which $\varrho_c(X) \geq 1$ holds for every set X with $\emptyset \neq X \subseteq V - r_0$. Prove that there is a spanning r_0-arborescence which enters every c-tight set exactly once where a set X is c-tight if $\varrho_c(X) = 1$.*

Completing branchings

What happens if we want to complete existing branchings into edge-disjoint spanning arborescences? For the simplest case $k = 1$, the following result is easily obtained.

Theorem 10.2.9 *Let $D = (V, A)$ be a digraph with a root-node r_0 admitting no entering edges. A branching (V, B) with $B \subseteq A$ can be extended to a spanning arborescence of root r_0 if and only if there is no subset $Z \subseteq V - r_0$ for which $\varrho_B(Z) = 0$ and $\varrho_B(v) > 0$ for every edge $uv \in A$ entering Z.*

Proof. If the set Z with the described property exists, then at least one edge uv entering Z must be added to B in order to get a spanning arborescence, but then the in-degree of v in the arborescence will be at least two, a contradiction. That is, the condition is necessary.

To see sufficiency, delete all edges uv from $A - B$ for which v is the head of an edge of B. If there is a spanning r_0-arborescence in the resulting digraph D', then this automatically includes B and we are done.

If there is no such an arborescence in D', then there is a subset $Z \subseteq V - r_0$ of zero in-degree. Then Z violates the condition of the theorem since Z can be entered by an edge $e = uv$ of D only if v is the head of an edge in B. •

The completion of just one branching is therefore a simple problem. If, however, one wants to complete 2 branchings to edge-disjoint spanning arborescences, then the problem becomes **NP**-complete even in the very special case when B_1 consists of just two edges and B_2 has no edge at all. Indeed, we show that if this special completion problem can be solved in polynomial time, then so can the two edge-disjoint paths problem in digraphs, a known **NP**-complete problem.

To see the reduction, suppose we want to find edge-disjoint $s_i t_i$-paths for $(i = 1, 2)$ in a digraph H. Add new nodes r_0 and t to H along with edges $r_0 s_i$ and $t_i t$ for $i = 1, 2$. Finally, add two parallel edges from t to every node. Let B_1 consist of the two edges $r_0 s_1$ and $t_1 t$ while B_2 has no edges. It is easy to check that the resulting digraph D has two edge-disjoint spanning arborescences of root r_0 such that one of them includes B_1 if and only if H includes edge-disjoint $s_i t_i$-paths $(i = 1, 2)$.

10.2.2 Packing maximal branchings

The **NP**-completeness of the 2 disjoint Steiner arborescences problem indicates that no much room is available for finding proper extensions of Edmonds' theorem on disjoint branchings or its equivalent form given in Theorem 10.2.1. Yet, such a generalization is possible.

Let $D = (V, A)$ be a directed graph and $R \subseteq V$ a non-empty set. By a **maximal R-branching** of D we mean a subgraph of D which is a branching the node-set of which consists of the nodes of D reachable from R in D and R is the root-set of B_i.

Notation For non-empty subsets Z and $X \subseteq V$, let $Z \mapsto X$ denote the property that Z and X are disjoint and X is reachable from Z (meaning that there is a directed path from Z to X).

Theorem 10.2.10 (Kamiyama, Katoh, and Takizawa [229]) *In a digraph $D = (V, A)$, let $\mathcal{R} = \{R_1, \dots, R_k\}$ be a family of k non-empty* (not necessarily disjoint) *subsets of V, called root-sets. There are edge-disjoint maximal R_i-branchings $(i = 1, \dots, k)$ if and only*

$\varrho_D(X) \geq p(X)$ for every non-empty subset $X \subseteq V$ where $p(X)$ denote the number of root-sets R_i for which $R_i \mapsto X$.

Proof. Proving necessity is straightforward. In the proof of sufficiency, we refer to a set $X \subseteq V$ as **tight** if $\varrho_D(X) = p(X)$.

Lemma 10.2.11 *Let X and Y be two intersecting tight sets and suppose that $X \cap Y$ is reachable from every node of the symmetric difference $X \ominus Y$. Then $X \cap Y$ is tight and there is no R_i for which $R_i \cap X \neq \emptyset$, $R_i \cap Y \neq \emptyset$, and $R_i \cap X \cap Y = \emptyset$.*

Proof. We claim that $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$. To see this, consider the contribution of one set R_i to the two sides. By the hypothesis of the lemma, $R_i \mapsto X$ implies (and $R_i \mapsto Y$ also) that $R_i \mapsto (X \cap Y)$. Hence if R_i contributes 1 to the left-hand side, then it contributes at least 1 to the right as well. If R_i contributes 2 to the left, then $R_i \mapsto X$ and $R_i \mapsto Y$, then $R_i \mapsto (X \cap Y)$ and $R_i \mapsto (X \cup Y)$ so R_i contributes 2 to the right-hand side.

We have $\varrho_D(X) + \varrho_D(Y) = p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \leq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \leq \varrho_D(X) + \varrho_D(Y)$ from which $p(X \cap Y) = \varrho_D(X \cap Y)$ (and also $p(X \cup Y) = \varrho_D(X \cup Y)$), furthermore $p(X) + p(Y) = p(X \cap Y) + p(X \cup Y)$. That is, $X \cap Y$ is tight and a set R_i mentioned in the lemma cannot exist since its contribution to the left-hand side is 0 and to the right-hand side is 1. •

Lemma 10.2.12 *Let X and Y be two intersecting sets for which X is tight, $\varrho_D(Y) = 0$, and suppose that $X \cap Y$ is reachable from every node in $Y - X$. Then $X \cap Y$ is tight and there is no R_i for which $R_i \cap X \neq \emptyset$, $R_i \cap Y \neq \emptyset$, and $R_i \cap X \cap Y = \emptyset$.*

Proof. We claim that $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$. From $0 = \varrho_D(Y) \geq p(Y) \geq 0$, we have $p(Y) = 0$. Therefore if a set R_j contributes to the left-hand side, then its contribution is 1 and $R_j \mapsto X$. If R_j is disjoint from Y, then R_j is disjoint from $X \cup Y$, too, and hence $R_j \mapsto (X \cup Y)$. If X is not disjoint form Y, then $R_j \mapsto (X \cap Y)$ since $X \cap Y$ is reachable from every element of $Y - X$. In both cases, R_j contributes to the right-hand side of the supermodular inequality.

We obtain $\varrho_D(X) + \varrho_D(Y) = p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \leq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \leq \varrho_D(X) + \varrho_D(Y)$ from which $p(X \cap Y) = \varrho_D(X \cap Y)$ and $p(X) + p(Y) = p(X \cap Y) + p(X \cup Y)$. The set R_i in the statement of the lemma cannot exist since its contribution is 0 to the left-hand side and 1 to the right. •

We use induction on $\sum_i |S_i - R_i|$ where S_i denotes the set of nodes reachable from R_i. If this number is zero, then $p \equiv 0$ in which case the empty branchings satisfy the requirements. Suppose now that $S_1 - R_1$ is non-empty. We define a tight set X as **dangerous** if $X - R_1 \neq \emptyset$ and $X \cap R_1 \neq \emptyset$. It suffices to prove that there is an edge $e = uv$ leaving R_1 which does not enter any dangerous set. Indeed, in this case we revise R_1 by adding v to it, delete e, and apply induction.

Lemma 10.2.13 *Let $t \in V - R_1$ be a node for which there is an edge st with $s \in R_1$ and let X be a smallest dangerous set containing t. Then* (A) *t is reachable form every node in X, and* (B) *there is an edge uv with $u \in X \cap R_1$ and $v \in X - R_1$.*

Proof. Let Y denote the set of nodes from which t is reachable. Then $\varrho_D(Y) = 0$, $t \in X \cap Y$, $s \in Y$ and thus $Y \cap R_1 \neq \emptyset$. By Lemma 10.2.12, $X \cap Y$ is tight and we cannot have $X \cap Y \cap R_1 = \emptyset$, and hence $X \cap Y$ is dangerous. By the minimality of X, we have $X \cap Y = X$, and hence $X \subseteq Y$, from which Part (A) follows.

To see Part (B), suppose indirectly that the edge uv in question does not exist. Then $t \in X$ implies that $s \notin X$ and hence $R_1 \mapsto (X - R_1)$, while $R_1 \cap X \neq \emptyset$ implies $R_1 \not\mapsto X$. Furthermore, for $i \geq 2$, if $R_i \mapsto X$, then $R_i \mapsto (X - R_1)$ since t is reachable from every node in $X \cap R_1$. Hence $p(X - R_1) > p(X)$ from which $p(X - R_1) > p(X) = \varrho_D(X) \geq \varrho_D(X - R_1)$, contradicting the hypothesis of the theorem. •

Let X be a smallest dangerous set for which there is an edge $e = uv$ with $v \in X - R_1$, $u \in R_1$. Here u can be in X or not in X but, by Part (B) of Lemma 10.2.13, we can assume that the first case occurs, that is, $u \in X$. We claim that e does not enter any dangerous set. Suppose indirectly it does and let Y be a minimal dangerous set entered by e. Apply Part (A) of Lemma 10.2.13 with v in place of t, first to X, and second to Y in place of X. We obtain that v is reachable from every element of $X \ominus Y$. By Lemma 10.2.11, $X \cap Y$ is tight and $X \cap Y \cap R_1 \neq \emptyset$, that is, $X \cap Y$ dangerous, contradicting the minimal choice of X.

Since e does not enter a tight set, the digraph $D' := D - e$ and the family $\mathcal{R}' := \{R'_1 := R_1 + v, R_2, \dots R_k\}$ of root-sets satisfy the condition of the theorem. By induction, there are edge-disjoint branchings $B'_1, B_2, \dots, B_k$ in D' satisfying the requirements. But then $B_1 := B'_1 + e, B_2, \dots, B_k$ form the requested edge-disjoint branchings in D. • •

Note that the proof of Theorem 10.2.10 also used the approach of Lovász, but it was certainly more tricky and technical than the basic version. In the special case when each root-set is a singleton in Theorem 10.2.10, we obtain the following.

Theorem 10.2.14 *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, r_2, \dots, r_k\} \subseteq V$ be a family of k* (possibly not distinct) *root-nodes. Let S_i denote the set of nodes reachable from r_i. There are edge-disjoint r_i-arborescences A_i spanning S_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{10.17}$$

where $p_1(Z)$ denotes the number of sets S_i's for which $S_i \cap Z \neq \emptyset$ and $r_i \notin Z$. •

Remark 10.2.2 In Remark 10.2.1 we noted that no reduction is known to derive Edmonds' disjoint branchings theorem from its special case when each root-set R_i is a singleton (which is actually equivalent to the disjoint arborescences theorem). In this light, it is interesting that Theorem 10.2.10 trivially follows from its special case when each root-set is a singleton as formulated in Theorem 10.2.14. To see this, add k new nodes $r_1, \dots, r_k$ to the digraph D and an edge from r_i to each element of R_i for $i = 1, \dots, k$. Theorem 10.2.10 is now clearly equivalent for D and root-sets R_i and for the extended digraph D' with singleton root-sets $R'_i := \{r_i\}$. We also note that Theorem 10.2.6 arises as a special case of Theorem 10.2.14 if we take each element r_i in $m(r_i)$ copies.

10.3 Further generalizations

Theorem 10.2.10 is still not the most general extension of Edmonds' theorem on disjoint arborescences. In order to arrive at this 'last' result, it is worth turning our attention to the proof of Theorem 10.2.1 developed by Lovász and attempting to better understand its true nature. The best way to distil its essence is to try formulating abstract extensions of disjoint arborescences. The next result can be considered an abstract version of the weak form of Edmonds' theorem.

10.3.1 Abstract forms

We say that a set F of directed edges **covers** a family $\mathcal{F}$ of subsets of nodes if $\varrho_F(X) \geq 1$ for every member X of $\mathcal{F}$.

Theorem 10.3.1 (Frank, [111]) *Let $D = (V, A)$ be a digraph and let $\mathcal{F}$ be an intersecting family of subsets of V. If the in-degree of every $X \in \mathcal{F}$ is at least k, then it is possible to partition the edge-set of D into k classes in such a way that each class covers $\mathcal{F}$.*

When $\mathcal{F}$ consists of all non-empty subsets of $V - r_0$, we are back at Theorem 10.1.1 since a subset of edges that enters every non-empty $X \subseteq V - r_0$ includes a spanning arborescence of root r_0. When $\mathcal{F}$ consists of all $t\bar{s}$-sets, we obtain the directed edge-version of Menger's theorem since a subset of edges that enters every $t\bar{s}$-set includes a directed st-path. Yet another special case comes from a rooted ℓ-edge-connected digraph $H = (V, F)$. Let $\mathcal{F}$ consist of the subsets of $V - r_0$ admitting exactly ℓ entering edges of H. Then $\mathcal{F}$ is an intersecting set-system for which the theorem characterizes digraphs $D = (V, A)$ whose edge-set can be partitioned into subsets $A_1, \ldots, A_k$ in such a way that $H + A_i$ is rooted $(\ell + 1)$-edge-connected for each $i = 1, \ldots, k$.

Problem 10.3.1 *Prove Theorem* 10.3.1.

We will derive a generalization of Theorem 10.3.1 below, but the reader may find it useful to devise a proof her/himself. Theorem 10.3.1 implies the weak form of Edmonds' theorem but not the strong one. The next result of Szegő [353] is an abstract extension of the strong form of Edmonds' theorem.

Theorem 10.3.2 (Szegő) *Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting families of subsets of nodes of a digraph $D = (V, A)$ with the following mixed intersection property:*

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \neq \emptyset \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then A can be partitioned into k subsets $A_1, \ldots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if $\varrho_D(X) \geq p_1(X)$ for all non-empty $X \subseteq V$ where $p_1(X)$ denotes the number of $\mathcal{F}_i$'s containing X. •

When the k families are identical, we are back at Theorem 10.3.1. When $\mathcal{F}_i = 2^{V-R_i} - \{\emptyset\}$, we obtain Theorem 10.2.1. The proof of Theorem 10.3.2 is based on the observation that the mixed intersection property implies that p_1 is a positively intersecting supermodular function and this is why Lovász' approach works again.

Since Theorem 10.3.1 is still not general enough to imply Theorem 10.2.10, we exhibit a further extension from [19] to bi-set families, and will include the proof of this extension only. We say that the bi-set families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ satisfy the **mixed intersection** property if

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset \Rightarrow\ X \sqcap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

For a bi-set X, let $p_2(X)$ denote the number of indices i for which $\mathcal{F}_i$ contains X. For $X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j$, the inclusion $X \sqsubseteq Y$ implies $X = X \sqcap Y \in \mathcal{F}_j$ and hence p_2 is non-increasing in the sense that $X \sqsubseteq Y,\ p_2(X) > 0$, and $p_2(Y) > 0$ imply $p_2(X) \geq p_2(Y)$.

Lemma 10.3.3 *If $p_2(X) > 0,\ \ p_2(Y) > 0$ and $X_I \cap Y_I \neq \emptyset$, then $p_2(X) + p_2(Y) \leq p_2(X \sqcap Y) + p_2(X \sqcup Y)$. Moreover, if there is an $\mathcal{F}_i$ for which $X \sqcap Y \in \mathcal{F}_i$ and $X, Y \notin \mathcal{F}_i$, then strict inequality holds.*

Proof. Consider the contribution of one family $\mathcal{F}_i$ to the two sides of the claimed inequality. If this contribution to the left-hand side is 2, that is, if both X and Y are in $\mathcal{F}_i$, then so are $X \sqcap Y$ and $X \sqcup Y$ and hence the contribution to the right-hand side is also 2. Suppose now that X belongs to $\mathcal{F}_i$ but Y does not. Since $p_2(Y) > 0$ is assumed, Y belongs to an $\mathcal{F}_j$. But then $X \sqcap Y$ belongs to $\mathcal{F}_i$ because of the mixed intersection property, and in this case the contribution of $\mathcal{F}_i$ to the right-hand side is at least one. An $\mathcal{F}_i$ with the properties in the second part of the theorem contributes only to the right-hand side, ensuring in this way the strict inequality. •

Theorem 10.3.4 *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ intersecting families of bi-sets on ground-set V satisfying the mixed intersection property. The edges of D can be partitioned into k parts $F_1, \ldots, F_k$ in such a way that F_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p_2(X) \text{ for every bi-set } X. \tag{10.18}$$

Proof. The condition is clearly necessary. We prove the sufficiency by induction on $\sum_i |\mathcal{F}_i|$. There is nothing to prove if this sum is zero so we can assume that $\mathcal{F}_1$, say, is non-empty. Let U be a maximal member of $\mathcal{F}_1$. Call a bi-set **tight** if $\varrho(X) = p_2(X) > 0$.

Claim 10.3.5 *There is an edge e entering U in such a way that each tight bi-set covered by e is in $\mathcal{F}_1$.*

Proof. Suppose indirectly that no such edge exists. Then each edge e entering U enters some tight bi-set $M \notin \mathcal{F}_1$. By the mixed intersection property, we cannot have $M \subseteq U$. Select a minimal tight bi-set $M \notin \mathcal{F}_1$ which intersects U. Since p_2 is non-increasing, we know that $p_2(U \cap M) \geq p_2(M)$. Here, in fact, strict inequality must hold since $U \cap M \in \mathcal{F}_1$ and $M \notin \mathcal{F}_1$. The inequality $p_2(U \cap M) > p_2(M)$ implies that D has an edge $f = uv$ for which $u \in M - U,\ v \in U \cap M$. By the indirect assumption, f enters some tight bi-set $Z \notin \mathcal{F}_1$. Lemma 10.3.3 implies that the intersection of M and Z is tight. Since neither M nor Z is in $\mathcal{F}_1$, the second part of the lemma implies that $M \cap Z$ is not in $\mathcal{F}_1$ either, contradicting the minimal choice of M. •

Let e be an edge ensured by the claim. Let $\mathcal{F}'_1 := \{X \in \mathcal{F}_1 : e \text{ does not enter } X\}$. Then $\mathcal{F}'_1$ is an intersecting family of bi-sets. We claim that the mixed intersection property holds for the families $\mathcal{F}'_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$. Indeed, let $X \in \mathcal{F}'_1$ and $Y \in \mathcal{F}_i$ be two intersecting bi-sets for some $i = 2, \ldots, k$. Since $\mathcal{F}'_1 \subseteq \mathcal{F}_1$, one has $X \cap Y \in \mathcal{F}_i$. If indirectly $X \cap Y$ is not in $\mathcal{F}'_1$, then e enters $X \cap Y$. Since e enters U and U was selected to be maximal in $\mathcal{F}_1$, it follows that $X \subseteq U$. But then e must enter X as well, contradicting the assumption $X \in \mathcal{F}'_1$.

Let $p'_2(X)$ denote the number of those families containing X (that is, $p'_2(X) = p_2(X) - 1$ if $X \in \mathcal{F}_1$ and e enters X and $p'_2(X) = p_2(X)$ otherwise). Let ϱ' denote the in-degree function on bi-sets with respect to $D' := D - e$. The choice of e implies $\varrho' \geq p_2'$. By induction, the edge-set of D' can be partitioned into k parts $F_1', \ldots, F_k$ in such a way that F_1' covers $\mathcal{F}_1$ and F_i covers $\mathcal{F}_i$ for $i = 2, \ldots, k$. By letting $F_1 := F_1' + e$, we obtain a partition of A requested by the theorem. • •

Theorem 10.3.4 can be formulated in terms of families of sets rather than bi-sets. For a subset $T \subseteq V$, we say that a family $\mathcal{F}$ of subsets of V is **T-intersecting** if $X, Y \in \mathcal{F}$ and $X \cap Y \cap T \neq \emptyset$ imply $X \cap Y, X \cup Y \in \mathcal{F}$.

Theorem 10.3.6 *Let $D = (V, A)$ be a digraph with a specified subset T of nodes containing the head of every edge of D. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be T-intersecting families of subsets of nodes of a digraph D with the following mixed intersection property: $X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \cap T \neq \emptyset \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j$. Then A can be partitioned into k subsets $A_1, \ldots, A_k$ such that A_i covers $\mathcal{F}_i$ for each $i = 1, \ldots, k$ if and only if $\varrho_D(X) \geq p_1(X)$ for all non-empty $X \subseteq V$ where $p_1(X)$ denotes the number of $\mathcal{F}_i$'s containing X.* •

Problem 10.3.2 $(*)$ *Derive Theorem* 10.3.6 *which is a reformulation of Theorem* 10.3.4 *in terms of families of sets.*

10.3.2 Disjoint arborescences spanning convex sets

Our next goal is to apply Theorem 10.3.4 and derive an extension of Theorem 10.2.14 which is due to Fujishige [162]. The proof below is taken from [20]. For two disjoint subsets X and Y of nodes of a digraph $D = (V, A)$, we say that Y is **reachable** from X if there is a directed path in D whose first node is in X and last node is in Y. We call a subset U of nodes **convex** if there is no node v in $V - U$ so that U is reachable from v and v is reachable from U.

Theorem 10.3.7 (Fujishige) *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, r_2, \ldots, r_k\} \subseteq V$ be a list of k* (possibly not distinct) *root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. There are edge-disjoint r_i-arborescences A_i spanning U_i for $i = 1, \ldots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{10.19}$$

where $p_1(Z)$ denotes the number of sets U_i's for which $U_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

Proof. If the node-set of an arborescence F of root r_i intersects a subset $Z \subseteq V - r_i$, then F contains an edge entering Z. Therefore if the k edge-disjoint arborescences exist, then Z admits as many entering edges as the number of sets U_i for which $Z \cap U_i \neq \emptyset$ and $r_i \notin Z$, and hence (10.19) is indeed necessary.

Sufficiency. For brevity, we call a strongly connected component of D an **atom**. The atoms form a partition of the node-set of D and we pointed out in Corollary 2.2.6 that there is a topological ordering of the atoms by which we mean that no edge from a later atom can go to an earlier one. By a **subatom** we mean a subset of an atom. Clearly, a subset $Z \subseteq V$ is a subatom if and only if any two elements of Z are reachable in D from each other. The following observation is obvious from the definitions.

Proposition 10.3.8 *If a subatom Z intersects a convex set U, then $Z \subseteq U$.* •

Define a bi-set family $\mathcal{F}_i$ for each $i = 1, \ldots, k$ as follows. Let $\mathcal{F}_i := \{(X_O, X_I) : X_O \subseteq V - r_i,\ X_I = X_O \cap U_i \neq \emptyset,\ X_I$ a subatom$\}$. For each bi-set X, let $p_2(X)$ denote the number of $\mathcal{F}_i$'s containing X. It follows immediately that $\mathcal{F}_i$ is an intersecting bi-set family.

Proposition 10.3.9 *The bi-set families $\mathcal{F}_i$ satisfy the mixed intersection property.*

Proof. Let $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$ be members of $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, and suppose that X and Y are intersecting, and hence $X_I \cap Y_I \neq \emptyset$. By Proposition 10.3.8 we have $X_I = X_O \cap U_i \subseteq U_i \cap U_j$ and $Y_I = Y_O \cap U_j \subseteq U_i \cap U_j$. These imply for the sets $Z_O := X_O \cap Y_O$ and $Z_I := X_I \cap Y_I$ that $Z_O \cap U_i = Z_I = Z_O \cap U_j$ from which $X \sqcap Y = (Z_O, Z_I) \in \mathcal{F}_i \cap \mathcal{F}_j$, as required. •

Proposition 10.3.10 $\varrho_D(X) \geq p_2(X)$ *for each bi-set X.*

Proof. Let $q := p_2(X)$ and suppose that X belongs to $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q$. Let $V' := V - (U_1 \cup \cdots \cup U_q)$ and $Z := X_I \cup \{v \in V' : X_I$ is reachable from $v\}$.

Let $e = uv$ be an edge of D entering the set Z. Then u cannot be in $V' - Z$ for otherwise X_I would be reachable from u and then u should belong to Z. Therefore u is in $(U_1 \cup \cdots \cup U_q) - Z$. Let U_i be one of the sets $U_1, \ldots, U_q$ containing u. We claim that the head v of e must be in X_I. For otherwise we are in a contradiction with the hypothesis that U_i is convex since v is reachable from U_i (along the edge uv) and U_i is also reachable from v since $X_I \subseteq U_i$ is reachable from v.

It follows that the edge e entering the set Z also enters the bi-set $X = (X_O, X_I)$. Therefore $\varrho_D(X) \geq \varrho_D(Z)$. By (10.19), we have $\varrho_D(Z) \geq p_1(Z)$. It follows from the definition of Z that $p_1(Z) \geq q = p_2(X)$, and hence $\varrho_D(X) \geq p_2(X)$ •

Therefore Theorem 10.3.4 applies and hence the edges of D can be partitioned into sets $F_1, \ldots, F_k$ such that F_i covers $\mathcal{F}_i$ for $i = 1, \ldots, k$.

Proposition 10.3.11 *Each F_i includes an r_i-arborescence A_i which spans U_i.*

Proof. If the requested arborescence does not exist for some i, then there is a non-empty subset Z of $U_i - r_i$ such that F_i contains no edge from $U_i - Z$ to Z. Consider a topological ordering of the atoms and let Q be the earliest one intersecting Z. Since no edge leaving a later atom can enter Q, no edge with tail in Z enters Q.

Let $X_O := (V - U_i) \cup (Z \cap Q)$ and $X_I := X_O \cap U_i$. Then $X_I = Z \cap Q$ is a subatom and $X = (X_O, X_I)$ belongs to $\mathcal{F}_i$. Therefore there is an edge $e = uv$ in F_i which enters X. It follows that $v \in X_I \subseteq Z$ and that $u \in U_i - X_I$. Since u is not in Z and not in $V - U_i$, it must be in $U_i - Z$, and hence e is an edge from $U_i - Z$ to $X_I \subseteq Z$, contradicting the assumption that no such an edge exists. • •

Research problem 10.3.3 *Develop a common generalization of Theorems* 10.3.7 *and* 10.1.3.

Research problem 10.3.4 *Let $D = (V, A)$ be a digraph with a set $\{r_1, \ldots, r_k\}$ of root-nodes. Let $U_1, \ldots, U_k$ be convex subsets of nodes such that $r_i \in U_i$. Find a necessary and sufficient condition for the existence of k arborescences $F_i, \ldots, F_k$ such that F_i is rooted at r_i, F_i spans U_i, and for each v the paths from r_i to v $(i = 1, \ldots, k)$ determined by the arborescences are node-disjoint apart from v.*

10.4 Covering by branchings, trees, and forests

So far we have concentrated on packing problems. In this section we discuss results on covering the edge-set of a digraph by k branchings and by k arborescences, and the edge-set of a undirected graph by k forests.

10.4.1 Covering by branchings and arborescences

Theorem 10.1.1 on packing arborescences can be used to obtain theorems on arborescences and branchings that cover the edge set.

Covering branchings

The following pretty consequence of Edmonds' theorem on disjoint arborescences appeared in [112].

Theorem 10.4.1 *The edge-set of a digraph $D = (V, A)$ can be covered by k branchings if and only if*

(A) *the in-degree of each node is at most k, and*

(B)

$$i_D(X) \leq k(|X| - 1) \text{ for every } \emptyset \subset X \subseteq V \tag{10.20}$$

where $i_D(X)$ denotes the number of edges induced by X.

Proof. Since the in-degree of each node in one branching is at most 1, Condition (A) is necessary. Since a forest can have at most $|X| - 1$ edges induced by X, k forests, and hence k branchings, can have at most $k(|X| - 1)$, and hence (B) is also necessary.

To prove sufficiency, we use the following elementary construction. Extend the digraph by adding a new node r_0 and by adding $k - \varrho_D(v)$ parallel edges from r_0 to v for each node $v \in V$. In the resulting digraph D', we have

$$\varrho'(X) = \varrho_D(X) + \sum[k - \varrho_D(v) : v \in X] = \varrho_D(X) - \varrho_D(X) - i_D(X) + k|X| \geq k$$

for every non-empty set $X \subseteq V$ and also $\varrho'(v) = k$ for every node $v \in V$. By Theorem 10.1.1, the edge-set of D' partitions into k edge-disjoint spanning arborescences of root r_0. By restricting these arborescences to the edge-set of D we obtain the requested partition of A into k branchings.

Problem 10.4.1 *Derive Theorem* 10.1.1 *from Theorem* 10.4.1.

Problem 10.4.2 (Kareyan [231]) *Prove that the edge-set of a digraph D not containing loops or parallel edges can be covered by $K + 1$ branchings where K denotes the maximum in-degree of a node of D. More generally, the edge-set of a loopless digraph can be covered $K + \ell$ branchings where ℓ denotes the maximum multiplicity of parallel uv-arcs.*

Covering arborescences

When can a digraph $D = (V, E)$ be covered by k spanning arborescences of root r_0? For any subset X of nodes, let $\Gamma^-(X) := \{v \in X$: there is an edge $uv \in A$ for which $u \in V - X\}$ and call this set the **entrance** of X. Informally, the entrance consists of the head nodes of edges entering X. The following result of Vidyasankar [373] can be considered as kind of covering counterpart of the disjoint arborescences theorem.

Theorem 10.4.2 (Vidyasankar) *Let r_0 be a root-node of a digraph $D = (V, A)$ such that no edge enters r_0. It is possible to cover the edge-set of D by k spanning arborescences of root r_0 if and only if*

$$\varrho_D(v) \leq k \text{ for every } v \in V - r_0 \tag{10.21}$$

and

$$k - \varrho_D(X) \leq \sum[k - \varrho_D(v) : v \in \Gamma^-(X)] \text{ for every } X \subseteq V - r_0. \tag{10.22}$$

Proof. Necessity. Since an arborescence contains one edge entering v, the necessity of (10.21) is evident. Suppose now that there are k arborescences covering A. Let $z(e)$ denote the number of arborescences covering e minus 1. Then $z \geq 0$, furthermore $\varrho_z(X) + \varrho_D(X) \geq k$ $(X \subseteq V - r_0)$ and $\varrho_z(v) + \varrho_D(v) = k$ $(v \in V - r_0)$. Since each edge entering X has its head in $\Gamma^-(X)$, we have $\varrho_z(X) \leq \sum[\varrho_z(v) : v \in \Gamma^-(X)]$ and these imply $k - \varrho_D(X) \leq \varrho_z(X) \leq \sum[\varrho_z(v) : v \in \Gamma^-(X)] = \sum[k - \varrho_D(v) : v \in \Gamma^-(X)]$.

The sufficiency is proved with the help of an elementary construction. For every node $v \in V - r_0$, add a copy of v to D which is denoted by v'. Add k parallel edges from v to v' and $k - \varrho_D(v)$ parallel edges from v' to v. (Note that $k - \varrho_D(v) \geq 0$ holds by (10.21).) In addition, add k parallel edges from u to v' for every edge $uv \in A$.

If the resulting digraph D' includes k edge-disjoint spanning arborescences, then these determine k arborescences of D covering all the edges. By Theorem 10.1.1, if these k edge-disjoint arborescences do not exist in D', then there is a subset $X' \subseteq V' - r_0$ for which $\varrho'(X') < k$ where $\varrho' := \varrho_{D'}$. Let $X := \{v \in X'\}$, $Z := \{v \in X', v' \notin X'\}$.

Due to the construction, if $v' \in X'$, then $v \in X'$ and if uv enters X, then $v \in Z$ and thus $\Gamma^-(X) \subseteq Z$. Therefore $k > \varrho'(X') = \varrho_D(X) + \sum[k - \varrho_D(v) : v \in Z] \geq \varrho_D(X) + \sum[k - \varrho_D(v) : v \in \Gamma^-(X)]$, contradicting Condition (10.22). •

10.4.2 Covering by forests

With the help of orientation theorems, we can derive results on packing and covering in undirected graphs. In Section 2.3, we defined an undirected graph $G = (V, E)$ as k**-sparse** for a positive integer k if

$$i_G(X) \leq k(|X| - 1) \text{ for every } \emptyset \neq X \subseteq V. \tag{10.23}$$

When equalilty holds, the set X will be referred to as **tight**.

Theorem 10.4.3 (Nash-Williams [305]) *The edge-set of an undirected graph $G = (V, E)$ can be covered by k forests if and only if G is k-sparse.*

Proof. The necessity is clear since any forest can have at most $|X| - 1$ edges induced by X.

For proving sufficiency, we claim that $G = (V, E)$ has an orientation D in which the in-degree of each node is at most k. Indeed, (10.23) implies that $i_G(X) \leq k|X|$ holds for $X \subseteq V$, and then Theorem 2.3.5, when applied to $g \equiv k$, states the existence of such an orientation. By applying Theorem 10.4.1 to D we are done. •

Problem 10.4.3 (∗) *Prove that the edge-set of a simple connected planar graph can be covered by three trees. If in addition G is bipartite, then there are two covering forests.*

The proof of Nash-Williams' theorem relied on the weak form of Edmonds' theorem on disjoint arborescences. Not surprisingly, by making use of the strong form one gets the following extension.

Theorem 10.4.4 *Suppose that the edge-set of an undirected graph $G = (V, E \cup F)$ is partitioned into subsets E and F where F is the union of k trees $T_i = (R_i, F_i)$ $(i = 1, \dots, k,)$ for which $\emptyset \neq R_i \subseteq V$* (where any F_i may be empty). *It is possible to extend the k trees from the elements of E into k forests which cover $E \cup F$ if and only if*

$$i_E(X) \leq \ddot{p}(X) - p(X) \text{ for every } \emptyset \subset X \subseteq V \tag{10.24}$$

where $i_E(X)$ denotes the number of edges in E induced by X, $p(X)$ denotes the number of sets R_i disjoint from X.

Proof. Necessity. Let X be a non-empty subset of V. If $R_i \cap X \neq \emptyset$, then a forest including T_i can contain at most $|X - R_i|$ elements of E induced by X. Therefore the total number of edges of this type is at most

$$\sum_{R_i \cap X \neq \emptyset} |X - V_i| = \sum_{1}^{k} |X - V_i| - p(X)|X| = \sum_{v \in X} p(v) - p(X)|X|.$$

If $R_i \cap X = \emptyset$, then a forest including T_i can contain at most $|X| - 1$ elements of E induced by X. Therefore the total number of these type of edges is at most $p(X)(|X| - 1)$. By combining the two upper bounds, (10.24) follows.

Sufficiency. We can assume that G is maximal in the sense that the addition of any possible new edge to E would destroy (10.24). Define a set-function b by $b(X) := \ddot{p}(X) - p(X)$. Then b is intersecting submodular. We refer to a set X as tight if $i_E(X) = b(X)$. The submodularity of b and (10.24) imply that the union (and intersection) of two intersecting tight sets X and Y is tight since $i_E(X) + i_E(Y) = b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \geq i_E(X \cap X) + i_E(X \cup Y) \geq i_E(X) + i_E(Y)$ from which one must have $i_E(X \cup Y) = b(X \cup Y)$. Therefore the maximal tight sets are disjoint and hence the maximality of G implies that V itself is tight, and hence $|E| = \sum[p(v) : v \in X]$.

Let $m(v) := p(v)$ $(v \in V)$. We have $|E| = \widetilde{m}(V)$ and $i_E(X) \leq \widetilde{m}(X)$ for every $X \subseteq V$ by (10.24). By the Orientation lemma (Theorem 2.3.2), there is an orientation $\vec{E}$ of E for which $\varrho_{\vec{E}}(v) = m(v)$ for every $v \in V$. Condition (10.24) implies for this orientation that

$$\varrho_D(X) = \sum[\varrho_D(v) : v \in X] - i_E(X) = \sum[p(v) : v \in X] - i_E(X) \geq p(X).$$

By Theorem 10.2.1 there are k edge-disjoint spanning branchings $\vec{B}_1, \dots, \vec{B}_k$ with root-sets $R_1, \dots, R_k$, respectively. By this construction each underlying forest B_i along with the initial tree T_i form a spanning tree of G. Since $\varrho_D(v) = p(v)$ for each node v, every edge of G must belong to one of these trees. •

Notice that Nash-Williams' theorem is indeed a special case: let $T_i = (\{r\}, \emptyset)$ where r is an arbitrary node. Then $\sum[p(v) : v \in X] - p(X) = k(|X| - 1) - 0 = k(|X| - 1)$

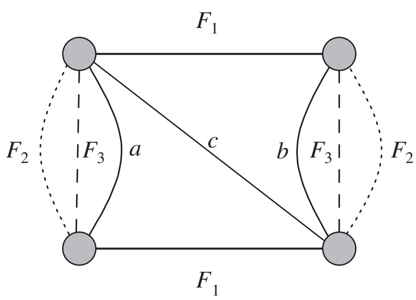


Figure 10.2 No extension of F_1, F_2, and F_3 to 3 covering forests exists

whenever $r \in X$ while $\sum[p(v) : v \in X] - p(X) = k|X| - k = k(|X| - 1)$ whenever $v \notin X$. In other words, (10.24) and (10.23) are equivalent in this case.

Question 10.4.1 *When is it possible to extend k given forests* (rather than just trees) *to forests covering all edges of a graph?*

The three forests F_1, F_2, F_3 in Figure 10.2 cannot be extended to forests that cover the edges a and b since both a and b could be added only to F_3, but $F_3 \cup \{a, b\}$ is not a forest. On the other hand, no subset of nodes induces 'too many' edges. With the help of matroids, a neat answer will be obtained in Section 13.3.2.

Bernáth and Király [29] proved the problem of deciding whether the edge-set of a connected graph $G = (V, E)$ can be partitioned into a spanning tree and a tree is **NP**-complete.

Conjecture 10.4.5 *Call the edge-disjoint union of k branchings a k-**branching**. Let $D = (V, A)$ be a digraph and $f : A \to \{1, \ldots, \ell\}$ a function. There are ℓ k-branchings covering f if and only if $i_f(X) \le k\ell(|X| - 1)$ for every non-empty subset $X \subseteq V$ and $\varrho_f(v) \le k\ell$ for every node $v \in V$, where $i_f(X) = \sum[f(uv) : uv \in A,\ u, v \in X]$.*

10.5 Packing trees and forests

10.5.1 Packing trees

By combining Edmonds' theorem on disjoint arborescences with an appropriate orientation theorem, we arrive at a classical result of Tutte [368]. Recall that a graph $G = (V, E)$ is k-partition-connected if

$$e_G(\mathcal{P}) \ge k(q - 1) \text{ for every partition } \mathcal{P} := \{V_1, \ldots, V_q\} \text{ of } V \tag{10.25}$$

where $e_G(\mathcal{P})$ denotes the number of edges connecting distinct parts (that is, $e_G(\mathcal{P}) = \sum_i d_G(V_i)/2$). We referred to G as k-tree-connected if G includes k disjoint spanning trees.

Theorem 10.5.1 (Tutte) *An undirected graph $G = (V, E)$ is k-tree-connected if and only if it is k-partition-connected. In other words, G includes k edge-disjoint spanning trees if and only if* (10.25) *holds.*

Proof. Necessity. If each member of a partition $\mathcal{P}$ of the node-set of a connected graph is shrunk into a single node, then the resulting graph is connected. Therefore each spanning tree must contain at least $q-1$ cross-edges to $\mathcal{P}$ and hence the union of k edge-disjoint spanning trees contains at least $k(q-1)$ cross-edges, from which the necessity of (10.25) follows.

To see sufficiency, observe first that, for an arbitrarily chosen root-node $r_0 \in V$, Theorem 9.1.8 implies the existence of a rooted k-edge-connected orientation D of G. Second, by applying Theorem 10.1.1, one obtains k edge-disjoint spanning arborescences of D which correspond to k edge-disjoint spanning trees of G. •

Note that the proofs of both Theorems 9.1.8 and 10.1.1 were constructive so we have an algorithm to compute the disjoint trees or a violating partition in Tutte's theorem. Based on a different approach, Baïou, Barahona, and Mahjoub [7] developed an efficient algorithm for finding a violating partition in the edge-capacitated case. A simpler algorithm was offered by Preissmann and Sebő [322]. Cunningham [61] described interesting applications of the most violating partitions.

Corollary 10.5.2 *Every $2k$-edge-connected graph $G=(V,E)$ has k edge-disjoint spanning trees.*

Proof. We claim that G is k-partition-connected. Indeed, for any partition $\mathcal{P}=\{V_1,\dots,V_q\}$ of V, the number of cross-edges to $\mathcal{P}$ is $\sum_i d_G(V_i)/2 \geq 2kq/2 = kq > k(q-1)$. Hence Theorem 10.5.1 applies. •

The second part of Theorem 9.1.8 implies the following extension or Tutte's theorem.

Theorem 10.5.3 *The maximum cardinality of a subset of edges of a graph $G=(V,E)$ that can be covered by k forests is equal to*

$$\min\{k(|V|-q)+e_G(\mathcal{P}) : \mathcal{P}=\{V_1,\dots,V_q\} \text{ a partition of } V\}. \tag{10.26}$$

Proof. Let $\mathcal{P}=\{V_1,\dots,V_q\}$ be a partition of V. Then the union F of k forests can have at most $k(|V_i|-1)$ edges induced by a class V_i and hence $|F| \leq e_G(\mathcal{P})+\sum_i k(|V_i|-1) \leq e_G(\mathcal{P})+k(|V|-q)$, from which max $\leq$ min follows. To see the reverse inequality, let μ denote the minimum in (10.26). Consider the maximum M in (9.3) and observe that $\mu + M = k(|V|-1)$. By Theorem 9.1.8, it is possible to add M edges to G to get a k-partition-connected graph G^+. Then G^+ includes k edge-disjoint spanning trees. By leaving out the M new edges from the union of these spanning trees, we obtain a subset F of edges of G that is the union of k forests and $|F| \geq k(|V|-1)-M=\mu$. •

Theorem 10.5.3 will be largely generalized to matroids: see the rank-formula of the sum of k matroids (Theorem 13.3.1). We say that a partition $\mathcal{P}^*$ is **k-tight** if it minimizes (10.26). A non-empty subset $X \subseteq V$ of nodes in a graph $G=(V,E)$ is **k-rich** if the subgraph $G'=(X, I_G(X))$ induced by X includes k edge-disjoint spanning trees. X is **strictly k-rich** if $G'-e$ includes k edge-disjoint spanning trees for every edge $e \in I_G(X)$. The following result describes interesting links between these concepts.

Theorem 10.5.4 (Jackson and Jordán [225]) *The maximal strictly k-rich subsets of a graph $G=(V,E)$ form a partition $\mathcal{P}_0$ of V, and the graph arising from G by shrinking* *each class into a node is k-sparse. The maximal k-rich subsets of a graph G form a partition $\mathcal{P}_1$ of V, and the graph arising by shrinking each class into a node is strictly k-sparse. Both $\mathcal{P}_0$ and $\mathcal{P}_1$ are k-tight partitions. $\mathcal{P}_0$ is the unique finest k-tight partition, while $\mathcal{P}_1$ is the unique coarsest one.* •

The theorem will be a consequence of the matroid partition algorithm, see Corollary 13.3.7.

Problems

10.5.1 *Prove Theorem* 10.5.4. *Develop an algorithm to compute the partitions $\mathcal{P}_0$ and $\mathcal{P}_1$ occurring in the theorem.*

10.5.2 *Let Z be a non-empty subset of a graph $G = (V, E)$. Prove that it is possible to delete a* (possibly empty) *subset $X \subseteq V - Z$ in such a way that the resulting graph includes k edge-disjoint spanning trees if and only if Z is a subset of a class of $\mathcal{P}_1$.*

The orientation approach helps us in finding a constructive characterization of k-tree-connected (= k-partition-connected) graphs.

Theorem 10.5.5 *A graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if it can be built up from a node by the following operations.*

(A') *Add a new edge connecting existing nodes.*
(B') *Add a new node z along with k* (possibly parallel) *edges connecting z with some existing nodes.*
(C') *Pinch j edges* $(0 < j < k)$ *together with a new node z and add $k - j$* (possibly parallel) *edges connecting z with some existing nodes.*

Proof. It is easy to check that each of the tree operations preserves k-tree-connectivity. For the converse, let r_0 be an arbitrary node of G. Since there are k edge-disjoint spanning trees, G has a rooted k-edge-connected orientation.

By Theorem 8.2.11, the resulting digraph arises from a node by applying operations (A), (B), and (C) as described in that theorem. Those operations just correspond to (A'), (B'), and (C'). •

Problem 10.5.3 *Prove that if the edge-set of a graph G can be partitioned into k edge-disjoint spanning trees, then there is a node z of degree at most* $2k - 1$. *Moreover, G can be partitioned into k spanning trees in such a way that the degree of z in each of the k trees is at most* 2.

Problem 10.5.4 $(*)$ *For safety reasons, engineers require a minimum list of spanning trees of a graph G such that if an arbitrary edge of G is destroyed, at least one of these trees remains intact. We can certainly help if G has two disjoint spanning trees because the approach above lends itself to finding these trees efficiently. What would you suggest if we are not that lucky and no two disjoint spanning trees exist?*

Conjecture 10.5.6 (Kriesell [249]) *Let $G = (V, E)$ be a graph with a specified terminal set $T \subseteq V$. Suppose that every cut separating T has at least* $2k$ *edges. Then there are k edges-disjoint trees so that each contains every terminal node.*

978–0–19–920527–1 10-Andras-Frank-c10-drv Andras Frank (Typeset by SPi, Publisher Services, India) 366 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

By Corollary 10.5.2, the conjecture holds for $T = V$. It also holds when $|T| = 2$ since then its statement follows from the undireced edge-version of Menger's theorem. At the end of this section we shall derive a weaker result on packing trees that can be considered a step toward solving Kriesell's conjecture.

Conjecture 10.5.7 (Kalai [228]) *Suppose that the edge-set A of a digraph $D = (V, A)$ can be partitioned into two spanning trees* (in the undirected sense). *If there is no 2-element cut so that the two edges are oppositely oriented* (that is, if every two-element cut is a dicut), *then A can be partitioned into 2 spanning trees F_1 and F_2 in such a way that reorienting each edge of F_1 results in a strongly connected digraph.*

Connection between coverings and packings

We show that the non-trivial parts of Tutte's theorem (Theorem 10.5.1) and of Nash-Williams' theorem (Theorem 10.4.3) can be derived from each other.

Proof of Tutte's theorem using Nash-Williams' theorem.

Suppose that G is k-partition-connected. We can assume that G is minimal in the sense that the removal of any edge destroys k-partition-connectivity. Let r_0 be an arbitrarily selected root-node. By Theorem 9.1.8, there is a rooted k-edge-connected orientation $\vec{G}$ of G. The minimality of G and Lemma 7.4.1 imply that $\varrho(r_0) = 0$ and $\varrho(v) = k$ for every node $v \in V - r_0$. Therefore the number $|E|$ of edges of $\vec{G}$ is $k(|V| - 1)$.

For a non-empty set $X \subseteq V - r_0$, we have $k \leq \varrho(X) = \ddot{\varrho}(X) - i_G(X) = k|X| - i_G(X)$ while for a subset X containing r_0, we have $0 \leq \varrho(X) = \ddot{\varrho}(X) - i_G(X) = 0 + k(|X| - 1) - i_G(X)$. In both cases $i_G(X) \leq k(|X| - 1)$ follows and therefore the edge-set of G can be partitioned into k forests by Theorem 10.4.3. Since $|E| = k(|V| - 1)$, each of the k forests must have exactly $|V| - 1$ edges, and hence these forests are spanning trees.•

Proof of Nash-Williams' theorem using Tutte's theorem.

We can assume that the graph is maximally k-sparse in the sense that adding any new edge to G would destroy the condition $i_G(X) \leq k(|X| - 1)$ for a set $X \subseteq V$. This means that every pair of distinct nodes belongs to a tight set. Since the union of two intersecting tight sets is tight again and each pair $\{s, v\}$ $(v \in V - s)$ belongs to a tight set, we conclude that the union of these tight sets is tight and equals V.

We claim that G is k-partition-connected. Indeed, for any partition $\mathcal{P} = \{V_1, \ldots, V_q\}$ of V, one has $e_G(\mathcal{P}) = |E| - \sum[i(V_j) : j = 1, \ldots, q] \geq k(|V| - 1) - \sum[k(|V_j| - 1) : j = 1, \ldots, q] = k(|V| - 1)$. By Tutte's theorem, G contains k edge-disjoint spanning trees. Since $|E| = k(|V| - 1)$, these trees partition the edge-set E.

An extension to tree packings

The orientation approach used to derive Tutte's theorem helps in obtaining an extension. We say that a graph (and its edge-set) is (≤ 1)**-orientable** (respectively, 1-orientable) if it has an orientation in which the in-degree of every node is at most 1 (exactly 1).

Theorem 10.5.8 (Whiteley [380]) *Let k and k_1 be non-negative integers. The edge-set of an undirected graph $G = (V, E)$ can be partitioned into k forests and k_1 (≤ 1)-orientable subgraphs if and only if*

$$i_G(X) \leq b(X) := k(|X|-1) + k_1|X| \textit{ for every } \emptyset \subset X \subseteq V. \tag{10.27}$$

Proof. The necessity follows by observing that any non-empty subset $X \subseteq V$ can induce at most $k(|X|-1)$ edges from the union of k forests and at most $k_1|X|$ edges from the union of k_1 (≤ 1)-orientable subgraphs.

Sufficiency. We can assume that the graph is saturated in the sense that no new edges can be added to G without destroying (10.27). This is equivalent to saying that every pair of distinct nodes is included in a tight set where a set X is tight if $i_G(X) = b(X)$. By noting that b is intersecting submodular, we can apply the standard submodular technique and conclude that the union of two intersecting tight sets is tight. Therefore the entire ground-set V is also tight, and hence $|E| = i_G(V) = b(V)$.

Let r_0 be an arbitrarily picked root-node of G. By the Orientation lemma, there is an orientation $\vec{G}$ of G in which $\varrho(r_0) = k_1$ and $\varrho(v) = k + k_1$ for every $v \in V - r_0$. Condition (10.27) implies that the in-degree of every non-empty subset of nodes not containing r_0 is at least k. By the disjoint arborescences theorem of Edmonds, $\vec{G}$ includes k disjoint spanning arborescences of root r_0 and they correspond to k disjoint spanning trees of G. The removal of the k arborescences from $\vec{G}$ leaves a digraph D in which every in-degree is exacly k_1. We can colour, without any conflict, the edges of D with k_1 colours so that every node admits exactly one entering edge from each colour. Accordingly, the colouration determines k_1 disjoint 1-orientable subsets of G. •

By using another orientation theorem, we can derive the following packing counterpart of Theorem 10.5.8.

Theorem 10.5.9 *Let k and k_1 be non-negative integers. An undirected graph $G = (V, E)$ includes k spanning trees and k_1 1-orientable spanning subgraphs which are pairwise disjoint if and only if*

$$e_G(\mathcal{P}) + i_G(V_0) \geq k(q-1) + (k+k_1)|V_0|$$

$$\textit{holds for every partition } \mathcal{P} := \{V_0, V_1, \dots, V_q\} \textit{ of } V \tag{10.28}$$

where only V_0 may be empty (allowing the degenerate cases when $V_0 = V, q = 0$ or when $\emptyset \subset V_0 \subset V, q = 1$).

Proof. (Outline) The necessity is left to the reader. For sufficiency, pick an arbitrary node r_0 as a root-node and define $f : V \to \mathbf{Z}_+$ as follows.

$$f(v) := \begin{cases} k_1 & \text{if } v = r_0 \\ k + k_1 & \text{if } v \in V - r_0. \end{cases}$$

It can be checked that condition (10.28) is equivalent to condition (9.6). By Theorem 9.1.10, there is a rooted k-edge-connected orientation $\vec{G}$ of G so that the in-degree of every node v is at least $f(v)$. By the disjoint arborescences theorem of Edmonds, $\vec{G}$ includes k disjoint spanning arborescences of root r_0 and they correspond to k disjoint spanning trees of G.

After removing the k arborescences from $\vec{G}$, we are left with a digraph D in which every in-degree is at least k_1. Delete edges so that every in-degree of the remaining digraph is exactly k_1, and colour the remaining edges with k_1 colours so that every node admits exactly one entering edge from each colour. This colouration determines k_1 disjoint 1-orientable subsets of G. •

Mixed graphs

Since we have a theorem on orienting even a mixed graph to be rooted k-edge-connected (Theorem 9.5.1), Tutte's theorem can be extended to mixed graphs. We call a mixed tree T a **mixed arborescence** of root r_0 if it is possible to orient its undirected edges so that the resulting directed tree is an arborescence. This is clearly equivalent to requiring that each directed edge of T be oriented away from r_0. The following result appeared in [113].

Theorem 10.5.10 *In a mixed graph $M = (V, A + E)$ with a root-node r_0 there are k edge-disjoint spanning mixed arborescences of root r_0 if and only if*

$$e_G(\mathcal{P}) \geq \sum_{i=1}^{p} [k - \varrho_A(V_i)] \text{ holds for every partition } \mathcal{P} := \{V_0, V_1, \ldots, V_p\} \text{ of } V \quad (10.29)$$

where $r_0 \in V_0$ and $e_G(\mathcal{P})$ denotes the number of edges from E connecting distinct parts of $\mathcal{P}$.

Proof. (Outline) The necessity is left to the reader. The sufficiency follows immediately by combining Theorems 9.5.1 and 10.1.1. •

A connection game

We exhibit a pretty application of Tutte's theorem. Two players—called short and cut players—are playing on a connected undirected graph $G = (V, E)$ having at least two nodes. In the connection game the players select alternately (not yet selected) edges of G. To be explicit, cut player moves first. The cut player wins once he has selected all edges of a non-trivial cut of G, the short player wins once he has selected all edges of a spanning tree of G. One of the two cases must occur by the time when every edge of G is selected since if the set F of edges selected by the short player does not induce a connected graph on V, then any component of it determines a cut disjoint from F, and hence the edges in this cut have been selected by the cut player.

Theorem 10.5.11 *The short player has a winning strategy if and only if there are two edge-disjoint spanning trees of G. The cut player has a winning strategy if and only if there is a partition $\{V_1, \ldots, V_t\}$ of V into non-empty sets so that the number of cross-edges is at most $2t - 3$.*

Proof. By applying Tutte's theorem to $k = 2$, we obtain that G contains two edge-disjoint spanning trees if and only if the number of cross-edges is at least $2t - 2$ for every partition $\{V_1, \ldots, V_t\}$. Therefore exactly one of the two configurations can occur.

Suppose first that there are two edge-disjoint spanning trees F_1, F_2. If G has exactly two nodes, then both trees consists of one edge. If the cut player selects one of them, the other one stays there and the short player wins. Suppose that $|V| \geq 3$. If the cut player selects in his first move an edge e from $F_1 \cup F_2$, say from F_1, then short player should select an edge f from F_2 which connects the two components of $F_1 - e$. In the case when e is not in $F_1 \cup F_2$ then f can be selected arbitrarily by the short player.

Let G' be the graph obtained from G by deleting e and contracting f. By the construction, it is easy to obtain from F_1 and F_2 two edge-disjoint spanning trees F_1' and F_2' of D'. Therefore short player wins by induction.

Suppose now that there is a partition for which the number of cross-edges is at most $2t-3$. The strategy for the cut player is that he always selects a cross-edge to this partition. After his very first move there are at most $2t-4$ cross-edges. During the entire game the short player can select at most half of these edges. Therefore, the set of all edges selected by the short player contains at most $t-2$ cross-edges to partition $\{V_1, \ldots, V_t\}$ implying that this set cannot form a spanning connected subgraph of G. •

Problem 10.5.5 *Develop winning strategies and a counterpart of Theorem* 10.5.11 *for the variation of the connection game when the short player moves first.*

How to play in a losing position

Suppose we are playing the connection game on a given graph G as a cut player. At the beginning, we realize that G includes two edge-disjoint spanning trees and hence we are in a losing position, provided our opponent does know the theorem above and follows the strategy suggested in the proof. But if the short player does not know the strategy, then intuitively it may be felt that one situation can be less hopeless for us than another.

For example, if there is a partition of V into t parts so that the number of cross-edges is $2t-2$, then the short player must always select a cross-edge to this partition for otherwise we can win. Therefore the presence of such a configuration is intuitively advantageous for us. More generally, the smaller the number of cross-edges in a t-partition (compared with $2t-2$) the better our chance is that our opponent fails to select sufficiently many cross-edges. Therefore a hopeful tactics for us could be choosing a partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ at the beginning of the game for which $e_G(\mathcal{P}) - (2t-2)$ is as small as possible and then selecting a cross-edge to this partition in each move.

Problem 10.5.6 *Based on results on* (k, ℓ)*-edge-connected orientations, figure out an algorithm for computing the best partition described above.*

10.5.2 Packing partition-connected hypergraphs

Given Tutte's elegant theorem on disjoint spanning trees of a graph, one may feel tempted to try extending it to hypergraphs. In order to do so, a first task is to figure out the right hypergraph counterpart of the notion of a spanning tree of a graph. A natural choice could be considering ordinary hypergraph connectivity. It was observed, however, that the problem of packing spanning connected hypergraphs is **NP**-complete for each $k \geq 2$ (see [144]). It turns out that the appropriate concept to generalize Tutte's theorem to hypergraphs is partition-connectivity. We referred to a hypergraph as k-partition-connected if the number $e(\mathcal{P})$ of hyperedges intersecting at least two parts is at least $k(q-1)$ for every q-partite partition $\mathcal{P}$ of V. When $k = 1$, we simply speak of a partition-connected hypergraph. Note that the two notions are equivalent for graphs. For general hypergraphs, however, though a partition-connected hypergraph is always connected but partition-connectivity is typically stronger than connectivity. For example, the hypergraph $(V, \{V\})$ with $|V| \geq 3$ is connected

978–0–19–920527–1 10-Andras-Frank-c10-drv Andras Frank (Typeset by SPi, Publisher Services, India) 370 of 640 December 11, 2010 15:18
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

but not partition-connected. Armed with the notion of partition-connectivity, we can follow the same line as we did for graphs and derive the following result of Frank, Király, and Kriesell [144].

Theorem 10.5.12 *A hypergraph H can be decomposed into k spanning partition-connected subhypergraphs if and only if H is k-partition-connected.*

Proof. The necessity is immediately clear from the definitions. For sufficiency, apply first Theorem 9.4.4 which states that H has a rooted k-edge-connected orientation $\vec{H}$. Second, choose any node r_0 of V as a root and apply Theorem 10.1.2 to $\vec{H}$. The theorem implies that $\vec{H}$ decomposes into k root-connected spanning dypergraphs and these dypergraphs, in the undirected sense, form the requested k disjoint spanning partition-connected subhypergraphs of H. •

By the **rank** of a hypergraph we mean the cardinality of its largest hyperedge. Corollary 10.5.2 can be generalized to hypergraphs as follows.

Theorem 10.5.13 *A (qk)-edge-connected hypergraph H of rank at most q can be decomposed into k partition-connected subhypergraphs and hence into k connected spanning subhypergraphs.*

Proof. By Theorem 10.5.12, it suffices to show that H is k-partition-connected. Let $\mathcal{P}$ be a partition of V. By the (qk)-edge-connectivity of H, there are at least qk hyperedges of H intersecting both V_i and $V - V_i$ for each class V_i of $\mathcal{P}$. Since one hyperedge can intersect at most q classes, we obtain that the total number of hyperedges intersecting more than one class is at least $(qk)|\mathcal{P}|/q > k(|\mathcal{P}| - 1)$, and hence H is indeed k-partition-connected. Therefore, by Theorem 10.5.12, H decomposes into k partition-connected subhypergraphs. •

Application to packing Steiner trees

Let $G = (V_0, E)$ be an undirected graph with a terminal set $V \subseteq V_0$. By a **Steiner tree** of G (spanning V) we mean a subtree (V', E') for which $E' \subseteq E$ and $V \subseteq V' \subseteq V_0$. The **disjoint Steiner trees** problem consists of finding k edge-disjoint Steiner trees of G. We say that G is **k-edge-connected in** V if every cut of G separating two elements of V has at least k edges. By Menger's theorem this is equivalent to requiring for every two elements u and v of V that there are k edge-disjoint paths in G connecting u and v. Conjecture 10.5.6 stated that there are k edge-disjoint Steiner trees if G is 2k-edge-connected within V.

Theorem 10.5.14 *Let $G = (V_0, E)$ be an undirected graph and $V \subset V_0$ a subset of nodes so that $U := V_0 - V$ is stable and G is $(3k)$-edge-connected in V. Then G includes k edge-disjoint Steiner trees spanning V.*

Proof. We use induction on the value $\mu_G := \sum[\max\{0, d_G(v) - 3\} : v \in U]$. Suppose first that μ_G is 0, that is, the degree of each node in U is at most 3. We can assume that V is also stable for otherwise each edge induced by V can be subdivided by a new node. Such an operation may add new nodes of degree 2 to the complement of V but it does not affect $(3k)$-edge-connectivity in V, and k disjoint Steiner trees in the new graph determine k disjoint Steiner trees in G.

978–0–19–920527–1 10-Andras-Frank-c10-drv Andras Frank (Typeset by SPi, Publisher Services, India) 371 of 640 December 11, 2010 15:18

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Let $H = (V, \mathcal{E})$ be the hypergraph corresponding to G, that is, for each element u of U, there is a corresponding hyperedge of H consisting of the neighbours of u in G. Since the degree of each element of U is at most 3, the rank of H is at most 3.

For any non-empty, proper subset X of V, let X' denote the set of those elements of U which have at most one neighbour in $V - X$ in the graph G. Since every degree in U is at most 3, we have $d_G(X \cup X') = d_H(X)$ and hence the $(3k)$-edge-connectivity of G implies the $(3k)$-edge-connectivity of H.

By Corollary 10.5.13, U can be partitioned into k disjoint subsets $U_1, \dots, U_k$ so that $V \cup U_i$ induces a connected subgraph $G_i = (V \cup U_i, E_i)$ of G for each $i = 1, \dots, k$. By choosing a spanning tree F_i from each G_i, we obtain the required edge-disjoint Steiner trees of G.

Suppose now that μ_G is positive and that the theorem holds for each graph G' with $\mu_{G'} < \mu_G$. Let $z \in U$ be a node with $d_G(z) \geq 4$. If there is a cut-edge e of G, then the elements of V belong to the same component of $G - e$ as G is at least k-edge-connected in V and then we can discard the other component of $G - e$ without destroying $(3k)$-edge-connectivity in V. Therefore we can assume that G is 2-edge-connected.

By Mader's undirected splitting theorem (Theorem 8.1.7), there are two edges $e = uz$, $f = vz$ in E so that replacing e and f by a new edge uv the local edge-connectivities do not drop. In particular, the resulting graph G' continues to be $(3k)$-edge-connected in V. By induction there are k edge-disjoint Steiner trees in G'. If one of these trees contains the split-off edge uv, we replace it by e and/or f in order to obtain a Steiner tree of G. Therefore we have proved the existence of k edge-disjoint Steiner trees of G. •

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

11
Preserving and improving connections

Relations between subgraph problems and augmentation problems

Deleting an edge or splitting off a pair of edges certainly does not increase the edge-connection of a graph or digraph. In connectivity preservation problems, we are interested in reductions that do not *decrease* edge-connection, or more generally, that preserve some specified property. For example, Lovász' splitting lemma (Theorem 8.1.1) stated that there is a splitting at a node of even degree that preserves k-edge-connectivity between all the other nodes ($k \geq 2$). If the operation is edge-deletion, then we arrive at the subgraph problem. The cheapest (spanning) tree problem can also be interpreted in this way, since it is tantamount to deleting a subset of edges with maximum total weight so that the remaining graph is still connected. Recall that the dual greedy algorithm, for example, always provides an optimal solution. The cheapest path problem can also be formulated as a preservation problem: delete a subset of edges with maximum total weight so that in the remaining digraph t is still reachable from s. More generally, when the connectivity property to be preserved is k-edge-connectivity from s to t, we are at the problem of minimum cost flows with unit capacities. In a certain sense even orientation problems can be considered as subgraph problems: if we replace each edge of the initial undirected graph by two opposite directed edges, then the orientation problem is equivalent to deleting exactly one member of these pairs so as to maintain a connectivity property (k-edge-connectivity, for example, in Nash-Williams' orientation theorem).

In connectivity improvements the situation is the reverse. Here the initial graph or digraph does not have the desired property and with specified operations we want to transform the graph to have it. In an augmentation problem, the operation is typically adding a new edge (or hyperedge or node), but other operations can also be interesting, such as shrinking two nodes or reversing a directed edge. For example, if the target connection is plain connectivity, then in the augmentation problem we want to make a disconnected graph $G = (V, E)$ connected by adding a set of new edges with minimum total cost. Let $H = (V, F)$ denote the graph of edges usable for the augmentation and let $c : F \to \mathbf{R}$ be a cost function. In order to have a solution at all we assume that the graph $G + H = (V, E + F)$ is connected. The solution in this case is straightforward: extend first the cost function c to the edges of $G + H$ so that $c(e) := 0$ for every edge e in E and then construct (say, by Kruskal) a cheapest spanning tree T of $G + H$. The edges of T belonging to H form the requested optimal augmentation.

This kind of reduction of the augmentation problem always works provided the minimum-cost version of the corresponding subgraph problem is tractable. For example, with the help of the min-cost flow algorithm of Ford and Fulkerson, one can solve the augmentation problem in which $\lambda_D(s,t) < k$ for the initial digraph D (that is, D does not have k edge-disjoint st-paths), and our task is to add a cheapest set of new edges so that $\lambda_{D^+}(s,t) \geq k$ holds for the augmented digraph D^+. Analogously, the algorithm of Chu and Liu we encount in Section 3.2.2 for computing a cheapest spanning arborescence can be used for finding optimal augmentation when rooted-connectivity is the target connectivity property.

Conversely, suppose that the minimum-cost augmentation problem is tractable for a specific property. Then the corresponding optimal subgraph problem is also tractable. For example, if an algorithm is available for making a digraph $D = (V, A)$ rooted edge-connected by adding a cheapest subset of edges of another digraph $H = (V, F)$, then one is able to compute a cheapest spanning arborescence of root r_0, since this problem can be reformulated as the optimal augmentation of the edgeless digraph $D = (V, \emptyset)$ with some edges of H.

This equivalence, in general, holds in the sense that the subgraph problem is solvable for every non-negative cost function exactly for which the augmentation problem is solvable for every non-negative cost function. There are, however, properties when a subgraph problem is **NP**-complete while the corresponding augmentation problem is tractable, at least for free augmentations. We speak of a **free** augmentation when every possible edge connecting the nodes of the initial graph or digraph is allowed to be added in any number of parallel copies and the cost of the new edges is identically 1. For example, finding a smallest cardinality spanning 2-edge-connected subgraph of a graph is **NP**-complete, while the corresponding free augmentation problem (consisting of adding a minimum number of new edges to get a 2-edge-connected graph) was solved by Eswaran and Tarjan; see Section 2.1.

11.1 Augmenting edge-connectivity of undirected graphs

We investigate the general problem when the task is to make an initial undirected graph $G = (V, E)$ k-edge-connected ($k \geq 2$) by adding a minimum number of new edges. The main trick of the approach is that we first solve the corresponding degree-specified augmentation problem and this solution is then used for minimal augmentations.

11.1.1 Degree-specified augmentations

Suppose that $k \geq 2$ and the graph $G = (V, E)$ is not k-edge-connected. Given a degree specification $m : V \to \mathbf{Z}_+$, we are interested in finding a graph $H = (V, F)$ for which $G + H$ is k-edge-connected and $d_H(v) = m(v)$ for every node v. We say that m is an (undirected edge-connectivity) k**-augmentation vector** if there is such an H. Recall the definition of (proper) solid sets from Section 7.3.

Theorem 11.1.1 *Given a graph $G = (V, E)$, an integer $k \geq 2$, and a degree specification $m : V \to \mathbf{Z}_+$, there exists a graph $H = (V, F)$ for which $d_H(v) = m(v)$ for every $v \in V$ and $G + H$ is k-edge-connected if and only if $\widetilde{m}(V)$ is even and*

$$\widetilde{m}(X) \geq k - d_G(X) \text{ for every set } \emptyset \subset X \subset V. \tag{11.1}$$

Furthermore, it suffices to require (11.1) *for solid subsets X.*

Proof. If the requested H exists, then $k \leq d_{G+H}(X) = d_G(X) + d_H(X) \leq d_G(X) + \sum[d_H(v) : v \in X] = d_G(X) + \widetilde{m}(X)$ from which (11.1) follows. Since the sum of the degrees of nodes is twice the number of edges, $\widetilde{m}(V)$ must be even.

To see sufficiency, add a new node z to G and $m(v)$ parallel zv-edges for every node v of G. By (11.1), we have $d_{G'}(X) = d_G(X) + \widetilde{m}(X) \geq k$ for the extended graph G', while $d_{G'}(z) = \widetilde{m}(V)$ is even. Hence we can apply Theorem 8.1.2 to G' and obtain that there is a complete splitting at z resulting in a k-edge-connected graph on node-set V. This means that $H = (V, F)$ satisfies the requirements of the theorem where F denotes the set of $d_{G'}(z)/2 = \widetilde{m}(V)/2$ edges arising from the splittings.

The last part of the theorem follows at once from Proposition 7.3.2. •

Problem 11.1.1 *Prove the converse: Theorem* 11.1.1 *implies Theorem* 8.1.2.

The following lemma will be useful for proving results on degree-constrained and minimum cardinality augmentations. We define a set X ($\emptyset \subset X \subset V$) to be **tight** with respect to a function $m : V \to \mathbf{Z}_+$ satisfying (11.1) if $\widetilde{m}(X) = k - d_G(X)$.

Lemma 11.1.2 *Let $m : V \to \mathbf{Z}_+$ be a function satisfying* (11.1) *and let T be a subset of nodes v for which $m(v) > 0$. Suppose that m is minimal in T in the sense that reducing $m(v)$ for any $v \in T$ destroys* (11.1). *Then there is a subpartition $\{X_1, \ldots, X_t\}$ of V consisting of tight proper solid sets which cover T.*

Proof. The minimality of m implies that each node $v \in T$ belongs to a tight set. Let $T(v)$ be a minimal tight set containing v. We claim that $T(v)$ is solid. For if not, then there is a proper subset $Z \subset T(v)$ with $d_G(Z) \leq d_G(T(v))$. Then $\widetilde{m}(Z) \geq k - d_G(Z) \geq k - d_G(T(v)) = \widetilde{m}(T(v)) = \widetilde{m}(Z) + \widetilde{m}(T(v) - Z)$, from which $\widetilde{m}(Z) = k - d_G(Z)$ and $\widetilde{m}(T(v) - Z) = 0$. Hence both Z and $T(v) - Z$ are tight, contradicting the minimal choice of $T(v)$.

Therefore each $v \in T$ belongs to a tight solid set. Let $X_1, \ldots, X_t$ denote the maximal tight proper solid sets. These sets are disjoint and cover T since by Proposition 7.3.1 solid sets form a laminar set-system. •

11.1.2 Minimal augmentations

The following fundamental theorem is due to Watanabe and Nakamura [375]. The proof below appeared in [125].

Theorem 11.1.3 (Watanabe and Nakamura) *An undirected graph $G = (V, E)$ can be made k-edge-connected ($k \geq 2$) by adding at most γ new edges if and only if*

$$2\gamma \geq \sum_i [k - d_G(X_i)] \textit{ for every subpartition } \{X_1, \ldots, X_t\} \textit{ of } V. \tag{11.2}$$

In other words, the minimum number of new edges for which addition to G results in a k-edge-connected graph is equal to $\max\lceil \sum_i (k - d_G(X_i))/2 \rceil$ where the maximum is taken over all subpartitions of V. In addition, it suffices to consider subpartitions consisting of proper solid sets.

Proof. We consider the first form of the theorem. In a k-edge-connected augmentation of G the degree of every non-empty proper subset X of V is at least k so if $d_G(X) < k$, then there must be at least $k - d_G(X)$ new edges leaving X. Since one edge can contribute to at most two of the cuts detemined by the disjoint sets X_i, it follows that the double of the total number of new edges is at least $\sum_i [k - d_G(X_i)]$, and hence (11.2) is indeed necessary.

In order to prove sufficiency, choose a function $m : V \to \mathbf{Z}_+$ for which (11.1) and $\widetilde{m}(V) \geq 2\gamma$ hold, and $\widetilde{m}(V)$ is as small as possible.

Claim 11.1.4 $\widetilde{m}(V) = 2\gamma$.

Proof. Suppose indirectly that $\widetilde{m}(V) > 2\gamma$. By the minimality of M, every element of $T := \{v : m(v) > 0\}$ belongs to a tight set. By Lemma 11.1.2, there is a subpartition $\{X_1, \ldots, X_t\}$ of V consisting of tight solid sets which cover every node v with $m(v) > 0$. By (11.2) we have $\widetilde{m}(V) = \sum_i \widetilde{m}(X_i) = \sum_i [k - d_G(X_i)] \leq 2\gamma$, contradicting the indirect assumption $\widetilde{m}(V) > 2\gamma$. •

Apply Theorem 11.1.1 to G and m. By the claim, the resulting graph H has γ edges. To see the last part of the theorem, consider a partition violating (11.2) for which $\sum_i |X_i|$ is minimum. Then each member X_i is solid. • •

Note that the statement of the theorem of Watanabe and Nakamura fails to hold for $k = 1$, as is shown by the empty graph on four nodes when one chooses γ to be 2. It is also worth mentioning that the proof above gives rise to a polynomial algorithm, since a complete splitting can be computed in polytime as pointed out in Section 8. For more efficient algorithms using MA orderings, see the book of Nagamochi and Ibaraki [300].

As a generalization of k-edge-connectivity, we introduced the concept of (k, ℓ)-partition-connectivity in Section 1.2 for $0 \leq \ell \leq k$ and derived characterizations for (k, ℓ)-partition-connected orientability in Section 9.2. The question naturally emerges: when is it possible to make a graph (k, ℓ)-partition-connected by adding γ new edges? A good characterization was presented in [142] but it is significantly more complicated than Theorem 11.1.3. Recall that (k, k)-partition-connectivity is equivalent to k-edge-connectivity while $(k, 0)$-partition-connectivity is equivalent, by Tutte's theorem (Theorem 10.5.1), to the existence of k-edge-disjoint spanning trees. In Part III, matroid theory will help to show that in the latter case even the minimum-cost augmentation problem is solvable.

Minimum cost k-edge-connected augmentation for node-induced cost functions

Suppose that every possible new edge has a specified cost and that we are interested in a cheapest k-edge-connected augmentation of a given graph G. This problem is **NP**-complete even in the special case when $k = 2$ and G has no edges at all since the Hamilton circuit problem can be formulated in this way. In the paper [125], however, it was shown that there is a special class of cost functions for which the min-cost augmentation problem is tractable. We say that a cost function c^* on the edge-set E^* of the complete graph on V is **node-induced** if there is a cost function $c : V \to \mathbf{R}$ on the node-set for which

$$c^*(uv) = c(u) + c(v) \text{ for every } uv \in E^*.$$

Here c may have negative values as well. Suppose we are interested in finding a cheapest set F of γ edges so that $G + F$ is k-edge-connected. By choosing $c \equiv 1/2$, we have $c^* \equiv 1$ and we are back at the problem solved by Watanabe and Nakamura. Let $\mathcal{S}_G$ denote the set-system of solid sets and let $\mathcal{S}' := \mathcal{S}_G - \{V\}$. Let $p : \mathcal{S}' \to \mathbf{Z}_+$ be defined by

$$p(X) := (k - d_G(X))^+ \text{ for } X \in \mathcal{S}'.$$

In order for a graph G to admit a k-edge-connected augmentation with γ new edges, it was necessary and sufficient by Theorem 11.1.3 that

$$\sum_i p(X_i) \leq 2\gamma \text{ for every subpartition } \{X_1, \dots, X_t\} \text{ of } V \text{ consisting of proper solid sets} \tag{11.3}$$

so we assume that (11.3) holds.

By Theorem 11.1.1, a vector m with $\widetilde{m}(V) = 2\gamma$ is a k-augmentation vector if $\widetilde{m}(X) \geq p(X)$ for every proper solid set X. Moreover, if F is a subset of edges for which $G + F$ is k-edge-connected and $d_F(v) = m(v)$ for every $v \in V$, then $|F| = \gamma$ and $\widetilde{c}^*(F) = cm$. Based on these observations, all we need to do is to find a function $m : V \to \mathbf{Z}_+$ which minimizes cm subject to

$$\widetilde{m}(X) \geq p(X) \text{ for every } X \in \mathcal{S}' \tag{11.4}$$

and

$$\widetilde{m}(V) = 2\gamma.$$

This can be accomplished with the help of a reverse greedy algorithm. Suppose that the elements of V are ordered so that $c(v_1) \geq c(v_2) \geq \cdots \geq c(v_n)$. At the beginning, let $m(v_i) := k + 1$ for $i = 1, \dots, n - 1$ and let $m(v_n) := \max\{k + 1, 2\gamma\}$. Clearly, this m satisfies (11.4) and $\widetilde{m}(V) \geq 2\gamma$. Starting with v_1, consider the elements of V in this order. At step i $(i = 1, \dots, n)$, reduce $m(v_i)$ as much as possible without destroying (11.4) and $\widetilde{m}(V) \geq 2\gamma$.

Theorem 11.1.5 *Let m_{alg} denote the final m obtained by the algorithm above. Then*

$$\widetilde{m}_{alg}(X) \geq p(X) \textit{ for every } X \in \mathcal{S}', \tag{11.5}$$

$$\widetilde{m}_{alg}(V) = 2\gamma,$$

$$cm_{alg} = \min\{cm : m \textit{ is integral, } m \textit{ satisfies } (11.4)\textit{, and } \widetilde{m}(V) = 2\gamma\}. \tag{11.6}$$

Proof. Due to the construction of m_{alg}, (11.5) holds. A solid set $Z \subset V$ is m_{alg}-tight if $\widetilde{m}_{alg}(Z) = p(Z)$. Since every singleton is solid, the rule of the algorithm implies that every node v_i belongs to an m_{alg}-tight set $Z_i \subseteq \{v_1, \dots, v_i\}$. By Proposition 7.3.1, $\mathcal{S}'$ is laminar, and hence the maximal m_{alg}-tight members of $\mathcal{S}'$ form a partition $\{X_1, \dots, X_q\}$ of V. Due to (11.3), we have

$$2\gamma \leq \widetilde{m}_{alg}(V) = \sum[\widetilde{m}_{alg}(X_i) : i = 1, \dots, q] = \sum[p(X_i) : i = 1, \dots, q] \leq 2\gamma$$

from which $\widetilde{m}_{alg}(V) = 2\gamma$.

Let m_{opt} denote an optimal solution to the minimization problem in (11.6). We say that a subset $X \subset V$ is m_{opt} − **tight** if $\widetilde{m}_{opt}(X) = k - d_G(X)$. We claim that $cm_{alg} = cm_{opt}$. Clearly, $cm_{alg} \geq cm_{opt}$. Suppose, indirectly, that $cm_{alg} > cm_{opt}$. Let i be the smallest index for which $m_{alg}(v_i) \neq m_{opt}(v_i)$ and assume that m_{opt} is chosen in such a way that i is as large as possible and, subject to this, $|m_{alg}(v_i) - m_{opt}(v_i)|$ is minimal.

Since $\widetilde{m}_{opt}(V) = 2\gamma = \widetilde{m}_{alg}(V)$, we must have $i \leq n-1$. It follows from the rule of the algorithm that $m_{alg}(v_i) < m_{opt}(v_i)$. If v_i belongs to an m_{opt}-tight solid set, then the smallest such set $T(v_i)$ cannot be a subset of $\{v_1, \ldots, v_i\}$ since then $\widetilde{m}_{alg}(T(v_i)) < \widetilde{m}_{opt}(T(v_i)) = p(T(v_i))$ would contradict (11.4). Therefore, there is a $j > i$ so that $v_j \in T(v(i))$. If v_i does not belong to any m_{opt}-tight solid set, let $j = i+1$. Decrease $m_{opt}(v_i)$ by 1 and increase $m_{opt}(v_j)$ by 1. The resulting m' also satisfies (11.4). Since $c(i) \geq c(j)$, it follows that $cm' \leq cm_{opt}$ and hence $cm' = cm_{opt}$. Therefore m' is another optimal solution, contradicting the special choice of m_{opt}. •

Remark In Subsection 5.5.5, we described the polymatroid greedy algorithm. In Section 14.5 of Part III, we will see that the greedy algorithm can be extended to base-polyhedra as well, even in the case when the base-polyhedron is defined by an intersecting supermodular function. The greedy algorithm above may be considered as a specialization of that greedy algorithm.

11.1.3 Degree- and size-constrained augmentations

The simple structure of solid sets (namely, their laminarity) enables us to combine augmentation problems: there can be restrictions on the total number of new edges as well as on the number of edges incident to each node. In the theorem of Watanabe and Nakamura it does not make any difference if the number of new edges is expected to be exactly γ or at most γ since in the latter case new edges can be added freely so as to have exactly γ new edges. When there are upper bounds on the degrees, this is not the case anymore, and one must distinguish between the two variations.

Degree-constrained augmentations with exactly γ edges

Consider first the case when γ is the exact number of the newly added edges.

Theorem 11.1.6 *Let $G = (V, E)$ be an undirected graph, $k \geq 2$ and $\gamma \geq 0$ integers, and $f : V \to \mathbf{Z} \cup \{-\infty\}$, $g : V \to \mathbf{Z} \cup \{\infty\}$ two integer-valued functions for which $f \leq g$. Let p denote a set-function defined by*

$$p(X) := \begin{cases} (k - d_G(X))^+ & if\ \emptyset \subset X \subset V \\ 0 & otherwise. \end{cases} \tag{11.7}$$

There is a k-augmentation vector m for which

$$f \leq m \leq g \text{ and } \widetilde{m}(V) = 2\gamma \tag{11.8}$$

if and only if

$$\widetilde{f}(V) \leq 2\gamma \leq \widetilde{g}(V), \tag{11.9}$$

$$\widetilde{g}(X) \geq p(X) \text{ for every } X \subseteq V, \tag{11.10}$$

and

$$\widetilde{f}(X_0) + \sum_{i=1}^{t} p(X_i) \leq 2\gamma \tag{11.11}$$

for every partition $\{X_0, X_1, \ldots, X_t\}$ $(t \geq 1)$ *of* V *in which only* X_0 *may be empty. In addition, it suffices to require* (11.10) *only when* $X \in \mathcal{S}_G$ *and* (11.11) *when* $X_i \in \mathcal{S}_G$ *for* $i = 1, \ldots, t$.

Proof. Necessity. Suppose that there is a graph $H = (V, F)$ for which the function m defined by $m(v) = d_H(v)$ $(v \in V)$ satisfies (11.8). Then $\widetilde{m}(V) = \sum[d_H(v) : v \in V] = 2|F| = 2\gamma$ and $f \leq m \leq g$ imply $\widetilde{f}(V) \leq 2\gamma \leq \widetilde{g}(V)$, and hence (11.9) holds. Also,

$$k \leq d_{G+H}(X) = d_G(X) + d_H(X) \leq d_G(X) + \sum[d_H(v) : v \in X]$$

$$= d_G(X) + \widetilde{m}(X) \leq d_G(X) + \widetilde{g}(X),$$

from which $\widetilde{g}(X) \geq k - d_G(X)$ and hence (11.10) follows. Finally, $d_H(X) \geq p(X)$ implies that $2\gamma = 2|F| \geq \sum[d_H(X_i) : i = 1, \ldots, t] + \sum[d_H(v) : v \in X_0] \geq \sum[p(X_i) : i = 1, \ldots, t] + \sum[f(v) : v \in X_0] = \sum[p(X_i) : i = 1, \ldots, t] + \widetilde{f}(X_0)$ and (11.10) follows.

In order to prove sufficiency, we can assume that g is finite-valued since any infinite value of g can be replaced by a big number (and, actually, k will suffice) without destroying the conditions. Due to (11.9) and (11.10), we can choose a function $m : V \to \mathbf{Z}_+$ for which (11.1) holds. Furthermore $f \leq m \leq g$, $\widetilde{m}(V) \geq 2\gamma$, and $\widetilde{m}(V)$ is as small as possible.

Claim 11.1.7 $\widetilde{m}(V) = 2\gamma$.

Proof. Suppose indirectly that $\widetilde{m}(V) > 2\gamma$. Then every element of $T := \{v : m(v) > f(v)\}$ belongs to a tight set. By Lemma 11.1.2, there is a subpartition $\{X_1, \ldots, X_t\}$ of V consisting of tight solid sets which cover every node v with $m(v) > f(v)$. Let $X_0 := V - (X_1 \cup \cdots \cup X_t)$. By (11.11), we have $2\gamma \leq \widetilde{m}(V) = \sum[\widetilde{m}(X_i) : i = 0, 1, \ldots t] = \sum[k - d_G(X_i) : i = 1, \ldots, t] + \widetilde{m}(X_0) = \sum[p(X_i) : i = 1, \ldots, t] + \widetilde{f}(X_0) \leq 2\gamma$, from which $\widetilde{m}(V) = 2\gamma$, contradicting the indirect assumption $\widetilde{m}(V) > 2\gamma$. •

By Theorem 11.1.1, m is an augmentation vector. • •

Remark In Theorem 11.1.6, we allowed the lower bound function f to take $-\infty$ on some nodes. The meaning of $f(v) = -\infty$ is simply that there is no lower bound for $m(v)$. In this case, it suffices to require (11.11) only for subpartitions in which X_0 contains no such v. Analogously, it suffices to require (11.10) for subsets X containing no node v with $g(v) = \infty$.

Linking property of augmentation vectors

Corollary 11.1.8 *Let* $G = (V, E)$ *be an undirected graph,* $k \geq 2$ *and* $\gamma \geq 0$ *integers, and* $f : V \to \mathbf{Z} \cup \{-\infty\}$, $g : V \to \mathbf{Z} \cup \{\infty\}$ *two integer-valued functions for which* $f \leq g$. *If there are k-augmentation vectors* m' *and* m'' *for which* $\widetilde{m}''(V) = \widetilde{m}'(V) = 2\gamma$, $f \leq m'$ *and* $m'' \leq g$, *then there is an augmentation vector for which* $\widetilde{m}(V) = 2\gamma$ *and* $f \leq m \leq g$.

Proof. By the existence of m' and m'', the necessary conditions in Theorem 11.1.6 hold and hence the theorem implies the existence of m. •

Corollary 11.1.9 *If it is possible to make a graph G k-edge-connected by adding at most* γ' *edges and it is possible to make G k-edge-connected by adding a graph* $H' = (V, F')$

satisfying $d_{H'}(v) \leq g(v)$ for every node $v \in V$, then there is a graph $H = (V, F)$ so that $G + H$ is k-edge-connected, $|F| \leq \gamma'$, and $d_H(v) \leq g(v)$ for every node $v \in V$.

Proof. Let $\gamma := \min\{\gamma', \lfloor \widetilde{g}(V)/2 \rfloor\}$ and $f \equiv 0$. By the hypotheses, all the conditions in Theorem 11.1.6 are satisfied and hence there is the desired H (with exactly γ edges). •

Degree-constrained augmentations with at most γ edges

The situation when γ is only an upper bound for the number of new edges (and not the expected exact value) is more complicated because parity comes into the picture. For example, if we want to make a graph k-edge-connected so that there is an upper bound $g : V \to \mathbf{Z}_+$ on the degrees of the graph of new edges, then the straightforward necessary condition (11.10) is not sufficient: let G be a graph with node-set $\{u, v_1, v_2, v_3\}$ and edge-set $\{uv_1, uv_2, uv_3\}$, let $k = 2$, and let $g(u) = 0$ and $g(v_1) = g(v_2) = g(v_3) = 1$.

Theorem 11.1.10 *Given a graph $G = (V, E)$, an integer $k \geq 2$, and a function $g : V \to \mathbf{Z}_+ \cup \{\infty\}$, there exists a graph $H = (V, F)$ for which $G + H$ is k-edge-connected and $d_H(v) \leq g(v)$ for every $v \in V$ if and only if*

$$\widetilde{g}(X) \geq k - d_G(X) \text{ holds for every set } \emptyset \subset X \subset V, \tag{11.12}$$

and there is no partition $\{X_0, X_1, \ldots, X_t\}$ of V $(t \geq 1)$ (in which only X_0 may be empty) *so that $\widetilde{g}(X_0) = 0$, $\widetilde{g}(X_i) = k - d_G(X_i)$ for each $i = 1, \ldots, t$, and $\widetilde{g}(V)$ is odd.*

Proof. Necessity. Suppose that there is a requested graph $H = (V, F)$. Then

$$k \leq d_{G+H}(X) = d_G(X) + d_H(X) \leq d_G(X) + \sum[d_H(v) : v \in X]$$
$$= d_G(X) + \widetilde{m}(X) \leq d_G(X) + \widetilde{g}(X),$$

implying that $\widetilde{g}(X) \geq k - d_G(X)$ from which (11.12) follows. If there is a partition with the given properties, then $\widetilde{g}(X_0) = 0$ implies that $d_H(v) = g(v)$ $(= 0)$ for every $v \in X_0$. Furthermore, for $i = 1, \ldots, t$ we have

$$\widetilde{g}(X_i) = k - d_G(X_i) \leq d_H(X) \leq \sum[d_H(v) : v \in X_i] \leq \sum[g(v) : v \in X_i] = \widetilde{g}(X_i)$$

and hence $d_H(v) = g(v)$ for every $v \in X_i$. But then $\widetilde{g}(V) = \sum[d_H(v) : v \in V]$ is even.

For sufficiency, we can assume that $\widetilde{g}(V)$ is finite since an infinite value $g(v)$ can be replaced by a large number without destroying the necessary conditions. Suppose first that $\widetilde{g}(V)$ is even. By applying Theorem 11.1.1 to $m \equiv g$, we are done.

Consider now the case when $\widetilde{g}(V)$ is odd. Assume that every element v with $g(v) > 0$ belongs to a g-tight set. By applying Lemma 11.1.2 to $m \equiv g$, we obtain that there is a subpartition $X_1, \ldots, X_t$ of tight solid sets covering all nodes v with $g(v) > 0$. Let $X_0 := V - \cup(X_i : i = 1, \ldots, t)$. Then $\{X_0, X_1, \ldots, X_t\}$ is a partition of V for which $\widetilde{g}(X_0) = 0$ and $\widetilde{g}(X_i) = k - d_G(X_i)$ for $i = 1, \ldots, t$, and hence the partition violates the condition of the theorem. Therefore there must be an element v with $g(v) > 0$ which does not belong to any tight set. By reducing the value of $g(v)$ by 1 the necessary conditions remain intact and we are back at the even case. •

Problem 11.1.2 *By refining the proof technique above, prove the following result.*

978–0–19–920527–1 11-Andras-Frank-c11-drv Andras Frank (Typeset by SPi, Publisher Services, India) 380 of 640 December 11, 2010 15:19
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Theorem 11.1.11 *Let G, k, γ, f, g, p be the same as in Theorem* 11.1.6. *There is a k-augmentation vector m for which $f \leq m \leq g$ and $\widetilde{m}(V) \leq 2\gamma$ if and only if*

$$\widetilde{f}(V) \leq 2\gamma, \tag{11.13}$$

$$\widetilde{g}(X) \geq p(X) \text{ for every } X \subseteq V, \tag{11.14}$$

and

$$\widetilde{f}(X_0) + \sum_{i=1}^{t} p(X_i) \leq 2\gamma k \tag{11.15}$$

for every partition $\{X_0, X_1, \dots, X_t\}$ $(t \geq 1)$ of V in which only X_0 may be empty and, in the case $\widetilde{g}(V)$ is finite and odd there is no partition $\{X_0, X_1, \dots, X_t\}$ $(t \geq 1)$ of V (in which only X_0 may be empty) *so that $\widetilde{f}(X_0) = \widetilde{g}(X_0)$ and $\widetilde{g}(X_i) = p(X_i)$ for $i = 1, \dots, t$.* •

11.1.4 Augmenting local edge-connectivities

In the same way that Lovász' splitting theorem could be used to increase global edge-connectivity of graphs, we can use Mader's more general undirected splitting theorem (Theorem 8.1.8) to increase local edge-connectivities. Let $G = (V, E)$ be an undirected graph. Let $r(x, y)$ $(x, y \in V)$ be a non-negative integer-valued **demand** function which is symmetric in the sense that $r(x, y) = r(y, x)$ for every $x, y \in V$. We are also given a function $m : V \to Z_+$ for which $\widetilde{m}(V)$ is even. We assume that

$$G \text{ has no component } C \text{ for which } \widetilde{m}(C) = 1. \tag{11.16}$$

As shown in [125], it is possible to work out a characterization for the general case when the existence of components given in (11.16) is not forbidden, but that characterization is technically a bit more complicated. Note that the theorem of Watanabe and Nakamura is true only when $k \geq 2$, and in the special case when $r(u, v) \equiv k \geq 2$ (11.16) holds automatically.

Theorem 11.1.12 *Let $G = (V, E)$ be a graph, m a degree specification with $\widetilde{m}(V)$ even, and $r(u, v)$ a demand function on the pairs of nodes satisfying* (11.16). *There is a graph $H = (V, F)$ for which*

$$d_H(v) = m(v) \text{ for every } v \in V \tag{11.17}$$

and

$$\lambda_{G^+}(x, y) \geq r(x, y) \text{ whenever } x, y \in V, \tag{11.18}$$

where $G^+ = G + H = (V, E \cup F)$, if and only if

$$\widetilde{m}(X) \geq R_r(X) - d_G(X) \text{ for every } X \subseteq V \tag{11.19}$$

where $R_r(\emptyset) := R_r(V) := 0$ and $R_r(X) := \max\{r(x, y) : x \in X, y \in V - X\}$ for $\emptyset \subset X \subset V$.

Proof. Necessity. If there is a graph H for which $G^+ = G + H$ satisfies (11.18), then $d_G(X) + d_H(X) = d_{G^+}(X) \geq R_r(X)$ from which $\widetilde{m}(X) \geq d_H(X) \geq R_r(X) - d_G(X)$, and hence (11.19) holds.

For proving sufficiency, add a new node z to G and $m(v)$ parallel zv-edges for every node $v \in V$. In the resulting graph G', $\lambda_{G'}(x, y) \geq r(x, y)$ holds for every pair $x, y \in V$ of nodes due to (11.19). Note that (11.16) implies that there is no cut-edge of G' incident to z. Therefore we can apply Theorem 8.1.8, which asserts that there is a complete splitting at z that preserves the local edge-connectivities. This means that $H = (V, F)$ satisfies the requirements of the theorem where F denotes the set of the $d_{G'}(z)/2 = \widetilde{m}(V)/2$ edges arising from the splittings. •

We say that a component C of G is **marginal** (with respect to r) if $R_r(C) = 1$. Note that if the values of r are different from 1, then there can be no marginal components. The following result is an extension of the theorem of Watanabe and Nakamura.

Theorem 11.1.13 (Frank [125]) *Suppose that G has no marginal components. There exists a graph $H = (V, F)$ of at most γ edges for which $\lambda_{G^+}(x, y) \geq r(x, y)$ holds for every pair x, y of nodes where $G^+ = G + H = (V, E \cup F)$ if and only if*

$$\sum [R_r(X_i) - d_G(X_i)] \leq 2\gamma \textit{ for every subpartition } \{X_1, \dots, X_t\} \textit{ of } V. \tag{11.20}$$

Proof. Let the deficit $q(X)$ of a subset X be defined by $q(X) := R_r(X) - d_G(X)$. The necessity of (11.20) is clear since at least $q(X)$ new edges must be added to join X and $V - X$ so the total number of new edges is at least $\sum_i q(X_i)/2$.

For sufficiency, we can assume that there is no component C with $R_r(C) = 0$, for otherwise the augmentation problem can be decomposed into two subproblems on the graphs induced by C and by $V - C$. It follows that $R_r(C) \geq 2$ for every component C of G.

In Lemma 8.1.9, we proved that R_r is skew supermodular which implies that q is also skew supermodular, that is, for every $X, Y \subset V$, at least one of the following two inequalities holds:

$$\begin{cases} q(X) + q(Y) \leq q(X \cap Y) + q(X \cup Y) & (\alpha) \\ q(X) + q(Y) \leq q(X - Y) + q(Y - X) & (\beta) \end{cases} \tag{11.21}$$

Let $m : V \to Z_+$ be chosen in such a way that (11.19) is satisfied, $\widetilde{m}(V) \geq 2\gamma$ and $\widetilde{m}(V)$ is minimal in the sense that reducing any positive $m(v)$ by one destroys (11.19) or $\widetilde{m}(V) \geq 2\gamma$.

Claim 11.1.14 $\widetilde{m}(V) = 2\gamma$.

Proof. Suppose indirectly that $\widetilde{m}(V) > 2\gamma$. Then every node v with $m(v) > 0$ belongs to a tight set where a set X is **tight** if $\widetilde{m}(X) = q(X)$. Let $\mathcal{F} := \{X_1, \dots, X_t\}$ be a system of tight sets which covers each node v with $m(v) > 0$, in which $|\mathcal{F}|$ is minimal, and with respect to this, in which $\sum[|Z| : Z \in \mathcal{F}]$ is minimal.

Suppose that $\mathcal{F}$ contains two intersecting members X and Y. If (11.21α) holds, then $X \cup Y$ is tight, in which case X and Y could be replaced by $X \cup Y$, contradicting the minimality of $|\mathcal{F}|$. Therefore (11.21β) must hold which implies

$$\begin{aligned} \widetilde{m}(X) + \widetilde{m}(Y) &= q(X) + q(Y) \leq q(X - Y) + q(Y - X) \leq \widetilde{m}(X - Y) + \widetilde{m}(Y - X) \\ &= \widetilde{m}(X) + \widetilde{m}(Y) - 2\widetilde{m}(X \cap Y) \end{aligned}$$

from which we can conclude that both $X - Y$ and $Y - X$ are tight and $\widetilde{m}(X \cap Y) = 0$. That is, in $\mathcal{F}$ we could replace X and Y by $X - Y$ and $Y - X$, contradicting the minimality of $\sum[|Z| : Z \in \mathcal{F}]$.

Therefore $\mathcal{F}$ must be a subpartition. Then

$$2\gamma \le \widetilde{m}(V) = \sum \widetilde{m}(X_i) = \sum[R_r(X_i) - d_G(X_i)] \le 2\gamma,$$

and hence $\widetilde{m}(V) = 2\gamma$, as required. •

Since there are no marginal components in G, (11.16) holds and Theorem 11.1.12 applies. • •

Corollary 11.1.15 *Let G and r be the same as in Theorem (11.1.13) and assume again that there are no marginal components. The minimum number of new edges whose addition to G results in a graph G^+ for which $\lambda_{D^+}(x, y) \ge r(x, y)$ for every pair $x, y \in V$ is equal to*

$$\max\left\{\left\lceil \left(\sum[R_r(X_i) - d_G(X_i)]\right)/2 \right\rceil : \{X_1, \ldots, X_t\} \textit{ a subpartition of } V\right\}. \tag{11.22}$$

When fractional edges are allowed (meaning that the edges endowed with fractional capacities)*, the minimum of the total sum of the new capacities is equal to*

$$\max\left\{\left(\sum[R_r(X_i) - d_G(X_i)]\right)/2 : \{X_1, \ldots, X_t\} \textit{ a subpartition of } V\right\}. \tag{11.23}$$

Moreover, the fractional optimum can be chosen to be half-integral.

Proof. The first part is just a reformulation of Theorem 11.1.13. For the fractional augmentation, the maximum in (11.23) is clearly a lower bound. To see that it is achievable, double the edges of the initial graph G and also double the demand function r. By the first part of the Corollary, there is an augmenting graph H' with respect to r' for which the number of edges is $\max\{\left\lceil \left(\sum[2R_r(X_i) - 2d_G(X_i)]\right)/2 \right\rceil : \{X_1, \ldots, X_t\}$ a subpartition of $V\}$. By taking each edge of H with half capacity we obtain a half-integral augmentation of G with respect to r whose total capacity is $\max\{\left(\sum[R_r(X_i) - d_G(X_i)]\right)/2 : \{X_1, \ldots, X_t\}$ a subpartition of $V\}$. •

Note that this result is a significant extension of the synthesis problem discussed in Section 7.2.

Augmenting hypergraphs

The theorem of Watanabe and Nakamura can be extended to hypergraphs. For a hypergraph H, let $c(H)$ denote the number of connected components of H. If $\mathcal{F}$ is a subset of hyperedges of H, then $H - \mathcal{F}$ denotes the hypergraph arising from H by deleting the hyperedges in F. The following result is due to T. Király, while its special case for $t = 2$ was proved earlier by Bang-Jensen and Jackson [9].

Theorem 11.1.16 (Király [236]) *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph and $t \ge 2$, $\gamma \ge 0$ integers. There is a t-uniform hypergraph H on node-set V with at most γ hyperedges so that $H_0 + H$ is k-edge-connected ($k \ge 1$) if and only if*

$$\sum_{X \in \mathcal{P}} [k - d_{H_0}(X)] \le t\gamma \textit{ for every subpartition } \mathcal{P} \textit{ of } V,$$

$$k - d_{H_0}(X) \geq \gamma \text{ for every } X \subset V,$$

$$c\left(H_0 - \mathcal{E}_0'\right) - 1 \leq (t-1)\gamma \text{ for every } \mathcal{E}_0' \subseteq \mathcal{E}_0,\ \left|\mathcal{E}_0'\right| = k-1$$

where $c(H)$ denotes the number of connected components of a hypergraph H. •

We do not include the proof here because it is significantly more complex than that of Theorem 11.1.3. In this sense the global edge-connectivity augmentation problem for hypergraphs is tractable. For local edge-connectivities, the situation for hypergraphs is radically different from that for graphs.

Theorem 11.1.17 (Cosh, Jackson, and Király [58]) *Let H be a hypergraph and $r(u,v)$ a demand for each unordered pair of nodes. The problem of deciding whether H can be augmented by γ graph edges so that the local edge-connectivity is at least $r(u,v)$ for every pair $\{u,v\}$ of nodes is* **NP***-complete.* •

There is, however, a nicely tractable related hypergraph augmentation problem where the total size of new hyperedges is the measure to be minimized.

Theorem 11.1.18 (Szigeti [354]) *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph and $r(u,v)$ a demand for each unordered pair of nodes. Then there is a hypergraph $H = (V, \mathcal{E})$ so that $\lambda_{H_0+H}(u,v) \geq r(u,v)$ for every $u, v \in V$ and so that $\sum[|Z| : Z \in \mathcal{E}] \leq \gamma$ if and only if*

$$\sum_{X \in \mathcal{P}} \left[R_r(X) - d_{\mathcal{H}_0}(X)\right] \leq \gamma \text{ for every subpartition } \mathcal{P} \text{ of } V. \bullet$$

11.2 Augmenting edge-connectivity of directed graphs

With the help of Mader's directed splitting theorem (Theorem 8.2.1), we can solve the directed edge-connectivity augmentation problem.

11.2.1 Degree-specified augmentations

Theorem 11.2.1 *Let $D = (V, A)$ be a digraph and $m_{in} : V \to \mathbf{Z}_+$, $m_{out} : V \to \mathbf{Z}_+$ degree specifications. There is a digraph $H = (V, F)$ for which*

$$\varrho_H(v) = m_{in}(v) \text{ and } \delta_H(v) = m_{out}(v) \text{ for every } v \in V \tag{11.24}$$

and $D + H$ is k-edge-connected if and only if $\widetilde{m}_{in}(V) = \widetilde{m}_{out}(V)$,

$$\widetilde{m}_{in}(X) \geq k - \varrho_D(X) \text{ for } \emptyset \neq X \subset V \tag{11.25}$$

and

$$\widetilde{m}_{out}(X) \geq k - \delta_D(X) \text{ for } \emptyset \neq X \subset V. \tag{11.26}$$

Proof. Necessity. If the requested digraph H exists, then $k \leq \varrho_{D+H}(X) = \varrho_D(X) + \varrho_H(X) \leq \varrho_D(X) + \widetilde{m}_{in}(X)$ from which (11.25) follows and (11.26) arises analogously. Since both $\widetilde{m}_{in}(V)$ and $\widetilde{m}_{out}(V)$ counts the number of edges of F they are equal.

Sufficiency. Add a new node z to D along with $m_{in}(v)$ parallel edges from z to v and $m_{out}(v)$ parallel edges from v to z for every $v \in V$. Due to (11.25), (11.26), and to the fact that $\widetilde{m}_{in}(V) = \widetilde{m}_{out}(V)$, the condition in Theorem 8.2.3 is satisfied. Therefore there is a complete splitting at z which results in a k-edge-connected digraph on V. The set of edges resulting from the complete splitting form a digraph H satisfying the requirements of the theorem. •

Problem 11.2.1 *Derive Theorem* 8.2.3 *from Theorem* 11.2.1.

Problem 11.2.2 *Prove that in Theorem* 11.2.1 *if the requested digraph H exists, then it can be chosen to be loopless if and only if $m_{in}(v) + m_{out}(v) \le \widetilde{m}_{in}(V)$ for every node v.*

11.2.2 Minimal augmentations

The directed counterpart of the theorem of Watanabe and Nakamura is as follows.

Theorem 11.2.2 (Frank [125]) *A digraph $D = (V, A)$ can be made k-edge-connected by adding at most γ new edges if and only if*

$$\gamma \ge \sum_i [k - \varrho_D(X_i)] \ \textit{ for every subpartition } \{X_1, \dots, X_t\} \textit{ of } V \tag{11.27}$$

and

$$\gamma \ge \sum_i [k - \delta_D(X_i)] \ \textit{ for every subpartition } \{X_1, \dots, X_t\} \textit{ of } V. \tag{11.28}$$

Proof. If $D + H$ is k-edge-connected, then $\varrho_H(X) \ge k - \varrho_D(X)$ and $\delta_H(X) \ge k - \delta_D(X)$ from which (11.27) and (11.28) follow, respectively.

For sufficiency, let $m_{in} : V \to \mathbf{Z}_+$ be a function satisfying (11.25) for which $\widetilde{m}_{in}(V) \ge \gamma$ and suppose that m_{in} is minimal in the sense that no value $m_{in}(v) > 0$ can be reduced without violating these conditions.

Lemma 11.2.3 $\widetilde{m}_{in}(V) = \gamma$.

Proof. Suppose indirectly that $\widetilde{m}_{in}(V) > \gamma$. A set $X \subset V$ is **in-tight** if $k - \varrho_D(X) = \widetilde{m}_{in}(X)$. By the minimality of m_{in}, each node v with $m_{in}(v) > 0$ belongs to an in-tight set.

Claim 11.2.4 *Let $\mathcal{M} := \{X_1, \dots, X_t\}$ be a minimal system of in-tight sets covering all nodes v with $m_{in}(v) > 0$. Then $\mathcal{M}$ is a subpartition.*

Proof. Let X and Y be two members of $\mathcal{M}$. We claim that $X \cup Y \ne V$. Indeed, in the case when $X \cup Y = V$, by applying (11.28) to the subpartition $\{\overline{X}, \overline{Y}\}$ (where $\overline{X} := V - X$, $\overline{Y} := V - Y$), we would obtain that

$$\gamma \ge k - \delta_D(\overline{X}) + k - \delta_D(\overline{Y}) = k - \varrho_D(X) + k - \varrho_D(Y) = \widetilde{m}_{in}(X) + \widetilde{m}_{in}(Y) \ge \widetilde{m}_{in}(V),$$

contradicting the indirect assumption. Therefore $X \cup Y \ne V$ from which we must have $X \cap Y = \emptyset$, for otherwise

$$k - \widetilde{m}_{in}(X) + k - \widetilde{m}_{in}(Y) = \varrho_D(X) + \varrho_D(Y) \ge \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \ge$$

$$k - \widetilde{m}_{in}(X \cap Y) + k - \widetilde{m}_{in}(X \cup Y) = k - \widetilde{m}_{in}(X) + k - \widetilde{m}_{in}(Y)$$

showing that $X \cup Y$ is in-tight. But in this case X and Y could be replaced by $X \cup Y$, contradicting the minimality of $\mathcal{M}$. •

From (11.27) we have $\widetilde{m}_{in}(V) = \sum[\widetilde{m}_{in}(X) : X \in \mathcal{M}] = \sum[k - \varrho_D(X) : X \in \mathcal{M}] \leq \gamma$, a contradiction. • •

A function m_{out} satisfying (11.26) for which $\widetilde{m}_{out}(V) = \gamma$ can be obtained in an analogous way. An application of Theorem 11.2.1 completes the proof. • • •

There is an important feature of this proof technique. In constructing the function m_{in}, we can start with any function m'_{in} satisfying (11.25) for which $m'_{in}(V) \geq \gamma$ and can then decrease its values as long as it satisfies these properties until the final m_{in} becomes minimal. Hence one obtains the following extension of Theorem 11.2.2.

Theorem 11.2.5 *Let $D = (V, A)$ be a digraph, $g_{in} : V \to \mathbf{Z}_+ \cup \{\infty\}$ and $g_{out} : V \to \mathbf{Z}_+ \cup \{\infty\}$ functions and $\gamma \geq 0$ an integer for which $\gamma \leq \min\{\widetilde{g}_{in}(V), \widetilde{g}_{out}(V)\}$. There is a digraph $H = (V, F)$ for which $D + H$ is k-edge-connected, $|F| \leq \gamma$ and*

$$\varrho_H(v) \leq g_{in}(v) \text{ and } \delta_H(v) \leq g_{out}(v) \text{ for every } v \in V \tag{11.29}$$

if and only if

$$\widetilde{g}_{in}(X) \geq k - \varrho_D(X) \text{ for every } \emptyset \subset X \subset V \tag{11.30}$$

and

$$\widetilde{g}_{out}(X) \geq k - \delta_D(X) \text{ for every } \emptyset \subset X \subset V. \tag{11.31}$$

Moreover, both (11.27) *and* (11.28) *hold.* •

Note that it is natural to assume in the theorem that $\gamma \leq \min\{\widetilde{g}_{in}(V), \widetilde{g}_{out}(V)\}$ since $\min\{\widetilde{g}_{in}(V), \widetilde{g}_{out}(V)\}$ is an upper bound for the number of new edges anyway. A certain linking property holds again.

Corollary 11.2.6 *Let D be a digraph, $\gamma \geq 0$ an integer, and g_{in}, g_{out} upper bound functions. If D can be made k-edge-connected by adding at most γ new edges and D can be made k-edge-connected by adding a digraph satisfying* (11.29), *then there is a digraph H for which $D + H$ is k-edge-connected and H satisfies both requirements.*

Proof. Since the two digraphs exist, the conditions in Theorem 11.2.5 hold, and therefore the requested H exists. •

A simple consequence of Theorem 11.2.5 is the case when only upper bounds are imposed on the degrees.

Theorem 11.2.7 *Let $D = (V, A)$ be a digraph, $g_{in} : V \to \mathbf{Z}_+ \cup \{\infty\}$ and $g_{out} : V \to \mathbf{Z}_+ \cup \{\infty\}$ functions. There is a digraph $H = (V, F)$ satisfying* (11.29) *for which $D + H$ is k-edge-connected if and only if both* (11.30) *and* (11.31) *hold, moreover,*

$$\sum_i [k - \varrho_D(V_i)] \leq \widetilde{g}_{out}(V) \text{ for each subpartition } \{V_1, \ldots, V_t\} \text{ of } V \tag{11.32}$$

and

$$\sum_i [k - \delta_D(V_i)] \le \widetilde{g}_{in}(V) \text{ for each subpartition } \{V_1, \dots, V_t\} \text{ of } V. \tag{11.33}$$

Proof. Proving necessity of the conditions is a simple exercise and is left to the reader. For proving sufficiency, let $\gamma := \min\{\widetilde{g}_{in}(V), \widetilde{g}_{out}(V)\}$. Observe that (11.31) implies that $\sum_i [k - \delta_D(V_i)] \le \sum_i g_{out}(V_i) \le g_{out}(V)$ from which, by (11.33), we have $\sum_i [k - \delta_D(V_i)] \le \gamma$, and hence (11.28) holds. Condition (11.27) is obtained analogously and hence Theorem 11.2.5 can be applied. •

Problem 11.2.3 *Prove that in Theorem* 11.2.7 *if* $\widetilde{g}_{in}(V) \le \widetilde{g}_{out}(V)$, *then* (11.32) *is redundant as it is implied by* (11.30).

It is interesting to state Theorem 11.2.7 in the following equivalent form.

Theorem 11.2.8 *Let* $D = (V, A)$ *be a digraph,* $g_{in} : V \to \mathbf{Z}_+ \cup \{\infty\}$ *and* $g_{out} : V \to \mathbf{Z}_+ \cup \{\infty\}$ *functions. There is a digraph* $H = (V, F)$ *such that* $D + H$ *is* k*-edge-connected and*

(A) *for which* $\varrho_H(v) \le g_{in}(v)$ *for every* $v \in V$ *if and only if both* (11.30) *and* (11.33) *hold,*
(B) *for which* $\delta_H(v) \le g_{out}(v)$ *for every* $v \in V$ *if and only if both* (11.31) *and* (11.32) *hold,*
(C) *for which* $\varrho_H(v) \le g_{in}(v)$ *and* $\delta_H(v) \le g_{out}(v)$ *for every* $v \in V$ *if and only if both* (A) *and* (B) *have solutions.* •

Augmenting directed hypergraphs

In Section 9.4 we investigated hypergraph orientation problems and showed, among other things, how Nash-Williams' orientation theorem can be extended to characterize hypergraphs which admit a k-edge-connected orientation. Analogously, connectivity augmentation problems can also be extended to directed hypergraphs. We cite just one important result.

Theorem 11.2.9 (Berg, Jackson, and Jordán [30]) *Let* $D = (V, \mathcal{A})$ *be a dypergraph,* t *and* γ *non-negative integers. Then* D *can be made* k*-edge-connected by adding* γ *dyperedges of size at most* t *if and only if*

$$\gamma \ge \sum_i [k - \varrho_D(X_i)] \text{ for every subpartition } \{X_1, \dots, X_t\} \text{ of } V \tag{11.34}$$

and

$$(t-1)\gamma \ge \sum_i [k - \delta_D(X_i)] \text{ for every subpartition } \{X_1, \dots, X_t\} \text{ of } V. \bullet \tag{11.35}$$

Note that when D is a digraph and $t = 2$, we are back to Theorem 11.2.2.

11.3 Augmenting ST-edge- and node-connectivity of digraphs

In undirected graphs not only the global edge-connectivity augmentation problems could be solved, but local ones as well (Theorem 11.1.13). Although no such a general result

978–0–19–920527–1 11-Andras-Frank-c11-drv Andras Frank (Typeset by SPi, Publisher Services, India) 387 of 640 December 11, 2010 15:19

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

is known for directed graphs, we present an extension of the global edge-connectivity augmentation theorem. As a consequence, we shall be able to solve the optimal node-connectivity augmentation problem in digraphs.

11.3.1 Augmenting ST-edge-connection

Let $D = (V, A)$ be a digraph and let S and T be two non-empty (but not necessarily disjoint) subsets of V. One may be interested in an augmentation of D in which every node of T is reachable from every node of S. This certainly generalizes the problem of making a digraph strongly connected ($S = T = V$) which was solved by Eswaran and Tarjan (see Section 2.2.7). Unfortunately the extent of this generalization is too great in the sense that the problem becomes **NP**-complete even in the two rather special cases of $|S| = 1$ or $S = T$.

Theorem 11.3.1 *It is* **NP**-*complete to decide if a digraph $D = (V, A)$ can be augmented by adding at most γ new edges in such a way that* (A) *each node in $T \subseteq V$ becomes reachable from a root-node r_0, or that* (B) *each node in T is reachable from every other node of T.*

Proof. Consider the set covering problem in which one is given a system $\mathcal{T} = \{V_1, \dots, V_q\}$ of subsets of a ground-set U and we want to decide if the sets in $\mathcal{T}$ can be covered by γ nodes. Let r_0 be a new node and $T = \{t_1, \dots, t_q\}$ a set of additional new nodes. Let D be a digraph on $V := U \cup T + r_0$ in which there is an edge from u to t_i if $u \in V_i$ $(i = 1, \dots, q)$.

Suppose first that there is a solution to Problem (A). Then there is a solution in which the tail of each new edge is r_0 since any new edge uv can be replaced by r_0v without destroying reachability of the elements of T from r_0. Also, we can assume that the head of each new edge is in U since a new edge r_0t_i can be replaced by an edge r_0v for any element v of V_i, again without violating the reachability. It follows that new edges are of type r_0u $(u \in U)$ and hence their heads from a subset of size at most γ which covers all members of $\mathcal{T}$.

Conversely, if $C \subseteq U$ is a subset covering $\mathcal{T}$, then the set $\{r_0u : u \in C\}$ forms a set of new edges whose addition to D makes each t_i reachable from r_0. Therefore the **NP**-completeness of the set covering problem implies the **NP**-completeness of Problem (A).

To see the **NP**-completeness of Problem (B), consider Problem (A) for digraph $D = (V, A)$ with a root-node r_0 and terminal set T. Extend D by adding an edge from t to r_0 for each $t \in T$. Let D' denote the resulting digraph and let $T' := T - r_0$. Due to this construction, Problem (B) for D' and T' is equivalent to Problem (A) showing that Problem (B) is also **NP**-complete. •

On the other hand, we shall show that if one is only allowed to add ST-edges, then the augmentation problem is tractable even for k-ST-edge-connectivity. A directed st-edge is an ST**-edge** if $s \in S, t \in T$. Let A^* denote the set of all ST-edges and let $m := |A^*|$. Clearly, $m = |S||T|$. We say that a subset X of nodes is ST**-non-trivial**, or non-trivial for short, if $X \cap T \neq \emptyset$ and $S - X \neq \emptyset$, which is equivalent to requiring that there is an ST-edge entering X. A digraph is k**-**ST**-edge-connected** if the number of edges entering $X \subseteq V$ is at least k for every non-trivial X. By Menger's theorem, this is equivalent to requiring that there be k edge-disjoint st-paths for every possibly choice of $s \in S$ and $t \in T$. Note that this property is much stronger than requiring only the existence of k edge-disjoint paths from S to T.

We say that two sets X and Y are ST**-crossing** if none of the sets $X \cap Y \cap T$, $S - (X \cup Y)$, $X - Y$, or $Y - X$ is empty. In the special case when $S = T = V$ this coincides with the standard notion of crossing. A family $\mathcal{L}$ is ST**-crossing** if both the intersection and the union of any two ST-crossing members of $\mathcal{L}$ belong to $\mathcal{L}$. If $\mathcal{L}$ does not include two ST-crossing members, it is said to be ST**-cross-free**. A family $\mathcal{I}$ of sets is ST**-independent** or just independent if, for any two members X and Y of $\mathcal{I}$, at least one of the sets $X \cap Y \cap T$ and $S - (X \cup Y)$ is empty. Note that the relation between two sets can be of three types. Either they are ST-crossing, or one includes the other, or they are ST-independent. A set F of ST-edges (or the digraph (V, F)) **covers** $\mathcal{L}$ if each member of $\mathcal{L}$ is entered by a member of F.

For an initial digraph $D = (V, A)$, that we want to make k-ST-edge-connected, define the deficiency function h on sets as follows.

$$h(X) := \begin{cases} (k - \varrho_D(X))^+ & \text{if } X \text{ is } ST\text{-non-trivial} \\ 0 & \text{otherwise.} \end{cases} \tag{11.36}$$

Therefore the addition of a digraph $H = (V, F)$ of ST-edges to D results in an k-ST-edge-connected digraph exactly when F **covers** h in the sense that $\varrho_H(X) \geq h(X)$ for every $X \subseteq V$.

Recall that for a set-function h we introduced a function $\widetilde{h}$ defined on families of subsets by

$$\widetilde{h}(\mathcal{I}) := \sum[h(X) : X \in \mathcal{I}].$$

Theorem 11.3.2 *A digraph $D = (V, A)$ can be made k-ST-edge-connected by adding at most γ new* (possibly parallel) *ST-edges* (or equivalently, h can be covered by γ ST-edges) *if and only*

$$\widetilde{h}(\mathcal{I}) \leq \gamma \text{ for every } ST\text{-independent family } \mathcal{I} \text{ of sets.} \tag{11.37}$$

Equivalently, the minimum number $\tau_h = \tau_h(D)$ of ST-edges whose addition to D results in a k-ST-edge-connected digraph is equal to the maximum $\nu_h = \nu_h(D)$ of the sum of h-values over all families of ST-independent sets.

Proof. We prove the second form. Since one ST-edge cannot cover two or more ST-independent sets the max $\leq$ min inequality follows. For the reverse direction we proceed by induction on ν_h. If this value is 0, that is, if there is no deficient set, then D itself is k-ST-edge-connected and hence no new edge is needed. So suppose that ν_h is positive.

Case 1 There is an edge $e \in A^*$ for which $\nu_h(D') \leq \nu_h(D) - 1$ for $D' = D + e$. Then it follows by induction that D' can be made k-ST-edge-connected by adding $\nu_h(D')$ ST-edges. But then the original D can be made k-ST-edge-connected by adding at most $\nu_h(D') + 1$ ST-edges and hence we have $\tau_h(D) \leq \tau_h(D') + 1 = \nu_h(D') + 1 \leq \nu_h(D) \leq \tau_h(D)$ from which equality follows throughout and, in particular, $\tau_h(D) = \nu_h(D)$.

Case 2 For every ST-edge e, there is an ST-independent system $\mathcal{F}_e$ for which $\widetilde{h}(\mathcal{F}_e) = \nu_h$ and e does not enter any member of $\mathcal{F}_e$.

Let $\mathcal{J}'$ denote the union of all of these systems $\mathcal{F}_e$ in the sense that as many copies of a set X are put into $\mathcal{J}'$ as the number of edges e for which X is in $\mathcal{I}_e$. Since m denotes the number of ST-edges, we have $\widetilde{h}(\mathcal{J}') = m\nu_h$ and

$$\text{every } ST\text{-edge enters at most } m-1 \text{ members.} \tag{11.38}$$

We can assume that the h-value of each member of $\mathcal{J}'$ is strictly positive since the members of zero h-values can be discarded. Apply the following uncrossing procedure as long as possible. If there are two ST-crossing members, replace them by their intersection and union. When a new member has zero h-value, remove it. The submodularity of the in-degree function implies that such an exchange preserves (11.38) and also that the h-value of the revised system is at least $\widetilde{h}(\mathcal{J}')$.

This uncrossing procedure terminates after a finite number of steps since the number of sets can never increase, and hence it can decrease only a finite number of times. Moreover, when it does not decrease the sum of squares of the sizes in the family strictly increases. Since the number of such steps is also finite, we can conclude that we arrive at an ST-cross-free family $\mathcal{J}$ after a finite number of uncrossing steps . Therefore we have $\widetilde{h}(\mathcal{J}) \geq \widetilde{h}(\mathcal{J}') = m\nu_h$.

We emphasize that a set $X \subseteq V$ can occur in several copies in $\mathcal{J}$. Let $s(X)$ denote the number of these copies. Evidently, the sum of the s-values over the subsets of V is exactly $|\mathcal{J}|$.

Claim 11.3.3 *The partial order on $\mathcal{J}$ defined by $X \subseteq Y$ admits no chain of s-weight larger than $m-1$.*

Proof. If indirectly there is a chain $\mathcal{C}$ of s-weight at least m, then there are m (not-necessarily-distinct) members of $\mathcal{J}$ which are pairwise comparable. Since the members of a chain of ST-non-trivial sets can be covered by a single ST-edge, we are in contradiction with property (11.38). •

We can apply the weighted polar-Dilworth theorem (Theorem 2.4.30) asserting that the maximum weight of a chain is equal to the minimum number of antichains covering each element as many times as its weight is. It follows that $\mathcal{J}$ contains $m-1$ antichains such that $s(X)$ of them contain X for every $X \in \mathcal{J}$. Since $\widetilde{h}(\mathcal{J}) \geq m\nu_h$, the h-sum of at least one of these antichains is larger than ν_h. However, $\mathcal{J}$ is ST-cross-free and hence this antichain is ST-independent, contradicting the definition of ν_h. This contradiction shows that Case 2 cannot occur. • •

Note that in the special case of $S = T = V$, ST-independence of a family simply means that any two members are either disjoint or co-disjoint (: their union is the ground-set).

Proposition 11.3.4 *If $S = T = V$, then the members of an ST-independent family $\mathcal{I}$ of non-empty proper subsets of V are either pairwise disjoint or pairwise co-disjoint.*

Proof. Suppose indirectly that $\mathcal{I}$ has two members which are disjoint and has two members which are co-disjoint. This implies, since any two members of $\mathcal{I}$ are disjoint or co-disjoint, that there is an $X \in \mathcal{I}$ which is disjoint from some $Y \in \mathcal{I}$ and co-disjoint from some $Z \in \mathcal{I}$. But then we must have $Y \subseteq Z$, contradicting the independence of $\mathcal{I}$. •

The proposition shows that in the special case $S = T = V$ Theorem 11.3.2 immediately implies Theorem 11.2.2 on global edge-connectivity augmentation.

11.3.2 Augmenting node-connectivity of digraphs

We have seen in Section 2.5 that the directed node-version of Menger's theorem can be derived easily from its edge-version by applying the node duplication technique. Node duplication does not work for obtaining global node connectivity results from theorems concerning global edge-connectivity. For example, if someone attempts to get a characterization for digraphs that can be made k-node-connected by adding at most γ edges, then the natural approach of using node duplication and applying the corresponding k-edge-connectivity augmentation theorem (Theorem 11.2.2) fails, since by duplicating a node we introduce a new node of in-degree 1, completely destroying k-edge-connection in this way. The great advantage of the concept of k-ST-edge-connection is that it is not sensitive of node duplication. Here we only indicate this application.

Let $D = (V, A)$ be a directed graph. Recall from Section 1.1 the definitions concerning bi-sets. Also, in Section 1.2 we defined a digraph D to be k-connected if

$$\varrho_D(X) + w(X) \geq k \text{ for every non-trivial bi-set } X, \tag{11.39}$$

and proved in Theorem 2.5.26 that D is k-connected if and only if there are k openly disjoint paths from each node to each other. Suppose now that D is not k-connected and define the deficit of a non-trivial bi-set $X = (X_O, X_I)$ by

$$h(X) := (k - \varrho_D(X) - w(X))^+. \tag{11.40}$$

Theorem 11.3.5 (Frank and Jordán [139]) *A directed graph $D = (V, A)$ can be made k-node-connected by adding at most γ new edges if and only if*

$$\widetilde{h}(\mathcal{F}) := \sum[h(X) : X \in \mathcal{F}] \leq \gamma \text{ holds for every independent set } \mathcal{F} \text{ of non-trivial bi-sets.} \bullet \tag{11.41}$$

Problem 11.3.1 $(*)$ *Relying on the node duplicating technique and on Theorem* 11.3.2*, derive Theorem* 11.3.5.

Remark Although the approach suggested in Problem 11.3.1 for deriving a node-connectivity augmentation theorem from an appropriate edge-connectivity augmentation result by using node duplication is logical, the solution to Problem 11.3.1 is a bit technical. In fact, this derivation can be completely avoided by following the original approach of [139]. This approach will be followed in Section 17.3 where Theorem 17.3.1, a common generalization of Theorems 11.3.5 and 11.3.2, will be formulated and proved. Theorem 17.3.1 is an abstract form in which h is a positively crossing supermodular *bi-set* function. This generality makes it possible to obtain a proof which is even simpler than that of Theorem 11.3.2. Also, we will be able to solve the augmentation problem in the more general case when the target is (k, ℓ)-hybrid-connectivity (see Theorem 17.3.14). Theorem 17.3.1 will have several other special cases and consequences, including a deep result of Győri on covering polyominoes with rectangles.

Problem 11.3.2 $(*)$ *A bi-set is* **one-way** *if no edge of D enters it. Show that in Theorem* 11.3.5 *if $|V| \geq k + 1$, then it suffices to require* (11.41) *only for independent one-way bi-sets.*

The conditions (11.27) and (11.28) in Theorem 11.2.2 can be reformulated as follows.

$$\gamma \geq \sum_{X \in \mathcal{F}} [k - \varrho_D(X)] \text{ for every subpartition and co-subpartition } \mathcal{F} \text{ of } V. \tag{11.42}$$

One difference between this condition and the one in Theorem 11.3.5 is that sets occur in the first one and bi-sets in the second. This is natural in light of the fact that edge-connectivity problems are typically handled with subsets of nodes (or cuts defined by subsets) while bi-sets are the handy tools for node-connectivity problems. There is a less natural difference between the two conditions as well. To formulate this other difference, observe that the form of (11.42) suggests the following necessary condition for bi-sets:

$$\sum [h(X) : X \in \mathcal{F}] \leq \gamma$$

for every set of non-trivial bi-sets $\mathcal{F}$ for which either the inner sets are pairwise disjoint or the outer sets are pairwise co-disjoint.

Note that a set of bi-sets is certainly independent if the inner sets are pairwise disjoint or if the outer sets are pairwise co-disjoint. The following example, however, shows that it is not sufficient to require (11.41) only for this restricted type of independence. In Figure 11.1 the digraph on node-set $V = \{a, b, c, d, e\}$ is strongly connected and we want to make it 2-connected. Consider the following 3 bi-sets X, Y, and Z defined by

$$X_I = \{d\},\ X_O = \{d, e\},\quad Y_I = \{a, e\},\ Y_O = \{a, b, e\},\quad Z_I = \{a, b, c\},\ Z_O = \{a, b, c, d\}.$$

Then $\varrho(X) = \varrho(Y) = \varrho(Z) = 0$ and $w(X) = w(Y) = w(Z) = 1$. Moreover, $\{X, Y, Z\}$ is independent since

$$X_I \cap Y_I = \emptyset, \quad X_I \cap Y_I = \emptyset, \quad Y_O \cup Z_O = V.$$

On the other hand, a simple case-checking shows that there are not three deficient bi-sets such that their inner sets are pairwise disjoint or such that their outer sets are pairwise co-disjoint.

Exercise 11.3.3 *Show that the digraph D in Figure* 11.1 *can be made 2-connected by adding 3 new edges. Show that D can be made 2-edge-connected by adding 2 new edges.*

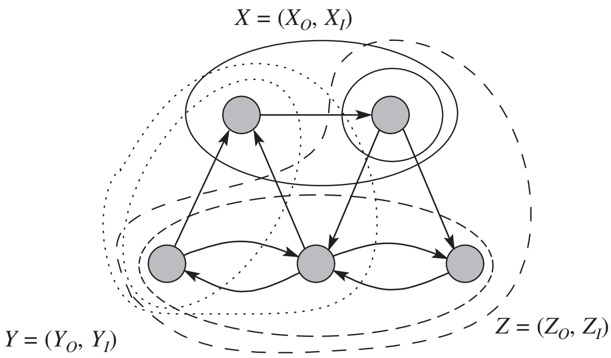


Figure 11.1 Three independent bi-sets

978–0–19–920527–1 11-Andras-Frank-c11-drv Andras Frank (Typeset by SPi, Publisher Services, India) 392 of 640 December 11, 2010 15:19
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

The proof of Theorem 11.2.2 relied on the splitting-off technique which gives rise to a polynomial algorithm and in the capacitated case the algorithm can even be made strongly polynomial. The proof of Theorem 11.3.5 on node-connectivity augmentation relied on the uncrossing procedure, which does not yield a polynomial algorithm. Using a different and complicated approach, Benczúr and Végh [18] constructed such an algorithm. In the special case of the node-connectivity augmentation problem when the initial digraph is $(k-1)$-connected, [155] describes a conceptually much simpler algorithm. In the special case $k = 1$, Enni [91] described an algorithmic proof of Theorem 11.3.2. Note that no strongly polynomial algorithm is known for the capacitated version of Theorem 11.3.2.

11.4 Rooted k-connections

Finding a cheapest strongly connected subgraph of a digraph is known to be **NP**-complete while the cheapest rooted subgraph problem in a digraph is solvable in polynomial time (see Section 3.2.2). More generally, it will turn out that the minimum cost rooted k-edge-connected subgraph problem as well as its rooted k-node-connected counterpart can be solved in polynomial time. These general algorithms, however, rely on the weighted matroid intersection algorithm and will be discussed only in Part III, similarly to the polyhedral description of these kinds of subgraphs. In the introduction to this chapter, we indicated that these algorithms can be used to solve the cheapest augmentation problems as well, since the min-cost augmentation problem is equivalent to the min-cost subgraph problem. In this section we shall show that the augmentation problem admits a much simpler solution algorithm in the special case when we want to increase the rooted (edge- or node-) connectivity of the digraph by one.

11.4.1 Minimum-cost arborescences revisited

Let us start by investigating the special case when $k = 1$, when the goal is to find a cheapest root-connected spanning subgraph of a digraph with a specified root-node r_0. For non-negative costs, this is equivalent to the minimum-cost spanning arborescence problem. In Section 3.2.2 we already described a polynomial time algorithm by Chu and Liu [50]. Here we show that a slight modification of the algorithm gives rise to a min-max theorem and to a totally dual integral description of the polyhedra of arborescences.

Let $D = (V, A)$ be a root-connected digraph from a root r_0 and let $c : A \to \mathbf{R}_+$ be a cost function. Assume that no edge enters the root. We refer to a set-function $z : 2^{V-r_0} \to \mathbf{R}_+$ as c**-feasible** if

$$c(a) \geq \sum \big[z(X) : X \text{ is entered by } a\big] \quad \text{for every } a \in A. \tag{11.43}$$

Theorem 11.4.1 (Fulkerson [170]) *The minimum cost of a spanning r_0-arborescence is equal to*

$$\max \Big\{ \sum [z(X) : X \subseteq V - r_0] : z \textit{ is } c\textit{-feasible} \Big\}.$$

Furthermore, if c is to be integer-valued, an optimal z can be chosen integer-valued.

Proof. Let F be a spanning r_0-arborescence and z a c-feasible function. Since each non-empty $X \subseteq V - r_0$ is entered by an element of F, we have

$$\widetilde{c}(F) = \sum[c(a) : a \in F] \geq \sum\left[\sum[z(X) : a \text{ enters } X] : a \in F\right] \geq \sum[z(X) : X \subseteq V - r_0]$$

from which max $\leq$ min follows. If we have equality throughout, then the given arborescence F is certainly of minimum cost and the given z is an optimal dual solution. These equalities require the following two properties called **optimality criteria**.

$$c(a) = \sum[z(X) : X \text{ is entered by } a] \text{ for every } a \in F, \tag{11.44}$$

$$z(X) > 0 \text{ implies } \varrho_F(X) = 1. \tag{11.45}$$

The algorithm below finds a spanning r_0-arborescence F and a feasible z satisfying the optimality criteria. It consists of two phases. The first phase, described by Fulkerson [170], constructs z in a greedy manner, while the second phase, due to Frank [111], also constructs F in a greedy manner by making use of the available z. During the course of the Phase 1, we revise the cost function from time to time. The current cost function is denoted by c'. We call an edge a a **0-edge** if $c'(a) = 0$.

Phase 1 Iterate the following step. Choose a minimal non-empty subset $X \subseteq V - r_0$ with no entering 0-edge. Define $z(X) := \min\{c'(a) : a \text{ enters } X\}$ and revise c' as follows. Let $c'(a) := c'(a) - z(X)$ when a enters X. The revised c' is non-negative and its value is zero on at least one more edge of D. Phase 1 terminates when every set $X \subseteq V - r_0$ admits an entering 0-edge, or equivalently, when there is a spanning r_0-arborescence consisting of 0-edges.

Phase 2 Starting at r_0, build up a spanning r_0-arborescence F by consecutively adding 0-edges in such a way that the currently added edge leaves the node-set of the already constructed arborescence. If, during this building process, there is more than one 0-edge leaving the arborescence already constructed, choose the one which became a 0-edge earliest during Phase 1.

Obviously, function z constructed in Phase 1 is c-feasible. Furthermore, z and F satisfy (11.44) since F consists of 0-edges.

Lemma 11.4.2 *z and F satisfy* (11.45).

Proof. Let X be a set with $z(X) > 0$. If, indirectly, $\varrho_F(X) > 1$, then there is a moment during Phase 2 when an arborescence F' is at hand for which $\varrho_{F'}(X) = 1$ and the edge e currently added to F' enters X. Due to the selection rule of Phase I, at the moment when $z(X)$ became positive during Phase I, no (at that time current) 0-edge entered X but every proper non-empty subset of X admitted an entering 0-edge. In particular, there was a 0-edge f entering $X - V(F')$ which did not enter X. Therefore when $z(X)$ became positive during Phase 1, f had already been a 0-edge while e was not. This contradicts the selection rule of Phase 2: edge e could not have been chosen because of f. •

Since F and z satisfy the optimality criteria, the proof of the theorem and the correctness of the algorithm is complete. • •

Remark Phase 1 of the algorithm requires finding a minimal non-empty set X with no entering 0-edge. In Section 3.2, we pointed out (in Lemma 3.2.8) that the current digraph D_0 of 0-edges during Phase 1 always includes a source-component X not containing r_0 and such an X serves as a minimal subset of $V - r_0$ with no entering 0-edge.

Fulkerson's theorem can be formulated in an equivalent form.

Theorem 11.4.3 *The linear system described for $x \in \mathbf{R}^A$ by*

$$\{x \geq 0,\ \varrho_x(X) \geq 1 \textit{ for every } \emptyset \subset X \subseteq V - r_0\} \tag{11.46}$$

is totally dual integral. In particular, the polyhedron described by (11.46) *is the convex hull* (of the characteristic vectors) *of all root-connected subgraphs of D.*

By using Cook's theorem (Theorem 4.1.26), we immediately obtain the following.

Corollary 11.4.4 *The linear system*

$$\{x \geq 0,\ \varrho_x(X) \geq 1 \textit{ for every } \emptyset \subset X \subseteq V - r_0,\ \varrho_x(v) = 1 \textit{ for every } v \in V - r_0\} \tag{11.47}$$

is TDI. In particular, the polytope described by (11.47) *is the convex hull* (of the characteristic vectors of) *all spanning r_0-arborescences of D.* •

Problems

11.4.1 ($*$) *Suppose that $T \subseteq V - r_0$ is a given set of terminals such that the head of every edge with positive cost is in T. Construct an algorithm for computing a cheapest arborescence F of root r_0 such that $T \subseteq V(F)$.*

11.4.2 ($*$) *Let $\{T_1, \dots, T_q\}$ be a system of non-empty subsets of $V - r_0$. Devise an algorithm to decide if there is a spanning r_0-arborescence that enters each T_i exactly once.*

11.4.3 *Suppose that there is a spanning r_0-rooted arborescence entering each T_i (i=1,2,...,q) once. Devise an algorithm which constructs such an arborescence of minimum cost with respect to a given cost function.*

Heaviest branchings

A related problem is as follows. Given a digraph $D = (V, A)$ and a weight function w on A, find a maximum weight branching. We call a pair (π, y) a **covering**, where $\pi : V \to R_+$ is a non-negative function on V and $y : 2^V \to \mathbf{R}_+$ a non-negative function on the subsets of V, if $w(uv) \leq \pi(v) + \sum[y(Z) : u, v \in Z \subseteq V]$ for every $uv \in A$. The **value** of a covering is $\sum[\pi(v) : v \in V] + \sum[y(Z)(|Z| - 1) : Z \subseteq V]$.

Theorem 11.4.5 (Edmonds [78]) *The maximum weight of a branching of D is equal to the minimum value of a covering. If w is integer-valued, an optimal covering can be chosen to be integer-valued.*

Proof. Obviously, max $\leq$ min. To see the other direction, extend D by a new node r_0 and new r_0v-edges for every $v \in V$. Define a cost function c on the edges of the extended digraph D' as follows. Let the cost of all the new edges of D' be $M := \max\{w(a) : a \in A\}$, and $c(a) := M - w(a)$ for every $a \in A$. By Theorem 11.4.1, there is a c-feasible vector $z : 2^V \to \mathbf{R}_+$ and a spanning r_0-arborescence F of D' satisfying the optimality criteria (11.44) and (11.45). Define $\pi(v) := M - \sum[z(Z) : v \in Z]$ and, for $|Z| > 1$, define $y(Z) := z(Z)$ $(Z \subseteq V)$. It is straightforward that (π, y) is a covering and that its value is equal to the w-weight of the branching $F \cap A$ of D. •

11.4.2 A general two-phase greedy algorithm

In Section 10.3, we could see the benefits of extending Edmonds' theorem on disjoint arborescences to an abstract setting on covering intersecting families of bi-sets by disjoint digraphs. Our goal in this section is to show that the two-phase approach exhibited in Section 11.4.1 for finding a cheapest spanning arborescence is not an individual and isolated algorithm either. The paper [111] included a straight generalization of this 2-phase greedy algorithm to compute a cheapest subset of edges of a digraph covering an intersecting supermodular *set*-system. Our present goal is to describe a two-phase greedy algorithm [133] for a more general problem in which the goal is to compute a cheapest subset of edges of a digraph covering an intersecting system of *bi-sets*. As a consequence, we shall have a simple and efficient algorithm for making a rooted $(k-1)$-edge-connected (respectively, $(k-1)$-node-connected) digraph rooted k-edge-connected (k-node-connected) by adding a set of new edges whose total cost is as small as possible. The cheapest arborescence problem is a special case when $k = 1$ and the starting digraph has no edges.

We emphasize, however, that the algorithm below is not a straight generalization of the algorithm of Section 11.4.1 (or the algorithm of [111]). The reason why we describe a different two-phase greedy algorithm is that the proof of a direct extension of the algorithm of Section 11.4.1 to bi-set families would become a bit messy. On the other hand the specialization of the algorithm below to arborescences is less efficient than the variant above. Therefore both versions have their own advantages.

Let $D = (V, A)$ be a directed graph endowed with a non-negative cost function $c : A \to \mathbf{R}_+$. Let $\mathcal{F}$ be an intersecting family of bi-sets on V so that each member of $\mathcal{F}$ is covered by D, that is, $\varrho_D(X) \geq 1$ for every $X \in \mathcal{F}$ where $\varrho_D(X)$ denotes the number of edges in D covering X. Recall that an edge e is said to cover or enter a bi-set $X = (X_O, X_I)$ if e enters both X_O and X_I. Given a covering F, a bi-set X is F**-tight** if $\varrho_F(X) = 1$.

A non-negative bi-set-function $y : \mathcal{F} \to \mathbf{R}_+$ is c**-feasible** or, simply **feasible**, if

$$c(a) \geq m(y, a) \quad \text{for every } a \in A \tag{11.48}$$

where

$$m(y, a) := \sum[y(X) : X \in \mathcal{F} \text{ is covered by } a].$$

(The function y is sometimes referred to as a dual feasible solution.) An edge a satisfying (11.48) with equality is said to be y**-tight**. In the proof, the following basic properties of bi-sets will be used.

$$\begin{cases} \textbf{(E1)} \quad \text{If } A \sqsubseteq B \sqsubseteq C \text{ and } a \text{ covers both } C \text{ and } A, \text{ then } a \text{ covers } B. \\ \textbf{(E2)} \quad \text{If } a \text{ covers } A \sqcap B \text{ or } A \sqcup B, \text{ then } a \text{ covers } A \text{ or } B. \end{cases}$$

The following result and its algorithmic proof appeared in [133].

Theorem 11.4.6 *Let $D = (V, A)$ be a digraph endowed with a non-negative cost function $c : A \to \mathbf{R}_+$ and let $\mathcal{F}$ be an intersecting family of bi-sets on V so that D covers $\mathcal{F}$. The minimum cost of a subset $F \subseteq A$ of edges covering $\mathcal{F}$ is equal to*

$$\max\left\{\sum[y(X) : X \in \mathcal{F}] : y \geq 0 \textit{ is c-feasible}\right\}.$$

Furthermore, if c is integer-valued, the optimal y can be chosen to be integer-valued.

Proof. Let $F \subseteq A$ be a subset of edges covering $\mathcal{F}$ and y a c-feasible function. We have

$$\widetilde{c}(F) = \sum[c(a) : a \in F] \geq \sum\left[\sum[y(X) : X \in \mathcal{F} \text{ is covered by } a] : a \in F\right]$$
$$\geq \sum[y(X) : X \in \mathcal{F}]$$

from which max $\leq$ min follows. If we have equality throughout, then the given set F is a cheapest covering of $\mathcal{F}$ and the given y is an optimal c-feasible vector. These equalities are equivalent to requiring the following two properties, called **optimality criteria**:

$$\text{every element of } F \text{ is } y\text{-tight}, \tag{11.49}$$

$$y(X) > 0 \text{ implies that } X \text{ is } F\text{-tight}. \tag{11.50}$$

The algorithm below finds an F covering $\mathcal{F}$ and a feasible y satisfying the optimality criteria. It consists of two phases. In the first one, we construct a dual feasible solution y in a greedy way. In the second phase, we construct a covering $F \subseteq A$ of $\mathcal{F}$, in a greedy way, by using only y-tight edges.

Phase 1 Iterate the following step for $i = 1, 2, \ldots$ for determining a member Z_i of $\mathcal{F}$ along with an edge $a_i \in A$ entering Z_i and $y(Z_i)$. At the beginning, let $y \equiv 0$, and assume that during the first $i-1$ steps, a $Z_j \in \mathcal{F}$, an edge a_j entering Z_j, and $y(Z_j)$ have been determined for each j $(1 \leq j \leq i-1)$. In step i $(i = 1, 2, \ldots)$, decide first if there is a $Z \in \mathcal{F}$ not covered by $\{a_1, \ldots, a_{i-1}\}$. If no such a Z exists, then the first phase terminates by letting $t := i-1$ and $\mathcal{Z} := \{Z_1, \ldots, Z_t\}$, and we turn to Phase 2. If there is such a Z, then choose a smallest one, denoted by Z_i. Let $\mu_i := \min\{c(a) - m(y, a) : a \text{ covers } Z_i\}$ and let $y(Z_i) = \mu_i$. Furthermore, let a_i be an edge covering Z_i where the minimum is attained and continue Phase 1 with $i := i+1$. Observe that the final y is dual feasible and that each a_i is y-tight. (It can be the case that $y(Z) = 0$ for some $Z \in \mathcal{Z}$.)

Phase 2 At the beginning, let $F = \emptyset$. Starting with a_t, consider the elements a_i in reverse order and add the current element a_i to F if F does not cover Z_i.

Note that the algorithm can be used in concrete situations if a subroutine is available in Phase 1 to find a minimal member Z of $\mathcal{F}$ not covered by a given subset of edges. See Problem 11.4.5 below.

To prove the validity of the algorithm we need to show two things. First, the resulting F covers $\mathcal{F}$ and, second, each Z_i is F-tight. The following property is implied by the selection rule of Phase 1.

Claim A *If $X \sqsubset Z_j$ for some $j = 1, \ldots, t$, then there is an $i < j$, for which a_i covers X.* •

Claim B *$\mathcal{Z}$ is laminar.*

Proof. Suppose indirectly that Z_j and Z_h are intersecting and let $j < h$. Applying Claim A to $X := Z_j \sqcap Z_h$, we obtain that there is an edge a_i for which $i < j$ and a_i covers $Z_j \sqcap Z_h$. But then a_i must cover Z_j or Z_h by **(E2)** and this contradicts the selection rule of Phase 1 (since $i < j < h$). •

From the selection rule of Phase 1 and from the laminar property of $\mathcal{Z}$ we obtain:

Claim C *Z_i is F-tight for every $i = 1, \dots, t$.*

Proof. By the selection rule of Phase 2, we have $\varrho_F(Z_i) \geq 1$. Suppose indirectly that there are two elements a_j and a_k $(j < k)$ in F which cover Z_i. Then $i \leq j < k$ by the selection rule of Phase 1. Due to edge a_j, Z_i and Z_j are intersecting, and due to edge a_k, Z_i and Z_k are intersecting. The laminarity of $\mathcal{Z}$ and $j < k$ imply $Z_i \sqsubseteq Z_j \sqsubset Z_k$. By **(E1)**, a_k covers Z_j, and hence (by the rule of Phase 2) a_j cannot be in F, a contradiction. •

Claim D *For every $Z \in \mathcal{F}$ there is a subscript j for which a_j covers Z and $Z_j \sqsubseteq Z$.*

Proof. At the end of Phase 1, there is an edge a_j covering Z. Assume that j is as small as possible. We show that $Z_j \sqsubseteq Z$. If this were not true, then $X \sqsubset Z$ for $X := Z_j \sqcap Z$. By Claim A, there is a subscript $i < j$ for which a_i covers X. Then a_i must cover at least one of Z_j and Z. On the other hand a_i cannot cover Z_j by the selection rule of Phase 1 and it cannot cover Z either by the minimal choice of j, and this contradiction proves the claim. •

Claim E *F covers $\mathcal{F}$.*

Proof. Suppose indirectly that $\mathcal{F}$ has a member Z not covered by F and let Z be maximal. By Claim D, there is a subscript j for which a_j covers Z and $Z_j \sqsubseteq Z$. Since Z_j is F-tight by Claim C, we cannot have $Z_j = Z$, and hence $Z_j \sqsubset Z$. Since Z is not covered by F, a_j is not in F and hence (by the selection rule of Phase 2) there is an edge $a_k \in F$ for which $k > j$, a_k covers Z_j, and $Z_j \sqsubset Z_k$. Now Z and Z_k are properly intersecting since a_k does not cover Z. The maximal choice of Z implies $\varrho_F(Z_k \sqcup Z) \geq 1$, and hence there is an edge $a_\ell \in F$ covering $Z_k \sqcup Z$. But then a_ℓ covers at least one of Z_k and Z which is impossible since Z is not covered by F, and if a_ℓ covers Z_k, then Z_k is covered by both a_k and a_ℓ, contradicting Claim C. •

Therefore we proved that the covering $F \subseteq A$ of $\mathcal{F}$ and the dual feasible solution y constructed by the two-phase greedy algorithm satisfy the optimality criteria. Moreover, if c is integer-valued then so is the constructed y. • •

Faigle and Peis [97] described extensions and further applications of the two-phase greedy algorithm. For other greedy algorihms concerning partially ordered sets, see the work [65] of Dietrich and Hoffman.

Linear programming formulation

Let Q be a (0, 1)-matrix whose rows correspond to the elements of $\mathcal{F}$ while its columns to the edges of D. An entry corresponding to a bi-set $Z \in \mathcal{F}$ and to an edge $a \in A$ is 1 when a covers Z and zero otherwise.

Consider the following pair of primal and dual programs.

$$\min\{cx : x \geq 0,\ Qx \geq \underline{1}\} \tag{11.51}$$

$$\max\{\underline{1}y : y \geq 0,\ yQ \leq c\}. \tag{11.52}$$

Observe that the characteristic vector $x = \underline{\chi}_F$ of a subset $F \subseteq A$ meets the primal constraints $Qx \geq \underline{1}$ if and only if F is a covering or $\mathcal{F}$. Also, an y is c-feasible if and only if $yQ \leq c$. Therefore Theorem 11.4.6 can be formulated in the following equivalent form.

Theorem 11.4.7 *Let D and $\mathcal{F}$ be the same as in Theorem* 11.4.6. *The linear system* $\{x \geq 0,\ Qx \geq \underline{1}\}$ *described for $x \in \mathbf{R}^A$ is totally dual integral. In particular, the convex hull of* (the characteristic vectors of) *coverings of $\mathcal{F}$ is described by* $\{x \in \mathbf{R}^A : x \geq 0,\ Qx \geq \underline{1}\}$.

This result will be considerably generalized in Section 17.1.

Application to augmenting rooted k-connection

In the special case when $k = 1$ and $\mathcal{F}$ consists of all simple non-trivial bi-sets for which $X_O = X_I \subseteq V - r_0$, the problem above specializes to the cheapest arborescence problem. (As mentioned above, the algorithm obtained in this way is different from the one exhibited in Section 11.4.1.) The much more general optimization problem of computing a cheapest rooted k-edge- or k-node-connected subgraph of a digraph will be solved via weighted matroid intersections in Section 13.5. In this way, we shall have an algorithm for solving the cheapest augmentation problem in which the target is rooted k-edge- or k-node-connectivity. Our present goal is to show that in the special case when the goal is to augment the rooted node- or edge-connectivity only by 1, the general algorithm can be avoided since the 2-phase algorithm discussed above can be applied. Actually, we describe a more general problem, in which case no matroidal approach is known.

Cheapest augmentation of rooted *k*-edge-connectivity by 1 Let $D_0 = (V, A_0)$ be a directed graph with a root-node r_0 and let $T \subseteq V - r_0$ be a non-empty set of terminal nodes so that D_0 is rooted $(k-1)$-edge-connected from r_0 to T, which means that $\varrho_{D_0}(X) \geq k-1$ holds for every subset $X \subseteq V - r_0$ intersecting T. By Menger's theorem this is equivalent to requiring that there are $k-1$ edge-disjoint r_0t-paths for every $t \in T$. Let $D = (V, A)$ be a digraph endowed with a non-negative cost function $c : A \to \mathbf{R}_+$ on its edges so that $D_0 + D$ is k-edge-connected from r_0 to T. If there are no restrictions on D, then the problem of finding a cheapest subset of edges of D the addition of which to D_0 results in a digraph that is rooted k-edge-connected from r_0 to T is **NP**-complete even in the special case when $k = 1$ and $A_0 = \emptyset$ since this is the Steiner arborescence problem. However, if

$$\text{the head of each edge } a \in A \text{ with } c(a) > 0 \text{ is in } T, \tag{11.53}$$

then the algorithm above can be used. To see how, first observe that each edge of zero cost can be added freely to D_0. Therefore we can assume that $(*)$ the head of every edge of D is in T.

Let

$$\mathcal{F} := \{X = (X_O, X_I) : X_O \subseteq V - r_0,\ \varrho_{D_0}(X_O) = k-1,\ X_I = T \cap X_O \neq \emptyset\}.$$

Standard submodular technique implies that $\mathcal{F}$ is an intersecting family of bi-sets. Furthermore, for a subset $F \subseteq A$ of edges $D_0 + F$ is k-edge-connected from r_0 to T if and only if F covers $\mathcal{F}$. (This is where $(*)$ is exploited.) Therefore we can apply the two-phase algorithm developed above. Note that in the special case when $T = V - r_0$, the restriction formulated in (11.53) holds automatically.

Exercise 11.4.4 *Show that in the special case when* $k = 1$ *the foregoing problem can be reduced to that of finding a cheapest spanning arborescence.*

Cheapest augmentation of rooted *k*-node-connectivity by 1 Given D_0, D, T, and c again with the same properties as above, consider now the augmentation problem when D_0 is rooted $(k-1)$-node-connected from r_0 to T, meaning that $\varrho_{D_0}(X) + w(X) \geq k - 1$ holds for every bi-set $X = (X_O, X_I)$ for which $X_I \cap T \neq \emptyset$ and $X_O \subseteq V - r_0$ where $w(X) = |X_O| - |X_I|$. By Menger's theorem, this property is equivalent to requiring that there be $k - 1$ openly disjoint r_0t-paths for every $t \in T$.

Let

$$\mathcal{F} := \{X = (X_O, X_I \cap T) : X_O \subseteq V - r_0,\ X_I \cap T \neq \emptyset,\ \varrho_{D_0}(X) + w(X) = k - 1\}.$$

It can be shown that $\mathcal{F}$ is intersecting and that a subset F of edges of D covers $\mathcal{F}$ if and only if $D_0 + F$ is k-node-connected from r_0 to T. Therefore, the general algorithm can be applied again.

In particular, in the special case when $T = V - r_0$, the algorithm above computes a cheapest subset of edges of D for which addition to D_0 (which is supposed to be rooted $(k-1)$-node-connected) results in a rooted k-node-connected digraph.

Problem 11.4.5 *Step* i *of Phase 1 of the general algorithm above requires deciding if there is a bi-set* $Z \in \mathcal{F}$ *not covered by* $\{a_1, \ldots, a_{i-1}\}$*, and if there is one, a smallest* Z_i *is to be computed. Show that such a subroutine can be constructed via MFMC computations in the special cases of rooted edge- and node-connectivity augmentation.*

Part III
Semimodular Optimization

In Chapter 5, we considered basic notions and properties of matroids, including the matroid greedy algorithm and some polyhedra related to submodular functions. Part I also covered the most important results on connectivity, such as the various versions of Menger's theorem or Hoffman's circulation theorem. Part II studied more advanced results and algorithms concerning higher-order connectivity of directed and undirected graphs, of which the proofs often relied on the use of submodular and supermodular functions.

The goal of Part III is to provide a synthesis in which semimodular functions are not merely useful tools in proofs but also become the central target of investigation. It is often the case that a theorem on semimodular functions is inspired by a connectivity result from Part I or II, and that it in turn has new special cases useful in solving other connectivity problems. In this sense, we say that a specific connectivity result is extended to an abstract form involving submodular or supermodular functions. For example, Theorem 11.2.7 included a characterization of digraphs that can be made k-edge-connected by adding a given number of new edges. This result serves as a motivation for finding a characterization of supermodular functions that can be covered by a given number of directed edges, and this extension makes it possible to solve the connectivity augmentation problem for other connectivity targets, such as (k, ℓ)-edge-connectivity or k-edge-connectivity within a specified subset. Another corollary of this approach will be a characterization of mixed graphs that have k-edge-connected orientations.

One may ponder why submodular functions are so successful in discrete optimization. One possible explanation is that convex functions are fundamental in continuous optimization, and there is a surprisingly strong analogy, formal and informal, between convex and submodular functions.

12

Setting the stage—aspects and approaches

Prior to the systematic study of semimodular functions in later chapters, we discuss here some fundamental results of the topic. Though their discussion seems to be a bit out of line at this point, there are two reasons why we highlight them. First, these results will have several far-reaching extensions and yet their proofs are technically quite simple. More importantly, they will provide significant help in proving and understanding later results.

12.1 Two aspects of investigation

There are two major aspects to studying semimodular functions. The first one is about exploring properties and structures, including algorithms and polyhedral descriptions, and also the abstract semimodular forms of specific connectivity results. Matroid and polymatroid greedy algorithms are typical examples, and these types of result provide the main body of the entire third part of the book. We will understand the true background of many previous results on flows, matroids, dijoins, graph orientations, and rooted k-connected digraphs. The general view will clarify some unexpected links such as the one between node-connectivity augmentations of digraphs and a deep min-max theorem of Győri [198] on covering polyominoes with rectangles.

The other aspect of investigation is of particular importance from an applicational point of view. This is about the various ways in which matroids and more complex semimodular frameworks can be given. In applications, quite often it is not at all straightforward to recognize that a certain structure is a matroid, and this necessitates the study of various relaxations of sub- or supermodularity. Let us start with this latter aspect.

12.1.1 Intersecting versus fully submodular functions

Given a set-function b, by its **lower truncation** $b^\vee$ we mean the following set-function.

$$b^\vee(Z) := \min \left\{ \sum_i^t b(X_i) : \{X_1, \dots, X_t\} \text{ a partition of } Z \text{ where } t \geq 1 \right\}. \qquad (12.1)$$

Since in this case the one-element partition $\{Z\}$ is also allowed, we see that $b^\vee \leq b$. In the literature, sometimes the term Dilworth truncation is used.

The **upper truncation** $p^\wedge$ of a set-function p is defined analogously by replacing min with max, that is,

$$p^\wedge(Z) := \max\left\{\sum_i^t p(X_i) : \{X_1, \ldots, X_t\} \text{ a partition of } Z \text{ where } t \geq 1\right\}. \tag{12.2}$$

Again, since $\{Z\}$ is a partition of Z, we have $p^\wedge \geq p$.

Since lower truncation is used only for submodular functions, we often leave out the attribute lower, in which case the truncation of a submodular function means its lower truncation. Analogously, by the truncation of a supermodular function we mean its upper truncation.

Theorem 12.1.1 (Dunstan [75]) *The truncation $b^\vee$ of an intersecting submodular function b is fully submodular. The truncation $p^\wedge$ of an intersecting supermodular function p is fully supermodular.*

Proof. By symmetry, it suffices to show the submodularity of $b^\vee$. For a family $\mathcal{F}$ of subsets, we use the notation $\widetilde{b}(\mathcal{F}) := \sum[b(X) : X \in \mathcal{F}]$. Let $A, B \subseteq S$. There is a partition $\{A_1, \ldots, A_k\}$ of A and a partition $\{B_1, \ldots, B_\ell\}$ of B for which $b^\vee(A) = \sum_i b(A_i)$ and $b^\vee(B) = \sum_j b(B_j)$. Then $\mathcal{F} = \{A_1, \ldots, A_k, B_1, \ldots, B_\ell\}$ is a family of subsets for which $\widetilde{b}(\mathcal{F}) = b^\vee(A) + b^\vee(B)$ and

$(*)$ $\mathcal{F}$ covers the elements of $A \cap B$ twice and the elements of $(A - B) \cup (B - A)$ once.

Select a family $\mathcal{F}_1$ of subsets (in which a set can occur in two copies) satisfying $(*)$ so that $\widetilde{b}(\mathcal{F}_1)$ is minimal and, subject to this, $\sum[|X|^2 : X \in \mathcal{F}_1]$ is maximal. Evidently, $\widetilde{b}(\mathcal{F}) \geq \widetilde{b}(\mathcal{F}_1)$.

We claim that $\mathcal{F}_1$ is laminar. For if $\mathcal{F}_1$ has two properly intersecting members, then replacing them by their intersection and union, we would get a family $\mathcal{F}_2$ satisfying $(*)$ for which $\widetilde{b}(\mathcal{F}_1) \geq \widetilde{b}(\mathcal{F}_2)$ and $\sum[|X|^2 : X \in \mathcal{F}_1] < \sum[|X|^2 : X \in \mathcal{F}_2]$, contradicting the choice of $\mathcal{F}_1$.

Since $\mathcal{F}_1$ is laminar, it can be partitioned into two parts $\mathcal{P}_1$ and $\mathcal{P}_2$ where $\mathcal{P}_1$ is a partition of $A \cap B$ and $\mathcal{P}_2$ is a partition of $A \cup B$. By the definition of $b^\vee$, we get $b^\vee(A \cap B) \leq \widetilde{b}(\mathcal{P}_1)$ and $b^\vee(A \cup B) \leq \widetilde{b}(\mathcal{P}_2)$. Therefore $b^\vee(A) + b^\vee(B) = \widetilde{b}(\mathcal{F}) \geq \widetilde{b}(\mathcal{F}_1) = \widetilde{b}(\mathcal{P}_1) + \widetilde{b}(\mathcal{P}_2) \geq b^\vee(A \cap B) + b^\vee(A \cup B)$, and hence $b^\vee$ is indeed fully submodular. •

Let b be a submodular function on ground-set S and let $g : S \to \mathbf{R} \cup \{\infty\}$ be a function on S. Define their **convolution** (more precisely, **lower convolution**) $b \bigtriangledown g$ by

$$(b \bigtriangledown g)(Z) := \min\{b(X) + \widetilde{g}(Z - X) : X \subseteq Z\}. \tag{12.3}$$

The next result shows that the convolution of a submodular and a modular function may be viewed as a truncation.

Theorem 12.1.2 *The convolution $b \bigtriangledown g$ of a submodular function b and a function $g : S \to \mathbf{R} + \{\infty\}$ is equal to the truncation $(b')^\vee$ of b' where*

$$b'(X) := \begin{cases} b(X) & \text{if } |X| \geq 2 \\ \min\{b(v), g(v)\} & \text{if } X = \{v\}. \end{cases} \tag{12.4}$$

Proof. It follows from the definition that b' is intersecting submodular. Apply the formula (12.1) to the lower truncation of b' and consider a minimizer partition $\{X_i\}$ for which the number of parts is as small as possible. Then b' does not satisfy the submodular inequality for any pair X_i, X_j. Hence the partition can have at most one member, say X_1, for which $b'(X_1) = b(X_1)$ while $b'(X_i) < b(X_i)$ holds for the other members X_i. Then X_i is a singleton and $b'(X_i) = g(v_i)$ where v_i denotes the single element of X_i. Therefore, $(b')^\vee(Z) = \min\{\sum_i b'(X_i) : \{X_i\}$ a partition of $Z\} = \min\{b(X_1) + g(v_2) + \cdots + g(v_k) : \{X_1, \{v_2\}, \ldots, \{v_k\}\}$ a partition of $Z\} = \min\{b(X) + \widetilde{g}(Z - X) : X \subseteq Z\} = (b \bigtriangledown g)(Z)$. •

Let b be a non-negative submodular function. An easy exercise shows that the function b_{imin} defined by

$$b_{imin}(Z) := \min\{b(X) : X \subseteq Z\} \tag{12.5}$$

is fully submodular and non-increasing. (Here *imin* refers to inner minimization.) Note that in the special case when $g \equiv 0$, the convolution of b and g is exactly b_{imin}. Analogously, we can define a function b_{omin} by

$$b_{omin}(Z) := \min\{b(X) : X \supseteq Z\} \tag{12.6}$$

and observe that b_{omin} is fully submodular and non-decreasing, and hence b_{omin} is a polymatroid function. (Here *omin* refers to outer minimization.)

Proposition 12.1.3 *Let b be a submodular function and k a non-negative number. Then the set-function b_k defined for non-empty subsets $X \subseteq S$ by*

$$b_k(X) = \min\left\{\sum_{i=1}^{q} b(X_i) - kq : \{X_1, \ldots, X_q\} \text{ a partition of } X\right\}$$

is fully submodular.

Proof. Define b' by

$$b'(X) := \begin{cases} b(X) - k & \text{if } \emptyset \subset X \subseteq S \\ 0 & \text{if } X = \emptyset. \end{cases} \tag{12.7}$$

Then b' is intersecting submodular and b_k is exactly the truncation of b'. Hence Theorem 12.1.1 applies. •

Truncation will be further explored in Chapter 15 and will turn out to be an indispensable tool in several applications.

12.1.2 Feasibility of submodular flows

Our next preliminary result shows an example for the paradigm of specific connectivity versus abstract submodularity results. Recall the theorem of Hoffman on the existence of a feasible modular flow (Theorem 3.4.5).

Let $D = (V, A)$ be a directed graph, $f : A \to \mathbf{R} \cup \{-\infty\}$, $g : A \to \mathbf{R} \cup \{\infty\}$ two bounding functions for which $f \le g$. Moreover, we are given a crossing submodular set-function $b : 2^V \to \mathbf{Z} \cup \{\infty\}$ for which $b(\emptyset) = 0$ and $b(V)$ is finite. We call a function (or vector) $x : A \to \mathbf{R}$ a **submodular flow** or a **subflow** if

$$\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z) \le b(Z) \text{ for every } Z \subseteq V. \tag{12.8}$$

Since $\Psi_x(V) = 0$ for an arbitrary x, the value $b(V)$ can be revised to be zero. Therefore we assume throughout that $b(V) = 0$. A subflow x is **feasible** if

$$f \le x \le g. \tag{12.9}$$

The set $Q(f, g; b)$ of feasible subflows is called a **submodular flow** (or **subflow**) **polyhedron**. We will say that the subflow or the subflow polyhedron is **confined** by function b. Subflows are sometimes given in the equivalent form in which b is defined only on a crossing family $\mathcal{F}$ of subsets, b is finite-valued and crossing submodular on $\mathcal{F}$, and (12.8) is required for the members Z of $\mathcal{F}$.

Consider the special case when b is finite-valued, modular, and $b(V) = 0$. This means that b arises from a function $m : V \to \mathbf{R}$ with $\widetilde{m}(V) = 0$ so that $b := \widetilde{m}$. For a submodular flow x, one has $\Psi_x(Z) \le b(Z)$ and $-\Psi_x(Z) = \Psi_x(V - Z) \le b(V - Z) = b(V) - b(Z) = -b(Z)$ from which $\Psi_x(Z) = b(Z)$ for every $Z \subseteq V$, and hence x is a modular flow. We show how the feasibility theorem of Hoffman for modular flows (in particular, for circulations) extends to submodular flows confined by a *fully* submodular function. It is worth realizing that the present proof follows almost word for word the one given for Theorem 3.4.5. For intersecting submodular functions b, the feasibility theorem will easily be handled by truncating b, but for the general case of crossing submodular functions the formulation of the feasibility result is more complex and will be postponed to Chapter 16.

Theorem 12.1.4 (Frank [119]) *Let f and g be functions on the edge-set of a digraph $D = (V, A)$ for which $f \le g$ and let b be a fully submodular function. There is a feasible subflow if and only if $\varrho_f - \delta_g \le b$, that is, if*

$$\varrho_f(Z) - \delta_g(Z) \le b(Z) \textit{ for every } Z \subseteq V. \tag{12.10}$$

If each of f, g, b is integral, then (12.10) *implies the existence of an integral feasible subflow.*

Proof. For a feasible subflow z, we have $\varrho_f(Z) - \delta_g(Z) \le \varrho_z(Z) - \delta_z(Z) \le b(Z)$, and hence (12.10) is necessary.

To see sufficiency, define p_{fg} by

$$p_{fg}(X) := \varrho_f(X) - \delta_g(X) \text{ for } X \subseteq V.$$

Recall Proposition 1.2.3 which stated

$$p_{fg}(X) + p_{fg}(Y) = p_{fg}(X \cap Y) + p_{fg}(X \cup Y) - d_{g-f}(X, Y) \text{ for every } X,\ Y \subseteq V. \tag{12.11}$$

Note that condition (12.10) can be concisely written in the form $p_{fg} \le b$. Define a subset Z of nodes to be **tight** if $p_{fg}(Z) = b(Z)$, while an edge e of D is **tight** if $f(e) = g(e)$.

If $g(e) = \infty$ for some edge e, then the value of $g(e)$ can be revised to be a suitable large finite value so as to preserve both (12.10) and $f \le g$. Therefore, we can assume for each edge e that $g(e) < \infty$, and analogously, that $f(e) > -\infty$.

Suppose indirectly that the theorem fails for D, and choose a counterexample (D is given) in which the total number of tight edges and tight sets is maximum. It cannot be the case that every edge is tight since then $x := f$ is finite-valued and $\varrho_x(Z) - \delta_x(Z) = \varrho_f(Z) - \delta_f(Z) = \varrho_f(Z) - \delta_g(Z) \le b(Z)$, and hence x would be a feasible subflow, contradicting the indirect assumption.

Let $h = st$ be an edge of D for which $f(h) < g(h)$.

Claim 12.1.5 *The edge h enters a tight set T and leaves a tight set S.*

Proof. We prove only the first statement as the second one is analogous. If h did not enter any tight set, then $f(h)$ could be increased so that the revised f' also satisfies $f' \le g$ and $\varrho_{f'}(Z) - \delta_g(Z) \le b(Z)$ holds for every $Z \subseteq V$, moreover, due to the assumption that $g(h)$ is finite, either the edge h becomes tight or a subset entered by h becomes tight (or both). Since this modification keeps every tight set tight, the maximality assumption about the total number of tight edges and tight sets implies that there is a feasible subflow x' with respect to f' and g. But then x' is feasible with respect to f and g, a contradiction. •

Since $f(h) < g(h)$ and h connects $S - T$ and $T - S$, it follows from (12.11) and (12.10) that $b(S) + b(T) = p_{fg}(S) + p_{fg}(T) < p_{fg}(S \cap T) + p_{fg}(S \cup T) \le b(S \cap T) + b(S \cap T) \le b(S) + b(T)$, and this contradiction shows that no counterexample may exist.

The same argument also shows that there is an integer-valued feasible subflow provided that each of f, g, and b is integer-valued. • •

By combining this result with Theorem 12.1.1, we obtain the following result [119].

Theorem 12.1.6 *Let $f \le g$ be functions on the edge-set of $D = (V, A)$ and let b be an intersecting submodular function for which $b(V) = 0$. There is a feasible subflow if and only if*

$$\varrho_f(Z) - \delta_g(Z) \le \sum_i b(V_i) \tag{12.12}$$

for every $Z \subseteq V$ and for every partition $\{V_1, \ldots, V_t\}$ of Z. If each of f, g, and b is integer-valued and there is a feasible subflow, then there is one which is integer-valued.

Proof. For a feasible subflow z, one has $\varrho_f(Z) - \delta_g(Z) \le \varrho_z(Z) - \delta_z(Z) = \Psi_z(Z) = \sum_i \Psi_z(V_i) \le \sum_i b(V_i)$, and hence the condition is indeed necessary. For the sufficiency, let $b^\vee$ denote the lower truncation of b. This is fully submodular and (12.12), when applied to $Z = V$, implies that $b^\vee(V) = b(V)$. Furthermore, we claim that $Q(f, g; b) = Q(f, g; b^\vee)$. Since $b^\vee \le b$, we clearly have $Q(f, g; b) \supseteq Q(f, g; b^\vee)$. For the reverse direction, let $z \in Q(f, g; b)$ and let $Z \subseteq V$. There is a partition $\{Z_1, \ldots, Z_k\}$ of Z so that $b^\vee(Z) = \sum_i b(Z_i)$. Since Ψ_z is modular, we have $\Psi_z(Z) = \sum_i \Psi_z(Z_i) \le \sum_i b(Z_i) = b^\vee(Z)$ and hence $z \in Q(f, g; b^\vee)$ from which $Q(f, g; b) = Q(f, g; b^\vee)$.

978–0–19–920527–1 12-Andras-Frank-c12-drv Andras Frank (Typeset by SPi, Publisher Services, India) 408 of 640 December 11, 2010 15:19
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

The result follows from Theorem 12.1.4 once we observe that (12.10) described for $b^\vee$ is just equivalent to the condition in (12.12). •

Submodular flows will turn out to be an extremely flexible framework with numerous applications: see Chapter 16.

12.2 Convexity and submodularity

The following result of Frank [119] may be considered as a discrete counterpart of the classical separation theorem of convex and concave functions.

Theorem 12.2.1 (Discrete separation theorem) *Let $p^* : 2^S \to \mathbf{R} \cup \{-\infty\}$ be a supermodular function and $b^* : 2^S \to \mathbf{R} \cup \{\infty\}$ a submodular function on a ground-set S* (for which $p^*(\emptyset) = b^*(\emptyset) = 0$). *There is a function $m : S \to \mathbf{R}$ so that the modular function $\widetilde{m}$ satisfies $p^* \leq \widetilde{m} \leq b^*$ if and only if*

$$p^* \leq b^*. \tag{12.13}$$

If p^* and b^* are integer-valued and $p^* \leq b^*$, then the separating m can also be chosen integral. •

Proof. The necessity of $p^* \leq b^*$ is evident. For proving the sufficiency, let S' and S'' be two disjoint copies of S. Let $V := S' \cup S''$, $A := \{s's'' : s \in S\}$, and $D = (V, A)$. Furthermore, let $f := -\infty$, $g := \infty$. Define b on V by $b(X' \cup Y'') := b^*(X) - p^*(Y)$ where X and Y are subsets of S while X' denotes the subset of S' corresponding to X and Y'' denotes the subset of S'' corresponding to Y.

The function b is clearly fully submodular. Apply Theorem 12.1.4 to this subflow feasibility problem. We claim that (12.10) is fulfilled. Indeed, it holds automatically when $\varrho_f(Z) - \delta_g(Z) = -\infty$, so we can assume that $Z = X' \cup X''$ for some set $X \subseteq S$. In this case, $\varrho_f(Z) - \delta_g(Z) = 0$, and hence (12.10) follows indeed from the hypothesis $p^* \leq b^*$. By Theorem 12.1.4, there is a feasible subflow x. Moreover, x is integer-valued if both p^* and b^* are integer-valued. Let $m : S \to \mathbf{R}$ be defined by $m(s) := x(s's'')$ $(s \in S)$. Then for any subset $X \subseteq S$, we have $\widetilde{m}(X) = \varrho_x(X'') = \varrho_x(X'') - \delta_x(X'') \leq b(X'') = b^*(X)$ and $-\widetilde{m}(X) = -\delta_x(X') = \varrho_x(X') - \delta_x(X') \leq b(X') = -p^*(X)$ from which $p \leq \widetilde{m} \leq b$ follows. •

Theorem 12.2.2 *Let p and b be intersecting super- and submodular functions, respectively, on a ground-set S. There is a function $m : S \to \mathbf{R}$ for which $p \leq \widetilde{m} \leq b$ if and only if*

$$p^\wedge \leq b^\vee, \tag{12.14}$$

that is, if

$$\sum_i p(X_i) \leq \sum_j b(Y_j) \text{ for every two subpartitions } \{X_i\} \text{ and } \{Y_j\} \text{ of } S \text{ with } \cup_i (X_i) = \cup_j (Y_j). \tag{12.15}$$

If p and b are integer-valued and (12.15) *holds, then m can be chosen integer-valued.*

Proof. Apply Theorem 12.2.1 to the lower truncation $b^\vee$ of b and to the upper truncation $p^\wedge$ of p. Observe that the inequality $p^\wedge \leq b^\vee$ is equivalent to (12.15). Furthermore, $p \leq \widetilde{m} \leq b$ if and only if $p^\wedge \leq \widetilde{m} \leq b^\vee$. •

The Discrete separation theorem suggests that there is an analogy between convex functions and submodular set-functions. This analogy is thoroughly worked out in a book entitled *Discrete Convex Analysis* by Murota [294]. Below, we briefly touch on some further indications of this link. It is well known that a differentiable function is convex if and only if its derivative is increasing. The following tiny result can be interpreted as a kind of discrete analogue.

Theorem 12.2.3 *A set-function b is submodular if and only if the difference function $b(X+s)-b(X)$ is non-increasing for every element s of S, meaning that $X \subset Y \subseteq S-s$ implies*

$$b(X+s)-b(X) \geq b(Y+s)-b(Y). \tag{12.16}$$

Proof. Suppose first that b is submodular. Then $b(X+s)+b(Y) \geq b[(X+s)\cap Y]+b(X+s\cup Y) = b(X)+b(Y+s)$ and (12.16) follows.

Conversely, suppose that $b(X+s)-b(X)$ is non-decreasing. We prove the submodular inequality for $X, Y \subseteq S$ by induction on $|X-Y|$. If this number is zero, that is, if $X \subseteq Y$, then the inequality holds (with equality). Let $s \in X-Y$ and $X' := X-s$. By induction $b(X')+b(Y) \geq b(X'\cap Y)+b(X'\cup Y) = b(X\cap Y)+b(X'\cup Y)$. By exploiting the monotonicity, we get $b(X'+s)-b(X') \geq b(X'\cup Y+s)-b(X'\cup Y)$. A combination of the two inequalities results in $b(X')+b(Y) \geq b(X\cap Y)+b(X'\cup Y+s)-b(X'+s)+b(X') = b(X\cap Y)+b(X\cup Y)-b(X)+b(X')$, that is, $b(X)+b(Y) \geq b(X\cap Y)+b(X\cup Y)$. •

Problem 12.2.1 *Prove that if b satisfies the submodular inequality for X and Y whenever $|X-Y| = |Y-X| = 1$, then b is fully submodular.*

Corollary 12.2.4 *A finite-valued submodular function b is modular if and only if $b(v)+b(S-v) = b(S)$ holds for every $v \in S$.*

Proof. If b is modular, then $b(v)+b(S-v) = b(S)$. To see the converse, let $Z \subseteq S$ and $s \in Z$. Since $b(X+s)-b(s)$ is non-increasing, we get $b(s) = b(s)-b(\emptyset) \geq b(Z)-b(Z-s) \geq b(S)-b(S-s) = b(s)$ from which $b(Z) = b(Z-s)+b(s)$ and hence, $b(Z) = \sum_{z\in Z} b(z)$ follows by induction for every $Z \subseteq S$. •

There is yet another link between submodularity and convexity. In Section 14.5, we shall prove (Theorem 14.5.6) that a finite-valued set-function b is submodular if and only if its linear extension $\hat{b}$ is convex (where $\hat{b}$ is a function on $\mathbf{R}^S$).

The minimum of two submodular functions

The minimum of two submodular functions is not necessarily submodular. However, if one of them is non-decreasing while the other one is non-increasing, then their minimum is submodular. Even more general is the following result of Lovász [272].

Theorem 12.2.5 *Let b_1 and b_2 be submodular functions for which $b_1 - b_2$ is non-decreasing. Then the function b defined by*

$$b(X) := \min\{b_1(X), b_2(X)\} \tag{12.17}$$

is submodular.

Proof. If $b_1(X) \leq b_2(X)$ and $b_1(Y) \leq b_2(Y)$, then $b(X) + b(Y) = b_1(X) + b_1(Y) \geq b_1(X \cap Y) + b_1(X \cup Y) \geq b(X \cap Y) + b(X \cup Y)$, and hence the submodular inequality holds. The situation is the same when $b_1(X) \geq b_2(X)$ and $b_1(Y) \geq b_2(Y)$. Therefore, we can assume that $b(X) = b_1(X)$ and $b(Y) = b_2(Y)$. Since $b_1 - b_2$ is non-decreasing and $X \cap Y \subseteq X$, one gets $b_1(X) - b_2(X) \geq b_1(X \cap Y) - b_2(X \cap Y)$ from which

$$\begin{aligned} b(X) + b(Y) = b_1(X) + b_2(Y) &\geq [b_1(X \cap Y) - b_2(X \cap Y) + b_2(X)] + b_2(Y) \\ &\geq b_2(X \cap Y) + b_2(X \cup Y) + b_1(X \cap Y) - b_2(X \cap Y) \\ &= b_2(X \cup Y) + b_1(X \cap Y) \geq b(X \cap Y) + b(X \cup Y). \bullet \end{aligned}$$

Corollary 12.2.6 *Let b_1 and b_2 be submodular functions. If b_1 is non-decreasing and b_2 is non-increasing, then their minimum b defined by $b(X) := \min\{b_1(X), b_2(X)\}$ is submodular.* $\bullet$

It follows the same way that b is intersecting (crossing) submodular if both b_1 and b_2 are intersecting (crossing) submodular and their difference is non-decreasing.

12.3 Abstract extensions of the Kőnig–Hall theorem

In order to clarify better what we mean by an abstract form of a specific connectivity result, we close this introductory chapter by exhibiting two early abstract extensions of the Kőnig–Hall theorem on bipartite matchings.

12.3.1 An extension concerning submodular functions

Let $G = (S, T; E)$ be a simple bipartite graph and M a matroid on S. This determines a matroid M_E on E in which a subset F is independent if $d_F(v) \leq 1$ for every $v \in S$ and the subset of S covered by F is independent in M. Recall that M_E is a matroid which arises from M by parallel multiplication.

The following result is a significant extension of Hall's theorem. For a subset $X \subseteq T$, $\Gamma_E(X)$ denotes the set of neighbours in S, or more formally, $\Gamma_E(X) := \{v \in S :$ there is an edge $uv \in E$ with $u \in X\}$.

Theorem 12.3.1 (Rado [326]) *Let $M = (S, r)$ be a matroid and $G = (S, T; E)$ a bipartite graph. G has an M_E-independent matching covering T if and only if the following Rado condition holds:*

$$r(\Gamma_E(X)) \geq |X| \text{ for every } X \subseteq T. \tag{12.18}$$

Proof. To see the necessity, suppose that $F \subseteq E$ is a requested subset and $X \subseteq T$. Then $r(\Gamma(X)) \geq r(\Gamma_F(X)) = |X|$.

978–0–19–920527–1 12-Andras-Frank-c12-drv Andras Frank (Typeset by SPi, Publisher Services, India) 411 of 640 December 11, 2010 15:19

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

We derive the sufficiency from the Discrete separation theorem. Define functions b and p on ground-set E as follows. For $F \subseteq E$ let $b(F)$ denote the rank of the subset of S covered by F, and let $p(F)$ denote the number of nodes t in T for which each edge of G incident to t belongs to F. An easy consideration shows that b is submodular while p is supermodular. We claim that $b \geq p$. For a contradiction, suppose that there is a subset $F \subseteq E$ for which $b(F) < p(F)$ and assume that F is as small as possible. If an edge $e = ts$ with some $t \in T$ belongs to F, then all edges incident to t must belong to F, for otherwise we could have $p(F - e) = p(F) > b(F) \geq b(F - e)$, contradicting the choice of F. Therefore there is a subset $X \subseteq T$ so that F consists of all edges incident to a node of X. Hence $p(F) > b(F) = r(\Gamma(X)) \geq |X| = p(F)$, a contradiction.

By Theorem 12.2.1, there is a function $m : E \to \mathbf{Z}$ for which $p \leq \widetilde{m} \leq b$. We can choose m to be minimal. Since p is non-negative, so is m. Moreover, $b(e) \leq 1$ for every $e \in E$, implying that m is (0, 1)-valued. Let $F := \{e_E : m(e) = 1\}$. Then the inequality $\widetilde{m} \geq p$ implies that $d_F(t) \geq 1$ for every $t \in T$, and by the minimal choice of m, we have in fact, $d_F(t) = 1$. Furthermore, $\widetilde{m} \leq b$ implies that $d_F(s) \leq 1$ for every $s \in S$, moreover, the subset S_F of S consisting of nodes incident to F has rank at least $|F|$, that is, S_F is independent in the matroid. •

There are direct proofs of Rado's theorem and we shell present one in Section 13.1. With the proof presented above we wanted to indicate in a specific case how the general tool of submodular flows works. Welsh [377] found the following extension of Rado's theorem.

Theorem 12.3.2 (Welsh) *Let $G = (S, T; E)$ a bipartite graph and let b be a polymatroid function on S. There is a subset $F \subseteq E$ so that $d_F(t) = 1$ for every $t \in T$ and*

$$b(\Gamma_F(X)) \geq |X| \text{ for every } X \subseteq T \tag{12.19}$$

if and only if

$$b(\Gamma_E(X)) \geq |X| \text{ for every } X \subseteq T. \bullet \tag{12.20}$$

Problem 12.3.1 *Relying on the Discrete separation theorem, derive Theorem* 12.3.2.

12.3.2 An extension concerning supermodular functions

As another abstract extension of the Kőnig–Hall theorem, Lovász [263] proved the following.

Theorem 12.3.3 *Let $p : 2^S \to \mathbf{Z}_+$ be a positively intersecting supermodular function which is* **element-subadditive** *in the sense that*

$$p(X) + p(v) \geq p(X + s) \text{ holds whenever } X \subseteq S \text{ and } s \in S - X. \tag{12.21}$$

Suppose that $G = (S, T; E)$ is a simple bipartite graph covering p in the sense that $|\Gamma_E(X)| \geq p(X)$ for every $X \subseteq S$. If $G' = (S, T; F)$ is a minimal subgraph of G (with respect to inclusion) *which covers p, then $d_F(v) = p(v)$ for every $v \in S$.*

Proof. For brevity we use the notation $\gamma(X) := |\Gamma_F(X)|$. Refer to a subset $X \subseteq S$ as **tight** if $\gamma(X) = p(X)$.

Proposition 12.3.4 *Tight sets form an intersecting family.*

Proof. Let X and Y be two intersecting tight sets. By the submodularity of γ, we have

$$p(X) + p(Y) = \gamma(X) + \gamma(Y) \geq \gamma(X \cap Y) + \gamma(X \cup Y) \geq p(X \cap Y) + p(X \cup Y)$$

from which the estimations must hold with equality and, in particular, $\gamma(X \cap Y) = p(X \cap Y)$ and $\gamma(X \cup Y) = p(X \cup Y)$. •

Let s be an arbitrary element of S. Since p is non-negative, the minimality of G' implies that s belongs to a tight set. It follows from the proposition that the intersection $T(s)$ of tight sets containing s is also tight. We are going to prove that $\{s\}$ is tight.

Suppose indirectly that $d_F(s) > p(s)$ and let $Z \subseteq T$ denote the F-neighbours of s. Then $\{s\} \subset T(s)$ and $d_F(s) = |Z| > p(s)$. The minimality of G' implies for each node $t \in Z$ that there is a tight set T_t containing s for which t has no F-neighbour in $T_t - s \subseteq S$. Since $T(s) \subseteq T_t$, t has no F-neighbour in $T(s) - s$, either. Therefore $\Gamma_F(T(s) - s) = \Gamma_F(T_s) - Z$ and for $X := T(s) - s$ we have

$$p(X + s) = p(T(s)) = \gamma(T(s)) = \gamma(X) + |Z| \geq p(X) + |Z| > p(X) + p(s),$$

contradicting (12.21). • •

For example, if $m : S \to \mathbf{Z}_+$ is a function and $\ell \geq 0$ an integer, then the set-function p defined by $p(\emptyset) = 0$ and

$$p(X) := \widetilde{m}(X) + \ell \text{ for } \emptyset \subset X \subseteq S \tag{12.22}$$

fulfils the property required in the theorem. When $m \equiv 1$ and $\ell = 0$, one is back at the Kőnig–Hall theorem. When $m \equiv 1$ and $\ell = 1$, we obtain Theorem 2.4.13 that was a characterization of bipartite graphs admitting a forest whose degree is two at every node in S.

We mention that an extension of Theorem 12.3.3 in which 12.21 is not assumed will be exhibited in Theorem 17.1.24.

Problem 12.3.2 *Show that the statement of Theorem* 12.3.3 *fails to hold when* (12.21) *is dropped from the assumptions. Figure out a necessary condition for this case.* (See Theorem 17.1.24.)

13
Matroid optimization

We continue our investigation by exploring matroids, the simplest submodular framework for which TDI-ness results can be formulated. Actually, in Section 5.5, we have already verified the correctness of the greedy algorithm for matroids which gave rise to a total dual integrality (TDI) description of the convex hull of independent sets of a matroid. Those results however do not cover such basic problems as optimization over bipartite matchings or arborescences. It was major revelation by Edmonds that these graph problems can be treated uniformly in terms of *two* matroids. He developed fundamental results and algorithms concerning the intersection of two matroids. Among these are his matroid intersection theorem, a TDI description of the common independent sets of two matroids, and an algorithm for computing a maximum weight common basis of two matroids. In addition to network flows, matroid intersections form one of the most important cornerstones of semimodular optimization.

13.1 Unweighted matroid intersection

A precursor of Edmonds' (unweighted) matroid intersection theorem is a fundamental result of Rado [326] which was already formulated and derived in Section 12.3 from the Discrete separation theorem. Here we exhibit a direct proof.

13.1.1 Rado's theorem

Let $G = (S, T; E)$ be a simple bipartite graph and M a matroid on S. They determine a matroid M_E on E in which a subset F is independent if $d_F(s) \leq 1$ for every $s \in S$ and the subset of S covered by F is independent in M. Recall that M_E can be interpreted as a matroid that arises from M by parallel multiplication.

Theorem 13.1.1 (Rado [326]) *Let $M = (S, r)$ be a matroid and $G = (S, T; E)$ a bipartite graph. G has an M_E-independent matching covering T if and only if the following Rado condition holds:*

$$r(\Gamma_G(X)) \geq |X| \text{ for every } X \subseteq T. \bullet \tag{13.1}$$

Rather than proving directly Theorem 13.1.1, we exhibit the proof of the following slight extension.

Theorem 13.1.2 (Extended Rado) *Let $M = (S, r)$ be a matroid, $G = (S, T; E)$ a bipartite graph, and $z : T \to \mathbf{Z}_+$ a non-negative integer-valued function. There is an M_E-independent subset $F \subseteq E$ for which $d_F(v) = z(v)$ for every $v \in T$ if and only if*

$$r(\Gamma_G(X)) \geq \widetilde{z}(X) \text{ for every } X \subseteq T. \tag{13.2}$$

Proof. To see necessity, suppose that $F \subseteq E$ is a requested subset. Then $r(\Gamma_G(X)) \geq r(\Gamma_F(X)) = \widetilde{z}(X)$.

For proving sufficiency we use induction on $|T|$. Since the statement is clear for $|T| \leq 1$, we assume that $|T| \geq 2$ and that the result holds for smaller cases.

Case 1 *There is a proper non-empty subset T' of T for which* (13.2) *holds with equality, that is, $r(S') = \widetilde{z}(T')$, where $S' := \Gamma_G(T')$.* Let $T'' := T - T'$, $S'' := S - S'$. Furthermore, let $G' = (S', T'; E')$ be the subgraph of G induced by $S' \cup T'$ and let $G'' = (S'', T''; E'')$ be the subgraph of G induced by $S'' \cup T''$. Let M' be the submatroid of $M|S'$ and M'' the matroid arising from M by contracing T'. We abbreviate the functions $\Gamma_{G'}$ and $\Gamma_{G''}$ to Γ' and Γ'', respectively.

By induction there is an $M_{E'}$-independent subset F' of G' for which $d_F(v) = z(v)$ for every $v \in T'$. We claim that, with respect to M'', condition (13.2) holds in G''. Indeed, let $X := T' \cup X''$ for every subset $X'' \subseteq T''$. Then $\Gamma_G(X) = S' \cup \Gamma''(X'')$. Since X satisfies (13.2), we get $r''(\Gamma''(X'')) = r(\Gamma''(X'') \cup S') - r(S') = r(\Gamma_G(X)) - r(S') \geq \widetilde{z}(X) - \widetilde{z}(T') = \widetilde{z}(X'')$. By induction, there is an $M''_{E''}$-independent subset F'' of edges in G'' for which $d_F(v) = z(v)$ for every $v \in T''$. But then $F := F' \cup F''$ is an M_E-independent subset for which $d_F(v) = z(v)$ for every $v \in T$.

Case 2 *For every proper non-empty subset X of T* (13.2) *holds with strict inequality.* Let ab $(a \in S, b \in T)$ be an edge of G for which $r(a) = 1$ and $z(b) \geq 1$. Let M' be the matroid arising from M by contracting a, let $G' = G - \{a\}$, and z' be a function on T arising from z by reducing the value $z(b)$ by one. We claim that G', M', and z' meet (13.2). Indeed, $r'(\Gamma'(X)) = r(\Gamma'(X) + a) - 1 \geq r(\Gamma(X)) - 1 \geq \widetilde{z}(X) \geq \widetilde{z}'(X)$ for each non-empty set $X \subset T$ while for $X = T$ one has $r'(\Gamma'(X)) = r(\Gamma'(X) + a) - 1 \geq r(\Gamma(X)) - 1 \geq \widetilde{z}(X) - 1 \geq \widetilde{z}'(X)$.

By induction there is an $M_{E'}$-independent subset F' of edges of G' for which $d_{F'}(v) = z'(v)$ for every $v \in T$. But then $F' + ab$ satisfies the requirements of the theorem. •

Analogously to Hall's theorem, Theorem 13.1.1 can also be formulated in an equivalent form.

Theorem 13.1.3 (Rado) *Let $M = (S, r)$ be a matroid and $\mathcal{T} := \{T_1, T_2, \ldots, T_k\}$ a family of k subsets of S. It is possible to select an element from each T_i in such a way that the k selected elements are distinct and form an independent set of M* (or concisely, $\mathcal{T}$ has an M-indpendent system of distinct representatives) *if and only if the union of every j members $(1 \leq j \leq k)$ of $\mathcal{T}$ is of rank at least j.*

Proof. Let $G = (S, T; E)$ denote the bipartite graph associated with the hypergraph $(S, \mathcal{T})$. The theorem follows by applying Theorem 13.1.1 to G and M. •

The condition in Theorem 13.1.3 is also called the Rado condition. Theorem 13.1.2 can also be equivalently rephrased in terms of subsets.

Theorem 13.1.4 (extended Rado) *Let $M = (S, r)$ be a matroid, $\mathcal{T} := \{T_1, T_2, \ldots, T_k\}$ a family of k subsets of S, and let $z_1, \ldots, z_k$ be non-negative integers. It is possible to select z_i elements from each T_i in such a way that the selected $\sum_i z_i$ elements are distinct and form an independent set of M if and only if*

$$r(T_{i_1} \cup \cdots \cup T_{i_j}) \geq z_{i_1} + \cdots + z_{i_j} \tag{13.3}$$

for every choice $\{i_1, \ldots, i_j\} \subseteq \{1, 2, \ldots, k\}$ of indices. •

Problem 13.1.1 *Derive Theorem* 13.1.1 *from its special case when the degree of each node in S is exactly one. Show that Theorem* 13.1.3 *follows from its special case when $\mathcal{T}$ is a partition.*

Defect form of Rado's theorem

How large can an M_E-independent matching be when the entire T cannot be covered? The content of the next result of Perfect [318] is that the answer depends only on the extent of the violation of (13.1) which is defined by the quantity $\Delta := \max\{|X| - r(\Gamma(X)) : X \subseteq T\}$.

Theorem 13.1.5 (Rado, defect form) *Let $G = (S, T; E)$ be a bipartite graph and M a matroid on S. The maximum cardinality of an M-independent matching is equal to*

$$\mu := \min_{X \subseteq T} \{r(\Gamma(X)) + |T - X|\}. \tag{13.4}$$

Proof. Obviously, an M-independent matching leaves exposed at least Δ elements of T, and hence its cardinality is at most $|T| - \Delta = \mu$, from which max $\leq$ min follows.

To see the reverse direction, extend G by adding a set S' of Δ new nodes and connecting each of them with all elements of T. Let M' be the matroid on $S \cup S'$ arising as the direct sum of M and the free matroid on S'. It follows from this construction that the resulting graph G' and matroid M' satisfy (13.1). By Rado's theorem there is an M'-independent matching of G' covering T. It has at most Δ new edges whose removal results in an M-independent matching of G having at least $|T| - \Delta = \mu$ edges. •

The rank function of homomorphic image

In Section 5.4, we introduced the operation of a homomorphic image $\varphi(M)$ of a matroid M. Now we derive the rank function of $\varphi(M)$ with the help of Rado's theorem.

Theorem 13.1.6 (Nash-Williams [306]) *The rank function r_φ of the homomorphic image $\varphi(M) = (T, r_\varphi)$ of a matroid $M = (S, r)$ is given by the following formula.*

$$r_\varphi(Z) = \min\{r(\varphi^-(X)) + |Z - X| : X \subseteq Z\} \tag{13.5}$$

for $Z \subseteq T$.

Proof. It suffices to prove the formula for $Z = T$. Let $G = (S, T; E)$ denote the bipartite graph in which $st \in E$ if $t = \varphi(s)$. Then $\varphi^-(X) = \Gamma(X)$ and hence $r_\varphi(T) = \min\{r(\varphi^-(X)) + |T - X| : X \subseteq T\}$ is equivalent to (13.4) and thus Theorem 13.1.5 applies. •

As a corollary, we immediately obtain the following characterization of independence in the image.

Theorem 13.1.7 *A subset $I \subseteq T$ is independent in the homomorphic image $\varphi(M)$ of $M = (S, r)$ if and only if*

$$r(\varphi^{-}(X)) \geq |X| \text{ for every } X \subseteq I. \bullet \tag{13.6}$$

Polymatroidal sets from matroids

In order to extend the notion of independent sets of a matroid to integral vectors, we introduced the concept of polymatroidal sets in Subsection 5.5.3 of Part I and exhibited in Theorem 5.5.17, without a proof, a possible way to construct polymatroidal sets from matroids. Let $M = (S, \mathcal{F})$ be a matroid and $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ a partition of its ground-set S into non-empty sets. Let $T = \{t_1, t_2, \ldots, t_k\}$ be a set whose elements correspond to the members of $\mathcal{T}$.

Theorem 13.1.8 *The set $\{z \in \mathbf{Z}^T : z(t_i) = |F \cap T_i| \text{ for some } F \in \mathcal{F},\ i = 1, \ldots, k\}$ of integral vectors is polymatroidal.*

Proof. Consider the aggregate $b : 2^T \to \mathbf{Z}^+$ of the rank function r of M (with respect to partition $\mathcal{T}$). As a direct consequence of Theorem 13.1.4, we conclude that P is equal to the polymatroidal set $P' := \{z \in \mathbf{Z}^S : z \geq 0, \widetilde{z}(X) \leq b(X)$ for every $X \subseteq S\}$. $\bullet$

13.1.2 Edmonds' intersection theorem

A central result of combinatorial optimization is Edmonds' matroid intersection theorem [80].

Theorem 13.1.9 (Edmonds' intersection theorem) *Given two matroids $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ on a common ground-set S, the maximum cardinality of a common independent set of M_1 and M_2 is equal to*

$$\min_{X \subseteq S} \{r_1(X) + r_2(S - X)\}. \tag{13.7}$$

Edmonds' theorem is sometimes formulated in the following equivalent form.

Theorem 13.1.10 *There is a k-element common independent set of matroids M_1 and M_2 if and only if*

$$r_1(X_1) + r_2(X_2) \geq k \tag{13.8}$$

holds for every bipartition $\{X_1, X_2\}$ of S.

Proof. The inequality (13.8) is necessary since, for a common independent set F and a bipartition $\{X_1, X_2\}$ of S, one has $r_1(X_1) \geq |X_1 \cap F|$ and $r_2(X_2) \geq |X_2 \cap F|$ from which $r_1(X_1) + r_2(X_2) \geq |X_1 \cap F| + |X_2 \cap F| = |F|$ follows. This also implies the max $\leq$ min direction in Theorem 13.1.10.

We prove algorithmically the non-trivial direction max $\geq$ min .

Edmonds' matroid intersection algorithm

We can assume that both matroids are given by an oracle which tells for any input independent set I and an element $s \in S - I$ whether $I + s$ is independent or not, and in the latter case, the oracle returns the fundamental circuit of s belonging to I.

The following useful lemma on simultaneous exchange has already been posed in Problem 5.3.1.

Lemma 13.1.11 *Let $x_1, x_2, \ldots, x_\ell$ be some elements of an independent set I of a matroid M on S, and let $y_1, y_2, \ldots, y_\ell$ be elements of $S - I$ such that $I + y_i$ is dependent for each $i = 1, \ldots, \ell$. Suppose that each x_i belongs to the fundamental circuit $C(I, y_i)$ of y_i but $x_h \notin C(I, y_j)$ whenever $h > j$. Then $I - \{x_1 \ldots, x_\ell\} \cup \{y_1, \ldots, y_\ell\}$ is independent.*

Proof. Induction on ℓ. For $\ell = 1$, the statement is evident since one element can always be exchanged with any element of its fundamental circuit. Suppose that $\ell > 1$ and assume the truth of the statement for $\ell - 1$. By the hypothesis, the set $I' := I - x_\ell + y_\ell$ is independent, and $I' + y_i \quad (i \leq \ell - 1)$ includes the circuit $C(I, y_i)$, and hence, $C(I', y_i) = C(I, y_i)$. Therefore I' along with $\{y_1, \ldots, y_{\ell-1}\}$ and $\{x_1, \ldots, x_{\ell-1}\}$ meets the hypotheses of the lemma, and hence we are done by induction. •

In order to show the non-trivial direction max $\geq$ min of the theorem, one has to find a common independent set F and a subset $X \subseteq S$ for which the following optimality criteria hold.

$$r_1(X) = |X \cap F| \text{ and } r_2(S - X) = |(S - X) \cap F|. \tag{13.9}$$

The algorithm consists of at most $|S|$ phases. It starts with an arbitrary common independent set F which can, for example, be the empty set. One phase either returns a common independent set F' for which $|F'| = |F| + 1$, in which case the procedure is iterated with F', or else it returns a subset X satisfying the optimality criteria (13.9). In this case the algorithm terminates. Since there can be at most $n = |S|$ phases, such an algorithm verifies the theorem. We describe one phase.

Let $S_i := \{s \in S - F : F + s \in \mathcal{F}_i\} \quad (i = 1, 2)$, or informally, S_i consists of those elements of $S - F$ which can be individually added to F in matroid M_i without destroying independence. If there is a common element v of S_1 and S_2, then $F + v$ is a common independent set and the phase terminates. Therefore we assume that S_1 and S_2 are disjoint. Construct an auxiliary digraph on node-set S as follows. It has two types of edges. Let uv be an edge if $u \in S - (F \cup S_1)$, $v \in F$, and $v \in C_1(F, u)$. Similarly, let xy be an edge if $x \in F$, $y \in S - (F \cup S_2)$, and $x \in C_2(F, y)$. By applying breadth-first search, compute the set X of nodes reachable from S_2. There can be two cases.

Case 1 $S_1 \cap X = \emptyset$. Since no edge leaves X, the fundamental circuit $C_1(F, u)$ in M_1 of each element $u \in X - F$ is included completely in X, and similarly, $C_2(F, y)$ is included completely in $S - X$ for every element $y \in S - (F \cup X)$. This implies on that $F \cap X$ is a maximal M_1-independent subset in X or more formally, $r_1(X) = |F \cap X|$, and that $F - X$ is a maximal M_2-independent set in $S - X$ or more formally, $r_2(S - X) = |F - X|$. Hence (13.9) holds and F is a common independent set of maximum cardinality.

Case 2 $S_1 \cap X \neq \emptyset$. This means that there is a directed path from S_2 to S_1 and the path P found by BFS is a shortest one. Let $F' := F \ominus P$ be the symmetric difference of F and

P. Clearly, $|F'|$ is one bigger than $|F|$. We claim that F' is independent both in M_1 and in M_2. We only show that F' is independent in M_1 since the proof of its independence in M_2 is analogous.

Let $y_1, x_1, y_2, x_2, \dots, y_\ell, x_\ell, p_0$ denote the nodes of P (where $y_1 \in S_2$ and $p_0 \in S_1$) and let $I := F + p_0$. The hypothesis of Lemma 13.1.11 holds for matroid M_1 since P is a shortest path. Hence the lemma implies that $F' = I - \{x_1, \dots, x_\ell\} \cup \{y_1, \dots, y_\ell\}$ is independent in M_1. • •

Gabow and Xu [174] constructed an efficient and practical matroid intersection algorithm for linear matroids .

Exercises

13.1.2 *Show that one obtains an equivalent condition when* (13.8) *is required only for such bipartitions of S in which* (A) *X_1 is closed in M_1, or in which* (B) *X_1 contains no cut element of M_1.*

13.1.3 *Show that one obtains an equivalent condition when* (13.8) *is required for pairs of not necessarily disjoint sets X_1, X_2 whose union is S and X_i is closed in M_i* $(i = 1, 2)$.

Problems

13.1.4 *Show that* $\mathrm{cl}_i(F) \subseteq \mathrm{cl}_i(F')$ *for* $i = 1, 2$ *where* cl_i *denotes the closure operator in matroid M_i where F and F' are the common independent sets ocurring in the algorithm above.*

13.1.5 $(*)$ *Suppose that the common ground-set S of two matroids M_1 and M_2 can be partitioned into k bases of M_i for $i = 1, 2$. Prove that there is a common basis of M_1 and M_2. Show that S need not have a partition into k common bases.*

13.1.6 *Prove for a digraph D on n nodes that the maximum cardinality of a branching of D is equal to n minus the number of source-components of D.*

13.1.7 $(*)$ *Prove for matroids $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ that* $\max\{r_1(X) + r_2(X) - |X| : X \subseteq S\} = \min\{r_1(Y) + r_2(S - Y) : Y \subseteq S\}$.

Research problem 13.1.1 *Find a good characterization for the existence of k common bases of two matroids. Open even for $k = 2$.*

Conjecture 13.1.12 [Aharoni and Berger [1]] *Is it true that if $M_1, \dots, M_k$ are matroids on the same ground-set S and $\sum[r_i(S_i) : i = 1, \dots, k] \geq \ell(k-1)$ holds for every partition $\{S_1, \dots, S_k\}$ of S, then there exists a common independent set of size ℓ?*

13.1.3 Variations and applications

Bases intersections

A variation of the matroid intersection problem is as follows. We are given again two matroids M_1 and M_2 on S. A common independent subset $F \subseteq S$ is called a **basis intersection** if it arises as the intersection of a basis of M_1 and a basis of M_2.

Obviously, a basis intersection is a common independent set but the converse statement fails to hold even if both matroids are free since in this case each subset is a common independent set but S is the only basis intersection. Furthermore, a maximal common independent set is always a basis-intersection but the converse of this is not true either: if there are two disjoint bases of two loopless matroids, then the empty set is a basis-intersection but it is not a maximal common independent set.

Problem 13.1.9 *Based on the matroid intersection algorithm, devise a method for deciding whether a common independent set is a basis intersection.*

What can be said about the possible cardinalities of basis intersections? Let μ_{min} and μ_{max} denote their minimum and maximum. Since a common independent set is of maximum cardinality if and only if it is a largest basis intersection, μ_{max} can be computed with the intersection algorithm. With a simple trick, μ_{min} can also be computed. To this end, consider the dual matroid M_2^* of M_2 and observe that the intersection of an M_1-basis B_1 and an M_2-basis B_2 is a basis intersection of minimum cardinality if and only if the basis-intersection $B_1 \cap B_2^*$ of the M_2^*-basis $B_2^* := S - B_2$ is of maximum cardinality with respect to matroids M_1 and M_2^*. Therefore we compute first a largest common independent set F' of M_1 and M_2^*. Second, extend F' to an M_1-basis B_1 and to an M_2^*-basis B_2^*. Then $F' = B_1 \cap B_2^*$ and thus $F := B_1 - F'$ is a smallest basis intersection of M_1 and M_2, namely, $F = B_1 \cap B_2$ where $B_2 := S - B_2^*$.

Theorem 13.1.13 *If μ_{min} and μ_{max} denote the cardinalities of the smallest and the largest basis intersection of matroids M_1 and M_2, then there is a j-element basis intersection for every value of j with $\mu_{min} \leq j \leq \mu_{max}$.*

Proof. Let $B_1 \cap B_2$ be a smallest and $B_1' \cap B_2'$ a largest basis intersection. In matroid M_1, we can arrive at B_1' from B_1 by exchanging the elements of $B_1' - B_1$ one by one with the elements of $B_1 - B_1'$ in such a way that each intermediate set is an M_1-basis. Similarly, we can arrive at B_2' from B_2 through M_2-bases. Since an exchange can change the cardinality of the current basis intersection by one, we obtain a j-element basis-intersection for all value of j between μ_{min} and μ_{max}. •

Strongly independent matchings of bipartite graphs

Kőnig's theorem (Theorem 2.4.1) can easily be derived from Edmonds' matroid intersection theorem. Brualdi [38] observed that Edmonds' theorem immediately implies a formulation which includes the theorems of Kőnig, Rado, and Edmonds as special cases. Let us be given a bipartite graph with a matroid on both of its colour classes. We call a matching **strongly independent** if it covers an independent set of nodes in both matroids.

Theorem 13.1.14 (Brualdi) *Let $G = (S, T; E)$ be a bipartite graph endowed with matroids M_1 and M_2 on ground-sets S and T, respectively. The largest cardinality of a strongly independent matching of G is equal to*

$$\min\{r_1(X) + r_2(Y) :\ X \subseteq S,\ Y \subseteq T,\ X \cup Y \text{ covers each edge of } G\}. \qquad (13.10)$$

Proof. Define a matroid M_1' on edge-set E in such a way that F is independent if it covers each node in S at most once and the set of covered nodes in S is independent in M_1. This is clearly a matroid which can be considered as one obtained from M_1 by parallel multiplication, namely, by replacing each element s of S with $d_G(s)$ parallel copies. Define analogously M_2' from M_2. Obviously, a subset F of edges is a strongly independent matching if and only if F is a common independent set of M_1' and M_2'. From a variant of Edmonds' intersection theorem, the maximum cardinality of common independent sets is the minimum of

$$r_1'(E_1) + r_2'(E_2) \tag{13.11}$$

where $E_1 \cup E_2 = E$, E_1 is closed in M_1' and E_2 is closed in M_2'. Every closed set E_1 of M_1' is obtained from a subset X_1 of S by taking all edges having one end-node in X_1, and then $r_1'(E_1) = r(X_1)$. Similarly, a closed set E_2 of M_2' is the subset of edges having one end-node of a subset $X_2 \subseteq T$. It follows that the minima in (13.10) and in (13.11) are the same. •

When $|S| = |T|$ and G consists of a single perfect matching, we are back at Edmonds' theorem. For a general bipartite graph, when one of the two matroids is free, we arrive at Rado's theorem (Theorem 13.1.1) or even at its defect form given in Theorem 13.1.5. When both matroids are free, Kőnig's theorem is obtained.

Exercise 13.1.10 *Derive Theorem* 13.1.2 *from the Intersection theorem.*

Induced independent sets

Let $M = (S, r)$ be a matroid and assume that the elements of the ground-set S are arranged into disjoint couples $(s_1, \bar{s}_1), \ldots, (s_k, \bar{s}_k)$ where $|S| = 2k$. The intersection theorem characterizes the maximum cardinality of an independent set containing at most one element of each couple. In the following related problem we call a set F **induced independent** or i-independent for short if F is independent in M and $s_i \in F$ implies $\bar{s}_i \in F$. How large can an i-independent set be? Let $S_1 := \{s_1, \ldots, s_k\}$ and $S_2 := S - S_1$. For a subset A of S_1, let $\bar{A} := \{\bar{s} : s \in A\}$. (In particular, $S_2 = \bar{S}_1$. Note that $\bar{A}$ is *not* the complement). The following result is taken from a paper of Fleiner, Frank, and Iwata [100].

Theorem 13.1.15 *For a matroid M and couples $(s_1, \bar{s}_1), \ldots, (s_k, \bar{s}_k)$ given above,*

$$\max\{|F| : F \textit{ i-independent}\} = \min\{r(S - A) + r(\bar{A}) : A \subseteq S_1\}. \tag{13.12}$$

Proof. An i-independent set F can have at most $r(S - A)$ elements from $S - A$ and at most $r(\bar{A})$ elements from A and hence $|F| \le r(S - A) + r(\bar{A})$, from which we have max $\le$ min.

To see the other direction, one has to find an i-independent set and a subset $A \subseteq S_1$ for which $|F| = r(\bar{A}) + r(S - A)$. Define two matroids on S_1. Let $M_1 := M \cdot S_1$ be a matroid arising from M by contracting S_2, and let M_2 be a matroid in which a subset A is independent if $\bar{A}$ is independent in M (that is, M_2 is the submatroid $M|S_2$ copied to S_1).

Let T be a largest common independent set of M_1 and M_2. Then $T \in \mathcal{F}_2$ implies that $\bar{T}$ is independent in M. $\bar{T}$ can be extended with elements of S_2 to a maximal (of rank $r(S_2)$) M-independent set $B \subseteq S_2$. Since T is independent in M_1, we get that $F := T \cup B$ is independent in M and hence F is i-independent by the construction.

By the Intersection theorem, there is a subset $A \subseteq S_1$ for which $|T| = r_2(A) + r_1(S_1 - A)$. We have $r_2(A) = r(\bar{A})$ and $r_1(S_1 - A) = r((S_1 - A) \cup S_2) - r(S_2) =$ r(S-A) $- r(S_2)$ from which $|F| = |B| + |T| = r(S_2) + [r(\bar{A}) + r(S - A) - r(S_2)] = r(\bar{A}) +$ r(S-A), as required. •

Remark 13.1.1 *In light of Theorem 13.1.15, one naturally considers the problem when a maximum independent set is needed which contains either none or both elements of each couple. This is the famous* **matroid parity** *or* **matroid matching** *problem which is known to be intractable for general matroids but tractable for linear matroids, as was shown by Lovász [271]. A complex and difficult theory has been developed for matroid parity which is not even touched on in the present book.*

Partition-bounded bases

Let M be matroid on S and $\mathcal{T}$ a partition of S. Furthermore, let us be given two functions $f : \mathcal{T} \to \mathbf{Z}_+$ and $g : \mathcal{T} \to \mathbf{Z}_+$ for which $f(X) \leq g(X) \leq |X|$ for each $X \in \mathcal{T}$. The partition-bounded basis problem shows up in several applications and it consists of finding an M-basis B for which $f(X) \leq |B \cap X| \leq g(X)$ for every $X \in \mathcal{T}$. In order to formulate the results concisely, let $\mathcal{T}^*$ denote the family of sets which arise as the union of some members of $\mathcal{T}$. We extend any function z on $\mathcal{T}$ to $\mathcal{T}^*$ by letting $\widetilde{z}(T_{i_1} \cup \cdots \cup T_{i_j}) := z(T_{i_1}) + \cdots + z(T_{i_j})$. The problem has already been solved for the special case when $f \equiv g$, meaning that exact values are prescribed for the intersections. In this case, Theorem 13.1.4 implies the following.

Theorem 13.1.16 *Let $\mathcal{T}$ be a partition of the ground-set S of a matroid M and let $z : \mathcal{T} \to \mathbf{Z}_+$ be a function. There is a basis B of M for which $|B \cap X| = z(X)$ for every $X \in \mathcal{T}$ if and only if $\widetilde{z}(S) = r(S)$ and $\widetilde{z}(X^*) \leq r(X^*)$ for every $X^* \in \mathcal{T}^*$.* •

The general result is as follows.

Theorem 13.1.17 *Let $\mathcal{T}$ be a partition of S endowed with two functions $f : \mathcal{T} \to \mathbf{Z}_+$ and $g : \mathcal{T} \to \mathbf{Z}_+$ for which $f(X) \leq g(X) \leq |X|$ for each $X \in \mathcal{T}$. A matroid M* (of rank function r and co-rank function t) *has a basis B for which*

(A) $|X \cap B| \leq g(X)$ *for every $X \in \mathcal{T}$ if and only if*

$$\widetilde{g}(Z) \geq t(Z) \text{ (or equivalently, } \widetilde{g}(Z) + r(S - Z) \geq r(S)) \text{ for every } Z \in \mathcal{T}^*, \tag{13.13}$$

(B) $|X \cap B| \geq f(X)$ *for every $X \in \mathcal{T}$ if and only if*

$$\widetilde{f}(Z) \leq r(Z) \text{ for every } Z \in \mathcal{T}^*, \tag{13.14}$$

(C) $f(X) \leq |X \cap B| \leq g(X)$ *for every $X \in \mathcal{T}$ if and only if there is a basis satisfying* (A) *and there is a basis satisfying* (B).

Proof. If a given basis B exists in Part (A), then $\widetilde{g}(Z) \geq |B \cap Z| \geq t(Z)$ and hence (13.13) is indeed necessary. For proving sufficiency, let M_1 be a partition matroid defined by partition $\mathcal{T}$ and the upper bound function g, and let $M_2 := M$. It suffices to show that there is a common independent set of $k := r(S)$ elements.

If no such a set exists, then the Intersection theorem ensures a subset violating $r_1(Z) + r_2(S - Z) \geq k$. Suppose that there is a member T of $\mathcal{T}$ for which $T \cap Z \neq \emptyset$ and $T - Z \neq \emptyset$. If $|T \cap Z| \leq g(T)$, then $Z' := Z - T$ is also violating since $r_1(Z') = r_1(Z) - |T \cap Z|$ and $r_2(S - Z') \leq r_2(S - Z) + |T \cap Z|$. If $|T \cap Z| > g(T)$, then $Z'' := Z \cup T$ is also violating since $r_1(Z') = r_1(Z)$ and $r_2(S - Z') \leq r_2(S - Z)$. Therefore we can assume that Z is chosen in such a way that either $|T \cap Z| = \emptyset$ or $T \subseteq Z$ holds for each $T \in \mathcal{T}$. But then $g(Z) + r(S - Z) = r_1(Z) + r_2(S - Z) < k$, contradicting (13.13). Note that the matroid intersection algorithm can be used to compute a requested basis B or a violating set Z.

To prove Part (B), observe that the given basis exists if and only if there is a basis B^* of the dual matroid M^* for which $|X \cap B^*| \leq g^*(X)$ for every $X \in \mathcal{T}$ where $g^*(X) := |X| - f(X)$. Therefore Part (B) follows from (in fact, equivalent to) Part (A) once we observe that condition (13.13), when applied to M^* and g^* is equivalent to (13.14). But the equivalence of $\widetilde{g}^*(Z) \geq t^*(Z)$ and $\widetilde{f}(Z) \leq t(Z)$ is indeed clear since $\widetilde{g}^*(Z) := |Z| - \widetilde{f}(Z)$ and $t^*(Z) = |Z| - r(Z)$.

To see Part (C), we use induction on the sum $\sum[g(T) - f(T) : T \in \mathcal{T}]$. If this number is zero, then we are back at Theorem 13.1.16. Suppose now that $f(T) < g(T)$ for some $T \in \mathcal{T}$. If the value $f(T)$ can be increased by 1 without destroying (13.14), then by induction we are done. Therefore such a modification is not possible, and hence there is a subset $X \in \mathcal{T}^*$ for which $T \subseteq X$ and $\widetilde{f}(X) = r(X)$. We claim that there is no subset $Y \in \mathcal{T}^*$ for which $T \subseteq Y$ and $\widetilde{g}(Y) = t(Y)$. Indeed, for such a Y we would have $\widetilde{f}(X) - \widetilde{g}(Y) = r(X) - t(Y) \geq r(X - Y) - t(Y - X) \geq \widetilde{f}(X - Y) - \widetilde{g}(Y - X) = [\widetilde{f}(X) - \widetilde{f}(X \cap Y)] - [\widetilde{g}(Y) - \widetilde{g}(X \cap Y)] = \widetilde{f}(X) - \widetilde{g}(Y) + [\widetilde{g}(X \cap Y) - \widetilde{f}(X \cap Y)] > \widetilde{f}(X) - \widetilde{g}(Y)$ where $\widetilde{g}(X \cap Y) - \widetilde{f}(X \cap Y) > 0$ holds since $T \subseteq X \cap Y$ and $f(T) < g(T)$. Therefore the value $g(T)$ can be decreased by 1 without violating (13.13), and the theorem follows by induction. •

This inductive proof for Part (C) is pretty simple but not algorithmic. The following alternative proof relies on Edmonds' matroid intersection algorithm and hence provides an efficient way to compute a partition-bounded basis of M.

Algorithmic proof for Part (C) Assume that both (13.13) and (13.14) hold. Then there is an M-independent set F_0 for which $|F_0 \cap X| = f(X)$ for every $X \in \mathcal{T}$. By Part (B), such an F_0 exists. Apply Edmonds' matroid intersection algorithm to the partition matroid M_1 determined by g and to $M_2 := M$. Since (A) implies that these two matroids have a common independent set B of $k := r(S)$ elements the algorithm finds such a B when it starts with F_0. A simple property of Edmonds' algorithm is that, at a change of a common independent set F to a one larger F', the closure of F in both matroids are included in the closure of F'. That is, $\mathrm{cl}_i(F) \subseteq \mathrm{cl}_i(F')$ for $i = 1, 2$ (a property formulated in Problem 13.1.4). Applying this observation to the partition matroid M_1, we conclude that $|F' \cap X| \geq |F \cap X|$ for every $X \in \mathcal{T}$. Therefore the final common independent set also satisfies $|B \cap X| \geq |F_0 \cap X| = f(X)$ for every $X \in \mathcal{T}$. •

Problem 13.1.11 *Derive Theorem* 9.1.16 *on degree-constrained spanning trees.*

13.2 Weighted matroid intersection

In Part I, we saw how the greedy algorithm computed a maximum-weight basis of a single matroid with respect to a weight function c. In the preceding section, Edmonds' min-max result was derived for the maximum cardinality of a common independent set of two matroids. The question naturally emerges: what can be said about the maximum weight of a common independent set? Actually, there are some closely related variations of this question. One may be interested in a heaviest common basis or, for any integer i, in a heaviest i-element common independent set. Our first goal is to exhibit a relatively short proof for a min-max theorem on the maximum weight of a common basis in the case when the weight function is integer-valued. The approach can be considered a straight extension of Egerváry's original proof for Theorem 3.3.1 on maximum-weight perfect matchings (see Remark 3.3.1). Second, a strongly polynomial algorithm will be given which, incidentally, gives rise to a proof of the min-max theorem for arbitrary weight functions.

Preparations

Claim 13.2.1 *For any weight function $c : S \to \mathbf{R}$, the maximum c-weight bases of a matroid M form a matroid M_c.*

Proof. To see the exchange axiom for bases, let B_1 and B_2 be two bases of M with maximum weight, and let $x \in B_1 - B_2$. By the Theorem 5.3.3 on symmetric basis exchange, there is a $y \in B_2 - B_1$ for which both $B'_1 := B_1 - x + y$ and $B'_2 := B_2 - y + x$ are bases of M. Hence we must have $c(x) = c(y)$, and hence B'_1 and B'_2 are also bases of maximum weight. •

Let us denote the rank function of matroid M_c by r_c. The maximum c-weight of a basis of M was called the vector-rank of c, denoted by $\hat{r}(c)$. Theorem 5.5.5 included the following simple formula:

$$\hat{r}(c) = r(S)c(s_n) + \sum_{i=1}^{n-1} r(S_i)[c(s_i) - c(s_{i+1})], \tag{13.15}$$

where $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$ and $S_i := \{s_1, \ldots, s_i\}$.

Claim 13.2.2 *Let $c : S \to \mathbf{Z}_+$ be integer-valued and suppose that $Z \subset S$ is a non-empty set for which $c(x) > c(y)$ whenever $x \in Z$ and $y \in S - Z$. Then*

$$\hat{r}(c - \underline{\chi}_Z) = \hat{r}(c) - r(Z). \tag{13.16}$$

Proof. By the hypothesis, Z must be one of the sets S_i in formula (13.15) for which $c(s_i) > c(s_{i+1})$. Hence (13.15) immediately imples (13.16). •

The following lemma tells us how the vector-rank may change when c is increased or decreased by the incidence vector $\underline{\chi}_Z$ of an arbitrary subset Z of S.

Lemma 13.2.3 *Let $M = (S, r)$ be a matroid. For an integral vector $c : S \to \mathbf{Z}$ and for a subset $Z \subseteq S$, let $c^+ := c + \underline{\chi}_Z$ and $c^- := c - \underline{\chi}_Z$. Then*

$$\hat{r}(c^+) = \hat{r}(c) + r_c(Z), \tag{13.17}$$

978–0–19–920527–1 13-Andras-Frank-c13-drv Andras Frank (Typeset by SPi, Publisher Services, India) 424 of 640 December 11, 2010 15:20
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

and

$$\hat{r}(c^-) = \hat{r}(c) - r(S) + r_c(S - Z). \tag{13.18}$$

Proof. To see the first identity, arrange the elements of S by non-increasing order of c in such a way that, with regard to this rule, the elements of Z are as early as possible. Since c is integer-valued, this ordering is also non-increasing with respect to c^+. Therefore the basis B provided by the greedy algorithm is of maximum c-weight and maximum c^+-weight, too. Then $r_c(Z) \geq |B \cap Z|$ but here we must have equality for otherwise there would exist another basis B' of maximum c-weight for which $|B' \cap Z| > |B \cap Z|$ and then $\widetilde{c}^+(B') = \widetilde{c}(B') + |Z \cap B'| > \widetilde{c}(B) + |Z \cap B| = \widetilde{c}^+(B)$, contradicting the assumption that B is of maximum c^+-weight. Therefore $r_c(Z) = |B \cap Z|$ and hence $\hat{r}(c^+) = \widetilde{c}^+(B) = \widetilde{c}(B) + |B \cap Z| = \hat{r}(c) + r_c(Z)$.

To prove the second identity, observe first that $r_c = r_{c'}$ holds for $c' := c - \underline{\chi}_S$. Apply then the first identity with c' in place of c and with $(S - Z)$ in place of Z. One gets $\hat{r}(c^-) = \hat{r}(c - \underline{\chi}_S + \underline{\chi}_{(S-Z)}) = \hat{r}(c') + r_{c'}(S - Z) = \hat{r}(c) - r(S) + r_c(S - Z)$. •

13.2.1 Common bases and independent sets of maximum weight

Let us start with the problem of heaviest common basis of two rank-k matroids M_1 and M_2 on a common ground-set S. We assume that they have a common basis. Let $c : S \to \mathbf{Z}$ be an integer-valued (!) weight function. We say that functions c_1 and c_2 form a **weight-splitting** of c if $c = c_1 + c_2$. The following result is due to Frank [116].

Theorem 13.2.4 (Weight-splitting theorem for matroid intersection) *The maximum c-weight of a common basis of matroids M_1 and M_2 is equal to*

$$\min \left\{ \hat{r}_1(c_1) + \hat{r}_2(c_2) : \ c_1 + c_2 = c, \ c_i \ \textit{integer-valued} \right\}. \tag{13.19}$$

Proof. For a common basis B and for a weight-splitting c_1, c_2 of c, one obviously has $\widetilde{c}(B) = \widetilde{c}_1(B) + \widetilde{c}_2(B) \leq \hat{r}_1(c_1) + \hat{r}_2(c_2)$ from which max $\leq$ min follows. This estimation shows that in order to prove the reverse inequality max $\geq$ min, one must find a weight-splitting $c = c_1 + c_2$ along with a common basis B of M_1 and M_2 in such a way that they satisfy the following optimality criterion.

$$B \text{ is a maximum } c_1\text{-weight basis of } M_1, \text{ and } B \text{ is a maximum } c_2\text{-weight basis of } M_2. \tag{13.20}$$

Select an integral weight-splitting $c = c_1 + c_2$ of c such that $\hat{r}_1(c_1) + \hat{r}_2(c_2)$ is as small as possible. (This minimum does exist since c_i is required to be integral and any common basis B provides a lower bound: $\widetilde{c}(B) \leq \hat{r}_1(c_1) + \hat{r}_2(c_2)$.) Let M'_i denote the matroid formed by the maximum c_i-weight bases of M_i $(i = 1, 2)$ and let r'_i denote the corresponding rank function.

Lemma 13.2.5 *The matroids M'_1 and M'_2 have a common independent set of k elements.*

Proof. If there is no k-element common independent set, then there is a subset $Z \subseteq S$ set for which

$$r'_1(Z) + r'_2(S - Z) < k \tag{13.21}$$

by Edmonds' matroid intersection theorem.

Let $c_1^+ := c_1 + \underline{\chi}_Z$ and $c_2^- := c_2 - \underline{\chi}_Z$. By applying first formula (13.17) to M_1 and to c_1, and then formula (13.18) to M_2 and to c_2, we obtain $\hat{r}_1\left(c_1^+\right) = \hat{r}_1(c_1) + r_1'(Z)$ and $\hat{r}_2\left(c_2^-\right) = \hat{r}_2(c_2) + r_2'(S-Z) - r_2(S) = \hat{r}_2(c_2) + r_2'(S-Z) - k$. These imply $\hat{r}_1\left(c_1^+\right) + \hat{r}_2\left(c_2^-\right) = \hat{r}_1(c_1) + \hat{r}_2(c_2) + r_1'(Z) + r_2'(S-Z) - k < \hat{r}_1(c_1) + \hat{r}_2(c_2)$, contradicting the minimal choice of c_1 and c_2. •

The common basis B ensured by the lemma satisfies the optimality criterion formulated in (13.20) and therefore the proof of the theorem is complete. • •

Corollary 13.2.6 *Suppose that there is a common basis of matroids M_1 and M_2. An arbitrary integer-valued weight function c admits an integral weight-splitting $c = c_1 + c_2$ and there is a common basis B of the two matroids such that B is a maximum c_1-weight basis of M_1 and B is a maximum c_2-weight basis of M_2. A common basis B is a maximum c-weight common basis if and only if c admits an integer-valued weight-splitting $c = c_1 + c_2$ satisfying* (13.20). •

Remark 13.2.1 Although Theorem 13.2.4 provides an elegant characterization for the maximum weight of a common basis of two matroids, it formally cannot be considered a good characterization since in principle it can be the case that some components of the optimal vectors c_i are so terribly big that their size is not polynomial in the input size of c. However, this danger can be avoided, and the content of the next result is that the optimal c_i's can be chosen of moderate size. In addition, the theorem will be useful in the proof of a theorem of Gröflin and Hoffman [195] (Theorem 13.2.23).

We refer to an integral function c on S as **Δ-narrow** if the difference of any two consecutive values of c is at most Δ. The **span** of c is the difference of its maximum and minimum values.

Theorem 13.2.7 *If the span of the weight function c in Theorem* 13.2.4 *is Δ, then c admits an optimal integral weight-splitting $c = c_1 + c_2$* (that is, $\hat{r}_1(c_1) + \hat{r}_2(c_2)$ is minimum) *for which both c_1 and c_2 are Δ-narrow.*

Proof. Let us choose an optimal weight-splitting $c = c_1 + c_2$ for which the sum of the spans of c_1 and c_2 is as small as possible. We are going to show that both c_1 and c_2 are Δ-narrow. Suppose indirectly that c_1, for example, is not Δ-narrow, which means that there is a subscript $h < t$ for which $\gamma_h - \gamma_{h+1} > \Delta$ where $\gamma_1 > \gamma_2 > \cdots > \gamma_t$ denote the distinct values of c_1. Let $Z_h := \{s : c_1(s) \geq \gamma_h\}$, $c_1' := c_1 - \underline{\chi}_{Z_h}$ and $c_2' := c - c_1'$. It follows from the definition that the span of c_1' is one less than that of c_1. By applying Lemma 13.2.2 to M_1, c_1, and Z_h, we get $\hat{r}_1\left(c_1'\right) = \hat{r}_1(c_1) - r_1(Z_h)$. For every pair of elements $x \in S - Z_h$, $y \in Z_h$ we have $c_1(y) \geq c_1(x) + \Delta + 1$. Furthermore the span of c is Δ from which $c(y) \leq c(x) + \Delta$ and hence $c_2(y) \leq c_2(x) - 1$. This implies that the span of c_2' is smaller than that of c_2. Moreover, we can apply Lemma 13.2.2 to M_2, c_2, and $Z := S - Z_h$ and formula (13.16) implies $\hat{r}_2\left(c_2'\right) = \hat{r}_2(c_2 + \underline{\chi}_{Z_h}) = \hat{r}_2(c_2 + \underline{\chi}_S - \underline{\chi}_{(S-Z_h)}) = r_2(S) + \hat{r}_2(c_2) - r_2(S - Z_h)$.

Since there is a common basis (of cardinality $r_2(S)$), we have $r_1(Z_h) + r_2(S - Z_h) - r_2(S) \geq 0$. Combining all of these, one gets $\hat{r}_1\left(c_1'\right) + \hat{r}_2\left(c_2'\right) = [\hat{r}_1(c_1) - r_1(Z_h)] + [\hat{r}_2(c_2) - r_2(S - Z_h) + r_2(S)] = \hat{r}_1(c_1) + \hat{r}_2(c_2) - [r_1(Z_h) + r_2(S - Z_h) - r_2(S)] \leq \hat{r}_1(c_1) + \hat{r}_2(c_2)$. By the optimal choice of c_1 and c_2, here equality must hold, and hence c_1' and c_2'

form another optimal weight-splitting of c in contradiction with the minimality of the sum of the spans of c_1 and c_2. •

Exercise 13.2.1 *Develop a min-max theorem for the maximum weight of a common independent set of i elements.*

Maximum-weight common independent sets

One may be interested in a common independent set of maximum weight. Since an element of negative weight does not play any role, we can assume that c is non-negative.

Theorem 13.2.8 *Let c be a non-negative integral weight function on the ground-set S of two matroids. The maximum weight of a common independent set is equal to*

$$\min\{\hat{r}_1(c_1) + \hat{r}_2(c_2) : c_1 + c_2 = c,\ c_1 \geq 0,\ c_2 \geq 0,\ c_i \text{ integral}\}. \tag{13.22}$$

Proof. Let F be a common independent set and let $c_1 \geq 0$, $c_2 \geq 0$ satisfy $c_1 + c_2 = c$. Then $\widetilde{c}(F) = \widetilde{c}_1(F) + \widetilde{c}_2(F) \leq \hat{r}_1(c_1) + \hat{r}_2(c_2)$ from which max $\leq$ min follows.

For the other direction, we will show that there is an integral weight-splitting $c = c_1 + c_2$ with non-negative functions c_i and there is a common independent set F for which F is a maximum c_i-weight independent set in both M_i for $i = 1, 2$.

We can assume that the c-weight of every element is strictly positive and that both matroids are loop-free. Let $R := \max\{r_1(S), r_2(S)\} + 1$ and let S' be a set of R elements which is disjoint from S. Let M_i^+ $(i = 1, 2)$ be two matroids arising in such a way that we take the R-shortening of the direct sum of M_i and the free matroid on S'. Extend c to $S \cup S'$ in such a way that its values on S' are identically 0. It can easily be checked that the rank of M_i^+ is R and an R-element set B is a basis of M_i^+ if and only if $B \cap S$ is independent in M_i. Clearly $\widetilde{c}(B) = \widetilde{c}(B \cap S)$ for such a B. (In particular S' is a basis of zero weight.) Hence the maximum weight of a common independent set of M_1 and M_2 is equal to the maximum weight of a common basis of M_1^+ and M_2^+.

By Corollary (13.2.6), there is a common basis B and there is an integral weight-splitting $c = c_1 + c_2$ for which B is a maximum c_i-weight basis of M_i^+ for $i = 1, 2$.

Claim 13.2.9 c_i $(i = 1, 2)$ *can be chosen in such a way that it is identically* 0 *on the elements of* S'.

Proof. First we show that the c_1-value of each element of S' is the same. By the choice of value R, $B \cap S'$ is non-empty. The set $S' - B$ cannot be empty either for if it were, then $|S'| = |B|$ implies $S' = B$ and hence we would have $\widetilde{c}(B) = 0$. This is, however, impossible since there is a common basis of positive weight due to the assumption that $c(s)$ is positive, $\{s\}$ is a common independent set and it can be extended to a common basis of M_1^+ and M_2^+.

Since neither $B \cap S'$ nor $S' - B$ is empty, it suffices to show that $c_1(x) = c_1(y)$ whenever $x \in B \cap S'$ and $y \in S' - B$. Since $B - x + y$ is also a common basis, we have $c_1(y) \leq c_1(x)$ and $c_2(y) \leq c_2(x)$. This and $c_1(x) + c_2(x) = 0 = c_1(y) + c_2(y)$ imply $c_1(y) = c_1(x)$.

Let α denote the common c_1-weights of the elements of S'. By decreasing these values by α and increasing the c_2-weights of the elements of S' by α, we obtain an optimal weight-splitting of c satisfying the requirement. •

Assuming that c_1 and c_2 are identically zero on S', we derive that $c_i(x) \geq 0$ for every $x \in S \cap B$. Indeed, as $B - x + y$ is also a basis of M_i^+ for any $y \in S' - B$, we infer $c_i(x) \geq c_i(y) = 0$.

Consider now an element $x \in S - B$ for which $c_1(x)$, say, is negative. Increase $c_1(x)$ to zero and decrease $c_2(x)$ to $c(x)$. B is a maximum-weight basis of M_2^+ with respect to the revised c_2' since c_2 has been reduced on an element not in B. Similarly, B is a maximum-weight basis of M_1^+, too, with respect to the revised c_1' since the c_1-weights of the elements of B are non-negative and the negative $c_1(x)$ has been increased to zero.

With such modifications we can ensure that c_i is non-negative. It follows that, with respect to the restriction $c_i|S$ of c_i to S, the $F := B \cap S$ is a maximum $c_i|S$-weight independent set of M_i for $i = 1, 2$. • •

Total dual integrality

We are now in a position to formulate Edmonds' original theorem on weighted matroid intersection [80, 82].

Theorem 13.2.10 *Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two matroids. The linear system*

$$\{x \geq 0,\ \widetilde{x}(Z) \leq r_1(Z),\ \widetilde{x}(Z) \leq r_2(Z) \text{ for every } Z \subseteq S\} \tag{13.23}$$

for $x \in \mathbf{R}^S$ is totally dually integral.

Proof. Let c be an integral weight function on S. We can assume that c is non-negative since if $c(v) < 0$ for some v, then the deletion of v does not change the common optimum value of the primal and the dual linear programs, furthermore a solution to the revised dual will be a solution to the original one. Consider the dual linear program

$$\min\left\{\sum[y_1(Z)r_1(Z) : Z \subseteq S] + \sum[y_2(Z)r_2(Z) : Z \subseteq S]\right\} \text{ subject to}$$

$$(y_1, y_2) \geq 0,\ \sum[y_1(Z) + y_2(Z) :\ s \in Z] \geq c(s) \text{ for every } s \in S. \tag{13.24}$$

By Theorem 13.2.8 there is an optimal integer-valued weight-splitting $c = c_1 + c_2$ where $c_i \geq 0$ and a common independent set F such that F is a maximum c_i-weight independent set in M_i for $i = 1, 2$.

Apply theorem 5.5.7 to M_i and c_i for $i = 1, 2$. There exist non-negative integer-valued dual optimal solutions y_1 and y_2 for which $\sum[y_i(Z) : s \in Z] \geq c_i(s)$ for $s \in S$ and $\widetilde{c}_i(F) = \sum[y_i(Z)r_i(Z) : Z \subseteq S]$. It follows that (y_1, y_2) form a solution to (13.24) and it is an optimal dual solution since $\widetilde{c}(F) = \widetilde{c}_1(F) + \widetilde{c}_2(F) = \sum[y_1(Z)r_1(Z) : Z \subseteq S] + \sum[y_2(Z)r_2(Z) : Z \subseteq S]$. •

We proved the TDI-ness of the linear system (13.23) from the weight-splitting theorem. But the converse derivation is also possible, and will be exhibited in a more general setting: a weight-splitting theorem for base-polyhedra (Theorem 16.1.8) will be obtained from the TDI-ness of a linear system describing the intersection of two base-polyhedra (Theorem 16.1.7).

By combining Theorems 4.1.26 and 13.2.10 we obtain the following.

Theorem 13.2.11 (Edmonds) *Suppose that the rank-r matroids M_1 and M_2 have a basis in common. The linear system*

$$\{x \geq 0,\ \widetilde{x}(Z) \leq r_1(Z),\ \widetilde{x}(Z) \leq r_2(Z) \text{ for every } Z \subset S,\ \widetilde{x}(S) = r\}$$

for $x \in \mathbf{R}^S$ is TDI. •

Corollary 13.2.12 *The convex hull of common independent sets of matroids M_1 and M_2 is the intersection of the independence polyhedra of matroids M_1 and M_2. Similarly, the convex hull of common bases is the intersection of the base-polyhedra of M_1 and M_2.* •

13.2.2 A weighted matroid-intersection algorithm

In [82] Edmonds developed an algorithm for the weighted matroid intersection problem. Both the optimality criteria and the algorithm itself use dual variables assigned to subsets of S. Here we exhibit another algorithm, developed by Frank [116], which is simpler as it uses weight-splittings rather than dual variables.

Let $M_1 = (S, \mathcal{F}_1)$ and $M_2 = (S, \mathcal{F}_2)$ be matroids and $c : S \to \mathbf{R}$ an arbitrary weight function. Where $\mathcal{H}$ is a family of subsets of S, we say a member $X \in \mathcal{H}$ to be c-maximal in $\mathcal{H}$ if $\widetilde{c}(X) \geq \widetilde{c}(Y)$ for every $Y \in \mathcal{H}$. Let $\mathcal{H}^k$ denote the subfamily of $\mathcal{H}$ consisting of k-element sets. In particular, the set of k-element common independent sets of the two matroids will be denoted by $\mathcal{F}_{12}^k$.

Preparations

In Section 5.5, we proved the following lemma with the help of the greedy algorithm.

Lemma 13.2.13 *A k-element independent set F of a matroid $M = (S, \mathcal{F})$ is c-maximal in $\mathcal{F}^k$ if and only if*

$$y \in S - F,\ F + y \notin \mathcal{F},\ x \in C(F, y) \text{ imply } c(y) \leq c(x), \tag{13.25}$$

and

$$y \in S - F,\ F + y \in \mathcal{F},\ x \in F \text{ imply } c(y) \leq c(x), \tag{13.26}$$

where $C(F, y)$ denotes the unique circuit in $F + y$. •

The following lemma generalizes Lemma 13.1.11

Lemma 13.2.14 *Let I be a k-element independent set of a matroid $M = (S, \mathcal{F})$ which is c-maximal in $\mathcal{F}^k$. Let $x_1, x_2, \ldots, x_\ell$ be elements of I and $y_1, y_2, \ldots, y_\ell$ elements of $S - I$ for which $I + y_i$ is dependent for each $i = 1, \ldots, \ell$. Suppose that $x_i \in C(I, y_i)$ and $c(x_i) = c(y_i)$ for $i = 1, \ldots, \ell$ and*

$$h > j \text{ and } c(x_h) = c(y_j) \text{ imply } x_h \notin C(I, y_j). \tag{13.27}$$

Then $I - \{x_1 \ldots, x_\ell\} \cup \{y_1, \ldots, y_\ell\}$ is also independent and c-maximal in $\mathcal{F}^k$.

Proof. Induction on ℓ. Since the case $\ell = 1$ is evident, we assume that $\ell \geq 2$ and that the lemma holds for smaller values. Let μ denote the minimum of the values $c(x_i) = c(y_i)$ and let h be the largest subscript for which $c(y_h) = \mu$.

978–0–19–920527–1 13-Andras-Frank-c13-drv Andras Frank (Typeset by SPi, Publisher Services, India) 429 of 640 December 11, 2010 15:20
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Claim 13.2.15 *$x_h \notin C(I, y_j)$ holds whenever $j \neq h$.*

Proof. Suppose indirectly that $x_h \in C(I, y_j)$ for some $j \neq h$. Since I is c-maximal in $\mathcal{F}^k$, we have $c(y_j) \leq c(x_h)$ from which $\mu \leq c(y_j) \leq c(x_h) = c(y_h) = \mu$, and hence equality must hold throughout. The hypothesis of the theorem implies $h < j$, contradicting the maximal choice of h. •

It follows that $I' := I - x_h + y_h$ is also a c-maximal member of $\mathcal{F}^k$ and Claim 13.2.15 implies that $C(I', y_j) = C(I, y_j)$ for every subscript j distinct from h. The lemma follows when the inductive statement is applied to I'. • •

The following claim is obvious.

Claim 13.2.16 *If $F \in \mathcal{F}^k_{12}$ and $c = c_1 + c_2$ is a weight-splitting of c for which F is c_i-maximal in $\mathcal{F}^k_i$ $(i = 1, 2)$, then F is c-maximal in $\mathcal{F}^k_{12}$.* •

The procedure

The algorithm computes a k-element common independent set F for each possible k and a series of weight-splittings $c = c_1 + c_2$ each of which satisfy the optimality criteria in Claim 13.2.16. In addition, the c_i's will be integer-valued when c itself is integer-valued. The procedure starts with $k = 0$ and increases the value of k one by one. It terminates when a bipartition $\{T, S - T\}$ of S is found for which $r_1(T) + r_2(S - T) = k$, showing that the current k-element independent set is of maximum cardinality. An essential feature of the procedure is that the current set $F \in \mathcal{F}^k_{12}$ is c-maximal in $\mathcal{F}^k_{12}$ for each k.

In order to describe one phase of the algorithm, suppose for an intermediate k that we have found a set $F \in \mathcal{F}^k_{12}$ and a weight-splitting $c = c_1 + c_2$ which satisfy the optimality criteria formulated in Claim 13.2.16. Starting from these, we construct a set $F' \in \mathcal{F}^{k+1}_{12}$ and a weight-splitting $c = c'_1 + c'_2$ which also satisfy the optimality criteria with $k + 1$ in place of k. For the initial case when $k = 0$, we can start with $F := \emptyset,\ c_1 := 0,\ c_2 := c$.

For $i = 1, 2$, let

$$m_i := \max\{c_i(y) :\ y \in S - F,\ F + y \in \mathcal{F}_i\} \tag{13.28}$$

and

$$Y_i := \{y \in S - F :\ F + y \in \mathcal{F}_i,\ c_i(y) = m_i\}. \tag{13.29}$$

(Note that Y_i is the set of those elements which would be added next to F by the greedy algorithm in M_i with respect to c_i.) Define a digraph $D = (S, A)$ as follows. There are two types of edges.

$$\text{Let } yx \text{ be an edge if } y \in S - F,\ F + y \notin \mathcal{F}_1,\ x \in C_1(F, y),\ c_1(x) = c_1(y) \tag{13.30}$$

and

$$\text{let } xy \text{ be an edge if } y \in S - F, F + y \notin \mathcal{F}_2, x \in C_2(F, y), c_2(x) = c_2(y). \tag{13.31}$$

With the help of a breadth-first search, compute the set T of nodes reachable in D from Y_2. There are two cases.

Case 1 *There is a path from Y_2 to Y_1*. Let U be such a path having a minimum number of nodes. A BFS automatically produces such a path. (We identify U with its node-set. What we actually will need is only that U is minimal with respect to inclusion.) Let F' be the symmetric difference of F and U, and let $c'_i := c_i$ $(i = 1, 2)$.

Claim 13.2.17 *F', c'_1, c'_2 satisfy the optimality criteria in Claim* 13.2.16 *for* $k + 1$.

Proof. Let the nodes of U be denoted by $y_1, x_1, y_2, x_2, \dots, y_\ell, x_\ell, p_0$ where $y_1 \in Y_2$ and $p_0 \in Y_1$, and the indices reflect the order of the nodes along U. By Lemma 13.2.13, $I := F + p_0$ is c_1-maximal in $\mathcal{F}_1^{k+1}$. Observe that $F' = I - \{x_1, x_2, \dots, x_\ell\} \cup \{y_1, y_2, \dots, y_\ell\}$. The hypotheses of Lemma 13.2.14 hold for the choice $k + 1$, $I \in \mathcal{F}_1^{k+1}$, $x_1, x_2, \dots, x_\ell$, and $y_1, y_2, \dots, y_\ell$. (Property (13.27) follows from the minimality of U.) Therefore Lemma 13.2.14 implies that F' is c_1-maximal in $\mathcal{F}_1^{k+1}$. It can be verified analogously that F' is c_2-maximal in $\mathcal{F}_2^{k+1}$ with the only difference that the nodes of U should be indexed in an opposite order. •

The following claim follows obviously from the construction. In fact, we will use it only for proving a corollary due to Krogdahl and not for the correctness of the algorithm.

Claim 13.2.18 $\widetilde{c}(F') - \widetilde{c}(F) = m_1 + m_2$. •

Case 2 *The set T of nodes reachable from Y_2 is disjoint from Y_1*. Then $Y_2 \subseteq T$, $Y_1 \subseteq S - T$, and no edge of D leaves T. Let

$$c'_1(z) := \begin{cases} c_1(z) + \delta & \text{if } z \in T \\ c_1(z) & \text{if } z \in S - T \end{cases} \tag{13.32}$$

and let $c'_2 := c - c'_1$ where

$$\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}, \tag{13.33}$$

in which

$$\begin{aligned}
\delta_1 &:= \min\{c_1(x) - c_1(y) : \ y \in T - F, \ F + y \notin \mathcal{F}_1, \ x \in C_1(F, y) - T\}, \\
\delta_2 &:= \min\{m_1 - c_1(y) : \ y \in T - F, \ F + y \in \mathcal{F}_1\}, \\
\delta_3 &:= \min\{c_2(x) - c_2(y) : \ y \in S - (T \cup F), \ F + y \notin \mathcal{F}_2, \ x \in C_2(F, y) \cap T\}, \\
\delta_4 &:= \min\{m_2 - c_2(y) : \ y \in S - (T \cup F), \ F + y \in \mathcal{F}_2\}.
\end{aligned}$$

The minimum taken over the empty set is defined to be ∞. Consider first the case when $\delta = \infty$, that is, the case when each of the four δ_i's is infinite. As $\delta_2 = \infty$, $F + y \notin \mathcal{F}_1$ for every $y \in T \cap (S - F)$, that is, there is a fundamental circuit $C_1(F, y)$ in M_1. In addition, $\delta_1 = \infty$ implies that $C_1(F, y) \subseteq T$, and hence $F \cap T$ spans T in matroid M_1 showing that $r_1(T) = |F \cap T|$. Similarly, $\delta_4 = \infty$ implies that $F + y \notin \mathcal{F}_2$ for every $y \in S - (T \cup F)$, and hence there is a fundamental circuit $C_2(F, y)$ in M_2. In addition, $\delta_3 = \infty$ implies that $C_2(F, y) \subseteq S - T$. Consequently, $F - T$ spans $S - T$ in matroid M_2 and hence $r_2(S - T) = |F - T|$. Therefore $|F| = |F \cap T| + |F - T| = r_1(T) + r_2(S - T)$ which indicates that the current common independent set F is of maximum cardinality. In such a case the algorithm terminates. Suppose now that δ is finite.

Claim 13.2.19 $\delta > 0$.

Proof. We prove only $\delta_1 > 0$ and $\delta_4 > 0$. The proof of $\delta_2 > 0$ and $\delta_3 > 0$ can be carried out analogously. If $x \in C_1(F, y) - T$, then Lemma 13.2.13 implies $c_1(x) \geq c_1(y)$. Here we cannot have equality since then yx would be an edge of D leaving T but such an edge does not exist. Therefore $\delta_1 > 0$. If $y \in S - (T \cup F)$, then $y \notin Y_2$ and hence $\delta_4 > 0$ follows from the definintion of m_2. •

Claim 13.2.20 *The set $F' := F$ and the weight-splitting c'_1, c'_2 of c satisfy the hypotheses of Lemma* 13.2.13.

Proof. We prove only that F' is c'_1-maximal in $\mathcal{F}_1^k$. Its c'_2-maximality in $\mathcal{F}_2^k$ can be proved similarly.

To see the c'_1-maximality of F', we have to prove that the hypotheses of Lemma 13.2.13 hold for c'_1. Select first elements x, y for which $x \in C_1(F, y)$. If, indirectly, $c'_1(x) < c'_1(y)$, then $c_1(x) \geq c_1(y)$ would imply $c'_1(x) = c_1(x) + \delta$ and $c'_1(y) = c_1(y)$ and also $y \in T - F$ and $x \in F - T$. But then $\delta \leq \delta_1 \leq c_1(x) - c_1(y)$, from which $c'_1(x) \leq c'_1(y)$, contradicting the indirect assumption. Therefore (13.25) holds for F and c'_1.

Select now x and y for which $x \in F, y \notin F$, and $F + y \in \mathcal{F}_1$. If, indirectly, $c'_1(x) < c'_1(y)$, then $c_1(x) \geq c_1(y)$ would imply $c'_1(y) = c_1(y) + \delta$ and $c'_1(x) = c_1(x)$ on the one hand, and $y \in T - F$ and $x \in F - T$ on the other. Hence $\delta \leq \delta_2 \leq m_1 - c_1(y)$. Since F satisfies (13.26), we have $c_1(x) \geq m_1$. From these, one gets $c'_1(y) = c_1(y) + \delta \leq m_1 \leq c_1(x) = c'_1(x)$, contradicting the indirect assumption. Therefore (13.26) also holds for F and c'_1. •

Iterate the algorithm for F', c'_1, c'_2. Observe that the definitions of δ_2 and δ_4 imply that $m'_1 = m_1$ and $m'_2 = m_2$, furthermore $Y_1 \subseteq Y'_1$ and $Y_2 \subseteq Y'_2$. It follows from the definition of the auxiliary digraph that $T' \subseteq T$. If $\delta = \delta_2$, then a node of $T - F$ gets into Y'_1 showing that Case 1 must occur in D'. If $\delta = \delta_1$ or $\delta = \delta_3$, then D' will have an edge leaving T showing that T' is strictly larger than T. Finally, if $\delta = \delta_4$, then Y'_2 will have an element outside T in which case we also have $T' \supset T$.

We can conclude that during the iterations of the algorithm, after at most $|S|$ consecutive occurrences of Case 1 either Case 2 occurs or δ gets infinite when the algorithm terminates. This completes the description of the algorithm and the proof of its correction. • •

What can be said about the complexity of the algorithm? Suppose that the matroids are given by an oracle which decides for any input independent set F and element $y \notin F$ whether $F + y$ is independent or not and in the latter case it returns the fundamental circuit $C(F, y)$. Let g denote the complexity of one call of the oracle. Consider the summation, subtraction, and comparision of two numbers as a single step. (The algorithm does not use multiplication.) Let $n := |S|$ és let K be the maximum cardinality of a common independent set.

The auxiliary digraph can be constructed with $O(n)$ calls of the subroutine. BFS needs $O(n^2)$ steps. When Case 2 occurs, the existing labels indicating the reachability situation can be reused since $T' \supset T, Y'_1 \supseteq Y_1, Y'_2 \supseteq Y_2$. It follows that between two consecutive occurrences of Case 1 the algorithm needs at most $O(gn^2)$ steps. Hence the overall comlexity is at most $O(gKn^2) \leq O(gn^3)$.

Remark If the algorithm starts with $c_1 \equiv 0$, then we will have throughout $m_1 = 0$ and $\delta_2 = \infty$. In the description above we did not make use of the possible simplification coming from this observation because a symmetry between the two matroids could be preserved.

978–0–19–920527–1 13-Andras-Frank-c13-drv Andras Frank (Typeset by SPi, Publisher Services, India) 432 of 640 December 11, 2010 15:20
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Moreover, in this way one has the option to start with and arbitrary F, c_1, c_2 which satisfy the optimality criteria in Lemma 13.2.16.

The algorithm incidentally proves the following result, as well.

Theorem 13.2.21 *A k-element common independent set F is c-maximal in $\mathcal{F}^k_{12}$ if and only if there is a weight-splitting $c = c_1 + c_2$ for which F is c_i-maximal in $\mathcal{F}^k_i$ for $i = 1, 2$. If, in addition, c is integer-valued, then c_i can also be chosen to be integer-valued.* •

Note that in the special case when $k = r_1(S) = r_2(S) = r_{12}(S)$ and c is integer-valued, this result is equivalent to Corollary 13.2.6.

Since the values of m_1 and m_2 never increase during the run of the algorithm, the following pretty result of Krogdahl [250] follows immediately form Claim 13.2.18.

Corollary 13.2.22 (Krogdahl) $c^{(k+1)} - c^{(k)} \leq c^{(k)} - c^{(k-1)}$*, where $c^{(j)}$ denotes the maximal c-weight of a j-element common independent set.* •

Localization of common bases

As an application of the weighted matroid intersection theorem, we derive an interesting 'unweighted' result of Gröflin and Hoffman [195] on the localization of common bases.

Theorem 13.2.23 (Gröflin and Hoffman) *Let M_1 and M_2 be two rank-k matroids on a common ground-set S and suppose they have a common basis. For a given subset $R \subseteq S$,*

$$\min\{|R \cap B| : B \textit{ a common basis}\} = \max\left\{\sum_{i=1}^{t}[k - r_{12}(S - R_i)]\right\} \tag{13.34}$$

where the maximum is taken over all partitions $\{R_1, \ldots, R_t\}$ of R and $r_{12}(T)$ denotes the maximum cardinality of a common independent subset of T.

Remark By the intersection theorem $r_{12}(T) = \min_{X \subseteq T}\{r_1(X) + r_2(T - X)\}$. Hence (13.34) can be written in the following equivalent form.

$$\min\{|R \cap B| : B \text{ a common basis}\} = \max\left\{\sum_{i=1}^{t}[k - (r_1(X_i) + r_2(Y_i))]\right\} \tag{13.35}$$

where $\{R_i\}$ is a partition of R and $\{R_i, X_i, Y_i\}$ is a 3-partition of S for each index i.

Proof. A common independent set can contain at most $r_{12}(S - R_i)$ elements of $S - R_i$. Therefore a common basis has at least $k - r_{12}(S - R_i)$ elements from R_i and hence it contains at least $\sum_{i=1}^{t}[k - r_{12}(S - R_i)]$ elements of R from which max $\leq$ min follows.

For the other direction, we must show that there is a common independent basis B and 3-partitions $\{R_i, X_i, Y_i\}$ of S for each $i = 1, \ldots, t$ such that $\{R_1, R_2, \ldots, R_t\}$ is a partition of R and

$$|R \cap B| = \sum_{i=1}^{t}[k - r_1(X_i) - r_2(Y_i)]. \tag{13.36}$$

For $R = \emptyset$, the choice $R_1 := X_1 := \emptyset$, $Y_1 := S$ and any common basis B will suffice while for $R = S$ the choice $R_1 := S$, $X_1 := Y_1 := \emptyset$ satisfies (13.36) with B. Hence we can assume that $\emptyset \subset R \subset S$.

Let $c := \underline{\chi}_{S-R}$ and apply Theorem 13.2.4. Since the span of c is 1, the theorem implies that there is a weight-splitting $c = c_1 + c_2$ such that both c_1 and c_2 are 1-narrow. By translating c_1 with a constant, we can assume that the distinct values of c_1 are $1, 2, \dots, t$. For $i = 1, \dots, t$ we use the following notation. Let $R_i := \{s \in R : c_1(s) = i\}$, $X_i := \{s \in S : c_1(s) \geq i+1\}$, $Y_i := \{s \in S : c_2(s) \geq 1 - i\}$. Then $\{R_1, \dots, R_t\}$ forms a partition of R and $\{R_i, X_i, Y_i\}$ is a 3-partition of S for each i.

It follows from formula (5.13) that $\hat{r}_1(c_1) = r_1(S) + \sum_{i=1}^t r_1(X_i)$ and $\hat{r}_2(c_2) = (-t)r_2(S) + \sum_{i=1}^t r_2(Y_i)$. (This is clear when $R_t \neq \emptyset$ while if R_t is empty, then $S = Y_t$ and hence $\hat{r}_2(c_2) = (-t+1)r_2(S) + \sum_{i=1}^{t-1} r_2(Y_i) = (-t)r_2(S) + \sum_{i=1}^t r_2(Y_i)$.)

By Theorem 13.2.4 theorem, there is a common basis B for which $\widetilde{c}(B) = |B - R| = \hat{r}_1(c_1) + \hat{r}_2(c_2) = \big[r_1(S) + \sum_{i=1}^t r_1(X_i)\big] + \big[(-t)r_2(S) + \sum_{i=1}^t r_2(Y_i)\big] = (1-t)k + \sum_{i=1}^t [r_1(X_i) + r_2(Y_i)]$ from which $|R \cap B| = k - |B - R| = k - \{(1-t)k + \sum_{i=1}^t [r_1(X_i) + r_2(Y_i)]\} = \sum_{i=1}^t [k - r_1(X_i) - r_2(Y_i)]$. •

13.3 Sum of matroids

In Theorem 5.4.1, we proved that the homomorphic image $\varphi(M)$ of a matroid M is a matroid. In the proof of Theorem 5.4.2, by a simple elementary construction, we showed that the sum M_Σ of k matroids $M_1, \dots, M_k$ on the same ground-set S can be obtained as a homomorphic image. (Namely, we considered the direct sum M_{big} of copies of the matroids M_i on disjoint sets S_i $(i = 1, \dots, k)$ along with the map φ of $S_1 \cup S_2 \cup \dots \cup S_k$ to S in which the image of every element of S_i is the corresponding element of S, and observed that $M_\Sigma = \varphi(M_{big})$.) Recall that in M_Σ a set $F \subseteq S$ is independent by definition if it is partitionable into independent sets of the k matroids. The same elementary construction shows that the rank formula given in (13.5) for $\varphi(M)$ can be applied to M_Σ. In this way, we obtain the following result of Edmonds and Fulkerson [85].

Theorem 13.3.1 (Matroid partition theorem) *The rank function r_Σ of the sum M_Σ of k matroids $M_i = (S, r_i)$ is given by the following formula*

$$r_\Sigma(Z) = \min_{X \subseteq Z} \left\{ |Z - X| + \sum_i r_i(X) \right\}. \tag{13.37}$$

In particular, the largest cardinality of a partitionable subset of S is equal to $\min\{\sum_i r_i(X) + |S - X| : X \subseteq S\}$.•

Below we shall discuss Edmonds' matroid partition algorithm which provides a direct proof for Theorem 13.3.1, without referring to any earlier results.

Consider the rank formula given in (13.37) when Z is the ground-set S. We say that a subset $X \subseteq S$ is a **minimizer for** M_Σ if $r_\Sigma(S) = |S - X| + \sum_i r_i(X)$. The following result is an immediate consequence of (13.37).

Corollary 13.3.2 *Let $F_1, \dots, F_k$ be disjoint subsets of S such that F_i is M_i-independent $(i = 1, \dots, k)$ and $|F_1 \cup \dots \cup F_k|$ is maximum. A subset X is a minimizer for M_Σ if and only if $|X \cap F_i| = r_i(X)$ for each $i = 1, \dots, k$ and $X \cup F_1 \cup \dots \cup F_k = S$.* •

Theorem 13.3.3 *The set-system of minimizers for M_Σ is closed under taking intersection and union. In particular, there is a unique smallest minimizer and there is a unique largest minimizer.*

Proof. Define a set-function b by $b(X) := |S - X| + \sum_i r_i(X)$ for $X \subseteq S$. Then b is clearly submodular and $r_\Sigma(S) = \min\{b(X) : X \subseteq S\}$. For two minimizers Y and Y', we have $r_\Sigma(S) + r_\Sigma(S) = b(Y) + b(Y') \geq b(Y \cap Y') + b(Y \cup Y') \geq r_\Sigma(S) + r_\Sigma(S)$ from which equality follows everywhere. In particular, $b(Y \cap Y') = r_\Sigma(S)$ and $b(Y \cup Y') = r_\Sigma(S)$, and hence both $Y \cap Y'$ and $Y \cup Y'$ are minimizers. •

Note that Rado's theorem was used, via the rank-formula of the homomorphic image $\varphi(M)$, to derive the Matroid partition theorem. On the other hand, the Intersection theorem was a generalization of Rado's theorem. Our present goal is to show that the matroid partition theorem and the matroid intersection theorem are actually equivalent.

Equivalence of matroid intersection and sum

Proof of Theorem 13.3.1 from Theorem 13.1.9. It suffices to derive the formula (13.37) for $Z = S$. Let $S_1, \dots, S_k$ be k disjoint copies of the ground-set S and let S_0 be their union. We define two matroids on S_0. Let N_1 be a partition matroid in which a subset is independent if it contains at most one of the k copies corresponding to s in S_0 for each element $s \in S$. Let N_2 be a direct sum in which a subset $F_1 \cup \cdots \cup F_k$ with $F_i \subseteq S_i$ is independent if the subset corresponding to F_i in S is independent in M_i for each $i = 1, \dots, k$. Let R_1 and R_2 denote the rank functions of N_1 és N_2.

There is a one-to-one correspondence between the common independent sets of N_1 and N_2 and the subpartitions $\{I_1, I_2, \dots, I_k\}$ of S in which I_i is M_i-independent. It follows from the intersection theorem that $r_\Sigma = \min\{R_1(X_0) + R_2(S_0 - X_0)\}$ where the minimum is taken over all subsets X_0 of S_0. Note that the minimum remains the same if only N_1-closed subsets X_0 are taken into consideration. But a subset $X_0 \subseteq S_0$ is closed in the partition matroid N_1 exactly if X_0 arises as the union of the k copies (each in an S_i) of a subset X of S. In this case, $R_1(X_0) = |X|$. By the rank of the direct sum, $R_2(S_0 - X_0) = \sum r_i(S - X)$ from which (13.37) follows.

Proof of Theorem 13.1.9 from Theorem 13.3.1. Let $\mathcal{B}_i$ denote the set of bases of M_i and consider the dual matroid M_2^* of M_2. Finding a largest common independent set of M_1 and M_2 is equivalent to finding bases $B_i \in \mathcal{B}_i$ for ($i = 1, 2$) whose intersection is maximum. But this is the same as finding a basis B_1 of M_1 and a basis B_2^* of M_2^* whose union is maximum.

By applying formula (13.37) to r_1 and r_2^*, we get that the maximum in question is $\min_{X \subseteq S}\left\{r_1(X) + r_2^*(X) + |S - X|\right\} = \min_{X \subseteq S}\{r_1(X) + |X| - r_2(S) + r_2(S - X) + |S - X|\} = \min_{X \subseteq S}\{r_1(X) + |S| - r_2(S) + r_2(S - X)\}$. From $|B_1 \cap B_2| = \left|B_1 \cup B_2^*\right| - \left|B_2^*\right|$ one gets

$$\begin{aligned}\max\left\{|B_1 \cap B_2| :\ B_1 \in \mathcal{B}_1,\ B_2^* \in \mathcal{B}_2^*\right\} &= \min_{X \subseteq S}\{r_1(X) \\ &\quad +|S| - r_2(S) + r_2(S - X)) - (|S| - r_2(S)\} \\ &= \min_{X \subseteq S}\{r_1(X) + r_2(S - X)\}. \bullet\end{aligned}$$

Remark 13.3.1 The technique described above for reducing the matroid partition problem to matroid intersection can be used algorithmically to solve a weighted version of the parttion problem in which one is given a weight function $c_i : S \to \mathbf{R}$ for each matroid $M_1, \dots, M_k$ and the goal is to find disjoint M_i-independent sets F_i such that $\sum_i \widetilde{c}_i(F_i)$ is maximum. The only difference in the reduction is that we also introduce a weight function c on S_0 for which $c(s_i) := c_i(s)$ for $s \in S$ where s_i denotes the element of S_i corresponding to s. Then the weighted matroid intersection algorithm can be applied to N_1 and N_2.

The theorem of Rado, the Partition theorem and Edmonds' matroid intersection theorem are three basic results of matroid optimization. Their equivalence was pointed out first in a paper by Welsh [377].

An application to basis exchange

As an interesting application of matroid sums, we derive the following strengthening of the symmetric exchange property of bases due to Greene [191].

Theorem 13.3.4 (Greene) *Let B_1 and B_2 be two bases of matroid M. For every subset $X_1 \subseteq B_1$ there is a subset $X_2 \subseteq B_2$ in such a way that both $B_1 - X_1 \cup X_2$ and $B_2 - X_2 \cup X_1$ are bases of M.*

Proof. It suffices to prove the theorem for the special case when the two bases are disjoint since by contracting the intersection we can reduce the problem to this case. Let $X_1' := B_1 - X_1,\ M_1 := (M/X_1)|B_2$, and $M_1' := \left(M/X_1'\right)|B_2$. The theorem is equivalent to stating that B_2 can be partitioned into sets X_2 and X_2' in such a way that both $X_1' \cup X_2$ and $X_1 \cup X_2'$ are bases of M, or in other word, B_2 can be partitioned into independent sets of M_1 and M_1'. To this end, it suffices to prove by Theorem 13.3.9 that $r_1(X) + r_1'(X) \geq |X|$ holds for every subset X of B_2. This is indeed the case since $r_1(X) = r(X \cup X_1) - r(X_1) = r(X \cup X_1) - |X_1|$ and $r_1'(X) = r\left(X \cup X_1'\right) - r\left(X_1'\right) = r\left(X \cup X_1'\right) - \left|X_1'\right|$ from which

$$r_1(X) + r_1'(X) = r(X \cup X_1) + r\left(X \cup X_1'\right) - \left[|X_1| + \left|X_1'\right|\right]$$

$$= r(X \cup X_1) + r\left(X \cup X_1'\right) - |B_1| \geq r\left(X \cup X_1 \cup X_1'\right) + r\left((X \cup X_1) \cap \left(X \cup X_1'\right)\right) - |B_1|$$

$$= r(B_1) + r(X) - |B_1| = |B_1| + |X| - |B_1| = |X|,$$

as required. •

Problem 13.3.1 *By applying repeatedly Theorem* 13.3.4*, derive the following result of Greene and Magnanti* [194]. *Why is this result an extension of Theorem* 13.3.4*?*

Theorem 13.3.5 (Greene and Magnanti) *Let B_1 and B_2 be two bases of a matroid M and let $\{Z_1, Z_2, \dots, Z_m\}$ be a partition of basis B_1. Then there is a partition $\{Y_1, \dots, Y_m\}$ of B_2 for which $(B_1 - Z_i) \cup Y_i$ is a basis for each $i = 1, \dots, m$.* •

13.3.1 Two algorithms for matroid partition

Based on the reduction above to matroid intersection, one can already compute a basis of the sum of k matroids with the help of Edmonds' matroid intersection algorithm. Edmonds [76], however, also developed a direct algorithm which works on the original ground-set S and hence is more efficient. This algorithm, like his intersection algorithm, can be considered a matroidal version of the alternating-path method. We will also exhibit an algorithm which is the matroidal counterpart of the push–relabel technique developed for network flows.

The partition algorithm of Edmonds

The matroid partition theorem states that the rank of the sum of k matroids is equal to $\min\{\sum_i r_i(X) + |S - X| : X \subseteq S\}$. The algorithm provides a new proof (of the non-trivial direction $r_\Sigma(S) \geq \min$) of this result by computing M_i-independent sets F_i for $i = 1, \dots, k$ which are disjoint and a set $X \subseteq S$ such that $S = X \cup (\cup_i F_i)$ and $F_i \cap X$ spans X in M_i for each i (where S is said to span M_i if $|X \cap F_i| = r_i(X)$). These two properties will be referred to as optimality criteria.

The algorithm, which is a variant of Edmonds' algorithm due to Knuth [241], may start with any set of k disjoint M_i-independent sets F_i. Let $R \subseteq S$ denote the set of elements not belonging to any F_i. If R is empty, then the algorithm terminates by conluding that ground-set S is independent in the sum. Suppose that $R \neq \emptyset$ and construct an auxiliary digraph D as follows. For each matroid M_i, let t_i be a new element and let T be the set of the k new elements. Let xt_i be an edge of D if $x \in S - F_i$ and $F_i + x$ is M_i-independent. Furthermore, let xy be an edge of D if $x \in S - F_i$, $y \in F_i$, $F_i + x$ is M_i-dependent, and y belongs to the unique M_i-circuit of $F_i + x$. There are two cases.

Case 1 *There is no path from R to T, that is, there is a subset $X \subseteq S$ for which $R \subseteq X$ and $\delta_D(X) = 0$.* Such an X can easily be computed: a breadth-first-search, for example, determines the set X_0 of nodes reachable from R in D, and this X_0 is the unique smallest set including R for which $\delta_D(X_0) = 0$. Or, a reverse search algorithm can compute the set $X_1 \subseteq S$ of nodes from which T is not reachable in D. This X_1 is the unique largest subset including R for which $\delta_D(X_1) = 0$.

Since no edge of D leaves X, $F_i + x$ is M_i-dependent whenever $x \in X - F_i$, moreover, the fundamental M_i-circuit of $F_i + x$ lies completely in X. This means that $F_i \cap X$ spans X in matroid M_i, and hence the optimality criteria hold for X and for $F_1, \dots, F_k$. Therefore, when Case 1 occurs, the algorithm terminates.

Case 2 *There is a path P from R to T.* Let P be a shortest path from R to T. Let st_j be the last edge of P. Extend F_j by s, that is, let $F_j := F_j + s$. For each $i = 1, \dots, k$ consider all the edges xy of P entering F_i and revise F_i by replacing y with x. Since P is a shortest path, Lemma 13.1.11 can be applied and it implies that the resulting sets F'_i are M_i-independent. In this way we have obtained disjoint M_i-independent sets whose union is one element larger (namely, with the starting node of P) than the union of the initial F_i-s.

It follows that by iterating this procedure we shell arrive at Case 1 after at most $|S|$ occurrences of Case 2. •

The work [173] by Gabow and Westermann includes an efficient implementation of the matroid sum algorithm along with several applications.

Problem 13.3.2 *Develop an algorithm to decide if the vector of $\mathbf{R}^S$ having each component $1/2$ belongs to the independence polyhedron of a matroid on S.*

Cunningham [60] developed a strongly polynomial time algorithm to decide for an arbitrary element of $\mathbf{R}^S_+$ whether it belongs to the independence polyhedron of a matroid on S.

Remark 13.3.2 Edmonds observed in his paper [76] that the algorithm immediately implies that, given k matroids on a common ground-set S, the partitionable sets satisfy the matroid independence axioms. Indeed, the only non-trivial axiom to be checked requires for each subset S' of S that the cardinality of the maximal partitionable subsets of S' depends only on S'. It suffices to prove this property only for $S' = S$ since the general case follows if we restrict the matroids M_i to S'. Now the axiom follows from the following feature of the algorithm. By starting with an arbitrary partitionable set $F = F_1 \cup \cdots \cup F_k$, the algorithm either finds a partitionable set $F' = F'_1 \cup \cdots \cup F'_k$ that includes F as a subset, or finds a certificate to show that F is a partitionable set of maximum size (where the certificate is a set Z for which $|F| = \sum_i r_i(Z) + |S - Z|$). Therefore if we start with any *maximal* partitionable set F, then the first alternative cannot occur, implying that any maximal independent set of M_Σ is of maximum cardinality.

Theorem 13.3.6 *Let M_Σ be the sum of matroids $M_1, \dots, M_k$. Suppose that the matroid partition algorithm terminates with a digraph D and with independent sets $F_1, \dots, F_k$. Let X_0 be the set of elements of S reachable from R in D, and X_1 the set of elements of S from which T is not reachable.*

(A) *A subset $X \subseteq S$ is a minimizer for M_Σ if and only if $R \subseteq X$ and $\delta_D(X) = 0$. Furthermore, X_0 is the unique smallest minimizer set while X_1 is the unique largest minimizer set. Any minimizer set X includes disjoint bases of the submatroids $M_1|X, \dots, M_k|X$, while $S - X$ partitions into independent sets of the contracted matroids $M_1/X, \dots, M_k/X$.*

(B1) *$X_0 - s$ includes disjoint bases of the submatroids $M_1|X_0, \dots, M_k|X_0$ for every $s \in X_0$.*

(B2) *$(S - X_1) + s'$ can be partitioned into independent sets of $M'_1/X_1, \dots, M'_k/X_1$ for every $s \in S - X_1$, where s' is a copy of s and M'_i is a matroid on $S + s'$ arising from M_i by duplicating s in parallel.*

Proof. Part (A) is a direct consequence of the argument used in Case 1 above.

(B1) If $s \in R$, then $F_i \cap X_0$ is a basis of $M_i|X_0$ and does not contain s for each $i = 1, \dots, k$. If s is in $X_0 - R$, then there is a directed path form R to s. Performing a sequence of exchanges along this path, analogously to the way how we did in Case 2 above, we obtain disjoint subsets $F'_1, \dots, F'_k$ of S such that F'_i is M_i-independent, $\left|F'_1 \cup \cdots \cup F'_k\right| = |F_1 \cup \cdots \cup F_k|$, and s is not in $F'_1 \cup \cdots \cup F'_k$. In this case, $F'_i \cap X_0$ is a basis of $M_i|X_0$ and does not contain s for each $i = 1, \dots, k$.

(B2) Since there is a directed path from s to T in D, we can perform a sequence of exchanges along this path and obtain disjoint subsets $F'_1, \dots, F'_k$ of S such that F'_i is M'_i-independent, $F_i \cap X_1 = F'_i \cap X_1$, and $F'_1 \cup \cdots \cup F'_k = F_1 \cup \cdots \cup F_k + s'$. Now $\left\{F'_1 - X_1, \dots, F'_k - X_1\right\}$ is a requested partition of $(S - X_1) + s'$. •

Corollary 13.3.7 *Let $G = (V, E)$ be an undirected graph and k a positive integer. Then there is a partition $\{V_1, \dots, V_q\}$ of V $(q \geq 1)$ such that each of the subgraphs $G_i = (V_i, E_i)$ induced by V_i is k-rich, while the graph G' arising from G by shrinking each V_i into a node is k-sparse. There is a finest partition $\mathcal{P}_0$ and there is a coarsest partition $\mathcal{P}_1$ with this property. Each member of $\mathcal{P}_0$ induces a strictly k-rich graph. By shrinking each member of $\mathcal{P}_1$, we obtain a strictly k-sparse graph.*

Proof. Apply Theorem 13.3.6 in the special case when each M_i is the circuit matroid of G and observe that if the matroids are identical, then the sets X_0 and X_1 in that theorem are closed. In the graphic matroid of G, the closed sets are exactly the union of edge-sets induced by the members of a partition of V. Hence X_0 (respectively, X_1) is the set of edges induced by the members of $\mathcal{P}_0$ (respectively, $\mathcal{P}_1$). •

Note that Corollary 13.3.7 implies the theorem of Jackson and Jordán (Theorem 10.5.4).

A push–relabel algorithm

Next, we show how the push–relabel technique can be used for proving algorithmically Theorem 13.3.1. The approach was developed in [147]. We will need the following simple observation.

Lemma 13.3.8 *Let B be a basis of a matroid $M = (S, \mathcal{B})$ where S is an n-element ground-set. Let $\Theta : S \to \{0, 1, \dots, n\}$ be a level function for which*

$$\Theta(v) \leq \Theta(u) + 1 \textit{ holds whenever } u \in S - B \textit{ and } v \in C(B, u). \tag{13.38}$$

Let $s \in S - B$ and $t \in C(B, s)$ be elements for which $\Theta(t) = \Theta(s) + 1$. Then (13.38) *holds for the basis $B' := B - t + s$.*

Proof. Let C_z and C'_z denote the fundamental circuits of any element z with respect to bases B and B'. For a subset $X \subseteq B$, let $\Theta(X) := \max\{\Theta(v) : v \in X\}$. Then (13.38) is equivalent to $\Theta(C_u) \leq \Theta(u) + 1$ for $u \in S - B$. If $u = t$, then $C'_u = C_s$ and hence $\Theta\left(C'_u\right) = \Theta(C_s) = \Theta(t) < \Theta(u) + 1$. Suppose now that $u \neq t$. If $t \notin C_u$, then $C'_u = C_u$ and hence $\Theta\left(C'_u\right) = \Theta(C_u) \leq \Theta(u) + 1$. If $t \in C_u$, then $\Theta(C_s) = \Theta(t) \leq \Theta(u) + 1$ from which $\Theta(C_u \cup C_s) \leq \Theta(u) + 1$. By the circuit axiom, there is a circuit $C' \subseteq C_s \cup C_u - t$ containing u and we must have $C'_u = C'$ from which $\Theta\left(C'_u\right) \leq \Theta(C_u \cup C_s) \leq \Theta(u) + 1$. •

Our goal is to construct M_i-bases B_i for $i = 1, \dots, k$ along with a subset $Z \subseteq S$ for which

$$S - Z \subseteq \cup_i B_i \tag{13.39}$$

$$B_i \cap B_j \cap Z = \emptyset \text{ for } 1 \leq i < j \leq k \tag{13.40}$$

$$B_i \cap Z \text{ spans } Z \text{ in } M_i \text{ for } i = 1, \dots, k. \tag{13.41}$$

At an intermediate stage of the algorithm, we are given M_i-bases B_i for $i = 1, \dots, k$ and a level function $\Theta : S \to \{0, 1, \dots, n = |S|\}$ for which the following level properties hold.

(L1) $u \in S - \cup_i B_i$ implies $\Theta(u) = 0$, or informally, every element not in any B_i has level 0.

(L2) $u \in S - B_i$ and $v \in C_i(B_i, u)$ imply $\Theta(v) \leq \Theta(u) + 1$,

If there is no multiply covered element, that is, if the bases are disjoint, then $Z := S$ meets the optimality criteria while if the bases cover all elements, than $Z := \emptyset$ does.

Suppose now that there is a multiply covered element and let t be one whose level is minimum. Among the k given bases, let $B_1, \dots, B_\ell$ $(2 \leq \ell \leq k)$ denote those containing t.

Assume first that there is a subscript $1 \leq i \leq \ell$ and an element $s \in S - B_i$ which satisfy $t \in C_i(B_i, s)$ and $\Theta(t) = \Theta(s) + 1$. In this case, we replace B_i by $B_i - t + s$ which is another M_i-basis. If no such a subscript and element exist, then lift t, that is, increase the level $\Theta(t)$ by 1.

The algorithm terminates when the level of the currently lifted t becomes empty. This certainly will occur in at most n^2 lifts since if there is an element of level n, then there must be an empty level.

Consider the situation when level j gets empty after lifting t. For the subset $Z := \{z : \Theta(z) \leq j\}$ of elements below t, Property **(L2)** implies the inclusion $C_i(B_i, u) \subseteq Z$ for every element $u \in S - B_i$ which is in Z, and hence (13.41) is met. Property **(L1)** implies that each element of $S - Z$ is covered by a basis B_i, and hence (13.39) is also fulfilled. Finally, (13.40) holds since t was chosen to be a lowest-level multiply covered element.

At a basis change, the union of the k bases is either unchanged or becomes one element larger. The set of steps of the algorithm while both this union and Θ are unchanged will be called a **phase**. There are at most $n + n^2$ phases, and within one phase there can be at most n changes of basis (as the level of the lowest multiply covered element gets one less each time a basis is changed). We can conclude that the algorithm terminates after $O(n^3)$ steps. •

13.3.2 Completing coverings and packings

The matroid partition theorem and algorithm can be used to derive packing and covering results concerning bases and independent sets of matroids (see the paper of Edmonds [77]).

Covering by independent sets

One of the important consequences of the matroid partition theorem is as follows.

Theorem 13.3.9 (Edmonds) *Let $M_1 = (S, r_1), \ldots, M_k = (S, r_k)$ be k matroids on a common ground-set S. It is possible to partition S into M_i-independent sets F_i $(i = 1, \ldots, k)$ if and only if*

$$\sum_i r_i(X) \geq |X| \text{ for every } X \subseteq S. \tag{13.42}$$

Proof. S is partitionable precisely if the rank of the sum of the k matroid is $|S|$. By Theorem 13.3.1 this is equivalent to requiring that $\min_{X \subseteq S}\{\sum_i r_i(X) + |S - X|\} \geq |S|$, and hence $\sum_i r_i(X) \geq |X|$ holds for every $X \subseteq S$. •

Theorem 13.3.10 *For a matroid $M = (S, r)$, the following are equivalent.*

(A) *The ground-set S can be covered by k bases,*
(B1) *$kr(X) \geq |X|$ for every set $X \subseteq S$,*
(B2) *$kr(X) \geq |X|$ for every non-separable and closed set $X \subseteq S$.*

Proof. The equivalence of (A) and (B1) is a direct consequence of Theorem 13.3.9. (B2) is a special case of (B1). To see that (B2) implies (B1), suppose indirectly that there is a set X for which $kr(X) < |X|$. We can assume that X is closed for otherwise X could be replaced by its closure. There is a unique partition $\{X_1, \ldots, X_t\}$ of X into non-separable sets. We claim that each X_i is closed. Indeed, let F_j be a maximal independent subset of X_j. If X_1, say, is not closed, then there is an element $s \in S - X_1$ for which $F_1 + s$ includes a circuit. Then s cannot be in X since $M|S$ is the direct sum of the matroids $M|X_i$. Since $F = \cup F_j$ is a maximal independent subset of X, we get $r(X+s) = r(F+s) = |F| = r(X)$ contradicting the closedness of X.

Therefore each set X_i is closed and non-separable and hence (B2) implies $kr(X) = k\sum_j r(X_j) \geq \sum_j |X_j| = |X|$, contradicting the indirect assumption. •

For the special case of the graphic matroid of a graph $G = (V, E)$, Theorem 13.3.10 specializes to Nash-Williams' theorem (Theorem 10.4.3) once we observe that a closed and non-separable set of elements in a graphic matroid always arises as the set of edges induced by a subset of nodes. Theorem 10.4.4 was an extension of Nash-Williams' theorem and gave a characterization of situations when k given subtrees of G could be extended to forests covering all edges of G. The following theorem generalizes this result to arbitrary matroids. Specializing to graphic matroids again, one obtains a characterization for the situation when k given *forests* can be extended to forests covering all edges of G. It should be noted, however, that in this case the condition is more complicated than the one in Theorem 10.4.3.

The matroid partition theorem can be used to derive an extension of Theorem 13.3.9 when some specified M_i-independent sets I_i $(i = 1, \ldots, k)$ should be completed to get a covering of S by independent sets.

Let $S' := S - \cup_i I_i$ and let M'_i be a matroid on S' arising from $M_i|(S' \cup I_i)$ by contracting I_i. The completion problem is equivalent to finding and an M'_i-independent set for each i whose union is S'. Since the rank function of M'_i is given by $r'_i(X) = r_i(X \cup I_i) - |I_i|$ $(X \subseteq S')$, Theorem 13.3.9 implies the following.

Theorem 13.3.11 *Given M_i-independent sets I_i $(i = 1, \ldots, k)$ can be completed to M_i-independent sets covering S if and only if*

$$\sum_i [r_i(X \cup I_i) - |I_i|] \geq |X| \tag{13.43}$$

holds for every $X \subseteq S'$.•

Packing bases

Recall that the co-rank $t(X)$ of a set X was defined as the minimum of $|B \cap X|$ over the bases of M and hence $t(X) = r(S) - r(S - X)$.

Theorem 13.3.12 (Edmonds) *We are given k matroids on a common ground-set S with rank and co-rank functions r_i and t_i $(i = 1, \ldots, k)$. There are disjoint M_i-bases for $i = 1, \ldots, k$ if and only if*

$$\sum_i t_i(X) \leq |X| \text{ for every } X \subseteq S, \tag{13.44}$$

which, in turn, is equivalent to

$$\sum_i [r_i(S) - r_i(Y)] \leq |S - Y| \text{ for every } Y \subseteq S. \tag{13.45}$$

Proof. Since $t(X) = r(S) - r(S - X)$, the equivalence of (13.44) and (13.45) follows by the substitution $Y = S - X$. Since the necessity of (13.44) is evident, we prove only the sufficiency of (13.45). To this end, all we need to show is that (13.45) implies the existence of a partitionable set of size $\sum_i r_i(S)$. By Theorem 13.3.1 this holds if $\sum_i r_i(Y) + |S - Y| \geq \sum_i r_i(S)$ for every Y but this last inequality is just (13.45). •

Theorem 13.3.13 *A matroid $M = (S, r)$ has k disjoint bases if and only if*

$$k(r(S) - r(Y)) \leq |S - Y| \text{ for every closed } Y \subseteq S \tag{13.46}$$

or equivalently

$$kt(X) \leq |X| \text{ for every open } X \subseteq S. \tag{13.47}$$

Proof. If a set Y' violates (13.46), then so does its closure and hence the result follows from Theorem 13.3.12. •

Theorem 13.3.12 can also be generalized into a completion form. Suppose that $I_1, \ldots, I_k$ are given disjoint sets such that I_i is M_i-independent. Let $S' := S - \cup_i I_i$ and let M_i' be the matroid on S' arising from $M_i|(S' \cup I_i)$ by contracting I_i. The basis completion problem is tantamount to finding disjoint bases of matroids M_i'. Theorem 13.3.12 implies the following.

Theorem 13.3.14 *The disjoint M_i-independent sets I_i $(i = 1, \ldots, k)$ can be completed to disjoint bases of M_i if and only if*

$$\sum_i [r_i(S' \cup I_i) - r_i(Y \cup I_i)] \leq |S' - Y| \tag{13.48}$$

for every $Y \subseteq S'$. •

Problem 13.3.3 $(*)$ *Prove the following theorem.*

Theorem 13.3.15 *Let $M = (S, r)$ be a loop-free matroid and $J \subseteq S$ a subset of at most k elements.*

(A) *If S can be partitioned into k independent sets, then S can be partitioned into k independent sets in such a way that each of them contains at most one element of J.*
(B) *If there are k disjoint bases, then there are k disjoint bases in such a way that each of them contains at most one element of J and the union of them includes J.* •

13.3.3 Bounded coverings and packings

Theorem 13.3.9 and 13.3.12 provided characterizations for the existence of k bases containing each element at least once and at most once, respectively. More general bounds on the elements can also be imposed, moreover, instead of using bases, one can use independent sets with upper and lower bounds on their sizes.

Bounds on the covering numbers

Given a prescription $m : S \to \mathbf{Z}_+$, when do k bases exist such that each element s belongs to exactly $m(s)$ of them? Theorem 13.3.9 immediately provides the following answer.

Theorem 13.3.16 *Let M_i be k matroids on a common ground-set S and let $m : S \to \mathbf{Z}_+$ be a function. There are M_i-bases B_i for $i = 1, \ldots, k$ such that each $s \in S$ is contained by $m(s)$ of them if and only if $\widetilde{m}(S) = \sum_i r_i(S)$ and*

$$\textstyle\sum_i r_i(X) \geq \widetilde{m}(X) \text{ for every } X \subseteq S. \tag{13.49}$$

Proof. The conditions are clearly necessary. To see the sufficiency, replace each element s by $m(s)$ parallel copies in each of the k matroids. Let S' denote the extended ground-set and M_i' the arising matroids. Since condition (13.49) is equivalent to (13.42), the extended ground-set can be covered by k independent sets, one from each M_i'. Therefore there are M_i-independent sets I_i such that each element s of S belongs to at least $m(s)$ of them. But the condition $\sum_i r_i(S) = \widetilde{m}(S)$ implies that each of these independent sets I_i must be a basis of M_i and that each s is in exactly $m(s)$ of them. •

Corollary 13.3.17 *The base polyhedron of a matroid admits the integer decomposition property.*

Proof. For a matroid M on ground-set S, we proved in Theorem 5.5.8 that the polytope B of the bases of M can be given in the form $B = \{x :\ x \geq 0,\ \widetilde{x}(X) \leq r(X)$ for every $X \subset S$ and $\widetilde{x}(S) = r(S)\}$. Suppose now that m is an integer element of $B_k := \{x :\ x \geq 0,\ \widetilde{x}(X) \leq kr(X)$ for every $X \subset S$ and $\widetilde{x}(S) = kr(S)\}$. By Theorem 13.3.16, there are bases $B_1, \dots, B_k$ of M such that each element s of S belongs to exactly $m(s)$ of these bases. Hence $m = \underline{\chi}_{B_1} + \cdots \underline{\chi}_{B_k}$ and $\underline{\chi}_{B_i} \in B$ for $i = 1, \dots, k$ and the corollary follows. •

Suppose now that we are given lower and upper bound functions $f : S \to \mathbf{Z}$ and $g : S \to \mathbf{Z}$ for which $0 \leq f \leq g$.

Theorem 13.3.18 *Let M_i be matroids with rank and co-rank functions r_i and t_i, respectively, for $i = 1, \dots, k$. It is possible to select a basis of each M_i such that*

(A) *each element s belongs to at least $f(s)$ of them if and only if*

$$b(X) := \textstyle\sum_i r_i(X) \geq \widetilde{f}(X) \text{ for every } X \subseteq S, \tag{13.50}$$

(B) *each element s belongs to at most $g(s)$ of them if and only if*

$$p(X) := \textstyle\sum_i t_i(X) \leq \widetilde{g}(X) \text{ for every } X \subseteq S, \tag{13.51}$$

(C) *each element s belongs to at most $g(s)$ and at least $f(s)$ of the k selected bases if and only if problems* (A) *and* (B) *can be solved separately, that is, if both* (13.50) *and* (13.51) *hold.*

Proof. The first two parts of the theorem follows immediately by parallel multiplication of the elements from Theorems 13.3.9 and 13.3.12, respectively.

For proving the sufficiency in Part (C), notice that p and b satisfy the following cross-inequality for every X, Y:

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X). \tag{13.52}$$

We proceed by induction on $\sum[g(s) - f(s) : s \in S']$. When this number is 0, that is, $f = g$, then $m := g$ satisfies (13.49) and hence the required bases exist by Theorem 13.3.16. Suppose now that there is an element $s \in S'$ for which $f(s) < g(s)$. We can assume that $f(s)$ cannot be increased by 1 without destroying (13.50). Therefore there is a set X containing s

978–0–19–920527–1 13-Andras-Frank-c13-drv Andras Frank (Typeset by SPi, Publisher Services, India) 443 of 640 December 11, 2010 15:20

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

for which $\widetilde{f}(X) = b(X)$. Similarly, we can assume that $g(s)$ cannot be decreased by 1 without destroying (13.51), implying that there is a set Y containing s for which $\widetilde{g}(Y) = p(Y)$. But then we have $\widetilde{f}(X) - \widetilde{g}(Y) = b(X) - p(Y) \geq b(X - Y) - p(Y - X) \geq \widetilde{f}(X - Y) - \widetilde{g}(Y - X) = \widetilde{f}(X) - \widetilde{f}(X \cap Y) - [\widetilde{g}(Y) - \widetilde{g}(X \cap Y)]$ from which $\widetilde{f}(X \cap Y) \geq \widetilde{g}(X \cap Y)$ contradicting $f \leq g$ and $f(s) < g(s)$. •

Bounds on the sizes

Theorem 13.3.19 *Let $M_1 = (S, r_1), \ldots, M_k = (S, r_k)$ be matroids and let $f_i \leq g_i$ be non-negative integer bounds for $i = 1, \ldots, k$. It is possible to choose sets $I_1, \ldots I_k$ such that I_i is M_i-independent and such that these sets*

(A) *cover S and $|I_j| \leq g_j$ for $j = 1, \ldots, k$ if and only if*

$$|X| \leq \textstyle\sum_i \min\{r_i(X), g_i\} \text{ for every } X \subseteq S, \tag{13.53}$$

(B) *are disjoint and $|I_j| \geq f_j$ for $j = 1, \ldots, k$ if and only if*

$$|X| \geq \textstyle\sum_i (f_i - r_i(S - X))^+ \text{ for every } X \subseteq S, \tag{13.54}$$

(C) *partition S and $f_j \leq |I_j| \leq g_j$ for $j = 1, \ldots, k$ if and only if both* (13.53) *and* (13.54) *hold.*

Proof. Define a matroid M_i' to be the g_i-shortening of M_i. That is, a subset of S is independent in M_i' if it is independent in M_i and has at most g_i elements. Part (A) follows from Theorem 13.3.9, when it is applied to matroids M_i'.

In Part (B), the necessity of (13.54) is obvious. For proving the sufficiency, assume (13.54). This implies $f_i \leq r_i(S)$ for each i for otherwise $X = \emptyset$ would violate (13.54). Let M_i' denote the f_i-shortening of M_i. Its rank function is given by $r_i'(X) = \min\{f_i, r_i(X)\}$. The packing in question exists if and only if there are disjoint M_i'-bases. By Theorem 13.3.12 this is the case precisely if

$$|X| \geq \textstyle\sum_i \left[r_i'(S) - r_i'(S - X)\right] \text{ for every } X \subseteq S. \tag{13.55}$$

Since $r_i'(S) = \min\{r_i(S), f_i\} = f_i$, we get $\sum_i \left[r_i'(S) - r_i'(S - X)\right] = \sum_i [f_i - \min\{r_i(S - X), f_i\}] = \sum_i (f_i - r_i(S - X))^+$, that is, (13.54) and (13.55) are equivalent.

For proving Part (C), we invoke Edmonds' matroid partition algorithm. Recall that the algorithm may start with any choice of M_i-independent sets F_i. It was an important feature of the algorithm that, at an augmenting step, the size of the current M_i-independent sets never decreases.

By Part (B), there are disjoint M_i-independent sets F_i for which $|F_i| = f_i$ and the partition algorithm is able to construct them (since one has to find disjoint bases in the f_i-shortened matroids). Continue to run the algorithm with these sets F_i with respect to the g_i-shortened matroids. Since S can be covered by independent set of these matroids, the algorithm will find such a covering and this will be the requested partition as the size of the current M_i-sets cannot get smaller than f_i. •

13.3.4 Applications to graphs and digraphs

In this section we exhibit some applications of the matroid partition and the (weighted) matroid intersection algorithm.

Finding a cheapest rooted k-edge-connected digraph

Let $D = (V, A)$ be a rooted k-edge-connected digraph with respect to a given root-node $r_0 \in V$. We will assume that no edge of D enters r_0. Given a non-negative cost function $c : A \to \mathbf{R}_+$ on the edge set, the **rooted k-edge-connection** problem consists of finding a cheapest spanning subgraph $D' = (V, A')$ of D such that D' is rooted k-edge-connected.

When $k = 1$, this problem amounts to finding a minimum-cost spanning arborescence for which a relatively simple algorithmic solution was described in Section 3.2.2. For the general case when $k \geq 1$, Edmonds suggested a reduction to weighted matroid intersection. The key is the following consequence of his disjoint arborescences theorem.

Theorem 13.3.20 *A digraph $D' = (V, B)$ is the union of k edge-disjoint spanning arborescences of root r_0 if and only if*

$$\varrho(r_0) = 0 \text{ and } \varrho(v) = k \text{ for every } v \in V - r_0 \tag{13.56}$$

and the underlying undirected graph of D' is the union of k edge-disjoint spanning trees.

Proof. The necessity of the conditions is straightforward. Their sufficiency follows from Theorem 10.1.1 once one shows that D' is rooted k-edge-connected. Indeed, for a non-empty subset $X \subseteq V - r_0$, we have $i(X) \leq k(|X| - 1)$ since the underlying graph is the union of k forests and hence $\varrho(X) = \sum[\varrho(v) : v \in X] - i(X) = k|X| - i(X) \geq k|X| - k(|X| - 1) = k$, as required. •

Theorem 5.4.2 on the sum of matroids implies that the subsets of edges of an undirected graph which are the union of k edge-disjoint spanning trees form the set of bases of a matroid to be denoted by M_1. Let M_2 denote the partition matroid whose set of bases is defined by (13.56). Theorem 13.3.20 implies that finding a cheapest subgraph of a digraph which is the union of k edge-disjoint spanning arborescences is equivalent to computing a cheapest common basis of matroids M_1 and M_2. This can be done with the help of a weighted matroid intersection algorithm provided that the requested independence oracles (or equivalent) for the two matroids are indeed available. This is obviously the case for the partition matroid M_2. As far as M_1 is concerned, Edmonds' matroid partition algorithm (described in Subsection 13.3.1) for computing the rank of the sum of matroids provides the requested oracle.

Summing up, this method for computing a cheapest rooted k-edge-connected subgraph of a digraph needs not only the matroid intersection algorithm but it relies on the disjoint arborescence theorem, on the matroid sum construction, and on the matroid partition algorithm. Since these ingredients are polynomially tractable, the entire algorithm is of polynomial complexity. A disadvantage of the approach, however, is that apparently closely-related problems cannot be handled in this way. For example, one may be interested in finding a cheapest rooted k-node-connected subgraph of a digraph. In

Section 13.5, by using a different approach, this problem will also be reduced to matroid intersection.

Disjoint trees of maximum total weight

Given an undirected graph $G = (V, E)$ including k edge-disjoint spanning trees. Suppose we are given k weight functions $c_1, \dots, c_k$ on E, and the problem is finding k edge-disjoint spanning trees $F_1, \dots, F_k$ of G such that $\sum_i \widetilde{c}_i(F_i)$ is maximium. This is a special case of the matroid optimization problem where the problem is finding disjoint bases $B_1, \dots, B_k$ such that $\sum_i \widetilde{c}_i(B_i)$ is minimum.

In order to formulate this optimization problem as a weighted matroid intersection, consider k copies of the matroid M on disjoint ground-sets $S_1, \dots, S_k$. Let $S_0 := S_1 \cup \cdots \cup S_k$ and let $c : S_0 \to \mathbf{R}$ be the direct sum of the weight functions c_i's. Define two matroids on S_0 as follows. Let N_1 be a partition matroid in which a subset is independent if it contains at most one of the k copies corresponding to s in S_0 for each element $s \in S$. Let N_2 be a direct sum in which a subset $F_1 \cup \cdots \cup F_k$ with $F_i \subseteq S_i$ is independent if the subset corresponding to F_i in S is independent in M_i for each $i = 1, \dots, k$. Let R_1 and R_2 denote the rank functions of N_1 és N_2. There is a one-to-one correspondence between the common independent sets of N_1 and N_2 having $kr(S)$ elements and the subpartitions $\{B_1, B_2, \dots, B_k\}$ of S in which B_i is an M_i-basis. By the definition of c, a maximum weight common basis determines the optimal disjoint bases of M.

Cheapest degree-constrained trees

Let $G = (V, E)$ be an undirected graph in which S is a stable set endowed with bounding functions $f : S \to \mathbf{R}_+$ and $g : S \to \mathbf{R}_+$ with $f \leq g$. In Theorem 9.1.16, a necessary and sufficient condition was given for the existence of a spanning tree F for which $f(s) \leq d_F(s) \leq g(s)$ for every $s \in S$. Suppose now that there is such a degree-constrained spanning tree and we want to find a cheapest one with respect to a cost function c on E. Since the degree-constrained spanning trees are exactly the common $(|V| - 1)$-element independent sets of the circuit matroid of G and the generalized partition matroid defined by f and g, a cheapest degree-constrained spanning tree can be computed with the help of a weighted matroid intersection algorithm.

13.4 Matroids from intersecting sub- and supermodular functions

In Part I, Theorem 5.5.14 stated for an (integer-valued) polymatroid function b that the set-system

$$\mathcal{I}(b) := \{I \subseteq S : b(Y) \geq |Y \cap I| \text{ for every } Y \subseteq S\} \tag{13.57}$$

satisfies the independence axioms of matroids, and in this sense b defines a matroid. The goal of this section is to show that in order for (13.57) to define a matroid not all properties of a polymatroid function are really needed. This fact will be particularly useful since weaker set-functions frequently occur in applications. We assume throughout that the set-functions in question are integer-valued and take zero on the empty set. The following proposition is an easy example for such relaxations.

Proposition 13.4.1 *If b is non-negative and submodular* (but not necessarily non-decreasing)*, then $\mathcal{I}(b)$ determines a matroid whose rank function r_b is given by $r_b(Z) = \min\{b(X) + |Z - X| : X \subseteq S\}$.*

Proof. It can be easily checked that $\mathcal{I}(b) = \mathcal{I}(b_{omin})$ where b_{omin} is defined in (12.6). Furthermore, we observed in Section 12.1.1 that b_{omin} is a polymatroid function. It also follows easily from the definition of b_{omin} that $\min\{b(X) + |Z - X| : X \subseteq S\} = \min\{b_{omin}(X) + |Z - X| : X \subseteq Z\}$. Therefore Theorem 5.5.14, when applied to b_{omin} in place of b, shows that $(S, \mathcal{I}(b))$ is a matroid and implies the rank formula given in the proposition. •

13.4.1 Intersecting submodular functions

By combining Theorems 5.5.14 and 12.1.1, we obtain the following generalization due to Edmonds [80].

Theorem 13.4.2 *Let b be a non-negative intersecting submodular function. Then $\mathcal{I}(b)$ satisfies the matroid independence axioms.*

(A) *The rank function of the matroid $M = (S, \mathcal{I}(b))$ is given by:*

$$r_b(Z) = \min\left\{\sum_{i=1}^{t} b(X_i) + |Z - (X_1 \cup X_2 \cup \cdots \cup X_t)| : \{X_1, \ldots, X_t\} \textit{ a subpartition of } S\right\}. \tag{13.58}$$

(B) *If b is intersecting submodular and non-decreasing, then it suffices to take the minimum over the subpartitions of Z, that is,*

$$r_b(Z) = \min\left\{\sum_{i=1}^{t} b(X_i) + |Z - (X_1 \cup X_2 \cup \cdots \cup X_t)| : \{X_1, \ldots, X_t\} \textit{ a subpartition of } Z\right\}. \tag{13.59}$$

(C) *If b is fully submodular, then*

$$r_b(Z) = \min\{b(X) + |Z - X| : X \subseteq S\}. \tag{13.60}$$

(D) *If b is a polymatroid function, then*

$$r_b(Z) = \min\{b(X) + |Z - X| : X \subseteq Z\}. \tag{13.61}$$

Proof. Let $b^\vee$ denote the (lower) truncation of b, defined in (12.1). Theorem 12.1.1 implies that $b^\vee$ is fully submodular. We claim that $\mathcal{I}(b^\vee) = \mathcal{I}(b)$. On the one hand, $b^\vee \leq b$ implies that $\mathcal{I}(b^\vee) \subseteq \mathcal{I}(b)$. To see the reverse inclusion, let $F \in \mathcal{I}(b)$ and let X be any subset of S. By the definition of $b^\vee$, there exists a partition $\{X_1, \ldots, X_t\}$ $(t \geq 1)$ of X such that $b^\vee(X) = \sum_i b(X_i)$. Then $|F \cap X| = \sum_i |F \cap X_i| \leq \sum_i b(X_i) = b^\vee(X)$, implying that $X \in \mathcal{I}(b^\vee)$ from which $\mathcal{I}(b^\vee) \subseteq \mathcal{I}(b)$. By applying Proposition 13.4.1 to $b^\vee$ in place of b, we conclude that in Part (A) $(S, \mathcal{I}_b)$ is a matroid with rank function given by (13.58). This implies the rank formula in (B) since if b is non-decreasing then $\sum_{i=1}^{t} b(Y_i) \geq \sum_{i=1}^{t} b(F \cap Y_i)$ holds for each subpartition $\{Y_1, \ldots, Y_t\}$ of S.

Parts (C) and (D) are just repetitions of the rank formulas in Proposition 13.4.1 and Theorem 5.5.14, respectively. •

We shall say that the matroid $(S, \mathcal{I}_b)$ given in the theorem **is defined by** or **belongs to** b.

An application to source location

Let $D = (V, A)$ be a digraph with at least $k + 1$ nodes in which there are no parallel edges. For a subset $Z \subseteq V$ and a node $v \in V$, let $\kappa^+(Z, v)$ denote the maximum number of paths from Z to v which are disjoint apart from their joint terminal node v. We call a subset Z a k**-source** if $\kappa^+(Z, v) \geq k$ holds for every node $v \in V - Z$. By this definition V is always a k-source. The following statement can easily be derived from the directed node-version of Menger's theorem.

Proposition 13.4.3 *A subset $Z \subseteq V$ is a k-source if and only if $|\Gamma^-(X)| \geq k$ holds for every non-empty subset $X \subseteq V - Z$ where $\Gamma^-(X) := \{u \in V - X :$ there is an edge $uv \in A$ with $v \in X\}$.* •

The problem of finding a minimum cardinality, or more generally a minimum-weight k-source, was solved by Nagamochi, Ishii, and Ito [301], who proved that the k-sources form the generators of a matroid. (A subset of the ground-set is called a generator of the matroid if it includes a basis.) See also the book of Nagamochi and Ibaraki [300, p. 298]. Once this reduction is made, the matroid greedy algorithm can be applied since an independence oracle for the matroid in question can be constructed relying on Menger's theorem and on a Max-flow Min-cut (MFMC) routine. The theorem of Nagamochi, Ishii, and Ito is as follows. The following short proof is taken from [15].

Theorem 13.4.4 *The set $\{I : I = V - Z,\ Z$ a k-source$\}$ satisfies the independence axioms of a matroid. In other words, the k-sources form the generators of a matroid.*

Proof. Let S_k denote the set of nodes with in-degree at least k. Since a k-source contains every node of in-degree smaller than k, the complements of k-sources are subsets of S_k. Define a set-function $b_k : 2^{S_k} \to \mathbf{Z}$ as follows. Let $b_k(\emptyset) = 0$ and

$$b_k(X) := |\Gamma^-(X)| + |X| - k \text{ when } X \neq \emptyset. \tag{13.62}$$

Proposition 13.4.5 *Set function b_k is non-negative, non-decreasing, and intersecting submodular.*

Proof. For any edge $uv \in A$ with $v \in X \subseteq S_k$, the tail u is either in X or in $\Gamma^-(X)$. Since there are at least k edges entering v, we obtain that $|X| + |\Gamma^-(X)| \geq k$ and hence b_k is indeed non-negative.

Let $X \subseteq Y \subseteq S_k$. The tail of an edge entering X is either in $Y - X$ or in $V - Y$. Therefore $|\Gamma^-(X)| \leq |\Gamma^-(Y)| + |Y - X|$ from which $|\Gamma^-(X)| + |X| \leq |\Gamma^-(Y)| + |Y|$ follows, and hence b_k is indeed non-decreasing.

The intersecting submodularity of b_k follows from the fact that the set-function $|\Gamma^-(X)|$ is (fully) submodular. •

Apply Theorem 13.4.2 to $b := b_k$, and consider the matroid $M_{b_k} = (S_k, \mathcal{F}_{b_k})$ determined by the theorem. A subset $I \subseteq S_k$ is independent in M_{b_k} precisely if $b_k(X) \geq |X|$ holds for every subset X of I, and hence $|\Gamma^-(X)| \geq k$. By Proposition 13.4.3, this is just equivalent to requiring that $V - I$ is a k-source. Therefore the independent sets of matroid M_{b_k} are the complements of k-sources, as required. • •

The rank formula (13.59) in Theorem 13.4.2 allows one to derive a min-max formula for the minimum cardinality of a k-source.

Theorem 13.4.6 *In a directed graph $D = (V, A)$, the minimum cardinality of a k-source is equal to*

$$|V - S_k| + \max\left\{\sum_i [k - |\Gamma^-(X_i)|] : \{X_1, \ldots, X_t\} \textit{ a subpartition of } S_k\right\} \tag{13.63}$$

where S_k is the set of nodes with in-degree at least k.

Proof. Consider the matroid M_{b_k} determined on ground-set S_k by function (13.62). The rank of M_{b_k} is given by formula (13.59): $\min\{\sum_i b_k(X_i) + |S_k - \cup X_i| : \{X_1, \ldots, X_t\}$ a subpartition of $S_k\}$. By Theorem 13.4.4, the minimal k-sources are the complements (with respect to V) of the bases of matroid M_{b_k}. Therefore the minimum cardinality of a k-source equals

$$|V| - \min\left\{\sum_i b_k(X_i) + |S_k - \cup X_i| : \{X_1, \ldots, X_t\} \text{ a subpartitition of } S_k\right\} =$$

$$|V - S_k| - \min\left\{\sum_i [|\Gamma^-(X_i)| - k + |X_i|] - |\cup X_i| : \{X_1, \ldots, X_t\} \text{ a subpartition of } S_k\right\} =$$

$$|V - S_k| + \max\left\{\sum_i [k - |\Gamma^-(X_i)|] : \{X_1, \ldots, X_t\} \text{ a subpartition of } S_k\right\}. \bullet$$

Remark 13.4.1 Quite analogously to k-sources, one can define a k**-sink** of a digraph as a subset Z of nodes such that, for every node $v \in V - Z$ there are k directed paths from v to Z which are pairwise disjoint apart from the common initial node v. Obviously a set Z is a k-sink if and only if Z is a k-source in the digraph arising by reversing each edge of D. Therefore k-sinks also form the generators of a matroid. By applying the matroid intersection theorem, one can formulate a necessary and sufficient condition for the existence of a subset of given size that is both a k-source and a k-sink. Also, the matroid partition theorem can be used to characterize digraphs in which there is a k-source whose complement is a k-sink.

13.4.2 Closure systems

A set-system $\mathcal{K}$ on ground-set S is a **closure system** if $X \cap Y \in \mathcal{K}$ whenever $X, Y \in \mathcal{K}$ and $X \cap Y \neq \emptyset$. For technical simplicity, we shall assume that $S \in \mathcal{K}$. For example, an intersecting family is a closure system and so is the set-system of flats of a matroid. For every subset $X \subseteq S$, there is a unique smallest subset in $\mathcal{K}$ including X. This set is called the **closure** of X and is denoted by $\underline{X}$.

For intersecting members X and Y of $\mathcal{K}$, let $X \wedge Y := X \cap Y$ and $X \vee Y := \underline{X \cup Y}$. We say that a function $b : \mathcal{K} \to \mathbf{Z}_+$ is intersecting submodular on $\mathcal{K}$ if

$$b(X) + b(Y) \geq b(X \wedge Y) + b(X \vee Y)$$

holds for every pair of intersecting members of $\mathcal{K}$. The following result is due to Edmonds [80].

Theorem 13.4.7 *Let $b \geq 0$ be an intersecting submodular function on a closure system $\mathcal{K}$. Then the set-function b' defined on the ground-set S by $b'(Z) := \min\{b(X) : Z \subseteq X \in \mathcal{K}\}$ is non-decreasing and intersecting submodular. Furthermore, $\mathcal{I}_{\mathcal{K}}(b) := \{I \subseteq S : b(Y) \geq |Y \cap I|$ for every $Y \in \mathcal{K}\}$ satisfies the independence axioms of matroids.*

Proof. The definition of b' immediately shows that b' is non-decreasing. Let X_1 and X_2 be two intersecting subsets of S. Then there are members Y_1, Y_2 of $\mathcal{K}$ for which $X_i \subseteq Y_i$ and $b'(X_i) = b(Y_i)$ for $i = 1, 2$. Evidently, Y_1 and Y_2 are intersecting. Since $X_1 \cap X_2 \subseteq Y_1 \cap Y_1 \in \mathcal{K}$ and $X_1 \cup X_2 \subseteq \underline{Y_1 \cup Y_2} \in \mathcal{K}$, we have $b'(X_1) + b'(X_2) = b(Y_1) + b(Y_2) \geq b(Y_1 \cap Y_2) + b(\underline{Y_1 \cup Y_2}) \geq b'(X_1 \cap X_2) + b'(X_1 \cup X_2)$, as required for the first part.

Since $b'(X) \leq b(X)$ for $X \in \mathcal{K}$, $\mathcal{I}(b') \subseteq \mathcal{I}_{\mathcal{K}}(b)$ and we claim that here we have equality. Indeed, suppose that I is not in $\mathcal{I}(b')$. Then there exists a subset $X \subseteq S$ for which $b'(X) < |X \cap I|$. By the definition of b' there is a member Y of $\mathcal{K}$ for which $X \subseteq Y$ and $b'(X) = b(Y)$. Then $|Y \cap I| \geq |X \cap I| > b'(X) = b(Y)$ showing that I is not in $\mathcal{I}(b')$ either, and hence $\mathcal{I}(b') = \mathcal{I}_{\mathcal{K}}(b)$. It follows from Theorem 13.4.2 that $\mathcal{I}_{\mathcal{K}}(b)$ satisfies the independence axioms. •

For a hypergraph $H = (V, \mathcal{E})$ (and in particular, for a graph), we can define a closure system $\mathcal{K}_H$ on $\mathcal{E}$ as follows.

$$\mathcal{K}_H := \{I_H(X) : X \subseteq V,\ I_H(X) \neq \emptyset\}$$

where $I_H(X)$ denotes the set of hyperedges induced by X. For a digraph $D = (V, A)$, we can define a closure system $\mathcal{K}_D$ on A as follows.

$$\mathcal{K}_D := \{I_D(X) : X \text{ a bi-set on } V,\ I_D(X) \neq \emptyset\}$$

where $I_D(X)$ denotes the set of arcs induced by X. Recall that a bi-set (X_O, X_I) is said to induce a directed edge $e = uv$ if $u \in X_O$ and $v \in X_I$.

Proposition 13.4.8 *For any hypergraph $H = (V, \mathcal{E})$, $\mathcal{K}_H$ is a closure system. For any digraph $D = (V, A)$, $\mathcal{K}_D$ is a closure system.*

Proof. Let X and Y be two subsets of nodes such that $X \cap Y$ induces at least one hyperedge. Then a hyperedge Z is a subset of $X \cap Y$ if and only if Z is a subset of both X and Y from which $I_H(X \cap Y) = I_H(X) \cap I_H(Y)$ follows and hence $\mathcal{K}_H$ is closed under intersection.

Let X and Y be two bi-sets on V such that $X \sqcap Y$ induces at least one edge of D. Then an edge $e = uv$ of D is induced by $X \sqcap Y$ if and only if e is induced by both X and Y from which $I_D(X \sqcap Y) = I_D(X) \cap I_D(Y)$ follows and hence $\mathcal{K}_D$ is closed under intersection. •

Theorem 13.4.9 *Let $\mathcal{K}_H$ be a closure system defined by a hypergraph $H = (V, \mathcal{E})$. Let h be a set-function on V for which $h(Z) \geq 0$ for every $Z \in \mathcal{E}$ and the submodular inequality* *holds for every pair of subsets X and Y of V for which $X \cap Y$ induces a hyperedge of H. Then the set*

$$\mathcal{I}^* := \{\mathcal{I} \subseteq \mathcal{E} : i_{\mathcal{I}}(Z) \leq h(Z) \text{ holds for every } Z \subseteq V \text{ with } i_H(Z) \geq 1\} \tag{13.64}$$

satisfies the independence axioms of matroids.

Proof. Let $Z \subseteq V$ be a set for which $i_H(Z) \geq 1$ and consider the subset $I_H(Z)$ of hyperedges induced by Z. Then $I_H(Z)$ is a member of $\mathcal{K}_H$. Define b on $\mathcal{K}_H$ by $b(I_H(Z)) := h(Z)$. The assumptions about h imply that b is non-negative and intersecting submodular on $\mathcal{K}_H$ and Theorem 13.4.7 applies. •

Corollary 13.4.10 *Let $\mathcal{K}_H$ be a closure system defined by a hypergraph $H = (V, \mathcal{E})$. Let $m : V \to \mathbf{Z}_+$ be a non-negative, integer-valued function and ℓ a* (possible negative) *integer such that $\widetilde{m}(Z) \geq \ell$ holds for every hyperedge $Z \in \mathcal{E}$. Then*

$$\mathcal{I}^\# := \{\mathcal{I} \subseteq \mathcal{E} : i_{\mathcal{I}}(Z) \leq \widetilde{m}(Z) - \ell \text{ holds for every } Z \subseteq V \text{ with } i_H(Z) \geq 1\} \tag{13.65}$$

satisfies the independence axioms of matroids.

Proof. Define h by $h(Z) := \widetilde{m}(Z) - \ell$ and apply Theorem 13.4.9. •

The matroid on ground-set $\mathcal{E}$ arising in the corollary is called a **count matroid** of hypergraph H. Count matroids on digraphs can be introduced in an analogous way which will be discussed in the next section where several applications of count matroids will also be discussed.

Covering supermodular functions by bipartite graphs

Recall Theorem 12.3.3 on minimal subgraphs of a bipartite graph $G = (S, T; E)$ covering a positively intersecting supermodular function $p : 2^S \to \mathbf{Z}_+$ which, in addition, was element-subadditive in the sense that p satisfied

$$p(X) + p(s) \geq p(X + s) \text{ whenever } X \subseteq S \text{ and } s \in S - X.$$

Element-subadditivity implies that $p(X) \leq \ddot{p}(X)$ for every $X \subseteq S$ where $\ddot{p}(X) = \sum[p(s) : s \in X]$.

Suppose now that $c : E \to \mathbf{R}_+$ is a cost function and we are interested in finding a cheapest subgraph of G that covers p. Based on Theorems 12.3.3 and 13.4.9, we are going to show that if p is intersecting supermodular, then this optimization problem is a matroid intersection problem, and hence a weighted matroid intersection algorithm can be applied.

Theorem 13.4.11 *Let p be an intersecting supermodular and element-subadditive set-function on S and let $r := \ddot{p}(S)$. Let $G = (S, T; E)$ be a bipartite graph covering p in the sense that $|\Gamma_E(X)| \geq p(X)$ for every $X \subseteq S$. The edge-sets of minimal subgraphs of G that cover p form the r-element common independent sets of two matroids.*

Proof. Let M_1 be a partition matroid on E in which $F \subseteq E$ is independent if $d_F(v) \leq p(v)$ for each $v \in S$. Define a set-function h on ground-set $S \cup T$ as follows.

$$h(Z) := \begin{cases} \ddot{p}(Z \cap S) - p(Z \cap S) + |Z \cap T| & \text{if } Z \subseteq S \cup T \text{ and } i_G(Z) \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{13.66}$$

Function h is non-negative since $\ddot{p}(Z \cap S) \geq p(Z \cap S)$. It follows from the definition of h that G and h satisfy the hypothesis of Theorem 13.4.9. Consider the matroid M_2 on E ensured by the theorem. In M_2 a subset $F \subseteq E$ is independent if $i_F(Z) \leq h(Z)$ holds for every $Z \subseteq S \cup T$ with $i_G(Z) \geq 1$.

Claim 13.4.12 *For an r-element subset $F \subseteq E$, the subgraph $G_F = (S, T; F)$ of G is a minimal subgraph covering p if and only if F is a common independent set of M_1 and M_2.*

Proof. Suppose first that F is an r-element common independent set. Since F is independent in M_1 and $|F| = r = \sum[p(v) : v \in S]$, we obtain that $d_F(s) = p(s)$ for every $s \in S$. Suppose now indirectly that F is not independent in M_2, which means that there exists a subset $X \subseteq S$ for which $|\Gamma_F(X)| < p(X)$. Consider the set $Z := X \cup \Gamma_F(X)$. On the one hand we have

$$i_F(Z) = \sum[d_F(s) : s \in X] = \sum[p(s) : s \in X] = \ddot{p}(X).$$

On the other hand, since F is independent in M_2, we have

$$i_F(Z) \leq \ddot{p}(Z \cap S) - p(Z \cap S) + |Z \cap T| = \ddot{p}(X) - p(X) + |\Gamma_F(X)|$$

from which $p(X) \leq |\Gamma_F(X)|$ follows, contradicting the indirect assumption.

Second, suppose that G_F is a minimal subgraph for which $|\Gamma_F(X)| \geq p(X)$ for every $X \subseteq S$. Theorem 12.3.3 implies that $d_F(s) = p(s)$ for every $s \in S$ showing that F is a basis of M_1. Suppose indirectly that F is dependent in M_2. Then there exists a subset Z of $S \cup T$ for which

$$i_F(Z) > \ddot{p}(Z \cap S) - p(Z \cap S) + |Z \cap T|. \tag{13.67}$$

Assume that Z is minimal with respect to (13.67). Let $X := Z \cap S$ and $Y := Z \cap T$. We claim that $Y \subseteq \Gamma_F(X)$. Indeed, if there is an element $t \in Y - \Gamma_F(X)$, then for $Z' := Z - t$ we would have

$$i_F(Z') = i_F(Z) > \ddot{p}(Z \cap S) - p(Z \cap S) + |Z \cap T| = \ddot{p}(Z' \cap S) - p(Z' \cap S) + |Z' \cap T| + 1$$

contradicting the minimal choice of Z. Now $Y \subseteq \Gamma_F(X)$ implies $|\Gamma_F(X)| = |\Gamma_F(X) \cap Z| + |\Gamma_F(X) - Y| = |Y| + |\Gamma_F(X) - Y|$. Let α denote the number of edges in F connecting X and $\Gamma_F(X) - Y$. Then $\alpha \geq |\Gamma_F(X) - Y| = |\Gamma_F(X)| - |Y| \geq p(X) - |Y|$ and hence $i_F(X) = \ddot{p}(X) - \alpha \leq \ddot{p}(X) - p(X) + |Y|$ contradicting (13.67). This contradiction completes the proof of the claim and the theorem. •

For the special class of functions p defined by (12.22), an independence oracle for M_2 can rather easily be constructed with the help of an MFMC-subroutine (see e.g. [135]).

13.4.3 Intersecting supermodular functions

Intersecting supermodular functions can also lead to matroids but in such cases it is more convenient to describe the generators rather than the independent sets. Let $p : 2^S \to \mathbf{Z}$ be a (not-necessarily non-negative) intersecting supermodular function (with the usual assumption $p(\emptyset) = 0$) and suppose that p is subcardinal, that is,

$$p(X) \leq |X| \text{ for every set } X \subseteq S. \tag{13.68}$$

Consider the set-system $\mathcal{G}^p := \{Z \subseteq S : |Z \cap X| \geq p(X) \text{ for every } X \subseteq S\}$. Since (13.68) implies $p(S) \leq |S|$, the ground-set S is necessarily in $\mathcal{G}^p$.

Theorem 13.4.13 *Let p be an intersecting supermodular function for which $p(X) \leq |X|$ for every $X \subseteq S$. Then $\mathcal{G}^p$ is the set of generators of a matroid M^p. The co-rank of M^p is*

$$\max \left\{ \sum_i p(X_i) : \{X_1, \ldots, X_t\} \textit{ a subpartition of } S \right\}. \tag{13.69}$$

Proof. Let $b(X) := |X| - p(X)$. Then b is intersecting submodular and (13.68) implies that b is non-negative. A subset I is a member of the system $\mathcal{F}_b$ if $|I \cap X| \leq b(X)$ for every $X \subseteq S$ which is equivalent to $|X - I| \geq p(X)$, or equivalently, to $|Z \cap X| \geq p(X)$ where $Z := S - I$. This implies that I is in $\mathcal{F}_b$ exactly if $S - I$ is in $\mathcal{G}^p$. Since $\mathcal{F}(b)$ is the set of independent sets of matroid M_b, it follows that $\mathcal{G}^p$ is the set of generators of the dual matroid.

By (13.58), the rank of matroid M_b is $\min\{\sum_i b(X_i) + |S - \cup X_i| : \{X_1, \ldots, X_t\}$ subpartition$\}$. Hence the rank of M^p is $|S| - (\min\{\sum_i b(X_i) + |S - \cup X_i| : \{X_1, \ldots, X_t\}$ a subpartition$\}) = \max\{-\{\sum_i [|X_i| - p(X_i)] : \{X_1, \ldots, X_t\}$ a subpartition$] - |\cup X_i|\}\} = \max\{\sum_i p(X_i) : \{X_1, \ldots, X_t\}$ a subpartition of $S\}$. •

Theorem 13.4.14 *The co-rank function t^p of M^p is given by the following formula:*

$$t^p(Z) = \max \left\{ \sum_i p(X_i) - |\cup_i X_i - Z| : \{X_1, \ldots, X_q\} \textit{ a subpartition of } S \right\}. \tag{13.70}$$

Proof. For an arbitrary subpartition $\{X_1, \ldots, X_q\}$, a basis B contains at least $p(X_i)$ elements from each X_i and hence it contains at least $p(X_i) - |X_i - Z|$ elements from $Z - X_i$. Hence $t^p(Z) \geq \max$.

A set X is said to be tight with respect to basis B if $|B \cap X| = p(X)$. The preceding estimation implies that in order to show equality one has to find a subpartition $\{X_1, \ldots, X_q\}$ and a basis B for which $X_i - Z \subseteq B \cap X_i$, $|B \cap X_i| = p(X_i)$, and $B \cap Z \subseteq \cup X_i$. In other words, $B \cap Z$ must be covered by tight sets lying in $B \cup Z$.

Let B be a basis minimizing $|B \cap Z|$. For each element $v \in Z \cap B$, the smallest (unique) tight set $P(v)$ containing v must be included in $B \cup Z$ for if there were an element $u \in P(v) - (B \cup Z)$, then $B' := B - v + u$ would be a basis with $|B' \cap Z| < |B \cap Z|$, contradicting the minimal choice of B. Therefore there are disjoint tight sets $X_1, \ldots, X_q$ in $B \cup Z$ covering $B \cap Z$ (namely, the components of the hypergraph $\{P(v) : v \in B \cap Z\}$). •

An application to rooted k-edge-connected digraphs

Let $D = (V, A)$ be a rooted k-edge-connected digraph with respect to a root-node r_0. Let A_0 denote the set of edges leaving r_0 and let $A^* := A - A_0$. Furthermore, let $\mathcal{G}$ denote the set-system consisting of those subsets F of A_0 for which the digraph $(V, F \cup A^*)$ is a rooted k-edge-connected spanning subgraph of D.

Theorem 13.4.15 *$\mathcal{G}$ is the set of generators of a matroid.*

Proof. The definition implies that $F \in \mathcal{G}$ if and only if

$$\varrho_F(X) \geq k - \varrho_{A^*}(X) \text{ for every non-empty subset } X \text{ of } V. \tag{13.71}$$

Define a set-function p on A_0 as follows. Let $p(\emptyset) := 0$ and for a non-empty subset $X \subseteq A_0$, let

$$p(X) := \max\{k - \varrho_{A^*}(Z) : Z \subseteq V - r_0,\ Z \text{ includes the head of each element of } X\}.$$

Based on the submodularity of ϱ_{A^*} and on (13.71), it can be rather easily seen that p is an intersecting supermodular function. Moreover, (13.71) is equivalent to requiring $|F \cap X| \geq p(X)$ for every subset X of A_0. By applying Theorem 13.4.13, we conclude that $\mathcal{G}$ is the set of generators of a matroid. •

Problem 13.4.1 *Let $D = (V, A)$ be a rooted k-edge-connected digraph. The maximum number of edges of D leaving r_0 whose deletion does not destroy rooted k-edge-connectivity is equal to* $\max\{\sum[k - \varrho_{D-r_0}(Z_i)] : \{Z_1, \dots, Z_q\}$ *a partition of* $V - r_0\}$.

13.5 Count matroids

In Section 13.3.4 we already discussed optimization problems regarding graphs and hypergraphs which were tractable via matroids. In these applications, we often need count matroids on (hyper)graphs, a notion introduced in Section 13.4.2. Here we provide a direct proof to show that they are indeed matroids. The present approach facilitates the construction of an independence oracle for count matroids.

Let $G = (V, E)$ be a loopless undirected graph and $m : V \to \mathbf{Z}_+$ a non-negative, integer-valued function. Recall that $V(F)$ denotes the set of nodes covered by a subset F of edges. More generally, in a hypergraph $H = (V, \mathcal{E})$, $V(\mathcal{F})$ denotes the set of nodes covered by a subset $\mathcal{F}$ of hyperedges, or formally, $V(\mathcal{F}) = \cup(Z : Z \in \mathcal{F})$. We also introduced the set-function n_m on the set of (hyper)edges, namely $n_m(F) = \widetilde{m}(V(F))$ for graphs and $n_m(\mathcal{F}) = \widetilde{m}(V(\mathcal{F}))$ for hypergraphs, and pointed out in Proposition 1.2.8 that n_m is a polymatroid function. Beside m, we are given a (possibly negative) integer ℓ.

13.5.1 Count matroids on undirected graphs

Consider first the graph case and assume that

$$m(u) + m(v) \geq \ell \text{ for every edge } uv \in E. \tag{13.72}$$

Define a set-function $b^\#$ on ground-set E as follows. Let $b^\#(\emptyset) = 0$ and

$$b^\#(F) := \widetilde{m}(V(F)) - \ell \quad (\emptyset \subset F \subseteq E). \tag{13.73}$$

Since n_m is a polymatroid function, $b^\#$ is non-decreasing and intersecting submodular, furthermore, (13.72) ensures its non-negativity as well. (Note that $b^\#$ is a polymatroid function when $\ell \leq 0$.) Let $M^\#$ denote the matroid belonging to $b^\#$, as defined by (13.57).

Let

$$\mathcal{I}^{\#} := \{F \subseteq E : F \text{ satisfies (13.75) for every } X \subseteq V \text{ with } i_F(X) \geq 1\} \tag{13.74}$$

$$i_F(X) \leq \widetilde{m}(X) - \ell. \tag{13.75}$$

Theorem 13.5.1 *$\mathcal{I}^{\#}$ is the set of independent sets of the matroid $M^{\#}$ belonging to $b^{\#}$.*

Proof. Let F be independent in $M^{\#}$. Let $X \subseteq V$ be a set for which $i_F(X) \geq 1$ and let $J := I_F(X)$ be the subset of F induced by X. Then

$$i_F(X) = |J| \leq \widetilde{m}(V(J)) - \ell = \widetilde{m}(X) - \ell$$

implying that $F \in \mathcal{I}^{\#}$.

Conversely, if F is dependent in $M^{\#}$, that is, if there is a subset $J \subseteq F$ for which $|J| > b^{\#}(J)$, then, for $X := V(J)$, we have $i_F(X) \geq |J| > b^{\#}(J) = \widetilde{m}(X) - \ell$ showing that $F \notin \mathcal{I}^{\#}$. •

The matroid $M^{\#}$ is called a **count matroid** of graph G.

Examples

1. *k*-sparse graphs For a positive integer k, let $m \equiv k$ and $\ell := k$. In the corresponding count matroid $M_k(G) := M^{\#}$, a subset F of edges is independent if every non-empty subset X of nodes induces at most $k(|X| - 1)$ elements of F. These graphs were previously referred to as k-sparse graphs (Section 2.3). For $k = 1$, the count matroid in question is just the circuit matroid of G. By a theorem of Nash-Williams (Theorem 10.4.3), the independent sets in M_k are those subsets of edges which can be partitioned into k forests. In other words, the count matroid for parameter k is just the sum of k copies of the circuit matroid. Therefore a matroid partition algorithm can be used to check a graph for k-sparsity but in Section 2.3 we described a simpler and more efficient one relying on degree-constrained orientations.

Remark 13.5.1 In Section 13.3.4, we outlined matroid intersection approach to finding a cheapest rooted k-edge-connected spanning subgraph of a digraph. The use of count matroid $M_k(G)$ allows us to obtain another, more flexible framework for handling such problems. Namely, let M_2 be the same partition matroid as before, while M_1' is the direct sum of the free matroid on A_0 (the set of edges leaving r_0) and the restriction of $M_k(G)$ to $A^* := A - A_0$. Although this matroid is more free than the matroid M_1 used in Edmonds' reduction, it is still true that the minimal rooted k-edge-connected subgraphs of D are exactly the common independent sets of $k(|V| - 1)$ elements of M_1' and M_2. Yet another advantage of this approach is that even a given upper bound γ can be imposed for the out-degree of r_0 in the cheapest rooted k-edge-connected subgraph of D. The only difference is that M_1' should be modified to be the direct sum of the uniform matroid of rank γ on A_0 and the restriction of $M_k(G)$ to $A^* := A - A_0$. Even more generally, any other matroid can be imposed on A_0.

2. Rigidity matroids Let $G^* = (V, E^*)$ denote the complete graph on node-set V and consider the count matroid M^* defined by the parameters $m \equiv 2$ and $\ell = 3$. For an arbitrary subgraph $G = (V, E)$ of G^*, the restriction of M^* to E is called the **rigidity matroid** of G. In Section 8.3 we cited a theorem of Laman stating that a simple graph $G = (V, E)$ is minimally rigid if and only if $|E| = 2n - 3$ and $i_G(X) \leq 2|X| - 3$ for every $X \subseteq V,\ |X| \geq$

2. This is equivalent to saying that the rigidity matroid of G is a free matroid of rank $2n-3$. Let b^* be defined by $b^*(\emptyset)=0$ and $b^*(F):=2|V(F)|-3$ for a non-empty $F\subset E^*$. By Theorem 13.5.1, E is independent in the rigidity matroid if and only if $b^*(F)\geq|F|$ for every non-empty $F\subseteq E$. By applying the rank formula in Theorem 13.4.2, we obtain the following result.

Theorem 13.5.2 (Lovász and Yemini [274]) *A graph $G=(V,E)$ is rigid if and only if the rank of its rigidity matroid is $r(M_G)=2n-3$. This is equivalent to requiring that*

$$\textstyle\sum_i b^*(E_i)\geq 2n-3 \text{ for every partition } \{E_1,\dots,E_k\} \text{ of } E \tag{13.76}$$

where b^* is defined by $b^*(F):=2|V(F)|-3$. If G is not rigid, then the minimum number of new edges whose addition to G results in a rigid graph is

$$2n-3-\min\left\{\sum_i b^*(E_i):\{E_1,\dots,E_k\} \text{ a partition of } E\right\}. \tag{13.77}$$

If $\{E_1,\dots,E_k\}$ is a minimizer partition, then $|V(E_i)\cap V(E_j)|\leq 1$ for $1\leq i<j\leq k$.

Proof. The first part is a direct consequence of Theorems 13.5.1 and 13.4.2. The second part follows from the obvious property that any subset X of a matroid on S can be extended to a set of rank $r(S)$ by adding a subset of $r(S)-r(X)$ elements. Specifically, E can be extended to a set of rank $2n-3$ by adding a subset of E^*-E of cardinality $(2n-3)-r(E)$. The last part follows from the observation that, for two disjoint subsets E_i and E_j of edges, $|V(E_i)\cap V(E_j)|\geq 2$ implies that $b^*(E_i)+b^*(E_j)>b^*(E_i\cup E_j)$, and hence by replacing E_i and E_j with their union we would obtain a better partition of E. •

3. Scene analysis Let $G=(S,T;E)$ be a simple bipartite graph. Let d be a positive integer and $\ell:=d+1$. Let $m:S\cup T\to\mathbf{Z}_+$ be defined as follows.

$$m(v):=\begin{cases} d & \text{if } v\in S\\ 1 & \text{if } v\in T.\end{cases} \tag{13.78}$$

The count matroid on G belonging to these parameters plays a fundamental role in an elegant theory of interpreting line drawings of three-dimensional objects. (See, for example, the book by Sugihara [357].) Intuitively, a line drawing whose vertices are in general positions can be combinatorially described with the help of an incidence structure which is a bipartite graph $G=(S,T;E)$. Here S is the set of vertices, T is the set of faces, and E describes the incidences. A fundamental theorem of Whiteley [381] asserts that G is 'generic reconstructible' in $d=3$ dimensions if and only if the edge-set of $G=(S,T;E)$ is independent in the count matroid.

13.5.2 Count matroids on hypergraphs

The construction of count matroids easily extends to hypergraphs $H=(V,\mathcal{E})$. Let m and ℓ be the same as before and, as an extension of (13.72), assume that

$$\widetilde{m}(Z)\geq\ell \text{ for every hyperedge } Z\in\mathcal{E}. \tag{13.79}$$

Define a set-function $b^{\#}$ on ground-set $\mathcal{E}$ as follows. Let $b^{\#}(\emptyset) = 0$ and

$$b^{\#}(\mathcal{F}) := \widetilde{m}(V(\mathcal{F})) - \ell \quad (\emptyset \subset \mathcal{F} \subseteq \mathcal{E}). \tag{13.80}$$

Again, $b^{\#}$ is a non-decreasing, intersecting submodular function which is, in fact, fully submodular for $\ell \leq 0$. Note that (13.79) ensures that $b^{\#}$ is non-negative.

Let

$$\mathcal{I}^{\#} := \{\mathcal{F} \subseteq \mathcal{E} : I \text{ satisfies (13.82) for every } X \subseteq V \text{ with } i_{\mathcal{I}}(X) \geq 1\} \tag{13.81}$$

$$i_{\mathcal{I}}(X) \leq \widetilde{m}(X) - \ell, \tag{13.82}$$

or informally, the number of hyperedges from $\mathcal{F}$ induced by node-set X is at most $\widetilde{m}(X) - \ell$ whenever X induces at least one hyperedge.

Theorem 13.5.3 *$\mathcal{I}^{\#}$ is the set of independent sets of the matroid $M^{\#}$ belonging to $b^{\#}$.*

Proof. Let $\mathcal{F}$ be independent in $M^{\#}$. Let $X \subseteq V$ be a set for which $i_{\mathcal{F}}(X) \geq 1$ and let $\mathcal{J} := I_{\mathcal{F}}(X)$ be the subset of $\mathcal{F}$ induced by X. Then

$$i_{\mathcal{F}}(X) = |\mathcal{J}| \leq \widetilde{m}(V(\mathcal{J})) - \ell = \widetilde{m}(X) - \ell$$

implying that $\mathcal{F} \in \mathcal{I}^{\#}$.

Conversely, if $\mathcal{F}$ is dependent in $M^{\#}$, that is, if there is a subset $\mathcal{J} \subseteq \mathcal{F}$ for which $|\mathcal{J}| > b^{\#}(\mathcal{J})$, then, for $X := V(\mathcal{J})$, we have $i_{\mathcal{F}}(X) \geq |\mathcal{J}| > b^{\#}(\mathcal{J}) = \widetilde{m}(X) - \ell$ showing that $\mathcal{F} \notin \mathcal{I}^{\#}$. •

The matroid $M^{\#}$ is called a **count matroid** of hypergraph H.

Examples

1. Transversal matroids One of the equivalent ways for defining transversal matroids used the edge-set of a hypergraph $H = (V, \mathcal{E})$ to be the ground-set and declared a subset of $\mathcal{F}$ edges independent if $\mathcal{F}$ has a system of distinct representatives. By Hall's theorem this is equivalent to requiring that the union of every j members of $\mathcal{F}$ has at least j elements. Therefore a transversal matroid is the count matroid on H belonging to the parameters $m \equiv 1$, $\ell = 0$.

2. Hypergraphic matroids In Section 5.2, hypergraphic matroids were defined on the edge-set of a hypergraph H in such a way that a subset of hyperedges is independent if the union of any $j \geq 1$ hyperedges has at least $j + 1$ elements. Therefore a hypergraphic matroid is a count matroid belonging to the parameters $m \equiv 1$ and $\ell = 1$.

A paper of Whiteley [380] exhibits several special cases and applications of count matroids on graphs and hypergraphs.

Problem 13.5.1 *Prove that the rank of the hypergraphic matroid of a hypergraph $H = (V, \mathcal{E})$ is equal to*

$$\min\{|V| - |\mathcal{P}| + e_H(\mathcal{P}) : \mathcal{P} \text{ a partition of } V\} \tag{13.83}$$

where $e_H(\mathcal{P})$ denotes the number of hyperedges intersecting at least two members of the partition.

Problem 13.5.2 *Prove that a hypergraph $H = (V, \mathcal{E})$ is partition-connected if and only if the rank of its hypergraphic matroid is $|V| - 1$.*

13.5.3 Count matroids on directed graphs

We are going to extend the notion of count matroids for digraphs (see [135]) in such a way that the extension will include count matroids on undirected graphs. The idea is that bi-sets of nodes take over the role of subsets. Recall that a directed edge $e = uv$ is said to be induced by a bi-set $X = (X_O, X_I)$ if the head v of e is in X_I while its tail u is in X_O. For a digraph $D = (V, A)$, let $I_D(X) := I_A(X)$ denote the set of edges induced by X and let $i_D(X) := i_A(X) := |I_D(X)|$.

Let $D^* = (V^*, A^*)$ be a digraph, $m_O : V^* \to \mathbf{Z}_+$ and $m_I : V^* \to \mathbf{Z}_+$ two functions and ℓ an integer such that

$$m_I(v) + m_O(u) + m_O(v) \geq \ell \quad \text{for every edge } uv \in A^*. \tag{13.84}$$

For a subset $J \subseteq A^*$ of edges of digraph D^*, let $H(J) := \{v : v$ is the head of some edges in $J\}$ and $V(J) := \{u : u$ is the head or the tail of some edges in $J\}$. Note that $V(J)$ is actually a set-function on the underlying undirected edge-set and independent of the orientation of the edges. Let

$$b_I(J) := \widetilde{m}_I(H(J)) \text{ and } b_O(J) := \widetilde{m}_O(V(J)). \tag{13.85}$$

The proof of the following proposition is an easy exercise and is left to the reader.

Proposition 13.5.4 *Both b_I and b_O are non-decreasing submodular functions on ground-set A^*.* •

Define the set-function $b^\#$ to be 0 on the empty set and, for $\emptyset \subset J \subseteq A^*$,

$$b^\#(J) := b_I(J) + b_O(J) - \ell. \tag{13.86}$$

By Proposition 13.5.4, $b^\#$ is intersecting submodular. Furthermore, if $uv \in A^*$ is an edge, then (13.84) implies for the singleton $J := \{uv\}$ that $b^\#(J) = \widetilde{m}_I(H(J)) + \widetilde{m}_O(V(J)) - \ell = m_I(v) + m_O(u) + m_O(v) - \ell \geq 0$. This and the non-negativity of m_I and m_O imply that $b^\#$ is non-decreasing (and hence non-negative). Let $M^\# := M_{b^\#}$ denote the matroid of $b^\#$ defined in Theorem 13.4.2.

Let

$$\mathcal{I}^\# := \{F \subseteq A^* : F \text{ satisfies (13.88) for every bi-set } X = (X_O, X_I) \text{ with } i_F(X) \geq 1\}: \tag{13.87}$$

$$i_F(X) \leq \widetilde{m}_I(X_I) + \widetilde{m}_O(X_O) - \ell. \tag{13.88}$$

Theorem 13.5.5 *$\mathcal{I}^\#$ is the set of independent sets of matroid $M^\#$ on A^* belonging to $b^\#$.*

Let F be independent in $M^\#$. Let X be a bi-set for which $i_F(X) \geq 1$ and let $J := I_F(X)$ be the subset of F induced by X. Since $H(J) \subseteq X_I$ and $V(J) \subseteq X_O$, we have $i_F(X) = |J| \leq b^\#(J) = \widetilde{m}_I(H(J) + \widetilde{m}_O(V(J) - \ell \leq \widetilde{m}_I(X_I) + \widetilde{m}_O(X_O) - \ell$, and hence (13.88) holds and $F \in \mathcal{F}$.

Conversely, if F is dependent in $M^{\#}$, then it has a subset J for which $|J| > b^{\#}(J)$. Let $X_I := H(J)$ and $X_O := V(J)$. Then every element of J is induced by bi-set $X = (X_O, X_I)$ and hence $i_F(X) \geq |J| > b^{\#}(J) = \widetilde{m}_I(H(J)) + \widetilde{m}_O(V(J)) - \ell$, and hence (13.88) is violated. •

Theorem 13.5.6 *Every count matroid on an undirected graph arises as a count matroid on a digraph.*

Proof. Consider a count matroid M on an undirected graph $G = (V, E)$ defined by the parameters m and ℓ. To see that M arises as a count matroid on a digraph, let $D = (V, A)$ be an arbitrary orientation of G. Define $m_I \equiv 0$ and $m_O = m$.

Then (13.87) and (13.88) read as $\mathcal{I}^{\#} := \{F \subseteq A^* : i_F(X) \leq \widetilde{m}_O(X_O) - \ell$ for every bi-set $X = (X_O, X_I)$ with $i_F(X) \geq 1\}$. Also, (13.84) is equivalent to (13.72). Let M' be the count matroid on D defined by m_I, m_O, and ℓ. Since $i_F(X_O, X_O) \geq i_F(X_O, X_I)$, the inequality $i_F(X) \leq \widetilde{m}_O(X_O) - \ell$ holds for every bi-set with $i_F(X) \geq 1$ if it holds for simple bi-sets of type (X_O, X_O), and hence $\mathcal{I}^{\#} = \{F \subseteq A^* : i_F(X_O) \leq \widetilde{m}_O(X_O) - \ell$ for every subset $X_O \subseteq V^*\}$. Since in this case the orientation of the edges does not play any role, we conclude that M and M' are isomorphic. •

An application to computing a cheapest rooted k-node-connected subgraph of a digraph

Let $D = (V, A)$ be a digraph with a specified root-node r_0 and let $c : A \to \mathbf{R}_+$ be a cost function. We indicated in Remark 13.5.1 how count matroids on undirected graphs can be used to compute a cheapest rooted k-edge-connected subgraph of D.

As an interesting application of count matroids on digraphs, we show that the problem of computing a cheapest rooted k-node-connected subgraph of a rooted k-node-connected digraph D (on the same node-set V) can be formulated as a matroid intersection problem [135]. Hence this optimization problem becomes solvable in strongly polynomial time with the help of a weighted matroid intersection algorithm, once an independence oracle is available for the count matroid. Such an oracle will be described in the next section.

Recall (p. 26) that a digraph $D = (V, A)$ is said to be rooted k-node-connected with respect to root-node r_0 if $\varrho_D(X) + w(X) \geq k$ holds for every non-trivial bi-set $X = (X_O, X_I)$ with $X_O \subseteq V - r_0$ (where $w(X) = |X_O| - |X_I|$). By Menger's theorem (Theorem 2.5.7), this is equivalent to requiring that $\kappa_D(r_0, v) \geq k$ for every $v \in V - r_0$. By a k**-foliage**, we mean a rooted k-node-connected digraph which is minimal with respect to edge-deletion. Such a digraph has no edges entering the root-node r_0, and Theorem 7.4.7 implies that the in-degree of every node $t \in V - r_0$ is exactly k.

Let A^* denote the set of edges of D induced by $V^* := V - r_0$, that is, $D^* = (V^*, A^*)$ is the digraph $D - r_0$. Let $A_0 := A - A^*$, that is, A_0 is the set of edges leaving r_0. Consider the count matroid $M^{\#}$ on A^* determined by $b_2(X) := m_O(X_O) + m_I(X_I) - \ell$ where $\ell := k$ and, for $v \in V^*$, $m_O(v) := 1$, $m_I(v) := k - 1$. For brevity, we shall refer to $M^{\#}$ as the **master** matroid. A simple calculation shows that $b_2(X) = k(|X_I| - 1) + w(X)$, and hence a subset $F \subseteq A^*$ is independent in $M^{\#}$ if and only if

$$i_F(X) \leq b_2(X) \tag{13.89}$$

for every bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq V^*$.

Define a matroid M_1 on A to be the direct sum of the free matroid on A_0 (in which, by definition, every subset is independent) and the master matroid $M^{\#}$. Let M_2 denote the partition matroid on ground-set A in which a subset $I \subseteq A$ is independent if $\varrho_I(v) \leq k$ for every node $v \in V^*$ (and $\varrho_I(r_0) = 0$).

Theorem 13.5.7 *A subgraph $D_B = (V, B)$ of digraph $D = (V, A)$ is a k-foliage if and only if B is a common independent set of matroids M_1 and M_2 and $|B| = k(n-1)$ where $n = |V|$.*

Proof. If D_B is a k-foliage, then $\varrho_B(v) = k$ and $\varrho_B(r_0) = 0$. Hence D_B has exactly $k(n-1)$ edges and B is a basis in M_2. For any bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq V^*$, one has $\varrho_B(X) + w(X) \geq k$ and hence $i_B(X) = \sum[\varrho_B(v) : v \in X_I] - \varrho_B(X) \leq k|X_I| + w(X) - k = b_2(X)$ and thus B is independent in M_1.

Conversely, suppose that a $k(n-1)$-element subset $B \subseteq A$ of edges is independent in both M_1 and M_2. Then $\varrho_B(v) = k$ for every $v \in V^*$ and $\varrho_B(r_0) = 0$. Furthermore, for a bi-set $X = (X_O, X_I)$ with $\emptyset \subset X_I \subseteq X_O \subseteq V^*$, one has $i_B(X) \leq k(|X_I| - 1) + w(X)$. Therefore $\varrho_B(X) + w(X) = \sum[\varrho_B(v) : v \in X_I] - i_B(X) + w(X) = k|X_I| - i_B(X) + w(X) \geq k|X_I| - k(|X_I| - 1) = k$ and hence $D_B = (V, B)$ is a k-foliage. •

Edmonds' matroid intersection theorem, combined with the rank-formula (13.59), provides a necessary and sufficient condition for the existence of a rooted k-node-connected subgraph of a digraph in which there is an upper bound on the out-degree of r_0.

Problem 13.5.3 *Derive the following theorem.*

Theorem 13.5.8 *A digraph $D = (V, A)$ admits a rooted k-node-connected subgraph in which the out-degree of root r_0 is at most a given integer γ if and only if $\sum_i [k - (\varrho_{D^*}(X_i) + w(X_i))] \leq \gamma$ for every set of non-void bi-sets $X_1, \ldots, X_q$ whose outer members are subsets of V^* and inner members are pairwise disjoint, where $D^* = D - r_0$. There is a rooted k-edge-connected subgraph of D in which the out-degree of r_0 is at most γ if and only if $\sum_i [k - \varrho_{D^*}(X_i)] \leq \gamma$ holds for every set of pairwise disjoint subsets X_i of V^*.* •

Note that if we impose upper bound restrictions for the out-degree of nodes $v \in V^*$, then the problem becomes **NP**-complete even in the special case when $k = 1$ since the Hamilton path problem can be formulated in this way.

The same approach can be used to handle a common generalization of rooted k-edge- and k-node-connected subgraphs. Let $D = (V, F)$, r_0, V^* be the same as before and let $g : V^* \to \{1, 2, \ldots, k\}$ be a function. We say that D is rooted (k, g)**-connected** if

$$\varrho_D(X) + w_g(X) \geq k \text{ for every bi-set } X = (X_O, X_I) \text{ with } \emptyset \subset X_I \subseteq X_O \subseteq V^* \tag{13.90}$$

where

$$w_g(X) := \widetilde{g}(X_O - X_I).$$

Recall that a set of edge-disjoint r_0t-paths are g-bounded if every node $v \in V^* - t$ belongs to at most $g(v)$ paths from this set. Furthermore, a hybrid version of Menger's theorem

(Theorem 2.5.12) implies that D is rooted (k, g)-connected if and only if

$$\lambda_g(r_0, t; D) \geq k \text{ holds for every } t \in V^* \tag{13.91}$$

where $\lambda_g(r_0, t; D)$ denotes the maximum number of g-bounded r_0t-paths. Note that rooted (k, g)-connectivity specializes to rooted k-connectivity when $g \equiv 1$ and to rooted k-edge-connectivity when $g \equiv k$. A digraph is called a (k, g)-**foliage** (of root r_0) if it is rooted (k, g)-connected but deleting any edge destroys this property. In Theorem 7.4.7, we showed that the in-degree of every node $v \in V^*$ of a (k, g)-foliage is exactly k.

Problem 13.5.4 *Show that the problem of finding a cheapest (k, g)-foliage of a digraph can be formulated as a weighted matroid intersection problem.*

Further details on the use of count matroids on digraphs can be found in [135].

13.5.4 Independence oracle for count matroids

In order to apply count matroids in algorithms, one must have an independence oracle to decide for any input subset X whether X is independent or not in $M^{\#}$. Sugihara and Imai [215, 358, 357] developed such an algorithm based on MFMC computations.

Here we follow another approach suggested by Berg and Jordán [22] who applied a degree-constrained orientation algorithm for rigidity matroids and noted that the method works for any count matroid on graphs. We describe the method for count matroids on digraphs, and show then how it can be adapted to undirected graphs.

The following lemma is a slight extension of a theorem of Hakimi (Theorem 2.3.5) which is obtained in the special case when $\gamma = 0$.

Lemma 13.5.9 *Let $H = (V, F)$ be an undirected graph, $g' : V \to \mathbf{Z}_+$ an upper-bound function and $\gamma \geq 0$ an integer. It is possible to remove at most γ edges from H in such a way that the remaining graph H' has an orientation with in-degree function ϱ' satisfying $\varrho'(v) \leq g'(v)$ for every node v if and only if*

$$g'(X) + \gamma \geq i_H(X) \tag{13.92}$$

holds for every subset X of nodes.

Proof. Necessity. If a subgraph $H' = (V, F')$ with $|F - F'| \leq \gamma$ has the required orientation, then $0 \leq \varrho'(X) = \sum[\varrho'(v) : v \in X] - i_{H'}(X) \leq \sum[g'(v) : v \in X] - (i_H(X) - \gamma)$ from which (13.92) follows.

Sufficiency. Starting with an arbitrary orientation of H, we gradually reduce the 'error-sum' $\sum[(\varrho(v) - g'(v))^+ : v \in V]$ by successively reorienting certain paths. Let Z_0 denote the set of bad nodes z where bad menas that $\varrho(z) > g'(z)$. If Z_0 is empty, then the current orientation of H itself is good. For a non-empty Z_0, compute the set Z of nodes from which Z_0 is reachable along a directed path in the current orientation. Then $Z_0 \subseteq Z$ and no edge enters Z. If there is a node $u \in Z$ with $\varrho(u) < g'(u)$, then by reorienting any path from u to Z_0 the error-sum becomes smaller. If no such node u exists, then remove $\varrho(z) - g'(z)$ entering edges at every bad node z. In the remaining digraph the in-degree of every node v is at most $g'(v)$. The number of removed edges is $\sum[\varrho(v) - g'(v) : v \in Z_0] = \sum[\varrho(v) - g'(v) : v \in Z] = \varrho(Z) + i_H(Z) - g'(Z) = i_H(Z) - g'(Z) \leq \gamma$. •

Note that the proof of the lemma gives rise to an algorithm of complexity $O(|V||E|^2)$ and since it can be considered as a variation of the alternating path algorithm for flows, the bound can actually be reduced to $O(|V|^3)$.

For the digraph $D_F = (V^*, F)$, construct a bipartite undirected graph $G = (V', V''; E)$ as follows. To every node $v \in V^*$, assign a node $v' \in V'$ and a node $v'' \in V''$ which are connected by $m_O(v)$ parallel edges. The set of these edges is denoted by E_1. Furthermore, with every directed edge $e = uv \in F$, we associate an edge $e_G = u'v''$ of G. The set of these edges is denoted by E_2. Let $E := E_1 \cup E_2$ and $V_G := V' \cup V''$. We use the convention that the subsets of V' and V'' corresponding to a subset $X \subseteq V^*$ will be denoted by X' and X'', respectively.

By (13.84), there are integers $0 \leq \ell(s) \leq m_O(s)$ and $0 \leq \ell(z) \leq m_O(z) + m_I(z)$ for which $\ell = \ell(s) + \ell(z)$. For example, $\ell(z) := \min\{\ell, m_O(z) + m_I(z)\}$ and $\ell(s) := \ell - \ell(z)$ will suffice.

Define a function $g' : V_G \to \mathbf{Z}_+$ as follows. Let $g'(s') := m_O(s) - \ell(s)$ and $g'(z'') := m_O(z) + m_I(z) - \ell(z)$. For $v' \in V' - s'$, let $g'(v') := m_O(v)$, and for $v'' \in V'' - z''$, let $g'(v'') := m_O(v) + m_I(v)$.

Lemma 13.5.10 *For the independent set $F' \subseteq A^*$ of $M^\#$ and for the edge $f = sz \in A^* - F'$, the set $F := F' + sz$ is independent in $M^\#$ if and only if G has an orientation in which the in-degree of each node x is at most $g'(x)$.*

Proof. Assume first that the required orientation does not exist. By Lemma 13.5.9 (when applied for $\gamma = 0$) there is a subset $X' \cup Y'' \subseteq V_G$ of nodes for which $i_G(X' \cup Y'') > g'(X' \cup Y'')$. Let $J \subseteq F$ denote the set of those edges $e = uv$ for which $u' \in X'$ and $v'' \in Y''$. Since $X' \cup Y''$ induces $\widetilde{m}_O(X \cap Y)$ edges from E_1, we have $|J| + \widetilde{m}_O(X \cap Y) = i_G(X' \cup Y'') > g'(X' \cup Y'') = g'(X') + g'(Y'') \geq [\widetilde{m}_O(X) - \ell(s)] + [\widetilde{m}_O(Y) + \widetilde{m}_I(Y) - \ell(z)] = \widetilde{m}_O(X) + \widetilde{m}_O(Y) + \widetilde{m}_I(Y) - \ell$, from which

$$|J| > \widetilde{m}_O(X \cup Y) + \widetilde{m}_I(Y) - \ell. \tag{13.93}$$

If, indirectly, F were independent in $M^\#$, then we would have $|J| \leq b^\#(J) = \widetilde{m}_I(H(J)) + \widetilde{m}_O(V(J)) - \ell \leq \widetilde{m}_I(Y) + \widetilde{m}_O(X \cup Y) - \ell$, contradicting (13.93).

To see the converse, assume that F is dependent in $M^\#$, that is, there is a bi-set $X = (X_O, X_I)$ for which

$$|J| = i_F(X) > \widetilde{m}_O(X_O) + \widetilde{m}_I(X_I) - \ell \tag{13.94}$$

where J denotes the subset of F induced by X.

Since F' is independent in $M^\#$, X must induce $f = sz$, that is, $s \in X_O$ and $z \in X_I$. Hence $g'\left(X''_I\right) = \widetilde{m}_O(X_I) + \widetilde{m}_I(X_I) - \ell(z)$ and $g'\left(X'_O\right) = \widetilde{m}_O(X_O) - \ell(s)$ from which $g'\left(X'_O \cup X''_I\right) = \widetilde{m}_I(X_I) + \widetilde{m}_O(X_I) + \widetilde{m}_O(X_O) - \ell$. The set $X'_O \cup X''_I \subseteq V_G$ induces (in G) $\widetilde{m}_O(X_I)$ edges from E_1. If, indirectly, the requested orientation does exist, then $|J| + \widetilde{m}_O(X_I) = i_G\left(X'_O \cup X''_I\right) \leq g'\left(X'_O \cup X''_I\right) = \widetilde{m}_I(X_I) + \widetilde{m}_O(X_I) + \widetilde{m}_O(X_O) - \ell$, and hence $|J| \leq \widetilde{m}_I(X_I) + \widetilde{m}_O(X_O) - \ell$, contradicting (13.94). •

We can conclude that with the help of the orientation lemma the necessary independence oracle for $M^\#$ is available (and this does not rely on the matroid partition algorithm).

Remark 13.5.2 In the oracle above, we needed the orientation result only in the special case when $\gamma = 0$, which is Hakimi's original theorem. The general form has only been included in order to outline an independence oracle for count matroids in the case when $\ell < 0$, which is even simpler than the one above for non-negative ℓ. Although this is not required for our present purposes, for the sake of completeness we include it.

For a negative ℓ, let $\gamma := -\ell$. For a subset $F \subseteq A^*$, we consider the same bipartite graph as before. (Now there is no special element of F and no a priori assumption is made about the independence of any subset of F.) Define a function $g' : V_G \to \mathbf{Z}_+$ as follows. For $v' \in V'$, let $g'(v') := m_O(v)$, and for $v'' \in V''$, let $g'(v'') := m_O(v) + m_I(v)$. Similarly to the proof of Lemma 13.5.11, it can be shown that F is independent in $M^\#$ if and only if $\gamma + i_G(X' \cup Y'') \leq g'(X' \cup Y'')$ for every subsets $X' \subseteq V'$, $Y'' \subseteq V''$, and this condition can be checked with the help of the orientation lemma 13.5.9.

Adaptation for undirected graphs

The algorithm can be adapted to the undirected case as follows. By (13.72), $m(s) + m(z) \geq \ell$. For $\alpha := \min\{\ell, m(s)\}$, we have $0 \leq \alpha \leq m(s)$ and $0 \leq \ell - \alpha \leq m(z)$. Define $g : V \to \mathbf{Z}_+$ as follows.

$$g(v) := \begin{cases} m(v) & \text{if } v \in V - \{s, z\} \\ m(s) - \alpha & \text{if } v = s \\ m(z) - (\ell - \alpha) & \text{if } v = z. \end{cases} \tag{13.95}$$

Obviously, $0 \leq g(s) \leq m(s)$, $0 \leq g(z) \leq m(z)$, and $g(s) + g(z) = m(s) + m(z) - \ell$.

Lemma 13.5.11 *For an independent set $F' \subseteq E$ of $M^\#$ and for an edge $f = sz \in E - F'$, the set $F := F' + f$ is independent in $M^\#$ if and only if $H = (V, F)$ has an orientation in which the in-degree of each node v is at most $g(v)$.*

Proof. By the choice of α, $\widetilde{m}(X) - \ell \leq \widetilde{g}(X)$ holds for every $X \subseteq V$. Therefore if F is independent in $M^\#$ and $i_F(X) \geq 1$, then $i_F(X) \leq \widetilde{m}(X) - \ell \leq \widetilde{g}(X)$. If $i_F(X) = 0$, then also $i_F(X) \leq \widetilde{g}(X)$ and the lemma implies that H admits the requested orientation.

Conversely, assume that the orientation exists and let X be a subset of nodes with $i_F(X) \geq 1$. If $|\{s, z\} \cap X| \leq 1$, then $i_F(X) = i_{F'}(X) \leq -\widetilde{m}(X) - \ell$ since F' was required to be independent in $M^\#$. If both s and z belong to s, then $i_F(X) \leq \widetilde{g}(X) = \widetilde{m}(X) - \ell$. Therefore F is independent in $M^\#$. •

Starting with an orientation of F', obtained in a preceding stage, for which the in-degree of each node v is at most $m(v)$, we can decide if $F = F' + f$ is independent in $M^\#$ by applying path reorientations at most ℓ times. Therefore, altogether, we need at most $\ell|E|$ paths reorientations.

Problem 13.5.5 *Develop an algorithm for testing independence in a count matroid defined on a hypergraph.*

14
Generalized polymatroids

In Part I, we introduced the notion of independence polyhedra of a matroid as well as its extension: polymatroids. It was proved that the greedy algorithm works correctly not only for matroids but also for polymatroids. The proof gave rise to a theorem asserting that each polymatroid $P(b)$ is integral and uniquely determines its defining polymatroid function b.

In graph theory, however, we encounter set-functions which are submodular but not necessarily non-decreasing. Supermodular functions also play an important role in various applications. Furthermore, in the preceding two chapters we could see the benefit arising from relaxing even submodularity by requiring the submodular inequality only for intersecting pairs of sets. It will turn out that we can go even further by way of relaxation: results on near and crossing sub- and supermodular functions will prove indispensable in several applications. In order to handle the various cases uniformly, we are going to generalize the notion of polymatroids, following the treatment of [118] and [152].

14.1 Semimodular functions and their polyhedra

14.1.1 Paramodular pairs

In what follows, we consider real-valued set-functions on a ground-set S for which ∞ and $-\infty$ are also allowed, but not both. It is assumed throughout that the value of a set-function on the empty set is zero. A set-function h is **non-decreasing** if $h(X) < \infty$ and $X \subseteq Y \subseteq S$ imply $h(X) \leq h(Y)$. h is **non-increasing**, if $-h$ is non-decreasing. The assumption $h(\emptyset) = 0$ implies that a finite-valued non-decreasing set-function h is non-negative. We say that a set-function h is **symmetric** if $h(X) = h(S - X)$ for every $X \subseteq S$.

A function $m : S \to \mathbf{R} + \{\infty\}$ or $m : S \to \mathbf{R} + \{-\infty\}$ can be extended to all subsets of S with the formula $\widetilde{m}(X) := \sum[m(s) : s \in X]$. This set-function is **modular** in the sense that $\widetilde{m}(X) + \widetilde{m}(Y) = \widetilde{m}(X \cap Y) + \widetilde{m}(X \cup Y)$ holds for each $X, Y \subseteq S$. It is an easy exercise to show that every finite-valued modular function with $\widetilde{m}(\emptyset) = 0$ arises in this way.

A set-function $b : 2^S \to \mathbf{R} + \{\infty\}$ is said to be **fully submodular** or just submodular if the submodular inequality

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{14.1}$$

holds for every pair $X, Y \subseteq S$ of subsets. A set-function p is **supermodular** if $-p$ is submodular. Recall that a polymatroid function is a finite-valued non-decreasing and fully submodular function.

Observe that a finite-valued supermodular function p is non-negative if and only if p is non-decreasing. Indeed, $p(\emptyset) = 0$ implies that a non-decreasing function is non-negative while if p is non-negative on every set, then $X \subseteq Y$ implies $p(X) \leq p(X) + p(Y - X) \leq p(X \cap (Y - X)) + p(X \cup (Y - X)) = p(\emptyset) + p(Y) = p(Y)$.

For an arbitrary set-function $h : 2^S \to \mathbf{R} \cup \{+\infty, -\infty\}$ for which $h(S)$ is finite, we define the **complement** $\overline{h}$ of set-function h by the following formula.

$$\overline{h}(X) := h(S) - h(S - X). \tag{14.2}$$

Obviously, $\overline{h}(\emptyset) = 0$, $\overline{h}(S) = h(S)$ and $\overline{\overline{h}} = h$. The complement of a submodular function is supermodular. The complement of a modular function h is h itself. The rank function and the co-rank function of a matroid are complements of each other.

We say that a pair (p, b) of set-functions is **fully paramodular**, or simply **paramodular** (or that (p, b) is a **strong pair**) if $p(\emptyset) = b(\emptyset) = 0$, p is fully supermodular, b is fully submodular, and they are **compliant** in the sense that the **cross-inequality**

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X) \tag{14.3}$$

holds for every pair of subsets $X, Y \subseteq S$.

If b is fully submodular, then the set-system $\{X : b(X) < \infty\}$ is a ring-family, and conversely, if b is a finite-valued function on a ring-family $\mathcal{F}$ which is submodular on the members of $\mathcal{F}$, then by defining the b-value of each subset of S not in $\mathcal{F}$ to be $+\infty$, one gets a submodular set-function. In other words, there is a one-to-one correspondence between the submodular functions defined on all subsets and finite-valued functions defined on a ring-family.

Exercises

14.1.1 *Prove that (t, r) is paramodular where r is the rank function and t is the co-rank function of a matroid.*

14.1.2 *Prove that if b is submodular and $b(S)$ is finite, then $(\overline{b}, b)$ is paramodular.*

14.1.3 *For a hypergraph H on node-set V, the number of hyperedges disjoint from $X \subset V$ determines a supermodular function. The number of hyperedges intersecting both X and $V - X$ determines a symmetric supermodular function.*

Problems

14.1.4 *If b is submodular, p is supermodular, and the function defined by $b(X) + p(S - X)$ is non-decreasing, then (p, b) is paramodular.*

14.1.5 *Prove for a graph $G = (V, E)$ that the pair (i_G, e_G) is paramodular, where $i_G(X)$ is the number of edges induced by X and $e_G(X)$ is the number of edges with at least one end-node in X.*

14.1.6 *Let $D = (V, A)$ be a directed graph and $f : A \to \mathbf{R} \cup \{-\infty\}$, $g : A \to \mathbf{R} \cup \{\infty\}$ two functions for which $f \le g$. Prove that the set-function p defined by $p(X) := \varrho_f(X) - \delta_g(X)$ is fully supermodular.*

14.1.2 Polyhedra of sub- and supermodular functions

In order to have a unified view, let us now consider the several types of closely-related polyhedra assigned to a set-function. The following list of definitions may seem a bit dry but these polyhedra often show up in applications and it can be useful to have them listed at one place. The polyhedra are assigned to set-functions p or b where p serves as a lower bound and its value can be $-\infty$ while b is an upper bound in the definitions and it can take $+\infty$. In the definition of $B(b)$ and $B'(p)$, we assume that $b(S)$ and $p(S)$ are finite.

$$\begin{cases} P(b) := \{x \in R^S : \ x \ge 0, \ \widetilde{x}(A) \le b(A) \text{ for every} A \subseteq S\} \\ S(b) := \{x \in R^S : \ \widetilde{x}(A) \le b(A) \text{ for every} A \subseteq S\} \\ B(b) := \{x \in R^S : \ \widetilde{x}(S) = b(S), \ \widetilde{x}(A) \le b(A) \text{ for every} A \subseteq S\} \end{cases}$$

$$\begin{cases} C(p) := \{x \in R^S : \ x \ge 0, \ \widetilde{x}(A) \ge p(A) \text{ for every} A \subseteq S\} \\ S'(p) := \{x \in R^S : \ \widetilde{x}(A) \ge p(A) \text{ for every} A \subseteq S\} \\ B'(p) := \{x \in R^S : \ \widetilde{x}(S) = p(S), \ \widetilde{x}(A) \ge p(A) \text{ for every} A \subseteq S\} \end{cases}$$

$$Q(p, b) := \{x \in R^S : \ p(A) \le \widetilde{x}(A) \le b(A) \text{ for every } A \subseteq S\}. \tag{14.4}$$

We will consider the following set systems associated with b and p. In the definition of $\mathcal{B}(b)$ and $\mathcal{B}'(p)$, we assume the finiteness of $b(S)$ and $p(S)$.

$$\begin{cases} \mathcal{I}(b) := \{I \subseteq S : \ |Y \cap I| \le b(Y) \text{ for every} Y \subseteq S\}. \\ \mathcal{B}(b) := \{I \subseteq S : \ |I| = b(S) \text{ and } |Y \cap I| \le b(Y) \text{ for every} Y \subseteq S\}. \end{cases}$$

$$\begin{cases} \mathcal{G}(p) := \{I \subseteq S : \ |Y \cap I| \ge p(Y) \text{ for every} Y \subseteq S\}. \\ \mathcal{B}'(p) := \{I \subseteq S : \ |I| = p(S) \text{ and } |Y \cap I| \ge p(Y) \text{ for every} Y \subseteq S\}. \end{cases}$$

$$\mathcal{F}(p, b) := \{Z \subseteq S : \ p(Y) \le |Y \cap Z| \le b(Y) \text{ for every } Y \subseteq S\}.$$

Note that the characteristic vectors of the members of $\mathcal{I}(b)$ are the (0, 1)-vectors of $P(b)$ or $S(b)$. In the same sense, the (0, 1)-vectors of $C(p)$ or $S'(p)$ correspond to the members of $\mathcal{G}(p)$. Analogous statements hold for $B(b)$, $B'(p)$, and $Q(p, b)$.

When b is a polymatroid function, $P(b)$ was previously called a polymatroid where b is the border function pf $P(b)$. For technical reasons, it is convenient to consider the empty set a polymatroid even though it cannot be obtained from a polymatroid function. It was proved in Theorem 5.5.14 that $(S, \mathcal{I}(b))$ is a matroid.

If b is fully submodular, $S(b)$ is called a **submodular polyhedron** whose submodular (or upper) border function is b. (Though it is not particularly helpful to use the same letter to denote both the ground-set S and the submodular polyhedron $S(b)$, hopefully this will not be the source of any confusion.) If b is fully submodular and $b(S)$ is finite, $B(b)$ is called a **base-polyhedron** whose border function is b. When $b(S) = 0$, we speak of a **0-base-polyhedron**. Again, by definition, the empty set is a base-polyhedron.

When p is a supermodular function, the complementary function $\overline{p}$ (for which $\overline{p}(\emptyset) = 0$ and $\overline{p}(S) = p(S)$) is submodular and obviously $B(\overline{p}) = B'(p)$. Therefore the polyhedron $B'(p)$ will also be called a base-polyhedron whose lower border function is the supermodular p. We will prove later that even for a crossing submodular function b, $B(b)$ is a (possibly empty) base-polyhedron.

If p is fully supermodular, $S'(p)$ is called a **supermodular polyhedron**. If, in addition, p is non-negative (which implies that p is finite-valued and non-decreasing), then $S'(p) = C(p)$ and it is called a **contra-polymatroid**. We will see soon that $C(p)$, for any supermodular function p, is a contra-polymatroid.

Note that the role of $S(b)$ and $S'(p)$ is symmetric in the sense that $S(b)$ and $S'(-b)$ are the reflections of each other through the origin. An analogous statement holds for $B(b)$ and $B'(p)$. On the other hand there is no such correspondence between polymatroids and contra-polymatroids or between $P(b)$ and $C(p)$, as each of these is located in the non-negative orthant.

Finally, if (p, b) is paramodular, the polyhedron $Q(p, b)$ is called a **generalized polymatroid** (for short, a **g-polymatroid**). Here (p, b) is called the **border pair** of Q while p is the **lower** and b is the **upper border function** of Q. If, in addition, (p, b) is integral, p is non-negative and b is subcardinal, then we say that the pair (p, b) is **g-matroidal** and that the pair $(S, \mathcal{F}(p, b))$ is a **generalized matroid** (for short, a **g-matroid**). The subsets belonging to $\mathcal{F}(p, b)$ are **feasible**. For a given matroid M with rank function r and co-rank function t, we mentioned above that (t, r) is a paramodular pair. Evidently, $\mathcal{F}(t, r)$ is the set of bases of M. Moreover, if m_0 denotes the identically 0 set-function on S, then both (m_0, r) and (t, m_0) are paramodular, and $\mathcal{F}(m_0, r)$ is the set of independent sets of M while $\mathcal{F}(t, m_0)$ is the set of its generators.

By definition we consider the empty set as a g-polymatroid although it will turn out that it cannot be defined with a paramodular pair, that is, every paramodular pair defines a non-empty g-polymatroid. Also, $(S, \emptyset)$ is considered a g-matroid.

We will see that not-necessarily paramodular pairs can also define g-polymatroids (for example, if the submodularity of b is required only for intersecting sets), but every non-empty g-polymatroid uniquely determines its paramodular border pair.

Theorem 14.1.1 *Polymatroids, sub- and supermodular polyhedra, contra-polymatroids, and base-polyhedra are all g-polymatroids.*

Proof. Let b be a polymatroid function. Define p to be the identically 0 function. Since b is non-decreasing, the cross-inequality holds for b and p and hence (p, b) is paramodular. It follows from the definition that $P(b) = Q(p, b)$.

Let b be fully submodular. Define p to be zero on the empty set and $-\infty$ everywhere else. Then (p, b) is paramodular and $S(b) = Q(p, b)$. Similarly, for a fully supermodular p, define b to be zero on the empty set and $+\infty$ everywhere else. Then (p, b) is paramodular and $S'(p) = Q(p, b)$. A contra-polymatroid was defined to be a special supermodular polyhedron.

Finally let $B(b)$ be a base-polyhedron. Let p be the complement of b (that is, $p(X) := b(S) - b(S - X)$). Then p is supermodular, p and b are compliant, and $Q(p, b) = B(b)$. •

Problem 14.1.7 *A g-polymatroid $Q(p, b)$ defined by a paramodular pair (p, b) is a base-polyhedron if and only if $p(S) = b(S)$.*

Box and plank

Let $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{\infty\}$ be functions for which $f \leq g$. The polyhedron $T(f, g) := \{x \in \mathbf{R}^S : f \leq x \leq g\}$ is called a **box**. In the special case when $g \equiv \infty$, we use the notation $T(f\leq)$ while for $f \equiv -\infty$ the term $T(\leq g)$ is used. These boxes are called, respectively, an **upper** and a **lower orthant**. Accordingly, when we do not want to refer to the ground-set S, the non-negative orthant $\mathbf{R}^S_+$ will be denoted by $T(0\leq)$. When $f \equiv 0$ and $g \equiv 1$, the box $T(f, g)$ is the unit-cube in $\mathbf{R}^S$ and is denoted by U_S. Let $\alpha \leq \beta$ be two numbers. The polyhedron $K(\alpha, \beta) := \{x \in \mathbf{R}^S : \alpha \leq \widetilde{x}(S) \leq \beta\}$ is called a **plank**. If $\alpha = \beta$, then the plank forms a hyperplane which is denoted by $K(= \alpha)$. If $\alpha = -\infty$ or if $\beta = \infty$, the corrseponding planks are denoted by $K(\leq \beta)$ and $K(\alpha \leq)$. In this case the plank is a halfspace.

Exercise 14.1.8 *Prove the following proposition.*

Proposition 14.1.2 *Both a box and a plank form a g-polymatroid.* •

More generally, it will be proved in Section 14.3 that the intersection of a g-polymatroid with a box and with a plank is itself a g-polymatroid.

14.2 Constructions and operations

14.2.1 Simple operations

Direct sum

Let b_1 and b_2 be submodular functions defined on disjoint sets S_1 and S_2 respectively. Then the function b defined on ground-set $S = S_1 \cup S_2$ by $b(X) := b_1(X \cap S_1) + b_2(X \cap S_2)$ is clearly submodular. This is called the **direct sum** of b_1 and b_2 and is denoted by $b = b_1 \oplus b_2$. The notion of direct sum can be extended to several summands in the natural way. The submodular polyhedron $S(b)$ is called the **direct sum** of $S(b_1)$ and $S(b_2)$. For g-polymatroids the notion of direct sum is defined analogously. It follows that the direct sum of base-polyhedra is a base-polyhedron.

Translation and reflection

Proposition 14.2.1 *The translation of a g-polymatroid $Q(p, b)$ by a vector $v \in \mathbf{R}^S$ as well as the reflection of $Q(p, b)$ through the origin are g-polymatroids.*

Proof. The definitions show immediately that the pair (p_1, b_1) is paramodular where $p_1(X) := p(X) + \widetilde{v}(X)$, $b_1(X) := b(X) + \widetilde{v}(X)$ and also that for the translation of $Q(p, b)$ by v one has $Q(p, b) + v = Q(p_1, b_1)$.

The pair $(-b, -p)$ is also paramodular and the reflection of $Q(p, b)$ through the origin is just $Q(-b, -p)$. •

Exercise 14.2.1 *Show that both a translation and the reflection of a base-polyhedron are also base-polyhedra.*

Problem 14.2.2 *Prove that the independence polyhedron of the dual of a matroid M can be obtained by reflecting first the independence polyhedron of M through the origin and translating the resulting polyhedron by $v = (1, 1, \ldots, 1)$.*

Restriction and contraction

Let b be a submodular function on ground-set S and let $T \subseteq S$. The restriction of b to the subsets of T is clearly submodular and is denoted by $b|T$. Suppose that $b(T)$ is finite for a subset $T \subseteq S$. It can easily be seen that the function b' defined on the subsets of $S - T$ by the formula $b'(X) := b(X \cup T) - b(T)$ is submodular. This is called a **contraction** of b to $S - T$ and is denoted by b/T or by $b \cdot (S - T)$. The name comes from matroids.

Parallel multiplication of elements

Let b be a set-function on S. By the **parallel multiplication** of an element $t \in S$ we mean the operation when t is replaced by a non-empty set T and a set-function b' is defined on the resulting ground-set $S' = (S - t) \cup T$ as follows:

$$b'(X) := \begin{cases} b(X) & \text{if } X \subseteq S' - T \\ b(X - T + t) & \text{if } X \cap T \neq \emptyset. \end{cases} \tag{14.5}$$

Obviously, b' is submodular.

14.2.2 Projection

Recall the definition of projection (p. 157). Let $Q = Q(p, b)$ be a g-polymatroid defined by a paramodular pair (p, b). Let $T \subset S$ and $S' := S - T$. Let p' and b' denote the restrictions of p and b, respectively, to S'. Then (p', b') is paramodular and hence $Q(p', b')$ is a g-polymatroid.

Theorem 14.2.2 (Projection theorem) *The projection Q' along a subset $T \subset S$ of a g-polymatroid $Q = Q(p, b)$ is the g-polymatroid $Q(p', b')$. When (p, b) is integral, then every integral element of Q' arises as the projection of an integral element of Q.*

Proof. By induction, it suffices to prove the theorem for the special case when T is a singleton. Let s denote the single element of T. It is evident that each element of Q' is in $Q(p', b')$, that is, $Q' \subseteq Q(p', b')$.

For the reverse inclusion we show that an arbitrary element $x' \in Q(p', b')$ arises as a projection of some $x \in Q$ along s. To this end, let X, Y be two subsets of S containing s. We claim that

$$b(X) - \widetilde{x}'(X - s) \geq p(Y) - \widetilde{x}'(Y - s). \tag{14.6}$$

Indeed, the cross-inequality implies

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X) \geq \widetilde{x}'(X - Y) - \widetilde{x}'(Y - X) = \widetilde{x}'(X - s) - \widetilde{x}'(Y - s),$$

978–0–19–920527–1 14-Andras-Frank-c14-drv Andras Frank (Typeset by SPi, Publisher Services, India) 469 of 640 December 11, 2010 15:20

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

and hence (14.6) holds. From this we have $m := \min\{b(X) - \widetilde{x}'(X - s) : s \in X \subseteq S\} \geq M := \max\{p(X) - \widetilde{x}'(X - s) : s \in X \subseteq S\}$. By taking any value α for which $m \geq \alpha \geq M$, the vector $x := (x', \alpha)$ is in Q. In addition, if p, b, x' are all integral, then α can also be chosen an integer. •

Note that this proof is nothing but the Fourier–Motzkin elimination when it is applied to the linear system given in (14.4).

Corollary 14.2.3 *A g-polymatroid $Q(p, b)$ defined by a paramodular pair (p, b), and in particular a base-polyhedron $B(b)$ defined by a submodular function b, is never empty. In addition, when (p, b) is integral, $Q(p, b)$ contains an integral element.* •

Base-polyhedra versus g-polymatroids

We have already seen that every base-polyhedron is a g-polymatroid. Now we show that each g-polymatroid $Q(p, b)$ arises as the projection of a base-polyhedron. To this end, extend S by a new element s^*. Define the set-functions b^* and p^* on the ground-set $S^* := S + s^*$ as follows.

$$b^*(X) = \begin{cases} b(X) & \text{if } X \subseteq S \\ -p(S - X) & \text{if } s^* \in X, \end{cases} \tag{14.7}$$

$$p^*(X) = \begin{cases} p(X) & \text{if } X \subseteq S \\ -b(S - X) & \text{if } s^* \in X. \end{cases} \tag{14.8}$$

It follows immediately from the definitions:

Proposition 14.2.4 $b^*(S^*) = p^*(S^*) = 0$ *and* $B(b^*) = B'(p^*) = Q(p^*, b^*)$. *Furthermore, $Q(p, b)$ is the projection of $B(b^*)$ along s^*. The function b^* is submodular if and only if the pair (p, b) is paramodular.* •

Theorem 14.2.5 *A g-polymatroid $Q := Q(p, b)$ defined by a paramodular pair (p, b) arises as the projection of a 0-base-polyhedron along one element, namely, Q is the projection of $B(b^*)$ along s^*.* •

This simple connection is important since it will help in carrying over results obtained for base-polyhedra to g-polymatroids. For example, we shall see soon that the greedy algorithm can be extended to base-polyhedra. Theorem 14.2.5 ensures that the greedy algorithm can be used for arbitrary g-polymatroids, as well.

14.2.3 Face

Let $B(b)$ be a base-polyhedron defined by a submodular function b and let $T \subseteq S$ be a subset. Consider the face of $B(b)$ determined by T which is defined by $B_T := \{x \in B(b),\ \widetilde{x}(T) = b(T)\}$. Let b_T denote the direct sum of the restriction $b_1 := b|T$ of b and the contraction $b_2 := b \cdot (S - T)$ of b. Then $b_T(X) = b(T \cap X) + [b(X \cup T) - b(T)] = b(T \cap X) + b(T \cup X) - b(T)$ and we have seen already that b_T is submodular for which $b_T(T) = b(T)$.

Theorem 14.2.6 *The face B_T of a base-polyhedron $B(b)$ determined by a subset $T \subseteq S$ is a non-empty base-polyhedron, namely, B_T is the direct sum of base-polyhedra $B(b_1)$ and $B(b_2)$, that is, $B_T = B(b_T)$.*

Proof. Since $B(b_T)$ is a base-polyhedron, it is non-empty, and hence it suffices to prove $B_T = B(b_T)$.

Let $x \in B_T$ and $X \subseteq S$. We have $\widetilde{x}(S) = b(S)$, $\widetilde{x}(T) = b(T) = b_T(T)$ from which

$$\widetilde{x}(X) = \widetilde{x}(T \cap X) + \widetilde{x}(T \cup X) - \widetilde{x}(T) \leq b(T \cap X) + b(T \cup X) - b(T) = b_T(X),$$

implying that $x \in B(b_T)$ and hence that $B_T \subseteq B(b_T)$.

Conversely, $x \in B(b_T)$ implies $\widetilde{x}(T) = b_T(T) = b(T)$. Furthermore,

$$b_T(X) = b(T \cap X) + b(T \cup X) - b(T) \leq b(X)$$

from which $\widetilde{x}(X) \leq b_T(X) \leq b(X)$, implying that $x \in B_T$ and hence that $B(b_T) \subseteq B_T$. •

Analogous result holds for g-polymatroids.

Theorem 14.2.7 *The face $Q_T := \{x \in Q : \widetilde{x}(T) = b(T)\}$ of a g-polymatroid $Q = Q(p, b)$ determined by a subset $T \subseteq S$ is a non-empty g-polymatroid which is integral, provided that (p, b) is integral.*

Proof. By Theorem 14.2.5, Q is the projection of a base-polyhedron $B(b^*)$. Hence Q_T is the projection of the face $B' := \{x^* \in B(b^*) : x^*(T) = b^*(T)\}$ of $B(b^*)$. By Theorem 14.2.6, B' is a base-polyhedron, therefore every projection of B' is a (non-empty) g-polymatroid. •

The following result is a direct consequence.

Theorem 14.2.8 *A g-polymatroid $Q = Q(p, b)$ uniquely determines its border paramodular pair, namely,*

$$b(Z) = \max\{\widetilde{x}(Z) : x \in Q\} \text{ and } p(Z) = \min\{\widetilde{x}(Z) : x \in Q\}. \tag{14.9}$$

For an integral (p, b), the optimizer x can also be chosen integral. •

Later we shall see that g-polymatroids can also be defined by pairs (p, b) which are not necessarily fully paramodular. It follows from Theorem 14.2.8 that a non-empty g-polymatroid Q, no matter how it is defined, determines its unique fully paramodular border pair. Note that this result was already proved for the special case of polymatroids in Corollary 5.5.13.

It follows from Theorems 14.2.8 and 14.1.1 that the border function of a submodular polyhedron $S(b)$ is unique. Similarly, the border function of a non-empty base-polyhedron is uniquely determined, and a supermodular polyhedron $S'(p)$ also uniquely determines its border function. Note, however, that a contra-polymatroid $C(p)$ defined by a (fully) supermodular function determines uniquely its border function only if p is non-negative. (E.g., $C(p)$ is the non-negative orthant for any nowhere positive supermodular function p).

As a special case of Theorem 14.2.8, we obtain the following.

Corollary 14.2.9 *Let b be a* (fully) *submodular function. Then the submodular polyhedron $S(b)$ uniquely determines its border function, namely, $b(Z) = \max\{\widetilde{x}(Z) : x \in S(b)\}$ for* *every $Z \subseteq S$. If, in addition, $b(S)$ is finite, then the base-polyhedron $B(b)$ uniquely determines its border function, namely, $b(Z) = \max\{\widetilde{x}(Z) : x \in B(b)\}$ for every $Z \subseteq S$.* •

Theorem 14.2.10 *Every g-polymatroid $Q = Q(p, b)$ defined by an integral paramodular pair (p, b) is an integral polyhedron.*

Proof. Each face $\{x \in Q, \widetilde{x}(Z) = b(Z)\}$ or $\{x \in Q, \widetilde{x}(Z) = p(Z)\}$ of Q is non-empty by Theorem 14.2.8 and contains an integral point. •

Exercise 14.2.3 *Prove that the in-degree vectors of orientations of an undirected graph G are exactly the integral elements of the base-polyhedron $B(e_G)$. Show that $B(e_G) = B'(i_G)$.*

Problem 14.2.4 *Let $S \subseteq V$ be a subset of nodes of a graph $G = (V, E)$. A vector $m' : S \to$ **Z** is the in-degree vector of an orientation of G* (that, is $\varrho(v) = m'(v)$ for every $v \in S$) *if and only if $m' \in Q'$, that is, if $i_G(X) \leq \widetilde{m}'(X) \leq e_G(X)$ holds for every subset $X \subseteq S$.*

14.2.4 Aggregation and sum

In Section 5.5.4 we have already introduced the notion of aggregation of a polymatroid function. Here we extend aggregation to arbitrary submodular functions and to g-polymatroids. Let t_1, t_2 be two elements of S, $T := \{t_1, t_2\}$, and t an element not in S. For a set-function b on S, define a set-function b' on the ground-set $S' := S - T + t$ as follows.

$$b'(X) := \begin{cases} b(X) & \text{if } X \subseteq S' - t \\ b'(X) := b((X - t) \cup T) & \text{if } t \in X \subseteq S'. \end{cases} \tag{14.10}$$

We say that b' arises from b by aggregating elements t_1 and t_2. Evidently, if b is submodular, then so is b', and if the pair (p, b) is paramodular, then so is (p', b').

Theorem 14.2.11 *If b is submodular, then for every element x' of $S(b')$ there is an $x \in S(b)$ for which $\widetilde{x}(T) = x'(t)$ and $s \in S - T$ implies $x(s) = x'(s)$. If b, x' are integral, then x can also be chosen integral.*

Proof. Let $x(s) := x'(s)$ for $s \in S - T$. Let $\alpha_1 := \min\{b(X) - \widetilde{x}'(X - t_1) : t_1 \in X \subseteq S - t_2\}$ and $\alpha_2 := \min\{b(X) - \widetilde{x}'(X - t_2) : t_2 \in X \subseteq S - t_1\}$. These are integers when b and x' are integer-valued. Then $\alpha_1 + \alpha_2 \geq x'(t)$ for otherwise there is a subset $X_1 \subseteq S - t_2$ containing t_1 and there is a subset $X_2 \subseteq S - t_1$ containing t_2 for which $[b(X_1) - \widetilde{x}'(X_1 - t_1)] + [b(X_2) - \widetilde{x}'(X_2 - t_2)] < x'(t)$ and then $x'(t) + \widetilde{x}'(X_1 - t_1) + \widetilde{x}'(X_2 - t_2) > b(X_1) + b(X_2) \geq b(X_1 \cap X_2) + b(X_1 \cup X_2) \geq$ $\widetilde{x}'(X_1 \cap X_2) + \widetilde{x}'((X_1 - t_1) \cup (X_2 - t_2) + t) = x'(t) + \widetilde{x}'(X_1 - t_1) + \widetilde{x}'(X_2 - t_2)$, which is not possible. Therefore there are numbers $x(t_1) \leq \alpha_1$ and $x(t_2) \leq \alpha_2$ (which are, in addition, integers, when b and x' are integer-valued) so that their sum is $x'(t)$. By the definition of α_i, the vector x obtained in this way is in $S(b)$. •

One can simultaneously aggregate more than two elements and the set-function obtained in this way is independent of the order of the aggregation. Even more generally, for a given partition $\{S_1, \ldots, S_t\}$ of S we can separately aggregate each S_i. Another interpretation of this last operation is as follows.

A given mapping $\varphi : S \to S'$ determines a partition $\{S_1, \ldots, S_q\}$ of S where S_i denotes the set of elements $v \in S$ for which $\varphi(v) = s_i$ and $S' := \{s_1, \ldots, s_q\}$. The aggregate $\varphi(b) = b'$ of b on the ground-set S' is defined by the formula $b'(X) := b(\varphi^-(X))$. For a vector $x \in \mathbf{R}^S$, let $x' = \varphi(x)$ be defined by $x'(s_i) := \widetilde{x}(S_i)$. For a set $R \subseteq \mathbf{R}^S$, let $\varphi(R) := \{\varphi(x) : x \in R\}$.

By repeatedly using Theorem 14.2.11, one gets the following.

Theorem 14.2.12 *If b is a submodular function, then $\varphi(S(b)) = S(\varphi(b))$. Moreover, if b is integer-valued, then for each integral element $x' \in S(\varphi(b))$ there is an integral element $x \in S(b)$ for which $x' = \varphi(x)$.* •

The analogous result holds for base-polyhedra, and hence, by the Projection theorem, for g-polymatroids as well.

Theorem 14.2.13 (Aggregation theorem) *If $Q = Q(p, b)$ is a g-polymatroid defined by a paramodular pair (p, b), then $\varphi(Q) = Q(\varphi(p), \varphi(b))$. Moreover, if (p, b) is integral, then for each integral element $x' \in \varphi(Q)$ there is an integral element $x \in Q$ for which $x' = \varphi(x)$.* •

Remark It is tempting to call $\varphi(Q)$ the homomorphic image of Q, rather than the aggregate. However, this terminology would not fit the well-established notion of homomorphic image of a matroid (pp. 186 and 415 above). Note that the aggregate of a matroid rank function is a polymatroid function, which is not a matroid rank function in general. We will prove that the independence polyhedron of the homomorphic image $\varphi(M)$ of a matroid M is the intersection of the (0, 1)-cube of $\mathbf{R}^S$ and the aggregate of the independence polyhedron of M determined by φ.

Aggregation of matroids

Let M be a matroid on S with rank function r and co-rank function t. Let $\mathcal{P} := \{S_1, \ldots, S_q\}$ be a subpartition of S. Let $S' = \{s_1, \ldots, s_q\}$ be a set whose elements correspond to the members of $\mathcal{P}$. A vector $(m_1, \ldots, m_q)$ is called a **base-aggregate** if there is a basis B of M for which $|B \cap S_i| = m_i$ for every $i = 1, \ldots, q$.

Corollary 14.2.14 *An integral vector $(m_1, \ldots, m_q)$ is a base-aggregate if and only if $t(X) \le m(X') \le r(X)$ for every $X' \subseteq S'$ where X denotes the union of members of $\mathcal{P}$ corresponding to the elements of X'. The base-aggregates are the integral points of a g-polymatroid* (namely, the integral points of the projection of the aggregate of $B(r)$). •

Note that the special case of the first part of the corollary when $\{S_1, \ldots, S_q\}$ is a partition of S is a reformulation of Theorem 13.1.16 which was a consequence of Rado's theorem (Theorem 13.1.4). To see this, observe that the requirement $t(X) \le m(X') \le r(X)$ in the corollary for every $X' \subseteq S'$ is equivalent to requiring $m(X') \le r(X)$ for every $X' \subseteq S'$ and $m(S') = r(S)$.

Sum of g-polymatroids

McDiarmid [285, 286] showed how the results on sum of matroids can be extended to polymatroids? His result can further be generalized to g-polymatroids. Recall the notion of a (Minkowski) sum of polyhedra. Let (p_i, b_i) $(i = 1, \ldots, k)$ be paramodular pairs. Obviously, their sum is also paramodular.

Theorem 14.2.15 (Sum theorem) $\sum_i Q(p_i, b_i) = Q\left(\sum_i p_i, \sum_i b_i\right)$. *Moreover, if each* (p_i, b_i) *is integral, then every integral element* x' *of* $Q\left(\sum_i p_i, \sum_i b_i\right)$ *arises as* $x' = x_1 + \cdots + x_k$ *where each* $x_i \in Q_i$ *is integral.*

Proof. Let $S_1, \ldots, S_k$ denote k disjoint copies of S and let S^* be their union. Define a mapping $\varphi : S^* \to S$ in such a way that the map of each element is the element of S corresponding to it. Let $\left(p'_i, b'_i\right)$ denote a copy of (p_i, b_i) on S_i. Let (p^*, b^*) be the direct sum of pairs $\left(p'_i, b'_i\right)$ while Q^* the direct sum of g-polymatroids $Q\left(p'_i, b'_i\right)$. It follows from the definitions that $Q\left(\sum p_i, \sum b_i\right) = \varphi(Q(p^*, b^*))$ and that $\sum p_i = \varphi(p^*)$ and $\sum b_i = \varphi(b^*)$, hence Theorem 14.2.13 applies. •

Partitionable sets in matroids

As a corollary, we derive the non-trivial part of Theorem 13.3.9 stating that, given matroids $M_1 = (S, r_1), \ldots, M_k = (S, r_k)$ on a common ground-set S, it is possible to partition S into M_i-independent sets F_i $(i = 1, \ldots, k)$ if and only if $(*)$ $\sum_i r_i(X) \geq |X|$ for every $X \subseteq S$.

Proof. Condition $(*)$ is equivalent to requiring that the identically 1 vector $z = \underline{\chi}_S$ is in the polymatroid $P\left(\sum_i r_i\right)$. By the Sum theorem (Theorem 14.2.15), there are integral elements $z_i \in P(r_i)$ so that $z = \sum_i z_i$. But such a z_i is the characteristic vector of an M_i-independent set F_i and hence $S = \cup_i F_i$. •

14.3 Intersection with a box and with a plank

It is known that the system of common independent sets of two matroids does not form a matroid in general, and similarly, the intersection of two g-polymatroids is typically not a g-polymatroid. There are, however, exceptions, and in this section we show two of them by proving that the intersection of a g-polymatroid with a box and with a plank is a g-polymatroid.

14.3.1 Monotonization, convolution, divolution

Monotonizations

Let h be a set-function on a ground-set S for which $h(\emptyset) = 0$. In Section 12 (see (12.5) and (12.6)), we assigned the set-functions h_{imin} and h_{omin} to h where

$$h_{imin}(Z) := \min\{h(X) : X \subseteq Z\}$$

$$h_{omin}(Z) := \min\{h(X) : X \supseteq Z\}.$$

These set-functions are non-decreasing and non-increasing, respectively. Analogously to h_{omin} and $h_{imin}(Z)$, the set-functions h_{omax} and h_{imax} can be defined as follows.

$$h_{omax}(Z) := \max\{h(X) : X \supseteq Z\}, \tag{14.11}$$

$$h_{imax}(Z) := \max\{h(X) : X \subseteq Z\}. \tag{14.12}$$

978–0–19–920527–1 14-Andras-Frank-c14-drv Andras Frank (Typeset by SPi, Publisher Services, India) 474 of 640 December 11, 2010 15:20
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Note that $h_{omin}(\emptyset)$ is negative if h has any negative value, therefore h_{omin} will be used only if $h \geq 0$. Similarly, in order to have $h_{omax}(\emptyset) = 0$, one must assume that h is non-positive.

Problem 14.3.1 (A) *If $b \geq 0$ is submodular, then b_{omin} is submodular, non-decreasing, and $b_{omin}(\emptyset) = 0$.* (B) *If p is supermodular, then p_{imax} is supermodular and non-decreasing.*

Theorem 14.3.1 (A) *If b is submodular, then b_{imin} is submodular. If, in addition, $b \geq 0$, then $b_{omin}(\emptyset) = 0$ and b_{omin} is submodular.*
(B) *If p is supermodular, then p_{imax} is supermodular. If, in addition, $p \leq 0$, then $p_{omax}(\emptyset) = 0$ and p_{omax} is supermodular.*

Proof. The $b \geq 0 \Rightarrow b_{omin}(\emptyset) = 0$ implication is evident. Let X and Y be two subsets of S. By definition there are $X' \supseteq X$ and $Y' \supseteq Y$ for which $b_{omin}(X) = b(X')$ and $b_{omin}(Y) = b(Y')$. Therefore $b_{omin}(X) + b_{omin}(Y) = b(X') + b(Y') \geq b(X' \cap Y') + b(X' \cup Y') \geq b_{omin}(X \cap Y) + b_{omin}(X \cup Y)$, and hence b_{omin} is indeed submodular. The submodularity of b_{imin} is seen analogously, as well as the supermodularity of p_{imax} and p_{omax}. •

Proposition 14.3.2 (A) *$P(b) = P(b_{omin})$ holds whenever b is a non-negative set-function. In particular, $\mathcal{I}(b) = \mathcal{I}(b_{omin})$.*
(B) *$C(p) = S'(p_{imax})$ holds for any set-function p. In particular, $\mathcal{G}(p) = \mathcal{G}(p_{imax})$.*

Proof. (A) $b_{omin} \leq b$ implies $P(b) \supseteq P(b_{omin})$. On the other hand, there is a superset $X \supseteq Z$ for each $Z \subseteq S$ such that $b_{omin}(Z) = b(X)$. Then for $x \in P(b)$ we have $\widetilde{x}(Z) = \widetilde{x}(X) - \widetilde{x}(Z - X) \leq \widetilde{x}(X) - 0 \leq b(X) = b_{omin}(Z)$ from which $x \in P(b_{omin})$, showing that $P(b) \subseteq P(b_{omin})$ and hence $P(b) = P(b_{omin})$.

(B) Observe first that $x \in S'(p_{imax})$ implies $\widetilde{x}(Z) \geq p_{imax}(Z) \geq \max\{0, p(Z)\}$ and hence $x \in C(p)$, from which $C(p) \supseteq S'(p_{imax})$. On the other hand Z admits a subset X for which $p_{imax}(Z) = p(X)$. Therefore $x \in C(p)$ implies $\widetilde{x}(Z) = \widetilde{x}(X) + \widetilde{x}(Z - X) \geq p(X) + 0 = p_{imax}(Z)$ from which $x \in S'(p_{imax})$ and hence $C(p) \subseteq S'(p_{imax})$, implying that $C(p) = S'(p_{imax})$. •

By combining these observations with Theorem 14.3.1, one gets the following.

Theorem 14.3.3 (A) *If $b \geq 0$ is submodular and $b(S)$ is finite, then $P(b)$ is a polymatroid whose unique defining border function is b_{omin}. In other words, b_{omin} is a polymatroid-function and $P(b) = P(b_{omin})$.*
(B) *If p is supermodular, then $C(p)$ is a contra-polymatroid for which the unique border function is p_{imax}. In other words, p_{imax} is non-negative, supermodular, and $C(p) = S'(p_{imax})$.* •

Convolution and divolution

Let $m : S \to \mathbf{R} + \{\pm\infty\}$ be a function which can take either $+\infty$ or $-\infty$ but not both. Let h be a set-function on S. Define the **lower divolution** $h_{\downarrow m}$ and the **upper divolution** $h_{\uparrow m}$ of h and m by

978–0–19–920527–1 14-Andras-Frank-c14-drv Andras Frank (Typeset by SPi, Publisher Services, India) 475 of 640 December 11, 2010 15:20
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

$$h_{\downarrow m}(Z) := \min\{h(X) - \widetilde{m}(X - Z) : X \supseteq Z\}, \tag{14.13}$$

$$h_{\uparrow m}(Z) := \max\{h(X) - \widetilde{m}(X - Z) : X \supseteq Z\}. \tag{14.14}$$

Define the **lower convolution** $h \bigtriangledown m$ and the **upper convolution** $h \bigtriangleup m$ of h and m by

$$(h \bigtriangledown m)(Z) := \min\{h(X) + \widetilde{m}(Z - X) : X \subseteq Z\}, \tag{14.15}$$

$$(h \bigtriangleup m)(Z) := \max\{h(X) + \widetilde{m}(Z - X) : X \subseteq Z\}. \tag{14.16}$$

Note that for $m \equiv 0$, one has $h_{\downarrow m} = h_{omin}$, $h \bigtriangledown m = h_{imin}$, $h_{\uparrow m} = h_{omax}$, $h \bigtriangleup m = h_{imax}$. Let $f : S \to \mathbf{R} + \{-\infty\}$ and $g : S \to \mathbf{R} + \{\infty\}$ be two functions.

Theorem 14.3.4 (A) *If b is submodular, then so is $b \bigtriangledown g$. If, in addition, $\widetilde{f}(Y) \le b(Y)$ for every $Y \subseteq S$, then $b_{\downarrow f}(\emptyset) = 0$ and $b_{\downarrow f}$ is submodular.*
(B) *If p is supermodular, then so is $p \bigtriangleup f$. If, in addition, $\widetilde{g}(Y) \ge p(Y)$ for every $Y \subseteq S$, then $p_{\uparrow g}(\emptyset) = 0$ and $p_{\uparrow g}$ is submodular.*

Proof. Consider the function b' defined by $b'(Z) := b(Z) - \widetilde{g}(Z)$. Then $(b \bigtriangledown g)(Z) = \min\{b(X) + \widetilde{g}(Z - X) : X \subseteq Z\} = \min\{b(X) - \widetilde{g}(X) : X \subseteq Z\} + \widetilde{g}(Z) = (b')_{imin}(Z) + \widetilde{g}(Z)$, and hence $b \bigtriangledown g = (b')_{imin} + g$. Since $(b')_{imin}$ is submodular by Theorem 14.3.1, $b \bigtriangledown g$ is also submodular.

Consider the function b'' defined by $b''(Z) := b(Z) - \widetilde{f}(Z)$. By the hypothesis, $b'' \ge 0$. We have $b_{\downarrow f}(Z) = \min\{b(X) - \widetilde{f}(X - Z) : X \supseteq Z\} = \min\{b(X) - \widetilde{f}(X) : X \supseteq Z\} + \widetilde{f}(Z) = \min\{b''(X) : X \supseteq Z\} + \widetilde{f}(Z)$, and hence $b_{\downarrow f} = (b'')_{omin} + f$. Since $(b'')_{omin}$ is submodular by Theorem 14.3.1, $b_{\downarrow f}$ is also submodular.

Part (B) follows analogously. •

Proposition 14.3.5 *Let $g : S \to \mathbf{R} + \{\infty\}$ be a function.*

(A) *If b is an arbitrary set-function, then $S(b) \cap T(\le g) = S(b \bigtriangledown g)$.*
(B) *If b and g are non-negative, then $P(b) \cap T(\le g) = P(b \bigtriangledown g)$.*
(C) *If $b(S)$ is finite and $(b \bigtriangledown g)(S) = b(S)$, then $B(b) \cap T(\le g) = B(b \bigtriangledown g)$.*

Proof. (A) Since $(b \bigtriangledown g)(Z) \le \min\{b(Z), \widetilde{g}(Z)\}$, we have $S(b) \cap T(\le g) \supseteq S(b \bigtriangledown g)$. For the reverse inclusion, let $x \in S(b) \cap T(\le g)$. Then for any subset $X \subseteq Z$, one has $\widetilde{x}(Z) = \widetilde{x}(X) + \widetilde{x}(Z - X) \le b(X) + \widetilde{g}(Z - X)$, from which $\widetilde{x}(Z) \le \min\{b(X) + \widetilde{g}(Z - X) : X \subseteq Z\} = (b \bigtriangledown g)(Z)$ follows. Hence $S(b) \cap T(\le g) \subseteq S(b \bigtriangledown g)$ holds completing the proof of Part (A). Part (B) follows from (A) since for non-negative b and g one has $P(b \bigtriangledown g) = S(b \bigtriangledown g) \cap T(\ge 0)$ and $P(b) \cap T(\le g) = S(b) \cap T(\le g) \cap T(\ge 0)$. Part (C) also follows from (A) since $B(b \bigtriangledown g) = S(b \bigtriangledown g) \cap \{x : \widetilde{x}(S) = (b \bigtriangledown g)(S)\}$ and $B(b) = S(b) \cap \{x : \widetilde{x}(S) = b(S)\}$. •

By combining this proposition with Theorem 14.3.4, we obtain the following result.

Theorem 14.3.6 *Let b be a submodular function and $g : S \to \mathbf{R} + \{\infty\}$.*

(A) *Then $S(b) \cap T(\le g)$ is a submodular polyhedron whose unique submodular border function is $b \bigtriangledown g$, that is, $S(b) \cap T(\le g) = S(b \bigtriangledown g)$.*
(B) *If b is a polymatroid function and $g \ge 0$, then $P(b) \cap T(\le g)$ is a polymatroid whose border function is $b \bigtriangledown g$, that is, $P(b) \cap T(\le g) = P(b \bigtriangledown g)$.*

(C) *If b is submodular, $b(S)$ is finite and $(b \triangledown g)(S) = b(S)$, then $B(b) \cap T(\leq g)$ is a non-empty base-polyhedron whose upper border function is $b \triangledown g$, that is, $B(b) \cap T(\leq g) = B(b \triangledown g)$.* •

Corollary 14.3.7 *If b is an integer-valued polymatroid function, then $\mathcal{I}(b)$ is the set of independent sets of a matroid. If p is an integer-valued supermodular set-function, then $\mathcal{G}(p)$ is the set of generators of a matroid.* •

Making a set-function finite-valued

With the help of convolution we show that an arbitrary submodular function can be transformed into a finite-valued submodular function without changing its originally finite values.

Corollary 14.3.8 *Let b be a submodular function and let $\mathcal{F} := \{x : b(X) < \infty\}$. For an arbitrary big integer $K \geq 0$ there is a finite-valued submodular function b' such that $b'(Z) = b(Z)$ for every $Z \in \mathcal{F}$ and $b'(Z) \geq K$ for every $Z \notin \mathcal{F}$.*

Proof. Let $\mu := K + \max\{|b(X)| : X \in \mathcal{F}\}$ and let $m : S \to \mathbf{Z}$ be the constant function of value μ. By theorem 14.3.4, the function $b' := b \triangledown m$ is submodular and $b'(Z) = \min\{b(X) + \mu|Z - X|\}$. Due to the suitably big choice of μ, the minimum for a set $Z \in \mathcal{F}$ is attained at $X = Z$ from which $b'(Z) = b(Z)$. For $Z \notin \mathcal{F}$ we have $b'(Z) = b(X) + \mu|Z - X|$ for some $X \subset Z$ and hence $b'(Z) \geq b(X) + \mu \geq K$. •

Problem 14.3.2 *Let b be a submodular function. Let $f : S \to \mathbf{R} + \{-\infty\}$ and $g : S \to \mathbf{R} + \{\infty\}$ be functions for which $f \leq g$ and $\widetilde{f}(X) \leq b(X)$ for every $X \subseteq S$. Prove that the function $b_{\downarrow[f,g]}$ defined by $b_{\downarrow[f,g]}(Z) = \min\{b(X) - \widetilde{f}(X - Z) + \widetilde{g}(Z - X) : X \subseteq S\}$ is fully submodular.*

Intersection with a box

Let (p, b) be a paramodular pair. Let $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{\infty\}$ be functions for which $f \leq g$.

Theorem 14.3.9 *The intersection M of a g-polymatroid $Q = Q(p, b)$ and a box $T(f, g)$ is a g-polymatroid. If M is non-empty, its border pair (p', b') is given by*

$$p'(Z) = \max\{p(X) - \widetilde{g}(X - Z) + \widetilde{f}(Z - X) : X \subseteq S\}, \tag{14.17}$$

$$b'(Z) = \min\{b(X) - \widetilde{f}(X - Z) + \widetilde{g}(Z - X) : X \subseteq S\}. \tag{14.18}$$

If p, b, f, and g are each integer-valued, then M is integral.

Proof. As the empty set is a g-polymatroid by definition, we can assume that $M \neq \emptyset$. Theorem 14.3.6 implies that the intersection of a base-polyhedron and a lower orthant is a (possibly empty) base-polyhedron. Since the reflection of a base-polyhedron through the origin is a base-polyhedron, it follows that the intersection of B with an upper orthant $T(\geq f)$ is also a base-polyhedron. By applying these two operations consecutively, we obtain that $B \cap T(f, g)$ is a base-polyhedron.

By Theorem 14.2.5, Q can be obtained as the projection of a 0-base-polyhedron $B(b^*)$ along an element s^* where b^* is a set-function defined on ground-set $S^* := S + s^*$ by (14.7). Define f^* and g^* as follows.

$$f^*(s) := \begin{cases} f(s) & \text{if } s \in S \\ -\infty & \text{if } s = s^* \end{cases}, \tag{14.19}$$

$$g^*(s) := \begin{cases} g(s) & \text{if } s \in S \\ \infty & \text{if } s = s^* \end{cases}. \tag{14.20}$$

Let $M^* := B(b^*) \cap T(f^*, g^*)$. Then M^* is a base-polyhedron and M is the projection of M^* along s^*. The Projection theorem (Theorem 14.2.2) implies that M is indeed a g-polymatroid.

We are going to prove only that the upper border function of M is b' since (14.17) arises analogously. Let $\mu(Z)$ denote the minimum occurring in (14.18). It follows from Theorem 14.2.8 that M has an element x for which $b'(Z) = \widetilde{x}(Z)$. Therefore $b'(Z) = \widetilde{x}(Z) = \widetilde{x}(X) - \widetilde{x}(X - Z) + \widetilde{x}(Z - X) \le b(X) - \widetilde{f}(X - Z) + \widetilde{g}(Z - X)$ holds for every $X \subseteq S$ from which $b'(Z) \le \mu(Z)$.

In order to prove $b'(Z) = \mu(Z)$, we have to find a subset X for which the following optimality criteria hold:

$$\widetilde{x}(X) = b(X), \;\; \widetilde{x}(X - Z) = \widetilde{f}(X - Z) \;\text{ and }\; \widetilde{x}(Z - X) = \widetilde{g}(Z - X). \tag{14.21}$$

We refer to a subset X as b'-tight if $\widetilde{x}(X) = b'(X)$ and p'-tight if $\widetilde{x}(X) = p'(X)$. By using the standard submodular technique, we see that the b'-tight sets are closed under union and intersection. Moreover the cross inequality implies that the difference of a b'-tight set and a p'-tight set is b'-tight.

Let $z \in Z$ be an element for which $x(z) < g(z)$. Then z belongs to a b'-tight set for otherwise $x' := x + \Delta \cdot \underline{\chi}_z$ would belong to M for small $\Delta > 0$, contradicting the maximal choice of x. The intersection $T_x(z)$ of b'-tight sets containing z is b'-tight. Let $u \in T_x(z)$ and let Y be p'-tight set containing u. Then z must be in Y for otherwise $T_x(z) - Y$ would be a b'-tight set containing z, contradicting the minimality of $T_x(z)$. Therefore

$$t \in T_x(z) - Z \text{ implies } x(t) = f(t), \tag{14.22}$$

for if we had $x(t) > f(t)$, then $x' := x + \Delta \cdot (\underline{\chi}_z - \underline{\chi}_t)$ would be in M for small Δ, contradicting again the maximal choice of x.

Since $X := \cup\{T_x(z) : z \in Z, x(z) < g(z)\}$ is the union of b'-tight sets, it is itself b'-tight for which $\widetilde{x}(Z - X) = \widetilde{g}(Z - X)$ follows from the definition of X, furthermore (14.22) implies $\widetilde{x}(X - Z) = \widetilde{f}(X - Z)$, and hence the optimality criteria hold.

The integrality part of the theorem follows by applying Theorem 14.2.10 to the integral paramodular pair (p', b'). •

Generalized matroids

Consider the intersection of a polymatroid $Q = P(b)$ with the $(0, 1)$-cube. In this case, b is a polymatroid function, $p \equiv 0$, $f \equiv 0$, and $g \equiv 1$. By (14.17), $p'(Z) = \max\{p(X) - \widetilde{g}(X - Z) + \widetilde{f}(Z - X) : X \subseteq S\} = \max\{0 - |X - Z| + 0 : X \subseteq S\} = 0$. By (14.18), $b'(Z) = \min\{b(X) - \widetilde{f}(X - Z) + \widetilde{g}(Z - X) : X \subseteq S\} = \min\{b(X) - 0 + |Z - X| : X \subseteq S\} = \min\{b(X) + |Z - X| : X \subseteq Z\}$. Therefore Theorem 14.3.9 can be regarded as a generalization of Theorem 5.5.14 which stated that every polymatroid function b

defines a matroid M_b in which a subset F is independent if $|F \cap X| \leq b(X)$ holds for every $X \subseteq F$.

More generally, one may consider the intersection of a g-polymatroid with the (0, 1)-cube.

Proposition 14.3.10 *Let (p, b) be a g-matroidal pair. Then both p and b are non-decreasing.*

Proof. Let $X \subseteq Y \subseteq S$. Since p is non-negative and supermodular, we have $p(X) \leq p(X) + p(Y - X) \leq p(Y)$. Since p is non-negative, by applying the cross inequality to Y and $X' := Y - X$, we obtain $b(Y) \geq b(Y) - p(X') \geq b(Y - X') - p(X' - Y) = b(X)$. •

Let (p, b) be an integral paramodular pair for which the set-system $\mathcal{F}(p, b)$ is non-empty. It is immediate from the definitions that the incidence vectors of the members of $\mathcal{F}(p, b)$ are the integral elements of the intersection Q_1 of the g-polymatroid $Q(p, b)$ with the unit-cube U_S. By Theorem 14.3.9, Q_1 is an integral g-polymatroid whose unique paramodular border pair (p', b') is given by

$$p'(Z) = \max\{p(X) - |X - Z| : X \subseteq S\} \text{ and } b'(Z) = \min\{b(X) + |Z - X| : X \subseteq S\}. \tag{14.23}$$

This immediately implies that both p' is non-negative and b' is subcardinal, and hence (p', b') is g-matroidal. Thus we have obtained the following construction for g-matroids.

Theorem 14.3.11 *Let (p, b) be an integral paramodular pair for which the set-system $\mathcal{F} = \mathcal{F}(p, b)$ is non-empty. Then (p', b') defined in* (14.23) *is g-matroidal and $(S, \mathcal{F})$ is the g-matroid $(S, \mathcal{F}(p', b'))$.* •

Theorem 14.3.12 *Let (p, b) be an integral paramodular pair for which the g-polymatroid $Q(p, b)$ is in the unit cube. Then (p, b) is g-matroidal.*

Proof. The hypothesis implies that the intersection of $Q(p, b)$ with the (0,1) box is $Q(p, b)$ itself. This and Theorem 14.2.8 imply $p = p'$ and $b = b'$ for the functions defined in (14.23). Hence (p, b) is g-matroidal since (p', b') is g-matroidal by Theorem 14.3.11. •

Intersection with a plank

Our next goal is to show that the intersection of a g-polymatroid with a plank is also a g-polymatroid.

Theorem 14.3.13 *The intersection $M := Q(p, b) \cap K(\alpha, \beta)$ is a g-polymatroid. If M is non-empty, then the border pair (p', b') of M is given by the following:*

$$p'(Z) := \max\{p(Z),\ \alpha - b(S - Z)\} \tag{14.24}$$

and

$$b'(Z) := \min\{b(Z),\ \beta - p(S - Z)\}. \tag{14.25}$$

If p, b, α, and β are integer-valued, then M is integral.

Proof. Since the empty set is by definiton a g-polymatroid, we can assume that M is non-empty. Then $\beta \geq p(S)$ and $\alpha \leq b(S)$ from which $p'(\emptyset) = 0$ and $b'(\emptyset) = 0$. By Theorem

14.2.5, Q arises as the projection of the 0-base-polyhedron $Q(p^*, b^*)$ where b^* and p^* are defined on $S^* = S + s^*$ by (14.7) and (14.8). Define functions $f^* : S^* \to \mathbf{R} + \{-\infty\}$ and $g^* : S^* \to \mathbf{R} + \{\infty\}$ as follows.

$$f^*(s) := \begin{cases} -\infty & \text{if } s \in S \\ -\beta & \text{if } s = s^* \end{cases}, \tag{14.26}$$

$$g^*(s) := \begin{cases} \infty & \text{if } s \in S \\ -\alpha & \text{if } s = s^* \end{cases}. \tag{14.27}$$

Consider the intersection $M' := Q(p^*, b^*) \cap T(f^*, g^*)$. By Theorem 14.3.9, $Q(p', b')$ is a g-polymatroid, and the projection of $Q(p', b')$ along s^* is just M. By applying Theorem 14.3.9 to S^*, b^*, f^*, g^*, we obtain for a subset $Z \subseteq S$ that the values considered in the minimum formula (14.18) can be finite only if $X = Z$ or $X = Z + s^*$, and in these two cases these values are, respectively, $b^*(Z) = b(Z)$ and $b^*(Z + s^*) - f^*(s^*) = -p(S - Z) - (-\beta) = \beta - p(S - Z)$, from which (14.25) follows. (14.17) implies similarly (14.24).

The integrality of M follows from the integrality of (p', b'). •

14.3.2 Feasibility and the linking property

Our next goal is to exhibit conditions for the non-emptiness of the intersection of a g-polymatroid with a box and a plank.

Theorem 14.3.14 *Let (p, b) be a paramodular pair. Suppose for functions $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{\infty\}$ that $f \le g$. The intersection $M := Q(p, b) \cap T(f, g)$ is non-empty if and only if*

$$\widetilde{f}(X) \le b(X) \text{ for every } X \subseteq S, \quad \text{and} \tag{14.28}$$

$$\widetilde{g}(X) \ge p(X) \text{ for every } X \subseteq S. \tag{14.29}$$

Proof. The conditions are obviously necessary. It is enough to prove sufficiency only for finite-valued f and g since if, for example, $g(v) = \infty$ for some $v \in S$, then revising the value of $g(v)$ to a suitably large (finite) number the box gets smaller while (14.29) continues to hold. We can also assume that f is maximal in the sense that the value of $f(s)$ cannot be increased for any element s without violating $f \le g$ or condition (14.28). This implies that either $f(s) = g(s)$ or there is a set X containing s for which $\widetilde{f}(X) = b(X)$. Indeed, if $f(s)$ were increasable for some s, then increasing its value to $\min\{g(s), \min\{b(X) - \widetilde{f}(X - s) : s \in X \subseteq S\}\}$ we would get a smaller box which continues to meet both (14.28) and (14.29). Analogously, we can assume that g is minimal.

If $f(s) = g(s)$ for every $s \in S$, then (14.28) and (14.29) imply that $p(X) \le \widetilde{g}(X) = \widetilde{f}(X) \le b(X)$ for each $X \subseteq S$, and hence $f \in M$. So assume that there is an element $s \in S$ for which $f(s) < g(s)$. The maximality of f implies that there is a set X containing s for which $\widetilde{f}(X) = b(X)$, and the minimality of g implies that there is a set Y containing s for which $\widetilde{g}(Y) = p(Y)$. But then $\widetilde{f}(X) - \widetilde{g}(Y) = b(X) - p(Y) \ge b(X - Y) - p(Y - X) \ge \widetilde{f}(X - Y) - \widetilde{g}(Y - X) = \widetilde{f}(X) - \widetilde{f}(X \cap Y) - [\widetilde{g}(Y) - \widetilde{g}(X \cap Y)]$ from which $\widetilde{f}(X \cap Y) \ge \widetilde{g}(X \cap Y)$ contradicting the assumptions $f \le g$ and $f(s) < g(s)$. •

In the special case when $f \equiv 0$ and $g \equiv 1$, Theorem 14.3.14 states that the intersection of $Q(p, b)$ with the unit cube is non-empty if and only if $p(X) \leq |X|$ and $b(X) \geq 0$ for every $X \subseteq S$. This and Theorem 14.3.11 imply the following.

Corollary 14.3.15 *Let (p, b) be an integral paramodular pair. Then $\mathcal{F} = \mathcal{F}(p, b)$ is non-empty if and only if p is subcardinal and b is non-negative.* •

It is worth formulating Theorem 14.3.14 for the special case of base-polyhedra.

Theorem 14.3.16 *For a submodular function b for which $b(S)$ is finite, the intersection $B(b) \cap T(f, g)$ is non-empty if and only if*

$$\widetilde{f}(Y) \leq b(Y) \text{ for every } Y \subseteq S \tag{14.30}$$

and

$$\widetilde{g}(Y) \geq \overline{b}(Y) \ \ [= b(S) - b(S - Y)] \text{ for every } Y \subseteq S. \tag{14.31}$$

For a supermodular p for which $p(S)$ is finite, the intersection $B'(p) \cap T(f, g)$ is non-empty if and only if

$$\widetilde{f}(X) \leq \overline{p}(X) \ \ [= p(S) - p(S - X)] \text{ for every } X \subseteq S \tag{14.32}$$

and

$$\widetilde{g}(X) \geq p(X) \text{ for every } X \subseteq S. \tag{14.33}$$

When the defining functions are integer-valued, the polyhedra in questions are integral.

Proof. It follows from Theorem 14.1.1 that $B(b) = Q(p, b)$ where $p := \overline{b}$. Then the conditions (14.28) and (14.30) are equivalent, moreover the conditions (14.29) and (14.31) for $X = S - Y$ are also equivalent. The formula for supermodular functions arises analogously. •

Theorem 14.3.16 has a kind of self-refining nature because it gives rise to a condition for the non-emptiness of the intersection of a g-polymatroid with a box and a plank.

Theorem 14.3.17 *Let (p, b) be a paramodular pair and let $\alpha \leq \beta$ be two numbers* (here α can be $-\infty$ and β can be ∞). *Let $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{\infty\}$ be two functions for which $f \leq g$. The intersection $M := Q(p, b) \cap T(f, g) \cap K(\alpha, \beta)$ is non-empty if and only if*

$$\widetilde{f}(X) \leq b(X) \text{ and } \widetilde{f}(X) \leq \beta - p(S - X) \text{ for every } X \subseteq S \tag{14.34}$$

and

$$\widetilde{g}(X) \geq p(X) \text{ and } \widetilde{g}(X) \geq \alpha - b(S - X) \text{ for every } X \subseteq S. \tag{14.35}$$

Proof. If $x \in M$, then $\widetilde{f}(X) \leq \widetilde{x}(X) \leq b(X)$ and $\widetilde{f}(X) \leq \widetilde{x}(X) = \widetilde{x}(S) - \widetilde{x}(S - X) \leq \beta - p(S - X)$, from which the necessity of (14.34) follows. Furthermore, $\widetilde{g}(X) \geq \widetilde{x}(X) \geq p(X)$ and $\widetilde{g}(X) \geq \widetilde{x}(X) = \widetilde{x}(S) - \widetilde{x}(S - X) \geq \alpha - b(S - X)$, and hence (14.35) is also necessary.

To prove sufficiency, we invoke again Theorem 14.2.5 stating that $Q(p, b)$ arises as the projection of a 0-base-polyhedron $B(b^*)$ where b^* is defined on $S^* = S + s^*$ by (14.7). Let f^* denote the extension of the f to S^* where $f^*(s^*) := -\beta$, and let g^* be the extension of g to S^* where $g^*(s^*) := -\alpha$.

We claim that f^* and b^* meet (14.30). Indeed, for $Y \subseteq S$ (14.34) implies $f^*(Y) = \widetilde{f}(Y) \le b(Y) = b^*(Y)$, while in the case when $s^* \in Y$ we have $f^*(Y) = \widetilde{f}(X) - \beta \le -p(S - X) = b^*(Y)$ for $X := Y - s^*$. Similarly, g^* and b^* meet (14.31), since for $Y \subseteq S$ (14.35) implies $g^*(Y) = \widetilde{g}(Y) \ge p(Y) = -b^*(S^* - Y) = \overline{b^*}(Y)$, while in the case when $s^* \in Y$ we have $g^*(Y) = \widetilde{g}(X) - \alpha \ge -b(S - X) = -b^*(S - X) = \overline{b^*}(Y)$ for $X := Y - s^*$. Therefore Theorem 14.3.16 implies that there is an element $x^* \in B(b^*)$ and the restriction of x^* to S is in M by the construction. •

Corollary 14.3.18 *Let (p, b) be a paramodular pair and let $\alpha \le \beta$ be two numbers* (where α can be $-\infty$ and β can be ∞). *The intersection M of a g-polymatroid $Q(p, b)$ and a plank $K(\alpha, \beta)$ is non-empty if and only if*

$$\beta \ge p(S) \text{ and } \alpha \le b(S). \tag{14.36}$$

Proof. Apply Theorem 14.3.17 in the special case when $f \equiv -\infty$ and $g \equiv \infty$. Note that condition (14.34) is met automatically for all sets X but $X = \emptyset$ when it requires $\widetilde{f}(\emptyset) \le \beta - p(S - \emptyset)$. Similarly, (14.35) is met automatically apart from the inequality $\widetilde{g}(\emptyset) \ge \alpha - b(S - \emptyset)$ which required $X = \emptyset$. But (14.36) requires exactly these two inequalities. •

Finally, formulate Theorem 14.3.17 for the special cases of sub- and supermodular polyhedra. Let b and p be sub- and supermodular functions, respectively. Let $\alpha \le \beta$ be two numbers (possibly $\alpha = -\infty$ and $\beta = \infty$). Suppose for functions $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{\infty\}$ that $f \le g$.

Theorem 14.3.19 *The intersection $M := S(b) \cap T(f, g) \cap K(\alpha, \beta)$ is non-empty if and only if*

$$\widetilde{f}(S) \le \beta, \tag{14.37}$$

$$\widetilde{f}(X) \le b(X) \text{ for every } X \subseteq S, \tag{14.38}$$

and

$$\widetilde{g}(X) \ge \alpha - b(S - X) \text{ for every } X \subseteq S. \tag{14.39}$$

The intersection $M := S'(p) \cap T(f, g) \cap K(\alpha, \beta)$ is non-empty if and only if

$$\widetilde{g}(S) \ge \alpha, \tag{14.40}$$

$$\widetilde{g}(X) \ge p(X) \text{ for every } X \subseteq S, \tag{14.41}$$

and

$$\widetilde{f}(X) \le \beta - p(S - X) \text{ for every } X \subseteq S. \tag{14.42}$$

Proof. Define p to be zero on the empty set and $-\infty$ everywhere else. Then $S(b) = Q(p, b)$ and Theorem 14.3.17 can be applied. The first half of (14.34) is exactly (14.38). The second half of (14.34) holds automatically whenever $X \subset S$, while for $X = S$ it is just (14.37). The first half of (14.35) holds automatically, while its second half is just (14.39). The supermodular case follows analogously. •

The linking property

Corollary 14.3.20 (Linking property) *Let $p, b, f,$ and g be the same as in Theorem* 14.3.14. *If there is an element x' of Q for which $x' \geq f$ and there is an element x'' of Q for which $x'' \leq g$, then there is an $x \in Q$ for which $f \leq x \leq g$. If, in addition, each of $p, b, f,$ and g is integer-valued, then x can also be chosen to be integral.*

Proof. $\widetilde{f}(X) \leq \widetilde{x}(X) \leq b(X)$ implies (14.28) while $\widetilde{g}(X) \geq y(X) \geq p(X)$ implies (14.29). Hence Theorem 14.3.14 applies. •

Theorem 14.3.21 (Strong linking property) *If a g-polymatroid Q has an element x' for which $x' \geq f$ and $\widetilde{x}'(S) \leq \beta$, furthermore Q has an element x'' for which $x'' \leq g$ and $\widetilde{x}''(S) \geq \alpha$, then there is an $x \in Q$ for which $f \leq x \leq g$ and $\alpha \leq \widetilde{x}(S) \leq \beta$. If $p, b, f, g, \alpha,$ and β are integral, then x can also be chosen to be integeral.* •

In the special case when $f \equiv -\infty$ and $\alpha = -\infty$, one gets the following.

Corollary 14.3.22 *If a g-polymatroid Q has an element x with $\widetilde{x}'(S) \leq \beta$, and it has an element x'' with $x'' \leq g$, then there is an $x \in Q$ for which $\widetilde{x}(S) \leq \beta$ and $x \leq g$.* •

Remark Even if there are elements x' and x'' of Q for which $x' \geq f$, $x'(S) \geq \alpha$ and $x'' \leq g$, $x''(S) \leq \beta$, it is not necessarily true that there is an $x \in Q$ for which $f \leq x \leq g$ and $\alpha \leq \widetilde{x}(S) \leq \beta$. Indeed, let the ground-set be a singleton $S := \{s\}$, let $f(s) := g(s) := 0$ and $\alpha := \beta := 1$.

14.4 Intersection of two g-polymatroids

When the intersection is a g-polymatroid

The common independent sets of two matroids typically do not form a matroid, and similarly the intersection of two g-polymatroids is not necessarily a g-polymatroid either. There are, however, special cases when it is. For example, we proved that the intersection of a g-polymatroid with a box or with a plank is a g-polymatroid. Also, the very definition of g-polymatroids shows that the intersection $S(b) \cap S'(p)$ of a submodular and a supermodular polyhedron is a g-polymatroid provided the cross inequality holds for their border functions b and p. Yet another case arises from Theorem 12.2.5 which stated that the minimum of submodular functions b_1 and b_2 is submodular provided that $b_1 - b_2$ is non-decreasing.

Corollary 14.4.1 *Let b_1 and b_2 be submodular functions for which $b_1 - b_2$ is non-decreasing* (in particular, if b_1 is non-decreasing and b_2 is non-increasing). *Then the intersection of the submodular polyhedra $S(b_1)$ and $S(b_2)$ is a submodular polyhedron whose border function b is described by* (12.17) (namely, $b(X) = \min\{b_1(X), b_2(X)\}$). •

14.4.1 Feasibility results for intersection

Recall a version of Edmonds' matroid intersection theorem (Theorem 13.1.10) which provided a necessary and sufficient condition for the existence of a common basis of two matroids. Edmonds extended his result to polymatroids as well [80]. Not surprisingly, the intersection theorem can be further extended to g-polymatroids [118].

Theorem 14.4.2 (Intersection theorem for g-polymatroids) *Let (p_1, b_1) and (p_2, b_2) be paramodular pairs. The intersection $M := Q_1 \cap Q_2$ of the g-polymatroids $Q_1 := Q(p_1, b_1)$ and $Q_2 := Q(p_2, b_2)$ is non-empty if and only if*

$$p_1 \le b_2 \text{ and } p_2 \le b_1. \tag{14.43}$$

If each border function is integral and M is non-empty, then M contains an integral element.

Proof. The conditions in (14.43) are obviously necessary. For the sufficiency, let Q'_2 denote the reflection of Q_2 through the origin, that is, $Q'_2 = Q(-b_2, -p_2)$. Consider the sum Q of the g-polymatroids Q_1 and Q'_2. Theorem 14.2.15 implies that $Q = Q(p_1 - b_2, b_1 - p_2)$ and a vector $m \in \mathbf{R}^S$ is in Q precisely if $p_1(X) - b_2(X) \le \widetilde{m}(X) \le b_1(X) - p_2(X)$ for every $X \subseteq S$. But for $m \equiv 0$ this is exactly the conditions in (14.43), and hence Q contains the origin O. Therefore O can be obtained as the sum of some vectors $x_1 \in Q_1$ and $x'_2 \in Q'_2$, which, in addition, can be chosen to be integral when (p_i, b_i) are integral. It follows that $x_2 := -x'_2 \in Q_2$, and hence $x_1 = x_2$ is a common element of Q_1 and Q_2. •

Exercise 14.4.1 *Derive Theorem* 14.3.14 *from the Intersection theorem.*

The theorem has a self-refining nature since it implies that the intersection is actually an integral polyhedron.

Theorem 14.4.3 *Let (p_1, b_1) and (p_2, b_2) be integer-valued paramodular pairs. The intersection $M := Q_1 \cap Q_2$ of g-polymatroids $Q_1 := Q(p_1, b_1)$ and $Q_2 := Q(p_2, b_2)$ is an integral polyhedron.*

Proof. Each non-empty face M' of M is the intersection of a face Q'_1 of Q_1 and a face Q'_2 of Q_2. Since Q'_1 and Q'_2 are integral g-polymatroids by Theorem 14.2.7, the g-polymatroid intersection theorem implies that their intersection M' contains an integral point and hence M is integral. •

It is worth formulating the Intersection theorem for some special cases.

Theorem 14.4.4 *Let $B_1 = B(b_1)$ and $B_2 = B(b_2)$ be two base-polyhedra and k an integer for which $b_1(S) = b_2(S) = k$. The intersection $M := B_1 \cap B_2$ is non-empty if and only if*

$$b_1(X) + b_2(S - X) \ge k. \tag{14.44}$$

for every $X \subseteq S$. If b_1 and b_2 are integer-valued and M is non-empty, then M is an integral polyhedron.

Proof. Let p_i be the complement of b_i. We have seen that $B_i = Q(p_i, b_i)$. By Theorem 14.4.2, $B_1 \cap B_2$ is non-empty if and only if $p_1 \le b_2$ and $p_2 \le b_1$ which is equivalent to requiring that $k - b_1(S - X) \le b_2(X)$ and $k - b_2(S - X) \le b_1(X)$ for every $X \subseteq S$. But these are equivalent to each other and to (14.44) as well. •

In the special case of matroid base-polyhedra we are back at the following version of Edmonds' matroid intersection theorem.

Theorem 14.4.5 *Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two rank-k matroids. They have a common basis if and only if $r_1(X) + r_2(S - X) \geq k$ for every $X \subseteq S$.*

Discrete separation versus g-polymatroid intersection

Consider Theorem 15.1.7 in the special case when $Q_1 = S(b)$ is a submodular polyhedron defined by a submodular function and $Q_2 = S'(p)$ is a supermodular polyhedron defined by a supermodular function p. Then the theorem is nothing but a reformulation of the Discrete separation theorem (Theorem 12.2.1). Conversely, Theorem 15.1.7 can easily be derived from Theorem 12.2.1. To this end, extend S by a new element s^*. Define the set-functions b^* and p^* on ground-set $S^* := S + s^*$ as follows.

$$b^*(X) = \begin{cases} b_1(X) & \text{if } X \subseteq S \\ -p_1(S - X) & \text{if } s^* \in X, \end{cases} \tag{14.45}$$

$$p^*(X) = \begin{cases} p_2(X) & \text{if } X \subseteq S \\ -b_2(S - X) & \text{if } s^* \in X. \end{cases} \tag{14.46}$$

Then (14.43) is equivalent to requiring $p^* \leq b^*$. By Theorem 12.2.1, there is a function m^* on S^* for which $p^* \leq \widetilde{m}^* \leq b^*$. Hence the restriction m of m^* to S is in $Q_1 \cap Q_2$.

14.5 The greedy algorithm

In Section 5.5.4 we saw how Edmonds extended the matroid greedy algorithm to polymatroids. Our present goal is to show that the same idea carries over to g-polymatroids as well.

14.5.1 Greedy algorithm for base-polyhedra

Let b be a fully submodular function on ground-set S for which $b(S)$ is finite. Let $c : S \to \mathbf{R}$ be a function and consider the optimization problem $\max\{cx : x \in B(b)\}$. The optimum value will be denoted by $\hat{b}(c)$. This function was already introduced in subsection 5.5.2 for the special case when b is the rank function of a matroid. When $c = \underline{\chi}_Z$ for some subset $Z \subseteq S$, then $cx = \widetilde{x}(Z)$ and Corollary 14.2.9 implies that $\hat{b}(\underline{\chi}_Z) = \hat{b}(c) = \max\{cx : x \in B(b)\} = \max\{\widetilde{x}(Z) : x \in B(b)\} = b(Z)$. In this sense, the function $\hat{b} : \mathbf{R}^S \to \mathbf{R}$ is an extension of b which will be called the **upper linear extension** or just the **linear extension** of b. (Lower linear extension could be analogously introduced by replacing max with min.) In the literature the name 'Lovász extension' is also used since the notion was introduced and studied in a paper by Lovász [272].

Consider first the technically simpler special case when b is finite-valued: the general case will soon be treated. Determining $\hat{b}(c)$ is the same as solving the following linear program:

$$\max\{cx : \widetilde{x}(S) = b(S) \text{ and } \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subset S\}, \tag{14.47}$$

978–0–19–920527–1 14-Andras-Frank-c14-drv Andras Frank (Typeset by SPi, Publisher Services, India) 485 of 640 December 11, 2010 15:20

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

for which the dual program is as follows.

$$\min\{yb : y \in D(c)\}, \tag{14.48}$$

where $yb := \sum_{Z\subseteq S} y(Z)b(Z)$ and

$$D(c) := \left\{y \in \mathbf{R}^{2^S} : y(Z) \geq 0 \text{ for } Z \subset S \text{ and } \sum_{Z\subseteq S} y_Z \underline{\chi}_Z = c\right\} \tag{14.49}$$

is the dual polyhedron. We are going to show that optimal solutions to the primal and to the dual programs can be constructed in a greedy way.

It can be assumed that the components of c are arranged in a non-increasing order, that is, $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$. Let S_i denote the set of the first i elements. Define the vector x_{alg} as follows.

$$x_{alg}(s_1) := b(s_1), \text{ and } x_{alg}(s_i) := b(S_i) - b(S_{i-1}) \text{ for } i = 2, \ldots, n. \tag{14.50}$$

This definition implies that $\widetilde{x}_{alg}(S_i) = b(S_i)$, for each $i = 1, \ldots, n$ where $\widetilde{x}_{alg}(Z) := \sum_{z\in Z} x_{alg}(z)$. In particular, $\widetilde{x}_{alg}(S) = b(S)$. It is also clear that x_{alg} is integral whenever b is.

Lemma 14.5.1 *The vector x_{alg} given in* (14.50) *is in the base-polyhedron $B(b)$.*

Proof. Since $\widetilde{x}_{alg}(S) = b(S)$, we have to show that

$$\widetilde{x}_{alg}(Z) \leq b(Z) \tag{14.51}$$

for every proper subset Z of S. Let $\mu(Z)$ denote the largest subscript i for which $s_i \in Z$. We proceed by induction on $\mu(Z)$. If $\mu(Z) = 1$, then $Z = \{s_1\}$ and hence $\widetilde{x}_{alg}(Z) = b(Z)$. Suppose now that $i := \mu(Z) \geq 2$. By using submodularity, we get $b(Z) + b(S_{i-1}) \geq b(Z \cap S_{i-1}) + b(Z \cup S_{i-1})$. By applying the inductive hypothesis to $Z \cap S_{i-1}$ and by observing that $Z \cup S_{i-1} = S_i$, we obtain $b(Z \cap S_{i-1}) + b(Z \cup S_{i-1}) \geq \widetilde{x}_{alg}(Z \cap S_{i-1}) + b(S_i)$ from which $b(Z) \geq \widetilde{x}_{alg}(Z \cap S_{i-1}) + b(S_i) - b(S_{i-1}) = \widetilde{x}_{alg}(Z \cap S_{i-1}) + \widetilde{x}_{alg}(s_i) = \widetilde{x}_{alg}(Z)$, as required. •

Define now the vector y^* as follows.

$$y^*(S_n) := c(s_n) \text{ and } y^*(S_i) := c(s_i) - c(s_{i+1}) \text{ for } i = 1, 2, \ldots, n-1, \tag{14.52}$$

while $y^*(Z) := 0$ for every other subset Z of S. It is easily seen that y^* is a solution to the dual program (14.48), and if c is integral, then so is y^*.

Theorem 14.5.2 *x_{alg} is an optimal solution to the primal program* (14.47) *while y^* is an optimal solution to the dual program* (14.48).

Proof. We have already shown that x_{alg} is a primal solution, y^* is a dual solution, and max $\leq$ min. Therefore the only thing we need to prove is that $cx_{alg} = y^*b$. But this follows immediately from the optimality criteria of linear programming since $y^*(Z)$ can only be positive if $Z = S_i$ and these sets meet the corresponding primal inequality $\widetilde{x}(Z) \leq b(Z)$ by equality, due to the definition of x_{alg}. (Without referring to the optimality criteria of linear programming, one can alternatively show the optimality: $cx_{alg} = c(s_1)b(S_1) + \sum_{i=2}^{n} c(s_i)[b(S_i) - b(S_{i-1})] = c(s_n)b(S_n) + \sum_{i=1}^{n-1}[c(s_i) - c(s_{i+1})]b(S_i) = \sum_{Z\subseteq S} y^*(Z)$ $b(Z) = y^*b$.) •

Corollary 14.5.3 *For a finite-valued submodular function b and a weight function c,*

$$\hat{b}(c) = b(S)c(s_n) + \sum_{i=1}^{n-1} b(S_i)[c(s_i) - c(s_{i+1})], \tag{14.53}$$

where $c(s_1) \geq c(s_2) \geq \cdots \geq c(s_n)$ *and* $S_i := \{s_1, \ldots, s_i\}$. •

Note that the greedy algorithm also shows that a base-polyhedron $B(b)$ defined by a submodular function b is never empty. Moreover, $B(b)$ is integral provided that b is integral. When the greedy algorithm is applied to the cost function $c := \underline{\chi}(A)$ $(A \subseteq S)$, we can conclude that $B(b)$ has an element x_0 for which $\widetilde{x}_0(A) = b(A)$ and if b, in addition, is integral, then so is x_0. This result was proved in Theorem 14.2.8.

Exercises

14.5.1 *Prove that* $\hat{b}$ *is* **positively homogeneous**, *meaning that* $\hat{b}(\alpha c) = \alpha\hat{b}(c)$ *for every non-negative number* α. *Prove* $\hat{b}(\underline{\chi}_X + \underline{\chi}_Y) = b(X \cap Y) + b(X \cup Y)$ *for* $X, Y \subseteq S$.

14.5.2 *Prove that* $\hat{b}(\sum_i \underline{\chi}_{Z_i}) = \sum_i b(Z_i)$ *where* $Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_m$ *is a chain of not-necessarily-distinct subsets of S.*

14.5.3 *Let* $\lambda_1 > \lambda_2 > \cdots > \lambda_t$ *be the distinct values of a weight function c and let* $T_i := \{s : c(s) \geq \lambda_i\}$ *for* $i = 1, \ldots, t$. *Then* $\hat{b}(c) = \lambda_t b(S) + \sum_{i=1}^{t-1}(\lambda_i - \lambda_{i+1})b(T_i)$.

Properties of linear extensions

Theorem 14.5.4 *If b is submodular, then* $\hat{b}$ *is* **subadditive**, *in the sense that*

$$\sum_{i=1}^{\ell} \hat{b}(c_i) \geq \hat{b}\left(\sum_{i=1}^{\ell} c_i\right). \tag{14.54}$$

for every set of vectors $c_1, c_2, \ldots, c_\ell \in \mathbf{R}^S$

Proof. Let $c_0 := \sum_i c_i$. By Corollary 14.5.3, for every c_i there is an element x_i of base-polyhedron $B(b)$ for which $c_i x_i = \hat{b}(c_i) \geq c_i x$ holds for every $x \in B(b)$. Hence

$$\hat{b}\left(\sum_{i=1}^{\ell} c_i\right) = \hat{b}(c_0) = c_0 x_0 = \left(\sum_{i=1}^{\ell} c_i\right) x_0 = \sum_{i=1}^{\ell} c_i x_0 \leq \sum_{i=1}^{\ell} c_i x_i = \sum_{i=1}^{\ell} \hat{b}(c_i),$$

as required. •

By specializing Theorem 14.5.4 to (0, 1)-valued vectors, one obtains the following extension of the submodular inequality to several terms.

Corollary 14.5.5 *Let b be a submodular function on ground-set S and let* $X_1, X_2, \ldots, X_m$ *be subsets of S. Then*

$$\sum_{i}^{m} b(X_i) \geq \hat{b}\left(\sum_{i}^{m} \underline{\chi}_{X_i}\right). \quad \bullet \tag{14.55}$$

Theorem 14.5.6 (Lovász [272]) *A finite-valued set-function b is submodular if and only if its linear extension $\hat{b}$ is convex.*

Proof. Suppose first that b is submodular. Since $\hat{b}$ is positively homogeneous and subadditive, we have $\sum_i \hat{b}(c_i) \geq \hat{b}(\sum_i c_i) = k\hat{b}((\sum_i c_i)/k)$, for vectors $c_1, \ldots, c_k$, and hence $\hat{b}$ is convex.

Conversely, let $\hat{b}$ be convex. For two given sets X and Y, define $c := (\underline{\chi}_X + \underline{\chi}_Y)/2$. Then $\hat{b}(c) = [b(X \cap Y) + b(X \cup Y)]/2$ from which $b(X) + b(Y) = \hat{b}(\underline{\chi}_X) + \hat{b}(\underline{\chi}_Y) \geq 2\hat{b}(c) = b(X \cap Y) + b(X \cup Y)$. •

Exercise 14.5.4 *A face of a base-polyhedron $B(b)$ determined by the subsets $X_1, \ldots, X_q \subseteq S$ is non-empty if and only if $\sum b(X_i) = \hat{b}(\sum_i \underline{\chi}(X_i))$.*

Infinite values

Consider now the optimization problem $\max\{cx : x \in B(b)\}$ for the general case when b can have ∞ values (though the finiteness of $b(S)$ is assumed). Let $\mathcal{F}$ denote the system of sets Z for which $b(Z)$ is finite. Then $\mathcal{F}$ is closed under intersection and union and $S \in \mathcal{F}$. Therefore $\mathcal{F}$ has a unique smallest member $T(s)$ containing s for each $s \in S$.

We claim that $\{cx : x \in B(b)\}$ is not bounded from above if there is an element $v \in T(u)$ for which $c(u) > c(v)$. Indeed, for any element $z \in B(b)$, when the component $z(u)$ is increased by any big δ while the component $z(v)$ is decreased by the same δ, the resulting z' continues to belong to $B(b)$ and $cz' = cz + \delta(c(u) - c(v))$. Therefore we can assume that

$$v \in T(u) \text{ implies } c(v) \geq c(u). \tag{14.56}$$

Two elements u and v of S are said to be equivalent if $c(u) = c(v)$ and $b(X) = \infty$ holds for every $X \subset S$ for which $|X \cap \{u, v\}| = 1$. By aggregating u and v, we obtain an optimization problem equivalent to the initial one. Therefore we can assume that there are no equivalent elements. It follows that

$$v \in T(u) \text{ implies either } c(v) > c(u) \text{ or } c(v) = c(u) \text{ and } T(v) \subseteq T(u) - u. \tag{14.57}$$

Lemma 14.5.7 *If there are no two equivalent elements, then there is an ordering $s_1, \ldots, s_n$ such that $c(s_1) \geq \cdots \geq c(s_n)$ and $b(S_i) < \infty$ for every starting segment $S_i = \{s_1, \ldots, s_i\}$.*

Proof. Arrange the elements of S in a decreasing order in such a way that, for any two elements u and v, if $c(u) = c(v)$ and $v \notin T(u)$, then u has a smaller subscript than v. Due to (14.57), the ordering $s_1, \ldots, s_n$ obtained in this way has the property that $T(s_i) \subseteq S_i$.

Claim 14.5.8 *$b(S_i) < \infty$ for every $i = 1, \ldots, n$.*

Proof. Let $i = 1$. $T(s_1)$ is a singleton for if there is a $v \in T(s_1) - s_1$, then $c(s_1) \geq c(v)$ follows from the maximality of $c(s_1)$ and hence (14.56) implies $c(s_1) = c(v)$. Since there are no two equivalent elements, we have $T(v) \subset T(s_1)$, contradicting the selection rule. Therefore $T(s_1) = \{s_1\} = S_1$, and hence $b(S_1)$ is indeed finite.

Suppose now that the claim holds for $i - 1 < n$, and hence $b(S_{i-1}) < \infty$. Then $\infty > b(S_{i-1}) + b(T(s_i)) \geq b(S_{i-1} \cup T(s_i)) + b(S_{i-1} \cap T(s_i))$ from which $S_i = S_{i-1} \cup T(s_i)$ implies that $b(S_i)$ is finite. • •

The greedy algorithm described above for the finite case can be applied to the ordering ensured by the lemma.

Corollary 14.5.9 $\{cx : x \in B(b)\}$ *is bounded from above if and only if* (14.56) *holds.*

Proof. We can assume that there are no two equivalent elements. It has already been proved above that $\{cx : x \in B(b)\}$ is not bounded from above if (14.56) does not hold. Conversely, if (14.56) holds, then the greedy algorithm provides a finite maximum. •

Geometric view

In several applications, the base-polyhedron B is given in a certain way, but its unique fully submodular border function is not explicitly available, and therefore the greedy algorithm above cannot be applied directly. Still, the following geometric interpretation of the greedy algorithm can often help.

Suppose that the elements are arranged in a non-increasing order according to the weight function c. Starting at s_1, the greedy algorithm passes through the elements in this order and establishes the values $x(s_i)$ one by one. The current value $x_i = x(s_i)$ is to be taken as large as possible under the restriction that there is a vector $(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)$ in $B(b)$. Here $x_1, x_2, \ldots, x_{i-1}$ denote the components already computed.

This interpretation has the advantage that the algorithm is independent of the form in which the base-polyhedron is given. Indeed, there are important cases when the current x_i can be computed without the explicit knowledge of the unique fully submodular border function.

14.5.2 Variants of the greedy algorithm

Generalized polymatroid

We proved that a g-polymatroid $Q = Q(p, b)$ can always be obtained as the projection of a 0-base-polyhedron $B(b^*)$ along one element. Therefore the problem of maximizing cx over the elements of Q can easily be transformed into a maximization problem over a base-polyhedron.

The resulting algorithm can be applied to special g-polymatroids like submodular polyhedra, polymatroids or contra-polymatroids but it is more instructive if we deduce a variant directly from the version above concerning base-polyhedra.

Submodular polyhedron

How can we maximize cx over a submodular polyhedron $S(b)$? For simplicity, assume again that b is finite-valued. In the linear programming dual of the maximization problem (14.47), apart from $y(S)$, the non-negativity of each dual variable $y(Z)$ was required. By the algorithm $y^*(S)$ will be negative exactly if the smallest value $c(s_n)$ of the weight function is negative. Therefore if $c(s_n) \geq 0$, then the optimum solution for $B(b)$ provided by the greedy algorithm will automatically be an optimal solution for $S(b)$, too.

If $c(s_n) < 0$, then the problem $\max\{cx : x \in S(b)\}$ is not bounded from above since taking any element x of $S(b)$, we can decrease the component $x(s_n)$ arbitrarily without leaving $S(b)$, showing that cx is indeed not bounded.

Polymatroid

We can show that the polymatroid greedy algorithm may be viewed as a special case of the greedy algorithm for submodular polyhedra. Let b be a polymatroid function. Let S^+ denote the set of elements s for which $c(s) > 0$. Let b' and c', respectively, be the restriction of b and c to S^+. Apply the greedy algorithm to the submodular polyhedron $S(b')$ and let $x' \in \mathbf{R}^{S^+}$ and y^* denote the optimal primal and dual solution. Then $c'x' = y^*b'$. Let $x^* \in \mathbf{R}^S$ be the vector obtained from x' by extending it with zero components. Since $c(s)$ is positive for $s \in S^+$, we see that x^* is non-negative and hence $x^* \in P(b)$. Since y^* is also in the dual polyhedron $\left\{y \in \mathbf{R}^{2^S} : y(Z) \geq 0, \sum_{Z \subseteq S} y_Z \underline{\chi}_Z \geq c\right\}$, we conclude that x^* is a primal optimum and y^* is a dual optimum since $cx^* = c'x' = y^*b' = y^*b$.

Contra-polymatroid

We want to minimize cx over a contra-polymatroid C. Arrange the elements of S by non-increasing order of the components of c. The current value $x_i = x(s_i)$ is chosen to be the smallest in such a way that there is a vector $(x_1, \dots, x_i, y_{i+1}, \dots, y_n)$ in C for some appropriate $y_{i+1}, \dots, y_n$.

How can one compute this minimal $x(s_i)$? Let k denote the largest value of $p(X)$. It is easy to verify that if $(x_1, \dots, x_i, y_{i+1}, \dots, y_n)$ is in C, then so is $(x_1, \dots, x_i, k, k, \dots, k)$. Accordingly, the greedy algorithm can be modified in such a way that the current x_i is chosen to be minimal subject to the requirement that $(x_1, \dots, x_i, k, k, \dots, k)$ belongs to C.

15
Relaxing semimodularity

In the introduction to the first chapter of Part III, we have already indicated that there are two major aspects of investigating semimodular functions. One is about exploring their properties and structure, and we did prove already some of these results such as the Sum theorem of g-polymatroids (Theorem 14.2.15).

The other aspect is about the ways in which matroids or polymatroids can be specified with functions of relaxed submodularity properties. In this chapter, we continue this line of investigation and extend the notion of truncation. At a first sight, some of these definitions may seem a bit artificial but each of them has been motivated by applications to be discussed later.

Definitions and elementary properties

Recall from Sections 1.1 and 1.4 the definitions given for a hypergraph to be intersecting, crossing, laminar, cross-free. A set-function $b : 2^S \to \mathbf{R} + \{\infty\}$ was previously defined to be intersecting, (co-intersecting, crossing) submodular if the submodular inequality (1.2) holds for every intersecting (co-intersecting, crossing) pair $X, Y \subseteq S$ of subsets.

Proposition 15.0.1 *Let b be a fully submodular function. By decreasing its values arbitrarily on singletons, one obtains an intersecting submodular function. By decreasing its values on all non-empty subsets by the same value, one obtains an intersecting submodular function. Analogous statements hold for supermodular functions when the values in question are increased.*

A pair (p, b) of set-functions is an **intersecting paramodular** (or **weak**, for short) pair if p and b are intersecting super- and submodular functions, respectively, and the cross-inequality is required only for properly intersecting subsets.

Let b be a set-function on ground-set S. A subset $X \subseteq S$ is b**-separable from below** (or said to be simply separable) if X can be partitioned into non-empty, disjoint subsets $X_1, X_2, \ldots, X_t$ for which $\sum_i b(X_i) \leq b(X)$ (where $t \geq 2$). We say that b is **near submodular** if the submodular inequality holds for every intersecting pair of non-separable subsets X, Y. An intersecting submodular function, by definition, is near submodular. Although Theorem 12.1.1 on truncation, for example, will be extended to near submodular functions, a disadvantage of this latter notion is that it is not closed under sum.

For a function p, one may analogously speak of subsets which are p-separable from above and introduce the notion of p being **near supermodular**. Finally, we say that a pair (p, b) of set-functions is **near paramodular** if p and b are respectively near super- and submodular and the cross-inequality holds for X and Y whenever X and Y are properly intersecting, X is not b-separable from below, and Y is not p-separable from above.

In Section 1.1 we introduced the following concept. A non-negative set-function p is **positively intersecting** (**crossing**) **supermodular** if the supermodular inequality holds for X and Y whenever $p(X) > 0$, $p(Y) > 0$ and X, Y are intersecting (respectively, crossing). In all applications we discuss, positively intersecting supermodular functions can be constructed from an intersecting supermodular function by replacing the negative values with zero. Note, however, that not every positively intersecting supermodular function arises in this way. That is, from an applicational point of view this notion does not seem to be important, but their use often helps in simplifying proofs.

In Subsection 8.1.3 we defined a function p to be skew supermodular if, for every pair of subsets X and Y, at least one of the following two inequalities holds:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

When p is non-negative and at least one of these inequalities is required to hold only for sets X and Y with $p(X) > 0$ and $p(Y) > 0$, we speak of a **positively skew supermodular** function. Note that the second inequality automatically holds for disjoint subsets and hence an intersecting supermodular function is skew supermodular. Also, the first inequality holds automatically whenever one of X and Y includes the other. That is, for intersecting subsets, it suffices to require that only one of the two inequalities holds. It follows that every positively supermodular function is skew supermodular. In Lemma 8.1.9 we described an important construction of skew supermodular functions which proved useful in providing a relatively simple proof of a deep theorem of Mader.

Lemma 15.0.2 *A symmetric and positively crossing supermodular function p is skew supermodular. A positively skew supermodular function p is near supermodular.*

Proof. In the first part, if X and Y are crossing, $p(X) > 0$, and $p(Y) > 0$, then the supermodular inequality holds for X and Y. Therefore, we can assume that $X \cup Y = S$. The symmetry of p implies that $p(X) + p(Y) = p(S - X) + p(S - Y) = p(Y - X) + p(Y - X)$, and hence p is indeed skew supermodular.

For the second part, let X and Y be properly intersecting sets which are not p-separable from above. Then $p \geq 0$ implies $p(X) > 0$ and $p(Y) > 0$. We have $p(X) > p(X - Y) + p(X \cap Y) \geq p(X - Y)$ and $p(Y) > p(Y - X) + p(X \cap Y) \geq p(Y - X)$. Therefore $p(X) + p(Y) > p(X - Y) + p(Y - X)$ and hence the skew supermodularity of p implies $p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$. •

15.1 Truncation

In Section 12 we defined the lower truncation $b^{\vee}$ of a set-function b by the formula

$$b^{\vee}(Z) := \min \left\{ \sum_{i}^{t} b(X_i) : \{X_1, \ldots, X_t\} \text{ a partition of } Z \ (t \geq 1) \right\}. \tag{15.1}$$

We observed that $b^\vee \leq b$ and proved that $b^\vee$ is fully submodular provided that b is intersecting submodular. This property was then used in Section 13.4 to show how matroids can be constructed from intersecting submodular functions.

By taking the outer minimization of the truncation of b, we obtain $(b^\vee)_{omin}$ and call it the **monotone lower truncation** of b. That is,

$$(b^\vee)_{omin}(Z) = \min\left\{\sum_i^t b(X_i) : \{X_1, \dots, X_t\} \text{ a subpartition of } S\ (t \geq 1) \text{ and } Z \subseteq \cup X_i\right\}. \tag{15.2}$$

The upper truncation $p^\wedge$ of p was defined as follows.

$$p^\wedge(Z) := \max\left\{\sum_i^t p(X_i) : \{X_1, \dots, X_t\} \text{ a partition of } Z, \text{ with } t \geq 1\right\}. \tag{15.3}$$

By taking the inner maximization $(p^\wedge)_{omax}$ of the upper truncation of p, we obtain the **monotone upper truncation** of p, where

$$(p^\wedge)_{\uparrow mon}(Z) = \max\left\{\sum_i^t p(X_i) : \{X_1, \dots, X_t\} \text{ a subpartition of } Z\ (t \geq 1)\right\}. \tag{15.4}$$

The following result is an extension of Dunstan's theorem (Theorem 12.1.1).

Theorem 15.1.1 (Truncation theorem) *The truncation $b^\vee$ of a near submodular function b is fully submodular. If in addition $b \geq 0$, then the monotone lower truncation of b is fully submodular and non-decreasing.*

The truncation $p^\wedge$ of a near supermodular function p is fully supermodular, its monotone upper truncation is fully supermodular and non-decreasing.

Proof. By Theorem 14.3.1, the assertion for the monotone truncation follows from the one on truncation. By symmetry, it suffices to show the submodularity of $b^\vee$.

For a family $\mathcal{F}$ of subsets, we use the notation $\widetilde{b}(\mathcal{F}) := \sum[b(X) : X \in \mathcal{F}]$. Let $A, B \subseteq S$. There is a partition $\{A_1, \dots, A_k\}$ of A and a partition $\{B_1, \dots, B_l\}$ of B for which $b^\vee(A) = \sum b(A_i)$ and $b^\vee(B) = \sum b(B_j)$. Then $\mathcal{F} = \{A_1, \dots, A_k, B_1, \dots, B_l\}$ is a family of subsets for which $\widetilde{b}(\mathcal{F}) = b^\vee(A) + b^\vee(B)$ and

($*$) $\mathcal{F}$ covers the elements of $A \cap B$ twice and the elements of $(A - B) \cup (B - A)$ once.

Pick a family $\mathcal{F}_1$ of subsets (in which a set may occur in two copies) satisfying ($*$) such that $\widetilde{b}(\mathcal{F}_1)$ is minimal, and subject to this, $|\mathcal{F}_1|$ is maximal and subject to this, $\sum[|X|^2 : X \in \mathcal{F}_1]$ is maximal. Evidently, $\widetilde{b}(\mathcal{F}) \geq \widetilde{b}(\mathcal{F}_1)$. It also follows easily from the maximality of $|\mathcal{F}_1|$ that the members of $\mathcal{F}_1$ are not separable. Then the submodular inequality holds for every two intersecting members of $\mathcal{F}_1$. We claim that $\mathcal{F}_1$ is laminar for if it had two properly intersecting members, then replacing these two by their intersection and union, we would get a family $\mathcal{F}_2$ satisfying ($*$) for which $\widetilde{b}(\mathcal{F}_1) \geq \widetilde{b}(\mathcal{F}_2)$, $|\mathcal{F}_1| = |\mathcal{F}_2|$, and $\sum[|X|^2 : X \in \mathcal{F}_1] < \sum[|X|^2 : X \in \mathcal{F}_2]$, contradicting the choice of $\mathcal{F}_1$.

Since $\mathcal{F}_1$ is laminar, it can be partitioned into two parts $\mathcal{P}_1$ and $\mathcal{P}_2$ where $\mathcal{P}_1$ is a partition of $A \cap B$ and $\mathcal{P}_2$ is a partition of $A \cup B$. By the definition of $b^\vee$, we get $b^\vee(A \cap B) \leq \widetilde{b}(\mathcal{P}_1)$

978–0–19–920527–1 15-Andras-Frank-c15-drv Andras Frank (Typeset by SPi, Publisher Services, India) 493 of 640 December 11, 2010 15:21

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

and $b^\vee(A \cup B) \leq b(\mathcal{P}_2)$. Therefore $b^\vee(A) + b^\vee(B) = \widetilde{b}(\mathcal{F}) \geq \widetilde{b}(\mathcal{F}_1) = \widetilde{b}(\mathcal{P}_1) + \widetilde{b}(\mathcal{P}_2) \geq b^\vee(A \cap B) + b^\vee(A \cup B)$, and hence $b^\vee$ is indeed fully submodular. •

15.1.1 G-polymatroids from relaxed semimodular functions

Polyhedra of semimodular functions

Proposition 15.1.2 (A) *$S(b^\vee) = S(b)$ and in particular, $\mathcal{I}(b^\vee) = \mathcal{I}(b)$ holds for any set-function b. If we assume that $b^\vee(S) = b(S)$, or equivalently that*

$$\sum_i b(S_i) \geq b(S) \text{ for every partition } \{S_1, \ldots, S_k\} \text{ of } S, \tag{15.5}$$

then $B(b) = B(b^\vee)$ and $\mathcal{B}(b^\vee) = \mathcal{B}(b)$.

(B) *$S'(p^\wedge) = S'(p)$ and $C(p^\wedge) = C(p)$ hold for any set-function p. In particular, $\mathcal{G}(p^\wedge) = \mathcal{G}(p)$. If we assume $p^\wedge(S) = p(S)$, then $B'(p^\wedge) = B'(p)$ and $\mathcal{B}'(p^\wedge) = \mathcal{B}'(p)$.*

Proof. $b^\vee \leq b$ implies $S(b^\vee) \subseteq S(b)$. On the other hand, let $x \in S(b)$ and consider a partition$\{Z_i\}$ of a subset $Z \subseteq S$ for which $b^\vee(Z) = \sum_i b(Z_i)$. Then $\widetilde{x}(Z) = \sum_i \widetilde{x}(Z_i) \leq \sum_i b(Z_i) = b^\vee(Z)$ from which $x \in S(b^\vee)$ and thus $S(b) = S(b^\vee)$. Therefore $b^\vee(S) = b(S)$ implies $B(b) = S(b) \cap \{x : \widetilde{x}(S) = b(S)\} = S(b^\vee) \cap \{x : \widetilde{x}(S) = b^\vee(S)\} = B(b^\vee)$. The second part follows analogously. •

By combining this result with Theorem 15.1.1, one gets the following.

Theorem 15.1.3 *Let b be near submodular. Then $S(b)$ is a submodular polyhedron for which the fully submodular border function is $b^\vee$. If $b(S)$ is finite, then $B(b)$ is a* (possibly empty) *base-polyhedron. When* (15.5) *holds, the fully submodular upper border function of $B(b)$ is $b^\vee$. If, in addition, $b \geq 0$, then $P(b)$ is a polymatroid for which the border function b' is the monotone truncation of b, that is,*

$$b'(Z) := \min\left\{\sum_i b(X_i) : \{X_1, \ldots, X_t\} \text{ a subpartition of } S \text{ and } Z \subseteq \cup X_i \subseteq S\right\}. \tag{15.6}$$

Let p be near supermodular (in particular, skew supermodular). *Then $S'(p)$ is a supermodular polyhedron for which the unique fully supermodular border function is $p^\wedge$. If $p(S)$ is finite, then $B'(p)$ is a base-polyhedron. When $p^\wedge(S) = p(S)$, the lower border function of $B'(p)$ is $p^\wedge$. Moreover, $C(p)$ is a contra-polymatroid for which the border function p'* (that is, for which, *$C(p) = S'(p')$*) *is the monotone upper truncation of p:*

$$p'(Z) := \max\left\{\sum_i p(X_i) : \{X_1, \ldots, X_t\} \text{ a subpartition of } Z\right\}. \bullet \tag{15.7}$$

It should be noted that $(p^\wedge, b^\vee)$ is not necessarily a paramodular pair when (p, b) is intersecting paramodular since the cross inequality does not necessarily hold for every subset X and Y. Still, it is true that every intersecting (even every near) paramodular pair does define a g-polymatroid, although the formula expressing its (unique paramodular) border pair is significantly more complicated than the one in the definition of truncation.

15.1.2 Truncation algorithmically

Given a near (in particular, intersecting) submodular function b on a ground-set S of n elements, how is it possible to determine algorithmically its (lower) truncation? That is, we are interested in computing $b^\vee(X)$ for any given subset X of S. Such an algorithm can naturally be used for computing the upper truncation of a near (in particular, intersecting or skew) supermodular function. Since the restriction of a near submodular function to X is also near submodular, it suffices to concentrate on computing $b^\vee(S)$. The algorithm relies on an oracle for minimizing certain submodular functions specified at the end of this description. In particular, this oracle can decide if there is a subset X with $b(X) < \infty$ containing a given element v.

If there is an element v such that $b(X) = \infty$ for every subset X containing v, then $b^\vee(S) = \infty$ follows from the very definition of truncation, and in this case, the algorithm terminates.

Therefore we can assume that S can be covered by sets of finite b-value. Since $b(X \cup Y) < \infty$ holds for intersecting X and Y with $b(X) < \infty, b(Y) < \infty$, we can obtain a partition $\{V_1, \dots, V_q\}$ of S for which $b(V_i) < \infty$ for $i = 1, \dots, q$. Therefore we can assume that $b(S)$ itself is finite since in the case when $b(S) = \infty$ we could reduce the value of $b(S)$ to $\sum_i b(V_i)$ and the set-function b' obtained in this way would also be near submodular for which $b^\vee(S) = (b')^\vee(S)$.

If u and v are two equivalent elements of S in the sense that $b(X) = \infty$ whenever $|X \cap \{u, v\}| = 1$, then u and v can be aggregated into one element. Therefore we assume that there are no equivalent elements, in which case, by Lemma 14.5.7, there is an ordering $s_1, \dots, s_n$ of the elements of S such that $b(S_i) < \infty$ for every starting segment $S_i := \{s_1, \dots, s_i\}$, $i = 1, 2, \dots, n$.

By Proposition 15.1.2 $B(b) = B(b^\vee)$ and by Corollary 14.2.9 we have $b^\vee(S) = \max\{\widetilde{x}(S) : x \in B(b^\vee)\} = \max\{\widetilde{x}(S) : x \in B(b)\}$. We are going to show that it is possible to run the greedy algorithm on $B(b^\vee)$ even if an evaluation oracle for $b^\vee$ is not explicitly available. Suppose that the algorithm has already determined $x^*(s_1), \dots, x^*(s_{k-1})$. The geometric view of the greedy algorithm defines $x^*(s_k)$ to be the largest value α for which $\widetilde{x}^*(X) \leq b(X)$ holds for every $X \subseteq S_k$, which implies that

$$\alpha = \min\{b(Z) - \widetilde{x}^*(Z - s_k) : s_k \in Z \subseteq S_k\}. \tag{15.8}$$

The final vector x^* computed in this way has the property that $\widetilde{x}^*(S) = b^\vee(S) = \min\{\sum_i b(T_i) : \{T_1, \dots, T_t\}$ a partition of $S\}$.

The computation can be carried out provided an oracle is available for computing α in (15.8). If the oracle computes not only the value of the minimum in question but a minimizer set Z as well, then a partition $\{T_1, \dots, T_t\}$ of S can also be computed for which $b^\vee(S) = \sum_i b(T_i)$. To this end, consider the corresponding minimizer sets $Z_1, Z_2, \dots, Z_n$. Since x^* belongs to $B(b^\vee)$ and each Z_i is tight (in the sense that $\widetilde{x}^*(Z_i) = b(Z_i)$). Since the union of intersecting tight sets is tight again, we know that the components of the hypergraph with hyperedges $\{Z_1, \dots, Z_n\}$ form a partition $\{T_1, \dots, T_t\}$ of S into tight sets from which

$$b^\vee(S) = \widetilde{x}^*(S) = \sum_i \widetilde{x}^*(T_i) = \sum_i b(T_i)$$

and hence $b^\vee(S) = \sum_i b(T_i)$, as required.

15.1.3 Feasibility

By combining Corollary 14.2.9 and Lemma 15.1.2, we obtain the following characterization of non-emptiness of a base-polyhedron defined by near sub- or supermodular functions.

Theorem 15.1.4 *Let b be a near submodular function for which $b(S)$ is finite. The base-polyhedron $B(b)$ is non-empty if and only if $\sum_i b(S_i) \geq b(S)$ for every partition $\{S_1, \ldots, S_k\}$ of S* (which is equivalent to $b^{\vee}(S) = b(S)$). *For a near supermodular function p, a base-polyhedron $B'(p)$ is non-empty if and only if $\sum_i p(S_i) \leq p(S)$ for every partition of S.* •

Intersecting a base-polyhedron with a box

By combining the supermodular part of Theorem 14.3.16 and the Truncation theorem, one gets the following generalization.

Theorem 15.1.5 *Let p_1 be near supermodular for which $p_1(S)$ is finite and let $f : S \to \mathbf{R} \cup \{-\infty\}$, $g : S \to \mathbf{R} \cup \{\infty\}$ be two functions for which $f \leq g$.*

(A) *The base-polyhedron $B'(p_1) \cap T(f \leq)$ is non-empty if and only if*

$$\widetilde{f}(S) \leq p_1(S) \tag{15.9}$$

and

$$\widetilde{f}(X_0) + \sum_{i=1}^{t} p_1(X_i) \leq p_1(S) \text{ for every partition } \{X_0, X_1, \ldots, X_t\} \ (t \geq 1) \text{ of } S \tag{15.10}$$

where only X_0 may be empty.

(B) *The base-polyhedron $B'(p_1) \cap T(\leq g)$ is non-empty if and only if*

$$\widetilde{g}(X) \geq p_1(X) \text{ for every } X \subseteq S \tag{15.11}$$

and

$$\sum_{i=1}^{t} p_1(X_i) \leq p_1(S) \text{ for every partition } \{X_1, \ldots, X_t\} \ (t \geq 1) \text{ of } S. \tag{15.12}$$

(C) *The base-polyhedron $B'(p_1) \cap T(f, g)$ is non-empty if and only if neither $B'(p_1) \cap T(f \leq)$ nor $B'(p_1) \cap T(\leq g)$ is empty.*

If, in addition, each of p_1, f, and g is integer-valued, then the corresponding base-polyhedra are integral.

Proof. The necessity of the conditions is straightforward. Let p denote the upper truncation of p_1. Then p is fully supermodular. Condition (15.10) in the special case when $X_0 = \emptyset$ is just equivalent to (15.12). Therefore both in Case (A) and in Case (B), $p(S) = p_1(S)$ and $B'(p) = B'(p_1)$.

Theorem 14.3.16 implies that $B'(p) \cap T(f \leq)$ is non-empty if and only if

$$p_1(S) \geq p(X) + \widetilde{f}(S - X) \text{ for every } X \subseteq S. \tag{15.13}$$

For $X = \emptyset$, (15.13) is just (15.9), while for a non-empty X, (15.13) is equivalent to (15.10). Hence Part (A) follows.

By Theorem 14.3.16, $B'(p_1) \cap T(\leq g) = B'(p) \cap T(\leq g)$ is non-empty if $p(X) \leq \widetilde{g}(X)$ for every $X \subseteq S$. This is equivalent to requiring that $\sum_i p_1(X_i) \leq \widetilde{g}(X)$ for every partition $\{X_i\}$ of X, and this holds indeed since (15.11) implies $\widetilde{g}(X) = \sum \widetilde{g}(X_i) \geq \sum p_1(X_i)$. Hence Part (B) follows.

Part (C) is a direct consequence of Theorem 14.3.16. •

Theorem 15.1.6 *Let $p_1 \geq 0$ be near supermodular. Let $\beta \geq 0$ be a number and $g : S \to \mathbf{R}_+ \cup \{\infty\}$ a function. The polyhedron $Q := C(p_1) \cap T(\leq g) \cap K(\leq \beta)$ is a g-polymatroid, which is integral provided p_1, g, β are integral. Q is non-empty if and only if*

$$\widetilde{g}(X) \geq p_1(X) \text{ for every } X \subseteq S \tag{15.14}$$

and

$$\sum_i p_1(X_i) \leq \beta \text{ for every subpartition } \{X_i\} \text{ of } S. \tag{15.15}$$

Proof. Let p denote the upper truncation of p_1. We claim for $p, g, \alpha = -\infty$, $f \equiv -\infty$ that the conditions of Theorem 14.3.19 hold. The inequality $\widetilde{g}(S) \geq \alpha$ is automatically met. Also, $\widetilde{g}(X) \geq p(X)$ holds since $\widetilde{g}(X) = \sum \widetilde{g}(X_i) \geq \sum p_1(X_i) = p(X)$ holds for those partitions $\{X_1, \ldots, X_q\}$ for which $p(X) = \sum p_1(X_i)$. Finally, the condition $\widetilde{f}(X) \leq \beta - p(S - X)$ automatically holds for every non-empty set X, while $X = \emptyset$ implies $0 \leq \beta - p(S)$ and this is just equivalent to (15.15).

By applying Theorem 14.3.19, one has only to observe that the polyhedron $S'(p) \cap T(f, g) \cap K(\alpha, \beta)$ in the theorem is just Q. •

Variations for discrete separation

The Discrete separation theorem (Theorem 12.2.1) can be considered as a feasibility theorem stating that the intersection of a submodular polyhedron $S(b)$ and a supermodular polyhedron $S'(p)$ defined by a fully submodular b and a fully supermodular p contains an integer element if $p \leq b$. By combining the Discrete separation theorem with the Truncation theorem, we get the following slight extension of Theorem 12.2.2.

Theorem 15.1.7 *Let b be near submodular and p near supermodular. The intersection $M = S(b) \cap S'(p)$ is non-empty if and only if*

$$p^\wedge \leq b^\vee, \tag{15.16}$$

that is, if

$$\sum_i p(X_i) \leq \sum_j b(Y_j) \tag{15.17}$$

holds for every two subpartitions $\{X_i\}$ and $\{Y_j\}$ of S for which $\cup_i(X_i) = \cup_j(Y_j)$. If p and b are integer-valued, then M is an integral polyhedron. •

Condition (15.17) in Theorem 15.1.7 can be simplified when (p, b) is near paramodular.

Theorem 15.1.8 *Let (p, b) be a near* (in particular, intersecting) *paramodular pair. The polyhedron $Q(p, b)$ is non-empty* (in other words, p and b are separable by a modular function m) *if and only if*

$$p(\cup Z_i) \le \sum_i b(Z_i) \text{ and } \sum_i p(Z_i) \le b(\cup Z_i) \tag{15.18}$$

for every subpartition $\{Z_1, \ldots, Z_t\}$ *of* S. *If, in addition,* (p, b) *is integral and* (15.18) *holds, then* $Q(p, b)$ *contains an integral element* (or in other words, m can be chosen to be integral).

Proof. The condition is clearly necessary. For the sufficiency, assume that no separating m exists. By Theorem 15.1.7, there is a subset $Z \subseteq S$ and two partitions $\mathcal{X} := \{X_1, \ldots, X_k\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_l\}$ of Z for which

$$\sum_i p(X_i) > \sum_j b(Y_j). \tag{15.19}$$

It can be assumed that no X_i is b-separable and no Y_j is p-separable. Choose Z to be minimal. We claim that the two partitions of Z together form a laminar family $\mathcal{F}$. Indeed, if an X_i and a Y_j were properly intersecting, then we could replace X_i and Y_j by $X_j - Y_j$ and $Y_j - X_i$, respectively, and obtain in this way two partitions of the set $Z - (X_i \cap Y_j)$. Due to the cross inequality, these partitions would also violate (15.19), contradicting the minimal choice of Z.

Consider now a maximal member of $\mathcal{F}$. By symmetry, we can suppose that this is X_1. Since $\mathcal{F}$ is laminar and covers every element of Z twice, X_1 partitions into some members $Y_{i_1}, \ldots, Y_{i_h}$ of $\mathcal{Y}$. By the first part of Condition (15.18) we have $p(X_1) \le b(Y_{i_1}) + \cdots + b(Y_{i_h})$. But in this case the set $Z' := Z - X_1$ and its two partitions $\{X_2, \ldots, X_k\}$ and $\mathcal{Y} - \{Y_{i_1}, \ldots, Y_{i_h}\}$ meet (15.19), contradicting the minimal choice of Z. •

Problem 15.1.1 *Derive Theorem* 14.3.17 *on the non-emptiness of* $Q(p, b) \cap T(f, g) \cap K(\alpha, \beta)$ *from Theorem* 15.1.8.

15.1.4 Supermodular colourings

As an application of g-polymatroids defined by intersecting paramodular pairs, we derive an interesting theorem of Schrijver [337] on colourings that cover supermodular functions. Let h be an integer-valued, intersecting supermodular function on a ground-set S and k a positive integer. We say that a k-colouring of S covers h if each subset $X \subseteq S$ contains at least $h(X)$ distinct colours. The condition

$$h(X) \le \min\{k, |X|\} \text{ for every } X \subseteq S \tag{15.20}$$

is clearly necessary for the existence of a k-colouring covering h. We can assume that

$$h(v) = 1 \text{ for every } v \in V \tag{15.21}$$

since (15.20) requires $h(v) \le 1$ and in the case when $h(v) = 0$ we can increase $h(v)$ to 1 without changing the colourability or intersecting supermodularity. We can also assume that $k = \max\{h(X) : X \subseteq S\}$.

Schrijver proved that (15.20) is not only sufficient but there always exists a common k-colouring covering two such functions when both meet (15.20). We are going to prove this result by using the approach of Tardos [360], who studied first the properties of one colour-class in a k-colouring covering a single intersecting supermodular function h and proved the following result. The present proof is a variant of that of Schrijver, which appeared in [340].

Theorem 15.1.9 (Tardos) *Let h be an intersecting supermodular function satisfying* (15.20) *and let $k = \max\{h(X) : X \subseteq S\}$. There exists a subset $C \subseteq S$ such that*

$$C \text{ intersects each subset } Y \subseteq S \text{ for which } h(Y) = k, \tag{15.22}$$

and

$$|X - C| \geq h(X) - 1 \text{ for every } X \subseteq S \text{ intersecting } C. \tag{15.23}$$

The collection of these subsets forms the feasible sets of a g-matroid.

Proof. We are going to define a certain non-empty g-polymatroid Q for which its integral elements are (0, 1)-vectors and prove that these vectors are exactly the incidence vectors of subsets satisfying (15.22) and (15.23).

Let $\mathcal{T} = \{T_1, \ldots, T_\ell\}$ be the set of minimal sets X for which $h(X) = k$. Standard supermodular technique shows that $\mathcal{T}$ is a subpartition of S. Indeed, if X and Y were two properly intersecting members of $\mathcal{T}$, then $k + k = h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) \leq k + k$ would imply $h(X \cap Y) = k$ contradicting the minimality of X. Let $T_0 := S - (T_1 \cup \cdots \cup T_\ell)$. Define a set-function b for $X \neq \emptyset$ by

$$b(X) := |X| - h(X) + 1$$

and define b' as follows. Let $b'(\emptyset) = 0$ and for a non-empty X, let

$$b'(X) := \begin{cases} b(X) & \text{if } X \subseteq T_i \text{ for some } i = 0, \ldots, \ell \\ \infty & \text{for otherwise.} \end{cases} \tag{15.24}$$

Note that $b(X)$ (and thus $b'(X)$) is positive on every non-empty set X, and $b(X) = 1$ for every singleton $X = \{v\}$. Define p as follows.

$$p(Y) := \begin{cases} 1 & \text{if } Y \in \mathcal{T} \\ 0 & \text{if } |Y| \leq 1 \text{ and } Y \notin \mathcal{T} \\ -\infty & \text{for otherwise.} \end{cases} \tag{15.25}$$

We claim that (p, b') is an intersecting paramodular pair. Indeed, since h is intersecting supermodular, b is intersecting submodular and hence so is b'. Also, p is clearly intersecting supermodular. The cross-inequality for two properly intersecting sets X and Y holds automatically since in this case $b(X) = \infty$ or $p(Y) = -\infty$.

By Theorem 15.3.11, $Q = Q(p, b')$ is an integral g-polymatroid. Since $p(v) \geq 0$ and $b'(v) = 1$ for each $v \in S$, every integral element of Q is a (0, 1)-vector. Moreover, Q is non-empty since the characteristic vector $\underline{\chi}(Z)$ of a subset $Z \subseteq S$ for which $|Z \cap T_i| = 1$ for each $i = 1, \ldots, \ell$ belongs to Q. In other words, $(S, \mathcal{Q}(p, b'))$ is a generalized matroid.

Proposition 15.1.10 $\widetilde{x}(X) \leq b(X)$ *for each $x \in Q$ and $X \subseteq S$, that is, $Q(p, b') = Q(p, b)$.*

Proof. There is nothing to prove when $b'(X) = b(X)$ so suppose that $b'(X) = \infty$, that is, there is a member T of $\mathcal{T}$ such that both $X - T$ and $X \cap T$ are non-empty. Since $h(T) = k \geq h(X \cup T)$ and $h(X) + h(T) \leq h(X \cap T) + h(X \cup T)$, we have $h(X) \leq h(X \cap T)$ from which

$$\widetilde{x}(X) = \widetilde{x}(X \cap T) + \sum[x(v) : v \in (X - T)] \le b'(X \cap T) + \sum[b'(v) : v \in (X - T)] \le$$

$$[|X \cap T| - h(X \cap T) + 1] + |X - T| \le |X| - h(X) + 1 = b(X). \bullet$$

The theorem follows once we observe that a subset C satisfies (15.22) and (15.23) if and only if $\underline{\chi}_C$ belongs to $Q(p, b)$. • •

Remark 15.1.1 It is easy to check that (p, b) is a near paramodular pair, and we have already mentioned, without a proof, that $(*)$ any near paramodular pair defines a g-polymatroid. Therefore if this latter result had been allowed to be applied, then we could have avoided introducing b' and proving that $Q(p, b) = Q(p, b')$. The proof of $(*)$, however, is a bit technical so we did not follow that line.

A k-colouring of S is **equitable** if the size of each colour-class is $\lfloor |S|/k \rfloor$ or $\lceil |S|/k \rceil$.

Exercise 15.1.2 *Prove that a k-colouring* $\mathcal{S}' := \{S_1, \dots, S_k\}$ *of S is equitable if* $\lfloor |S|/k \rfloor \le |S_1| \le \lceil |S|/k \rceil$ *and* $\{S_2, \dots, S_k\}$ *is an equitable* $(k-1)$*-colouring of* $S - S_1$.

Theorem 15.1.11 (Schrijver [337]) *Let k be a positive integer,* p_1 *and* p_2 *two positively intersecting supermodular functions on a ground-set S for which* $p(X) \le \min\{k, |X|\}$ *for every* $X \subseteq S$ *where*

$$p(X) := \max\{p_1(X), p_2(X)\}.$$

Then there is a k-colouring of S covering p. Moreover, the colouring can be chosen to be equitable.

Proof. Induction on k. The theorem is trivial for $k = 1$ so we assume that $k \ge 2$.

Lemma 15.1.12 *There is a subset* $S_1 \subset S$ *such that* (A) S_1 *intersects every X with* $p(X) = k$, (B) $|X - S_1| \ge p(X) - 1$ *for every* $X \subseteq S$, *and* (C) $\lfloor |S|/k \rfloor \le |S_1| \le \lceil |S|/k \rceil$.

Proof. Let Q_i denote the intersection of the g-polymatroid in Theorem 15.1.9 assigned to p_i $(i = 1, 2)$ and the plank $K(\alpha, \beta)$ where $\alpha = \lfloor |S|/k \rfloor$ and $\beta = \lceil |S|/k \rceil$. Then Q_i is an integral g-polymatroid. Let $m : S \to \mathbf{R}$ be the vector having each component of value $|1|/k$. We claim that $m \in Q_i$. Indeed, we clearly have $m \in K(\alpha, \beta)$. Moreover, if $|X| \le k$, then $p_i(X) \le |X|$ implies $\widetilde{m}(X) = |X|/k \le 1 \le |X| - p_i(X) + 1$, while if $k \le |X|$, then $p_i(X) \le k$ implies $p_i(X) \le k = (k-1)k/k + 1 \le (k-1)|X|/k + 1 = |X| - |X|/k + 1$, from which $\widetilde{m}(X) = |X|/k \le |X| - p_i(X) + 1$. Finally, $p_i(Z) = k$ implies $|Z| \ge p_i(Z) = k$ from which $\widetilde{m}(Z) = |Z|/k \ge 1$. It follows that $m \in Q_1 \cap Q_2$.

Since the intersection of two integral g-polymatroids is an integral polyhedron, and in our case the intersection is non-empty, there is an integral element of $Q_1 \cap Q_2$, and this element is, by Theorem 15.1.9, the incidence vector of a subset $S_1 \subseteq S$ satisfying properties (A), (B), and (C). •

The existence of set S_1 ensured by Lemma 15.1.12 implies the theorem as follows. Define set-functions p'_i $(i = 1, 2)$ on $S' = S - S_1$ by $p'_i(Z) := \max\{p_i(Z \cup X) - 1 : X \subseteq S_1\}$ $(i = 1, 2)$. These are clearly positively intersecting supermodular for which $p'_i(Z) \le k - 1$ holds by (A), and $p'_i(Z) \le |Z|$ holds for every $Z \subseteq S'$ by (B). By the inductive hypothesis,

S' has an equitable $(k-1)$-colouring $\{S_2, \ldots, S_k\}$ covering both p'_1 and p'_2. But then $\{S_1, \ldots, S_k\}$ is an equitable k-colouring of S covering both p_1 and p_2. • •

Note that in Schrijver's theorem intersecting supermodularity can be weakened to skew supermodularity, a result of Bernáth and Király [28], who also provided applications of this extension.

15.2 Applications to branchings and augmentations

15.2.1 Disjoint bibranchings

Schrijver [337] used his supermodular colouring theorem to derive an extensions of Edmonds' theorem on disjoint arborescences (Theorem 10.1.1). Let $D = (V, A)$ be a directed graph with a bipartition $\{S, T\}$ of its node-set. A **bibranching** is a digraph that includes a directed path from S to every $t \in T$ and includes a directed path to T from every $s \in S$. For example, when $|S| = 1$, the notion of a bibranching is equivalent to root-connectivity.

Theorem 15.2.1 (Schrijver's disjoint bibranchings theorem) *A digraph D includes k edge-disjoint bibranchings if and only if $\varrho_D(X) \geq k$ holds for every $\emptyset \subset X \subseteq T$ and $\delta_D(X) \geq k$ holds for every $\emptyset \subset X \subseteq S$.*

Proof. The proof of necessity is straightforward. For proving the sufficiency, consider the set F of edges of D entering T. For a non-empty subset $X \subseteq F$, define

$$p_1(X) := \max\{k - \varrho_{D-S}(Z) : Z \subseteq T,\ Z \text{ includes the head of each element of } X\}.$$

It is easy to check that p_1 is an intersecting supermodular function on ground-set F.

Let D' denote the digraph arising from D by shrinking S into a single node s and consider a given k-colouration $\{F_1, \ldots, F_k\}$ of F. The strong form of Edmonds' disjoint arborescences theorem (Theorem 10.2.4) implies that $\{F_1, \ldots, F_k\}$ can be extended to disjoint spanning arborescences of D' if and only if $\varrho_{D-S}(Z)$ plus the number of colour-classes entering Z is at least k for every $\emptyset \subset Z \subseteq T$. This is equivalent to requiring that the number of colour-classes intersecting X is at least $p_1(X)$ for every non-empty subset X of F.

Analogous arguments show that $\{F_1, \ldots, F_k\}$ can be extended to disjoint spanning reverse arborescences of D'' if and only if the number of colour-classes intersecting X is at least $p_2(X)$ for every non-empty subset X of F where D'' arises from D by shrinking T into a single node and

$$p_2(X) := \max\{k - \varrho_{D-T}(Z) : Z \subseteq S,\ Z \text{ includes the tail of each element of } X\}.$$

Therefore all we need to show is that there is a k-colouring of F covering both p_1 and p_2. Theorem 15.1.11 just ensures the existence of such a colouring. •

Note that the same proof shows that the k disjoint bibranchings can be chosen in such a way that they define an equitable colouring of the set of edges entering T. This result is interesting even in the special case when $|S| = 1$ when it asserts that if a digraph includes k

disjoint spanning arborescences of a given root r_0, then the k arborescences can be chosen in such a way that their out-degrees at r_0 differ (pairwise) by at most one.

Remark 15.2.1 While the proof above required the application of the *strong* form of the disjoint arborescences theorem, Theorem 15.2.1 includes only the weak form. It is not known if there is a common generalization of Theorems 15.2.1 and 10.2.4.

Schrijver [335] applied the disjoint bibranchings theorem to settle Woodall's conjecture (Conjecture 2.2.10) in a special case. An acyclic digraph is referred to as **source-sink connected** if there is a path from every source-node to every sink-node. Recall that a dijoin is a set of edges covering all dicuts.

Theorem 15.2.2 (Schrijver) *If every dicut of a source-sink connected digraph has at least k elements, then there are k disjoint dijoins.* •

A detailed proof can be found in [340]. We also recall a conjecture of Guenin (Conjecture 2.2.11) stating the truth of Woodall's conjecture for digraphs having a supersink and a supersource.

15.2.2 Edge-connectivity augmentation of undirected graphs

Augmenting global edge-connectivity

In Chapter 11 we discussed several results concerning optimal connectivity augmentation problems. For example, a theorem of Watanabe and Nakamura (Theorem 11.1.3) characterized undirected graphs that can be made k-edge-connected by adding γ new edges. Furthermore we solved the degree-constrained k-edge-connectivity augmentation problem and pointed out that the linking property holds in this case. The minimum cost augmentation problem could also be handled for node-induced cost functions. As a matter of fact, it was possible to apply a reverse greedy algorithm to find an augmentation vector of minimum cost. Looking back at these results, the reader may suspect that there should be a g-polymatroid in the background. Indeed, we shall prove that k-augmentation vectors of given modulus span a base-polyhedron.

Suppose that $k \geq 2$ and the graph $G = (V, E)$ is not k-edge-connected. We called a degree specification $m : V \to \mathbf{Z}_+$ an (undirected edge-connectivity) **k-augmentation vector** if there is a graph $H = (V, F)$ for which $G + H$ is k-edge-connected and $d_H(v) = m(v)$ for every node v. Recall the definition of solid sets from Section 7.3 where we observed that the set-system $\mathcal{S} = \mathcal{S}_G$ of solid sets is laminar. Furthermore, Theorem 11.1.1 stated that m is a k-augmentation vector if and only if $\widetilde{m}(V)$ is even and

$$\widetilde{m}(X) \geq k - d_G(X) \text{ for every solid set } X \subset V.$$

Let γ be a non-negative integer and define set-functions p and $p^{2\gamma}$ as follows.

$$p(X) := \begin{cases} (k - d_G(X))^+ & \text{if } X \subseteq V,\ X \in \mathcal{S} \\ 0 & \text{otherwise,} \end{cases}$$

$$p^{2\gamma}(X) := \begin{cases} p(X) & \text{if } X \subset V,\ X \in \mathcal{S} \\ 2\gamma & \text{if } X = V. \end{cases}$$

Since p is non-negative and $p(X)$ is positive only on the members X of a laminar family, p is certainly intersecting supermodular and the base-polyhedron $B'(p^{2\gamma})$ is in $\mathbf{R}_+^V$. Furthermore, $B'(p^{2\gamma})$ is the intersection of the contra-polymatroid $C(p)$ with the hyperplane $\{x : \widetilde{x}(V) = 2\gamma\}$. We conclude that the integral points of $C(p)$ with even modulus are exactly the k-augmentation vectors and, in particular, the integral points of $B'(p^{2\gamma})$ are the degree-vectors of graphs $H = (V, F)$ for which $G + H$ is k-edge-connected and H has γ edges. Therefore we can apply the feasibility theorems of Subsection 15.1.3 to obtain all degree- and size-constrained edge-connectivity augmentation results proved in Section 11.1.

For example, by Theorem 15.1.4, $B'(p^{2\gamma})$ is non-empty if and only if $\sum_i p(V_i) \leq 2\gamma$ for every partition $\{V_1, \ldots, V_t\}$ of V. This theorem, when combined with Theorem 11.1.1, immediately implies the theorem of Watanabe and Nakamura. Theorem 11.1.6 on degree-constrained augmentation is just a special case of Theorem 15.1.5 on the intersection of a base-polyhedron and a box. Also, the linking property of k-augmentation vectors is a special case of that of g-polymatroids. Finally, we remark that the greedy algorithm described in Section 11.1 for finding an optimal k-augmentation vector is also a special case of the greedy algorithm for contra-polymatroids. Note that the g-polymatroid of k-augmentation vectors is particularly simple since the bounding function is positive only on the members of a laminar set-system.

Augmenting local edge-connectivity

Consider now the general case of optimal local connectivity augmentations (Theorem 11.1.12). We show that even in this case the augmentation vectors span a g-polymatroid, although the bounding function here is skew supermodular. This application has been one of the main motivations for considering near supermodular functions as bounding functions of g-polymatroids.

Let $G = (V, E)$ be again an undirected graph. For every unordered pair $\{u, v\}$ of nodes we are given a demand $r(u, v) \neq 1$ and we are interested in augmentations of G in which the local edge-connectivity for $\{u, v\}$ is at least $r(u, v)$ for each pair. (The local edge-connectivity, by definition, is the minimum cardinality of a cut separating u and v.) Define a set-function R_r by $R_r(X) := \max\{r(u, v) : |\{u, v\} \cap X| = 1\}$. In Lemma 8.1.9, we proved that R_r is skew supermodular and hence near supermodular.

We say that a graph $G^+ = G + H$ covers R_r if $d_{G^+} \geq R_r$. This is easily seen to be equivalent to requiring that

$$\lambda_{G^+}(u, v) \geq r(u, v) \text{ for every pair } \{u, v\} \text{ of nodes.}$$

The degree-vector of such a graph H is called an $\mathbf{r}$-augmentation vector. Theorem 11.1.12 provided a characterization of $\mathbf{r}$-augmentation vectors asserting that m is such a vector if and only if $\widetilde{m}(V)$ is even and

$$\widetilde{m}(X) \geq R_r(X) - d_G(X) \text{ for every } X \subseteq V.$$

Let γ be a non-negative integer and define set-functions p an $p^{2\gamma}$ as follows.

$$p(X) := \begin{cases} (R_r(X) - d_G(X))^+ & \text{if } X \subseteq V, \\ 0 & \text{otherwise,} \end{cases}$$

$$p^{2\gamma}(X) := \begin{cases} p(X) & \text{if } X \subset V, \\ 2\gamma & \text{if } X = V. \end{cases}$$

Both functions are skew (and hence near) supermodular. By Theorem 15.1.3, $C(p)$ is a contra-polymatroid and $B'(p^{2\gamma})$ is a base-polyhedron. We can conclude that the integral points of $C(p)$ with even modulus are exactly the **r**-augmentation vectors and, in particular, the integral points of $B'(p^{2\gamma})$ are the degree-vectors of graphs $H = (V, F)$ for which $G + H$ satisfies the local edge-connectivity demand r and H has γ edges.

Therefore one can apply the feasibility theorems of Subsection 15.1.3 to obtain degree- and size-constrained local edge-connectivity augmentation results. In this view, Theorem 11.1.13 on minimal augmentations, for example, is a direct corollary of Theorem 15.1.4. Also, Theorem 15.1.5 gives rise to characterizations for degree-constrained augmentations. The linking property for local edge-connectivity augmentations holds again and we can also use the greedy algorithm developed for contra-polymatroids to find a cheapest augmentation in the special case when the cost function is a node-induced function.

We formulate only one consequence of the several results on g-polymatroids. Namely, Corollary 14.3.22 implies the following.

Theorem 15.2.3 *Let $G = (V, E)$ be a connected undirected graph and $g : V \to \mathbf{Z}_+$ a function. If there exists a graph $H = (V, F)$ so that $d_H(v) \leq g(v)$ for every $v \in V$ and*

$$\lambda_{G+H}(u, v) \geq r(u, v) \text{ for every pair } \{u, v\} \text{ of nodes} \tag{15.26}$$

and there exists a graph H' with at most γ edges which satisfies (15.26) *with H' in place of H, then there is a graph satisfying the requirements simultaneously.* •

As far as directed edge- and node-connectivity augmentation problems are concerned, the necessary abstract extensions regarding digraphs covering crossing supermodular functions will be developed in Sections 17.2 and 17.3. For example, Theorem 17.2.9 will imply that the in-degree vectors of k-edge-connected augmentation of a digraph are the integral elements of a contra-polymatroid.

15.3 Full truncation

Crossing sub- or supermodular functions are indispensable in several applications, and this section is offered to develop the necessary tools. In particular, we shall investigate set systems and polyhedra associated with such functions. The set-system $\mathcal{I}(b) = \{I \subseteq S : b(Y) \geq |Y \cap I| \text{ for every } Y \subseteq S\}$ was shown in Theorem 13.4.2 to satisfy the independence axioms of matroids when b is a non-negative intersecting submodular function. The following example shows that this is not necessarily the case when b is crossing submodular. On a 3-element ground-set $S := \{u, v, z\}$, let

$$b(X) := \begin{cases} 1 & \text{if } X = \{u, v\} \text{ or } X = \{v, z\} \\ |X| & \text{otherwise.} \end{cases} \tag{15.27}$$

Then b is crossing submodular (since any set-function on a 3-element set is crossing submodular) and both $\{v\}$ and $\{u, z\}$ are maximal members of $\mathcal{I}(b)$ showing that $(S, \mathcal{I}(b))$

is not a matroid. The same function shows that $P(b)$ is not a polymatroid and neither is $S(b)$ a submodular polyhedron. In contrast, we are going to show that a crossing submodular function leads to a base-polyhedron and, in particular, to the set of bases of a matroid.

In Section 14.3, we proved that the intersection of a base-polyhedron and a box is a base-polyhedron. As an extension, we are going to prove that the (possibly empty) polyhedron

$$Q := B(b) := \{x \in \mathbf{R}^S : \widetilde{x}(S) = b(S),\ \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subset S\}$$

is a base-polyhedron whenever b is a crossing submodular set-function on a ground-set S for which $b(S)$ is finite. However, determining the (unique, fully submodular) border function of a non-empty $B(b)$ requires a formula more complicated than the truncation which was applicable for intersecting (or near) submodular functions. This is why we need the concept of composition of a set.

15.3.1 Composition of sets

Let $K = (S, \mathcal{K})$ be a regular hypergraph on ground-set S for which $\mathcal{K}$ is non-empty. We call $\mathcal{K}$ (or K) a **composition** of S. The common degree of the nodes is the **ground-degree** of the composition. Both a partition and a co-partition of S form a special composition.

Lemma 15.3.1 *A cross-free composition $\mathcal{K}$ of S can be partitioned into partitions and co-partitions of S.*

Proof. The lemma follows by induction once we show that $\mathcal{K}$ includes a partition or a co-partition. By Theorem 1.4.2, $\mathcal{K}$ can be represented by a directed tree $F = (U, A)$ and a mapping $\varphi : V \to V(F)$ in such a way that every member X of $\mathcal{K}$ corresponds to an edge a of F in such a way that $X = \varphi^{-1}(U_a)$ where U_a denotes that component of $F - a$ which is entered by a.

Since $\mathcal{K}$ is regular, Lemma 1.4.3 implies that all but one level of F consists of empty nodes. Therefore at least one of the lowest level and the highest level consists of empty nodes. If u is a node on the lowest level for which $\varphi^{-1}(u)$ is empty, then the members of $\mathcal{K}$ corresponding to the tree edges leaving u form a partition of S, while if v is node on the highest level for which $\varphi^{-1}(v)$ is empty, then the members of $\mathcal{K}$ corresponding to the tree edges entering v form a co-partition of S. •

A hypergraph $(S, \mathcal{K})$ is said to be a **composition** of a non-empty subset $Z \subset S$ if $\mathcal{K}^+ := \mathcal{K} \cup \{S - Z\}$ forms a regular hypergraph on S . The **ground-degree** of $\mathcal{K}$ is defined to be one less than the degree of $\mathcal{K}^+$. In particular, a partition of Z is a composition of Z with ground-degree zero. For a partition $\{Z_1, \dots, Z_t\}$ of $S - Z$, we previously called the set-system $\{S - Z_1, \dots, S - Z_t\}$ a co-partition of Z. This is also a composition of Z for which the ground-degree is $t - 1$.

To introduce a common generalization of partitions and co-partitions, let $\{Z_1, \dots, Z_t\}$ again be a partition of Z, and for each Z_i, let $\left\{Z_i^1, \dots, Z_i^{t_i}\right\}$ be a partition of $S - Z_i$ ($t_i \geq 1$). Then the set-system $\mathcal{D} = \left\{S - Z_i^j\right\}$ is called a **double-partition** of Z. In other words, a double-partition consists of the sets occurring in the co-partitions of the members of a partition of Z. It can be checked easily that a double-partition of Z is a composition of Z

for which the ground-degree is $\sum_{i=1}^{t}(t_i - 1)$. When $t = 1$, $\mathcal{D}$ is a co-partition of Z while if each t_i is one, $\mathcal{D}$ is a partition of Z.

For a set $Z \subset S$, we call a cross-free double-partition of Z a **tree-composition** Z if each edge of the representing directed tree has its tail in $\varphi(S - Z)$ and its head in $\varphi(Z)$ and, furthermore, the preimage $\varphi^{-1}(v)$ of each node v of the tree is non-empty. In other words, a tree-composition of Z can be described by providing a partition $\{Z_1, \dots, Z_k\}$ $(k \geq 1)$ of Z, a partition $\{U_1, \dots, U_\ell\}$ of $S - Z$ $(\ell \geq 1)$, and a directed tree F on node-set $\{z_1, \dots, z_k, u_1, \dots, u_\ell\}$ in which every edge is of type $u_i z_j$. There is a one-to-one correspondence between the members of a tree-composition and the edges of F. Namely, for any edge e of the tree, let Z_e denote the node-set of the component of $F - e$ that is entered by e. The union of the sets Z_i's and U_j's corresponding to the nodes z_i's and u_j's in Z_e is the member of the tree-composition corresponding to e.

In Figure 15.1 φ is the identity map. The three edges of the directed tree define the sets Z_1, Z_2, Z_3 which form a tree-composition of $Z = \{v_1, v_2\}$. By convention, a tree-composition of the ground-set S is meant to be a partition or a co-partition of S.

Exercise 15.3.1 *Show that a tree-composition of Z is a double-partition.*

Theorem 15.3.2 *Let $\mathcal{T}$ be a crossing family on a ground-set S and let $Z \subset S$ be a non-empty subset such that there exists a $u\bar{v}$-set in $\mathcal{T}$ for every pair of nodes for which $u \in Z$, $v \in S - Z$. Then there is a tree-composition $\mathcal{H} \subseteq \mathcal{T}$ of Z.*

Proof. For each pair $\{u, v\}$ with $u \in Z$, $v \in S - Z$, let $T(u, \bar{v})$ be a $u\bar{v}$-set in $\mathcal{T}$. For every $v \in S - Z$ consider the set $\mathcal{K}(v)$ of connected components of the hypergraph $(S, \{T(u, \bar{v}) : u \in Z\})$. Since $\mathcal{T}$ is a crossing family, the members of $\mathcal{K}(v)$ are pairwise disjoint sets for which their union covers Z and does not contain v. Let $K = (S, \mathcal{K})$ denote the hypergraph where $\mathcal{K}$ is the multi-union of families $\mathcal{K}(v)$ over $v \in S - Z$. Note that $\mathcal{K}$ has at most $|S - Z||Z| \leq |S|^2$ members. It follows from this construction for the degree function d_K of K that

$$d_K(u) = \Delta \text{ for } u \in Z \text{ and } d_K(v) < \Delta \text{ for } v \in S - Z \tag{15.28}$$

where $\Delta = |S - Z|$.

Apply the uncrossing procedure to $\mathcal{K}$; that is, as long as there are two crossing members of the current family, replace them by their union and intersection. Since such an uncrossing step increases the auxiliary parameter $\sum[|X|^2 : X \in \mathcal{K}]$, the procedure terminates after at

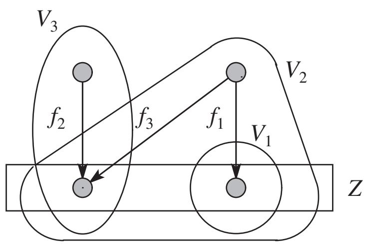


Figure 15.1 Tree-composition of $Z = \{v_1, v_2\}$

most $\sum[|S|^2 : X \in \mathcal{K}] = |S|^2|\mathcal{K}| \leq |S|^4$ uncrossing steps. Let $H = (S, \mathcal{H})$ denote the final cross-free hypergraph. Since $\dot{d}_H = \dot{d}_K$, (15.28) is satisfied by H in place of K for some Δ.

Consider the tree-representation (F, φ) of $\mathcal{H}$ ensured by Theorem 1.4.2 and recall the notion of levels of F. Let L^* and L_* denote the nodes of F with highest and lowest levels, respectively.

Lemma 1.4.3 and (15.28) imply for the elements s of S that $\varphi(s)$ is in L^* if and only if $s \in Z$. If $\varphi^{-1}(w)$ is empty for a node $w \in L^*$, then the members of $\mathcal{H}$ corresponding to the tree edges entering w form a co-partition of S. Revise H by removing the members of this co-partition. Since a co-partition is a regular hypergraph, the revised $\mathcal{H}$ continues to fulfil (15.28) for some Δ.

If $W := \varphi^{-1}(w)$ is empty for a node $w \in L_*$, then the members of $\mathcal{H}$ corresponding to the tree edges leaving w form a partition of S. Revise H by removing the members of this partition. The revised $\mathcal{H}$ continues to fulfil (15.28) for some Δ. Consider now the case when W is non-empty and F has more than two levels. Then $d_H(s) \leq \Delta - 2$ for every $s \in W$ since the difference of the highest and the lowest levels is at least two. Now the members of $\mathcal{H}$ corresponding to the tree edges leaving w form a partition of $S - W$. Revise H again by removing the members of this subpartition. The revised $\mathcal{H}$ continues to fulfil (15.28).

In this way we arrive at a hypergraph $\mathcal{H}$ for which the representing directed tree has two levels and no empty nodes, which means that $\mathcal{H}$ is a tree-composition of Z. • •

The proof above gives rise to a polynomial algorithm to compute the tree-composition of Z. The algorithm requires a subroutine that is able to find a $u\bar{v}$-set $T(u, \bar{v})$ in $\mathcal{T}$ if one exists. We also remark that typically we use the uncrossing procedure as a theoretical tool in a non-constructive proof. But in the present case it is an essential part of an algorithm (and that was the main reason why we provided the polynomial bound $O(|S|^4)$ for the possible numbers of uncrossing steps).

15.3.2 Base-polyhedra from crossing semimodular functions

Let b be a crossing submodular function with finite $b(S)$ and suppose that $B(b)$ is non-empty. In order to prove that $B(b)$ is a base-polyhedron, we are going to assign a fully submodular function $b^{\downarrow}$ to b for which $B(b) = B(b^{\downarrow})$.

***Constructing* $\mathbf{b}^{\downarrow}$ *from* b**

We will construct $b^{\downarrow}$ in four steps using the operations of upper and lower truncation and complementing a set-function.

First, let p denote the complement of b, that is, $p(X) = b(S) - b(S - X)$. Clearly, p is crossing supermodular for which $p(\emptyset) = 0$, $p(S) = b(S)$ and $B'(p) = B(b)$.

Second, let $p^\wedge$ denote the upper truncation of p (see Section 15.1). We claim that $p^\wedge(S) = p(S)$. Indeed, $p^\wedge(S) \geq p(S)$ is implied by the definition of upper truncation. If, indirectly, we have strict inequality here, then there is a partition $\{S_1, \dots, S_q\}$ of S for which $p(S) < \sum_i p(S_i)$. But this would imply for an element z of $B(p)$ that $\widetilde{z}(S) = p(S) < \sum_i p(S_i) \leq \sum_i \widetilde{z}(S_i) = \widetilde{z}(S)$, and this contradiciton shows that $p^\wedge(S) = p(S)$ indeed holds. This in turn implies that $B'(p^\wedge) = B'(p) = B(b)$.

Third, let b' denote the complement of $p^\wedge$. Since $p^\wedge(S) = p(S) = b(S)$, it follows that $b'(\emptyset) = 0$, $b'(S) = b(S)$ and $B(b') = B'(p^\wedge)$.

Finally, let $b^{\downarrow}$ denote the lower truncation of b'. We claim that $b^{\downarrow}(S) = b'(S)$. Indeed, if indirectly there were a partition $\{S_1, \dots, S_q\}$ of S so that $b'(S) > \sum_i b'(S_i)$, then, for any element z of $B(b')$, we would have $\widetilde{z}(S) = b'(S) > \sum_i b'(S_i) \geq \sum_i \widetilde{z}(S_i) = \widetilde{z}(S)$. It follows that $b^{\downarrow}(\emptyset) = 0,\ b^{\downarrow}(S) = b(S)$ and $B(b^{\downarrow}) = B(b)$.

We have thus proved the following proposition.

Proposition 15.3.3 *For the functions defined above, one has* $b(S) = p(S) = p^{\wedge}(S) = b'(S) = b^{\downarrow}(S)$ *and* $B(b) = B'(p) = B'(p^{\wedge}) = B(b') = B(b^{\downarrow})$. •

Summing up, $b^{\downarrow}$ is the set-function arising from b by the following sequence of four operations: complement b, upper truncate, complement, and lower truncate. In notation,

$$b^{\downarrow} = (\overline{(\overline{b})^{\wedge}})^{\vee}. \tag{15.29}$$

Obviously, if b is integral, then so is $b^{\downarrow}$. It is possible to express the value of $b^{\downarrow}(Z)$ in a single minimization formula but for the sake of simplicity we do this only in the special case when $b(S) = 0$.

Corollary 15.3.4 *Let b be a crossing submodular function with* $b(S) = 0$ *for which* $B(b)$ *is non-empty. Then*

$$b^{\downarrow}(Z) = \min \left\{ \sum_{\{i,j\}} b\left(Z_i^j\right) : \left\{Z_i^j\right\} \text{ a double-partition of } Z \right\} \tag{15.30}$$

or more concisely,

$$b^{\downarrow}(Z) = \min\{\widetilde{b}(\mathcal{D}) : \mathcal{D} \text{ a double-partition of } Z\}. \bullet \tag{15.31}$$

Problem 15.3.2 *Prove for the general case:* $b^{\downarrow}(Z) = \min\{\widetilde{b}(\mathcal{D}) - \Delta b(S)\}$, *where the minimum is taken over the double-partitions* $\mathcal{D}$ *of* Z *and* Δ *denotes the ground-degree of* $\mathcal{D}$.

Theorem 15.3.5 *Let b be a crossing submodular function for which* $b(S)$ *is finite. Then* $B(b)$ *is a base-polyhedron. If* $B(b)$ *is non-empty, then its unique fully submodular upper border function is* $b^{\downarrow}$, *that is,* $B(b) = B(b^{\downarrow})$. *In particular, when b is integer-valued,* $B(b)$ *is an integral base-polyhedron.*

Proof. By Proposition 15.3.3, the only thing we need to prove is that $b^{\downarrow}$ is fully submodular.

First, we claim that $p^{\wedge}$ is co-intersecting supermodular. Indeed, for an arbitrary proper subset S' of S, the restriction $p|S'$ of p to S' is an intersecting supermodular function on ground-set S'. Obviously, the upper truncation of $p|S'$ is the same as the restriction of $p^{\wedge}$ to S'.

Therefore the Truncation theorem (Theorem 15.1.1), when applied to $p|S'$, implies that $p^{\wedge}|S'$ is fully supermodular on the ground-set S'. Hence $p^{\wedge}$ satisfies the supermodular inequality for X and Y if $X \cup Y$ is a proper subset of S, and hence $p^{\wedge}$ is indeed co-intersecting supermodular.

It follows that the complement b' of $p^{\wedge}$ is intersecting submodular and a second application of Theorem 15.1.1 shows that the lower truncation $b^{\downarrow}$ of b' is fully submodular. •

We call the function $b^{\downarrow}$ the **full** (**lower**) **truncation** of b. Note that the full upper truncation of a crossing supermodular function p can be defined analogously, namely, $p^{\uparrow} := -[(-p)^{\downarrow}]$. By relying on tree-compositions, the formula (15.31) for $b^{\downarrow}$ can be simplified. The following result appears in [130].

Theorem 15.3.6 *Let b be a crossing submodular function for which $b(S) = 0$ and $B(b)$ is non-empty. Then $b^{\downarrow}(Z) = \min\{\widetilde{b}(\mathcal{F}) : \mathcal{F}$ a tree-composition of $Z\}$.*

Proof. Since a tree-composition is a special double-partition, (15.31) implies that $b^{\downarrow}(Z) \leq \min\{\widetilde{b}(\mathcal{F}) : \mathcal{F}$ a tree-composition of $Z\}$. Therefore it suffices to show that there is a tree-composition of Z for which $b^{\downarrow}(Z) = \widetilde{b}(\mathcal{F})$.

By Theorem (15.3.5) $B(b) = B(b^{\downarrow})$. Corollary 14.2.3 implies that there is an element m of $B(b)$ for which $\widetilde{m}(Z) = b^{\downarrow}(Z)$. A subset $X \subset V$ is said to be **tight** if $\widetilde{m}(X) = b(X)$. Let $\mathcal{T}$ denote the set-system of tight sets. Then the standard submodular technique shows that $\mathcal{T}$ is a crossing set-system.

Claim 15.3.7 *There exists a tight $u\bar{v}$-set $T(u, \bar{v})$ for every pair of nodes for which $u \in Z, v \in S - Z$.*

Proof. If no tight $u\bar{v}$-set exists for nodes $u \in Z$ and $v \in S - Z$, then the vector m' defined by

$$m'(z) := \begin{cases} m(z) + \varepsilon & \text{if } z = u \\ m(z) - \varepsilon & \text{if } z = v \\ m(z) & \text{otherwise} \end{cases}$$

would belong to $B(b) = B(b^{\downarrow})$ for a small positive ε, but m' cannot belong to $B(b^{\downarrow})$ since $\widetilde{m}'(Z) = \widetilde{m}(Z) + \varepsilon > b^{\downarrow}(Z)$. •

By Theorem 15.3.2, there is a tree-composition $\mathcal{F}$ of Z consisting of tight sets. Let Δ denote the ground-degree of $\mathcal{F}$. Then

$$\widetilde{b}(\mathcal{F}) = \sum[\widetilde{m}(X) : X \in \mathcal{F}] \leq \Delta\widetilde{m}(S) + \widetilde{m}(Z) = \Delta b(S) + \widetilde{m}(Z) = 0 + b^{\downarrow}(Z),$$

as required. • •

Algorithmic aspects

The proof of Theorem 15.3.2 gave rise to a polynomial algorithm to compute the optimal tree-composition $\mathcal{F}$ of Z for which $b^{\downarrow}(Z) = \widetilde{b}(\mathcal{F})$ provided that a member $m \in B(b)$ for which $\widetilde{m}(Z) = b^{\downarrow}(Z)$ is already available and that a tight $u\bar{v}$-set can be computed if one exists.

In Section 16.2.5 an algorithm will be described for computing a k-edge-connected orientation of a mixed graph if one exists. When no such an orientation exists, the obstacle, as will be proved later, is a certain tree-composition which can be computed by making use of the procedure above. Also, in Section 16.3, we will describe an optimization algorithm which, in a special case, computes a cheapest k-edge-connected orientation of a $2k$-edge-connected graph. Here the computation of optimal dual solution will also need the procedure above.

Problem 15.3.3 *Prove for a crossing submodular function b for which $b(S)$ is finite and $B(b)$ is non-empty that $b^{\downarrow}(Z) = \min\{\widetilde{b}(\mathcal{F}) - \Delta b(S) : \mathcal{F}$ a tree-composition of Z and Δ is its ground-degree$\}$.*

Matroids from crossing semimodular functions

Since the intersection of a base-polyhedron with a (0, 1)-cube, if non-empty, is the base-polyhedron of a matroid, crossing sub- and supermodular functions can be used to define matroids. Namely, for a non-negative crossing submodular function b and integer k, the set-system $\mathcal{B}(b) := \{Z : |Z| = k, |X \cap Z| \leq b(X)$ for every $X \subseteq S\}$, if non-empty, forms the set of bases of a matroid [151].

Problem 15.3.4 *Let $D = (V, A)$ be a directed graph with a non-negative cost function c on A. Prove that the cheapest dijoin problem can be formulated as that of finding a minimum weight common bases of two matroids.*

15.3.3 Feasibility for crossing submodular functions

In what follows, we assume that $|S| \geq 2$, as the results are trivial for $|S| \leq 1$.

Theorem 15.3.8 (Fujishige) *Let b be a crossing submodular function for which $b(S) = 0$. The base-polyhedron $B(b)$ is non-empty if and only if*

$$\widetilde{b}(\mathcal{P}) \geq 0, \tag{15.32}$$

for every partition and co-partition $\mathcal{P}$ of S, or equivalently, if

$$\sum_i b(S_i) \geq 0 \tag{15.33}$$

and

$$\sum_i b(S - S_i) \geq 0. \tag{15.34}$$

for every partition $\{S_1, \dots, S_t\}$

Proof. Let z be an element of $B(b)$. Then $0 = b(S) = \widetilde{z}(S) = \sum_i \widetilde{z}(S_i) \leq \sum_i b(S_i)$ and $0 = b(S) = \widetilde{z}(S) = \sum_i \widetilde{z}(S_i) = \sum_i [\widetilde{z}(S) - \widetilde{z}(S - S_i)] = -\sum_i \widetilde{z}(S - S_i) \geq -\sum_i b(S - S_i)$ implying the necessity of (15.33) and (15.34).

For the sufficiency, let s be any element of S. Let b' denote the restriction of b to $S' = S - s$. Define p' on $S' = S - s$ by $p'(X) := -b(S - X)$. An easy consideration shows that b is crossing if and only if (p', b') is an intersecting paramodular pair and $Q(p', b')$ is just the projection of $B(b)$ along s.

Let $\{S_1, \dots, S_t\}$ be a partition of S for which $s \in S_1$. Then the inequality $\sum_i b(S_i) \geq 0$ in (15.33) is equivalent to $b'(S_2) + b'(S_3) + \cdots + b'(S_t) \geq p'(S_2 \cup S_3 \cup \cdots \cup S_t)$, and similarly the inequality $\sum_i b(S - S_i) \geq 0$ in (15.34) is equivalent to $p'(S_2) + p'(S_3) + \cdots + p'(S_t) \geq b'(S_2 \cup S_3 \cup \cdots \cup S_t)$. By this correspondence, we obtain that the theorem is just equivalent to the special case of Theorem 15.1.8 concerning intersecting paramodular pairs. ●

This proof of Fujishige's theorem relied on a series of earlier results: Theorem 12.1.4 on feasibility of submodular flows implied the Discrete separation theorem (Theorem 12.2.1). The latter was combined with the Truncation theorem to obtain Theorem 15.1.7 on separating near sub- and supermodular functions. Finally, Theorem 15.1.7 was used to derive Theorem 15.1.8, which immediately implied Fujishige's theorem. Although the proofs of each of these preliminary results were pretty clear, even simple, the need for a direct, self-contained proof of Fujishige's theorem is natural.

Direct proof of the sufficiency in Theorem 15.3.8.

Proposition 15.3.9 *If* (15.33) *and* (15.34) *hold for every partition of S, then $\widetilde{b}(\mathcal{K}) \geq 0$ holds for every composition $\mathcal{K}$ of S.*

Proof. Suppose indirectly that there is a violating composition and choose one for which $|\mathcal{K}|$ is minimum and, subject to this, $\sum[|X|^2 : X \in \mathcal{K}]$ is maximum. We claim that $\mathcal{K}$ is cross-free. Indeed, if X and Y were two crossing members of $\mathcal{K}$, then $\mathcal{K}' := \mathcal{K} - \{X, Y\} \cup \{X \cap Y, X \cup Y\}$ is also a composition of S for which $\widetilde{b}(\mathcal{K}') \leq \widetilde{b}(\mathcal{K}) < 0$, $|\mathcal{K}'| = |\mathcal{K}|$, and $\sum[|X|^2 : X \in \mathcal{K}'] > \sum[|X|^2 : X \in \mathcal{K}]$, contradicting the choice of $\mathcal{K}$.

By Lemma 15.3.1, $\mathcal{K}$ can be divided into partitions and co-partitions and hence $\widetilde{b}(\mathcal{P}) < 0$ must hold for at least one of these, contradicting (15.32). •

A composition $\mathcal{P}$ is **tight** if $\widetilde{b}(\mathcal{P}) = 0$. Crossing submodularity cannot be destroyed by reducing the value of b on a singleton $\{s\}$. Therefore we can assume that $b(s)$ is as small as possible subject to the inequality in (15.32) for each partition and co-partition $\mathcal{P}$. The minimality of $b(s)$ implies that the singleton $\{s\}$ belongs to a tight partition or co-partition. Since a co-partition including a singleton $\{s\}$ must have exactly two members, namely $\{s\}$ and $S - \{s\}$, and a two-element co-partition is a (two-element) partition, it follows that $\{s\}$, in fact, belongs to a tight partition $\mathcal{P}_s$ for every element $s \in S$.

Define a vector $z \in \mathbf{R}^S$ by letting $z(s) := b(s)$ for every $s \in S$. We are going to prove that $z \in B(b)$, which is equivalent to requiring that $\widetilde{z}(S) = 0$ and $\widetilde{z}(Z) \leq b(Z)$ for every $Z \subset S$.

Let $(S, \mathcal{K})$ be a hypergraph such that $\mathcal{K}$ is the multi-union of partitions $X \in \mathcal{P}_s$ over $s \in S$. Then K is a composition of S, $\widetilde{b}(\mathcal{K}) = 0$, and $\mathcal{P}_S \subseteq \mathcal{K}$ where $\mathcal{P}_S := \{\{s\} : s \in S\}$ is the finest partition of S. Therefore $\mathcal{K}' := \mathcal{K} - \mathcal{P}_S$ is also a composition of S. Proposition 15.3.9 implies $\widetilde{b}(\mathcal{K}') \geq 0$ from which $0 \leq \widetilde{b}(\mathcal{P}_S) = \widetilde{b}(\mathcal{K}) - \widetilde{b}(\mathcal{K}') \leq 0$, and hence $\mathcal{P}_S$ is a tight partition implying $\widetilde{z}(S) = 0$.

For an arbitrary non-empty set $Z \subset S$, let $\mathcal{K}_Z$ be the union of partitions $\mathcal{P}_s$ over the elements $s \in Z$ and let $\mathcal{P}_Z := \{\{s\} : s \in Z\}$ be the finest partition of Z. Then $\mathcal{K}_Z$ is a composition of S for which $\widetilde{b}(\mathcal{K}_Z) = 0$ and $\mathcal{P}_Z \subseteq \mathcal{K}_Z$. Hence $\mathcal{K}' := \mathcal{K}_Z - \mathcal{P}_Z \cup \{Z\}$ is also a composition of S. Proposition 15.3.9 implies $0 \leq \widetilde{b}(\mathcal{K}') = \widetilde{b}(\mathcal{K}_Z) - \widetilde{b}(\mathcal{P}_Z) + b(Z) = 0 - \widetilde{z}(Z) + b(Z)$, from which $\widetilde{z}(Z) \leq b(Z)$, as required. • •

Fujishige proved his result in the following slightly more general (though equivalent) form.

Theorem 15.3.10 (Fujishige) *Let b be a crossing submodular function for which $b(S)$ is finite. The polyhedron $B(b)$ is non-empty if and only if*

$$\sum_i b(S_i) \geq b(S) \tag{15.35}$$

and

$$\sum_i b(S - S_i) \geq (t-1)b(S) \quad \text{(equivalently: } \sum_i \overline{b}(S_i) \geq b(S)) \tag{15.36}$$

for every partition $\mathcal{F} = \{S_1, \ldots, S_t\}$ of S. Let p be crossing supermodular for which $p(S)$ is finite. The polyhedron $B'(p)$ is non-empty if and only if

$$\sum_i p(S_i) \leq p(S) \tag{15.37}$$

and

$$\sum_i p(S - S_i) \leq (t-1)p(S) \quad \text{(equivalently: } \sum_i \overline{p}(S_i) \leq p(S)). \tag{15.38}$$

for every partition $\mathcal{F} = \{S_1, \ldots, S_t\}$ of S.

Proof. The two parts are equivalent so we prove only the first one. Select an arbitrary element s of S and reduce $b(X)$ by $b(S)$ for each set $X \subseteq S$ containing s. The resulting set-function b^* is also crossing submodular for which $b^*(S) = 0$. The conditions (15.33) and (15.34), when applied to b^*, transform into (15.35) and (15.36), respectively. On the other hand $B(b^*)$ is the translate of $B(b)$ by the vector $z \in \mathbf{R}^S$ for which $z(s) = b(S)$ and $z(v) = 0$ for each $v \in S - s$. •

Problem 15.3.5 ($*$) *Derive Theorem* 14.3.16 *on the non-emptiness of the intersection of a base-polyhedron with a box from Theorem* 15.3.10.

15.3.4 G-polymatroids from intersecting paramodular pairs

Theorem 15.3.11 *A polyhedron $Q = Q(p, b)$ defined by an intersecting paramodular pair (p, b) is a g-polymatroid, which is, in addition, integral whenever (p, b) is integral.* •

Proof. The empty set is a g-polymatroid by definition so we assume that $Q \neq \emptyset$. Consider the set-function b^* given by (14.7) on ground-set $S^* = S + s^*$. Since (p, b) is intersecting paramodular, b^* is crossing submodular and furthermore $Q(p, b)$ is the projection of $B(b^*)$ along s^*. By Theorem 15.3.5, $B(b^*)$ is a base-polyhedron and by Theorem 14.2.2, $Q(p, b)$ is a g-polymatroid. If b is integer-valued, then so is b^* and hence both $B(b^*)$ and its projection are integral polyhedra. •

It is possible to extend Theorem 15.3.11 to near paramodular pairs as well, but the proof is significantly more complicated. Therefore here we only state the result without its proof.

Theorem 15.3.12 *A polyhedron $Q = Q(p, b)$ defined by a near paramodular pair (p, b) is a g-polymatroid, which is, in addition, integral whenever (p, b) is integral.*•

Problem 15.3.6 *Derive from Theorem* 15.3.11 *that the intersection of a g-polymatroid with a box and with a plank is a g-polymatroid.*

Theorem 15.1.8, when applied to an intersecting paramodular pair, specializes to the following.

Theorem 15.3.13 *A g-polymatroid $Q = Q(p, b)$ defined by an intersecting paramodular pair (p, b) is non-empty if and only if*

$$p(\cup Z_i) \leq \sum_i b(Z_i) \text{ and } \sum_i p(Z_i) \leq b(\cup Z_i). \tag{15.39}$$

for every subpartition $\{Z_1, \ldots, Z_t\}$ of the ground-set S. •

Our first proof for Fujishige's theorem used Theorem 15.1.8. A similar approach gives rise to a formula of the (fully) paramodular border function of a non-empty g-polymatroid $Q = (p, b)$ defined by an intersecting paramodular pair (p, b). We only state the formula since it is not particularly simple and will not be used later. For set-functions p and b for which $p \leq b$, define the lower divolution b_p of b and p as follows: $b_p(Z) := \min\{b(X) - p(X - Z) : X \subseteq Z\}$. The upper divolution p_b of p and b as follows: $p_b(Z) := \max\{p(X) - b(X - Z) : X \subseteq Z\}$.

Theorem 15.3.14 *Let (p, b) be an intersecting paramodular pair and suppose that the g-polymatroid $Q = Q(p, b)$ is non-empty. Then the border pair (p', b') of Q is given by the following formula.*

$$b' = (b_{\downarrow(p^\wedge)})^\vee, \quad p' = (p_{\uparrow(b^\vee)})^\wedge. \bullet \tag{15.40}$$

15.4 Graph orientations via base-polyhedra

15.4.1 Orientation problems

The goal of this section is to show how the theory of semimodular functions provides a unified treatment of almost all orientation problems discussed so far: the single result not fitting in this framework is the strong orientation theorem of Nash-Williams (Theorem 9.8.1). We call these earlier results connectivity-type since the property to be achieved was always a connectivity or an in-degree requirement. Let us briefly survey these results.

Connectivity orientations

There have been two types of problems: feasibility and optimal reorientation. Two initial feasibility results discussed in Part I concerning the existence of certain orientations are Robbins' theorem on strong orientability of a 2-edge-connected graph and Hakimi's Orientation lemma (Theorem 2.3.2) on in-degree specified orientations. Robbins' theorem was extended to mixed graphs, while the Orientation lemma was extended to in-degree constrained orientations, including the first appearence of the linking property. In Chapter 9, these results were generalized and unified. Nash-Williams' theorem on k-edge-connected orientability of $2k$-edge-connected graphs was a significant generalization of Robbins' theorem. A result of Frank and Gyárfás (Theorem 9.2.9) treated in-degree constraints and strong connectivity simultaneously, while Theorem 9.2.6 extended this result so as to involve k-edge-connectivity. The linking property in this situation remained valid. As a rooted counterpart of Nash-Williams' theorem, a characterization was given in Theorem 9.1.8 for graphs having rooted k-edge-connected orientations. Even more, Theorem 9.3.1 on (k, ℓ)-edge-connected orientations described a common generalization. As far as optimal reorientations are concerned, optimal degree-constrained reorientations could be handled with network flows while the deep theorem of Lucchesi and Younger was interpreted as a theorem on optimal strongly connected reorientations of a digraph. In Section 9.4 we also considered hypergraph orientation problems and showed that the orientation lemma as well as results on degree-constrained orientations could rather easily be extended to hypergraphs. Robbins' theorem was also extended to hypergraphs in Theorem 9.4.9, but the characterization was trickier than the one in the graph case since the notion of (1, 1)-partition connectivity of the hypergraph came into the picture.

Rooted k-edge-connected orientability of mixed graphs was also characterized in Theorem 9.5.1. This result could be considered as a straight extension of Theorem 9.1.8 concerning undirected graphs. An uncomfortable difference, however, was that the linking property, which was shown in Theorem 9.1.10 to be valid for the undirected case, did not hold for mixed graphs. It did not hold for strong orientability of mixed graphs either. It is also worth mentioning that the path-reorientation technique could be used in each case above concerning orientation of undirected graphs, while mixed graphs required other techniques.

The following problems and questions remained unanswered in Chapter 9. What is a necessary and sufficient condition for a mixed graph to have a k-edge-connected orientation? How is it possible to compute a cheapest strongly connected (more generally: k-edge-connected) reorientation of a digraph? What is behind the phenomenon that the linking property holds in certain cases and not in others? Why is it that a simple cut condition is sufficient in some cases (e.g., Nash-Williams' theorem) while the more complicated partition-type conditions are required in others? Also, the theorems for characterizing hypergraphs admitting a (k, ℓ)-edge-connected orientation were already formulated in Theorems 9.4.12 and 9.4.13, but their proofs were postponed to the present section. Here we shall focus on generalizations of feasibility orientation problems while optimal reorientations will be discussed in Chapter 16.

Abstract orientations

Let $G = (V, E)$ be an undirected graph and h an integer-valued set-function defined on V, called a **demand function**. The **abstract orientation** problem consists of finding an orientation of G covering h in the sense that $\varrho(X) \geq h(X)$ for every $X \subseteq V$ where ϱ denotes the in-degree function of the given orientation of G. For example, if h is identically k on non-empty proper subsets of V and $h(\emptyset) = h(V) = 0$, then an orientation of G covers h precisely if it is k-edge-connected. This abstract formulation allows one to incorporate the digraph D into h when a mixed graph $M = D + G$ is to be oriented. Indeed, by defining h to be $h(X) = k - \varrho_D(X)$ for $\emptyset \subset X \subset V$ and $h(\emptyset) = h(V) = 0$, it follows that M has a k-edge-connected orientation if and only if G has an orientation covering h.

There is an optimization version of the abstract orientation problem when the goal is to find a cheapest orientation of G covering h where costs are assigned to the two possible orientations of each edge of G. Here one can assume that the minimum of the costs $c(uv)$ and $c(vu)$ is zero for each edge $e = uv$ of G where uv and vu denote the two possible orientations of e. Indeed, by adding a constant to both $c(uv)$ and $c(vu)$ one obtains an equivalent optimization problem. Therefore the optimization problem can equivalently be formulated as finding a cheapest reorientation of a digraph that covers h.

One can consider other orientation-type problems as well. For example, when is it possible to orient the elements of a specified subset E' of edges of a graph $G = (V, E)$ in such a way that the resulting mixed graph is k-edge-connected? It will turn out that a natural necessary cut condition proves to be sufficient. In the optimization version of this problem, one wants to find the largest subset E' for which such an orientation exists. Although this problem can be proved to be **NP**-complete, its rooted k-edge-connected variation is nicely tractable. Yet another problem is when a smallest (or cheapest) subgraph or supergraph of an undirected graph is to be found which has an orientation with given properties. And finally, the deorientation problem is about finding a minimum number of edges of a digraph whose deorientation results in a mixed graph of specified connectivity properties.

In the introduction to Chapter 9, we already indicated that the abstract orientation problem is too general when no further restrictions are imposed on h, since the **NP**-complete problem of 2-colourability of hypergraphs could be formulated as an orientation problem in which h is actually (0, 1)-valued. Apart from Theorem 9.8.1, all orientation problems discussed in the first two parts can be formulated as one with a crossing supermodular demand function h. It is worth considering a slightly more general class of demand functions. We refer to a set-function h as **crossing** (**intersecting**) G**-supermodular** if

$$h(X) + h(Y) \leq h(X \cup Y) + h(X \cap Y) + d_G(X, Y)$$

holds for every crossing (intersecting) subsets X and Y of V where $d_G(X, Y)$ denotes the number of edges of G connecting $X - Y$ and $Y - X$.

Reduction to base-polyhedra

There is a natural way to formulate abstract orientation problems in terms of semimodular functions. It is based on the observation that the in-degree sequence (or vector) of an orientation of G (defined in Section 2.3 as a sequence consisting of the in-degrees of the nodes) uniquely determines the in-degrees of subsets of nodes. Namely,

$$\varrho(Z) = \sum[\varrho(v) : v \in Z] - i_G(Z), \tag{15.41}$$

showing that there is an orientation of G covering h if and only if there is an in-degree vector $m : V \to \mathbf{Z}$ for which

$$\widetilde{m}(Z) \geq p(Z) := h(Z) + i_G(Z) \text{ for every } Z \subseteq V. \tag{15.42}$$

By the Orientation lemma an integer-valued m is an in-degree vector if and only if m is an element of the base-polyhedron $B'(i_G)$, that is,

$$\widetilde{m}(V) = |E| \text{ and } \widetilde{m}(Z) \geq i_G(Z) \text{ for every } Z \subseteq V. \tag{15.43}$$

Recall that i_G is a (fully) supermodular function. When the demand function h is crossing G-supermodular, then $p := h + i_G$ is crossing supermodular and $B'(p)$ is a base-polyhedron. Therefore there is a one-to-one correspondence between the integral elements of $B'(i_G) \cap B'(p)$ and the in-degree vectors of orientations of G covering h. This reduction to base-polyhedra allows one to understand why the linking property holds in some orientation problems but not in others.

978–0–19–920527–1 15-Andras-Frank-c15-drv Andras Frank (Typeset by SPi, Publisher Services, India) 515 of 640 December 11, 2010 15:21
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

15.4.2 Non-negative crossing G-supermodular demand functions

A particularly friendly situation is when h is not only crossing G-supermodular but non-negative (in particular, finite valued) as well. In this case $p = h + i_G \geq i_G$ and hence $B'(p) \subseteq B'(i_G)$, and hence $B'(p) \cap B'(i_G) = B'(p)$. Consequently, there is an orientation of G covering h if and only if a base-polyhedron bordered by a crossing supermodular function is non-empty. Fujishige's theorem (Theorem 15.3.10) is tailored exactly for this case.

Theorem 15.4.1 (Frank [114]) *Let h be a non-negative crossing G-supermodular function for which $h(\emptyset) = h(V) = 0$. G has an orientation covering h if and only if*

$$e_G(\mathcal{P}) \geq \sum_j h(V_j) \text{ for every partition } \mathcal{P} = \{V_1, \ldots, V_t\} \text{ of } V \tag{15.44}$$

and

$$e_G(\mathcal{P}) \geq \sum_j h(V - V_j) \text{ for every partition } \mathcal{P} = \{V_1, \ldots, V_t\} \text{ of } V \tag{15.45}$$

where $t \geq 2$ and $e_G(\mathcal{P})$ denotes the number of cross-edges to $\mathcal{P}$.

Proof. Necessity. If an orientation of G covers h, then $e_G(\mathcal{P}) = \sum_j \varrho(V_j) \geq \sum_j h(V_j)$ and (15.44) follows. By reorienting each edge, one obtains an orientation covering h', where h' is defined by $h'(X) := h(V - X)$, from which $e_G(\mathcal{P}) = \sum_j \varrho(V_j) \geq \sum_j h'(V_j) = \sum_j h(V - V_j)$ and (15.45) follows.

Sufficiency. As mentioned above, the function $p := h + i_G$ is crossing supermodular. By (15.44), we have

$$\sum_j p(V_j) = \sum_j h(V_j) + |E| - e_G(\mathcal{P}) \leq |E| = p(V).$$

By (15.45), we have

$$\sum_j p(V - V_j) = \sum_j h(V - V_j) + (t-1)|E| - e_G(\mathcal{P}) \leq (t-1)|E| = (t-1)p(V).$$

Theorem 15.3.10 implies that there exists an integer-valued function $m : V \to \mathbf{Z}$ for which $\widetilde{m}(V) = |E|$ and $\widetilde{m}(X) \geq p(X)$ for every $X \subseteq V$. It follows from $h \geq 0$ that $\widetilde{m}(X) \geq i_G(X)$ holds automatically, which in turn implies by the Orientation lemma that m is the in-degree vector of an orientation of G. This orientation covers h since $\varrho(X) = \sum[\varrho(v) : v \in X] - i_G(X) = \widetilde{m}(X) - i_G(X) \geq h(X)$ for every $X \subset V$. •

Corollary 15.4.2 (A) *If the non-negative crossing G-supermodular function h in Theorem* 15.4.1 *is non-increasing, then condition* (15.44) *alone is sufficient.* **(B)** *If h is non-negative crossing G-supermodular and symmetric in Theorem* 15.4.1*, then condition* (15.44) *requested only for $t = 2$ is sufficient. This latter condition is equivalent to requiring*

$$d_G(X) \geq 2h(X) \text{ for every subset } X \subset V. \tag{15.46}$$

Proof. (A) Since $V_{j+1} \subset V - V_j$ for $j = 1, \ldots, t$ (where $V_{t+1} := V_1$), we have $h(V_{j+1}) \geq h(V - V_j)$ from which $\sum_{j=1}^t h(V_j) = \sum_{j=1}^t h(V_{j+1}) \geq \sum_{j=1}^t h(V - V_j)$. Therefore (15.44) implies (15.45).

(B) The symmetry of h implies that (15.44) and (15.45) are equivalent. Suppose now that (15.44) holds for every 2-partite partition and consider a t-partite partition $\mathcal{P} := \{V_1, V_2, \ldots, V_t\}$ of V for some $t \geq 3$. The number $e_G(\mathcal{P})$ of cross-edges is $\sum_j d_G(V_j)/2$ and hence the condition $d_G(X) \geq 2h(X)$ implies $e_G(\mathcal{P}) = \sum_j d_G(V_j)/2 \geq \sum_j h(V_j)$, and hence (15.44) holds indeed. •

In the special case when $h(X) = k$ for every non-empty and proper subset X, we obtain Nash-Williams' (weak) orientation theorem by Part **(B)** of the corollary. Relying on g-polymatroid intersection, we shall see soon that the statement in Part **(B)** remains true when the non-negativity of h is dropped. If h is defined by (9.21), then Part **(A)** of Corollary 15.4.2 implies Theorem 9.3.1 on (k, ℓ)-edge-connected orientations of graphs.

Problem 15.4.1 (Kovács [244]) $(*)$ *Let $G = (V, E)$ be a $2k$-edge-connected graph and $F \subset E$ a specified subset of at most k edges. Prove that any orientation of F can be extended to a k-edge-connected orientation of G. Formulate and prove a generalization for (k, ℓ)-partition-connected graphs.*

We obtained Theorem 15.4.1 from Fujishige's theorem quite easily, but the two results are actually equivalent, and there is an equally easy way to show the reverse implication, at least in the case when b is finite-valued.

Proof of Theorem 15.3.8 from Theorem 15.4.1. Suppose that b is a finite-valued (and integer-valued) crossing submodular function for which $b(S) = 0$ which satisfies (15.32) for every partition and co-partition of S. Let $G = (S, E)$ be an Euler graph for which $d_G/2 \geq b$. Then $h := d_G/2 - b$ is a non-negative, integer-valued, and G-crossing supermodular function for which $h(S) = 0$. Let $\mathcal{P} := \{S_1, \ldots, S_q\}$ be a partition or a co-partition of S. Then $\sum_i b(S_i) \geq 0$ implies that

$$e_G(\mathcal{P}) = \sum_i d_G(S_i)/2 = \sum_i h(S_i) + \sum_i b(S_i) \geq \sum_i h(S_i).$$

By Theorem 15.4.1, there is an orientation D of G that covers h. Let $z : S \to \mathbf{Z}_+$ be defined by $z(v) := d_G(v)/2 - \varrho_D(v)$. We claim that $z \in B(b)$. Indeed,

$$\widetilde{z}(X) = \sum [d_G(v)/2 - \varrho_D(v) : v \in X] = [d_G(X)/2 - i_G(X)] - [\varrho_D(X) - i_D(X)]$$

$$= d_G(X)/2 - \varrho_D(X) \leq d_G(X)/2 - h(X) = b(X)$$

for $X \subseteq S$, and, in addition, for $X = S$ the last inequality is fulfilled with equality from which $\widetilde{z}(S) = b(S) = 0$, as required. •

Orientation of hypergraphs

Before proceeding to other graph orientation results, we make a detour into hypergraphs. An orientation of a hypergraph is said to cover a set-function h if the in-degree of every subset of nodes is at least $h(X)$ for every subset $X \subseteq V$. The following results are due to Frank, Király, and Király [145].

Theorem 15.4.3 *Let $H = (V, \mathcal{E})$ be a hypergraph and let h be a non-negative crossing supermodular function on V for which $h(\emptyset) = h(V) = 0$. There is an orientation of H covering h if and only if*

$$e_H(\mathcal{P}) \geq \sum_j h(V_j) \textit{ for every partition } \mathcal{P} = \{V_1, \ldots, V_t\} \textit{ of } V \tag{15.47}$$

where $e_H(\mathcal{P})$ denotes the number of cross-hyperedges of H to $\mathcal{P}$, and

$$\sum_{Z \in \mathcal{E}} [e_{\mathcal{P}}(Z) - 1] \geq \sum_{j=1}^{t} h(V - V_j) \textit{ for every partition } \mathcal{P} = \{V_1, \ldots, V_t\} \textit{ of } V \tag{15.48}$$

where $e_{\mathcal{P}}(Z)$ denotes the number of members of $\mathcal{P}$ intersecting Z.

Proof. Necessity. Suppose that there is an orientation of H covering h and let $\mathcal{P} = \{V_1, \ldots, V_t\}$ be a partition of V. Then every cross-hyperedge to $\mathcal{P}$ is oriented to a dyperedge that enters exactly one member of $\mathcal{P}$. Hence $e_H(\mathcal{P}) = \sum_j \varrho(V_j) \geq \sum_j h(V_j)$, and hence (15.47) is necessary.

A hyperedge Z of H intersects $e_{\mathcal{P}}(Z)$ members of $\mathcal{P}$ and thus the oriented Z leaves exactly $e_{\mathcal{P}}(Z) - 1$ members of $\mathcal{P}$. Therefore $\sum_j h(V - V_j) \leq \sum_j \varrho(V - V_j) = \sum_j \delta(V_j) = \sum_{Z \in \mathcal{P}} [e_{\mathcal{P}}(Z) - 1]$, and hence (15.48) is also necessary.

For sufficiency, consider the set-function $p := h + i_H$. This function is crossing supermodular and we are going to show that the conditions (15.37) and (15.38) in Theorem 15.3.10 are fulfilled. Indeed, by (15.47), we have

$$\sum_j p(V_j) = \sum_j h(V_j) + \sum_j i_H(V_j) \leq e_H(\mathcal{P}) + \sum_j i_H(V_j) = |\mathcal{E}| = p(V).$$

Similarly, (15.48) implies

$$\sum_j p(V - V_j) = \sum_j h(V - V_j) + \sum_j i_H(V - V_j) \leq \sum_{Z \in \mathcal{E}} [e_{\mathcal{P}}(Z) - 1] + \sum_j i_H(V - V_j)$$

$$= \sum_{Z \in \mathcal{E}} [e_{\mathcal{P}}(Z) - 1] + \sum_{Z \in \mathcal{E}} [t - e_{\mathcal{P}}(Z)] = \sum_{Z \in \mathcal{E}} [t - 1] = (t-1)|\mathcal{E}| = (t-1)p(V).$$

Theorem 15.3.10 implies that there exists an integer-valued function $m : V \to \mathbf{Z}$ for which $\widetilde{m}(V) = |\mathcal{E}|$ and $\widetilde{m}(X) \geq p(X)$ for every $X \subseteq V$. It follows from $h \geq 0$ that $\widetilde{m}(X) \geq i_H(X)$ holds automatically which in turn implies, by the Hypergraph orientation lemma (Theorem 9.4.2), that m is the in-degree vector of an orientation of H. This orientation covers h since $\varrho(X) = \sum[\varrho(v) : v \in X] - i_H(X) = \widetilde{m}(X) - i_H(X) \geq h(X)$ for every $X \subset V$. •

Observe that in the special case when the hypergraph H is a graph G, the sum $\sum_{Z \in \mathcal{E}}^{t} [e_{\mathcal{P}}(Z) - 1]$ in (15.48) is equal to the number $e_G(\mathcal{P})$ of cross-edges to $\mathcal{P}$ and hence (15.48) transforms to (15.45).

Theorem 15.4.4 *Let $H = (V, \mathcal{E})$ be a hypergraph with a root-node r_0 and let k and ℓ be non-negative integers. H has a (k, ℓ)-edge-connected orientation if and only if*

$$e_H(\mathcal{P}) \geq k(|\mathcal{P}| - 1) + \ell \textit{ for every partition } \mathcal{P} \textit{ of } V \tag{15.49}$$

and

$$\sum_{Z \in \mathcal{E}} [e_{\mathcal{P}}(Z) - 1] \geq \ell(|\mathcal{P}| - 1) + k \textit{ for every partition } \mathcal{P} \textit{ of } V \tag{15.50}$$

where $e_{\mathcal{P}}(Z)$ *denotes the number of members of* $\mathcal{P}$ *intersecting* Z. *If* $\ell \leq k$, *then* (15.49) *implies* (15.50)*, and hence, in this case* (15.49) *itself is sufficient for* (k, ℓ)*-edge-connected orientability.*

Proof. Define a set-function $h_{k\ell}$ as follows.

$$h_{k\ell}(X) := \begin{cases} k & \text{if } \emptyset \subset X \subseteq V - r_0 \\ \ell & \text{if } r_0 \in X \subset V \\ 0 & \text{if } X = \emptyset \text{ or } X = V. \end{cases} \tag{15.51}$$

Function $h_{k\ell}$ is clearly crossing supermodular. An orientation of H is (k, ℓ)-edge-connected if and only if it covers $h_{k\ell}$. Hence the first part of the theorem is just a special case of Theorem 15.4.3.

To see the second part, assume that $\ell \leq k$ and that (15.49) holds. Then

$$\ell(|\mathcal{P}| - 1) + k \leq k(|\mathcal{P}| - 1) + \ell \leq e_H(\mathcal{P}) \leq \sum_{Z \in \mathcal{E}} [e_{\mathcal{P}}(Z) - 1],$$

that is, (15.50) is fulfilled. •

Note that Theorem 15.4.4 includes Theorems 9.4.12 and 9.4.13.

The linking property for orientations

Since the in-degree vectors of orientations covering a non-negative G-supermodular demand function h are the integer-valued elements of a single base-polyhedron, we can apply the linking property of g-polymatroids formulated in Theorem 14.3.21 and obtain the following.

Corollary 15.4.5 *Let h be a non-negative crossing G-supermodular set-function on V. Let f and g be integral functions on the node-set of a graph $G = (V, E)$ $(0 \leq f \leq g)$. Suppose that there is an orientation covering h for which $\varrho'(v) \geq f(v)$ for every node v and there is an orientation covering h for which $\varrho''(v) \leq g(v)$ for every node v. Then there is an orientation covering h for which $f(v) \leq \varrho(v) \leq g(v)$ for every node v.* •

Theorem 15.4.1 has a self-refining feature since additional upper and lower bounds for the in-degree of nodes can be incorporated into the demand function.

Theorem 15.4.6 *Let f and g be integral functions on the node-set of a graph $G = (V, E)$ for which $0 \leq f(v) \leq g(v) \leq d_G(v)$ for every $v \in V$ and let h be a non-negative crossing G-supermodular function. There exists an orientation covering h for which*

$$f(v) \leq \varrho(v) \leq g(v) \text{ for every node } v \tag{15.52}$$

if and only if

$$e_G(\mathcal{P}) + i_G(U_0) \geq \sum [h(U_j) : j = 1, \ldots, t] + \widetilde{f}(U_0) \tag{15.53}$$

and

$$e_G(\mathcal{P}) - e_G(U_0) \geq \sum [h(V - U_j) : j = 1, \ldots, t] - \widetilde{g}(U_0) \tag{15.54}$$

holds for every partition $\mathcal{P} = \{U_0, U_1, \ldots, U_t\}$ of V where only U_0 can be empty and $e_G(X)$ denotes the number of edges of G with at least one end-node in X.

Proof. Define a set-function h' as follows.

$$h'(X) := \begin{cases} h(X) & \text{if } X \subseteq V \text{ and } 2 \leq |X| \leq |V| - 2 \\ \max\{h(v),\ f(v)\} & \text{if } X = \{v\} \text{ for some } v \in V \\ \max\{h(V - v), d_G(v) - g(v)\} & \text{if } X = V - v \text{ for some } v \in V. \end{cases} \tag{15.55}$$

Then h' is also crossing G-supermodular and non-negative. An easy exercise shows that (15.44) written up for h' is just equivalent to (15.53) and (15.45) written up for h' is just equivalent to (15.54). •

Note that Theorem 15.4.6 implies at once Corollary 15.4.5 since g does not occur in (15.53) and f does not occur in (15.54).

When $h \equiv 0$, we are back at Theorem 2.3.5. If h is defined for given integers $0 \leq \ell \leq k$ by

$$h(X) := \begin{cases} k & \text{if } \emptyset \subset X \subseteq V - r_0 \\ \ell & \text{if } r_0 \in X \subset V \\ 0 & \text{if } X = \emptyset \text{ or } X = V, \end{cases} \tag{15.56}$$

then h is non-decreasing and Corollary 15.4.2 implies Theorem 9.3.1 on (k, ℓ)-edge-connected orientations of G. Naturally, Theorem 15.4.6 can be used to characterize graphs having (k, ℓ)-edge-connected orientation satisfying $f(v) \leq \varrho(v) \leq g(v)$ for every node v. For the special case when $k = \ell$, this result was already formulated and proved in Theorem 9.2.6. We can also conclude that the linking property holds for (k, ℓ)-edge-connected orientations of an undirected graph.

15.4.3 Symmetric demand functions

We indicated above that in Part (B) of Corollary 15.4.2 the non-negativity of h can actually be dropped. We are now going to prove sufficiency of the cut condition even for a more general class of symmetric functions.

Theorem 15.4.7 *Let h_1 and h_2 be two symmetric crossing G-supermodular functions on V with $h_1(V) = h_2(V) = 0$ and suppose that h_2 is non-negative. Let h be defined by $h(X) := \max\{h_1(X), h_2(X)\}$. An undirected graph $G = (V, E)$ has an orientation covering h if and only if $d_G(X) \geq 2h(X)$ for every $X \subseteq V$.*

Proof. The condition is clearly necessary for an arbitrary symmetric demand function since an orientation covering h satisfies $d_G(X) = \varrho(X) + \varrho(V - X) \geq h(X) + h(V - X) = 2h(X)$.

For proving sufficiency, it suffices to find an in-degree vector m for which $\widetilde{m}(Z) \geq h(Z) + i_G(Z)$ holds for every $Z \subset V$. In other words, we have to show that the polyhedron

$$P := \{x \in \mathbf{R}^V : x \geq 0, \widetilde{x}(V) = |E|,\ \widetilde{x}(Z) \geq i_G(Z) \text{ and } \widetilde{x}(Z) \geq h(Z) + i_G(Z) \text{ for every } Z \subset V\}$$

contains an integral element.

Since h_2 is non-negative, so is h, and hence the inequalities $\widetilde{x}(Z) \geq i_G(Z)$ are redundant. Therefore, $P = \{x \in \mathbf{R}^V : x \geq 0, \widetilde{x}(V) = |E|,\ \widetilde{x}(Z) \geq p_1(Z)$ and $\widetilde{x}(Z) \geq p_2(Z)$ for every

$Z \subset V\}$ where $p_j(Z) := h_j(Z) + i_G(Z)$ $(j = 1, 2)$. Since h_1 and h_2 are crossing G-supermodular, the functions p_1 and p_2 are crossing supermodular, implying that P is the intersection of two base-polyhedra. Hence P is an integral polyhedron by the g-polymatroid intersection theorem, and thus it suffices to prove that P is non-empty. We claim that the vector x^* defined by $x^*(v) := d_G(v)/2$ belongs to P. Indeed, we have

$$\widetilde{x}^*(V) = \sum [x^*(v) : v \in V] = \sum [d_G(v)/2 : v \in V] = |E|,$$

and also

$$\widetilde{x}^*(Z) = \sum [x^*(v) : v \in Z] = \sum [d_G(v)/2 : v \in Z] = d_G(Z)/2 + i_G(Z) \geq h(Z) + i_G(Z)$$

for each $Z \subset V$. That is, P is non-empty and hence it contains an integral element. •

The theorem implies the following sharpening of Nash-Williams' orientation theorem.

Corollary 15.4.8 *Let G be a $(2k)$-edge-connected graph and H an Eulerian subgraph of G. Then any Eulerian orientation of H can be extended to a k-edge-connected orientation of G.*

Proof. Consider the subgraph $G' = (V, E')$ of G consisting of the undirected edges. Define h_1 by $h_1(Z) := k - d_H(Z)/2$ for $\emptyset \subset Z \subset V$ and let $h_2 \equiv 0$. The result follows by applying Theorem 15.4.7 to G'. •

Corollary 15.4.9 *Let E' be a subset of edges of a k-edge-connected graph $G = (V, E)$. It is possible to orient the elements of E' such that the resulting mixed graph is k-edge-connected if and only if $d_{E'}(X) \leq 2(d_E(X) - k)$ holds for every non-empty subset $X \subset V$.*

Proof. First replace each element of $E - E'$ by two oppositely oriented edges and then apply the preceding corollary. •

By relying on the Sum theorem and the Intersection theorem of g-polymatroids, we can use the same approach to obtain the following extension. It was proved originally by Király and Szigeti [240] in a significantly more general form using the Pairing theorem (Theorem 9.8.3).

Theorem 15.4.10 (Király and Szigeti) *Let $G = (V, E)$ be a $2k$-edge-connected graph and let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_t = (V_t, E_t)$ be subgraphs of G such that $\{E_1, \ldots, E_t\}$ is a partition of E and $G_j = (V_j, E_j)$ is $(2k_j)$-edge-connected for $j = 1, \ldots, t$. Then G has a k-edge-connected orientation whose restriction to E_j is a k_j-edge-connected orientation of G_j for each $j = 1, \ldots, t$.*

Proof. Define a set-function p_0 by $p_0(X) := k + i_G(X)$ when $\emptyset \subset X \subset V$, and $p_0(\emptyset) = 0$, $p_0(V) := |E|$. Then $B_0 := B'(p_0)$ is a base-polyhedron. For $j = 1, 2, \ldots, t$, define set-functions p_j on ground-set V_j by $p_j(X) := i_{G_j}(X) + k_j$ if $\emptyset \subset X \subset V_j$ and $p_j(\emptyset) = 0$, $p_j(V_j) := |E_j|$. The functions p_j are crossing supermodular (on ground-set V_j) so we can consider the base-polyhedra $B'_j := B'(p_j) \subseteq \mathbf{R}^{V_j}$. Let $B_j \subseteq \mathbf{R}^V$ be the direct sum of B'_j and the zero vector of $\mathbf{R}^{V - V_j}$. It follows that an integer vector z is in B_j if and only if $z(v) = 0$ for $v \in V - V_j$ and there is a k_j-edge-connected orientation of G_j such that the in-degree of each $v \in V_j$ is $z(v)$. Let B^* denote the sum of $B_1, \ldots, B_t$. Note that both B^* and B_0 are integral base-polyhedra.

We claim that the intersection of B^* and B_0 is non-empty. Indeed, let $m(v) := d_E(v)/2$ and $m_j(v) := d_{E_j}(v)/2$ for each $v \in V$. Note that $m_j(v) = 0$ when $v \in V - V_j$. Since $\{E_1, \dots, E_t\}$ is a partition of E, we have $m(v) = \sum[m_j(v) : j = 1, \dots, t]$, $\widetilde{m}(V) = |E|$, and $\widetilde{m}_j(V) := |E_j|$. Furthermore $\widetilde{m}(X) = \sum[m(v) : v \in X] = \sum[d_E(v)/2 : v \in X] == d_G(X)/2 + i_G(X) \geq k + i_G(X) = p_0(X)$ for each $X \subset V$, and hence $m \in B_0$.

In order to prove that $m_j \in B_j$ for $j = 1, \dots, t$, it suffices to show that the restriction $m'_j := m_j|V_j$ is in B'_j. For $X \subseteq V_j$, we have $\widetilde{m}_j(X) = \sum[m_j(v) : v \in X] = \sum[d_{G_j}(v)/2 : v \in X] = d_{G_j}(X)/2 + i_{G_j}(X) \geq k_j + i_{G_j}(X) = p_j(X)$, and hence $m_j \in B_j$ from which $m \in B^*$ follows.

Since the intersection of two integral base-polyhedra is an integral polyhedron, it follows that $B_0 \cap B^*$ has an integer-valued element m. By the Sum theorem of g-polymatroids, m can be expressed as $m = m_1 + m_2 + \cdots + m_t$ where each m_j is an integral element of B_j.

Now each m_j is an in-degree vector of a k_j-edge-connected orientation of G_j. These vectors determine an orientation of G whose in-degree function is $m = m_1 + m_2 + \cdots + m_t$. But m belongs to B_0 as well and hence this orientation of G is k-edge-connected. •

What Király and Szigeti actually proved was the following extension of the strong form of Nash-Williams' orientation theorem.

Theorem 15.4.11 (Király and Szigeti) *Let $G = (V, E)$ be an undirected graph and let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \dots, G_t = (V_t, E_t)$ be subgraphs of G such that $\{E_1, \dots, E_t\}$ is a partition of E. Then G has a best-balanced orientation whose restriction to E_j is a best-balanced orientation of G_j for each $j = 1, \dots, t$.* •

Problem 15.4.2 *Derive the following common generalization of Theorem* 15.4.10 *and Corollary* 15.4.8.

Theorem 15.4.12 *Let $G = (V, E)$ be a $2k$-edge-connected graph and let $\{E_0, E_1, \dots, E_t\}$ be a partition of E such that $G_0 = (V, E_0)$ is Eulerian and each subgraph $G_j = (V, E_j)$ is $(2k_j)$-edge-connected $(j = 1, \dots, t)$. Then G has a k-edge-connected orientation such that its restriction to E_0 is an Euler orientation of G_0 and its restriction to E_j is a k_j-edge-connected orientation of G_j for $j = 1, \dots, t$.*

15.4.4 Intersecting G*-supermodular demand functions*

Finding a rooted k-edge-connected orientation of an undirected graph is a special case of the abstract orientation problem when h is crossing (and, in fact, intersecting) G-supermodular and non-increasing. This explains why the linking property held in this case. In Section 9.5 we proved Theorem 9.5.1 that characterized mixed graphs admitting a k-edge-connected orientation. This theorem is not a special case of Theorem 15.4.1 but it still has the following abstract extension [113].

Theorem 15.4.13 *Let $G = (V, E)$ be an undirected graph and h an integer-valued intersecting G-supermodular function* (with possible negative values). *There is an orientation of G covering h if and only if*

$$e'_G(\mathcal{P}) \geq \sum[h(V_i) : i = 1, \dots, t] \text{ holds for every subpartition } \mathcal{P} = \{V_1, \dots, V_t\} \text{ of } V \tag{15.57}$$

where $e'_G(\mathcal{P})$ denotes the number of edges of G entering at least one member of $\mathcal{P}$.

Proof. For an orientation (with in-degree function ϱ) covering h, we have $e'_G(\mathcal{P}) \geq \sum[\varrho(V_j) : j = 1, \dots, t] \geq \sum[h(V_j) : j = 1, \dots, t]$, and hence the condition is necessary.

To prove sufficiency, consider the set-functions $p := h + i_G$ and $b := e_G$. Then b is fully submodular, p is intersecting supermodular, and $p(V) = b(V) = |E|$. Moreover, (15.57) implies for any subpartition $V_1, \dots, V_t$ of V that $\sum p(V_j) = \sum[h(V_j) + i_G(V_j)] \leq e_G(Z) = b(Z)$ where $Z = V_1 \cup \dots \cup V_t$. It follows from Theorem 15.1.7 that there is an integer-valued function $m : V \to \mathbf{Z}$ for which $p(X) \leq \widetilde{m}(X) \leq b(X)$ for every $X \subseteq V$. It follows that $\widetilde{m}(V) = |E|$ and hence the Orientation lemma implies that G has an orientation for which $\varrho(v) = m(v)$ for every $v \in V$. But then $\widetilde{m}(X) \geq p(X)$ implies $\varrho(X) = \sum[\varrho(v) : v \in X] - i_G(X) = \widetilde{m}(X) - i_G(X) \geq [h(X) + i_G(X)] - i_G(X) = h(X)$, and hence the orientation covers h. •

Remark 15.4.1 One may be wondering whether the intersecting supermodularity of h in the theorem could be weakened to near supermodularity. The proof above certainly cannot be carried over since the sum of a near-supermodular function h and a fully supermodular function i_G need not be near-supermodular.

Recall Theorem 9.5.1, which asserted that *a mixed graph $M = D + V$ with a designated root-node $r_0 \in V$ has a rooted k-edge-connected orientation if and only if*

$$e_G(\mathcal{P}) \geq \sum_{j=1}^{p}[k - \varrho_D(V_j)] \text{ holds for every partition } \mathcal{P} := \{V_0, V_1, \dots, V_p\} \text{ of } V \quad (15.58)$$

where $r_0 \in V_0$.

To see that this result is indeed a special case of Theorem 15.4.13, define h by

$$h(X) := \begin{cases} k - \varrho_D(X) & \text{if } \emptyset \subset X \subseteq V - r_0 \\ 0 & \text{if } X = \emptyset \\ -\infty & \text{otherwise.} \end{cases} \quad (15.59)$$

Then h is intersecting supermodular and an easy calculation shows that (15.58) is equivalent to (15.57).

The linking property did hold for (k, ℓ)-edge-connected orientations of undirected graphs since the in-degree vectors of such orientations span a base-polyhedron. The reason why the linking property does not hold in general for (k, ℓ)-edge-connected orientations of mixed graphs is that the in-degree vectors in this case span the intersection of *two* base-polyhedra.

Connectivity orientation problems remained to be settled

In this section we proved two abstract orientation results that were not comparable. In the first one, the demand set-function h to be covered was crossing supermodular and non-negative while in the second one h was intersecting supermodular (with possible negative values). These two abstract theorems include almost all specific orientation results (concerning feasiblity) in the first two parts of the book. In the light of this generality, it is a bit strange, however, that one of the simplest and earliest results on connectivity orientations is not a special case. Namely, Theorem 2.2.12 on strong orientability of a mixed graph cannot be formulated in terms of intersecting supermodular or non-negative crossing supermodular functions. What is worse is that we have not been able so far even to formulate a conjecture for the existence of a k-edge-connected orientation of a mixed graph. This specific question motivates us to try to combine the two abstract theorems above into one in which the demand set-function is crossing supermodular without the non-negativity assumption.

Beside this goal, we also want to develop methods for considering minimum-cost orientations of given properties. The right tool to achieve these goals is the framework of submodular flows, the content of the next chapter.

16
Submodular flows

In Chapter 12 we introduced the notion of submodular flows and proved a feasibility theorem (Theorem 12.1.4) for characterizing non-empty submodular flow polyhedra confined by a fully submodular function. This result implied the Discrete separation theorem which turned out to be equivalent to the g-polymatroid intersection theorem. Combined with the Truncation theorem, this latter result gave rise to Fujishige's theorem on the non-emptiness of a base-polyhedron bounded by a crossing submodular function.

As far as optimization problems are concerned, we studied the greedy algorithm for g-polymatroids and the much deeper problem of maximum-weight common independent sets of two matroids. We proved Edmonds' theorem stating that the solution set of the linear system

$$\{x \geq 0,\ \widetilde{x}(Z) \leq r_1(Z),\ \widetilde{x}(Z) \leq r_2(Z) \text{ for every } Z \subseteq S\}$$

is the convex hull of common independent sets; moreover, the system is totally dual integral (see p. 427). In this chapter, we extend this result to g-polymatroid intersections and to submodular flows.

16.1 Submodular and supermodular flows

A general goal of polyhedral combinatorics is to find polyhedral descriptions of integral polytopes. Any projection of an integral polytope P is always integral, but even if we do know a polyhedral description for P, finding one for any non-trivial projection can be a difficult task. A notorious example is the matching polytope P of a non-bipartite graph: a classic theorem of Edmonds (not discussed in this book) provides a polyhedral description for P, but no such description is known for the projection of P. In sharp contrast with this, we proved in Theorem 14.2.2 that the projection of a g-polymatroid (along a coordinate axis) is itself a g-polymatroid. Theorem 14.4.3 asserted that the intersection Q of two integral g-polymatroids is an integral polyhedron. The question naturally arises: what is a polyhedral description for the projection of Q? As an elegant link, we shall prove that submodular flow polyhedra are exactly these projections.

Let $D = (V, A)$ be a directed graph, $f : A \to \mathbf{Z} \cup \{-\infty\}$, $g : A \to \mathbf{Z} \cup \{\infty\}$ two integer-valued bounding functions for which $f \leq g$. Moreover, we are given an integer-valued

978–0–19–920527–1 16-Andras-Frank-c16-drv Andras Frank (Typeset by SPi, Publisher Services, India) 525 of 640 December 11, 2010 15:21
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

crossing submodular set-function $b : 2^V \to \mathbf{Z} \cup \{\infty\}$ for which $b(\emptyset) = 0$ and $b(V)$ is finite. Recall from Section 12 that a function (or vector) $x : A \to \mathbf{R}$ was called a submodular flow or just a subflow confined by b if

$$\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z) \leq b(Z) \text{ for every } Z \subseteq V. \tag{16.1}$$

Recall that $\dot{\Psi}$ denotes the restriction of Ψ to singletons. Note that $\dot{\Psi}_x = Q_D x$ where Q_D denotes the $(0, \pm 1)$-valued incidence matrix of D. Since $\Psi_x(V) = 0$ for an arbitrary x, the value $b(V)$ can be revised to be zero. Therefore we assume throughout that $b(V) = 0$. Recall that a subflow x is feasible if

$$f \leq x \leq g. \tag{16.2}$$

The set $Q(f, g; b)$ of feasible subflows was previously called a submodular flow polyhedron. Recall that in Proposition 3.4.1 we observed that the set-function Ψ_x on V is modular for every $x \in \mathbf{R}^A$.

16.1.1 Subflows in different forms

The use of crossing submodular functions in the definition of subflows ensures large flexibility and wide applicability of the framework. On the other hand, for the sake of easier proofs and simpler algorithmic descriptions, it is useful to define subflows with stronger confining functions. We are going to show that the feasibility restriction $f \leq x \leq g$ can be built into the confining crossing submodular function b. Moreover, b can be replaced by a fully submodular function.

Subflows in free form

We say that a subflow polyhedron is given in a **free form** if D consists of disjoint directed edges, $f \equiv -\infty$, and $g \equiv \infty$.

Theorem 16.1.1 *Every subflow polyhedron can be given in a free form.*

Proof. Let $Q(f, g; b)$ be a subflow polyhedron defined on digraph $D = (V, A)$. Suppose that there is a node v of D which is incident to exactly one edge e of D. If e enters v, then $\Psi_x(v) = x(e)$ and hence the inequality $\Psi_x(v) \leq b(v)$ is equivalent to $x(e) \leq b(v)$. Therefore we can assume for such an edge that $g(e) \leq b(v)$. Similarly, if e leaves v, then $\Psi_x(v) = -x(e)$ and hence the inequality $\Psi_x(v) \leq b(v)$ is equivalent to $x(e) \geq -b(v)$. Therefore we can assume for such an edge that $f(e) \geq -b(v)$. These properties are implicitly assumed in definition (16.3) below.

Create a digraph $D^* = (V^*, A^*)$ as follows. With each edge $e = uv \in A$, associate two new nodes: a head-node e_v and a tail-node e_u. Let V^* denote the set of these nodes and let $A^* := \{e_u e_v : e = uv \in A\}$. Define f^* and g^* on A^* to be identically $-\infty$ and ∞, respectively.

For every subset $Z \subseteq V$, let

$$F(Z) := \{e_v \in V^* : e = uv \in A,\ v \in Z\} \cup \{e_u \in V^* : e = uv \in A,\ u \in Z\}.$$

Let

$$\mathcal{F} := \{F(Z) : Z \subseteq V\} \cup \{e_v : e = uv \in A\} \cup \{V^* - e_u : e = uv \in A\}.$$

Then $\mathcal{F}$ is a crossing family. (Intuitively, D^* arises from D by detaching every node v into $\varrho_D(v) + \delta_D(v)$ new nodes. $F(Z)$ denotes the set of new nodes corresponding to the $Z \subseteq V$.) Define a function b^* on $\mathcal{F}$ as follows.

$$b^*(X) := \begin{cases} b(Z) & \text{if } X = F(Z) \text{ for some } Z \subset V,\ 1 < |X| < |V^*| - 1 \\ g(e) & \text{if } X = \{e_v\} \text{ for some } e = uv \in A \\ -f(e) & \text{if } X = V^* - \{e_u\} \text{ for some } e = uv \in A \end{cases} \tag{16.3}$$

Then b^* is a crossing submodular function on $\mathcal{F}$, and the subflow polyhedron in free form confined by b^* on D^* is identical with $Q(f, g; b)$. •

Subflows versus base-polyhedra

It was proved in Theorem 15.3.5 that $B(b)$ is a base-polyhedron whenever b is crossing submodular. Therefore Proposition 3.4.1 implies that a function $x : A \to \mathbf{R}$ is a subflow if and only if the vector $\dot{\Psi}_x : V \to \mathbf{R}$ is in the base-polyhedron $B(b)$. A base-polyhedron can be given in other forms. For example, if p is a crossing supermodular function, then $B'(p)$ is a base-polyhedron and hence the polyhedron defined by (16.2) and by

$$\varrho_x(Z) - \delta_x(Z) \geq p(Z) \text{ for every } Z \subseteq V \tag{16.4}$$

is also a subflow polyhedron, denoted by $Q'(f, g; p)$. Based on this observation, one can also speak of **supermodular flows**. In the special case when $f \equiv -\infty$ and $g \equiv \infty$, $Q(f, g; b)$ and $Q'(f, g; p)$ are denoted by $Q(D; b)$ and $Q'(D; p)$, respectively.

In addition, if (p, b) is an intersecting paramodular pair, then the intersection of the g-polymatroid $Q(p, b)$ with the hyperplane $\{z \in \mathbf{R}^V : \widetilde{z}(V) = 0\}$ is also a 0-base-polyhedron and hence

$$\{x \in \mathbf{R}^A : \dot{\Psi}_x \in Q(p, b)\} = \{x \in \mathbf{R}^A : p(Z) \leq \varrho_x(Z) - \delta_x(Z) \leq b(Z) \text{ for every } Z \subseteq V\}$$

is also submodular flow polyhedron. In what follows, the term **semimodular flow** will be used when we do not want to specify how the base-polyhedron in question is given.

Yet another consequence of Theorem 15.3.5 is that each non-empty subflow polyhedron can be defined by a fully submodular confining function. Namely, if $Q(f, g; b)$ is non-empty for a crossing submodular function b, then $B(b)$ is non-empty and $B(b) = B(b^{\downarrow})$, from which $Q(f, g; b) = Q(f, g; b^{\downarrow})$ where $b^{\downarrow}$ denotes the full truncation of b.

As an application, we derive the following useful property of subflow polyhedra [62].

Theorem 16.1.2 *A face of a subflow polyhedron Q is a subflow polyhedron.*

Proof. We can assume that Q is given in free form on a digraph $D = (V, A)$ and confined by a fully submodular function b with $b(V) = 0$. Let Z be a subset of nodes for which the face of Q defined by $Q_Z = \{x \in Q : \Psi_x(Z) = b(Z)\}$ is non-empty. It suffices to prove that Q_Z is a subflow polyhedron. To this end, consider the face of the base-polyhedron $B(b) \subseteq \mathbf{R}^V$ defined by $B' := \{z \in B(b) : \widetilde{z}(Z) = b(Z)\}$. By Theorem 14.2.6, B' is a base-polyhedron. Let b' denote the unique fully submodular border function of B'. Since x is in Q_Z if and only if the vector $\dot{\Psi}_x : V \to \mathbf{R}$ is in B', it follows that Q_Z is indeed a subflow polyhedron, namely, $Q_Z = \{x : \Psi_x(X) \leq b'(X) \text{ for every } X \subseteq V\}$. •

16.1.2 Total dual integrality

A fundamental result of Edmonds and Giles [86] on subflows is as follows.

Theorem 16.1.3 *The linear system given by* (16.2) *and by one of* (16.1) *or* (16.4) *is totally dual integral.*

Proof. We mentioned already that a subflow polyhedron can be defined by (16.4) in which p is a crossing supermodular function. Therefore it suffices to prove the theorem only for (16.1).

Let Q denote the $(0, \pm 1)$-valued matrix the columns of which columns correspond to the edges of D, while the rows correspond to the non-empty subsets $Z \subset V$ for which $b(Z)$ is finite, and an entry of Q corresponding to edge e and subset Z is 1 if e enters Z, -1 if e leaves Z, and 0 otherwise.

For the sake of technical simplicity, we assume that each of f, g, b is finite-valued. The proof goes along the same line in the general case with the remark that a primal inequality (as well as the corresponding dual variable) appears in the linear systems only if the right-hand side is finite-valued.

Consider the primal linear program

$$\max\{cx:\ Qx \le b,\ x \le g,\ -x \le -f\} \tag{16.5}$$

along with its dual program:

$$\min\left\{\sum_Z y(Z)b(Z) + \sum_{e\in A} y_g(e)g(e) - \sum_{e\in A} y_f(e)f(e)\right\} \text{ subject to} \tag{16.6}$$

$$(y, y_f, y_g) \ge 0,$$

$$\beta_y(e) + y_g(e) - y_f(e) = c(e) \text{ for every edge } e, \tag{16.7}$$

where

$$\beta_y(e) := \sum[y(Z) : Z \text{ is entered by } e] - \sum[y(Z) : Z \text{ is left by } e]. \tag{16.8}$$

Note that y uniquely determines the optimal dual variables y_g and y_f, namely, $y_f(e) := \max\{0, \beta_y(e) - c(e)\}$, $y_g(e) := \max\{0, c(e) - \beta_y(e)\}$.

Let $y_0 : 2^V - \{\emptyset\} \to \mathbf{R}_+$ be an optimal dual solution. Since any linear program has an optimal basic solution, we can assume that y_0 is rational. As long as there are two incomparable subsets X and Y with positive y_0-values for which b satisfies the submodular inequality, revise y_0 by applying the following uncrossing step. Let $\alpha := \min\{y_0(X), y_0(Y)\}$. Decrease the values of $y_0(X)$ and $y_0(Y)$ by α and increase the values of $y_0(X \cap Y)$ and $y_0(X \cup Y)$ by α. For the resulting y' we have $\beta_{y'} = \beta_{y_0}$ and hence y' is also dual feasible. Moreover, y' is also dual optimal since b was assumed to meet the submodular inequality for X and Y.

By Lemma 1.1.2, this uncrossing procedure terminates after a finite number of steps. Therefore we can assume that y_0 itself is an optimal dual solution for which the system $\mathcal{F}$ of sets with positive y_0-values is cross-free. By Theorem 4.2.10 the submatrix of Q formed by the rows corresponding to the members of $\mathcal{F}$ is totally unimodular and hence Lemma 4.2.18 implies the theorem. •

The proof of the theorem gives rise to the following useful property.

Corollary 16.1.4 *The dual linear program to the primal program defined by* (16.1) *and* (16.2) *admits an integer-valued optimum y such that the submodular inequality fails to hold for X and Y whenever X and Y are not comparable and* $y(X) > 0,\ \ y(Y) > 0$. •

16.1.3 Feasibility

Our next goal is to extend Theorem 12.1.4 to a characterization for the non-emptiness of a subflow polyhedron confined by intersecting and by crossing submodular flows. The next result is due to Frank [121].

Theorem 16.1.5 (Subflow feasibility) *Let* $f \leq g$ *be functions on the edge-set of a digraph* $D = (V, A)$ *and let b be a* (fully, intersecting, or crossing) *submodular set-function on V for which* $b(\emptyset) = b(V) = 0$. *A necessary and sufficient condition for the existence of a feasible submodular flow,*

(A) *when b is fully submodular, is that*

$$\varrho_f(Z) - \delta_g(Z) \leq b(Z) \text{ for every } Z \subseteq V, \tag{16.9}$$

(B) *when b is intersecting submodular, is that*

$$\varrho_f(Z) - \delta_g(Z) \leq \textstyle\sum_i b(V_i) \text{ holds for every } Z \subseteq V \text{ and every partition } \{V_1, \ldots, V_t\} \text{ of } Z, \tag{16.10}$$

(C) *when b is crossing submodular, is that*

$$\varrho_f(Z) - \delta_g(Z) \leq \sum[b(X) : X \in \mathcal{F}] \text{ for every subset } Z \subseteq V$$
$$\text{and every tree-composition } \mathcal{F} \text{ of } Z. \tag{16.11}$$

If each of f, g, b is integral and there is a feasible subflow, then there is an integral feasible subflow too.

Proof. Part (A) is just the repetition of Theorem 12.1.4.

(B) For a feasible subflow z, one has $\varrho_f(Z) - \delta_g(Z) \leq \varrho_z(Z) - \delta_z(Z) = \Psi_z(Z) = \sum_i \Psi_z(V_i) \leq \sum_i b(V_i)$, and hence Condition (16.10) is necessary. For sufficiency, let $b^{\vee}$ denote the lower truncation of b. This is fully submodular and (16.10), when applied to $Z = V$, implies that $b^{\vee}(V) = b(V)$ from which $B(b^{\vee}) = B(b)$.

The result follows from Theorem 16.1.5 once we observe that (16.9) described for $b^{\vee}$ is just equivalent to the condition in (16.10).

(C) Let x be a feasible subflow. Now the base-polyhedron $B(b)$ is non-empty since the vector $\dot{\Psi}_x$ is in $B(b)$. Then the full truncation $b^{\downarrow}$ of b exists and $B(b) = B(b^{\downarrow})$. Therefore $\Psi_x(Z) \leq b^{\downarrow}(Z)$ and hence $\varrho_f(Z) - \delta_g(Z) \leq \varrho_x(Z) - \delta_x(Z) = \Psi_x(Z) \leq b^{\downarrow}(Z)$ from which the necessity of (16.11) follows.

For sufficiency, observe first that Condition (16.11), when applied to $Z = V$, requires that $\sum[b(X) : X \in \mathcal{F}]$ for every partition and co-partition $\mathcal{F}$ of V. Fujishige's theorem (Theorem 15.3.8) implies that the base-polyhedron $B(b)$ is non-empty. Then the full truncation $b^{\downarrow}$ exists and $B(b) = B(b^{\downarrow})$. By Theorem 15.3.6, the Condition (16.9), when applied to $b^{\downarrow}$, is just equivalent to Condition (16.11). •

978–0–19–920527–1 16-Andras-Frank-c16-drv Andras Frank (Typeset by SPi, Publisher Services, India) 529 of 640 December 11, 2010 15:21

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Remark 16.1.1 The condition in Part (A) is particularly friendly since it requires an inequality for subsets. The partition-type condition in Part (B) is still palatable as it includes partitions. But the inequality concerning tree-compositions in Part (C) is rather complicated and may make the reader ponder whether a simpler inequality would not be enough in this case too. In Subsection 16.1.5, however, we shall point out that tree-compositions cannot be avoided.

Projection of subflows

As an application of the feasibility theorem, we prove the following result from [152].

Theorem 16.1.6 *Any projection of a subflow polyhedron Q* (along co-ordinate axis) *is a subflow polyhedron.*

Proof. By theorem 16.1.1, we can assume that Q is given in free form and Q is confined by a fully submodular function b. Suppose that $A' \subset A$ is a subset of edges and Q is projected to $\mathbf{R}^{A''}$ along the elements of A' where $A'' = A - A'$. Let Q'' denote the projection of Q. A vector x'' is in Q'' by definition if there is an $x' \in \mathbf{R}^{A'}$ such that $x = (x', x'')$ is in Q. Define f and g on A by $f(e') = -\infty$ and $g(e') = \infty$ for every $e' \in A'$, and $f(e'') := g(e'') := x''(e'')$ for every $e'' \in A''$. Then $x'' \in Q''$ if and only if the subflow polyhedron $Q(f, g; b)$ is non-empty. By Theorem 16.1.5, $Q(f, g; b)$ is non-empty if and only if $\varrho_f - \delta_g \leq b$, which is, for the given choice of f and g, equivalent to requiring that $\varrho_{x''}(Z) - \delta_{x''}(Z) \leq b(Z)$ for every $Z \in \mathcal{F}$ where $\mathcal{F} := \{Z \subseteq V : \varrho_{A'}(Z) = 0 = \delta_{A'}(Z)\}$. Since $\mathcal{F}$ is a crossing family, these vectors x'' form a subflow polyhedron. •

16.1.4 Intersection of two g-polymatroids revisited

In Section 14.4 we exhibited feasibility results concerning the intersection of two g-polymatroids. Here we explore the exact relationship between these intersections and subflow polyhedra.

Theorem 16.1.7 *Let (p_1, b_1) and (p_2, b_2) be two intersecting paramodular pairs. The intersection of g-polymatroids $Q_1 = Q(p_1, b_1)$ and $Q_2 = Q(p_2, b_2)$ is a subflow polyhedron. Moreover, the linear system $\{p_1(Z) \leq \widetilde{x}(Z) \leq b_1(Z)$ and $p_2(Z) \leq \widetilde{x}(Z) \leq b_2(Z)$: for every $Z \subseteq S\}$ is totally dual integral.*

Proof. Let S' and S'' be two disjoint copies of ground-set S and let $V := S' \cup S''$. For every element $s \in S$, let $s's''$ be a directed edge and let $D = (V, A)$ denote the digraph obtained in this way. Let $f \equiv -\infty$ and $g \equiv \infty$. Define a set-function b on V as follows. For $X \subseteq S$, let $b(X'') := b_2(X)$, $b(V - X'') := -p_1(X)$, $b(X') := -p_2(X)$, and $b(V - X') := b_1(X)$. It can easily be checked that b is crossing submodular and the subflow polyhedron $Q(f, g; b)$ is equal to the intersection $Q_1 \cap Q_2$ once we identify the elements of S and the edges of D. Similarly, the inequalities of the linear system $\{\varrho_x(X) - \delta_x(X) \leq b(X) : X \subseteq S\}$ describing the subflow polyhedron correspond to the inequalities in the system $\{p_1(Z) \leq \widetilde{x}(Z) \leq b_1(Z),\ p_2(Z) \leq \widetilde{x}(Z) \leq b_2(Z)$: for every $Z \subseteq S\}$ describing the intersection. •

Recall that for a finite-valued fully submodular set-function b and a cost function c, $\hat{b}(c)$ denoted $\max\{cx : x \in B(b)\}$ and the function $\hat{b}$ was called the linear extension of b.

978–0–19–920527–1 16-Andras-Frank-c16-drv Andras Frank (Typeset by SPi, Publisher Services, India) 530 of 640 December 11, 2010 15:21
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Corollary 14.5.3 provided a simple formula for $\hat{b}(c)$ and an optimizer x could be computed with the polymatroid greedy algorithm. Based on this, we derive the following min-max theorem for the intersection of two base-polyhedra.

Theorem 16.1.8 (Weight-splitting for intersection of base-polyhedra) *Suppose that b_1 and b_2 are finite-valued fully submodular functions for which $b_1(S) = b_2(S)$ and for which $B(b_1) \cap B(b_2)$ is non-empty. Then*

$$\max\{cx : x \in B(b_1) \cap B(b_2)\} = \min\{\hat{b}_1(c_1) + \hat{b}_2(c_2) : c_1 + c_2 = c\}. \tag{16.12}$$

If c is integer-valued, the optimal c_i can be chosen integer-valued.

Proof. By applying Theorem (16.1.7) to the base-polyhedra $B(b_1)$ and $B(b_2)$, we obtain that

$$\max\ \{cx : x \in B(b_1) \cap B(b_2)\}$$

$$= \min\ \left\{\sum[y_1(X)b_1(X) : X \subseteq S] + \sum[y_2(X)b_2(X) : X \subseteq S]\right\} \text{ subject to}$$

$$y_1(X) \geq 0 \text{ and } y_2(X) \geq 0 \text{ for every } X \subset S,$$

and

$$\sum[y_1(X)\underline{\chi}_X : X \subseteq S] + \sum[y_2(X)\underline{\chi}_X : X \subseteq S] = c.$$

Let y_1 and y_2 denote an optimal dual solution. Let $X_1, \ldots, X_h$ be the subsets of S for which $y_1(X) \neq 0$ and let $Y_1, \ldots, Y_k$ be the subsets of S for which $y_2(Y) \neq 0$. Since b_1 is fully submodular, by Corollary 16.1.4 we can assume that the sets X_i's form a chain: $X_1 \subset \cdots \subset X_h$. Similarly, we can assume that $Y_1 \subset \cdots \subset Y_k$. Let $c_1 := y_1(X_1)\underline{\chi}_{X_1} + \cdots + y_1(X_h)\underline{\chi}_{X_h}$ and $c_2 := y_2(Y_1)\underline{\chi}_{Y_1} + \cdots + y_2(Y_k)\underline{\chi}_{Y_k}$. Then the dual feasibility of (y_1, y_2) implies $c = c_1 + c_2$. Moreover, by applying formula (14.53) to b_1 and c_1 in place of b and c, we obtain that $\hat{b}_1(c_1) = \sum[y_1(X_i)b_1(X_i) : i = 1, \ldots, h]$. Similarly, we get $\hat{b}_2(c_2) = \sum[y_2(Y_i)b_2(Y_i) : i = 1, \ldots, k]$ from which (16.12) follows. •

Remark 16.1.2 Theorem 16.1.8 is a direct extension of the weight-splitting theorem on matroid intersection (Theorem 13.2.4). In Section 13.2, we gave a direct proof of Theorem 13.2.4 and used the theorem to derive the total dual integrality of the linear system

$$\{x \geq 0,\ \widetilde{x}(Z) \leq r_1(Z),\ \widetilde{x}(Z) \leq r_2(Z) \text{ for every } Z \subset S,\ \widetilde{x}(S) = r_1(S)\ (= r_2(S))\}.$$

Here we followed an opposite path by deriving the weight-splitting theorem for the intersection of base-polyhedra from the corresponding TDI-ness (total-dual-integrality) result.

Problem 16.1.1 *Show for fully submodular functions b_1 and b_2 that* $\max\{cx : x \in S(b_1) \cap S(b_2)\}$ *is bounded from above if and only if $c \geq 0$. If $c \geq 0$, then*

$$\max\ \{cx : x \in S(b_1) \cap S(b_2)\} = \min\ \{\hat{b}_1(c_1) + \hat{b}_2(c_2) : c_1 \geq 0, c_2 \geq 0,\ c_1 + c_2 = c\}.$$

If, in addition, b_1 and b_2 are non-negative and $c \geq 0$, then

$$\max\ \{cx : x \in P(b_1) \cap P(b_2)\} = \min\ \{\hat{b}_1(c_1) + \hat{b}_2(c_2) : c_1 \geq 0, c_2 \geq 0,\ c_1 + c_2 = c\}.$$

In both cases, when c is integral, the optimal c_1 and c_2 can be chosen integral too.

Theorem 16.1.7 shows that the intersection of two g-polymatroids is always a subflow polyhedron. The converse, however, is not true. To see this, let D be a digraph on two nodes consisting of two oppositely directed edges e and e'. Let $Q = \{x \in \mathbf{R}^2 : x(e) - x(e') = 0,\ 0 \leq x(e) \leq 1,\ 0 \leq x(e') \leq 1\}$, or informally, Q is a set of feasible circulations with respect to $f \equiv 0$ and $g \equiv 1$. (Note that Q is the line segment in $\mathbf{R}^2$ connecting $(0,0)$ and $(1,1)$.) This Q cannot be the intersection of g-polymatroids, since any g-polymatroid containing the origin and $(1,1)$ must contain each point (x,y) with $0 \leq x \leq 1,\ 0 \leq y \leq 1$. The next theorem from [118] shows that the situation radically changes when projection is also allowed.

Theorem 16.1.9 *Every subflow polyhedron $Q \subseteq \mathbf{R}^A$ defined on a digraph $D = (V, A)$ can be obtained as a projection of the intersection of two $2|A|$-dimensional base-polyhedra.*

Proof. We can assume that Q is non-empty, given in free from, and its confining function b is fully submodular. By definition $D = (V, A)$ consists of disjoint edges. Let $H(A)$ and $T(A)$ denote the sets of head-nodes and tail-nodes of the edges in A, respectively. Then there is a one-to-one correspondence between the elements of A and $H(A)$. Consider the base-polyhedron $B_1 = B(b)$ and let B_2 be another base-polyhedron defined by $B_2 = \{z \in \mathbf{R}^V : z(u) + z(v) = 0 \text{ for each } uv \in A\}$. Let Q' denote the projection of $B_1 \cap B_2$ into $\mathbf{R}^{H(V)}$.

For every element $x \in Q$, define a vector $z : H(V) \to \mathbf{R}$ as follows. Let $z(v) := x(e)$ and $z(u) := -x(e)$ for $e = uv \in A$. Then $z \in B_1 \cap B_2$. Conversely, for a vector $z \in B_1 \cap B_2$, define $x \in A \to \mathbf{R}$ by $x(e) := z(v)$ for $e = uv \in A$. Then we have $x \in Q$. Hence Q is identical with Q'. •

By combining Theorems 16.1.6 and 16.1.9, we obtain the following result of [152].

Theorem 16.1.10 *A polyhedron Q is a subflow polyhedron if and only if Q can be obtained as a projection of the intersection of two g-polymatroids.*•

Application to covering directed cuts

We show that Theorem 9.7.2 of Lucchesi and Younger is an easy consequence.

Theorem 16.1.11 (Lucchesi and Younger) *In a digraph the minimum cardinality of a dijoin is equal to the maximum number of edge-disjoint dicuts.*

Proof. Let $\mathcal{F}$ denote the family of non-empty proper subsets X of V for which $\delta_D(X) = 0$. Let p be the identically 1 function on $\mathcal{F}$. Then $\mathcal{F}$ is a crossing family and p is crossing supermodular. By applying the supermodular part of Theorem 16.1.3 to $c \equiv 1$, the theorem follows. •

One can formulate an analogous min-max theorem concerning a maximum packing of dicuts of minimum cardinality.

Theorem 16.1.12 *Suppose that each dicut of a digraph has at least k edges. The minimum number of edges covering all k-element dicuts is equal to the maximum number of disjoint k-element dicuts.*

Proof. Let $\mathcal{F}$ denote the family of non-empty proper subsets X of V for which $\delta_D(X) = 0$ and $\varrho_D(X) = k$. Let p be the identically 1 function on $\mathcal{F}$. Then $\mathcal{F}$ is a crossing family and p is crossing supermodular. The theorem follows by applying the supermodular part of Theorem 16.1.3 to $c \equiv 1$. •

Problem 16.1.2 *Prove that in a simple planar digraph the maximum number of edge-disjoint directed triangles is equal to the minimum number of edges covering all directed triangles.*

16.1.5 Graph-orientations via submodular flows

In Section 15.4 we discussed abstract orientation problems in which we wanted to orient an undirected graph $G = (V, E)$ in such a way that the resulting digraph covers a set-function h. With the help of base-polyhedra, the orientation problem has been solved in two non-comparable special cases. In the first one h was non-negative and crossing G-supermodular, in the second one h was intersecting G-supermodular with possible negative values.

Our present goal is to show that the common generalization, when h is crossing G-supermodular (with possible negative values), can be treated with the help of submodular flows. To this end, consider an arbitrary orientation $D = (V, A)$ of G. This will serve as a reference orientation of G and the requested orientation is obtained by appropriately reorienting some of the edges of D. A reorientation will be described with the help of a vector $x : A \to \{0, 1\}$ where $x(a) = 1$ means that the directed edge a gets reoriented while $x(a) = 0$ means that a is left alone. Therefore, after the reorientation defined by x, the in-degree of a subset Z will be $\varrho_A(Z) - \varrho_x(Z) + \delta_x(Z)$. It follows that an $x : A \to \{0, 1\}$ determines an orientation of G covering a set-function h if and only if $\varrho_A(Z) - \varrho_x(Z) + \delta_x(Z) \geq h(Z)$ holds for each $Z \subseteq V$. This is, in turn, equivalent to requiring

$$\varrho_x(Z) - \delta_x(Z) \leq b(Z) := \varrho_A(Z) - h(Z). \tag{16.13}$$

When h is crossing G-supermodular, the set-function b defined in (16.13) is crossing submodular. Therefore the solution set of the linear system $\{\varrho_x(Z) - \delta_x(Z) \leq b(Z),\ 0 \leq x \leq 1\}$ is a subflow polyhedron Q. Since h is integral-valued, so is b and hence Q is an integral polyhedron by Theorem 16.1.3. It follows that there is an orientation of G covering h precisely if Q is non-empty.

For example, if h is identically k on the non-empty proper subsets of V, then the vector $x \in \mathbf{R}^A_+$ with all components $1/2$ belongs to Q if and only if G is $2k$-edge-connected. Therefore in this case Q is non-empty implying that it has a $(0, 1)$-valued element, and thus a $(2k)$-edge-connected graph has a k-edge-connected orientation. This result is the weak form of the orientation theorem of Nash-Williams.

The same approach can be used to solve the following optimization problem. Given a digraph $D = (V, A)$ for which the underlying undirected graph is $2k$-edge-connected, determine the minimum number of edges of D for which the reorientation results in a k-edge-connected digraph. By using the duality theorem of linear programming and Theorem 16.1.3, it is possible to formulate a min-max theorem for the minimum in question, though this is not so elegant than the one for the special case $k = 1$ which is a reformulation of the theorem of Lucchesi and Younger.

For general crossing G-supermodular functions, the feasiblity theorem for subflows (Theorem 16.1.5) can be applied as well as the algorithms for subflows to be developed. For example, Part (B) immediately gives rise to Theorem 15.4.13 which was earlier derived from g-polymatroid intersection. With the help of Part (C), we arrive at the promised common generalization of Theorems 15.4.1 and 15.4.13.

Theorem 16.1.13 *Let $G = (V, E)$ be undirected graph and h a crossing G-supermodular function for which $h(\emptyset) = h(V) = 0$. There is an orientation of G covering h if and only if*

$$\widetilde{h}(\mathcal{F}) \leq \sum_{e \in E} w_e(\mathcal{F}) \tag{16.14}$$

holds for every subset Z of V and for every tree-composition $\mathcal{F}$ of Z where, for an edge $e = uv \in E$, $w_e(\mathcal{F})$ denotes the maximum of the number of $u\bar{v}$-sets in $\mathcal{F}$ and the number of $v\bar{u}$-sets in $\mathcal{F}$.

Proof. Consider an arbitrary reference orientation $D = (V, A)$ of G. A function $x : A \to \{0, 1\}$ determines an orientation of G covering h if $\varrho_A(Z) - \varrho_x(Z) + \delta_x(Z) \geq h(Z)$ for every $Z \subseteq V$, which is equivalent to

$$\varrho_x(Z) - \delta_x(Z) \leq b(Z) := \varrho_A(Z) - h(Z). \tag{16.15}$$

Since this function b is crossing submodular the system $\{\varrho_x(Z) - \delta_x(Z) \leq b(Z),\ 0 \leq x \leq 1\}$ describes a subflow polyhedron. It can be checked easily that Condition (16.11) in Theorem 16.1.5 is equivalent to (16.14). Therefore Q has an integral element which defines an orientation of G covering h. •

Application: k-edge-connected orientations of mixed graphs

As an easy extension of Robbins' classic orientation theorem, Theorem 2.2.12 asserted roughly that the natural cut condition is necessary and sufficient for a mixed graph to admit a strong orientation. For $k \geq 2$, however, we pointed out in Subsection 9.5.2 that not only a simple cut condition is not sufficient but neither are the more complicated partition or co-partition type conditions. In this light we must get accustomed to the fact that a necessary and sufficient condition in this case unavoidably includes families of subsets more complicated than partitions or co-partitions.

As a main application of the theory, we are going to answer Question 9.5.2 on the existence of a k-edge-connected orientation of mixed graphs. Let $M = D + G$ be a mixed graph consisting of $G = (V, E)$ and $D = (V, A)$. Define a demand function h by $h(\emptyset) = h(V) = 0$ and $h(X) = k - \varrho_D(X)$ for $\emptyset \subset X \subset V$. Then h is crossing supermodular and an orientation $\vec{G} = (V, \vec{E})$ of G covers h precisely if $(V, A + \vec{E})$ is k-edge-connected. Theorem 16.1.13, when applied to this h, provides a necessary and sufficient condition for a mixed graph to admit a k-edge-connected orientation. Obviously, (k, ℓ)-edge-connected orientability of mixed graphs can also be reduced to Theorem 16.1.13 in an analogous way.

Return to the example given in Subsection 9.5.2 (see Figure 9.3). In the example the sets $V_1 = \{v_2\},\ V_2 = \{v_1, v_2, v_3\},\ V_3 = \{v_1, v_4\}$ form a tree-composition of $Z := \{v_1, v_2\}$. Indeed, consider the partition $\{\{v_1\}, \{v_2\}\}$ of Z and the partition $\{\{v_3\}, \{v_4\}\}$ of $V - Z$. Then the three directed edges $f_1 = v_3v_2,\ \ f_2 = v_4v_1,\ \ f_3 = v_3v_1$ form a directed tree which defines the tree-composition $\mathcal{F} := \{V_1, V_2, V_3\}$ of Z. Now we have $\widetilde{h}(\mathcal{F}) = \sum_{i=1}^{3}$

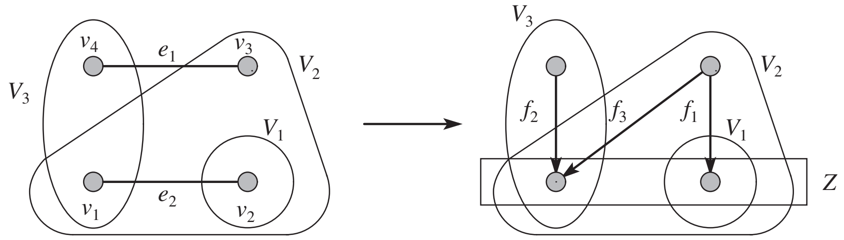


Figure 16.1 $w_{e_1}(\mathcal{F}) + w_{e_2}(\mathcal{F}) = 1 + 1 \not\geq 1 + 1 + 1 = h(V_1) + h(V_2) + h(V_3)$

$[k - \varrho_D(V_i)] = \sum_{i=1}^{3}[2-1] = 3$ on the one hand, and $\sum_{e\in E} w_e(\mathcal{F}) = 1 + 1 = 2$ on the other, and hence the condition in Theorem 16.1.13 fails to hold. See Figure 16.1.

16.2 A push–relabel algorithm for submodular flows

Fujishige and Zhang [165] developed a push–relabel algorithm for finding a feasible submodular flow. Our present goal is to exhibit a simplified push–relabel algorithm [147] that is a straight generalization of the one concerning modular flows (see Section 6.1). The algorithm either finds a feasible submodular flow or constructs a certificate for the non-existence. Such a certificate was described by the feasibility theorem for subflows (Theorem 16.1.5). Suppose that $D = (V, A)$ is a digraph on n nodes endowed with bounding functions $f : A \to \mathbf{R} \cup \{-\infty\}$ and $g : A \to \mathbf{R} \cup \{\infty\}$ for which $f \leq g$. Recall that a function $x : A \to \mathbf{R}$ is feasible if $f \leq x \leq g$. The net-inflow Ψ_x was defined as a set-function by

$$\Psi_x(Z) = \varrho_x(Z) - \delta_x(Z) \quad (Z \subseteq V)$$

and we noticed that Ψ_x is modular. Finally, x was called a subflow confined by b if $\Psi_x \leq b$ for a given crossing submodular function b. Recall that for a function $m : V \to \mathbf{R}$, we defined its modular extension $\widetilde{m}$ by $\widetilde{m}(X) := \sum[m(v) : v \in X]$ while for a set-function h restriction $\dot{h}$ of h to the elements of V was defined by $\dot{h}(v) := h(\{v\})$.

Due to the modularity of Ψ_x, there is a feasible subflow precisely if there is an element m of the base-polyhedron $B(b)$ for which there is a feasible m-flow, and our approach will rest on this observation. Namely, the basic frame of the algorithm is that we separately maintain a feasible vector x and a member m of $B(b)$. During the course of the algorithm, we revise suitably both x and m to make $\dot{\Psi}_x$ and m eventually equal. Once this goal has been achieved, the algorithm terminates by returning the final x as a feasible subflow. To qualify the relationship of the current x and m, we refer to a node v as **Ψ-larger**, **Ψ-smaller**, or **neutral** according to whether $\Psi_x(v) - m(v)$ is positive, negative, or zero, respectively.

First we concentrate on the special case when the *confining function b is fully submodular*. We can assume that $b(V) = 0$ for if $b(V) < 0$, then Q is certainly empty while if $b(V) > 0$, then lowering $b(V)$ to zero does not change Q and preserves submodularity. The algorithm is a constructive proof of the non-trivial direction of Part (A) of Theorem 16.1.5 which stated the following.

When the confining function b is fully submodular, there is a feasible subflow if and only if

$$\varrho_f(Z) - \delta_g(Z) \leq b(Z) \text{ for every } Z \subseteq V. \tag{16.16}$$

If each of f, g, b is integral, then (16.16) *implies the existence of an integral feasible subflow.*

The procedure is a straight extension of the push–relabel algorithm exhibited in Section 6.1 for computing a feasible m-flow where $m : V \to \mathbf{R}$ was a given vector with $\widetilde{m}(V) = 0$. An edge e is **decreasable** or **increasable** when $x(e) > f(e)$ or $x(e) < g(e)$, respectively. When b is a modular set-function determined by m, that is, $b(X) = \widetilde{m}(X)$ for every $X \subseteq V$, then the base-polyhedron $B(b)$ consists of the single element m showing that the polyhedron of m-flows is a special subflow polyhedron. Like in the special case of m-flows, we can assume that there are no parallel edges of D and hence $|A| \leq n^2$.

In the present algorithm, as indicated above, not only the current x is changing (while preserving feasibility) but $m \in B(b)$ as well. Given such an m, a subset $X \subseteq V$ is m**-tight** (or just **tight**) if $\widetilde{m}(X) = b(X)$. The ground-set V is certainly tight as $m \in B(b)$. Since b is fully submodular, the set of tight sets is closed under taking intersection and union. Therefore there is a unique smallest tight set $T(u)$ containing u for every node $u \in V$. For any two distinct elements u and v of V, define

$$\Delta(u, v) := \min\{b(X) - \widetilde{m}(X) : u \in X \subseteq V - v\}. \tag{16.17}$$

Then Δ is non-negative since $m \in B(b)$, moreover, $\Delta(u, v) > 0$ if and only if $v \in T(u) - u$. The algorithm below can be used when a subroutine is available for computing Δ.

16.2.1 Preparations: level properties and stopping rules

At a general stage of the algorithm, a triplet (x, m, Θ) is available where x is a feasible vector, m is a member of $B(b)$, and $\Theta : V \to \{0, 1, \ldots, n\}$ is a level function. The value $\Theta(v)$ is the level of node v. The algorithm advances stage by stage by basic operations to be described below.

By the **level** $\Theta(X)$ **of a** (non-empty) **subset** $X \subseteq V$, we mean the lowest level of its nodes: $\Theta(X) = \min\{\Theta(v) : v \in X\}$. For a level $\ell \in \{0, 1, \ldots, n\}$, the set $L_\ell := \{v \in V : \Theta(v) = \ell\}$ is called a **level set**. Consider the following **level properties**.

(LP1) Every Ψ-smaller node is on the lowest level, that is, in L_0.

(LP2′) $\Theta(v) \geq \Theta(u) - 1$ for every increasable edge uv. (Intuitively, each increasable edge steps down at most one level.)

(LP2″) $\Theta(v) \leq \Theta(u) + 1$ for every decreasable edge uv. (Intuitively, each decreasable edge steps up at most one level.)

(LP3) $\Theta(T(v)) \geq \Theta(v) - 1$ for every node v.

A triplet (x, m, Θ) is **feasible** if $f \leq x \leq g$, $m \in B(b)$, and Θ fulfils the level properties.

Lemma 16.2.1 *Let (x, m, Θ) be a feasible triplet. Suppose that L_ℓ is an empty level and $Z := \{v : \Theta(v) \geq \ell\}$ is non-empty. Then Z is m-tight and $\Psi_x(Z) = \varrho_f(Z) - \delta_g(Z)$.*

Proof. The emptiness of L_ℓ and Property (LP3) imply that $T(v) \subseteq Z$ for every $v \in Z$. It follows that the union $\cup(T(u) : u \in Z)$ is equal to Z. Since the union of tight sets is tight, we can conclude that $\widetilde{m}(Z) = b(Z)$.

The emptiness of L_ℓ implies that every edge e leaving Z steps down at least two levels and hence (LP2′) implies $x(e) = g(e)$, that is, $\delta_x(Z) = \delta_g(Z)$. Similarly, every edge e entering Z steps up at least two levels and hence (LP2″) implies $x(e) = f(e)$, that is, $\varrho_x(Z) = \varrho_f(Z)$. Therefore $\Psi_x(Z) = \varrho_f(Z) - \delta_g(Z)$. •

The algorithm terminates when one of the following two **stopping rules** occurs.

(A) There is no more Ψ-larger node.
(B) There exists a Ψ-larger node z and an empty level set L_ℓ under z (where $\ell < \Theta(z)$).

Lemma 16.2.2 *Let (x, m, Θ) be a feasible triplet. Then* **(A)** *implies that x is a feasible subflow while* **(B)** *implies that the set $Z := \{v \in V : \Theta(v) > \ell\}$ violates* (16.16).

Proof. If there is no Ψ-larger node, then $\Psi_x(Y) = \sum[\Psi_x(v) : v \in Y] \le \sum[m(v) : v \in Y] = \widetilde{m}(Y) \le b(Y)$ for every $Y \subseteq V$, and hence x is a feasible subflow.

Suppose now that **(B)** holds. By Lemma 16.2.1, we have $\widetilde{m}(Z) = b(Z)$ and $\Psi_x(Z) = \varrho_f(Z) - \delta_g(Z)$. Since all the Ψ-smaller nodes are in L_0 by (LP1), Z contains no Ψ-smaller node. But Z does contain the Ψ-larger z and therefore $\Psi_x(Z) = \sum[\Psi_x(v) : v \in Z] > \widetilde{m}(Z) = b(Z)$, implying that $\varrho_f(Z) - \delta_g(Z) = \Psi_x(Z) > b(Z)$, and hence (16.16) is violated. •

16.2.2 The algorithm for fully submodular b

At an intermediate stage, a feasible triplet (x, m, Θ) is available for which neither of the two stopping rules holds. The core subroutine of the algorithm is a **treatment** of a Ψ-larger node z that suitably changes x, m, and Θ. A treatment of z ends up either by making z neutral or by lifting the level of z by 1. It consists of applying three basic operations that are used several times. Two of them, edge-push and node-lift, have already been introduced at the push–relabel algorithm for m-flows. The new one, called node-push, is about manipulating m.

Three basic operations at a ψ-larger node z

1. **Edge-push at** z changes $x(e)$ on an edge e entering or leaving z, as follows.
(Increasing) If $e = zu$ is an increasable edge stepping down from z, then let $x' := x + \alpha \underline{\chi}_e$ where $\alpha := \min\{g(e) - x(e),\ \Psi_x(z) - m(z)\}$. Informally, we increase $x(e)$ by α.
(Decreasing) If $e = uz$ is a decreasable edge stepping up to z, then let $x' := x - \alpha \underline{\chi}_e$ where $\alpha := \min\{x(e) - f(e),\ \Psi_x(z) - m(z)\}$. Informally, we decrease $x(e)$ by α.
2. **Node-push at** z to change m applies only when $\Theta(T(z)) = \Theta(z) - 1$. It selects an *arbitrary* element $t \in T(z)$ for which $\Theta(t) = \Theta(z) - 1$ and revises m by $m' := m + \alpha(\underline{\chi}_z - \underline{\chi}_t)$ where $\alpha := \min\{\Psi_x(z) - m(z),\ \Delta(t, z)\}$. Informally, we decrease the value $m(t)$ with α and increase the value $m(z)$ with α. We will say that the node-push is carried out **along** (z, t). Also, this choice of t will be said to be **generic**, to be distinguished from a specific choice introduced and analysed later.

3. **Node-lift of** z is defined by $\Theta' := \Theta + \underline{\chi}_z$. Informally, we increase the level of z by 1, or simply we lift z.

An edge- or node-push at z is **neutralizing** if it converts z neutral. Otherwise (when z remains Ψ-larger), the push is **non-neutralizing**. The core subroutine of the algorithm is as follows.

Treating a Ψ-larger node z

Step 1 (edge-pushes at z) As long as z stays Ψ-larger and there is an increasable edge stepping down from z or a decreasable edge stepping up to z, apply an edge-push at z.

Step 2 (node-pushes at z) Suppose that no more edge-push is possible at z. As long as z stays Ψ-larger and $\Theta(T(z)) = \Theta(z) - 1$, apply a node-push at z.

Step 3 (lifting z) When z is still Ψ-larger but no more edge- or node-push is possible at z, apply a node-lift at z.

As mentioned above, a treatment of z terminates when either z becomes neutral or z is lifted.

Description of the algorithm

The algorithm starts with an arbitrary feasible triplet (x, m, Θ). This can be obtained by choosing an arbitrary x with $f \leq x \leq g$, by taking any arbitrary element m of $B(b)$ (which can, for example, be determined by the greedy algorithm for base-polyhedra, as described in Subsection 14.5.1), and by taking Θ to be identically 0.

At an intermediate stage, a feasible triplet (x, m, Θ) is available. Suppose that neither of the two Stopping rules holds. Then there are Ψ-larger nodes but none of them can be in L_n since $\Theta(z) = n$ for a Ψ-larger node z would imply that at least one level set under z is empty, and hence Stopping rule **(B)** would occur. As long as there are Ψ-larger nodes, the algorithm selects one with highest level (**highest level rule**) and treats it.

One way of termination is that the current treatment neutralizes z and no more Ψ-larger node remains. In this case, Stopping rule **(A)** holds and hence the resulting x is a feasible subflow by Lemma 16.2.2. The other way of termination is that the current treatment lifts z and leaves its (original) level set empty. In this case, Stopping rule **(B)** holds and hence the set $Z = \{v : \Theta(v) \geq \Theta(z)\}$ violates (16.16) by Lemma 16.2.2 showing that no feasible subflow exists.

16.2.3 Correctness and complexity

Consider a transition from a stage to the subsequent one which transforms the current functions x, m, Θ, T, Ψ, respectively, into $x', m', \Theta', T', \Psi'$. It is evident from the definition of the corresponding α that the vector x' arising from x by an edge-push is feasible and the vector m' arising from m by a node-push is in $B(b)$.

Lemma 16.2.3 *The basic operations preserve the level properties.*

Proof. An edge-push has no effect on (LP3). It creates no new Ψ-smaller nodes, so Property (LP1) is also preserved. Because it operates on an edge uv for which $|\Theta(u) - \Theta(v)| = 1$, Property (LP2) cannot break down either.

A node-push along (z, t) does not affect x (that is, $x' = x$). Hence it has no effect on (LP2). The definition of α at a node-push implies that $\alpha \leq \Psi_x(z) - m(z)$. Therefore z cannot become Ψ-smaller and thus Property (LP1) is also preserved. Property (LP3) for m' in place of m follows from the next claim.

Claim 16.2.4 *When a node-push is carried out along* (z, t)*, then* ($\Theta' = \Theta$ and)

$$\Theta(T'(v)) \geq \Theta(T(v)). \tag{16.18}$$

for every node v.

Proof. If $T(v)$ is m'-tight, then $T'(v) \subseteq T(v)$ from which (16.18) follows. If $T(v)$ is not m'-tight, then $t \in T(v)$ from which $\Theta(T(v)) \leq \Theta(t) = \Theta(T(z))$. This implies for $T := T(v) \cup T(z)$ that $\Theta(T) = \min\{\Theta(T(v)), \Theta(T(z))\} = \Theta(T(v))$. Now T is m-tight as it is the union of two m-tight sets. Since $z, t \in T$, it follows that T is m'-tight, too. Hence $T'(v) \subseteq T$ from which $\Theta(T'(v)) \geq \Theta(T) = \Theta(T(v))$. •

A node-lift of z keeps (LP1) intact since $\Theta'(z) > \Theta(z)$ only if z is Ψ-larger. It preserves (LP2), since node-lift was applied only when there was no increasable edge stepping down from z and no decreasable edge stepping up to z. It preserves (LP3), since a node-lift was applied at z only when $\Theta(T(z)) = \Theta(z)$. • •

The non-trivial direction of Theorem 16.1.5 follows once we show that one of Stopping rules (A) and (B) occurs after a finite number of operations. In fact, we shell prove the following polynomial bound.

Theorem 16.2.5 *The total number of basic operations is* $O(n^4)$.

Proof. We claim that a single treatment of a node z needs $O(n)$ basic operations. Indeed, there may be just one neutralizing push, one node-lift, $O(n)$ non-neutralizing edge-pushes since we excluded parallel edges in D, and finally $O(n)$ non-neutralizing node-pushes since if one is carried out along (z, t), then $T'(z) \subseteq T(z) - t$.

By a **phase** of the algorithm we mean the set of operations carried out between two consecutive node-lifts. Since the level of a single node can increase at most n times, the total number of phases is at most n^2.

A treatment of a node z is completed when z becomes neutral or when z is lifted. Due to the highest level rule, z will not be Ψ-larger again within the same phase. Therefore, in one phase there can be at most n treatments and hence the total number of basic operations is $O(n^4)$. •

Improving the complexity

We describe now a tiny modification of the algorithm above that gives rise to a $O(n^3)$ bound. The proof, however, will need a little more care. The idea is that we make the generic choice of t during a node-push specific. To this end, we choose a total ordering of the nodes of D in advance. This is arbitrary but fixed and is specified by a one-to-one function $\varphi : V \to \{1, \dots, n\}$. We will say that u is **earlier than** v if $\varphi(u) < \varphi(v)$.

Recall that a node-push at z is generic when only

$$t \in T(z) \text{ and } \Theta(t) = \Theta(z) - 1 \tag{16.19}$$

were required for t, meaning that when more than one such node existed it did not matter which of them has been selected. A node-push operation is **φ-consistent** if t is selected to be the earliest node meeting (16.19). In addition to the highest level rule, the revised algorithm selects t throughout φ-consistently.

Theorem 16.2.6 *The total number of basic operations, when φ-consistent node-pushes are applied throughout, is $O(n^3)$.*

Proof. In the foregoing proof, we have already pointed out that within one phase there are at most n neutralizing pushes and hence their total number is at most n^3.

Claim 16.2.7 *The total number of non-neutralizing edge-pushes is at most n^3.*

Proof. A non-neutralizing edge-push on e either increases $x(e)$ to $g(e)$ making e non-increasable or decreases $x(e)$ to $f(e)$ making e non-decreasable. Therefore, after a non-neutralizing edge-push on e, the next edge-push on e can occur only when the sign of $\Theta(z) - \Theta(u)$ has changed. By that time the sum $\Theta(z) + \Theta(u)$ must have been increased by at least two. Hence the number of non-neutralizing edge-pushes on a single edge is at most n and thus the total number of non-neutralizing edge-pushes is at most $|A|n \leq n^3$. •

So far we have not used the φ-consistent selection rule. It is needed only for the next lemma which will complete the proof.

Lemma 16.2.8 *The total number of non-neutralizing node-pushes is at most n^3.*

Proof. Given a feasible triplet (x, m, Θ) at a stage of the algorithm, let

$$B(v) := \{u : u \in T(v),\ \Theta(u) = \Theta(v) - 1\} \quad \text{and} \quad \beta(v) := \min\{\varphi(u) : u \in B(v)\}$$

where $\beta(v)$ is meant to be ∞ if $B(v) = \emptyset$. We use the notational convention that the corresponding functions after performing a basic operation will be denoted by primed letters.

Claim 16.2.9 *If a φ-consistent node-push is carried out along (z, t), then*

$$\beta'(v) \geq \beta(v) \tag{16.20}$$

for every node v. Moreover, the inequality is strict when $v = z$ and the node-push is non-neutralizing.

Proof. The definition of φ-consistency shows that $\beta(z) = \varphi(t)$. When $v = z$ and the node-push is non-neutralizing, we have $B'(v) \subseteq B(v) - t$. This and the φ-consistent choice of t imply that $\beta'(v) > \varphi(t) = \beta(z) = \beta(v)$.

Suppose now that $v \neq z$. If $B'(v) \subseteq B(v)$, then (16.20) follows immediately so we can assume that

$$B'(v) \not\subseteq B(v) \tag{16.21}$$

and, in paricular, $B'(v) \neq \emptyset$. Then $T'(v) \not\subseteq T(v)$ and hence $T(v)$ is not m'-tight implying that $t \in T(v)$. The set $T := T(v) \cup T(z)$ is m-tight and it is m'-tight, too, since $z, t \in T$. Therefore $T'(v) \subseteq T$. We cannot have $\Theta(t) \geq \Theta(v)$ since then $\Theta(T(z)) = \Theta(t)$ would imply $B'(v) \subseteq B(v)$ contradicting (16.21). Hence $t \in B(v)$, $\Theta(v) = \Theta(z)$, and $B'(v) \subseteq B(v) \cup B(z)$ from which $\beta'(v) \geq \min\{\beta(v), \beta(z)\} = \min\{\beta(v), \varphi(t)\} = \beta(v)$. •

Observe that lifting a node z results in $T'(v) = T(v)$ and $B'(v) \subseteq B(v)$ for every node $v \neq z$ from which $\beta'(v) \geq \beta(v)$ follows. This and Claim 16.2.9 imply that, as long as the level of a node is fixed, the number of non-neutralizing node-pushes at this node is at most n. Since one node can be lifted at most n times and there are n nodes, the total number of non-neutralizing node-pushes is at most n^3, completing the proof of the lemma and the theorem. • •

Remark 16.2.1 The technique of φ-consistent changes was originally introduced by Schönsleben [334] under the name lexicographic selection rule in an augmenting path-type polynomial time algorithm for finding a maximum element of the intersection of two polymatroids. This approach has later been applied to other submodular frameworks, such as submodular flow feasibility in [119], testing membership in a matroid polyhedron [60] by Cunningham, optimization over polymatroidal flows [257, 258] by Lawler and Martel, or submodular function minimization [339] by Schrijver. It was also used in a push–relabel algorithm for submodular flows by Fujishige and Zhang [165] and for submodular function minimization by Fleischer and Iwata [102].

The φ-consistent node-push operation above can be considered as a lexicographic rule for a push–relabel algorithm. In this environment, as we pointed out above, the φ-consitent selection rule is avoidable if our goal is only to have a polynomial time algorithm. Its main role was to help improving on the running time. Interestingly, each polynomial time augmenting path-type algorithm known so far for polymatroid intersection, say, does use the lexicographic selection rule.

16.2.4 Computing a most violating set and maximizing $x(e_0)$

The algorithm above, in the case when no feasible subflow existed, found a subset Z violating (16.16). With a slight modification of the procedure even a most violating subset can be computed, where X is most violated if $\varrho_f(X) - \delta_g(X) - b(X)$ is as large as possible.

To this end, the algorithm does not terminate when a node-lift operation leaves a level set empty. Therefore there can be Ψ-larger nodes in level set L_n and this motivates the other modification: z is to be chosen a Ψ-larger node of highest level *under* n. The algorithm terminates when there are no more Ψ-larger nodes under level n. When there is no Ψ-larger node at all, then the current x is a feasible subflow.

Theorem 16.2.10 *When there are Ψ-larger nodes at termination of the revised algorithm, the set Z of nodes above an empty level violates* (16.16) *most, or formally,*

$$\varrho_f(Z) - \delta_g(Z) - b(Z) \geq \varrho_f(X) - \delta_g(X) - b(X) \text{ for every } X \subseteq V. \tag{16.22}$$

Proof. Relying on Lemma (16.2.1) and using that Z contains every Ψ-larger node but no Ψ-smaller ones, we have $\varrho_f(Z) - \delta_g(Z) - b(Z) = \Psi_x(Z) - \widetilde{m}(Z) = \sum[\Psi_x(v) - m(v) : v \in Z] \geq \sum[\Psi_x(v) - m(v) : v \in X] = \Psi_x(X) - \widetilde{m}(X) \geq \varrho_f(X) - \delta_g(X) - b(X)$. •

We apply a simple trick which was already used in obtaining a maximum feasible flow push–relabel algorithm from that for modular flows. Theorem 16.2.10 can be used to compute a feasible submodular flow x, when one exists, for which $x(e_0)$ is maximum for a

978–0–19–920527–1 16-Andras-Frank-c16-drv Andras Frank (Typeset by SPi, Publisher Services, India) 541 of 640 December 11, 2010 15:21
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

specified edge $e_0 = st$ of the digraph. To this end, we can assume, without loss of generality, that there is no upper bound on e_0, and hence $g(e_0) = +\infty$.

Our first task is to check whether $x(e_0)$ is bounded from above over the elements x of $Q(f, g; b)$.

Theorem 16.2.11 *Let D, f, g, b be the same as before and suppose that there is a feasible subflow x_0. For a specified edge e_0, the value $x(e_0)$ is bounded from above over $Q(f, g; b)$ if and only if there is an $s\bar{t}$-set Z for which $b(Z)$ is finite, $g(e)$ is finite for every edge e leaving Z, and $f(e)$ is finite for every edge but possibly e_0 entering Z.*

Proof. Suppose first that there is a set Z as described in the theorem and let x be an arbitrary element of Q. Then $\varrho_x(Z) - \delta_x(Z) \leq b(Z)$ and hence

$$x(e_0) \leq b(Z) + \delta_x(Z) - \sum[f(e) : e \neq e_0 \text{ enters } Z] < \infty, \tag{16.23}$$

and hence $x(e_0)$ is indeed bounded in this case.

To see the reverse direction, define $f' : A \to \mathbf{Z} \cup \{-\infty\}$, $g' : A \to \mathbf{Z} \cup \{\infty\}$, and $b' : 2^V \to \mathbf{Z} \cup \{\infty\}$ as follows.

$$f'(e) := \begin{cases} 1 & \text{if } e = e_0 \\ 0 & \text{if } e \neq e_0,\ f(e) > -\infty \\ -\infty & \text{if } e \neq e_0,\ f(e) = -\infty, \end{cases} \tag{16.24}$$

$$g'(e) := \begin{cases} 0 & \text{if } g(e) < \infty \\ \infty & \text{if } g(e) = \infty, \end{cases} \tag{16.25}$$

$$b'(X) := \begin{cases} 0 & \text{if } b(X) < \infty \\ \infty & \text{if } b(X) = \infty. \end{cases} \tag{16.26}$$

Obviously, b' is fully submodular. For this case, the feasibility theorem asserts that

$Q' := Q(f', g'; b')$ *is empty if and only if there is a subset Z for which* $\varrho_{f'}(Z) - \delta_{g'}(Z) > b'(Z)$.

In terms of the original f, g, b, this is just equivalent to the properties listed in the theorem for Z. •

Therefore the algorithm described above, when applied to Q', can be used to decide if $x(e_0)$ is bounded over Q. (Actually, the feasibilty problem when the subflow polyhedron is confined by a $(0, \infty)$-valued submodular function can be reduced to ordinary flow feasibility.) Suppose now that $x(e_0)$ is bounded. In this case the algorithm constructs a subset Z meeting (16.23) and such a subset provides a concrete upper bound K. By defining $f'(e)$ to be $K + 1$, clearly, there is no feasible subflow and the algorithm above computes a most violating subset. It follows that such a subset determines the maximum value M of $x(e_0)$ over Q. By defining $f''(e)$ to be M, another run of the algorithm will compute a feasible subflow for which $x(e_0) = M$.

Exercise 16.2.1 *Specialize this approach to compute an element of the intersection of two g-polymatroids in $\mathbf{R}^S$ for which $\widetilde{x}(S)$ is maximum?*

16.2.5 The algorithm for a crossing submodular function

How can one find a member of a subflow polyhedron $Q = Q(f, g; b)$ confined by a crossing submodular function b? We can assume that $b(V) = 0$ for if $b(V) < 0$, then Q is certainly empty while if $b(V) > 0$, then lowering $b(V)$ to zero does not change Q and preserves crossing submodularity. Recall Parts (B) and (C) of Theorem 16.1.5.

Theorem 16.2.12 *Let b be crossing submodular for which $b(V) = 0$. There is a feasible subflow if and only if*

$$\varrho_f(Z) - \delta_g(Z) \leq \sum[b(X) : X \in \mathcal{F}]. \tag{16.27}$$

for every subset $Z \subseteq V$ and every tree-composition $\mathcal{F}$ of Z. When b is intersecting submodular, $\mathcal{F}$ can be restricted to partitions of Z. If each of f, g, b is integer-valued and there is a feasible subflow, then there is one which is integer-valued. •

First we check whether $B(b)$ is empty or not. This can be done, for example, by the algorithm outlined in the proof of Theorem 15.3.8. If $B(b)$ turns out to be empty, then we obtain a partition or a co-partition $\mathcal{F}$ of V for which $0 > \sum[b(X) : X \in \mathcal{F}]$. In this case, the algorithm for testing $Q = Q(f, g; b)$ for emptiness concludes that Q is empty and returns $\mathcal{F}$ as a tree-composition of V which violates (16.27) since $\varrho_f(V) - \delta_g(V) = 0$.

Suppose now that $B(b)$ is non-empty and a member m of $B(b)$ has already been computed. In this case, the full (lower) truncation $b^\downarrow$ of b exists. By Theorem 15.3.5, $B(b)$ is a base-polyhedron for which the unique fully submodular upper border function is $b^\downarrow$, and hence $B(b) = B(b^\downarrow)$. Recall that $b^\downarrow$ was called the full truncation of b. Obviously $Q(f, g; b) = Q(f, g; b^\downarrow)$ and therefore the algorithm above for fully submodular border functions can, in principle, be applied to $b^\downarrow$ in place of b. There are two issues here to be settled.

First, the algorithm above required a subroutine for computing $\Delta_{(b^\downarrow - \widetilde{m})}(u, v) := \min\{b^\downarrow(X) - \widetilde{m}(X) : u \in X \subseteq V - v\}$. The next lemma shows that it suffices to have a subroutine for computing

$$\Delta_{(b-\widetilde{m})}(u, v) := \min\{b(X) - \widetilde{m}(X) : u \in X \subseteq V - v\}. \tag{16.28}$$

which is simpler since $\Delta_{(b-\widetilde{m})}(u, v)$ is a function depending directly on the original b and not on its full truncation $b^\downarrow$. In applications to connectivity problems, (16.28) is typically available via a Max-flow Min-cut computation.

Lemma 16.2.13 *Let b be a crossing submodular function for which $b(V) = 0$ and m a member of $B(b)$. Then $\Delta_{(b^\downarrow - \widetilde{m})}(u, v) = \Delta_{(b-\widetilde{m})}(u, v)$.*

Proof. We need two observations.

Claim 16.2.14 $B(b - \widetilde{m}) = B(b^\downarrow - \widetilde{m})$ *and*

$$(b - \widetilde{m})^\downarrow = b^\downarrow - \widetilde{m}. \tag{16.29}$$

Proof. For a vector x with $\widetilde{x}(V) = 0$, we have the following sequence of equivalences. $x \in B(b - \widetilde{m}) \iff \widetilde{x} \leq b - \widetilde{m} \iff \widetilde{x} + \widetilde{m} \leq b \iff x + m \in B(b) \iff x + m \in B(b^\downarrow) \iff \widetilde{x} + \widetilde{m} \leq b^\downarrow \iff \widetilde{x} \leq b^\downarrow - \widetilde{m} \iff x \in B(b^\downarrow - \widetilde{m})$. Hence $B((b - \widetilde{m})^\downarrow) = B(b^\downarrow - \widetilde{m})$.

978–0–19–920527–1 16-Andras-Frank-c16-drv Andras Frank (Typeset by SPi, Publisher Services, India) 543 of 640 December 11, 2010 15:21

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Now $B(b - \widetilde{m})$ is non-empty since the origin belongs to it. Therefore the full truncation $(b - \widetilde{m})^{\downarrow}$ of $b - \widetilde{m}$ esists and then $B(b - \widetilde{m}) = B((b - \widetilde{m})^{\downarrow})$. This and the first part imply that $B((b - \widetilde{m})^{\downarrow}) = B(b^{\downarrow} - \widetilde{m})$. Since both $(b - \widetilde{m})^{\downarrow}$ and $b^{\downarrow} - \widetilde{m}$ are fully submodular for which $B(b - \widetilde{m}) = B(b^{\downarrow} - \widetilde{m})$, we conclude by Corollary 14.2.9 that (16.29) holds. •

Claim 16.2.15 *Let h be a non-negative crossing submodular function on ground-set V for which $h(V) = 0$. Then the full truncation $h^{\downarrow}$ of h exists and $\Delta_{h^{\downarrow}} = \Delta_h$.*

Proof. Note that $B(h)$ is non-empty since the non-negativity of h implies that the origin is in $B(h)$. Hence the full truncation of h indeed exists. Since $h^{\downarrow} \leq h$, we have $\Delta_{h^{\downarrow}} \leq \Delta_h$. Let u and v be two elements of V and let Z be a $u\bar{v}$-set for which $\Delta_{h^{\downarrow}}(u, v) = h^{\downarrow}(Z)$.

By formula (15.31), there exists a composition (which is in fact a double partition) $\mathcal{D}$ of Z such that $h^{\downarrow}(Z) = \sum[h(X) : X \in \mathcal{D}]$. By the definition of a composition, $d_{\mathcal{D}}(u) = d_{\mathcal{D}}(v) + 1$. Therefore there is a $u\bar{v}$-member Z' of $\mathcal{D}$ and the non-negativity of h implies that $h(Z') \leq h^{\downarrow}(Z)$ from which $\Delta_h(u, v) \leq \Delta_{h^{\downarrow}}(u, v)$ follows implying $\Delta_{h^{\downarrow}}(u, v) = \Delta_h(u, v)$. •

The lemma follows by applying Claim 16.2.15 to $h := b - \widetilde{m}$. • •

The second problem to be overcome arises when the subflow polyhedron is empty and the algorithm terminates by returning a subset Z for which $\varrho_f(Z) - \delta_g(Z) > b^{\downarrow}(Z)$. In this situation, the algorithm outlined in the proof of Theorem 15.3.6 for computing a tree-composition $\mathcal{F}$ of Z for which $b^{\downarrow}(Z) = \sum[b(X) : X \in \mathcal{F}]$ is to be used.

16.3 Optimization over submodular flows

After working out an algorithm for solving subflow feasibility problems, we turn our attention to optimization questions. We are given again a digraph $D = (V, A)$ and a subflow polyhedron $Q = Q(f, g; b)$ in $\mathbf{R}^A$. Furthermore, $c : A \to \mathbf{R}$ is a weight (or cost) function. In the optimization problem we consider, the input functions f, g, and b are integer-valued and the goal is to compute an integer-valued feasible subflow x for which cx is maximum or minimum. The integrality of c is not a priori assumed.

In some special cases minimization is the goal. For example, minimum-cost circulations, cheapest dijoins, and cheapest k-edge-connected reorientations belong to this class. In other cases, like weighted matroid or g-polymatroid intersection, maximization is our concern. The two versions for subflows are clearly equivalent since the negative of a subflow polyhedron is again a subflow polyhedron. In what follows, we describe an algorithm for maximization [119]. For other polynomial algorithms for submodular flows, see the work by Fujishige, Röck, and Zimmermann [163], and by Iwata, McCormick, and Shigeno [223]. A detailed description of various polynomial time algorithms concerning otimization over submodular flows can be found in the book of Fujishige [161].

For technical simplicity, we assume that b, f, and g are finite-valued and hence the submodular polyhedron Q is bounded. The algorithm can be extended to the general case but we do not discuss this extension since it requires overcoming only technical difficulties and no new ideas are needed. With standard transformations, we can reduce the problem to the special case when $f \equiv 0 \leq g$ and c is non-negative. Such reductions for modular flows were exhibited in Section 3.4.2.

Furthermore, we shall consider only the special case when $g \equiv 1$. From the point of view of deriving the corresponding min-max theorem, the problem with general g can be reduced to the special case when $g \equiv 1$ by replacing each edge e by $g(e)$ parallel copies. Such a reduction, however, is not satisfactory from an algorithmic point of view even in the special case of ordinary flows. The difficulty can be overcome, for example, by the scaling technique [62].

Summing up, we assume that

$$f \equiv 0,\ g \equiv 1,\ c \geq 0,\ b(V) = 0, \text{ and } b \text{ is integer-valued.}$$

In Section 16.1.5, we pointed out that this special case includes the abstract graph orientation problem in which the demand function h to be covered by the orientation of G is crossing G-supermodular. Namely, the orientations of G covering h are in a one-to-one correspondence with the (0, 1)-vectors of a subflow polyhedron. In particular, by applying the algorithm below, one is able to find a cheapest k-edge-connected reorientation of a digraph for which the underlying graph is $2k$-edge-connected, and it can be used even for finding a cheapest (k, ℓ)-edge-connected orientation of a graph.

The algorithm is based on three major ideas: each has been introduced earlier. One of them makes use of Theorem 15.3.5 which stated that a non-empty base-polyhedron B bordered by a crossing submodular function admits a unique fully submodular function defining the same B. Therefore it suffices to work out first the algorithm for fully submodular functions and show after how its steps can be carried out when b is crossing submodular. This approach was already applied in Section 16.2 in the feasibility algorithm for subflows. The second crucial idea of the algorithm is that we work with a certain potential function on V rather than with dual variables assigned to subsets of V. This approach was already used in the weight-splitting technique of the weighted matroid intersection algorithm and also in the algorithmic proof of the theorem of Lucchesi and Younger (see Section 9.7). The third idea is that in the auxiliary digraph for computing an augmenting path not only the original edges of D determine forward or backward edges (as they did in ordinary flow algorithms) but a third type of edges, coming from the submodular constraint and called jumping edges, are also introduced. These were already used in the algorithmic proof of the theorem of Lucchesi and Younger theorem and in the weighted matroid intersection algorithm.

Optimality criteria

Let Q denote the $(0, \pm 1)$-valued matrix the columns of which columns correspond to the edges of D, while the rows correspond to the non-empty subsets $Z \subset V$, and an entry of Q corresponding to edge $e \in A$ and subset Z is 1 if e enters Z, -1 if e leaves Z, and 0 otherwise. The column of Q corresponding to edge e is denoted by q_e. (It causes no confusion that the same letter Q is used to denote both the matrix and the subflow polyhedron in question.)

The following result is a direct consequence of Theorems 16.1.3 and 4.1.24 by Edmonds and Giles.

Theorem 16.3.1 *The primal linear programming problem*

$$\max\{cx : 0 \leq x \leq \underline{1},\ Qx \leq b\} \tag{16.30}$$

has an integral optimal solution provided it has a solution at all. The maximum is equal to the minimum in the following dual problem:

$$\min\left\{\sum_{Z\subset V} y(Z)b(Z) + \sum_{e\in A} z(e) : (y,z)\geq 0,\ yq_e + z(e) \geq c(e) \text{ for every edge } e\in A\right\}. \tag{16.31}$$

If c is integer-valued, there exists an integer-valued dual optimum.

Note that $Qx \leq b$ is the matrix form of the inequality system $\Psi_x \leq b$ defining submodular flows, while the scalar product yq_e can be expressed as follows.

$$yq_e = \sum[y(Z) : Z \text{ is entered by } e] - \sum[y(Z) : Z \text{ is left by } e].$$

The optimality criteria for a primal solution x and for a dual solution (y, z) are as follows.

$$\begin{cases} (\mathbf{A1'}) & y(Z) > 0 \ \Rightarrow\ \Psi_x(Z) = b(Z) \\ (\mathbf{A2'}) & x(e) > 0 \ \Rightarrow\ yq_e + z(e) = c(e) \\ (\mathbf{A3'}) & z(e) > 0 \ \Rightarrow\ x(e) = 1. \end{cases}$$

We call a subset Z of V x**-tight** or just tight if $\Psi_x(Z) = b(Z)$. Since $\Psi_x(V) = 0 = b(V)$, we consider V tight. Note that y uniquely determines the optimal dual variable z, namely, $z(e) := \max\{0, c(e) - yq_e\}$ for $e \in A$. Therefore we shall refer to a dual solution only by the vector y without explicitly mentioning z. Since we are interested in (0, 1)-valued primal solutions, the optimality criteria can be rewritten as follows.

$$\begin{cases} (\mathbf{A1}) & y(Z) > 0 \Rightarrow Z \text{ is tight} \\ (\mathbf{A2}) & x(e) = 1 \Rightarrow yq_e \leq c(e) \\ (\mathbf{A3}) & x(e) = 0 \Rightarrow yq_e \geq c(e). \end{cases} \tag{16.32}$$

The algorithm below provides an alternative proof of Theorem 16.3.1 by constructing a (0, 1)-valued x and a y fulfilling the optimality criteria.

Problem 16.3.1 *Prove that the optimality criteria for general bounds f and g are as follows.*

$$\begin{cases} (\mathbf{A1}) & y(Z) > 0 \Rightarrow Z \text{ is tight} \\ (\mathbf{A2''}) & x(e) > f(e) \Rightarrow yq_e \leq c(e) \\ (\mathbf{A3''}) & x(e) < g(e) \Rightarrow yq_e \geq c(e). \end{cases} \tag{16.33}$$

16.3.1 Algorithm and proof when b is fully submodular

We describe first an algorithmic proof of Theorem 16.3.1 for the special case when b is fully submodular. It is assumed that a (0, 1)-valued element of the subflow polyhedron Q is available at the beginning. Such an x can by computed with the help of the feasiblity algorithm described in Section 16.2. Since b is submodular and Ψ_x is modular, it follows by the standard submodular technique that the system of tight sets is closed under taking intersection and union. Let $T(u) = T_x(u)$ denote the intersection of all tight sets containing a node $u \in V$, that is, $T(u)$ is the unique smallest tight set containing u.

Potentials

Let $\pi : V \to \mathbf{R}$ be a function and let $p_0 < p_1 < \cdots < p_k$ denote the distinct values of π. We will refer to such a function as a potential on V. Let $V_i := \{v : \pi(v) \geq p_i\}$ for $i = 1, 2, \ldots, k$ and associate a set-function (or dual variable) y on ground-set V with π as follows. If π is constant, then let $y \equiv 0$, while if $k \geq 1$, then let

$$y(Z) := \begin{cases} p_i - p_{i-1} & \text{if } Z = V_i, \quad i = 1, \ldots, k \\ 0 & \text{otherwise.} \end{cases} \tag{16.34}$$

Note that if we translate π by a constant, meaning that the same number is added to each component of π, then the associated y does not change. As usual, the function $c_\pi : A \to \mathbf{R}$ is defined for a directed edge uv by $c_\pi(uv) = c(uv) - (\pi(v) - \pi(u))$. Consider the following properties.

$$\begin{cases} \textbf{(B1)} \quad v \in T(u) \quad \Rightarrow \quad \pi(v) \geq \pi(u) \\ \textbf{(B2)} \quad x(uv) = 1 \ \Rightarrow \ c_\pi(uv) \geq 0 \\ \textbf{(B3)} \quad x(uv) = 0 \ \Rightarrow \ c_\pi(uv) \leq 0. \end{cases} \tag{16.35}$$

Observe that the translation of π by a constant does not affect these properties and thus we can assume that the smallest value p_0 of π is zero.

Lemma 16.3.2 *For π and y defined above, the optimality criteria* (16.32) *and* (16.35) *are equivalent.*

Proof. The definition of y implies that $\pi(v) - \pi(u) = yq_e$ for every edge $e = uv \in A$. Hence **(A2)** is equivalent to **(B2)** and **(A3)** is equivalent to **(B3)**.

Suppose now that **(A1)** of (16.32) holds. This is clearly equivalent to requiring that each V_i is tight. Let v be an element of $T(u)$ and let i be the index for which $\pi(u) = p_i$. If indirectly **(B1)** fails, meaning that $\pi(v) < \pi(u)$ for some $v \in T(u)$, then V_i contains u but not v and hence the smallest tight set containing u would not contain v, contradicting that $v \in T(u)$. Conversely, suppose that **(B1)** holds. Then $u \in V_i$ implies $T(u) \subseteq V_i$ for each $i = 1, \ldots, k$. Since the union of tight sets is tight, we conclude that $V_i = \cup\{T(u) : u \in V_i\}$ is tight. Furthermore $y(Z) > 0$ if and only if $Z = V_i$ for some $i = 1, \ldots, k$ and hence y satisfies **(A1)** of (16.32). •

Inner Algorithm

By Lemma 16.3.2, in order to prove Theorem 16.3.1, it suffices to construct a feasible subflow x and a potential π satisfying (16.35). The core of the entire procedure is the following subroutine, called the Inner Algorithm. The input of the Inner Algorithm is a (0, 1)-valued subflow x, a potential π, and an edge $a_1 = u_1v_1 \in A$ such that **(B1)** and **(B2)** hold but a_1 violates **(B3)**, which means that $x(u_1v_1) = 0$ and $c_\pi(u_1v_1) > 0$. The output is a (0, 1)-valued subflow x' and a potential π' such that **(B1)** and **(B2)** continue to hold, a_1 does not violate **(B3)** anymore, and an edge of D can violate **(B3)** only if it violated **(B3)** with respect to (x, π).

Provided this Inner Algorithm is available, the procedure is as follows. At the beginning, let $\pi \equiv 0$ and let x be a (0, 1)-valued subflow. Since c is non-negative, $c_\pi = c \geq 0$, and hence x and π satisfy **(B1)** and **(B2)**. Apply repeatedly the Inner Algorithm as long as there are edges violating **(B3)**. After no more than $|A|$ applications of the Inner Algorithm the final output will fulfil each of the three criteria in (16.35).

To describe the Inner Algorithm, we define an auxiliary digraph $H = (V, A')$ depending on the current x and π. Call an edge $e \in A$ a **0-edge** or a **1-edge** depending on $x(e)$ is 0 or 1. Let $A' := A_F \cup A_B \cup A_J$ where

$$\begin{cases} A_F := \{uv \ : \ uv \text{ is a 0-edge and } c_\pi(uv) \geq 0\} \\ A_B := \{vu \ : \ uv \text{ is a 1-edge and } c_\pi(uv) \leq 0\} \\ A_J := \{uv \ : \ v \in T(u) \text{ and } \pi(u) = \pi(v)\}. \end{cases}$$

The elements of A_F, A_B, and A_J are called **forward**, **backward**, and **jumping** edges, respectively. Note that a jumping edge never leaves a tight set. Compute the subset S of nodes that can be reached from v_1 in H along a dipath.

Case 1 $u_1 \notin S$. By the definition of S,

$$\text{no edge of } H \text{ leaves } S. \tag{16.36}$$

Revise the potential as follows.

$$\pi'(v) := \begin{cases} \pi(v) + \delta & \text{if } v \in S \\ \pi(v) & \text{if } v \in V - S \end{cases} \tag{16.37}$$

where $\delta := \min\{\delta_a, \delta_F, \delta_B, \delta_J\}$, where

$$\begin{cases} \delta_a := c_\pi(u_1 v_1) \\ \delta_F := \min\{-c_\pi(uv) : uv \text{ is a 0-edge leaving } S\} \\ \delta_B := \min\{c_\pi(uv) : uv \text{ is a 1-edge entering } S\} \\ \delta_J := \min\{\pi(v) - \pi(u) : u \in S, v \in T(u) - S)\}. \end{cases}$$

Here the minimum is defined to be ∞ when it is taken over the empty set.

Claim 16.3.3 $\delta > 0$.

Proof. Since a_1 violates **(B3)**, $\delta_a > 0$ follows. Let uv be a 0-edge leaving S. If we had $c_\pi(uv) \geq 0$, then uv would be a forward edge of H leaving S, in a contradiction with Property (16.36). Hence $c_\pi(uv) < 0$ for every 0-edge uv entering S from which $\delta_F > 0$.

Let uv be a 1-edge entering S. If we had $c_\pi(uv) \leq 0$, then vu would be a backward edge of H leaving S, in a contradiction with Property (16.36). Hence $c_\pi(uv) > 0$ for every 1-edge leaving S from which $\delta_B > 0$.

Finally, let $u \in S$ and $v \in T(v) - S$ be two nodes. Then $\pi(v) \geq \pi(u)$ by **(B1)** and we cannot have $\pi(v) = \pi(u)$ since then uv would be a jumping edge in H leaving S, contradicting again Property (16.36). Hence $\pi(v) > \pi(u)$ whenever $u \in S$ and $v \in T(v) - S$ from which $\delta_J > 0$ •

Note that $c_{\pi'}$ can be expressed as follows.

$$c_{\pi'}(uv) = \begin{cases} c_\pi(uv) + \delta & \text{if } uv \in A \text{ leaves } S \\ c_\pi(uv) - \delta & \text{if } uv \in A \text{ enters } S \\ c_\pi(uv) & \text{otherwise.} \end{cases} \tag{16.38}$$

Claim 16.3.4 **(B1)** *continues to hold.*

Proof. Note that $T(u)$ depends only on x and not on the current potential. Let $v \in T(u)$ and suppose indirectly that $\pi'(v) < \pi'(u)$. Then $v \in V - S$ and $u \in S$ from which $\pi'(v) = \pi(v)$ and $\pi'(u) = \pi(u) + \delta$. Hence $\pi(v) - \pi(u) < \delta$. On the other hand, $\pi(v) - \pi(u) \geq \delta_J \geq \delta$, a contradiction. •

Claim 16.3.5 **(B2)** *continues to hold.*

Proof. For any 1-edge uv, we have $c_\pi(uv) \geq 0$. If, indirectly, $c_{\pi'}(uv) < 0$, then uv enters S by (16.38). Now $c_\pi(uv) \geq \delta_B \geq \delta$, and hence $c_{\pi'}(uv) \geq 0$, a contradiction. •

Claim 16.3.6 *If* **(B3)** *holds for an edge uv with respect to (x, π), then it holds for uv with respect to (x, π').*

Proof. Let uv be a 0-edge for which $c_\pi(uv) \leq 0$. If, indirectly, $c_{\pi'}(uv) > 0$, then uv leaves S by (16.38). Now $-c_\pi(uv) \geq \delta_F \geq \delta$, from which $c_{\pi'}(uv) = c_\pi(uv) + \delta \leq 0$, a contradiction. •

If $\delta = \delta_a$, then $a_1 = u_1v_1$ fulfils **(B3)** with respect to (x, π') and hence x and π' satisfy the requirements made for the output of the Inner Algorithm.

If $\delta < \delta_a$, then repeat the steps of the Inner Algorithm described above with respect to the input a_1, x, and $\pi := \pi'$. Observe that the edges induced by S in the revised auxiliary digraph H' is the same as in H. Moreover, the definition of δ ensures that H' contains at least one edge leaving S (which is forward, backward or jumping according to whether δ is equal to δ_F, δ_B or δ_J). This implies that the set S' of nodes that can be reached in H' from v_1 properly includes S. Consequently, after at most $|V| - 1$ iterations, either $\delta = \delta_a$ will occur, in which case the Inner Algorithm terminates, or else node u_1 becomes reachable from v_1. This is Case 2.

Case 2 There is a directed path in H from v_1 to u_1. Let P be a shortest one. Actually, we shall use only the property that there is no jumping edge that 'shortcuts' P. This latter property requires that if the nodes of P are $v_1 = z_0, z_1, \ldots, z_k = u_1$, then z_iz_{i+j} $(j \geq 2)$ must not be a jumping edge.

Since u_1v_1 is a forward edge of H, $C := P + u_1v_1$ is a directed circuit in H which may include forward, backward, and jumping edges. Let A_1 denote the set of edges of D which correspond to forward and to backward edges of C. Define a new vector x' as follows.

$$x'(e) := \begin{cases} 1 - x(e) & \text{if } e \in A_1 \\ x(e) & \text{if otherwise.} \end{cases} \tag{16.39}$$

That is, a 1-edge in A_1 becomes a 0-edge and vice versa. In particular $a_1 = u_1v_1$ becomes a 1-edge. We shall prove that x' and the unchanged potential π fulfils the requirements imposed on the output of the Inner algorithm. For a subset $X \subset V$, let $\varrho^j(X)$ (respectively, $\delta^j(X)$) stand for the number of jumping edges of path P that enter (resp., leave) X.

Lemma 16.3.7 *x' is a solution to the primal program* (16.30)*, meaning that $\Psi_{x'}(Z) \leq b(Z)$ for every $Z \subseteq V$.*

Proof. We need some simple observations.

Claim 16.3.8 $\Psi_{x'}(Z) = \Psi_x(Z) - \varrho^j(Z) + \delta^j(Z)$ *for* $Z \subseteq V$.

Proof. The formula is evident when $\varrho^j(Z) = \delta^j(Z) = 0$ and it follows easily in the general case by induction on $\varrho^j(Z) + \delta^j(Z)$. •

It follows from the claim that in order to prove $\Psi_{x'}(Z) \leq b(Z)$ it suffices to show that $\delta^j(Z) \leq \varepsilon(Z)$ holds for $Z \subset V$ where $\varepsilon(Z) := b(Z) - \Psi_x(Z)$. Since b is submodular and Ψ_x is modular, the set-function ε is also submodular. Observe that if $\delta^j(Z) = 0$, then $\delta^j(Z) \leq \varepsilon(Z)$ holds automatically. Suppose now that $\delta^j(Z) > 0$ and let uv be a jumping edge of P leaving Z such that $\pi(u)$ $(= \pi(v))$ is as large as possible, and in addition, if there is more than one such edge, then let uv be the first one along path P (starting from v_1).

Claim 16.3.9 $\delta^j(Z \cup T(u)) = \delta^j(Z) - 1$.

Proof. Since no jumping edge leaves $T(u)$ and uv leaves Z but not $Z \cup T(u)$, we have $\delta^j(Z \cup T(u)) \leq \delta^j(Z) - 1$. If we have indirectly strict inequality, then there is a jumping edge rz of P, distinct from uv, that leaves Z but not $Z \cup T(u)$. Then $z \in T(u) - Z$. Hence $\pi(z) \geq \pi(u)$ by **(B1)**, and the maximal choice of $\pi(u)$ would imply $\pi(r) = \pi(z) = \pi(u)$ showing that uz is also a jumping edge that leaves Z. Due to the selection rule of uv, the edge uv precedes rz along P and thus uz is a jumping shortcut edge to P, a contradiction. •

Claim 16.3.10 $\delta^j(Z) \leq \varepsilon(Z)$ *for every* $Z \subset V$.

Proof. By induction on $\delta^j(Z)$. Observe that the non-negativity of set-function ε is equivalent to the primal feasibility of x, and $\varepsilon(Z)$ is zero precisely if Z is tight with respect to x. Consider now a subset Z for which $\varepsilon(Z) > 0$. Then

$$\varepsilon(Z) = \varepsilon(Z) + \varepsilon(T(u)) \geq \varepsilon(Z \cap T(u)) + \varepsilon(Z \cup T(u)) \geq$$
$$1 + \varepsilon(Z \cup T(u)) \geq 1 + \delta^j(Z \cup T(u)) = \delta^j(Z).$$

Here we made use of the submodularity of ε, the inductive hypothesis for $Z \cup T(u)$, and Claim 16.3.9. •

By combining Claims 16.3.8 and 16.3.10, we obtain $\Psi_{x'}(Z) = \Psi_x(Z) - \varrho^j(Z) + \delta^j(Z) \leq \Psi_x(Z) + \delta^j(Z) \leq \Psi_x(Z) + \varepsilon(Z) = b(Z)$, making the proof of the lemma complete. • •

After proving that x' is indeed a feasible subflow, let us investigate the optimality criteria. In Lemma 16.3.2, we proved that **(B1)** and **(A1)** of (16.32) are equivalent. Since $y(Z) > 0$ if and only if $Z = V_i$ for some $i = 1, \ldots, k$, it follows that **(B1)** is equivalent to requiring that each V_i is tight. Since the change of x does not effect the V_i's, we can conclude that **(B1)** holds for (x', π).

If uv is a new 1-edge, that is, if $x(uv) = 0$ and $x'(uv) = 1$, then uv is a forward edge of H from which $c_\pi(uv) \geq 0$ showing that **(B2)** holds with respect to x'.

If uv is a new 0-edge, that is, if $x(uv) = 1$ and $x'(uv) = 0$, then vu is a backward edge of H from which $c_\pi(uv) \leq 0$ showing that **(B2)** holds for uv with respect to x'. Moreover, we assumed that the edge $a_1 = u_1v_1$ violates **(B3)**, that is, $x(u_1v_1) = 0$ and $c_\pi(u_1v_1) > 0$. Since $x'(u_1v_1) = 1$, we can conlude that a_1 does not violate **(B3)** anymore.

By now the description of the Inner Algorithm is complete, as well as the proof of its correctness. As mentioned earlier, at most $|A|$ applications of the Inner Algorithm yields a $(0, 1)$-valued primal solution x along with a potential π which satisfy (16.35). Note that if c is integer-valued, the algorithm preserves the integrality of π throughout. Since the dual solution y associated with π satisfies the optimality criteria (16.32), the proof of the theorem is complete for the special case when b is fully submodular. • • •

Steps of the algorithm

The optimization algorithm above requires an initial feasible submodular flow which can be constructed with the help of a feasibility algorithm. Recall that the feasiblity algorithm in Section 16.2 required a subroutine for computing:

$$\Delta_{(b-\widetilde{m})}(u, v) := \min\{b(X) - \widetilde{m}(X) : u \in X \subseteq V - v\} \tag{16.40}$$

for any pair of nodes u, v where m is an element of the base-polyhedron $B(b)$.

In order to run the algorithm, we must be able to compute the jumping edges. For a $(0, 1)$-valued subflow x, let $m : V \to \mathbf{Z}$ be defined by $m(v) := \Psi_x(v)$ for $v \in V$. Then $\widetilde{m}(Z) = \Psi_x(Z)$ and $v \in T(u)$ holds if and only if $\Delta_{(b-\widetilde{m})}(u, v) > 0$. Once Δ in (16.40) is available, the jumping edges can be computed. It should be noted that if an initial solution is available somehow, then in order to compute the jumping edges we do not need to compute the values $\Delta_{(b-\widetilde{m})}(u, v)$: all we need is a subroutine to decide if there is a tight $u\bar{v}$-set or not. Let γ denote the complexity of computing one jumping edge. Since the set of nodes properly increases in the Inner Algorithm when Case 2 occurs, the number of steps before Case 1 occurs is $O(\gamma|A|)$. Since the Inner Algorithm is applied at most $|A|$ times the overall complexity of the algorithm is $O(\gamma|A|^2)$.

16.3.2 When b *is crossing submodular*

Consider now the optimization problem in the general case when b is crossing submodular and suppose again that an initial $(0, 1)$-valued subflow x is available. Then the base-polyhedron $B(b)$ is non-empty since the vector $m := \dot{\Psi}_x$ belongs to $B(b)$. Theorem 15.3.5 implies that there is a (unique) fully submodular function $b^\downarrow$ for which $B(b) = B(b^\downarrow)$.

Therefore, in order to handle the optimization problem for a crossing submodular function b, we can simply run the algorithm described above with $b^\downarrow$ in place of b. This can be done if an oracle is available to check whether there is a $u\bar{v}$-set which is tight with respect to the full truncation $b^\downarrow$ of b. Equivalently, we need an oracle to check whether there is a $u\bar{v}$-set X for which $\widetilde{m}(X) = \Psi_x(X) = b^\downarrow(X)$.,

Lemma 16.2.13 asserted that $\Delta_{(b^\downarrow-\widetilde{m})}(u, v) = \Delta_{(b-\widetilde{m})}(u, v)$. This implies that there is a $u\bar{v}$-set which is tight with respect to b if and only if there is one which is tight with respect to $b^\downarrow$. Therefore, in order to compute a $(0, 1)$-valued subflow that maximizes cx over a subflow polyhedron confined by a crossing submodular function, we can apply the optimization algorithm developed for subflow polyhedra confined by a fully submodular function without any changes.

There is, however a difference in finding the optimal dual solution. Suppose that the algorithm terminated by producing a (0, 1)-primal solution x and a potential π which fulfil the optimality criteria in (16.35). In (16.34), an optimal dual solution y could be easily established from π when the confining function is fully submodular.

Let us see how an optimal dual solution can be computed in the crossing submodular case. Let $V_i := \{v : \pi(v) \geq p_i\}$ for $i = 1, 2, \ldots, k$ be the same as before. Select an index $i = 1, \ldots, k$, and investigate V_i. We know that V_i is tight with respect to $b^{\downarrow}$, that is, $\widetilde{m}(V_i) = b^{\downarrow}(V_i)$. Therefore the algorithm described in the proof of Theorem 15.3.6 can be applied to compute a tree-composition $\mathcal{F}_i$ of V_i for which $b^{\downarrow}(V_i) = \widetilde{b}(\mathcal{F}_i)$. Define a set-function y^i on V by

$$y^i(X) := \begin{cases} p_i - p_{i-1} & \text{if } X \in \mathcal{F}_i \\ 0 & \text{otherwise} \end{cases}$$

and let $y := y^1 + \cdots + y^k$. This definition of y implies that $y(X)$ can be positive only if X is a tight set with respect to b, and hence optimality criterion **(A1)** of (16.32) holds.

Let $e = uv \in A$ be an edge of D. It follows from the definition of compositions that the number of sets in $\mathcal{F}_i$ entered by e minus the number of sets in $\mathcal{F}_i$ left by e is $+1$, -1, or 0. Namely, it is $+1$ if e enters V_i, -1 if e leaves V_i, and 0 otherwise. Hence $yq_e = \sum[y(X) : X \subseteq V \text{ is entered by } e] - \sum[y(X) : X \subseteq V \text{ is left by } e] = \pi(v) - \pi(u)$. Therefore **(B2)** of (16.32) implies **(A2)** of (16.32), and **(B3)** implies **(A3)**.

We can conclude that y is an optimal dual solution, completing this way both the description of the algorithm and the proof of Theorem 16.3.1 in the general case when the confining function b is crossing submodular.

In the general case when f and g are arbitrary, then a subflow polyhedron may not be bounded. The next result is a characterization for directional boundedness. Let $Q(f, g; b)$ be a subflow polyhedron and $c : A \to \mathbf{R}$ a cost function. Let $D' = (V, A')$ be a digraph where $A' := A_F \cup A_B \cup A_J$ and

$$\begin{cases} A_F := \{uv : uv \in A, g(uv) = \infty\} \\ A_B := \{vu : uv \in A, f(uv) = -\infty\} \\ A_J := \{uv : b(X) = \infty \text{ for every } u\bar{v}\text{-set} X\}. \end{cases}$$

Let $c' : A' \to \mathbf{R}$ be defined as follows.

$$c'(e) := \begin{cases} c(uv) & \text{if } e = uv \in A_F \\ -c(uv) & \text{if } e = vu \in A_B \\ 0 & \text{if } uv \in A_J \end{cases} \tag{16.41}$$

Theorem 16.3.11 *Let $Q = Q(f, g; b)$ a subflow polyhedron and c a cost function. Then $\{cx : x \in Q\}$ is bounded from above if and only if c' is conservative.* •

Problem 16.3.2 *Derive Theorem* 16.3.11.

978–0–19–920527–1 16-Andras-Frank-c16-drv Andras Frank (Typeset by SPi, Publisher Services, India) 552 of 640 December 11, 2010 15:21
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

The optimality criteria formulated above for a primal solution x can be considered an **NP**-characterization since a dual solution y satisfying the optimality critera is a certificate for the optimality of x. One can be interested in a co-**NP**-characterization as well which is an easily verifyable certificate that a certain element of Q is not optimal.

Let x^* be an element of Q. Define a digraph $D^* = (V, A^*)$ as follows. Let $A^* = A_F \cup A_B \cup A_J$ and

$$\begin{cases} A_F := \{uv : uv \in A, x^*(uv) < g(uv)\} \\ A_B := \{vu : uv \in A, x^*(uv) > f(uv)\} \\ A_J := \{uv : uv \text{ leaves no tight set}\}. \end{cases}$$

Let $c^* : A' \to \mathbf{R}$ be defined as follows.

$$c^*(e) := \begin{cases} c(uv) & \text{if } e = uv \in A_F \\ -c(uv) & \text{if } e = vu \in A_B \\ 0 & \text{if } uv \in A_J \end{cases} \tag{16.42}$$

Theorem 16.3.12 *For an element x^* of a subflow polyhedron $Q = Q(f, g; b)$, $cx^* \geq \max\{cx : x \in Q\}$ if and only c^* is conservative in the digraph D^*.* •

Problem 16.3.3 *Derive Theorem* 16.3.12.

17

Covering supermodular functions by digraphs

In the previous chapters, semimodular flows have formed a framework on a digraph in which the net in-flow $\Psi_x(Z) = \varrho_x(Z) - \delta_x(Z)$ was constrained from above by a crossing submodular function or, in an equivalent formulation, from below by a crossing supermodular function. This framework proved to be extremely useful since it included the theorem of Lucchesi and Younger, the g-polymatroid intersection theorem, and almost all specific results on connectivity and degree-constrained orientations from Parts I and II (where the characteristic exception is the best-balanced orientability of graphs).

Despite their generality, semimodular flows do not include some important results on connectivity of a similar sort. For example, in Section 11.4.1 of Part II, we discussed Fulkerson's elegant theorem on the minimum cost of a spanning arborescence of a digraph (Theorem 11.4.1) along with a polyhedral description of the convex hull of spanning arborescences (Corollary 11.4.4). Moreover, it is natural to want to generalize Fulkerson's results and find a polyhedral description for the rooted k-edge-connected (or rooted k-node-connected) subgraphs of a digraph even if $k \geq 2$. It is interesting to notice that the primal side of this problem (that is, finding a cheapest rooted k-edge- or k-node-connected subgraph of a digraph) could be handled neatly with the help of weighted matroid intersection which is part of submodular flows. The dual problem in Theorem 11.4.1, however, does not seem to fit the framework of subflows.

This problem is one of the main motivations for introducing a new framework concerning supermodular functions on digraphs. Actually we will consider two strongly related (and, in fact, equivalent) models. A decisive feature of them is that the linear systems in question are totally dual integral. Moreover, we shall show that the polyhedron of the framework is a submodular flow polyhedron, at least in all the special cases that are used in applications. This property is important in light of the fact that efficient algorithms for submodular flows are available. The first half of the chapter is devoted to examining these links.

Connectivity augmentation problems, discussed in Chapter 11, form a large class where the theory of submodular flows cannot help at all. Theorem 2.2.7, for example, provided a min-max formula for the minimum number of new edges required to make a digraph strongly connected. An explanation why this theorem falls outside the scope of subflows is that its weighted version is **NP**-complete. Therefore we cannot expect a min-max result concerning weighted versions or a total dual integrality (TDI) describing system. None the less, in the second half of the chapter, we shall consider abstract frameworks on covering supermodular functions by digraphs that include not only Theorem 2.2.7 but its extension to higher-order edge- and node-connectivity augmentation problems.

In the new frameworks, instead of constraining the net in-flow Ψ_x, we impose a supermodular lower bound on the in-flow $\varrho_x(Z)$. Mechanically following the definition of submodular flows, one may feel tempted to consider a polyhedron Q described by a linear system

$$\{x \in \mathbf{R}^A_+ : x \leq g \text{ and } \varrho_x(Z) \geq p(Z) \text{ for every } Z \subseteq V\} \tag{17.1}$$

where $g : A \to \mathbf{Z}_+\{\infty\}$ is an upper bound function and p is a *crossing* supermodular function. Here and throughout the chapter we assume that the set-functions in question are integer-valued. This problem formulation, however, is too general since Q is not necessarily integral and optimization over the integral elements of Q may include **NP**-complete problems. Consider, for example, the function p to be identically 1 on non-empty proper subsets of V and $p(V) = p(\emptyset) = 0$. This function is clearly crossing supermodular and the (0, 1)-valued elements of Q are exactly the characteristic vectors of strongly connected subgraphs of D. This implies that optimizing over the (0, 1)-elements of Q is hopeless since even the special problem of deciding whether there is an n-element strong subgraph of a digraph is **NP**-complete, being equivalent to the directed Hamilton circuit problem.

The situation, however, becomes tractable in two special cases. In one of them, *crossing* supermodularity is kept but the underlying digraph is the complete digraph $D^* = (V, A^*)$ in which every possible edge uv $(u, v \in V)$ occurs, and there is no upper bound on x. In this framework, to be discussed in Section 17.2, we shall be able to obtain min-max results which can be considered as abstract versions of connectivity augmentation problems. During the last two decades, several exciting results have been obtained in this area. We will discuss only some of them; the interested reader can find further material in two survey papers: one by Frank and Király [143] and one by Szigeti [355].

In the next section, we develop abstract frameworks concerning *intersecting* supermodular functions on digraphs that allow TDI polyhedral descriptions.

17.1 Supermodular frameworks with TDI systems

Recall the definition of positively intersecting supermodular functions introduced in Section 1.1, where we noticed that such a function can be obtained by replacing each negative value of an intersecting supermodular function by zero. It can be shown that not every positively supermodular function arises in this way. This notion can be further relaxed as follows. Let T be a subset of the ground-set. A set-function p is **positively T-intersecting supermodular** if the supermodular inequality (1.3) is required only for pairs of subsets X, Y for which $X \cap Y \cap T \neq \emptyset$ and $p(X) > 0,\ p(Y) > 0$.

17.1.1 Intersecting supermodular functions

We start by formulating an initial result due to Frank [111, 122].

Theorem 17.1.1 *Let $D = (V, A)$ be a digraph, $p : 2^V \to \mathbf{Z}_+$ a positively intersecting supermodular function, and $g : A \to \mathbf{Z}_+ \cup \{\infty\}$ a function for which $\varrho_g(Z) \geq p(Z)$ for every $Z \subseteq V$. Then the linear system*

$$\{0 \leq x \leq g \text{ and } \varrho_x(Z) \geq p(Z) \text{ for every } Z \subset V\} \tag{17.2}$$

described for $x \in \mathbf{R}^A$ is totally dual integral. In particular, the linear programming problem

$$\min\{cx : x \text{ satisfies } (17.2)\} \tag{17.3}$$

has an integer-valued optimum solution, and so has its linear programming dual, provided c is integer-valued. Furthermore, there is an optimal dual solution for which the set of subsets of V admitting a strictly positive dual variable is laminar. •

We will prove this result soon in a more general setting, but it is worth formulating now two consequences which serve as a main motivation.

Rooted k-edge-connected subgraphs

Theorem 17.1.2 *Let $D = (V, A)$ be a rooted k-edge-connected digraph with respect to a root-node r_0, where $k \geq 1$ is an integer. The linear system on $x \in \mathbf{R}^A$ given by*

$$\{0 \leq x \leq \underline{1},\ \varrho_x(X) \geq k \text{ for every } \emptyset \subset X \subseteq V - r_0\} \tag{17.4}$$

is TDI. In particular, the polyhedron described by (17.4) *is the convex hull* (of the characteristic vectors) *of all spanning r_0-rooted k-edge-connected subgraphs of D, or equivalently the linear programming problem*

$$\min\{cx : x \text{ satisfies } (17.4)\} \tag{17.5}$$

takes its optimum at a rooted k-edge-connected subgraph of D, and if c is integer-valued, the dual linear program also has an integer-valued optimum.

Proof. Let $g \equiv 1$ and define p by

$$p(X) := \begin{cases} k & \text{if } \emptyset \subset X \subseteq V - r_0 \\ 0 & \text{otherwise.} \end{cases} \tag{17.6}$$

Then p is intersecting supermodular and Theorem 17.1.1 implies that (17.4) is TDI. It follows that the polyhedron Q determined by (17.4) is an integral polyhedron. Since each integral element of Q is in fact (0, 1)-valued, the integral elements of Q are precisely the characteristic vectors of rooted k-edge-connected spanning subgraphs of D. •

As a straight extension to the polyhedral description of spanning arborescences (Corollary 11.4.4), we provide a polyhedral description of the minimally rooted k-edge-connected spanning subgraphs of D. Note that due to Theorem 10.1.1 these subgraphs are exactly the ones partitionable into k spanning arborescences.

Theorem 17.1.3 *The linear system on $z \in \mathbf{R}^A$ given by*

$$\{0 \leq x \leq \underline{1},\ \varrho_x(v) = k \text{ for every } v \in V - r_0 \text{ and } \varrho_x(X) \geq k \text{ for every } \emptyset \subset X \subseteq V - r_0\} \tag{17.7}$$

is TDI. In particular, the polyhedron described by (17.7) *is the convex hull of* (the characteristic vectors) *of all minimal spanning r_0-rooted k-edge-connected subgraphs of D, or equivalently, the linear programming problem*

$$\min\{cx : x \text{ satisfies } (17.7)\} \tag{17.8}$$

takes its optimum at a minimal rooted k-edge-connected spanning subgraph of D, and in the case when c is integer-valued the dual linear program also has an integer-valued optimum.

Proof. The TDI-ness of the system in question follows from Theorem 17.1.2 by Theorem 4.1.26. Therefore each vertex of the polyhedron Q defined by (17.7) is the characteristic vector of a minimal r_0-rooted k-edge-connected subgraph of D. Conversely, such a subgraph has in-degree k at every node distinct from the root by Lemma 7.4.1 and hence its characteristic vector belongs to Q. •

Lucchesi and Younger again

The theorem of Lucchesi and Younger was an immediate consequence of Theorem 16.1.3. Here we deduce it from Theorem 17.1.1. In the special case when $g \equiv \infty$ and $c \equiv 1$, Theorem 17.1.1 implies the following result.

Corollary 17.1.4 *The minimum number of* (not necessarily distinct) *edges of a digraph $D = (V, A)$ covering a positively intersecting supermodular function p is equal to the maximum of $\sum[p(X) : X \in \mathcal{F}]$ over laminar families $\mathcal{F}$ which are independent in the sense that no edge of D enters more than one member of $\mathcal{F}$.* •

To obtain the theorem of Lucchesi and Younger, define a set-function p as follows,

$$p(X) := \begin{cases} \sigma(X) & \text{if } \emptyset \subset X \subseteq V \text{ and } \delta_D(X) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{17.9}$$

where $\sigma(X)$ denotes the number of weak components of $D - X$. It follows from (9.34) that p is positively intersecting supermodular. Furthermore, a family F of edges covers p if and only if F enters each subset Z for which $p(Z) = 1$, that is, if F covers each dicut of D. Therefore a minimal family of edges covering p consists of distinct edges of D. Moreover, an independent family $\mathcal{F}$ of sets determines $\sum[p(X) : X \in \mathcal{F}]$ disjoint dicuts, namely, the sets of components of $D - X$ for which $X \in \mathcal{F}$ form the in-shores of the dicuts. Hence the corollary indeed implies the theorem of Lucchesi and Younger.

17.1.2 T-intersecting supermodular functions and orientation constraints

We are going to prove Theorem 17.1.1 in a form which is a generalization from two distinct perspectives. One motivation for extending Theorem 17.1.1 arises from the need to handle, at least in special cases, the problem of finding and describing minimum cost subgraphs of a digraph in which there are k edge-disjoint paths from a root-node r_0 to every element of a designated terminal set T. This problem includes the one answered by Theorem 17.1.2 in the special case $T = V$. On the other hand, the problem formulation is too general in the sense that it is **NP**-complete even in the special case $k = 1$ when it reduces to the Steiner arborescence problem. We will see, however, that under the (severe) restriction that every edge of D has its head in T, there is a TDI-describing linear system. Note that in the special case $T = V$ the restriction is automatically fulfilled.

A second motivation for extending Theorem 17.1.1 comes from the intuitive feeling that there should exist a common background for (cheapest) rooted k-edge-connected orientations of an undirected graph and for the polyhedral description of rooted k-edge-connected subgraphs of a digraph. For example, we may want to find a cheapest subgraph of a mixed graph that has a rooted k-edge-connected orientation. In another related problem we seek to orient a maximum number of edges of an undirected graph such that the resulting mixed graph is rooted k-edge-connected. The idea here is that the notion of orienting an undirected uv-edge e can be interpreted in such a way that we must select exactly one of the two opposite directed edges uv and vu. This is equivalent to assigning (0, 1)-valued variables $x(a)$ and $x(\bar{a})$ to the opposite edges $a = uv$ and $\bar{a} = vu$ such that $x(a) + x(\bar{a}) = 1$. More generally, we call an inequality of type $\alpha \leq x(a) + x(\bar{a}) \leq \beta$ an **orientation constraint** where $\alpha \leq \beta$ are specified integers.

This notion and an extension of Theorem 17.1.1 including orientation constraints are due to Khanna, Naor, and Shepherd [233], while the extension to T-intersecting supermodular functions was discussed in [135]. The next result integrates the two frameworks. Let $D = (V, A)$ be a directed graph with a specified subset T of nodes such that

$$T \text{ contains the head of every edge of } D.$$

Let $p \geq 0$ be a positively T-intersecting supermodular function and $g : A \to \mathbf{Z}_+ \cup \{\infty\}$ a bounding function on the edge-set. Furthermore, let $\mathcal{A} = \{\{a_1, \bar{a}_1\}, \ldots, \{a_h, \bar{a}_h\}\}$ be a set of h pairs consisting of opposite edges of D where the $2h$ edges $a_1, \bar{a}_1, \ldots, a_h, \bar{a}_h$ are distinct. The assumption about T implies that each a_i is induced by T. Furthermore, we are given non-negative integers $\alpha_i \leq \beta_i \leq \infty$ for $i = 1, \ldots, h$ where each α_i is finite.

Theorem 17.1.5 *Suppose that the polyhedron $R \subseteq \mathbf{R}^A$ defined by*

$$\begin{cases} 0 \leq x \leq g \\ \varrho_x(Z) \geq p(Z) \text{ for every } Z \subset V \\ \alpha_i \leq x(a_i) + x(\bar{a}_i) \leq \beta_i \text{ for } i = 1, \ldots, h \end{cases} \tag{17.10}$$

is non-empty. Then the linear system in (17.10) *is totally dual integral. In particular, the linear programming problem*

$$\min\{cx : x \text{ satisfies } (17.10)\} \tag{17.11}$$

has an integer-valued optimum solution and so has its linear programming dual provided c is integer-valued.

Proof. Let $c : A \to \mathbf{Z}$ be integer-valued function for which $\{cx : x \in R\}$ is bounded from below (and hence the dual polyhedron is non-empty). Let $\mathcal{T}$ denote the set-system consisting of subsets of V intersecting T. It follows from the hypotheses that $p(X) > 0$ implies that $X \in \mathcal{T}$. Let Q denote a (0, 1)-matrix in which the rows correspond to the members of $\mathcal{T}$ while the columns correspond to the edges of D. An entry of Q corresponding to X and edge e is 1 if e enters X and zero otherwise. In what follows, we also denote by p a vector for which the components correspond to the members X of $\mathcal{T}$ with value $p(X)$. Furthermore, let

Q_2 denote a (0, 1)-matrix in which the rows correspond to the pairs in $\mathcal{A}$ while the columns correspond to the edges of D. An entry of Q_2 corresponding to a pair $\{a_i, \bar{a}_i\}$ and edge e is 1 for $e = a_i$ and for $e = \bar{a}_i$, and zero otherwise. Let $\alpha = (\alpha_1, \ldots, \alpha_h)$ and $\beta = (\beta_1, \ldots, \beta_h)$. For technical simplicity, we assume that each β_i is finite and that g is also finite-valued. Then the primal linear programming problem is

$$\min\{cx : 0 \leq x,\ -Ix \geq -g,\ Qx \geq p,\ -Q_2 x \geq -\beta,\ Q_2 x \geq \alpha\}$$

In the dual program, the dual variables corresponding to the rows of Q, $-I$, Q_2, and $-Q_2$ are denoted by $y, y_g, y_\alpha, y_\beta$, respectively. The dual constraints are

$$(y, y_g, y_\alpha, y_\beta) \geq 0 \text{ and } yQ - y_g I + y_\alpha Q_2 - y_\beta Q_2 \leq c$$

while the dual objective is maximizing

$$yp - y_g g + y_\alpha \alpha - y_\beta \beta.$$

What we need to prove is that the optimum to the dual program is attained by an integer-valued vector.

Two incomparable sets X and Y are **properly T-intersecting** if $X \cap Y \cap T \neq \emptyset$. A subset $\mathcal{F}$ of $\mathcal{T}$ is **T-laminar** if $\mathcal{F}$ contains no properly T-intersecting members.

Claim 17.1.6 *There exists an optimal dual solution in which the set-system $\mathcal{F}_y := \{X : y(X) > 0,\ p(X) > 0\}$ is T-laminar.*

Proof. Consider an arbitrary optimal dual basic solution y. Since the constraint matrix is integral, y is certainly a rational vector. Suppose that there exist two properly T-intersecting members X, Y of $\mathcal{F}_y$. Define $\gamma := \min\{y(X), y(Y)\}$, decrease both $y(X)$ and $y(Y)$ by γ, and increase both $y(X \cap Y)$ and $y(X \cup Y)$ by γ. Let y' denote the vector obtained in this way from y. Due to the submodularity of the in-degree function ϱ, we have $y'Q \leq yQ$ and hence $(y', y_g, y_\alpha, y_\beta)$ is also a dual solution. Since p is positively T-intersecting supermodular, we have $y'p \geq yp$. Since $(y, y_g, y_\alpha, y_\beta)$ is an optimal dual solution, here $y'p = yp$ must hold, implying that $(y', y_g, y_\alpha, y_\beta)$ is another optimal dual solution. Let us call such a change in the dual solution an uncrossing step. As long as the current $\mathcal{F}_y$ contains two properly T-intersecting members, apply an uncrossing step.

We are going to prove that after a finite number of uncrossing steps $\mathcal{F}_y$ become T-laminar and this will prove the claim. Consider a linear ordering of the members of $\mathcal{T}$ in which X precedes Y if $X \subset Y$. In this ordering, for arbitrary $X, Y \in \mathcal{T}$, $X \cap Y$ precedes both X and Y while $X \cup Y$ follows both of X and Y. Therefore Lemma 1.1.2 implies that the number of uncrossing steps cannot be infinite. •

Let Q' denote the submatrix of Q consisting of the rows of Q corresponding to the members of $\mathcal{F}_y$.

Lemma 17.1.7 *The matrix $M := \begin{pmatrix} Q' \\ Q_2 \end{pmatrix}$ is a network matrix.*

Proof. For an arbitrary directed tree $H = (U, F)$ and a system $\mathcal{P} = \{P_1, \ldots, P_k\}$ of directed subpaths of H, define a (0, 1)-matrix K as follows. The rows of K correspond to the edges of H, the columns of K correspond to the members of $\mathcal{P}$, and an entry of K

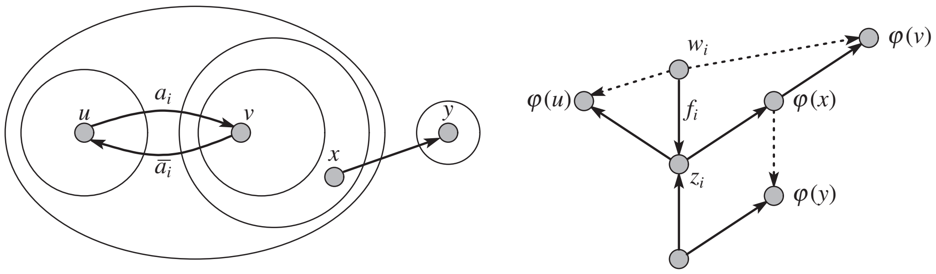


Figure 17.1 Directed tree representation

corresponding to $f \in F$ and $P_i \in \mathcal{P}$ is 1 if $f \in P_i$ and 0 otherwise. In other words, K is the transpose of the incidence matrix of $\mathcal{P}$ with respect to the ground-set F. We will refer to K as the **transposed incidence matrix** of $\mathcal{P}$. By this construction, K is a network matrix.

We are going to construct a directed tree $H = (U, F' \cup F_2)$ and a system $\mathcal{P}_A$ of directed subpaths of H (see Figure 17.1). The elements of F' correspond to the members of $\mathcal{F}_y$ (that is, to the rows of Q'), while the elements of F_2 correspond to pairs in $\mathcal{A}$ (that is, to the rows of Q_2). Furthermore, the members or $\mathcal{P}_A$ correspond to the edges of D in such a way that

$$\text{the transposed incidence matrix of } \mathcal{P}_A \text{ is exactly } M. \tag{17.12}$$

By Claim 17.1.6, $\mathcal{F}_y$ is T-laminar and hence the family $\mathcal{F}'_y := \{X \cap T : X \in \mathcal{F}_y\}$ is laminar. By Theorem 1.4.1, there exists an arborescence $H = (U', F')$ along with a map $\varphi : V \to U'$ such that the members of $\mathcal{F}_y$ and the elements of F' are in a one-to-one correspondence as follows. For every edge $e \in F'$, the member X of $\mathcal{F}_y$ corresponding to e has the property that $X \cap T = \varphi^{-1}(V_e)$ where V_e denotes the component of $H - e$ that is entered by e.

Consider first an arbitrary edge $e = uv \in A$ which does not belong to any pair of $\mathcal{A}$. By the hypothesis, $v \in T$ and the members of the laminar $\mathcal{F}'_y$ which are entered by e form a chain of sets for which the members correspond to the edges of a directed subpath $P(e)$ of F'.

Consider now a pair $\{a_i, \bar{a}_i\} \in \mathcal{A}$ $(i = 1, \ldots, h)$ of opposite edges where a_i is a uv-edge and $\bar{a}_i$ is a vu-edge. Let P denote the unique path in F' connecting $\varphi(u)$ and $\varphi(v)$ and let z_i be the closest node of P to the root of H'. Then the subpaths P'_{a_i} of P from z_i to $\varphi(v)$ is a directed path as well as the subpath $P'_{\bar{a}_i}$ from z_i to $\varphi(u)$. The edges of P'_{a_i} and $P'_{\bar{a}_i}$ correspond to the members of $\mathcal{F}'_y$ entered by a_i and by $\bar{a}_i$, respectively. Let w_i be a new node and $f_i := w_i z_i$ a new directed edge. The set $\{f_i : i = 1, \ldots, h\}$ of new edges is denoted by F_2. Then $F' \cup F_2$ is a directed tree on a node-set U. Furthermore $P_{a_i} := f_i + P'_{a_i}$ and $P_{\bar{a}_i} := f_i + P'_{\bar{a}_i}$ are directed subpaths of the tree $H = (U, F' \cup F_2)$.

By virtue of this construction, the property in (17.12) holds for the directed tree $H = (U, F' \cup F_2)$ and the system $\mathcal{P}_A := \{P_e : e \in A\}$ of directed subpaths of H. •

Since a network matrix is totally unimodular, there is an integer-valued optimal dual solution to the original problem, and hence the linear system in (17.10) is TDI. • •

Relation to submodular flows

In order to find an algorithm for the optimization problem given in Theorem 17.1.5, we are going to prove that the polyhedron R defined by the linear system (17.10) is a submodular flow polyhedron, at least in the special case when p is the non-negative part of a T-intersecting supermodular fuction. Since there are efficient combinatorial algorithms for submodular flows, in this way we shall have one for finding optimal coverings of intersecting supermodular functions. The approach was suggested by Schrijver [336] for the framework of Theorem 17.1.1 and was further developed in [145] for handling orientation constraints as well.

Theorem 17.1.8 *Let $p : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be a T-intersecting supermodular function. Then the polyhedron R determined by* (17.10) *is a submodular flow polyhedron.*

Proof. Since the intersection of a subflow polyhedron with a box is also a subflow polyhedron, it suffices to prove the theorem for the special case when $g = \infty$.

With each pair $\{a_i, \bar{a}_i\} \in \mathcal{A}$, we associate a new node s_i. Let $S_2 := \{s_1, \dots, s_h\}$. Let $A_1 \subseteq A$ denote the set of edges of D which do not belong to any pair in $\mathcal{A}$. Note that each edge entering T belongs to A_1. For each $e \in A_1$, let s_e denote a new node associated with e. Let $S_1 := \{s_e : e \in A_1\}$ and let $S := S_1 \cup S_2$. Let $\hat{A}_1 := \{s_e v : e = uv \in A_1\}$ and $\hat{A}_2 := \{s_i u, s_i v : a_i = uv, \bar{a}_i = vu, i = 1, \dots, h\}$. Define a directed graph $\hat{D} = (\hat{V}, \hat{A})$ where $\hat{V} := S \cup T$ and $\hat{A} = \hat{A}_1 \cup \hat{A}_2$. Note that the $\delta_{\hat{D}}(s_e) = 1$ for every $s_e \in S_1$ and $\delta_{\hat{D}}(s_i) = 2$ for every $s_i \in S_2$. Furthermore, the edges of D and the edges of $\hat{D}$ are in a one-to-one correspondence.

For a subset $Z \subseteq V$, let $S_2(Z)$ denote the set of nodes in S_2 corresponding to pairs of opposite edges of D induced by $Z \cap T$, and let $S_1(Z)$ denote the set of nodes in S_1 corresponding to the elements of A_1 induced by Z, and let $S(Z) := S_1(Z) + S_2(Z)$.

Define a set-function $\hat{p}$ on ground-set $\hat{V}$ as follows.

$$\hat{p}(X) := \begin{cases} p(Z) & \text{if } X = (Z \cap T) \cup F,\ Z \subseteq V,\ F \subseteq S(Z) \\ -\beta_i & \text{if } Z = \{s_i\},\ s_i \in S_2 \\ \alpha_i & \text{if } Z = \hat{V} - s_i,\ s_i \in S_2 \\ 0 & \text{if } Z = \emptyset \text{ or } Z = \hat{V} \\ -\infty & \text{otherwise.} \end{cases}$$

Claim 17.1.9 *$\hat{p}$ is a crossing supermodular function.*

Proof. Let X and X' be two crossing subsets for which $\hat{p}(X)$ and $\hat{p}(X')$ are finite.

Then there are sets $Z \subseteq V$ and $F \subseteq S$ such that $X = (Z \cap T) \cup F$ and $\hat{p}(X) = p(Z)$, and there are sets $Z' \subseteq V$ and $F' \subseteq S$ such that $X' = (Z' \cap T) \cup F'$ and $\hat{p}(X') = p(Z')$.

If $Z \cap T$ and $Z' \cap T$ are disjoint, then there is an element $s \in S$ for which $s \in X \cap X'$ and this s cannot be in S_1 since $\delta_{\hat{D}}(s) = 1$. But s cannot be in S_2 either since if $s = s_i$ for some $i = 1, \dots, h$, and a_i is a uv-edge, then u is in $Z \cap T$ and v is in $Z' \cap T$ contradicting that $s \in F \cap S_2(Z)$. Therefore $Z \cap T$ and $Z' \cap T$ are intersecting.

978–0–19–920527–1 17-Andras-Frank-c17-drv Andras Frank (Typeset by SPi, Publisher Services, India) 561 of 640 December 11, 2010 15:21

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Since $I_D(Z) \cap I_D(Z') = I_D(Z \cap Z')$, we have $F \cap F' \subseteq S(Z \cap Z')$ and similarly $I_D(Z) \cup I_D(Z') \subseteq I_D(Z \cap Y')$ implies that $F \cup F' \subseteq S(Z \cup Y')$. Therefore

$$\hat{p}(X) + \hat{p}(X') = p(Z) + p(Z') \leq p(Z \cap Z') + p(Z \cup Z') =$$

$$\hat{p}((Z \cap Z' \cap T) \cup (F \cup F')) + \hat{p}(((Z \cup Z') \cap T) \cup (F \cap F')) = \hat{p}(X \cap X') + \hat{p}(X \cup X'). \bullet$$

Consider now the supermodular flow polyhedron $\hat{P} := \{x : x \geq 0, \varrho_x - \delta_x \geq \hat{p}\}$. Due to the construction of $\hat{D}$ and $\hat{p}$, $\hat{P}$ is equal to the polyhedron defined by (17.10) for $g \equiv \infty$. $\bullet\,\bullet$

Corollary 17.1.10 *Let $D = (V, F_0 \cup A)$ be a digraph with a specified root-node r_0 and terminal set $T \subseteq V - r_0$ such that the head of each edge in A is in T. Suppose that D is k-edge-connected from r_0 to T. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph $(V, F_0 \cup F)$ is k-edge-connected from r_0 to T is equal to the polyhedron*

$$\{x \in \mathbf{R}^A : 0 \leq x \leq \underline{1} \text{ and } \varrho_x(X) \geq k - \varrho_{F_0}(X) \text{ for each } X \subseteq V - r_0 \text{ with } X \cap T \neq \emptyset\}. \tag{17.13}$$

Furthermore, the linear system in (17.13) *is TDI and determines a submodular flow polyhedron.*

Proof. Let $g : A \to \mathbf{Z}_+$ be identically 1 and define a set-function p_1 as follows.

$$p_1(Z) := \begin{cases} k - \varrho_{F_0}(Z) & \text{if } Z \cap T \neq \emptyset \text{ and } Z \subseteq V - r_0 \\ 0 & \text{otherwise.} \end{cases} \tag{17.14}$$

Then p_1 is positively T-intersecting supermodular and the corollary follows from the special case of Theorem 17.1.5 when there are no orientation inequalities. $\bullet$

Problem 17.1.1 *What is the maximum number of edges of an undirected graph $G = (V, E)$ which can be oriented in such a way that the resulting mixed graph is rooted k-edge-connected? Based on Theorem* 17.1.5*, develop a min-max theorem.* (Note that Theorem 9.1.8 characterized those graphs where this maximum is $|E|$.)

17.1.3 Intersecting supermodular bi-set functions

With an elementary construction, Theorem 17.1.5 can be converted into a result on bi-set functions. We consider only the case when no orientation constraints are imposed. The following result appears in [135].

Theorem 17.1.11 *Let $D = (V, A)$ be a digraph. Let $p : \mathcal{P}_2 \to \mathbf{Z}_+$ be a positively intersecting supermodular bi-set-function and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ a non-negative upper bound on the edges of D that covers p. The linear system*

$$\{0 \leq x \leq g_A \text{ and } \varrho_x(Z) \geq p(Z) \text{ for every bi-set } Z \in \mathcal{P}_2\} \tag{17.15}$$

described for $x \in \mathbf{R}^A$ is totally dual integral. In particular, the linear programming problem

$$\min\{cx : x \text{ satisfies } (17.15)\} \tag{17.16}$$

has an integer-valued optimum solution and so has its linear programming dual provided c is integer-valued. When p is the non-negative part of an intersecting supermodular bi-set-function, then (17.15) *determines a subflow polyhedron.*

Proof. Let S and T be two disjoint copies of V. For a node $u \in V$, let u^S and u^T denote the corresponding elements of S and T, respectively. Define a digraph D' on node-set $V' = S \cup T$ such that for every edge $uv \in A$, let $u^S u^T$ be an edge of D'. With every bi-set $X = (X_O, X_I)$, we associate a subset $X_O^S \cup X_I^S$. An edge $uv \in A$ enters X if and only if the corresponding edge $u^S u^T$ of D' enters $X_O^S \cup X_I^S$. Furthermore, let p' be a set-function on V' for which $p'(Z) = p(X)$ if $Z = X_O^S \cup X_I^S$ and zero otherwise. then p' is a positively T-intersecting supermodular set-function. By applying Theorem 17.1.5 to D' and p' (without any orientation constraint), the theorem follows. •

Note that in the special case when $g_A \equiv \infty$ and p is identically 1 on a given intersecting family of bi-sets and zero otherwise, Theorem 17.1.11 was algorithmically proved in Subsection 11.4.2 with the help of a two-phase greedy algorithm.

Theorem 17.1.11 has a certain self-refining nature. Given a subset $T \subseteq V$, we say that a bi-set-function p is (**positively**) **T-intersecting supermodular** if the supermodular inequality holds for bi-sets X and Y whenever $X_I \cap Y_I \cap T \neq \emptyset$ (and $p(X) > 0,\ p(Y) > 0$).

Proposition 17.1.12 *For bi-set-function p_1, define a bi-set-function p on bi-sets $Z = (Z_O, Z_I)$ by*

$$p(Z) := \begin{cases} \max\{p_1(Z_O, Z_I \cup K) : K \subseteq Z_O - T\} & if\, Z_I \subseteq T \\ 0 & otherwise. \end{cases} \tag{17.17}$$

If p_1 is (positively) *T-intersecting supermodular, then so is p.*

Proof. Let X and Y be two intersecting bi-sets (for which $p(X) > 0,\ p(Y) > 0$ in the case when p_1 is positively T-intersecting supermodular). There are subsets $K \subseteq X_O - T,\ L \subseteq Y_O - T$ for which $p(X) = p_1(X')$ and $p(Y) = p_1(Y')$ where $X' = (X_O, X_I \cup K)$ and $Y' = (Y_O, Y_I \cup L)$. Since $(X_I \cup K) \cap (Y_I \cup L) \neq \emptyset$, $K \cap L \subseteq (X_O \cap Y_O) - T$ and $K \cup L \subseteq (X_O \cup Y_O) - T$, therefore $p_1(X' \cap Y') \leq p(X \cap Y)$ and $p_1(X' \cup Y') \leq p(X \cup Y)$. Hence $p(X) + p(Y) = p_1(X') + p_1(Y') \leq p_1(X' \cap Y') + p_1(X' \cup Y') \leq p(X \cap Y) + p(X \cup Y)$, as required. •

Theorem 17.1.13 *Let $D = (V, A)$ be a digraph and $g_A : A \to \mathbf{Z}_+ \cup \{\infty\}$ an integer-valued function. Let $T \subseteq V$ be a subset of nodes containing the head of every edge of D. Let p_1 be a positively T-intersecting supermodular bi-set-function covered by g_A. Then the linear system*

$$\{0 \leq x \leq g_A \text{ and } \varrho_x(X) \geq p_1(X) \text{ for every bi-set } X\} \tag{17.18}$$

described for $x \in \mathbf{R}^A$ is totally dual integral. When p_1 is the non-negative part of an intersecting T-supermodular function, the polyhedron defined by (17.18) *is a submodular flow polyhedron.*

Proof. By Proposition 17.1.12 the bi-set-function p defined in (17.17) is positively intersecting supermodular. Since every edge has its head in T, a vector $x : A \to \mathbf{R}$ covers p_1 if and only if x covers p. Furthermore, a dual solution y to (17.16) determines a dual solution y_1 to the $\{\min cx : x$ satisfies (17.18)$\}$, as follows. For $X = (X_O, X_I)$ with $X_I \subseteq T$ let Y be the bi-set for which $Y_O = X_O$, $X_I \subseteq Y_I$ and $p(X) = p_1(Y)$. Define $y_1(Y) := y(X)$ if Y arises in this way and $y_1(Y) := 0$ otherwise. Then y_1 is a dual feasible solution to (17.18) having the same value as y does. Therefore Theorem 17.1.11 implies that the system (17.18) is also TDI. •

Application to root-connectivity

Let $k \geq 1$ be an integer and $g : V \to \{1, \ldots, k\}$ a function. A set of edge-disjoint st-paths in a digraph $D = (V, A)$ is g-bounded if every node $v \in V - \{s, t\}$ is used by at most $g(v)$ of these paths. A digraph is rooted (k, g)-connected with respect to a root-node r_0 if there are k g-bounded paths from r_0 to every node of D.

Theorem 17.1.14 *Let $D = (V, A)$ be a rooted (k, g)-connected digraph. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph (V, F) is rooted (k, g)-connected is equal to the polyhedron*

$$\{x \in \mathbf{R}^A : 0 \leq x \leq \underline{1} \textit{ and } \varrho_x(Z) \geq k - w_g(Z)$$
$$\textit{for each bi-set } Z \textit{ with } \emptyset \subset Z_I \subseteq Z_O \subseteq V - r_0\}. \tag{17.19}$$

Furthermore, the linear system in (17.19) *is TDI and describes a submodular flow polyhedron.*

Proof. Let $g_A : A \to \mathbf{Z}_+$ be the identically 1 function and define a bi-set-function p as follows.

$$p(Z) := \begin{cases} k - \varrho_A(Z) - w_g(Z) & \text{if } \emptyset \subset Z_I \subseteq Z_O \subseteq V - r_0 \\ 0 & \text{otherwise.} \end{cases} \tag{17.20}$$

This p is positively intersecting supermodular. By Theorem 17.1.11, the linear system in (17.19) is TDI. Therefore, the polyhedron Q defined by (17.19) is integral. The theorem follows by the easy observation that the integral elements of Q are exactly the characteristic vectors of those subsets F of A for which the subgraph (V, F) of D is rooted (k, g)-connected. •

In the special cases of $g \equiv k$ and $g \equiv 1$, we obtain the convex hull of rooted k-edge-connected and rooted k-node-connected subgraphs, respectively. By using Theorem 17.1.13, the following more general result can also be deduced.

Theorem 17.1.15 *Let $H = (V, F_0 \cup A)$ be a digraph with a specified root-node r_0 and terminal set $T \subseteq V - r_0$ such that the head of each edge in A is in T. Suppose that H is (k, g)-connected from r_0 to T. The convex hull of incidence vectors of the edge-sets $F \subseteq A$ for which the subgraph $(V, F_0 \cup F)$ is (k, g)-connected from r_0 to T is equal to the polyhedron*

$$\{x \in \mathbf{R}^A : 0 \leq x \leq \underline{1} \textit{ and } \varrho_x(Z) \geq p_1(Z) \textit{ for every bi-set } Z\} \tag{17.21}$$

where p_1 *is defined for each bi-set* $Z = (Z_O, Z_I)$ *by*

$$p_1(Z) := \begin{cases} k - \varrho_{F_0}(Z) - w_g(Z) & if Z_I \cap T \neq \emptyset and Z_O \subseteq V - r_0 \\ -\infty & otherwise. \end{cases} \tag{17.22}$$

Furthermore, the linear system in (17.21) *is TDI and determines a submodular flow polyhedron.* •

17.1.4 In-degree bounded covers with applications

We are going to prove a result on in-degree constrained coverings of *set-functions* but the formulation and the proof is technically simpler if we use bi-set functions. Recall that the wall $W(X)$ of a bi-set $X = (X_O, X_I)$ was defined as the set $W(X) = X_O - X_I$. For a function $m : V \to \mathbf{Z}$, we observed that the bi-set-function w_m given by $w_m(X) := \sum[m(v) : v \in W(X)]$ is modular.

Theorem 17.1.16 *Let* $D = (V, A)$ *be a digraph with a specified subset* T *of nodes such that* T *contains the head of every edge of* D.

Let $g_V : V \to \mathbf{Z}_+ \cup \infty$ *and* $g : A \to \mathbf{Z}_+ \cup \infty$ *be two integer-valued functions, and* p *a non-negative, integer-valued, positively* T*-intersecting supermodular set-function on* V *with* $p(\emptyset) = 0$. *There is an integer-valued* $z : A \to \mathbf{Z}$ *for which* $0 \leq z \leq g$, $\dot{\varrho}_z \leq g_V$ *and* $\varrho_z \geq p$ *if and only if*

$$p(X_O) \leq \varrho_g(X) + w_{g_V}(X) \textit{ for every bi-set } X = (X_O, X_I) \textit{ of } V. \tag{17.23}$$

In the special case when $g \equiv \infty$, (17.23) *specializes to*

$$p(Z) \leq \widetilde{g}_V(\Gamma^-(Z)) \textit{ for every subset } Z \subseteq V \tag{17.24}$$

where $\Gamma^-(Z)$ *denotes the entrance of* Z *defined by* $\Gamma^-(Z) := \{v \in Z$ *there is an edge* $uv \in A$ *with* $u \in V - Z\}$.

Proof. Suppose there is a z in the theorem, and let $X = (X_O, X_I)$ be a bi-set. Then the set of edges of D entering X_O can be divided into two disjoint parts according as their head is in X_I or in $W(X)$. Hence $\varrho_z(X_O) = \varrho_z(X) + \sum[z(uv) : uv \in A,\ v \in W(X),\ u \in V - X_O] \leq \varrho_g(X) + w_{g_V}(X)$, and hence (17.23) is indeed necessary.

Suppose now that (17.23) holds. We can assume that g is finite-valued since each infinite component of g can be reduced to the maximum value of p without destroying (17.23). A bi-set is **tight** if it satisfies (17.23) with equality.

Claim 17.1.17 *If* X *and* Y *are* T*-intersecting tight bi-sets for which* $p(X) > 0$ *and* $p(Y) > 0$, *then* $X \sqcap Y$ *and* $X \sqcup Y$ *are also tight.*

Proof. Let $b(X) := \varrho_g(X) + w_{g_V}(X)$. Since ϱ_g is a submodular and w_{g_V} is a modular bi-set-function, b is also submodular. Therefore $p(X_O) + p(Y_O) = b(X) + b(Y) \geq b(X \sqcap Y) + b(X \sqcup Y) \geq p(X_O \cap Y_O) + p(X_O \cup Y_O) \geq p(X_O) + p(Y_O)$ from which we infer $b(X \sqcap Y) = p(X_O \cap Y_O)$ and $b(X \sqcup Y) = p(X_O \cup Y_O)$, as required. •

We can assume that g is minimal in the sense that every edge e with $g(e) > 0$ enters a tight bi-set. Indeed, if e does not enter any tight bi-set, then $g(e)$ can be reduced by 1 without destroying (17.23). We can also assume that g is positive everywhere since an edge e with $g(e) = 0$ can be deleted form D without destroying (17.23).

Claim 17.1.18 *If $\varrho_D(v) > 0$ for a node $v \in T$, then there is a tight bi-set X such that $v \in X_I$ and each edge entering v enters X.*

Proof. Let X be a minimal tight bi-set entered by an edge with head v. We show that every edge entering v enters X. Suppose, indirectly, that an edge $e = uv$ does not enter X. Since e enters a tight bi-set Y with $p(Y_O) > 0$ and since $p(X_O) = \varrho_g(X) + w_{g_V}(X) > 0$, we obtain from Claim 17.1.17 that $X \sqcap Y$ is also tight. But this contradicts the minimality of X since e enters $X \sqcap Y$ and $X \sqcap Y \sqsubset X$. •

Claim 17.1.19 $\dot{\varrho}_g \le g_V$.

Proof. Suppose indirectly that there is a node v for which $\varrho_g(v) > g_V(v)$. Then $v \in T$ and by Claim 17.1.18 there is a tight bi-set X such that $v \in X_I$ and each edge entering v enters X. Let $X'_I := X_I - v$ and consider the bi-set $X' = \left(X_O, X'_I\right)$. Then $p(X_O) = \varrho_g(X) + w_{g_V}(X) = \varrho_g(X') + \varrho_g(v) + w_{g_V}(X) = \varrho_g(X') + \varrho_g(v) + w_{g_V}(X') - g_V(v) > \varrho_g(X') + w_{g_V}(X')$ and this contradiction to (17.23) verifies the claim. •

By taking $z := g$, the main part of the theorem follows. The necessity of (17.24) in the special case when $g \equiv \infty$ follows the same way as that of (17.23). For the sufficiency, we prove that (17.24) implies (17.23) when $g \equiv \infty$. To this end, suppose indirectly that there is a bi-set X for which $p(X_O) > \varrho_g(X) + w_{g_V}(X)$. Let $Z := X_O$. Since $g \equiv \infty$, we must have $\varrho_D(X) = 0$ from which $\Gamma^-(Z) \subseteq X_O - X_I$ and hence $p(Z) = p(X_O) > \varrho_g(X) + w_{g_V}(X) = w_{g_V}(X) \ge \widetilde{g}_V(\Gamma^-(Z))$, contradicting (17.24). • •

Remark 17.1.1 What happens if we impose upper bounds on the out-degrees rather than on the in-degrees? Perhaps a bit surprisingly, this version of the problem includes **NP**-complete problems. To see this, let r_0 be a special node of a digraph $D = (V, A)$ for which $\varrho_D(r_0) = 0$. Define a set-function p as follows. Let $p(X) = 1$ if $\emptyset \subset X \subseteq V - r_0$ and $p(X) = 0$ otherwise. Let $g_V \equiv 1$. If $x \in \mathbf{R}^A_+$ is an integral vector covering p for which $\delta_x(v) \le 1$ for every node v, then x is the incidence vector of the edge-set of a subgraph (V, F) of D in which every out-degree is most 1 and which is an s-rooted spanning arborescence. In other words, F is the edge-set of a Hamilton path starting at r_0. But the Hamilton path problem is **NP**-complete.

Application to arborescences

Recall the theorem of Vidyasankar (Theorem 10.4.2) on the existence of k spanning arborescences of a digraph D covering all edges of D. We will apply Theorem 17.1.16 in the special case when $T = V - r_0$ and derive the following generalization [154] in which both lower and upper bounds are imposed on the edges. The second, special part of the next result is just a reformulation of Vidyasankar's theorem.

Theorem 17.1.20 *Let r_0 be a root-node of digraph $D = (V, A)$ in which no edge enters r_0. Let $f : A \to \{0, 1, \ldots, k\}$ and $g' : A \to \mathbf{Z}_+ \cup \infty$ be two functions on A for which*

$f \le g'$. Let $g := g' - f$ and define $g_V : (V - r_0) \to \mathbf{Z}$ by $g_V(v) := k - \varrho_f(v)$. There are k spanning arborescences of root r_0 such that each edge e belongs to at least $f(e)$ and at most $g'(e)$ of the k arborescences if and only if $g_V \ge 0$ and

$$k - \varrho_f(Z) \le \widetilde{g}_V(Y) + \sum[g(e) : e \in A \text{ a } (V - Z, Z - Y)\text{-edge}]$$
$$\text{whenever } Y \subseteq Z \subseteq V - r_0, Z \ne \emptyset\}. \tag{17.25}$$

In the special case when $g' \equiv \infty$, (17.25) *specializes to*

$$k - \varrho_f(Z) \le \widetilde{g}_V(\Gamma^-(Z)) \text{ whenever } 0 \subset Z \subseteq V - r_0 \tag{17.26}$$

where $\Gamma^-(Z)$ is the entrance of Z.

Proof. Suppose that the k arborescences exist with the required properties. The necessity of $\varrho_f(v) \le k$ is evident. Let $x(e)$ denote the number of these arborescences containing e and let $z(e) := x(e) - f(e)$. Then $0 \le z \le g$ and we have

$$k \le \varrho_x(Z) = \varrho_f(Z) + \varrho_z(Z) = \varrho_f(Z) + \sum[z(e) : e \text{ a } (V - Z, Y)\text{-edge}$$
$$+ \sum[z(e) : e \text{ a } (V - Z, Z - Y)\text{-edge}] \le \varrho_f(Z) + \sum[\varrho_z(v) : v \in Y]$$
$$+ \sum[g(e) : e \text{ a } (V - Z, Z - Y)\text{-edge}] \le \varrho_f(Z)$$
$$+ \sum[k - \varrho_f(v) : v \in Y] + \sum[g(e) : e \text{ a } (V - Z, Z - Y)\text{-edge}]$$
$$= \varrho_f(Z) + \widetilde{g}_V(Y) + \sum[g(e) : e \text{ a } (V - Z, Z - Y)\text{-edge}]$$

showing the necessity of (17.25).

Sufficiency. Suppose now that (17.25) holds. Define a set-function p as follows.

$$p(X) := \begin{cases} (k - \varrho_f(X))^+ & \text{if } \emptyset \subset X \subseteq V - r_0 \\ 0 & \text{otherwise.} \end{cases} \tag{17.27}$$

Then p is positively intersecting supermodular.

Claim 17.1.21 *There is an integer-valued $z : A \to \mathbf{Z}$ for which $0 \le z \le g$, $\dot{\varrho}_z \le g_V$ and $\varrho_z \ge p$.*

By Theorem 17.1.16, it suffices to show that (17.23) is fulfilled. Suppose indirectly that a bi-set X violates (17.23), and hence $p(X_O) > \varrho_g(X) + w_{g_V}(X)$. Let $Z := X_O$ and $Y := X_O - X_I$. Since $k - \varrho_f(X_O) = p(X_0)$, $w_{g_V}(X) = \widetilde{g}_V(Y)$ and $\varrho_g(X) = \sum[g(e) : e \in A$ a $(V - Z, Z - Y)$-edge], we can conclude that Y and Z violate (17.25), a contradiction. •

Replace each edge $a \in A$ by $f(a) + z(a)$ parallel edges and let D' denote the resulting digraph. Then $\varrho_{D'}(X) \ge k$ for every non-empty subset X of $V - r_0$ since $\varrho_{D'}(X) = \varrho_f(X) + \varrho_z(X) \ge k$ follows from $\varrho_z \ge p$. Moreover, $\varrho_{D'}(v) = k$ for every node $v \in V - r_0$ since $k \le \varrho_{D'}(v) = \varrho_z(v) + \varrho_f(v) \le k$ follows from $\varrho_z(v) \le g_V(v) = k - \varrho_f(v)$. By Edmonds' disjoint arborescences theorem the edge-set of D' partitions into k spanning arborescences of root r_0. Due to the construction of D', these arborescences correspond to k arborescences of D satisfying the requirements.

The second part of the theorem, when $g' \equiv \infty$, follows in the same way from the second part of Theorem 17.1.16. • •

Application to degree-constrained dijoins

The theorem of Lucchesi and Younger provided a min-max formula for the minimum cardinality of a dijoin of a digraph where a dijoin is a subset of edges covering all directed cuts. We are now interested in dijoins having an upper bound on the in-degrees.

Theorem 17.1.22 *In a digraph $D = (V, A)$ there is dijoin which is a branching if and only if, for every non-empty subset $X \subseteq V$, the number of components K of $D - X$ for which $\varrho_D(K) = 0$ is at most $|X|$.*

Proof. Necessity. Let $F \subseteq A$ be a branching covering every dicut and let K be a component of $D - X$ with $\varrho_D(K) = 0$. Then K determines a dicut of D such F contains at least one edge leaving K. Since F is a branching, there is at most one element of F entering any element of X. Therefore the number of components in question is at most $|X|$.

Sufficiency. Define a set-function p as follows.

$$p(X) := \begin{cases} 0 & \text{if } X = \emptyset \text{ or } \delta_D(X) > 0 \\ \sigma(X) & \text{if } X \neq \emptyset,\ \delta_D(X) = 0 \end{cases} \tag{17.28}$$

where $\sigma(X)$ denotes the number of components induced by $V - X$ (that is, the number of components of $D - X$). By Proposition 1.2.6, p is a positively intersecting supermodular function.

Let $g_V \equiv 1$. If there is an integral x for which $\varrho_x \geq p$ and $\varrho_x(v) \leq 1$ for every node v, then x is $(0, 1)$-valued and the set $F := \{e : x(e) = 1, e \in A\}$ is a dijoin. Since $\varrho_F(v) \leq 1$ for every node v, we can assume that F is a branching, for if F includes a directed circuit C, then $F - X$ is also a dijoin.

Consider now the case when the x in question does not exist. By the second part of Theorem 17.1.16, there is a subset Z for which $p(Z) > \widetilde{g}_V(\Gamma^-(Z)) = |\Gamma^-(Z)|$. Let $X := \Gamma^-(Z)$. Since $p(Z) > 0$, no edge leaves Z.

Therefore every component of $D - Z$ is a component of $D - X$, too. Hence the number of components K of $D - X$ for which $\varrho_D(K) = 0$ is at least $p(Z)$ which in turn is larger than $|X|$, contradicting the hypothesis. •

Problem 17.1.2 *Given $g_V : V \to \mathbf{Z}_+$, a digraph $D = (V, A)$ admits a dijoin F for which $\varrho_F(v) \leq g_V(v)$ for every node $v \in V$ if and only if the number of components K of $D - X$ for which $\varrho_D(K) = 0$ is at most $\widetilde{g}_V(X)$ for every non-empty $X \subset V$.*

Covering supermodular functions by bipartite graphs

As another application of Theorem 17.1.16, we are going to derive an extension of a theorem of Lovász (Theorem 12.3.3) which appeared in [154]. In order to have better notational compliance, we rewrite Theorem 12.3.3 in a form where the letters S and T are interchanged:

Theorem 17.1.23 *Let $p : 2^T \to \mathbf{Z}_+$ be a positively intersecting supermodular function and element-subadditive. Suppose that $G = (S, T; E)$ is a simple bipartite graph covering p in* *the sense that* $|\Gamma_E(X)| \geq p(X)$ *for every* $X \subseteq T$. *If* $G' = (S, T; F)$ *is a minimal subgraph of* G *which covers* p, *then* $d_F(v) = p(v)$ *for every* $v \in T$. •

Theorem 17.1.24 *Let* $p : 2^T \to \mathbf{Z}_+$ *be a positively intersecting supermodular function with* $p(\emptyset) = 0$, $g_T : T \to \mathbf{Z}_+$ *an upper bound function, and* $G = (S, T; E)$ *a simple bipartite graph. There is a subgraph* $G_F = (S, T; F)$ *of* G *such that*

$$|\Gamma_F(X)| \geq p(X) \textit{ for every } X \subseteq T \tag{17.29}$$

and

$$d_F(v) \leq g_T(v) \textit{ for every } v \in T \tag{17.30}$$

if and only if

$$p(Y) \leq |\Gamma_E(X)| + \widetilde{g}_T(Y - X) \textit{ whenever } X \subseteq Y \subseteq T. \tag{17.31}$$

Proof. Necessity. Suppose that there is an F satisfying (17.29) and (17.30). Then $p(Y) \leq |\Gamma_F(Y)| \leq |\Gamma_F(X)| + d_F(Y - X) \leq |\Gamma_E(X)| + \widetilde{g}_T(Y - X)$, showing the necessity of (17.31).

For the sufficiency, define a set-function p' on $V := S \cup T$ by $p'(X) := (p(X \cap T) - |S \cap X|)^+$. Then p' is positively T-intersecting. Let g' be defined by

$$g_V(v) := \begin{cases} g_T(v) & \text{if } v \in T \\ 0 & \text{if } v \in S. \end{cases}$$

Let D' denote the digraph arising from G by orienting each edge toward T.

Claim 17.1.25 $p'(Z) \leq \widetilde{g}_V(\Gamma^-(Z))$ *for every subset* $Z \subseteq V$.

Proof. We can assume that $p'(Z)$ is positive. Let $Y := Z \cap T$ and $X := Y - \Gamma^-(Z)$ where Γ^- is the entrance function of digraph D. Then $\Gamma^-(Z) = Y - X$ and hence $g_T(Y - X) = g_V(\Gamma^-(Z))$. Furthermore, $\Gamma_E(X) \subseteq Z \cap S$ and hence $|\Gamma_E(X)| \leq |Z \cap S|$. By (17.31) we have $p(Y) \leq |\Gamma_E(X)| + \widetilde{g}_T(Y - X)$. By combining these observations, we conclude that $p'(Z) = p(Y) - |Z \cap S| \leq p(Y) - |\Gamma_E(X)| \leq g_T(Y - X) = g_V(\Gamma^-(Z))$, as required. •

The second part of Theorem 17.1.16, when applied to p', implies that there is an integer-valued $z : A \to \mathbf{Z}_+$ for which $\dot{\varrho}_z \leq g_V$ and $\varrho_z \geq p'$. Suppose that z is minimal with respect to this property.

Claim 17.1.26 z *is* $(0, 1)$*-valued.*

Proof. Suppose indirectly that there is an edge $e = st$ of D for which $z(e) \geq 2$ where $s \in S$ and $t \in T$. The minimality of z implies that e enters a subset Z for which $\varrho_z(Z) = p'(Z)$. But then for set $Z' := Z + s$ we have

$$\varrho_z(Z') \geq p'(Z') = p'(Z) - 1 = \varrho_z(Z) - 1 \geq \varrho_z(Z') + z(e) - 1 \geq \varrho_z(Z') + 1,$$

a contradiction. •

Let $F := \{e = st \in E : z(st) = 1\}$. We claim that $|\Gamma_F(X)| \geq p(X)$ for every $X \subseteq T$. Indeed, for $Z := X \cup \Gamma_F(X)$ we have $0 = \varrho_z(Z) \geq p'(Z) = p(X) + |\Gamma_F(X)|$, and hence F satisfies the requirements. • •

Problem 17.1.3 *Derive Theorem* 17.1.23 *from Theorem* 17.1.24.

17.2 Non-TDI frameworks for covering set-functions

So far we have investigated two large classes of semimodular frameworks on directed graphs. In submodular flows the net inflow $\Psi_x = \varrho_x - \delta_x$ was bounded from above by a *crossing* submodular function. In the previous section, ϱ_x was bounded by an *intersecting* supermodular function from below. Both models shared the decisive feature that the linear inequality system in question was totally dual integral. Precisely the presence of this strong property indicates that these frameworks are not suitable to capture specific connectivity problems where the minimum cost version is **NP**-complete. For example, Theorem 2.2.7 provided a min-max formula for the minimum number of new edges required to make a digraph strongly connected while the min-cost version is **NP**-complete.

In the present section, we develop a new framework which involves several connectivity augmentation problems. This time ϱ_x is bounded from below by a *crossing supermodular* function. The underlying directed graph is, however, very special being the complete digraph or, more generally, a digraph consisting of all st-edges with $s \in S$ and $t \in T$ where S and T are two specified subsets of nodes. Recall that a digraph $D = (V, A)$ is said to cover a set-function p if $\varrho_D \geq p$, that is,

$$\varrho_D(X) \geq p(X) \text{ for every } X \subseteq V, \tag{17.32}$$

and similarly a function $x : A \to \mathbf{Z}$ covers p if $\varrho_x \geq p$. We also recall that $\dot{\varrho}_H$ denotes the restriction of set-function ϱ_H to the elements of V. For example, $\dot{\varrho}_H \leq g$ is shorthand for the requirement that $\varrho_H(v) \leq g(v)$ for every node $v \in V$.

17.2.1 Digraphs covering crossing supermodular set-functions

Recall Theorem 11.2.1 on degree-specified edge-connectivity augmentation:

Theorem 17.2.1 *Let $D = (V, A)$ be a digraph and $m_{in} : V \to \mathbf{Z}_+$, $m_{out} : V \to \mathbf{Z}_+$ degree specifications. There is a digraph $H = (V, F)$ for which $D + H$ is k-edge-connected and*

$$\dot{\varrho}_H = m_{in} \text{ and } \dot{\delta}_H(v) = m_{out} \tag{17.33}$$

if and only if

$$\widetilde{m}_{in}(V) = \widetilde{m}_{out}(V),$$

$$\widetilde{m}_{in}(X) \geq k - \varrho_D(X) \text{ for } \emptyset \neq X \subset V \tag{17.34}$$

and

$$\widetilde{m}_{out}(X) \geq k - \delta_D(X) \text{ for } \emptyset \neq X \subset V. \tag{17.35}$$

This result was actually a reformulation of Mader's directed splitting-off lemma. We are going to extend this theorem on covering supermodular functions by digraphs. For a set-function p on ground-set V for which $p(\emptyset) = 0 = p(V)$, let $\hat{p}$ denote the set-function for which

$$\hat{p}(X) := p(V - X) \text{ for every } X \subseteq V. \tag{17.36}$$

Obviously $\hat{p}(\emptyset) = \hat{p}(V) = 0$, and if p is positively crossing supermodular, then so is $\hat{p}$. The next result of Frank [128] is an abstract extension: in the special case when p is defined by $p(X) := [k - \varrho_D(X)]^+$ we are back at Theorem 17.2.1.

Theorem 17.2.2 *Let p be a non-negative, integer-valued, and positively crossing supermodular set-function on a ground-set V for which $p(\emptyset) = p(V) = 0$. Let m_{in} and m_{out} be two non-negative integer-valued functions on V and γ an integer so that $\widetilde{m}_{in}(V) = \gamma = \widetilde{m}_{out}(V)$. There exists a digraph $H = (V, F)$ covering p for which*

$$\dot{\varrho}_H = m_{in} \text{ and } \dot{\delta}_H = m_{out} \tag{17.37}$$

if and only if

$$p \leq \widetilde{m}_{in} \tag{17.38}$$

and

$$\hat{p} \leq \widetilde{m}_{out} \tag{17.39}$$

If, in addition, H is requested to be loopless, then the conditions above must be supplemented by

$$m_{in}(v) + m_{out}(v) \leq \gamma \text{ for every } v \in V. \tag{17.40}$$

Proof. Necessity. Suppose there is a digraph H with the required properties. Then $\widetilde{m}_{in}(X) = \sum[\varrho_H(v) : v \in X] \geq \varrho_H(X) \geq p(X)$ and $\widetilde{m}_{out}(V - X) = \sum[\delta_H(v) : v \in V - X] \geq \delta_H(V - X) = \varrho_H(X) \geq p(X)$, and hence (17.38) and (17.39) hold. Moreover, if H is loopfree, then every edge of head v is distinct from every edge of tail v and hence $m_{in}(v) + m_{out}(v) = \varrho_H(v) + \delta_H(v) \leq \gamma$.

Sufficiency. Assume indirectly that no digraph exists with the required properties. Since $\widetilde{m}_{in}(V) = \widetilde{m}_{out}(V)$, it can easily be seen that there is a digraph H (possibly with loops and parallel edges) satisfying (17.37). Let $q_H(X) := p(X) - \varrho_H(X)$ and $\mu_H := \max\{q_H(X) : X \subseteq V\}$. In fact, H can be chosen loopfree when (17.40) holds. Let $\mathcal{F}_H := \{X \subset V : q_H(X) = \mu_H\}$. By $p(\emptyset) = 0 = \varrho_H(\emptyset)$, we have $\mu_H \geq 0$ and since $\mu_H = 0$ is equivalent to (17.32), μ_H must be positive. This implies that $p(X) > 0$ for every member X of $\mathcal{F}_H$.

Claim 17.2.3 *Let X and Y be two crossing members of $\mathcal{F}_H$. Then both $X \cap Y$ and $X \cup Y$ belong to $\mathcal{F}_H$.*

Proof. Since ϱ_H is submodular, p is positively crossing supermodular, and $p(X) > 0$ and $p(Y) > 0$, we have $\mu_H + \mu_H = q_H(X) + q_H(Y) \leq q_H(X \cap Y) + q_H(X \cup Y) \leq \mu_H + \mu_H$ from which $q_H(X \cap Y) = \mu_H$ and $q_H(X \cup Y) = \mu_H$, and the claim follows. •

978–0–19–920527–1 17-Andras-Frank-c17-drv Andras Frank (Typeset by SPi, Publisher Services, India) 571 of 640 December 11, 2010 15:21

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Assume now that H is chosen in such a way that $(*)$ μ_H *is as small as possible* and, subject to this, $(**)$ $|\mathcal{F}_H|$ *is as small as possible.*

Let K be a minimal member of $\mathcal{F}_H$ and $L \supseteq K$ a maximal member of $\mathcal{F}_H$. There is an edge $e = uv$ of H with $u, v \in K$ for otherwise $\widetilde{m}_{in}(K) = \sum[\varrho_H(z) : z \in K] = \varrho_H(K) = p(K) - \mu_H < p(K)$, contradicting (17.38). There is an edge $f = xy$ of H with $x, y \in V - L$ for otherwise $\widetilde{m}_{out}(V - L) = \sum[\delta_H(z) : v \in V - L] = \delta_H(V - L) = \varrho_H(L) = p(L) - \mu_H < p(L)$, contradicting (17.39).

Revise H by replacing edges e and f with edges uy and xv, and let H' denote the resulting digraph. Clearly, H' satisfies (17.37), as well. The following is immediate.

Claim 17.2.4 *If $\varrho_{H'}(X) < \varrho_H(X)$ for some subset $X \subseteq V$, then either uv enters X and xy leaves X or else uv leaves X and xy enters X* (and, in particular, none of e and f is a loop). •

This implies $\varrho_{H'}(X) \geq \varrho_H(X) - 1$ for every subset $X \subseteq V$. There is no set $X \in \mathcal{F}_H$ for which $\varrho_{H'}(X) = \varrho_H(X) - 1$, for otherwise X and K are crossing by Claim 17.2.4 and then $X \cap K \in \mathcal{F}_H$ by Claim 17.2.3, contradicting the minimal choice of K. It follows that $\mu_{H'} \leq \mu_H$ and actually here we have equality by assumption $(*)$.

Since $\varrho_{H'}(K) > \varrho_H(K)$, the subset K is not in $\mathcal{F}_{H'}$. By assumption $(**)$, there must be a set X which is in $\mathcal{F}_{H'} - \mathcal{F}_H$. Then $\varrho_{H'}(X) < \varrho_H(X)$, and hence $\varrho_{H'}(X) = \varrho_H(X) - 1$. Apply Claim 17.2.4. By symmetry we can assume that uv enters X and xy leaves X.

We have $q_H(X) + 1 = q_{H'}(X) = \mu_{H'} = \mu_H$. Since $q_{H'}(X) = \mu_{H'}$ and $q_H(K) = \mu_H$ are positive numbers, so are $p(X)$ and $p(K)$. Since K and X are crossing, and by the minimal choice of K we have

$$\mu_H + (\mu_H - 1) = q_H(K) + q_H(X) \leq q_H(K \cap X) + q_H(K \cup X) \leq (\mu_H - 1) + \mu_H,$$

from which $q_H(K \cup X) = \mu_H$. That is, $X' := K \cup X$ belongs to $\mathcal{F}_H$. Since L and X' are either crossing or $L \subset X'$, we obtain by using Claim 17.2.3 that $X' \cup L \in \mathcal{F}_H$, in a contradiction with the maximal choice of L. • •

Research problem 17.2.1 *Develop a necessary and sufficient condition for the existence of a simple digraph H that covers a crossing supermodular set-function and satisfies $\dot{\varrho}_H = m_{in}$, $\dot{\delta}_H = m_{out}$.*

What happens if only the in-degree vector m_{in} is specified and no restriction is made for the out-degree function δ_H of H? That is, our goal is to find a digraph $H = (V, F)$ covering p for which $\dot{\varrho}_H = m_{in}$. The inequality (17.38) remains necessary at any rate. Since (17.39) in Theorem 17.2.2 has nothing to do with m_{in}, one may hope that (17.38) is perhaps sufficient in the present case, too, when only m_{in} is prescribed. Unfortunately, and perhaps surprisingly, this is not the case. For example, let p and m_{in} be defined on a ground-set $V := \{a, b, c\}$ as follows.

$$p(X) := \begin{cases} 1 & \text{if } X = \{a, b\} \\ 1 & \text{if } X = \{b, c\} \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$m_{in}(v) := \begin{cases} 1 & \text{if } v = b \\ 0 & \text{otherwise.} \end{cases}$$

Now p is crossing supermodular and $p \leq \widetilde{m}_{in}$ holds, yet there is no digraph H covering p for which $\dot{\varrho}_H = m_{in}$.

In what follows we not only overcome this difficulty, but by making use of feasibility theorems of base-polyhedra we solve the covering problem when upper bounds are given on $\dot{\delta}_H$, on $\dot{\varrho}_H$, and on the number of edges of H as well.

For a given integer $\gamma \geq 0$ and set-function p, define set-functions p^γ and $\hat{p}^\gamma$ as follows.

$$p^\gamma(X) := \begin{cases} p(X) & \text{if } X \subset V, \\ \gamma & \text{if } X = V, . \end{cases} \tag{17.41}$$

$$\hat{p}^\gamma(X) := \begin{cases} p(V - X) & \text{if } X \subset V, \\ \gamma & \text{if } X = V. \end{cases} \tag{17.42}$$

Proposition 17.2.5 *Let $p \geq 0$ be a positively crossing, integer-valued supermodular set-function with $p(V) = 0$ and γ an integer so that*

$$\gamma \ \geq \ \hat{p}(X) + \hat{p}(Y) \ \ [= p(V - X) + p(V - Y)] \ \ \textit{for disjoint sets } X, Y \subset V. \tag{17.43}$$

Let $g : V \to \mathbf{Z}_+ \cup \{\infty\}$ be a function for which $\gamma \leq \widetilde{g}(V)$ and let $T(\leq g) := \{x \in \mathbf{R}^V : x \leq g\}$. Then p^γ is positively intersecting supermodular and both $B'(p^\gamma)$ and $B'(p^\gamma) \cap T(\leq g)$ are integral base-polyhedra. Furthermore, $B'(p^\gamma) \cap T(\leq g)$ is non-empty if and only if

$$p(X) \leq \widetilde{g}(X) \textit{ for every } X \subseteq V \tag{17.44}$$

and

$$\sum_i p(V_i) \leq \gamma \textit{ for every partition } \{V_1, \ldots, V_t\} \textit{ of } V. \tag{17.45}$$

Proof. Since p is positively crossing supermodular, so is p^γ, and (17.43) ensures the supermodular inequality for co-disjoint sets, p^γ is positively intersecting supermodular. By Theorem 15.1.3, $B'(p^\gamma)$ is a base-polyhedron and so is $B'(p^\gamma) \cap T(\leq g)$ by Theorem (14.3.9).

The proof of necessity of (17.44) and (17.45) is an easy exercise and left to the reader. For their sufficiency, consider Part (B) of Theorem 15.1.5 with p^γ in place of p_1. Now $p^\gamma \leq \widetilde{g}$ holds since $p^\gamma(X) = p(X) \leq \widetilde{g}(X)$ for every $X \subset V$ by (17.44) and also $p^\gamma(V) = \gamma \leq \widetilde{g}(V)$. By (17.45), we also have $\sum_i p^\gamma(V_i) \leq \sum_i p(V_i) \leq \gamma = p^\gamma(V)$ for every partition $\{V_1, \ldots, V_t\}$ of V. Hence Theorem 15.1.5 implies that $B'(p^\gamma) \cap T(\leq g)$ is non-empty. •

Theorem 17.2.6 *Let $p \geq 0$ be a positively crossing, integer-valued supermodular set-function on ground-set V with $p(V) = 0$. Let $g_{in} : V \to \mathbf{Z}_+ \cup \{\infty\}$ and $g_{out} : V \to \mathbf{Z}_+ \cup \{\infty\}$ be upper bound functions. Let $\gamma \geq 0$ be an integer for which*

$$\gamma \leq \min\{\widetilde{g}_{in}(V),\ \widetilde{g}_{out}(V)\}.$$

There exists a digraph $H = (V, F)$ covering p for which

$$\dot{\delta}_H \leq g_{out}, \quad \dot{\varrho}_H \leq g_{in}, \quad |F| \leq \gamma$$

if and only if

$$p \leq \widetilde{g}_{in}, \tag{17.46}$$

$$\hat{p} \leq \widetilde{g}_{out}, \tag{17.47}$$

$$\sum_i p(V_i) \leq \gamma \text{ for every partition } \{V_1, \ldots, V_t\} \text{ of } V, \tag{17.48}$$

$$\sum_i \hat{p}(V_i) \leq \gamma \text{ for every partition } \{V_1, \ldots, V_t\} \text{ of } V. \tag{17.49}$$

Proof. Suppose first that there is a digraph $H = (V, A)$ satisfying the requirements. Then $\widetilde{g}_{in}(Z) \geq \sum_{v \in Z} \varrho_H(v) \geq \varrho_H(Z) \geq p(Z)$ for every $Z \subseteq V$, and hence (17.46) holds. Similarly, $\widetilde{g}_{out}(Z) \geq \sum_{v \in Z} \delta_H(v) \geq \delta_H(Z) = \varrho_H(V - Z) \geq p(V - Z) = \hat{p}(Z)$ for every $Z \subseteq V$, and hence (17.47) holds.

For a partition $\{V_1, \ldots, V_t\}$ of V, we have $\gamma \geq |F| \geq \sum_i \varrho_H(V_i) \geq \sum_i p(V_i)$, and hence (17.48) holds. Also, $\gamma \geq |F| \geq \sum_i \delta_H(V_i) = \sum_i \varrho_H(V - V_i) \geq \sum_i p(V - V_i) = \sum_i \hat{p}(V_i)$, and hence is, (17.47) holds.

Suppose now that the four conditions in the theorem hold. For proving their sufficiency, we can assume that both g_{in} and g_{out} are finite-valued since replacing an infinite component with a sufficiently large integer does not affect the conditions.

Claim 17.2.7 *$\hat{p}(X) + \hat{p}(Y) \leq \gamma$ holds for disjoint subsets X, Y of V.*

Proof. If $\{X, Y\}$ is a partition, then (17.49) implies $\hat{p}(X) + \hat{p}(Y) \leq \gamma$. If $X \cup Y \subset V$, then by using the non-negativity of $\hat{p}$ and applying (17.49) to the partition $\{V_1, V_2, V_3\}$, where $V_1 := X$, $V_2 := Y$, $V_3 := V - (X \cup Y)$, we obtain that $\hat{p}(X) + \hat{p}(Y) \leq \sum_{i=1}^{3} \hat{p}(V_i) \leq \gamma$. •

For $g := g_{in}$, condition (17.44) is implied by (17.46). Furthermore (17.45) is the same as (17.48). Therefore Proposition 17.2.5 implies that there is an integral vector m_{in} for which $p \leq \widetilde{m}_{in}$, $m_{in} \leq g_{in}$, and $m_{in}(V) = \gamma$.

Since the role of $\hat{p}$ and p is symmetric, we can apply in the same way Proposition 17.2.5 to $\hat{p}$ in place of p and to $g := g_{out}$ and conclude that there is an integral vector m_{out} for which $\hat{p} \leq \widetilde{m}_{out}$, $m_{out} \leq g_{out}$, and $m_{out}(V) = \gamma$. Finally, Theorem 17.2.2 ensures the existence of the requested digraph H. • •

Problem 17.2.2 *Prove that in Theorem* 17.2.6 *if $\widetilde{g}_{in}(V) \leq \widetilde{g}_{out}(V)$, then conditions* (17.46), (17.47)*, and* (17.49) *are already sufficient. If $\widetilde{g}_{in}(V) = \widetilde{g}_{out}(V)$, then conditions* (17.46) *and* (17.47) *are already sufficient.*

It is useful to reformulate Theorem 17.2.6 in the following equivalent form because in this way the linking property gets special emphasis.

Theorem 17.2.8 *Let $p \geq 0$ be a positively crossing supermodular set-function on groundset V with $p(V) = 0$. Let $g_{in} : V \to \mathbf{Z}_+ \cup \{\infty\}$ and $g_{out} : V \to \mathbf{Z}_+ \cup \{\infty\}$ be upper bound functions. Let $\gamma \geq 0$ be an integer. There exists a digraph $H = (V, F)$ covering p*

(A) *for which $\dot{\varrho}_H \leq g_{in}$ if and only if* (17.46) *holds and*

$$\sum_i \hat{p}(V_i) \leq \widetilde{g}_{in}(V) \text{ for every partition } \{V_1, \dots, V_t\} \text{ of } V, \tag{17.50}$$

(B) *for which $\dot{\delta}_H \leq g_{out}$ if and only if* (17.47) *holds and*

$$\sum_i p(V_i) \leq \widetilde{g}_{out}(V) \text{ for every partition } \{V_1, \dots, V_t\} \text{ of } V, \tag{17.51}$$

(C) *for which $|F| \leq \gamma$ if and only if*

$$\textstyle\sum_i p(V_i) \leq \gamma \quad \text{and} \quad \sum_i \hat{p}(V_i) \leq \gamma \quad \text{for every partition } \{V_1, \dots, V_t\} \text{ of } V. \tag{17.52}$$

(D) *If there is a digraph fulfilling* (A)*, and there is a digraph fulfilling* (B)*, and there is a digraph fulfilling* (C)*, then there is one that satisfies simultaneously each of* (A), (B)*, and* (C). •

Let γ^* denote the minimum cardinality of the edge-set of a digraph covering p. By Theorem 17.2.6, γ^* is the maximum value of $\sum_i p(V_i)$ and $\sum_i \hat{p}(V_i)$ over all partitions $\{V_1, \dots, V_t\}$ of V. By Proposition 17.2.5, p^{γ^*} is positively intersecting supermodular and hence $C(p^{\gamma^*})$ is a contra-polymatroid.

Theorem 17.2.9 *A vector $m_{in} \in \mathbf{Z}_+^V$ is the in-degree vector of a digraph covering p if and only if $p \leq \widetilde{m}_{in}$ and*

$$\sum_i \hat{p}(V_i) \leq \widetilde{m}_{in}(V) \text{ for every partition } \{V_1, \dots, V_t\} \text{ of } V. \tag{17.53}$$

The in-degree vectors of digraphs covering p are exactly the integral elements of contra-polymatroid $C(p^{\gamma^*})$.

Proof. The first part is a direct consequence of Part (A) of Theorem 17.2.8, since a digraph H covering p for which $\dot{\varrho}_H \leq m_{in}$ can be enlarged to H' so that $\dot{\varrho}_{H'} = m_{in}$ and then H' also covers p.

For the second part, let m_{in} be the in-degree vector of a digraph $H = (V, F)$ covering p. Then $\widetilde{m}_{in}(V) = |F| \geq \gamma^*$, furthermore $\widetilde{m}_{in}(X) = \sum_{v \in X} \varrho_H(v) \geq \varrho_H(X) \geq p(X)$ for every $X \subseteq V$ and hence $m_{in} \in C(p^{\gamma^*})$. Conversely, if m_{in} is an integral element of $C(p^{\gamma^*})$, then $p \leq p^{\gamma^*} \leq \widetilde{m}_{in}$ and $\sum_i \hat{p}(V_i) \leq \gamma^* \leq \widetilde{m}_{in}(V)$ for every partition of V. Therefore the first half of the theorem implies that m_{in} is the in-degree vector of a digraph covering p •

Remark 17.2.1 Since p is assumed only positively crossing supermodular, the polyhedron $C(p^\gamma) = \{x : x(Z) \geq p^\gamma(Z) \text{ for } Z \subseteq V\}$ need not be a contra-polymatroid for every integer $\gamma \geq 0$. However, if γ is large enough in the sense that $\gamma \geq \hat{p}(X) + \hat{p}(Y)$ holds for every pair of disjoint sets X, Y, then p^γ is positively intersecting supermodular, in which case $C(p^\gamma)$ is a contra-polymatroid. No combinatorial meaning is known for the integral elements of $C(p^\gamma)$ in general. However, if $\gamma \geq \sum_i \hat{p}(V_i)$ for every partition $\{V_i\}$ of V, then the integral points of $C(p^\gamma)$ are exactly the in-degree vectors of digraphs with at least γ edges that cover p.

Cheapest covers

Let us consider now the problem of finding a cheapest digraph covering a crossing supermodular function. As mentioned earlier, the directed travelling salesman problem is a special case. Therefore we consider only a special class of cost functions.

Let c_{out} and c_{in} be two cost functions defined on V. By definition, the cost $c(e)$ of a uv-edge e is defined by the sum $c_{out}(v) + c_{in}(u)$. We call such a c a **node-induced** cost function. Consider the problem of finding a cheapest digraph of γ edges covering p where γ is a specified number. All we need to do is find first an element m_{in} of base-polyhedron $B'(p^\gamma)$ of minimum c_{in}-cost, find next an element m_{out} of $B'(\hat{p}^\gamma)$ of minimum c_{out}-cost, and apply finally the algorithmic proof of Theorem 17.2.2. The elements m_{in} and m_{out} can be found by the version of the greedy algorithm concerning base-polyhedra bounded by positively intersecting supermodular functions.

17.2.2 Applications to directed edge-connectivity

Extending splitting-off theorems

The directed splitting-off lemma of Mader (Theorem 8.2.1) can be extended more than one way. One possibility is as follows.

Theorem 17.2.10 *Let $D = (V + z, A)$ be a digraph with a root-node $r_0 \in V$ and with a special node z for which $\varrho_D(z) = \delta_D(z)$. Assume that D is rooted (k, ℓ)-edge-connected apart from z, meaning that $\varrho_D(X) \ge k$ and $\delta_D(X) \ge \ell$ hold for every $\emptyset \ne X \subseteq (V + z) - r_0$, $X \ne \{z\}$. Then the edges entering z can be paired with the edges leaving z in such a way that simultaneously splitting off the $\varrho_D(z)$ pairs results in a rooted (k, ℓ)-edge-connected digraph on node-set V.*

Proof. For every node $v \in V$, let $m_{in}(v)$ and $m_{out}(v)$ denote the number of parallel zv-edges and vz-edges, respectively. Let $D' = (V, A')$ be the subgraph of D induced by V. Define a set-function p on ground-set V as follows.

$$p(X) := \begin{cases} (k - \varrho_{D'}(X))^+ & \text{if } \emptyset \subset X \subseteq V - r_0 \\ (\ell - \varrho_{D'}(X))^+ & \text{if } r_0 \in X \subset V \\ 0 & \text{otherwise.} \end{cases} \tag{17.54}$$

Then p is positively intersecting supermodular. Moreover it follows from the hypotheses of the theorem that $\widetilde{m}_{in}(V) = \widetilde{m}_{out}(V)$, furthermore $\widetilde{m}_{in}(X) \ge k - \varrho_{D'}(X)$ for $\emptyset \ne X \subseteq V - r_0$ and $\widetilde{m}_{out}(X) \ge \ell - \delta_{D'}(X)$ for $r_0 \in X \subset V$.

Therefore (17.38) and (17.39) hold. Hence there exists a digraph $H = (V, F)$ for which $\dot{\delta}_H = m_{out}$ and $\dot{\varrho}_H = m_{in}$ and H covers p. This last property can be easily verified to be equivalent to the rooted (k, ℓ)-edge-connectivity of $D' + H$. Moreover, due to the in- and out-degree specifications, it is possible to assign a zv-edge and a uz-edge of D to every uv-edge of H. Therefore H can be obtained by a complete splitting of the edges at z. •

The following generalization of Mader's directed splitting-off lemma was already formulated and proved in Theorem 8.2.6.

Theorem 17.2.11 *Suppose that in a digraph $D = (V + z, A)$, z is a special node for which $\varrho_D(z) = \delta_D(z)$. Let $V' \subseteq V$ be a subset of V so that V' contains each node which is a (in- or out-) neighbour of z. Suppose that D is k-edge-connected within V' ($k \geq 1$). Then there is a uz-edge f for every zt-edge e so that the digraph D^{ef} obtained by splitting off the pair $\{e, f\}$ is also k-edge-connected within V'.*

Proof. Let $m_{in}(v)$, $m_{out}(v)$, and D' be the same as in the preceding proof. Define a set-function p' on ground-set V' as follows.

$$p'(X') := \begin{cases} \max\{(k - \varrho_{D'}(X))^+ : X \subseteq V, X' = X \cap V'\} & \text{if } \emptyset \subset X' \subset V' \\ 0 & \text{otherwise} \end{cases} \tag{17.55}$$

It is not difficult to check that p' is positively crossing supermodular. Therefore Theorem 17.2.2 applies. •

Exercise 17.2.3 *Prove that the set-function p' in the proof of Theorem* 17.2.11 *is positively crossing supermodular.*

Goemans and Bertsimas [185] investigated other abstract versions of splitting-off theorems.

Augmenting directed edge-connectivity

The results of the section can be applied to solve degree- and size-constrained edge-connectivity augmentation problems. Let $D = (V, A)$ be a digraph with a root-node r_0 and let $\ell \leq k$ be integers. Define a set-function $p_{k\ell}$ as follows.

$$p_{k\ell}(X) := \begin{cases} (k - \varrho_D(X))^+ & \text{if } \emptyset \subset X \subseteq V - r_0 \\ (\ell - \varrho_D(X))^+ & \text{if } r_0 \in X \subset V \\ 0 & \text{otherwise.} \end{cases} \tag{17.56}$$

Then $p_{k\ell}$ is positively crossing supermodular and a digraph $H = (V, F)$ covers $p_{k\ell}$ precisely if $D + H$ is rooted (k, ℓ)-edge-connected. Therefore Theorem 17.2.2 provides a characterization for the existence of in- and out-degree specified rooted (k, ℓ)-edge-connected augmentation of a digraph. When $k = \ell$, we are back at Theorem 17.2.1. Also, Theorem 17.2.6 (and its equivalent form Theorem 17.2.8), when applied to $p_{k\ell}$ yields generalizations of Theorems 11.2.2 and 11.2.5 to (k, ℓ)-edge-connectivity.

We proved in Theorem 11.3.1 that the problem of augmenting a digraph $D = (V, A)$ by adding a minimum number of new edges so that the resulting digraph is k-edge-connected within a specified subset $T \subseteq V$ of nodes is **NP**-complete. We are going to show that the problem becomes tractable if every new edge must be induced by T.

Theorem 17.2.12 *Let $D = (V, A)$ be a digraph with a specified non-empty subset $T \subseteq V$ of nodes. D can be made k-edge-connected within T by adding at most γ new edges connecting the nodes of T if and only if*

$$\gamma \geq \sum_i [k - \varrho_D(X_i)] \text{ and } \gamma \geq \sum_i [k - \delta_D(X_i)] \tag{17.57}$$

hold for every set-system $\{X_1, \dots, X_t\}$ *for which* $\emptyset \subset X_i \cap T \subset T$ *and* $\{X_1 \cap T, \dots, X_t \cap T\}$ *is a subpartition.*

Proof. The necessity of the conditions is evident since in an augmentation requested in the theorem $\varrho_{D+H}(X) \geq k$ must hold for every subset X for which $\emptyset \subset X \cap T \subset T$. The sufficiency of the conditions follows by applying Part (C) of Theorem 17.2.8 to the set-function p_T defined on ground-set T by

$$p_T(X) := \begin{cases} \max\{(k - \varrho_D(X'))^+ : X' \subseteq V, X = X' \cap T\}, & \text{if } \emptyset \subset X \subset T \\ 0 & \text{otherwise.} \end{cases} \tag{17.58}$$

Function p_T can be checked to be positively crossing supermodular and (17.52) in this case holds if and only if (17.57) is fulfilled. •

When $T = V$ we are back at Theorem 11.2.2.

The minimum-cost versions of the problems above are **NP**-complete. Grötschel and Monma [196] investigated polyhedral techniques that help in finding good bounds for the optimum.

17.3 Non-TDI frameworks for covering bi-set functions

Supermodular set-functions played a central role in handling edge-connectivity problems on an abstract level. Bi-sets and bi-set functions have been useful in solving node-connectivity problems. For example, in Section 17.1.3, we proved the total dual integrality of a linear system regarding *intersecting* supermodular bi-set functions. This result implied a polyhedral description of the polytope of rooted k-node-connected subgraphs of a digraph. In this section, we study digraphs covering a *crossing* supermodular bi-set-function. A consequence of the main result will be Theorem 11.3.5 on optimal node-connectivity augmentation.

17.3.1 Minimum coverings

Let $\mathcal{P}_2 = \mathcal{P}_2(V)$ denote the set of bi-sets on a ground-set V and let p be a non-negative, integer-valued, positively crossing supermodular function on $\mathcal{P}_2$ such that $p(X) = 0$ for every trivial bi-set X. Let $D^* = (V, A^*)$ denote the complete digraph on V. As before, a function $z : A^* \to \mathbf{Z}_+$ is said to cover p if

$$\varrho_z(X) \geq p(X) \text{ for every } X \in \mathcal{P}_2.$$

Since the value of p on trivial bi-sets is zero, p can always be covered by a z.

For a family $\mathcal{F}$ of bi-sets, the sum $\sum[p(X) : X \in \mathcal{F}]$ will be called the p-sum of $\mathcal{F}$ and denoted by $\widetilde{p}(\mathcal{F})$. Let τ_p be that minimum total value of a covering of p, that is,

$$\tau_p := \min\{\widetilde{z}(A^*) : z \geq 0 \text{ an integer-valued covering of } p\},$$

and let ν_p denote the maximum p-sum of an independent subset of $\mathcal{P}_2$, that is,

$$\nu_p := \max\{\widetilde{p}(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{P}_2,\ \mathcal{F} \text{ independent}\}.$$

Theorem 17.3.1 (Frank and Jordán) *Let $D^* = (V, A^*)$ be a complete directed graph and p a positively crossing supermodular function on $\mathcal{P}_2$ such that $p(X) = 0$ for every trivial bi-set X. Then $\tau_p = \nu_p$.*

Proof. In order to prove $\nu_p \leq \tau_p$, let z be a covering of p and $\mathcal{F}$ an independent subset of $\mathcal{P}_2$. Since no edge of D^* can cover more than one member of $\mathcal{F}$, we have

$$\widetilde{z}(A^*) \geq \sum[\varrho_z(X) : X \in \mathcal{F}] \geq \sum[p(X) : X \in \mathcal{F}] = \widetilde{p}(\mathcal{F})$$

and hence $\nu_p \leq \tau_p$.

The reverse inequality is trivial when $|V| = 1$, so we assume that $|V| \geq 2$ from which $|A^*| \geq 2$. Define a bi-set-function p_e for every $e \in A^*$ as follows.

$$p_e(X) := \begin{cases} (p(X) - 1)^+ & \text{if } e \text{ covers } X \\ p(X) & \text{otherwise.} \end{cases} \tag{17.59}$$

Since the bi-set-function ϱ_{D^*} is submodular, p_e is also positively crossing supermodular. Furthermore $p_e \leq p$ implies $\nu_p - 1 \leq \nu_{p_e} \leq \nu_p$. An edge $e \in A^*$ is said to be **reducing** if $\nu_{p_e} < \nu_p$. The next lemma lies at the heart of the proof.

Lemma 17.3.2 *If $\nu_p > 0$, then there is a reducing edge.*

Proof. Suppose indirectly that there is no reducing edge, that is, $\nu_{p_e} = \nu_p$ for every $e \in A^*$. This implies that there exists an independent bi-set-system $\mathcal{J}_e \subseteq \mathcal{P}_2$ for every edge $e \in A^*$ such that $\widetilde{p}(\mathcal{J}_e) = \nu_p$ and e covers no member of $\mathcal{J}_e$. Let $\mathcal{J}'$ denote the multi-union of the bi-set systems $\mathcal{J}_e$ $(e \in A^*)$. Then $\widetilde{p}(\mathcal{J}') = m^*\nu_p$ where $m^* := |A^*| \geq 2$ and

$$\text{every edge of } D^* \text{ covers at most } m^* - 1 \text{ members of } \mathcal{J}'. \tag{17.60}$$

We can assume that $p(X) > 0$ for every $X \in \mathcal{J}'$ since members X of $\mathcal{J}'$ with $p(X) = 0$ could be deleted. Apply the following uncrossing procedure as long as possible. If there are two crossing members X and Y with positive $p(X)$ and $p(Y)$ in the current $\mathcal{J}'$, then replace them by $X \sqcap Y$ and $X \sqcup Y$. If $p(X \sqcap Y) = 0$, then remove $X \sqcap Y$, and if $p(X \sqcup Y) = 0$, then remove $X \sqcup Y$. The submodularity of ϱ_{D^*} on bi-sets ensures that such an uncrossing step preserves (17.60). Furthermore, it follows from the crossing supermodularity of p that the p-value of the revised family is at least $\widetilde{p}(\mathcal{J}')$. Due to Lemma 1.1.2, after a finite number of uncrossing steps, we obtain a cross-free family $\mathcal{J}$ of bi-sets for which $\widetilde{p}(\mathcal{J}) \geq \widetilde{p}(\mathcal{J}') = m\nu_p$ and for which (17.60) holds with $\mathcal{J}$ in place of $\mathcal{J}'$. Note that $\widetilde{p}(\mathcal{J}) = \sum[p(X)s(X) : X \in \mathcal{P}_2]$ where $s := \underline{\chi}_{\mathcal{J}}$ denotes the indicator function of $\mathcal{J}$.

Consider the poset on $\mathcal{P}_2$ defined by the relation $\sqsubseteq$. We claim that the total s-weight of every chain of $\mathcal{P}_2$ is at most $m^* - 1$. Indeed, if indirectly there is a chain $\mathcal{C} \subseteq \mathcal{P}_2$ with total s-weight at least m^*, then the family $\mathcal{J}$ has m^* members which are pairwise comparable. Then there is an edge $e \in A^*$ that covers all of these m^* bi-sets. (Namely, if X^1 is the smallest member of $\mathcal{C}$ and X^k is the largest member, then any uv-edge e with $u \in V - X^1_O$ and $v \in X^k_I$ will suffice.) But the existence of such an edge would contradict property (17.60).

Recall the polar-Dilworth theorem (Theorem 2.4.30), which implies that if the total s-weight of a chain in $\mathcal{P}_2$ is at most $m^* - 1$, then there are $m^* - 1$ antichains such that each member X of $\mathcal{P}_2$ belongs to $s(X)$ antichains. This implies that the members of $\mathcal{J}$ can be partitioned into $m^* - 1$ antichains. Therefore, the p-value of at least one of these antichains is at least $\widetilde{p}(\mathcal{J})/(m^* - 1) \geq m^* \nu_p/(m^* - 1) > \nu_p$ but this contradicts the definition of ν_p since the cross-freeness of $\mathcal{J}$ implies that any antichain consisting of members of $\mathcal{J}$ is an independent set of bi-sets, and this contradiction proves the lemma. •

To prove inequality $\nu_p \geq \tau_p$, we proceed by induction on ν_p. If this value is 0, then p is identically 0 in which case $z :\equiv 0$ covers p, that is, $\tau_p = 0$. Suppose now that $\nu_p > 0$. Lemma 17.3.2 ensures the existence of a reducing edge $e \in A^*$. Now $\nu_{p_e} \leq \nu_p - 1$ and by induction there is a z' covering p_e for which $\widetilde{z}'(A^*) = \nu_{p_e}$. Increase $z'(e)$ by 1 and let z denote the revised vector. Then z covers p from which $\tau_p \leq \tau_{p_e} + 1$. By using the inductive hypothesis, we obtain $\tau_p - 1 \leq \tau_{p_e} = \nu_{p_e} \leq \nu_p - 1$ and hence $\tau_p \leq \nu_p$, as required. • •

It is worth formulating Theorem 17.3.1 in the following equivalent form. A digraph is said to cover p if the in-degree of every $X \in \mathcal{P}_2$ is at least $p(X)$.

Theorem 17.3.3 *Let p be positively crossing supermodular function on $\mathcal{P}_2(V)$. For a given integer γ, it is possible to cover p by a digraph $H = (V, F)$ for which $|F| \leq \gamma$ if and only if $\widetilde{p}(\mathcal{F}) \leq \gamma$ holds for every independent $\mathcal{F} \subset \mathcal{P}_2$.* •

The theorem has a self-refining nature in the sense that it implies a generalization when only ST-edges are allowed to be used in the covering of p. Let S and T be two non-empty subsets of V. Suppose that p is a positively crossing supermodular function on $\mathcal{P}_2$ such that $p(X) > 0$ implies that there is an ST-edge covering X. We say that a set $\mathcal{F}$ of bi-sets is ST**-independent** if no ST-edge can cover more than one member of $\mathcal{F}$.

Theorem 17.3.4 *It is possible to cover p by γ ST-edges if and only if $\widetilde{p}(\mathcal{F}) \leq \gamma$ holds for every ST-independent $\mathcal{F} \subseteq \mathcal{P}_2$.*

Proof. Since an ST-edge can cover at most one member of an ST-independent $\mathcal{F}$ and each member X of $\mathcal{F}$ must be covered by at least $p(X)$ edges, we conclude that γ is at least $\sum[p(X) : X \in \mathcal{F}] = \widetilde{p}(\mathcal{F})$.

Consider the set $\mathcal{P}_2' := \{X' = (X_O', X_I') : X_I' \subseteq T,\ V - S \subseteq X_O\}$. Note that if an edge e covers a member of $\mathcal{P}_2'$, then e is an ST-edge. Define a mapping $\varphi : \mathcal{P}_2 \to \mathcal{P}_2'$ as follows. Let $\varphi(X) := X' = (X_O', X_I')$ where $X_O' = X_O \cup (V - S)$ and $X_I' = X_I \cap T$. Then a set $\mathcal{F}$ of bi-sets is ST-independent if and only if the set $\varphi(\mathcal{F}) := \{\varphi(X) : X \in \mathcal{F}\}$ is independent. Define a bi-set-function p' as follows.

$$p'(X') := \begin{cases} \max\{p(X) : X' = \varphi(X),\ X \in \mathcal{P}_2\} & \text{if } X' \in \mathcal{P}_2' \\ 0 & \text{otherwise.} \end{cases}$$

It is a little exercise to verify that p' is a positively crossing supermodular bi-set-function. It follows from these definitions that, for every independent subset $\mathcal{F}' \subseteq \mathcal{P}_2'$, there is an ST-independent set $\mathcal{F} \subseteq \mathcal{P}_2$ such that $\mathcal{F}' = \varphi(\mathcal{F})$ and $\widetilde{p}'(\mathcal{F}') = \widetilde{p}(\mathcal{F})$. Also, a digraph is a minimal covering of p' if and only if it consists of ST-edges and it is a minimal covering of p. Consequently, the result follows by applying Theorem 17.3.3 to p'. •

For a (0, 1)-valued function p, $\mathcal{L}_p := \{X \in \mathcal{P}_2 : p(X) = 1\}$ is a crossing subset of $\mathcal{P}_2$ and Theorem 17.3.3 specializes as follows.

Theorem 17.3.5 *Let $\mathcal{L} \subseteq \mathcal{P}_2$ be a crossing set of non-trivial bi-sets. The minimum number $\tau = \tau(\mathcal{L})$ of directed edges covering $\mathcal{L}$ is equal to the maximum number $\nu = \nu(\mathcal{L})$ of independent members of $\mathcal{L}$.* •

Back to set-functions

A bi-set-function $p \geq 0$ which is positive only on simple bi-sets can be identified with a set-function. Therefore, the theorems above can be specialized to set-functions. For example, Theorem 17.3.3 yields the following.

A positively crossing supermodular set-function p can be covered by γ edges if and only if $\widetilde{p}(\mathcal{F}) \leq \gamma$ holds for every independent set-system $\mathcal{F}$.

Here independence means that no edge can cover two members of $\mathcal{F}$ which is equivalent to requiring that every two members of $\mathcal{F}$ are disjoint or co-disjoint. In Proposition 11.3.4 we pointed out that in this case either $\mathcal{F}$ consists of pairwise disjoint sets or $\mathcal{F}$ consists of pairwise co-disjoint sets, and therefore Theorem 17.3.3 includes Part (C) of Theorem 17.2.8.

Theorem 17.3.4 can also be specialized to set-functions. Two sets X and Y are **ST-crossing** if none of the sets $X \cap Y \cap T$, $S - (X \cup Y)$, $X - Y$, $Y - X$ is empty. A set-system $\mathcal{I}$ is **ST-independent** if at least one of $X \cap Y \cap T$ and $S - (X \cup Y)$ is empty for every two members X and Y of $\mathcal{I}$. A set-function $p : 2^V \rightarrow \mathbf{Z}_+$ is **positively ST-crossing supermodular** if the supermodular inequality holds for every pair of ST-crossing sets for which the p-value is positive.

Theorem 17.3.6 *Let p be a positively ST-crossing supermodular set-function for which $p(X)$ can be positive only if $T \cap X \neq \emptyset$ and $S - X \neq \emptyset$. It is possible to cover p by γ ST-edges if and only if $\widetilde{p}(\mathcal{F}) \leq \gamma$ holds for every ST-independent set-system $\mathcal{F}$.* •

17.3.2 In-degree vectors of coverings

Theorem 17.2.9 asserts that the in-degree vectors of digraphs covering a crossing supermodular set-function span a contra-polymatroid. In fact, it was this property that played the decisive role in the background in establishing results on degree- and size-constrained edge-connectivity augmentations as well as an algorithm to treat minimum cost augmentations for node-induced cost functions.

Our present goal is to prove analogous results for bi-set functions. There is, however, a significant difference between the approaches. In studying set-functions, we started with a result on degree-specified coverings and used it to derive results on size-constrained or minimum coverings. Here we follow an opposite path. In Theorem 17.3.1 we formulated and proved already a min-max result on minimum coverings of a crossing supermodular bi-set-function and this result will be used for handling degree-constrained situations.

Let p be again a positively crossing supermodular bi-set-function on $\mathcal{P}_2(V)$. For a bi-set-function b on $\mathcal{P}_2(V)$, we define a set-function p_{in} on ground-set V as follows. For $Z \subseteq V$, let

$$p_{in}(Z) := \max\{\widetilde{p}(\mathcal{F}) : \mathcal{F} \subset \mathcal{P}_2,\ \mathcal{F} \text{ independent},\ X_I \subseteq Z \text{ for every } X \in \mathcal{F}\}. \qquad (17.61)$$

Theorem 17.3.7 *Let $p \geq 0$ be an integer-valued and positively crossing supermodular bi-set-function on $\mathcal{P}_2(V)$ which is zero on trivial bi-sets. For a given vector $m_{in} \in \mathbf{Z}_+^V$, there exists a digraph $H = (V, A)$ covering p for which $\dot{\varrho}_H = m_{in}$ if and only if*

$$\widetilde{m}_{in}(Z) \geq p_{in}(Z) \text{ for every } Z \subseteq V. \tag{17.62}$$

Proof. If there is a digraph H with the requested properties, then the definition of p_{in} implies that the head of at least $p_{in}(Z)$ edges of H must be in Z and hence $p_{in}(Z) \leq \widetilde{m}_{in}(Z)$ follows, proving the necessity of (17.62).

For proving the sufficiency, suppose that (17.62) holds. By applying (17.62) to each singleton $Z := \{v\}$, we get $m_{in}(v) \geq p(Z, Z)$. Revise p as follows. For every singleton $Z = \{v\}$ $(v \in V)$ for which $m_{in}(v) > p(Z, Z)$, increase the value of $p(Z, Z)$ to $m_{in}(v)$. The resulting bi-set-function p' keeps to be positively crossing supermodular. If there is a digraph $H = (V, F)$ with at most $\gamma := \widetilde{m}_{in}(V)$ edges that covers p', then

$$\gamma \geq |F| = \ddot{\varrho}_H(V) \geq \sum [p'(\{v\}, \{v\}) : v \in V] \geq \widetilde{m}_{in}(V) = \gamma$$

and hence we must have $\varrho_H(v) = m_{in}(v)$ for every node v, and in this case we are done. If no such a digraph exists, then Theorem 17.3.3 implies that there is an independent $\mathcal{F} \subset \mathcal{P}_2$ for which $\widetilde{p}'(\mathcal{F}) := \sum[p'(X) : X \in \mathcal{F}] > \gamma = \widetilde{m}_{in}(V)$. Let Y denote the set of nodes $v \in V$ for which $m_{in}(v) > p(\{v\}, \{v\})$ and the bi-set $(\{v\}, \{v\})$ belongs to $\mathcal{F}$. Let $\mathcal{F}_1 := \{(\{v\}, \{v\}) : v \in Y\}$ and let $\mathcal{F}_2 := \mathcal{F} - \mathcal{F}_1$. Then

$$\widetilde{m}_{in}(V) < \widetilde{p}'(\mathcal{F}) = \widetilde{p}'(\mathcal{F}_1) + \widetilde{p}'(\mathcal{F}_2) = \widetilde{m}_{in}(Y) + \widetilde{p}(\mathcal{F}_2).$$

The independence of $\mathcal{F}$ implies that the inner set of each member of $\mathcal{F}_2$ must be disjoint from Y, that is, these inner sets are included in $Z := V - Y$. Therefore, $p_{in}(Z) \geq \widetilde{p}(\mathcal{F}_2)$ and hence $\widetilde{m}_{in}(Z) = \widetilde{m}_{in}(V) - \widetilde{m}_{in}(Y) < \widetilde{p}(\mathcal{F}_2) \leq p_{in}(Z)$, contradicting (17.62). •

Theorem 17.3.8 *Where p is the same as in Theorem* 17.3.7*, the set-function p_{in} defined in* (17.61) *is fully supermodular.*

Proof. For two subset $A, B \subseteq V$, let $\mathcal{A}$ and $\mathcal{B}$ be independent bi-set systems for which $p_{in}(A) = \widetilde{p}(\mathcal{A}) := \sum[p(X) : X \in \mathcal{A}]$ and $p_{in}(B) = \widetilde{p}(\mathcal{B})$, furthermore, the inner set of each member of $\mathcal{A}$ is included in A and the inner set of each member of $\mathcal{B}$ is included in B. Let $\mathcal{F}'$ be the multi-union of $\mathcal{A}$ and $\mathcal{B}$.

Apply the uncrossing procedure to family $\mathcal{F}'$, that is, as long as there are two crossing members X and Y in the current family with $p(X) > 0$ and $p(Y) > 0$, replace them by $X \sqcap Y$ and $X \sqcup Y$ but remove $X \sqcap Y$ or $X \sqcup Y$ immediately if $p(X \sqcap Y) = 0$ or $p(X \sqcup Y) = 0$, respectively. After a finite number of uncrossing steps (again, by Lemma 1.1.2), we arrive at a cross-free collection $\mathcal{F}$.

Since both $\mathcal{A}$ and $\mathcal{B}$ are independent, no directed edge covers more than two members of $\mathcal{F}'$. Moreover,

$$\text{if an edge } uv \text{ covers a member of } \mathcal{F}', \text{ then } v \in A \cup B \tag{17.63}$$

and

$$\text{if an edge } uv \text{ covers two members of } \mathcal{F}', \text{ then } v \in A \cap B. \tag{17.64}$$

Since an uncrossing step preserves these properties, the final $\mathcal{F}$ includes no chain of more than two members. Let $\mathcal{F}_1$ denote the maximal members of $\mathcal{F}$ and let $\mathcal{F}_2 := \mathcal{F} - \mathcal{F}_1$. (To be more precise, if two copies of a bi-set X occurs in $\mathcal{F}$, then one copy will belong to $\mathcal{F}_1$ while the other one to $\mathcal{F}_2$.) Due to this definition, both $\mathcal{F}_1$ and $\mathcal{F}_2$ are antichains with respect to the relation $\sqsubseteq$, and hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent.

Let $X = (X_O, X_I)$ be a member of $\mathcal{F}_2$. We claim that $X_I \subseteq A \cap B$. Indeed, by the definition of $\mathcal{F}_2$, there is a member $Y = (Y_O, Y_I)$ of $\mathcal{F}_1$ for which $X \sqsubseteq Y$, that is, $X_O \subseteq Y_O$ and $X_I \subseteq Y_I$. Then Property (17.64) implies that $X_I \subseteq A \cap B$, that is, the inner set of each member of $\mathcal{F}_2$ is included in $A \cap B$ from which $\widetilde{p}(\mathcal{F}_2) \leq p_{in}(A \cap B)$. Property (17.63) implies that the inner set of each member of $\mathcal{F}_1$ is included in $A \cup B$ from which $\widetilde{p}(\mathcal{F}_1) \leq p_{in}(A \cup B)$. By combining these observations, we obtain

$$p_{in}(A) + p_{in}(B) = \widetilde{p}(\mathcal{A}) + \widetilde{p}(\mathcal{B}) \leq \widetilde{p}(\mathcal{F}) = \widetilde{p}(\mathcal{F}_1) + \widetilde{p}(\mathcal{F}_2) \leq p_{in}(A \cap B) + p_{in}(A \cup B),$$

as required for the supermodularity of p_{in}. •

Combining Theorems 17.3.7 and 17.3.8, we obtain the following result.

Theorem 17.3.9 *Let p be a positively crossing supermodular bi-set-function. The in-degree vectors of digraphs covering p are the integral elements of the contra-polymatroid $C_{in} := C(p_{in})$.* •

Define a set-function p_{out} as follows. For $Z \subseteq V$, let

$$p_{out}(Z) = \max\{\widetilde{p}(\mathcal{F}) : \mathcal{F} \subset \mathcal{P}_2,\ \mathcal{F} \text{ independent}, (V - X_O) \subseteq Z \text{ for every } X \in \mathcal{F}\}. \tag{17.65}$$

There is an equivalent way of defining p_{out}. For a bi-set X let $\overline{X} := (V - X_I, V - X_O)$ and define a bi-set-function $\hat{p}$ by $\hat{p}(X) := p(\overline{X})$. Now $\hat{p}$ is also a positively crossing supermodular bi-set-function and $p_{out} = \hat{p}_{in}$. Hence p_{out} is fully supermodular by Theorem 17.3.8. Since a digraph covers p if and only if the reversed digraph covers $\hat{p}$, we conclude that Theorem 17.3.9 implies its counterpart for out-degrees.

Theorem 17.3.10 *Let p be a positively crossing supermodular bi-set-function. The out-degree vectors of digraphs covering p are the integral elements of the contra-polymatroid $C_{out} := C(p_{out})$.* •

Theorem 17.3.11 *Let p be a positively crossing supermodular bi-set-function p. Let $m_{in} \in C_{in}$ and $m_{out} \in C_{out}$ be two integral vectors for which $\widetilde{m}_{in}(V) = \gamma = \widetilde{m}_{out}(V)$. Then there exists a loopless digraph $H = (V, F)$ for which $\varrho_H = m_{in}$ and $\delta_H = m_{out}$ if and only if*

$$m_{in}(v) + m_{out}(v) + \widetilde{p}(\mathcal{F}') \leq \gamma \tag{17.66}$$

for every $v \in V$ and for every independent set $\mathcal{F}'$ of bi-sets X for which $v \in X_O - X_I$.

Proof. Suppose that there is a digraph H with the required properties. Let v be a node and $\mathcal{F}'$ an independent family of bi-sets X for which $v \in X_O - X_I$. Then no edge leaving or entering v can cover any member of $\mathcal{F}'$. Hence $\gamma = |F| \geq m_{in}(v) + m_{out}(v) + \widetilde{p}(\mathcal{F}')$, showing the necessity of (17.66).

Suppose now that (17.66) holds. Since $m_{in} \in C_{in}$, we have $m_{in}(v) \geq p(Z, Z)$ for each singleton $Z = \{v\}$. If $m_{in}(v) > p(Z, Z)$, increase the value of $p(Z, Z)$ to $m_{in}(v)$.

Analogously, since $m_{out} \in C_{out}$, we have $m_{out}(v) \geq p(Y, Y)$ for every $v \in V$ where $Y = V - v$. If $m_{out}(v) > p(Y, Y)$, increase the value of $p(Y, Y)$ to $m_{out}(v)$. The resulting bi-set-function p' keeps to be positively crossing supermodular.

Suppose first that there is a digraph H with at most $\gamma = \widetilde{m}_{in}(V)$ edges that covers p'. Then we must have $\dot{\varrho}_H = m_{in}$ and $\dot{\delta}_H = m_{out}$. Moreover $\varrho_H(\{v\}, \{v\}) = m_{in}(v)$ means that there are $m_{in}(v)$ edges with head v and tail in $V - v$. That is, these edges are not loops and hence H is simple, and in this case we are done.

Suppose now no such a digraph exists. Then Theorem 17.3.3 implies that there is an independent $\mathcal{F} \subset \mathcal{P}_2$ for which $\widetilde{p}'(\mathcal{F}) := \sum[p'(X) : X \in \mathcal{F}] > \gamma = \widetilde{m}_{in}(V)$. Let Y_1 denote the set of nodes $v \in V$ for which $m_{in}(v) > p(\{v\}, \{v\})$ and the bi-set $(\{v\}, \{v\})$ belongs to $\mathcal{F}$, and let $\mathcal{F}_1 := \{(\{v\}, \{v\}) : v \in Y_1\}$. Let Y_2 denote the set of nodes $v \in V$ for which $m_{out}(v) > p(V - v, V - v)$ and the bi-set $(V - v, V - v)$ belongs to $\mathcal{F}$, and let $\mathcal{F}_2 := \{(V - v, V - v) : v \in Y_2\}$. Finally, let $\mathcal{F}_3 := \mathcal{F} - (\mathcal{F}_1 \cup \mathcal{F}_2)$.

If $Y_2 = \emptyset$, then $\widetilde{m}_{in}(V) = \gamma < \widetilde{p}'(\mathcal{F}) = \widetilde{p}'(\mathcal{F}_1) + \widetilde{p}'(\mathcal{F}_3) = \widetilde{m}_{in}(Y_1) + \widetilde{p}(\mathcal{F}_3)$. The independence of $\mathcal{F}$ implies that the inner set of each member of $\mathcal{F}_3$ must be disjoint from Y_1, that is, these inner sets are included in $Z := V - Y_1$. Therefore $p_{in}(Z) \geq \widetilde{p}(\mathcal{F}_3)$, and hence $\widetilde{m}_{in}(Z) = \widetilde{m}_{in}(V) - \widetilde{m}_{in}(Y_1) < \widetilde{p}(\mathcal{F}_3) \leq p_{in}(Z)$, that is, (17.62) fails to hold, contradicting the hypothesis that $m_{in} \in C_{in}$. An analogous argument shows that if $Y_1 = \emptyset$, then m_{out} would not be in C_{out}.

Investigate now the remaining case when neither Y_1 nor Y_2 is empty. If there are distinct nodes v and u such that $v \in Y_1$ and $u \in Y_2$, then a uv-edge covers both bi-sets $(\{v\}, \{v\}) \in \mathcal{F}_1$ and $(V - u, V - u) \in \mathcal{F}_2$ contradicting the assumption that $\mathcal{F}$ is independent. Therefore $Y_1 = \{v\} = Y_2$ holds for a node $v \in V$. The independence of $\mathcal{F}$ implies that $v \in X_O - X_I$ for every bi-set $(X_O, X_I) \in \mathcal{F}_3$. Furthermore, $\gamma < \widetilde{p}'(\mathcal{F}) = \widetilde{p}'(\mathcal{F}_1) + \widetilde{p}'(\mathcal{F}_2) + \widetilde{p}'(\mathcal{F}_3) = m_{in}(v) + m_{out}(v) + \widetilde{p}(\mathcal{F}_3)$, contradicting (17.66). •

Theorem 17.3.12 *Let $p \geq 0$ be an integer-valued, positively crossing supermodular bi-set-function. Let $m_{in} \in C_{in}$ and $m_{out} \in C_{out}$ be two integral vectors for which $\widetilde{m}_{in}(V) = \widetilde{m}_{out}(V)$. Then there exists a digraph $H = (V, F)$ for which $\dot{\varrho}_H = m_{in}$ and $\dot{\delta}_H = m_{out}$.*

Proof. Let V' and V'' be two disjoint copies of V and let $V^* := V' \cup V''$. For each subset X of V, let X' and X'' denote the corresponding subsets of V' and V'', respectively. For each element v of V, v' and v'' denote the corresponding elements of V' and V'', respectively. For a uv-edge e, let e^* be a $u'v''$-edge. For a digraph $H = (V, F)$ let $H^* := (V^*, F^*)$ where $F^* := \{e^* : e \in F\}$.

For a bi-set $X = (X_O, X_I) \in \mathcal{P}_2(V)$, let $\varphi(X) := \big(V'' \cup X'_O,\ X'_I\big)$. Define a bi-set-function p^* on $\mathcal{P}_2(V^*)$, as follows.

$$p^*(X^*) := \begin{cases} p(X) & \text{if } X^* = \varphi(X) \\ 0 & \text{if otherwise.} \end{cases} \tag{17.67}$$

Define m^*_{in} and m^*_{out} on V^* as follows.

$$m^*_{in}(v^*) := \begin{cases} m_{in}(v) & \text{if } v^* = v',\ v \in V \\ 0 & \text{if } v^* = v'',\ v \in V. \end{cases} \tag{17.68}$$

$$m^*_{out}(v^*) := \begin{cases} m_{out}(v) & \text{if } u = v'',\ v \in V \\ 0 & \text{if } v^* = v',\ v \in V. \end{cases} \tag{17.69}$$

The bi-set-function p^* is positively crossing supermodular and a digraph $H = (V, F)$ covers p if and only H^* covers p^*. Moreover, $\dot{\varrho}_{H^*} = m^*_{in}$ if and only if $\dot{\varrho}_H = m_{in}$, and $\dot{\delta}_{H^*} = m^*_{out}$ if and only if $\dot{\delta}_H = m_{out}$.

When we write condition (17.66) for p^*, m^*_{in}, and m^*_{out}, it requires that $m^*_{in}(v^*) + m^*_{out}(v^*) + \widetilde{p}^*(\mathcal{F}^*) \leq \gamma$ holds for every independent set $\mathcal{F}^*$ of bi-sets X^* for which $v^* \in X^*_O - X^*_I$ for every $v^* \in V^*$. In terms of p, m_{in}, and m_{out}, this is equivalent to requiring that $m_{in}(v) + \widetilde{p}(\mathcal{F}) \leq \gamma$ and $m_{out}(v) + \widetilde{p}(\mathcal{F}) \leq \gamma$ hold whenever $v \in V$ and $\mathcal{F}$ is an independent family of bi-sets $X \in \mathcal{P}_2$ for which $v \in X_O - X_I$. But this latter property does hold since $m_{in} \in C_{in}$ and $m_{out} \in C_{out}$. Therefore Theorem 17.3.11 implies the existence of a loopless digraph $H^* = (V^*, F^*)$ covering p^* for which $\dot{\varrho}_{H^*} = m^*_{in}$ and $\dot{\delta}_{H^*} = m^*_{out}$. Since H^* can have only $V''V'$-edges, it corresponds to a digraph $H = (V, F)$ covering p for which $\dot{\varrho}_H = m_{in}$ and $\dot{\delta}_H = m_{out}$. (Note that H may have loops since a $v'v''$-edge of H^* determines a loop of H at v.) •

The following result is an extension of Theorem 17.2.8 to bi-set functions. Recall the definition of p_{in} and p_{out} given in (17.61) and (17.65).

Theorem 17.3.13 *Let $p \geq 0$ be a positively crossing supermodular bi-set-function on ground-set V such that $p(X) = 0$ for every trivial bi-set X. Let $g_{in} : V \to \mathbf{Z}_+ \cup \{\infty\}$ and $g_{out} : V \to \mathbf{Z}_+ \cup \{\infty\}$ be upper bound functions. Let $\gamma \geq 0$ be an integer. There exists a digraph $H = (V, F)$ covering p*

(A) *for which $\dot{\varrho}_H \leq g_{in}$ if and only if $p_{in} \leq \widetilde{g}_{in}$,*
(B) *for which $\dot{\delta}_H \leq g_{out}$ if and only if $p_{out} \leq \widetilde{g}_{out}$,*
(C) *for which $|F| \leq \gamma$ if and only if $\widetilde{p}(\mathcal{F}) \leq \gamma$ for every independent set of bi-sets.*
(D) *If there is a digraph H_1 for which $\dot{\varrho}_{H_1} \leq g_{in}$, and there is a digraph H_2 for which $\dot{\delta}_{H_2} \leq g_{out}$, and there is a digraph H_3 with at most γ edges such that each H_i covers p, then there is one covering p that satisfies simultaneously each of* (A), (B), *and* (C). •

Proof. We can assume that both g_{in} and g_{out} are finite-valued. Parts (A), (B), and (C) were proved earlier in Theorems 17.3.7, 17.3.10 17.3.3, respectively. To see (D), let γ^* denote the minimum number of edges of a digraph covering p. Then $p_{in}(V) = \gamma^* = p_{out}(V)$ and $\gamma^* \leq \gamma$. Since $g_{in} \in C_{in}$ and C_{in} is a contra-polymatroid, there is an element $m_{in} \in C_{in}$ for which $m_{in} \leq g_{in}$ and $\widetilde{m}_{in}(V) = \gamma^*$. Since $g_{out} \in C_{out}$ and C_{out} is a contra-polymatroid, there is an element $m_{out} \in C_{out}$ for which $m_{out} \leq g_{out}$ and $\widetilde{m}_{out}(V) = \gamma^*$.

By Theorem 17.3.12, there is a digraph $H = (V, F)$ covering p for which $|F| = \gamma^* \leq \gamma$, $\dot{\varrho}_H = m_{in} \leq g_{out}$, and $\dot{\delta}_H = m_{out} \leq g_{out}$. •

Problem 17.3.1 *Derive Theorem* 17.2.8 *from Theorem* 17.3.13.

Augmenting node-connectivity

One of the main motivations behind Theorem 17.3.3 has been Theorem 11.3.5 on the possibility of making a digraph $D = (V, A)$ k-node-connected by adding γ new edges. In (11.40) we introduced a bi-set-function h as follows: $h(X) := (k - \varrho_D(X) - w(X))^+$

978–0–19–920527–1 17-Andras-Frank-c17-drv Andras Frank (Typeset by SPi, Publisher Services, India) 585 of 640 December 11, 2010 15:21

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

when X is non-trivial and $h(X) := 0$ when X is trivial. This function is positively crossing supermodular and a digraph H covers h precisely if $D + H$ is k-node-connected. Therefore, if we apply Theorem 17.3.3 with h in place of p, we obtain Theorem 11.3.5 as a special case.

The same approach enables us to obtain more general connectivity augmentation results. Let k and ℓ be non-negative integers. Recall that a digraph is said to be (k, ℓ)-hybrid-connected if

$$\varrho(X) + \ell w(X) \geq k\ell \text{ for every non-trivial bi-set } X.$$

By a simple reformulation of the definition, we obtain that a digraph is (k, ℓ)-hybrid-connected if and only if the removal of any j $(0 \leq j \leq k-1)$ nodes leaves an $\ell(k-j)$-edge-connected digraph. Corollary 2.5.15 included a characterization of (k, ℓ)-hybrid-connected digraphs in terms of k-bundles. For $k = 1$, the notion is equivalent to ℓ-edge-connectivity, while for $\ell = 1$, we are at k-node-connectivity.

Consider the augmentation problem where we want to make a digraph $D = (V, A)$ (k, ℓ)-hybrid-connected by adding γ new directed edges. Define a bi-set-function h as follows.

$$h(X) := \begin{cases} (\ell(k - w(X)) - \varrho_D(X))^+ & \text{if } X \text{ is non-trivial} \\ 0 & \text{if } X \text{ is trivial.} \end{cases} \tag{17.70}$$

Theorem 17.3.14 *A digraph $D = (V, A)$ can be made (k, ℓ)-hybrid-connected by adding at most γ new edges if and only if*

$$\widetilde{h}(\mathcal{F}) \leq \gamma \text{ holds for every independent set } \mathcal{F} \text{ of non-trivial bi-sets} \tag{17.71}$$

where h is defined in (17.70).

Proof. Observe that the bi-set-function h in question is positively crossing supermodular, and a digraph H covers h if and only if $D + H$ is (k, ℓ)-hybrid-connected. By applying Theorem 17.3.3 with h in place of p, the theorem follows. •

Theorem 17.3.13 can be used in a similar way to deduce characterizations for a given digraph $D = (V, A)$ to have a k-node-connected augmentation obeying in-degree and out-degree constraints. The theory can also be used for obtaining similar results concerning other target connectivities such as k-edge-connectivity from S to T or k-node-connectivity from S to T. Because of the contra-polymatroid property of the in- and out-degree functions, the corresponding minimum-cost augmentation problems are also tractable for node-induced cost functions.

17.3.3 An application to rectilinear polygons

Our last goal is to exhibit one of the most astonishing links between the combinatorial optimization problems touched on in the present book. Starting with directed connectivity augmentation problems, we arrived at Theorem 17.3.1, an abstract result on covering bi-set functions. We are going to show that this theorem can be used to derive the beautiful min-max theorem of Győri [198] on the minimum number of rectangles covering a rectilinear polygon, a result which apparently has nothing to do with connectivity augmentations or with supermodular functions. To this end, we need a little preparation.

Reformulating Theorem 17.3.5 in terms of set-pairs

We will need Theorem 17.3.1 only in the special case of (0, 1)-valued supermodular bi-set functions. In this was formulated in Theorem 17.3.5 and reads as follows.

Let $\mathcal{L} \subseteq \mathcal{P}_2$ be a crossing set of non-trivial bi-sets. The minimum number $\tau = \tau(\mathcal{L})$ of directed edges covering $\mathcal{L}$ is equal to the maximum number $\nu = \nu(\mathcal{L})$ of independent members of $\mathcal{L}$.

For technical reasons, it is worth reformulating Theorem 17.3.5. Let V be a ground-set. For two disjoint subsets X_T and X_H of V, we call the pair $X = [X_T, X_H]$ a **set-pair**. The first member X_T of the pair is the **tail** of X while the second member X_H is the **head**. Let $\mathcal{D}_2$ denote the system of set-pairs.

Consider the operation φ on $\mathcal{D}_2$ that replaces the tail of a member X of $\mathcal{D}_2$ by its complement. Then $\varphi(X)$ can be considered as a bi-set where the head of X is the inner set of $\varphi(X)$ while the complement of the tail of X is the outer set of $\varphi(X)$. Clearly, φ defines a one-to-one correspondence between set-pairs and bi-sets. Accordingly, we can carry over all definitions and operations concerning bi-sets to those concerning set-pairs. For example, a set-pair X is **non-trivial** if $\varphi(X)$ is a non-trivial bi-set, that is, neither the tail nor the head of X is empty. Or, a directed edge e covers $[X_T, X_H]$ if e covers the bi-set $(V - X_T, X_H)$, that is, if e is an $X_T X_H$-edge. Also, for set-pairs X and Y, we write $X \preceq Y$ if $\varphi(X) \preceq \varphi(Y)$ holds for the corresponding bi-sets. Similarly, we can define $X \sqcap Y$ and $X \sqcup Y$, for example, $X \sqcap Y := \varphi^{-1}(\varphi(X) \sqcap \varphi(Y)) = [X_T \cup Y_T, X_H \cap Y_H]$. With this convention in mind, we speak of independent, crossing, cross-free, and comparable set-pairs without any further reference to their definition. A reformulation of Theorem 17.3.5 is as follows.

Theorem 17.3.15 *Let $\mathcal{E} \subset \mathcal{D}_2$ be a crossing system of non-trivial set-pairs. The minimum number of directed edges covering $\mathcal{E}$ is equal to the maximum number independent members of $\mathcal{E}$.* •

Generating a path system

Let $P = (v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n)$ be a simple directed path or circuit where e_i is a $v_i v_{i+1}$-edge, with the convention that $v_n = v_1$ if P is a circuit. Let V denote the set of nodes of P. In what follows, by a path we mean the edge-set of the path. The underlying P will also be identified with its edge-set. The first node and the last node of a path U will be denoted by $s(U)$ and $t(U)$, respectively. In particular, for an edge $e \in P$, $s(e)$ and $t(e)$ denote the tail and the head of e, respectively.

Let $\mathcal{U}$ be a system of subpaths of P. A system $\mathcal{G}$ of subpaths of P is said to **generate** $\mathcal{U}$ (and $\mathcal{G}$ is called a **generator** of $\mathcal{U}$) if every member of $\mathcal{U}$ can be obtained as the union of some members of $\mathcal{G}$. For example, $\mathcal{U}$ is a generator of itself, and the system $\{e_1, \dots, e_n\}$ consisting of one-element paths is also a generator of $\mathcal{U}$. Let $\gamma(\mathcal{U})$ denote the minimum cardinality of a generator of $\mathcal{U}$.

Let $\mathcal{I} := \{I_1, \dots, I_q\}$ be a family of subpaths of P and let $\mathcal{R} := \{f_1, f_2, \dots, f_q\} \subseteq P$ be a system of distinct representatives of $\mathcal{I}$, meaning that the edges f_i are distinct elements of P and $f_i \in I_i$ holds for every $i = 1, \dots, q$. We say that $\mathcal{R}$ is a **strong system of distinct representatives** if $I_i \cap I_j$ does not contain both f_i and f_j $(1 \leq i < j \leq q)$. $\mathcal{I} := \{I_1, \dots, I_q\}$

is **strongly representable** if it admits a strong system of distinct representatives. Let $\sigma(\mathcal{U})$ denote the maximum number of strongly representable paths in $\mathcal{U}$.

Proposition 17.3.16 *If $\mathcal{I} := \{I_1, \dots, I_q\}$ is a strongly representable system of paths, then $\mathcal{I}$ cannot be generated by less than $|\mathcal{I}|$ paths.*

Proof. Let $\{f_1, \cdots, f_q\}$ be a strong system of distinct representatives of $\mathcal{I}$ and let $\mathcal{G}$ be a generator of $\mathcal{I}$. For each $j = 1, \dots, q$, there is a member G_j of $\mathcal{G}$ such that $f_j \in G_j \subseteq I_j$. For $1 \leq j < k \leq q$, G_j and G_k are distinct for otherwise both f_j and f_k would belong to $I_j \cap I_k$ which is not possible since $\{f_1, \dots, f_q\}$ is a strong system of distinct representatives. Therefore $|\mathcal{G}| \geq q$, as required. •

The proposition implies that $\sigma(\mathcal{U}) \leq \gamma(\mathcal{U})$, and Győri proved that here, in fact, equality holds when P is a path.

Theorem 17.3.17 (Győri [198]) *For a system $\mathcal{U}$ of subpaths of a directed path P, the minimum number $\gamma(\mathcal{U})$ of subpaths of P generating $\mathcal{U}$ is equal to the maximum cardinality $\sigma(\mathcal{U})$ of strongly representable subsets of $\mathcal{U}$.* •

We are going to derive this result from Theorem 17.3.15 even in the more general case when P is a di-circuit. This is indeed a generalization since a path $P = (v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n)$ can be extended to a di-circuit by adding an edge $v_n v_0$, without affecting the input system $\mathcal{U}$ of subpaths.

Theorem 17.3.18 (Frank and Jordán [139]) *For a system $\mathcal{U}$ of subpaths of a directed circuit P, the minimum number $\gamma(\mathcal{U})$ of subpaths of P generating $\mathcal{U}$ is equal to the maximum cardinality $\sigma(\mathcal{U})$ of strongly representable subsets of $\mathcal{U}$.*

Proof. For a path U and edge $e \in U$, we say that the pair (U, e) is a **represented path**. Let $(U, e)^-$ and $(U, e)^+$ denote the sets of nodes of U preceding and following e, respectively. We emphasize that a U is considered a set of edges, while $(U, e)^-$ and $(U, e)^+$ are sets of nodes which form a bipartition of the node-set of U. The set-pair $[(U, e)^-, (U, e)^+]$ is said to be **associated with** the represented path (U, e). With a slight abuse of notation, henceforth we identify a represented path with its associated set-pair, and write $(U, e) = [(U, e)^-, (U, e)^+]$.

Therefore all notions concerning set-pairs can be carried over to represented paths. For example, we can speak of the independence of a subset of represented paths, or say that two represented paths are crossing or non-crossing. (See Figure 17.2.) Let $\mathcal{U}^{(r)} := \{(U, e) : e \in U \in \mathcal{U}\}$, and defined a member (U, f) of $\mathcal{U}^{(r)}$ to be **essential** if there is no member U' of $\mathcal{U}$ for which $f \in U' \subset U$. Let $\mathcal{E}$ denote the set of essential represented paths belonging to the path system $\mathcal{U}$, or formally

$$\mathcal{E} := \{(U, f) = [(U, f)^-,\ (U, f)^+] \in \mathcal{U}^{(r)} :\ (U, f) \text{ essential}\}.$$

Proposition 17.3.19 *Let $\mathcal{U}$ be a system of subpaths of a di-circuit P. Then the set $\mathcal{E}$ of essential represented paths belonging to $\mathcal{U}$ is a crossing system of set-pairs.*

Proof. Let (U, f) and (U', f') be two crossing members of $\mathcal{E}$. By symmetry, we can assume that $s(U') \in (U, f)^-$. It follows that the set-pair $[(U, f)^- \cup (U', f')^-, (U, f)^+ \cap$

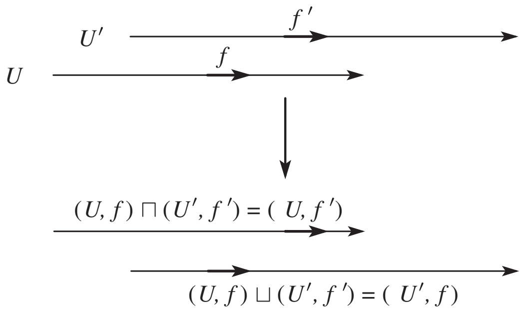


Figure 17.2 Crossing represented paths

$(U', f')^+]$ is the one associated with the represented path (U, f'), or concisely, $(U, f) \sqcap (U', f') = (U, f')$. Similarly, the set-pair $[(U, f)^- \cap (U', f')^-,\ (U, f)^+ \cup (U', f')^+]$ is the one associated with represented path (U', f), and hence $(U, f) \sqcup (U, f') = (U', f)$.

What remained to show is that both (U', f) and (U, f') are essential. We prove this only for (U, f') since the proof for (U', f) is analogous. Suppose indirectly that there is a member Z of $\mathcal{U}$ for which $f' \in Z \subset U$. Then $f \notin Z$ since (U, f) is essential. This in turn implies that $Z \subset U'$, contradicting the hypothesis of (U', f') being essential. •

Turning to the proof of the theorem, note that we have already proved the trivial direction $\sigma \leq \gamma$ in Proposition 17.3.16, and hence we deal with the reverse inequality $\sigma \geq \gamma$. Observe, that an independent subset of $\mathcal{E}$ corresponds to a a subset of $\mathcal{U}$ having a strong representation. We show how a covering of $\mathcal{E}$ defines a generator of $\mathcal{U}$.

Claim 17.3.20 *Let $C := \{c_1, \ldots, c_q\}$ be a set of directed edges on V covering $\mathcal{E}$. Let G_i be a subpath of P from $s(c_i)$ to $t(c_i)$ $(i = 1, \ldots, q)$, and let $\mathcal{G}_C := \{G_1, \ldots, G_q\}$. Then $\mathcal{G}_C$ generates $\mathcal{U}$.*

Proof. If $\mathcal{G}_C$ does not generate $\mathcal{U}$, then there is a minimal member U of $\mathcal{U}$ which cannot be obtained as the union of some members of $\mathcal{G}_C$. Then U has an element f such that

$$\text{there is no } G \in \mathcal{G}_C \text{ with } f \in G \subseteq U. \tag{17.72}$$

Since C covers all pairs of sets associated with essential represented path, (U, f) is not essential. Therefore there exists a member U' of $\mathcal{U}$ for which $f \in U' \subset U$. By the minimal choice of U, U' is the union of some members of $\mathcal{G}_C$ contradicting (17.72). •

By Proposition 17.3.19, Theorem 17.3.15 applies, and hence the theorem follows. • •

Covering rectilinear polygons by rectangles

Let T be a rectilinear polygon (a polyomino) which is by definition a closed region in the plane bounded by horizontal and vertical line segments. We want to cover T by a minimum number of rectangles lying in T, where a rectangle is meant throughout to be a closed one bounded by horizontal and vertical lines. It is evident that the number of covering rectangles is at least the number of 'independent' points in T where two points are independent if no rectangle lying in T contains both. (See Figure 17.3.) In a survey paper [199], Győri exhibits

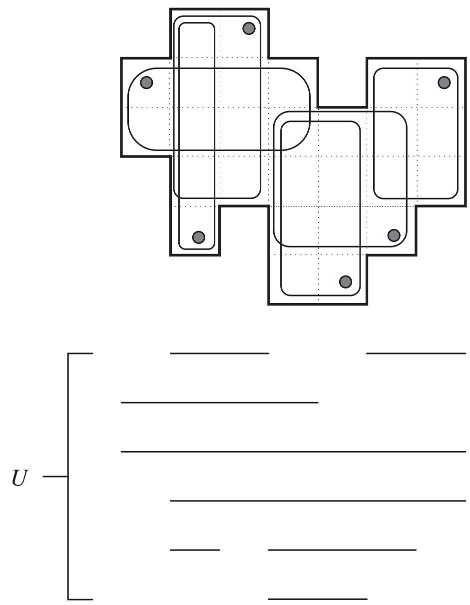


Figure 17.3 Path system assigned to a rectilinear polygon

a rectilinear polygon T, due to Szemerédi, in which the minimum number of rectangles covering T is 8 while the maximum number of independent points in T is 7. Moreover, T contains no holes. The existence of such an example is not so surprising since Culberson and Reckhow [59] proved that it is **NP**-complete to find a minimum number of rectangles covering a rectilinear polygon T even if T contains no holes. In this light, it is particularly valuable that the min-max relationship does hold for vertically (or horizontally) convex rectilinear polygons. A region T of the plane is said to be **vertically convex** if its intersection with a vertical line is a line segment or empty.

Theorem 17.3.21 (Győri [198]) *Let T be a vertically convex rectilinear polygon. The minimum number of rectangles covering T is equal to the maximum number of independent points of T.*

Proof. It can be assumed that the coordinates of the boundary horizontal and vertical lines are integers and the coordinates of the leftmost and the rightmost boundary vertical lines are 0 and n, respectively. Let $\{v_0, \dots, v_n\}$ be the node-set of a directed path P from v_0 to v_n.

We say that a horizintally maximal rectangle lying in T is **elementary** if it is of unit height and the coordinates of the boundary lines are integers. The elementary rectangles cover T, and two of them can have points in common only when their lower or upper horizontal boundary segments are on the same straight line. With every elementary rectangle Z, we associate a subpath P_Z of P from v_i to v_j where i and j are the coordinates of the left and the right boundary lines of Z, respectively. Let $\mathcal{U}$ denote the system of subpaths of P obtained in this way. Let $y(P_Z)$ be one half bigger than the coordinate of the lower horizontal side of Z.

Consider a set $\{(I_1, f_1), (I_2, f_2), \ldots, (I_q, f_q)\}$ of strongly represented subpaths of P for which $q = \sigma(\mathcal{U})$ is maximum where each I_i is in $\mathcal{U}$. Let $y_i := y(I_i)$ and let x_i one half bigger that the coordinate of the tail of edge f_i. Due to this construction, the points (x_i, y_i) $(i = 1, \ldots, q)$ are all in T. We claim that these points are independent. Indeed, if (x_i, y_i) and (x_j, y_j) were not independent, then the unique smallest rectangle containing them would lie in T contradicting the definition of strong representability. Therefore the maximum number of independent points of T is at least $\sigma(\mathcal{U})$.

Let $\mathcal{G}$ be a minimum set of subpaths of P generating $\mathcal{U}$. That is, the cardinality of $\mathcal{G}$ is $\gamma(\mathcal{U})$. With each member of $G \in \mathcal{G}$, we can associate a unique vertically maximal rectangle lying in T for which the 'vertical projection' to path P is G. We claim that the $\gamma(\mathcal{U})$ rectangles obtained in this way cover T. Indeed, an arbitrary point (x, y) of T belongs to an elementary rectangle Z. Let P_Z denote the member of $\mathcal{U}$ assigned to Z. Since P_Z is the union of some members of $\mathcal{G}$, there is a member $G \in \mathcal{G}$ which is included in P_Z and contains the edge $v_{\lfloor x \rfloor} v_{\lceil x \rceil}$ of P. Since T is vertically convex, the rectangle associated with G contains the point (x, y).

Summing up, we proved that there are $\sigma(\mathcal{U})$ independent points in T and that T can be covered by $\gamma(\mathcal{U})$ rectangles from which the theorem follows by Theorem 17.3.18. •

Remark In the proof of Győri's theorem on rectilinear polygons, we relied on Theorem 17.3.17, which is a special case of Theorem 17.3.18. But the same proof works when the underlying P is a directed circuit if we use Theorem 17.3.18. In this way a min-max formula can be obtained for the minimum number of 'cylindric rectangles' to cover a vertically convex rectilinear polygon on the surface of a vertically standing cylinder.

17.3.4 An algorithm for generating paths

The proof of Theorem 17.3.18 is not algorithmic, but there is a polynomial time algorithm, developed by Franzblau and Kleitman [156], for the special case when P is a path. Another algorithm, relying on Dilworth's theorem, was described by Frank [131], and this latter one has the advantage that it extends to the case when the underlying P is a directed circuit. A weighted extension of Győri's theorem along with an algorithmic approach was given by Lubiw [276]. An algorithm was described in [132] for a further extension of the weighted version in the general case when P is a directed circuit.

In this subsection, we exhibit the algorithm that appeared in [131]. For technical simplicity, however, here we consider only the special case when P is a path. The algorithm can be considered as an alternative proof of the non-trivial direction $\sigma \geq \gamma$ of Theorem 17.3.17. The algorithm constructs a covering C of $\mathcal{E}$ (which corresponds to a generator of $\mathcal{U}$ by Claim 17.3.20) and an independent subset $\mathcal{I}$ of $\mathcal{E}$ for which $|C| = |\mathcal{I}|$.

Recall how we introduced a partial order $\preceq$ on $\mathcal{E}$. Furthermore, two represented paths (U, f) and (U', f'), for which $s(f') \in (U, f)^-$, are crossing if and only if $s(U) < s(U') \leq s(f) < s(f') < t(U) < t(U')$ (see Figure 17.2).

The algorithm consists of three phases. In the first one, we construct a cross-free subset $\mathcal{K}$ of $\mathcal{E}$. In the second phase, we apply Dilworth's theorem to the poset $(\mathcal{K}, \preceq)$, and compute a minimum chain-decomposition of $\mathcal{K}$ along with a maximum subset $\mathcal{I}$ of pairwise incomparable members of $\mathcal{K}$. The chain-decomposition will correspond to a set C of directed edges covering every member of $\mathcal{K}$. In the third phase, we modify C, without changing its cardinality, so as to obtain a covering of $\mathcal{E}$.

Phase 1 Consider the edges $e_1, \dots, e_n$ of the underlying path P in this order one by one, and assign each member of $\mathcal{E}$ to one of two groups $\mathcal{K}$ and $\mathcal{T}$. Once a member is assigned to $\mathcal{K}$ or to $\mathcal{T}$, its status will never be changed. The final $\mathcal{K}$ and $\mathcal{T}$ will form a bipartition of $\mathcal{E}$.

At the beginning, both groups $\mathcal{K}$ and $\mathcal{T}$ are empty. At a general step, when edge $f \in P$ is processed, consider each member (U, f) of $\mathcal{E}$ which has not been assigned to $\mathcal{T}$ and put it into $\mathcal{K}$. At the same time, put every essential pair $(U', f') \in \mathcal{E}$ into $\mathcal{T}$ for which f precedes f' and (U, f) crosses (U', f'). The rule of Phase 1 implies that the final $\mathcal{K}$ is cross-free.

Phase 2 Apply Dilworth's theorem (Theorem 2.4.25) to the partially ordered set $(\mathcal{K}, \preceq)$. We obtain that there exists an antichain $\mathcal{I} \subseteq \mathcal{K}$ and a chain-decomposition $\{\mathcal{C}_1, \dots, \mathcal{C}_q\}$ of $\mathcal{K}$ such that $|\mathcal{I}| = q$. Note that the proof of Dilworth's theorem was based on a clever reduction to Kőnig's matching theorem, and hence a bipartite matching algorithm can be used to compute the extrema in question. Since $\mathcal{K}$ is cross-free, so is $\mathcal{I}$, and therefore $\mathcal{I}$ is an independent family of represented paths.

Claim 17.3.22 *The members of a chain $\mathcal{C}$ of $\mathcal{K}$ can be covered by one directed edge.*

Proof. Let $(U_1, f_1) \succeq \cdots \succeq (U_k, f_k)$ be the members of $\mathcal{C}$. Then $(U_1, f_1)^- \subseteq (U_2, f_2)^- \subseteq \cdots \subseteq (U_k, f_k)^-$ and $(U_1, f_1)^+ \supseteq (U_2, f_2)^+ \supseteq \cdots \supseteq (U_k, f_k)^+$, and hence a directed edge from $s(U_1)$ to $t(U_k)$ covers every member of $\mathcal{C}$. •

By the claim, each chain $\mathcal{C}_i$ $(i = 1, \dots, q)$ can be covered by a directed edge c_i. Then $C := \{c_1, \dots, c_q\}$ covers every member of $\mathcal{K}$ and $|C| = |\mathcal{I}|$.

Phase 3 We say that two elements $g_1 = x_1y_1$ and $g_2 = x_2y_2$ of C are **exchangeable** if $x_2 < x_1 < y_1 < y_2$ and

$$C' = C - \{g_1, g_2\} \cup \left\{g_1', g_2'\right\} \tag{17.73}$$

is also a covering of $\mathcal{K}$ where $g_1' := x_2y_1$, $g_2' := x_1y_2$. Replacing C by C' is called an **exchange step**. Phase 3 consists of applying exchange steps as long as there are exchangeable members of the current covering of $\mathcal{K}$.

For a directed edge $c = xy$, let $h(c)$ denote the number of edges of the underlying path P from x to y. It is easy to see that $\sum[h^2(c) : c \in C'] < \sum[h^2(c) : c \in C]$. Therefore Phase 3 terminates after at most $|C|n^2 \leq n^3$ exchange steps.

At termination, we are left with a covering C of $\mathcal{K}$ possessing no exchangeable edges. The crucial point of the proof is the fortunate fact that such a covering of $\mathcal{K}$ will automatically cover the entire $\mathcal{E}$.

Lemma 17.3.23 *If C is a covering of $\mathcal{K}$ with no exchangeable edges, then C covers $\mathcal{E}$.*

Proof. Suppose indirectly that C does not cover a member (U, f) of $\mathcal{E}$. Let us choose U and f such that $|(U, f)^-|$ is minimum, and, subject to that, $|(U, f)^+|$ is minimum.

Since C covers $\mathcal{K}$, (U, f) does not belong to $\mathcal{K}$. By the rule of Phase 1, there is a member (U', f') of $\mathcal{K}$ for which f' precedes f and (U', f') crosses (U, f). Let us choose (U', f') in such a way that $|(U', f')^+|$ is minimum. Since both (U, f) and (U', f') are essential, so

978–0–19–920527–1 17-Andras-Frank-c17-drv Andras Frank (Typeset by SPi, Publisher Services, India) 592 of 640 December 11, 2010 15:21
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

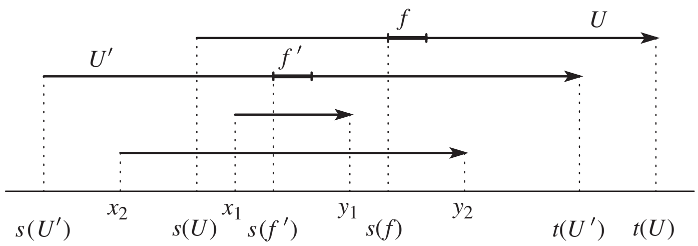


Figure 17.4 Proof of Lemma 17.3.23

are (U, f') and (U', f). It follows from the minimality of $|(U, f)^-|$ that C covers (U, f'), and hence there is an edge $g_1 = x_1y_1 \in C$ covering (U, f').

Claim 17.3.24 $(U', f) \in \mathcal{K}$.

Proof. If, indirectly, (U', f) belongs to $\mathcal{T}$, then the rule of Phase 1 implies the existence of a member (Z, z) of $\mathcal{K}$ such that z precedes f and (Z, z) crosses (U', f). The edge z cannot precede f' for otherwise (Z, z) would cross (U', f') and hence (U', f') would belong to $\mathcal{T}$. Therefore either $z = f'$ or else f' precedes z. In both cases, (Z, z) crosses (U, f) and $(Z, z)^+ \subset (U', f')^+$, contradicting the minimal choice of $|(U', f')^+|$. •

Since C covers $\mathcal{K}$ and $(U', f) \in \mathcal{K}$ by the claim, there is an element $g_2 = x_2y_2$ of C covering (U', f) (see Figure 17.4). Since neither g_1 nor g_2 covers (U, f), we have

$$s(U') \le x_2 < s(U) \le x_1 \le s(f') < y_1 \le s(f) < y_2 \le t(U') < t(U). \qquad (17.74)$$

By the hypothesis of the lemma, g_1 and g_2 are not exchangeable, and hence there is a member (K, k) of $\mathcal{K}$ which is not covered by C', where C' is defined in (17.73).

If g_2 covers (K, k), then at least one of g_1' and g_2' would also cover (K, k), but this cannot be the case since C' does not cover (K, k). Therefore g_2 does not cover (K, k), and hence g_1 does. Since neither g_1' nor g_2' covers (K, k), we must have $s(U') \le x_2 < s(K) \le x_1 < y_1 \le t(K) < y_2 \le t(U')$, and hence $f' \in K \subset U'$, contradicting that (U', f') is essential. This contradiction proves the lemma. • •

Associate with each edge uv in C the subpath of P from u to v and let $\mathcal{G}_C$ denote the family of these associated subpaths. By Claim 17.3.20, $\mathcal{G}_C$ is a generator of $\mathcal{U}$ and $|\mathcal{G}_C| = |\mathcal{I}|$, as required for Győri's theorem. • • •

The constructive proof above for Győri's theorem gives rise to a polynomial time algorithm for computing a minimum generator of a path system and a maximum independent family of represented paths. A detailed analysis of the algorithm was carried out in the work of Benczúr, Förster, and Király [17] where they proved that there is an implementation of complexity $O(n^3)$.

Remark By developing further the idea of the algorithm above, Frank and Végh [155] constructed an algorithmic proof for Theorem 17.3.15. This enables one algorithmically to make a $(k - 1)$-connected digraph k-connected by adding a smallest set of new edges.

18
Solutions to selected problems

Problem 1.3.3 *Let $G = (V, E)$ be an undirected graph and let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \dots, G_t = (V_t, E_t)$ be subgraphs of G such that $\{E_1, \dots, E_t\}$ is a partition of E. Then G has a smooth orientation for which the restriction to E_j is a smooth orientation of G_j for each $j = 1, \dots, t$.*

Induction on the number of edges of G. If an E_i includes a circuit C, we cen delete the edges of C. So each E_i is a forest. We can divide each forest into its components. So each E_i is a tree. If each E_i is a single edge, then any smooth orientation of G will suffice. If an E_i is not a single edge, then let s and t be two leaf nodes of E_i and P the unique path in E_i connecting s and t. By deleting the edges of P from G and from E_i and adding a new edge $e = st$, we obtain a graph G' form G and a forest F_i from E_i. Clearly $d_{G'}(v)$ and $d_G(v)$ have the same parity for every node v of G. By induction, there is a requested orientation D' of G'. Suppose that the new edge e is oriented from s to t. Orient P from t to s and let $\vec{P}$ be the resulting directed path. By removing the new edge from D' and adding $\vec{P}$, we obtain a requested orientation of G.

Problem 2.2.1 *Prove that a strongly connected digraph D has a flat covering B of di-circuits such that each node is reachable in $D - B$ from a specified root-node r_0.*

Let B be a flat covering of di-circuits such that the set Z of nodes reachable from r_0 in $D - B$ is as large as possible. We claim that $Z = V$. Suppose indirectly that $Z \subset V$. In the proof of Theorem 2.2.2, we pointed out that in such a situation every edge of D' leaving Z is in B and no edge entering Z belongs to B. Revise B by removing all the edges leaving Z and adding all the edges entering Z. The resulting set B' is a flat covering of di-circuits and the set of nodes reachable from r_0 in $D - B'$ includes properly Z, a contradiction.

Problem 2.2.2 *Let $D = (V, A)$ be a strongly connected orientation of a 3-edge-connected undirected graph G. Let $F \subseteq A$ be a subset of edges such that the digraph obtained from D by reversing all elements of F is strongly connected. Then there is an element f of F such that reversing only f results in a strong digraph.*

Suppose indirectly that there is a subset $P_f \subset V$ entered by f for every $f \in F$ such that $\varrho_D(P_f) = 1$. Choose each P_f to be minimal and let $\mathcal{P} := \{P_f : f \in F\}$. Show that $\mathcal{P}$ is cross-free.

Since the reversal of F results in a strong digraph, there is an $f' \in F$ for every edge $f \in F$ such that f' leaves P_f. Show that $P_f \cap P_{f'} = \emptyset$. Show that $P_f \cup P_{f'} \neq V$. Show that the tail of f is not in $P_{f'}$.

Let P_e be a maximal member of $\mathcal{P}$. Consider a longest sequence $e, e', e'', \dots, e^{(k)}$ for which the sets $P_{e^{(i)}}$ are pairwise disjoint and do not contain the tail of e. Then $k \geq 2$. Let $f := e^{(k+1)}$ be the edge leaving $P_{e^{(k)}}$ and let Z denote the union of the sets $P_{e^{(i)}}$ for $(i = 0, \dots, k+1)$.

If P_f is disjoint from each $P_{e^{(i)}}$, then the maximality of k implies that Z contains the tail of e. Therefore, $\varrho_D(Z) = 0$ and hence $Z = V$. But in this case $d_G(P_{e^{(i)}}) = 2$ for each i, contradicting the 3-edge-connectivity of the underlying G.

Therefore, P_f must intersect a $P_{e^{(j)}}$ for some $j \in \{0, \dots, k\}$. Suppose that j is minimal. Then $j \neq 0$, for otherwise P_e and P_f would cross each other. Hence $j \geq 1$, implying $P_f \supset P_{e^{(j)}}$. But the minimal choice of j implies that P_f and $P_{e^{(j-1)}}$ are disjoint and hence $e^{(j)}$ also enters P_f, a contradiction.

Problem 2.2.3 *Prove that an acyclic digraph $D = (V, A)$ has a flat covering of directed cuts.*

Since D is acyclic, every edge belongs to a dicut. Let $B_1, B_2, \dots, B_k$ be a minimal set of dijoins covering all edges. Clearly, $k \leq |A|$. With every dijoin F, we associate a vector $v_F = (|B_1 \cap F|, |B_2 \cap F|, \dots, |B_k \cap F|)$. Let F be a dijoin for which v_F is minimal.

Then F is a minimal dijoin, for if $F' := F - f$ also covers all dicuts, then $v_{F'} < v_F$, contradicting the minimal choice of F. It follows, that $(*)$ every element of a dijoin, for which the associated vector is minimal, belongs to a directed cut covered exactly once.

Consider now an edge $e \in A - F$. If we contract F, then the resulting digraph D' is strongly connected, so there is a directed circuit C' in D' containing e. This determines a circuit C of D containing e. An edge of C is clockwise or anticlockwise if going around C its orientation is the same or the opposite as the orientation of e. By the construction of C, every element of $C - F$ is clockwise. We claim that every element f of $C \cap F$ is anticlockwise. Suppose indirectly that f is clockwise. Since f is in F, it belongs to a dicut B for which $|B \cap F| = 1$. But this is not possible since $B \cap C$ must contain an anticlockwise edge, and this is another element of F belonging to B.

Let F' be the symmetric difference of F and C. Since no clockwise edge of C belongs to F and every anticlockwise edge belongs to F, we can conclude that F' is also a dijoin. Moreover $|F \cap B| = |F' \cap B|$ for every dicut B, and hence $v_F = v_{F'}$. Since $e \in F'$, Property $(*)$ implies that e belongs to a dicut B for which $1 = |F' \cap B| = |F \cap B|$.

Problem 2.3.1 *At a chess tournament, the winner of a match gets one point, the loser no points, while both players get half a point at a draw. In order to avoid fractions, multiply everything by two, implying that the winner, for example, gets 2 points. Under this assumption, when can a sequence $m_1 \geq m_2 \geq \cdots \geq m_n$ be the final score of a chess tournament?*

Let G be a graph on n nodes in which every pair of nodes is connected by two parallel edges. If Player v beats Player u, then we orient both parallel edges toward v while at a draw these two edges are oriented oppositely. In this way, the in-degree of v measures exactly the

978–0–19–920527–1 18-Andras-Frank-c18-drv Andras Frank (Typeset by SPi, Publisher Services, India) 595 of 640 December 11, 2010 15:22
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

total number of points gained by v. Therefore the question is whether G has an orientation with a specified in-degree sequence. The Orientation lemma gives rise to the following result.

Theorem 1 *A sequence $m_1 \geq \cdots \geq m_n$ of non-negative integers is the final score of a chess tournament if and only if $2\sum_{i=1}^{n} m_i = n(n-1)$ and $2\sum_{i=1}^{k} m_i \leq k(k-1) + 2k(n-k)$ $(k = 1, \ldots, n)$.*

Problem 2.3.8 *Let $G = (V, E)$ be a simple maximal* (that is, tirangulated) *planar graph on at least five nodes. Prove that E can be partitioned into claws where a claw is the complete bipartite graph $K_{1,3}$.*

It follows from Euler's formula (Theorem 1.3.16) that $|E| = 3|V| - 6$. Since G has at least 5 nodes and G is planar, there are two non-adjacent nodes s and t. Let $m : V \to \mathbf{Z}_+$ be defined by $m(s) := m(t) := 0$ and $m(v) = 3$ for every $v \in V - \{s, t\}$. Then $\widetilde{m}(V) = 3|V| - 6 = |E|$. By Euler's formula, $i_G(X) \leq 3|X| - 6$ for every $X \subseteq V$ with $|X| \geq 2$. Hence $i_G \leq \widetilde{m}$ and, by the Orientation lemma, there exists an orientation of G for which $\dot{\varrho} = m$. Then the three edges entering v is a claw for every $v \in V - \{s, t\}$. Since s and t are non-adjacent, these $|V| - 2$ claws form a partition of E.

Problem 2.4.1 *Prove that the Hall condition implies* (2.19).

Suppose that there is a subset X violating (2.19), that is, $|X \cap S| > |X \cap T| + d_G(X, T)$. Let $Z := X \cap S$. We can assume that each $\Gamma_G(Z) \subseteq X$, for if there is an edge uv with $u \in Z$, $v \in T - X$, then $X' = X + v$ also violates (2.19). We can furthermore assume that $\Gamma_G(Z) = X \cap T$ for if there is a node $v \in (X \cap T) - \Gamma_G(Z)$, then $X - v$ also violates (2.19). Therefore, $X = Z \cup \Gamma_G(Z)$ and in this case $|X \cap S| > |X \cap T| + d_G(X, T)$ is just equivalent to $|Z| > |\Gamma_G(Z)|$, contradicting the Hall condition.

Problem 2.4.3 *Derive the following extension* [263] *of Theorem* 2.4.13.

Theorem 2 *Let $m : S \to \mathbf{Z}_+$ be a function. In a bipartite graph $G = (S, T; E)$, there exists a forest for which the degree of every node $s \in S$ is exactly $m(s)$ if and only if*

$$|\Gamma_G(X)| \geq \widetilde{m}(X) - |X| + 1 \tag{18.1}$$

for every non-empty subset $X \subseteq S$.

Proof. Necessity. A forest F in the graph induced by $X \cup \Gamma_G(X)$ can have at most $|X| + |\Gamma_G(X)| - 1$ edges, that is, $i_F(X \cup \Gamma_G(X)) \leq |X| + |\Gamma_G(X)| - 1$. On the other hand if every node s in S is incident to $m(s)$ edges from F, then $i_F(X \cup \Gamma_G(X)) = \widetilde{m}(X)$, from which $|\Gamma_G(X)| \geq \widetilde{m}(X) - |X| + 1$ follows.

Sufficiency. We can assume that m is strictly positive since a node s with $m(s) = 0$ can be left out. Define $z : S \to \mathbf{Z}_+$ by $z(s) := m(s) - 1$. By Theorem 2.4.6, there exists a subgraph $(S, T; M)$ of G such that $d_M(s) = z(s)$ for every $s \in S$ and $d_M(t) \leq 1$ for every $t \in T$. Let $R := \{t \in T : d_M(t) = 0\}$. Shrink R into a single new node r. Orient the elements of M toward T, while all the other edges toward S. We claim that every node in S is reachable from r in the resulting digraph D. Indeed, if the set X of nodes in S which are not reachable were non-empty, then $\Gamma_G(X) = \Gamma_M(X)$, from which $|\Gamma_G(X)| = |\Gamma_M(X)| = \widetilde{z}(X)$, and hence X

would violate (18.1). It follows from the construction that every node in T is also reachable from r and hence D has a spanning r-arborescence F. Since the in-degree of every node $t \in T - R$ is exactly one, F includes M. Since F enters every node but r exactly, we conclude that the edges in F form a forest in which the degree of every node s in S is $m(s)$.

Problem 2.4.8 *Consider the partial order on the power set of a set S defined by set inclusion. An antichain corresponds to a Sperner family of S* (no member includes another one). *Let $\mathcal{D}$ denote the highest D-antichain, ensured by Problem* 2.4.6. *Prove that if j denotes the largest cardinality of a subset of S belonging to $\mathcal{D}$, then every j-element subset of S must belong to $\mathcal{D}$. Derive Sperner's theorem which states that $\mathcal{D}$ can have at most $\binom{n}{\lceil n/2 \rceil}$ members.*

Suppose that $Z \subseteq S$ is a j-element set belonging to $\mathcal{D}$. Suppose indirectly that there is a j-element subset Z' of S which does not belong to $\mathcal{D}$. Consider a bijection $\varphi : S \to S$ for which $\varphi(Z) = Z'$. Then $\mathcal{D}' := \{\varphi(X) : X \in \mathcal{D}\}$ is another D-antichain and $\mathcal{D}'$ contains Z'. Since $\mathcal{D}$ is the highest D-antichain, there must be a member of $\mathcal{D}$ that includes Z', but this is impossible due to the the maximal choice of j.

It follows that $\mathcal{D}$ contains every j-element subset of S and hence it cannot contain any other subset of S implying that $|\mathcal{D}| = \binom{n}{j}$. We obtain from the maximality of $|\mathcal{D}|$ that $j = \lceil n/2 \rceil$.

Problem 2.5.3 *Suppose that a digraph $D = (V, A)$ includes no st-edges. Prove that there are two edge-disjoint st-paths after removing any node from $V - \{s, t\}$ if and only if there are three 2-bundles from s to t such that each edge belongs to at most two of them.*

If there is a node $v \in V - \{s, t\}$ such that $D' := D - v$ includes no 2 edge-disjoint st-paths, then the directed edge-Menger theorem implies that there is a $t\bar{s}$-set T such that $\varrho_{D'}(T) \le 1$ in which case we cannot have three 2-bundles using each edge at most twice. Conversely, suppose that there are two edge-disjoint st-paths after removing any node from $V - \{s, t\}$. Since there are no st-edges, it is not possible to cover all st-paths by two edges and hence $\lambda_D(s, t) \ge 3$ by the directed edge-Menger theorem (meaning that there are 3 edge-disjoint st-paths).

We claim that

$$2\varrho_D(X) + 3w(X) \ge 6 \text{ for every bi-set } X \text{ with } t \in X_I,\ s \in V - X_O. \tag{18.2}$$

Indeed, the inequality holds automatically if $w(X) \ge 2$. If $w(X) = 1$, then it requires that $\varrho_D(X) \ge (6 - 3)/2$, and hence $\varrho_D(X) \ge 2$, which certainly holds since there are two edge-disjoint st-paths after removing the single element of $X_O - X_I$. Finally, if $w(S) = 0$, then (18.2) requires $\varrho_D(X) \ge 3$ and this follows from $\lambda_D(s, t) \ge 3$.

Duplicate in parallel each edge of D and let D'' denote the resulting digraph. Condition (18.2) implies that $\varrho_{D''}(X) + 3w(X) \ge 6$ for every bi-set X with $t \in X_I,\ s \in V - X_O$. By Theorem 2.5.14, there are 3 edge-disjoint 2-bundles from s to t, and these define 3 edge-disjoint 2-bundles in D using every edge of D at most twice.

Problem 3.2.1 *Suppose that a graph $G = (V, E)$ can be partitioned into k spanning trees and $F \subseteq E$ is an arbitrary subset of at most k edges. Then G can be partitioned into k spanning trees $T_1, \dots, T_k$ in such a way that the trees are equitable in F in the sense that $|T_i \cap F| \le 1$ for every $1 \le i \le k$.*

Choose k disjoint spanning trees $T_1, \ldots, T_k$ in such a way that the number of T_i's disjoint from F is minimum. We are done if each T_i intersects F in at most one edge. If this is not the case, then there are two trees, say T_1 and T_2, such that $|T_1 \cap F| \geq 2$ and $|T \cap F_2| = 0$. Let $f_1 \in T_1 \cap F$. By Lemma 3.2.2 on mutual exchange, there is an edge $f_2 \in T_2$ such that both $T_1' := T_1 - f_1 + f_2$ and $T_2' := T_2 - f_2 + f_1$ are spanning trees but this contradicts the minimal choice of T_i's since $\left|T_1' \cap F\right| \geq 1$ and $\left|T_2' \cap F\right| \geq 1$.

Problem 3.4.5 *Hoffman's theorem has a self-refining nature: derive the following extension. Suppose that in addition to the bounds f and g on the edge-set of D, we are given bounds $f_V : V \to \mathbf{R} \cup \{-\infty\}$ and $g_V : V \to \mathbf{R} \cup \{\infty\}$ on the node-set, as well, for which $f_V \leq g_V$. There exists an $m : V \to \mathbf{R}$ for which $f_V \leq m \leq g_V$ such that there exists a feasible m-flow if and only if*

$$\varrho_f(X) - \delta_g(X) \leq \min\{-\widetilde{f}_V(V - X), \widetilde{g}_V(X)\} \textit{ for every } X \subseteq V. \tag{18.3}$$

Proof. Extend D by adding a new node s and new edges $a_v = vs$ for every node $v \in V$. Define $f(a_v) := f_V(v)$ and $g(a_v) := g_V(v)$. It can be seen that the original problem is solvable if and only if there is a feasible circulation in the extended digraph. Furtheremore, a little calculation shows that Hoffman's condition (requiring $\varrho_f(Y) \leq \delta_g(Y)$ for every $Y \subseteq V + s$) for the existence of a feasible circulation in the extended digraph is equivalent to (18.3).

Problem 3.6.1 *Prove that the maximum number of disjoint D-antichains is equal to the minimum number of elements covering all D-antichains. Can you find algorithmically these quantities?*

Consider the sequence in Theorem 3.6.9 and focus on the members $\mathcal{A}_{i_1} | \ \mathcal{C}_{a-1}$.

Problem 4.2.3 *Prove directly Corollary* 4.2.24 *by devising a simple greedy algorithm.*

Arrange the nodes of the tree into levels $V_1, V_2, \ldots, V_t$ in such a way that the head of each edge is one level higher than its tail. Colour an edge of the tree by i $(mod\ k)$ if its head is in V_i. This colouring will suffice.

Problem 5.2.2 *In Proposition* 1.2.6*, we proved for a graph $G = (V, E)$ that $c(X) + c(Y) \leq c(X \cap Y) + c(X \cup Y) + d_G(X, Y)$ holds whenever $X, Y \subseteq V$, where $c(Z)$ denotes the number of components induced by Z and $d_G(X, Y)$ is the number of edges connecting $X - Y$ and $Y - X$. Derive this inequality from the submodularity of the rank function of the circuit matroid of G.*

Let r denote the rank function of the circuit matroid of G. Consider the sets $I(X)$ and $I(Y)$ of edges induced by X and Y, respectively, and let $E(X, Y)$ denote the set of edges connecting $X - Y$ and $Y - X$. Then $I(X) \cap I(Y) = I(X \cap Y)$ from which $r((I(X) \cap I(Y)) = r(I(X \cap Y))$. Furthermore, $I(X) \cup I(Y) \cup E(X, Y) = I(X \cup Y)$ from which $r(I(X) \cup I(Y)) \geq r(I(X \cup Y)) - |E(X, Y)| = r(I(X \cup Y)) - d(X, Y)$. Since r is submodular and $r(I(Z)) = |Z| - c(Z)$ for $Z \subseteq V$, we conclude that c satisfies $c(X) + c(Y) \leq c(X \cap Y) + c(X \cup Y) + d_G(X, Y)$.

Problem 5.3.11 *Let X be a closed set for which $r(X) < r(S)$. Show that X arises as the intersection of $r(S) - r(X)$ hyperplanes. A non-empty set is open if and only if it is the union of cuts.*

The second part follows form the first one and from Problem 5.3.10. To see the first part, let F be a maximal independent subset of X and let B be a base including F. For every element $x \in B - F$, let H_x be the closure of $B - x$ and let $H := \cap_{x \in B-F} H_x$. We claim that $H = X$. Indeed, obviously $X \subseteq H$ holds. For the reverse inclusion, let $z \in S - X$. Then z is not a loop since every loop belongs to every closed set. If $z \in B$, then $z \notin H_z$ and thus $z \notin H$. If $z \notin B$, then consider the fundamental circuit $C := C(B, z)$. Since X is closed, there must be an element x in $C \cap (B - F)$. But then z is not in H_x and hence z is not in H either.

Problem 6.1.1 *Prove that during the algorithm of Goldberg and Tarjan the maximum level of every node is at most $2n - 1$.*

A node z is lifted only if z is Ψ-larger. In this case there is a path from s to t consisting of edges on which $x(e) > 0$. Since a decreasable edge can step up at most one level, the level of z can be at most $\Theta(s) + n - 1 = 2n - 1$.

Problem 6.1.2 *In Subsection 3.4.2 we showed that the m-flow feasiblity problem is equivalent to its special case when $f \equiv 0$ and $g \equiv \infty$. Work out the details of the push-relabel algorithm for this special case along with possible simplifications.*

Hoffman's feasibility theorem reduces to the following.

Theorem 3 *et $D = (V, A)$ be a digraph and $m : V \to \mathbf{R}$ a function for which $\widetilde{m}(V) = 0$. There exists a non-negative m-flow if and only if*

$$\widetilde{m}(X) \geq 0 \textit{ whenever } \delta_D(X) = 0. \tag{18.4}$$

For any $x \geq 0$, every edge is always increasable and an edge a is decreasable if $x(a)$ is positive. Level properties:

(LP1) Every Ψ-smaller node is in L_0.
(LP2′) $\Theta(v) \geq \Theta(u) - 1$ for every edge uv.
(LP2″) $\Theta(v) \leq \Theta(u) + 1$ for every positive edge uv.

Stopping rules:

(A) There is no more Ψ-larger node.
(B) There exists a Ψ-larger node z and an empty level set L_ℓ under z.

Lemma 18.0.25 *Suppose that x and Θ meet the level properties. Then* **(A)** *implies that x is a non-negative m-flow while* **(B)** *implies that the set $Z := \{v \in V : \Theta(v) > \ell\}$ violates* (18.4).

1. **Edge-push at z:**
   (Increasing) If $e = zu$ is an edge stepping down from z, then increase $x(e)$ by $\Psi_x(z) - m(z)\}$.

978–0–19–920527–1 18-Andras-Frank-c18-drv Andras Frank (Typeset by SPi, Publisher Services, India) 599 of 640 December 11, 2010 15:22
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

(Decreasing) If $e = uz$ is a positive edge stepping up to z, then decrease $x(e)$ by $\alpha := \min\{x(e) - f(e), \Psi_x(z) - m(z)\}$.

2. **Node-lift of z** increases $\Theta(z)$ by one provided that no edge-push operation is possible at z.

The algorithm starts with a non-negative $x : A \to \mathbf{R}+$ and $\Theta \equiv 0$. At an intermediate stage, a non-negative x and a Θ are available which satisfy the level properties. As long as there are Ψ-larger nodes, the algorithm selects one for which the level is as high as possible and treats it.

One way of termination is that the current treatment neutralizes z and no more Ψ-larger node remains. By stopping rule **(A)**, the resulting x is a feasible m-flow. The other way of termination is that the current treatment lifts z and leaves its (original) level set empty. By stopping rule **(B)**, the set $Z = \{v : \Theta(v) \geq \Theta(z)\}$ violates (3.32) (18.4).

Problem 7.2.3 *Prove the following sharpening of Theorem* 7.2.6.

Theorem 4 *If $\delta(r_0) > \varrho(r_0)$ for a node r_0 of a digraph D, then there is a node $v \in V - r_0$ for which $\lambda(r_0, v) > \lambda(v, r_0)$ and $\delta(v) < \varrho(v)$.*

Proof. Modify the proof of Theorem 7.2.6, as follows. Since $\delta(r_0) > \varrho(r_0)$, there is a node v for which $\delta(v) < \varrho(v)$. Let $\mathcal{T} = \{T_1, \dots, T_t\}$ be the set-system consisting of the maximal members of the family $\{T(v) : v \in V - s,\ \delta(v) < \varrho(v)\}$, and suppose that $T_i = T(v_i)$. Let $Z := \{v : \delta(v) \geq \varrho(v),\ v \in (V - r_0) - (\cup_i T_i)\}$, and let S_i denote the set of elements of T_i which do not belong to other members of $\mathcal{T}$. Clearly, $\mathcal{S} = \{S_1, \dots S_t\}$ is a subpartition.

By Lemma 7.2.7, $v_i \in S_i$ and hence

$$\delta(S_i) \geq \lambda(v_i, r_0) \geq \lambda(r_0, v_i) = \varrho(T_i) \tag{18.5}$$

and

$$\delta(Z) - \varrho_D(Z) = \sum[\delta(v) - \varrho(v) : v \in Z] \geq 0. \tag{18.6}$$

Consider the set F of edges that leave either r_0 or a member of $\mathcal{S}$, or Z. Then each element f of F enters either r_0 or a member of $\mathcal{T}$ or Z. Hence $\delta(r_0) + \delta(Z) + \sum_i \delta(S_i) = |F| \leq \varrho(r_0) + \varrho(Z) + \sum_i \varrho(T_i)$. By (18.5) and (18.6), we get $\varrho(r_0) - \delta(r_0) \geq \delta(Z) - \varrho(Z) + \sum_i[\delta(S_i) - \varrho(T_i)] \geq 0$ contradicting the hypothesis of the theorem.

Exercise 7.2.4 *Decide whether the following statement is true or not. If a digraph admits k edge-disjoint spanning arborescences of root r_0 and $\varrho(r_0) = \ell \leq k$, then there is a node v for which $\lambda(v, r_0) \geq \ell$.*

The statement is false for $k = \ell = 2$. Let $V = \{r_0, u, v\}$, and let A consist of two parallel r_0u-edges, two parallel r_0v-edges, one ur_0-edge, and one vr_0-edge. Then $\delta(u) = 1 = \delta(v)$ and $\lambda(v, r_0) = 1 = \lambda(u, r_0)$.

Problem 7.3.1 *If the intersection of two in-solid (out-solid) sets X and Y is non-empty, then $X \cup Y$ is in-solid (out-solid).*

978–0–19–920527–1 18-Andras-Frank-c18-drv Andras Frank (Typeset by SPi, Publisher Services, India) 600 of 640 December 11, 2010 15:22

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Proof. If indirectly $X \cup Y$ is not in-solid, then there is a maximal non-empty subset $Z \subset X \cup Y$ with $\varrho(Z) \leq \varrho(X \cup Y)$.

If Z includes one of X and Y, say X, then $Z \cap Y \subset Y$, $X \cup Y = Z \cup Y$ and hence $\varrho(Z \cap Y) > \varrho(Y)$, $\varrho(Z \cup Y) = \varrho(X \cup Y) \geq \varrho(Z)$ from which $\varrho(Y) + \varrho(Z) \geq \varrho(Z \cap Y) + \varrho(Z \cup Y) > \varrho(Y) + \varrho(Z)$ would follow. Therefore Z can include neither X nor Y.

If Z is disjoint from X or from Y, for example from X, then $Z \subseteq Y - X$ and $\varrho(Z) > \varrho(Y)$ which is not possible since $\varrho(X) + \varrho(Y) \geq \varrho(X \cap Y) + \varrho(X \cup Y) > \varrho(X) + \varrho(X \cup Y)$ implies $\varrho(Y) > \varrho(X \cup Y)$ from which we would have $\varrho(Z) > \varrho(X \cup Y)$, contradicting the assumption $\varrho(Z) \leq \varrho(X \cup Y)$. Therefore Z must intersect both X and Y.

It follows that $X \cap Z \neq \emptyset$ and $X \cap Z \subset X$ from which $\varrho(X \cap Z) > \varrho(X)$ as X is in-solid. Since $Z \subset X \cup Z$, the maximal choice of Z implies $\varrho(X \cup Z) \geq \varrho(Z)$. Therefore we have $\varrho(X) + \varrho(Z) \geq \varrho(X \cap Z) + \varrho(X \cup Z) > \varrho(X) + \varrho(Z)$, a contradiction. The proof for out-solid sets is analogous.

Problem 7.5.5 *Let F be a tree and $F_1, F_2, \ldots, F_n$ be a family of subtrees of F. Let G be a simple graph with node-set $\{v_1, \ldots, v_n\}$ in which v_i and v_j are adjacent if and only if subtrees F_i and F_j have a node in common. Prove that G is chordal and that every chordal graph can be obtained in this way.*

To prove that the line-graph of a subtree hypergraph is chordal is easy and left to the reader. For the converse, let $G = (V, E)$ be a chordal graph. By Theorems 7.5.13 and 7.5.14, has a simplicial node u. By induction, $G - u$ is the line-graph of a subtree hypergraph. Consider the underlying tree F and the $h := d_G(u)$ subtrees $F_1, \ldots, F_h$ corresponding to the neighbours of u in G. Since a subtree hypergraph admits the Helly property (actually, it is normal), these subtrees have a node z of F in common. Add a new node z' to F along with an edge zz'. Extend each of $F_1, \ldots, F_h$ by this new edge zz', and finally define the subtree F_u corresponding to u such that it consists of the single node z'. By this construction G is the line-graph of the subtree hypergraph arising in this way.

Problem 8.1.1 *Prove that in Lemma* 8.1.3 *there are actually at least two nodes of degree K.*

The proof of Lemma 8.1.3 shows that *every* subset X of degree K contains a node of degree K. Therefore the complement of a node of degree K contains another node of degree K.

Problem 8.3.4 *Prove that an undirected graph G admits a Hamilton circuit if and only if G can be obtained from a 2-element circuit by the following operations.*

(A) *Add an edge connecting two existing nodes,*

(B) *subdivide an existing edge incident to a node of degree 2.*

If a graph is Hamiltonian, then the graph obtained by (A) or by (B) is also Hamiltonian. Conversely, if G has a Hamilton circuit C, then apply first a sequence of Operation (B) as long as C is built up, and second, apply a sequence of Operation (A) to obtain the entire G.

Problem 10.1.1 *Let $D = (V, A)$ be a root-connected digraph from a root-node r_0. Suppose that every dicut of D has k elements. Prove that A can be coloured by k colours in such a way that each colour class covers every dicut of D.*

Since D is root-connected, every dicut is directed 'away' from r_0. For each edge uv of D add k parallel vu-edges to D. By the hypothesis of the statement, the resulting digraph D' is rooted k-edge-connected. By Theorem 10.1.1, it is possible to colour the edges of D' by k colours in such a way that each colour-class determines a root-connected subgraph of D'. The colour-classes restricted to the original edges cover all dicuts of D.

Problem 10.1.3 *Let $\{s_i, t_i\}$ be pairs of nodes in a k-edge-connected digraph D ($i = 1, 2, \dots, k$). Prove Schiloach's theorem* [350] *stating that there are $s_i t_i$-paths ($i = 1, 2, \dots, k$) which are edge-disjoint.*

Extend D by a new root-node r_0 and an $r_0 s_i$-edge for every $i = 1, \dots, k$ and apply Theorem 10.1.1. It ensures k disjoint spanning arborescences of root r_0. Since there are exactly k edges leaving r_0, each of the k arborescences contains one of them, and hence these arborescences include the requested $s_i t_i$-paths.

Problem 10.1.6 *Construct a strongly connected dypergraph which cannot be trimmed to a strongly connected digraph.*

On the node-set $V = \{a, b, c\}$, let $\mathcal{E} = \{ab, ac, bca\}$ where the heads of the three dyper-edges are, respectively, b, c, and a.

Problem 10.2.2 *Prove that Theorems* 10.2.1 *and* 10.2.5 *are equivalent.*

In both theorems the necessity of the conditions is evident, so we prove only the equivalence of their sufficiency. In order to derive Theorem 10.2.5, define $R_i := V(F_i)$ for $i = 1, \dots, k$ and apply Theorem 10.2.1 to digraph D.

Conversely, to derive Theorem 10.2.1, add a new node r_0 to D and as many parallel edges from r_0 to v as is the number of root-sets R_i containing v for each node v. In the resulting digraph D', let F_i be an arborescence containing one edge from r_0 to v for every $v \in R_i$. Theorem 10.2.5 implies that D contains edge-disjont subsets $B_1, \dots, B_k$ such that $F_1 \cup B_1, \dots, F_k \cup B_k$ are edge-disjoint spanning arborescences of D'. But then $B_1, \dots, B_k$ form the edge-sets of edge-disjoint branchings of D with root-sets $R_1, \dots, R_k$, respectively.

Problem 10.3.2 *Derive Theorem* 10.3.6 *which is a reformulation of Theorem* 10.3.4 *in terms of families of sets.*

Necessity is evident. For sufficiency, define a family $\mathcal{F}'_i$ of bi-sets as follows. For each set $X \in \mathcal{F}_i$ let the bi-set $(X, X \cap T)$ be a member of $\mathcal{F}'_i$. Then each $\mathcal{F}'_i$ is intersecting and they meet the mixed intersection property. Since the head of every edge is in T, an edge enters a subset X precisely when it enters the bi-set $(X, X \cap T)$. Hence the partition of A into k sets ensured by Theorem 10.3.4 meets the requirement of the theorem.

Problem 10.4.3 *Prove that the edge-set of a simple connected planar graph can be covered by three trees.*

We can assume that G has at least two nodes and that G is maximal in the sense that each of its regions in the plane is bounded by a triangle. Then for the number L of its regions we have $3L = 2m$ and $L + n = m + 2$ from which $m = 3n - 6$. Therefore, the number of edges in a simple planar graph is at most $3n - 6$, and this holds for every induced subgraph. Apply Nash-Williams' theorem (Theorem 10.4.3) to $k = 3$.

Problem 10.5.4 *Engineers require a minimum list of spanning trees of a graph G such that if an arbitrary edge of G is destroyed, at least one of these trees remains intact. We can certainly help if G has two disjoint spanning trees because the approach above lends itself to finding these trees efficiently. What would you suggest, if we are not that lucky?*

Multiply (in parallel) each edge $k - 1$ times and decide if the resulting graph G' admits k disjoint spanning trees. If so, then these trees correspond to trees of the original graph which have no element in common. Conversely, if G has k spanning trees with no edge in common, then they define k disjoint trees in G'.

Problem 11.3.1 *Relying on the node duplicating technique and on Theorem* 11.3.2*, derive Theorem* 11.3.5.

Suppose first that for some set F of new edges, the digraph $D^+ := (V, A + F)$ is k-connected. Then F must have at least $h(X)$ elements entering any bi-set X and since an edge can cover at most one member of an independent family of bi-sets, F must have at least $\widetilde{h}(\mathcal{F})$ elements, from which the necessity of (11.41) follows.

For deriving sufficiency, let S and T be two disjoint copies of V. For a subset $X \subseteq V$, let X_S and X_T denote the corresponding subsets of S and T, respectively, and analogously for $v \in V$ we denote the corresponding element of S and T by v_S and v_T. Construct an auxiliary digraph $D' = (V', A')$ on node-set $V' := S \cup T$ as follows. For every node $v \in V$, put an edge from v_S to v_T and k parallel edges from v_T to v_S. Moreover, for every edge uv of D, put an edge in D' from u_T to v_S.

Due to this construction, the initial digraph D is k-connected if and only if the auxiliary digraph D' is k-ST-connected. Furthermore, there is one-to-one correspondence between the deficient bi-sets of D and the deficient subsets of nodes of D' which intersect T and not include S (where a deficient subset is one with in-degree less than k). Namely, if $X = (X_O, X_I)$ is a deficient bi-set, then $\varrho_{D'}(X') = \varrho_D(X) + |X_O| - |X_I|$ for the set $X' = (X_I)_S \cup (X_O)_T$.

Conversely, let $X' \subseteq V'$ be a deficient set in D'. Denote by X_I the subset of V corresponding to $X' \cap S$ and by X_O the subset of V corresponding to $X' \cap T$.

It follows from the construction of D' that $X_I \subseteq X_O$ and $\varrho_{D'}(X') = \varrho_D(X) + |X_O| - |X_I|$. In this correspondence two bi-sets X and Y are independent if and only if the corresponding subsets X' and Y' of V' are ST-independent. Therefore we can apply Theorem 11.3.2 to get the desired k-node-connected augmentation of D.

Problem 11.3.2 *A bi-set is* **one-way** *if no edge of D enters it. Show that in Theorem* 11.3.5 *if* $|V| \geq k + 1$*, then it suffices to require* (11.41) *only for independent one-way bi-sets.*

Suppose that there is an edge xy of D entering a member $X \in \mathcal{F}$, and hence $\varrho_D(X) \geq 1$. Since $k - (\varrho_D(X) + w(X)) = h(X) > 0$ and $|V| \geq k + 1$, we have $w(X) < k - 1$ and hence at least one of the two sets X_I and $V - X_O$ has at least two elements. If $|X_I| \geq 2$, then the bi-set $X' := (X_O, X_I - y)$ is non-trivial and $h(X') \geq h(X)$. If $|V - X_O| \geq 2$, then bi-set $X' := (X_O + x, X_I)$ is non-trivial and $h(X') \geq h(X)$. In both cases, we can replace X in $\mathcal{F}$ by X'. This change does not destroy the independence, and the h-sum does not decrease either. By repeating this procedure, each bi-set in $\mathcal{F}$ can be made one-way.

Problem 11.4.1 *Suppose that $T \subseteq V - r_0$ is a given set of terminals such that the head of every edge with positive cost is in T. Construct an algorithm for computing a cheapest arborescence of root r_0 such that each element of T belongs to the arborescence.*

Since all nodes not reachable from r_0 can be removed without affecting the problem, and since each edge with head outside T is of cost zero, a cheapest spanning arborescence will suffice.

Problem 11.4.2 *Let $\{T_1, \dots, T_q\}$ be a system of non-empty subsets of $V - r_0$. Devise an algorithm to decide if there is a spanning r_0-arborescence that enters each T_i exactly once.*

Let $c(e)$ denote the number of T_i's which are entered by edge e. Compute a cheapest spanning arborescence F with respect to c. Prove that F enters each T_i exactly once if and only if there is a spanning r_0-arborescence that enters each T_i exactly once.

Problem 13.1.5 *Suppose that the common ground-set S of two matroids M_1 and M_2 can be partitioned in both matroids into k bases. Prove that there is a common basis of M_1 and M_2. Show that S need not have a partition into k common bases.*

Let r denote the common value of $r_1(S)$ and $r_2(S)$. Since S partitions into k bases of M_1, S can be covered by k M_1-independent sets and hence $kr_1(X) \geq |X|$ holds for every subset X. Since S partitions into k bases of M_2, S includes k disjoint M_2-bases and hence $kt_2(X) \leq |X|$ holds for every subset X where t_2 denotes the co-rank function of M_2 (that is, $t_2(X) = r_2(S) - r_2(S - X) = r - r_2(S - X)$). Hence $kr_1(X) \geq |X| \geq kt_2(X) = k(r - r_2(S - X))$. Therefore, $r_1(X) + r_2(S - X) \geq r$ holds for every $X \subseteq S$, and the matroid intersection theorem implies the existence of a common independent set of r elements, which is a common basis of M_1 and M_2.

Let M_1 be the circuit matroid of K_4, the complete graph on four nodes. Let M_2 be the partition matroid on the edge-set of K_4 in which a subset is independent if it contains at most one element from each of the three perfect matchings of K_4. In both matroids the ground-set partitions into two bases but an easy case-checking shows that there is no partition into two common bases.

Problems 13.1.7 *Prove for matroids $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ that* $\max\{r_1(X) + r_2(X) - |X| : X \subseteq S\} = \min\{r_1(Y) + r_2(S - Y) : Y \subseteq S\}$.

Let $X \subseteq S$ be a smallest subset that maximizes the left-hand side. Then X is a common independent set. For if X is not independent in M_1, say, then it has an element s for which $r_1(X') = r_1(X)$ where $X' = X - s$. Since $r_2(X') - |X'| \geq r_2(X) - |X|$, we can conclude that X' is also a maximizer set contrary to the minimality of X. Consequently, it suffices to consider the maximum on the left-hand side only for common independent sets. for such sets, however, the maximum is $r_1(X) + r_2(X) - |X| = |X|$, the cardinality of a maximum common independent set, and hence the matroid intersection theorem applies.

Problem 13.3.3 *Prove the following theorem.*

Theorem 5 *Let $M = (S, r)$ be a loop-free matroid and $J \subseteq S$ a subset of at most k elements.*

(A) *If S can be partitioned into k independent sets, then S can be partitioned into k independent sets in such a way that each of them contains at most one element of J.*
(B) *If there are k disjoint bases, then there are k disjoint bases in such a way that each of them contains at most one element of J and the union of them includes J.*

We prove the following statement that implies both results.

Let $\mathcal{F} := \{F_1, \dots, F_k\}$ be a set of k disjoint independent sets for which the union F of these sets is maximal, and subject to this, the number of F_i's disjoint from J is minimal. Then $J \subseteq F$ and each F_i contains at most one element of J.

If an element $s \in J$ is not in F, then $|J| \leq k$ implies for some i that $F_i \cap J = \emptyset$, and then s could be exchanged with an element of F_j, contradicting the assumption that the number of members of $\mathcal{F}$ disjoint from J is minimal.

For proving the second part, assume indirectly that there is a member of $\mathcal{F}$, say F_1, which is disjoint from J. Then there is another member of $\mathcal{F}$, say F_2, such that $|F_2 \cap J| \geq 2$. It follows from Theorem 5.3.3 on symmetric exchange that either $F_2 + s_1$ is independent or s_1 can be symmetrically exchanged with an element $s_2 \in F_2$. In both cases, we obtain a set $\mathcal{F}'$ of k independent sets that is better than $\mathcal{F}$.

Problem 15.3.5 *Derive Theorem* 14.3.16 *on the non-emptiness of the intersection of a base-polyhedron with a box from Theorem* 15.3.10.

Observe that reducing the values of a fully submodular function on singletons and on the complement of singletons gives rise to a crossing submodular function.

Problem 15.4.1 *Let $G = (V, E)$ be a $2k$-edge-connected graph and $F \subset E$ a specified subset of at most k edges. Prove that any orientation of F can be extended to a k-edge-connected orientation of G. Formulate and prove a generalization for (k, ℓ)-partition-connected graphs.*

Let $G = (V, E)$ be a (k, ℓ)-partition-connected graph and $F \subset E$ a specified subset of at most ℓ edges. Then any orientation of F can be extended to a (k, ℓ)-partition-connected orientation of G. More generally, we prove the following.

Let h be a non-negative, crossing supermodular function for which there is an orientation of G covering h. Let $\vec{F}$ be an orientation of a subset $F \subseteq A$ which is a subset of edges for which $\varrho_{\vec{F}}(X) \leq h(X)$ for every $X \subseteq V$. Then the elements of $A - F$ can be oriented in such a way that the resulting orientation of G covers h.

Let $G' := (V, E - F)$ and consider the set-function $h' := h - \varrho_{\vec{F}}$. By the hypothesis, h' is non-negative and crossing supermodular for which (15.44) and (15.45) hold with G' and h' in place of G and h, respectively. Therefore Theorem 15.4.1 applies.

As a corollary, we obtain that if $F \subseteq E$ is a subset of edges for which $d_F \leq h$, then any orientation of F can be extended to an orientation of G that covers h.

OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi

Bibliography

[1] R. Aharoni and E. Berger, *The intersection of a matroid with a simplicial complex*, Trans. Amer. Math. Soc. 358 (2006), 4895–917.

[2] R.K. Ahuja, T.L. Magnanti, and J.B. Orlin, Network Flows: Theory, Algorithms, and Applications, Prentice-Hall, New York, 1993.

[3] N. Alon, *Combinatorial Nullstellensatz*, in: Recent trends in combinatorics (Mátraháza, 1995). Combin. Prob. Comp. 8. no.1–2 (1999), 7–29.

[4] N. Alon, *Ranking tournaments*, SIAM J. Discrete Math., 20 (1) (2006) 137–42.

[5] S.R. Arikati and K. Mehlhorn, *A correctness certificate of the Stoer-Wagner min-cut algorithm*, Inform. Proc. Letters, 70 (1999) 251–4.

[6] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, M. Protasi, Complexity and Approximation, Combinatorial Optimization Problems and their Approximability Properties, Springer Verlag, Berlin, 1999.

[7] M. Baïou, F. Barahona, and R. Mahjoub, *Separation of partition inequalities,* Math. of Op. Res., 25, (2) (2000) 243–54.

[8] J. Bang-Jensen, A. Frank, and B. Jackson, *Preserving and increasing local edge-connectivity in mixed graph*, SIAM J. Disc. Math., 8 (2) (May 1995) 155–78.

[9] J. Bang-Jensen and B. Jackson, *Augmenting hypergraphs by edges of size 2*, Math. Prog., Ser. B, 84 (1999) 467–81.

[10] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer Monographs in Mathematics, Springer, Berlin, 2009, 2nd edn.

[11] M. Bárász, oral communication.

[12] M. Bárász, J. Becker, and A. Frank, *An algorithm for source location in directed graphs*, Op. Res. Letters, 33 (2005) 221–230.

[13] D.W. Barnette and B. Grünbaum, *On Steinitz's theorem concerning convex 3-polytopes and on some properties of planar graphs*, in: eds. G. Chartrand and S.F. Kapoor, The Many Facets of Graph Theory, Lecture Notes in Mathematics, vol. 110, Springer, Berlin, 1969, 27–40.

[14] S. Baum and L.E. Trotter, Jr, *Integer rounding and polyhedral decomposition for totally unimodular systems,* in: eds. R. Henn, B. Korte, W. Oettli, Optimization and Operations Research (Proceedings Workshop Bad Honnef, 1977) [Lecture Notes in Economics and Mathematical Systems 157], Springer, Berlin, 1978, 15–23.

[15] J. Becker and A. Frank, *A quick proof for the matroidal structure of a source location problem*, QP-2008-01, EGRES Quick-Proofs series, http://www.cs.elte.hu/egres/ (2008).

[16] R. Bellman, *On a routing problem*, Quart. of Appl. Math., 16 (1958) 87–90.

[17] A. Benczúr, J. Förster, and Z. Király, *Dilworth's theorem and its application for path system of a cycle – implementation and analysis*, in: ed. J. Nesetril, Lecture Notes in Computer Science, 1643, Algorithms—ESA '99, Springer, Berlin, 498–509.

[18] A. Benczúr and L. Végh, *Primal-dual approach for directed vertex connectivity augmentation and generalizations,* ACM Trans. Alg., 4 (2) (2008).

[19] K. Bérczi and A. Frank, *Variations for Lovász' submodular ideas*, in: eds. M. Grötschel and G.O.H. Katona, Building Bridges Between Mathematics and Computer Science, Bolyai Society, Series: Mathematical Studies, 19, Springer, Berlin, 2008, 137–64.

[20] K. Bérczi and A. Frank, *Disjoint arborescences*, RIMS (Kyoto) Kôkyûroku Bessatsu, to appear.

[21] C. Berge, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.

[22] A.R. Berg, and T. Jordán, *Algorithms for graph rigidity and scene analysis*, in: eds. G. Di Battista and U. Zwick, Proc. 11th Annual European Symposium on Algorithms (ESA), 2003, Springer Lecture Notes in Computer Science, 2832 (2003) 78–89.

[23] A.R. Berg, and T. Jordán, *Minimally k-edge-connected directed graphs of maximal size*, Graphs and Combin. 21 (1) (2005) 39–50.

[24] A.R. Berg, and T. Jordán, *Sparse certificates and removable cycles in l-mixed and p-connected graphs,* Op. Res. Letters, 33 (2005) 111–114.

[25] A. Bernáth, *A representation for intersecting families*, Matematikai Lapok, 1 (2006–7) 6–12 (in Hungarian). For an English version, see Egerváry Research Report 2004–12 (2004), www.cs.elte.hu/egres/.

[26] A. Bernáth, S. Iwata, T. Király, Z. Király, and Z. Szigeti, *Recent results on well-balanced orientations*, Disc. Optim., 5 (2008) 663–76.

[27] A. Bernáth and G. Joret, *Well-balanced orientations of mixed graphs*, Inform. Proc. Letters, 106 (2008) 149–151.

[28] A. Bernáth and T. Király, *Covering skew-supermodular functions by hypergraphs of minimum total size*, Op. Res. Letters, 37 (5) (2009) 345–50.

[29] A. Bernáth and Z. Király, *Finding edge-disjoint subgraphs in graphs,* QP-2010-04, EGRES Quick-Proofs series, http://www.cs.elte.hu/egres/ (2010).

[30] A. Berg, B. Jackson, and T. Jordán, *Edge splitting and connectivity augmentation in directed hypergraphs*, Disc. Math., 273 (2003) 71–84.

[31] D. Bertsimas and J.N. Tsitsiklis, Introduction to Linear Optimization, Athena Scientific, Boston, 1997.

[32] S. Bessy and S. Thomassé, *Spanning a strong digraph by α circuits: A proof of Gallai's conjecture*, Combinatorica, 27 (6) (2007) 659–67. A preliminary version has appeared in: eds. D. Bienstock and G.R. Nemhauser, Integer Programming and Combinatorial Optimization, vol. 10, New York, 2004.

[33] G. Birkhoff, *Tres observaciones sobre el algebra lineal*, Revista Facultad de Ciencias Exactas, Puras Aplicadas Universidad Nacional de Tucuman, Serie A (Matematicas y Fisica Teorica) 5 (1946) 147–51.

[34] F. Boesch and R. Tindell, *Robbins's theorem for mixed multigraphs,* Am. Math. Monthly, 87 (1980) 716–19.

[35] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.

[36] J.A. Bondy and U.S.R. Murty, Graph Theory, Springer, Berlin, 2008.

[37] O. Bо̊ruvka, *O jistém problému minimálním* [Czech, with German summary; On a Minimal Problem], Práce Moravské Přírodovědecké Společnosti Brno [Acta Societatis Scientiarum Naturalium Moravicae] 3 (1926) 37–58.

[38] R.A. Brualdi, *Common transversals and strong exchange systems*, J. Comb. Theory 8 (1970) 307–29.

[39] M.C. Cai, *Arc-disjoint arborescences of digraphs*, J. Graph Theory 7 (1983) 235–40.

[40] K. Cameron, Polyhedral and Algorithmic Ramifications of Antichains, Ph.D. Thesis, University of Waterloo, Canada, 1982. (Supervisor: J. Edmonds.)

[41] K. Cameron and J. Edmonds, *Coflow polyhedra*, Disc. Math., 101 (1992) 1–21.
[42] P. Camion, *Chemins et circuits hamiltoniens des graphes complets*, C. R. Acad. Sci. 249 (1959) 2151–2.
[43] A. Caprara, A. Panconesi, and R. Rizzi, *Packing cycles in undirected graphs*, J. Algor., 48 (1) (2003) 239–56.
[44] A. Caprara, A. Panconesi, and R. Rizzi, *Packing cuts in undirected graphs,* Networks, 44 (1) (2004) 1–11.
[45] P. Charbit, S. Thomassé, and A. Yeo, *The minimum feedback arc set problem is NP-hard for tournaments*, Comb., Prob. and Comp., 16, (1) (Jan. 2007) 1–4.
[46] J. Cheriyan and S.N. Maheswari, *Analysis of preflow push algorithms for maximum network flows*, SIAM J. Comp. 18 (1989) 1057–86.
[47] J. Cheriyan, M.-Y. Kao, and R. Thurimella, *Scan-first search and sparse certificates: an improved parallel algorithm for k-vertex connectivity*, SIAM J. Comp. 22 (1) (1993) 157–74.
[48] J. Cheriyan and K. Mehlhorn, *An analysis of the highest-level selection rule in the preflow-push max-flow algorithm*, Inform. Proc. Letters 69 (1999) 239–42.
[49] B.V. Cherkassky, *A solution of a problem of multicommodity flows in a network*, Ekon. Mat. Metody, 13 (1) (1977), 143–51 (in Russian).
[50] Y.-J. Chu and T.-H. Liu, *On the shortest arborescence of a directed graph,* Scientia Sinica [Peking], 14 (1965) 1396–1400.
[51] M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas, *The strong perfect graph theorem*, Ann. Math. 164 (2006) 51–229.
[52] C.J. Colbourn, The Combinatorics of Network Reliability, Oxford University Press, Oxford, 1986.
[53] L. Colussi, M. Conforti, and G. Zambelli, *Disjoint Paths in Arborescences,* Disc. Math. 292 (2005) 187–91.
[54] S.A. Cook, *The complexity of theorem-proving procedures*, in: Conference Record of Third Annual ACM Symposium on Theory of Computing (3rd STOC, Shaker Heights, Ohio, 1971), Association for Computing Machinery, New York, 1971, 151–8.
[55] W.J. Cook, *Operations that preserve total dual integrality*, Op. Res. Letters, 2 (1983) 31–5.
[56] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, and A. Schrijver, Combinatorial Optimization, John Wiley and Sons, New York, 1998.
[57] W.J. Cook, J. Fonlupt, and A. Schrijver, *An integer analogue of Caratheodory's theorem*, J. Comb. Theory, Ser. B, 40 (1986) 63–70.
[58] B. Cosh, B. Jackson, and Z. Király, *Local connectivity augmentation in hypergraphs is NP-complete*, Disc. Appl. Math., 158 (6) (2010) 723–7.
[59] J.C. Culberson and R.A. Reckhow, Covering polygons is hard, 29th Annual Symposium on Foundations of Computer Science (FOCS 1988) White Plains, NY, 601–11.
[60] W.H. Cunningham, *Testing membership in matroid polyhedra,* J. Comb. Theory, Ser. B, 36 (1984) 161–88.
[61] W.H. Cunningham, *Optimal attack and reinforcement of a network*, J. Assoc. Comp. Mach., 32 (1985) 549–61.
[62] W.H. Cunningham and A. Frank, *A primal-dual algorithm for submodular flows*, Math. Op. Res., 10 (2) (1985) 251–61.
[63] M. Dalmazzo, *Nombre d'arcs dans les graphes k-arc-fortement connexes minimaux*, C. R. Acad. Sci. Paris Sr. A-B 285, no. 5, (1977) A341–4.
[64] D. de Werra, *On some combinatorial problems arising in scheduling*, Can. Op. Res. Soc. J., 8 (1970) 165–75.

[65] B.L. Dietrich and A.J. Hoffman, *On greedy algorithms, partially ordered sets, and submodular functions*, IBM J. Res. and Dev., 47 (2003), 25–30.

[66] E.W. Dijkstra, *A note on two problems in connexion with graphs*, Num. Math., 1 (1959) 269–71.

[67] R. Diestel, Graph Theory, Graduate Texts in Mathematics 173, 2nd edn, Springer, Berlin, 2000.

[68] R.P. Dilworth, *A decomposition theorem for partially ordered sets*, Annals of Math. (2) 51 (1950), 161–166.

[69] E.A. Dinits, *Algoritm resheniya zadachi o maksimalnom potoke v seti so stepennoi otsenkoi* [Russian], Doklady Akademii Nauk SSSR 194 (1970) 754–7 [English translation: *Algorithm for solution of a problem of maximum flow in a network with power estimation*, Soviet Math. Doklady 11 (1970) 1277–80].

[70] E.A. Dinits, A.V. Karzanov, and M.V. Lomonosov, *On the structure of a family of minimal weighted cuts in graphs*, in: ed. A.A. Fridman, Studies in Discrete Mathematics (in Russian), Nauka, Moskva, 1976, 290–306.

[71] Y. Dinits and A. Vainshtein, *The connectivity carcass of a vertex subset in a graph and its incremental maintenance*, Proc. 26th Annual ACM Symp. on Theory of Comp., 716–25

[72] G.A. Dirac, *On rigid circuit graphs*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 25 (1961) 71–6.

[73] P. Duchet, Hypergraphs (Theorem 3.8) in: eds. R. Graham, M. Grötschel, L. Lovász, Handbook of Combinatorics, Elsevier Science BV, New York, 1995, 381–432.

[74] R.J. Duffin, *The extremal length of a network*, J. Math. Anal. and Appl., 5 (1962) 200–15.

[75] F.D.J. Dunstan, *Matroids and submodular functions*, Quart. J. Math. (Oxford), (2) 27 (1976) 339–348. 313

[76] J. Edmonds, *Minimum partition of a matroid into independent sets,* J. Res. Nat. Bur. Standards, B69 (1965) 67–72.

[77] J. Edmonds, *Lehman's switching game and a theorem of Tutte and Nash-Williams,* J. Res. Nat. Bur. Standards, B69 (1965) 73–7.

[78] J. Edmonds, *Optimum branchings*, J. Res. Nat. Bur. Standards, B71 (1967) 233–40.

[79] J. Edmonds, *System of distinct representatives and linear algebra*, J. Res. Nat. Bur. Standards, B71 (1967) 241–7.

[80] J. Edmonds, *Submodular functions, matroids, and certain polyhedra*, in: eds. R. Guy, H. Hanani, N. Sauer, and J. Schönheim, Combinatorial Structures and their Applications, Gordon and Breach, New York, 1970, 69–87.

[81] J. Edmonds, *Matroids and the greedy algorithm*, Math. Prog., 1 (1971) 127–36.

[82] J. Edmonds, *Matroid intersection*, Ann. Disc. Math., 4 (1979) 39–49.

[83] J. Edmonds, *Edge-disjoint branchings,* in: ed. B. Rustin, Combinatorial Algorithms, Academic Press, New York, 1973, 91–6.

[84] J. Edmonds and R.M. Karp, *Theoretical improvements in algorithmic efficiency for network flow problems,* J. ACM, 19 (1972) 248–64.

[85] J. Edmonds and D.R. Fulkerson, *Transversals and matroid partition*, J. Res. Nat. Bur. Standards, B69 (1965) 147–53.

[86] J. Edmonds and R. Giles, *A min-max relation for submodular functions on graphs*, Ann. Disc. Math., 1 (1977) 185–204.

[87] Y. Egawa, A. Kaneko, and M. Matsumoto, *A mixed version of Menger's theorem,* Combinatorica, 11 (1991) 71–4.

[88] Egerváry J., *Matrixok kombinatorius tulajdonságairól*, Matematikai és Fizikai Lapok, 38 (1931) 16–28. (In Hungarian. In English as *Combinatorial properties of matrices*, Math. and Phys. Letters, 38 (1931) 16–28.)

[89] Egerváry J., *Kombinatorikus módszer a szállítási probléma megoldására,* A Matematikai Kutató Intézet Közleményei, IV/1 (1958), 15–28. (In Hungarian. In English as *Combinatorial method for the transportation problem*, Reports of the Mathematical Research Institute, IV/1 (1958) 15–28.)

[90] K. Engel, Sperner Theory, Encyclopedia of Mathematics and its Applications 65, Cambridge University Press, Cambridge 1997.

[91] S. Enni, *A 1-(S,T)-edge-connectivity augmentation algorithm*, Math. Prog., 84 (3) (April, 1999) 529–35.

[92] K.P. Eswaran and R.E. Tarjan, *Augmentation problems,* SIAM J. Comp., 5 (4) (1976) 653–65.

[93] S. Even, G. Itkis, and S. Rajsbaum, *On mixed connectivity certificates*, Theor. Comp. Sci., 203 (2) (Aug. 1998) 253–69.

[94] U. Faigle, *The greedy algorithm for partially ordered sets*, Discrete Math. 28 (1979) 153–9.

[95] U. Faigle, *Matroids on ordered sets and the greedy algorithm,* in: eds. R.E. Burkard, R.A. Cuninghame-Green, U. Zimmermann, Algebraic and Combinatorial Methods in Operations Research, Proceedings Workshop on Algebraic Structures in Operations Research, North-Holland, Amsterdam, 1984, 115–28.

[96] U. Faigle and W. Kern, *Submodular linear programs on forests*, Math. Prog. 72 (1996) 195–206.

[97] U. Faigle and B. Peis, *Two-phase greedy algorithms for some classes of combinatorial linear programs*, Proceedings SODA 2008, San Francisco, 161–6.

[98] I. Fáry, *On the straight line representation of planar graphs*, Acta Sci. Math., 11/4 (1948) 229–33.

[99] T. Fleiner and A. Frank, *A quick proof for the cactus representation of mincuts* QP-2009-03, EGRES Quick-Proofs series, www.cs.elte.hu/egres/.

[100] T. Fleiner, A. Frank, and S. Iwata, *A constrained independent set problem for matroids,* Op. Res. Letters, 32 (2004) 23–6.

[101] T. Fleiner and T. Jordán, *Covering and structure of crossing families* in: ed. A. Frank, Connectivity Augmentation of Networks: Structures and Algorithms, Mathematical Programming, Ser. B, 84 (3) (1999) 505–18.

[102] L.K. Fleischer and S. Iwata, *A push-relabel framework for submodular function minimization and applications to parametric optimization*, Disc. Applied Math., 131 (2) (Sept. 2003) 311–22.

[103] J. Folkman and D.R. Fulkerson, *Edge-colorings in bipartite graphs*, in: eds. R.C. Bose and T.A. Dowling, Combinatorial Mathematics and its Applications (Proc. Conference Chapel Hill, North Carolina, 1967), University of North Carolina Press, Chapel Hill, 1969, 561–77.

[104] L.R. Ford and D.R. Fulkerson, *Maximal flow through a network* [Notes on Linear Programming: Part XX], Paper P-605 [= Research Memorandum RM-1400], The RAND Corporation, Santa Monica, 19 Nov. 1954.

[105] L.R. Ford, Jr and D.R. Fulkerson, *Maximal flow through a network*, Can. J. Math. 8 (1956) 399–404.

[106] L.R. Ford, Jr, *Network Flow Theory*, Paper P-923, RAND Corporation, Santa Monica, [Aug. 14] 1956.

[107] L.R. Ford and D.R. Fulkerson, Flows in Networks, Princeton Univ. Press, Princeton, 1962.

[108] S. Fortune, J. Hopcroft, and J. Wyllie, *The directed subgraph homeomorphism problem*, Theor. Comp. Sci., 10 (1980) 111–21.

[109] G.N. Frederickson and J. Ja'Ja', *Approximation algorithms for several graph augmentation problems*, SIAM J. Comp., 10 (1981) 270–83.

[110] A. Frank, *Kombinatorikus algoritmusok, algoritmikus bizonyítások,* Egyetemi doktori értekezés, ELTE TTK, Budapest (1976). (*Combinatorial algorithms, algorithmic proofs*, in Hungarian, University doctoral dissertation.)

[111] A. Frank, *Kernel systems of directed graphs*, Acta Scientiarum Mathematicarum (Szeged) 41 (1–2) (1979) 63–76.

[112] A. Frank, *Covering branchings*, Acta Scientiarum Mathematicarum (Szeged) 41 (1–2) (1979) 77–81.

[113] A. Frank, *On disjoint trees and arborescences,* in: Algebraic Methods in Graph Theory, Colloquia Mathematica Soc. J. Bolyai, 25 (1978) 159–69.

[114] A. Frank, *On the orientation of graphs*, J. Comb. Theory, Ser. B, 28 (3) (1980) 251–61.

[115] A. Frank, *On chain and antichain families of a partially ordered set*, J. Comb. Theory, Ser. B, 29 (2) (1980) 176–84.

[116] A. Frank, *A weighted matroid intersection algorithm*, J. Alg. 2 (1981) 328–36.

[117] A. Frank, *How to make a digraph strongly connected*, Combinatorica, 1 (2) (1981) 145–53.

[118] A. Frank, *Generalized polymatroids,* in: Finite and infinite sets, Colloquia Mathematica Soc. J. Bolyai, 37 (1981) 285–94.

[119] A. Frank, *An algorithm for submodular functions on graphs*, Ann. of Disc. Math., 16 (1982) 97–120.

[120] A. Frank, *A note on k-strongly connected orientations of an undirected graph*, Disc. Math., 39 (1982) 103–4.

[121] A. Frank, *Finding feasible vectors of Edmonds-Giles polyhedra,* J. Comb. Theory, Ser. B, 36 (4) (1984) 221–39.

[122] A. Frank, *Submodular flows,* in: ed. W. Pulleyblank, Progress in Combinatorial Optimization, Academic Press, New York, 1984, 147–65.

[123] A. Frank, *Edge-disjoint paths in planar graphs*, J. Comb. Theory, Ser. B., 2 (1985) 164–78.

[124] A. Frank, *On connectivity properties of Eulerian digraphs*, Ann. of Disc. Math., 41 (1989) 179–94.

[125] A. Frank, *Augmenting graphs to meet edge-connectivity requirements*, SIAM J. Disc. Math., (Feb. 1992) 5 (1) 22–53.

[126] A. Frank, *On a theorem of Mader*, Disc. Math., 101 (1992) 49–57.

[127] A. Frank, *Applications of submodular functions,* in: ed. K. Walker, Surveys in Combinatorics, London Mathematical Society Lecture Note Series 187, Cambridge University Press, Cambridge, 1993, 85–136.

[128] A. Frank, *Connectivity augmentation problems in network design,* in: eds. J.R. Birge and K.G. Murty, Mathematical Programming: State of the Art 1994, University of Michigan Press, Ann Arbor, 34–63.

[129] A. Frank, *On the edge-connectivity algorithm of Nagamochi and Ibaraki*, Laboratoire Artemis, IMAG, Université J. Fourier, Grenoble, March 1994. For an accessible electronic version, see QP-2009-01, EGRES Quick-Proofs series, www.cs.elte.hu/egres/ (2009).

[130] A. Frank, *Orientations of graphs and submodular flows,* Congressus Numerantium, 113 (1996) 111–42.

[131] A. Frank, *Finding minimum generators of path systems*, J. Comb. Theory, Ser. B, 75 (1999) 237–44.

[132] A. Frank, *Finding minimum weighted generators of a path system*, in: eds. R.L. Graham, J. Kratochvil, J. Nesetril, and F.S. Roberts, Contemporary Trends in Discrete

Mathematics, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 49 1999, 129–38.

[133] A. Frank, *Increasing the rooted connectivity of a digraph by one*, in: ed. A. Frank, Connectivity Augmentation of Networks: Structures and Algorithms, Mathematical Programming, Ser. B, 84 (3) (1999) 565–76.

[134] A. Frank, *A Magyar Módszer és általánosításai*, Szigma, XXXIII, 1–2 (2002) 13–44. (*The Hungarian method and its extensions*, in Hungarian.)

[135] A. Frank, *Rooted k-connections in digraphs*, Disc. Appl. Math., 157 (2009) 1242–54.

[136] A. Frank, *On Kuhn's Hungarian method—a tribute from Hungary*, Naval Res. Logistics, 52 (2005) 2–5.

[137] A. Frank and A. Gyárfás, *How to orient the edges of a graph,* in: Combinatorics, Coll. Math. Soc. J. Bolyai, 18 (1976) 353–64.

[138] A. Frank, T. Ibaraki, and H. Nagamochi, *On sparse subgraphs preserving connectivity properties*, J. Graph Theory, 17 (3) (1993) 275–281.

[139] A. Frank and T. Jordán, *Minimal edge-coverings of pairs of sets*, J. Comb. Theory, Ser. B, 65 (1) (1995, Sept.) 73–110.

[140] A. Frank, A. Karzanov, and A. Sebő, *On integer multiflow maximization*, SIAM J. Disc. Math., 10 (1) (1997) 158–70.

[141] A. Frank and Z. Király, *Graph orientations with edge-connection and parity constraints,* Combinatorica, 22 (1) (2002) 47–70.

[142] A. Frank and T. Király, *Combined connectivity augmentation and orientation problems,* in: ed. S. Fujishige, Submodularity, Discrete Applied Mathematics, 131 (2) (Sept. 2003) 401–19.

[143] A. Frank and T. Király, *A survey on covering supermodular functions*, in: eds. W. Cook, L. Lovász, and J. Vygen, Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, 87–126.

[144] A. Frank, T. Király, and M. Kriesell, *On decomposing a hypergraph into k connected sub-hypergraphs,* in: ed. S. Fujishige, Submodularity, Discrete Applied Mathematics, 131 (2) (Sept. 2003) 373–83.

[145] A. Frank, T. Király, and Z. Király, *On the orientation of graphs and hypergraphs,* in: ed. S. Fujishige, Submodularity, Discrete Applied Mathematics, 131 (2) (Sept. 2003) 385–400.

[146] A. Frank, L.C. Lao, and J. Szabó, *A note on degree-constrained subgraphs*, Disc. Math., 308 (12) (2008) 2647–8.

[147] A. Frank and Z. Miklós, *Push–relabel algorithms for matroids and submodular flows*, Technical Reports 2010-05 (2010) www.cs.elte.hu/egres/.

[148] A. Frank and Z. Miklós, *Simplified push–relabel algorithms with applications*, Technical Reports 2010-06 (2010) www.cs.elte.hu/egres/.

[149] A. Frank, A. Sebő, and É. Tardos, *Covering directed and odd cuts*, Math. Prog. Stud. 22 (1984) 99–112.

[150] A. Frank and L. Szegő, *Constructive characterizations for packing and covering with trees,* in: ed. S. Fujishige, Submodularity, Discrete Applied Mathematics, 131 (2) (Sept. 2003) 347–71.

[151] A. Frank and É. Tardos, *Matroids from crossing families*, in: Finite and Infinite Sets, Colloquia Math. Soc. J. Bolyai, 37 (1981) 295–304.

[152] A. Frank and É. Tardos, *Generalized polymatroids and submodular flows*, Math. Prog., Ser. B, 42 (1988) 489–563.

[153] A. Frank and É. Tardos, *An application of simultaneous diophantine approximation in combinatorial optimization*, Combinatorica, 7 (1) (1987) 49–65.

[154] A. Frank and É. Tardos, *An application of submodular flows*, Linear Algebra and its Applications, 114/115 (1989) 329–48.

[155] A. Frank and L. Végh, *An algorithm to increase the node-connectivity of a digraph by one*, Disc. Optim., 5 (2008) 677–84.

[156] D.S. Franzblau and D.J. Kleitman, *An algorithm for covering polygons with rectangles*, Inform. Control, 63 (3) (Dec. 1986) 164–89.

[157] F.G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, Sitzungsberichte der Königlich Preuischen Akademie der Wissenschaften zu Berlin (1912) 456–77. [Reprinted in: ed. J.-P. Serre, Ferdinand Georg Frobenius, Gesammelte Abhandlungen, Band III, Springer, Berlin, 1968, 546–67].

[158] S. Fujishige, *Algorithms for solving the independent-flow problem*, J. Op. Res. Soc. Japan, 21 (1978) 189–203.

[159] S. Fujishige, *Structures of polyhedra determined by submodular functions on crossing families*, Math. Prog., 29 (1984) 125–41.

[160] S. Fujishige, *A characterization of faces of the base polyhedron associated with submodular systems,* J. Op. Res. Soc. Japan, 27 (2) (1984) 112–28.

[161] S. Fujishige, Submodular Functions and Optimization, 2nd edn, Ann. Disc. Math., 58, Elsevier (2005).

[162] S. Fujishige, *A note on disjoint arborescences*, Combinatorica, 30 (2) (2010) 247–52.

[163] S. Fujishige, H. Röck, and U. Zimmermann, *A strongly polynomial algorithm for minimum cost submodular flow problems*. Math. Op. Res., 14 (1989) 60–9.

[164] S. Fujishige and X. Zhang, *New algorithms for the intersection problem of submodular systems*, Japan J. Ind. and Appl. Math. 9 (1992) 369–82.

[165] S. Fujishige and X. Zhang, *A push/relabel framework for submodular flows and its refinement for* 0–1 *submodular flows*, Optimization 38 (1996) 113–54.

[166] K. Fukuda, A. Prodon, and T. Sakuma, *Notes on acyclic orientations and the shelling lemma*, Theor. Comp. Sci., 263 (2001) 9–16.

[167] D.R. Fulkerson, *Note on Dilworth's decomposition theorem for partially ordered sets*, Proc. Amer. Math. Soc. 7 (1956) 701–2.

[168] D.R. Fulkerson, *On the equivalence of the capacity-constrained transshipment problem and the Hitchcock problem* [Notes on Linear Programming and Extensions 2014 Part 53], Research Memorandum RM-2480, RAND Corporation, Santa Monica, 13 Jan. 1960.

[169] D.R. Fulkerson, *An out-of-kilter method for minimal-cost flow problems,* J. Soc. Ind. Appl. Math., 9 (1961) 18–27.

[170] D.R. Fulkerson, *Packing rooted directed cuts in a weighted directed graph,* Math. Prog. 6 (1974) 1–13.

[171] O. Fülöp, *Sparse graph certificates for mixed connectivity*, Disc. Math. 294 (3) (2005) 285–290.

[172] H.N. Gabow, *A matroid approach to finding edge-connectivity and packing arborescences*, J. Comp. Sys. Sci., 50 (1995) 259–73.

[173] H.N. Gabow and H.H. Westermann, *Forests, frames and games: Algorithms for matroid sums and applications,* Algorithmica, 7 (1992), 465–97 (special graph algorithms issue).

[174] H.N. Gabow and Y. Xu, *Efficient theoretic and practical algorithms for linear matroid intersection problems*, J. Comp. Sys. Sci., 53 (1) (1996) 129–47.

[175] H.N. Gabow *Centroids, representations, and submodular flows*, J. Alg., 18 (1995) 586–628.

[176] H.N. Gabow and K.S. Manu, *Packing algorithms for arborescences (and spanning trees) in capacitated graphs*, Math. Prog., 82 (1998) 83–109.

[177] M.R. Garey and D.S. Johnson, Computers and intractability: A Guide to the Theory of NP-completeness, W.H. Freeman and Co., New York, 1979.

[178] T. Gallai (= Grünwald), *Ein neuer Beweis eines Mengerischen Satzes*, J. London Math. Soc. 13 (1938) 188–92.

[179] T. Gallai, *Maximum-Minimum Sätze über Graphen*, Acta Mathematica Academiae Scientiarum Hungariae, 9 (1958) 395–434.

[180] T. Gallai, *Problem 15*, in: ed. M. Fiedler, Theory of Graphs and it Applications, Proc. Sympos. Smolenice, 1963, 161.

[181] L. Georgiadis and R.E. Tarjan, *Dominator Tree Verification and Vertex-Disjoint Paths*, in: Proceedings of the 16th ACM-SIAM Symposium on Discrete Algorithms, 2005, 433–42.

[182] A. Ghouila-Houri, *Caractérisation des matrices totalement unimodulaires*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences [Paris] 254 (1962) 1192–4.

[183] F.R. Giles and W.R. Pulleyblank, *Total dual integrality and integer polyhedra,* Lin. Alg. Appl. 25 (1979) 191–6.

[184] M.X. Goemans, *Approximate Edge Splitting,* SIAM J. Disc. Math., 14 (2001) 138–41.

[185] M.X. Goemans and D. Bertsimas, *Survivable networks, linear programming relaxations and the parsimonious property*, Math. Prog., 60 (1993) 145–66.

[186] A.V. Goldberg and R.E. Tarjan, *A new approach to the maximum-flow problem*, J. Assoc. Comp. Mach., 35 (1988) 921–40.

[187] A.V. Goldberg and R.E. Tarjan, *Finding minimum-cost circulations by canceling negative cycles*, J. Assoc. Comp. Mach., 36 (1989) 873–86.

[188] R.E. Gomory and T.C. Hu, *Multi-terminal network flows*, J. Soc. Ind. Appl. Math. 9 (1961) 551–70.

[189] R.E. Gomory and T.C. Hu, *Synthesis of a communication network,* J. Soc. Ind. Appl. Math. 12 (1964) 348–69.

[190] R. Graham, M. Grötschel, and L. Lovász (eds.), Handbook of Combinatorics, Elsevier Science BV, New York 1995.

[191] C. Greene, *Another exchange property for bases*, Proc. Am. Math. Soc. 46 (1974) 155–6.

[192] C. Greene, *Some partitions associated with a partially ordered set*, J. Comb. Theory, Ser. A, 20 (1976) 60–79.

[193] C. Greene and D.J. Kleitman, *The structure of Sperner k-families,* J. Comb. Theory, Ser. A, 20 (1976) 41–68.

[194] C. Greene and T.L. Magnanti, *Some abstract pivot algorithms*, SIAM J. Appl. Math. 29 (1975) 530–9.

[195] H. Gröflin and A.J. Hoffman, *On matroid intersections*, Combinatorica 1 (1981), 43–7.

[196] M. Grötschel and C.L. Monma, *Integer Polyhedra arising from certain network design problems with Connectivity Constraints*, SIAM J. Disc. Math., 3 (4) (1990) 502–23.

[197] R.P. Gupta, *On edge-coloration of multigraphs*, Ann. Disc. Math., 8 (1980) 229–33.

[198] E. Győri, *A minimax theorem on intervals,* J. Comb. Theory, Ser. B, 37 (1984) 1–9.

[199] E. Győri, *Partitions and covers of rectilinear regions,* in: eds. P. Frankl, Z. Füredi, G.O.H. Katona, and D. Miklós, Extremal Problems for Finite Sets, Bolyai Society Mathematical Studies, János Bolyai Mathematical Society, Budapest, 1994, 289–304.

[200] S.L. Hakimi, *On the degrees of the vertices of a directed graph*, J. Franklin Inst. 279 (4) (1965), 290–308.

[201] P. Hall, *On representatives of subsets*, J. London Math. Soc., 10 (1935) 26–30.

[202] Y.O. Hamidoune and M.L. Las Vergnas, *Local edge-connectivity in regular bipartite graphs*, J. Comb. Theory, Ser. B, 44 (1986) 370–1.

[203] J. Hao and J.B. Orlin, *A faster algorithm for finding the minimum cut in a directed graph*, J. Alg. 17 (1994) 424–46.

[204] J. Hartmanis, *Lattice theory of generalized partitions*, Can. J. Math. 11 (1959) 97–106.

[205] T. Helgason, *Aspects of the theory of hypermatroids*, in: eds. C. Berge, D. Ray-Chaudhuri, Hypergraph Seminar (Proceedings Working Seminar on Hypergraphs, Columbus, Ohio, 1972) [Lecture Notes in Mathematics 411], Springer, Berlin, 1974, 191–213.

[206] L. Henneberg, Die graphische Statik der starren Systeme, Teubner, Leipzig, 1911.

[207] A.J. Hoffman, *Some recent applications of the theory of linear inequalities to extremal combinatorial analysis,* Proceedings of the Symposia of Applied Mathematics, 10 (1960) 113–127.

[208] A.J. Hoffman, *A generalization of max-flow min-cut,* Math. Prog., 6 (1974) 352–9.

[209] A.J. Hoffman and J.B. Kruskal, Jr, *Integral boundary points of convex polyhedra*, Linear Inequalities and Related Systems, Ann. Math. Study, 38 (1956) 223–346.

[210] I. Holyer, *The NP-completeness of edge-coloring,* SIAM J. Comp., 10 (1981) 718–20.

[211] A. Horn, *A characterizations of unions of linearly independent sets*, J. London Math. Soc. 30 (1955) 494–6.

[212] A. Huck, *Independent branchings in acyclic digraphs*, Disc. Math. 199 (1999) 245–9.

[213] A. Huck, *Disproof of a conjecture about independent branchings in k-connected directed graphs*, J. Graph Theory, 20 (1995) 235–9.

[214] C.A.J. Hurkens, A. Schrijver, and É. Tardos, *On fractional multicommodity flows and distance functions*, Disc. Math., 73 (1988/9) 99–109.

[215] H. Imai, *On combinatorial structures of line drawings of polyhedra*, Disc. Appl. Math., 10 (1985) 79–92.

[216] A.W. Ingleton and M.J. Piff, *Gammoids and transversal matroids*, J. Comb. Theory, Ser. B, 15 (1973) 51–68.

[217] H. Ito, K. Makino, K. Arata, S. Honami, Y. Itatsu, and S. Fujishige, *Source location problem with flow requirements in directed networks*, Optim. Methods and Software, 18 (4) (Aug. 2003) 427–35.

[218] A. Iványi, oral communication.

[219] S. Iwata, *A fully combinatorial algorithm for submodular function minimization*, J. Comb. Theory, B, 84 (2) (2002) 203–12.

[220] S. Iwata, L. Fleischer, and S. Fujishige, *A combinatorial, strongly polynomial-time algorithm for minimizing submodular functions*, J. of the ACM, 48 (4) (2001) 761–77.

[221] S. Iwata and T. Matsuda, *Finding coherent cyclic orders in strong digraphs*, Combinatorica, 28 (2008) 83–8.

[222] S. Iwata and Y. Kobayashi, *An algorithm for minimum cost arc-connectivity orientations,* Algorithmica, 56 (2010) 437–47.

[223] S. Iwata, S.T. McCormick, and M. Shigeno, *A strongly polynomial cut canceling algorithm for minimum cost submodular flow*, SIAM J. Discrete Math., 19 (2) (2005) 304–20.

[224] B. Jackson, *Some remarks on arc-connectivity, vertex splitting, and orientation in graphs and digraphs*, J. Graph Theory, 12 (1988) 429–36.

[225] B. Jackson and T. Jordán, *Brick partitions of graphs*, Disc. Math., 310 (2) (2010) 270–5.

[226] C.G.J. Jacobi (1804-1851), '*Looking for the order of a system of arbitrary ordinary differential equations*', *De investigando ordine systematis aequationum differentialium*

vulgarium cujuscunque, trans. François Ollivier, Applicable Algebra in Engineering, Communication and Computing, 20 (1) (2009) 7–32.

[227] A. Jüttner, *On the effieciency of Egerváry's perfect matching algorithm,* Technical Report TR-2004-13, Egerváry Research Group, Budapest, 2004, www.cs.elte.hu/egres.

[228] G. Kalai, oral communication, 2002, Oberwolfach.

[229] N. Kamiyama, N. Katoh, and A. Takizawa, *Arc-disjoint in-trees in directed graphs*, in: Proc. the Nineteenth Annual ACM-SIAM Symposium on Discrete Mathematics (SODA 2008), 218–526.

[230] G.R. Kampen,*Orienting planar graphs*, Disc. Math., 14 (4) (1976) 337–41.

[231] Z.A. Kareyan, *Drevesnost orientirovannyh grafov*, in: ed. A.V. Petrosjan, Matematichesskie Voprosi Kibernetiki i Vichislitelnoj Tehniki, Izdatelstvo Akademii Nauk Armjanskoj SSR, Yerevan, 1979, 59–63. (In Russian. *Arboritcity of directed graphs*, in: Mathematical Problems in Cybernetics and Computer Science, Publishing House of Armenian Academy of Sciences.)

[232] R.M. Karp, *On the computational complexity of combinatorial problems*, Networks, 5 (1975) 45–68.

[233] S. Khanna, J. Naor, and F.B. Shepherd, *Directed network design with orientation constraints*, SIAM J. Disc. Math., 19 (1) (2005) 245–57.

[234] R.M. Karp, *A characterization of the minimum cycle mean in a digraph*, Disc. Math. 23 (1978) 309–11.

[235] A.V. Karzanov and M.V. Lomonosov, *Systems of flows in undirected networks*, in: ed. O.I. Larichev, Mathematical Programming, (Inst. for System Studies, Moscow, iss. 1, 59-66) (in Russian) (1978).

[236] T. Király, *Covering symmetric supermodular functions by uniform hypergraphs*, J. Comb. Theory, Ser. B, 91 (2004) 185–200.

[237] T. Király, *Minimal feedback sets in binary oriented matroids*, EGRES Quick Proofs series, QP-2006-01, www.cs.elte.hu/egres/.

[238] T. Király, *Computing the minimum cut in hypergraphic matroids*, EGRES Quick Proofs series, QP-2010-05, www.cs.elte.hu/egres/.

[239] T. Király and L.C. Lau, *Approximate min-max theorems for Steiner rooted-orientations of graphs and hypergraphs*, J. Comb. Theory, Ser. B, 98 (2008) 1233–52.

[240] Z. Király and Z. Szigeti, *Simultaneous well-balanced orientations of graphs*, J. Comb. Theory, Ser. B, 96 (2006) 684–92.

[241] D.E. Knuth, *Matroid Partitioning*, Report STAN-CS-73-342, Computer Science Department, Stanford University, Stanford, 1973.

[242] D.E. Knuth, *Wheels within wheels*, J. Comb. Theory, Ser. B, 16 (1974) 42–6.

[243] D.E. Knuth, *The asymptotic number of geometries*, J. Comb. Theory, Ser. A, 17 (1974) 398–401.

[244] E. Kovács, oral communication.

[245] E. Kovács and L. Végh, *A constructive characterization of* (k, ℓ)*-edge-connected digraphs*, Combinatorica, to appear, Egres Tech. Report, 2008-14.

[246] Kőnig D., *Graphok és matrixok*, Matematikai és Fizikai Lapok, 38 (1931) 116–119. (*Graphs and matrices*, in Hungarian.)

[247] D. Kőnig, Theory der endlichen und unendlichen Graphen, Chelsea Publishing Company, Providence, 1935.

[248] B. Korte and J. Vygen, Combinatorial Optimization: Theory and Algorithms, 3rd edn, Springer, Berlin, 2005, Algorithms and Combinatorics 21.

[249] M. Kriesell, *Edge disjoint trees containing some given vertices in a graph*, J. Comb. Theory, Ser. B, 88 (2003) 53–65.
[250] S. Krogdahl, *A combinatorial proof for a weighted matroid intersection algorithm*, Computer Science Report 17, Institute of Mathematical and Physical Sciences, University of Tromso, Tromso, 1976.
[251] J.B. Kruskal, Jr, *On the shortest spanning subtree of a graph and the traveling salesman problem*, Proc. Am. Math. Soc., 7 (1956) 48–50.
[252] H.W. Kuhn, *The Hungarian method for the assignment problem,* Naval Res. Log. Quart., 2 (1955) 83–97.
[253] C. Kuratowski, *Sur le probléme des courbes gauches en Topologie*, Fundamenta Mathematicae 15 (1930) 271-283 [reprinted in: eds. K. Borsuk, R. Engelking, and C. Ryll-Nardzewski, Kazimierz Kuratowski—Selected Papers, PWN, Warszawa, 1988, 345–57].
[254] G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Eng. Math., 4 (1970) 331–40.
[255] H.G. Landau, *On dominance relations and the structure of animal societies III. The condition for the core structure*, Bull. Math. Biophys., 15 (1953) 143–8.
[256] E.L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart, and Winston, New York, 1976.
[257] E.L. Lawler and C.U. Martel, *Flow network formulations of polymatroid optimization problems*, in: eds. A. Bachem, M. Grötschel, B. Korte, Bonn Workshop on Combinatorial Optimization (Bonn, 1980) Annals of Discrete Mathematics 16, North-Holland, Amsterdam, 1982, 189–200.
[258] E.L. Lawler and C.U. Martel, *Computing maximal 'polymatroidal' network flows*, Math. Op. Res. 7 (1982) 334–47.
[259] Jon Lee, A First Course in Combinatorial Optimization, Cambridge Text in Applied Mathematics, Cambridge University Press, Cambridge, 2004.
[260] M.V. Lomonosov, *Combinatorial approaches to multiflow problems*, Graph Theory Newsletters, 9 (1979) 4.
[261] M. Lorea, *Hypergraphes et matroides*, Cahiers Centre Etud. Rech. Oper. 17 (1975) 289–91.
[262] L. Lovász, Solution to Problem 11, in: Report on the Memorial Mathematical Contest Miklós Schweitzer of the year 1968 (in Hungarian), *Matematikai Lapok,* 20 (1969) 145–71.
[263] L. Lovász, *A generalization of Kőnig's theorem,* Acta. Math. Acad. Sci. Hungar. 21 (1970) 443–6.
[264] L. Lovász, *A remark on Menger's theorem*, Acta Mathematica Academiae Scientiarum Hungaricae, 21 (1970) 365–8.
[265] L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Disc. Math. 2 (1972) 253–67 [reprinted as: *Normal hypergraphs and the weak perfect graph conjecture*, in: eds. C. Berge and V. Chvatal, Topics on Perfect Graphs [Annals of Discrete Mathematics 21], North-Holland, Amsterdam, 1984, 29–42].
[266] L. Lovász, *Connectivity in digraphs*, J. Comb. Theory, Ser. B, 15 (1973) 174–7.
[267] L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam, 1979.
[268] L. Lovász, *On two minimax theorems in graph theory*, J. Comb. Theory, Ser. B, 21 (1976) 96–103.
[269] L. Lovász, *On some connectivity properties of Eulerian graphs*, Acta Mat. Acad. Sci. Hungaricae 28 (1976) 129–38.
[270] Lovász, L., *Gráfelmélet és diszkrét programozás*, Matematikai Lapok, 27 (1–2) (1976–9) 69–86. (*Graph theory and discrete programming*, in Hungarian.)

[271] L. Lovász, *Selecting independent lines from a family of lines in a space*, Acta Sci. Univ. Szeged, 43 (1980) 121–31.
[272] L. Lovász, *Submodular functions and convexity*, in: eds. A. Bachem, M. Grötschel, and B. Korte, Mathematical Programming—The State of the Art, Springer, Berlin, 1983, 235–57.
[273] L. Lovász and M. Plummer, Matching Theory, North-Holland, Amsterdam, 1986. Repr. with corrections: American Mathematical Society, Chelsea Publishing, Providence, 2009.
[274] L. Lovász and Y. Yemini, *On generic rigidity in the plane*, SIAM J. Alg. Disc. Methods, 3 (1) (1982) 91–8.
[275] C.L. Lucchesi and D.H. Younger, *A minimax relation for directed graphs,* J. London Math. Soc., 17 (2) (1978) 369–74.
[276] A. Lubiw, *A weighted min-max relation for intervals*, J. Comb. Theory, Ser. B, 53 (2) (1991) 151–72.
[277] W. Mader, *Über minimal n-fach zusammenhängende, unendliche Graphen und ein extremal Problem*, Arch. Math. 23 (1972) 553–60.
[278] W. Mader, *A reduction method for edge-connectivity in graphs,* Ann. Disc. Math., 3 (1978) 145–64.
[279] W. Mader, *Über die Maximalzahl kantendisjunkter A-Wege*, Archiv der Mathematik (Basel) 30 (1978) 325–36.
[280] W. Mader, *Konstruktion aller n-fach kantenzusammenhängenden Digraphen,* Europ. J. Comb. (3) (1982) 63–7.
[281] W. Mader, *On n-edge-connected digraphs,* Ann. Disc. Math., 17 (1983) 439–41.
[282] W. Mader, *Minimal n-fach zusammenhängende Digraphen*, J. Comb. Theory, Ser. B, 38 (2) (1985) 102–17.
[283] D. Marx, *Eulerian disjoint paths problem in grid graphs is NP-complete*, Disc. Appl. Math., 143 (2004) 336–41.
[284] J.H. Mason, *On a class of matroids arising from paths in graphs*, Proc. London Math. Soc. 25 (3) (1972) 55–74.
[285] C.J.H. McDiarmid, *Rado's theorem for polymatroids*, Math. Proc. Cambridge Philosophical Soc., 78 (1975) 263–81.
[286] C. McDiarmid, *Blocking, antiblocking, and pairs of matroids and polymatroids,* J. Comb. Theory, Ser. B, 25 (1978) 313–25.
[287] N.S. Mendelsohn and A.L. Dulmage, *Some generalizations of the problem of distinct representatives*, Can. J. Math., 10 (1958) 230–41.
[288] K. Menger, *Zur allgemeinen Kurventheorie*, Fund. Math., 10 (1927) 96–115.
[289] C.A. Micchelli (ed.), Selected Papers of Alan Hoffman with Commentary, World Scientific, New York, 2003.
[290] M. Middendorf and F. Pfeiffer, *On the complexity of the disjont paths problems*, Comb., 13 (1) (1997) 97–107.
[291] L. Mirsky, Transversal Theory, Academic Press, New York, 1971.
[292] L. Mirsky and H. Perfect, *Applications of the notion of independence to problems of combinatorial analyzis*, J. Comb. Theory, 2 (1967) 327–57.
[293] J. Munkres, *Algorithms for the assignment and transportation problems,* J. Soc. Ind. Appl. Math., 5 (1957) 32–8.
[294] K. Murota, Discrete Convex Analysis, SIAM Monographs on Discrete Mathematics and Applications, 2003.
[295] H. Nagamochi, *Sparse connectivity certificates via MA orderings in graphs*, Disc. Appl. Math., 154 (16) (2006) 2411–17.
[296] H. Nagamochi and T. Ibaraki, *Computing edge-connectivity in multiple and capacitated graphs,* SIAM J. Disc. Math., 5 (1992) 54–66.

[297] H. Nagamochi and T. Ibaraki, *A linear time algorithm for finding a sparse k-connected spanning subgraph of a k-connected graph*, Algorithmica, 7 (1992) 583–96.
[298] H. Nagamochi and T. Ibaraki, *Deterministic* $\hat{O}(nm)$ *time edge-splitting in undirected graphs*, J. Comb. Optim., 1 (1997) 5–46.
[299] H. Nagamochi and T. Ibaraki, *Graph connectivity and its augmentation: applications of MA orderings*, Disc. Appl. Math., 123 (2002) 447–72.
[300] H. Nagamochi and T. Ibaraki, Algorithmic Aspects of Graph Connectivities, Cambridge University Press, Cambridge 2008, Encyclopedia of Mathematics and its Applications 123.
[301] H. Nagamochi, T. Ishii, and H. Ito, *Minimum cost source location problem with vertex-connectivity requirements in digraphs*, Inform. Proc. Letters, 80 (2001) 287–94.
[302] D. Naor, D. Gusfield, and C. Martel, *A fast algorithm for optimally increasing the edge connectivity,* SIAM J. Comp., 26, (1997) 1139–65.
[303] C.St.J.A. Nash-Williams, *On orientations, connectivity and odd vertex pairings in finite graphs*, Can. J. Math., 12 (1960) 555–67.
[304] C.St.J.A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc., 36 (1961) 445–50.
[305] C.St.J.A. Nash-Williams, *Decomposition of finite graphs into forests*, J. London Math. Soc., 39 (1964) 12.
[306] C.St.J.A. Nash-Williams, *An application of matroids to graph theory*, in: ed. P. Rosenstiehl, Theory of Graphs—International Symposium—Théorie des graphes—Journées internationales d'étude (Rome, 1966), Gordon and Breach, New York, and Dunod, Paris, 1967, 263–5.
[307] C.St.J.A. Nash-Williams, *Strongly connected mixed graphs and connected detachments of graphs*, J. Comb. Math. Comb. Comp., 19 (1995) 33–47.
[308] G. Naves and A. Sebő, *Multiflow feasibility: an annotated tableau*, in: eds. W. Cook, L. Lovász, and J. Vygen, Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, 261–83.
[309] J. von Neumann, *A certain zero-sum two-person game to the optimal assignment problem*, in: eds. H.W. Kuhn and A.W. Tucker Contributions to the Theory of Games II, Annals of Mathematical Studies, 28, Princeton University Press, Princeton, 1953, 5–12.
[310] T. Nishizeki and S. Poljak, *k-Connectivity and Decomposition of Graphs into Forests*, Disc. Appl. Math., 55 (3) (1994) 295–301.
[311] H. Okamura and P.D. Seymour, *Multicommodity flows in planar graphs,* J. Comb. Theory, Ser. B, 31 (1981) 75–81.
[312] H. Okamura, *Multicommodity flows in graphs,* Disc. Appl. Math., 6 (1983) 55–62.
[313] O. Ore, *Graphs and matching theorems*, Duke Math. J., 22 (1955) 625–39.
[314] O. Ore, *Studies on directed graphs, I*, Ann. Math., 63 (1956) 383–406.
[315] O. Ore, Theory of Graphs, American Mathematical Society, Providence, 1962.
[316] J.G. Oxley, Matroid Theory, Oxford University Press, Oxford, 2004.
[317] H. Perfect, *Applications of Menger's graph theorem*, J. Math. Anal. Appl., 22 (1968) 96–111.
[318] H. Perfect, *Independence spaces and combinatorial problems*, Proc. London Math. Soc. 19 (1969) 17–30.
[319] J. Petersen, *Die Theorie der regulären graphs,* Acta Math., 15 (1891) 193–220.
[320] J.-C. Picard and M. Queyranne, *On the structure of all minimum cuts in a network and applications*, Math. Prog. Study, 13 (1980) 8–16.
[321] J. Plesník, *Minimum block containing a given graph*, Archiv der Mathematik [Basel], 27 (1976) 668–72.

[322] M. Preissmann and A. Sebő, *Graphic submodular function minimization: a graphic approach and application*, in: eds. W. Cook, L. Lovász, and J. Vygen, Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, 365–85.
[323] R.C. Prim, *Shortest connection networks and some generalizations*, Bell System Tech. J., 36 (1957) 1389–1401.
[324] J.S. Pym, *The linking of sets in graphs*, J. London Math. Soc., 44 (1969) 542–50.
[325] J.S. Pym, *A proof of the linkage theorem*, J. Math. Analysis Appl., 27 (1969) 636–8.
[326] R. Rado, *A theorem on independence relations,* Quart. J. Math. (Oxford) 13 (2) (1942) 83–9.
[327] R. Rado, *Note on independence functions,* Proc. London Math. Soc., 7 (3) (1957) 300–20.
[328] A. Recski, Matroid Theory and its Applications in Electric Network Theory and in Statics, Springer, Berlin, 1989.
[329] R. Rizzi, *Kőnig's edge coloring theorem without augmenting paths*, J. Graph Theory, 29 (1998) 87.
[330] R. Rizzi, *On minimizing symmetric set functions*, Combinatorica 20 (2000) 445–50.
[331] N. Robertson and P.D. Seymour, *Graph minors: the disjoint paths problem*, J. Comb. Theory, Ser. B, 63 (1995) 65–110.
[332] H.E. Robbins, *A theorem on graphs with an application to a problem of traffic control*, Amer. Math. Monthly, 46 (1939) 281–3.
[333] B. Rothschild and A. Whinston, *Feasibility of two commodity network flows*, Op. Research, 14 (1966) 1121–9.
[334] P. Schönsleben, Ganzzahlige Polymatroid Intersections Algorithmen, Ph.D. thesis, Eidgenössischen Techn. Hochschule, Zürich, 1980.
[335] A. Schrijver, *Min-max relations for directed graphs*, in: eds. A. Bachem, M. Grötschel, B. Korte, Bonn Workshop on Combinatorial Optimization, Annals of Discrete Mathematics 16, North-Holland, Amsterdam, 1982, 261–80.
[336] A. Schrijver, *Total dual integrality from directed graphs, crossing families and sub- and supermodular functions*, in: ed. W.R. Pulleyblank, Progress in Combinatorial Optimization, Academic Press, New York, 1984 315–61.
[337] A. Schrijver, *Supermodular colourings,* in: eds. L. Lovász, A. Recski, Matroid Theory (Proceedings Colloquium on Matroid Theory, Szeged, 1982) [Colloquia Mathematica Societatis János Bolyai, 40], North-Holland, Amsterdam, 1985, 327–43.
[338] A. Schrijver, Theory of Linear and Integer Programming, Wiley, Chichester, 1986.
[339] A. Schrijver, *A combinatorial algorithm minimizing submodular functions in strongly polynomial time*, J. Comb. Theory, Ser. B, 80 (2000) 575–88.
[340] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, 3 vols, Springer, New York, 2003. Vol. 24 of the series Algorithms and Combinatorics.
[341] A. Schrijver, On the history of combinatorial optimization (till 1960), in: eds. K. Aardal, G.R. Nemhauser, R, Weismantel, Handbook of Discrete Optimization, Elsevier, Amsterdam, 2005, 1–68.
[342] W. Schwärzler, *On the complexity of the planar edge-disjoint paths problem with terminals on the outer boundary*, Comb., 29 (1) (2009) 121–6.
[343] A. Sebő, *Minmax relations for cyclically ordered digraphs*, J. Comb. Theory, Ser. B, (2007) 518–52.
[344] R. Sedgewick, Algorithms in C, Addison-Wesley, New York, 2002.
[345] P.D. Seymour, *Four-terminus flows*, Networks 10 (1980) 79–86.
[346] P.D. Seymour, *Disjoint paths in graphs*, Discrete Math., 29 (1980) 293–309.
[347] P.D. Seymour, *On odd cuts and plane multicommodity flows,* Proc. London Math. Soc., 42 (1981) 178–92.

[348] F.B. Shepherd and A. Vetta, *Visualizing, Finding and Packing Dijoins*, in: eds. D. Avis, A. Hertz, and O. Marcotte, Graph Theory and Combinatorial Optimization, Springer, Berlin, 2005, 219–254.

[349] F.B. Shepherd, *Single-sink multicommodity flow with side constraints*, in: eds. W. Cook, L. Lovász, and J. Vygen, Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, 429–50.

[350] Y. Shiloach, *Edge-disjoint branching in directed multigraphs,* Inform. Proc. Letters, 8 (1979) 24–7.

[351] H. Shirazi and J. Verstraëte, *A note on polynomials and f-factors of graphs*, Elec. J. Comb., 15 (22) (2008).

[352] E. Sperner, *Ein Satz über Untenmengen einer endliche Menge*, Math. Z., 27 (1928) 544–8.

[353] L. Szegő, *Note on covering intersecting set-systems by digraphs*, Disc. Math., 234 (2001) 187–9.

[354] Z. Szigeti, *Hypergraph connectivity augmentation*, in: ed. A. Frank, Connectivity Augmentation of Networks: Structures and Algorithms, Mathematical Programming, Ser. B, 84 (3) (1999) 519–27.

[355] Z. Szigeti, *Edge-connectivity Augmentations of Graphs and Hypergraphs*, in: eds. W. Cook, L. Lovász, and J. Vygen, Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, 483–521.

[356] Z. Szigeti, *Two remarks on local edge-connectivity of graphs*, QP-2010-05, EGRES Quick-Proofs series, www.cs.elte.hu/egres/ (2010).

[357] K. Sugihara, Machine interpretation of line drawings, MIT Press, Cambridge, MA, 1986.

[358] K. Sugihara, *Detection of structural inconsistency in systems of equations with degrees of freedom and its applications*, Disc. Appl. Math., 10 (1985) 312–28.

[359] É. Tardos, *A strongly polynomial mincost circulation algorithm*, Combin., 5 (3) (1985) 247–55.

[360] É. Tardos, *Generalized matroids and supermodular colourings,* in: eds. L. Lovász and A. Recski, Matroid Theory (Proceedings Colloquium on Matroid Theory, Szeged, 1982) [Colloquia Mathematica Societatis János Bolyai, 40], North-Holland, Amsterdam, 1985, 359–82.

[361] R.E. Tarjan, *Depth-first seach and linear graph algorithms*, SIAM J. Comp., 1 (1972) 146–60

[362] R.E. Tarjan, Data Structures and Network Algorithms, 1983, CBMS-NSF Regional Conferencs Series in Applied Mathematics, 44, SIAM, Philadelphia.

[363] R.E. Tarjan, *A good algorithm for edge-disjoint branching*, Inform. Proc. Letters, 3 (1974) 52–3.

[364] R.E. Tarjan and M. Yannakakis, *Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs*, SIAM J. Comp., 13 (3) (Aug. 1984) 566–79.

[365] C. Thomassen, *2-linked graphs*, Europ. J. Combin. 1 (1980) 371–8.

[366] C. Thomassen, *Kuratowski's theorem,* J. Graph Theory, 5 (1981) 225–41.

[367] W.T. Tutte, *The factorization of linear graphs*, J. London Math. Soc., 22 (1947) 107–11.

[368] W.T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. London Math. Soc., 36 (1961) 221–30.

[369] W.T. Tutte, *A theory of 3-connected graphs,* Indag. Math., 23 (1961) 441–55.

[370] W.T. Tutte, *How to draw a graph,* Proc. London Math. Soc., 13 (3) (1963) 743–68.

[371] W.T. Tutte, *Lectures on matroids*, J. Res. Nat. Bur. Standards, Section B, 69 (1965) 1–347 [reprinted in: eds. D. McCarthy and R.G. Stanton, Selected Papers of W.T. Tutte, Vol. II, Charles Babbage Research Centre, St. Pierre, Manitoba, 1979, 439–96.]

[372] W.T. Tutte, Connectivity in Graphs, University of Toronto Press, Toronto, 1966.

[373] K. Vidyasankar, *Covering the edge-set of a directed graph with trees*, Discrete Math., 24 (1978) 79–85.

[374] J. Vygen, *NP-completeness of some edge-disjoint paths problems,* Disc. Appl. Math., 61 (1995) 83–90.

[375] T. Watanabe and A. Nakamura, *Edge-connectivity augmentation problems,* Comp. System Sciences, 35 (1) (1987) 96–144.

[376] D.J.A. Welsh, *Some applications of a theorem of Rado*, Mathematika [London], 15 (1968) 199–203.

[377] D.J.A. Welsh, *On matroid theorems of Edmonds and Rado*, J. London Math. Soc., 2 (2) (1970) 251–6.

[378] D.J.A. Welsh, *Generalized versions of Hall's theorem*, J. of Combinatorial Theory, Ser. B, 10 (1971) 95–101.

[379] D.J.A. Welsh, Matroid Theory, Academic Press, New York, 1976.

[380] W. Whiteley, *Some matroids from discrete applied geometry,* in: eds. J.E. Bonin, J.G. Oxley, and B. Servatius, Matroid Theory Contemp. Math., 197, Amer. Math. Soc., Providence, 1996, 171–311.

[381] W. Whiteley, *A matroid on hypergraphs, with applications in scene analysis and geometry*, Disc. Comp. Geometry, 4 (1) (1989) 75–95.

[382] H. Whitney, *Non-separable and planar graphs,* Trans. Amer. Math. Soc., 34 (1932) 339–62.

[383] H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math., 57 (1935) 509–33.

[384] H. Whitney, *Two-isomorphic graphs*, Amer. J. Math., 55 (1933) 245–54.

[385] R.W. Whitty, *Vertex-disjoint paths and edge-disjoint branchings in directed graphs,* J. Graph Theory, 11 (1987) 349–58.

[386] D.R. Woodall, *Minimax theorems in graph theory*, In: eds. L.W. Beineke and R.J. Wilson, Selected Topics in Graph Theory, Academic Press, New York, 1978, 237–69.

[387] Y. Wu, K. Jain, and S.-Y. Kung, *A unification of network coding and tree-packing (routing) theorems*, IEEE/ACM Transactions on Networking (TON) vol. 14, Issue SI, Special issue on networking and information theory, June 2006, 2398–409.

[388] U. Zimmermann, *Minimization of some nonlinear functions over polymatroidal network flows*, in: eds. A. Bachem, M. Grötschel, and B. Korte, Bonn Workshop on Combinatorial Optimization, Annals of Discrete Mathematics 16, North-Holland, Amsterdam, 1982, 287–309.

[389] D.H. Younger, *Maximum families of disjoint directed cut sets*, in: ed. W.T. Tutte, Recent Progress in Combinatorics, Proceedings of the Third Waterloo Conference on Combinatorics, May 1968, Academic Press, New York, 1969, 329–33.

Author index

General index

978–0–19–920527–1 21-Andras-Frank-General-Index-drv Andras Frank (Typeset by SPi, Publisher Services, India) 639 of 640 December 11, 2010 15:23
OUP CORRECTED PROOF – REVISES, 11/12/2010, SPi