Graph Theory in Primary, Middle, and High School

Daniela Ferrarello and Maria Flavia Mammana

Abstract In this paper we present an experimental teaching activity conduced in some primary, middle and high schools in Sicily. The activity concerned several topics of graph theory. Here we highlight, in particular, the approach to teaching Eulerian graphs. The aim of the whole project was to present a fun, easy approach to mathematics in order to promote a good attitude towards mathematics in primary school children and to improve it in middle school kids and in high school young people. This goal is pursued also by showing some connections of mathematics with real life, making mathematics less abstract than the topics too often taught in school. Through this activity we also reach mathematical knowledge and practical abilities (related to graph theory), and above all mathematical competencies related to reasoning and mathematization, in particular by the use of graphs in mathematical models to solve problems. The teaching experiments were different, according to the different school level, but unified by the method, based on laboratorial activities, by presenting a problem to be solved together with classmates, by manipulating objects and guided by the teacher. These activities were realized by the use of artefacts: in the sense of Vygotskijan semiotic mediation, we used signs, symbols, maps, language and, in many cases, new technology's artefacts, to mediate mathematical concepts. Lessons involved also the body as a mean to learning, especially with children, according to embodied cognition theory.

Keywords Graph theory · Mathematical modelling · Eulerian trail
Eulerian cycle

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1 Introduction

The capability of using mathematical knowledge in solving real life problems is widely tested in various national and international tests (INVALSI, PISA, TIMSS) and, since last year, in the Italian final mathematics' exam, at the end of high school. But, according to the results obtained, students have difficulties in solving these kinds of problems.

Mathematics is more often seen as the *subject of numbers and rules* that is hard to understand and difficult to study: lots of students see it as something of no use and cannot see the connection with everyday life. The study of *real life problems* is becoming central in teaching at all levels:

> The National Council of Teachers of Mathematics (NCTM) is providing leadership in communicating to teachers, students, and parents what mathematical modeling looks like in K–12 levels. The 2015 Focus issue of NCTM's Mathematics Teaching in the Middle School was about mathematical modeling and the 2016 Annual Perspectives in Mathematics Education also focused on the topic (Levy 2015).

The idea is, given a real life problem, to translate it into a math problem, solve it with mathematical knowledge, and interpret the solution in terms of the given problem.

In this context graph theory is a good tool for modeling problems. Even if it is a quite new branch of mathematics (it was born at the end of 1700), over the years it has acquired a leading role for its use for applications in areas such as transportation, telecommunications, science experiments …

Graph theory presents, in mathematics education, several advantages: it permits students to *see applications of mathematics*, to start some argumentation and it boosts reasoning.

Graph theory is easy to understand, fun to use and intriguing to use in modeling real situations. It can be used to develop "a suitable vision of mathematics, not reduced to a set of rules to be memorized and applied, but recognized as a framework to address significant questions, to explore and perceive relationships and structures recurring in nature and in the creations of mankind" (MIUR 2012).

With this respect, in this paper we discuss an approach to graph theory that we carried out in primary, middle and high schools over the past 12 years. Countries other than Italy have a structured approach to Discrete mathematics (including graph theory, counting methods, recursion, iteration, induction, and algorithms), see DeBellis and Rosenstein (2004), Rosenstein (2014). The introduction of Discrete mathematics in school is, in fact, recommended by the National Council of Teachers of Mathematics (NCTM) in the USA since 1989, while in Italy it is not explicitly asked for in the curricula.

2 Königsberg Bridges Problem

In 1736 the Academy of Sciences in Petersburg published an article in which Leonard Euler solves the problem of the *bridges of Königsberg*: The city of Königsberg in Prussia (now Kaliningrad, Russia), crossed by the Pregel River, includes two large

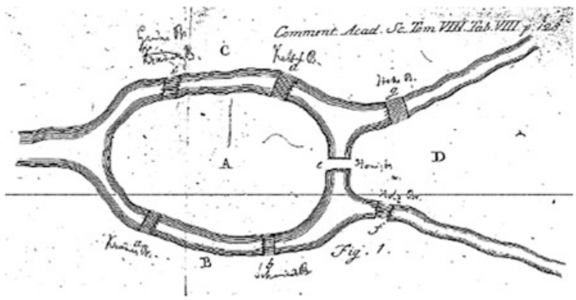


Fig. 1 Figure retrieved from Euler (1741)

islands. The islands and the mainland are connected to each other by seven bridges (Fig. 1). In 1700, the citizens of Königsberg used to have a nice walk around the city and wondered if they could find a walk that would cross each bridge once and only once, with the condition (optional) that they would be back to the same point they started.

Euler proposes to denote by A, B, C and D the four parts of the city, A and D are the islands and B and C the mainland (Fig. 1). He denotes a bridge between A and B by AB, one between B and D by BD: so, if a citizen starts from A, moves to B and then to D we can denote his/her walk by ABD. If the citizen then moves from D to C the whole walk would be ABDC. Then, the *word* ABDC states which walk the citizen did and how many bridges he crossed: three bridges in this case, one less than the number of letters, in general. And vice versa, if the citizen crosses n bridges then the number of letters of the word describing his/her walk is $n + 1$. Then, the word solving the problem should have 8 letters (among A, B, C, D) corresponding to the 7 bridges. Solving the problem means to find the *correct* word.

Euler showed that this problem cannot be solved by essentially using the following reasoning. If the region A is connected to another area with only one bridge, then the correct word contains A only once, no matter if the walk starts in A or not. If the region A is connected with another area with three (five, seven…) bridges, then the correct word contains A twice (respectively three times, four times…), no matter if the walk starts in A or not. In general, if an area is connected to other areas with an odd number of bridges, say n, then the letter associated to that area appears exactly $(n + 1)/2$ times. Then, in the specific case of Königsberg, the letter A must appear three times, the letter B must appear two times, so as letters C and D: so we should have a word with $3 + 2 + 2 + 2 = 9$ letters, other than the 8 letters that the correct word must have. The Königsberg citizen then cannot find the walk they were looking for. This problem was basically modelled with the aid of what has become known as a *graph* (Fig. 2).

A graph is a pair $G = (V, S)$, with V a finite set, whose elements are called vertices or nodes, and S is a set of pairs of elements of V, called edges with endpoints the vertices of the pair.

The Königsberg bridges graph (Fig. 2) is a graph with vertices A, B, C and D and edges represented with the drawn lines, that are $[A, B]_1$ $[A, B]_2$, $[A, C]_1$, $[A, C]_2$, $[A, D]$, $[B, D]$ and $[C, D]$. The problem of Königsberg's citizens can be seen as the problem of drawing the graph in Fig. 2 without lifting the pencil from the paper and without passing twice on the same line.

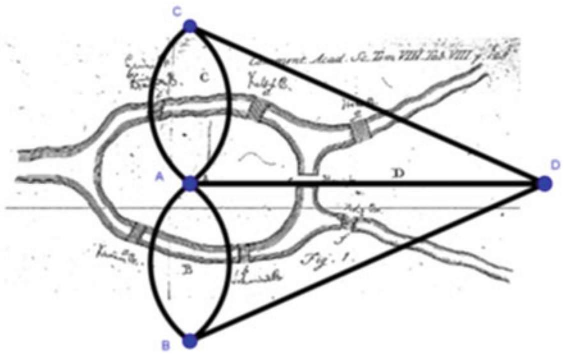


Fig. 2 Graph superimposed on map of Königsberg

3 Theoretical Framework

Several activities have been developed by the authors to introduce graph theory at all school levels.

The activities are realized in the perspective of horizontal teaching (Ferrarello et al. 2014) and with the use of technology. In horizontal teaching, the teacher enters the realm of the students' real life and offers activities fitting with the students' age, needs and the needs of the whole class. Technology helps to model situations by using "ad hoc" applications.

The activities on graphs we carried out are all based on a common theoretical framework: semiotic mediation in a Vygotskijan perspective supporting the activity of construction of mathematical concepts, through laboratorial activities, based in embodied cognition theory.

Mathematical content is mediated by the use of artefacts, in the Vygotskijan sense (Vygotskij 1981, p. 137): writing, speaking, using mathematical symbols, maps, diagrams are all involved in the mediation process that transforms situated signs into mathematical signs whenever a task is given. Our students (from children to adults) used written signs such as words and figures to represent the situation asked for by the task, to conjecture a possible solution, to test it, and then they used words to argue about possible solutions and to communicate it to the others, and finally transformed situated signs into mathematical meanings (Bartolini Bussi and Mariotti 2008).

Graphs perfectly fit in the *embodied mind* framework (Lakoff and Johnson 1999). Graphs can be used not only drawn on paper, but also to be manipulated as real objects: real strings in primary school, for instance, but also constructed by using special software; in such a way that nodes can be dragged and edges can be warped, as they were real. Body and mind, as a whole, participate in the construction of mathematical meaning, by using grounding metaphors (Lakoff and Nunez 2000), strings as edges, for instance.

Manipulation of objects is one the four components of a laboratory activity (Anichini et al. 2004; Reggiani 2008): (1) A problem to be solved; (2) Objects to be manipulated; (3) Interaction with people; (4) Role of the teacher.

1. A problem to be solved is a task that has to be not too hard (in the non-competence area of Vygotskij) nor too easy (in the competence area of Vygostskij), but accessible: a problem that can be solved in the interaction with others (peers or teacher), i.e. in the Zone of Proximal Development;
2. Objects, real or virtual, are to be manipulated;
3. Interaction with people refers to collaboration with mates to solve problems and mathematical discussion among teacher and students to strengthen concepts.
4. Role of the teacher is as a trainer, guiding and encouraging students to discover, to argue, to conjecture, to prove.

In fact,

> we can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together […] to the communication and sharing of knowledge in the classroom, either working in small groups in a collaborative and cooperative way, or by using the methodological instrument of the mathematic discussion, conveniently lead by the teacher (Anichini et al. 2004).

All the activities are oriented toward a good approach to mathematics supported by a positive interaction of affect and thinking in the learning process, because "affect influences thinking, just as thinking influences affect" (Brown 2012, p. 186).

The same topic (graphs) is treated, in our activities, in very different contexts by using different levels, as suggested by Van Hiele (1986), from visualization and description to rational and logical, in the view of vertical curriculum. In primary school, students receive tools to be able "to represent relationships and data and, in meaningful situations, use representations to obtain information, to express judgments and make decisions"—goals to be reached at the end of the fifth grade, in the "Relations, data and forecasts" part (MIUR 2012). In addition, significant time is given to playing, as a means for the "development of strategies suitable to different contexts." In middle school more attention towards formalization, generalization, argumentation is given, in order to "develop the ability to communicate and discuss, to properly argue." In high-school simple proofs and algorithms can be introduced (MIUR 2012). For high-school students it is important to understand the concept of a mathematical model. Generally, in Italian schools, it is proposed as a connection between mathematics and physics (for example the physical concept of velocity through the mathematical concept of derivative). It should also be used as an approach to reasoning with the help of graphical representation and referring to real contexts, as other mathematical models.

4 Topics

In this paper we present some activities we carried out in recent years that have been developed. The topic is the same for primary, middle and high schools, Eulerian graphs,[1] but the approach changes depending on level (Van Hiele 1986). We briefly present the contents here in an intuitive way. For a rigorous approach refer to West (2001), Wilson (1996).

Recall that a *graph* is a pair $G = (V, S)$, V being a finite set whose elements are called vertices or nodes, and S is a set of pairs of elements of V, called edges with endpoints being the vertices of the pair.

A graph is *connected* if from each vertex you can always reach any other vertex through adjacent edges. In this paper we always deal with connected graphs.

The *degree* of a vertex is the number of edges for which that vertex is an endpoint. For example, the graph in Fig. 2 is connected and the degree of C is 3, the degree of A is 5.

SemiEulerian graphs are those that *you can draw without lifting the pen from the paper and passing through each edge exactly once*. The taken walk is called an Eulerian trail. Eulerian graphs are SemiEulerian graphs such that there exists a closed Eulerian trail, i.e. such that the first vertex and the last vertex are the same. Such trail is called an Eulerian cycle.

The Königsberg Bridges Problem, mentioned at the beginning of the paper, consists in finding an Eulerian cycle/trail in the graph in Fig. 2. Euler found a necessary and sufficient condition for the existence of an Eulerian trail/cycle in a given graph, referring to *words* (Euler 1741). Referring to graphs, there is an Eulerian cycle in a connected graph if and only if each vertex has even degree; there is an Eulerian trail in a connected graph if and only if there are at most two vertices of odd degree.

Fleury's algorithm produces an Eulerian cycle (trail) in an Eulerian graph. The algorithm works as follows: if the graph is connected and with all vertices of even degree (at most two of odd degree), choose any vertex (a vertex of odd degree, if any) as starting vertex and select successively adjacent edges choosing a bridge only if there is no other choice, where a bridge is an edge which, if removed, produces a disconnected graph.

5 Eulerian Graphs in Primary, Middle and High School

Several activities have been carried out in several schools, but in different years.

In 2007, for the first time, we tested a graph theory activity in middle school, 8th grade, (Mammana and Milone 2009a, b) and the following year, a different one in 6th grade (Mammana and Milone 2010). In 2012 we brought graphs in primary

[1]Other graph theory topics have been proposed to students but are not presented here.

schools, 3rd, 4th and 5th grade. Then, a proposal for high-school was written (Aleo et al. 2009) and tested in 9th and 10th grade (Ferrarello and Mammana 2017).

The way of conducting the activity was the same for all levels: laboratorial activities (Chiappini 2007). We also used some technology. Specifically, we used technological artefacts to support teaching and learning at every level, and we employed both old and new technology, namely paper, pencils, strings, but also software and online games. The new technologies used were:

- *yEd*, graph editor (https://www.yworks.com/products/yed);
- *Cabri* (http://www.cabri.com/);
- *Icosien* (http://www.freewebarcade.com/game/icosien/);
- *Planarity* (http://planarity.net/);
- *Fly tangle* (http://www.giochigratisonline.it/giochi-online/giochi-puzzle/FlyTangle3/).

yEd is a free software developed to draw and manipulate graphs. With yEd we can import images or our own data from existing spreadsheets, easily create diagrams via an intuitive user interface, automatically (or manually) arrange our diagrams elements, and export images of the created graphs (see Fig. 3). yEd has been used in primary school and in high school, while in middle school we used Cabri geometre. We used Cabri just to draw graphs, not as a dynamic geometry system. Any other Dynamic geometry software can be used. Of course, the first mission of a DGS is not drawing graphs, but back then (2007) yEd was not developed yet.

Icosien is an online game. It is not an educational game, but we used it with didactic purpose. In fact, the aim of the game is to wrap a string around some nails to create the given shape in each level, by constructing Semieulerian and Hamiltonian graphs (graphs in which a trail can be constructed that uses every vertex exactly once, but not necessarily every edge) (see Fig. 4).

Planarity and *Fly-tangle* are games in which the player has to arrange the vertices of a graph so that none of the lines intersect, except at the vertices.

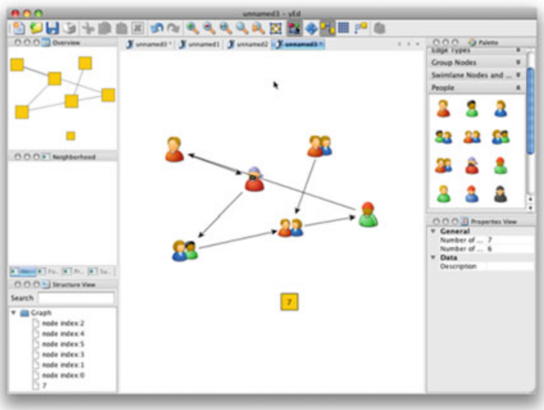


Fig. 3 Screen of a *yEd* developed graph

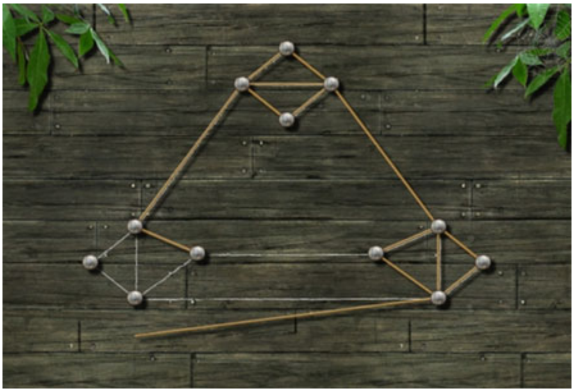


Fig. 4 Example of *Icosien* game

- Contents of the activities in primary school are: definition of graph, planar graph, vertex colouring, Eulerian graph, Hamiltonian graph (these activities lasted between 30 and 40 h).
- Contents of the activities in middle school are: definition of graph, graphs to model situations, Eulerian graphs (these activities lasted between 15 and 20 h).
- Contents of the activity in high school are: definition of graph, Eulerian graphs, Fleury algorithm, Spanning threes, Kruskal's algorithm, applications (these activities lasted about 20 h).

At all levels we proposed the Eulerian graph topic, that is the one we concentrate in this paper from this point forward.

In the activities we developed we had the following goals (in the following P stands for Primary, M for middle, H for High-school):

- recognize that a graph provides a possible modelling of a problem; (P, M, H)
- know how to go from a problem to its model as a graph; (P, M, H)
- recognize the essential elements of a graph; (P, M, H)
- identify similarities and differences between graphs; (P, M, H)
- recognize same graphs but with different representations; (P, M, H)
- know how to draw and represent a graph both with paper and pencil and by means of a suitable software; (P, M, H)
- formulate hypotheses on the characteristics of an Eulerian/SemiEulerian graph; (M, H)
- test formulated hypotheses on the characteristics of an Eulerian/SemiEulerian graph; (M, H)
- formulate conjectures, discussion supervised by teachers; (M, H)
- independently develop conjectures, argumentation; (H)
- compare hypotheses with classmates to achieve shared results; (P, M, H)

- use the condition of existence of a Semieulerian/Eulerian path for solving problems; (P, M, H)
- knowing how to apply an algorithm in order to find an Eulerian/Semieulerian path; (M, H)
- think about the fact that mathematical objects are "hidden" in various everyday situations. (P, M, H).

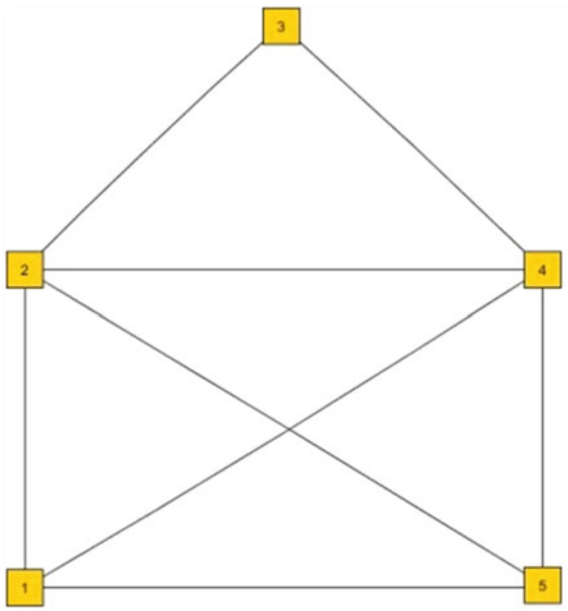
5.1 Eulerian Graphs in Primary School

Activities with 8 and 9 years old students were carried out (Ferrarello 2014, 2017). We aimed to make children enjoy the topic, rather than focus on mathematical competencies (that were achieved anyway). So the Eulerian activity was related to storytelling, playing with games, and a little of argumentation, as described below.

The first approach, using storytelling, was the problem of Königsberg seven bridges. Children tried to solve the problem by using a map of Königsberg, and then they were guided to construct the model of Fig. 2, by using *yEd*. We did not use standard definitions, but we called Eulerian and Semieulerian graphs *walkable* graphs, with the aim to recall in students' minds the activity to walk around the graph, by touching, just once, all the edges.

Together with the *impossible* graph of Königsberg, other *solvable* problems were given, as the classical *cabin* graph of Fig. 5.

Fig. 5 SemiEulerian cabin graph



In such a way pupils did not give up and carried on, trying and playing with solvable and non-solvable problems.

After nodes, edges and nodes degree were introduced, children were guided to notice similarities among walkable graphs: they discovered that such graphs had only nodes of even degree, we called them closed walkable, or just two nodes of odd degree, we called them open walkable. Students practiced with several graphs, with paper and pencil, by using the online game Icosien and with real string (see Fig. 6).

Activities with paper and pencil also included games of words and sentences, by using graphs whose nodes are letters and two letters are joined by an edge if you can use the two letters consecutively in a word. Semieulerian graphs hiding sentences were given: children found the right sentence by walking on each edge just once.

The use of the *Icosien game* was useful: it lets you know if you are wrong and pupils could try by themselves, even at home without the teacher. Pupils were happy to learn how to win the game, but they did not care at all about the motivation of why the strategy worked.

To argument about the reason behind the winning strategy we used the graph in Fig. 7, by orienting the edges in the direction of a *winning* path and by colouring every source (where the oriented edge starts) with a green chalk and every sink (when the oriented edges finish) with a red chalk. Then we focused on the first node of the winning path and reasoned on the number of going out and going in edges: two of them are used to pass through the path and one of them is used to go out at the beginning. Similarly, an even number of edges (two) are used in the last node to cross the trail and one is used to go in at the end of the trail. The other nodes are all used to cross the path, so they must have even degree. Similar considerations were given for Eulerian graphs.

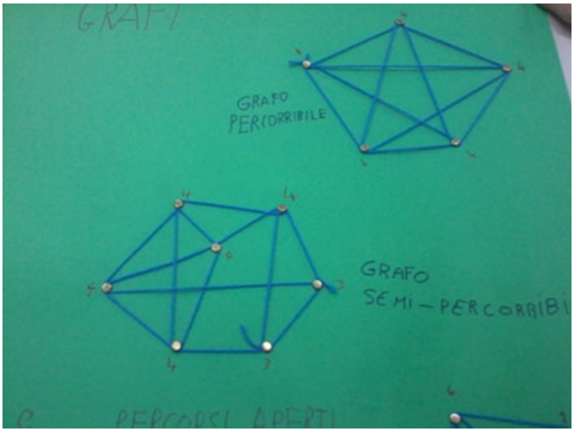


Fig. 6 Eulerian graphs made of string

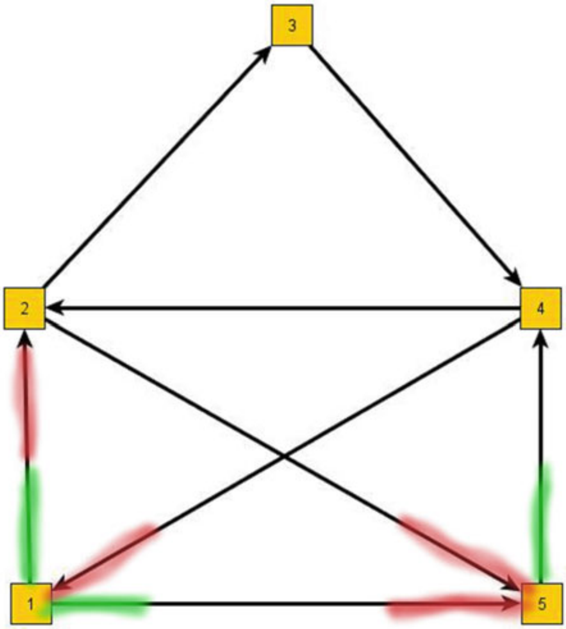


Fig. 7 Color-coding *in* and *out* parts of edges in a path

Table 1 Description of major components of Königsberg bridges activity

Problem to be solved	Objects	Interaction	Role of the teacher
Königsberg bridges problem; Necessary and sufficient condition for the existence of Eulerian cycle/trail	Paper and pencil; Maps; Strings; Icosien	Collaboration among students; Mathematics discussion; Groups formed by students themselves: Each group had a "task": writers, drawers, thinkers	Prepare the activity; Coordinate the groups and the Math discussion

The whole phase of argumentation was led by the teacher, who stimulated pupils with questions, encouraging them to express their thoughts, making them think on their own actions and claims. And finally exulting for their good insights and reasoning.

In Table 1, the major components of the lab (described in Sect. 3), related to the activity, are described.

5.2 Eulerian Graphs in Middle School

Again, to arouse students' interest, they were immediately given the real problem situation of *The Königsberg bridges problem*, which they modelled using the graph that simplifies the analysis of the problem. They were then asked to solve the same problem on other graphs, obtained from the Königsberg graph by adding or deleting or moving an edge (Fig. 8). Through the identification of similarities and differences, students were lead to discover the conditions that must be satisfied in order to solve the problem, that is, so that there exists an Eulerian cycle.

This condition is then applied to polygons: precisely, students were asked to find polygons that are possible to draw, together with their diagonals, without lifting the pencil from the paper. At the end of this phase, the algorithm of Fleury is presented. For a better understanding of the algorithm and for catching students interest, the

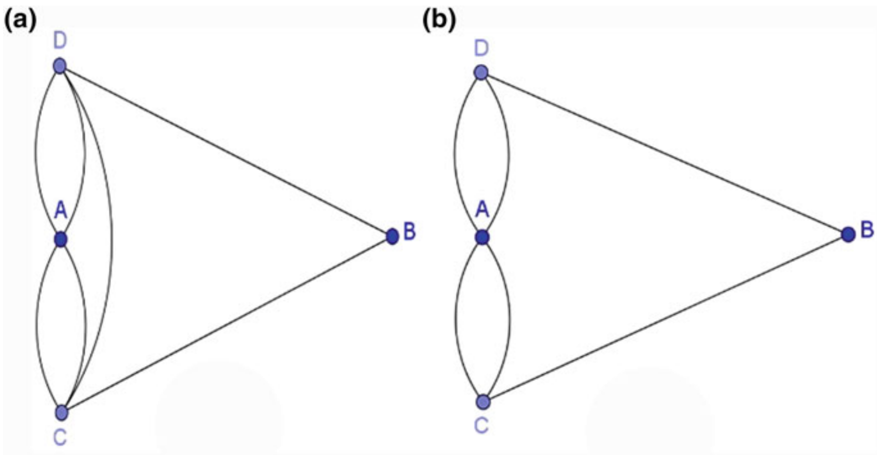


Fig. 8 Figure 2 with edges added and deleted

Fig. 9 Graph for problem 2 of worksheet 3

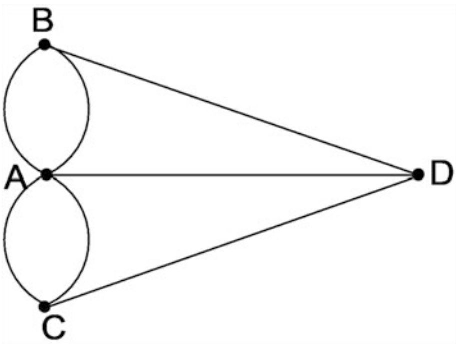


Fig. 10 Graph for high school activity

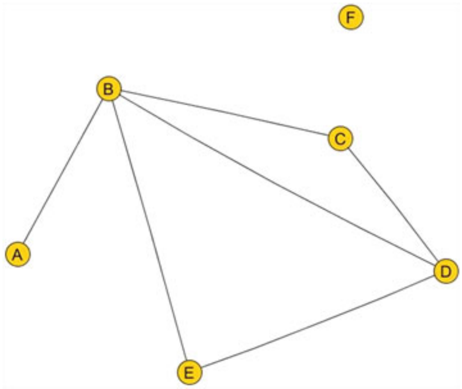


Table 2 Connections between various cities

	Florence	Pisa	Arezzo	Livorno	Lucca	Siena
Florence		YES		YES	YES	YES
Pisa	YES					YES
Arezzo				YES	YES	
Livorno	YES		YES			
Lucca	YES		YES			
Siena	YES	YES				

Cabri software was used—of course, Cabri was not used as a dynamic geometry software but for visualizing, step by step, the algorithm. They were also posed the problem of the existence of an Eulerian trail, and students were lead to discover the conditions that must be met for such a trail to exist. By using Cabri Geometre, students did find an Eulerian cycle/trail in several graphs. At the end of the whole activity students were asked to solve a problem (Table 2) that they had to model using a graph and to solve with graph theory tools.

Alice goes Florence, in Tuscany. She wants to visit some cities in Tuscany, Florence, Pisa, Arezzo, Livorno, Lucca, Siena, returning to Florence airport. Alice also wants to see as much scenery as possible along all the connections between the cities one and only one time because she does not have much time. Can you help Alice?

The table shows the existing connections between the various cities. (The empty boxes indicate that between the two cities there is not a direct link.)

Table 3 Worksheet 3 for Königsberg problem

WORKSHEET 3
(INDIVIDUAL WORK IN CLASS) Problem 3: It follows the graphs of Problem 1 (see Fig. 8a) and Problem 2 (see Fig. 9): For graph of Problem 1 you can find a solution and for graph of Problem 2 no. Why is that? What is the difference between the graphs? Try to give some answer: a) ……………………………………………………………………………… ………………………………………………………………… b) ……………………………………………………………………………… ………………………………………………………………………………..

We have chosen to provide concepts through descriptions in natural language using gradually increasingly formalized language: in this way some linguistic difficulties could be avoided. For example, we did not define the *degree of a vertex*, but we talked of *number of edges for a vertex*: in this manner, although not correct from the graph language point of view, we used an expression that is nearer to the natural language of the pupils.

The whole activity has been carried out by means of worksheets, prepared by the teacher. The students worked on the worksheets by themselves, in class or at home, but there was always a class discussion to make sure that everybody got to the same point.

Here we report Worksheet 3 (in Worksheet 1, the Königsberg problem was introduced and students were asked if it was solvable or not, and in Worksheet 2 students were asked if, given the graph on the left in Fig. 8, it was possible to find a walk passing through all the edges exactly once, starting and finishing at the same vertex) (Table 3).

In Table 4, the major components of the lab related to this activity are described.

Table 4 Description of major components to middle school lab

Problem to be solved	Objects	Interaction	Role of the teacher
Königsberg bridges problem; Finding the necessary and sufficient condition for the existence of Eulerian cycle/trail; Application to polygons; Application to “Alice’s Tuscany tour”	Paper and pencil; Cabri geometre; Guided worksheets	Collaboration among students; Mathematics discussion; Groups works	Prepare the activity; Coordinate the groups and the Math discussion

5.3 Eulerian Graphs in High School

The activity was carried out in a high-school by following the path described in Aleo et al. (2009). The contents were the same as in middle school but the approach slightly different. Since the whole terminology is new for the students and to list all definitions can be quite boring, we decided to have the students discover them, with the aid of a guided activity (an example in Table 5). A final worksheet, called Trip logbook, collects all definitions as a reference for students during their *tour* around graphs. Afterwards, after modelling some problems in terms of graphs and after understanding that their solution consists in finding an Eulerian cycle, we present and use the Fleury algorithm for finding possible solutions.

At the end of the activity the following real life problem was given:

> Air Transportation is a complex activity: there are big money investments (aircrafts and infrastructure), highly qualified workers (pilots and staff), real time information (booking service for example). Expenses on air traffic are huge and we have to avoid waste. For example, an airplane on the ground does not result in any revenue for the company: therefore, no-fly time has to be reduced. To this end, some airline companies, once the routes are decided, study «circular trails» for each aircraft (a circular trail is a trail that covers all the routes only once).
>
> Eurofly, has to organize a circular route for an aircraft that has to cover the routs indicated in Table 6. If there is an empty cell then there is not a direct link between the cities.

Most of the students were able to model the given problem with the aid of a graph and understood that they were asked to find an Euler cycle of that graph. Students were very happy to have mathematics *touch* real life. One of them thought to use graph theory to model some problem of his father's company (a logistic transportation company).

Table 5 Guided activity for high school

Observe the following graph (see Fig. 10):
We have: Degree of A = 1 Degree of B = 4 Degree of C = 2 Degree of D = 3 Degree of E = 2 Degree of F = 0
Definition: The Degree of a vertex is the number of …… whose the vertex is an endpoint.

Table 6 Routes for Eurofly aircraft

	Rome	Paris	London	Athen	Milan	Madrid
Rome		YES		YES	YES	YES
Paris	YES					YES
London				YES	YES	
Athen	YES		YES			
Milan	YES		YES			
Madrid	YES	YES				

Table 7 Description of major components of high school activity

Problem to be solved	Objects	Interaction	Role of the teacher
Königsberg bridges problem; Represent the network of an airline company with a graph; Finding necessary and sufficient condition for the existence of Eulerian cycle/trail; Application to Airline companies problem	Maps; yEd; Icosien; Worksheets	Collaboration among students; Mathematics discussion	Prepare the activity; Coordinate the groups and the Math discussion

In Table 7, the major components of the lab, related to the activity, are described.

6 Brief Conclusions

We strongly believe that graph theory deserves space in school teaching, because it permits an approach to modelling, argumentation, and connection with reality and, it may foster affection to mathematics from those students that have had a bad experience with it.

Students we worked with did benefit from this experience both for mathematical skills developed and a positive affective point of view. In particular:

Mathematical knowledge: Students, from primary to high school, were able to recognize the essential elements of a graph, to know how to draw, represent and manipulate a graph both with paper and pencil and by means of software, to identify similarities and differences between graphs and recognize the same graph drawn with different representations, to know the condition of existence of Semieulerian/ Eulerian paths in a graph.
Graphs were easily understood and manipulated as real and everyday-life objects (from cartoon princesses for pupils to social network friendships for teenagers), as suggested by horizontal teaching and embodied cognition theory.

Practical abilities: All the students used the condition of existence of a Semieulerian/Eulerian paths for solving problems, and most of the students in middle and high school knew how to apply an algorithm in order to find an Eulerian/Semieulerian path: Because when you discover properties, by touching them through laboratorial activities, instead of listening to the teacher, you get better and remember those properties.

Mathematization: Students, from primary to high school, achieved the ability to recognize in a graph a possible model of a problem, to know how to go from a problem to its model through a graph and to see mathematical objects hidden in various situations and everyday objects.

The mathematization process was driven by the use of speech, diagrams, gestures, … as situated signs later on transformed into mathematical signs.

Reasoning: Students of all grades formulated and tested hypotheses on Eulerian/SemiEulerian graphs and compared their own hypotheses with classmates to achieve shared results. In middle and high school, they formulated conjectures, with argumentation guided by teachers. In high school some students independently developed conjectures, through argumentation.

Affect: The most enthusiastic were primary school children who saw a beautiful and different mathematics, rich in games and cartoon characters. Middle school students organized an exhibition of the activity they did in class and invited parents and future students of the school to visit it.

Some of the high school students in the last experimentation came to the graphs meetings *forced* by a project carried out by their own school, but in the end they were happy to have participated and brought some modeling results home (one of them, for instance, thought to use graph theory for modelling problems of the transportation company of his father).

For all of them, fun and motivation was the first step towards learning. We aimed to promote a vision of mathematics not made of cold calculations, or of pseudo-problems to solve, which is often thought just to blindly apply those rules you read about some pages before.

The vision of mathematics we want to achieve is a mathematics included in everyday life, but hidden: to see it you should raise the veil that covers it.

Despite all of its usefulness and beauty, teachers are not prepared to teach it in class. Sometimes they do not even know the topic itself. Some teacher-training programs may include these topics in the future. More to come!

References

Aleo, M. A., Ferrarello, D., Inturri, A., Jacona, D., Mammana, M. F., Margarone, D., et al. (2009). *Guardiamo il mondo con i grafi [Let's look at the world with the graphs]*. Catania: Casa editrice La Tecnica della Scuola.

Anichini, G., Arzarello, F., Ciarrapico, L., & Robutti, O. (Eds.). (2004). *La matematica per il cittadino. Attività didattiche e prove di verifica per un nuovo curricolo di Matematica (ciclo secondario)*. Lucca: Matteoni Stampatore.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 720–749). Mahwah, NJ: Erlbaum.

Brown, T. (2012). Affective productions of mathematical experience. *Educational Studies in Mathematics, 80,* 185–199.

Chiappini, G. (2007). Il laboratorio didattico di matematica: Riferimenti teorici per la costruzione. *Innovazione educativa, Inserto allegato al numero, 8,* 9–12.

DeBellis, V. A., & Rosenstein, J. G. (2004). Discrete mathematics in primary and secondary schools in the United States. *ZDM Mathematics Education, 36*(2), 46–55.

Euler, L. (1741). Solutio Problematis ad geometriam situs pertinentis. *Commentarii academiae scientiarum Petropolitanae, 8*, 128–140.

Ferrarello, D. (2014). Primary graphs *Quaderni di Ricerca in Didattica. Proceedings CIEAEM* 66, *24*(1).

Ferrarello, D. (2017). Graphs in primary school: Playing with technology. In G. F. Hitt, L. Bazzini, & U. Gellert (Eds.), *Mathematics and technology.* A C.I.E.A.E.M. Sourcebook. Berlin: Springer.

Ferrarello, D., & Mammana, M. F. (2017). Teoria dei grafi: Come e perchè. *L'insegnamento Della Matematica E Delle Scienze Integrate, 40,* 249–271.

Ferrarello, D., Mammana, M. F., & Pennisi, M. (2014). Teaching by doing. QRDM. *Proceedings of CIEAEM 2013* (Vol. 24, no. 1, pp. 429–433). Torino, Italia, 22–26 Luglio 2013.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh. The embodied mind and its challenge to western thought*. New York: Basic Books.

Lakoff, G., & Nunez, R. R. (2000). *Where mathematics come from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Levy, R. (2015). 5 Reasons to teach mathematical modeling. *American Scientist*. http://www.americanscientist.org/blog/pub/5-reasons-to-teach-mathematical-modeling.

Mammana, M. F., & Milone, C. (2009a). I grafi: Un percorso possibile (parte prima). *L'insegnamento della Matematica e delle Scienze Integrate, 32A*(4), 427–440.

Mammana, M. F., & Milone, C. (2009b). I grafi: Un percorso possibile (parte seconda). *L'insegnamento della Matematica e delle Scienze Integrate, 32A*(2), 109–132.

Mammana, M. F., & Milone, C. (2010). Dal problema al modello: Una sperimentazione sul concetto di grafo nella scuola media. *L'educazione matematica, IX*(2), 23–36.

MIUR. (2012). *Indicazioni nazionali per il curricolo della scuola dell'infanzia e del primo ciclo d'istruzione*. http://www.indicazioninazionali.it/documenti_Indicazioni_nazionali/indicazioni_nazionali_infanzia_primo_ciclo.pdf.

Reggiani, M. (2008). *Il laboratorio come ambiente per l'insegnamento-apprendimento della matematica: Riflessioni*. L'insegnamento della Matematica e delle scienze integrate, 31B(6).

Rosenstein, J. G. (2014). *Problem solving and reasoning with discrete mathematics (preliminary edition, with accompanying activity book)*. Highland Park, NJ: Shiviti Publications.

Van Hiele, P. M. (1986). *Structure and insight*. Orlando: Academic Press.

Vygotskij, L. S. (1981). The genesis of higher mental functions. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 147–188). Armonk, NY: Sharpe.

West, D. B. (2001). *Introduction to graph theory*. New York: Prentice Hall.

Wilson, R. J. (1996). *Introduction to graph theory*. Essex: Longman Group Ltd.