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Discrete Applied Mathematics 82 (1998) 135–154

DISCRETE APPLIED MATHEMATICS

A strongly polynomial algorithm for the inverse shortest arborescence problem

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Received 26 June 1994; received in revised form 15 April 1997; accepted 28 July 1997

Abstract

In this paper an inverse problem of the weighted shortest arborescence problem is discussed. That is, given a directed graph G and a set of nonnegative costs on its arcs, we need to modify those costs as little as possible to ensure that T, a given v_1-arborescence of G, is the shortest one. It is found that only the cost of T needs modifying. An $O(n^3)$ combinatorial algorithm is then proposed. This algorithm also gives an optimal solution to the inverse weighted shortest path problem. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

In this paper, we define a weighted directed graph as the triple (V, A, c), where (V, A) is a directed graph with n vertices and m arcs, and where c is a set of nonnegative costs $\{c_g\colon g \in A\}$ associated with the arcs. The well-known weighted shortest arborescence problem (see [3] or [4] for instance) is, for a given weighted directed graph, to find the arborescence T, which has the shortest total cost. An inverse problem of the weighted shortest arborescence problem is as follows:

(IWSA) Given a weighted directed graph $G = (V, A, c)$, vertex $v_1 \in V$ and a v_1-arborescence T, find w, a new set of costs, such that

$$\min \|w - c\| \tag{1}$$

is achieved under the constraints that

$$w_g \geqslant 0 \quad (g \in A) \tag{2}$$

and the arborescence T is a shortest v_1-arborescence in $G' = (V, A, w)$.

Clearly, a number of interesting variants of this basic problem can be constructed by considering various norms in (1). In particular, the l_1, l_2, and l_∞ norms seem attractive.

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PII S0166-218X(97)00098-X

In this paper, we will restrict ourselves to the l_1 norm, mostly because it is widely used and leads to tractable computational methods.

Throughout this paper, we mostly use the start and end vertices to express an arc. For example, arc xy means an arc from x to y. Sometimes for notational simplicity we use a letter to call an arc such as arc g. Costs associated with these arcs are correspondingly denoted by c_{xy} or c_g. If H is a subgraph of G, then $c(H)=\sum_{g\in A(H)} c_g$. All arcs in this paper are directed. A directed walk in G is a finite sequence of vertices $W=v_0v_1v_2\cdots v_k$, such that $v_{i-1}v_i$, $1\leqslant i\leqslant k$, is an arc in G. If, in addition, all the vertices except v_0 are distinct and $v_0=v_k$, W is called a circuit. A cycle (resp. tree) in G is a subgraph whose underlying undirected graph is a cycle (resp. tree). A subgraph T in G is called a v_1-arborescence if it is a spanning tree of G and every vertex different from v_1 is the head of exactly one arc in T. If T is a v_1-arborescence of G, then for every vertex $x\in V\backslash\{v_1\}$, let $p_T(x)$ be the immediate predecessor of x in T. Eliminating the arc $p_T(x)x$ will divide T into two subarborescences, with $T^{\mathrm{R}}(x)$ containing the root node and $T(x)$ containing the node x.

When there is an v_1-arborescence T, for any vertex x, we associate a nonnegative integral number $h_T(x)$, called the generational number of x relating to T, where $h_T(x)$ is defined as follows:

If x is the root of T, let $h_T(x)=0$.

If $h_T(p_T(x))$ has been defined, then define $h_T(x)=h_T(p_T(x))+1$.

Since an arborescence is a spanning tree, all the properties of the spanning tree are true to the arborescence. In particular, when an arc $xy\in A(G)\backslash A(T)$ is added to T, there will appear a cycle and only a cycle denoted by C. If $x\in T(y)$, then C is a circuit. Otherwise, in G there exists another v_1-arborescence $T'=T-\{p_T(x)x\}+\{xy\}$. This is an important property for the proof of our algorithm.

2. Basic structure of the solution set

We denote by $\Omega(G,T)$ the optimal solution set of (IWSA).

Theorem 1. *Let $w\in\Omega(G,T)$. Then, for every arc $xy\in A(G)$, $w_{xy}\leqslant c_{xy}$ if $xy\in A(T)$ and $w_{xy}\geqslant c_{xy}$ otherwise.*

Proof. Let us suppose by contradiction that there exists $xy\in A(T)$ so that $w_{xy}>c_{xy}$. We will show that $w\notin\Omega(G,T)$. For this purpose, we construct w' as follows:

$$w'_g=\begin{cases} c_{xy} & \text{if } g=xy,\\ w_g & \text{otherwise.}\end{cases}$$

Then, for every v_1-arborescence T' of G, we have

$$w'(T')=\begin{cases} w(T') & \text{if } xy\notin A(T'),\\ w(T')+c_{xy}-w_{xy} & \text{otherwise.}\end{cases}$$

Noting that $w(T') \geqslant w(T)$ and $c_{xy} < w_{xy}$, we conclude $w'(T') \geqslant w(T) + c_{xy} - w_{xy} = w'(T)$. Thus, T is a shortest v_1-arborescence in $G' = (V, A, w')$. Now,

$$\begin{aligned} \|w' - c\| &= \sum_{g \in A} |w'_g - c_g| \\ &= \sum_{g \in A} |w_g - c_g| - |w_{xy} - c_{xy}| \\ &< \|w - c\|, \end{aligned}$$

contrary to $w \in \Omega(G, T)$. Thus, for every arc $xy \in A(T)$, $w_{xy} \leqslant c_{xy}$. Similarly, $w_{xy} \geqslant c_{xy}$ if $xy \in A(G) \backslash A(T)$. □

As a consequence of Theorem 1, we can see that if the problem (IWSA) has an optimal solution then it is equivalent to

$$\min \|w - c\| = \sum_{g \in A(T)} (c_g - w_g) + \sum_{g \in A \backslash A(T)} (w_g - c_g) \tag{3}$$

subject to

$$0 \leqslant w_g \leqslant c_g, \quad g \in A(T), \tag{4}$$

$$w_g \geqslant c_g, \quad g \in A \backslash A(T) \tag{5}$$

and the shortest v_1-arborescence constraint, which may be expressed as a (possibly large) set of linear constraints of the type

$$\sum_{g \in A(T)} w_g \leqslant \sum_{g \in A(T')} w_g, \tag{6}$$

where T' is any v_1-arborescence of G. So, if $\Omega(G, T) \neq \emptyset$, the problem (IWSA) is a classical linear programming (LP) problem. This LP is, however, quite special because its constraint set is (potentially) very large, very structured, and possibly involves a great amount of redundancy.

In Section 3, we shall give a combinatorial algorithm which shows that there exists an optimal solution $w \in \Omega(G, T)$ such that $w_g = c_g$ for all $g \in A \backslash A(T)$. But, unfortunately, we cannot prove it in a simple way until Theorem 6 is proved. Here we reach a slightly weaker conclusion in this direction, which is essential to Theorem 6.

Theorem 2. *There exists an optimal solution $w \in \Omega(G, T)$ such that for all $y \neq v_1$*

$$w_{p_T(y)y}(w_{xy} - c_{xy}) = 0, \quad \forall xy \in A \backslash A(T). \tag{7}$$

Proof. We first prove $\Omega(G, T) \neq \emptyset$. Consider the LP defined by (3)–(6), which is bounded and has a feasible solution w defined by

$$w_g = \begin{cases} 0 & \text{if } g \in A(T), \\ c_g & \text{otherwise.} \end{cases}$$

So, by the duality theorem of linear programming, it has an optimal solution $w^{(0)}$. We prove $w^{(0)} \in \Omega(G,T)$. Suppose $w = w^{(1)}$ is a nonnegative cost which satisfies (6), we need only to show that $\|w^{(1)} - c\| \geqslant \|w^{(0)} - c\|$. Define

$$X = \{g\colon g \in A(T),\ w_g^{(1)} > c_g\},$$

$$Y = \{g\colon g \in A \backslash A(T),\ w_g^{(1)} < c_g\},$$

and

$$w_g^{(2)} = \begin{cases} c_g & \text{if } g \in X \cup Y, \\ w_g^{(1)} & \text{otherwise.} \end{cases}$$

Then, $w = w^{(2)}$ satisfies (4) and (5), and for every v_1-arborescence T',

$$w^{(2)}(T') = w^{(1)}(T') + \sum_{g \in A(T') \cap X} (c_g - w_g^{(1)}) + \sum_{g \in A(T') \cap Y} (c_g - w_g^{(1)}).$$

But

$$g \in X \ \Rightarrow\ c_g - w_g^{(1)} < 0,$$

$$g \in Y \ \Rightarrow\ c_g - w_g^{(1)} > 0,$$

and $w = w^{(1)}$ satisfies (6), so

$$w^{(2)}(T') \geqslant w^{(1)}(T) + \sum_{g \in X} (c_g - w_g^{(1)}) = w^{(2)}(T).$$

Thus, $w = w^{(2)}$ satisfies (4)–(6), which implies $\|w^{(2)} - c\| \geqslant \|w^{(0)} - c\|$. So

$$\begin{aligned} \|w^{(1)} - c\| &\geqslant \sum_{g \in A \backslash (X \cup Y)} |w_g^{(1)} - c_g| \\ &= \sum_{g \in A \backslash (X \cup Y)} |w_g^{(2)} - c_g| \\ &= \|w^{(2)} - c\| \\ &\geqslant \|w^{(0)} - c\|. \end{aligned}$$

Hence, indeed, $w^{(0)} \in \Omega(G,T)$, and

$$\Omega(G,T) = \{w\colon w \text{ satisfies (4)–(6), and } \|w - c\| = \|w^{(0)} - c\|\}.$$

Again, by the duality theorem of linear programming, we can pick an optimal solution $w \in \Omega(G,T)$ such that $\sum_{g \in A \backslash A(T)} (w_g - c_g)$ reaches the minimum. We claim that w satisfies (7). By way of contradiction, assume that there exists $xy \in A \backslash A(T)$ such that $y \neq v_1$ and $w_{p_T(y)y}(w_{xy} - c_{xy}) \neq 0$. Then by Theorem 1, since $w \geqslant 0$,

$\min\{w_{xy}-c_{xy}, w_{p_T(y)y}\}=s>0$. Define $w'_{xy}=w_{xy}-s$, $w'_{p_T(y)y}=w_{p_T(y)y}-s$, and $w'_g=w_g$ otherwise. Then, $w'\geqslant 0$ and $w'_{xy}-c_{xy}=w_{xy}-c_{xy}-s\geqslant 0$. This together with (4) gives

$$|w'_{xy}-c_{xy}|=w_{xy}-c_{xy}-s=|w_{xy}-c_{xy}|-s$$

and

$$|w'_{p_T(y)y}-c_{p_T(y)y}|=c_{p_T(y)y}-(w_{p_T(y)y}-s)=|w_{p_T(y)y}-c_{p_T(y)y}|+s,$$

so

$$\begin{aligned}\|w'-c\|&=\sum_{g\in A'}|w'_g-c_g|+|w'_{xy}-c_{xy}|+|w'_{p_T(y)y}-c_{p_T(y)y}|\\&=\sum_{g\in A'}|w_g-c_g|+|w_{xy}-c_{xy}|+|w_{p_T(y)y}-c_{p_T(y)y}|\\&=\|w-c\|,\end{aligned}$$

where $A'=A\setminus\{xy, p_T(y)y\}$. We prove T is also a shortest v_1-arborescence under weight w'. Suppose T' is any other v_1-arborescence, then $|A(T')\cap\{xy, p_T(y)y\}|\leqslant 1$, which implies $w'(T')\geqslant w(T')-s\geqslant w(T)-s=w'(T)$. Hence, $w'\in\Omega(G,T)$. By the definition of w, we see that

$$\sum_{g\in A\setminus A(T)}(w'_g-c_g)=\sum_{g\in A\setminus A(T)}(w_g-c_g)-s<\sum_{g\in A\setminus A(T)}(w_g-c_g),$$

contrary to the choice of w. □

3. An algorithm

We show the algorithm through the following example:

Given a weighted directed graph $G=(V,A,c)$ and a v_1-arborescence T of G, as indicated in Fig. 1, where the numbers in brackets are the costs of the arcs, the arcs in T are given by double lines, and the costs of v_3v_4 and v_4v_5 are positive integers.

To show the algorithm clearly and briefly, we assume that $i\geqslant 4$ and define

$$f(i)=\begin{cases} i & \text{if } 4\leqslant i\leqslant 6,\\ 6 & \text{if } i>6\end{cases}$$

(the cases $0\leqslant i<4$ can be discussed similarly).

Step 0: $G_0=(V_0,A_0,c^{(0)})=(V,A,c)$, $T_0=T$, $k=0$.

Step 1: For each $y\in V_k\setminus\{v_1\}$, find vertex $a_k(y)$ and $b_k(y)$ such that

$$c^{(k)}_{a_k(y)y}=\min\{c^{(k)}_{xy}\colon xy\in A_k,\ x\in T_k(y)\},$$

and

$$c^{(k)}_{b_k(y)y}=\min\{c^{(k)}_{xy}\colon xy\in A_k,\ x\in T^{\mathrm{R}}_k(y)\}.$$

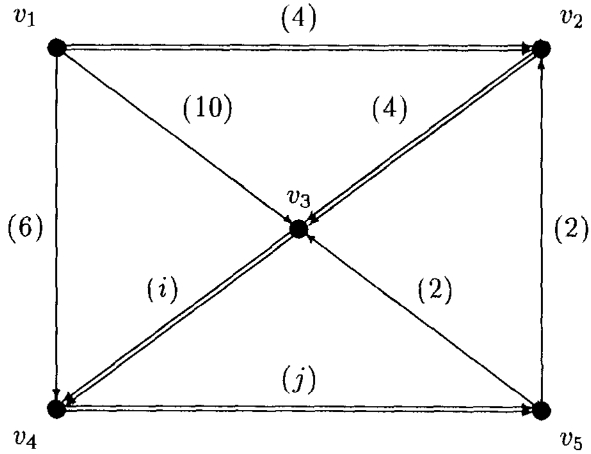

Fig. 1.

(if such vertices of $a_k(y)$ (resp. $b_k(y)$) are more than one, take one of them with $h_{T_k}(a_k(y))$ (resp. $h_{T_k}(b_k(y))$) biggest; if there exists no vertex $x \in T_k(y)$ such that $xy \in A_k$, take $a_k(y) = \emptyset$ and $c^{(k)}_{a_k(y)y} = w^{(k)}_{a_k(y)y} = +\infty$). Set

$$U(G_k) = \{y\colon\ y \in V_k \backslash \{v_1\},\ c^{(k)}_{a_k(y)y} < c^{(k)}_{b_k(y)y}\}.$$

Define the revised cost of G_k by

$$w^{(k)}_{xy} = \begin{cases} c^{(k)}_{b_k(y)y} & \text{if } x = p_{T_k}(y), \\ c^{(k)}_{xy} & \text{otherwise.} \end{cases}$$

In our example, $U(G_0) = \{v_2, v_3\}$. The revised cost of G_0 is given by $w^{(0)}_{v_3v_4} = c^{(0)}_{b_0(v_4)v_4} = \min\{c^{(0)}_{v_1v_4},\ c^{(0)}_{v_3v_4}\} = \min\{6, i\} = f(i)$ and $w^{(0)}_g = c_g$ otherwise.

Step 2: If $U(G_k) = \emptyset$, then $N \leftarrow k$, $l^{(k)}_g \leftarrow w^{(k)}_g$, $\forall g \in A(T_k)$. Goto Step 5. Otherwise, we take the vertex y_k such that

$$h_{T_k}(y_k) = \max\{h_{T_k}(u)\colon\ u \in U(G_k)\}$$

(if such vertices are more than one, take one of them arbitrarily). Let $x_k = a_k(y_k)$ and let C_k be the only circuit in graph $T_k \cup \{x_k y_k\}$. We contract $V(C_k)$ to a single vertex z_k and the new graph (resp. v_1-arborescence) obtained from G_k (resp. T_k) by such contraction is denoted by G_{k+1} (resp. T_{k+1}), in which the multiple arcs are replaced by a single arc and the loops are omitted.

In our example, $(y_0, x_0, C_0) = (v_3, v_5, v_3v_4v_5v_3)$, and the graphs G_1 and T_1 are shown by Fig. 2.

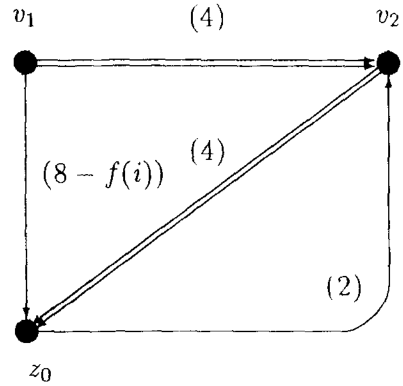

Fig. 2.

Step 3: We define the cost of G_{k+1} as follows:

$$c_{xy}^{(k-1)} = \begin{cases} w_{xy}^{(k)} & \text{if } x \neq z_k,\ y \neq z_k, \\ \min\{w_{ay}^{(k)}:\ a \in V(C_k),\ ay \in A_k\} & \text{if } x = z_k, \\ \min\{w_{xa}^{(k)} - w_{C_k(a)a}^{(k)} + w_{x_k y_k}^{(k)}:\ a \in V(C_k),\ xa \in A_k\} & \text{if } y = z_k. \end{cases}$$

where $C_k(a)$ is the predecessor of a in C_k.

In graph G_0, there exist two arcs v_1v_3 and v_1v_4 from v_1 to $C_0 = v_3v_4v_5v_3$. Hence in the contracted graph G_1 the cost of v_1z_0 is

$$\begin{aligned} c_{v_1z_0}^{(1)} &= \min\{w_{v_1v_3}^{(0)} - w_{v_5v_3}^{(0)} + w_{v_5v_3}^{(0)}, w_{v_1v_4}^{(0)} - w_{v_3v_4}^{(0)} + w_{v_5v_3}^{(0)}\} \\ &= \min\{10 - 2 + 2,\ 6 - f(i) + 2\} \\ &= 8 - f(i). \end{aligned}$$

Similarly, the cost of v_2z_0 is $w_{v_2v_3}^{(0)} - w_{v_5v_3}^{(0)} + w_{v_5v_3}^{(0)} = 4 - 2 + 2 = 4$. The complete cost of G_1 is shown in Fig. 2.

Step 4: $k \leftarrow k + 1$, $V_k \leftarrow V(G_k)$, $A_k \leftarrow A(G_k)$. Goto step 1.

In the example, when we execute the algorithm for $k = 1$, we have $U(G_1) = \{v_2\}$, and the revised cost of G_1 is given by $w_{v_2z_0}^{(1)} = \min\{c_{v_1z_0}^{(1)}, c_{v_2z_0}^{(1)}\} = \min\{8 - f(i), 4\} = 8 - f(i)$ and $w_g^{(1)} = c_g^{(1)}$ otherwise. Thus, $y_1 = v_2$, $x_1 = z_0$, and $C_1 = v_2z_0v_2$. When we contract $V(C_1)$ to a single vertex z_1, the arcs v_1v_2 and v_1z_0 become an arc from v_1 to z_1. So

$$\begin{aligned} c_{v_1z_1}^{(2)} &= \min\{w_{v_1v_2}^{(1)} - w_{z_0v_2}^{(1)} + w_{z_0v_2}^{(1)},\ w_{v_1z_0}^{(1)} - w_{v_2z_0}^{(1)} + w_{z_0v_2}^{(1)}\} \\ &= \min\{4 - 2 + 2,\ (8 - f(i)) - (8 - f(i)) + 2\} \\ &= 2. \end{aligned}$$

The complete cost of G_2 is shown in Fig. 3.

Executing the algorithm for $k = 2$, we have $w^{(2)} = c^{(2)}$ and $U(G_2) = \emptyset$. Hence, $N = 2$ and $l^{(2)} = w^{(2)}$. Then, we go to the following step.

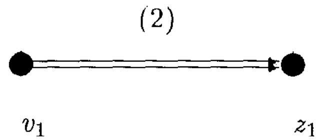

Fig. 3.

Step 5: If $k=0$, then set $l_g^{(0)}=c_g$, $\forall g\in A\backslash A(T)$. We obtain $l^{(0)}$ and the algorithm terminates. Otherwise, $k\leftarrow k-1$. We define $p_N(x)=p_{T_N(x)}$, $p_k(x)=p_{T_k(x)}$, and

$$l_{xy}^{(k)}=\begin{cases} l_{xy}^{(k+1)} & \text{if } x=p_k(y),\ \ y\in U(G_k)\backslash\{y_k\},\\ l_{p_{k+1}(z_k)z_k}^{(k+1)} & \text{if } xy=p_k(y_k)y_k,\\ w_{xy}^{(k)} & \text{if } x=p_k(y),\ \ y\notin U(G_k). \end{cases}$$

Goto Step 5.

In our example, we obtain $l^{(0)}$ as follows:

$$l_{v_1v_2}^{(0)}=l_{v_1v_2}^{(1)}=l_{v_1z_1}^{(2)}=w_{v_1z_1}^{(2)}=c_{v_1z_1}^{(2)}=2,$$

$$l_{v_2v_3}^{(0)}=l_{v_2z_0}^{(1)}=w_{v_2z_0}^{(1)}=8-f(i)=\begin{cases} 8-i & \text{if } 4\leqslant i\leqslant 6,\\ 2 & \text{if } i>6,\end{cases}$$

$$l_{v_3v_4}^{(0)}=w_{v_3v_4}^{(0)}=f(i)=\begin{cases} i & \text{if } 4\leqslant i\leqslant 6,\\ 6 & \text{if } i>6,\end{cases}$$

$$l_{v_4v_5}^{(0)}=w_{v_4v_5}^{(0)}=c_{v_4v_5}=j,$$

$$l_g^{(0)}=c_g \quad (g\in A\backslash\{v_1v_2,\ v_2v_3,\ v_3v_4,\ v_4v_5\}).$$

In the following, we want to prove $l^{(0)}\in\Omega(G,T)$, which shows our algorithm solves the (IWSA) correctly.

4. Proof of the algorithm

Let $k\in\{0,1,2,\ldots,N\}$. To prove the algorithm, we define

$$l_g^{(k)}=c_g^{(k)},\quad \forall g\in A_k\backslash A(T_k).$$

This together with Steps 1 and 5 implies

$$l_g^{(k)}=c_g^{(k)}=w_g^{(k)},\quad \forall g\in A_k\backslash A(T_k), \tag{8}$$

and

$$l_g^{(k)} = w_g^{(k)}, \quad \forall g \in A_k \setminus \{p_k(y)y\colon y \in U(G_k)\}. \tag{9}$$

Now we show some properties of the algorithm, the first three of which are trivial.

Property 1. *In each iteration* $k \geqslant 1$, $V_k = (V_{k-1} \setminus V(C_{k-1})) \cup \{z_{k-1}\}$. *Hence, the algorithm terminates at finite steps.*

Property 2. *In each iteration* k,

$$xy \in A_k \;\Rightarrow\; \min\{c_{vy}^{(k)}\colon vy \in A_k\} \leqslant w_{xy}^{(k)} \leqslant c_{xy}^{(k)}.$$

Property 3. *Let* $0 \leqslant k \leqslant N$. *If* $y \in V_k \setminus \{v_1\}$, *then,*

$$w_{a_k(y)y}^{(k)} = c_{a_k(y)y}^{(k)} = \min\{w_{xy}^{(k)}\colon xy \in A_k,\ x \in T_k(y)\},$$

and

$$w_{p_k(y)y}^{(k)} = c_{b_k(y)y}^{(k)} = w_{b_k(y)y}^{(k)} = \min\{w_{xy}^{(k)}\colon xy \in A_k,\ x \in T_k^{\mathrm{R}}(y)\}. \tag{10}$$

Hence,

$$\begin{aligned} U(G_k) &= \{y\colon y \in V_k \setminus \{v_1\},\ c_{a_k(y)y}^{(k)} < c_{b_k(y)y}^{(k)}\} \\ &= \{y\colon y \in V_k \setminus \{v_1\},\ w_{a_k(y)y}^{(k)} < w_{p_k(y)y}^{(k)}\}. \end{aligned} \tag{11}$$

Property 4. *Let* $0 \leqslant k < N$ *and* $y \in V_k \setminus (\{v_1\} \cup V(C_k))$. *Then,* $c_{b_{k+1}(y)y}^{(k+1)} = w_{b_k(y)y}^{(k)} = w_{p_k(y)y}^{(k)}$.

Proof. By the definition of $c^{(k+1)}$, we have for $xy \in A_{k+1}$

$$c_{xy}^{(k+1)} = \begin{cases} \min\{w_{uy}^{(k)}\colon u \in V(C_k),\ uy \in A_k\} & \text{if } x = z_k, \\ w_{xy}^{(k)} & \text{otherwise.} \end{cases} \tag{12}$$

But

$$z_k \in T_{k+1}^{\mathrm{R}}(y) \;\Rightarrow\; V(C_k) \subseteq T_k^{\mathrm{R}}(y),$$

and

$$x \in T_{k+1}^{\mathrm{R}}(y) \setminus \{z_k\} \;\Rightarrow\; x \in T_k^{\mathrm{R}}(y),$$

this together with (10) and (12) gives

$$xy \in A_{k+1},\ x \in T_{k+1}^{\mathrm{R}}(y) \;\Rightarrow\; c_{xy}^{(k+1)} \geqslant w_{b_k(y)y}^{(k)}. \tag{13}$$

Noting that the equality of (13) holds if either $x = b_k(y) \notin V(C_k)$ or $x = z_k$ and $b_k(y) \in V(C_k)$, we conclude that $c_{b_{k+1}(y)y}^{(k+1)} = w_{b_k(y)y}^{(k)} = w_{p_k(y)y}^{(k)}$. $\square$

Remark 1. By an argument similar to Property 4, we also have

$$y \in V_k \backslash (\{v_1\} \cup V(C_k)) \Rightarrow c^{(k+1)}_{a_{k+1}(y)y} = w^{(k)}_{a_k(y)y}, \tag{14}$$

and

$$y \in V_k \backslash (\{v_1\} \cup V(C_k)) \Rightarrow c^{(k+1)}_{p_{k+1}(y)y} = w^{(k)}_{p_k(y)y}. \tag{15}$$

This together with Properties 3 and 4 gives

$$U(G_{k+1}) \subseteq (U(G_k) \backslash \{y_k\}) \cup \{z_k\} \tag{16}$$

and

$$w^{(k+1)}_{p_{k+1}(y)y} = c^{(k+1)}_{b_{k+1}(y)y} = w^{(k)}_{b_k(y)y} = w^{(k)}_{p_k(y)y} = c^{(k+1)}_{p_{k+1}(y)y}$$

for $y \in V_k \backslash (\{v_1\} \cup V(C_k))$. This and the definition of $w^{(k+1)}$ imply that

$$c^{(k+1)}_g = w^{(k+1)}_g, \quad \forall g \in A_{k+1} \backslash \{p_{k+1}(z_k)z_k\} \tag{17}$$

Property 5. *Let* $0 \leqslant k \leqslant N-1$. *If* $y \in V(C_k)$, *then*

$$w^{(k)}_{C_k(y)y} = \min\{w^{(k)}_{xy}: xy \in A_k\}.$$

Proof. By the choice of y_k and the definition of $C_k(y)$, we have $V(C_k) \cap U(G_k) = y_k$ and

$$C_k(y) = \begin{cases} a_k(y_k) & \text{if } y = y_k, \\ p_k(y) & \text{if } y \in V(C_k) \backslash \{y_k\}. \end{cases}$$

This together with Property 3 gives

$$w^{(k)}_{C_k(y)y} = \min\{w^{(k)}_{a_k(y)y}, w^{(k)}_{p_k(y)y}\} = \min\{w^{(k)}_{xy}: xy \in A_k\}. \qquad \square$$

Property 6. *Let* $0 \leqslant k \leqslant N-1$. *If* $vy_k \in A_k$ *and* $v \notin V(C_k)$, *then*

$$w^{(k)}_{x_k y_k} \leqslant c^{(k+1)}_{vz_k} \leqslant w^{(k)}_{vy_k}.$$

Proof. Let $X = \{x: x \in V(C_k),\ vx \in A_k\}$. From the definition of $c^{(k+1)}$ and Property 5, it follows that

$$c^{(k+1)}_{vz_k} = \min_{x \in X}\{w^{(k)}_{vx} - w^{(k)}_{C_k(x)x} + w^{(k)}_{x_k y_k}\} \geqslant w^{(k)}_{x_k y_k}.$$

Similarly, $c^{(k+1)}_{vz_k} \leqslant w^{(k)}_{vy_k}$, since $y_k \in X$ and $C_k(y_k) = x_k$. $\square$

Remark 2. By similar argument as in Property 6, we also have

$$vz_k \in A_{k+1} \Rightarrow c^{(k+1)}_{vz_k} \geqslant w^{(k)}_{x_k y_k}. \tag{18}$$

Lemma 3. *Let* $0 \leqslant k \leqslant N$. *If* $y \in V_k \backslash \{v_1\}$, *then*

(a) $xy \in A_k \Rightarrow \min\{w^{(k)}_{vy}: vy \in A_k\} = \min\{c^{(k)}_{vy}: vy \in A_k\} \leqslant l^{(k)}_{xy} \leqslant w^{(k)}_{xy}$, *and if* $k < N$, *then*

(b) $y \in V(C_k) \Rightarrow l^{(k)}_{C_k(y)y} = w^{(k)}_{C_k(y)y} = \min\{l^{(k)}_{xy}\colon xy \in A_k\}$, *and*
(c) $g \in A_k \cap A_{k+1} \Rightarrow l^{(k+1)}_g = l^{(k)}_g$.

Proof. (a) Let us suppose by contradiction that we have chosen k and an associated xy such that this assertion fails and k is maximal. Then by Property 2, we have $\min\{w^{(k)}_{vy}\colon vy \in A_k\} = \min\{c^{(k)}_{vy}\colon vy \in A_k\}$, and $l^{(k)}_{xy} \neq w^{(k)}_{xy}$. This together with (9) implies $x = p_k(y)$ and $y \in U(G_k)$. Noting that $U(G_N) = \emptyset$, we have $k < N$.

Assume first $y = y_k$, then

$$l^{(k)}_{xy} = l^{(k)}_{p_k(y_k)y_k} = l^{(k+1)}_{p_{k+1}(z_k)z_k}. \tag{19}$$

By the choice of k, we see that

$$\min\{c^{(k+1)}_{vz_k}\colon vz_k \in A_{k+1}\} \leqslant l^{(k+1)}_{p_{k+1}(z_k)z_k} \leqslant w^{(k+1)}_{p_{k+1}(z_k)z_k}. \tag{20}$$

By Properties 2 and 6, we have

$$w^{(k+1)}_{p_{k+1}(z_k)z_k} \leqslant c^{(k+1)}_{p_{k+1}(z_k)z_k} = c^{(k+1)}_{p_k(y_k)z_k} \leqslant w^{(k)}_{p_k(y_k)y_k}.$$

This together with (18)–(20) gives

$$w^{(k)}_{x_k y_k} \leqslant \min\{c^{(k+1)}_{vz_k}\colon vz_k \in A_{k+1}\} \leqslant l^{(k)}_{p_k(y_k)y_k} \leqslant w^{(k+1)}_{p_{k+1}(z_k)z_k} \leqslant w^{(k)}_{p_k(y_k)y_k},$$

a contradiction.

Next, assume that $y \in U(G_k)\backslash\{y_k\}$, then $l^{(k)}_{p_k(y)y} = l^{(k+1)}_{p_k(y)y}$ and $p_k(y) = p_{k+1}(y)$. By the choice of k, we have

$$\min\{c^{(k+1)}_{vy}\colon vy \in A_{k+1}\} \leqslant l^{(k+1)}_{p_k(y)y} \leqslant w^{(k+1)}_{p_k(y)y}. \tag{21}$$

By Eqs. (12), (15), and (17), we get $w^{(k+1)}_{p_k(y)y} = c^{(k+1)}_{p_k(y)y} = c^{(k+1)}_{p_{k+1}(y)y} = w^{(k)}_{p_k(y)y}$ and

$$vy \in A_{k+1} \;\Rightarrow\; c^{(k+1)}_{vy} \geqslant \min\{w^{(k)}_{uy}\colon uy \in A_k\}.$$

This together with (21) and $l^{(k)}_{p_k(y)y} = l^{(k+1)}_{p_k(y)y}$ yields

$$\min\{w^{(k)}_{uy}\colon uy \in A_k\} \leqslant \min\{c^{(k+1)}_{vy}\colon vy \in A_{k+1}\} \leqslant l^{(k)}_{p_k(y)y} \leqslant w^{(k)}_{p_k(y)y},$$

a contradiction.

(b) By Property 5, we have $w^{(k)}_{C_k(y)y} = \min\{w^{(k)}_{vy}\colon vy \in A_k\}$. This and (a) imply (b).

(c) Let $g \in A_k \cap A_{k+1}$, then $g \notin \{p_k(y_k)y_k, p_{k+1}(z_k)z_k\}$. By the definition of $l^{(k)}$, we may assume that $g \notin \{p_k(y)y\colon y \in U(G_k)\}$ since otherwise $l^{(k)}_g = l^{(k+1)}_g$. By (16), since $\{p_k(y)y\colon y \in U(G_k)\backslash\{y_k\}\} \subseteq A(T^{\mathrm{R}}_k(y_k)) = A(T^{\mathrm{R}}_{k+1}(z_k))$, we have

$$g \notin \{p_k(y)y\colon y \in U(G_k)\} \cup \{p_{k+1}(y)y\colon y \in U(G_{k+1})\}.$$

This together with (9), (17) and the definition of $c^{(k+1)}$ gives

$$l^{(k)}_g = w^{(k)}_g = c^{(k+1)}_g = w^{(k+1)}_g = l^{(k+1)}_g. \qquad \square$$

Remark 3. From Remark 2 and Step 3, it follows that

$$c_{xy}^{(k+1)} \geqslant \begin{cases} \min\{w_{vy}^{(k)}: vy \in A_k\} & \text{if } y \neq z_k, \\ w_{x_k y_k}^{(k)} & \text{otherwise.} \end{cases}$$

This together with Property 2 and Lemma 3(a) gives

$$c^{(k)} \geqslant 0 \;\Rightarrow\; c^{(k)} \geqslant w^{(k)} \geqslant l^{(k)} \geqslant 0 \;\Rightarrow\; c^{(k+1)} \geqslant 0.$$

Therefore,

$$c^{(k)} \geqslant w^{(k)} \geqslant l^{(k)} \geqslant 0, \quad \forall k = 0, 1, \ldots, N.$$

Lemma 4. *For $0 \leqslant k \leqslant N$, define $G_k^* = (V_k, A_k, l^{(k)})$. Then, T_k is a shortest v_1-arborescence of G_k^*.*

Proof. We prove this assertion by induction. First we assume that $k = N$, then by (8) and Step 2 we have $l^{(N)} = w^{(N)}$. By our algorithm, $U(G_N) = \emptyset$. This and Property 3 imply that for all $y \in V_N \setminus \{v_1\}$, $\min\{w_{xy}^{(N)}: xy \in A_N\} = \min\{w_{a_N(y)y}^{(N)}, w_{p_N(y)y}^{(N)}\} = w_{p_N(y)y}^{(N)}$. Thus, T_N is a shortest v_1-arborescence of G_N^*.

Assume $k < N$, and the assertion holds for $k+1$, we shall show that it also holds for k. For this purpose, we let T_k' be a shortest v_1-arborescence of G_k^*, and choose $a \in V(C_k)$ so that $h_{T_k'}(a)$ is minimal. For $y \in V_k \setminus \{v_1\}$, let $p'(y) = p_{T_k'}(y)$. Define $J = \{p'(y)y: y \in V(C_k) \setminus \{a\},\ p'(y) \neq C_k(y)\}$. By replacing every arc $p'(y)y$ of J with $C_k(y)y$, we obtain a v_1-arborescence T_k^* of G_k^* which satisfies $A(C_k) \setminus \{C_k(a)a\} \subseteq A(T_k^*)$. By Lemma 3(b), we have

$$l^{(k)}(T_k') - l^{(k)}(T_k^*) = \sum_{p'(y)y \in J} (l_{p'(y)y}^{(k)} - l_{C_k(y)y}^{(k)}) \geqslant 0,$$

thus, $l^{(k)}(T_k') = l^{(k)}(T_k^*)$.

Let T_{k+1}^* be the graph obtained from T_k^* by shrinking $V(C_k)$ to a single vertex z_k. Then, since $A(C_k) \setminus \{C_k(a)a\} \subseteq A(T_k^*)$, T_{k+1}^* is a v_1-arborescence of G_{k+1}^*. Thus,

$$l^{(k+1)}(T_{k+1}) \leqslant l^{(k+1)}(T_{k+1}^*). \tag{22}$$

Let $X = \{x: x \notin V(C_k),\ p'(x) \in V(C_k)\}$. From Lemma 3(c), it follows that

$$\begin{aligned} l^{(k)}(T_k^*) = {} & l^{(k+1)}(T_{k+1}^*) + l_{p'(a)a}^{(k)} - l_{p'(a)z_k}^{(k+1)} + l^{(k)}(C_k) \\ & - l_{C_k(a)a}^{(k)} + \sum_{x \in X} \{l_{p'(x)x}^{(k)} - l_{z_k x}^{(k+1)}\}. \end{aligned} \tag{23}$$

By the choice of y_k, we see that $U(G_k) \setminus \{y_k\} \subseteq V(T_k^{\mathrm{R}}(y_k)) = V(T_{k+1}^{\mathrm{R}}(z_k))$. This together with (16) gives

$$x \in X,\ p'(x) = p_k(x) \;\Rightarrow\; z_k = p_{k+1}(x) \;\Rightarrow\; x \notin U(G_k) \cup U(G_{k+1}). \tag{24}$$

By (9), (17), (24) and the definition of $c^{(k+1)}$, we have

$$x \in X \;\Rightarrow\; l^{(k)}_{p'(x)x} - l^{(k+1)}_{z_k x} = w^{(k)}_{p'(x)x} - w^{(k+1)}_{z_k x} = w^{(k)}_{p'(x)x} - c^{(k+1)}_{z_k x} \geqslant 0. \tag{25}$$

Combining this with (23), we see that

$$l^{(k)}(T_k^*) \geqslant l^{(k+1)}(T_{k+1}^*) + l^{(k)}_{p'(a)a} - l^{(k+1)}_{p'(a)z_k} + l^{(k)}(C_k) - l^{(k)}_{C_k(a)a}. \tag{26}$$

By (15), (17), (24), and (25), we also have

$$x \in X,\; p'(x) = p_k(x) \in V(C_k) \;\Rightarrow\; l^{(k)}_{p_k(x)x} - l^{(k+1)}_{z_k x} = w^{(k)}_{p_k(x)x} - c^{(k+1)}_{p_{k+1}(x)x} = 0.$$

This together with an argument similar to (26) gives

$$\begin{aligned} l^{(k)}(T_k) &= l^{(k+1)}(T_{k+1}) + l^{(k)}_{p_k(y_k)y_k} - l^{(k+1)}_{p_k(y_k)z_k} + l^{(k)}(C_k) - l^{(k)}_{C_k(y_k)y_k} \\ &= l^{(k+1)}(T_{k+1}) + l^{(k)}(C_k) - l^{(k)}_{C_k(y_k)y_k}. \end{aligned}$$

Combining this with (22) and (26), we get

$$l^{(k)}(T_k^*) - l^{(k)}(T_k) \geqslant l^{(k)}_{p'(a)a} - l^{(k+1)}_{p'(a)z_k} - l^{(k)}_{C_k(a)a} + l^{(k)}_{C_k(y_k)y_k}. \tag{27}$$

We now wish to show $l^{(k)}(T_k^*) \geqslant l^{(k)}(T_k)$. This is clear if $p'(a)a = p_k(y_k)y_k$; otherwise, $a \in V(C_k)\setminus\{y_k\}$ or $p'(a) \neq p_k(a)$. From the choice of y_k, it follows that either $a \notin U(G_k)$ or $p'(a) \neq p_k(a)$. This and (9) imply

$$l^{(k)}_{p'(a)a} = w^{(k)}_{p'(a)a}. \tag{28}$$

By Remark 3, we have $l^{(k+1)}_{p'(a)z_k} \leqslant w^{(k+1)}_{p'(a)z_k} \leqslant c^{(k+1)}_{p'(a)z_k}$. This together with (27), (28), and Lemma 3(b) gives

$$l^{(k)}(T_k^*) - l^{(k)}(T_k) \geqslant w^{(k)}_{p'(a)a} - c^{(k+1)}_{p'(a)z_k} - w^{(k)}_{C_k(a)a} + w^{(k)}_{C_k(y_k)y_k}.$$

Here $l^{(k)}(T_k^*) \geqslant l^{(k)}(T_k)$ follows since $C_k(y_k) = x_k$ and

$$c^{(k+1)}_{p'(a)z_k} = \min\{w^{(k)}_{p'(a)x} - w^{(k)}_{C_k(x)x} + w^{(k)}_{x_k y_k} : x \in V(C_k),\; p'(a)x \in A_k\}.$$

Hence, indeed, $l^{(k)}(T_k^*) \geqslant l^{(k)}(T_k)$, implying that $l^{(k)}(T_k) = l^{(k)}(T_k^*) = l^{(k)}(T_k')$. □

For $0 \leqslant k \leqslant N$, define

$$\alpha_k = \begin{cases} w^{(k)}_{p_k(y_k)y_k} - c^{(k+1)}_{p_{k+1}(z_k)z_k} & \text{if } k < N, \\ 0 & \text{otherwise,} \end{cases}$$

$$X_k = \{x\colon x \in V_k\setminus\{v_1\},\; w^{(k)}_{p_k(x)x} < c^{(k)}_{p_k(x)x}\}, \tag{29}$$

and if $X_k \neq \emptyset$, then for each $x \in X_k$, let

$$\beta_{k,x} = c^{(k)}_{p_k(x)x} - w^{(k)}_{p_k(x)x}.$$

By Step 1, we have

$$\|c^{(k)} - w^{(k)}\| = \sum_{x \in X_k} \beta_{k,x}. \tag{30}$$

Assume now $k<N$, then $p_k(y_k) = p_{k+1}(z_k)$. From Lemma 3(c) and the definition of $c^{(k+1)}$, it follows that

$$l_g^{(k+1)} - c_g^{(k+1)} = l_g^{(k)} - w_g^{(k)}, \quad \forall g \in A_k \cap A_{k+1}. \tag{31}$$

By $\{p_k(y)y\colon\ y \in U(G_k)\backslash\{y_k\}\} \subseteq A(T_k^{\mathrm{R}}(y_k)) \subseteq A_k \cap A_{k+1}$, we have from (9)

$$l_g^{(k)} \neq w_g^{(k)}, \quad g \in A_k \backslash A_{k+1} \ \Rightarrow\ g = p_k(y_k)y_k. \tag{32}$$

Similarly, we have from (9) and (16)

$$l_g^{(k+1)} \neq w_g^{(k+1)}, \quad g \in A_{k+1} \backslash A_k \ \Rightarrow\ g = p_{k+1}(z_k)z_k.$$

This together with (17), (31), and (32) gives

$$\begin{aligned}
&\|l^{(k)} - w^{(k)}\| - \|l^{(k+1)} - c^{(k+1)}\| \\
&\quad = \sum_{g \in A_k \backslash A_{k+1}} |l_g^{(k)} - w_g^{(k)}| - \sum_{g \in A_{k+1} \backslash A_k} |l_g^{(k+1)} - c_g^{(k+1)}| \\
&\quad = |l^{(k)}_{p_k(y_k)y_k} - w^{(k)}_{p_k(y_k)y_k}| - |l^{(k+1)}_{p_{k+1}(z_k)z_k} - c^{(k+1)}_{p_{k-1}(z_k)z_k}| \\
&\quad = \alpha_k.
\end{aligned}$$

But by Remark 3, we have $l^{(k)} \leqslant w^{(k)} \leqslant c^{(k)}$, therefore

$$\begin{aligned}
\|l^{(k)} - c^{(k)}\| &= \|l^{(k)} - w^{(k)}\| + \|w^{(k)} - c^{(k)}\| \\
&= \|l^{(k+1)} - c^{(k+1)}\| + \alpha_k + \sum_{x \in X_k} \beta_{k,x}.
\end{aligned} \tag{33}$$

Lemma 5. *Let $0 \leqslant k \leqslant N$. Then, there exist an integer $q_k > 0$ and a sequence of v_1-arborescence $T_k^{(1)}, T_k^{(2)}, \ldots, T_k^{(q_k)}$ of G_k so that for all $1 \leqslant i \neq j \leqslant q_k$,*
(a) $A(T_k) \subseteq A(T_k^{(i)}) \cup A(T_k^{(j)})$.
(b) $\sum_{i=1}^{q_k} (c^{(k)}(T_k) - c^{(k)}(T_k^{(i)})) = \|l^{(k)} - c^{(k)}\|$.

Proof. We shall prove this lemma by induction. For convenience, first we give some constructions of the v_1-arborescence.

Construction 1. Construct $T_k^{[x]}$ from T_k. Let $x \in X_k$, then $w^{(k)}_{p_k(x)x} = c^{(k)}_{b_k(x)x} < c^{(k)}_{p_k(x)x}$. Define $T_k^{[x]} = T_k + \{b_k(x)x\} - \{p_k(x)x\}$.

Construction 2. Construct T_k^* from T_{k+1}^*. Let T_{k+1}^* be a v_1-arborescence of G_{k+1} and let x be the immediate predecessor of z_k in T_{k+1}^*, and then choose $a \in V(C_k)$ so that

$xa \in A_k$ and

$$c_{xz_k}^{(k+1)} = w_{xa}^{(k)} - w_{C_k(a)a}^{(k)} + w_{x_k y_k}^{(k)}. \tag{34}$$

Construct T_k^* as follows: the vertex set of T_k^* is V_k; the arc set of T_k^* is that of T_{k+1}^* with xz_k omitted, with all arcs of $(A(C_k)\setminus\{C_k(a)a\}) \cup \{xa\}$ added to replace xz_k, and with all arcs $z_k y$ in T_{k+1}^* replaced by $t(y)y$, where $t(y)$ is chosen so that $t(y) \in V(C_k)$, $t(y)y \in A_k$, and $w_{t(y)y}^{(k)} = \min\{w_{ty}^{(k)}: t \in V(C_k), ty \in A_k\} = c_{z_k y}^{(k+1)}$. Then by (15), we may assume that $t(y) = p_k(y)$ when $z_k = p_{k+1}(y)$.

Construction 3. Construct $T_k^{(i)}$ from $T_{k-1}^{(i)}$. This construction occurs when $T_{k+1}^{(i)}$ is a v_1-arborescence of G_{k+1} and $p_{k+1}(z_k)z_k \in A(T_{k-1}^{(i)})$. In this case, we replace xa with $p_k(y_k)y_k$, and construct $T_k^{(i)}$ in a way similar to Construction 2.

It is easy to see that all the graphs $T_k^{[x]}$, T_k^*, $T_k^{(i)}$ are v_1-arborescences of G_{k-1}, and

$$c^{(k)}(T_k) - c^{(k)}(T_k^{[x]}) = \beta_{k,x}. \tag{35}$$

Noting that $c_g^{(k+1)} = w_g^{(k)}$, $\forall g \in A_k \cap A_{k+1}$, we have from Construction 2

$$w^{(k)}(T_k^*) = c^{(k+1)}(T_{k+1}^*) - c_{xz_k}^{(k+1)} + w_{xa}^{(k)} + w^{(k)}(C_k) - w_{C_k(a)a}^{(k)}.$$

This together with (34) gives

$$w^{(k)}(T_k^*) = c^{(k+1)}(T_{k+1}^*) + w^{(k)}(C_k) - w_{x_k y_k}^{(k)}. \tag{36}$$

By $p_{k+1}(z_k) = p_k(y_k)$ and $C_k(y_k) = x_k$, we also have

$$\begin{aligned} w^{(k)}(T_k^{(i)}) &= c^{(k+1)}(T_{k+1}^{(i)}) - c_{p_{k+1}(z_k)z_k}^{(k+1)} + w_{p_k(y_k)y_k}^{(k)} + w^{(k)}(C_k) - w_{C_k(y_k)y_k}^{(k)} \\ &= c^{(k+1)}(T_{k+1}^{(i)}) + \alpha_k + w^{(k)}(C_k) - w_{x_k y_k}^{(k)}. \end{aligned} \tag{37}$$

Noting that T_k can be constructed from T_{k+1} by Construction 3, we conclude that (37) also holds when $T_k^{(i)} = T_k$ and $T_{k+1}^{(i)} = T_{k+1}$. This together with (36) and (37) gives

$$w^{(k)}(T_k) - w^{(k)}(T_k^*) = c^{(k+1)}(T_{k+1}) - c^{(k+1)}(T_{k+1}^*) + \alpha_k, \tag{38}$$

and

$$w^{(k)}(T_k) - w^{(k)}(T_k^{(i)}) = c^{(k+1)}(T_{k+1}) - c^{(k+1)}(T_{k+1}^{(i)}). \tag{39}$$

With observations above, we now begin the proof of Lemma 5. Assume first $k = N$, then since $U(G_N) = \emptyset$, by (9) we have $l^{(N)} = w^{(N)}$. By (30) and (35), we see that $\{T_N^{[x]}\}_{x \in X_N}$ satisfies Lemma 5 (if $X_N = \emptyset$, then we let $q_N = 1$, and $T_N^{(1)} = T_N$).

Assume now $0 \leqslant k < N$ and the assertion holds for $k+1$. We shall show that it also holds for k. Let $\{T_{k+1}^{(i)}\}_{i=1}^{q_{k+1}}$ be a sequence of v_1-arborescence of G_{k+1} which meets Lemma 5, then

$$\sum_{i=1}^{q_{k+1}} (c^{(k+1)}(T_{k+1}) - c^{(k+1)}(T_{k+1}^{(i)})) = \|l^{(k+1)} - c^{(k+1)}\| \tag{40}$$

and

$$1 \leqslant i \neq j \leqslant q_{k+1} \;\Rightarrow\; A(T_{k+1}) \subseteq A(T_{k+1}^{(i)}) \cup A(T_{k+1}^{(j)}), \tag{41}$$

which yields

$$|D_k| = |\{i:\ 1 \leqslant i \leqslant q_{k+1},\ p_{k+1}(z_k)z_k \notin A(T_{k+1}^{(i)})\}| \leqslant 1.$$

Without loss of generality, assume that $D_k \subseteq \{1\}$. Let $T_k^{(1)}$ (resp. $T_k^{(j)}$, $2 \leqslant j \leqslant q_{k+1}$) be the graph obtained from $T_{k+1}^{(1)}$ (resp. $T_{k+1}^{(j)}$) by Construction 2 (resp. Construction 3). From (29), (38), (39), and the definition of $w^{(k)}$, it follows that

$$\begin{aligned} & c^{(k)}(T_k) - c^{(k)}(T_k^{(i)}) \\ &\quad = w^{(k)}(T_k) - w^{(k)}(T_k^{(i)}) + \sum_{x \in X_k^{(i)}} \beta_{k,x} \\ &\quad = c^{(k+1)}(T_{k+1}) - c^{(k+1)}(T_{k+1}^{(i)}) + \alpha_k \delta_{1,i} + \sum_{x \in X_k^{(i)}} \beta_{k,x}, \end{aligned} \tag{42}$$

where $X_k^{(i)} = \{x:\ x \in X_k,\ p_k(x)x \notin A(T_k^{(i)})\}$ and

$$\delta_{1,i} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We now claim that

$$1 \leqslant i \neq j \leqslant q_{k+1} \;\Rightarrow\; A(T_k) \subseteq A(T_k^{(i)}) \cup A(T_k^{(j)}). \tag{43}$$

Suppose not, then there exists $y \in V_k \backslash \{v_1\}$ so that $p_k(y)y \notin A(T_k^{(i)}) \cup A(T_k^{(j)})$. By Construction 3, $(A(C_k) \backslash \{x_k y_k\}) \cup \{p_k(y_k)y_k\} \subseteq \bigcap_{r=2}^{q_{k+1}} A(T_k^{(r)})$ implying that $y \in V_k \backslash (\{v_1\} \cup V(C_k)) = V_{k+1} \backslash \{v_1, z_k\}$. By (41), we may assume that $p_{k+1}(y)y \in A(T_{k+1}^{(i)})$. This together with $p_k(y)y \notin A(T_k^{(i)})$ implies $p_{k+1}(y) = z_k$. By our construction, we have $t(y)y = p_k(y)y \in A(T_k^{(i)})$, a contradiction.

Let $Y_k = \bigcup_{i=1}^{q_{k+1}} X_k^{(i)}$. By (43), we have $X_k^{(i)} \cap X_k^{(j)} = \emptyset$, $\forall i \neq j$. This together with (40) and (42) gives

$$\begin{aligned} & \sum_{i=1}^{q_{k+1}} \{c^{(k)}(T_k) - c^{(k)}(T_k^{(i)})\} \\ &\quad = \sum_{i=1}^{q_{k+1}} \{c^{(k+1)}(T_{k+1}) - c^{(k+1)}(T_{k+1}^{(i)})\} + \alpha_k + \sum_{i=1}^{q_{k+1}} \left\{ \sum_{x \in X_k^{(i)}} \beta_{k,x} \right\} \\ &\quad = \|l^{(k+1)} - c^{(k+1)}\| + \alpha_k + \sum_{x \in Y_k} \beta_{k,x}. \end{aligned} \tag{44}$$

Noting $X_k \backslash Y_k = \{x:\ x \in X_k,\ p_k(x)x \in \bigcap_{i=1}^{q_{k+1}} A(T_k^{(i)})\}$ and using (33), (35), (43), and (44), we conclude that $\{T_k^{(i)}\}_{i=1}^{q_{k+1}}$ together with $\{T_k^{[x]}\}_{x \in X_k \backslash Y_k}$ satisfies Lemma 5. □

Theorem 6. *For all* $k=0,1,\ldots,N$, $l^{(k)} \in \Omega(G_k, T_k)$.

Proof. By Lemma 4, T_k is a shortest v_1-arborescence of $G_k^* = (V_k, A_k, l^{(k)})$. As $\Omega(G_k, T_k) \neq \emptyset$, we need only to show

$$l^* \in \Omega(G_k, T_k) \;\Rightarrow\; \|l^* - c^{(k)}\| \geqslant \|l^{(k)} - c^{(k)}\|. \tag{45}$$

To deduce (45), we let $\{T_k^{(i)}\}_{i=1}^{q_k}$ be a sequence of v_1-arborescence of G_k which meets Lemma 5. Then,

$$\sum_{i=1}^{q_k} \{c^{(k)}(T_k) - c^{(k)}(T_k^{(i)})\} = \|l^{(k)} - c^{(k)}\|. \tag{46}$$

Define

$$\bigcup_{i=1}^{q_k} (A(T_k)\backslash A(T_k^{(i)})) = \{e_1, e_2, \ldots, e_p\}, \tag{47}$$

$$\bigcup_{i=1}^{q_k} (A(T_k^{(i)})\backslash A(T_k)) = \{f_1, f_2, \ldots, f_{p'}\}, \tag{48}$$

and for $1 \leqslant t \leqslant p$, let

$$D_t = \{i\colon\ 1 \leqslant i \leqslant q_k,\ e_t \notin A(T_k^{(i)})\}.$$

Then, $|D_t| = 1$; otherwise, there exists $i, j \in D_t$ with $i \neq j$, implying that $e_t \in A(T_k)\backslash (A(T_k^{(i)}) \cup A(T_k^{(j)}))$, which is contrary to Lemma 5(a). Hence, indeed, for each arc e_t, there exists only one entry i so that $1 \leqslant i \leqslant q_k$ and $e_t \in A(T_k)\backslash A(T_k^{(i)})$. Thus,

$$\sum_{i=1}^{q_k} c^{(k)}(A(T_k)\backslash A(T_k^{(i)})) = c^{(k)}\left(\bigcup_{i=1}^{q_k} (A(T_k)\backslash A(T_k^{(i)})\right) = \sum_{t=1}^{p} c_{e_t}^{(k)}. \tag{49}$$

Noting that $f_t = xy \in A(T_k^{(i)})$ implies $p_k(y)y \notin A(T_k^{(i)})$, we also have

$$|\{i\colon\ 1 \leqslant i \leqslant q_k,\ f_t \in A(T_k^{(i)})\}| = 1, \quad \forall t = 1, 2, \ldots, p'.$$

This together with $|D_t| = 1$ and (47)–(49) gives $p = p'$ and

$$\sum_{i=1}^{q_k} c^{(k)}(A(T_k^{(i)})\backslash A(T_k)) = c^{(k)}\left(\bigcup_{i=1}^{q_k} (A(T_k^{(i)})\backslash A(T_k)\right) = \sum_{t=1}^{p} c_{f_t}^{(k)}.$$

Combining this with (49), we get

$$\sum_{i=1}^{q_k} \{c^{(k)}(T_k) - c^{(k)}(T_k^{(i)})\} = \sum_{t=1}^{p} c_{e_t}^{(k)} - \sum_{t=1}^{p} c_{f_t}^{(k)}. \tag{50}$$

Let $l^* \in \Omega(G_k, T_k)$, then

$$l^*(T_k) \leqslant l^*(T_k^{(i)}), \quad \forall i = 1, 2, \ldots, q_k. \tag{51}$$

In a way similar to (50), we have

$$\sum_{i=1}^{q_k}(l^*(T_k)-l^*(T_k^{(i)}))=\sum_{t=1}^{p} l^*_{e_t}-\sum_{t=1}^{p} l^*_{f_t}.$$

This together with (51) yields

$$\sum_{t=1}^{p} l^*_{e_t}\leqslant\sum_{t=1}^{p} l^*_{f_t}. \tag{52}$$

By (46), (50), (52), and Theorem 1, we conclude that

$$\begin{aligned}\|l^*-c^{(k)}\| &= \sum_{g\in A(T_k)}(c_g^{(k)}-l_g^*)+\sum_{g\in A_k\setminus A(T_k)}(l_g^*-c_g^{(k)})\\ &\geqslant \sum_{t=1}^{p}(c_{e_t}^{(k)}-l^*_{e_t})+\sum_{t=1}^{p}(l^*_{f_t}-c^{(k)}_{f_t})\\ &= \left(\sum_{t=1}^{p}c^{(k)}_{e_t}-\sum_{t=1}^{p}c^{(k)}_{f_t}\right)+\left(\sum_{t=1}^{p}l^*_{f_t}-\sum_{t=1}^{p}l^*_{e_t}\right)\\ &\geqslant \|l^{(k)}-c^{(k)}\|. \quad\square\end{aligned}$$

Noting that in our algorithm the complexity of iteration k is $O(|V_k|^2)$, we have

Theorem 7. *The algorithm in Section 3 is correct and its complexity is* $O(|V(G)|^3)$.

5. An application

Consider the following inverse problem of the weighted shortest path problem:

(IWSP) Given a weighted directed graph $G=(V,A,c)$, vertices $v_1,v_p\in V$, and a path P from v_1 to v_p, find w, a new set of nonnegative costs, such that

$$\min\sum_{g\in A}|w_g-c_g|$$

is achieved under the constraint that P is a shortest (v_1,v_p)-path in $G'=(V,A,w)$.

We denote by $\Omega(G,P)$ the optimal solution set of (IWSP). To derive an optimal solution of (IWSP) from (IWST), we define $B=\{v_pv\colon v\in V\setminus\{v_p\}\}$, and construct the graph $G^*=(V,A^*,c^*)$ so that $A^*=A\cup B$ and

$$c_g^*=\begin{cases}0 & \text{if } g\in B,\\ c_g & \text{otherwise.}\end{cases} \tag{53}$$

Let T be the graph obtained from P by adding all nodes of $V\setminus V(P)$ and all arcs from v_p to $V\setminus V(P)$. Then, T is a v_1-arborescence of G^*.

By our algorithm in Section 3, there exists $l^*\in\Omega(G^*,T)$ so that

$$0\leqslant l_g^*\leqslant c_g^* \quad (g\in A(T))$$

and

$$l_g^* = c_g^* \quad (g \in A^* \backslash A(T)).$$

This together with (53) gives

$$l_g^* = 0 = c_g^*, \quad \forall g \in B. \tag{54}$$

Define

$$l_g = \begin{cases} c_g & \text{if } g \in B, \\ l_g^* & \text{otherwise.} \end{cases} \tag{55}$$

Then by (53)–(55), we have

$$\begin{aligned} \|l^* - c^*\| &= \sum_{g \in A \backslash B} |l_g^* - c_g^*| + \sum_{g \in B} |l_g^* - c_g^*| \\ &= \sum_{g \in A \backslash B} |l_g - c_g| \\ &= \sum_{g \in A \backslash B} |l_g - c_g| + \sum_{g \in A \cap B} |l_g - c_g| \\ &= \|l - c\|. \end{aligned} \tag{56}$$

Theorem 8. $l \in \Omega(G, P)$.

Proof. Let $\Omega_1 = \{w$: P is a shortest (v_1, v_p)-path in $G[w] = (V, A, w)\}$, and $\Omega_2 = \{w^*$: T is a shortest v_1-arborescence in $G^*[w^*] = (V, A^*, w^*)\}$. We first show that $l \in \Omega_1$. For this purpose, we assume that P' is a (v_1, v_p)-path in G, and construct the v_1-arborscence T' of G^* in a way similar to T. Then, since $l^* \in \Omega_2$, $l(P) = l^*(T) \leqslant l^*(T') = l(P')$, which implies $l \in \Omega_1$.

Assume now $w \in \Omega(G, P)$, and define w^* in a way similar to c^*. Then, by (53), we have

$$\|w^* - c^*\| = \sum_{g \in A \backslash B} |w_g - c_g| \leqslant \|w - c\|. \tag{57}$$

To deduce $w^* \in \Omega_2$, we let T' be a v_1-arborescence of G^* and P' the only (v_1, v_p)-path of T'. Then, $A(P') \cap B = \emptyset$, which implies P' is also a (v_1, v_p)-path of G. This together with $w \in \Omega(G, P)$ gives

$$w^*(T) = w(P) \leqslant w(P') = w^*(P') \leqslant w^*(T').$$

Therefore, $w^* \in \Omega_2$. As $l^* \in \Omega(G^*, T)$,

$$\|l^* - c^*\| \leqslant \|w^* - c^*\|. \tag{58}$$

Combining (56)–(58), we obtain $\|l - c\| \leqslant \|w - c\|$. This together with $l \in \Omega_1$ and $w \in \Omega(G, P)$ implies $l \in \Omega(G, P)$. $\square$

Acknowledgements

We wish to thank the referees for their many helpful suggestions and comments. Especially, we thank the referee who suggested the use of Theorem 2.

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