Problem set 2 (April 27, 2025)

Lab Exercises

Question 2: Rolling Forecast Evaluation (2-step-ahead forecasts)

Our findings are summed up in the table below:

Method	RMSE	p_value (vs. RW)
RW	0.00657	
AR_Iter	0.00603	0.26808
AR_Direct	0.01103	

Table 1: Root Mean Square Forecast Error (RMSE) and DM test p-value for each forecasting method

Therefore, we find that AR_direct significantly underperforms Random Walk model due to his completely different DGP that is used:
To sum up, the two forecasts will differ as: - Iterated forecasts: We use a one-step-ahead model repeatedly: you forecast y_{T+1}, then plug that forecast back in to predict y_{T+2}, and so on. - Direct forecasts: estimate a separate model for each horizon h; e.g. for $h = 2$ you regress y_{t+2} directly on past lags of y_t. The key trade-off is that Iterated forecasts are asymptotically efficient if the one-step model is correctly specified, but they can accumulate error when you roll forward. Direct forecasts avoid error propagation and can be more robust to misspecification at the cost of estimating many separate models.
Particularly, for the Iterated (Recursive) Forecasts, we have: 1. Fit a one-step-ahead model: (e.g. AR(p)) on your training data.
2. Forecast: $\hat{y}_{T+1}$.
3. Append: $\hat{y}_{T+1}$ to the sample and forecast $\hat{y}_{T+2}$ using the same AR(p) coefficients.
4. Repeat until you reach your desired horizon.

Defined more formally, if our AR(p) model is $y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$,
then
$\hat{y}_{T+2|T} = \phi_1 \hat{y}_{T+1|T} + \phi_2 y_T + \cdots + \phi_p y_{T+2-p}$.
On the other hand, the Direct Forecasts follows: 1. For each horizon h, re-estimate a model of the form :

$$y_{t+h} = \alpha_0^{(h)} + \alpha_1^{(h)} y_t + \alpha_2^{(h)} y_{t-1} + \cdots + \alpha_p^{(h)} y_{t+1-p} + u_{t+h}^{(h)}.$$

2. Predict: $\hat{y}_{T+h}$ directly from that horizon-h regression.
So the big advantage is that there is no error accumulation between steps, but on the

other hand, you must estimate a new model for each horizon (e.g. $h = 2$, $h = 3$, ...), so each horizon-specific regression uses fewer effective observations (reducing precision).
Under an infinite information set and a correctly specified linear model, iterated and direct forecasts are theoretically identical.
In our case AR_iterated marginally improves RW model thanks to the richer dynamic structure but p-value=0.268 so the difference is not statistically significant.
On the other hand, empirical studies—both for advanced and emerging economies—find that inflation follows an integrated process with high persistence, making a random-walk model (ARIMA(0,1,0)) a surprisingly strong benchmark (see "As good as a random walk: Inflation forecasting in emerging market" (https://cepr.org/voxeu/columns/good-random-walk-inflation-forecasting-emerging-market-economies) and "Why Has U.S. Inflation Become Harder to Forecast?"(https://www.princeton.edu/ mwatson/papers/Stock_Watson_JMCB_2007.pdf)). The two studies show that when variables exhibit near-unit-root behavior, even well-specified AR(p) models struggle to add forecast value beyond simply carrying forward the last observed value.

Question 3: Nonstationary time series model and cointegration

For some reasons, both pp.test and adf.test raise a warning saying "Warning message: In pp.test(policy_ts) : p-value smaller than printed p-value" so we don't manage to get the proper p-value, but we know that it is lower than p<0.01 in all cases, confirming that the original time series are I(1), so that the differentiated ones are weakly stationary. So we get the following table: Thanks to this property, we can get VAR consistent estimator,

Series	ADF_Level	ADF_Diff	PP_Diff
GDP	0.990	<0.010	<0.010
Inflation	0.990	<0.010	<0.010
Policy Rate	<0.010	<0.010	<0.010

Table 2: Unit Root Test P-Values for Time Series (levels and first differences)

which selects 5 as lag order based on AIC criterion.
In the following table I also report the Johansen Trace Test Statistics and Critical Values:

	10pct	5pct	1pct
r <= 2 |	7.52000	9.24000	12.97000
r <= 1 |	17.85000	19.96000	24.60000
r = 0 |	32.00000	34.91000	41.07000

Table 3: Critical Values for Johansen Trace Test

From the table above we see that for $r \leq 2$: $12.08 > 9.24$ (reject at 5%) but $12.08 < 12.97$ (fail to reject at 1%). Therefore, at the 1% level do not reject, and we can conclude

$r = 2$ at 1% significance.

Based on the previous results, we estimate the VECM parameters, meanly we extract β (cointegration vector) and α (adjustment coefficients) and we get the following tables:

	GDP.l5	INF.l5
GDP.l5	1.00000	1.00000
INF.l5	-0.81581	1.07109
RATE.l5	-0.00831	0.08863
constant	-0.00596	-0.01199

Table 4: Normalized Cointegrating Vectors

	GDP.l5	INF.l5
GDP.d	-1.20008	-0.10782
INF.d	0.12902	-0.01132
RATE.d	9.13275	-3.62844

Table 5: Adjustment Coefficients

Finally we show the estimates of the OLS regression coefficients for each of the VECM equations:

	GDP.d	INF.d	RATE.d
ect1	-1.30791	0.11771	5.50431
ect2	0.86355	-0.11738	-11.33698
GDP.dl1	-1.24705	0.02738	-0.14179
INF.dl1	0.35431	-0.52036	8.69085
RATE.dl1	0.00272	-0.00238	-0.19261
GDP.dl2	-1.29629	0.06230	1.10433
INF.dl2	0.73815	-0.34464	-10.54692
RATE.dl2	0.00123	0.00077	-0.43669
GDP.dl3	-1.19645	0.11361	1.83279
INF.dl3	0.70647	-0.46123	-6.90044
RATE.dl3	0.00524	-0.00207	-0.35647
GDP.dl4	-1.24656	0.10045	3.45810
INF.dl4	0.51052	0.28844	-8.04665
RATE.dl4	0.00129	-0.00096	-0.46192

Table 6: Regression Coefficients for VECM Equations

Question 4: Spectral Analysis

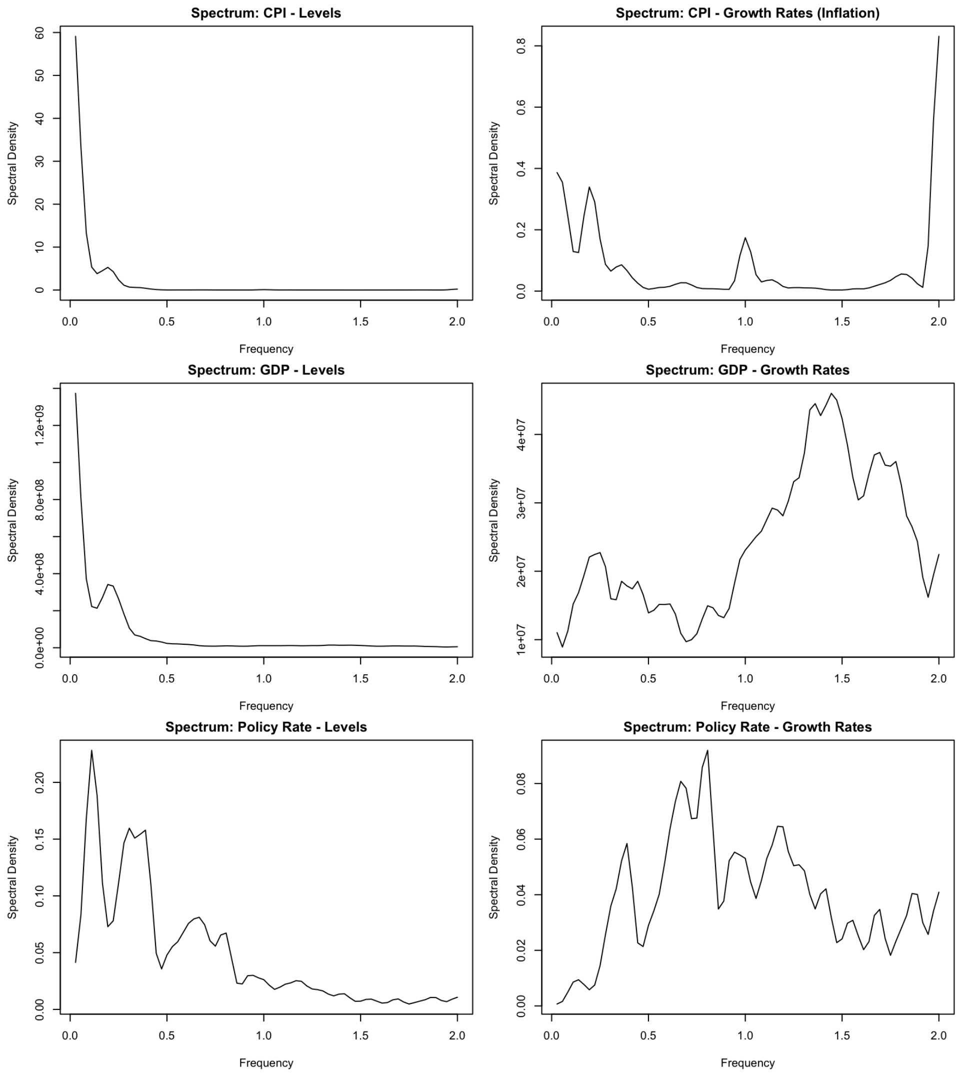

Figure 1: Spectral Analysis of UK Economic Indicators

The figure 1 shows the smoothed periodograms for the UK CPI, GDP, and policy rate series in both levels (left panels) and growth-rate form (right panels). In levels, each series is dominated by very large spectral density at near-zero frequencies—confirming the strong trending, nonstationary behavior of CPI, GDP, and interest rates. After converting to growth rates (log-differences for CPI and GDP), the low-frequency peak disappears and the spectra flatten out, revealing modest cyclical power around business-cycle frequencies (roughly 0.2–0.4 cycles per quarter). The stationarity of the logged growth series is evident in these more uniform periodograms, which justifies our decision to apply log-differencing before VAR estimation. Consequently, using log (CPI) and log (GDP) growth rates both removes spurious trend effects and isolates the genuine cyclical dynamics we wish to analyze.

Question 4.

(a) We have

$$s_t = \alpha + s_{t-1} + e_t, \quad \mathcal{F}_t = \sigma(s_j : j \le t).$$

In a random walk with drift we assume:

$$\mathbb{E}[e_{t+1}|\mathcal{F}_t] = 0$$

Then we get:

$$\mathbb{E}[s_{t+1} \mid \mathcal{F}_t] = \mathbb{E}[\alpha + s_t + e_{t+1} \mid \mathcal{F}_t] = \alpha + s_t + \underbrace{\mathbb{E}[e_{t+1} \mid \mathcal{F}_t]}_{=0} = \alpha + s_t,$$

Thus we obtain: $\mathbb{E}[s_{t+1} \mid \mathcal{F}_t] - s_t = \alpha \neq 0$.
By assumption $\{e_t\}$ satisfies $\mathbb{E}[e_t|\mathcal{F}_{t-1}] = 0$, hence it is a martingale-difference sequence.

(b) Under the DGP implied by the model, $\Delta s_t := s_t - s_{t-1} = \alpha + e_t$, with $e_t \overset{\text{iid}}{\sim} N(0, \sigma_e^2)$ and $s_0 = 0$.

One possible test can be constructed as the following regression:

$$\Delta s_t = \alpha + e_t, \quad t = 1, \ldots, T,$$

Here $H_0 \colon \alpha = 0$. The OLS estimator is then given by:

$$\hat{\alpha} = \frac{1}{T} \sum_{t=1}^{T} \Delta s_t,$$

By the CLT:

$$\sqrt{T}\,(\hat{\alpha} - \alpha) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_t \xrightarrow{d} N(0, \sigma_e^2).$$

Hence under H_0 the expression above becomes:

$$\sqrt{T}\,\hat{\alpha} \xrightarrow{d} N(0, \sigma_e^2)$$

Then the t-statistic is given by:

$$t_{\hat{\alpha}} = \frac{\hat{\alpha}}{\hat{\sigma}_e/\sqrt{T}} \xrightarrow{d} N(0,1).$$