Problem set 3 (May 8, 2025)

Theory Exercise

Question 4

$$y_t = B_0^{-1}\varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, D_t)$$

where $D_t = diag(e^{h_{1,t}}, \dots, e^{h_{n,t}})$ and $h_t = (h_{1,t}, \dots, h_{n,t})$ with

$$h_{i,t} = \phi_i h_{i,t-1} + \eta_{i,t}, \quad \eta_{i,t} \sim \mathcal{N}(0, \omega^2) \quad \forall i \in \{1, \dots, n\}$$

(1) Here we assume that $D_t = I_n$ and B_0 is lower traingular with ones on the diagonal. Since we have VAR with no lags the reduced form errors are simply given by

$$u_t = y_t = B_0^{-1}\varepsilon_t \sim \mathcal{N}\left(0, B_0^{-1}\left(B_0^{-1}\right)'\right)$$

Let's denote $M \equiv B_0^{-1}$, where M is also a lower triangular matrix and notice that

$$u_{i,t} = \sum_{k=1}^{n} M_{ik}\varepsilon_{k,t}$$

Using $\varepsilon_{k,t} \overset{iid}{\sim} \mathcal{N}(0,1)$ and the fact that all the elements M_{ij} with $j > i$ are zeros we get:

$$Var(u_{i,t}) = \sum_{k=1}^{n} M_{ik}^2 = \sum_{k=1}^{i} M_{ik}^2$$

Note that the above variance is constant in time (due to the assumption on D_t that we made). It's immediately clear that increasing i by 1 unit adds another quadratic and hence non negative term into this sum. Hence by induction:

$$Var(u_{i,t}) \leq Var(u_{j,t}) \quad \forall i < j$$

(2) Now, getting back to original formulation of the model we get:

$$y_t \sim \mathcal{N}(0, \Sigma_t) \quad \Sigma_t \equiv M D_t M'$$

The density function will then look like:

$$f(y_t) = (2\pi)^{-n/2}\,|\Sigma_t|^{-1/2}\,e^{\frac{1}{2}y_t'\Sigma_t^{-1}y_t}$$

The joint likelihood function is the given by

$$L(y|B_0, h) = \prod_{t=1}^{T} f(y_t)$$

Let's now consider a permutation matrix P. The variance of y_t will then be given by:

$$\tilde{\Sigma} = \tilde{B}_0^{-1}\tilde{D}_t(\tilde{B}_0^{-1})'$$

$$\tilde{B}_0^{-1} = (PB_0P')^{-1} = PB_0^{-1}P', \quad \tilde{D}_t = PD_tP'$$

hence for the variance we obtain:

$$\tilde{\Sigma} = PB_0^{-1}P'PD_tP'P(B_0^{-1})'P' = PB_0^{-1}D_t(B_0^{-1})'P' = P\Sigma_tP'$$

Then density function for the permuted data will be:

$$f(\tilde{y}_t) = (2\pi)^{-n/2}\ |\tilde{\Sigma}_t|^{-1/2}\ e^{\frac{1}{2}\tilde{y}_t'\tilde{\Sigma}_t^{-1}\tilde{y}_t}$$

$$L(\tilde{y}|\tilde{B}_0, \tilde{h}) = \prod_{t=1}^{T} f(\tilde{y}_t)$$

Let's now show that this boils down to the same expression as in (1). Using the fact that matrices P, Σ, P' are square and conformable (their dimensions are $n \times n$ which allows for multiplication) and by the properties of permutation matrix P:

$$|\tilde{\Sigma}_t| = |P\Sigma_tP'| = |P||\Sigma_t||P'| = |\Sigma_t|$$

$$\tilde{y}_t'\tilde{\Sigma}_t^{-1}\tilde{y}_t = y_t'P'(P\Sigma_tP')^{-1}Py_t = y_t'P'P\Sigma_t^{-1}P'Py_t = y_t'\Sigma_t^{-1}y_t$$

Hence $f(y_t) = f(\tilde{y}_t)$ and as a result:

$$L(\tilde{y}|\tilde{B}_0, \tilde{h}) = \prod_{t=1}^{T} f(\tilde{y}_t) = \prod_{t=1}^{T} f(y_t) = L(y|B_0, h)$$

Lab Exercises

Question 5: Time-varying parameter model

The Taylor rule is a policy-rule framework that prescribes how a central bank should set its short-term interest rate in response to deviations of inflation and economic activity from their targets. The usual form is:

$$r_t = r^* + \pi_t + \tfrac{1}{2}(\pi_t - \pi^*) + \tfrac{1}{2}(y_t - y^*),$$

where r_t is the nominal policy rate, π_t is inflation, y_t is real output (or its gap), and $(\,\cdot\,)^*$ denotes a target or equilibrium value. Intuitively, it embeds two principles:

(a) Inflation responsiveness: Raises the nominal rate by more than one-to-one with inflation surprises to stabilize real rates (the Taylor principle).

(b) Output stabilization: Lowers (raises) rates when growth is below (above) potential, dampening business-cycle swings.

It is usually employed as a benchmark or assumption about how policymakers might trade off inflation control and output stabilization.
In our case, we rewrite the Taylor rule specification as:

$$R_t = \beta_{0,t} \ + \ \beta_{R,t}\, R_{t-1} \ + \ \beta_{y,t}\, y_t \ + \ \beta_{\pi,t}\, \pi_t \ + \ \varepsilon_t$$

such that, with $\beta_t = (\beta_{0,t}, \beta_{R,t}, \beta_{y,t}, \beta_{\pi,t})'$, we can rewrite as:

$$\beta_t = \beta_{t-1} + u_t, \qquad u_t \sim \mathcal{N}(0, \Sigma_u),$$

In time-varying parameter models, this specification has multiple advantages:

(a) Time-Varying Coefficients: In a TVP framework, we allow the weights on inflation and output $(\beta_{\pi,t}, \beta_{y,t})$ to evolve, capturing changing policy priorities or regimes.

(b) Identification: The Taylor rule provides a structural policy equation within a state-space model, letting us identify shifts in the policy reaction function over time

(c) Policy Analysis: By estimating β_t dynamically, we can assess how aggressively the central bank responded to shocks in different eras (e.g. pre-2008 vs. post-2008)

0.0.1 Part 1.

Below is the **state–space representation** of the time-varying Taylor rule.
1. **Observation equation**

$$\underbrace{R_t}_{y_t} = \underbrace{\begin{bmatrix} 1 & R_{t-1} & y_t & \pi_t \end{bmatrix}}_{Z_t} \underbrace{\begin{pmatrix} \beta_{0,t} \\ \beta_{R,t} \\ \beta_{y,t} \\ \beta_{\pi,t} \end{pmatrix}}_{\alpha_t} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2).$$

2. **State (transition) equation**: This is the equation describing the underlying process of the unobserved variables

$$\alpha_t = \alpha_{t-1} + u_t, \quad u_t \sim N\big(0, \Sigma_u\big), \quad \Sigma_u = \mathrm{diag}(\sigma_0^2, \sigma_R^2, \sigma_y^2, \sigma_\pi^2).$$

Part 2.-3.

We estimated the model by Kalman filter and we obtained the following point estimates of the filtered time-varying coefficients:

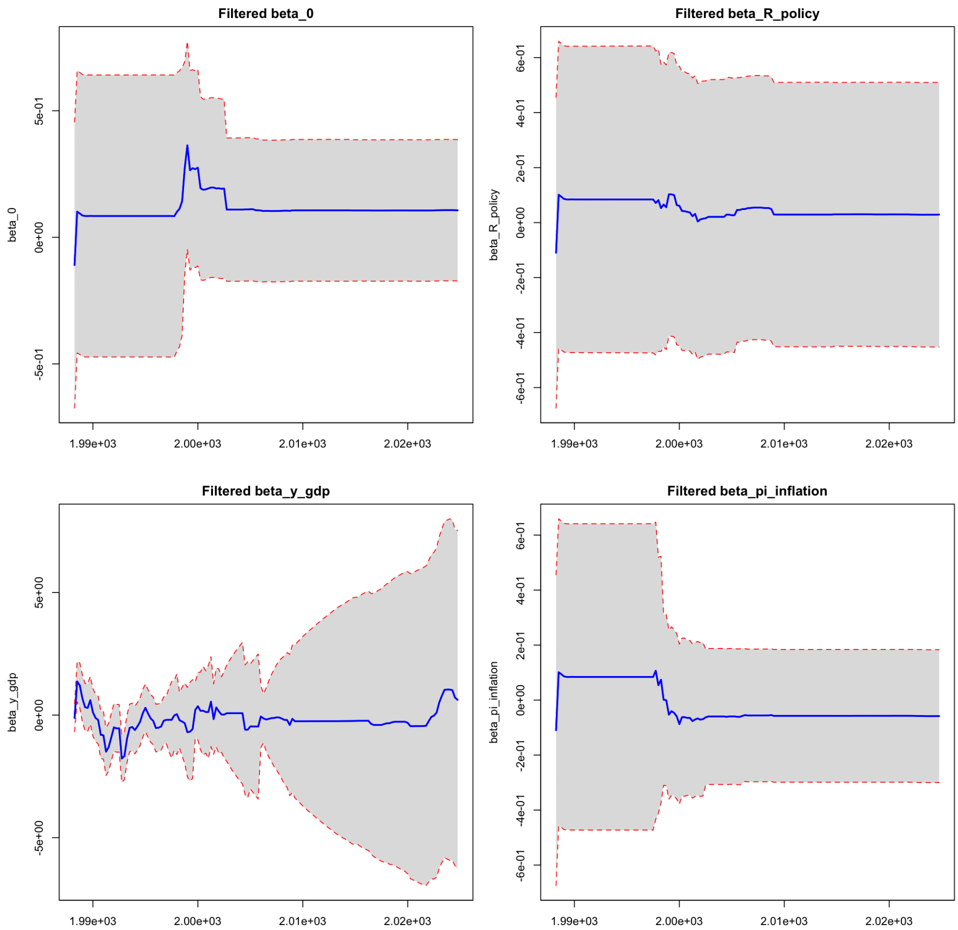

Figure 1: Filtered time-varying coefficients of the state variables

In the GDP (aggregate economic activity) we can see that the coefficients are significantly varying over time. On the other hand, the coefficients of the other model parameters seem to vary less, especially when compared to the confidence interval, but they all seem to show a structural break around the 1997 and beginning of 1998. Infact, while the UK did not join the euro, two significant economic and policy events occurred around 1997–1999 that could plausibly cause structural breaks or increased volatility in macroeconomic relationships:

(a) **Operational Independence of the Bank of England**: In May 1997, the Bank of England was granted operational independence to set interest rates. This was a major policy shift affecting monetary transmission and expectations [13].

(b) **Tightening and Subsequent Easing of Policy**: The Bank of England raised interest rates by 150 basis points between May 1997 and June 1998 to counter overheating, then cut rates by 200 basis points between October 1998 and February 1999 as the economy slowed.

On the other hand, in the smoothed time-varying coefficients, only the GDP coefficient shows a significant time variation:

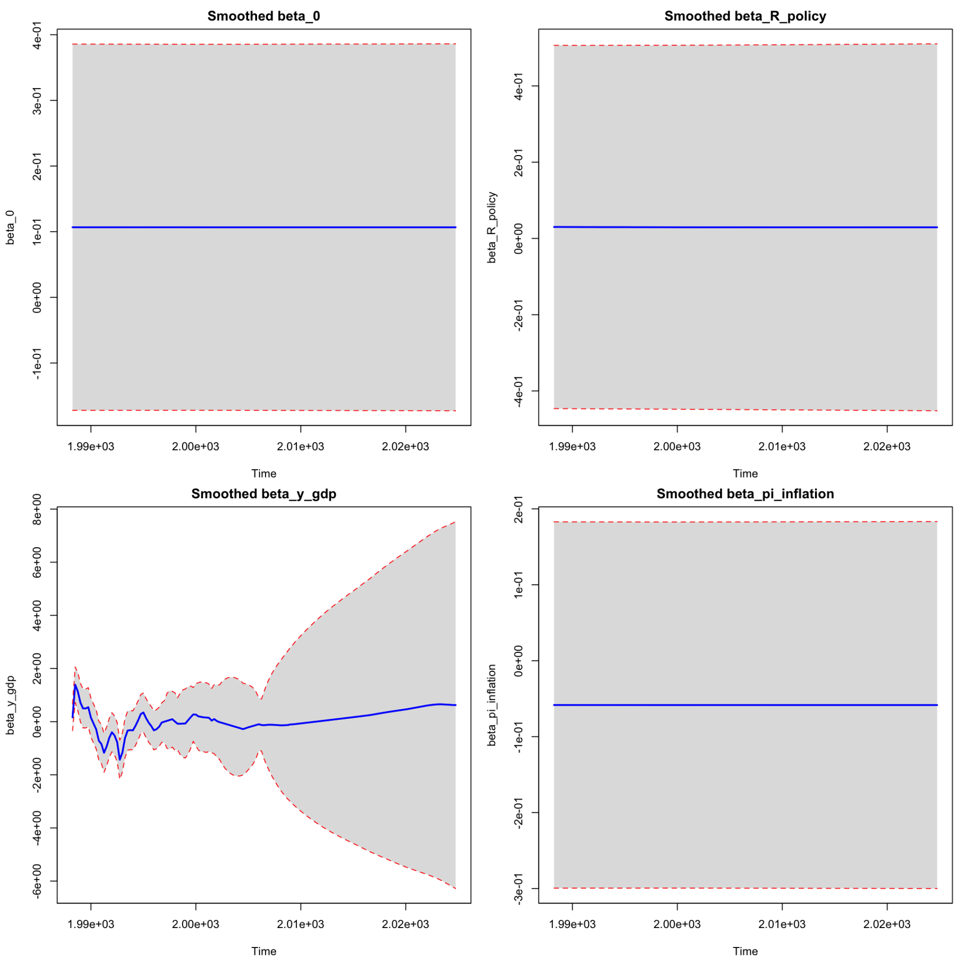

Figure 2: Smoothed time-varying coefficients of the state variables

Part 4.

If we consider the Taylor rule with constant coefficients, we have:

$$R_t = \beta_0 \; + \; \beta_R \, R_{t-1} \; + \; \beta_y \, y_t \; + \; \beta_\pi \, \pi_t \; + \; \varepsilon_t$$

From this we obtain the following coefficients and std. errors:

Parameter	Estimate	Std_Error	p_value	t_value
(Intercept)	-6.88E-02	5.62E-02	2.23E-01	-1.22E+00
R_lag	5.35E-01	6.86E-02	1.21E-12	7.80E+00
y	5.69E+00	2.04E+00	6.01E-03	2.79E+00
pi	-6.75E-01	5.50E+00	9.02E-01	-1.23E-01

Table 1: Constant-coefficient Taylor Rule Model Coefficients

These seem quite different from the coefficients for the Kalman filter where R_lag had a coefficient around 0.1. Also all the other coefficients are smaller in the Kalman filter graphs. On the other hand, this is plausible because the Kalman filter in a TVP model implicitly shrinks coefficients toward their prior (or previous) values when the process-noise variances are small, yielding much flatter, more stable parameter paths—even if the OLS average is larger. In contrast, OLS with no shrinkage fits a single coefficient that best explains all historical variation, typically producing larger estimates when the true parameter is stable on average [8].
From an economic point of view we can draw two important conclusions.

Stability of UK Policy Rule

Empirical studies of UK monetary policy have consistently found **little time variation** in the core reaction-function parameters, particularly the inflation response and interest-rate persistence. For example, Taylor's (2000) guide using Taylor rules for 1972–1997 shows that after the formal adoption of inflation targeting in 1997, the inflation-weight remained essentially unchanged [9]. Our **smoothed** TVP estimates—leveraging full-sample information—confirm that both the intercept and the inflation coefficient β_π have stayed virtually flat through the late 1990s and early 2000s, with only a modest structural shift evident around 1998 when the Bank of England was granted operational independence to set interest rates [10]. In contrast, the output-growth weight β_y displays mild drift, possibly reflecting an increased focus on demand stabilization.

Constant vs. Time-Varying Specification

A constant-coefficient Taylor rule estimated by OLS is both more parsimonious and—according to Gerlach Mizen (2004)—almost as accurate in forecasting UK policy rate moves as more elaborate TVP models [11]. Our OLS fit yields $\beta_R \approx 0.54$ on the lagged rate and a significant output coefficient, capturing the average policy response across the sample. By contrast, the filtered TVP estimates are significantly shrunk toward initial priors (e.g. $\beta_{R,t} \approx 0.1$) because the estimated process-noise variances are small; smoothed coefficients remain tightly clustered around their means. This suggests that, outside of the modest 1998 break, the Bank of England's reaction function has been stable enough that a constant-parameter model suffices for both policy analysis and forecasting.

0.0.2 Conclusion

1. **Policy Credibility:** The flat inflation-response coefficient underscores the credibility of the Bank's 2% target, anchoring public expectations and reducing long-run inflation volatility [12]. 2. **Demand Stabilization:** The slight downward drift in β_π and in the intercept post-1998 indicates a marginally stronger emphasis on output stabilization, consistent with the MPC's dual-mandate remit during the transition to formal independence [10, 13]. 3. **Modeling Choice:** Given the high stability of core parameters—with only one economically meaningful regime shift in 1998—a constant-coefficient Taylor rule provides a robust and transparent benchmark for counterfactual policy simulations and stress testing [11].

APPENDIX: Checks for Weak Stationarity and Structural Breaks analysis

After the corrections in Problem Set 1, we went back and looked more into our data and possible violation of weak stationarity, and possible solutions to them.

1. Motivation and Approach

Before estimating our trivariate VAR, we investigated whether the *raw level* policy-rate series `policy_non_stationary_ts` is already weakly stationary, which would avoid the need for differencing. For this reason, we first look for structural breaks, and then we evaluate how significant they are and if there are methods to remove them or to control for them. Once identified, we used two complementary strategies to handle the breaks and check if the weak stationarity assumption could still hold:

(a) **Subsample stationarity tests:** isolate the interval between the two largest structural breaks (1993 Q2–2008 Q3) and apply ADF and KPSS tests, to see if stationarity holds in a "stable" regime.

(b) **Full-sample ARIMA with intervention dummies:** include step dummies for 1993 Q2 and 2008 Q3 in an ARIMA regression, then test residual stationarity.

Both strategies combine unit-root tests (ADF, KPSS), structural-break detection (Bai–Perron, Zivot–Andrews, Chow), and intervention-analysis via ARIMA errors.

2. Structural Break Detection

2.1 Inflation Series (`inflation_ts`)

- **Bai–Perron multiple break test** `breakpoints(inflation_ts 1,h=0.15)` estimates breaks at 1993.5 and 2019.0 (i.e. 1993 Q2, 2019 Q1) [1].
- **Zivot–Andrews unit-root with break** Test statistic $\tau = -4.5508$ vs. CVs $-5.57, -5.08, -4.82$ at 1%/5%/10% [2]. Detected break at 1991.5 at >10% significance level (1991 Q2).

2.2 Policy-Rate Series (`policy_non_stationary_ts`)

- **Bai–Perron multiple break test** Unrestricted breaks at 1993.5, 2001.25, 2008.75, 2019.25; with the two most significant breaks at: 1993.5 and 2008.75 (1993 Q2, 2008 Q3) [1].
- **Variance breaks** On squared series, we find breaks in the second moment at 1993.5 and 2008.5.
- **Zivot–Andrews unit-root with break** Test statistic $\tau = -4.1499$ vs. CVs $-5.57, -5.08, -4.82$, break at 2008.5 (2008 Q3) [2].

2.3 Historical events related to the detected breaks

Based on Bai–Perron and Zivot–Andrews procedures, we identified two principal level shifts in the policy rate, which may infact be related to possible shocks in the policy rate and inflation time series:

- **1993 Q2 (Erm Exit & Inflation Targeting):** After "Black Wednesday" (Sept 1992), the Bank of England abandoned the ERM, hiked rates to defend sterling, and in 1997 adopted formal 2% CPI inflation targeting under the MPC framework .
- **2008 Q3 (Global Financial Crisis):** The MPC slashed Bank Rate from 5.75% to 0.5% by March 2009 and implemented quantitative easing to stabilize financial markets.

3. Stationarity Tests

After identifying the 2 structural breaks, we then employed two methods to test if we could make the time serie weakly stationary:

3.1 Subsample (1993 Q3–2008 Q4)

First, we extracted the window from 1993 Q2 through 2008 Q3 and conducted the following tests: The ADF fails to reject nonstationarity (p=0.2007), and the KPSS rejects

Test	Statistic	p-value
ADF (unit root)	−2.9247	0.2007
KPSS (level station.)	0.8039	0.01

Table 2: Stationarity Tests on Policy Rate Subsample

level stationarity (p=0.01), indicating a stochastic trend remains and the process is still not weakly stationary [3, 4].
We also evaluated the Covid shock to test if it could be a significant structural break, even though it was not detected as one by the previous test (as described in Subsection 2):

(a) **Chow test at 2020 Q2** `sctest(inflation_ts 1,type="Chow",point=2020.25)` yields $F = 10.003$, p=0.0019, rejecting no-break [5]

So the Chow test shows that the Covid shock could infact be a significant structural break in inflation timeserie.

3.2 Full-Sample ARIMA with Break Dummies

Secondly, we fitted two ARIMA-type models on the entire level series, using `auto.arima()` from the `forecast` package:

- **Benchmark:** ARIMA(1,1,1) (i.e. first-differenced ARMA errors).
- **Dummy-augmented:** Regression–ARIMA with ARIMA(2,0,2) errors and two step dummies for 1993 Q2 and 2008 Q3.

Model	(p, d, q)	AIC	Dummy Coeffs	ADF(Res)	LB p
No dummies	(1,1,1)	154.41	–	<0.01	0.7361
With 2 dummies	(2,0,2)	152.56	0.7531 (0.3509), 1.3118 (0.3559)	<0.01	0.3890

Table 3: ARIMA Intervention Models for Policy Rate

Including break dummies lowers AIC (152.56 vs. 154.41), even without differencing, and yields statistically significant dummy coefficients, while both models produce stationary residuals (ADF $p < 0.01$) and no residual autocorrelation (Ljung–Box $p > 0.05$). This demonstrates that explicit modeling of 1993 Q2 and 2008 Q3 shifts can render the level series stationary without differencing [6, 7].

4. Conclusion

Together, the previous results paint a contradictory picture: explicit dummies can render the level series stationary, but subsample tests still indicate a unit root. The subsample and KPSS results both indicate persistent stochastic trends, while the ARIMA+dummy approach effectively controls for the largest regime shifts but lies beyond the current VAR exercise's scope. Therefore, **we adopt the first-difference of the policy rate** ($d = 1$) in our VAR, ensuring weak stationarity. This choice reflects economic intuition that central banks typically adjust rates by increments (i.e. changes), rather than targeting an ex ante level, and simplifies the model without sacrificing statistical validity.

References

[1] Bai, J., & Perron, P. (1998). Estimating and testing linear models with multiple structural changes. *Econometrica*, 66(1), 47–78.

[2] Zivot, E., & Andrews, D. (1992). Further evidence on the great crash, the oil-price shock, and the unit-root hypothesis. *Journal of Business & Economic Statistics*, 10(3), 251–270.

[3] Dickey, D. A., & Fuller, W. A. (1979). Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association*, 74, 427–431.

[4] Kwiatkowski, D., Phillips, P. C. B., Schmidt, P., & Shin, Y. (1992). Testing the null hypothesis of stationarity against the alternative of a unit root. *Journal of Econometrics*, 54(1–3), 159–178.

[5] Zeileis, A., Leisch, F., & Hornik, K. (2005). Monitoring structural change in dynamic econometric models. *Journal of Applied Econometrics*, 20(1), 99–121.

[6] Box, G. E. P., & Tiao, G. C. (1975). Intervention analysis with applications to economic and environmental problems. *Journal of the American Statistical Association*, 70(349), 70–79.

[7] Hyndman, R. J., & Athanasopoulos, G. (2024). *Forecasting: Principles and Practice* (3rd ed.). OTexts.

[8] D. Sirikanchanarak, W. Yamaka, C. Khiewgamdee, and S. Sriboonchitta, *Time-varying threshold regression model using the Kalman filter method*, Vol. 14, pp. 133–148, Jan. 2016.

[9] J.B.Taylor, *UK Monetary Policy 1972–97: A Guide Using Taylor Rules,* Stanford University Working Paper, July 2000.

[10] C.Bean, *Prospects for Inflation and Monetary Policy,* Speech, Bank of England, February 2007.

[11] S.Gerlach and P.Mizen, *Predicting Directional Changes in the UK Interest Rate,* Proceedings, Asia-Pacific Applied Economics Association Conference, 2004.

[12] R.Clarida, J.Galí, and M.Gertler, *Monetary Policy Rules and Macroeconomic Stability, Quarterly Journal of Economics*, vol.115, no.1, pp.147–180, 2000.

[13] Gordon Brown, *Letter from the Chancellor to the Governor of the Bank of England*, 6 May 1997, Bank of England. https://www.bankofengland.co.uk/-/media/boe/files/letter/1997/chancellor-letter-060597