STRUCTURAL DUALITY

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Charles L. Dodgson, the eminent Oxford mathematician, once expounded, in the form of a conversation between an egg and a seven-and-a-half-year-old girl, the interesting concept of an "unbirthday." Conversely and perhaps complementarily, lexiphiles enjoy making lists of words like couth, plussed, ept, ebriety, peccable. Structural duality is, of course, more than whimsical playing with words, but it is concerned with such notions as contrastatus (versus status), antibalance (versus balance), and unliaison person (versus liaison person). It also involves the use of various kinds of graphs as a mathematical model, an approach whose value lies in the fact that it is not bound by physical units of measurement. The rigorous mathematical development of a duality theory may provide a source of fruitful concepts for behavioral science.

Our purpose is to develop, formalize, and discuss three basic laws for structural duality. In so doing, we hope to provide a unifying framework for several concepts which have already been discussed either explicitly or implicitly in the literature of the behavioral sciences. In addition we attempt to create a systematic theory which has the philosophical purpose of extending our conceptual resources rather than the relatively materialistic motive of providing a specific model for a specific setting. Intuitively the three laws involve interchanging the relational motions of presence and absence, forward and backward, and positive and negative. The present paper is a sequel to three earlier papers (13, 14, 6) in which there appear introductory expositions of the theory of graphs, directed graphs, and signed graphs respectively. The operations of complementation, conversion, and negation on these three types of graphs lead to the three laws for structural duality.

These notions of duality are not new. For example, we were anticipated by Freud in 1915, who wrote:

> Loving admits of not merely one, but of three antitheses. First there is the antithesis of loving-hating; secondly there is loving-being loved; and, in addition to these, loving and hating together are the opposite of the condition of neutrality or indifference (9, p. 76).

It will be seen that these three antitheses of Freud correspond respectively to our "antithetical," "directional," and "existential" duality laws.

In developing the laws for structural duality various kinds of graphs are used. A *graph* (13, 17) is a finite collection of *points* A, B, C, . . . together with a prescribed set of *unordered* pairs of distinct points, called *lines*. A *directed graph* or *digraph* (13, 17) also consists of a finite collection of *points*, but its *lines* are *ordered* pairs of distinct points. A *signed graph* (6, 11) is obtained from a graph by taking some of its lines as *negative* and the remaining lines as *positive*.

Graphs and directed graphs have been used as a mathematical model in various applications of communication theory to group problem solving by Bavelas *et al.* in the series

[1] The author wishes to thank D. Cartwright, C. H. Hubbell, and I. C. Ross for some stimulating and helpful discussions.

of papers (3, 15, 18). In addition Rapoport (20) has used graphs implicitly in his theorizing on pecking sequences. More recently, Ashby (1) has considered directed graphs under the name of "kinematic graphs" in introducing the notions of cybernetics. In the field of organization theory, Weiss *et al.* (16, 22, 29) have made use of graphs and directed graphs as models for the structure of an organization. We shall see that these results either implicitly allude to or may be enriched by the appropriate use of the laws of structural duality.

We begin with examples of duality taken from mathematics: set theory, logic, and group theory; and from physics: future-past, right-left, and interchanging the sign of charged particles. On turning to structural duality, we shall find that some new concepts are induced and we shall explore some of these, such as unliaison person, contrastatus, and antibalance. We then derive two further duality laws from the three basic laws, and show that no additional laws are derivable from these.

DUALITY

In general, any duality principle has the two properties: (1) the dual of the dual of a statement is the original statement, and (2) the dual of a true statement is true. A classic example of duality occurs in set theory (31, Chap. III), interchanging union and intersection, or equivalently in propositional logic (31, Chap. IX), interchanging disjunction and conjunction, i.e., the connectives "or" and "and." Thus the dual of the logical DeMorgan Law:

$$\text{not } (p \text{ or } q) = (\text{not } p) \text{ and } (\text{not } q) \quad [1]$$

is

$$\text{not } (p \text{ and } q) = (\text{not } p) \text{ or } (\text{not } q). \quad [2]$$

Similarly the dual of the commutative law for set-addition:[2]

$$A \cup B = B \cup A \quad [3]$$

is the commutative law for set-multiplication:

$$A \cap B = B \cap A. \quad [4]$$

A *self-dual* statement is one whose dual is the same statement. An example of a self-dual logical law is that of double negation:

$$\text{not } (\text{not } p) = p. \quad [5]$$

This law is self-dual since there are no "and" or "or" connectives to interchange. Another self-dual statement, taken from set theory, is given by the equation:

$$A \cap (A \cup B) = A \cup (A \cap B) \quad [6]$$

sometimes called the "law of absorption," since both expressions are equal to the set A. On interchanging the operations union and intersection each side of this equation is transformed into the other side.

In view of the DeMorgan laws, the interchange of the operations union and intersection can be applied by taking the complement or negation of both sides of a set theoretic or propositional equation, respectively. The law of double negation, which can be written for sets in the form:

$$(A')' = A, \quad [7]$$

then assures us that this process is a dual one.

In his celebrated paper (24) which introduced modern switching theory, Shannon demonstrated the isomorphism between switching functions on two-terminal networks and Boolean functions. In this model, the above logical and set theoretic dualities are realized by an interchange of series connections with parallel connections.

A duality theorem for (mathematical) groups (30, Chap. VII) is given by the transformation which interchanges the order of the terms in each group-product $a \cdot b$. In accordance with this notion, the dual of the left-cancellation law:

$$a \cdot b = a \cdot c \text{ implies } b = c \quad [8]$$

is the right-cancellation law:

$$b \cdot a = c \cdot a \text{ implies } b = c. \quad [9]$$

In the study of the calculus of binary relations by the logicians Tarski and McKinsey,

[2] The *union* of two sets A and B, written $A \cup B$, is that set whose elements lie in at least one of the sets A or B. Their *intersection*, $A \cap B$, is the set of all elements which are in both A and B. The *complement* A' of A is the set of all elements under consideration which are not in A.

several kinds of duality occur. These include complementary duality, converse duality, and the combination of these, which correspond precisely to our existential, directional, and (the combined) direxistential duality. It is these relational dualities which are mentioned in the following quotation from the beginning of an article by Gottschalk (10).

> It is well-known that every involution in a logical or mathematical system gives rise to a theory of duality; for example, negation in the sentential calculus and predicate calculus, complementation in the calculus of classes, complementation and conversion in the calculus of relations, etc. The purpose of this note is to call attention to the fact that every involution in a logical or mathematical system gives rise to a theory of *quarternality*. . . .

Gottschalk then describes the pairs of dual "constants": {*true*, *false*}, {*and*, *or*}, {implies, is not implied by}, {is implied by, does not imply}, {equivalent, inequivalent}, and an apparently unintentional pun {*stroke* and *dagger*}, where *a stroke b* means not-*a* and not-*b*, while *a dagger b* stands for not-*a* or not-*b*. The "constant" *not* is self-dual. His quaternality theorem for logic is that a formula yields three others on negating the "variables," "constants," or both.

There is a beautifully written exposition of three physical duality laws by Uhlenbeck (28), one of which is also discussed in a similar vein by Blatt (5). The first of these duality laws can be called the future-past law, or time reversal, or *temporal duality*. This law asserts that for each physical law involving time, the interchange of positive or future time with negative or past time results in another (possibly the same) law. A purely mathematical description of basically the same phenomenon is given by Feller (7) in his statement of the "backward equations" for Markov chains.

The second of these physical duality laws is the right-left law or *directional duality*. In accordance with this law, replacement of the distance abscissa $+x$ in a physical law by the variable $-x$ results in the same physical law. That is, the designation of whether right or left is to be called the positive position of the x-axis is physically immaterial.

The third physical duality law is that of *charge duality*, which proposes that to each charged particle, there corresponds another charged particle of the same mass but of opposite charge. It is this duality principle which predicted the existence of a positive electron or "positron," and of a negative proton or "antiproton" even before these two particles were experimentally observed. Thus these three physical duality laws have to do with changing the sign of time, distance, and charge respectively.

STRUCTURAL DUALITY

I. The first kind of structural duality is based on the operation of taking the complement of a graph or of a digraph. The *complement* of a graph $\mathcal{G}$ is that graph $\mathcal{G}'$ having the same set of points as $\mathcal{G}$, but in which two points are adjacent, or joined by a line, if and only if they are not adjacent in $\mathcal{G}$. Thus the lines that are present in $\mathcal{G}'$ are precisely those which are absent in $\mathcal{G}$. For example the complements of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in the top half of Figure 1 are the graphs $\mathcal{G}_1'$ and $\mathcal{G}_2'$ in the lower half. We note that the graph $\mathcal{G}_2$ is *self-complementary* in the sense that $\mathcal{G}_2'$ is isomorphic to $\mathcal{G}_2$. Also $\mathcal{G}_1'$ is a disconnected graph even though $\mathcal{G}_1$ is connected.

By the complementary concept of a concept about a graph $\mathcal{G}$, we mean the same concept for the graph $\mathcal{G}'$.

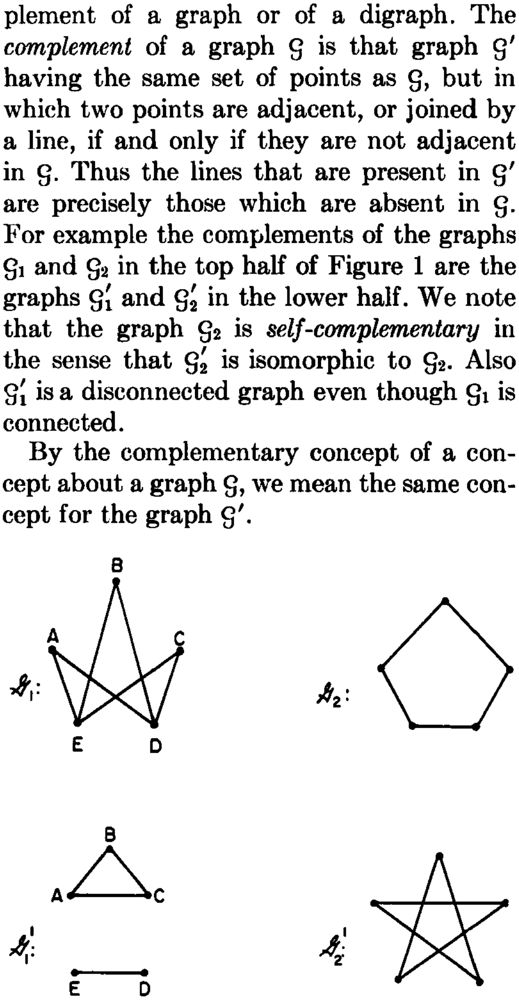

FIG. 1. Examples of two graphs ($\mathcal{G}_1$ and $\mathcal{G}_2$) and their complements ($\mathcal{G}_1'$ and $\mathcal{G}_2'$).

Existential duality

Each theorem about graphs yields another theorem when every concept in the original theorem is replaced by its complementary concept.

We note that the *complementary digraph* $\mathfrak{D}'$ to a digraph $\mathfrak{D}$ is defined by the same presence-absence consideration as for graphs. The only difference is that the lines of a digraph are *directed* from one point to another. Thus Figure 2 shows a digraph $\mathfrak{D}$ and its complement $\mathfrak{D}'$.

We now illustrate the idea of existential duality with the concepts dual to those of an isolated point of a graph, and an articulation point of a graph or equivalently a liaison person in an organization (16, 22, 29).

An *isolated point* of $\mathcal{G}$ is one which is not adjacent to any other points of $\mathcal{G}$. An *unisolated point of* $\mathcal{G}$ is an isolated point of $\mathcal{G}'$. An *articulation point* of a connected graph $\mathcal{G}$ is one whose removal results in a disconnected graph. An *unarticulation point* of $\mathcal{G}$ is an articulation point of $\mathcal{G}'$.

The *degree* of a point of a graph is the number of lines to which it is incident. If p is the number of points in a graph, it is clear that an unisolated point is one whose degree is $p - 1$.

Examples of articulation points are given by a liaison person in an organization (29), an overseas telephone operation (13), or an interpreter such as a unique bilingual person in a group whose remaining members are each monolingual and speak two different languages.

This last situation is used by Bavelas (2) to illustrate inner and outer members of a group. Similarly an isolated point of a graph corresponds to an isolate or isolated person in a group, e.g., one who is not in communication with any other group members. On the other hand the existential opposite, an unisolated point, is realized by a person who is in complete communication with the members of his group. Obviously any group with an unisolated person can have no isolate.

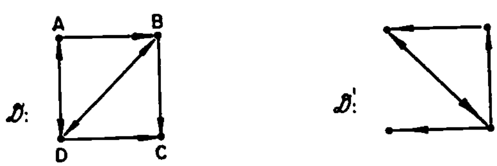

FIG. 2. A digraph and its complement.

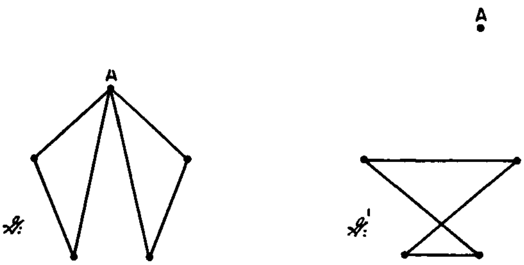

FIG. 3. Graphs showing an unisolated point and an isolated point.

The graphs $\mathcal{G}$ and $\mathcal{G}'$ of Figure 3 serve to illustrate both of these complementary concepts. For, the point A is both an unisolated point of $\mathcal{G}$ and an unarticulation point of $\mathcal{G}'$. A discussion of the existential duals of the theorems concerning graphs and digraphs is beyond the scope of the present article. It would not be entirely unexpected if existential duality should prove to be a fertile field for science fiction (as well as for science itself).

Since none of the concepts of existential duality have been explored empirically, it is not feasible to discuss actual applications. However a well-known anecdote uses these concepts implicitly.

> A man who had been shipwrecked on an uncharted island in the Pacific was found there ten years later, leading a Robinson Crusoe sort of existence. His rescuers found that he had built not only a comfortable home, but *two* churches. He explained that the first church was the one he attended while the second was the one he didn't attend.

A variation of the above description of existential duality is implied by the quotation from Freud on the first page of this article. In accordance with his third antithesis, it is ambivalence and indifference which constitute a pair of opposite relations rather than loving and indifference. This is represented graphically by interchanging the presence of both a positive and a negative line joining the same pair of points of a signed graph with the absence of any line between this point-pair.

II. The second kind of structural duality stems from the operation of finding the converse of a digraph. The *converse* $\mathfrak{D}^{\smile}$ of the

FIG. 4. Example of the converse of a digraph (see also Fig. 2).

digraph $\mathfrak{D}$ is that digraph with the same points as $\mathfrak{D}$, in which the (directed) line $\overrightarrow{AB}$ occurs if and only if the line $\overrightarrow{BA}$ is in $\mathfrak{D}$. Thus the converse $\mathfrak{D}^{\smile}$ of the digraph $\mathfrak{D}$ of Figure 2 is given in Figure 4. In addition, Figure 4 shows the converse of the digraph $\mathfrak{D}'$ of Figure 2. We denote the converse of $\mathfrak{D}'$ by $\mathfrak{D}'^{\smile}$. Similarly, we write $\mathfrak{D}^{\smile\prime}$ for the complement of the converse of $\mathfrak{D}$. One readily verifies the general theorem that the unary operations of converse and complement commute with each other, that is:

$$\mathfrak{D}'^{\smile} = \mathfrak{D}^{\smile\prime} \qquad [10]$$

for this digraph $\mathfrak{D}$ by referring to Figure 4.

Directional duality

To each theorem about digraphs, there is a corresponding theorem obtained on replacing every concept by its converse concept.

For example, a *point basis* of a digraph $\mathfrak{D}$ is a minimal collection of points from which all points are reachable. Dually, a point *contrabasis* of $\mathfrak{D}$ is a minimal collection of points of $\mathfrak{D}$ such that at least one of these points is reachable from any point of $\mathfrak{D}$, or equivalently, is a point basis of $\mathfrak{D}^{\smile}$. Thus, the digraph of Figure 5 has A, D as a point basis and B, E as a point contrabasis.

As an example of the use in the literature of a concept equivalent to that of a point contrabasis, we mention Ashby's idea of a "basin" (1). To do this in the terminology of digraph theory, we require some definitions. A *path* from A to E in a digraph is a collection of lines of the form $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\cdots$, $\overrightarrow{DE}$ where the points $A, B, C, \cdots, D, E$ are distinct. The *distance* from A to E is the minimum of the lengths of all paths from A to E; the *length* of a path being the number of lines in it. A digraph is *strong* or *strongly connected* if for every pair of distinct points A, E there is a path from A to E and a path from E to A. A strong component of a digraph is a maximal strong subgraph. The *condensed graph* $C(\mathfrak{D})$ of a digraph $\mathfrak{D}$ is that digraph obtained from $\mathfrak{D}$ on replacing each strong component of $\mathfrak{D}$ by a point and preserving the sense of the directed lines otherwise. A *receiver* of a digraph is a point P such that there are no lines from P to any other point. Then a *basin* [cf. Ashby (1)] is a strong component of $\mathfrak{D}$ which corresponds to a receiver in the condensed graph $C(\mathfrak{D})$. These concepts are evolved in all detail with motivation, illustrations, and applications in the forthcoming dimonograph (14).

We now exemplify directional duality with a theorem motivated by the study of pecking sequences in chickens. A formal theory for the systematic study of some of the properties of pecking sequences is developed by Rapoport (20), where further references to pecking studies may be found. We omit the proof of the following theorem, which will appear in (14).

Theorem. If every pair of points of $\mathfrak{D}$ are joined by a line in at least one direction, then there exists a point P such that the distance from P to any other point is less than 3.

To this theorem, there corresponds the:

Dual Theorem. If every pair of points of $\mathfrak{D}$ are joined by a line in at least one direction, then there exists a point of Q such that the distance from any other point to Q is less than 3.

Another example of directional duality, of interest in organization theory, involves the status of a person. In the paper (12), a formula is proposed for the status of a person P based not only on the total number of subordinates, but also on the distance from P to each of his subordinates. More precisely, the *status* $s(P)$ of a person P may be defined

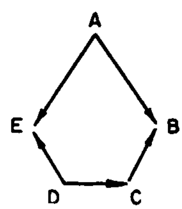

FIG. 5. Illustration of point basis and point contrabasis in a digraph.

by the following formula, in which $d(P, Q)$ denotes the distance from P to Q:

$$s(P) = \sum d(P, Q). \qquad [11]$$

where the summation ranges over all the persons Q in the organization. Then one can define the *contrastatus* $s^{\vee}(P)$ by the dual formula:

$$s^{\vee}(P) = \sum d(Q, P). \qquad [11']$$

As the formula indicates, the contrastatus of a person gives an indication of the amount of power or influence which all other persons have over him. It is our contention, as expressed in (12), that an accurate representation of the power position of a person in an organization must necessarily take not only status but also its directional dual, contrastatus, into consideration. In the paper (25), a similar theory using the terminology of "control" is developed by Tannenbaum. Here "having control over" (active) and "being controlled by" (passive) correspond approximately to our concepts of status and contrastatus respectively.

An application of directional duality to social psychological theory is provided by French (8) by means of his postulate to the effect that a liking relation on a group induces a converse influence relation. Expressed for pairs, whenever A likes B, B has (social) influence over A. As a consequence of this, each contra-concept in the social structure of a group is equivalent to a concept in its influence structure.

III. The third kind of structural duality has as its setting relations which are sometimes positive and sometimes negative. The appropriate kind of graph for such relations is introduced and studied in the papers (6, 11). In a signed graph, some of the lines are positive, and the remaining lines are negative. The *negation* $\mathcal{S}^-$ of the signed graph $\mathcal{S}$ is obtained from $\mathcal{S}$ by changing the sign of each line of $\mathcal{S}$. Clearly, the negation of the negation of $\mathcal{S}$ is $\mathcal{S}$ itself.

Antithetical duality

Each theorem on signed graphs is transformable into another theorem on signed graphs when one replaces every concept by its negation.

A *cycle* of a graph consists of a path AB, BC, $\cdots$, DE, together with the line EA joining the two endpoints of the path. A signed graph is *balanced* if every cycle is positive, i.e., has an even number of negative lines. The fundamental theorem on balanced signed graphs [proved in (11), and in disguised form in (17)] is the:

Structure Theorem for Balance. A signed graph is balanced if and only if its points can be separated into two mutually exclusive subsets such that each positive line joins two points of the same subset and each negative line joins points from different subsets.

The concept which is antithetical to balance is that of antibalance. A signed graph is *antibalanced* if every cycle has an even number of positive lines. Applying the law of antithetical duality, we immediately have without further proof the dual of the preceding structure theorem:

Structure Theorem for Antibalance. A signed graph is antibalanced if and only if its points can be separated into two mutually exclusive subsets such that each negative line joins two points of the same subset and each positive line joins points from different subsets.

One justification for the study of antibalance is that the relation under consideration may have negative connotations, e.g., the relation "dislikes," whereupon its opposite or antithetical relation (in this case "likes") would then be regarded as being represented by the negative lines of a graph. An interesting aspect of balance arises from the consideration of self-duality. We say that a signed graph is *duobalanced* provided it is both balanced and antibalanced. In this case, every cycle must have even length since it contains both an even number of negative lines and an even number of positive lines. Another example of two antithetical relations is that of "power" and "negative power," (as suggested in 8). One interpretation of duobalanced structures is that regardless of whether the original relation is intrinsically positive or negative, the structure is in a balanced state.

The diagram of Figure 6 presents a condensed form of a balanced signed graph in accordance with the Structure Theorem for

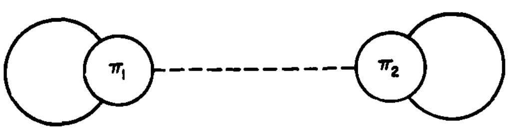

FIG. 6. A balanced signed graph.

FIG. 7. An antibalanced signed graph.

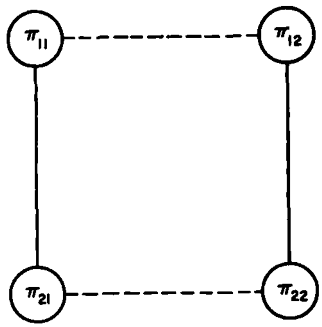

FIG. 8. An illustration of duobalance.

Balance. The points π_1 and π_2 represent the two subsets, the negative line joining them stands for any negative line of the graph, and the two positive lines joining π_1 with itself and π_2 with itself are condensations of all the positive lines. Similarly, Figure 7 is the prototype for all antibalanced signed graphs, following the Structure Theorem for Antibalance. Combining these two theorems, we are led directly to a corresponding theorem for duobalance, which is indicated in condensed form in Figure 8. If π' and π'' are disjoint point sets, a π'–π'' line means a line joining a point of π' with one of π''.

Structure Theorem for Duobalance. A signed graph is duobalanced if and only if its point set π can be partitioned into four mutually exclusive subsets π_{11}, π_{12}, π_{21}, and π_{22} such that

(*a*) there are no lines joining two points of the same subset, or π_{11}–π_{22} lines, or π_{12}–π_{21} lines,

(*b*) all positive lines are π_{11}–π_{21} lines or π_{12}–π_{22} lines, and

(*c*) all negative lines are π_{11}–π_{12} lines or π_{21}–π_{22} lines.

To prove this theorem, we use the definition of duobalance, which entitles us to apply both of the previous structure theorems. Let π_i and π_2 be the two point sets induced by balance and $\bar{\pi}_1$ and $\bar{\pi}_2$ be the two point sets induced by antibalance as in Figures 6 and 7 respectively. Then we define a partition of π into four mutually exclusive point sets as follows. Let

$$\pi_{11} = \bar{\pi}_1 \cap \pi_1,$$

$$\pi_{12} = \bar{\pi}_1 \cap \pi_2,$$

$$\pi_{21} = \bar{\pi}_2 \cap \pi_1,$$

$$\pi_{22} = \bar{\pi}_2 \cap \pi_2.$$

(*a*) There are no π_{ij}–π_{ij} lines (i.e., lines joining two points of the same subset) since any such line would have to be both negative since it joins two points of $\bar{\pi}_i$ and positive since it joins two points of π_j. Similarly if there is a π_{11}–π_{22} line or a π_{12}–π_{21} line, it is positive since it is a $\bar{\pi}_1$–$\bar{\pi}_2$ line and it also is negative since it is a π_1–π_2 line.

(*b*) Since all positive lines are $\bar{\pi}_1$–$\bar{\pi}_2$ lines, and since there are no π_{11}–π_{22} or π_{12}–π_{21} lines, every positive line must be a π_{11}–π_{21} line or a π_{12}–π_{22} line.

(*c*) Similarly all negative lines are π_{11}–π_{12} lines or π_{21}–π_{22} lines.

It is interesting to observe that there is a psychoanalytic school of thought which holds that the difference between loving and hating is less than that between loving and indifference. One possible support for this viewpoint is that both loving and hating are intense feelings while indifference involves no feeling at all.

DERIVED DUALITY LAWS

There are two additional duality laws derivable from the three previous kinds of duality. These are "uncontra-duality" which combines existential and directional duality, and "anticontra-duality" which is obtained from antithetical and directional duality. Before stating these explicitly, we show that existential and antithetical duality are not combinable. In view of this fact, it is clear that the two above-mentioned combined duality laws are the only ones derivable from the three basic kinds of duality treated in the preceding section.

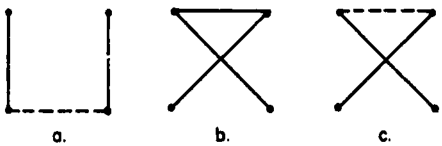

FIG. 9. A signed graph and two of its complements.

If existential and antithetical duality are combinable, then both the negation and the complement of a signed graph must be uniquely defined. The negation of a signed graph is completely determined since it is obtained from the given graph by changing the sign of every line. However, the complement of a signed graph is not uniquely determined since there are two possible choices for the sign of each line in it. This is illustrated in Figure 9, where both Figure 9b and Figure 9c (among others) are possible candidates for the complement of Figure 9a.

For "uncontra-duality," we recall the equation [10], $\mathfrak{D}'^{\smile} = \mathfrak{D}^{\smile\prime}$, which asserts that the unary operations of complement and converse of a digraph commute with each other. Both this and "anticontra-duality" are special cases of the:

Combined Duality Theorem. If S^* *and* $S^\#$ *are different unary operations on* S *which commute with each other:*

$$S^{*\#} = S^{\#*}, \qquad [12]$$

each of which induces a duality, then the product of these two operations, given by either member of equation [12] also induces a duality.

Proof. By hypothesis $S^{**} = S$, $S^{\#\#} = S$, and $S^{*\#} = S^{\#*}$. Therefore $(S^{*\#})^{*\#} = (S^{**})^{\#\#} = S^{\#\#} = S$. Further, if both the $*$-dual and the $\#$-dual of a statement concerning S are true, then so is the corresponding $*\#$-statement.

By this combined duality theorem, we know that "anticontra-duality" is a duality provided we know that negation and converse of a signed digraph commute with each other:

$$\mathfrak{D}^{-\smile} = \mathfrak{D}^{\smile -}. \qquad [13]$$

This follows immediately from the fact that both members of Equation 13 represent the signed digraph obtained from $\mathfrak{D}$ when both the direction and the sign of every line is changed.

Now that we have established that these two combinations of duality are themselves dualities, we state them formally.

IV. *Direxistential Duality Theorem. If one takes both the complement and the converse of each concept in a theorem about digraphs, the resulting statement is a theorem.*

We illustrate this kind of combined duality with the concept of a transmitter of a digraph. The *input* of a point P of a digraph is the number of lines to P; the *output* of P is the number of lines from P. A *transmitter* is a point of zero input; a *receiver* is a point of zero output. Thus a *contratransmitter* is a receiver. Hence P is an *uncontratransmitter* of $\mathfrak{D}$ whenever P is an unreceiver of $\mathfrak{D}$, that is, a receiver of $\mathfrak{D}'$. Such a point P is characterized by the property that the output of P is one less than the number of points of $\mathfrak{D}$.

V. *Directional-Antithetical Duality Theorem. If one takes both the converse and the negation of each concept in a theorem about signed digraphs, the resulting statement is a theorem.*

We can exemplify this last kind of duality for signed digraphs with the set of all points P such that there is a positive directed line from a given point A to P. If we denote this set by $\pi_+(A)$, then the dual set $\pi_-^{\smile}(A)$ is the set of all points Q such that there exists a negative line from Q to A.

MATRIX DUALITY

We have discussed five kinds of structural duality, namely: I. existential, II. directional, III. antithetical, IV. direxistential, and V. directional-antithetical. To each of these, there is a corresponding matrix duality. The matrix operations needed to describe these are as follows:

The matrix U is that n by n matrix in which every entry is a one.

The transpose, M^t, of the square matrix M is obtained from M by reflection about the main diagonal.

The negative, $-M$, of M is that matrix in which each entry is the negative of the corresponding entry of M.

For structural dualities I, II, and IV, we stipulate that the matrices have entries which are either 0 or 1. For the two remaining dualities, the entries may range over -1, 0, or 1. The five corresponding matrix dualities may then be described by the following one-to-one mappings (indicated by the double arrows):

$$\text{I. } M \leftrightarrow U - \mathrm{M}$$

$$\text{II. } M \leftrightarrow M^t$$

$$\text{III. } M \leftrightarrow -M$$

$$\text{IV. } M \leftrightarrow U - M^t$$

$$\text{V. } M \leftrightarrow -M^t$$

There is actually a correction factor which should be added to some of these if we exclude structures in which there is a line from a point to itself, but we omit this readily handled complication for simplicity.

DUOPROPERTIES

We have already encountered the concept of a duobalanced structure, namely one which is both balanced and antibalanced. Are there duoproperties for the other kinds of duality?

For existential duality, we consider the three concepts of connectedness, isomorphism, and liaison person. A *duoconnected* structure is one whose graph $\mathcal{G}$ is connected, and whose complementary graph $\mathcal{G}'$ is also connected. For example, in Figure 1 both graphs are connected but only the second one is duoconnected (see Figure 1 for the complementary graphs).

Two graphs are *isomorphic* if there is a one-to-one correspondence between their point sets which preserves adjacency. A group is *duostructured* if its graph $\mathcal{G}$ is isomorphic to its complement $\mathcal{G}'$. For example, the graph $\mathcal{G}_2$ of Figure 1 represents a duostructured group. Another example is given in Figure 10.

It can be readily be shown that there do

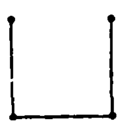

FIG. 10. A graph of a duostructured group.

not exist any duoliaison persons! This fact is contained in the:

Theorem. If A is an articulation point of $\mathcal{G}$, *then A is not an articulation point of* $\mathcal{G}'$.

We omit the proof, but remark that there do exist groups having both a liaison person and an unliaison person. The theorem does show that these two roles cannot be filled by the same person.

We mention briefly a trivial example of a duoproperty for directional duality. A point P is a duotransmitter (or a duoreceiver) if and only if P is both a transmitter and a receiver; hence duotransmitters coincide with isolated points of a digraph.

OTHER DUALITIES

There may exist additional kinds of structural duality of interest in the study of behavioral science. We sketch some of these briefly using the language of graph theory; the translation to the concepts of interpersonal and other relations offers no difficulty:

(*a*) A *sling* is a directed line from a point to itself. If some of the points of a thus generalized digraph $\mathfrak{D}$ have slings and the others do not, then the digraph obtained from $\mathfrak{D}$ by interchanging the presence and absence of slings provides the setting for a new kind of duality.

(*b*) If $\mathcal{S}$ is a signed digraph with slings (either positive or negative) permitted, then the operation of negating each sling is a duality.

(*c*) Related to this last idea is that of "signed points." If each point has a positive or negative "valence," then changing the sign of the valence of each point constitutes a dual process.

(*d*) If there are two different relations on the same set of points, i.e., if we have a *graph of type 2*, we can interchange these two relations to obtain a duality rule.

(*e*) These dualities are sometimes combinable to obtain new ones.

We note that there are other kinds of matrix duality, for example

(*a*) the inverse of a nonsingular matrix,

(*b*) the complex conjugate of a matrix whose entries are complex numbers, or

(*c*) the matrix obtained from a given matrix M all of whose entries are nonzero, on taking the reciprocal of each entry, etc. But none of these appears behaviorally applicable.

In matrix theory, if M is a matrix whose elements are complex numbers, then M is a symmetric matrix if $M = M^t$, M is "skew-symmetric" if $M = -M^t$, M is "Hermitian" if it is equal to the transpose of its complex conjugate. These are all examples of "self-duality." We have noted previously that $M = M^t$ and $M = -M^t$ correspond respectively to directional self-duality and, directional-antithetical self-duality.

An example of structural self-duality has been given by duostructured groups; these are self-dual existentially. An ordinary graph (i.e., one with no directed lines) is self-dual directionally. This leads us to the expected generalization that a structure is *self-dual* with regard to a given duality if it is its own dual. For example, any signed graph consisting of exactly one alternating cycle, as in Figure 11, is self-dual antithetically.

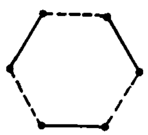

FIG. 11. A signed graph that is self-dual antithetically.

We mention as a question for possible future investigation the possibility of the existence of interesting kinds of structural triality, etc. Two kinds of structural triality which suggest themselves are:

(*a*) Given three binary relations (R_1, R_2, and R_3 on the same set of points, the process of transforming the three different types of lines cyclically: $R_1 \to R_2 \to R_3 \to R_1$ provides an example of structural triality.

(*b*) To obtain an analogy with existential duality, one can consider a kind of graph [of strength two; see (13)] in which any pair of points may be joined by either no lines, one line, or two lines. Then the procedure of adding one additional line joining each pair of points, with the stipulation that three lines are regarded as no lines, is a structural triality which can be properly called "existential triality."

Whitney (30) defined the dual of a planar map, e.g., the map of continental Europe, by essentially the following construction. The dual graph of the map of Europe has a point standing for each country, and two points are adjacent in this graph whenever the corresponding countries have a common boundary in the map. This kind of duality, which is of considerable importance in electrical network theory, was rediscovered by Bavelas (2) in his formalization of parts of Lewin's work (19).

We conclude by mentioning two more kinds of physical duality, only the first of which has been found empirically thus far. *Magnetic duality* is illustrated by the discovery of the "antineutron" (23), which has the same mass as the neutron, but opposite magnetic orientation. *Gravitational duality* has been hypothesized by some astrophysicists. This theory proposed the existence of antimatter which attracts other antimatter gravitationally, but repels ordinary matter.

We mention *en passant* part of a delightful letter from a well-known physicist to a well-known magazine (26). We note that the prefix "anti" in this quote refers always to physical charge duality:

> In a recent issue of *The New Yorker*, I found the following poem, describing the meeting of Dr. Edward Anti-Teller with an imagined person differing from Anti-Teller only in the sign of the charges carried by the particles in his body.
>
> Perils of Modern Living
>
> Well up beyond the tropostrata
> There is a region stark and stellar
> Where, on a streak of anti-matter,
> Lived Dr. Edward Anti-Teller.
>
> Remote from Fusion's origin,
> He lived unguessed and unawares
> With all his anti-kith and kin,
> And kept macassars on his chairs.
>
> One morning, idling by the sea,
> He spied a tin of monstrous girth
> That bore three letters: A. E. C.
> Out stepped a visitor from Earth.
>
> Then, shouting gladly o'er the sands,
> Met two who in their alien ways
> Were like as lentils. Their right hands
> Clasped, and the rest was gamma rays.
>
> —H. P. F.
>
> I was pleased that *The New Yorker* mentioned me. Come to think of it, only Anti-Teller was mentioned by name in the poem, but I am con-

fident that somewhere in an anti-galaxy *The Anti-New Yorker* devoted some pleasant lines to

Yours sincerely,
EDWARD TELLER

Finally, we mention a very recent development. A critical experiment appears to imply the necessity of abandoning the physical directional and charge duality laws (21)!

The two concepts definitely believed destroyed are the principle of parity and an allied principle called charge conjugation. This indicates that a particle and its corresponding antiparticle will decay in an identical manner (27).

Regardless of the potentially great effect of these results on physical theory, the derivation of measurement techniques for the behavioral sciences which are sufficiently reliable to perform analogous critical experiments testing the validity of structural dualities seems to be in the distant future.

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(Manuscript received January 30, 1957.)